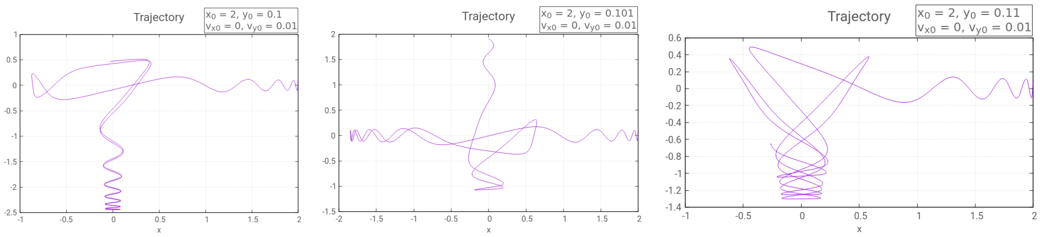
Given the potential $V(x, y) = \lambda x^2 y^2$, solve for the trajectory of the particle for various initial conditions.

Trajectory

Runge-Kutta method is used to calculate the trajectory of the particle for different initial conditions. The code is trajectory.cpp The trajectories are plotted below.



(a) (b) (c)

Figure 1: Trajectories for three close initial points, plotted for time range 70.

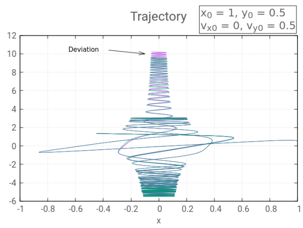


Figure 2: The two trajectories are plotted for t=90, and v_{y0} differ by 10^{-12}.

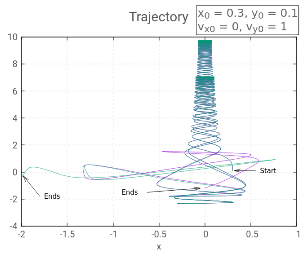


Figure 3: The two trajectories are plotted for t=95, and x_0 and y_0 differ by 10^{-10} and 10^{-12} respectively.

The above plots show that even if two trajectories differ by very small amount, they start to deviate after some time.

Lyapunov spectrum

Lyapunov spectrum is calculated using the code LyapunovSpectrum.cpp.
The following element calculates the difference in the final positions of the two trajectories and stores it in the array `final_diff`.

```
//calculate difference in final values of the two trajectories
trajectory[0].diffInFinalVal(trajectory[0], trajectory[1], final_diff);
```

The following element calculates the Lyapunov spectrum

```
LyapunovSpectrum spectrum;
//calculate Jacobian matrix
spectrum.setJacobian(initial_diff, final_diff);
//calculate Jacobian*Trans(Jacobian) matrix
```

```
spectrum.setLambda();
//calculate eigenvalues of Jacobian*Trans(Jacobian) matrix
spectrum.calcEigenVal();
//calculate Lyapunov spectrum
spectrum.calcSpectrum(time);

long double lspectrum[4];
//store the Lyapunov spectrum in the array lspectrum
spectrum.getSpectrum(lspectrum);
```

- Let X_0 and Y_0 be two initial coordinates(at $t = 0$) in phase space leading to two different trajectories $X(t)$ and $Y(t)$ respectively. Coordinates are ordered as $(x_0, y_0, v_{x0}, v_{y0})$.

$$\begin{aligned} X_0 &= (0.1, 0.5, 0, 0.1), \\ Y_0 &= (0.1 + 10^{-20}, 0.5 + 10^{-20}, 0 + 10^{-20}, 0.1 + 2 \times 10^{-20}). \end{aligned}$$

  So, `initial_diff` has element $(10^{-20}, 10^{-20}, 10^{-20}, 2 \times 10^{-20})$.

- After time 40000, the `final_diff` stores the values $(-2894.26, 2.34996 \times 10^{-196}, -0.144793, -2.966 \times 10^{-194})$.

- The Jacobian matrix is calculated to be

$$J = \begin{bmatrix} -2.89426 \times 10^{23} & 2.34996 \times 10^{-176} & -1.44793 \times 10^{19} & -2.966 \times 10^{-174} \\ -2.89426 \times 10^{23} & 2.34996 \times 10^{-176} & -1.44793 \times 10^{19} & -2.966 \times 10^{-174} \\ -2.89426 \times 10^{23} & 2.34996 \times 10^{-176} & -1.44793 \times 10^{19} & -2.966 \times 10^{-174} \\ -1.44713 \times 10^{23} & 1.17498 \times 10^{-176} & -7.23965 \times 10^{19} & -1.483 \times 10^{-174} \end{bmatrix}.$$

- The matrix JJ^T is calculated to be

$$JJ^T = 10^{46} \times \begin{bmatrix} 8.37677 & 8.37677 & 8.37677 & 4.18838 \\ 8.37677 & 8.37677 & 8.37677 & 4.18838 \\ 8.37677 & 8.37677 & 8.37677 & 4.18838 \\ 4.18838 & 4.18838 & 4.18838 & 2.09419 \end{bmatrix}.$$

- Eigenvalues of JJ^T matrix are: $-1.6876 \times 10^{31},\ 2.72245 \times 10^{47},\ 0,\ 0.$

- Eigenvalues of the matrix

$$\Lambda = \lim_{t \to \infty} \frac{1}{2t} \ln(J(t)J(t)^T),$$

  are obtained as,

  `nan, 0.00136529, -inf, -inf.`

  Here $t = 40000$.

As can be seen only one eigenvalue has a definite value and other three are not defined. However, one of the value is 0.00136529, which is positive so the systemm is chaotic.

Conclusion

- Even for very small difference in initial values the trajectories deviate arbitrarily after sufficiently long time.

- One of the Lyapunov exponent is positive. So, the system is chaotic.