Aditya Shirode, Rhythm Shah, Shrenuj Gandhi

3. [6 points] *Understanding asymptotic notation*

In proving big-oh and big-omega bounds there is a relationship between the c that is used and the smallest n_0 that will work (for big-oh, the smaller the c, the larger the n_0; for big-omega, the larger the c, the larger the n_0). In each of the following situations, describe (the smallest *integer*) n_0 as a function of c. You'll have to use the ceiling function to ensure that n_0 is an integer. Your solution should also give you a lower bound (for big-oh) or an upper bound (for big-omega) on the constant c.

(a) Let $f(n) = 3n^3 + 5n^2$ and prove that $f(n) \in \mathrm{O}(n^3)$.

(b) Let $f(n) = 2n^3 - 7n^2$ and prove that $f(n) \in \Omega(n^3)$.

(c) Let $f(n) = 5n^3 + 2n^2$ and prove that $f(n) \in \mathrm{O}(n^4)$. Note that the exponent on n is 4.

2 points for each part

Answer:

(a) $f(n) = 3n^3 + 5n^2$

$f(n) \in \mathrm{O}(g(n)) \quad$ iff $\quad f(n) \le cg(n), \qquad$ for some $c > 0$, $n \ge n_0$

$$\begin{aligned} 3n^3 + 5n^2 &\le cn^3 \\ 3n + 5 &\le cn \\ n(c-3) &\ge 5 \\ n &\ge \left\lceil \frac{5}{c-3} \right\rceil, \end{aligned}$$

which gives the lowest bound on c as $c > 3$

$\therefore$ Smallest $n_0 = 1$ and corresponding c can be 8.

As both values of c and n_0 are valid, we can say that $f(n) \in \mathrm{O}(n^3)$

(b) $f(n) = 2n^3 - 7n^2$

$f(n) \in \Omega(g(n)) \quad$ iff $f(n) \ge cg(n), \qquad$ for some $c > 0$, $n \ge n_0$

$$\begin{aligned} 2n^3 - 7n^2 &\ge cn^3 \\ 2n - 7 &\ge cn \\ n(2-c) &\ge 7 \\ n &\ge \left\lceil \frac{7}{2-c} \right\rceil, \end{aligned}$$

which gives the upper bound on c as $c < 2$

If we substitute c as 0.01, we get n as $\lceil 3.51 \rceil$

$\therefore$ Smallest $n_0 = 4$

As both values of c and n_0 are valid, we can say that $f(n) \in \Omega(n^3)$

(c) $f(n) = 5n^3 + 2n^2$

$$\begin{aligned} 5n^3 + 2n^2 &\le cn^4 \\ 5n + 2 &\le cn^2 \\ cn^2 - 5n - 2 &\ge 0 \end{aligned}$$

This is the relationship between n and c.
Lets take the smallest n_0 i.e. $n_0 = 1$.
By solving the above equation, we get: $7 \leq c$.
$\therefore$ Lower bound of c is 7.

As both values of c and n_0 are valid, we can say that $f(n) \in \mathrm{O}(n^4)$