Name: Adarsh Vishwakarma
Institute: The Institute of Mathematical Sciences

Intrtoduction

Inside a neutron star, the gravitational force at every point is balanced by the outward pressure due to what is called the *neutron degeneracy pressure*. The mass inside a radius r is given by

$$m(r) = \frac{4\pi}{c^2}\int_0^r \rho(r')r'^2 dr', \tag{1}$$

or in differential form,

$$\frac{dm}{dr} = \frac{4\pi}{c^2}\rho(r)r^2, \tag{2}$$

where ρ is the density. For an isotropic, non-rotating star, the radial pressure gradient is given as,

- Nonrelativistic case,
$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\frac{\rho(r)}{c^2}. \tag{3}$$
- Relativistic case, called *Tolman-Oppenheimer-Volkov* equation,
$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3 P}{mc^2}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1}. \tag{4}$$

In order to solve the above set of eqaquations, they need to be supplemented with another relation between pressure and density. The relation is obtained from the *Equation of State*. There are various models for EoS for the Neutron Star one of which is *Bethe-Johnson Model*,

$$\rho(n) = 236n^{2.54} + m_N n, \tag{5}$$

here m_N is the mass of neutron, $m_N = 938.926\,\mathrm{MeV/c^2}$, ρ is in units of $\mathrm{MeV/c^2fm^{-3}}$, and n is the number density with dimension $[n] = \mathrm{fm}^{-3}$. Using $P = -\partial E/\partial V|_N$ and $E = c^2 V\rho(n)$, one obtains

$$P(n) = n\frac{\partial\rho(n)}{\partial n} - \rho(n) = 363.44\times n^{2.54}. \tag{6}$$

Using eqs (5) and (6),

$$\rho = \frac{P}{1.54} + \frac{m_N}{363.44^{1/2.54}}P^{1/2.54}. \tag{7}$$

Aim

1. mass-radius relation for a Neutron star.
2. Mass-Radius relation for the Neutron stars.
3. Mass vs Central density relation.

Algorithm

1. Given $\rho_c = \rho_i, i = 0$, begin from $r_i = 0, i = 0$.
2. Calculate P_{i+1} and m_{i+1} at $r_{i+1} = r_i + dr$ using the previous values.
3. Calculate ρ_{i+1} at $r_{i+1} = r_i + dr$ using eq(7).
4. Repeat from step 2 for successive values of i.
5. Stop at $P = 0$.

Dimensionless Equations

In relation (3) and (7), ρ and P have the same dimension as mass density. Eq(2) does not have any other constants than c. So, it is suitable to absorb c in ρ and make its dimension that of mass density. In eq(7), ρ and P have dimension of mass-energy density in units of $\mathrm{MeV/c^2 fm^{-3}}$. Choosing units $\mathrm{MeV/c^2}$ for mass and $\mathrm{MeV/c^2 fm^{-3}}$ for density and pressure, eqs (2) and (3) become

$$\frac{dm}{dr} = 4\pi\rho(r)r^2, \quad \frac{dP}{dr} = -\frac{G}{c^2}\frac{m(r)\rho(r)}{r^2}, \tag{8}$$

here $G = 1.18976283 \times 10^5$ $(\mathrm{MeV/c^2})^{-1}\mathrm{fm^3 s^{-2}}$, and $c = 2.99792458 \times 10^{23}$ fm/s.

To make the equation s dimensionless, let $m = M_0\hat{m},\ r = R_0\hat{r},\ \rho = \rho_s\hat{\rho}, P = \rho_s\hat{P}$, where $\hat{m}, \hat{r}, \hat{\rho}, \hat{P}$ are dimensionless. Using these in eqs(8) and (4), and making the equations dimensionless, we get

$$\frac{d\hat{m}}{d\hat{r}} = \hat{\rho}\hat{r}^2, \tag{9}$$

non-relativistic relation,

$$\frac{d\hat{P}}{d\hat{r}} = -\frac{\hat{m}\hat{\rho}}{\hat{r}^2}, \tag{10}$$

relativistic relation,

$$\frac{d\hat{P}}{d\hat{r}} = -\frac{(\hat{\rho} + \hat{P})(\hat{r}^3\hat{P} + \hat{m})}{\hat{r}^2 - 2\hat{m}\hat{r}} \tag{11}$$

with the conditions on constants

$$\frac{M_0}{R_0} = \frac{c^2}{G}, \tag{12}$$

$$\rho_s R_0^2 = \frac{c^2}{4\pi G}. \tag{13}$$

Choose M_0 and R_0 to be of the order of dimension of Neutron Stars so that the magnitudes of the quantities in eqs (9), (10), and (11) do not differ too much. The choice is to take R_0 to be 10km, and M_0 to be in units of solar mass, i.e. $R_0 = 10^{19}$ fm, and using eq(12),

$$M_0 = 6.7722184\, M_\odot,$$

where $M_\odot = 1.11544986 \times 10^{60}\mathrm{MeV/c^2}$ is the solar mass.
Using R_0 and M_0 in eq(13),

$$\rho_s = 601.133796\,\mathrm{MeV/c^2 fm^{-3}}.$$

Realistic Model and Equation of State

A Neutron star consists of different regions:

1. The atmosphere, about few centimeters thick, with a maximum density of about 1 g cm^{-3}.
2. The envelope, about 10-100 m in thickness. The density is about $10^{10} - 10^{11}$ g cm^{-3}.
3. Below the envelope, the nuclei are neutron rich. For about $\rho \gtrsim 4 \times 10^{11}$ g cm^{-3}, some neutrons drip out of the nucleus.
4. The densest part is the core. The density is about $\rho_s/4 - \rho_s/2$.

Different layers have different composition, consequently their equation of states are different. So, it is not accurate to estimate the equation of state with a single polytropic equation. A simple yet more realistic approach is to consider piecewice equation of states depending on the density in the regions. There are many models for it one of which is to consider a set of seven layers of regions with the atmosphere having density ρ_0 and increasing density as one goes towards the center:

$$\rho_0 < \rho_1 < \rho_2 < \rho_3 < \rho_4 < \rho_5 < \rho_6 < \rho_c,$$

where ρ_c is the density at the center.

The equation of state is given as[1]

$$\begin{aligned} P(\rho) &= K_i \rho^{\Gamma_i}, \\ \epsilon(\rho) &= (1 + a_i)\rho + \frac{K_i}{\Gamma_i - 1}\rho^{\Gamma_i}, \\ a_i &= \frac{\epsilon(\rho_{i-1})}{\rho_{i-1}} - 1 - \frac{K_i}{\Gamma_i - 1}\rho_{i-1}^{\Gamma_i - 1}, \end{aligned} \tag{14}$$

where ρ is the rest-mass density in unit g cm^{-3}, ϵ is the energy density in unit g cm^{-3}, K_i and Γ_i, $i = 0 - 6$ are constants. The values of ρ_i and Γ_i are fixed to better fit the observed data and to satisfy the physical constraints.

For low density regions are given by a low-density equation of state. There many models for it one of which is "SLy" that uses four polytropes whose values are given below.

K_i	Γ_i	ρ_i
6.80110×10^9	1.584 25	
1.06186×10^6	1.287 33	2.44034×10^7
5.32697×10^1	0.62223	3.78358×10^{11}
3.99874×10^8	1.35692	2.62780×10^{12}

K_i is in unit for which the corresponding P is in unit dyne/cm^2. The last density changes depending on the density level of the high-density EoS. The relevant value here is calculated to be $\rho_3 = 5.572667 \times 10^{13}$ g cm^{-3}. The high density parameters are obtained to be

$$\log(P(\rho_4)) = 34.384,\ \Gamma_4 = 3.005,\ \Gamma_5 = 2.988,\ \Gamma_6 = 2.851.$$

The values of K_4, K_5, and K_6 are calculated by demanding the pressure to be continuous thoughout the regions. The values are

$$K_4 = 1.6236787 \times 10^{-10},\ K_5 = 2.8866896 \times 10^{-10},\ K_6 = 3.276424 \times 10^{-8}.$$

The dimensionless equations in this case are

$$\hat{P} = |K_i| \rho_s^{\Gamma_i - 1} (1.7826619 \times 10^{12})^{\Gamma_i} (6.241509 \times 10^{-34}) \hat{\rho}^{\Gamma_i}, \tag{15}$$

$$\hat{\epsilon} = \frac{1 + a_i}{|K_i|^{1/\Gamma_i}} |c|^{2/\Gamma_i} (9.3316801 \times 10^{-16})^{1 - 1/\Gamma_i} \hat{P}^{1/\Gamma_i} + \frac{1}{\Gamma_i - 1}\hat{P}, \tag{16}$$

where ρ_s is same as defined before.

Results

Non-Relativistic Model

For a star with central density $\rho_c = 10^{15}\text{gm/cm}^3$, or $560.95886\text{MeV/c}^2\text{fm}^{-3}$, the variation of mass enclosed with radial distance from the center is obtained using Rung-Kutta method with fixed step-size and adaptive Runge-Kutta method. From herein, unless otherwise specified, Runge-Kutta method means fixed step-size RK method.

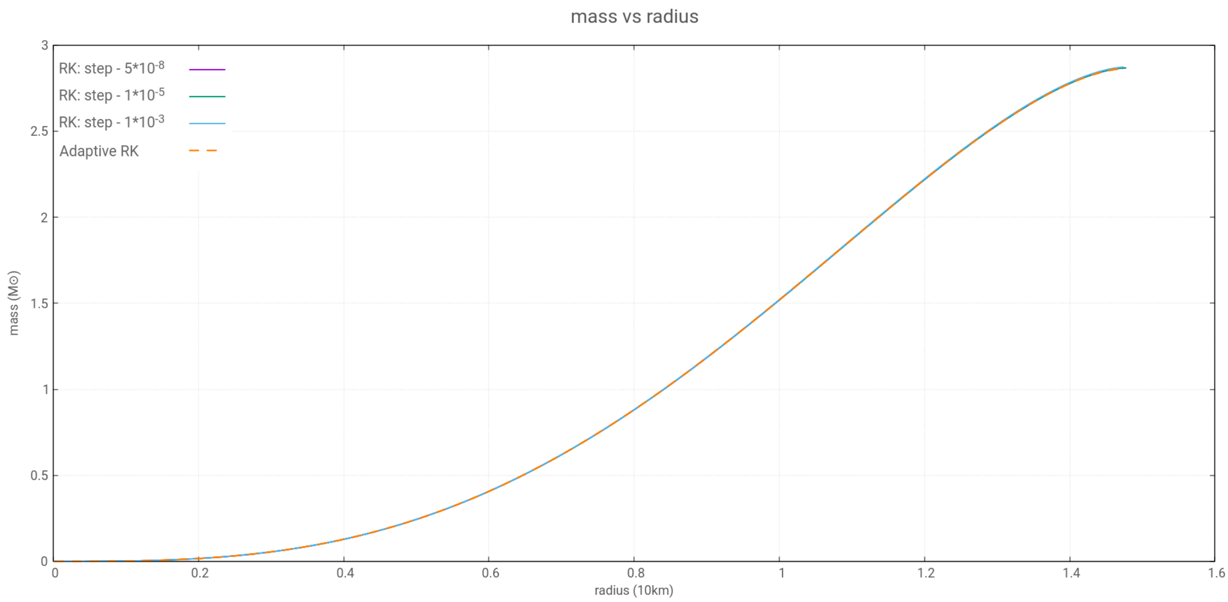

Figure 1: Variation of mass enclosed with the distance from the center.

A closeup of the above figure near the surface of the star.

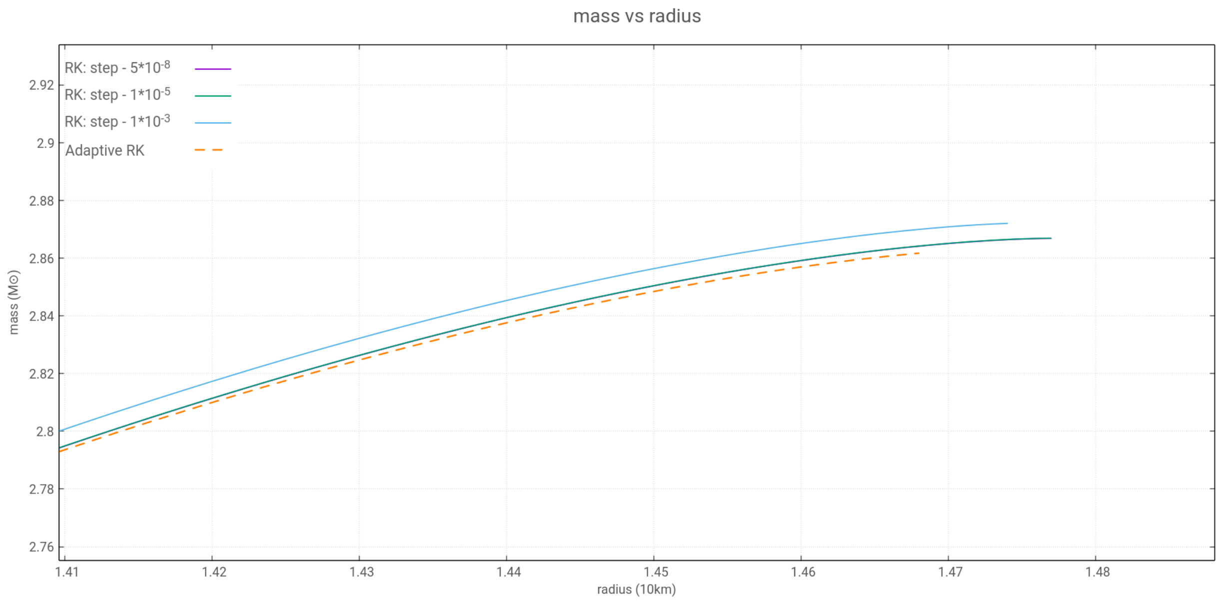

Figure 2: Variation of mass enclosed near the surface of the star.

Variation of density with the distance from the center of the star, obtained using RK method and adaptive RK, is plotted below.

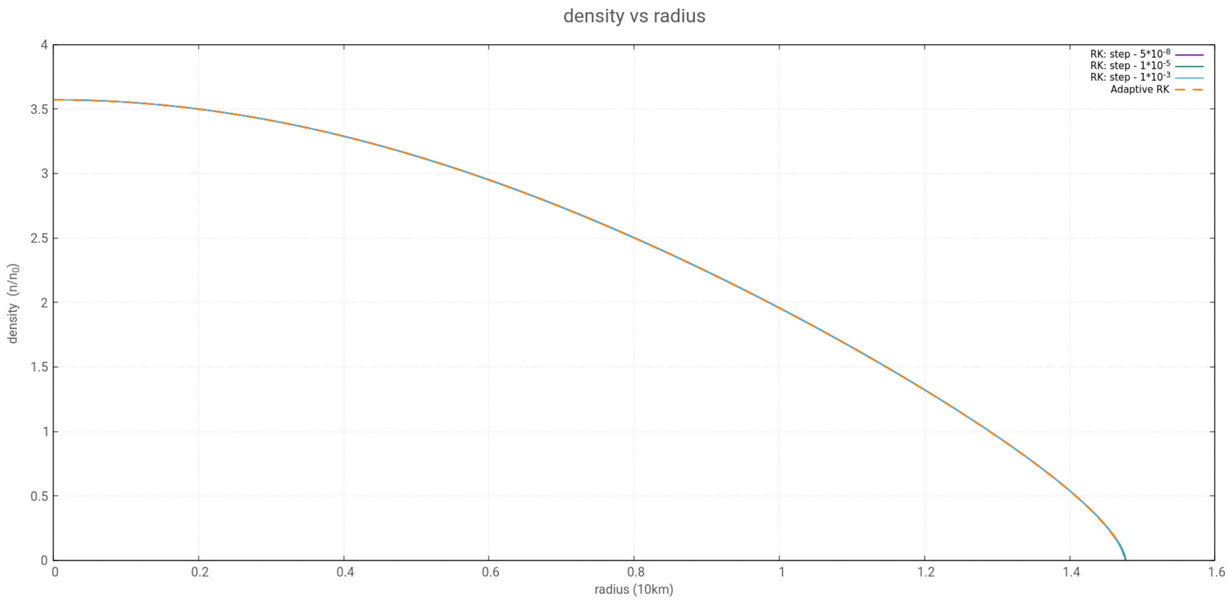

Figure 3: Variation of mass enclosed with the denisty.

A closeup of the above figure near the surface of the star.

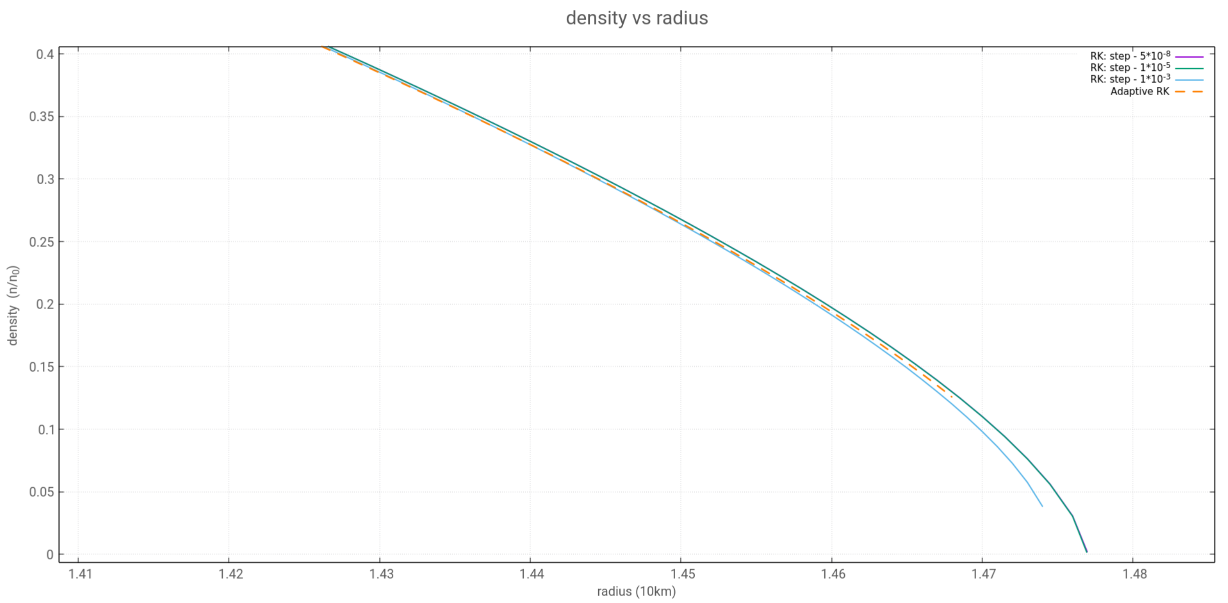

Figure 4: Variation of mass enclosed with the denisty near the surface of the star.

The central density parameterizes the total mass and radius of the Neutron star. Varying the central density one obtains different values of radius and mass of the star. Below the relation between total mass and radius of the Neutron star is plotted.

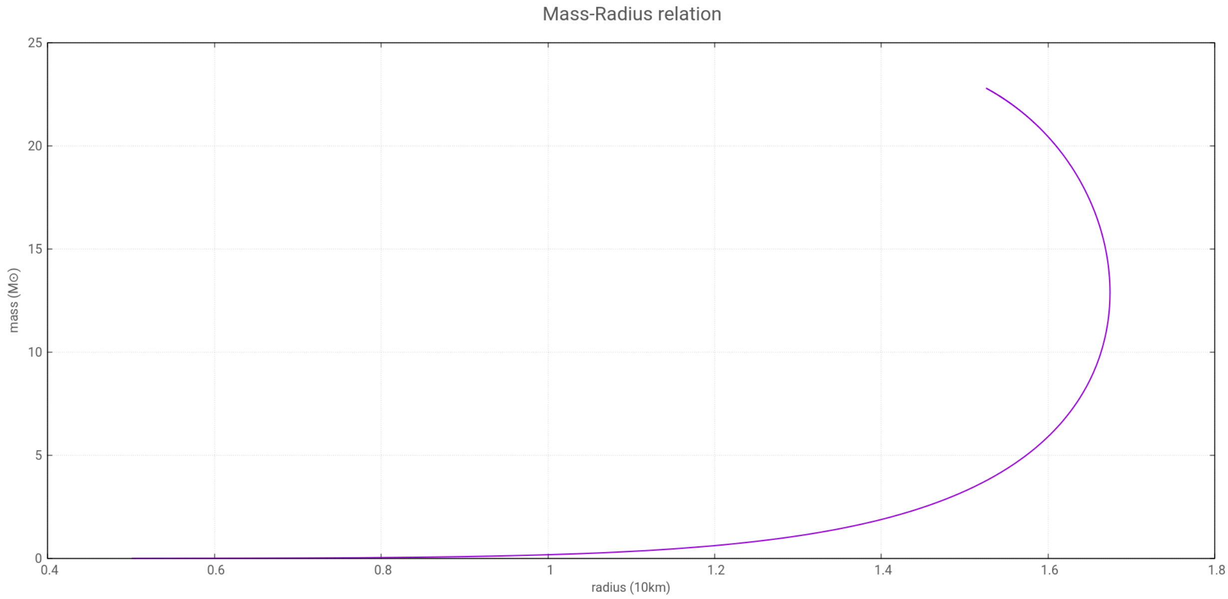

Figure 5: Relation between Mass of the star and its radius.

Variation of the mass of the star with the central density is plotted below.

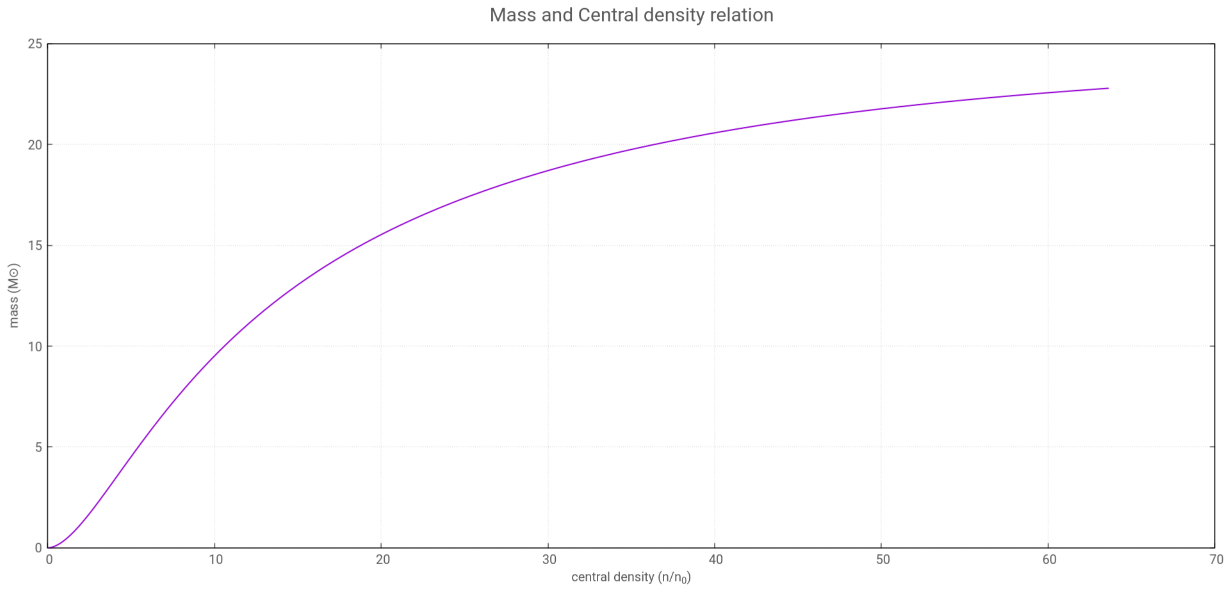

Figure 6: Relation between Mass of the star and its central density.

Variation of the radius with the central density of the star is plotted below.

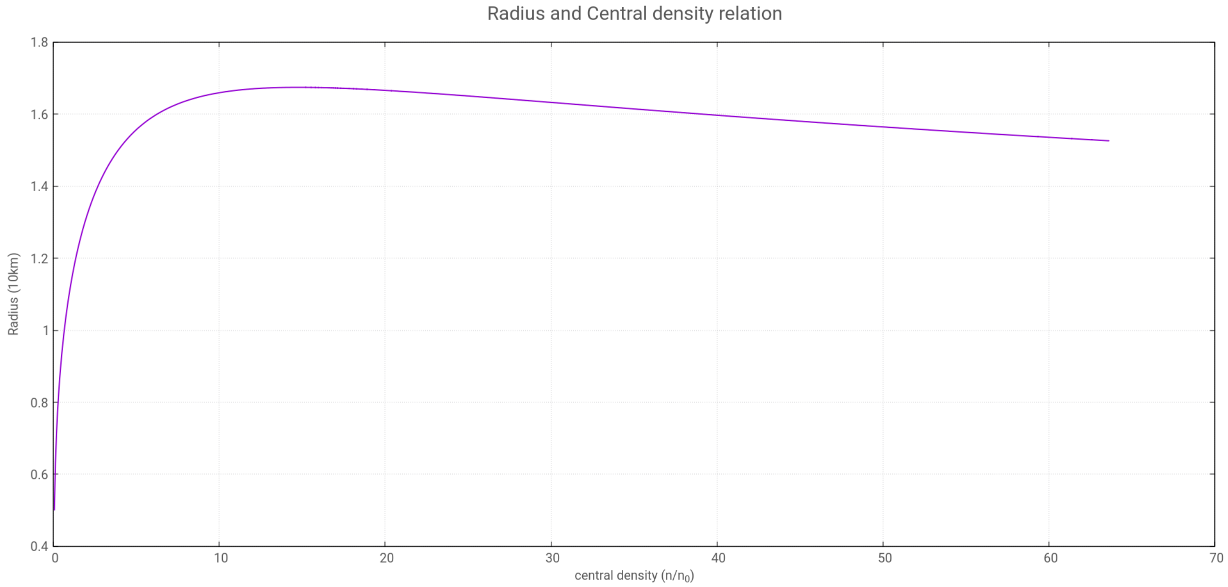

Figure 7: Relation between Radius and central density.

Relativistic Model

In this case, the TOV eq(11) is used to solve for the problem. For a fixed central density, 10^{18} kg/m^3, various relations are compared between Newtonian and Relativistic Models.

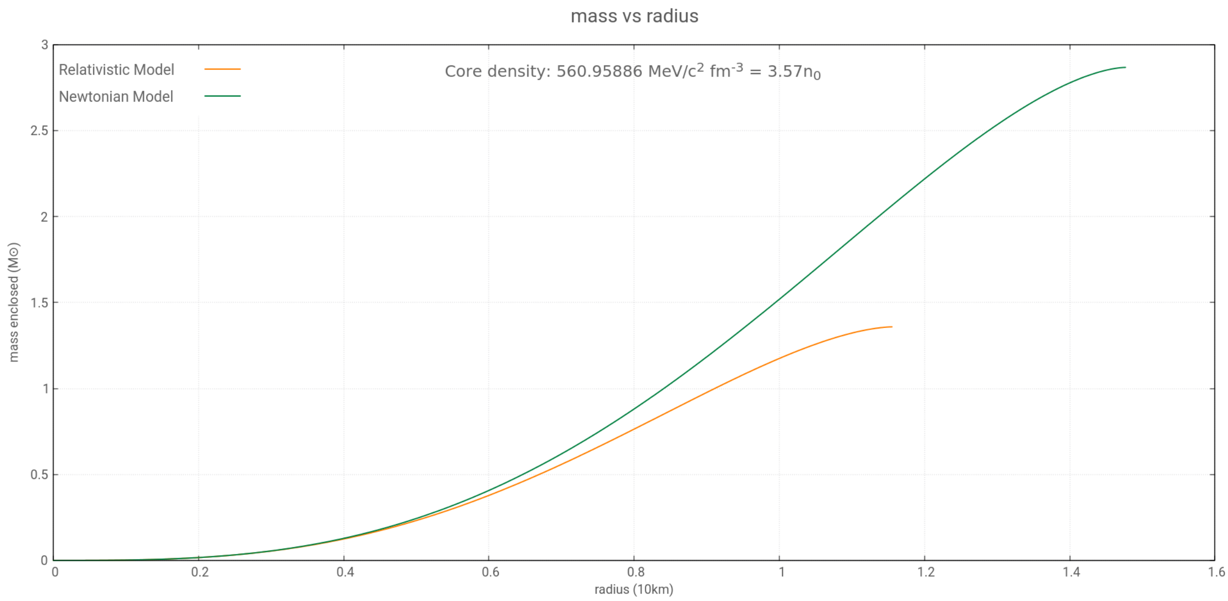

Figure 8: Variation of mass enclosed with disance from the center.

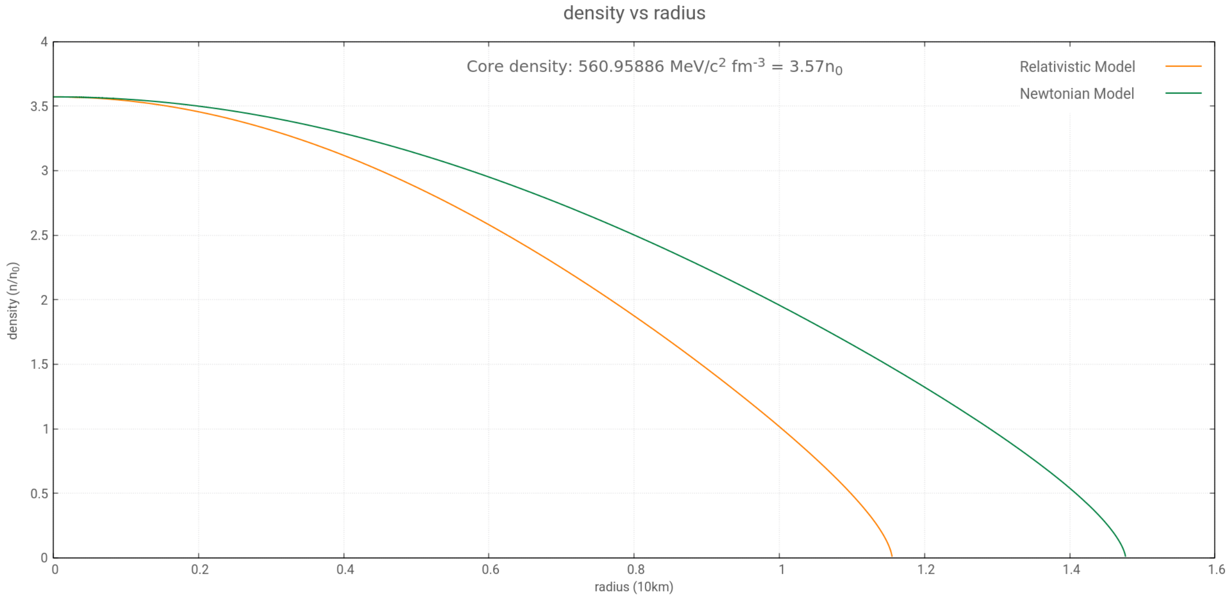

Figure 9: Radial variation of the density for the two models.

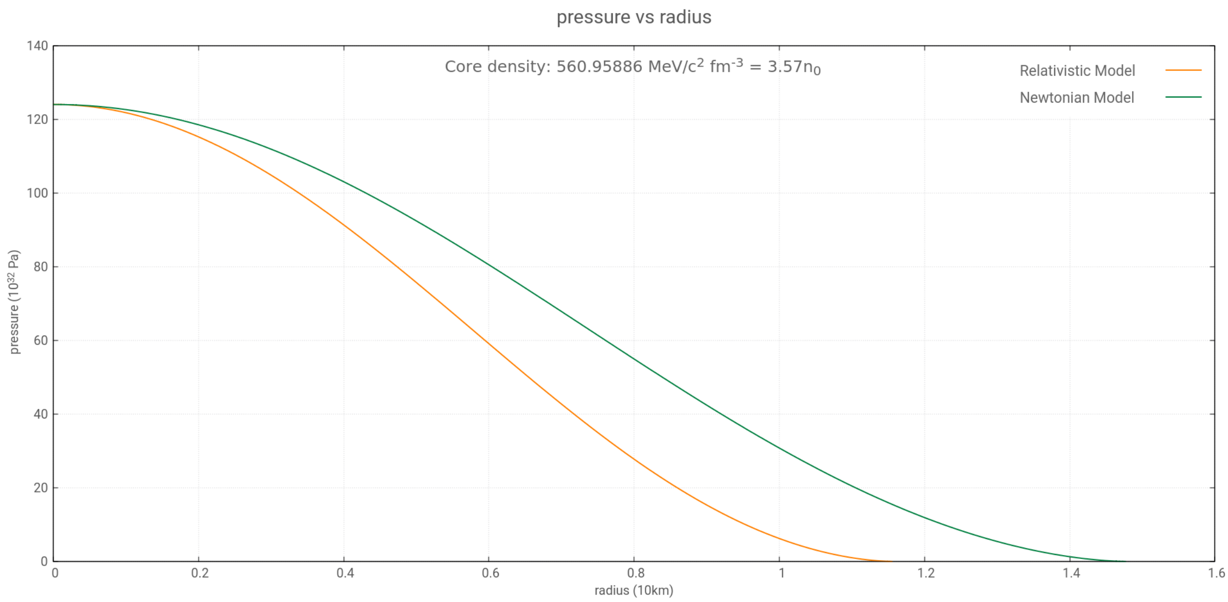

Figure 10: Radial variation of the pressure for the two models.

For the central density 10^{18} kg/m^3 inside a Neutron Star, the two models predict following for the Neutron star

	Mass ($M_\odot$)	Radius (10 km)
Non-relativistic	2.867	1.477
Relativistic	1.155	1.358

The Mass-Radius relation in the relativistic model is plotted below.

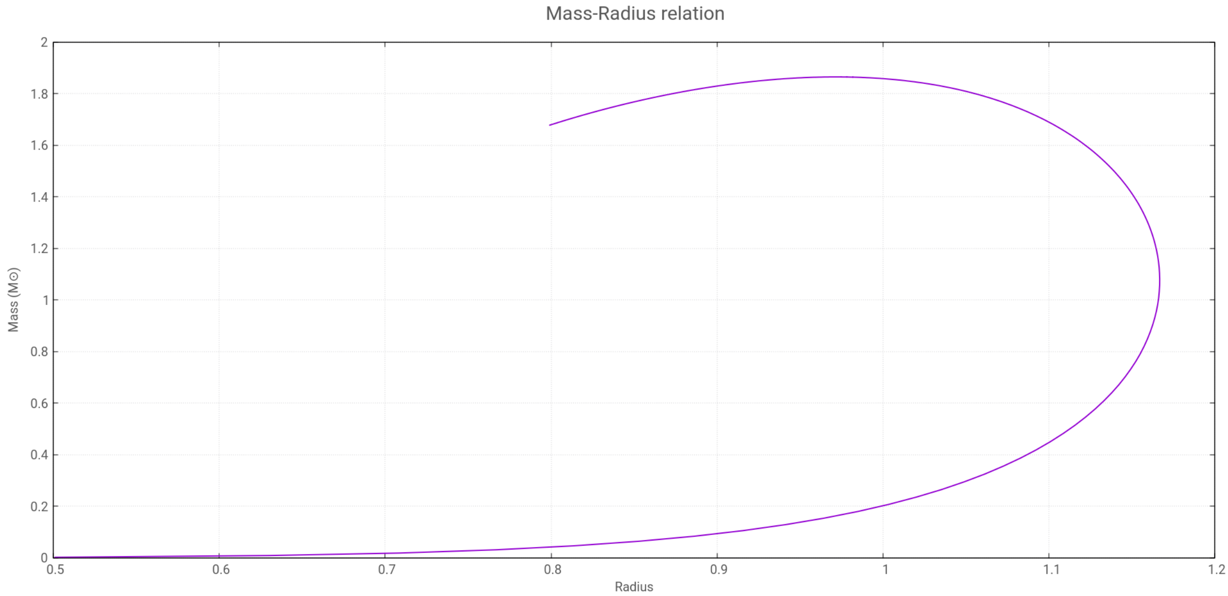

Figure 11: Mass-Radius relation obtained using TOV equation.

Relations between Radius and central density, and mass and central density are plotted below.

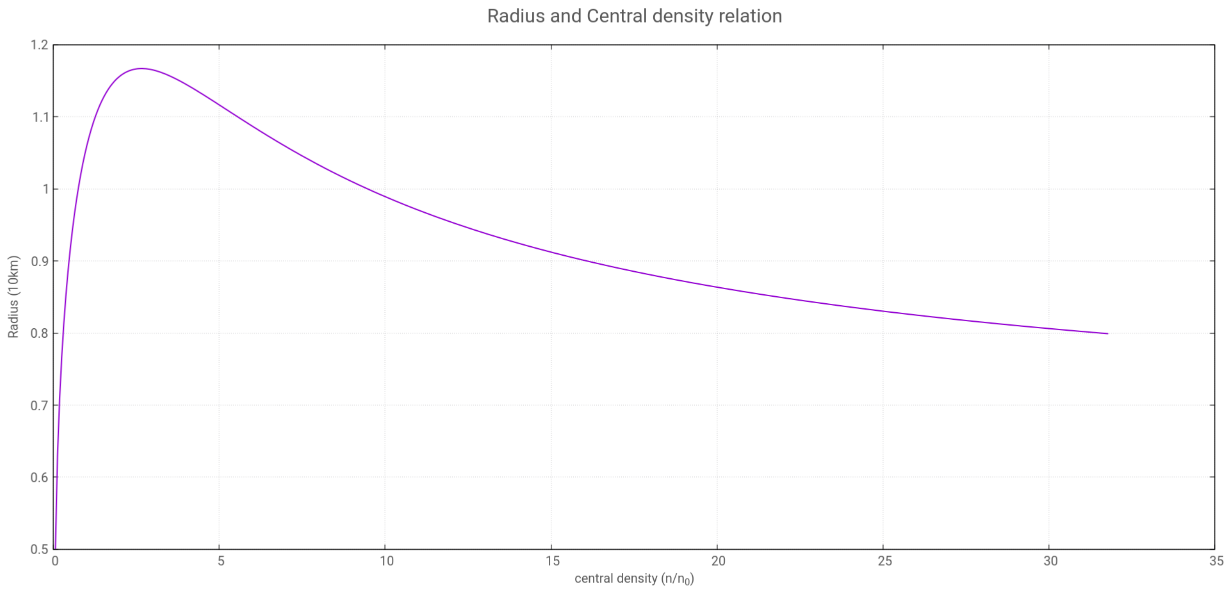

Figure 12: Relation between radius and central density for relativistic model.

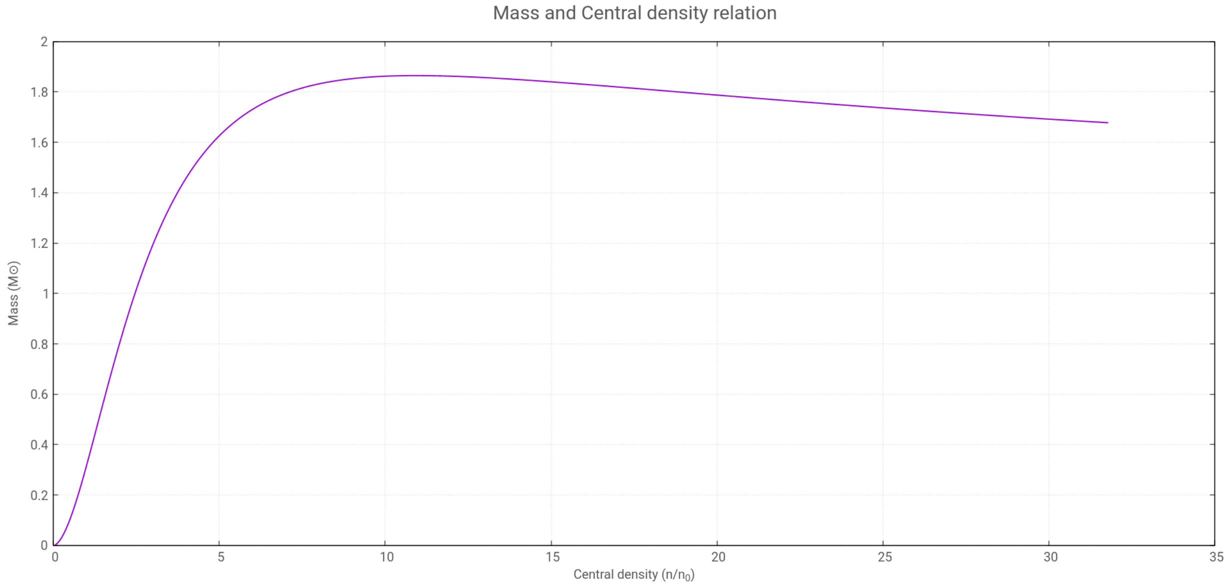

Figure 13: Mas and central density relation in the relativistic model.

Upper limit for the mass of Neutron Star and the Maximum Radius, as predicted by the relativistic model, are obtained below

Central density (n_0)	
10.95	Max Mass 1.86464 $M_{\odot}$
2.67	Max Radius 1.667 km

Conclusions

Algorithm

1. The adaptive RK method changes the step size in each step of the integration to give precise result. From figs (2) and (4), the result of simple RK method approaches that of the adaptive case as the step size reduces.

2. For a given tolerance for pressure near surface of the star, the coarser step-size gives smaller radius and higher mass of the star.

Family of Stars

With a given Equation of State, following are the predictions of the non-relativistic and relativistic equations:

1. For the non-relativistic case, the higher the central density the larger is the mass of the Neutron star, however, such is not the case in the relativistic model; after a certain critical central density, the mass decreases with increasing central density.

2. There seemed to be no upper limit for the Neutron star as predicted by the Newtonian Model whereas the relativistic model predicts an upper limit for the mass of the neutron star.

3. Both Models predict an upper limit for the radius of the Neutron star. The relativistic model gives tighter limit than the non-relativistic one.

Single Star predictions

1. For a given central density, the relativistic equation predicts a compact and lighter neutron star than the one predicted by the non-relativistic equation.

2. The form of the relation between quantities like pressure, density, radial distance, and mass enclosed seems to be similar for both relativistic and non-relativistic cases.

Questions

(1) In figures (5) and (11), why the radius start decreasing after certain maximum value?

The decrease in radius after a certain critical density is seen in both models, Newtonian and Relativistic, so, I took the simpler case where the same is happening.

Below is a figure that shows pressure variation in newtonian model for four different central densities. The central density $2310.006147\mathrm{MeV/c^2fm^{-3}}$ corresponds to the the maximum radius.

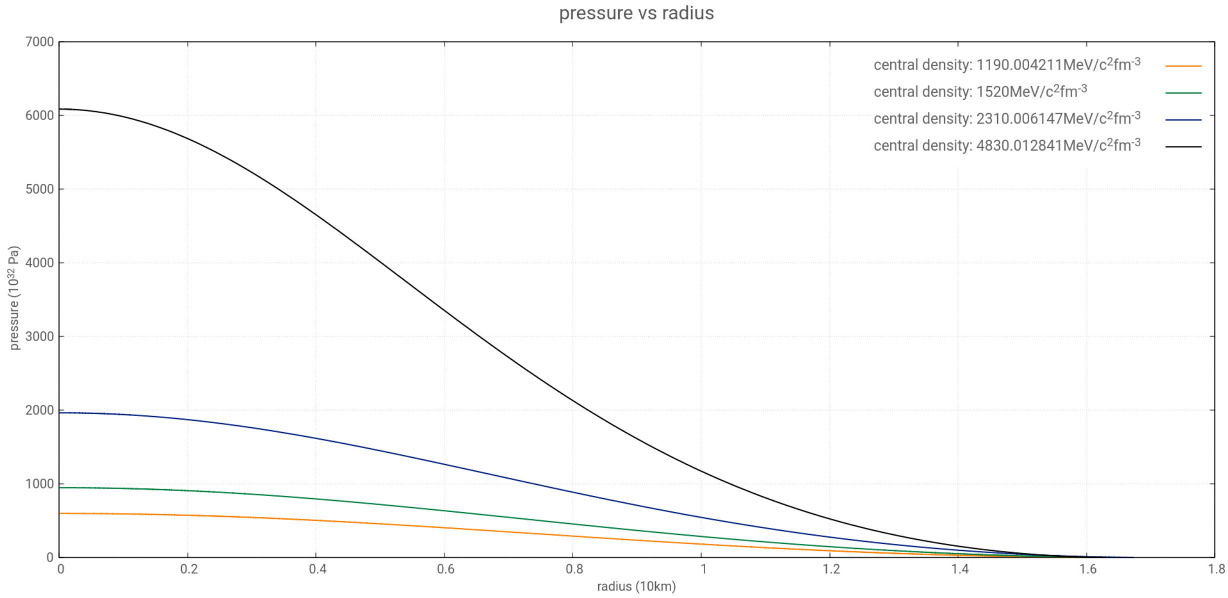

Figure 14: Variation of pressure with radius for different central densities.

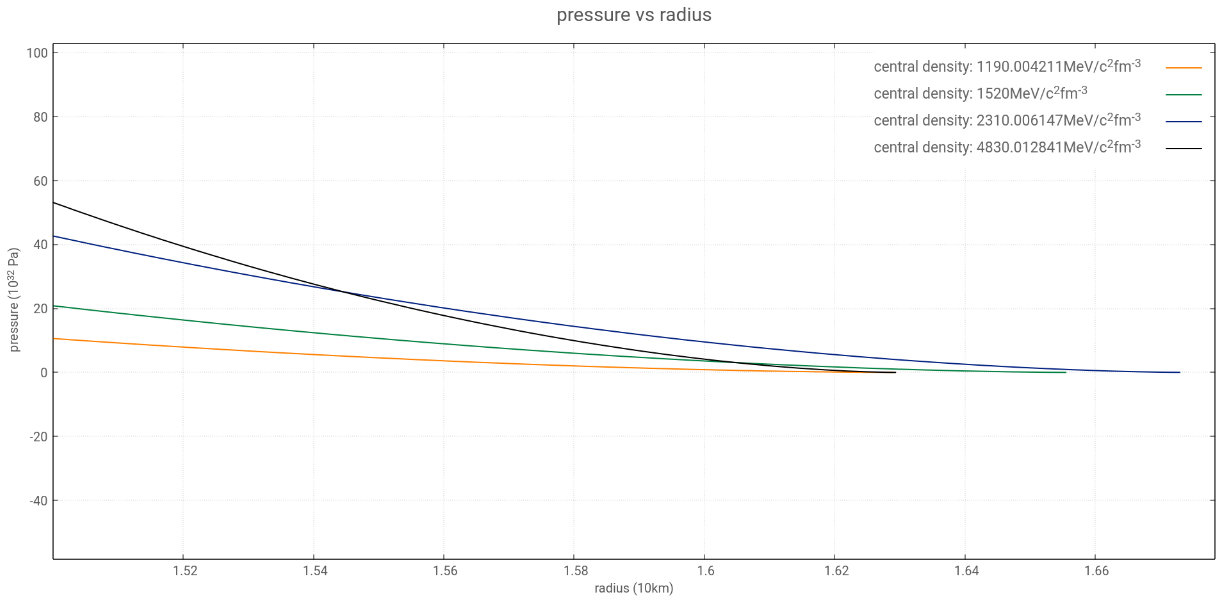

Figure 15: A closeup of the fig(14) near the surface of the star.

From the above figure, starting from the density $1190.004211\mathrm{MeV/c^2fm^{-3}}$, the radius at which pressure drops to zero is increasing with increasing central density upto a central density $2310.006147\mathrm{MeV/c^2fm^{-3}}$ after which the radius starts decreasing showing that the absolute value of the pressure gradient increases rapidly with central density so that at higher densities the pressure drops to zero rapidly leading to the shrinking of star after certain critical mass.

I tried to see what effect the EOS has in this Mass-Radius relation. In the below I plotted below the Mass-Radius relation for an unphysical EOS: $\rho = P/1.54$. This is unphysical because P cannot be linear in ρ without temperature dependence. Moreover, using thermodynamic relation on this equation would yield contradictory result. I used this only for mathematical purpose: to see how a linear relation in ρ and P would translate into the Mass-Radius relation.

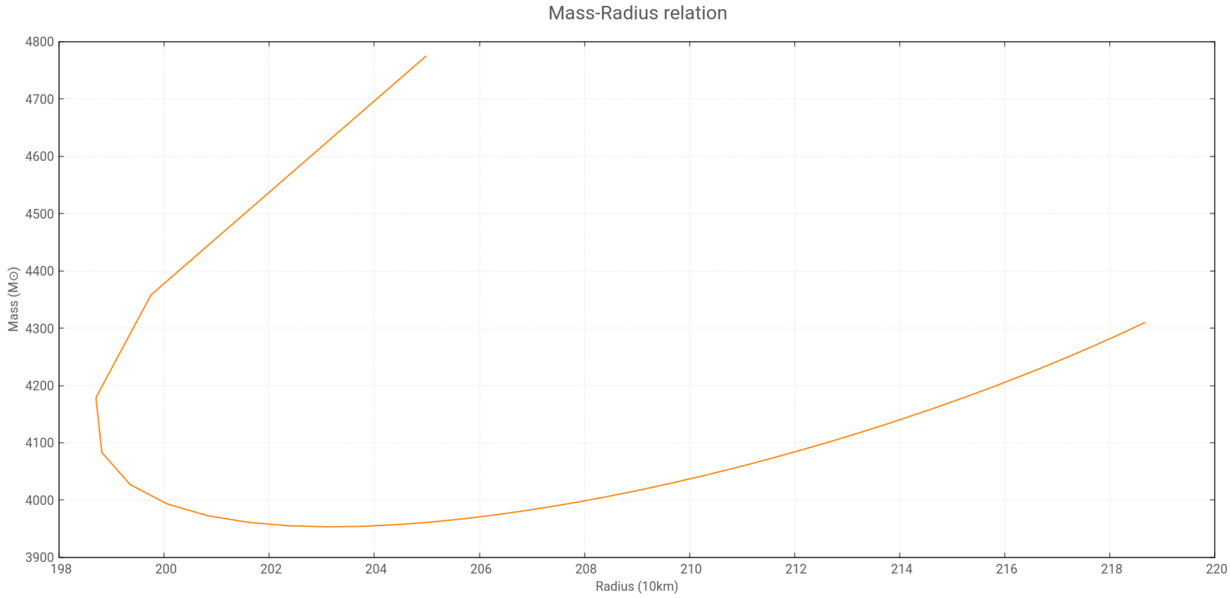

Figure 16: Mass-radius relation for the EOS $\rho = P/1.54$.

The above figure is showing even more wierder result. In the above figure the central density is very small starting from the top end of the curve. As the central density is increased, the mass start to decrease! until a point after which the mass and radius both seem to show monotonic increment with central density.

To obtain the above figure I had to keep the pressure tolerance relatively high than the previous cases because it was taking so much time to reach that tolerance. So, it is still not clear if this decrease in mass is a numerical error or correct for the EOS $\rho = P/1.54$.

In any case, it seems that the decrease in radius after certain critical central density has to do with the equation of state, which I have to explore yet.

References

1. Jocelyn S. Read, Benjamin D. Lackey, Benjamin J. Owen, and John L. Friedman Phys. Rev. D 79, 124032 – Published 22 June 2009.

2. COMPUTATIONAL PHYSICS - Morten Hjorth-Jensen.