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5. [4 points]

The *closest point heuristic* for finding a traveling salesperson tour when distances satisfy the triangle inequality works as follows. Start with a trivial cycle C consisting of an arbitrary single vertex. At each step, identify the vertex u that is not part of C and insert it into C immediately after the vertex of C to which it is closest. Repeat until all vertices are in C. Prove that the closest point heuristic find a tour whose cost is no more than twice that of the optimum tour.

Answer:

Let T be the cost of tour found by *closest point heuristic*.
Let T^* be the cost of the optimal traveling salesperson tour.
Let S be the total cost of edges added at each step.

Consider three vertices u, v, *and* w, where v is closer to u than w.
As the problem states, by definition of *closest point heuristic*,
v will be added to the trivial cycle consisting of u, before w. ($u \to v$, $v \to w$)

The increase in the cost by going $u - v - w$ over $u - w$ is equal to $c(v, w) - c(u, w)$.

Using the triangle inequality, we get -

$c(v, w) \leq c(v, u) + c(u, w)$ where $c(x, y) \leftarrow$ cost for edge from x to y

$\implies c(v, w) - c(w, v) \leq c(v, u)$

$\therefore c(v, u) + c(u, w) - c(w, v) \leq 2c(v, u)$ which is S + increase in the cost of the tour

$\therefore T \leq 2S$ (1)

We can see that *closest point heuristic* is adding new vertices to the cycle like *Prim's* algorithm. And we know that, in case of *Prim's*, the cost of the MST formed (by summing the cost of each added edge) $\leq$ the optimal tour.

$\therefore S \leq T^*$ (2)

From (1) and (2), $T \leq T^*$.