# THE STABLE ADJUNCTION IN $\mathbb{A}^1$-HOMOTOPY THEORY

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## Introduction

This article applies the newly developed framework in [KMZ24] to motivic homotopy theory to write down an $\mathbb{A}^1$-homotopical analog of Beck monadicity of the kind presented in [May09]. The objective is to study the loop-suspension adjunction between spaces and spectra in stable $\mathbb{A}^1$-homotopy theory, and the monad that arises from this adjunction.

The starting point for the homotopy theory of schemes is Morel and Voevodsky's definition of the closed model category of algebraic spaces (we will denote this category by $\mathscr{T}$, and refer to its objects as *spaces*). As for spectra, one might choose to use Voevodsky's category of $\mathbb{T}$-spectra (created in analogy with sequential $\Omega$-spectra in topology) which is symmetric monoidal under the smash product. However, the result of G. Lewis in [Lew91], renders this choice incompatible with the loop-suspension adjunction such that the sphere spectrum is a unit for the smash product. The choice we make is that of the category of coordinate-free spectra defined by P. Hu in [Hu03] (we will denote this category by $\mathscr{S}$, and refer to its objects as *spectra*). This is the analogous notion to the topological coordinate-free spectra in [EKMM97] and [LMSM86]. An essential point of P. Hu's coordinate-free spectra is that they are indexed on cofinite subspaces of a universe $\mathcal{U} \cong \mathbb{A}^\infty$ (as opposed to finite-dimensional subspaces, as in EKMM and LMS spectra). The loop-suspension adjunction that is naturally built into the context of [Hu03], and in complete analogy with the topological story, will be our starting point for this program. We will denote this adjunction by $(\Sigma^\infty, \Omega^\infty)$, where $\Sigma^\infty \colon \mathscr{T} \longrightarrow \mathscr{S}$ is given by the suspension spectrum, and $\Omega^\infty \colon \mathscr{S} \longrightarrow \mathscr{T}$ is given by the zeroth space. The objects in the image of $\Omega^\infty$ are referred to as *infinite loop spaces*. Let $\Gamma = \Omega^\infty\Sigma^\infty$ denote the adjunction monad in our context. $\Gamma$ acts naturally on the

*Date*: October 30, 2025.

image of $\Omega^\infty$ in $\mathscr{T}$. Let $\Gamma[\mathscr{T}]$ denote the category of $\Gamma$-algebras in $\mathscr{T}$. There is a more structured variant of the adjunction that incorporates the $\Gamma$-action in $\Gamma[\mathscr{T}]$. Namely, the zeroth space functor $\Omega^\infty_\Gamma\colon \mathscr{S} \longrightarrow \Gamma[\mathscr{T}]$ as before is a right adjoint, with a left adjoint $\Sigma^\infty_\Gamma\colon \Gamma[\mathscr{T}] \longrightarrow \mathscr{S}$ (we think of $\Sigma^\infty_\Gamma$ as a coequalized version of $\Sigma^\infty$ with respect to the $\Gamma$-action). The goal of this article is to be able to collect enough homotopical preliminaries to fit into the framework of [KMZ24] and obtain the following results[1].

**Theorem 0.1.** *There is a functor* $\mathrm{Bar}\colon \Gamma[\mathscr{T}] \longrightarrow \mathscr{T}$ *written* $Y \mapsto \overline{Y}$*, a natural homotopy equivalence* $\zeta\colon \overline{Y} \longrightarrow Y$*, and a functor* $\mathbb{E}\colon \Gamma[\mathscr{T}] \longrightarrow \mathscr{S}$ *such that there is a weak equivalence* $\overline{Y} \longrightarrow \Omega^\infty_\Gamma \mathbb{E} Y$*.*

There is a hint of a conceptual gap here. The functor Bar in the above theorem alludes to the two-sided monadic bar resolution of $\Gamma$-algebras—and as such we would have liked a functor $\mathrm{Bar}\colon \Gamma[\mathscr{T}] \longrightarrow \Gamma[\mathscr{T}]$ in the theorem statement. Similarly, we may have also liked for the weak equivalence the theorem produces to be a map of $\Gamma$-algebras. However, as we discuss in Section 3, in the adjunction context there is a priori no $\Gamma$-algebra structure on the two-sided bar construction of a $\Gamma$-algebra. A natural question to ask is what can one say about the homotopy category of $\Gamma$-algebras and the homotopy category of spectra? As we have pointed out, Theorem 0.1 does not guarantee an equivalence between these homotopy categories.

We resolve this gap by demonstrating that although $\overline{Y}$ is not a $\Gamma$-algebra, it is the realization of a simplicial $\Gamma$-algebra that is *linked* to the $\Gamma$-algebra $\Omega^\infty_\Gamma \mathbb{E} Y$ in a very precise sense laid out in Section 3. A natural extension of weak equivalences between $\Gamma$-algebras to the linked setting, allows us to prove this statement about homotopy categories.

**Theorem 0.2.** *The pair* $(\mathbb{E}, \Omega^\infty_\Gamma)$ *induces an equivalence between the localization of the full subcategory of* $\Gamma$*-algebras in* $\mathscr{T}$ *at the linked weak equivalences, and the homotopy category of connective spectra.*

The lack of necessity for grouplike objects and group completions in these results is also curious and tells us that the category of $\Gamma$-algebras (allowing for linked maps) and that of connective spectra are not homotopically distinct. This is the homotopical version of Beck monadicity in our context. A sequel paper [SM25] will discuss an analog of the *operadic* monads that appear in the space-level story in topology—towards obtaining an operadic recognition principle of motivic infinite loop spaces that has remained elusive. These monads turn out to be the interesting examples in nature since they provide information about the *structure* of infinite loop spaces invisible to $\Gamma$. They also have the added advantage that the monadic bar resolution of algebras has a natural algebra structure—and that the analogs of maps in the statement of Theorem 0.1 are naturally maps of algebras. This will allow us to circumvent the linked machinery required for the adjunction monad.

The paper is structured as follows. In Section 1 we discuss preliminaries about the motivic stable homotopy category. In Section 2 we recall the construction of coordinate-free spectra in [Hu03], and define the stable adjunction that is the crux to our story. Assuming a list of properties about simplicial spaces and simplicial

[1]The term *connective* in Theorem 0.2 is often replaced by *very effective* in literature but we use connective here to emphasize the axiomatized meaning

spectra, in Section 3 we prove strengthened forms of Theorems 0.1 and 0.2 using the bar resolution of $\Gamma$-algebras. Our strengthening relies on a newly developed notion of linked algebras [KMZ25] that we provide a generalized conceptual treatment for here. The properties of simplicial objects needed for the proof of homotopical monadicity are then proved in Section 4. The hardest and perhaps conceptually most insightful result in this section is Proposition 4.6 which describes the weak commutativity of realization and $\Omega^\infty$. We resolve this hurdle using Rezk's reformulation of the compatibility of realizations with homotopy pullbacks [Rez14]. As we emphasize in Section 3, the weak commutativity of $\Omega^\infty$ and realization results in the weak commutativity of $\Gamma$ and $|-|$. This is precisely the cause of the conceptual gap we briefly described below Theorem 0.1—and exactly the kind of gap that the new linked notion allows to bridge.

## Acknowledgements

Much of this article was developed while I was a visitor at the University of Chicago in the summers of 2024 and 2025. I am indebted to Peter May for his support through the summer program at Chicago, and for his efforts in this project, which was very much joint work with him. My heartfelt gratitude also goes out to Alicia Lima, who co-advised me through the summer of 2024. I thank Paul Goerss for clarifications on path-space fibrations in simplicial model categories and Aravind Asok for pointing me to references on the commutativity of realization and the homotopy pullback (and for continued discussion over the year on the sociology of mathematics). Much thanks goes out to fellow visitors the first summer, Yoyo Jiang (for helpful categorical and simplicial remarks) and Akash Narayanan (for great mathematical conversation throughout that summer). Any errors in this article are mine alone.

## 1. Preliminaries

This section presents the Morel-Voevodsky construction of the model category of algebraic spaces, developed in [Voe96b], [MV99], and [Mor12]—and the properties of this category that we will find useful throughout the article. Let $k$ be a perfect field and $S$ a smooth affine scheme of finite type over $k$ (in particular, we ask that $S = \mathrm{Spec}(R)$ for a Noetherian integral domain $R$). We define $\mathrm{Sm}_S$ to be the category of smooth schemes of finite type over $S$. Whenever we refer to a scheme, we mean an object of $\mathrm{Sm}_S$. The category $\mathrm{Sm}_S$ is infeasible for homotopy theory since it is not cocomplete. Therefore we cannot use the objects of $\mathrm{Sm}_S$ to mean "spaces." The fix for this is to consider simplicial sheaves over $\mathrm{Sm}_S$ as our spaces. To speak of sheaves, we must first endow $\mathrm{Sm}_S$ with a Grothendieck topology. [MV99] identified the Nisnevich topology, which is finer than the Zariski topology but coarser than the Étale topology on $\mathrm{Sm}_S$, to be a convenient choice. It allows for our site to enjoy some nice properties of both the Étale and Zariski topologies. It is defined so that the covering sieves are those families of étale morphisms $\varphi_i \colon \{U_i\} \longrightarrow X$ such that for any $x \in X$, there exists an $i$ and a $u \in U_i$ with the corresponding map on residue fields being an isomorphism (that maps to $x$ with the same residue field).

Equivalently, the Nisnevich coverings are generated by the diagrams

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"'] & X \end{tikzcd},$$

with $p\colon V \longrightarrow X$ an étale morphism and $i\colon U \longrightarrow X$ an open embedding such that $p^{-1}(X \setminus U) \longrightarrow X \setminus U$ is an isomorphism. These furnish the elementary covering diagrams of the Nisnevich topology on $\mathrm{Sm}_S$, so that a presheaf $\mathcal{F}$ on $\mathrm{Sm}_S$ is a sheaf iff $\mathcal{F}$ takes each diagram as above to a pullback square ([MV99] Proposition 3.1.4). We write $(\mathrm{Sm}_S)_{Nis}$ to denote the site of $\mathrm{Sm}_S$ equipped with the Nisnevich topology. The promised definition of the category of spaces over $S$ is then[2]:

$$\mathrm{Spc}(S) := \mathrm{sSh}((\mathrm{Sm}_S)_{Nis}).$$

We can also define the category of based spaces over $S$ to be $\mathrm{Spc}(S)_\bullet = S \downarrow \mathrm{Spc}(S)$. We write $\mathscr{T}$ to mean this category and we call its objects spaces. The presheaf represented by an object $X \in \mathrm{Sm}_S$ is in fact a sheaf in the Nisnevich topology (this is due to [SGA4V2] VII.2a). We also use $\mathscr{T}_k$ to mean the category $\mathrm{Spc}(k)_\bullet$ of spaces over $k$.

We note some important properties of the objects of $\mathscr{T}$ before moving on to its homotopical properties and its model structure. $\mathscr{T}$ is a complete and cocomplete category. We also have the following characterization of each space as a small colimit.

**Proposition 1.1.** *Every object of $\mathscr{T}$ is a small colimit of smooth schemes in* $\mathrm{Sm}_S$.

This is a standard fact, but we sketch a rough idea of the proof here. A space $\mathcal{F} \in \mathrm{Spc}(S)$ is a sheaf on $\mathrm{Sm}_S$. Define $\mathrm{Sm}_{S(\mathcal{F})}$ to be the category whose objects are pairs $(X, t)$ with $X \in \mathrm{Sm}_S$ and $t \in \mathcal{F}(X)$ and morphisms $f\colon (X,t) \longrightarrow (Y,z)$ are morphisms $f\colon X \longrightarrow Y$ such that $\mathcal{F}(f)$ takes $z$ to $t$. The Yoneda lemma then yields a presheaf isomorphism $\mathcal{F} \cong \mathrm{colim}_{(X,t)\in \mathrm{Sm}_{S(\mathcal{F})}} \mathrm{Hom}(-, X)$. Passing to sheafification, and noting that $\mathrm{Sm}_S$ is a small site, provides the required result.

The following is vital to the spectrification defined in Sec. 2.

**Proposition 1.2.** *Every object of* $\mathrm{Sm}_S$ *is compact in* $\mathscr{T}$

*Proof.* Let $\mathcal{I}$ be a small directed category and $D\colon \mathcal{I} \longrightarrow \mathrm{Spc}(S)$ be a functor. Let $X \in \mathrm{Sm}_S$ be a smooth scheme (i.e. a representable sheaf). At the presheaf level, the Yoneda lemma provides: $\mathrm{Hom}(X, \varinjlim D) \cong (\varinjlim D)(X)$ and $\varinjlim \mathrm{Hom}(X, D) \cong \varinjlim D(X)$. Again, as presheaves we know $(\varinjlim D)(X) \cong \varinjlim D(X)$. It suffices to show that $\varinjlim D$ (with the direct limit taken at the presheaf level) is in fact a Nisnevich sheaf. To do this, it suffices to show that $\varinjlim D$ takes each elementary covering diagram (defined above) to a pullback square. We know that each $D(i)$ for $i \in \mathcal{I}$ certainly does (as a sheaf). Since the pullback squares here are finite limits of sets, they are stable under small directed colimits. Therefore $\varinjlim D$ also takes each elementary covering diagram to a pullback square. ∎

[2]Notation: for a Grothendieck site $(\mathcal{C}, \tau)$ we write $\mathrm{sSh}(\mathcal{C}_\tau)$ to mean the category of simplicial **Set**-valued sheaves on $\mathcal{C}$. When there is no confusion, we will call the objects of $\mathrm{sSh}(\mathcal{C}_\tau)$ simplicial sheaves on $\mathcal{C}$.

There is a small caveat in the above proof/s. We had forgotten to say anything about basepoints. Now we say all that needs to be said, following [Hu03]. A direct limit of spaces in the unbased category is in fact a direct limit in $\mathscr{T}$. Given an $X \in \mathscr{T}$, thinking of it as an unbased space, there is a small directed category $\mathcal{I}$ and a functor $D\colon \mathcal{I} \longrightarrow \mathrm{sSh}((\mathrm{Sm}_S)_{Nis})$ such that $X \cong \varinjlim D$ in $\mathrm{Spc}(S)$. The datum of a basepoint of $X$ is just a morphism $S \longrightarrow X$. Since $S$ is a compact object as a point, this morphism lifts to one of the $S_i$ for some $i \in \mathcal{I}$. Consider the comma category $\mathcal{J}$ of objects in $\mathcal{I}$ with a map from $i$. Writing $E$ for the restriction of $D$ to $\mathcal{J}$, we have (in $\mathscr{T}$) $X \cong \varinjlim E$.

We now get to the Quillen model structure on $\mathscr{T}$ and the notion of an $\mathbb{A}^1$-homotopy in $\mathscr{T}$, following [MV99]. We write $\mathbb{A}^1$ to mean $\mathbb{A}^1_S$, the affine line over $S$. Two maps $f, g\colon X \longrightarrow Y$ in $\mathscr{T}$ are said to be $\mathbb{A}^1$-homotopic if there is a map $h\colon \mathbb{A}^1 \times X \longrightarrow Y$ in $\mathrm{Spc}(k)$, such that the composite $X \xrightarrow{\iota_0} \mathbb{A}^1 \times X \xrightarrow{h} Y$ is $f$, and $X \xrightarrow{\iota_1} \mathbb{A}^1 \times X \longrightarrow Y$ is $g$ (where $\iota_j$ is induced by the inclusion of $S$ into $\mathbb{A}^1_S$ as $j$).

**Definition 1.3.** A map $f\colon X \longrightarrow Y$ in $\mathscr{T}$ is said be a:

1. simplicial cofibration, if it is a monomorphism.
2. simplicial weak equivalence if for every $x^* \in \mathrm{Sm}_S$, the induced map $f^*\colon x^*(X) \longrightarrow x^*(Y)$ is a weak equivalence of simplicial sets.[3]
3. simplicial fibration if it has the right lifting property (RLP) with respect to all trivial simplicial cofibrations.

We do not provide a proof here, but the above definitions provide a Quillen model structure on $\mathscr{T}$ known as the simplicial model structure (see [Jar87, Corollary 2.7]). Denote by $\mathcal{H}_s(\mathscr{T})$ the homotopy category of $\mathscr{T}$ with respect to the simplicial model structure. We now localize this model structure with respect to $\mathbb{A}^1$-homotopies. The first steps towards such a localization are the following definitions.

**Definition 1.4.** We say $X \in \mathscr{T}$ is $\mathbb{A}^1$-local if for every $Y \in \mathscr{T}$, the map on homotopy classes $[Y, X]_{\mathcal{H}_s(\mathscr{T})} \longrightarrow [Y \times \mathbb{A}^1, X]_{\mathcal{H}_s(\mathscr{T})}$ induced by the projection $Y \times \mathbb{A}^1 \longrightarrow Y$ is a bijection.

**Definition 1.5.** A map $f\colon X \longrightarrow Y$ in $\mathscr{T}$ is said to be an

1. $\mathbb{A}^1$-cofibration if it is a monomorphism.
2. $\mathbb{A}^1$-weak equivalence if for any $\mathbb{A}^1$-local $Z \in \mathscr{T}$, the map on homotopy classes $f^*\colon [Y, Z]_{\mathcal{H}_s(\mathscr{T})} \longrightarrow [X, Z]_{\mathcal{H}_s(\mathscr{T})}$ is a bijection.
3. $\mathbb{A}^1$-fibration if it has the right lifting property with respect to all trivial cofibrations.

[MV99, Theorem 2.21] shows that the above $\mathbb{A}^1$-local definitions prescribe a model structure on $\mathscr{T}$ that admits functorial factorization. For the most part, the verification of the model category axioms follow directly from the results of Section 2 in [Jar87]. We also have the following characterization of spaces upto simplicial weak equivalence. We denote by $\mathcal{H}(\mathscr{T})$ the homotopy category of $\mathscr{T}$ associated to this model structure.

**Proposition 1.6.** *Every object in $\mathscr{T}$ is simplicially weak equivalent to a homotopy colimit of finite colimits of smooth schemes.*

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[3] Recall that one may think of each object of a site $\mathcal{S}$ as a functor $\mathrm{Sh}(\mathcal{S}) \longrightarrow \mathbf{Set}$.

*Proof.* The proof is immediate from the work of [BK72] (see XII.3.4), forgetting basepoints using the fix following Proposition 1.2. A simplicial sheaf $X = \{X_n\}$ is a diagram in $\mathrm{Sh}((\mathrm{Sm}_S)_{Nis})$ indexed by $\Delta^{op}$. By [BK72] XII.3.4, $\mathrm{hocolim}_{\Delta^{op}} X_n \longrightarrow X$ is a simplicial weak equivalence. ∎

$\mathscr{T}$ is also tensored and cotensored over itself. That is to say, there is a smash product $\wedge \colon \mathscr{T} \times \mathscr{T} \longrightarrow \mathscr{T}$, and there are internal Hom-space functors $\underline{\mathrm{Hom}}_{\mathscr{T}}(X, -)$ (right adjoint to $- \wedge X \colon \mathscr{T} \longrightarrow \mathscr{T}$) defined in the usual way.

We close by discussing the $\mathbb{A}^1$-homotopy sheaves, $\pi_n^{\mathbb{A}^1}(-, *)$ of an (unbased) space. Let $X \in \mathrm{Spc}(S)$. For $n = 0$, define $\pi_0^{\mathbb{A}^1}(X)$ to be the sheaf associated to the presheaf $U \mapsto [U, X]_{\mathcal{H}(\mathscr{T})}$ (for $U \in \mathrm{Sm}_S$). For $n \geq 1$, and a choice $*$ of a basepoint $S \hookrightarrow X$, define $\pi_n^{\mathbb{A}^1}(X, *)$ to be the sheaf associated to the presheaf $U \mapsto [\Sigma^n U_+, X]_{\mathcal{H}(\mathscr{T})}$ (here $\Sigma^n$ denotes the simplicial suspension, and we conflate $X$ with the based space $X$ equipped with the basepoint $*$). We have the following motivic analog of Whitehead's theorem, proven in [MV99].

**Theorem 1.7.** *A map of spaces $f \colon X \longrightarrow Y$ in $\mathscr{T}$ is an $\mathbb{A}^1$-weak equivalence iff it induces isomorphisms of sheaves $\pi_0^{\mathbb{A}^1}(X) \xrightarrow{\sim} \pi_0^{\mathbb{A}^1}(Y)$, and $\pi_n^{\mathbb{A}^1}(X, *) \xrightarrow{\sim} \pi_n^{\mathbb{A}^1}(Y, \star)$ for every choice of basepoints $*$ for $X$ and $\star$ for $Y$ (as objects in* $\mathrm{Spc}(S)$*).*

A most important property of the $\mathbb{A}^1$-motivic homotopy sheaves is that of the long exact sequence associated to a homotopy fiber sequence analogous to the one in topology.

**Theorem 1.8.** *If $F \longrightarrow E \longrightarrow B$ is a homotopy fiber sequence of spaces, then there is a natural long exact sequence of homotopy sheaves:*

$$\cdots \longrightarrow \pi_{n+1}^{\mathbb{A}^1}(B) \longrightarrow \pi_n^{\mathbb{A}^1}(F) \longrightarrow \pi_n^{\mathbb{A}^1}(E) \longrightarrow \pi_n^{\mathbb{A}^1}(B) \longrightarrow \cdots$$

This is the natural long exact sequence of presheaves, passed through Nisnevich sheafification (which is exact). In the particular case of an inclusion of a subspace $\iota \colon A \hookrightarrow Y$, and the homotopy fiber of this inclusion, we get an analog of the long exact sequence of relative homotopy groups.

Fixing terminology for the rest of the paper, we say that $X \in \mathscr{T}$ is *connected* if $\pi_0^{\mathbb{A}^1}(X)$ is terminal in the category of sheaves of sets (i.e. a constant sheaf at a singleton set). Similarly we say $X \in \mathscr{T}$ is *$n$-connected* if $\pi_q^{\mathbb{A}^1}(X)$ is terminal for $q \leq n$ (in the category of sheaves of sets for $q = 0$, of groups for $q = 1$, and of abelian groups for $q \geq 2$). We say that a map $f \colon X \longrightarrow Y$ in $\mathscr{T}$ is *$n$-connected* if its homotopy fiber $Ff$ is $(n-1)$-connected (by Theorem 1.8 this means that $f$ induces isomorphisms $\pi_q^{\mathbb{A}^1}(X) \xrightarrow{\sim} \pi_q^{\mathbb{A}^1}(Y)$ for $q < n$ and a surjection $\pi_n^{\mathbb{A}^1}(X) \twoheadrightarrow \pi_n^{\mathbb{A}^1}(Y)$). We say that a pair $(X, A)$ (i.e. an inclusion $\iota \colon A \longrightarrow X$ in $\mathscr{T}$) is *$n$-connected* if $\iota$ is $n$-connected.

## 2. Coordinate-free spectra and the $(\Sigma^\infty, \Omega^\infty)$ adjunction

In this section, we recall the construction of the coordinate-free spectra of [Hu03], and define the adjunction central to this paper. Recall that our base scheme was $S = \mathrm{Spec}(R)$ for a Noetherian integral domain $R$. A natural choice for our indexing universe would be the infinite-dimensional affine space over $S$ defined as follows. We define the universe to be the countably infinite-dimensional $R$-module $\mathcal{U} = \bigoplus_{\infty} R$. [4] We say $Z$ is a *finite dimensional subspace* of $\mathcal{U}$ if it is a finitely generated projective submodule of $\mathcal{U}$ such that the inclusion $Z \hookrightarrow \mathcal{U}$ splits. Fix a basis $\{e_1, e_2, \cdots\}$ for $\mathcal{U}$ and define $T_n$ to be the free submodule generated by $\{e_1, \cdots, e_n\}$ so that $\mathcal{U} \cong \mathrm{colim}_n T_n$ as an $R$-module. Now to think of $\mathcal{U}$ as a space over $S$, it suffices to note that every finitely generated projective $R$-module can be thought of as a vector bundle over $S$. Each $T_n$ then corresponds to the trivial bundle $\mathbb{A}^n_S$, the $n$-dimensional affine space over $S$. To define $\mathcal{U}$ as a space (over $S$), we only need to write $\mathcal{U} \cong \mathrm{colim}_n T_n \cong \mathrm{colim}_n \mathbb{A}^n_S \cong \mathbb{A}^\infty_S$. Note that $\mathcal{U}$ is not really an object of $\mathrm{Sm}_S$, but it is an ind-scheme over $S$. Similarly, a finitely generated projective submodule $Z$ of $\mathcal{U}$ can be thought of as a finite-dimensional subbundle of $\mathcal{U}$ over $S$. When the inclusion $Z \hookrightarrow \mathcal{U}$ is split, we say that $Z$ is a subspace of $\mathcal{U}$.

We had promised to index our spectra over the cofinite subspaces of $\mathcal{U}$. These subspaces, in words, are the subspaces of $\mathcal{U}$ of finite codimension (i.e. the split projective submodules of finite codimensions thought of as vector bundles over $S$). To put this definition on rigorous footing, we first define the Grassmannian $Gr^{cof}_S(\mathcal{U})$ of cofinite subspaces of $\mathcal{U}$. To do this, we first define the Grassmannian $Gr^{cof}_k(\mathcal{U}_k)$ over $k$ of cofinite subspaces of $\mathbb{A}^\infty_k$, and write $Gr^{cof}_S(\mathcal{U}) := Gr^{cof}_k(\mathcal{U}_k \times_{\mathrm{Spec}(k)} S)$. For $N \in \mathbb{N}$ and $m \leq N$, define the Grassmannian $Gr^m_k(\mathbb{A}^N)$ to be the space (over $k$) of subspaces of $\mathbb{A}^N$ whose direct sum with $\mathbb{A}^m$ is $\mathbb{A}^N$ (here $\mathbb{A}^m \hookrightarrow \mathbb{A}^N$ by the inclusion into the first $m$ coordinates of $\mathbb{A}^N$). Identifying $\mathcal{U}_k$ with $\mathrm{colim}_N \mathbb{A}^N$, we then define $Gr^{cof}_k(\mathcal{U}_k)$ to be the colimit $\mathrm{colim}_m \lim_{N \geq m} Gr^m_k(\mathbb{A}^N)$. The limit is taken over the restriction maps $Gr^m_k(\mathbb{A}^{N+1}) \longrightarrow Gr^m_k(\mathbb{A}^N)$ taking $V$ to $V \cap \mathbb{A}^N$ (here we note $\mathbb{A}^N \subset \mathbb{A}^{N+1}$ by inclusion into the first $N$ coordinates). We say that a map $V\colon S \longrightarrow Gr^{cof}_k(\mathcal{U}_k)$ over $k$ is a cofinite subspace of $\mathcal{U}$, in other words the cofinite subspaces of $\mathcal{U}$ precisely correspond to the split projective submodules of $\mathcal{U}$ of finite codimension. We similarly define an $n$-dimensional subspace of $\mathbb{A}^N_S$ to be a map $Z\colon S \longrightarrow Gr_k(n, \mathbb{A}^N_k)$ (here $Gr_k(n, \mathbb{A}^N_k)$ is the Grassmannian of $n$-dimensional subspaces over $k$ of $\mathbb{A}^N_k$). Again, this is just to say that the finite-dimensional subspaces are the finite-dimensional split projective submodules. For $U$ an affine space over $\mathbb{S}$, and subspaces $V, W$ of $U$ with trivial intersection, we denote by $V \oplus W$ the internal direct sum of $V$ and $W$.

We are now ready to define our indexing category. Define the category of cofinite subspaces of $\mathcal{U}$, written $\mathcal{C}(\mathcal{U})$, to be the category with objects cofinite subspaces of $\mathcal{U}$ and morphisms $Z\colon U \longrightarrow V$ given by finite-dimensional subspaces $Z \subset U$ such that $V \oplus Z = U$. The composite of morphisms $Z\colon U \longrightarrow V$ and $T\colon V \longrightarrow W$ is the finite-dimensional subspace $Z \oplus T$ of $U$ (note that $W \oplus (Z \oplus T) = U$. [Hu03, Lemma 2.3] shows that $\mathcal{C}(\mathcal{U})$ is a small directed category.

Let $Z$ be a finite-dimensional space (alias, finite-dimensional projective module) over $S$. Thought of as a space, $Z$ looks like a vector bundle over $S$ equipped with a rational point $0\colon S \longrightarrow Z$ (alias the zero-section of $Z$ as a vector bundle). We

[4] It is perfectly acceptable for our purposes to fix the universe here. [Hu03] presents the theory for arbitrary universes.

define the space $S^Z$ by the pushout

$$\begin{tikzcd} Z \setminus \{0\} \arrow[r] \arrow[d] & Z \arrow[d] \\ S \arrow[r] & S^Z. \end{tikzcd}$$

We think of $S^Z$ as the one-point compactification of $Z$ over $S$. For $X \in \mathscr{T}$, we write $\Omega^Z X$ to mean $\underline{\mathrm{Hom}}_{\mathscr{T}}(S^Z, X)$.

**Definition 2.1.** A *prespectrum* $D$ is a family of spaces $\{D_U\}_{U \in \mathcal{C}(\mathcal{U})}$, together with morphisms $\rho^U_{V,Z} \colon D_U \longrightarrow \Omega^Z D_V$ (for every morphism $Z \colon U \longrightarrow V$ in $\mathcal{C}(\mathcal{U})$) satisfying the following conditions:

a) $\rho^U_{U,0} = \mathrm{id}_{D_U}$
b) For every $Z \colon U \longrightarrow V$, and $T \colon V \longrightarrow W$ in $\mathcal{C}(\mathcal{U})$,

$$\rho^U_{W,Z\oplus T} = (\Omega^Z \rho^V_{W,T}) \circ \rho^U_{V,Z}.$$

The morphisms $\rho^U_{V,Z}$ are called the *structure maps* of $D$. Given prespectra $D$ and $D'$, we call a collection of maps $\{f_U \colon D_U \longrightarrow D'_U\}_{U \in \mathcal{C}(\mathcal{U})}$ whose elements are compatible with the structure maps, a map of prespectra from $D$ to $D'$ (written $f \colon D \longrightarrow D'$). Prespectra defined this way form a category (under the usual composition) that we denote by $p\mathscr{S}$

**Definition 2.2.** We say that a prespectrum $E$ is a *spectrum* if all of its structure maps are isomorphisms. We call a map of prespectra between two spectra, a map of spectra. Spectra defined this way form a category (under the usual composition) that we denote by $\mathscr{S}$

All limits in $\mathscr{S}$ (and all limits and colimits in $p\mathscr{S}$) can be computed spacewise, just as in the topological case. Denote by $R \colon \mathscr{S} \longrightarrow p\mathscr{S}$ the forgetful functor obtained from 2.2, obtained by forgetting that the structure maps are isomorphisms. $R$ admits a left adjoint $L$ that we now describe. Thanks to the Nisnevich topology, the story here is much simpler than in topology where one must first pass through inclusion prespectra and Freyd's adjoint functor theorem.

**Proposition 2.3.** *There exists a functor $L \colon p\mathscr{S} \longrightarrow \mathscr{S}$ called spectrification, defined by $D \mapsto LD$ where $(LD)_U = \mathrm{colim}_{(V,Z)\in U\downarrow\mathcal{C}(\mathcal{U})} \Omega^Z D_V$ with prescribed structure maps $\rho^U_{W,T} : (LD)_U \xrightarrow{\sim} \Omega^T (LD)_W$. Here the colimit is taken over the maps $\Omega^Z D_V \longrightarrow \Omega^{Z'} D_{V'}$ given by $\Omega^Z \rho^V_{V',T}$ for maps $T \colon (V,Z) \longrightarrow (V',Z')$.*

*Proof.* We show that $\{(LD)_U\}_{U\in\mathcal{C}(\mathcal{U})}$ defines a spectrum. The functoriality of $L$ and the fact that it is left adjoint to $R$ are easy to see. It suffices to pick the structure maps $\rho^U_{W,T} : (LD)_U \xrightarrow{\sim} \Omega^T (LD)_W$ for $T \colon U \longrightarrow W$ in $\mathcal{C}(\mathcal{U})$. We begin with $(LD)_U$.

$$\begin{aligned}
(LD)_U &= \operatorname{colim}_{(V,Z)\in U\downarrow\mathcal{C}(\mathcal{U})} \Omega^Z D_V \\
&\cong \operatorname{colim}_{(V,Z)\in W\downarrow\mathcal{C}(\mathcal{U})} \Omega^{T\oplus Z} D_V \\
&\cong \operatorname{colim}_{(V,Z)\in W\downarrow\mathcal{C}(\mathcal{U})} \Omega^T(\Omega^Z D_V) \\
&\cong \operatorname{colim}_{(V,Z)\in W\downarrow\mathcal{C}(\mathcal{U})} \underline{\operatorname{Hom}}_{\mathscr{T}}(S^T, \Omega^Z D_V) \\
&\cong \underline{\operatorname{Hom}}_{\mathscr{T}}(S^T, \operatorname{colim}_{(V,Z)\in W\downarrow\mathcal{C}(\mathcal{U})} \Omega^Z D_V) \\
&= \Omega^T \operatorname{colim}_{(V,Z)\in W\downarrow\mathcal{C}(\mathcal{U})} \Omega^Z D_V \\
&= \Omega^T (LD)_W
\end{aligned}$$

In the above chain of isomorphisms, we have used Proposition 1.2 to see that $S^T$ is compact in $\mathscr{T}$ (thus allowing us to commute the internal Hom and the colimit). We now pick $\rho^U_{W,T}$ to be the composite of the above isomorphisms. ■

As a left adjoint, spectrification commutes with colimits. Therefore we may compute colimits in $\mathscr{S}$ by first computing them at the prespectrum level and then spectrifying. The smash product $X \wedge E$ of a space $X$ with a spectrum $E$ is given by taking smash products spacewise and then spectrifying. We also have the function spectrum $F(X, E)$ defined by $F(X, E)_U = \underline{\operatorname{Hom}}_{\mathscr{T}}(X, E_U)$ for each $U \in \mathcal{C}(\mathcal{U})$. These constructions show that $\mathscr{S}$ is tensored and cotensored over $\mathscr{T}$. One might wonder how the category $\mathscr{S}$ compares with Voevodsky's original construction of the category of sequential $\mathbb{T}$-spectra in [Voe96b]. These concerns are addressed in [Hu03, Proposition 2.3 and Theorem 4.5] which shows that these categories are in fact equivalent.

We now define our adjunction $(\Sigma^\infty, \Omega^\infty)$. For any $V \in \mathcal{C}(\mathcal{U})$, there is the $V$-th space functor $\Omega^{\mathcal{U}}_V : \mathscr{S} \longrightarrow \mathscr{T}$ given by $\Omega^{\mathcal{U}}_V E = E_V$. This functor has a left adjoint $\Sigma^{\mathcal{U}}_V : \mathscr{T} \longrightarrow \mathscr{S}$, called the $V$-th shift desuspension of the suspension spectrum, which we now construct. Let $X \in \mathscr{T}$. For each finite-dimensional subspace of $\mathcal{U}$, denote by $\Sigma^Z X$ the smash product $S^Z \wedge X$. For each $U \in \mathcal{C}(\mathcal{U})$, define the space $D_V(X)_U = \bigvee_{Z\oplus U=V} \Sigma^Z X$, and $D_V(X)_U = S$ (a point) otherwise. If $W \subset U \subset V$ (so that $V \longrightarrow U \longrightarrow W$ in $\mathcal{C}(\mathcal{U})$) such that $W \oplus T = U$, then for any $Z$ with $Z \oplus U = V$, we have $(T \oplus Z) \oplus W = V$. This furnishes compatible maps:

$$\Sigma^T D_V(X)_V = \bigvee_{Z\oplus U=V} \Sigma^{T\oplus Z} X \longrightarrow \bigvee_{Z'\oplus W=V} \Sigma^{Z'} X = D_V(X)_W.$$

By the $(\Sigma^Z, \Omega^Z)$ adjunction, this furnishes a prespectrum $D_V(X) = \{D_V(X)_U\}_{U\in\mathcal{C}(\mathcal{U})}$. We define $\Sigma^{\mathcal{U}}_V X := LD_V(X)$.

It remains to show that $\Sigma^{\mathcal{U}}_V$ is left adjoint to $\Omega^{\mathcal{U}}_V$. It suffices to work on the level of prespectra. Let $f : D_V(X) \longrightarrow E$ be a map of prespectra (with $D_V(X)$ as defined above), i.e. a collection of compatible maps $f_U : D_V(X)_U \longrightarrow E_U$. The $V$-th map $f_V$ is then just a map $X \longrightarrow E_V$. Conversely, let $g : X \longrightarrow E_V$ be a map of spaces. Let $U$ be a cofinite subspace of $V$. For each map $Z : V \longrightarrow U$ (that is, a finite subspace $Z \subset V$ such that $Z \oplus U = V$), we obtain the map $\Sigma^Z X \xrightarrow{\Sigma^Z g} \Sigma^Z E_V \longrightarrow E_U$ (where the last map is furnished by the structure maps of $E$). This prescription provides compatible maps $D_V(X)_U \longrightarrow E_U$, as needed.

We denote by $\Sigma^\infty$ the functor $\Sigma^{\mathcal{U}}_{\mathcal{U}}\colon \mathscr{T} \longrightarrow \mathscr{S}$, and by $\Omega^\infty$ the functor $\Omega^{\mathcal{U}}_{\mathcal{U}}\colon \mathscr{S} \longrightarrow \mathscr{T}$. We call $\Sigma^\infty X$ the *suspension spectrum* of $X$, and $\Omega^\infty E$ the *$\mathcal{U}$-th space* of $E$. We have just shown that the pair $(\Omega^\infty, \Sigma^\infty)$ is an adjunction between $\mathscr{T}$ and $\mathscr{S}$. We clear out some categorical preliminaries and notation now. Denote by $\Gamma$ the adjunction monad $\Omega^\infty\Sigma^\infty$. The adjunction provides a unit map that we call $\eta\colon \mathrm{id}_{\mathscr{T}} \longrightarrow \Gamma$, and a counit map that we call $\varepsilon\colon \Sigma^\infty\Omega^\infty \longrightarrow \mathrm{id}_{\mathscr{S}}$. We write $\Gamma[\mathscr{T}]$ for the category of $\Gamma$-algebras in $\mathscr{T}$ (i.e. spaces in $\mathscr{T}$ with prescribed $\Gamma$-actions). It is easy to note that $\Omega^\infty E$ is a $\Gamma$-algebra for every $E \in \mathscr{S}$. The $\Gamma$-action is provided by the composite $\Gamma\Omega^\infty E = \Omega^\infty\Sigma^\infty\Omega^\infty E \xrightarrow{\Omega^\infty\varepsilon_E} \Omega^\infty E$. On the other hand, $\Sigma^\infty$ is a $\Gamma$-functor in the sense that there is a natural map $\beta\colon \Sigma^\infty\Gamma \longrightarrow \Sigma^\infty$ defined by $\varepsilon_{\Sigma^\infty X}\colon \Sigma^\infty\Gamma X \longrightarrow \Sigma^\infty X$ for every $X \in \mathscr{T}$ (such that $\beta\circ\Sigma^\infty\mu = \beta\circ\beta$ and $\beta\circ\Sigma^\infty\eta = \mathrm{id}$). We denote by $\Omega^\infty_\Gamma$ the functor $\mathscr{S} \longrightarrow \Gamma[\mathscr{T}]$ obtained by just applying $\Omega^\infty$. For each $X \in \Gamma[\mathscr{T}]$ with $\Gamma$-action $\theta$, define $\Sigma^\infty_\Gamma X$ to be the coequalizer

$$\Sigma^\infty\Gamma X \underset{\Sigma^\infty\theta}{\overset{\beta_X}{\rightrightarrows}} \Sigma^\infty X \longrightarrow \Sigma^\infty_\Gamma X.$$

This construction defines a functor $\Sigma^\infty_\Gamma\colon \Gamma[\mathscr{T}] \longrightarrow \mathscr{S}$ that is left adjoint to $\Omega^\infty_\Gamma$, and in doing so provides a coequalized adjunction $(\Sigma^\infty_\Gamma, \Omega^\infty_\Gamma)$ between $\Gamma[\mathscr{T}]$ and $\mathscr{S}$.

Getting back to spectra, we now define the stable $\mathbb{A}^1$-local model structure on $\mathscr{S}$. We begin with the stable simplicial model structure on $\mathscr{S}$, which itself comes from the levelwise and stable simplicial model structures on $p\mathscr{S}$.

**Definition 2.4.** We say a map $f\colon D \longrightarrow E$ in $p\mathscr{S}$ is a levelwise simplicial weak equivalence or fibration if for every $V \in \mathcal{C}(\mathcal{U})$, the $V$-th space map $f_V\colon D_V \longrightarrow E_V$ is a simplicial weak equivalence or simplicial fibration of spaces (over $k$) respectively. We say $f$ is a levelwise simplicial cofibration if it satisfies the left lifting property (LLP) with respect to every levelwise simplicial acyclic fibrations.

[Hu03] shows, using the small objects argument that the distinguished classes of maps in the above definitions form a model structure on $p\mathscr{S}$. We call this model structure the *levelwise simplicial model structure* on $p\mathscr{S}$.

**Definition 2.5.** Let $f\colon D \longrightarrow E$ be a map in $p\mathscr{S}$. We say $f$ is a:

(1) stable simplicial cofibration if it is a levelwise simplicial cofibration.
(2) stable simplicial weak equivalence if it induces a bijection of colimits:

$$\mathrm{colim}_{(W,Z)\in V\downarrow\mathcal{C}(\mathcal{U})}[\Sigma^Z X_+, D_W]_{\mathcal{H}_s(\mathscr{T}_k)} \cong \mathrm{colim}_{(W,Z)\in V\downarrow\mathcal{C}(\mathcal{U})}[\Sigma^Z X_+, E_W]_{\mathcal{H}_s(\mathscr{T}_k)}$$

for every $V \in \mathcal{C}(\mathcal{U})$ and every $X \in \mathrm{Sm}_S$.
(3) stable simplicial fibration if it satisfies the RLP with respect to all stable simplicial acyclic cofibrations.

The Bousfield-Friedlander theorem of [**BF78**] shows that the above definitions form a model structure on $p\mathscr{S}$. We call this model structure the *stable simplicial model structure* on $p\mathscr{S}$. We can now define the stable simplicial model structure on $\mathscr{S}$ by the following definitions.

**Definition 2.6.** Let $f\colon D \longrightarrow E$ be a map in $\mathscr{S}$. We say $f$ is a stable simplicial cofibration, weak equivalence, or fibration if it is one in the stable simplicial model structure on $p\mathscr{S}$.

The small objects argument used in [Hu03] can be applied to the above definitions again to show that they prescribe a model structure on $\mathscr{S}$. We call this model structure the stable simplicial model structure on $\mathscr{S}$, and we denote the associated homotopy category of $\mathscr{S}$ by $\mathcal{H}_s(\mathscr{S})$.

To $\mathbb{A}^1$-localize the stable simplicial model structure on $\mathscr{S}$, we first introduce a notion of $\mathbb{A}^1$-homotopy in $\mathscr{S}$. Consider two maps $f, g\colon E \longrightarrow G$ in $\mathscr{S}$. We say $f$ and $g$ are $\mathbb{A}^1$-homotopic if there is a map $h\colon E \wedge \mathbb{A}^1_+ \longrightarrow G$ whose composition with the maps $\iota_0, \iota_1\colon E = E \wedge S^0 \longrightarrow E \wedge \mathbb{A}^1_+$ (one sending the non-base-point of $S^0$ to 0, and the other sending that point to 1 in $\mathbb{A}^1$) can be identified with $f$ and $g$ respectively. We define $\mathbb{A}^1$-local spectra analogous to $\mathbb{A}^1$-local spaces.

**Definition 2.7.** A spectrum $G \in \mathscr{S}$ is termed $\mathbb{A}^1$-local if for every $E \in \mathscr{S}$, the map $E \wedge \mathbb{A}^1_+ \longrightarrow E$ induced by the projection map $\mathbb{A}^1_+ \longrightarrow S^0$ induces a bijection of homotopy classes:

$$[E, G]_{\mathcal{H}_s(\mathscr{S})} \longrightarrow [E \wedge \mathbb{A}^1_+, G]_{\mathcal{H}_s(\mathscr{S})}$$

**Definition 2.8.** Let $f\colon E \longrightarrow E'$ be a map in $\mathscr{S}$. We say that $f$ is an:

1. $\mathbb{A}^1$-cofibration if it is a stable simplicial cofibration.
2. $\mathbb{A}^1$-weak equivalence if for every $\mathbb{A}^1$-local $G$, the map of homotopy classes $[E', G]_{\mathcal{H}_s(\mathscr{S})} \longrightarrow [E, G]_{\mathcal{H}_s(\mathscr{S})}$ is a bijection.
3. $\mathbb{A}^1$-fibration if it satisfies the RLP with respect to every acyclic $\mathbb{A}^1$-cofibration.

The above definitions form a model structure on $\mathscr{S}$ that we call the *stable* $\mathbb{A}^1$-*local model structure* on $\mathscr{S}$. The proof of this fact (as shown in [Hu03]) is mainly due to [MV99, Theorem 2.2.21], the Joyal trick presented in [Jar87], and small object arguments. The first three model structure axioms (denoted CM1-CM3 classically) are easy to see (these corresponds to bicompleteness, the two out of three property of weak equivalences, and closure under retracts). By definition, any $\mathbb{A}^1$-fibration has the RLP with respect to acyclic $\mathbb{A}^1$-cofibrations. The Joyal trick is used to show that acyclic $\mathbb{A}^1$-fibrations have the RLP with respect to $\mathbb{A}^1$-cofibrations. To see that every map factors into an acyclic $\mathbb{A}^1$-cofibration and an $\mathbb{A}^1$-fibration requires the small object arguments. The factorization of every map into an $\mathbb{A}^1$-cofibration and an acyclic $\mathbb{A}^1$-fibration follows from the corresponding fact in the simplicial model structure. We denote by $\mathcal{H}(\mathscr{S})$ the homotopy category of $\mathscr{S}$ associated to the stable $\mathbb{A}^1$-local model structure. We will recall properties of homotopy colimits in $\mathscr{S}$ as the need comes up. We record the following key characterization of $\mathbb{A}^1$-local spectra and of $\mathbb{A}^1$-weak equivalences due to [Hu03, Lemma 6.12 and Proposition 6.13].

**Theorem 2.9.** *A spectrum $G$ is $\mathbb{A}^1$-local iff for every $X \in \mathrm{Spc}(S)$ that is a finite colimit of smooth schemes over $S$, and for every $V \in \mathcal{C}(\mathcal{U})$, the map*

$$[\Sigma^{\mathcal{U}}_V X_+, G]_{\mathcal{H}_s(\mathscr{S})} \longrightarrow [\Sigma^{\mathcal{U}}_V X_+ \wedge \mathbb{A}^1_+, G]_{\mathcal{H}_s(\mathscr{S})}$$

*induced by the projection $\mathbb{A}^1_+ \longrightarrow S^0$ is a bijection.*

**Theorem 2.10.** *A map $f\colon G \longrightarrow G'$ in $\mathscr{S}$ is an $\mathbb{A}^1$-weak equivalence iff for every $X \in \mathrm{Spc}(S)$ that is a finite colimit of smooth schemes over $S$, and $V \in \mathcal{C}(\mathcal{U})$, the induced map: $[\Sigma^{\mathcal{U}}_V X_+, G]_{\mathcal{H}(\mathscr{S})} \longrightarrow [\Sigma^{\mathcal{U}}_V X_+, G']_{\mathcal{H}(\mathscr{S})}$ is a bijection.*

We now define connective spectra. Let $\mathscr{S}_C$ denote the smallest full subcategory of $\mathscr{S}$ containing all suspension spectra of smooth schemes over $S$ that is closed under homotopy colimits and extensions. Equivalently $\mathscr{S}_C$ is generated by colimits and extensions of $\Sigma^\infty_+ X$ for $X \in \mathrm{Sm}_S$ (here $\Sigma^\infty_+ X$ is used as shorthand for $\Sigma^\infty X_+$). The objects of $\mathscr{S}_C$ are referred to as very effective spectra in literature (see [SØ12, Definition 5.5]), but we call them $\Omega^\infty$-*connective spectra* or *connective spectra* for short. $\mathscr{S}_C$ is not a triangulated category itself, but it forms the homologically positive part of a $t$-structure on $\mathscr{S}$. It is symmetric monoidal however under the smash product. It is of course trivial, by Proposition 1.1, that $\Sigma^\infty\colon \mathscr{T} \longrightarrow \mathscr{S}$ takes values in $\mathscr{S}_C$. We now show that $\Omega^\infty$ and $\Sigma^\infty$ preserve weak equivalences.

**Proposition 2.11.** *If $f\colon X \longrightarrow Y$ be a weak equivalence in $\mathscr{T}$, then $\Sigma^\infty f\colon \Sigma^\infty X \longrightarrow \Sigma^\infty Y$ is a weak equivalence in $\mathscr{S}$.*

*Proof.* Let $f\colon X \longrightarrow Y$ be a weak equivalence in $\mathscr{T}$, so that for every $\mathbb{A}^1$-local space $Z$, the induced map $[Y,Z]_{\mathcal{H}_s(\mathscr{T})} \longrightarrow [X,Z]_{\mathcal{H}_s(\mathscr{T})}$ is an isomorphism. Let $E$ be an arbitrary $\mathbb{A}^1$-local spectrum in $\mathscr{S}$. Note that $\Omega^\infty E = E_{\mathcal{U}}$ is $\mathbb{A}^1$-local in $\mathscr{T}$. This is because for every $W \in \mathscr{T}$, $[\Sigma^\infty W, E] \longrightarrow [\Sigma^\infty W \wedge \mathbb{A}^1, E]$ (induced by the projection) is an isomorphism. By the adjunction then, we can see that $[W, E_{\mathcal{U}}] \longrightarrow [W \wedge \mathbb{A}^1, E_{\mathcal{U}}]$ (again induced by the projection) is an isomorphism. Now we know that $[Y, E_{\mathcal{U}}]_{\mathcal{H}_s(\mathscr{T})} \longrightarrow [X, E_{\mathcal{U}}]_{\mathcal{H}_s(\mathscr{T})}$ is an isomorphism. Passing to the adjunction, we have that $[\Sigma^\infty Y, E]_{\mathcal{H}_s(\mathscr{S})} \longrightarrow [\Sigma^\infty X, E]_{\mathcal{H}_s(\mathscr{S})}$ is an isomorphism as well. ∎

The analogous result for $\Omega^\infty$ follows from Theorem 2.10 and Theorem 1.7

**Proposition 2.12.** *If $g\colon E \longrightarrow E'$ is a weak equivalence in $\mathscr{S}$, then $g_{\mathcal{U}}\colon E_{\mathcal{U}} \longrightarrow E'_{\mathcal{U}}$ is one in $\mathscr{T}$.*

Moreover, a stronger result holds in the case of connective spectra.

**Proposition 2.13.** *Let $g\colon E \longrightarrow E'$ be a map between connective spectra in $\mathscr{S}$. The map $g$ is a weak equivalence in $\mathscr{S}$ iff $g_{\mathcal{U}}\colon E_{\mathcal{U}} \longrightarrow E'_{\mathcal{U}}$ is one in $\mathscr{T}$.*

*Proof.* It only remains to prove the backward direction, beginning with a map $g\colon E \longrightarrow E'$ such that $g_{\mathcal{U}}\colon E_{\mathcal{U}} \longrightarrow E'_{\mathcal{U}}$ is a weak equivalence in $\mathscr{T}$. To reduce the problem, we invoke [Hu03, Proposition 6.10]–a descent for weak equivalences of spectra which states that (small) directed colimits of spectra preserve weak equivalences. In light of the definition of connective spectra, and [Hu03, Proposition 6.10], we can restrict ourselves to the case of a weak equivalence $g_{\mathcal{U}}\colon (\Sigma^\infty Y_+)_{\mathcal{U}} \longrightarrow (\Sigma^\infty Y'_+)_{\mathcal{U}}$ (where $Y$ and $Y'$ are finite colimits of smooth schemes).

Let $X$ be a finite colimit of smooth schemes. By assumption, $[\Sigma^{\mathcal{U}}_{\mathcal{U}} X_+, \Sigma^\infty Y_+]_{\mathcal{H}(\mathscr{S})} \longrightarrow [\Sigma^{\mathcal{U}}_{\mathcal{U}} X_+, \Sigma^\infty Y'_+]_{\mathcal{H}(\mathscr{S})}$ is a bijection. Let $V \in \mathcal{C}(\mathcal{U}) \setminus \{\mathcal{U}\}$. We observe that for any finite colimit of smooth schemes $\mathcal{Y}$, $[\Sigma^{\mathcal{U}}_V X_+, \Sigma^\infty \mathcal{Y}_+]_{\mathcal{H}(\mathscr{S})}$ is trivial. Note the following chain of isomorphisms:

$$\begin{aligned}
[\Sigma^{\mathcal{U}}_V X_+, \Sigma^\infty \mathcal{Y}_+]_{\mathcal{H}(\mathscr{S})} &\cong [X_+, (\Sigma^\infty \mathcal{Y}_+)_V]_{\mathcal{H}(\mathscr{S})} \\
&\cong \left[X_+, \operatorname{colim}_{(W,Z)\in V\downarrow\mathcal{C}(\mathcal{U})} \Omega^Z \bigvee_{T\oplus W=\mathcal{U}} \Sigma^T \mathcal{Y}_+\right]_{\mathcal{H}(\mathscr{S})},
\end{aligned}$$

where $T\colon \mathcal{U} \longrightarrow V$ in $\mathcal{C}(\mathcal{U})$. Now using Proposition 1.2 on $X_+$, we see that:

$$\begin{aligned}[\Sigma^{\mathcal{U}}_V X_+, \Sigma^\infty \mathcal{Y}_+]_{\mathcal{H}(\mathscr{S})} &\cong \operatorname{colim}_{(W,Z)\in V\downarrow \mathcal{C}(\mathcal{U})} \left[ X_+, \Omega^Z \bigvee_{T\oplus W = \mathcal{U}} \Sigma^T \mathcal{Y}_+ \right]_{\mathcal{H}(\mathscr{S})} \\ &\cong \operatorname{colim}_{(W,Z)\in V\downarrow \mathcal{C}(\mathcal{U})} \left[ \Sigma^Z X_+, \bigvee_{T\oplus W = \mathcal{U}} \Sigma^T \mathcal{Y}_+ \right]_{\mathcal{H}(\mathscr{S})}.\end{aligned}$$

To finish the argument, note that $\dim Z < \dim(T) = \operatorname{codim}(W)$ for every $T$ such that $T \oplus W = \mathcal{U}$ (since $V \neq \mathcal{U}$), and $\Sigma^T \mathcal{Y}_+$ is $(\dim(T)-1)$-connected[5]. We conclude from Theorem 2.10 that $g$ is a weak equivalence in $\mathscr{S}$. ■

## 3. The homotopical monadicity theorem

The proof of the homotopical monadicity theorem, just as in [KMZ25], will rely on a derived homotopy theory defined by the bar resolution of $\Gamma$-algebras. The two-sided monadic bar construction used here is built simplicially, and as such we list properties of simplicial objects in $\mathscr{T}$ and $\mathscr{S}$ that will be essential for the proof. These properties are proved in Section 4.

As before, we will write $s\mathcal{C}$ for the simplicial objects in a category $\mathcal{C}$. For a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$, we will write $F_*$ for the induced functor on simplicial objects $F_*\colon s\mathcal{C} \longrightarrow s\mathcal{D}$ given by applying $F$ objectwise on simplices.

1. There exist realization functors $|-|\colon s\mathscr{T} \longrightarrow \mathscr{T}$ and $|-|\colon s\mathscr{S} \longrightarrow \mathscr{S}$ that are both left adjoints. The composite $|c_*|$ where $c_*$ is the constant simplicial object functor $\mathcal{C} \longrightarrow s\mathcal{C}$, is the identity for $\mathcal{C} = \mathscr{T}, \mathscr{S}$.
2. $|-|$ preserves homotopies on $s\mathscr{T}$ and $s\mathscr{S}$.
3. $|-|$ preserves weak equivalences between Reedy cofibrant objects in both $s\mathscr{T}$ and $s\mathscr{S}$.
4. For $K_* \in s\mathscr{T}$, there is a natural isomorphism $\Sigma^\infty |K_*| \cong |\Sigma^\infty_* K_*|$.
5. $|-|\colon s\mathscr{S} \longrightarrow \mathscr{S}$ takes levelwise connective spectra to connective spectra.
6. For $K_* \in s\mathscr{S}$, the natural map $\gamma\colon |\Omega^\infty_* K_*| \longrightarrow \Omega^\infty |K_*|$, given by the adjoint to the composite $\Sigma^\infty |\Omega^\infty_* E_*| \cong |\Sigma^\infty \Omega^\infty E_*| \xrightarrow{|\varepsilon_*|} |E_*|$ is, for levelwise connective $E_*$, a weak equivalence in $\mathscr{T}$.

For the rest of this section, we assume that these assumptions hold and we treat them formally going forth. We will also use the following observation.

**Observation 3.1.** For a simplicial object $X_* \in s\mathscr{T}$ and an object $J \in \mathscr{T}$, a map $f\colon X_0 \longrightarrow J$ such that $fd_0 = fd_1$ induces a map $\xi_*\colon X_* \longrightarrow c_* J$ such that $\xi_0 = f$, and therefore induces a map $\xi := |\xi_*|\colon |X_*| \longrightarrow J$ in $\mathscr{T}$.

The proof of the monadicity theorem presented here is due to an upcoming article [KMZ25] and discussions with J.P. May. First, we define the two-sided bar constructions that are essential for the proof. Let us return to the monad $(\Gamma, \mu, \eta)$, where $\Gamma = \Omega^\infty \Sigma^\infty$. We noted in Section 2 that $\Sigma^\infty$ was a $\Gamma$-functor in the sense that there was a natural map $\beta\colon \Sigma^\infty \Gamma \longrightarrow \Sigma^\infty$ defined by $\beta_X = \varepsilon_{\Sigma^\infty X}$. We can also consider $\Gamma$ as a $\Gamma$-functor with the action map $\mu = \Omega^\infty \varepsilon\colon \Gamma\Gamma \longrightarrow \Gamma$. For any $\Gamma$-algebra $(Y, \theta)$, we have the two-sided bar constructions:

$$\overline{Y} = B(\Gamma, \Gamma, Y) \quad \text{and} \quad \mathbb{E}Y = B(\Sigma^\infty, \Gamma, Y),$$

[5]See remarks following Theorem 18 in [Mor12].

where $\overline{Y} \in \mathscr{T}$, and $\mathbb{E}Y \in \mathscr{S}$. Recall that $B(\Gamma,\Gamma,Y)$ is the realization of the simplicial object in $\mathscr{T}$ (denoted $B_*(\Gamma,\Gamma,Y)$) whose $q$-simplices are given by $\Gamma^{q+1}Y$, and $B(\Sigma^\infty,\Gamma,Y)$ is the realization of the simplicial object in $\mathscr{S}$ (denoted $B_*(\Sigma^\infty,\Gamma,Y)$) whose $q$-simplices are given by $\Sigma^\infty\Gamma^q Y$. The zeroth face is given by the action of $\Gamma$, the $i$-th face for $0 < i < q$ is given by $\mu$, and the $q$-th face is given by $\theta$. The degeneracies are all given by $\eta\colon \mathrm{id} \longrightarrow \Gamma$. An extra degeneracy argument shows that:

**Lemma 3.2.** *For $Y \in \Gamma[\mathscr{T}]$, the maps $\mu\colon \Gamma^{q+1} \longrightarrow \Gamma^q$ induce a natural map $\zeta\colon \overline{Y} \longrightarrow Y$ in $\mathscr{T}$. $\zeta$ is a homotopy equivalence with homotopy inverse $\nu\colon Y \longrightarrow \overline{Y}$ induced by the maps $\eta\colon Y \longrightarrow \Gamma^{q+1}Y$.*

It is in this sense that we can see $\overline{Y}$ as a derived resolution of $Y \in \Gamma[\mathscr{T}]$. However, there is a conceptual gap in the above result. $Y$ is a $\Gamma$-algebra, and the maps $\mu\colon \Gamma^{q+1}Y \longrightarrow \Gamma^q Y$ and $\eta\colon Y \longrightarrow \Gamma^{q+1}Y$ are both maps of $\Gamma$-algebras. We would like to say that $\overline{Y}$ is a $\Gamma$-algebra as well, and $\zeta$ and $\nu$ are maps of $\Gamma$-algebras—so that we can really think of $\overline{Y}$ as a derived resolution of $Y$ in the category of $\Gamma$-algebras. However, since $\Omega^\infty$ does not commute with $|-|$ on the nose, it is not at all clear that there is a canonically induced $\Gamma$-algebra structure on $\overline{Y}$. We will soon elaborate on the kind of structure that $\overline{Y}$ possesses.

Now we describe a conceptually dual procedure on $\mathscr{S}$. Let $\hat{\Gamma} := \Sigma^\infty\Omega^\infty$ and $E \in \mathscr{S}$. Observe that $B_*(\Sigma^\infty,\Gamma,\Omega^\infty E)$ has $q$-simplices $\hat{\Gamma}^{q+1}E$ and all faces given by $\varepsilon$. We define the following as a conceptual dual to $\overline{Y}$ for $Y \in \Gamma[\mathscr{T}]$.

$$\hat{E} := B(\Sigma^\infty,\Gamma,\Omega^\infty E) = \mathbb{E}\Omega^\infty_\Gamma E.$$

Since $\Sigma^\infty Z$ is always connective for $Z \in \mathscr{T}$, and $|-|$ takes levelwise connective spectra to connective spectra (by assumption (5)), we see that $\mathbb{E}Y$ is connective for any $Y \in \Gamma[\mathscr{T}]$. Therefore $\hat{E}$ is connective for any $E \in \mathscr{S}$. The following should be seen as the dual to Lemma 3.2.

**Lemma 3.3.** *For $E \in \mathscr{S}$, the face maps $\varepsilon\colon \hat{\Gamma}^{q+1} \longrightarrow \hat{\Gamma}^q$ induce a natural map $\xi\colon \hat{E} \longrightarrow E$ in $\mathscr{S}$. If $E$ is connective, then $\xi$ is a weak equivalence.*

*Proof.* The first part of the statement is a direct application of Observation 3.1 to $\varepsilon\colon \hat{\Gamma}E \longrightarrow E$. Now suppose $E$ is connective. Since $\hat{E}$ is connective, by Proposition 2.13, it suffices to show that $\Omega^\infty\xi\colon \Omega^\infty\hat{E} \longrightarrow \Omega^\infty E$ is a weak equivalence. Consider the following commuting diagram (due to Observation 3.1):

$$\begin{tikzcd}
\overline{\Omega^\infty E} = B(\Gamma,\Gamma,\Omega^\infty E) \arrow[rr, "\gamma"] \arrow[dr, "\zeta"'] & & \Omega^\infty B(\Sigma^\infty,\Gamma,\Omega^\infty E) = \Omega^\infty\hat{E} \arrow[dl, "\Omega^\infty\xi"] \\
& \Omega^\infty E &
\end{tikzcd}$$

where $\gamma$ and $\zeta$ are weak equivalences (by assumption (6) and Lemma 3.2). It follows by the two-out-of-three property that $\xi$ is a weak equivalence as well. ∎

Just as we had thought of $\overline{Y}$ as a derived resolution of $Y \in \Gamma[\mathscr{T}]$, we should think of $\hat{E}$ as a derived connective cover of $E$ (whose zeroth space is weak equivalent to the derived resolution of the zeroth space of $E$).

Piecing together the derived language from above, we get the following result—a crude version of what we had set out to prove.

**Theorem 3.4** (Crude homotopical monadicity)**.** *The functors $\mathbb{E}\colon \Gamma[\mathscr{T}] \longrightarrow \mathscr{S}_c$ and $\Omega^\infty_\Gamma\colon \mathscr{S}_c \longrightarrow \Gamma[\mathscr{T}]$ preserve weak equivalences. For $E \in \mathscr{S}_c$, $\xi\colon \hat{E} \longrightarrow E$ is a natural weak equivalence. For $Y \in \Gamma[\mathscr{T}]$, we have natural weak equivalences in $\mathscr{T}$*

$$Y \xleftarrow{\ \zeta\ } \overline{Y} \xrightarrow{\ \gamma\ } \Omega^\infty \mathbb{E} Y\,.$$

*Proof.* We first note that $\Omega^\infty_\Gamma$ and $\mathbb{E}$ preserve weak equivalences since $\Omega^\infty$ and $\Sigma^\infty$ preserve weak equivalences, and $|-|$ preserves weak equivalences between Reedy cofibrant spectra (this is assumption (3)). Lemma 3.3 provides a natural weak equivalence $\xi\colon \hat{E} \longrightarrow E$ for every connective spectrum $E$. For $Y \in \Gamma[\mathscr{T}]$, the weak equivalence $\overline{Y} \longrightarrow Y$ is given by Lemma 3.2, and the weak equivalence $\gamma\colon \overline{Y} \longrightarrow \Omega^\infty \mathbb{E} Y$ is furnished by assumption (6). ∎

The conceptual gap described following Lemma 3.2 is much more apparent now. We would really like to say that $\overline{Y}$ is a $\Gamma$-algebra such that $\zeta$ and $\gamma$ are $\Gamma$-algebra maps. As a result, it is a priori not clear that Theorem 3.4 provides an equivalence between the homotopy categories of $\Gamma$-algebras and connective spectra. Nonetheless, with some conceptual investigation into the relation between $\Gamma$-algebras and simplicial $\Gamma$-algebras, we will establish a sharpened result.

Moving to $s\mathscr{T}$, we have an adjunction $(\Sigma^\infty_*, \Omega^\infty_*)$ and its associated monad $\Gamma_*$ on $s\mathscr{T}$. We refer to $\Gamma_*$-algebras as simplicial $\Gamma$-algebras. Going forth, we conflate $\gamma$ with the natural map $\gamma\colon |\Omega^\infty_*\Sigma^\infty_*| \longrightarrow \Omega^\infty|\Sigma^\infty| \cong \Omega^\infty\Sigma^\infty|-|$. Realization connects the monad $\Gamma_*$ to $\Gamma$, and thereby 'links' the right adjoints $\Omega^\infty_*$ and $\Omega^\infty$.

**Proposition 3.5.** *$(|-|, \gamma)$ is an op-lax map of monads $\Gamma_* \longrightarrow \Gamma$. That is to say, $|-|\colon s\mathscr{T} \longrightarrow \mathscr{T}$ is a functor equipped with a natural transformation $\gamma\colon |\Gamma_*| \longrightarrow \Gamma|-|$ such that the following commute.*

$$\begin{tikzcd}
|\Gamma_*\Gamma_*| \arrow[r, "\gamma\Gamma_*"] \arrow[d, "|\mu_*|"'] & \Gamma|\Gamma_*| \arrow[r, "\Gamma\gamma"] & \Gamma\Gamma|-| \arrow[d, "\mu|-|"] \\
|\Gamma_*| \arrow[rr, "\gamma"'] & & \Gamma|-|
\end{tikzcd}
\qquad
\begin{tikzcd}
& |-| \arrow[dl, "|\eta_*|"'] \arrow[dr, "\eta|-|"] & \\
|\Gamma_*| \arrow[rr, "\gamma"'] & & \Gamma|-|
\end{tikzcd}$$

*Moreover, $(|-|, \gamma)\colon \Gamma_* \longrightarrow \Gamma$ links $\Omega^\infty_*$ to $\Omega^\infty$ in the sense that the following commute.*

$$\begin{tikzcd}
|\Gamma_*\Omega^\infty_*| \arrow[r, "\gamma\Omega^\infty_*"] \arrow[d, "|\Omega^\infty_*\varepsilon_*|"'] & \Gamma|\Omega^\infty_*| \arrow[r, "\Gamma\gamma"] & \Gamma\Omega^\infty|-| \arrow[d, "\Omega^\infty\varepsilon|-|"] \\
|\Omega^\infty_*| \arrow[rr, "\gamma"'] & & \Omega^\infty|-|.
\end{tikzcd}
\qquad
\begin{tikzcd}
|\Omega^\infty_*| \arrow[rr, "\gamma"] \arrow[d, "|\eta_*\Omega^\infty_*|"'] \arrow[dr, "\eta|\Omega^\infty_*|"] & & \Omega^\infty|-| \arrow[d, "\eta\Omega^\infty|-|"] \\
|\Gamma_*\Omega^\infty_*| \arrow[r, "\gamma\Omega^\infty_*"'] & \Gamma|\Omega^\infty_*| \arrow[r, "\Gamma\gamma"'] & \Gamma\Omega^\infty|-|
\end{tikzcd}$$

*Proof.* We prove the commutativity of the unit diagram for $(|-|, \gamma)$ (the top right diagram), and the product diagram for the linking of $\Omega^\infty_*$ to $\Omega^\infty$ (the bottom left diagram)—the rest follow automatically. For the unit diagram, we first pass to the adjoint:

$$\begin{tikzcd}
& {|\Sigma^\infty_*|} \arrow[dl, "{|\Sigma^\infty_* \eta_*|}"'] \arrow[dr, "{|\Sigma^\infty_*|}"] & \\
{|\Sigma^\infty_* \Omega^\infty_* \Sigma^\infty_*|} \arrow[rr, "{|\varepsilon_* \Sigma^\infty_*|}"'] & & {|\Sigma^\infty_*|.}
\end{tikzcd}$$

This diagram is just the realization of a triangle identity for the adjunction $(\Sigma^\infty_*, \Omega^\infty_*)$ in $s\mathscr{T}$, and therefore it commutes. As for the product diagram, passing to the adjoint again, we have the slightly larger diagram (where $\tilde{\gamma}\colon \Sigma^\infty|\Omega^\infty| \longrightarrow |-|$ denotes the adjoint map to $\gamma$):

$$\begin{tikzcd}
\Sigma^\infty|\Gamma_* \Omega^\infty_*| \arrow[r, "\cong"] \arrow[d, "\Sigma^\infty|\Omega^\infty_* \varepsilon|"'] & {|\Sigma^\infty_* \Omega^\infty_* \Sigma^\infty_* \Omega^\infty_*|} \arrow[r, "|\varepsilon_*|"] \arrow[d, "|\Sigma^\infty_* \Omega^\infty_* \Sigma^\infty_*|"'] & {|\Sigma^\infty_* \Omega^\infty_*|} \arrow[r, "\cong"] \arrow[d, "|\varepsilon_*|"'] & \Sigma^\infty|\Omega^\infty_*| \arrow[r, "\Sigma^\infty \gamma"] \arrow[dl, "\tilde{\gamma}"'] \arrow[dr, "\tilde{\gamma}"] & \Sigma^\infty \Omega^\infty|-| \arrow[d, "\varepsilon|-|"] \\
\Sigma^\infty|\Omega^\infty_*| \arrow[r, "\cong"'] & {|\Sigma^\infty_* \Omega^\infty_*|} \arrow[r, "|\varepsilon_*|"'] & {|-|} \arrow[rr, "="'] & & {|-|}
\end{tikzcd}$$

The left two squares commmute trivially. The left triangle in the rightmost square commutes by definition of $\gamma$, and the right triangle commutes since $\gamma$ and $\tilde{\gamma}$ are adjoint maps related by the counit $\varepsilon$. ∎

This is the first hint at how realization relates the simplicial adjunction $(\Sigma^\infty_*, \Omega^\infty_*)$ to the space-level adjunction $(\Sigma^\infty, \Omega^\infty)$. We would now like to say something about the algebras over these adjunctions (by which we mean algebras over the corresponding monads). We turn to the general story for a bit, studying conceptually how an op-lax map of monads may link its algebras.

**Definition 3.6.** Let $(\mathbb{C}, \mu, \eta)$ and $(\mathbb{D}, \nu, \delta)$ be monads on categories $\mathscr{V}$ and $\mathscr{W}$. Let $(F, \alpha)$ be an op-lax map of monads $\mathbb{C} \longrightarrow \mathbb{D}$. We say that $(X, \theta) \in \mathbb{C}[\mathscr{V}]$ and $(Y, \varphi) \in \mathbb{D}[\mathscr{W}]$ are $F$-linked by $\beta\colon FX \longrightarrow Y$ (called an $F$-linking map) if the following diagram commutes.

$$\begin{tikzcd}
F\mathbb{C}X \arrow[r, "\alpha_X"] \arrow[d, "F\theta"'] & \mathbb{D}FX \arrow[r, "\mathbb{D}\beta"] & \mathbb{D}Y \arrow[d, "\varphi"] \\
FX \arrow[rr, "\beta"'] & & Y
\end{tikzcd}$$

When there is no ambiguity in the op-lax map $(F, \alpha)$, we will just say that $X$ and $Y$ are linked, and that they are linked by a linking map $\beta$.

**Remark 3.7.** Consider the case where $\mathscr{V} = \mathscr{W}$, $\mathbb{C} = \mathbb{D}$, and the op-lax map of monads is $(\mathrm{id}, \mathrm{id})$ on $\mathbb{C}$. See that two algebras $(X, \theta)$ and $(Y, \varphi)$ in $\mathbb{C}[\mathscr{V}]$ are linked by $\beta\colon X \longrightarrow Y$ iff the following commutes.

$$\begin{tikzcd}
\mathbb{C}X \arrow[r, "\mathbb{C}\beta"] \arrow[d, "\theta"'] & \mathbb{C}Y \arrow[d, "\varphi"] \\
X \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

That is, iff $\beta$ is a map of $\mathbb{C}$-algebras! This tells us that maps of $\mathbb{C}$-algebras are already linking maps.

More generally, suppose we have an op-lax map $(F, \mathrm{id})$ from $\mathbb{C}$ to $\mathbb{D}$, so that $F\mathbb{C} = \mathbb{D}F$. In this case, for any $(X, \theta) \in \mathbb{C}[\mathscr{V}]$, we have a canonical $\mathbb{D}$-action on $FX$ given by $F\theta\colon \mathbb{D}FX = F\mathbb{C}X \longrightarrow FX$, so that $(FX, F\theta) \in \mathbb{D}[\mathscr{W}]$. For $(Y, \varphi) \in \mathbb{D}[\mathscr{W}]$, we note that a map $\beta\colon FX \longrightarrow Y$ is a linking map iff it is a map of $\mathbb{D}$-algebras. This is exactly what the picture would have been if $|\Gamma_*| = \Gamma|-|$ in our applications (and $\gamma$ would have been a map of $\Gamma$-algebras)—but the failure of this equality is what forces us to leave the world of maps of algebras, as this example has shown.

**Remark 3.8.** Still more generally, return to the case of an op-lax map $(F, \alpha)$ from $\mathbb{C}$ to $\mathbb{D}$. Suppose we have $(X, \theta) \in \mathbb{C}[\mathscr{V}]$, $(FX, \psi) \in \mathbb{D}[\mathscr{W}]$, and $(Y, \varphi) \in \mathbb{D}[\mathscr{W}]$. A map $\beta\colon FX \longrightarrow Y$ is a linking map iff the following commutes.

$$\begin{tikzcd} F\mathbb{C}X \arrow[r, "\alpha_X"] \arrow[d, "F\theta"'] & \mathbb{D}FX \arrow[r, "\mathbb{D}\beta"] & \mathbb{D}Y \arrow[d, "\varphi"] \\ FX \arrow[rr, "\beta"'] & & Y \end{tikzcd}$$

Note that if $\beta$ is a map of $\mathbb{D}$-algebras, and the following triangle commutes, then $\beta$ is a linking map.

$$\begin{tikzcd} F\mathbb{C}X \arrow[r, "\alpha_X"] \arrow[d, "F\theta"'] & \mathbb{D}FX \arrow[dl, "\psi"] \\ FX & \end{tikzcd}$$

Note that this triangle commutes iff $\mathrm{id}\colon FX \longrightarrow FX$ is a linking map linking the algebras $(X, \theta)$ and $(FX, \psi)$. Whenever this is the case, [KMZ25, Definition 2.30] calls $X$ and $FX$ linked. As a consequence of this remark, we obtain that $X$ and $FX$ are linked in the sense of [KMZ25] iff $X$ and $FX$ are linked via $\mathrm{id}\colon FX \longrightarrow FX$ in the sense of Definition 3.6.

**Remark 3.9.** In the loop-space story, we take $\mathbb{C} = \Gamma_*\colon s\mathscr{T} \longrightarrow s\mathscr{T}$, $\mathbb{D} = \Gamma\colon \mathscr{T} \longrightarrow \mathscr{T}$, and $(F, \alpha)$ to be the op-lax map $(|-|, \gamma)$ from Proposition 3.5. So $X_* \in \Gamma_*[s\mathscr{T}]$ and $Y \in \Gamma[\mathscr{T}]$ are linked by $\beta\colon |X_*| \longrightarrow Y$ iff the following diagram commutes.

$$\begin{tikzcd} {|\Gamma_* X_*|} \arrow[r, "\gamma"] \arrow[d, "|\varphi_*|"'] & \Gamma|X_*| \arrow[r, "\Gamma\beta"] & \Gamma Y \arrow[d, "\theta"] \\ {|X_*|} \arrow[rr, "\beta"'] & & Y \end{tikzcd}$$

In particular, by Proposition 3.5, $\Omega^\infty_* E_*$ and $\Omega^\infty |E_*|$ are linked by $\gamma\colon |\Omega^\infty_* E_*| \longrightarrow \Omega^\infty |E_*|$. Specializing furthermore, for the purposes of understanding the structure of $\overline{Y}$ in Theorem 3.4, we see that $B_*(\Gamma, \Gamma, Y)$ and $\Omega^\infty_\Gamma \mathbb{E}Y$ are linked by $\gamma\colon \overline{Y} \longrightarrow \Omega^\infty_\Gamma \mathbb{E}Y$.

**Example 3.10.** Consider the case where there exists an op-lax map $(G, \mathrm{id})$ from $\mathbb{D}$ to $\mathbb{C}$ such that $FG = \mathrm{id}$ and $\alpha_{GY} = \mathrm{id}$ for all $Y \in \mathscr{W}$. Intuitively $(G, \mathrm{id})$ should be thought of as a section of the op-lax map $(F, \alpha)$. If $(Y, \varphi)$ is a $\mathbb{D}$-algebra, then $(GY, G\varphi)$ is automatically a $\mathbb{C}$-algebra. In this case, $GY$ and $Y$ are trivially linked by the identity $\mathrm{id}\colon FGY \longrightarrow Y$.

$$\begin{tikzcd} F\mathbb{C}GY \arrow[r, equal] \arrow[d, "\varphi"'] & \mathbb{D}FGY \arrow[r, equal] & \mathbb{D}Y \arrow[d, "\varphi"] \\ FGY \arrow[rr, equal] & & Y \end{tikzcd}$$

More generally, when we have $(Y,\varphi) \in \mathbb{D}[\mathscr{W}]$, $(X,\theta) \in \mathbb{C}[\mathscr{V}]$, and $(FX,\psi) \in \mathbb{D}[\mathscr{W}]$, a map $\beta\colon Y \longrightarrow FX$ in $\mathscr{W}$ is a linking map iff it is a map of $\mathbb{D}$-algebras. Suppose $\beta = F(f)$ for some $F\colon GY \longrightarrow X$, a map of $\mathbb{C}$-algebras. Assume also that $(X,\theta)$ and $(FX,\psi)$ are linked by the identity $\mathrm{id}\colon FX \longrightarrow FX$. In this case, $\beta\colon Y \longrightarrow FX$ is a map of $\mathbb{D}$-algebras (and therefore also a linking map). Indeed, we have by assumption that the following diagrams commute:

$$\begin{tikzcd} \mathbb{C}GY \arrow[r, "\mathbb{C}f"] \arrow[d, "G\varphi"'] & \mathbb{C}X \arrow[d, "\theta"] \\ GY \arrow[r, "f"'] & X \end{tikzcd} \qquad \begin{tikzcd} F\mathbb{C}X \arrow[r, "\alpha_X"] \arrow[d, "F\theta"'] & \mathbb{D}FX \arrow[dl, "\psi"] \\ FX & \end{tikzcd}$$

Applying $F$ to the diagram on the left and composing the two diagrams, we get:

$$\begin{tikzcd} \mathbb{D}Y \arrow[r, "F\mathbb{C}f"] \arrow[d, "\varphi"'] & F\mathbb{C}X \arrow[r, "\alpha_X"] \arrow[d, "F\theta"] & \mathbb{D}FX \arrow[dl, "\psi"] \\ Y \arrow[r, "\beta"'] & FX & \end{tikzcd}$$

Using the naturality of $\alpha$, we know that $\alpha_X \circ F\mathbb{C}f = \mathbb{D}Ff = \mathbb{D}\beta$. Therefore $\beta$ is a map of $\mathbb{D}$-algebras. This is again the picture we would have had with $\zeta$ in Theorem 3.4 had $|B_*(\Gamma,\Gamma,Y)|$ been a $\Gamma$-algebra[6]. We will return to the case of $(G,\mathrm{id})$ soon. Note that in applications, we will take $G$ to be the constant simplicial object functor $G\colon \mathscr{T} \longrightarrow s\mathscr{T}$.

We elaborate on what it means for the algebras $(X,\theta) \in \mathbb{C}[\mathscr{V}]$ and $(Y,\varphi) \in \mathbb{D}[\mathscr{W}]$ to be linked. Consider the unit diagrams for the algebras $(X,\theta)$ and $(Y,\varphi)$.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & \mathbb{C}X \arrow[d, "\theta"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\delta_Y"] \arrow[dr, equal] & \mathbb{D}Y \arrow[d, "\varphi"] \\ & Y \end{tikzcd}$$

[6]By Remark 3.7, for any $(Y,\varphi) \in \mathbb{D}[\mathscr{W}]$ and $(X,\theta) \in \mathbb{C}[\mathscr{V}]$, a map $f\colon GY \longrightarrow X$ is in $\mathbb{C}[\mathscr{V}]$ iff it is a $G$-linking map. Therefore in general when $f$ is a map of $\mathbb{C}$-algebras, while it may not make sense to speak of $\beta = Ff\colon Y \longrightarrow FX$ as a map of $\mathbb{D}$-algebras or as an $F$-linking map, it is always the image under $F$ of a $G$-linking map.

Applying $F$ to the unit diagram for $X$, the map $\beta\colon FX \longrightarrow Y$ links together the unit diagrams to produce the commutative diagram below.

$$\begin{tikzcd}
 & & Y \arrow[r, "\delta_Y"] \arrow[dr, equal] & \mathbb{D}Y \arrow[d, "\varphi"] \\
 & & & Y \\
FX \arrow[r, "F\eta_X"'] \arrow[dr, equal] \arrow[uurr, "\beta"] & F\mathbb{C}X \arrow[d, "F\theta"] \arrow[uurr, "\mathbb{D}\beta\circ\alpha_X"'] & & \\
 & FX \arrow[uurr, "\beta"'] & &
\end{tikzcd}$$

To see that this prism commutes, it suffices to see that its top face commutes. We can write the top face as the composite:

$$\begin{tikzcd}
FX \arrow[rr, "\beta"] \arrow[d, "F\eta_X"'] \arrow[dr, "\delta_{FX}"] & & Y \arrow[d, "\delta_Y"] \\
F\mathbb{C}X \arrow[r, "\alpha_X"'] & \mathbb{D}FX \arrow[r, "\mathbb{D}\beta"'] & \mathbb{D}Y
\end{tikzcd}$$

The triangle in this diagram commutes since it is the unit diagram of the op-lax map $(F, \alpha)$, and the remaining trapezoid is a naturality square for $\delta$.

Similarly, we have the product diagrams for $X$ and $Y$:

$$\begin{tikzcd}
\mathbb{C}\mathbb{C}X \arrow[r, "\mathbb{C}\theta"] \arrow[d, "\mu_X"'] & \mathbb{C}X \arrow[d, "\theta"] \\
\mathbb{C}X \arrow[r, "\theta"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}\mathbb{D}Y \arrow[r, "\mathbb{D}\varphi"] \arrow[d, "\nu_Y"'] & \mathbb{D}Y \arrow[d, "\varphi"] \\
\mathbb{D}Y \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

Once more, applying $F$ to the product diagram for $X$, the map $\beta\colon FX \longrightarrow Y$ links together the product diagrams to produce the commuting diagram below.

$$\begin{tikzcd}
 & & & \mathbb{D}\mathbb{D}Y \arrow[r, "\mathbb{D}\varphi"] \arrow[d, "\nu_Y"'] & \mathbb{D}Y \arrow[d, "\varphi"] \\
 & & \mathbb{D}\mathbb{D}FX \arrow[ur, "\mathbb{D}\mathbb{D}\beta"] & \mathbb{D}Y \arrow[r, "\varphi"'] & Y \\
F\mathbb{C}\mathbb{C}X \arrow[r, "F\mathbb{C}\theta"'] \arrow[d, "F\mu_X"'] \arrow[urr, "\mathbb{D}\alpha_X\circ\alpha_{\mathbb{C}X}"] & F\mathbb{C}X \arrow[d, "F\theta"] \arrow[uurrr, "\mathbb{D}\beta\circ\alpha_X"'] & & & \\
F\mathbb{C}X \arrow[r, "F\theta"'] \arrow[uurrr, "\mathbb{D}\beta\circ\alpha_X"'] & FX \arrow[uurrr, "\beta"] & & &
\end{tikzcd}$$

The front and back faces of this cuboid commute since they are the product diagrams for $X$ and $Y$. The right face and the bottom face also commute since $X$ and $Y$ are linked by $\beta$ by assumption. We can decompose the top face into the following commutative diagrams:

$$\begin{tikzcd}
F\mathbb{C}\mathbb{C}X \arrow[r, "\alpha_{\mathbb{C}X}"] \arrow[d, "F\mathbb{C}\theta"'] & \mathbb{D}F\mathbb{C}X \arrow[r, "\mathbb{D}\alpha_X"] \arrow[d, "\mathbb{D}F\theta"] & \mathbb{D}\mathbb{D}FX \arrow[r, "\mathbb{D}\mathbb{D}\beta"] & \mathbb{D}\mathbb{D}Y \arrow[d, "\mathbb{D}\varphi"] \\
F\mathbb{C}X \arrow[r, "\alpha_X"'] & \mathbb{D}FX \arrow[rr, "\mathbb{D}\beta"'] & & \mathbb{D}Y.
\end{tikzcd}$$

The left square commutes as a naturality square for $\alpha$, and the right square commutes since it is $\mathbb{D}$ applied to the linking diagram for $\beta$. Similarly the left face of the cuboid also decomposes as:

$$\begin{tikzcd}
F\mathbb{C}\mathbb{C}X \arrow[r, "\alpha_{\mathbb{C}X}"] \arrow[d, "F\mu_X"'] & \mathbb{D}F\mathbb{C}X \arrow[r, "\mathbb{D}\alpha_X"] & \mathbb{D}\mathbb{D}FX \arrow[r, "\mathbb{D}\mathbb{D}\beta"] \arrow[d, "\nu_{FX}"'] & \mathbb{D}\mathbb{D}Y \arrow[d, "\nu_Y"] \\
F\mathbb{C}X \arrow[rr, "\alpha_X"'] & & \mathbb{D}FX \arrow[r, "\mathbb{D}\beta"'] & \mathbb{D}Y.
\end{tikzcd}$$

The left square commutes as the product diagram for the op-lax map $(F, \alpha)$, and the right square commutes since it is the naturality square for $\nu$.

In essence, what we have tried to motivate here is this: to say the algebras $(X, \theta)$ and $(Y, \varphi)$ are linked, is to say in a precise sense that their algebra structures are linked together via the linking map $\beta$.

Now that we have said something about the structures of $\overline{Y}$ and $\Omega^\infty\mathbb{E}Y$, and how $\gamma$ links the two—we say more about the structure of the map $\gamma\circ\zeta^{-1}\colon Y \longrightarrow \Omega^\infty\mathbb{E}Y$ in Theorem 3.4. Let us return again to the general story of an op-lax map $(F, \alpha)$ from $\mathbb{C}$ to $\mathbb{D}$ with a section $(G, \mathrm{id})$. Note that $G$ provides an embedding of $\mathbb{D}[\mathscr{W}]$ into a full subcategory $\mathbb{C}[\mathscr{V}]$. As a consequence, if $f\colon Y' \longrightarrow Y$ is a map of $\mathbb{D}$-algebras, then $f$ lifts uniquely to a map of $\mathbb{C}$-algebras $\tilde{f}\colon GY' \longrightarrow GY$ over $F$. If however $f : Y' \longrightarrow Y$ was only a map *between* $\mathbb{D}$-algebras, there is a priori no guarantee that such a lift exists. The best we can do is hope for an approximation by a map out of the source algebra $Y'$ that lifts to a map of $\mathbb{C}$-algebras. Such a map, pushed forward through $F$, lands in the image of a $\mathbb{C}$-algebra under $F$—so by an approximation, the only thing we could mean is a linking map to the target algebra $Y$ [7]. If such an approximation exists, we say that the map between $\mathbb{D}$-algebras we started out with is a linked map. We capture this intuition precisely in the following definition.

**Definition 3.11.** Let $(F, \alpha)$ be an op-lax map of monads from $(\mathbb{C}, \mu, \eta)$ (on $\mathscr{V}$) to $(\mathbb{D}, \nu, \delta)$ (on $\mathscr{W}$). We say that $(F, \alpha)$ admits a section if there exists an op-lax map of monads $(G, \mathrm{id})$ from $(\mathbb{D}, \nu, \delta)$ to $(\mathbb{C}, \mu, \eta)$ such that $FG = \mathrm{id}$ and $\alpha_{GY} = \mathrm{id}$ for every $Y \in \mathscr{W}$. Such an op-lax map is called a *section* of $F$. We say that a map $f\colon Y' \longrightarrow Y$ is a *linked map* if there exists a map $\rho\colon GY' \longrightarrow X$ of $\mathbb{C}$-algebras, and

[7] In applications, this linking map will be a weak equivalence, so the usage of 'approximation' will be more justified then.

a linking map $\beta\colon FX \longrightarrow Y$ such that $f = \beta \circ F\rho$.

$$\begin{tikzcd} & FX \arrow[dr, "\beta"] & \\ Y' \arrow[ur, "F\rho"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Heading back to the loop-space story, we note that our op-lax map of monads $(|-|, \gamma)$ from $\Gamma_*$ to $\Gamma$ admits a section $(c_*, \mathrm{id})$ where $c_*$ is the constant simplicial object functor[8]. We can therefore speak of linked maps in our setting. The specialization of linked maps to weak equivalences, the kinds of maps that appeared in Theorem 3.4, is clear now. We say that a map between algebras is a linked weak equivalence if it is a linked map that is a weak equivalence.

Our conceptual discussion so far on linked algebras and linked maps of algebras culminates seamlessly in the following sharpened version of the homotopical monadicity theorem.

**Theorem 3.12** (Sharpened homotopical monadicity)**.** *Let $(Y, \theta)$ be a $\Gamma$-algebra. Consider the maps:*

$$\begin{tikzcd} Y \arrow[d, "="'] & \overline{Y} \arrow[l, "\zeta"'] \arrow[r, "\gamma"] \arrow[d, "="] & \Omega^\infty \mathbb{E}Y \arrow[d, "="] \\ |Y_*| & |\Omega^\infty_* \mathbb{E}_* Y| \arrow[l, "|\zeta_*|"] \arrow[r, "\gamma"'] & \Omega^\infty |\mathbb{E}_* Y|. \end{tikzcd}$$

*We have:*

1. *Forgetting structures, $\zeta$ is a homotopy equivalence and $\gamma$ is a weak equivalence.*
2. *$\zeta = |\zeta_*|$ with homotopy inverse $\nu = |\nu_*|$ where $\zeta_*$ and $\nu_*$ are maps of simplicial $\Gamma$-algebras.*
3. *$\gamma$ is a linking map that links $\overline{Y}$ to the $\Gamma$-algebra $\Omega^\infty \mathbb{E}Y$.*
4. *The composite $\gamma \circ \nu$ is a linked weak equivalence between $Y$ and $\Omega^\infty \mathbb{E}Y$.*

The consequence for the homotopy categories of $\Gamma$-algebras and connective spectra that one would hope for, is provided by this corollary of the sharpened monadicity theorem.

**Corollary 3.13.** *The full subcategory of $\Gamma$-algebras in $\mathscr{T}$ localized at the linked weak equivalences is equivalent to the homotopy category of connective spectra.*

## 4. Simplicial objects

In this section we present the properties of simplicial objects in $\mathscr{T}$ and $\mathscr{S}$ assumed in Section 3. We have denoted by $s\mathcal{C}$ the category of simplicial objects in a category $\mathcal{C}$. In particular, $s\mathscr{T} = \mathrm{ssSh}((\mathrm{Sm}_S)_{Nis})$, the category of bisimplicial sheaves on $\mathrm{Sm}_S$. In order to prescribe realization functors $|-|\colon s\mathscr{T} \longrightarrow \mathscr{T}$ and $|-|\colon s\mathscr{S} \longrightarrow \mathscr{S}$, we must first record our choice of the cosimplicial objects in $\mathscr{T}$

[8]Note that $\gamma_{c_* Y}\colon \Gamma Y \longrightarrow \Gamma Y$ is the identity since the adjoint to the map $\varepsilon\colon \Sigma^\infty\Omega^\infty E \longrightarrow E$ is the identity.

and $\mathscr{S}$. There is a conventional choice in the case of $\mathscr{T}$. For each $\mathbf{n} \in \Delta$, let $\Delta_n$ denote the object in $\mathscr{T}$ representing $S \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[x_1, \cdots, x_n]/\left(\sum_i x_i = 1\right)$. Given $f\colon \mathbf{n} \longrightarrow \mathbf{m}$ in $\Delta$, there is a map $\Delta_f\colon \Delta_n \longrightarrow \Delta_m$ (considered as schemes) given by the ring homomorphism $\Delta_f(x_i) = \sum_{j\in f^{-1}(i)} x_j$ if $f^{-1}(i) \neq \emptyset$ and 0 otherwise. Together, this defines a cosimplicial object in $\mathscr{T}$ such that each $\Delta_n$ is isomorphic to $\mathbb{A}^n_S$. The typical prescription, noting that $\mathscr{T}$ is tensored over itself, then leads us to define $|K_*| = K_* \otimes_\Delta \Delta_*$ for $K_* \in s\mathscr{T}$. The realization functor here however takes a much simpler form due to the bisimplicial nature of simplicial spaces. For $K_* \in s\mathscr{T}$, thought of a bisimplicial sheaf $K_{*,*}$, we have that $|K_*|$ is isomorphic to the diagonal simplicial sheaf $d(K)_n = K_{n,n}$. By the usual yoga, since $\mathscr{T}$ is cotensored over itself, $|-|\colon s\mathscr{T} \longrightarrow \mathscr{T}$ is a left adjoint functor with right adjoint $\mathbb{S}\colon \mathscr{T} \longrightarrow s\mathscr{T}$ given by $\mathbb{S}(X)_q = F(\Delta_q, X)$ where $F(-,-)$ is the cotensor functor $\mathscr{T}^{\mathrm{op}} \times \mathscr{T} \longrightarrow \mathscr{T}$. Since $\mathscr{S}$ is tensored and cotensored over $\mathscr{T}$, our choice of the cosimplicial objects in $\mathscr{T}$ yields the realization functor $|-|\colon s\mathscr{S} \longrightarrow \mathscr{S}$ given by $|E_*| = E_* \otimes_\Delta \Delta_*$ for simplicial spectra and its right adjoint (which we also denote $\mathbb{S}\colon \mathscr{S} \longrightarrow s\mathscr{S}$) given by $\mathbb{S}(E)_q = F(\Delta_*, E)$ (as in the case of spaces, $F$ is used to denote the cotensor functor $\mathscr{T}^{op} \times \mathscr{S} \longrightarrow \mathscr{S}$).

**Proposition 4.1.** *If $h_*$ is a homotopy between simplicial maps $f_*, g_*\colon K_* \longrightarrow L_*$ in $s\mathscr{T}$, then $|h_*|$ determines a homotopy between maps $|f_*|, |g_*|\colon |K_*| \longrightarrow |L_*|$ in $\mathscr{T}$. The corresponding statement for $\mathscr{S}$ also holds true.*

*Proof.* The notion of a simplicial homotopy here is that of [May67]. It is important to first note that $|-|$ preserves finite products since $\mathscr{T}$ (likewise $\mathscr{S}$) is cartesian closed and since $|-|$ preserves products of representables. The proof of this fact is purely formal, beginning from the co-Yoneda lemma. By $\Delta[1]$, let us denote the standard simplicial set $\mathbf{n} \mapsto \operatorname{Hom}_\Delta(\mathbf{n}, \mathbf{1})$, considered a simplicial space using the constant scheme construction of [SGA3V1] levelwise. In the case of $\mathscr{T}$, by [May67, Proposition 6.2] there exists $H_*\colon \Delta[1] \times K_* \longrightarrow L_*$ defined as follows. For $x \in K_*(U)$ (for $U \in \mathrm{Sm}_S$), we define $H_*((0), x) = g_*(x)$, $H_*((1), x) = f_*(x)$ (here (0) denotes any simplex of $\partial_1\Delta[1]$ and (1) any simplex of $\partial_0\Delta[1]$), and $H_q(s_{q-1}\cdots s_{i-1}s_{i+1}\cdots s_0\mathbf{1}, x) = \partial_{i+1}h_i(x)$ for each $0 \leq i \leq q-1$. $H_*$ constructed in this way is not only a map of simplicial sets but also one of simplicial spaces since $h_i$ and $\partial_{i+1}$ are morphisms in $\mathscr{T}$. Finally, it remains to note that $|\Delta[1]|$ is isomorphic to $\mathbb{A}^1$, so that the composite $\mathbb{A}^1 \times |K_*| \xrightarrow{\sim} |\Delta[1]| \times |K_*| \xrightarrow{\sim} |\Delta[1] \times K_*| \xrightarrow{|H_*|} |L_*|$ provides the required homotopy. The story for spectra is similar.[9] ∎

Since $\Sigma^\infty$ is a left adjoint functor, it commutes with the colimits that build $|-|\colon s\mathscr{T} \longrightarrow \mathscr{T}$. Thanks to this observation, we obtain:

**Proposition 4.2.** *There exists a natural isomorphism $\Sigma^\infty|K_*| \cong |\Sigma^\infty K_*|$ for each $K_* \in s\mathscr{T}$ (where the suspension spectrum is taken levelwise on the right-hand side).*

Recall that a simplicial object $X_* \in s\mathscr{V}$ for a category $\mathscr{V}$ is called *Reedy cofibrant* if the latching maps $LX_q = \operatorname{colim}_{\phi\colon \mathbf{q+1} \twoheadrightarrow \mathbf{s}, \phi \neq \mathrm{id}} X_s \hookrightarrow X_{q+1}$ are cofibrations for each $q$. $LX_q$ here is the scheme-theoretic analog to the union of all degenerate $q$-simplices in the simplicial set context. As in [May09] we implicitly assume that our spaces are nondegenerately based (i.e. that for each based space $X$, the basepoint

[9] The proof here is almost identical to that of [May72, Corollary 11.10], and might be formalized to a more general context–for example to a cartesian closed concrete category with a notion of homotopy

inclusion $S \longrightarrow X$ is a cofibration). So in this case every simplicial space is Reedy cofibrant. The following is derived from a general model-theoretic fact first due to [Ree73]. The presentation here is inspired from [May74].

**Proposition 4.3.** $|-|$ *preserves weak equivalences between Reedy cofibrant objects in $s\mathscr{T}$ (resp. in $s\mathscr{S}$).*

We define weak equivalences in $s\mathscr{T}$ and $s\mathscr{S}$ as those maps that are level-wise weak equivalences (in the appropriate model category)[10]. The proof of the above theorem relies on the gluing theorem of Brown in [Bro06].

**Theorem 4.4.** *Suppose given a commutative diagram:*

$$\begin{tikzcd}
& X \arrow[rr, "g"] \arrow[dd, "j"] & & Z \arrow[dd, "\bar{j}"] \\
A \arrow[ur, "\alpha"] \arrow[rr, "f"] \arrow[dd, "i"] & & C \arrow[ur, "\beta"] \arrow[dd, "\bar{i}"] & \\
& Y \arrow[rr, "\bar{g}"] & & W \\
B \arrow[ur, "\gamma"] \arrow[rr, "\bar{f}"] & & D \arrow[ur, "\delta"] &
\end{tikzcd}$$

*that $i$ and $j$ are cofibrations, and that the front and back squares are pushouts. If $\alpha$, $\beta$, and $\gamma$ are homotopy equivalences, then so is $\delta$.*

*Proof of Proposition 4.3.* As before, we provide the proof for spaces first–the proof for spectra is nearly identical and is hence omitted. Let $f_*\colon X_* \longrightarrow X'_*$ be a map of Reedy cofibrant spaces. We have the following pushout square for each $q$:

$$\begin{tikzcd}
(LX_q \times \Delta_{q+1}) \sqcup_{LX_q \times \partial\Delta_{q+1}} (X_{q+1} \times \Delta_{q+1}) \arrow[r, "g"] \arrow[d] & F_q|X| \arrow[d] \\
X_{q+1} \times \partial\Delta_{q+1} \arrow[r] & F_{q+1}|X|.
\end{tikzcd}$$

Here the standard map $g(s_i x, u) = [x, \sigma_i u]$ and $g(x, \delta_i v) = [\partial_i x, v]$ is used (the prescription of the map here is interpreted to be given at an arbitrary element of the site $U \in \mathrm{Sm}_S$). $F_q|X|$ denotes the $q$th piece of the natural filtration on $|X|$, given by capping off the tensor product defining $|X|$ at the $q$-simplices. Alternatively, one might take the above to be the definition of $F_q|X|$ inductively (with $F_0|X| = X_0$) so that $|X| = \varinjlim F_q|X|$. There is also a similar square for $X'_*$ of course. Proceeding inductively and applying Theorem 4.4, we see it suffices to show that $f_q\colon LX_q \longrightarrow LX'_q$ is a homotopy equivalence for each $q$. For each $0 \leq k \leq q$ let $L^k X_q = \operatorname{colim}_{\phi\colon \mathbf{q+1} \twoheadrightarrow \mathbf{s}, s \leq k} X_s$. We obtain the following commutative diagram

[10]In a sense, this result records the extent to which $|-|$ induces an equivalence between the homotopy categories of $s\mathscr{T}$ and $\mathscr{T}$ (resp. $s\mathscr{S}$ and $\mathscr{S}$)

with a pushout square on the right side:

$$\begin{tikzcd}
L^{k-1}X_{q-1} \arrow[r, "L_k", "\sim"'] \arrow[d] & L^{k-1}X_q \times_{X_{q+1}} L_kX_q \arrow[r] \arrow[d] & L^{k-1}X_q \arrow[d] \\
X_q \arrow[r, "L_k", "\sim"'] & L_kX_q \arrow[r] & L^kX_q
\end{tikzcd}$$

Here $L_kX_q$ denotes the image of $X_k \hookrightarrow X_{q+1}$, the map seen in the colimit definition of $LX_q$. Proceeding by induction on $q$, assuming that each $L^{k-1}X_{q-1} \longrightarrow L^kX_{q-1}$ is a cofibration for each $0 < k < q$, we conclude that $L^{k-1}X_q \longrightarrow L^kX_q$ is a cofibration. The same is true of $X'$. Since we know $L_0\colon X_q \longrightarrow L_0X_q$ is an isomorphism, $f_{q+1}\colon L^0X_q \longrightarrow L^0X'_q$ must be a homotopy equivalence. By induction on $q$, and induction on $k$ having fixed $q$, and Theorem 4.4, we know that $f_{q+1}\colon L^kX_q \longrightarrow L^kX'_q$ is a homotopy equivalence for all $k$ and $q$. ■

The following is clear from Proposition 4.2, our definition of connective spectra, and the fact that $|-|$ is a left adjoint. Alternatively, if one defines homotopy sheaves of spectra as being certain direct limits of homotopy sheaves of spaces, then since $|-|$ is a left adjoint, it commutes with those direct limits. The proposition is clear once again if one takes connective spectra to be spectra with certain vanishing homotopy sheaves.

**Proposition 4.5.** *$|-|\colon s\mathscr{S} \longrightarrow \mathscr{S}$ takes levelwise connective spectra to connective spectra.*

We conclude with the most significant result of the section.

**Proposition 4.6.** *The map $\gamma\colon |\Omega^\infty E_*| \longrightarrow \Omega^\infty|E_*|$ given by the adjoint to the composite $\Sigma^\infty|\Omega^\infty E_*| \cong |\Sigma^\infty\Omega^\infty E_*| \xrightarrow{|\varepsilon|} |E_*|$ is, for levelwise connective $E_*$, a weak equivalence in $\mathscr{T}$.*

An important distinction from the theory of operadic monads must be made here. Generally speaking, a monad $\mathbb{C}$ coming from an operad commutes with $|-|$ up to isomorphism by the categorical Fubini theorem. For the adjunction monad however, as we will show now, a priori $\Omega^\infty$ commutes with $|-|$ only up to weak equivalence—and therefore $\Gamma$ commutes with $|-|$ only up to weak equivalence.

We prove Proposition 4.6 by reducing to $n$-fold and singlefold loop spaces first, and then passing to the colimit. For $X \in \mathscr{T}$ denote by $\Omega X$ the hom-space $\underline{\mathrm{Hom}}_{\mathscr{T}}(S^{\mathbb{A}^1}, X)$ (where $S^{\mathbb{A}^1} = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$). Denote by $\Omega^nX$ the hom-space $\underline{\mathrm{Hom}}_{\mathscr{T}}(S^{\mathbb{A}^n}, X)$ (where $S^{\mathbb{A}^n} = \mathbb{A}^n/(\mathbb{A}^n - \{0\})$. $\Omega^n$ (as seen by the loop-suspension adjunction) is clearly the $n$-fold composition of $\Omega$ with itself. There is a map $|\Omega^nX_*| \longrightarrow \Omega^n|X_*|$ again given by the adjoint to $\Sigma^n|\Omega^nX_*| \cong |\Sigma^n\Omega^nX_*| \xrightarrow{|\varepsilon|} |X_*|$ (here $\Sigma^n(-) = - \wedge S^{\mathbb{A}^n}$) which we also call $\gamma$.

**Lemma 4.7.** *If $X_* \in s\mathscr{T}$ is Reedy cofibrant and each $X_q$ is connected, then $\gamma\colon |\Omega X_*| \longrightarrow \Omega|X_*|$ is a weak equivalence in $\mathscr{T}$.*

We first note that Lemma 4.7 allows us to conclude Proposition 4.6. Of course, iteratively applying the above lemma tells us that $\gamma^n\colon |\Omega^nX_*| \longrightarrow \Omega^n|X_*|$ is a weak equivalence for Reedy cofibrant $X_*$ with connected $X_q$. The idea now is that (under the assumptions on $E_*$) the map $\gamma\colon |\Omega^\infty E_*| \longrightarrow \Omega^\infty|E_*|$ can be identified with the colimit of the maps $\gamma^n \colon |\Omega^n(E_n)_*| \longrightarrow \Omega^n|(E_n)_*|$, so that Proposition 4.6 follows by passing to colimits. We must clarify the notation used. We have written $E_n$ to mean $E_{\mathcal{U}/\mathbb{A}^n}$—the *sequential* indexing here is chosen purely for simplicity. Note that each $(E_n)_*$ is a simplicial space, and $\Omega^n(E_n)_*$ is compatibly isomorphic to $(E_0)_*$, so that the colimit of $|\Omega^n(E_n)_*|$ is identified with $|\Omega^\infty E_*|$. By a remark made after Proposition 2.3, for $E_* = LT_*$, we can compute the colimit of $\Omega^n|(E_n)_*|$ as $\Omega^\infty|LT_*| \cong \Omega^\infty L|T_*|$, where the realization of a simplicial prespectrum is defined as the levelwise realization–so that the needed observation follows by working at the prespectrum level.

There is, however, a caveat. The argument we use moving forward only works on fibrant objects—this requires us to pass to fibrant replacements. We introduce fibrant replacements in $s\mathscr{T}$ after proving Lemma 4.9 thus proving Lemma 4.7. The following is a precise reformulation of the claim that emphasizes what is needed.

**Lemma 4.8.** *Let $X_* \in s\mathscr{T}$, then $|PX_*|$ is contractible, and there are natural morphisms $\gamma$ and $\delta$ making the following diagram commute:*

$$\begin{tikzcd}
|\Omega X_*| \arrow[r, "\subset"] \arrow[d, "\gamma"'] & |PX_*| \arrow[r, "|p_*|"] \arrow[d, "\delta"'] & |X_*| \arrow[d, equal] \\
\Omega|X_*| \arrow[r, "\subset"] & P|X_*| \arrow[r, "p"] & |X_*|.
\end{tikzcd}$$

*Moreover, when $X_*$ is fibrant, Reedy cofibrant, and level-wise connected, the top row of the above diagra $|\Omega X_*| \longrightarrow |PX_*| \xrightarrow{|p_*|} |X_*|$ is a homotopy fiber sequence, and therefore $\gamma \colon |\Omega X_*| \longrightarrow \Omega|X_*|$ is a weak equivalence.*

We have used the standard notation $PY$ to denote the path-space of $Y \in \mathscr{T}$, defined by $\underline{\mathrm{Hom}}_{\mathscr{T}}(\mathbb{A}^1_+, Y)$ (the simplicial version $PY_*$ for $Y_* \in s\mathscr{T}$ is defined by taking the levelwise path-space). We outline the proof of the first part here. Firstly, we note that there is a natural contracting homotopy equivalence $h \colon PY \longrightarrow *$ given by $\varphi \mapsto \varphi \circ \iota_0$ (here $\iota_0 \colon S \longrightarrow \mathbb{A}^1_S$ denotes the inclusion of $S$ as the basepoint 0 of $\mathbb{A}^1_+$ and we identify the composite $S \longrightarrow Y$ with the basepoint of $Y$). This homotopy, applied to each $PX_q$, descends to a simplicial contracting homotopy $\Delta[1] \times PX_* \longrightarrow PX_*$. By Proposition 4.1, we conclude that $|PX_*|$ is indeed contractible. We are left with defining $\delta \colon |PX_*| \longrightarrow P|X_*|$. Everything that follows in this paragraph fixes an arbitrary $U \in \mathrm{Sm}_S$, and work over this $U$ implicitly. For $f \in PX_q$, $u \in \Delta_q$, and $t \in \mathbb{A}^1$, define $\delta[f,u](t) = [f(t), u]$. $\delta$ is a well-defined morphism in $\mathscr{T}$, and it indeed makes the diagram in question commute. In some sense, this definition of $\delta$ was the only sensible choice we had. There is categorical reason for this. The functors $- \wedge \mathbb{A}^1_+$ and $P(-)$ are adjoint to each other. The map $\delta \colon |PX_*| \longrightarrow P|X_*|$ is the one obtained from $|PX_*| \wedge \mathbb{A}^1_+ \cong |(PX \wedge \mathbb{A}^1)_*| \xrightarrow{|\epsilon_*|} |X_*|$ where $\epsilon \colon PX \wedge \mathbb{A}^1 \longrightarrow X$ is the counit map.

What remains is the following, an analog of J.P. May's result in [May72], and a specialization of Anderson's result in [And78], for the geometric realization of fibrations.

**Lemma 4.9.** *If $p_*\colon E_* \longrightarrow B_*$ is a fibration in $s\mathscr{T}$ with fiber $F_*$ and $B_*$ is Reedy cofibrant and level-wise connected then $|F_*| \longrightarrow |E_*| \xrightarrow{|p_*|} |B_*|$ is a homotopy fiber sequence. Moreover, when $X_*$ is fibrant, $p_*\colon PX_* \longrightarrow X_*$ is a fibration in $s\mathscr{T}$ with fiber $\Omega X_*$, leading us to conclude that $|\Omega X_*| \longrightarrow |PX_*| \xrightarrow{|p_*|} |X_*|$ is a homotopy fiber sequence (when $X_*$ is Reedy cofibrant and levelwise connected).*

We must first define what we mean by a fibration $p_*\colon E_* \longrightarrow B_*$ in $s\mathscr{T}$. This definition follows that of Reedy for the model structure on $s\mathcal{C}$ given a model category $\mathcal{C}$, using coskeletons. Namely a map $p_*\colon E_* \longrightarrow B_*$ is a fibration in $s\mathscr{T}$ if for each $m$, the induced map $E_{m+1} \longrightarrow \mathrm{cosk}_m(E)_{m+1} \times_{\mathrm{cosk}_m(B)_{m+1}} B_{m+1}$ is a fibration in $\mathscr{T}$. Together with the dually defined cofibrations in $s\mathscr{T}$, we obtain a model structure on $s\mathscr{T}$ that is in fact simplicially enriched. There is a pairing $-\otimes- \colon \mathbf{sSet} \times s\mathscr{T} \longrightarrow s\mathscr{T}$ given by $(A_* \otimes X_*)_k = X_k[A_k] := \coprod_{a\in A_k} X_k$. Appealing to the dual provides a pairing $\mathrm{Hom}(-,-)\colon \mathbf{sSet}^{\mathrm{op}} \times s\mathscr{T} \longrightarrow s\mathscr{T}$. Finally, there is the standard **sSet**-enrichment on $s\mathscr{T}$, $\mathrm{Hom}(-,-)\colon s\mathscr{T}^{\mathrm{op}} \times s\mathscr{T} \longrightarrow \mathbf{sSet}$, given by $\mathrm{Hom}(X_*, Y_*) = \mathrm{Hom}_{s\mathscr{T}}(\Delta[-] \otimes X_*, Y_*) \in \mathbf{sSet}$. Moreover, these pairings satisfy the pullback-power axiom[11]:

- If $i_*\colon A_* \longrightarrow B_*$ is a cofibration in **sSet** and $p_*\colon X_* \longrightarrow Y_*$ is a fibration in $s\mathscr{T}$, then the following is a fibration in **sSet**.

  $$\mathrm{Hom}(B_*, X_*) \longrightarrow \mathrm{Hom}(B_*, Y_*) \times_{\mathrm{Hom}(A_*, Y_*)} \mathrm{Hom}(A_*, X_*)$$

  Moreover, the above map is an acyclic fibration if $p_*$ is also a weak equivalence.

With this new enriched model structure in our toolbox, we now handle the simplicial path space fibration on fibrant objects. That $p_*\colon PX_* \longrightarrow X_*$ is a fibration in $s\mathscr{T}$ when $X_*$ is fibrant, is an instance of a well-known general fact (see [GJ99] for example). The idea is to begin with the cofiber sequence of pointed simplicial sets[12] $\partial\Delta[1] \hookrightarrow \Delta[1] \longrightarrow \Delta[1]/\partial\Delta[1]$. The pullback-power axiom for $s\mathscr{T}$ discussed above then ensures that when we power this sequence by a fibrant $X_*$, we obtain a fiber sequence $\mathrm{Hom}(\Delta[1]/\partial\Delta[1], X_*) \longrightarrow \mathrm{Hom}(\Delta[1], X_*) \longrightarrow \mathrm{Hom}(\partial\Delta[1], X_*)$. Noting that $\Omega X_* \simeq \mathrm{Hom}(\Delta[1]/\partial\Delta[1], X_*)$, $PX_* \simeq \mathrm{Hom}(\Delta[1], X_*)$, and $\mathrm{Hom}(\partial\Delta[1], X_*) \simeq X_*$, this yields a fiber sequence $\Omega X_* \longrightarrow PX_* \longrightarrow X_*$.

It remains to show that if $p_*\colon E_* \longrightarrow B_*$ is a fibration in $s\mathscr{T}$ with fiber $F_*$ and $B_*$ is Reedy cofibrant and level-wise connected, then $|F_*| \longrightarrow |E_*| \longrightarrow |B_*|$ is a homotopy fiber sequence. This is the real meat of Lemma 4.9. In [May72], the author employs the Dold-Thom criteria for quasifibrations to achieve the required analogous result. Anderson, in [And78], provides a conceptually clearer treatment of the result by passing to bisimplicial sets and solving a lifting problem with respect to the image of the left adjoint of the realization functor. The initial idea here was to make use of Anderson's argument, or the $\pi_*$-Kan condition for bisimplicial sets (due to Bousfield and Friedlander [BF78])—both of which, in a sense, rely on the construction of the simplicial groupoid $\Pi_\infty(-)$[13]. However, the conceptual reformulation, and the strengthened result pertaining to geometric realizations of

[11]They also satisfy an equivalent dual *pushout-copower* axiom.

[12]We consider $\Delta[1]$ to be pointed at 1 here.

[13]For completeness, the groupoid $\Pi_\infty(Z)$ is the wreath product of the fundamental groupoid with the product of the higher homotopy groups of $Z$.

homotopy pullbacks, in [Rez14] is more amenable to our purposes—and moreover is conceptually easier to approach. A specialized application of Proposition 5.5.6.17 in [Lur17] to the $\infty$-topos $L_W\mathscr{T}$ also yields the required result, but we expect that the generality of Lurie's statement will be more relevant in future studies where Rezk's argument is not readily accessible. We state the required result now:

**Theorem 4.10.** *Let $B_* \in s\mathscr{T}$ be a simplicial space such that the simplicial sheaf $\pi_0^{\mathbb{A}^1}(B_*)$, defined levelwise as $\pi_0^{\mathbb{A}^1}(B_*)_n = \pi_0^{\mathbb{A}^1}(B_n)$, is discrete. Let $f_*\colon Y_* \longrightarrow B_*$ be a map in $s\mathscr{T}$. Then for every homotopy pullback square,*

$$\begin{tikzcd} F_* \arrow[r] \arrow[d] & Y_* \arrow[d, "f_*"] \\ E_* \arrow[r] & B_* \end{tikzcd}$$

*the following square,*

$$\begin{tikzcd} {|F_*|} \arrow[r] \arrow[d] & {|Y_*|} \arrow[d, "|f_*|"] \\ {|E_*|} \arrow[r] & {|B_*|} \end{tikzcd}$$

*obtained by applying $|-|$ to each corner, is still a homotopy pullback.*

The notation used above is of course suggestive of the fact that in our applications, we will take $Y_*$ to be the terminal simplicial space—so that the level-wise connectedness of $B_*$ will give us Lemma 4.9. But one must take brief respite to make a conceptual remark about Theorem 4.10, to put us directly in the framework of [Rez14]. Implicitly restricting down to Reedy cofibrant objects, a routine check using the Bousfield-Kan map tells us that there is a weak equivalence $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} X_* \longrightarrow |X_*|$ for each $X_* \in s\mathscr{T}$. Under this lens, Theorem 4.10 is manifestly a question of the preservation of homotopy pullbacks under certain kinds of homotopy colimits. For the rest of this section, we will use $|X_*|$ and $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} X_*$ interchangeably (and by one, we may as well mean the other). By a *realization-fibration* (sometimes denoted RF), we will mean a map $f_*\colon Y_* \longrightarrow B_*$ such that the output of Theorem 4.10 holds true (i.e. a realization-fibration is a map such that homotopy colimits commute with homotopy pullbacks of the map). Our task then is to ascertain conditions under which a map of simplicial spaces is a realization-fibration.

The perspective due to Rezk allows us to deduce Theorem 4.10 from a local-to-global principle for realization-fibrations, applied to the homotopy colimit of representable sifted presheaves. Our starting point for the local-to-global point of view is the principle of descent, as presented in Section 6.5 and Proposition 6.6 in [Rez10] for model toposes. We say that a natural transformation $f\colon U \longrightarrow V$ of functors $U, V\colon \mathscr{J} \longrightarrow \mathscr{T}$ is equifibered if for every $J \longrightarrow J'$ in $\mathscr{J}$, the associated naturality square is a homotopy pullback square.

**Lemma 4.11** (Descent)**.** *Let $\mathscr{J}$ be a (small) category.*

*(1) Let $p\colon E \longrightarrow B$ be a fibration in $\mathscr{T}$ and $V\colon \mathscr{J} \longrightarrow \mathscr{T}$ be a diagram of spaces such that $h\colon \mathrm{hocolim}_{\mathscr{J}} V \longrightarrow B$ is a weak equivalence in $\mathscr{T}$. Let $U\colon \mathscr{J} \longrightarrow \mathscr{T}$ be the diagram of spaces defined by $U(J) = V(J) \times_B E$. The map $\mathrm{hocolim}_{\mathscr{J}} U \longrightarrow E$ is a weak equivalence.*

(2) *Let $U$ and $V$ be diagrams of spaces $\mathscr{J} \longrightarrow \mathscr{T}$, and $f\colon U \longrightarrow V$ an equifibered map in $[\mathscr{J}, \mathscr{T}]$. For each $J \in \mathscr{J}$, the square:*

$$\begin{tikzcd} U(J) \arrow[r] \arrow[d] & \operatorname{hocolim}_{\mathscr{J}} U \arrow[d] \\ V(J) \arrow[r] & \operatorname{hocolim}_{\mathscr{J}} V \end{tikzcd}$$

*is a homotopy pullback.*

(3) *Let $f\colon X \longrightarrow Y$ be an equifibered map, and*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

*a homotopy pullback square in $[\mathscr{J}, \mathscr{T}]$. Applying* $\operatorname{hocolim}_{\mathscr{J}}$ *at each corner of the above square still yields a homotopy pullback.*

The central local-to-global statement we seek states that under certain hypotheses (i.e. equifiberedness) realization-fibrations can be glued together to get a realization-fibration.

**Lemma 4.12.** *Let $\mathscr{J}$ be a small category, $f\colon W \longrightarrow V$ an equifibered map in $[\mathscr{J}, s\mathscr{T}]$ that is an object-wise realization-fibration. Then the map obtained by passing to the homotopy colimit* $\operatorname{hocolim}_{\mathscr{J}} h\colon \operatorname{hocolim}_{\mathscr{J}} W \longrightarrow \operatorname{hocolim}_{\mathscr{J}} V$ *is also a realization-fibration.*

*Proof of Lemma 4.12.* The argument in this proof is due to [Rez14]. Define $B := \operatorname{hocolim}_{\mathscr{J}} V$ and pick a factorization of $\operatorname{hocolim}_{\mathscr{J}} h\colon \operatorname{hocolim}_{\mathscr{J}} W \longrightarrow B$ as the composite $\operatorname{hocolim}_{\mathscr{J}} W \xrightarrow{\iota} E \xrightarrow{p} B$, where $\iota$ is a weak equivalence and $p$ is a fibration. Note that to show $\operatorname{hocolim}_{\mathscr{J}} h$ is an RF, it suffices to show $p$ is an RF. Toward that end, we make some initial remarks. For each $J \in \mathscr{J}$, we have a map $V(J) \longrightarrow B$. Denote by $U(J)$ the fiber product $E \times_B V(J)$.

$$\begin{tikzcd} U(J) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ V(J) \arrow[r] & B \end{tikzcd}$$

$h\colon W \longrightarrow V$ is equifibered, so descent tells us that there is a homotopy pullback square.

$$\begin{tikzcd} W(J) \arrow[r] \arrow[d] & \operatorname{hocolim}_{\mathscr{J}} W \arrow[d, "\operatorname{hocolim}_{\mathscr{J}} h"] \\ V(J) \arrow[r] & B \end{tikzcd}$$

Since $\iota\colon \operatorname{hocolim}_{\mathscr{J}} W \longrightarrow E$ is a weak equivalence, we see that there is a weak equivalence $W(J) \longrightarrow U(J)$ for each $J \in \mathscr{J}$. In particular, the maps $U(J) \longrightarrow V(J)$ are RF too. Descent also provides a chain of weak equivalences $\operatorname{hocolim}_{\mathscr{J}} W \longrightarrow \operatorname{hocolim}_{\mathscr{J}} U \longrightarrow E$. We are ready to show that $p$ is an RF now.

Let $\phi\colon B' \longrightarrow B$ be a map in $s\mathscr{T}$. Define $E'$, $U'$, and $V'$ by the following homotopy pullbacks.

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ B' \arrow[r, "\phi"'] & B \end{tikzcd} \qquad \begin{tikzcd} V'(J) \arrow[r] \arrow[d] & V(J) \arrow[d] \\ B' \arrow[r, "\phi"'] & B \end{tikzcd} \qquad \begin{tikzcd} U'(J) \arrow[r] \arrow[d] & U(J) \arrow[d] \\ B' \arrow[r, "\phi"'] & B \end{tikzcd}$$

By observing the middle and the right homotopy pullbacks, and using the homotopy pullback lemma, we get the following homotopy pullback square.

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

We already know that there is a weak equivalence $\operatorname{hocolim}_{\mathscr{J}} U \longrightarrow E$, and an equality $B = \operatorname{hocolim}_{\mathscr{J}} V$. But by descent, there are also weak equivalences $\operatorname{hocolim}_{\mathscr{J}} V' \longrightarrow B'$ and $\operatorname{hocolim}_{\mathscr{J}} U' \longrightarrow E'$. This tells us that applying $\operatorname{hocolim}_{\mathscr{J}}$ to each corner of the above square, we get a homotopy pullback

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ B' \arrow[r, "\phi"'] & B \end{tikzcd}$$

It suffices to know that applying $|-|$ at each corner, we still get a homotopy pullback square. But notice that $|\operatorname{hocolim}_{\mathscr{J}}(-)| \simeq \operatorname{hocolim}_{\mathscr{J}} |-|$ since colimits commute with colimits. So we might as well have realized down in our earlier homotopy pullback, and then applied $\operatorname{hocolim}_{\mathscr{J}}$. Note that since each $U(J) \longrightarrow V(J)$ is an RF, upon realization we get a homotopy pullback.

$$\begin{tikzcd} {|U'|} \arrow[r] \arrow[d] & {|U|} \arrow[d] \\ {|V'|} \arrow[r] & {|V|} \end{tikzcd}$$

Now we notice that $|U| \longrightarrow |V|$ is equifibered since each $U(J) \longrightarrow V(J)$ is an RF. Therefore by descent, applying $\operatorname{hocolim}_{\mathscr{J}}$ to each corner will yield yet another homotopy pullback square, which is what we needed. ∎

We now make use of the siftedness of $\Delta^{\mathrm{op}}$. When we say sifted, we implicitly mean the kind of homotopy siftedness (after a routine check in homotopy coend calculus using the Bousfield-Kan formula) that results in a canonical weak equivalence $|X \times Y| \longrightarrow |X| \times |Y|$ for any $X, Y$ in $s\mathscr{T}$. In simple terms, siftedness tells us that realization commutes with finite products in $s\mathscr{T}$. The primary utility of working over a sifted category is demonstrated in the following lemma.

**Lemma 4.13.** *All maps of the form $p\colon B \times C \longrightarrow B$ in $s\mathscr{T}$ are realization-fibrations.*

*Proof of Lemma 4.13.* Let $\phi\colon B' \longrightarrow B$ be a map in $s\mathscr{T}$. Any homotopy pullback of $p$ along $\phi$ must look like the following diagram.

$$\begin{tikzcd} B' \times C \arrow[r, "\phi\times \mathrm{id}_C"] \arrow[d, "\pi_{B'}"'] & B \times C \arrow[d, "p"] \\ B' \arrow[r, "\phi"'] & B \end{tikzcd}$$

where $\pi_{B'}\colon B' \times C \longrightarrow B'$ is the projection map onto $B'$. Applying $|-|$ on each corner, and using the canonical weak equivalences $|B \times C| \longrightarrow |B| \times |C|$ and $|B' \times C| \longrightarrow |B'| \times |C|$, we get a diagram.

$$\begin{tikzcd} |B'| \times |C| \arrow[r, "|\phi|\times \mathrm{id}_{|C|}"] \arrow[d, "\pi_{|B'|}"'] & |B| \times |C| \arrow[d, "|p|"] \\ |B'| \arrow[r, "|\phi|"'] & |B| \end{tikzcd}$$

where $\pi_{|B'|}\colon |B'| \times |C| \longrightarrow |B'|$ is the projection map onto $|B'|$. This is evidently another homotopy pullback square, and we are done. ■

Now we introduce the notions of weak and local projection maps. We will soon see that every local projection map is in fact a realization-fibration, and that the hypotheses in Theorem 4.10 ensure that $f_*\colon Y_* \longrightarrow B_*$ will be a local projection. A map $p\colon E \longrightarrow B$ in $s\mathscr{T}$ is called a *weak projection* if it is weakly equivalent over $B$ to a projection map $\tilde{p}\colon B \times C \longrightarrow B$ in $s\mathscr{T}$. We will say that a map $p\colon E \longrightarrow B$ in $s\mathscr{T}$ is a *local projection* if for every $n$, and every homotopy pullback square of the form:

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "b"'] & B \end{tikzcd}$$

where $\Delta[n]$ denotes the standard simplicial set $\mathbf{m} \mapsto \mathrm{Hom}_\Delta(\mathbf{m}, \mathbf{n})$ (thought of as a levelwise discrete object in $s\mathscr{T}$), the map $q$ is a weak projection map. Given $X \in s\mathscr{T}$, there is a simplicial presheaf $\pi_0^{\mathbb{A}^1}(X)$ given by $(\pi_0^{\mathbb{A}^1}(X))_n = \pi_0^{\mathbb{A}^1}(X_n)$. We can think of $\pi_0^{\mathbb{A}^1}(X)$ as a levelwise discrete object in $s\mathscr{T}$. Given a map $p\colon E \longrightarrow B$, let $\mathrm{lproj}(p)_n \subseteq (\pi_0^{\mathbb{A}^1}(B))_n$ denote the presheaf prescribed locally as the collection of sections of $(\pi_0^{\mathbb{A}^1}(B))_n$ represented by maps $b\colon \Delta[n] \longrightarrow B$ such that the pullback of $p$ along $b$ is a weak projection. This prescription defines a simplicial subpresheaf $\mathrm{lproj}(p) \subseteq \pi_0^{\mathbb{A}^1}(B)$.

Suppose $p\colon E \longrightarrow B$ and $\phi\colon B' \longrightarrow B$ are maps in $s\mathscr{T}$. Note that by the pullback lemma, the pullback of $p$ along $\phi$ is a local projection iff $\phi(\pi_0^{\mathbb{A}^1} B') \subseteq \mathrm{lproj}(p)$. Taking $B' = B$ and $\phi = \mathrm{id}_B$, this means in particular that $p$ is a local projection iff $\mathrm{lproj}(p) = \pi_0^{\mathbb{A}^1}(B)$. In applications, we typically demonstrate that $p$ is a local projection by showing that $\pi_0^{\mathbb{A}^1}(B) \subseteq \mathrm{lproj}(p)$.

**Lemma 4.14.** *All local projections in $s\mathscr{T}$ are realization-fibrations.*

*Proof of Lemma 4.14.* Let $p\colon E \longrightarrow B$ be a local projection in $s\mathscr{T}$. Note that we can write $B$ up to weak equivalence as the colimit of representable objects in $s\mathscr{T}$ (by which we mean $\Delta[n]$ for $n \geq 0$, considered as simplicial spaces). For each map $\Delta[n] \longrightarrow B$ forming the appropriate colimit, since $p$ is a local projection, we can form the homotopy pullback square below.

$$\begin{tikzcd} \Delta[n] \times C \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] & B \end{tikzcd}$$

Descent tells us that $E$ then is the homotopy colimit of objects of the form $\Delta[n] \times C$. In sum, $p$ can be written as the homotopy colimit of maps that each look like $\Delta[n] \times C \longrightarrow \Delta[n]$. By Lemma 4.13, we know that each of these maps is a realization-fibration. Lemma 4.12 then tells us that $p$ is a realization-fibration too. ■

*Proof of Theorem 4.10.* We return to the notation set up in the statement of Theorem 4.10. Namely, let $f_*\colon Y_* \longrightarrow B_*$ be a map in $s\mathscr{T}$. Note first that since $\Delta[0]$, as a simplicial space, is terminal, every map into $\Delta[0]$ in $s\mathscr{T}$ is a weak projection. As a result, any 0-simplex of $\pi_0^{\mathbb{A}^1}(B_*)$, thought of as being represented by a map out of $\Delta[0]$, is contained in $\mathrm{lproj}(f_*)$. Since $\pi_0^{\mathbb{A}^1}(B_*)$ is 2 by assumption, meaning $\pi_0^{\mathbb{A}^1}(B_*) \simeq \mathrm{const}(\pi_0^{\mathbb{A}^1}(B_0))$, we see in fact that $\pi_0^{\mathbb{A}^1}(B_*) \subseteq \mathrm{lproj}(f_*)$. We conclude that $f_*$ is a local projection, and therefore a realization-fibration. ■

We complete the proof of Lemma 4.7 by introducing, and passing to, fibrant replacements. Recall that $\mathscr{T}$ admits functorial factorization. It follows from a result on Reedy model structures that this property lifts to simplicial spaces as well, so that $s\mathscr{T}$ admits functorial factorization too (see [Hov99, Theorem 5.2.5] for the general statement due to transfinite induction). The existence of a fibrant replacement functor $\mathrm{Ex}\colon s\mathscr{T} \longrightarrow s\mathscr{T}$ with the following properties is now immediate.

(1) For every $X_* \in s\mathscr{T}$, $\mathrm{Ex}(X_*)$ is fibrant in $s\mathscr{T}$.
(2) For every $X_* \in s\mathscr{T}$, there is an acyclic cofibration $\mu\colon X_* \longrightarrow \mathrm{Ex}(X_*)$.
(3) Ex preserves (and in fact reflects) weak equivalences[14].

Let $X_*$ be an arbitrary simplicial space that is Reedy cofibrant and level-wise connected. Consider the following naturality square.

$$\begin{tikzcd} {|\Omega X_*|} \arrow[r, "\gamma"] \arrow[d, "|\Omega\mu|"'] & \Omega|X_*| \arrow[d, "\Omega|\mu|"] \\ {|\Omega \operatorname{Ex} X_*|} \arrow[r, "\gamma"'] & \Omega|\operatorname{Ex} X_*| \end{tikzcd}$$

The bottom arrow of this square (by Lemma 4.8) is a weak equivalence since $\operatorname{Ex} X_*$ is fibrant. The left and right arrows of the square are weak equivalences since $\Omega$ preserves weak equivalences and $|-|$ preserves weak equivalences between Reedy cofibrant objects. By the two-out-of-three property for weak equivalences, it follows that the top arrow $\gamma\colon |\Omega X_*| \longrightarrow \Omega|X_*|$ is a weak equivalence.

[14] This follows from applications of the two-out-of-three property for weak equivalences.

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