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1. [7 points] *Understanding asymptotic notation*

   For each of the following statements, prove that it is true or give a counterexample to prove that it is false. If you give a counterexample, you still have to prove that your example is, indeed, a counterexample.

   (a) If $f(n)$ is $\mathrm{O}(g(n))$ then $g(n)$ is $\mathrm{O}(f(n))$.

   (b) If $f(n)$ and $g(n)$ are both monotonically nondecreasing and asymptotically positive, and $f(n)$ is $\Theta(g(n))$, then $\lg f(n)$ is $\Theta(\lg g(n))$.

   (c) If $f(n)$ is $\Theta(g(n))$, then $2^{f(n)}$ is $\Theta(2^{g(n)})$.

   (d) If $f(n)$ is $\mathrm{O}(g(n))$ then $g(n)$ is $\Omega(f(n))$.

   (e) $f(n)$ is $\Theta(f(n/2))$.

   (f) If $g(n)$ is $\mathrm{o}(f(n))$ then $f(n)+g(n)$ is $\Theta(f(n))$

   (g) If $f(n)$ and $g(n)$ are both (asymptotically) positive, then $f(n)+g(n)$ is $\Theta(\max(f(n), g(n)))$.

Answer:

(a) $f(n) \in \mathrm{O}(g(n)) \quad \Longrightarrow \quad f(n) \leq c \cdot g(n) \quad$ for some $c > 0, n \geq n_0$
Let $g(n)$ be n^2.
$\therefore \; f(n) = \mathrm{O}(n^2) = 1$

Now, if $g(n) = \mathrm{O}(f(n)) \quad \Longrightarrow \quad g(n) \leq c' \cdot f(n) \quad$ for some $c' > 0, n \geq n_0$,
which might be true for some cases.
But, in our case above, $\quad n^2 \leq \mathrm{O}(1) \quad$ *i.e.* $\quad n^2 \leq c'$
which is not possible, as n^2 can be bigger than some constant c'.

Hence, given statement is FALSE.

(b) $f(n) \in \Theta(g(n)) \quad \Longrightarrow \quad c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$,
for some $c_1, c_2 > 0,\ n \geq n_0,\ c_1 < c_2$
Taking log: $\quad \lg c_1 \cdot g(n) \leq \lg f(n) \leq \lg c_2 \cdot g(n)$

As both $f(n)$ and $g(n)$ are monotonously nondecreasing and asymptotically positive, and both constants c_1 and c_2 are positive, this modified inequality will also hold true.

$\therefore$ We can rewrite it as -
$c_1' \cdot g(n) \leq \lg f(n) \leq c_2' \cdot \lg g(n) \quad \Longrightarrow \quad \lg f(n) = \Theta(g(n))$

Hence, given statement has been proved TRUE.

(c) $f(n) \in \Theta(g(n)) \quad \Longrightarrow \quad c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$,
for some $c_1, c_2 > 0,\ n \geq n_0,\ c_1 < c_2$

Lets split this into two bounds - Upper and lower.
If O() and Ω() holds true, then Θ() would hold true as well.

Let $g(n) = n, f(n) = 2n$

Hence, $2n = O(n)$
But, $2^{2n} \neq O(2^n)$
As the upper (O()) bound does not hold true, the $\Omega()$ bound will also not hold.

Hence, given statement is FALSE.

(d) $f(n) \in O(g(n)) \implies f(n) \leq c \cdot g(n)$, for some $c > 0, n \geq n_0$

Dividing by c: $\frac{1}{c} \cdot f(n) \leq g(n)$
Let $k = \frac{1}{c}$, $k \cdot f(n) \leq g(n)$
i.e. $g(n) \geq k \cdot f(n)$
$\implies$ $g(n) = \Omega(f(n))$

Hence, given statement has been proved TRUE.

(e) If $f(n) \in \Theta(f(n/2))$,
then, it will also be true that $f(n) = O(f(n/2))$, considering the upper bound.
Let$f(n) = n^2$
Then, $2^n \leq c \cdot 2^{n/2}$, for some $c > 0,\ n \geq n_0$
$\therefore$ $2^{n/2} \leq c$
which is not true for all cases,
as $2^{n/2}$ is an exponentially growing, unbounded term; whilst c is a fixed constant.

Hence, given statement is FALSE.

(f) $g(n) \in o(f(n)) \implies g(n) \leq c \cdot f(n)$, for some $c > 0,\ n \geq n_0$

$\therefore \frac{g(n)}{f(n)} < c$
Adding 1 to both sides, and let $c' = c + 1$,

$1 + \frac{g(n)}{f(n)} < c'$

$\frac{f(n)+g(n)}{f(n)} < c'$

Multiplying by $f(n)$: $c' \cdot f(n) \leq f(n) + g(n)$ or $f(n) + g(n) \leq c' \cdot f(n)$

$\therefore f(n) + g(n) = \Theta(f(n))$

Hence, given statement has been proved TRUE.

(g) $f(n)$ and $g(n)$ are both asymptotically positive. $\therefore f(n) > 0 \quad = \quad g(n) > 0$

Now, let $max(f(n), g(n)) = \begin{cases} f(n) & \text{if } f(n) \geq g(n) \\ g(n) & \text{if } g(n) > f(n) \end{cases}$

$\therefore$ If$f(n) \geq g(n)$, then $f(n) + g(n) \geq max(f(n), g(n)) > 0$, *i.e.* $f(n) + g(n) \geq h(n) > 0$
and if $g(n) \geq f(n)$, then $f(n)+g(n) \geq max(f(n), g(n)) > 0$, *i.e.* $f(n)+g(n) \geq g(n) > 0$

Now, if we assume $c_1 = 1$ in, $f(n) + g(n) \geq c_1 \cdot max(f(n), g(n))$,
then, we can write - $f(n) + g(n) \in \Omega(max(f(n), g(n)))$ (1)

Also, if $f(n) \geq g(n)$, then, $max(f(n), g(n)) \geq g(n) > 0$ (2)
and if $g(n) \geq f(n)$, then, $max(f(n), g(n)) \geq f(n) > 0$ (3)

Adding (2) and (3):
$0 < f(n) + g(n) \leq 2 \cdot max(f(n), g(n))$

Now, if we assume $c_2 = 2$ in, $f(n) + g(n) \geq c_2 \cdot max(f(n), g(n))$,
then, we can write - $f(n) + g(n) \in \mathrm{O}(max(f(n), g(n)))$ (4)

From (1) and (4):
$f(n) + g(n) \in \Theta(max(f(n), g(n)))$

Hence, given statement has been proved TRUE.