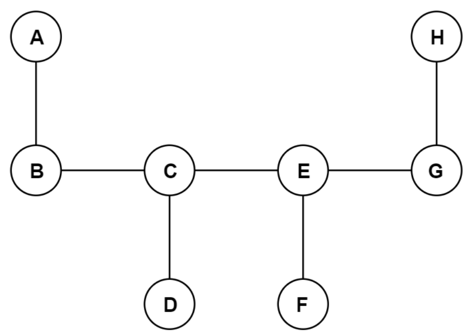
Aditya Shirode, Rhythm Shah, Shrenuj Gandhi

2. [4 points each part]

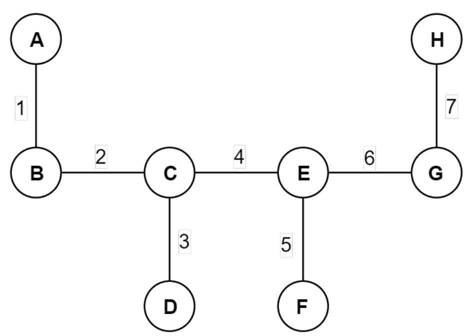
(a) Give an example of an unweighted free tree T with eight vertices, that, depending on how weights are assigned to the edges will cause Boruvka's algorithm to take one, two or three stages to find the minimum spanning tree (which will be T). Draw T or list its edges, give the three different weight assignments, and trace the stages of Boruvka's algorithm for each of the three assignments.

Answer:

Consider the below unweighted free tree T.

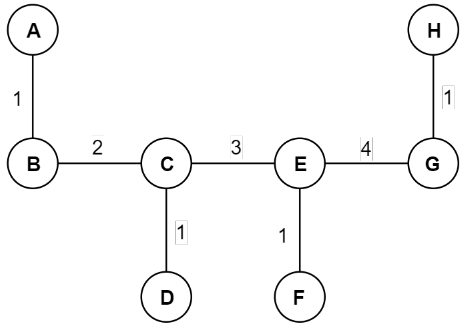


Case 1: When Boruvka's algorithm takes **one** stage to find MST

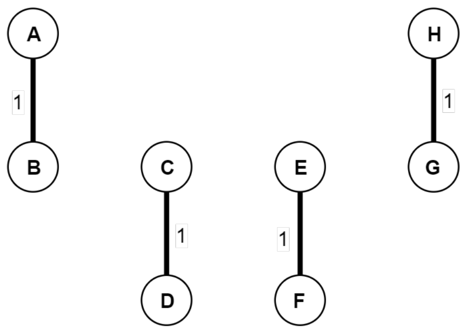


The trace of the above graph is same as the graph.

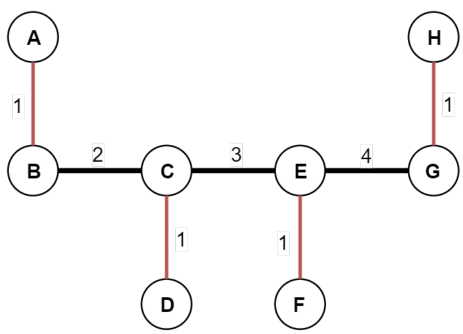
Case 2: When Boruvka's algorithm takes **two** stages to find MST



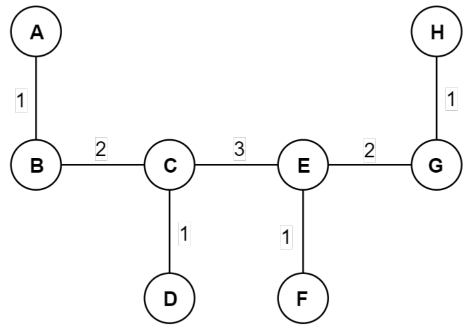
`Stage 1:`

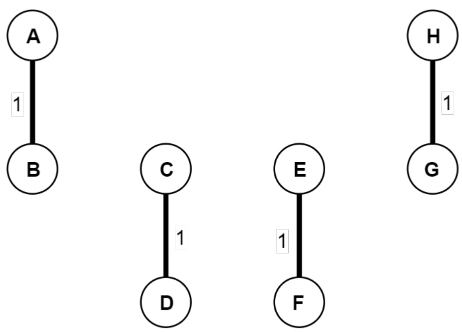


`Stage 2:`

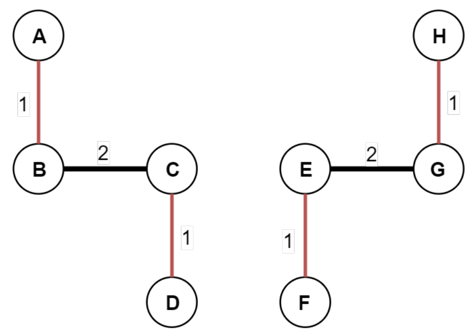


Case 3: When Boruvka's algorithm takes **three** stages to find MST

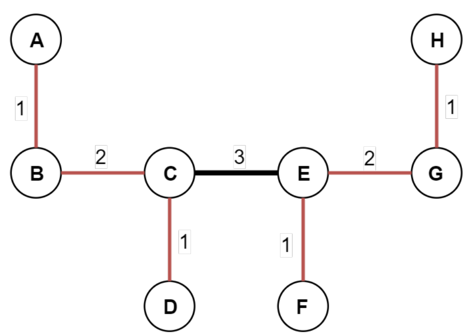




Stage 1



Stage 2

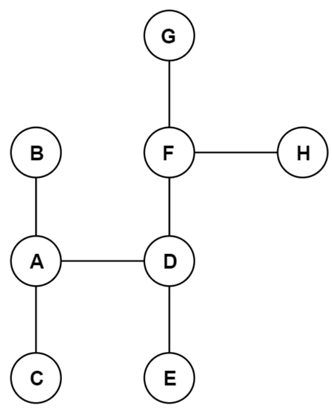


Stage 3

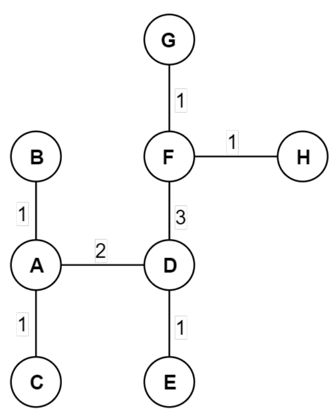
(b) Give an example of an unweighted free tree T with eight vertices, none of which have degree more than three, that, no matter how edge weights are assigned, cannot cause Boruvka's algorithm to require more than two stages. Draw T or list its edges, give weight assignments that result in one or two stages and do the tracing. In addition, prove, using the structure of T only (no algorithm details except for the main part of the time bound proof), that it is not possible for the algorithm to do three stages for any weight assignment.

Answer:

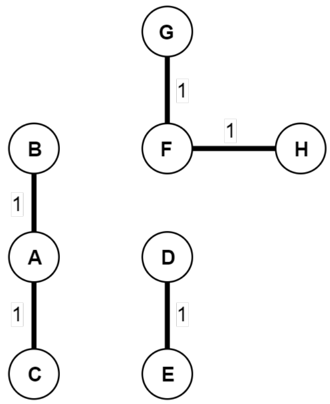
The following is an example of an unweighted free tree with 8 vertices (none of which have degree *more than 3*. For this tree no matter how edge weights are assigned, they cannot cause Boruvka's algorithm to require more than *two* stages.



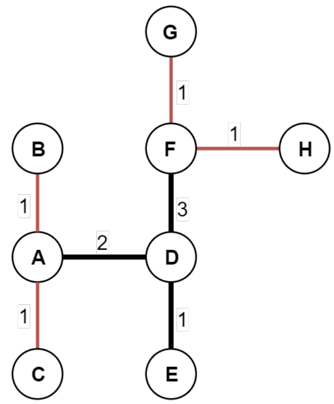
Assume the following weights assignment:
$AB = 1, AC = 1, AD = 2, DE = 1, DF = 3, FG = 1, FH = 1$



Traces:



`Stage 1`



`Stage 2`

In the first stage, at most 3 components are bound to be formed. (In our case, they are $[(a, b, c), (b, g, h), (d, e)]$. In the second stage, each component will connect to atleast one other component.
Hence even if two components connect to each other (the edge between them is minimum for them), the third component will connect to either of the two previous components, forming a MST.

Thus, from the above example, we can argue that if an unweighted free tree has **5 nodes** out of total 8 with **Degree 1**, then it will not cause Boruvka's to take more than two stages. We are bound to get at most three connected components in such a case at the end of first stage of Boruvka's, which leads to a MST at the end of second stage.