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2. [5 points] *Understanding asymptotic notation*

   An alternate definition of Ω, call it Ω^∞, is defined as follows: $f(n) \in \Omega^\infty(g(n))$ if there exists $c > 0$ such that $f(n) \geq cg(n) \geq 0$ for infinitely many integers $n > 0$. If $f(n) \in \Omega^\infty(g(n))$ then you can't hope to show $f(n) \in \mathrm{O}(h(n))$ for any $h(n)$ that grows more slowly than $g(n)$.

   (a) Obviously $f(n) \in \Omega(g(n))$ implies $f(n) \in \Omega^\infty(g(n))$. Prove that the other direction is not true, i.e., prove that $f(n) \in \Omega^\infty(g(n))$ *does not imply* $f(n) \in \Omega(g(n))$.

   (b) Prove that, for any two non-negative functions $f(n)$ and $g(n)$, either $f(n) \in \mathrm{O}(g(n))$ or $f(n) \in \Omega^\infty(g(n))$ (or both) and that the same is not true if we replace Ω^∞ with Ω.

   1 point for part (a), 3 points for the first part of (b), 1 points for the counterexample in part (b).

Answer:

(a) If for some $c > 0$, $\quad f(n) \in \Omega^\infty(g(n)), \quad$ then, $f(n) \geq cg(n) \geq 0$
This will hold true for infinite integer values of $n > 0$.

Now, if $f(n) \in \Omega(g(n)) \quad \Longrightarrow \quad f(n) \geq c_1 g(n), \quad c_1 > 0, n \geq n_0.$

As above inequality holds true for values of $n \geq n_0$, and not all $n > 0$,
it is a subset of Ω^∞ case.

Lets prove this with an example.
Assume $f(n) = n, g(n) = \frac{n}{\lg n}$

For this assumption, $f(n) \in \Omega^\infty(g(n))$ for many $n > 0$.
But $f(n) \notin \Omega(g(n))$ for values of $n < 2$.

Hence, $f(n) \in \Omega^\infty(g(n))$ *does not imply* $f(n) \in \Omega(g(n))$.

(b) $f(n)$ and $g(n)$ are non-negative functions.
$\therefore \; f(n) > 0, \quad , g(n) > 0$

Now, there will be some value of c and n_0,
for which, either

$$
\begin{aligned}
f(n) &\geq c_1 g(n) && \text{will not grow slower} && (1)\\
\text{or } f(n) &\leq c_1 g(n) && \text{will not grow faster} && (2)\\
\text{Hence, if (1), then } f(n) &\in \Omega^\infty(g(n)) && && (3)\\
\text{and if (2), then } f(n) &\in \mathrm{O}(g(n)) && && (4)
\end{aligned}
$$

In the rare case that $f(n) = cg(n)$, both (3) and (4) will hold true.

Now, for the same case, if we replace Ω^∞ with $\Omega()$, it does not hold true, as is proven by the example in (a).