Solve two body problem, specifically for Earth-Moon system, numerically using Runge-Kutta and Leap Frog methods and study the stability of the orbit.

Equation to be solved,

$$\frac{d^2\mathbf{r}_{moon}}{dt^2} = -\frac{Gm_{earth}}{r^3_{moon}}\mathbf{r}_{moon}, \quad \frac{d^2\mathbf{r}_{earth}}{dt^2} = -\frac{Gm_{moon}}{r^3_{earth}}\mathbf{r}_{earth}.$$

Reduce the problem to the one-body equivalent,

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -\frac{Gm_e m_m}{m}\frac{\mathbf{r}}{r^3}, \tag{1}$$

where m is the reduced mass.
Convert the problem to two first order equations,

$$\dot{\mathbf{r}} = \mathbf{v}, \ \dot{\mathbf{v}} = \mathbf{a}. \tag{2}$$

Runge-Kutta method

$$\mathbf{g}_1 = \mathbf{a}(\mathbf{r}), \ \mathbf{g}_2 = \mathbf{a}\left(\mathbf{r} + \mathbf{k}_1\frac{\delta t}{2}\right), \ \mathbf{g}_3 = \mathbf{a}\left(\mathbf{r} + \mathbf{k}_2\frac{\delta t}{2}\right), \ \mathbf{g}_4 = \mathbf{a}\left(\mathbf{r} + \mathbf{k}_3\delta t\right) \tag{3}$$

$$\mathbf{k}_1 = \mathbf{v}(t), \ \mathbf{k}_2 = \mathbf{v}(t) + \mathbf{g}_1\frac{\delta t}{2}, \ \mathbf{k}_3 = \mathbf{v}(t) + \mathbf{g}_2\frac{\delta t}{2}, \ \mathbf{k}_4 = \mathbf{v}(t) + \mathbf{g}_3\delta t \tag{4}$$

Use eqs 3 and 4 to calculate the position and velocity at the next time step,

$$\mathbf{v}(t+\delta t) = \mathbf{v}(t) + \frac{\delta t}{6}[\mathbf{g}_1 + 2\mathbf{g}_2 + 2\mathbf{g}_3 + \mathbf{g}_4], \tag{5}$$

$$\mathbf{r}(t+\delta t) = \mathbf{r}(t) + \frac{\delta t}{6}[\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4]. \tag{6}$$

The code implementing the method is orbit.cpp . The plot of the orbit is shown below.

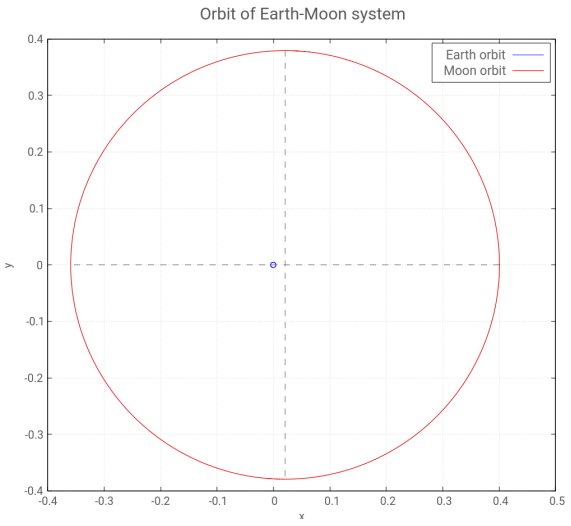

Figure 1: Earth Moon orbit. Distance measured in billion metres unit.

Stability

The same problem is solved using the Leap-Frog method. Energy distributions for Runge-Kutta and Leap Frog methods are plotted below (figs 2 and 3). The distribution is plotted over six time periods.

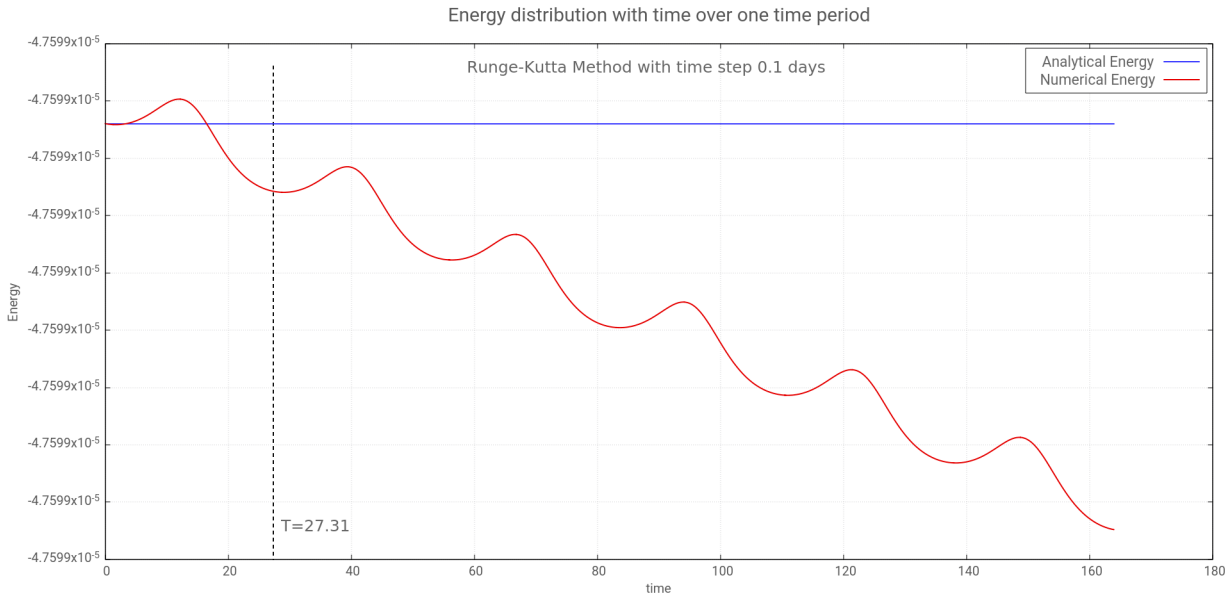

Figure 2: Energy is decreasing over time.

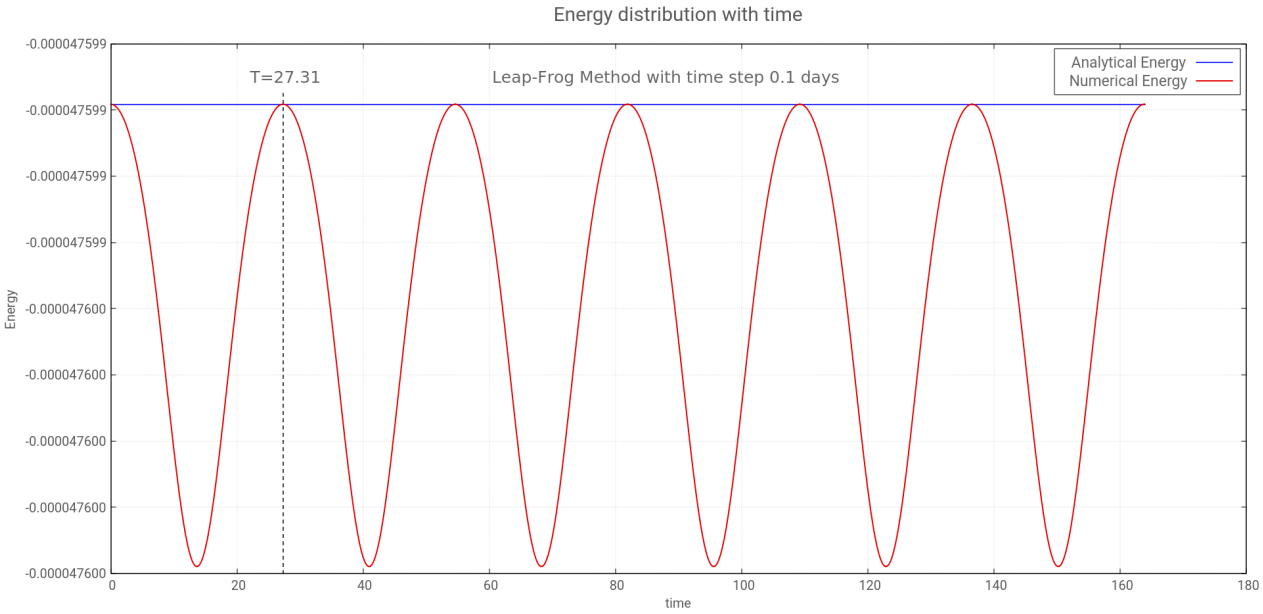

Figure 3: Energy remain conserved on average.

To study the energy conservation in the two methods with time step size, the average energy is calculated for one time period for both methods for time steps ranging from 0.001 to 2.731. The variation of average energy with step size is plotted below (fig 4), the corresponding code is stability.cpp.

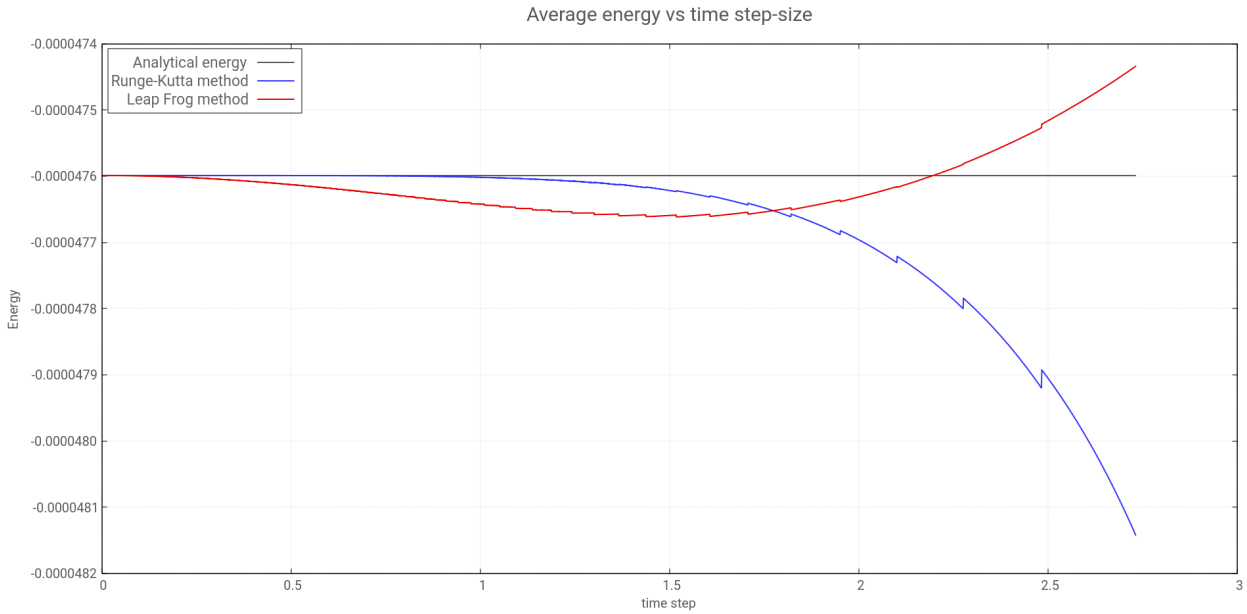

Figure 4: Average energy variation with time-step size for RK and LF methods.

From the figure it is clear that for sufficiently small time-steps the average energy in RK method is more accurate than that of LF method. However, as time passes the deviation cumulate in the case of RK method resulting in poor accuracy. Opposite happens as time-step is made coarser. The average energy over one time period deviates significantly in RK method compared to LF method, i.e. LF maintains energy conservation even for relatively large time-steps.

The extent of variation about average value over the orbit is studied by plotting the standard deviation of the energy distributions 2 and 3 for diffrent time-step sizes. The plot is shown below.

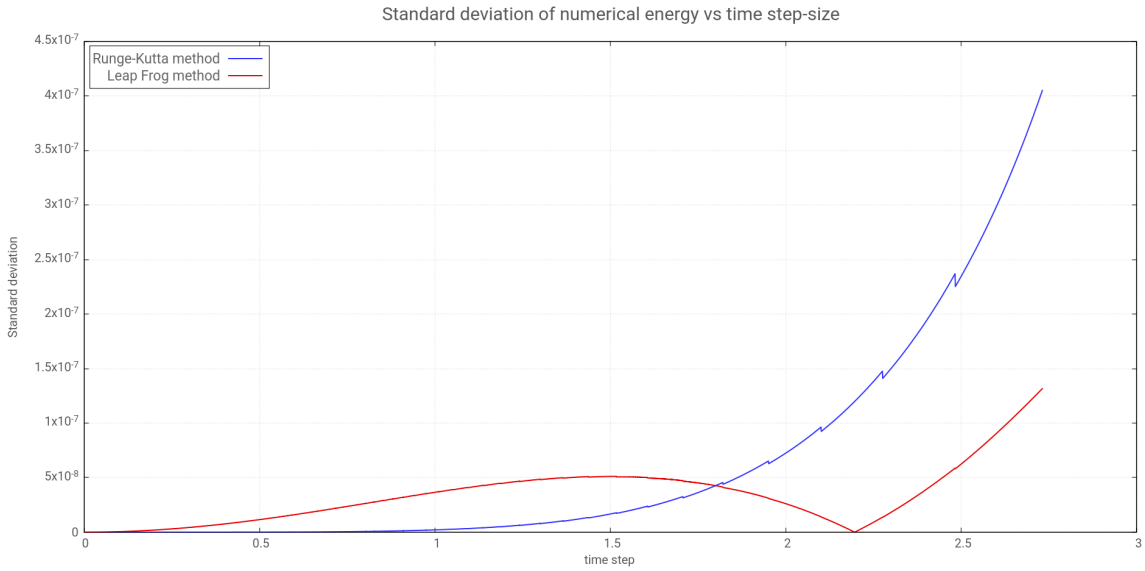

Figure 5: Energy remain conserved on average.

Again, for sufficiently small time-steps, the variation from average value is small in RK method than the LF method. For larger time steps the energy distribution has large spread than that of LF method.

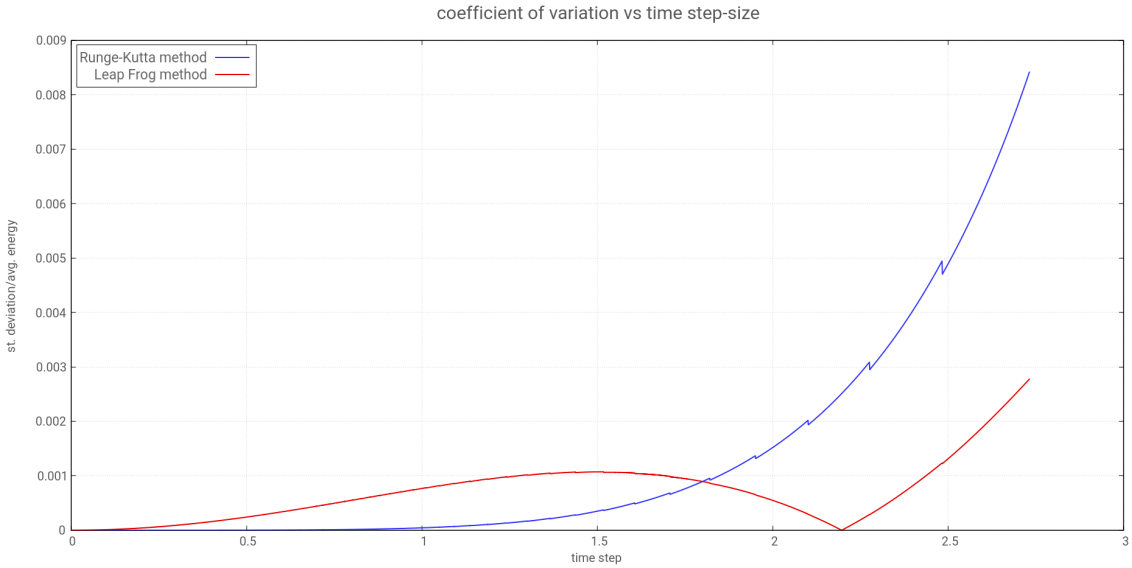

Figure 6: Energy remain conserved on average.

To compare the variation relative to the average energy(for one time period), the coefficient of variation is plotted (fig 6) against time-step size for both methods. It follows that for sufficiently small time-steps, RK gives more precise average energy than LF but for larger time steps it is significantly less accurate than LF.

Conclusion

- Over one time period, the RK method gives more precise average energy than LF method.
- For relatively large time-steps, LF gives more accurate energy than RK method.
- As time increases in step of one time period, the energy of the system continue to decrease in RK method whereas in LF the energy oscillates about a fixed value.