Unit 4mathsmtech - Graph Theory

Mathematics (Indiana Institute of Technology)

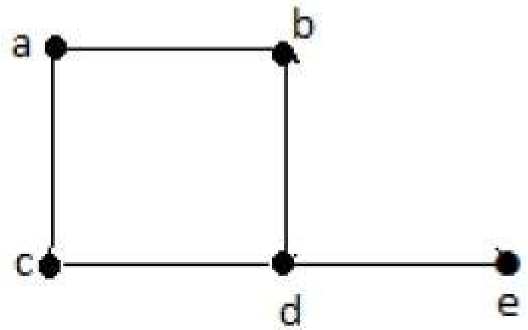
Unit-4 (MFCS)

Graph Theory:

What is a Graph?

A graph is a pictorial representation of a set of objects where some pairs of objects are connected by links. The interconnected objects are represented by points termed as **vertices,** and the links that connect the vertices are called **edges**.

Formally, a graph is a pair of sets **(V, E),** where **V** is the set of vertices and **E** is the set of edges, connecting the pairs of vertices. Take a look at the following graph –



In the above graph,

V = {a, b, c, d, e}

E = {ab, ac, bd, cd, de}

Applications of Graph Theory

Graph theory has its applications in diverse fields of engineering −

- **Electrical Engineering** − The concepts of graph theory is used extensively in designing circuit connections. The types or organization of connections are named as topologies. Some examples for topologies are star, bridge, series, and parallel topologies.
- **Computer Science** − Graph theory is used for the study of algorithms. For example,
    - Kruskal's Algorithm
    - Prim's Algorithm
    - Dijkstra's Algorithm
- **Computer Network** − The relationships among interconnected computers in the network follows the principles of graph theory.
- **Science** − The molecular structure and chemical structure of a substance, the DNA structure of an organism, etc., are represented by graphs.

- **Linguistics** − The parsing tree of a language and grammar of a language uses graphs.
- **General** – Routes between the cities can be represented using graphs. Depicting hierarchical ordered information such as family tree can be used as a special type of graph called tree.

Isomorphism:

A graph can exist in different forms having the same number of vertices, edges, and also the same edge connectivity. Such graphs are called isomorphic graphs. Note that we label the graphs in this chapter mainly for the purpose of referring to them and recognizing them from one another.

Isomorphic Graphs

Two graphs G_1 and G_2 are said to be isomorphic if −

- Their number of components (vertices and edges) are same.
- Their edge connectivity is retained.

Note − In short, out of the two isomorphic graphs, one is a tweaked version of the other. An unlabelled graph also can be thought of as an isomorphic graph.

There exists a function 'f' from vertices of G_1 to vertices of G_2
[f: $V(G_1) \Rightarrow V(G_2)$], such that
Case (i): *f* is a bijection (both one-one and onto)
Case (ii): *f* preserves adjacency of vertices, i.e., if the edge $\{U, V\} \in G_1$, then the edge $\{f(U), f(V)\} \in G_2$, then $G_1 \equiv G_2$.

Note

If $G_1 \equiv G_2$ then −

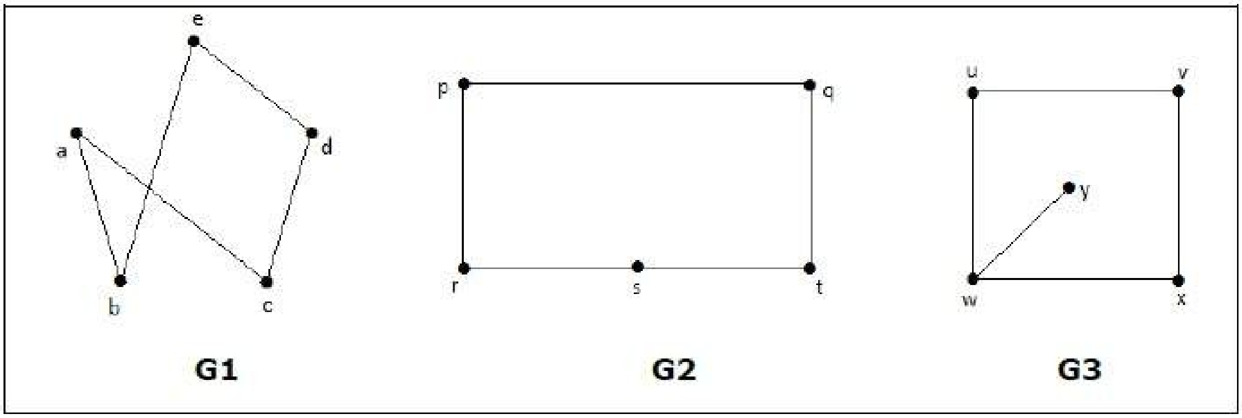
- $|V(G_1)| = |V(G_2)|$
- $|E(G_1)| = |E(G_2)|$
- Degree sequences of G_1 and G_2 are same.
- If the vertices $\{V_1, V_2, .. V_k\}$ form a cycle of length K in G_1, then the vertices $\{f(V_1), f(V_2),\dots f(V_k)\}$ should form a cycle of length K in G_2.

All the above conditions are necessary for the graphs G_1 and G_2 to be isomorphic, but not sufficient to prove that the graphs are isomorphic.

- $(G_1 \equiv G_2)$ if and only if $(G_1{-} \equiv G_2{-})$ where G_1 and G_2 are simple graphs.
- $(G_1 \equiv G_2)$ if the adjacency matrices of G_1 and G_2 are same.
- $(G_1 \equiv G_2)$ if and only if the corresponding subgraphs of G_1 and G_2(obtained by deleting some vertices in G_1 and their images in graph G_2) are isomorphic.

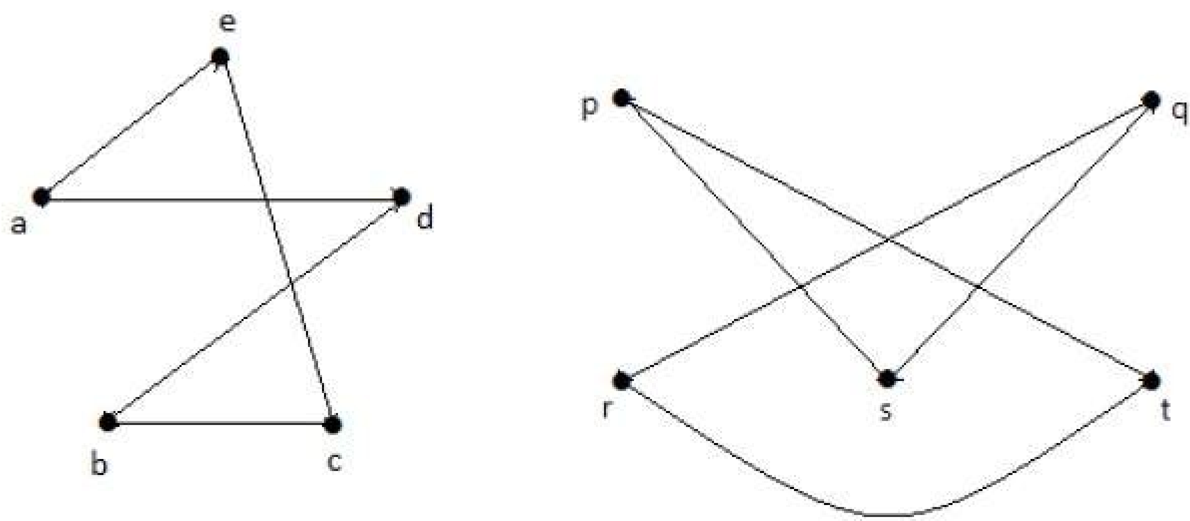
Example

Which of the following graphs are isomorphic?



In the graph G_3, vertex 'w' has only degree 3, whereas all the other graph vertices has degree 2. Hence G_3 not isomorphic to G_1 or G_2.

Taking complements of G_1 and G_2, you have −

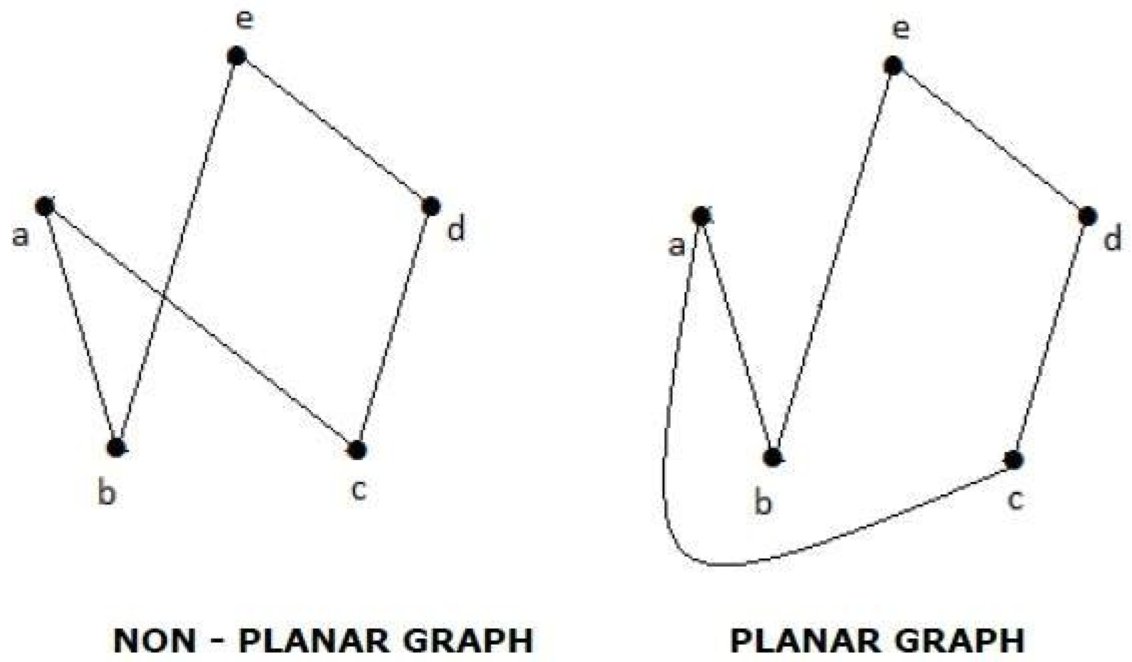


Here, ($G_1- \equiv G_2-$), hence ($G_1 \equiv G_2$).

Planar Graphs:

A graph 'G' is said to be planar if it can be drawn on a plane or a sphere so that no two edges cross each other at a non-vertex point.

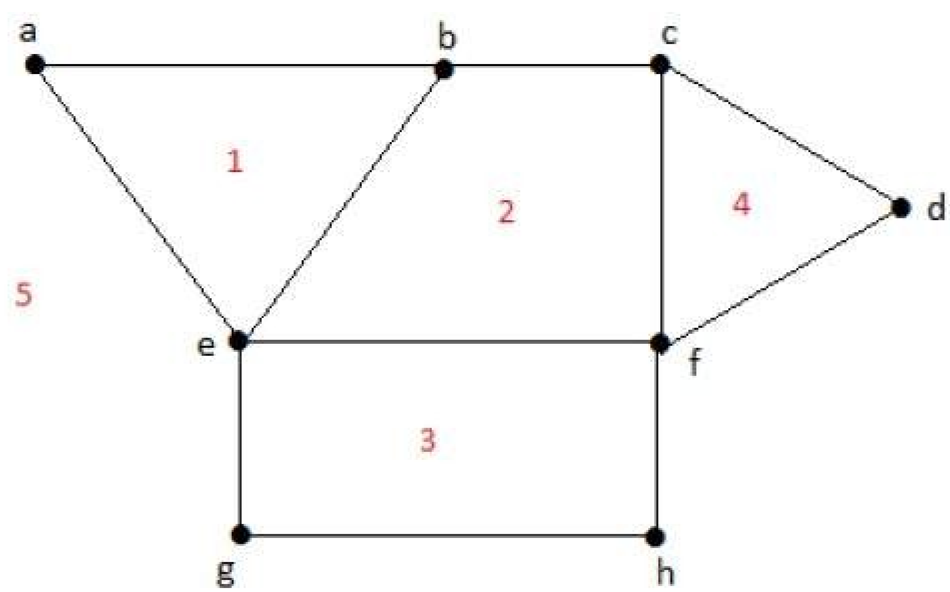
Example



Regions

Every planar graph divides the plane into connected areas called regions.

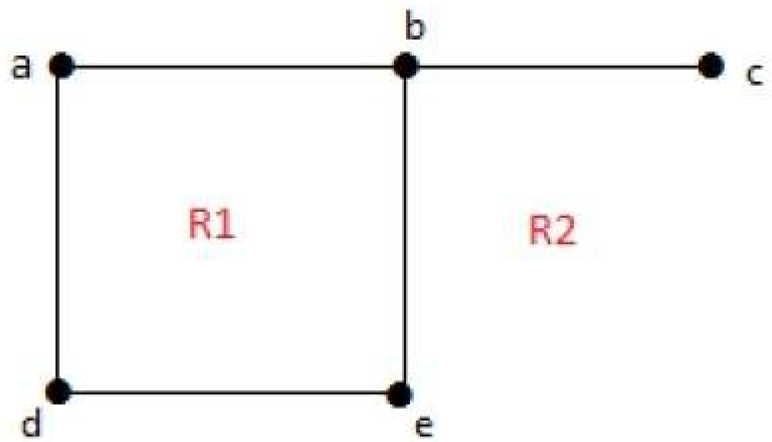
Example:



Degree of a bounded region **r = deg(r)** = Number of edges enclosing the regions **r**.

deg(1) = 3

```
deg(2) = 4
deg(3) = 4
deg(4) = 3
deg(5) = 8
```



Degree of an unbounded region **r = deg(r)** = Number of edges enclosing the regions **r**.

```
deg(R1) = 4
deg(R2) = 6
```

In planar graphs, the following properties hold good −

- **1.** In a planar graph with 'n' vertices, sum of degrees of all the vertices is

$$\sum_{i=1}^{n} \deg(V_i) = 2|E|$$

- **2.** According to **Sum of Degrees of Regions** Theorem, in a planar graph with 'n' regions, Sum of degrees of regions is −

$$\sum_{i=1}^{n} \deg(r_i) = 2|E|$$

Based on the above theorem, you can draw the following conclusions −

In a planar graph,

- If degree of each region is K, then the sum of degrees of regions is

$$K|R| = 2|E|$$

- If the degree of each region is at least K($\geq$ K), then

$$K|R| \leq 2|E|$$

- If the degree of each region is at most K($\leq$ K), then

$$K|R| \geq 2|E|$$

Note − Assume that all the regions have same degree.

3. According to **Euler's Formulae** on planar graphs,

- If a graph 'G' is a connected planar, then

$$|V| + |R| = |E| + 2$$

- If a planar graph with 'K' components then

$$|V| + |R|=|E| + (K+1)$$

Where, |V| is the number of vertices, |E| is the number of edges, and |R| is the number of regions.

4. Edge Vertex Inequality

If 'G' is a connected planar graph with degree of each region at least 'K' then,

$|E| \leq kk - 2\{|v| - 2\}$
You know, $|V| + |R| = |E| + 2$

$K.|R| \leq 2|E|$

$K(|E| - |V| + 2) \leq 2|E|$

$(K - 2)|E| \leq K(|V| - 2)$

$|E| \leq kk - 2\{|v| - 2\}$
5. If 'G' is a simple connected planar graph, then

$|E| \leq 3|V| - 6$
$|R| \leq 2|V| - 4$

There exists at least one vertex V ∈ G, such that $\deg(V) \leq 5$

6. If 'G' is a simple connected planar graph (with at least 2 edges) and no triangles, then

$|E| \leq \{2|V| - 4\}$

7. Kuratowski's Theorem

A graph 'G' is non-planar if and only if 'G' has a subgraph which is homeomorphic to K_5 or $K_{3,3}$.

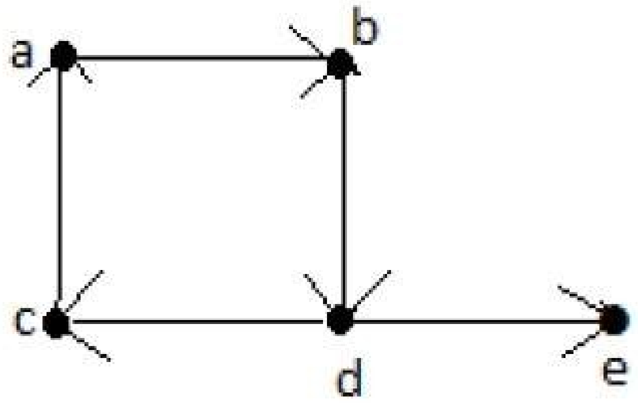
Euler Cycles:

A graph is traversable if you can draw a path between all the vertices without retracing the same path. Based on this path, there are some categories like Euler's path and Euler's circuit which are described in this chapter.

Euler's Path

An Euler's path contains each edge of 'G' exactly once and each vertex of 'G' at least once. A connected graph G is said to be traversable if it contains an Euler's path.

Example:

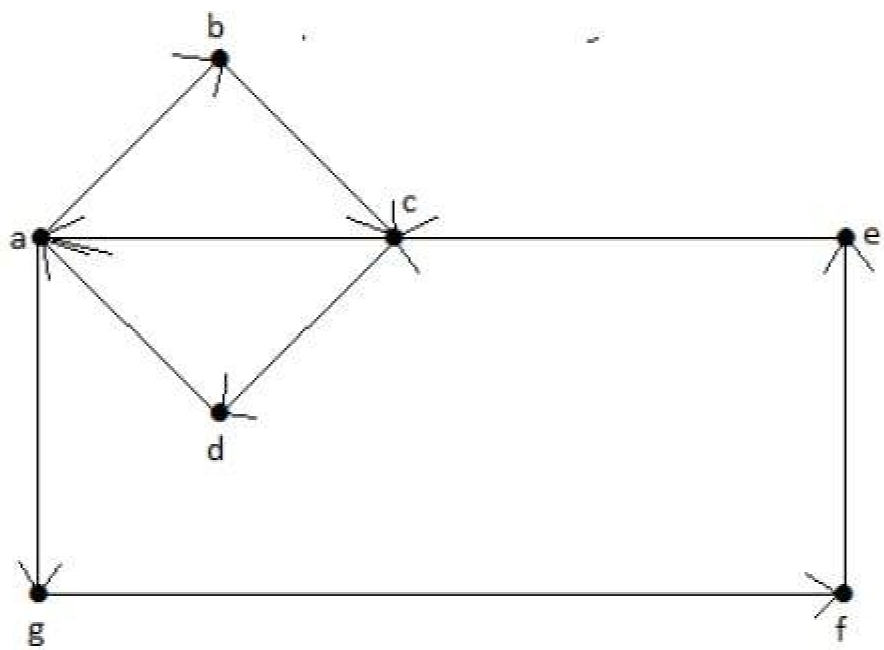


Euler's Path = d-c-a-b-d-e.

Euler's Circuit

In an Euler's path, if the starting vertex is same as its ending vertex, then it is called an Euler's circuit.

Example



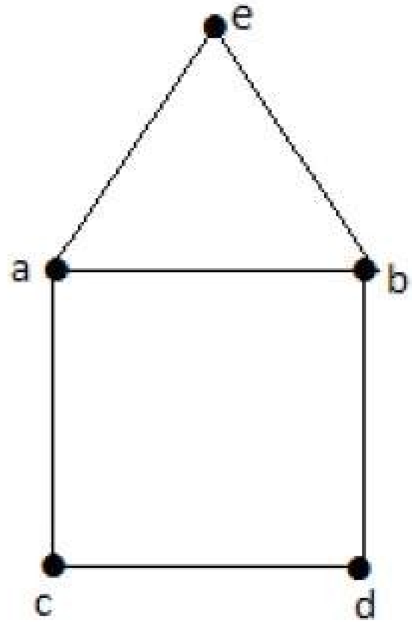
Euler's Path = a-b-c-d-a-g-f-e-c-a.

Euler's Circuit Theorem

A connected graph 'G' is traversable if and only if the number of vertices with odd degree in G is exactly 2 or 0. A connected graph G can contain an Euler's path, but not an Euler's circuit, if it has exactly two vertices with an odd degree.

Note − This Euler path begins with a vertex of odd degree and ends with the other vertex of odd degree.

Example



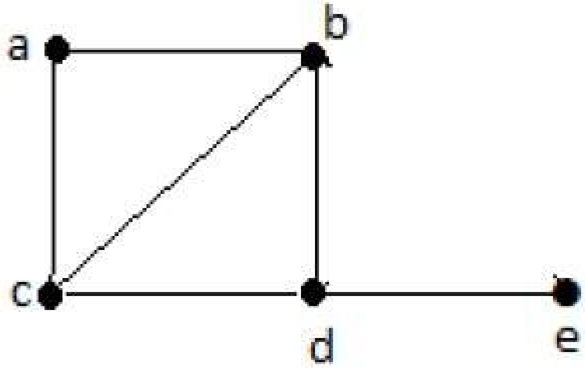
Euler's Path − b-e-a-b-d-c-a is not an Euler's circuit, but it is an Euler's path. Clearly it has exactly 2 odd degree vertices.

Note − In a connected graph G, if the number of vertices with odd degree = 0, then Euler's circuit exists.

Hamiltonian Circuits:

A connected graph is said to be Hamiltonian if it contains each vertex of G exactly once. Such a path is called a **Hamiltonian path**.

Example



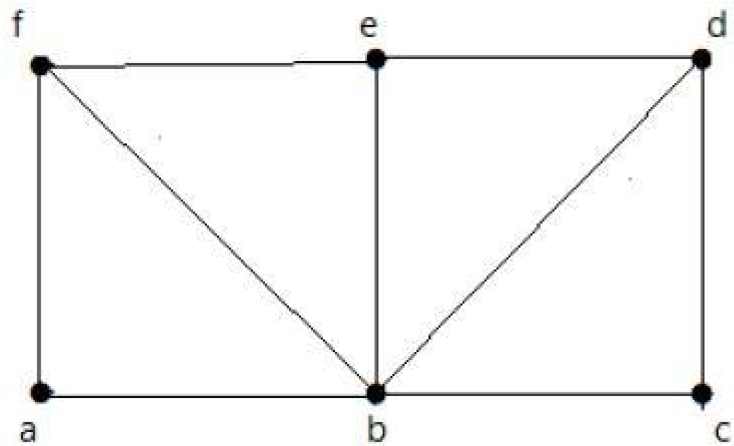
Hamiltonian Path − e-d-b-a-c.

Note −

- Euler’s circuit contains each edge of the graph exactly once.
- In a Hamiltonian cycle, some edges of the graph can be skipped.

Example

Take a look at the following graph



For the graph shown above −

- Euler path exists – false
- Euler circuit exists – false
- Hamiltonian cycle exists – true
- Hamiltonian path exists – true

G has four vertices with odd degree, hence it is not traversable. By skipping the internal edges, the graph has a Hamiltonian cycle passing through all the vertices.

Graph Coloring:

Graph coloring is nothing but a simple way of labelling graph components such as vertices, edges, and regions under some constraints. In a graph, no two adjacent vertices, adjacent edges, or adjacent regions are colored with minimum number of colors. This number is called the **chromatic number** and the graph is called a **properly colored graph**.

While graph coloring, the constraints that are set on the graph are colors, order of coloring, the way of assigning color, etc. A coloring is given to a vertex or a particular region. Thus, the vertices or regions having same colors form independent sets.

Vertex Coloring:

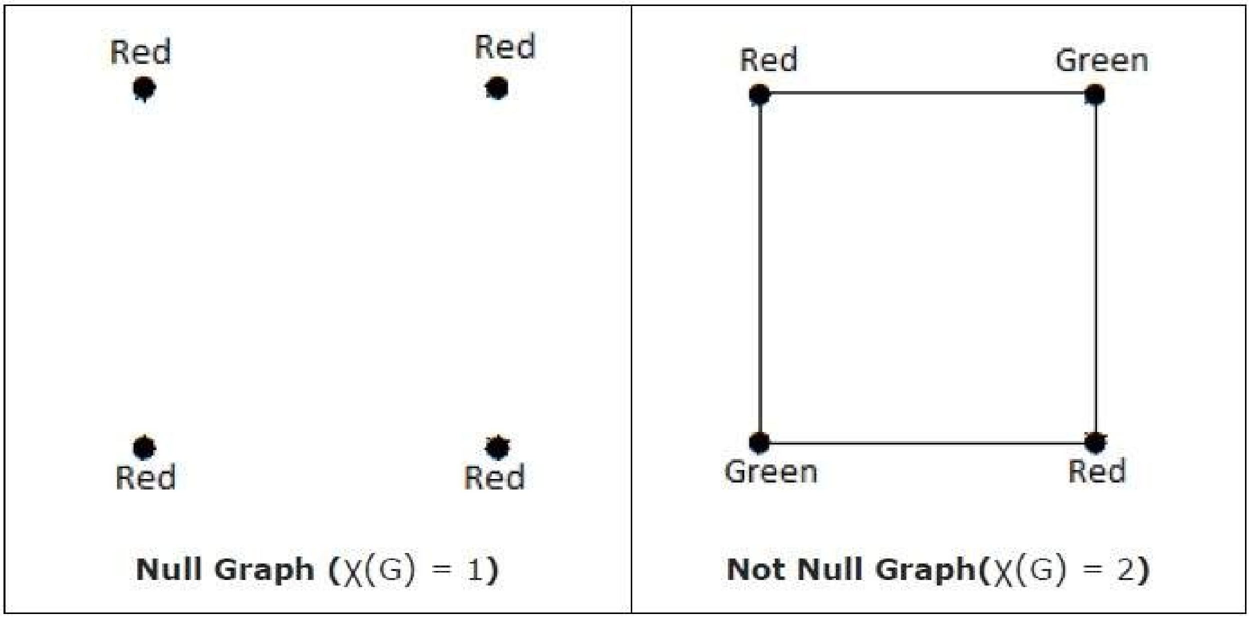
Vertex coloring is an assignment of colors to the vertices of a graph ‘G’ such that no two adjacent vertices have the same color. Simply put, no two vertices of an edge should be of the same color.

Chromatic Number:

The minimum number of colors required for vertex coloring of graph ‘G’ is called as the chromatic number of G, denoted by X(G).

$\chi(G) = 1$ if and only if 'G' is a null graph. If 'G' is not a null graph, then $\chi(G) \geq 2$

Example



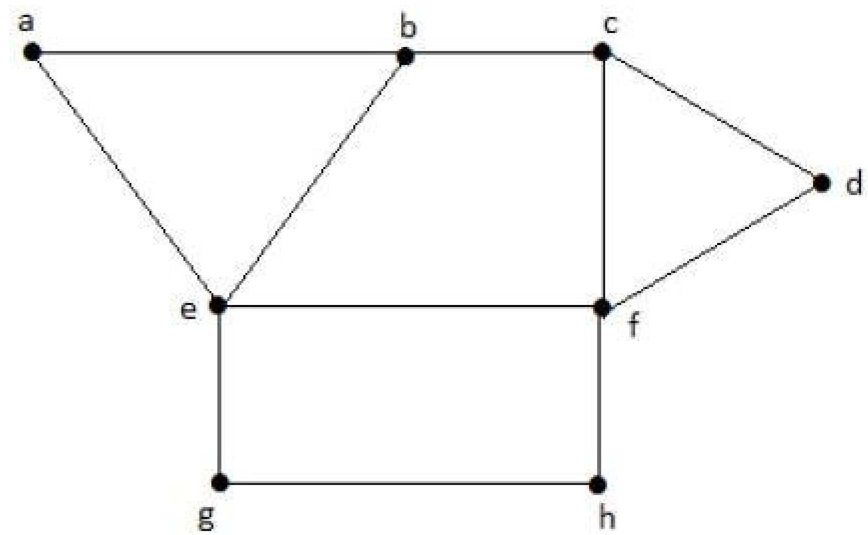
Note − A graph ‘G’ is said to be n-coverable if there is a vertex coloring that uses at most n colors, i.e., $X(G) \leq n$.

Region Coloring:

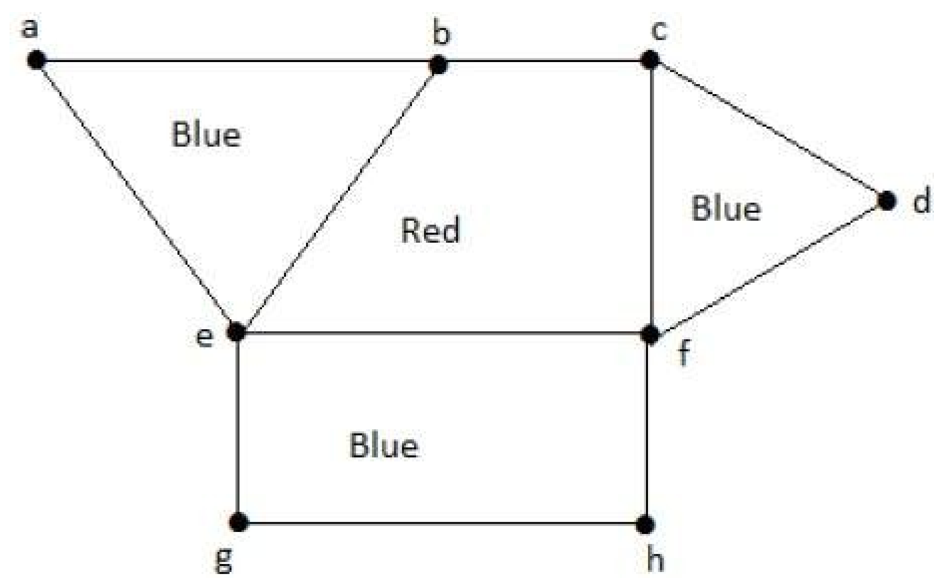
Region coloring is an assignment of colors to the regions of a planar graph such that no two adjacent regions have the same color. Two regions are said to be adjacent if they have a common edge.

Example

Take a look at the following graph. The regions ‘aeb’ and ‘befc’ are adjacent, as there is a common edge ‘be’ between those two regions.



Similarly the other regions are also coloured based on the adjacency. This graph is coloured as follows –



Example

The chromatic number of K_n is

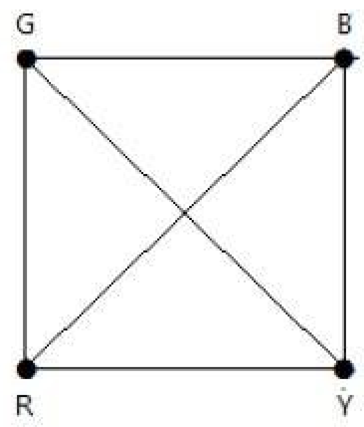
a) n

b) n–1

c) $\lfloor n\boxed{2}\rfloor$

d) $\lceil n\boxed{2}\rceil$

Consider this example with K_4.



In the complete graph, each vertex is adjacent to remaining (n – 1) vertices. Hence, each vertex requires a new color. Hence the chromatic number of K_n = n.

Applications of Graph Coloring

Graph coloring is one of the most important concepts in graph theory. It is used in many real-time applications of computer science such as −

- Clustering
- Data mining
- Image capturing
- Image segmentation
- Networking
- Resource allocation
- Processes scheduling

Permutations:

A permutation is an arrangement of all or part of a set of objects, with regard to the order of the arrangement. For example, suppose we have a set of three letters: A, B, and C. we might ask how many ways we can arrange 2 letters from that set.

Permutation is defined and given by the following function:

Formula

nPr=n!(n−r)!nPr=n!(n−r)!

Where −

- nn = of the set from which elements are permuted.
- rr = size of each permutation.

- n,rn,r are non negative integers.

Example

Problem Statement:

A computer scientist is trying to discover the keyword for a financial account. If the keyword consists only of 10 lower case characters (e.g., 10 characters from among the set: a, b, c... w, x, y, z) and no character can be repeated, how many different unique arrangements of characters exist?

Solution:

Step 1: Determine whether the question pertains to permutations or combinations. Since changing the order of the potential keywords (e.g., ajk vs. kja) would create a new possibility, this is a permutations problem.

Step 2: Determine n and r

n = 26 since the computer scientist is choosing from 26 possibilities (e.g., a, b, c... x, y, z).

r = 10 since the computer scientist is choosing 10 characters.

Step 2: Apply the formula

26P10=26!(26−10)! =26!16! =26(25)(24)...(11)(10)(9)...(1)(16)(15)...(1) =26(25)(24)...(17) =19275223968000

Permutation with repetition:

A combination is a selection of all or part of a set of objects, without regard to the order in which objects are selected. For example, suppose we have a set of three letters: A, B, and C. we might ask how many ways we can select 2 letters from that set.

Combination is defined and given by the following function:

Formula

C(n,r)=n!r!(n−r)!C(n,r)=n!r!(n−r)!

Where −

- nn = the number of objects to choose from.
- rr = the number of objects selected.

Example

Problem Statement:

How many different groups of 10 students can a teacher select from her classroom of 15 students?

Solution:

Step 1: Determine whether the question pertains to permutations or combinations. Since changing the order of the selected students would not create a new group, this is a combinations problem.

Step 2: Determine n and r

n = 15 since the teacher is choosing from 15 students.

r = 10 since the teacher is selecting 10 students.

Step 3: Apply the formula

15C10=15!(15−10)!10!=15!5!10!=15(14)(13)(12)(11)(10!)5!10!=15(14)(13)(12)(11)5!=15(14)(13)(12)(11)5(4)(3)(2)(1)=(14)(13)(3)(11)(2)(1)=(7)(13)(3)(11)=3003

Examples

1. How many three digit numbers can be formed with the digits: 1, 2, 3, 4, 5?

$n = 5 \quad k = 3$

The order of the elements does matter.

The elements are repeated.

$$PR(5,3) = 5^3 = 125$$

2. How many three digit numbers can be formed with the digits: 0, 1, 2, 3, 4, 5?

$n = 6 \quad k = 3$

The numbers must be separated into two blocks:

— — —

The first set, of one number, can occupy only one of 5 digits because a number does not begin with zero (except for license plates and other special cases).

$n = 5 \quad k = 1$

The second block, of two numbers, can occupy any digit.

$n = 6 \quad k = 2$

$$P(5,1) \cdot PR(6,2) = 5 \cdot 6^2 = 180$$

3. How many nine-digit numbers can be formed with the numbers 2, 2, 2, 3, 3, 3, 3, 4, 4?

$n = 9 \quad a = 3 \quad b = 4 \quad c = 2 \quad a + b + c = 9$

The order of the elements does matter.

The elements are repeated.

$$PR_9^{3,4,2} = \frac{9!}{3! \cdot 4! \cdot 2!} = 1{,}260$$

4. The signal mast of a ship can raise nine flags at one time (three red, two blue and four green). How many different signals can be communicated by the placement of these nine flags?

$n = 9 \quad r = 3 \quad b = 2 \quad g = 4 \quad r + b + g = 9$

The order of the elements does matter.

The elements are repeated.

$$PR_9^{3,2,4} = \frac{9!}{3! \cdot 2! \cdot 4!} = 1{,}260$$

Permutations without Repetition:

In this case, we have to **reduce** the number of available choices each time.

Example: what order could 16 pool balls be in?

After choosing, say, number "14" we can't choose it again.

So, our first choice has 16 possibilites, and our next choice has 15 possibilities, then 14, 13, 12, 11, ... etc. And the total permutations are:

$$\mathbf{16 \times 15 \times 14 \times 13 \times \ldots = 20{,}922{,}789{,}888{,}000}$$

But how do we write that mathematically? Answer: we use the " factorial function "

The **factorial function** (symbol: **!**) just means to multiply a series of descending natural numbers. Examples:

$4! = 4 \times 3 \times 2 \times 1 = 24$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$

$1! = 1$

Note: it is generally agreed that **0! = 1**. It may seem funny that multiplying no numbers together gets us 1, but it helps simplify a lot of equations.

So, when we want to select **all** of the billiard balls the permutations are:

$$\mathbf{16! = 20{,}922{,}789{,}888{,}000}$$

But when we want to select just 3 we don't want to multiply after 14. How do we do that? There is a neat trick: we divide by **13!**

$$\textit{16} \times \textit{15} \times \textit{14} \times \textit{13} \times \textit{12} \ldots \mathbf{13 \times 12 \ldots} = \mathbf{16 \times 15 \times 14}$$

That was neat. The **13 × 12 × ... etc** gets "cancelled out", leaving only **16 × 15 × 14**.

The formula is written:

$$\frac{n!}{(n-r)!}$$

where ***n*** is the number of things to choose from,
and we choose ***r*** of them,
no repetitions,
order matters.

Example Our "order of 3 out of 16 pool balls example" is:

$$\frac{\mathbf{16!}}{\mathbf{(16\text{-}3)!}} = \frac{\mathbf{16!}}{\mathbf{13!}} = \frac{\mathbf{20{,}922{,}789{,}888{,}000}}{\mathbf{6{,}227{,}020{,}800}} \mathbf{= 3{,}360}$$

(which is just the same as: **16 × 15 × 14 = 3,360**)

Example: How many ways can first and second place be awarded to 10 people?

$$\frac{\mathbf{10!}}{\mathbf{(10\text{-}2)!}} = \frac{\mathbf{10!}}{\mathbf{8!}} = \frac{\mathbf{3{,}628{,}800}}{\mathbf{40{,}320}} \mathbf{= 90}$$

(which is just the same as: **10 × 9 = 90**)

In other words, there are 3,360 different ways that 3 pool balls could be arranged out of 16 balls.

Combinations:

A combination is a selection of all or part of a set of objects, without regard to the order in which objects are selected. For example, suppose we have a set of three letters: A, B, and C. we might ask how many ways we can select 2 letters from that set.

Combination is defined and given by the following function:

Formula

C(n,r)=n!r!(n−r)!C(n,r)=n!r!(n−r)!

Where −

- nn = the number of objects to choose from.
- rr = the number of objects selected.

Example

Problem Statement:

How many different groups of 10 students can a teacher select from her classroom of 15 students?

Solution:

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n = 15 since the teacher is choosing from 15 students.

r = 10 since the teacher is selecting 10 students.

Step 3: Apply the formula

15C10
=15!(15−10)!10!
=15!5!10!
=15(14)(13)(12)(11)(10!)5!10!
=15(14)(13)(12)(11)5!
=15(14)(13)(12)(11)5(4)(3)(2)(1)
=(14)(13)(3)(11)(2)(1)
=(7)(13)(3)(11)
=3003

Combination with repetition:

Given a set of **n** elements, the combinations with repetition are different groups formed by the **k** elements of a subset such that:

The order of the elements does not matter.

The elements are repeated.

$$CR(n,k) = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

Examples

There are five different types of bottles in a wine cellar. How many ways can four bottles be chosen from the cellar?

The order of the elements does not matter.

The elements are repeated.

$$C(5,4)=\frac{(5+4-1)!}{4!(5-1)!}=\frac{8!}{4!\cdot 4!}=70$$

Combination without Repetition:

In the standard Combination case, no repeated elements are allowed, so you need to *choose* unique items from a group.

Combination Example
Due to budget cuts, there will only be **1** winner in this year's poetry contest. Luckily only 3 people (Anna, Bill and Charlie) entered the contest. How many ways can you choose 1 winner from **3** contestants?

There are **3** possible choices. (Duh.)

Real Example
The poetry club raised some more money and can now award 2 identical prizes (without differentiating between 1st and 2nd place). How many ways can you choose 2 winners from 3?
If it was a permutation, you would just do:
3 * 2 = 6

You could list the 6 possible permutations:

{Anna, Bill}	{Bill, Anna}
{Anna, Charlie}	{Charlie, Anna}
{Bill, Charlie}	{Charlie, Bill}

However, selections with the same elements (in a different order) should count as **1** combination. Choosing {Anna, Bill} is the same as {Bill, Anna}. As you can see, there are **2** *permutations* for each *combination*. This is because there are 2 ways to order every 2-element combination.

Enumerative combinatorics:

is an area of combinatorics that deals with the number of ways that certain patterns can be formed. Two examples of this type of problem are counting combinations and counting permutations. More generally, given an infinite collection of finite sets S_i indexed by the natural numbers, enumerative combinatorics seeks to describe a *counting function* which counts the number of objects in S_n for each n. Although counting the number of elements in a set is a rather broad mathematical problem, many of the problems that arise in applications have a relatively simple combinatorial description. The twelvefold way provides a unified framework for counting permutations, combinations and partitions.

The simplest such functions are *closed formulas*, which can be expressed as a composition of elementary functions such as factorials, powers, and so on. For instance, as shown below, the number of different possible orderings of a deck of n cards is $f(n) = n!$. The problem of finding a closed formula is known as algebraic enumeration, and frequently involves deriving a recurrence relation or generating function and using this to arrive at the desired closed form.

Often, a complicated closed formula yields little insight into the behavior of the counting function as the number of counted objects grows. In these cases, a simple asymptotic approximation may be preferable.