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3. [12 points – 4 points each part] *Analysis of linear-time median algorithms.*

   Assume $T(1) = c_1$, where c_1 is some constant. Prove Θ bounds for each of the following recurrences using the substitution method.

   (a) $T(n) = n + (1/n)\sum_{i=1}^{n-1} T(i)$

   (b) $T(n) = n + \sum_{i=1}^{n-1} T(i)$ (this one is tricky)

   (c) $T(n) = n + T(\lceil n/2 \rceil) + T(\lceil n/4 \rceil) + T(\lceil n/8 \rceil)$

Answer:

(a)

$$T(n) = n + \frac{1}{n}\sum_{i=1}^{n-1} T(i)$$

$$T(n-1) = (n-1) + \frac{1}{n-1}\sum_{i=1}^{n-2} T(i)$$

$$\begin{aligned}
nT(n) - (n-1)T(n-1) &= n^2 + \sum_{i=1}^{n-1} T(i) - (n-1)^2 - \sum_{i=1}^{n-2} T(i) \\
\therefore nT(n) - (n-1)T(n-1) &= T(n-1) + 2n - 1 \\
nT(n) &= nT(n-1) + 2n - 1 \\
T(n) &= T(n-1) + 2 - \frac{1}{n} \qquad (1) \\
\text{and} \quad T(1) &= c_1 \qquad \text{, where } c_1 \text{ is some constant}
\end{aligned}$$

BigO Analysis

Let us assume $T(n) \in O(n)$, then $T(n) \leq cn, c \geq 0$

Lets check the base case first i.e. for $n = 1$,

$T(1) = c_1$, from (1)

$T(1) \leq c$, from assumption

Lets check for recurrence relation, i.e. $T(n-1) \leq c(n-1)$

$$\begin{aligned}
T(n) &\leq c(n-1) + 2 - \frac{1}{n} \\
T(n) &\leq cn - c + 2 - \frac{1}{n} \\
T(n) &\leq cn - c + 2 \qquad \left(\text{since } n \geq 0 \text{ and } \frac{-1}{n} \text{ is negative}\right)
\end{aligned}$$

and we can say that $T(n) \leq cn$ (from our assumption)

$\therefore \quad cn - c + 2 \leq cn$

$\implies c \geq 2$

Our assumption has been proven true.
Hence, we can say that, $T(n) \in \mathrm{O}(n)$ if $c \geq \max(2, c_1)$

BigOmega Analysis
Let us assume that $T(n) \in \Omega(n)$,
then $T(n) \geq cn, \quad$ for $c \geq 0$

Lets check the base case first i.e. for $n = 1$,
$T(1) = c_1, \quad$ from (1)
$T(1) \geq cn, \quad$ from assumption
Hence, we can say our assumption is true, if $c \leq c_1$

Lets check the recurrence relation i.e. $T(n-1) \geq c(n-1)$

$$\begin{aligned}
T(n) &\geq c(n-1) + 2 - \frac{1}{n} \\
&= cn - c + 2 - \frac{1}{n} \\
&\qquad \text{and we can say that} \\
T(n) &\geq cn \qquad \text{(which is also our assumption)} \\
cn &\leq cn - c + 2 - \frac{1}{n} \\
\implies c &\leq 2 - \frac{1}{n} \\
\implies c &\leq \frac{3}{2} \qquad \text{(since min value of } n \text{ can be 2)}
\end{aligned}$$

Hence, we can say that our assumption is true, and $T(n) \in \Omega(n)$ if $c \leq \min(c, \frac{3}{2})$
Thus, we can say - $\quad T(n) \in \Theta(n)$

(b)

$$T(n) = n + \sum_{i=1}^{n-1} T(i)$$

$$T(n-1) = (n-1) + \sum_{i=1}^{n-2} T(i)$$

$$T(n) - T(n-1) = n + T(n-1) + \sum_{i=1}^{n-2} T(i) - \sum_{i=1}^{n-2} T(i)$$

$$T(n) - T(n-1) = T(n-1) + 1$$

$$T(n) = 2T(n-1) + 1 \tag{2}$$

$$T(1) = c_1, \qquad \text{where } c_1 \text{ is some constant}$$

BigO Analysis
Let us assume $T(n) \in \mathrm{O}(2^n)$, then

$T(n) \leq c2^n - b$, $c \geq 0$, and b is also a constant.

Lets check for base case first i.e. $n = 1$
$T(1) = c_1$ from (2)
$T(1) \leq 2c - b$ from assumption
Hence, our assumption is true, if $c \geq \left(\frac{c_1+b}{2}\right)$.

Lets check for recurrence relation -

$$
\begin{aligned}
T(n) &\leq 2(c2^{n-1} - b) + 1 \\
&\leq 2c2^{n-1} - 2b + 1 \\
&\leq c2^n - b - b + 1
\end{aligned}
$$

And we can say that

$$
\begin{aligned}
T(n) &\leq c2^n - b \qquad \text{if} \\
c2^n - 2b + 1 &\leq c2^n - b \\
\implies \qquad b &\geq 1
\end{aligned}
$$

Hence, our assumption is true, and
$T(n) \in \mathrm{O}(2^n)$, if $c \geq \frac{c_1+b}{2}$ and $b \geq 1$

BigOmega Analysis
Let us assume $T(n) \in \Omega(2^n)$, then
$T(n) \geq c2^n - b$, $c \geq 0$ and b is some constant.

Lets check base case first, i.e. $n = 1$
$T(1) = c_1$ from (2)
$T(1) \geq c * 2 - b$ from assumption
Hence, our assumption is true, if $c \leq \frac{c_1+b}{2}$

Lets check for the recurrence relation -

$$
\begin{aligned}
T(n) &\geq 2(c2^{n-1} - b) + 1 \\
&\geq 2c2^{n-1} - 2b + 1 \\
&\geq c2^n - b - b + 1
\end{aligned}
$$

And we can say that -

$$
\begin{aligned}
T(n) &\geq c2^n - b \qquad \text{if} \\
c2^n - 2b + 1 &\geq c2^n - b \\
\implies \qquad b &\leq 1
\end{aligned}
$$

Hence, our assumption is true and
$T(n) \in \Omega(2^n)$ if $c \leq \frac{c_1+b}{2}$ and $b \leq 1$

Thus, we can say - $T(n) \in \Theta(2^n)$

(c)

$$T(n) = n + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lceil \frac{n}{4} \right\rceil\right) + T\left(\left\lceil \frac{n}{8} \right\rceil\right)$$
$$T(1) = c_1 \tag{3}$$

BigO Analysis
Let us assume $T(n) \in \mathrm{O}(n \lg n)$,
then $T(n) \leq cn \lg n, \qquad c \geq 0$
Lets check the base case first i.e. $n = 1$
$T(1) = c_1$ from (3)
$T(1) \leq c(1)(\lg 1) = 0$ from assumption
Hence, our assumption is true, if $c_1 \leq 0$

Lets check for the recurrence relation -

$$\begin{aligned} T(n) &\leq n + \left\lceil c\frac{n}{2} \lg \frac{n}{2} \right\rceil + \left\lceil c\frac{n}{4} \lg \frac{n}{4} \right\rceil + \left\lceil c\frac{n}{8} \lg \frac{n}{8} \right\rceil \\ &\leq n + \left(c\frac{n}{2} \lg \frac{n}{2} + 1\right) + \left(c\frac{n}{4} \lg \frac{n}{4} + 1\right) + \left(c\frac{n}{8} \lg \frac{n}{8} + 1\right) \qquad \text{as } \lceil a \rceil \leq a + 1 \\ &\leq n + 3 + \frac{cn}{2}(\lg n - \lg 2) + \frac{cn}{4}(\lg n - \lg 4) + \frac{cn}{8}(\lg n - \lg 8) \\ &\leq n + 3 + cn \lg n \left(\frac{4}{8} + \frac{2}{8} + \frac{1}{8}\right) - cn\left(\frac{4}{8} + \frac{4}{8} + \frac{3}{8}\right) \\ &\leq n + 3 + \frac{7}{8}cn \lg n - \frac{11}{8}cn \\ &\leq n + 3 + \frac{7}{8}cn \lg n \qquad \text{(since } n \geq 1 \text{ and } c \geq 0) \end{aligned}$$

And we say that -

$$\begin{aligned} T(n) &\leq cn \lg n && \text{if} \\ n + 3 + \frac{7}{8}cn \lg n &\leq cn \lg n \\ \implies \qquad c &\geq \frac{8n + 24}{n \lg n} && \text{where } n \geq 1 \end{aligned}$$

Hence, we can say that our assumption is true and

$T(n) \in \mathrm{O}(n \lg n)$ if $c \geq \frac{8n+24}{n \lg n}$, $n \neq 1$

BigOmega Analysis
Let us assume that $T(n) \in \Omega(n \lg n)$,
then $T(n) \geq cn \lg n$, for $c \geq 0$

Lets check with base case first i.e. for $n = 1$,
$T(1) = c_1$ from (1)
$T(1) \geq c(1)(\lg 1) = 0$ from assumption

Hence our assumption is true if $c_1 \geq 0$

Lets check for the recurrence relation -

$$
\begin{aligned}
T(n) &\geq n + \left\lceil \frac{cn}{2} \lg \frac{n}{2} \right\rceil + \left\lceil \frac{cn}{4} \lg \frac{n}{4} \right\rceil + \left\lceil \frac{cn}{8} \lg \frac{n}{8} \right\rceil \\
&\geq n + \left\lceil \frac{cn}{2}(\lg n - \lg 2) + \frac{cn}{4}(\lg n - \lg 4) + \frac{cn}{8}(\lg n - \lg 8) \right\rceil \qquad \text{since } \lceil a \rceil + \lceil b \rceil \geq \lceil a + b \rceil \\
&\geq n + \left\lceil cn \lg n \left(\frac{4}{8} + \frac{2}{8} + \frac{1}{8} \right) - cn \left(\frac{4}{8} + \frac{4}{8} + \frac{3}{8} \right) \right\rceil \\
&\geq n + \left\lceil \frac{7}{8} cn \lg n - \frac{11}{8} cn \right\rceil \\
&\geq n + \left\lceil \frac{7}{8} cn \lg n \right\rceil - \left\lceil \frac{11}{8} cn \right\rceil \\
&\geq n + cn \lg n - 2cn \qquad \ldots \text{ taking ceiling}
\end{aligned}
$$

And we say that -

$$
\begin{aligned}
T(n) &\geq cn \lg n \qquad \text{if} \\
n + cn \lg n - 2cn &\geq cn \lg n \\
\implies \qquad c &\leq \frac{1}{2}
\end{aligned}
$$

Hence, we can say that our assumption is true and $T(n) \in \Omega(n \lg n)$ if $c \leq \frac{1}{2}$

Thus we have, $T(n) \in \Theta(n \lg n)$