Solve two dimensional diffusion equation for the distribution function $f(x, y, t)$ with source $S(x, y, t)$,

$$\frac{\partial f(x,y,t)}{\partial t} = D\nabla^2 f(x,y,t) + S(x,y,t). \tag{1}$$

Crank-Nicolson method

The time and space derivatives in the Crank-Nicolson's scheme are taken to be,

$$\begin{aligned}
\frac{\partial f(x,y,t)}{\partial t} &= \frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\delta t},\\
\frac{\partial^2 f(x,y,t)}{\partial x^2} &= \frac{1}{2(\delta x)^2}[f_{i+1,j}^{n+1} + f_{i-1,j}^{n+1} - 2f_{i,j}^{n+1} + f_{i+1,j}^{n} + f_{i-1,j}^{n} - 2f_{i,j}^{n}],\\
\frac{\partial^2 f(x,y,t)}{\partial y^2} &= \frac{1}{2(\delta y)^2}[f_{i,j+1}^{n+1} + f_{i,j-1}^{n+1} - 2f_{i,j}^{n+1} + f_{i,j+1}^{n} + f_{i,j-1}^{n} - 2f_{i,j}^{n}].
\end{aligned}$$

The eq 1 is discretized using the Crank-Nicolson method to obtain the recurrence relation,

$$a(f_{i+1,j}^{n+1} + f_{i-1,j}^{n+1}) - 2bf_{i,j}^{n+1} + a(f_{i,j+1}^{n+1} + f_{i,j-1}^{n+1}) = -a(f_{i+1,j}^{n} + f_{i-1,j}^{n}) + 2cf_{i,j}^{n} - a(f_{i,j+1}^{n} + f_{i,j-1}^{n}) - 2\delta t S(i,j,n), \tag{2}$$

where $a = D\delta t/(\delta x)^2$, $b = -(2a+1)$, and $c = 2a - 1$.

Key points used in the Crank-Nicolson algorithm

- δx and δy are taken to be equal.
- It is noted that the source or the initial profile in the problem is at the origin so the heat flow will be symmetric about $y = x$ line. So, the matrix equation for 2 is written only for $y < x$.
- At the boundary $x = 0$ and $y = 0$, the ghost point is chosen to be $f_{-1,j}^{n} = f_{1,j}^{n}$, $f_{i,-1}^{n} = f_{i,1}^{n}$

Forward in time and Central in space method

The time and space derivatives are

$$\begin{aligned}
\frac{\partial f(x,y,t)}{\partial t} &= \frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\delta t},\\
\frac{\partial^2 f(x,y,t)}{\partial x^2} &= \frac{1}{(\delta x)^2}[f_{i+1,j}^{n} + f_{i-1,j}^{n} - 2f_{i,j}^{n}],\\
\frac{\partial^2 f(x,y,t)}{\partial y^2} &= \frac{1}{(\delta y)^2}[f_{i,j+1}^{n} + f_{i,j-1}^{n} - 2f_{i,j}^{n}].
\end{aligned}$$

The eq 1 is discretized using the Forward-time-Central-space method to obtain the recurrence relation,

$$f_{i,j}^{n+1} = sD(f_{i+1,j}^{n} + f_{i-1,j}^{n} + f_{i,j+1}^{n} + f_{i,j-1}^{n}) + (1 - 4sD)f_{i,j}^{n} + \delta t S(i,j,n), \tag{3}$$

where $s = \delta t/(\delta x)^2$.

Key points used in the Forward-time-Central-space algorithm

- δx and δy are taken to be equal.
- It is noted that the source or the initial profile in the problem is at the origin so the heat flow will be symmetric about $y = x$ line. So, the matrix equation for 2 is written only for $y < x$.
- At the boundary $x = 0$ and $y = 0$, the ghost point is chosen to be $f_{-1,j}^{n} = f_{1,j}^{n}$, $f_{i,-1}^{n} = f_{i,1}^{n}$

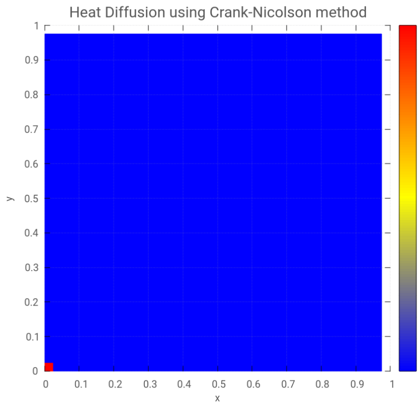
Part 1

Initial profile: $f(t = 0, x, y) = \delta_{x,0}\delta_{y,0}$, Source: $S(x, y, t) = 0$.

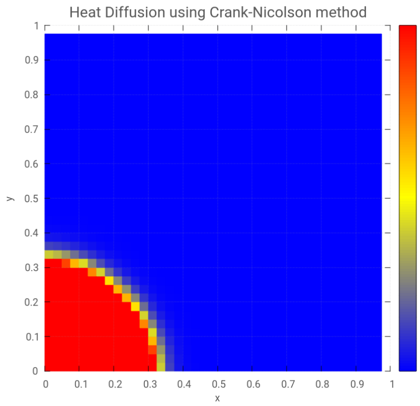
This is solved using the Crank-Nicolson scheme in the code 2dDiffusionCrank.cpp . Values used in the code

- $s = 0.3$, $D = 1$
- x-range: 0 - 1, y-range: 0 - 1, t-range: 0 - 0.08125 (1300 iterations)
- number of division in the grid along x and y axes: $N = 40$.

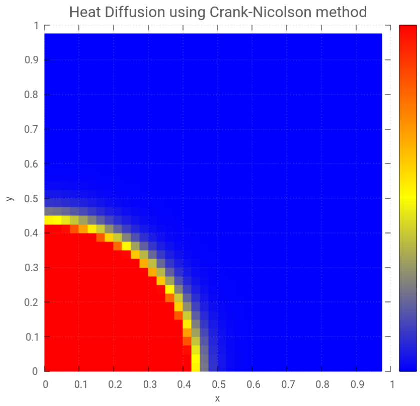
The solutions are plotted at different times in fig 1



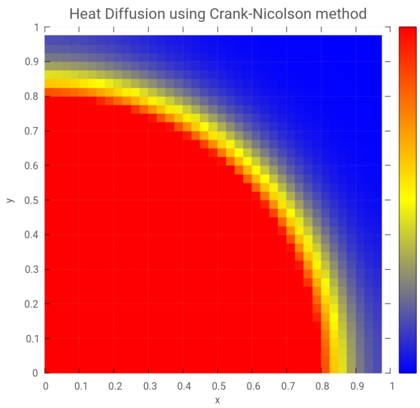
(a) t=0



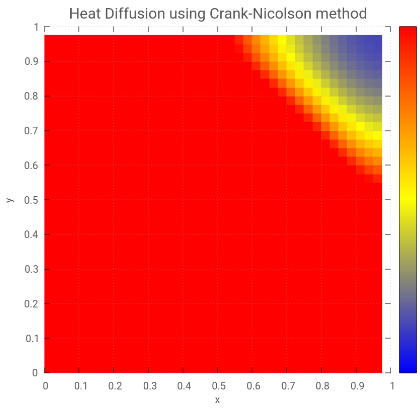
(b) t=0.003125



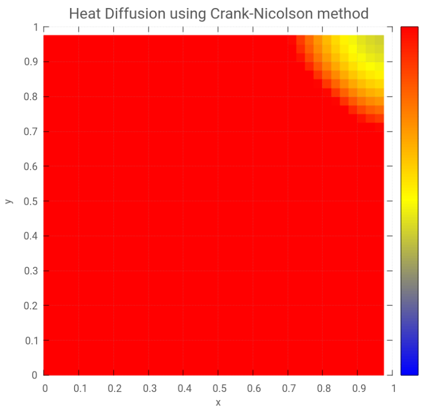
(c) t=0.00625



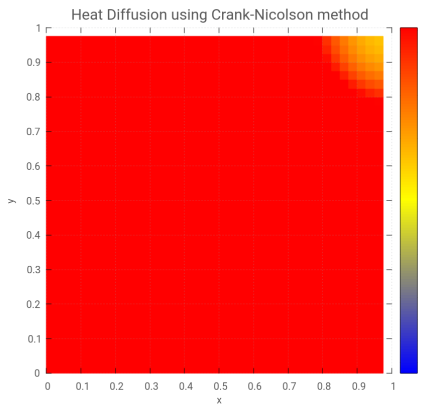
(d) t=0.03125



(e) t=0.0625



(f) t=0.075



(g) t=0.08125

Figure 1: Distributions at different times for $f(t=0,x,y)=\delta_{x,0}\delta_{y,0}$ and $S=0$.

Part 2

Initial profile: $f(t=0,x,y)=0$, Source: $S(x,y,t)=\delta_{x,0}\delta_{y,0}$.

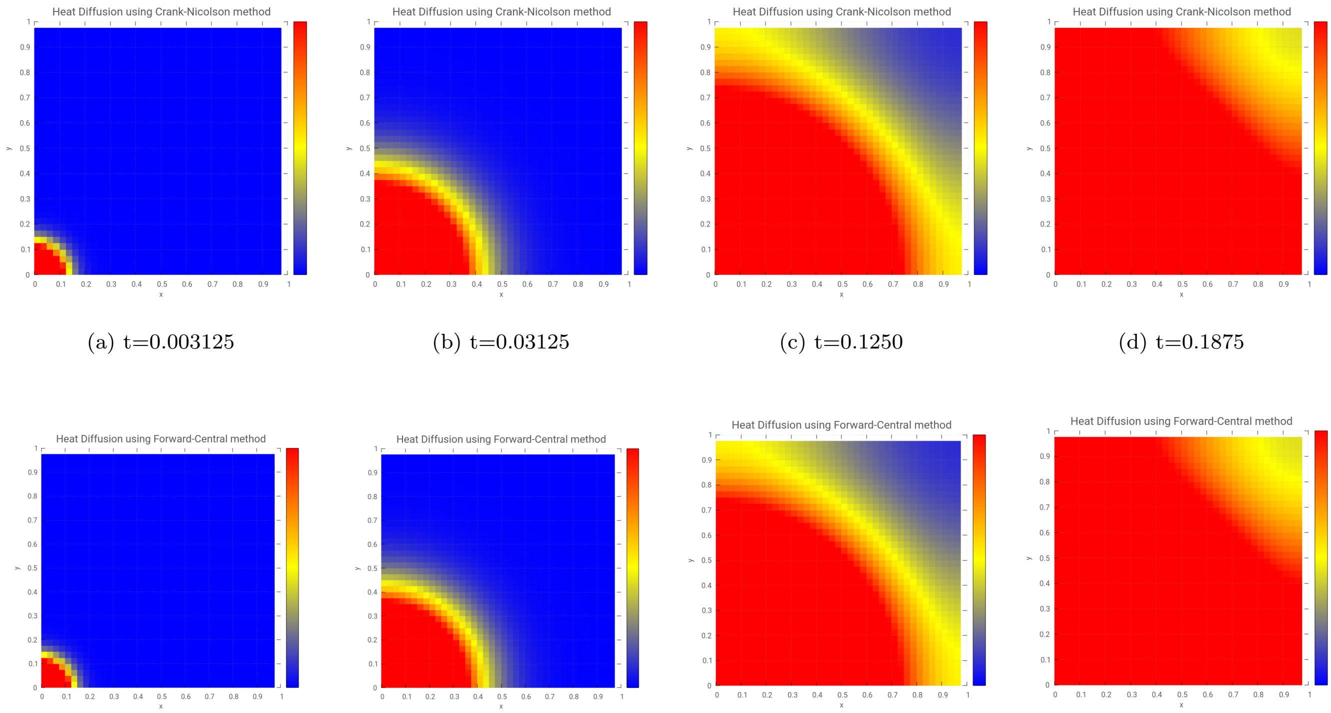
Solution using the Crank-Nicolson method is obtained using the same code 2dDiffusionCrank.cpp by uncommenting the line below

```
//gsl_matrix_add(Fn1, S);
```

which adds the contribution from the source.

Solution using Forward-time-Central-space method is obtained using the code 2dDiffusionForward.cpp .

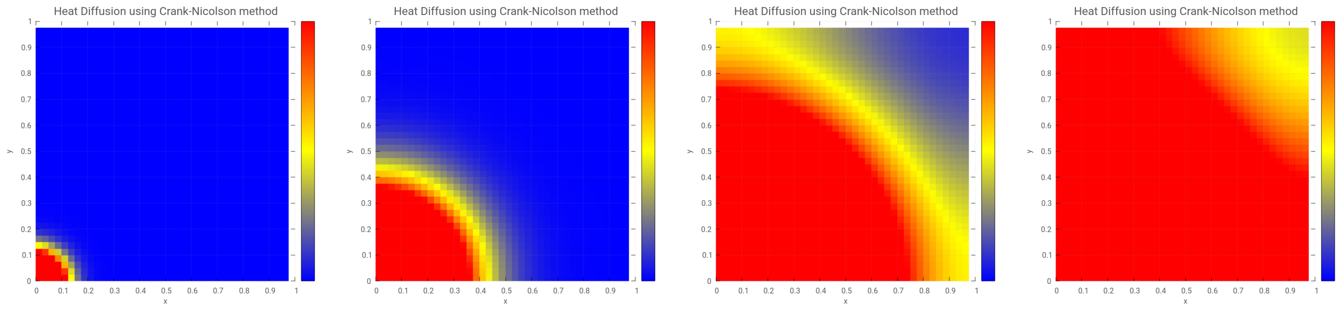
The distributions are plotted for both Crank-Nicolson and Forward-Central method, using various values of the parameter s ranging from 0.10 to 0.35. The diffusion constant, D, is taken to be one.



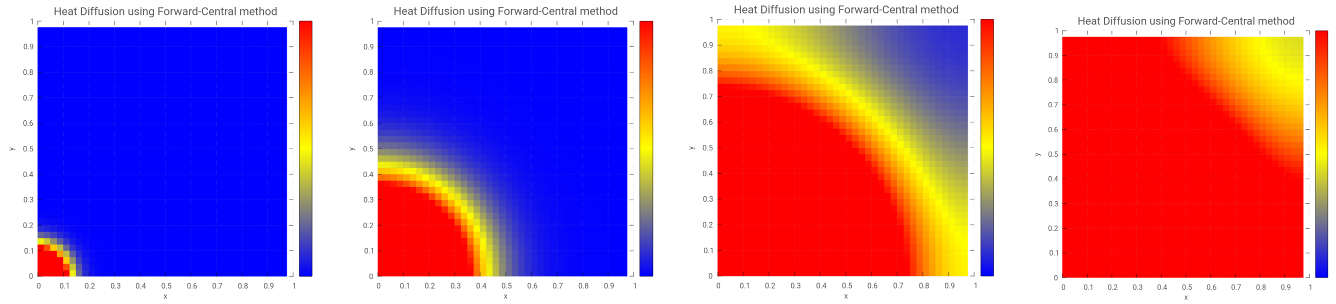
(a) t=0.003125 (b) t=0.03125 (c) t=0.1250 (d) t=0.1875

(e) t=0.003125 (f) t=0.03125 (g) t=0.1250 (h) t=0.1875

Figure 2: Distributions at different times for $f(t=0,x,y)=0$ and $S=\delta_{x,0}\delta_{y,0}$. $s=0.10$

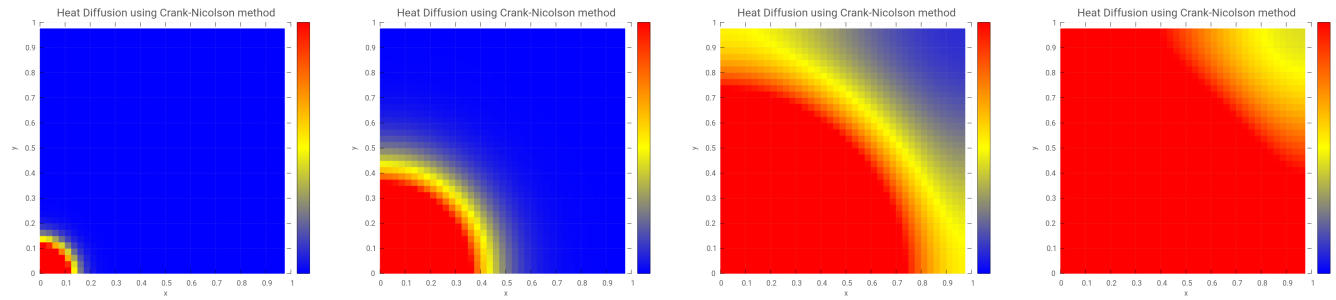


(a) t=0.003094 (b) t=0.03122 (c) t=0.1250 (d) t=0.1875

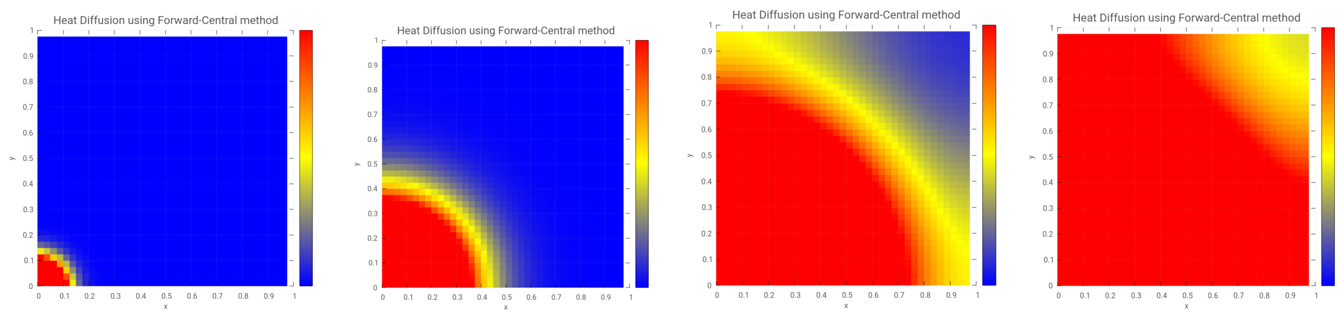


(e) t=0.003094 (f) t=0.03122 (g) t=0.1250 (h) t=0.1875

Figure 3: Distributions at different times for $f(t=0,x,y)=0$ and $S=\delta_{x,0}\delta_{y,0}$. $s=0.15$

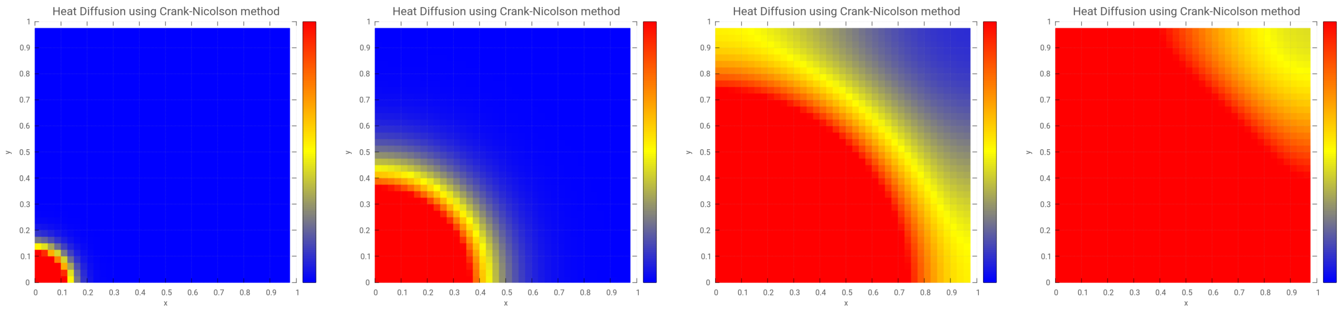


(a) t=0.003125 (b) t=0.03125 (c) t=0.1250 (d) t=0.1875

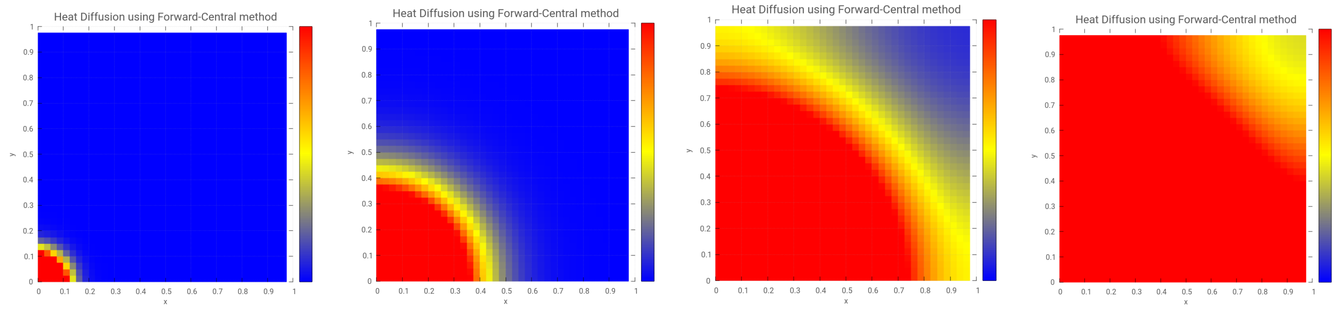


(e) t=0.003125 (f) t=0.03125 (g) t=0.1250 (h) t=0.1875

Figure 4: Distributions at different times for $f(t=0,x,y)=0$ and $S=\delta_{x,0}\delta_{y,0}$. $s=0.20$

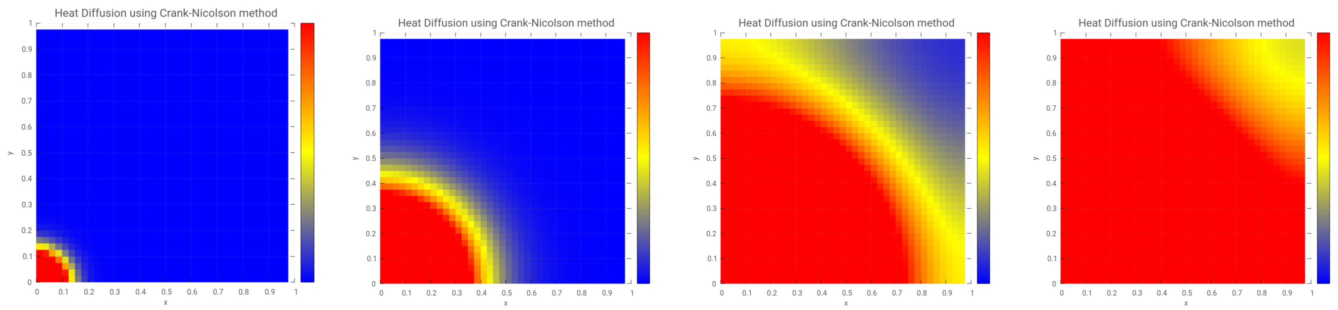


(a) t=0.003125 (b) t=0.03125 (c) t=0.1250 (d) t=0.1875

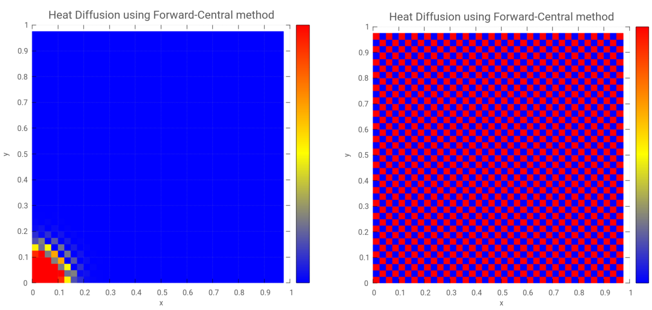


(e) t=0.003125 (f) t=0.03125 (g) t=0.1250 (h) t=0.1875

Figure 5: Distributions at different times for $f(t=0,x,y)=0$ and $S=\delta_{x,0}\delta_{y,0}$. $s=0.25$

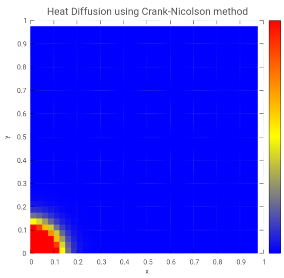


(a) t=0.003188 (b) t=0.03131 (c) t=0.12506 (d) t=0.1875

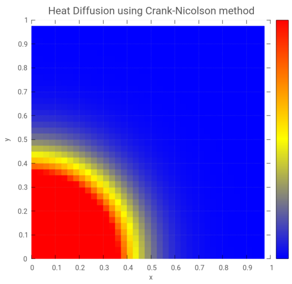


(e) t=0.003188 (f) t=0.03131, 0.12506, 0.1875

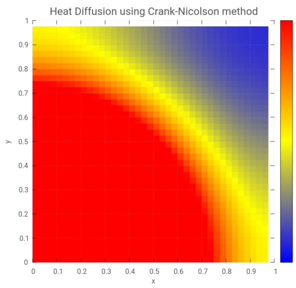
Figure 6: Distributions at different times for $f(t=0,x,y)=0$ and $S=\delta_{x,0}\delta_{y,0}$. $s=0.30$



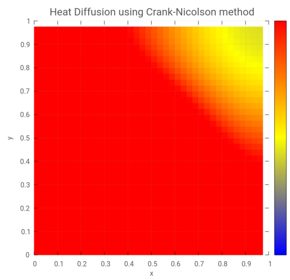
(a) t=0.003063



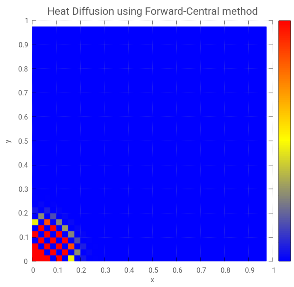
(b) t=0.03128



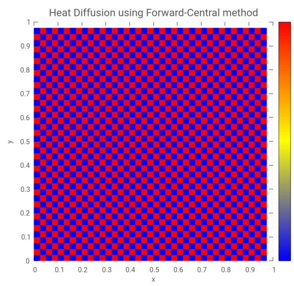
(c) t=0.1250



(d) t=0.18747



(e) t=0.003063



(f) t=0.03128, 0.1250, 0.18747

Figure 7: Distributions at different times for $f(t=0,x,y)=0$ and $S=\delta_{x,0}\delta_{y,0}$. $s=0.35$

From the above plots it is seen that for the Forward-Central method to be useful, the parameter s has to be sufficiently small else the solution is unstable. However, this is not the case with Crank-Nicolson method.

Conclusion

- When the initial distribution is at the origin and there is no source, the distribution $f(x, y, t)$ diffuse out to the whole region. Over time the distribution is getting more and more uniform, i.e. uniform distribution is the state of equilibrium.
- Forward-Central method works only for sufficiently small value of the parameter s while the solution using Crank-Nicolson method is stable even for coarser value of s.