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The Geometry of Iterated Loop Spaces

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Preface

This is the first of a series of papers devoted to the study of iterated loop spaces. Our goal is to develop a simple and coherent theory which encompasses most of the known results about such spaces. We begin with some history and a description of the desiderata of such a theory.

First of all, we require a recognition principle for n-fold loop spaces. That is, we wish to specify appropriate internal structure such that a space X possesses such structure if and only if X is of the (weak) homotopy type of an n-fold loop space. For the case $n = 1$, Stasheff's notion [28] of an A_∞ space is such a recognition principle. Beck [5] has given an elegant proof of a recognition principle, but, in practice, his recognition principle appears to be unverifiable for a space that is not given a priori as an n-fold loop space. In the case $n = \infty$, a very convenient recognition principle is given by Boardman and Vogt's notion [8] of a homotopy everything space, and, in [7], Boardman has stated a similar recognition principle for $n < \infty$.

We shall prove a recognition principle for $n < \infty$ in section 13 (it will first be stated in section 1) and for $n = \infty$ in section 14; the latter result agrees (up to language) with that of

Boardman and Vogt, but our proof is completely different. By generalizing the methods of Beck, we are able to obtain immediate non-iterative constructions of classifying spaces of all orders. Our proof also yields very precise consistency and naturality statements. In particular, a connected space X which satisfies our recognition principle (say for $n = \infty$) is not only weakly homotopy equivalent to an infinite loop space $B_0 X$, where spaces $B_i X$ with $B_i X = \Omega B_{i+1} X$ are explicitly constructed, but also the given internal structure on X agrees under this equivalence with the internal structure on $B_0 X$ derived from the existence of the spaces $B_i X$. We shall have various other consistency statements and our subsequent papers will show that these statements help to make the recognition principle not merely a statement as to the existence of certain cohomology theories, but, far more important, an extremely effective tool for the calculation of the homology of the representing spaces.

An alternative recognition principle in the case $n = \infty$ is due to Segal [27] and Anderson [1, 2]. Their approach starts with an appropriate topological category, rather than with internal structure on a space, and appears neither to generalize to the recognition of n-fold loop spaces, $1 < n < \infty$, nor to yield the construction of homology operations, which are essential to the most important presently known applications.

The second desideratum for a theory of iterated loop spaces is a useable geometric approximation to $\Omega^n S^n X$ and $\Omega^\infty S^\infty X = \varinjlim \Omega^n S^n X$.

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In the case n = 1, this was first obtained by James [15]. For $n<\infty$, Milgram [22] obtained an ingenious, but quite intricate, approximation for connected CW-complexes. In the case $n = \infty$, such an approximation was first obtained by Dyer and Lashof [unpublished] and later by Barratt [4], Quillen [unpublished], and Segal [27].

We shall obtain simple functorial approximations to $\Omega^n S^n X$ for all n and all connected X in section 6 (a first statement is in section 2). Our result shows that the homotopy type of $\Omega^n S^n X$ is built up from the iterated smash products $X^{[j]}$ of X with itself and the classical configuration spaces $F(R^n; j)$ of j-tuples of distinct points of R^n. Moreover, in our theory the approximation theorem, together with quite easy categorical constructions and some technical results concerning geometric realization of simplicial topological spaces, will directly imply the recognition principle. This is in fact not surprising since $\Omega^n S^n X$ and $\Omega^\infty S^\infty X$ are the free n-fold and infinite loop spaces generated by X and should play a central role in any complete theory of iterated loop spaces.

The third, and pragmatically most important, requirement of a satisfactory theory of iterated loop spaces is that it lead to a simple development of homology operations. The third paper in this series will study such operations on n-fold loop spaces, $n < \infty$, and will contain descriptions of $H_*(\Omega^n S^n X)$ for all n as functors of $H_*(X)$. The second paper in the series will study homology operations on E_∞ spaces and infinite loop spaces and will apply the present theory to the study of such spaces as F, F/O, BF, BTop, etc. It will be seen there that the precise geometry that allows the recognition principle to be applied to these spaces is not only well adapted to the construction of homology operations but can actually be used for their explicit evaluation. Statements of some of the results of these papers may be found in [20].

Our basic definitional framework is developed in sections 1, 2, and 3. The notion of "operad" defined in § 1 arose simultaneously in Max Kelly's categorical work on coherence, and conversations with him led to the present definition. Sections 4 through 8 are concerned with the geometry of iterated loop spaces and with the approximation theorem. The definition of the little cubes operads in §4 and of their actions on iterated loop spaces in §5 are due to Boardman and Vogt [8]. The results of §4 and §5 include all of the geometry required for the construction of homology operations and for the proofs of their properties (Cartan formula, stability, Adem relations, etc.). The observations of §8, which simplify and generalize results originally proven by Milgram [23], Tsuchiya [33], and myself within the geometrical framework developed by Dyer and Lashof [11], include all of the geometry required for the computation of the Pontryagin ring of the monoid F of based homotopy equivalences of spheres. Our key categorical construction is presented in §9, and familiar special cases of this construction are discussed in §10. This construction leads to simplicial spaces, and a variety of technical results on the geometric realization of simplicial spaces are proven in §11 and §12. The recognition theorems are proven in §13 and §14 and are discussed in §15. A conceptual understanding of these results can be obtained by reading §1-3 and §9 and then §13, referring back to the remaining sections for the geometry as needed.

The results of §10 and §11 will be used in [21] to simplify and generalize the theories of classifying spaces of monoids and of classification theorems for various types of fibrations.

It is a pleasure to acknowledge my debt to Saunders Mac Lane and Jim Stasheff, who read preliminary versions of this paper and made very many helpful suggestions. Conversations with Mike Boardman and Jim Milgram have also been invaluable.

1. Operads and $\mathcal{C}$-spaces

Our recognition principle will be based on the notion of an operad acting on a space. We develop the requisite definitions and give a preliminary statement of the recognition theorem in this section.

To fix notations, let $\mathcal{U}$ denote the category of compactly generated Hausdorff spaces and continuous maps, and let $\mathcal{J}$ denote the category of based compactly generated Hausdorff spaces and based maps. Base-points will always be denoted by $*$ and will be required to be non-degenerate, in the sense that $(X, *)$ is an NDR-pair for $X \in \mathcal{J}$. Products, function spaces, etc., are always to be given the compactly generated topology. Steenrod's paper [30] contains most of the point-set topology required for our work. In an appendix, we recall the definition of NDR-pairs and prove those needed results about such pairs which are not contained in [30].

An operad is a collection of suitably interrelated spaces $\mathcal{C}(j)$, the points of which are to be thought of as j-adic operations $X^j \to X$. Precisely, we have the following definitions.

Definition 1.1. An operad $\mathcal{C}$ consists of spaces $\mathcal{C}(j) \in \mathcal{U}$ for $j \geq 0$, with $\mathcal{C}(0)$ a single point $*$, together with the following data:

(a) Continuous functions $\gamma: \mathcal{C}(k) \times \mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k) \to \mathcal{C}(j)$,

$j = \sum j_s$, such that the following associativity formula is satisfied for all $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$, and $e_t \in \mathcal{C}(i_t)$:

$$\gamma(\gamma(c; d_1, \ldots, d_k); e_1, \ldots, e_j) = \gamma(c; f_1, \ldots, f_k),$$

where $f_s = \gamma(d_s; e_{j_1+\ldots+j_{s-1}+1}, \ldots, e_{j_1+\ldots+j_s})$, and

$f_s = *$ if $j_s = 0$.

(b) An identity element $1 \in \mathcal{C}(1)$ such that $\gamma(1;d) = d$ for $d \in \mathcal{C}(j)$ and

$\gamma(c;1^k) = c$ for $c \in \mathcal{C}(k)$, $1^k = (1,\ldots,1) \in \mathcal{C}(1)^k$.

(c) A right operation of the symmetric group Σ_j on $\mathcal{C}(j)$ such that the following equivariance formulas are satisfied for all $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$, $\sigma \in \Sigma_k$, and $\tau_s \in \Sigma_{j_s}$:

$$\gamma(c\sigma; d_1, \ldots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \ldots, d_{\sigma^{-1}(k)})\sigma(j_1, \ldots, j_k)$$

and $\gamma(c; d_1\tau_1, \ldots, d_k\tau_k) = \gamma(c; d_1, \ldots, d_k)(\tau_1 \oplus \ldots \oplus \tau_k)$,

where $\sigma(j_1, \ldots, j_k)$ denotes that permutation of j letters which permutes the k blocks of letters determined by the given partition of j as σ permutes k letters, and $\tau_1 \oplus \ldots \oplus \tau_k$ denotes the image of $(\tau_1, \ldots, \tau_k)$ under the evident inclusion of $\Sigma_{j_1} \times \ldots \times \Sigma_{j_k}$ in Σ_j.

An operad $\mathcal{C}$ is said to be Σ-free if Σ_j acts freely on $\mathcal{C}(j)$ for all j. A morphism $\psi : \mathcal{C} \to \mathcal{C}'$ of operads is a sequence of Σ_j-equivariant maps $\psi_j : \mathcal{C}(j) \to \mathcal{C}'(j)$ such that $\psi_1(1) = 1$ and the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(k) \times \mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k) \arrow[r, "\gamma"] \arrow[d, "\psi_k \times \psi_{j_1} \times \ldots \times \psi_{j_k}"'] & \mathcal{C}(j) \arrow[d, "\psi_j"] \\
\mathcal{C}'(k) \times \mathcal{C}'(j_1) \times \ldots \times \mathcal{C}'(j_k) \arrow[r, "\gamma'"] & \mathcal{C}'(j)
\end{tikzcd}$$

Definition 1.2. Let $X \in \mathcal{T}$ and define the endomorphism operad $\mathcal{E}_X$ of X as follows. Let $\mathcal{E}_X(j)$ be the space of based maps $X^j \to X$; $X^0 = *$, and $\mathcal{E}_X(0)$ is the inclusion $* \to X$. The data are defined by

(a) $\gamma(f; g_1, \ldots, g_k) = f(g_1 \times \ldots \times g_k)$ for $f \in \mathcal{E}_X(k)$ and $g_s \in \mathcal{E}_X(j_s)$.

(b) The identity element $1 \in \mathcal{E}_X(1)$ is the identity map of X.

(c) $(f\sigma)(y) = f(\sigma y)$ for $f \in \mathcal{E}_X(j)$, $\sigma \in \Sigma_j$, and $y \in X^j$, where Σ_j acts on X^j by $\sigma(x_1, \ldots, x_j) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(j)})$.

An operation θ of an operad $\mathcal{C}$ on a space X is a morphism of operads $\theta: \mathcal{C} \to \mathcal{E}_X$, and the pair (X, θ) is then said to be a $\mathcal{C}$-space. A morphism $f: (X, \theta) \to (X', \theta')$ of $\mathcal{C}$-spaces is a based map $f: X \to X'$ such that $f \circ \theta_j(c) = \theta'_j(c) \circ f^j$ for all $c \in \mathcal{C}(j)$. The category of $\mathcal{C}$-spaces is denoted by $\mathcal{C}[\mathcal{T}]$.

It should be clear that the associativity and equivariance formulas in the definition of an operad merely codify the relations that do in fact hold in $\mathcal{E}_X$. The notion of an operad extracts the essential information contained in the notion of a PROP, as defined by Adams and MacLane [16] and topologized by Boardman and Vogt [8].

Our recognition theorem, roughly stated, has the following form.

Theorem 1.3. There exist Σ-free operads $\mathcal{C}_n$, $1 \leq n \leq \infty$, such that every n-fold loop space is a $\mathcal{C}_n$-space and every connected $\mathcal{C}_n$-space has the weak homotopy type of an n-fold loop space.

In the cases $n = 1$ and $n = \infty$, the second statement will be valid with $\mathcal{C}_1$ and $\mathcal{C}_\infty$ replaced by any A_∞ operad and E_∞ operad, as defined in section 3.

Perhaps some plausibility arguments should be given. Let $\mathcal{C}$ be any operad, and let $(X, \theta) \in \mathcal{C}[\mathcal{J}]$. For $c \in \mathcal{C}(2)$, $\theta_2(c): X^2 \to X$ defines a product on X. If $\mathcal{C}(1)$ is connected, then $*$ is a two-sided homotopy identity for $\theta(c)$; indeed, the requisite homotopies are obtained by applying θ_1 to any paths in $\mathcal{C}(1)$ connecting 1 to $\gamma(c; *, 1)$ and 1 to $\gamma(c; 1, *)$. Similarly, if $\mathcal{C}(3)$ is connected, then $\theta(c)$ is homotopy associative since $\gamma(c; 1, c)$ can be connected to $\gamma(c; c, 1)$. If $\mathcal{C}(2)$ is connected, then $\theta(c)$ is homotopic to $\theta(c\tau)$, where $\tau \in \Sigma_2$ is the transposition, and therefore $\theta(c)$ is homotopy commutative. It should be clear that higher connectivity on the spaces $\mathcal{C}(j)$ will determine higher coherence homotopies. Stasheff's theory of A_∞-spaces [28] states essentially that an H-space X is of the homotopy type of a loop space (i.e., has a classifying space) if and only if it has all possible higher coherence homotopies for associativity. It is obvious that if X can be de-looped twice, then its product must be homotopy commutative. Thus higher coherence homotopies for commutativity ought to play a role in determining precisely how many times X can be de-looped. Fortunately, the homotopies implicitly asserted to exist in the statement that a suitably higher connected operad acts on a space will play no explicit role in any of our work.

The spaces $\mathcal{C}_n(j)$ in the operads of Theorem 1.3 will be (n-2)-connected. Thus, if $n = \infty$, it is plausible that there should be no obstructions to the construction of classifying spaces of all orders. In the cases $1 < n < \infty$, the higher homotopies guaranteed by the connectivity of the $\mathcal{C}_n(j)$ are only part of the story. It is not true that any $\mathcal{C}$-space, where $\mathcal{C}(j)$ is (n-2)-connected, is of the homotopy type of an n-fold loop space. Thus Theorem 1.3 is considerably deeper in these cases than in the degenerate case $n = 1$ (where commutativity plays no role) or in the limit case $n = \infty$.

Since the notion of an action θ of an operad $\mathcal{C}$ on a space X is basic to all of our work, it may be helpful to explicitly reformulate this notion in terms of the adjoints $\mathcal{C}(j) \times X^j \to X$ of the maps $\theta_j: \mathcal{C}(j) \to \mathcal{E}_X(j)$; these adjoints will also be denoted by θ_j.

Lemma 1.4. An action $\theta: \mathcal{C} \to \mathcal{E}_X$ determines and is determined by maps $\theta_j: \mathcal{C}(j) \times X^j \to X$, $j \geq 0$ $(\theta_0: * \to X)$, such that

(a) The following diagrams are commutative, where $\Sigma j_s = j$ and u denotes the evident shuffle homeomorphism:

$$\begin{tikzcd}
\mathcal{C}(k) \times \mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k) \times X^j \arrow[rr, "\gamma \times 1"] \arrow[d, "1 \times u"'] & & \mathcal{C}(j) \times X^j \arrow[dr, "\theta_j"] & \\
& & & X \\
\mathcal{C}(k) \times \mathcal{C}(j_1) \times X^{j_1} \times \ldots \times \mathcal{C}(j_k) \times X^{j_k} \arrow[rr, "1 \times \theta_{j_1} \times \ldots \times \theta_{j_k}"] & & \mathcal{C}(k) \times X^k \arrow[ur, "\theta_k"'] &
\end{tikzcd}$$

(b) $\theta_1(1; x) = x$ for $x \in X$, and

(c) $\theta_j(c\sigma; y) = \theta_j(c; \sigma y)$ for $c \in \mathcal{C}(j)$, $\sigma \in \Sigma_j$, and $y \in X^j$.

A morphism $f: (X, \theta) \to (X', \theta')$ in $\mathcal{C}[\mathcal{T}]$ is a map $f: X \to X'$ in $\mathcal{T}$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{C}(j) \times X^j & \xrightarrow{\theta_j} & X \\ {\scriptstyle 1 \times f^j}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{C}(j) \times (X')^j & \xrightarrow{\theta'_j} & X' \end{array}$$

We complete this section by showing that, for any operad $\mathcal{C}$, the category of $\mathcal{C}$-spaces is closed under several standard topological constructions and by discussing the product on $\mathcal{C}$-spaces. These results will yield properties of the Dyer-Lashof homology operations in the second paper of this series and will be used in the third paper of this series to study such spaces as F/O and F/Top. The proofs of the following four lemmas are completely elementary and will be omitted.

Lemma 1.5. Let $(X, \theta) \in \mathcal{C}[\mathcal{T}]$ and let (Y, A) be an NDR-pair in $\mathcal{U}$. Let $X^{(Y,A)} \in \mathcal{T}$ denote the space of maps $(Y, A) \to (X, *)$, with (non-degenerate) base-point the trivial map. Then $(X^{(Y,A)}, \theta^{(Y,A)}) \in \mathcal{C}[\mathcal{T}]$, where $\theta_j^{(Y,A)}: \mathcal{C}(j) \times (X^{(Y,A)})^j \to X^{(Y,A)}$ is defined pointwise:

$$\theta_j^{(Y,A)}(c; f_1, \ldots, f_j)(y) = \theta_j(c; f_1(y), \ldots, f_j(y)).$$

In particular, $(\Omega X, \Omega\theta)$ and $(PX, P\theta)$ are in $\mathcal{C}[\mathcal{T}]$, where $\Omega\theta = \theta^{(I, \partial I)}$ and $P\theta = \theta^{(I, 0)}$, and the inclusion $i: \Omega X \to PX$ and end-point projection $p: PX \to X$ are $\mathcal{C}$-morphisms.

<u>Lemma 1.6.</u> $(*, \theta) \in \mathcal{C}[\mathcal{T}]$, where each θ_j is the trivial map; if $(X, \theta) \in \mathcal{C}[\mathcal{T}]$, then the unique maps $* \to X$ and $X \to *$ in $\mathcal{T}$ are $\mathcal{C}$-morphisms.

<u>Lemma 1.7.</u> Let $f: (X,\theta) \to (B, \theta'')$ and $g: (Y, \theta') \to (B, \theta'')$ be $\mathcal{C}$-morphisms. Let $X \times^B Y \subset X \times Y$ denote the fibred product $\{(x, y) \mid f(x) = g(y)\}$ of f and g in $\mathcal{T}$. Then $(X \times^B Y, \theta \times^B \theta')$ is the fibred product of f and g in the category $\mathcal{C}[\mathcal{T}]$, where $(\theta \times^B \theta')_j : \mathcal{C}(j) \times (X \times^B Y)^j \to X \times^B Y$ is defined coordinatewise:

$$(\theta \times^B \theta')_j(c; (x_1, y_1), \ldots, (x_j, y_j)) = (\theta_j(c; x_1, \ldots, x_j), \theta'_j(c; y_1, \ldots, y_j)).$$

In particular, with $B = *$, $(X \times Y,\ \theta \times \theta')$ is the product of (X, θ) and (Y, θ') in the category $\mathcal{C}[\mathcal{T}]$, and the diagonal map $\Delta: X \to X \times X$ is thus a $\mathcal{C}$-morphism for any $(X, \theta) \in \mathcal{C}[\mathcal{T}]$.

The previous lemmas imply that any morphism in $\mathcal{C}[\mathcal{T}]$ can be replaced by a fibration in $\mathcal{C}[\mathcal{T}]$.

<u>Lemma 1.8.</u> Let $f: (X, \theta) \to (Y, \theta')$ be a morphism in $\mathcal{C}[\mathcal{T}]$. Define $(\tilde{X}, \tilde{\theta}) \in \mathcal{C}[\mathcal{T}]$ by letting $\tilde{X} = X \times^Y (Y^I)$ be the fibred product of f and g, where $g(w) = w(0)$ for $w \in Y^I$, and by letting $\tilde{\theta} = \theta \times^Y (\theta')^I$. Then the inclusion $i: X \to \tilde{X}$, the retraction $r: \tilde{X} \to X$, and the fibration $\tilde{f}: \tilde{X} \to Y$ are all $\mathcal{C}$-morphisms, where $i(x) = (x, w_{f(x)})$ with $w_{f(x)}(t) = f(x)$, $r(x, w) = x$, and $\tilde{f}(x, w) = w(1)$.

Finally, we consider the product on a $\mathcal{C}$-space. The following lemma is the only place in our theory where a less stringent (and more complicated) notion of $\mathcal{C}$-morphism would be of any service. Such a notion is crucial to Boardman and Vogt's work precisely because the H-space structure on a $\mathcal{C}$-space plays a central role in their theory. In contrast, our entire geometric theory could perfectly well be developed without ever explicitly mentioning the product on $\mathcal{C}$-spaces. The product is only one small part of the structure carried by an n-fold loop space, and there is no logical reason for it to play a privileged role.

<u>Lemma 1.9</u>. Let $(X, \theta) \in \mathcal{C}[\mathcal{T}]$ and let $\phi = \theta(c): X^2 \to X$ for some fixed $c \in \mathcal{C}(2)$. Let $\phi_2 = \phi$ and $\phi_j = \phi(1 \times \phi_{j-1}): X^j \to X$ for $j > 2$.

(i) If $\mathcal{C}(j)$ is connected and $d \in \mathcal{C}(j)$, then $\theta(d): X^j \to X$ is homotopic to the iterated product ϕ_j.

(ii) If $\mathcal{C}(j)$ is Σ_j-free and $\mathcal{C}(2j)$ is contractible, then the following diagram is Σ_j-equivariantly homotopy commutative:

$$\begin{tikzcd}
\mathcal{C}(j) \times (X \times X)^j \arrow[r, "(\theta \times \theta)_j"] \arrow[d, "1 \times \phi^j"'] & X \times X \arrow[d, "\phi"] \\
\mathcal{C}(j) \times X^j \arrow[r, "\theta_j"] & X
\end{tikzcd}$$

<u>Proof.</u> (i) $\phi_j = \theta(c_j)$, where $c_2 = c$ and $c_j = \gamma(c; 1, c_{j-1})$ for $j > 2$. Any path in $\mathcal{C}(j)$ connecting d to c_j provides the desired homotopy.

(ii) Define maps f and g from $\mathcal{C}(j)$ to $\mathcal{C}(2j)$ by $f(d) = \gamma(d; c^j)$ and

$g(d) = \gamma(c; d, d)v$, where $v \in \Sigma_{2j}$ gives the evident shuffle map $(X \times X)^j \to X^j \times X^j$ on X^{2j}. An examination of the definitions shows that if $d \in \mathcal{C}(j)$ and $z \in X^{2j}$, then

$$\theta_j(1 \times \phi^j)(d, z) = \theta_{2j}(f(d), z) \quad \text{and} \quad \phi(\theta \times \theta)_j(d, z) = \theta_{2j}(g(d), z).$$

If Σ_j is embedded in Σ_{2j} by $\sigma \to \sigma(2, \ldots, 2)$, in the notation of Definition 1.1(c), then f and g are Σ_j-equivariant. Our hypotheses guarantee that f and g are Σ_j-equivariantly homotopic, and the result follows.

2. Operads and monads

In this section, we show that an operad $\mathcal{C}$ determines a simpler mathematical structure, namely a monad, and that $\mathcal{C}$-spaces can be replaced by algebras over the derived monad. We shall also give a preliminary statement of the approximation theorem. The present reformulation of the notion of $\mathcal{C}$-space will lead to a simple categorical construction of classifying spaces for $\mathcal{C}_n$-spaces in section 9. We first recall the requisite categorical definitions.

Definition 2.1. A monad (C, μ, η) in a category $\mathcal{T}$ consists of a (covariant) functor $C: \mathcal{T} \to \mathcal{T}$ together with natural transformations of functors $\mu: C^2 \to C$ and $\eta: 1 \to C$ such that the following diagrams are commutative for all $X \in \mathcal{T}$:

$$\begin{tikzcd} CX \arrow[r, "C\eta(X)"] \arrow[dr, equal] & C^2X \arrow[d, "\mu(X)"] & CX \arrow[l, "\eta(CX)"'] \arrow[dl, equal] \\ & CX & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C^3X \arrow[r, "\mu(CX)"] \arrow[d, "C\mu(X)"'] & C^2X \arrow[d, "\mu(X)"] \\ C^2X \arrow[r, "\mu(X)"] & CX \end{tikzcd}$$

A morphism $\psi: (C, \mu, \eta) \to (C', \mu', \eta')$ of monads in $\mathcal{T}$ is a natural transformation of functors $\psi: C \to C'$ such that the following diagrams are commutative for all $X \in \mathcal{T}$:

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\ CX \arrow[rr, "\psi"] & & C'X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} CCX \arrow[r, "\psi^2"] \arrow[d, "\mu"'] & C'C'X \arrow[d, "\mu'"] \\ CX \arrow[r, "\psi"] & C'X \end{tikzcd}$$

Here squares (and higher iterates) of natural transformations $\psi: C \rightarrow C'$ are defined by means of the commutative diagrams

$$\begin{tikzcd} C^2X \arrow[r, "C\psi"] \arrow[d, "\psi"'] \arrow[dr, "\psi^2"] & CC'X \arrow[d, "\psi"] \\ C'CX \arrow[r, "C'\psi"] & (C')^2X \end{tikzcd}$$

Thus a monad (C, μ, η) is, roughly, a "monoid in the functor category" with multiplication μ and unit η, and a morphism of monads is a morphism of "monoids". Following MacLane, we prefer the term "monad" to the more usual term "triple". operationally, in our theory, the term monad is particularly apt; the use of monads allows us to replace actions by operads, which are sequences of maps, by monadic algebra structure maps, which are single maps.

Definition 2.2. An algebra (X, ξ) over a monad (C, μ, η) is an object $X \in \mathcal{J}$ together with a map $\xi: CX \to X$ in $\mathcal{J}$ such that the following diagrams are commutative:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, equal] & CX \arrow[d, "\xi"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} CCX \arrow[r, "\mu"] \arrow[d, "C\xi"'] & CX \arrow[d, "\xi"] \\ CX \arrow[r, "\xi"] & X \end{tikzcd}$$

A morphism $f: (X, \xi) \to (X', \xi')$ of C-algebras is a map $f: X \to X'$ in $\mathcal{J}$ such that the following diagram is commutative:

$$\begin{tikzcd} CX \arrow[r, "Cf"] \arrow[d, "\xi"'] & CX' \arrow[d, "\xi'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

The category of C-algebras and their morphisms will be denoted by $C[\mathcal{J}]$.

We now construct a functor from the category of operads to the category of monads in $\mathcal{J}$, where $\mathcal{J}$ is our category of based spaces. In order to handle base-points, we require some preliminary notation.

Notations 2.3. Let $\mathcal{C}$ be an operad. Define maps $\sigma_i: \mathcal{C}(j) \to \mathcal{C}(j-1)$, $0 \leq i < j$, by the formula $\sigma_i c = \gamma(c; s_i)$ for $c \in \mathcal{C}(j)$, where

$$s_i = 1^i \times * \times 1^{j-1-i} \in \mathcal{C}(1)^i \times \mathcal{C}(0) \times \mathcal{C}(1)^{j-1-i}.$$

Thus, in the endomorphism operad of $X \in \mathcal{J}$, $(\sigma_i f)(y) = f(s_i y)$ for $f: X^j \to X$ and $y \in X^{j-1}$, where $s_i: X^{j-1} \to X^j$ is defined by

$$s_i(x_1, \ldots, x_{j-1}) = (x_1, \ldots, x_i, *, x_{i+1}, \ldots, x_{j-1}).$$

Construction 2.4. Let $\mathcal{C}$ be an operad. Construct the monad (C, μ, η) associated to $\mathcal{C}$ as follows. For $X \in \mathcal{J}$, let $\approx$ denote the equivalence relation on the disjoint union $\sum_{j\geq 0} \mathcal{C}(j) \times X^j$ generated by

(i) $(\sigma_i c, y) \approx (c, s_i y)$ for $c \in \mathcal{C}(j)$, $0 \leq i < j$, and $y \in X^{j-1}$; and

(ii) $(c\sigma, y) \approx (c, \sigma y)$ for $c \in \mathcal{C}(j)$, $\sigma \in \Sigma_j$, and $y \in X^j$.

Define CX to be the set $\sum_{j\geq 0} \mathcal{C}(j) \times X^j/(\approx)$. Let $F_k CX$ denote the image of $\sum_{j=0}^{k} \mathcal{C}(j) \times X^j$ in CX and give $F_k CX$ the quotient topology. Observe that $F_{k-1}CX$ is then a closed subspace of $F_k CX$ and give CX the topology of the union of the $F_k CX$. $F_o CX$ is a single point and is to be taken as the base-point of CX. If $c \in \mathcal{C}(j)$ and $y \in X^j$, let $[c, y]$ denote the image of (c, y) in CX. For a map $f: X \to X'$ in $\mathcal{J}$, define $Cf: CX \to CX'$ by $Cf[c; y] = [c; f^j(y)]$. Define natural maps $\mu: C^2X \to CX$ and $\eta: X \to CX$ by the formulas

(iii) $\mu[c, [d_1, y_1], \ldots, [d_k, y_k]] = [\gamma(c; d_1, \ldots, d_k), y_1, \ldots, y_k]$
for $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$, and $y_s \in X^{j_s}$; and

(iv) $\eta(x) = [1, x]$ for $x \in X$.

The associativity and equivariance formulas of Definition 1.1 imply both that μ is well-defined and that μ satisfies the monad identity $\mu \cdot \mu = \mu \cdot C\mu$; the unit formulas of Definition 1.1 imply that $\mu \cdot C\eta = 1 = \mu\eta$. If $\psi: \mathcal{C} \to \mathcal{C}'$ is a morphism of operads, construct the associated morphism of monads, also denoted ψ, by letting $\psi: CX \to C'X$ be the map defined by $\psi[c, y] = [\psi_j(c), y]$ for $c \in \mathcal{C}(j)$ and $y \in X^j$.

The association of monads and morphisms of monads to operads and morphisms of operads thus constructed is clearly a functor. Of course, to validate the construction, we should verify that the spaces CX are indeed in $\mathcal{J}$ for $X \in \mathcal{J}$. We shall do this and shall examine the topology of the CX in the following proposition. We first fix notations for certain spaces, which are usually referred to in the literature as "equivariant half-smash products."

Notations 2.5. Let $W \in \mathcal{U}$ and let π act from the right on W, where π is any subgroup of Σ_j. Let $X \in \mathcal{J}$ and observe that the left action of Σ_j on X^j induces a left action of π on the j-fold smash product $X^{[j]}$. Let $e[W, \pi, X]$ denote the quotient space $W \times X^{[j]}/(\approx)$, where the equivalence relation $\approx$ is defined by $(w, *) \approx (w', *)$ for $w, w' \in W$ and $(w\sigma, y) \approx (w, \sigma y)$ for $w \in W$, $\sigma \in \pi$, and $y \in X^{[j]}$.

The spaces CX are built up by successive cofibrations from the spaces $e[\mathcal{C}(j), \Sigma_j, X]$. Precisely, we have the following result.

Proposition 2.6. Let $\mathcal{C}$ be an operad and let $X \in \mathcal{J}$. Then

(i) $(F_jCX, F_{j-1}CX)$ is an NDR-pair for $j \geq 1$, and $CX \in \mathcal{J}$;

(ii) $F_jCX/F_{j-1}CX$ is homeomorphic to $e[\mathcal{C}(j), \Sigma_j, X]$;

(iii) $C: \mathcal{J} \to \mathcal{J}$ is a homotopy and limit preserving functor.

Proof. It is immediate from the definitions that

$$F_jCX - F_{j-1}CX = \mathcal{C}(j) \times_{\Sigma_j} (X - *)^j .$$

It follows easily that each F_jCX is Hausdorff, hence, by [30, 2.6], compactly generated. Since $(X, *)$ is an NDR-pair by assumption, there is a representation (h_j, u_j) of $(X, *)^j$ as a Σ_j-equivariant NDR-pair by Lemma A.4. Define $\tilde{h}_j: I \times F_jCX \to F_jCX$ and $\tilde{u}_j: F_jCX \to I$ by the formulas

$$\tilde{h}_j(t, z) = z \quad \text{and} \quad \tilde{u}_j(z) = 0 \quad \text{for} \quad z \in F_{j-1}CX, \quad \text{and}$$

$$\tilde{h}_j(t, z) = [c, h_j(t, y)] \quad \text{and} \quad \tilde{u}_j(z) = u_j(y) \quad \text{for} \quad z = [c, y],\ c \in \mathcal{C}(j) \text{ and } y \in (X-*)^j.$$

Then $(\tilde{h}_j, \tilde{u}_j)$ represent $(F_jCX, F_{j-1}CX)$ as an NDR-pair. By [30, 9.2 and 9.4], $CX \in \mathcal{U}$ and each (CX, F_jCX) is an NDR-pair. Therefore $CX \in \mathcal{J}$. Part (ii) is now obvious. For (iii), if $h_t: X \to X'$ is a homotopy, then $Ch_t: CX \to CX'$ is a homotopy, and it is evident that C preserves limits on directed systems of inclusions in $\mathcal{J}$.

We shall see in a moment that the CX are $\mathcal{C}$-spaces, and our approximation theorem can be stated as follows.

<u>Theorem 2.7.</u> For the operads $\mathcal{C}_n$ of the recognition principle, there is a natural map of $\mathcal{C}_n$-spaces $\alpha_n: C_nX \to \Omega^n S^n X$, $1 \leq n \leq \infty$, and α_n is a weak homotopy equivalence if X is connected.

In fact, $\Omega^n S^n$ defines a monad in $\mathcal{J}$, and the natural transformations $\alpha_n: C_n \to \Omega^n S^n$ will be morphisms of monads. This fact will provide the essential link connecting the approximation theorem to the recognition principle.

We now investigate the relationship between $\mathcal{C}$-spaces and C-algebras, where C is the monad associated to the operad $\mathcal{C}$.

Proposition 2.8. Let $\mathcal{C}$ be an operad and let C be its associated monad. Then there is a one-to-one correspondence between $\mathcal{C}$-actions $\theta: \mathcal{C} \to \mathcal{E}_X$ and C-algebra structure maps $\xi: CX \to X$ defined by letting θ correspond to ξ if and only if the following diagrams are commutative for all j:

$$\begin{tikzcd} \mathcal{C}(j) \times X^j \arrow[rr, "\pi_j"] \arrow[dr, "\theta_j"'] & & CX \arrow[dl, "\xi"] \\ & X & \end{tikzcd}$$

(where π_j is the evident composite $\mathcal{C}(j) \times X^j \to F_j CX \to CX$). Moreover, this correspondence defines an isomorphism between the category of $\mathcal{C}$-spaces and the category of C-algebras.

Proof. By the definition of the spaces CX, a map $\xi: CX \to X$ determines and is determined (via the stated diagrams) by a sequence of maps $\theta_j: \mathcal{C}(j) \times X^j \to X$ such that $\theta_{j-1}(\sigma_i c, y) = \theta_j(c, s_i y)$ and $\theta_j(c\sigma, y) = \theta_j(c, \sigma y)$. Since $\sigma_i c = \gamma(c; s_i)$, the maps θ_j given by a $\mathcal{C}$-action θ do satisfy these formulas. For a given map $\xi: CX \to X$, the relation $\xi \cdot \mu = \xi \cdot C\xi$ is equivalent to the commutativity of the diagrams given in Lemma 1.4(a) for the corresponding maps θ_j, and the relation $\xi\eta = 1$ is equivalent to $\theta_1(1, x) = x$ for all $x \in X$. Thus a map $\xi: CX \to X$ is a C-algebra structure map if and only if the corresponding maps θ_j define an action of $\mathcal{C}$ on X. The last statement follows from the observation that if (X, ξ) and (X', ξ') are C-algebras and if $f: X \to X'$ is a map in $\mathcal{J}$, then $f \cdot \xi = \xi' \cdot Cf$ if and only if $f\theta_j = \theta'_j(1 \times f^j)$ for all j.

Henceforward, we shall use the letter θ both for $\mathcal{C}$-actions and for the corresponding C-algebra structure maps. Thus the maps $\theta_j\colon \mathcal{C}(j) \times X^j \to X$ which define a $\mathcal{C}$-action should now be thought of as components of the single map $\theta\colon CX \to X$.

We should observe that the previous proposition implies that CX is the free $\mathcal{C}$-space generated by the space X, in view of the following standard lemma in category theory.

<u>Lemma 2.9</u>. Let (C, μ, η) be a monad in a category $\mathcal{J}$. Then $(CX, \mu) \in C[\mathcal{J}]$ for $X \in \mathcal{J}$, and there is a natural isomorphism

$$\phi\colon \mathrm{Hom}_{\mathcal{J}}(X, Y) \to \mathrm{Hom}_{C[\mathcal{J}]}((CX, \mu), (Y, \xi))$$

defined by $\phi(f) = \xi \circ Cf$; ϕ^{-1} is given by $\phi^{-1}(g) = g \circ \eta$.

The preceding lemma states that the forgetful functor $U\colon C[\mathcal{J}] \to \mathcal{J}$ defined by $U(Y, \xi) = Y$ and the free functor $Q\colon \mathcal{J} \to C[\mathcal{J}]$ defined by $QX = (CX, \mu)$ are adjoint. We shall later need the following converse result, which is also a standard and elementary categorical observation.

<u>Lemma 2.10</u>. Let $\phi\colon \mathrm{Hom}_{\mathcal{J}}(X, UY) \to \mathrm{Hom}_{\mathcal{X}}(QX, Y)$ be an adjunction between functors $U\colon \mathcal{X} \to \mathcal{J}$ and $Q\colon \mathcal{J} \to \mathcal{X}$. For $X \in \mathcal{J}$, define $\eta = \phi^{-1}(1_{QX})\colon X \to UQX$ and define

$$\mu = U\phi(1_{UQX})\colon UQUQX \to UQX .$$

Then (UQ, μ, η) is a monad in $\mathcal{J}$. For $Y \in \mathcal{X}$, define

$$\xi = U\phi(1_{UY})\colon UQUY \to UY.$$

Then $(UY, \xi) \in UQ[\mathcal{J}]$, and $\xi\colon UQU \to U$ is a natural transformation of functors $\mathcal{Y} \to \mathcal{T}$. Thus there is a well-defined functor $V: \mathcal{Y} \to UQ[\mathcal{T}]$ given by $VY = (Y, \xi)$ on objects and $Vg = Ug$ on morphisms.

Of course, V is not an isomorphism of categories in general. However, if the adjunction ϕ is derived as in Lemma 2.9 from a monad C, with $\mathcal{Y} = C[\mathcal{T}]$, then it is evident that the monads UQ and C are the same and that V is the identity functor.

3. A_∞ and E_∞ operads

We describe certain special types of operads here and show that the constructions of the previous section include the James construction and the infinite symmetric product. Most important, we obtain some easy technical results that will allow us to transfer the recognition principle and approximation theorem from the particular operads $\mathcal{C}_1$ and $\mathcal{C}_\infty$ to arbitrary A_∞ and E_∞ operads, respectively.

We first define discrete operads $\mathcal{M}$ and $\mathcal{N}$ such that an $\mathcal{M}$-space is precisely a topological monoid and an $\mathcal{N}$-space is precisely a commutative topological monoid.

Definition 3.1. (i) Define $\mathcal{M}(j) = \Sigma_j$ for $j \geq 1$, and let e_j denote the identity element of Σ_j, $e_1 = 1$. Let $\mathcal{M}(0)$ contain the single element e_0. Define $\gamma(e_k; e_{j_1}, \ldots, e_{j_k}) = e_j$, $j = \sum j_s$, and extend the domain of definition of γ to the entire set $\Sigma_k \times \Sigma_{j_1} \times \ldots \times \Sigma_{j_k}$ by the equivariance formulas of Definition 1.1(c). With these data, the $\mathcal{M}(j)$ constitute a discrete operad $\mathcal{M}$.
(ii) Define $\mathcal{N}(j) = \{f_j\}$, a single point. Let $1 = f_1$, let Σ_j act trivially on $\mathcal{N}(j)$, and define $\gamma(f_k; f_{j_1}, \ldots, f_{j_k}) = f_j$, $j = \sum j_s$. With these data, the $\mathcal{N}(j)$ constitute a discrete operad $\mathcal{N}$.

Observe that if $\mathcal{C}$ is any operad with each $\mathcal{C}(j)$ non-empty, then the unique functions $\mathcal{C}(j) \to \mathcal{N}(j)$ define a morphism of operads $\mathcal{C} \to \mathcal{N}$, hence any $\mathcal{N}$-space is a $\mathcal{C}$-space.

A topological monoid G in $\mathcal{J}$ (with identity element $*$) determines and is determined by the action $\theta: \mathcal{M} \to \mathcal{E}_G$ defined by letting $\theta_j(e_j): G^j \to G$ be the iterated product and extending θ_j to all of Σ_j by equivariance. The permutations in $\mathcal{M}$ serve only to record the possibility of changing the order of factors in forming products in a topological monoid. Clearly a topological monoid G is commutative if and only if the corresponding action $\theta: \mathcal{M} \to \mathcal{E}_G$ factors through $\mathcal{N}$.

For $X \in \mathcal{J}$, the monoids MX and NX are called the James construction and the infinite symmetric product on X; it should be observed that the successive quotient spaces $e[\mathcal{M}(j), \Sigma_j, X]$ and $e[\mathcal{N}(j), \Sigma_j, X]$ are homeomorphic to the j-fold smash product $X^{[j]}$ and to the orbit space $X^{[j]}/\Sigma_j$, respectively. The arguments above and the results of the previous section yield the following proposition.

<u>Proposition 3.2.</u> The categories $\mathcal{M}[\mathcal{J}] = M[\mathcal{J}]$ and $\mathcal{N}[\mathcal{J}] = N[\mathcal{J}]$ are isomorphic to the categories of topological monoids and of commutative topological monoids, respectively. For $X \in \mathcal{J}$, MX and NX are the free topological monoid and the free commutative topological monoid generated by the space X, subject to the relation $* = 1$.

We shall only be interested in operads which are augmented over either $\mathcal{M}$ or $\mathcal{N}$, in a sense which we now make precise. Let $\mathcal{C}$ be any operad, and let $\pi_0\mathcal{C}(j)$ denote the set of path components of $\mathcal{C}(j)$. Define

$\delta_j\colon \mathcal{C}(j) \to \pi_0\mathcal{C}(j)$ by $\delta_j(c) = [c]$, where $[c]$ denotes the path component containing the point c. The data for $\mathcal{C}$ uniquely determine data for $\pi_0\mathcal{C}$ such that $\pi_0\mathcal{C}$ is a discrete operad and δ is a morphism of operads. Clearly π_0 defines a functor from the category of operads to the category of discrete operads. If $\mathcal{D}$ is any discrete operad and if $\varepsilon\colon \mathcal{C} \to \mathcal{D}$ is a morphism of operads, then ε factors as the composite $\pi_0\varepsilon \circ \delta$, where $\pi_0\varepsilon\colon \pi_0\mathcal{C} \to \pi_0\mathcal{D} = \mathcal{D}$. With these notations, we make the following definition.

<u>Definition 3.3</u>. An operad over a discrete operad $\mathcal{D}$ is an operad $\mathcal{C}$ together with a morphism of operads $\varepsilon\colon \mathcal{C} \to \mathcal{D}$ such that $\pi_0\varepsilon\colon \pi_0\mathcal{C} \to \mathcal{D}$ is an isomorphism of operads. ε is called the augmentation of $\mathcal{C}$. A morphism $\psi\colon (\mathcal{C}, \varepsilon) \to (\mathcal{C}', \varepsilon')$ of operads over $\mathcal{D}$ is a morphism of operads $\psi\colon \mathcal{C} \to \mathcal{C}'$ such that $\varepsilon'\psi = \varepsilon\colon \mathcal{C} \to \mathcal{D}$.

We shall say that an operad $\mathcal{C}$ is locally n-connected if each $\mathcal{C}(j)$ is n-connected. Clearly an operad $\mathcal{C}$ can be augmented over $\mathcal{N}$ if and only if it is locally connected, and $\mathcal{C}$ then admits a unique augmentation. An operad $\mathcal{C}$ can be augmented over $\mathcal{M}$ if and only if $\pi_0\mathcal{C}(j)$ is isomorphic to Σ_j, and an augmentation of $\mathcal{C}$ is then a suitably coherent choice of isomorphisms.

We shall say that a morphism of operads $\psi\colon \mathcal{C} \to \mathcal{C}'$ is a local equivalence, or a local Σ-equivalence, if each $\psi_j\colon \mathcal{C}(j) \to \mathcal{C}'(j)$ is a homotopy equivalence, or a Σ_j-equivariant homotopy equivalence (that is, the requisite homotopies are required to be Σ_j-equivariant). Of course, these are not equivalence relations since there need be no inverse morphism of operads $C' \to C$. The following proposition will be essential in passing from one operad over $\mathcal{M}$ or $\mathcal{N}$ to another.

Proposition 3.4. Let $\psi: C \to C'$ be a morphism of operads over $\mathcal{M}$ or $\mathcal{N}$. Assume either that ψ is a local Σ-equivalence or that ψ is a local equivalence and C and C' are Σ-free. Then the associated maps $\psi: CX \to C'X$ are weak homotopy equivalences for all connected spaces X.

Proof. Since $\psi: CX \to C'X$ is an H-map between connected H-spaces, it suffices to prove that ψ induces an isomorphism on integral homology. By Proposition 2.6 and the five lemma, this will hold if the maps $e[C(j),\Sigma_j,X] \to e[C'(j),\Sigma_j,X]$ determined by ψ_j induce isomorphisms on homology. These maps are homotopy equivalences if ψ_j is a Σ_j-equivariant homotopy equivalence. If $C(j)$ is Σ_j-free, then the map $C(j)\times X^{[j]} \to C(j)\times_{\Sigma_j} X^{[j]}$ is clearly a covering map and so determines a spectral sequence converging from $E^2 = H_*(\Sigma_j; H_*(C(j)\times X^{[j]}))$ to $H_*(C(j)\times_{\Sigma_j} X^{[j]})$. Thus if $C(j)$ and $C'(j)$ are Σ_j-free and ψ_j is a homotopy equivalence, then ψ_j induces an isomorphism on E^2, hence on $H_*(C(j)\times_{\Sigma_j} X^{[j]})$, hence on $H_*(e[C(j),\Sigma_j,X])$.

We now define and discuss A_∞ and E_∞ operads and spaces.

Definition 3.5. (i) An A_∞ operad is a Σ-free operad over $\mathcal{M}$ such that $\varepsilon: C \to \mathcal{M}$ is a local Σ-equivalence. An A_∞ space (X,θ) is a C-space over any A_∞ operad C.
(ii) An E_∞ operad is a Σ-free operad over $\mathcal{N}$ such that $\varepsilon: C \to \mathcal{N}$ is a local equivalence. An E_∞ space, or homotopy everything space, (X,θ) is a C-space over any E_∞ operad C.

We have not defined and shall not need any notion of an A_∞ or E_∞ morphism between A_∞ or E_∞ spaces over different operads.

An operad $\mathcal{C}$ is an E_∞ operad if and only if each $\mathcal{C}(j)$ is Σ_j-free and contractible. Thus the orbit space $\mathcal{C}(j)/\Sigma_j$ is a classifying space for Σ_j; its homology will give rise to the Dyer-Lashof operations on the homology of an E_∞ space. We have required an E_∞ operad to be Σ-free in order to have this interpretation of the spaces $\mathcal{C}(j)/\Sigma_j$ and in order to have that CX is weakly homotopy equivalent to $\Omega^\infty S^\infty X$ for any E_∞ operad $\mathcal{C}$ and connected space X. Note in particular that we have chosen not to regard $\mathcal{N}$ as an E_∞ operad, although a connected $\mathcal{N}$-space is evidently an infinite loop space. The following amusing result shows that, for non-triviality, we must not assume ε to be a local Σ-equivalence in the definition of an E_∞ operad.

<u>Proposition 3.6</u>. Let $\mathcal{C}$ be an operad over $\mathcal{N}$ such that $\varepsilon : \mathcal{C} \to \mathcal{N}$ is a local Σ-equivalence. Let (X, θ) be a $\mathcal{C}$-space, where X is a connected space. Then X is weakly homotopy equivalent to $\underset{n\geq 1}{\times} K(\pi_n(X), n)$.

<u>Proof</u>. We have the following commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, equal] & CX \arrow[d, "\theta"] \\ & X \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\eta"'] & CX \arrow[d, "\varepsilon"] \\ & NX \end{tikzcd}$$

By Proposition 3.4, ε is a weak homotopy equivalence. It is well-known and easy to prove that $\eta_*: \pi_*(X) \to \pi_*(NX) = \tilde{H}_*(X)$ may be taken as the definition of the Hurewicz homomorphism h. Thus $1 = \theta_*\eta_* = (\theta_* \varepsilon_*^{-1})h$, and h is a monomorphism onto a direct summand of $\tilde{H}_*(X)$. By the proof of [18, Theorem 24.5], this is precisely enough to imply the conclusion.

An operad $\mathcal{C}$ is an A_∞ operad if and only if each $\pi_0 \mathcal{C}(j)$ is isomorphic to Σ_j and each component of $\mathcal{C}(j)$ is contractible. In particular, $\mathcal{M}$ is itself an A_∞ operad. In contrast to the preceding result, we have the following observation concerning operads over $\mathcal{M}$.

<u>Lemma 3.7.</u> Any operad $\mathcal{C}$ over $\mathcal{M}$ is Σ-free and any local equivalence $\psi: \mathcal{C} \to \mathcal{C}'$ between operads over $\mathcal{M}$ is a local Σ-equivalence.

<u>Proof.</u> Each $\sigma \in \Sigma_j$ must act on $\mathcal{C}(j)$ by permuting components, carrying $\varepsilon_j^{-1}(\tau)$ homeomorphically onto $\varepsilon_j^{-1}(\tau\sigma)$ for $\tau \in \Sigma_j$. For the second statement, we may assume that $\varepsilon'\psi = \varepsilon$ (redefining ε by this equation if necessary), and then ψ_j must restrict to a homotopy equivalence $\varepsilon_j^{-1}(e_j) \to (\varepsilon_j')^{-1}(e_j)$. The resulting homotopies can be transferred by equivariance to the remaining components of $\mathcal{C}(j)$ and $\mathcal{C}'(j)$, and the result follows.

In the applications, it is essential that our recognition theorem apply, for $n = 1$ and $n = \infty$, to arbitrary A_∞ and E_∞ operads. However, there need be no morphism of operads between two A_∞ or two E_∞ operads. Fortunately, all that is needed to circumvent this difficulty is the observation that the category of operads has products.

Definition 3.8. Let $\mathcal{C}$ and $\mathcal{C}'$ be operads. Define an operad $\mathcal{C} \times \mathcal{C}'$ by letting $(\mathcal{C} \times \mathcal{C}')(j) = \mathcal{C}(j) \times \mathcal{C}'(j)$ and giving $\mathcal{C} \times \mathcal{C}'$ the following data:

(a) $(\gamma \times \gamma')(c \times c'; d_1 \times d_1', \ldots, d_k \times d_k') = \gamma(c; d_1, \ldots, d_k) \times \gamma'(c'; d_1', \ldots, d_k')$ for $c \times c' \in \mathcal{C}(k) \times \mathcal{C}'(k)$ and $d_s \times d_s' \in \mathcal{C}(j_s) \times \mathcal{C}'(j_s)$;

(b) $1 = 1 \times 1 \in \mathcal{C}(1) \times \mathcal{C}'(1)$; and

(c) $(c \times c')\sigma = c\sigma \times c'\sigma$ for $c \times c' \in \mathcal{C}(j) \times \mathcal{C}'(j)$ and $\sigma \in \Sigma_j$.

Then $\mathcal{C} \times \mathcal{C}'$ is the product of $\mathcal{C}$ and $\mathcal{C}'$ in the category of operads. The monad associated to $\mathcal{C} \times \mathcal{C}'$ will be denoted $C \times C'$ (by abuse of notation, since we do not assert that $C \times C'$ is the product of C and C' in the category of monads in $\mathcal{J}$).

The product of an operad over $\mathcal{B}$ and an operad over $\mathcal{B}'$ is evidently an operad over $\mathcal{B} \times \mathcal{B}'$. Since $\mathcal{M} \times \mathcal{M} \neq \mathcal{M}$, the above product is inappropriate for the study of operads over $\mathcal{M}$. Observe that the category of operads has fibred products as well as products.

Definition 3.9. Let $(\mathcal{C}, \varepsilon)$ and $(\mathcal{C}', \varepsilon')$ be operads over $\mathcal{M}$. Define an operad $(\mathcal{C} \nabla \mathcal{C}', \varepsilon \nabla \varepsilon')$ over $\mathcal{M}$ by letting $\mathcal{C} \nabla \mathcal{C}'$ be the fibred product of ε and ε' in the category of operads and letting $\varepsilon \nabla \varepsilon'$ be defined by commutativity of the following diagram:

$$\begin{tikzcd} \mathcal{C}\nabla\mathcal{C}' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\varepsilon\nabla\varepsilon'"] & \mathcal{C}' \arrow[d, "\varepsilon'"] \\ \mathcal{C} \arrow[r, "\varepsilon"] & \mathcal{M} \end{tikzcd}$$

Explicitly, $\mathcal{C}\nabla\mathcal{C}'$ is the sub operad of $\mathcal{C}\times\mathcal{C}'$ such that $(\mathcal{C}\nabla\mathcal{C}')(j)$ is the disjoint union of the spaces $\varepsilon_j^{-1}(\sigma)\times(\varepsilon_j')^{-1}(\sigma)$ for $\sigma \in \Sigma_j$. Then $(\mathcal{C}\nabla\mathcal{C}', \varepsilon\nabla\varepsilon')$ is the product of $(\mathcal{C}, \varepsilon)$ and $(\mathcal{C}', \varepsilon')$ in the category of operads over $\mathcal{M}$. The monad associated to $\mathcal{C}\nabla\mathcal{C}'$ will be denoted by $C\nabla C'$.

In conjunction with Proposition 3.4, the following result contains all the information about changes of operads that is required for our theory.

Proposition 3.10. (i) Let $\mathcal{C}$ be an A_∞ operad and let $\mathcal{C}'$ be any operad over $\mathcal{M}$. Then the projection $\pi_2: \mathcal{C}\nabla\mathcal{C}' \to \mathcal{C}'$ is a local Σ-equivalence.
(ii) Let $\mathcal{C}$ be an E_∞ operad and let $\mathcal{C}'$ be any Σ-free operad. Then the projection $\pi_2: \mathcal{C}\times\mathcal{C}' \to \mathcal{C}'$ is a local equivalence between Σ-free operads.

Proof. (i) follows from Lemma 3.7 since $\varepsilon_j^{-1}(\sigma)$ is contractible for $\sigma \in \Sigma_j$ and therefore $\pi_2: \varepsilon_j^{-1}(\sigma) \times (\varepsilon_j')^{-1}(\sigma) \to (\varepsilon_j')^{-1}(\sigma)$ is a homotopy equivalence. Part (ii) is immediate from the definitions.

Since (ii) depends only on the local contractibility (and not on the Σ-freeness) of $\mathcal{C}$, the proof of our recognition principle for E_∞ spaces will actually apply to $\mathcal{C}$-spaces over any locally contractible operad $\mathcal{C}$.

Corollary 3.11. Let $\mathcal{C}$ be an E_∞ operad. Then $\pi_2: \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is a local Σ-equivalence and therefore $\mathcal{C} \times \mathcal{M}$ is an A_∞ operad. If (X, θ) is a $\mathcal{C}$-space, then $(X, \theta\pi_1)$ is a $\mathcal{C} \times \mathcal{M}$-space, $\pi_1: \mathcal{C} \times \mathcal{M} \to \mathcal{C}$. Thus every E_∞ space is an A_∞ space.

Since A_∞ spaces are of interest solely inthe study of first loop spaces, where commutativity plays no role, a simpler theory of A_∞ spaces can be obtained by throwing out the permutations by means of the following definition and proposition. We have chosen to describe A_∞ spaces in terms of operads in order to avoid further special arguments, and no use shall be made of the theory sketched below.

Definition 3.12. A non-Σ operad $\mathcal{B}$ is a sequence of spaces $\mathcal{B}(j) \in \mathcal{U}$ for $j \geq 0$, with $\mathcal{B}(0) = *$, together with the data (a) and (b) in the definition of an operad. An operad $\mathcal{C}$ determines an underlying non-Σ operad $\mathcal{U}\mathcal{C}$ by neglect of permutations. An action of a non-Σ operad $\mathcal{B}$ on a space $X \in \mathcal{T}$ is a morphism of non-Σ operads $\theta: \mathcal{B} \to \mathcal{U}\mathcal{E}_X$, and $\mathcal{B}[\mathcal{T}]$ denotes the category of $\mathcal{B}$-spaces (X, θ). By omission of the equivariance relation (ii) in construction 2.4, a non-Σ operad $\mathcal{B}$ determines an associated monad B such that the categories $\mathcal{B}[\mathcal{T}]$ and $B[\mathcal{T}]$ are isomorphic. The notion of a non-Σ operad over a discrete non-Σ operad is defined by analogy with Definition 3.3. The product $\mathcal{B} \times \mathcal{B}'$ of non-Σ operads $\mathcal{B}$ and $\mathcal{B}'$ is defined by analogy with Definition 3.8.

Let $\mathcal{A}$ denote the sub non-Σ operad of $\mathcal{M}$ such that $\mathcal{A}(j) = \{e_j\}$. The categories $\mathcal{A}[\mathcal{T}]$ and $\mathcal{M}[\mathcal{T}]$ are evidently isomorphic. A non-Σ operad over $\mathcal{A}$ clearly admits a unique augmentation. A non-Σ operad determines an operad $\Sigma\mathcal{B}$ such that $\mathcal{U}\Sigma\mathcal{B} = \mathcal{B}$ by letting Σ_j act trivially on $\mathcal{B}(j)$. In particular, $\Sigma\mathcal{A}$ is isomorphic to $\mathcal{N}$.

Proposition 3.13. Let $(\mathcal{C}, \varepsilon)$ be an operad over $\mathcal{M}$ and define $w(\mathcal{C}, \varepsilon) = \varepsilon^{-1}(\mathcal{A})$; then $w(\mathcal{C}, \varepsilon)$ is a non-Σ operad over $\mathcal{A}$ and the monads associated to $\mathcal{C}$ and to $w(\mathcal{C}, \varepsilon)$ are isomorphic. Let $\mathcal{B}$ be a non-Σ operad over $\mathcal{A}$ and define $w^{-1}\mathcal{B} = (\Sigma\mathcal{B} \times \mathcal{M}, \pi_2)$; then $w^{-1}(\mathcal{B})$ is an operad over $\mathcal{M}$ and the monads associated to $\mathcal{B}$ and to $w^{-1}(\mathcal{B})$ are isomorphic. Moreover, w and w^{-1} are the object maps of an equivalence between the categories of operads over $\mathcal{M}$ and of non-Σ operads over $\mathcal{A}$.

Proof. The first two statements follow immediately from the definitions. For the last statement, it is obvious how to define w and w^{-1} on morphisms, and we must show that ww^{-1} and $w^{-1}w$ are naturally isomorphic to the respective identity functors. Now $ww^{-1}(\mathcal{B}) = \mathcal{B} \times \mathcal{A}$ is evidently naturally isomorphic to $\mathcal{B}$, and a natural isomorphism

$$v: (\mathcal{C}, \varepsilon) \rightarrow w^{-1}w(\mathcal{C}, \varepsilon) = (\Sigma\varepsilon^{-1}(\mathcal{A}) \times \mathcal{M}, \pi_2)$$

can be defined by $v_j(c) = (c\sigma^{-1}, \sigma)$ for $c \in \varepsilon_j^{-1}(\sigma)$ and $\sigma \in \Sigma_j$; v^{-1} is then given by $v_j^{-1}(c, \sigma) = c\sigma$ for $c \in \varepsilon_j^{-1}(e_j)$ and $\sigma \in \Sigma_j$.

It follows that the notion of an A_∞ operad is equivalent to the notion of a locally contractible non-Σ operad over $\mathcal{A}$, and the notion of an A_∞ space is equivalent to the notion of a $\mathcal{B}$-space over such a non-Σ operad $\mathcal{B}$.

Remark 3.14. The notion of A_∞ space originally defined by Stasheff [28] is included in our notion. Stasheff constructs certain spaces K_j for $j \geq 2$; with $K_0 = *$ and $K_1 = 1$, these K_j can be verified to admit structure maps γ so as to form a locally contractible non-Σ operad $\mathcal{K}$ such that an A_∞ space in Stasheff's sense is precisely a $\mathcal{K}$-space.

4. The little cubes operads $\mathcal{C}_n$

We define the Σ-free operads $\mathcal{C}_n$ and discuss the topology of the spaces $\mathcal{C}_n(j)$ in this section. I am indebted to M. Boardman for explaining to me the key result, theorem 4.8. The definition of the $\mathcal{C}_n$ (in the context of PROP's) is due to Boardman and Vogt [8].

Definition 4.1. Let I^n denote the unit n-cube and let J^n denote its interior. An (open) little n-cube is a linear embedding f of J^n in J^n, with parallel axes; thus $f = f_1 \times \ldots \times f_n$ where $f_i : J \to J$ is a linear function, $f_i(t) = (y_i - x_i)t + x_i$, with $0 \leq x_i < y_i \leq 1$. Define $\mathcal{C}_n(j)$ to be the set of those j-tuples $<c_1, \ldots, c_n>$ of little n-cubes such that the images of the c_r are pairwise disjoint. Let ${}^jJ^n$ denote the disjoint union of j copies of J^n, regard $<c_1, \ldots, c_j>$ as a map ${}^jJ^n \to J^n$, and topologize $\mathcal{C}_n(j)$ as a subspace of the space of all continuous functions ${}^jJ^n \to J^n$. Write $\mathcal{C}_n(0) = < >$, and regard $<>$ as the unique "embedding" of the empty set in J^n. The requisite data are defined by

(a) $\gamma(c; d_1, \ldots, d_k) = c \cdot (d_1 + \ldots + d_k): {}^{j_1}J^n + \ldots + {}^{j_k}J^n \to J^n$

for $c \in \mathcal{C}_n(k)$ and $d_s \in \mathcal{C}_n(j_s)$, where + denotes disjoint union;

(b) $1 \in \mathcal{C}_n(1)$ is the identity function; and

(c) $<c_1, \ldots, c_j>\sigma = <c_{\sigma(1)}, \ldots, c_{\sigma(j)}>$ for $\sigma \in \Sigma_j$.

By our functional interpretation of $<>$, (a) implies that

(d) $\sigma_i <c_1, \ldots, c_j> = <c_1, \ldots, c_i, c_{i+2}, \ldots, c_j>, \quad 0 \leq i < j.$

The associativity, unitary, and equivariance formulas required of an operad are trivial to verify, and the action of Σ_j on $\mathcal{C}_n(j)$ is free in view of the requirement that the component little cubes of a point of $\mathcal{C}_n(j)$ have disjoint images. Define a morphism of operads $\sigma_n: \mathcal{C}_n \to \mathcal{C}_{n+1}$ by

(e) $\sigma_{n,j}\langle c_1, \ldots, c_j\rangle = \langle c_1 \times 1, \ldots, c_j \times 1\rangle$, $1: J \to J$.

Each $\sigma_{n,j}$ is an inclusion, and $\mathcal{C}_\infty(j)$ denotes the space $\varinjlim \mathcal{C}_n(j)$, with the topology of the union. Clearly $\mathcal{C}_\infty$ inherits a structure of Σ-free operad from the $\mathcal{C}_n$.

The topology we have given the $\mathcal{C}_n(j)$ is convenient for continuity proofs and will be needed in our study of the Dyer-Lashof operations on F in the third paper of this series. The following more concrete description of this topology is more convenient for analyzing the homotopy type of the spaces $\mathcal{C}_n(j)$.

<u>Lemma 4.2.</u> Let $c = \langle c_1, \ldots, c_j\rangle \in \mathcal{C}_n(j)$. Observe that c determines and is determined by the point $c(\alpha, \beta) \in J^{2nj}$ defined by

$$c(\alpha, \beta) = (c_1(\alpha), c_1(\beta), \ldots, c_j(\alpha), c_j(\beta)),$$
$$\text{where } \alpha = (\tfrac{1}{4}, \ldots, \tfrac{1}{4}) \in J^n \text{ and } \beta = (\tfrac{3}{4}, \ldots, \tfrac{3}{4}) \in J^n .$$

Let $\mathcal{U}$ denote the topology on $\mathcal{C}_n(j)$ obtained by so regarding $\mathcal{C}_n(j)$ as a subset of J^{2nj} and let $\mathcal{V}$ denote the topology on $\mathcal{C}_n(j)$ defined in Definition 4.1. Then $\mathcal{U} = \mathcal{V}$.

<u>Proof.</u> Let $W(C, U)$ denote the $\mathcal{V}$-open set consisting of those c such that $c(C) \subset U$, where C is compact in ${}^jJ^n$ and U is open in J^n. Let α_r (resp. β_r) denote the point α (resp. β) in the r-th domain cube $J^n_r \subset {}^jJ^n$. If U_r and V_r are open subsets of J^n, $1 \leq r \leq j$, then

$$\mathcal{C}_n(j) \cap (U_1 \times V_1 \times \ldots \times U_j \times V_j) = \bigcap_{j=1}^{r} W(\alpha_r, U_r) \cap W(\beta_r, V_r).$$

It follows that any $\mathcal{U}$-open set is $\mathcal{V}$-open. Conversely, consider $W(C, U)$. We may assume that U is the image of an open little cube g and that C is contained in a single domain cube J_r^n. Let $C' \subset J_r^n$ be the image of the smallest closed little cube f containing C (f may be degenerate; that is, some of its intervals may be points). Then, by linearity, $W(C, U) = W(C', U)$. Clearly $c = \langle c_1, \ldots, c_j \rangle \in W(C', U)$ if and only if $c_r f(0) > g(0)$ and $c_r f(1) < g(1)$, with the inequalities interpreted coordinate-wise and with $0 = (0, \ldots, 0)$ and $1 = (1, \ldots, 1)$ in I^n. It is now easy to verify that $W(C', U)$ is $\mathcal{U}$-open.

Using this lemma, we can relate the spaces $\mathcal{C}_n(j)$ to the configuration spaces of R^n. We first review some of the results of Fadell and Neuwirth [12] on configuration spaces.

Definition 4.3. Let M be an n-dimensional manifold. Define the j-th configuration space $F(M; j)$ of M by

$$F(M; j) = \{ \langle x_1, \ldots, x_j \rangle \mid x_r \in M, \ x_r \neq x_s \text{ if } r \neq s \} \subset M^j,$$

with the subspace topology. $F(M; j)$ is a jn-dimensional manifold and $F(M; 1) = M$. Let Σ_j operate on $F(M; j)$ by

$$\langle x_1, \ldots, x_j \rangle \sigma = \langle x_{\sigma(1)}, \ldots, x_{\sigma(j)} \rangle.$$

This operation is free, and $B(M; j)$ denotes the orbit space $F(M; j)/\Sigma_j$.

Fadell and Neuwirth have proven the following theorem.

Theorem 4.4. Let M be an n-dimensional manifold, $n \geq 2$. Let $Y_o = \emptyset$ and $Y_r = \{y_1, \ldots, y_r\}$, $1 \leq r < j$, where the y_i are distinct points of M. Define $\pi_r : F(M - Y_r; j-r) \to M - Y_r$ by $\pi_r \langle x_1, \ldots, x_{j-r} \rangle = x_1$, $0 \leq r < j-1$. Then π_r is a fibration with fibre $F(M - Y_{r+1}; j-r-1)$ over the point y_{r+1}, and π_r admits a cross-section if $r \geq 1$.

Let ${}^rS^n$ denote the wedge of r copies of S^n; since $R^n - Y_r$ is homotopy equivalent to ${}^rS^{n-1}$, the theorem gives the following corollary.

Corollary 4.5. If $n \geq 3$, then $\pi_i F(R^n; j) = \sum_{r=1}^{j-1} \pi_i({}^rS^{n-1})$; $\pi_i F(R^2; j) = 0$ for $i \neq 1$ and $\pi_1 F(R^2; j)$ is constructed from the free groups $\pi_1({}^rS^1)$, $1 \leq r < j$, by successive split extensions.

The case $n = 2$ is classical. $B(R^2; j)$ is a $K(B_j, 1)$, where B_j is the braid group on j strings, and there is a short exact sequence $1 \to I_j \to B_j \to \Sigma_j \to 1$ which is isomorphic to the homotopy exact sequence of the covering projection $F(R^2; j) \to B(R^2; j)$. Detailed descriptions of $I_j = \pi_1 F(R^2; j)$ and of B_j may be found in Artin's paper [3]. Fox and Neuwirth [13] have used $F(R^2; j)$ to rederive Artin's description of B_j in terms of generators and relations.

Let $R^{\infty} = \lim\limits_{\rightarrow} R^n$ with respect to the standard inclusions. Since $F(M; j)$ is functorial on embeddings of manifolds, we can define $F(R^{\infty}; j) = \lim\limits_{\rightarrow} F(R^n; j)$.

Corollary 4.6. $F(R^{\infty}; j)$ is Σ_j-free and contractible.

We shall also need the degenerate case $n = 1$.

Lemma 4.7. $\pi_0 F(R^1; j)$ is isomorphic to Σ_j, and each component of $F(R^1; j)$ is a contractible space.

Proof. Let $F_o = \{(x_1, \ldots, x_j) | \; x_1 < \ldots < x_j\} \subset F(R^1; j)$. F_o is clearly homeomorphic to the interior of a simplex and is therefore contractible. F_o is one component of $F(R^1; j)$, and it is evident that the operation by Σ_j defines a homeomorphism from $F_o \times \Sigma_j$ to $F(R^1; j)$.

Theorem 4.8. For $1 \leq n \leq \infty$ and $j \geq 1$, $\mathcal{C}_n(j)$ is Σ_j-equivariantly homotopy equivalent to $F(R^n; j)$. Therefore $\mathcal{C}_1$ is an A_{∞} operad, $\mathcal{C}_n$ is a locally $(n-2)$-connected Σ-free operad over $\mathcal{N}$ for $1 < n < \infty$, and $\mathcal{C}_{\infty}$ is an E_{∞} operad.

Proof. The second statement will follow immediately from the first statement and the properties of the spaces $F(R^n; j)$. We first consider the case $n < \infty$. For convenience, we may as well replace R^n by J^n. Define a map $g: \mathcal{C}_n(j) \rightarrow F(J^n; j)$ by the formula

(i) $\quad g\langle c_1, \ldots, c_j\rangle = \langle c_1(\gamma), \ldots, c_j(\gamma)\rangle$, where $\gamma = (\frac{1}{2}, \ldots, \frac{1}{2}) \in J^n$.

For $c = \langle c_1, \ldots, c_j\rangle \in \mathcal{C}_n(j)$, write $c_r = c_{r1} \times \ldots \times c_{rn}$, where $c_{rs}: J \rightarrow J$

is given by $c_{rs}(t) = (y_{rs} - x_{rs})t + x_{rs}$. We say that c is equidiameter of diameter d if $y_{rs} - x_{rs} = d$ for all r and s (thus each c_r is actually a cube, and all c_r have the same size). Obviously, for each $b \in F(J^n; j)$, there is some equidiameter $c \in \mathcal{C}_n(j)$ such that $g(c) = b$; we can radially expand the little cubes of this c until some boundaries intersect. Thus define $f: F(J^n; j) \to \mathcal{C}_n(j)$ by the formula

(ii) $f(b) = c$, where $g(c) = b$ and c is the equidiameter element of $\mathcal{C}_n(j)$ with maximal diameter subject to the condition $g(c) = b$.

The continuity of f and g is easily verified by use of Lemma 4.2, and f and g are clearly Σ_j-equivariant. Obviously $gf = 1$. Define $h: \mathcal{C}_n(j) \times I \to \mathcal{C}_n(j)$ as follows. Let $c \in \mathcal{C}_n(j)$ be described as above, and let d be the diameter of $fg(c)$. Then define

$$h(c,u) = \langle \mathop{\times}_{s=1}^{n} c_{1s}(u), \ldots, \mathop{\times}_{s=1}^{n} c_{js}(u) \rangle, \quad 0 \le u \le 1, \quad \text{where}$$

$$c_{rs}(u)(t) = [(1-u)(y_{rs} - x_{rs}) + ud]t + \tfrac{1}{2}(uy_{rs} + (2-u)x_{rs} - ud) \; .$$

In words, h expands or contracts each coordinate interval c_{rs} linearly from its mid-point to a coordinate interval of length d. It is easy to verify that h is well-defined, Σ_j-equivariant, and continuous. Since $h(c,0) = c$, $h(c,1) = fg(c)$, and $h(f(b),u) = f(b)$, we see that $F(J^n; j)$ is in fact a strong Σ_j-equivariant deformation retract of $\mathcal{C}_n(j)$. Now embed J^n in J^{n+1} by $x \to (x, \frac{1}{2})$ and let $\sigma_{n,j}: F(J^n; j) \to F(J^{n+1}; j)$ be the induced inclusion. Write g_n for the map g defined in (i). Then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}_n(j) \arrow[r, "\sigma_{n,j}"] \arrow[d, "g_n"'] & \mathcal{C}_{n+1}(j) \arrow[d, "g_{n+1}"] \\
F(J^n; j) \arrow[r, "\sigma_{n,j}"] & F(J^{n+1}; j)
\end{tikzcd}$$

Thus we can define $g_\infty = \varinjlim g_n \colon \mathcal{C}_\infty(j) \to F(J^\infty; j)$. Clearly $\mathcal{C}_\infty(j)$ has trivial homotopy groups. It is tedious, but not difficult, to verify that $\mathcal{C}_\infty(j)$ is paracompact and $ELC X$ and therefore has the homotopy type of a CW-complex, by Milnor [25, Lemma 4]. Therefore $\mathcal{C}_\infty(j)$ is contractible and g_∞ is a Σ_j-equivariant homotopy equivalence.

We shall later need the following technical lemma, which is an easy consequence of the theorem.

<u>Lemma 4.9</u>. Define $\sigma'_{n-1,j} \colon \mathcal{C}_{n-1}(j) \to \mathcal{C}_n(j)$ by sending each little $(n-1)$-cube f to the little n-cube $1 \times f$, $1 \colon J \to J$. Then $\sigma'_{n-1,j}$ is Σ_j-equivariantly homotopic to $\sigma_{n-1,j}$.

<u>Proof.</u> It suffices to prove that $\sigma_{n-1} \simeq \sigma'_{n-1} \colon F(J^{n-1}; j) \to F(J^n; j)$, where $\sigma_{n-1}(x) = (x, \frac{1}{2})$ and $\sigma'_{n-1}(x) = (\frac{1}{2}, x)$ on points $x \in J^{n-1}$. Define maps $\tau, \tau' \colon F(J^n; j) \to F(J^n; j)$ by the following formulas on points $(s, x) \in J \times J^{n-1} = J^n$:

$$\tau(s,x) = (x,s) \quad \text{and} \quad \tau'(s,x) = \begin{cases} (s,x) & \text{if } n \text{ is odd} \\ (1-s,x) & \text{if } n \text{ is even} \end{cases}$$

then $\tau\sigma'_{n-1} = \sigma_{n-1}$ and $\tau'\sigma'_{n-1} = \sigma'_{n-1}$, hence it suffices to prove that τ is

Σ_j-equivariantly homotopic to τ'. Let

$$\psi_n : (I^n, \partial I^n) \to (S^n, e_o)$$

be the relative homeomorphism defined by Toda [31, p. 5], where $S^n \subset R^{n+1}$ is the standard n-sphere and $e_o = (1, 0, \ldots, 0)$. Toda has observed that, as based maps $S^n \to S^n$,

$$\psi_n \tau \psi_n^{-1}(s_1, \ldots, s_{n+1}) = (s_1, s_3, s_4, \ldots, s_{n+1}, s_2), \text{ and}$$

$$\psi_n \tau' \psi_n^{-1}(s_1, \ldots, s_{n+1}) = (s_1, (-1)^{n-1} s_2, s_3, \ldots, s_{n+1}).$$

Obviously, these maps lie in the same component of $O(n)$ since they both have degree $(-1)^{n-1}$. They are thus connected by a path $k: I \to O(n)$, where $O(n)$ acts as usual on (S^n, e_o). Define $h_t = \psi_n^{-1} k(t) \psi_n : J^n \to J^n$; then $h_o = \tau$ and $h_1 = \tau'$. Since each h_t is a homeomorphism, the product homotopy $(h_t)^j : (J^n)^j \to (J^n)^j$ restricts to give the desired Σ_j-equivariant homotopy $\tau \simeq \tau'$ on $F(J^n; j)$.

Remarks 4.10. Barratt, Mahowald, Milgram, and others (see [24] for a survey) have made extensive calculations in homotopy by use of the quadratic construction $e[S^n, Z_2, X]$ on a space X (see Notations 2.5 for the definition). Since $F(R^{n+1}; 2)$ is Z_2-equivariantly homotopy equivalent to S^n, $e[\mathcal{C}_{n+1}(2), Z_2, X]$ is homotopy equivalent to $e[S^n, Z_2, X]$. For odd primes p, Toda [32] has studied the extended p-th power $e[W^n, Z_p, X]$ on X, where W^n is the n-skeleton of S^∞ with its standard structure of a regular Z_p-free acyclic CW-complex. W^n clearly maps Z_p-equivariantly into $F(R^{n+1}; p)$

and we thus have a map

$$e[W^n, Z_p, X] \to e[F(R^{n+1}, p), \Sigma_p, X] \simeq e[\mathcal{C}_{n+1}(p), \Sigma_p, X].$$

It appears quite likely that the successive quotients $e[\mathcal{C}_n(j), \Sigma_j, X]$ of the filtered space $C_n X$ will also prove to be useful in homotopy theory.

5. Iterated loop spaces and the $\mathcal{C}_n$

We here show that $\mathcal{C}_n$ acts naturally on n-fold loop spaces and that this action leads to a morphism of monads $C_n \to \Omega^n S^n$. The first statement will yield the homology operations on n-fold loop spaces and the second statement is the key to our derivation of the recognition principle from the approximation theorem.

We must first specify our categories of loop spaces precisely. Let $\mathcal{L}_n$, $1 \leq n \leq \infty$, denote the following category of n-fold loop sequences. The objects of $\mathcal{L}_n$ are sequences $\{Y_i \mid 0 \leq i \leq n\}$, or $\{Y_i \mid i \geq 0\}$ if $n = \infty$, such that $Y_i = \Omega Y_{i+1}$ in $\mathcal{T}$. The morphisms of $\mathcal{L}_n$ are sequences $\{g_i \mid 0 \leq i \leq n\}$ or $\{g_i \mid i \geq 0\}$ if $n = \infty$, such that $g_i = \Omega g_{i+1}$ in $\mathcal{T}$. Let $U_n \colon \mathcal{L}_n \to \mathcal{T}$ denote the forgetful functor defined by $U_n\{Y_i\} = Y_o$ and $U_n\{g\} = g_o$. An n-fold loop space or map is a space or map in the image of U_n.

For $n < \infty$, an n-fold loop sequence has the form $\{\Omega^{n-i}Y\}$, and $U_n\{\Omega^{n-i}Y\} = \Omega^n Y$. $\mathcal{L}_n$ serves only to record the fact that the space $\Omega^n Y$ does not determine the space Y and that we must remember Y in order to have a well-defined category of n-fold loop spaces. We shall use the notation $\Omega^n Y$ ambiguously to denote both n-fold loop spaces and sequences, on the understanding that naturality statements refer to $\mathcal{L}_n$. Of course, $\mathcal{L}_n$ is isomorphic to $\mathcal{T}$.

For $n = \infty$, it is more usual to define an infinite loop space to be the initial space Y_o of a bounded Ω-spectrum $\{(Y_i, f_i) \mid i \geq 0\}$ and to define an infinite loop map to be the initial map g_o of a map $\{g_i\}: \{(Y_i, f_i)\} \rightarrow \{(Y_i', f_i')\}$ of bounded Ω-spectra (thus $f_i: Y_i \rightarrow \Omega Y_{i+1}$ is a homotopy equivalence and $\Omega g_{i+1} \cdot f_i$ is homotopic to $f_i' g_i$). The geometric and categorical imprecision of this definition is unacceptable for our purposes. I have proven in [19] that these two notions of infinite loop spaces and maps are entirely equivalent for all purposes of homotopy theory; we can replace bounded Ω-spectra and maps by objects and maps of $\mathcal{L}_\infty$, naturally up to homotopy, and via weak homotopy equivalences on objects. Precise statements and related results may be found in [19].

We regard $\Omega^n X$ as the space of maps $(S^n, *) \rightarrow (X, *)$, where S^n is identified with the quotient space $I^n/\partial I^n$.

Theorem 5.1. For $X \in \mathcal{T}$, define $\theta_{n,j}: \mathcal{C}_n(j) \times (\Omega^n X)^j \rightarrow \Omega^n X$ as follows. Let $c = \langle c_1, \ldots, c_j \rangle \in \mathcal{C}_n(j)$ and let $y = (y_1, \ldots, y_j) \in (\Omega^n X)^j$. Define $\theta_{n,j}(c, y)$ to be $y_r c_r^{-1}$ on $c_r(J^n)$ and to be trivial on the complement of the image of c; thus, for $v \in S^n$

$$\theta_{n,j}(c,y)(v) = \begin{cases} y_r(u) & \text{if } c_r(u) = v \\ * & \text{if } v \notin \operatorname{Im} c \end{cases}$$

Then the $\theta_{n,j}$ define an action θ_n of $\mathcal{C}_n$ on $\Omega^n X$. If $X = \Omega X'$, then $\theta_n = \theta_{n+1}\sigma_n$, where $\sigma_n: \mathcal{C}_n \rightarrow \mathcal{C}_{n+1}$ and θ_{n+1} is the action of $\mathcal{C}_{n+1}$ on $\Omega^{n+1} X'$. If $\{Y_i\} \in \mathcal{L}_\infty$, then the actions θ_n of $\mathcal{C}_n$ on $Y_o = \Omega^n Y_n$ define an action θ_∞ of $\mathcal{C}_\infty$ on Y_o. The actions θ_n, $1 \leq n \leq \infty$, are natural on maps in $\mathcal{L}_n$; precisely, if $W_n: \mathcal{L}_n \to C_n[\mathcal{T}]$ is defined by $W_nY = (U_nY, \theta_n)$ on objects, where $\theta_n: C_nU_nY \to U_nY$ is the C_n-algebra structure map determined by the $\theta_{n,j}$, and by $W_n(g) = U_ng$ on morphisms, then W_n is a functor from n-fold loop sequences to C_n-algebras.

<u>Proof</u>. The $\theta_{n,j}$ are clearly continuous and Σ_j-equivariant, and $\theta_{n,1}(1, y) = y$ is obvious. An easy inspection of the definitions shows that the diagrams of Lemma 1.4(a) commute, and the $\theta_{n,j}$ thus define an action θ_n of $\mathcal{C}_n$ on Ω^nX. If $X = \Omega X'$, then $\Omega^nX = \Omega^{n+1}X'$ via the correspondence $y \leftrightarrow y'$ where $y(u)(t) = y'(u,t)$ for $(u,t) \in I^n \times I$; since $\sigma_n(f) = f \times 1$ on little n-cubes f, $\theta_n = \theta_{n+1}\sigma_n$ follows. If $\{Y_i\} \in \mathcal{L}_\infty$, then $\theta_n = \theta_{n+1}\sigma_n: \mathcal{C}_n \to \mathcal{E}_{Y_o}$ and therefore $\theta_\infty = \varinjlim \theta_n: \mathcal{C}_\infty \to \mathcal{E}_{Y_o}$ is defined. The naturality statement is immediate from the definitions.

We next use the existence of the natural $\mathcal{C}_n$-action θ_n on n-fold loop spaces to produce a morphism of monads $C_n \to \Omega^nS^n$. We require some categorical preliminaries. We have the adjunction

(1) $\quad \phi: \mathrm{Hom}_{\mathcal{T}}(X, \Omega Y) \to \mathrm{Hom}_{\mathcal{T}}(SX, Y)$, $\phi(f)[x, s] = f(x)(s)$,

where $SX = X \times I/* \times I \cup X \times \partial I$ defines the suspension.

By iteration of ϕ, we have the further adjunctions

(2) $\quad \phi^n: \mathrm{Hom}_{\mathcal{T}}(X, \Omega^nY) \to \mathrm{Hom}_{\mathcal{T}}(S^nX, Y)$, $1 \leq n < \infty$.

It is conceptually useful to reinterpret (2) as follows. Define

(3) $\quad Q_nX = \{\Omega^{n-i}S^nX \mid 0 \leq i \leq n\} \in \mathcal{L}_n$; then $U_nQ_nX = \Omega^nS^nX$.

Since a morphism $\{g_i \mid 0 \le i \le n\}$ in $\mathcal{X}_n$ is determined by g_n, we have

$$\mathrm{Hom}_{\mathcal{J}}(S^n X, Y) = \mathrm{Hom}_{\mathcal{X}_n}(Q_n X, \{\Omega^{n-i} Y\}).$$

Therefore (2) may be interpreted as defining an adjunction

(4) $\quad \phi_n : \mathrm{Hom}_{\mathcal{J}}(X, U_n\{\Omega^{n-i}Y\}) \to \mathrm{Hom}_{\mathcal{X}_n}(Q_n X, \{\Omega^{n-i}Y\})$.

The ϕ_n pass to the limit case $n = \infty$. To see this, define

(5) $\quad \sigma_n = \Omega^n \phi^{-1}(1_{S^{n+1}X}) : \Omega^n S^n X \to \Omega^{n+1} S^{n+1} X$

Geometrically, if $\Omega^n S^n X$ is identified with $\mathrm{Hom}_{\mathcal{J}}(S^n, S^n X)$, then

(6) $\quad \sigma_n(f) = Sf = f \wedge 1 : S^n \wedge S^1 = S^{n+1} \to S^{n+1}X = S^n X \wedge S^1$, $\; f : S^n \to S^n X$.

Thus each σ_n is an inclusion, and we can define

(7) $\quad QX = \Omega^\infty S^\infty X = \varinjlim \Omega^n S^n X$, with the topology of the union.

We shall use the alternative notations QX and $\Omega^\infty S^\infty X$ interchangeably. Since a map $S^1 \to QSX$ lands in some $\Omega^n S^{n+1} X$, $\Omega QSX = QX$. Define

(8) $\quad Q_\infty X = \{QS^i X \mid i \ge 0\} \in \mathcal{X}_\infty$; then $U_\infty Q_\infty X = \Omega^\infty S^\infty X$.

If $\{Y_i \mid i \ge 0\} \in \mathcal{X}_\infty$ and if $f : X \to Y_o = U_\infty\{Y_i\}$ is a map in $\mathcal{J}$, then we have the commutative diagrams:

$$\begin{tikzcd}
\Omega^n S^{n+i} X \arrow[rr, "\Omega^n \phi^{n+i}(f)"] \arrow[d, "\sigma_n"'] & & \Omega^n Y_{n+i} \arrow[d, phantom, "\cong"] \\
 & & Y_i \arrow[d, phantom, "\cong"] \\
\Omega^{n+1} S^{n+i+1} X \arrow[rr, "\Omega^{n+1} \phi^{n+i+1}(f)"] & & \Omega^{n+1} Y_{n+i+1}
\end{tikzcd}$$

We therefore have the further adjunction

(9) $\quad \phi_\infty : \mathrm{Hom}_{\mathcal{J}}(X, U_\infty\{Y_i\}) \to \mathrm{Hom}_{\mathcal{X}_\infty}(Q_\infty X, \{Y_i\})$, where

$$\phi_\infty(f)_i = \varinjlim \Omega^n \phi^{n+i}(f) : QS^i X \to Y_i \,,\; i \ge 0, \text{ for } f : X \to Y_o \,.$$

Here $\phi_\infty^{-1}\{g_i\} = g_o \eta_\infty$, where $\eta_\infty: X \to QX$ is the evident inclusion. A pedantic proof that ϕ_∞ is an adjunction, together with categorical relationships between $Q_\infty X$ and the suspension spectrum of X, may be found in [19].

Clearly (4) and (9) state that $Q_n X$, $1 \leq n \leq \infty$, is the free n-fold loop sequence generated by the space X; it is in this sense that the $\Omega^n S^n X$ are free n-fold loop spaces. By Lemma 2.10, our adjunctions ϕ_n yield monads $(\Omega^n S^n, \mu_n, \eta_n)$ and functors $V_n: \mathcal{L}_n \to \Omega^n S^n[\mathcal{J}]$, with $V_n Y = (U_n Y, \xi_n)$ on objects. Explicitly, in terms of iterates of the adjunction ϕ, we have

(10) $\eta_n = \phi^{-n}(1_{S^n X}): X \to \Omega^n S^n X$ if $n < \infty$; $\eta_\infty = \varinjlim \eta_n$.

(11) $\mu_n = \Omega^n \phi^n(1_{\Omega^n S^n X}): \Omega^n S^n \Omega^n S^n X \to \Omega^n S^n X$ if $n < \infty$;

$\mu_\infty = \varinjlim \mu_n$ (which makes sense since $QQX = \varinjlim \Omega^n S^n \Omega^n S^n X$).

(12) $\xi_n = \Omega^n \phi^n(1_{\Omega^n Y}): \Omega^n S^n \Omega^n Y \to \Omega^n Y$ if $n < \infty$;

$\xi_\infty = \varinjlim \Omega^n \phi^n(1_{\Omega^n Y_n}): \Omega^\infty S^\infty Y_o \to Y_o$ for $\{Y_i\} \in \mathcal{L}_\infty$.

By (5), (10), and (11), each $\sigma_n: \Omega^n S^n \to \Omega^{n+1} S^{n+1}$ is a morphism of monads, and $\Omega^\infty S^\infty = \varinjlim \Omega^n S^n$ as a monad.

Let (C_n, μ_n, η_n) denote the monad associated to $\mathcal{C}_n$, and observe that $C_\infty = \varinjlim C_n$ as a monad. With these notations, we have the following theorem, which is in fact a purely formal consequence of Theorem 5.1 and the definitions.

Theorem 5.2. For $X \in \mathcal{J}$ and $1 \leq n \leq \infty$, define $\alpha_n : C_nX \to \Omega^n S^n X$ to be the composite map $C_nX \xrightarrow{C_n\eta_n} C_n\Omega^nS^nX \xrightarrow{\theta_n} \Omega^nS^nX$. Then $\alpha_n : C_n \to \Omega^nS^n$ is a morphism of monads, and the following diagram of functors commutes, where $\alpha_n^*(Y, \xi) = (Y, \xi \cdot \alpha_n)$:

$$\begin{tikzcd} & \mathcal{X}_n \arrow[dl, "V_n"'] \arrow[dr, "W_n"] & \\ \Omega^nS^n[\mathcal{J}] \arrow[rr, "\alpha_n^*"] & & C_n[\mathcal{J}] \end{tikzcd}$$

Moreover, the following diagrams of morphisms of monads are commutative for $n < \infty$, and α_∞ is obtained from the α_n for $n < \infty$ by passage to limits:

$$\begin{tikzcd} C_n \arrow[r, "\alpha_n"] \arrow[d, "\sigma_n"'] & \Omega^nS^n \arrow[d, "\sigma_n"] \\ C_{n+1} \arrow[r, "\alpha_{n+1}"] & \Omega^{n+1}S^{n+1} \end{tikzcd}$$

Proof. The fact that each μ_n, ξ_n, and σ_n for the monad Ω^nS^n is an n-fold loop map and that θ_n is natural on such maps, together with the very definition of a natural transformation and of an algebra over a monad, immediately yield the commutativity of the following diagrams for $X \in \mathcal{J}$:

$$\begin{tikzcd}
 & C_nX \arrow[dr, "C_n\eta_n"] \arrow[drr, "\alpha_n"] & & \\
X \arrow[ur, "\eta_n"] \arrow[dr, "\eta_n"'] & & C_n\Omega^nS^nX \arrow[r, "\theta_n"] & \Omega^nS^nX \\
 & \Omega^nS^nX \arrow[ur, "\eta_n"'] \arrow[urr, "1"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
C_nC_nX \arrow[r, "C_nC_n\eta_n"] \arrow[d, "\mu_n"] & C_nC_n\Omega^nS^nX \arrow[r, "C_n\theta_n"] \arrow[d, "\mu_n"] & C_n\Omega^nS^nX \arrow[r, shift left, "C_n\eta_n"] \arrow[d, "\theta_n"] & C_n\Omega^nS^n\Omega^nS^nX \arrow[l, shift left, dashed, "C_n\mu_n"] \arrow[d, "\theta_n"] \\
C_nX \arrow[r, "C_n\eta_n"] & C_n\Omega^nS^nX \arrow[r, "\theta_n"] & \Omega^nS^nX & \Omega^nS^n\Omega^nS^nX \arrow[l, "\mu_n"']
\end{tikzcd}$$

$$\begin{tikzcd}
C_n\Omega^nX \arrow[r, "C_n\eta_n"] \arrow[dr, "1"'] & C_n\Omega^nS^n\Omega^nX \arrow[r, "\theta_n"] \arrow[d, "C_n\xi_n"] & \Omega^nS^n\Omega^nX \arrow[d, "\xi_n"] \\
 & C_n\Omega^nX \arrow[r, "\theta_n"] & \Omega^nX
\end{tikzcd}$$

$$\begin{tikzcd}
C_nX \arrow[rr, "C_n\eta_n"] \arrow[dd, "\sigma_n"'] & & C_n\Omega^nS^nX \arrow[rr, "\theta_n"] \arrow[dl, "\sigma_n"'] \arrow[dr, "C_n\sigma_n"] & & \Omega^nS^nX \arrow[dd, "\sigma_n"] \\
 & C_{n+1}\Omega^nS^nX \arrow[dr, "C_{n+1}\sigma_n"] & & C_n\Omega^{n+1}S^{n+1}X \arrow[dl, "\sigma_n"'] \arrow[dr, "\theta_n"] & \\
C_{n+1}X \arrow[ur, "C_{n+1}\eta_n"] \arrow[rr, "C_{n+1}\eta_{n+1}"] & & C_{n+1}\Omega^{n+1}S^{n+1}X \arrow[rr, "\theta_{n+1}"] & & \Omega^{n+1}S^{n+1}X
\end{tikzcd}$$

The first diagram gives $\alpha_n\eta_n = \eta_n$, the second gives $\mu_n\alpha_n^2 = \alpha_n\mu_n$ ($C_n\mu_n$ is inserted solely to show commutativity), the third gives $\xi_n\alpha_n = \theta_n$, as required for $\alpha_n^*V_n = W_n$, and the last gives $\sigma_n\alpha_n = \alpha_{n+1}\sigma_n$. The first two diagrams are valid as they stand for $n = \infty$, and the third has an obvious analog in this case; consistency with limits is clear from the last diagram.

We next show that the morphisms of monads $\alpha_n : C_n \to \Omega^n S^n$ factor through $\Omega^i C_{n-i} S^i$ for $1 \leq i < n$. The following elementary categorical observation about adjunctions and monads in any category $\mathcal{J}$ implies that the natural transformations $\Omega^i\alpha_{n-i}S^i : \Omega^i C_{n-i} S^i \to \Omega^n S^n$ are in fact morphisms of monads.

<u>Lemma 5.3.</u> Let $\phi : \mathrm{Hom}_{\mathcal{J}}(X, \Lambda Y) \to \mathrm{Hom}_{\mathcal{J}}(\Sigma X, Y)$ be an adjunction, and let (C, μ, η) be a monad in $\mathcal{J}$. Then $(\Lambda C \Sigma, \tilde{\mu}, \tilde{\eta})$ is a monad in $\mathcal{J}$, where, for $X \in \mathcal{J}$, $\tilde{\mu}$ and $\tilde{\eta}$ are the composites

$$\Lambda C \Sigma \Lambda C \Sigma X \xrightarrow{\Lambda C \phi(1)} \Lambda C C \Sigma X \xrightarrow{\Lambda \mu \Sigma} \Lambda C \Sigma X$$

and

$$X \xrightarrow{\phi^{-1}(1)} \Lambda \Sigma X \xrightarrow{\Lambda \eta \Sigma} \Lambda C \Sigma X .$$

Moreover, if $\psi : C \to C'$ is a morphism of monads, then $\Lambda\psi\Sigma : \Lambda C \Sigma \to \Lambda C' \Sigma$ is also a morphism of monads.

We must still construct morphisms of monads $C_n \to \Omega^i C_{n-i} S^i$, and, by the lemma, it suffices to do this in the case $i = 1$.

Proposition 5.4. For $n > 1$, there is a morphism of monads $\beta_n : C_n \to \Omega C_{n-1} S$ such that $\alpha_n = (\Omega \alpha_{n-1} S)\ \beta_n$. Therefore α_n factors as a composite of morphisms of monads

$$C_n \to \Omega C_{n-1} S \to \dots \to \Omega^{n-1} C_1 S^{n-1} \to \Omega^n S^n .$$

Proof. Define $\beta_n : C_n X \to \Omega C_{n-1} SX$ as follows. Let $c = \langle c_1, \dots, c_j \rangle \in \mathcal{C}_n(j)$, let $x = (x_1, \dots, x_j) \in X^j$, and let $t \in I$. Write $c_r = c'_r \times c''_r$, where $c'_r : J \to J$ and $c''_r : J^{n-1} \to J^{n-1}$. Let $r_1, \dots, r_i$, in order, denote those indices r (if any) such that $t \in c'_r(J)$. Since the c_r have disjoint images, the little $(n-1)$-cubes c''_{r_q}, $1 \le q \le i$, have disjoint images. Thus we can define β_n by the formula

(1) $\qquad \beta_n[c, x](t) = *$ if $t \notin \bigcup_{r=1}^{j} c'_r(J)$, and

$$\beta_n[c, x](t) = [\langle c''_{r_1}, \dots, c''_{r_i} \rangle, [x_{r_1}, s_1], \dots, [x_{r_i}, s_i]]$$

$$\text{if } c'_{r_q}(s_q) = t,\ 1 \le q \le i, \text{ and } t \notin c'_r(J) \text{ for } r \notin \{r_q\} .$$

It is easily verified that β_n is well-defined and continuous. For $v \in S^{n-1}$, formula (1) and Theorem 5.1 give

$$(2) \qquad \Omega\alpha_{n-1} S \circ \beta_n[c, x](t, v) = \begin{cases} [x_r, s, u] & \text{if } c_r(s, u) = (t, v) \\ * & \text{if } (t, v) \notin \operatorname{Im} c. \end{cases}$$

Thus $\Omega\alpha_{n-1} S \circ \beta_n = \alpha_n : C_n X \to \Omega^n S^n X$. The fact that β_n is a morphism of monads can easily be verified from the definitions and also follows from the facts that β_n and $\Omega\alpha_{n-1} S$ are inclusions for all X and that α_n and $\Omega\alpha_{n-1} S$ are morphisms of monads.

We conclude this section with some consistency lemmas relating Theorem 5.1 to the lemmas at the end of section 1. These results will be needed in the study of homology operations; their proofs are easy verifications and will be omitted.

Lemma 5.5. Let $w\colon (\Omega^n X)^{(Y,A)} \to \Omega^n(X^{(Y,A)})$ be the homeomorphism defined by $w(f)(v)(y) = f(y)(v)$ for $y \in Y$ and $v \in S^n$. Then w is a $\mathcal{C}_n$-morphism with respect to the actions $\theta_n^{(Y,A)}$ on $(\Omega^n X)^{(Y,A)}$ and θ_n on $\Omega^n(X^{(Y,A)})$.

In particular, $w\colon (\Omega(\Omega^n X), \Omega\theta_n) \to (\Omega^n(\Omega X), \theta_n)$ is a $\mathcal{C}_n$-morphism, where $\theta_n = \theta_{n+1}\sigma_n$ on $\Omega^n(\Omega X)$. Observe that w transfers the first coordinate of $\Omega^{n+1}X$ (y above) to the last coordinate. Under the identity map on $\Omega^{n+1}X$, $\Omega\theta_n$ corresponds to $\theta_{n+1}\sigma'_n$, and Lemmas 1.5 and 4.9 therefore yield the following result.

Lemma 5.6. For $X \in \mathcal{J}$, the following diagram is commutative, and $\Omega\theta_{n,j} = \theta_{n+1,j} \cdot \sigma'_{n,j}$ is Σ_j-equivariantly homotopic to $\theta_{n,j} = \theta_{n+1,j}\sigma_{n,j}$:

$$\begin{tikzcd}
\mathcal{C}_n(j) \times (\Omega^{n+1}X)^j \arrow[r, "1 \times i^j"] \arrow[d, "\Omega\theta_{n,j}"] & \mathcal{C}_n(j) \times (P\Omega^n X)^j \arrow[r, "1 \times p^j"] \arrow[d, "P\theta_{n,j}"] & \mathcal{C}_n(j) \times (\Omega^n X)^j \arrow[d, "\theta_{n,j}"] \\
\Omega^{n+1}X \arrow[r, "i"] & P\Omega^n X \arrow[r, "p"] & \Omega^n X
\end{tikzcd}$$

Lemma 5.7. Let $f: X \to B$ and $g: Y \to B$ be maps in $\mathcal{J}$. Identify $\Omega^n(X \times^B Y)$ with $\Omega^n X \times^{\Omega^n B} \Omega^n Y$ as subspaces of $\Omega^n X \times \Omega^n Y$. Then the $\mathcal{C}_n$-actions θ_n and $\theta_n \times^{\Omega^n B} \theta_n$ are identical. In particular, θ_n agrees with $\theta_n \times \theta_n$ on $\Omega^n(X \times Y) = \Omega^n X \times \Omega^n Y$.

Remarks 5.8. Lemma 1.9(ii) is obviously inappropriate for the study of the product on n-fold loop spaces for $n < \infty$. Observe that $\Omega^n X$ may be given the product

$$\Omega^{n-1}\phi: \Omega^n X \times \Omega^n X = \Omega^{n-1}(\Omega X \times \Omega X) \to \Omega^n X,$$

where ϕ is the standard product. Clearly $\Omega^{n-1}\phi$ is then a $\mathcal{C}_{n-1}$-morphism. Similarly, we can give $\Omega^n X$ the inverse map $\Omega^{n-1}c: \Omega^n X \to \Omega^n X$, where $c: \Omega X \to \Omega X$ is the standard inverse, and then $\Omega^{n-1}c$ is a $\mathcal{C}_{n-1}$-morphism. The point is that the product and conjugation on $H_*(\Omega^n X)$ will commute with any homology operations which can be derived from the action θ_{n-1} of $\mathcal{C}_{n-1}$ on $\Omega^n X$.

6. The approximation theorem

This section and the next will be devoted to the proof of the approximation theorem (2.7) and related results. The following more detailed statement of the theorem contains an outline of the proof.

Theorem 6.1. For $X \in \mathcal{J}$ and $n \geq 1$, there is a space E_nX which contains C_nX and there are maps $\pi_n: E_nX \to C_{n-1}SX$ and $\tilde{\alpha}_n: E_nX \to P\Omega^{n-1}S^nX$ such that the following diagram commutes:

$$\begin{tikzcd}
C_nX \arrow[r, "\subset"] \arrow[d, "\alpha_n"'] & E_nX \arrow[r, "\pi_n"] \arrow[d, "\tilde{\alpha}_n"] & C_{n-1}SX \arrow[d, "\alpha_{n-1}"] \\
\Omega^nS^nX \arrow[r, "\subset"] & P\Omega^{n-1}S^nX \arrow[r, "p"] & \Omega^{n-1}S^nX
\end{tikzcd}$$

where, if $n = 1$, $C_0SX = SX$ and α_0 is the identity map. E_nX is contractible for all X and π_n is a quasi-fibration with fibre C_nX for all connected X. Therefore α_n is a weak homotopy equivalence for all connected X and all n, $1 \leq n \leq \infty$.

We shall construct the required diagram and give various consequences and addenda to the theorem in this section. The proof that E_nX is contractible and that π_n is a quasi-fibration for connected X will be deferred until the next section, where these results will be seen to be special cases of more general theorems.

Coupled with Propositions 3.4 and 3.10, the theorem yields the following corollaries, which transfer our approximations for $n = 1$ and $n = \infty$ from $\mathcal{C}_1$ and $\mathcal{C}_\infty$ to arbitrary A_∞ and E_∞ operads. The reader should recall that a map is said to be a weak homotopy equivalence if it induces isomorphisms on homotopy groups, and that two spaces X and Y are said to be weakly homotopy equivalent if there are weak homotopy equivalences from some third space Z to both X and Y. Thus the following corollary contains the statement that the James construction MX is naturally of the same weak homotopy type as ΩSX, for connected X; curiously, our proof of this fact uses neither classifying spaces nor associative loop spaces.

Corollary 6.2. Let $X \in \mathcal{J}$ be connected and let $\mathcal{C}$ be any A_∞ operad. Then the following natural maps are all weak homotopy equivalences:

$$MX \xleftarrow{\;\varepsilon\;} CX \xleftarrow{\;\pi_1\;} (C \triangledown C_1)X \xrightarrow{\;\pi_2\;} C_1 X \xrightarrow{\;\alpha_1\;} \Omega SX .$$

Corollary 6.3. Let $X \in \mathcal{J}$ be connected and let $\mathcal{C}$ be any E_∞ operad. Then the following natural maps are all weak homotopy equivalences:

$$CX \xleftarrow{\;\pi_1\;} (C \times C_\infty)X \xrightarrow{\;\pi_2\;} C_\infty X \xrightarrow{\;\alpha_\infty\;} \Omega^\infty S^\infty X$$

and, if $1 \leq n < \infty$, $(C \times C_n)X \xrightarrow{\;\pi_2\;} C_n X \xrightarrow{\;\alpha_n\;} \Omega^n S^n X.$

Of course, for arbitrary (non-connected) X, we can approximate $\Omega^n S^n X$ by $\Omega C_{n-1} SX$, since SX is connected.

Corollary 6.4. Let $X \in \mathcal{T}$ and let $\mathcal{C}$ be any E_∞ operad. Then the following natural maps are all weak homotopy equivalences:

$$\Omega CSX \xleftarrow{\Omega\pi_1 S} \Omega(C \times C_\infty)SX \xrightarrow{\Omega\pi_2 S} \Omega C_\infty SX \xrightarrow{\Omega\alpha_\infty S} \Omega^\infty S^\infty X$$

and, if $1 \leq n < \infty$, $\Omega(C \times C_n)SX \xrightarrow{\Omega\pi_2 S} \Omega C_n SX \xrightarrow{\Omega\alpha_n S} \Omega^{n+1}S^{n+1}X.$

In these corollaries, all maps are evidently given by morphisms of monads. Clearly this implies that these maps are H-maps, but the H-space structure is only one small portion of the total structure preserved.

Remarks 6.5. In [4], Barratt has constructed an approximation $|\Gamma^+X|$ to $\Omega^\infty S^\infty|X|$ for connected simplicial sets X. Implicitly, Barratt constructs a "simplicial operad" consisting of simplicial sets $D_*\Sigma_j$. If we define $\mathcal{D}(j) = |D_*\Sigma_j|$, then we obtain an E_∞ operad $\mathcal{D}$, and it is easily verified that $|\Gamma^+X|$ is homeomorphic to $D|X|$ (where D denotes the monad in $\mathcal{T}$ associated to $\mathcal{D}$). Thus Corollary 6.3 displays an explicit natural weak homotopy equivalence between $|\Gamma^+X|$ and $\Omega^\infty S^\infty|X|$, for connected X. For all X, Γ^+X is a simplicial monoid, and if ΓX denotes the simplicial group generated by Γ^+X, then $|\Gamma X|$ is homotopy equivalent to $\Omega^\infty S^\infty|X|$. We shall describe $\mathcal{D}$ explicitly in section 15.

We begin the proof of Theorem 6.1 with the definition of a functor E_n from pairs (X, A) to spaces. E_nX will be the space $E_n(TX, X)$, where TX denotes the cone on X.

<u>Construction 6.6.</u> Let (X, A) be a pair in $\mathcal{J}$, by which we understand a closed subspace A of X with $* \in A$. We construct a space $E_n(X, A)$ as follows. For a little n-cube f, write $f = f' \times f''$, where $f': J \to J$ and $f'': J^{n-1} \to J^{n-1}$; if $n = 1$, then $f = f'$. Define $\mathcal{E}_n(j; X, A)$ to be the subspace of $\mathcal{C}_n(j) \times X^j$ consisting of all points $(\langle c_1, \ldots, c_j \rangle, x_1, \ldots, x_j)$ such that if $x_r \notin A$, then the intersection in J^n of the sets $(c_r'(0), 1) \times c_r''(J^{n-1})$ and $c_s(J^n)$ is empty for all $s \neq r$. The equivalence relation $\approx$ defined on $\sum_{j \geq 0} \mathcal{C}_n(j) \times X^j$ in the construction, (2.4), of $C_n X$ restricts to an equivalence relation on $\sum_{j \geq 0} \mathcal{E}_n(j; X, A)$. Define $E_n(X, A)$ to be the set

$$E_n(X, A) = \sum_{j \geq 0} \mathcal{E}_n(j; X, A)/(\approx),$$

topologized as a subspace of $C_n X$. Since A is closed in X, $E_n(X, A)$ is closed in $C_n X$ and $E_n(X, A) \in \mathcal{U}$. $E_n(X, A)$ is a filtered space with filtration defined by

$$F_j E_n(X, A) = E_n(X, A) \cap F_j C_n X,$$

and $F_0 E_n(X, A) = *$. Clearly $\mathcal{C}_n(j) \times A^j \subset \mathcal{E}_n(j; X, A)$ and thus $C_n A \subset E_n(X, A)$. If $f: (X, A) \to (X', A')$ is a map of pairs, then $E_n f: E_n(X, A) \to E_n(X', A')$ is defined to be the restriction of $C_n f: C_n X \to C_n X'$ to $E_n(X, A)$.

The following results, particularly Lemmas 6.7 and 6.10, show that the definition of $E_n(X, A)$ is quite naturally dictated by the geometry. Observe that $E_n(X, X) = C_n X$; at the other extreme, $E_n(X, *)$ is closely related to $C_{n-1} X$.

Lemma 6.7. Let $X \in \mathcal{J}$ and let $[c, x] \in E_n(X, *)$, where $c = \langle c_1, \ldots, c_j \rangle$ and $x \in (X - *)^j$ for some $j \geq 1$. Then $j = 1$ if $n = 1$ and $c'' = \langle c_1'', \ldots, c_j'' \rangle \in \mathcal{C}_{n-1}(j)$ if $n > 1$. There is thus a natural surjective based map $v_n : E_n(X, *) \to C_{n-1}X$ defined by the following formulas on points other than $*$:

(1) $v_1[c, x] = x \in X = C_0 X$; and

(2) $v_n[c, x] = [c'', x] \in C_{n-1}X$ if $n > 1$.

Proof. Let $x = (x_1, \ldots, x_j)$, $x_r \in X - *$. Fix $r \neq s$, $1 \leq r \leq j$ and $1 \leq s \leq j$. For definiteness, assume that $c_r'(0) \leq c_s'(0)$. Let $t \in c_s'(J)$. If $n = 1$, then $t \in (c_r(0), 1) \cap c_s(J)$, which contradicts the definition of $\mathcal{E}_1(j; X, *)$; thus $r \neq s$ is impossible if $n = 1$ and therefore $j = 1$. If $n > 1$ and if $v \in c_r''(J^{n-1}) \cap c_s''(J^{n-1})$, then $(t, v) \in c_s(J^n)$ and $t \in (c_r'(0), 1)$, which contradicts the definition of $\mathcal{E}_n(j; X, *)$. Thus the little (n-1)-cubes c_r'' and c_s'' have disjoint images and $c'' \in \mathcal{C}_{n-1}(j)$.

Notations 6.8. Let $\pi : (X, A) \to (Y, *)$ be a map of pairs in $\mathcal{J}$. Then the composite map $E_n(X, A) \xrightarrow{E_n \pi} E_n(Y, *) \xrightarrow{v_n} C_{n-1}Y$ will be denoted π_n. Since E_n is a functor and v_n is a natural transformation, π_n is natural on commutative diagrams

$$\begin{tikzcd}
(X, A) \arrow[r, "\pi"] \arrow[d, "f"'] & (Y, *) \arrow[d, "g"] \\
(X', A') \arrow[r, "\pi'"] & (Y', *)
\end{tikzcd},$$

in the sense that $C_{n-1}g \circ \pi_n = \pi'_n \circ E_n f$ for any such diagram.

Lemma 6.9. For $X \in \mathcal{J}$ and $n \geq 1$, there is a natural commutative diagram

$$\begin{tikzcd}
C_n X \arrow[r, "\subset"] \arrow[d, "C_n \eta_n"'] & E_n(TX, X) \arrow[r, "\pi_n"] \arrow[d, "E_n \tilde{\eta}_n"] & C_{n-1}SX \arrow[d, "C_{n-1}\eta_{n-1}"] \\
C_n \Omega^n S^n X \arrow[r, "\subset"] & E_n(P\Omega^{n-1}S^n X, \Omega^n S^n X) \arrow[r, "p_n"] & C_{n-1}\Omega^{n-1} S^n X
\end{tikzcd}$$

Proof. Define the cone functor T by $TX = X \times I/* \times I \cup X \times 0$, and embed X in TX by $x \to [x, 1]$; $SX = TX/X$ and $\pi: TX \to SX$ denotes the natural map. Define $\tilde{\eta}_n: TX \to P\Omega^{n-1}S^n X$ by the formula

$$\tilde{\eta}_n[x, s](t)(v) = [x, st, v] \quad \text{for } [x, s] \in TX,\ t \in I, \text{ and } v \in S^{n-1}.$$

Then the following diagram commutes and the result follows:

$$\begin{tikzcd}
X \arrow[r, "\subset"] \arrow[d, "\eta_n"'] & TX \arrow[r, "\pi"] \arrow[d, dash, "\tilde{\eta}_n"] & SX \arrow[d, dash, "\eta_{n-1}"] \\
\Omega^n S^n X \arrow[r, "\subset"] & P\Omega^{n-1}S^n X \arrow[r, "p"] & \Omega^{n-1}S^n X
\end{tikzcd}$$

Since α_n factors as the composite $\theta_n \circ C_n \eta_n$, the lemma gives half of the diagram required for Theorem 6.1. The following sharpening of Lemma 5.6 will lead to the other half of the required diagram and will also be needed in the study of homology operations on n-fold loop spaces.

Lemma 6.10. For $X \in \mathcal{T}$, define $\tilde{\theta}_{n,j}\colon \mathcal{E}_n(j; P\Omega^{n-1}X, \Omega^n X) \to P\Omega^{n-1}X$ as follows. Let $(c, y) \in \mathcal{E}_n(j; P\Omega^{n-1}X, \Omega^n X)$, where $c = \langle c_1, \ldots, c_j\rangle$ and $y = (y_1, \ldots, y_j)$. For $t \in I$ and $v \in S^{n-1}$, define

$$\tilde{\theta}_{n,j}(c,y)(t)(v) = \begin{cases} y_r(s)(u) & \text{if } c_r(s,u) = (t,v) \\ y_r(1)(u) & \text{if } t \geq c_r'(1),\ c_r''(u) = v,\ y_r \notin \Omega^n X \\ * & \text{otherwise} \end{cases}$$

Then the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{C}_n(j) \times (\Omega^n X)^j \arrow[r, "\theta_{n,j}"] \arrow[d, "\subset"] & \Omega^n X \arrow[d, "\subset"] \\
\mathcal{E}_n(j; P\Omega^{n-1}X, \Omega^n X) \arrow[r, "\tilde{\theta}_{n,j}"] & P\Omega^{n-1}X \arrow[dd, "p"] \\
\mathcal{C}_{n-1}(j) \times (P\Omega^{n-1}X)^j \arrow[u, "\sigma'_{n-1,j} \times 1^j"] \arrow[ur, "P\theta_{n-1,j}"'] \arrow[d, "1 \times p^j"] & \\
\mathcal{C}_{n-1}(j) \times (\Omega^{n-1}X)^j \arrow[r, "\theta_{n-1,j}"] & \Omega^{n-1}X
\end{tikzcd}$$

Proof. The definition of $\mathcal{E}_n(j; P\Omega^{n-1}X, \Omega^n X)$ gives that if $q \neq r$ and $y_r \notin \Omega^n X$, then no element of $c_q(J^n)$ has the form (t, v) with $t \geq c_r'(1)$ and

$v \in c_r''(J^{n-1})$. Thus the first and second parts of the definition of $\tilde{\theta}_{n,j}$ have disjoint domains. Of course, $y_r(s)(u) = *$ for $u \in \partial I^{n-1}$, and it follows that $\tilde{\theta}_{n,j}$ is continuous. By comparison with Theorem 5.1, $\tilde{\theta}_{n,j} = \theta_{n,j}$ on $\mathcal{C}_n(j) \times (\Omega^n X)^j$. $P\theta_{n-1,j}$ is defined in Lemma 1.5, and the commutativity of the bottom square follows from that lemma. The commutativity of the triangle is immediate from the definitions, since $\sigma'_{n-1,j}$ is given by $f \to 1 \times f$ on little (n-1)-cubes f.

<u>Lemma 6.11.</u> For $X \in \mathcal{T}$, the maps $\tilde{\theta}_{n,j}: \mathcal{E}_n(j; P\Omega^{n-1}X, \Omega^n X) \to P\Omega^{n-1}X$ induce a map $\tilde{\theta}_n: E_n(P\Omega^{n-1}X, \Omega^n X) \to P\Omega^{n-1}X$ such that the following diagram is commutative (where, if $n = 1$, $\theta_o = 1: X \to X$):

$$\begin{tikzcd}
C_n\Omega^n X \arrow[r, "\subset"] \arrow[d, "\theta_n"'] & E_n(P\Omega^{n-1}X, \Omega^n X) \arrow[r, "p_n"] \arrow[d, "\tilde{\theta}_n"] & C_{n-1}\Omega^{n-1}X \arrow[d, "\theta_{n-1}"] \\
\Omega^n X \arrow[r, "\subset"] & P\Omega^{n-1}X \arrow[r, "p"] & \Omega^{n-1}X
\end{tikzcd}$$

<u>Proof.</u> $\tilde{\theta}_{n,j}(c\sigma, y) = \tilde{\theta}_{n,j}(c, \sigma y)$ and $\tilde{\theta}_{n,j-1}(\sigma_i c, y) = \tilde{\theta}_{n,j}(c, s_i y)$, in the notation of Construction 2.4, and therefore $\tilde{\theta}_n$ is well-defined. The previous lemma implies that $\tilde{\theta}_n = \theta_n$ on $C_n\Omega^n X$. Clearly

$$p\tilde{\theta}_n[c, y](v) = \begin{cases} y_r(1)(u) & \text{if } c_r''(u) = v \text{ and } y_r \notin \Omega^n X \\ * & \text{otherwise .} \end{cases}$$

By the definition $p_n = v_n \circ E_n p$ and by the definition of v_n in Lemma 6.7 and of θ_{n-1} in Theorem 5.1, $p\tilde{\theta}_n = \theta_{n-1}p_n$ follows.

Define $\tilde{\alpha}_n = \tilde{\theta}_n \circ E_n\tilde{\eta}_n : E_nX = E_n(TX, X) \to P\Omega^{n-1}S^nX$. Then the commutativity of the diagram in the statement of Theorem 6.1 results from Lemmas 6.9 and 6.11.

We complete this section by showing that our approximations relate nicely to the Hurewicz homomorphism h and to the homotopy and homology suspensions S_*. Recall that we have morphisms of monads $\varepsilon : C_n \to N$, where NX is the infinite symmetric product on X; by abuse, if $n = 1$, ε here denotes the evident composite $C_1 \to M \to N$. For connected spaces X, we may identify $\pi_i(NX)$ with $\tilde{H}_i(X)$, and then $h = \eta_* : \pi_i(X) \to \pi_i(NX)$ and $S_* = \partial^{-1} : \pi_i(NX) \to \pi_{i+1}(NSX)$, where ∂ denotes the connecting homomorphism of the quasi-fibration $N\pi : NTX \to NSX$ with fibre NX; proofs of these results may be found in [10].

Lemma 6.12. Let $\pi : (X, A) \to (Y, *)$ be a map of pairs in $\mathcal{J}$, and let $\tilde{\varepsilon}$ denote the composite $E_n(X, A) \xrightarrow{\subset} C_nX \xrightarrow{\varepsilon} NX$. Then the following diagram is commutative where, if $n = 1$, $\varepsilon = \eta : Y \to NY$.

$$\begin{tikzcd}
C_nA \arrow[r, "\subset"] \arrow[d, "\varepsilon"'] & E_n(X, A) \arrow[r, "\pi_n"] \arrow[d, "\tilde{\varepsilon}"] & C_{n-1}Y \arrow[d, "\varepsilon"] \\
NA \arrow[r, "\subset"] & NX \arrow[r, "N\pi"] & NY
\end{tikzcd}$$

Proof. The commutativity of the left-hand square is obvious and the commutativity of the right-hand square follows easily from the definition of π_n. For $n = 1$, the crucial fact is that at most one coordinate x_r of an element $[\langle c_1, \ldots, c_j\rangle, x_1, \ldots, x_j] \in E_1(X, A)$ is not in A.

<u>Corollary 6.13.</u> Let $X \in \mathcal{J}$ be connected. Then there is a natural commutative diagram, with isomorphisms as indicated:

$$\begin{tikzcd}
 & \pi_i(\Omega^n S^n X) \arrow[r, "\partial^{-n}", "\cong"'] & \pi_{i+n}(S^n X) \arrow[d, equal] \\
\pi_i(X) \arrow[ur, "\eta_{n*}"] \arrow[r, "\eta_{n*}"] \arrow[dr, "\eta_* = h"'] & \pi_i(C_n X) \arrow[u, "\alpha_{n*}", "\cong"'] \arrow[d, "\varepsilon_*"] \arrow[r, "\partial^{-n}", "\cong"'] & \pi_{i+n}(S^n X) \arrow[d, "\eta_* = h"] \\
 & \pi_i(NX) \arrow[r, "\partial^{-n} = S^n_*", "\cong"'] & \pi_{i+n}(NS^n X)
\end{tikzcd}$$

where $\partial^{-n} \circ \eta_{n_*} : \pi_i(X) \to \pi_{i+n}(S^n X)$ is the homotopy suspension.

<u>Proof.</u> The triangles commute since α_n and ε are morphisms of monads. The upper square commutes by the diagrams of Theorem 6.1 and the lower square commutes by the lemma applied to $X \subset TX$ and $\pi : TX \to SX$.

<u>Remark 6.14.</u> Let $M(X,A)$ denote the image of the space $E_1(X,A)$ under the augmentation $\varepsilon : C_1 X \to MX$; Gray [14] has made an intensive study of $M(X,A)$, which he calls $(X,A)_\infty$. The natural map $\pi : X \to X/A$ induces $\pi_1 : E_1(X,A) \to X/A$, and π_1 clearly factors (via ε) through a map $\rho : M(X,A) \to X/A$. If A is connected (and if the pair (X,A) is suitably nice), then, by Theorem 7.3 and [14], π_1 and ρ are quasi-fibrations with respective fibres $C_1 A$ and MA, and $\varepsilon : C_1 A \to MA$ is a weak homotopy equivalence by Proposition 3.4; therefore $\varepsilon : E_1(X,A) \to M(X,A)$ is also a weak homotopy equivalence.

7. Cofibrations and quasi-fibrations

We prove here that $E_n(X,A)$ is aspherical if (X,A) is an NDR-pair such that X is contractible and that, for appropriate NDR-pairs (X,A), the maps $\pi_n: E_n(X,A) \to C_{n-1}(X/A)$ and $C_\infty\pi: C_\infty X \to C_\infty(X/A)$ are quasi-fibrations with respective fibres C_nA and $C_\infty A$. Applied to the pairs (TX,X), these results will complete the proof of the approximation theorem. They will also imply that $\pi_*(C_\infty X)$ is a homology theory on connected spaces X (which, a fortiori, is isomorphic to stable homotopy theory).

Theorem 7.1. Let (X,A) be an NDR-pair in $\mathcal{T}$. Then

(i) $(F_jE_n(X,A), F_{j-1}E_n(X,A))$ is an NDR-pair for $j \geq 1$.

(ii) If X is contractible, then $E_n(X,A)$ is aspherical, and $E_n(X,A)$ is contractible if X is compact, or if X is the cone on A, or if n=1.

Proof. By Lemma A.5 applied to $(X,A,*)$, there is a representation (h,u) of $(X,*)$ as an NDR-pair such that $h(I \times A) \subset A$. By Lemma A.4, (h,u) determines a representation (h_j,u_j) of $(X,*)^j$ as an Σ_j-equivariant NDR-pair. Since any coordinate in A remains in A throughout the homotopy h_j, the representation $(\tilde{h}_j,\tilde{u}_j)$ of $(F_jC_nX, F_{j-1}C_nX)$ as an NDR-pair which was derived from (h_j,u_j) in the proof of Proposition 2.6 restricts to a representation of $(F_jE_n(X,A), F_{j-1}E_n(X,A))$ as an NDR-pair. The contractibility statement is more delicate. Indeed, my first proof was incorrect and the argument to follow is due to Vic Snaith. Let $g: I \times X \to X$ be a contracting homotopy, $g(0,x) = x$, $g(t,*) = *$, and $g(1,x) = *$. Clearly g cannot in general be so chosen that $g(I \times A) \subset A$. For $c = \langle c_1 \ldots, c_j \rangle \in \mathcal{C}_n(j)$, write $c_i = c_i' \times c_i''$, $c_i': J \to J$, and define

$$v_i(c) = 2 \max_{k \neq i} (c_k'(1) - c_i'(1))/\lambda(c), \text{ where } \lambda(c) = \min_{1 \leq k \leq j} (c_k'(1) - c_k'(0)).$$

Define a homotopy $G: I \times (F_jE_n(X,A) - F_{j-1}E_n(X,A)) \to F_jE_n(X,A)$ by

$$G(t,[c,x_1 \ldots, x_j]) = [c, g(t_1,x_1), \ldots, g(t_j,x_j)], \text{ where}$$

$$t_i = \begin{cases} t & \text{if } v_i(c) \le 0 \\ t(1-v_i(c)) & \text{if } 0 \le v_i(c) \le 1 \\ 0 & \text{if } v_i(c) \ge 1 \end{cases}$$

G is well-defined since, as is easily verified, $v_i(c) \le 1$ implies that $(c_i'(0),1) \times c_i''(J^{n-1}) \cap c_k(J^n)$ is empty for all $k \neq i$ (and thus that the i^{th} coordinate in X is unrestricted). G starts at the identity and ends in $F_{j-1}E_n(X,A)$ since $v_i(c) \le 0$ for at least one i and each c. Note, however, that G cannot be extended over all of $F_jE_n(X,A)$. Now assume that there exists $\varepsilon > 0$ such that $g(I \times u^{-1}[0,\varepsilon]) \subset u^{-1}[0,1)$. If X is compact, then there exists such an ε by an easy exercize in point-set topology; if $X = TA$, (j,v) represents $(A,*)$ as an NDR-pair, and $u[a,s] = v(a).s$, $h(t,[a,s]) = [j(t,a),s]$, and $g(t,[a,s]) = [a,s-st]$, then any $\varepsilon < 1$ suffices. Define a homotopy $H: I \times F_jE_n(X,A) \to F_jE_n(X,A)$ by $H(t,z) = z$ for $z \in F_{j-1}E_n(X,A)$ and by

$$H(t,[c,y]) = \begin{cases} G(t,[c,y]) & \text{if } u_j(y) \ge \varepsilon/2 \\ G(2t.u_j(y)/\varepsilon,[c,y]) & \text{if } u_j(y) \le \varepsilon/2 \end{cases}$$

for $[c,y] \in F_jE_n(X,A) - F_{j-1}E_n(X,A)$. Then H deforms $F_jE_n(X,A)$ into $\tilde{u}_j^{-1}[0,1)$ and, by the first part, $\tilde{u}_j^{-1}[0,1)$ can be deformed into $F_{j-1}E_n(X,A)$ in $F_jE_n(X,A)$. It follows that each $F_jE_n(X,A)$ is contractible, and the argument given by Steenrod in [30,9.4] shows that $E_n(X,A)$ is contractible. For arbitrary contractible X, a map $f: S^q \to E_n(X,A)$ has image in $F_jE_n(Y, A \cap Y)$ for some j and some compact $Y \subset X$; if ε is such that $g(I \times u^{-1}[0,\varepsilon] \cap Y) \subset u^{-1}[0,1)$ then the homotopy H above deforms $F_jE_n(Y, A \cap Y)$ into $\tilde{u}^{-1}[0,1)$ in $F_jE_n(X,A)$, and it follows that f is null-homotopic. Thus $E_n(X,A)$ is aspherical. Finally, if $n = 1$, then we can write points of $E_1(X,A)$ in the form $[c,y]$ where the intervals c_i of $c \in \mathcal{C}_1(j)$ are arranged in order (on the line); then the retracting homotopy for $(X,*)^j$ obtained from h_{j-1} on X^{j-1} and g on X by Lemma A.3 can be used to deform $F_jE_1(X,A)$ into $F_{j-1}E_1(X,A)$.

Recall that a map $p: E \to B$ is said to be a quasi-fibration if p is onto and if $p_*: \pi_i(E, p^{-1}(x), y) \to \pi_i(B, x)$ is an isomorphism (of pointed sets or groups) for all $x \in B$, $y \in p^{-1}(x)$, and $i \geq 0$. A subset U of B is said to be distinguished if $p: p^{-1}(U) \to U$ is a quasi-fibration. The following lemma, which results from the statements [10, 2.2, 2.10, and 2.15] of Dold and Thom, describes the basic general pattern for proving that a map is a quasi-fibration.

Lemma 7.2. Let $p: E \to B$ be a map onto a filtered space B. Then each F_jB is distinguished and p is a quasi-fibration provided that

(i) F_oB and every open subset of $F_jB - F_{j-1}B$ for $j > 0$ is distinguished.

(ii) For each $j > 0$, there is an open subset U of F_jB which contains $F_{j-1}B$ and there are homotopies $h_t: U \to U$ and $H_t: p^{-1}(U) \to p^{-1}(U)$ such that

(a) $h_o = 1$, $h_t(F_{j-1}B) \subset F_{j-1}B$, and $h_1(U) \subset F_{j-1}B$;

(b) $H_o = 1$ and H covers h, $pH_t = h_tp$; and

(c) $H_1: p^{-1}(x) \to p^{-1}(h_1(x))$ is a weak homotopy equivalence for all $x \in U$.

The notion of a strong NDR-pair used in the following theorem is defined in the appendix, and it is verified there that (M_f, X) is a strong NDR-pair for any map $f: X \to Y$.

Theorem 7.3. Let (X, A) be a strong NDR-pair in $\mathcal{J}$, and assume that A is connected. Let $\pi: X \to X/A$ be the natural map. Then

(i) $\pi_n: E_n(X, A) \to C_{n-1}(X/A)$ is a quasi-fibration with fibre C_nA;

(ii) $C_\infty\pi: C_\infty X \to C_\infty(X/A)$ is a quasi-fibration with fibre $C_\infty A$.

Proof. (i). The maps π_n are defined in Notations 6.8. For the case $n = 1$, recall that $C_0(X/A) = X/A$ and define $F_0(X/A) = *$ and $F_1(X/A) = X/A$. The proof for $n = 1$ will be exceptional solely in that we need only consider the first filtration, $j = 1$ below, and therefore no special argument will be given. $F_0C_{n-1}(X/A) = *$ is obviously distinguished, and we must first show that any open subset V of $F_jC_{n-1}(X/A) - F_{j-1}C_{n-1}(X/A)$ is distinguished. By use of permutations and the equivalence relation used to define $E_n(X, A)$, and by the definition of π_n, any point $y \in \pi_n^{-1}(V)$ may be written in the following form:

(2) $y = [\langle c, d\rangle, x, a]$, where $c = \langle c_1, \ldots, c_j\rangle \in \mathcal{C}_n(j)$, $d = \langle d_1, \ldots, d_k\rangle \in \mathcal{C}_n(k)$, $x \in (X-A)^j$, and $a \in A^k$; here if $c_r = c_r' \times c_r''$, $c_r' : J \to J$, then the intersection of $(c_r'(0), 1) \times c_r''(J^{n-1})$ and $d_s(J^n)$ is empty, and $\pi_n(y) = [c'', \pi^j(x)] \in V$, where $c'' = \langle c_1'', \ldots, c_j''\rangle \in \mathcal{C}_{n-1}(j)$.

Define $q: \pi_n^{-1}(V) \to C_nA$ by $q(y) = [d, a]$ for y as in (2). It is easy to verify that q is well-defined and continuous. We claim that $\pi_n \times q: \pi_n^{-1}(V) \to V \times C_nA$ is a fibre homotopy equivalence, and this will clearly imply that V is distinguished. Define morphisms of operads $\sigma^+: \mathcal{C}_{n-1} \to \mathcal{C}_n$ and $\tau^-: \mathcal{C}_n \to \mathcal{C}_n$ by the formulas

(3) $\sigma^+(f) = g^+ \times f$ on little $(n-1)$-cubes f, where $g^+(s) = \frac{1}{2}(1+s)$, $g^+(J) = (\frac{1}{2}, 1)$.

(4) $\tau^-(f) = (g^- \times 1^{n-1})f$ on little n-cubes f, where $g^-(s) = \frac{1}{2}s$, $g^-(J) = (0, \frac{1}{2})$.

Then define $w: V \times C_nA \to \pi_n^{-1}(V)$ by the formula

(5) $w([c'', \pi^j(x)], [d, a]) = [<\sigma^+(c''), \tau^-(d)>, x, a]$, where $c'' \in \mathcal{C}_{n-1}(j)$, $x \in (X - A)^j$, $d \in \mathcal{C}_n(k)$, and $a \in A^k$ (for any $k \geq 0$).

The definition of σ^+ and τ^- ensures that the little cubes on the right satisfy the requirements specified in (2) for points of $\pi_n^{-1}(V)$. Clearly w is continuous and fibrewise over V. Now $(\pi_n \times q)\ w$ is the map $1 \times \tau^-$, where $\tau^-: C_nA \to C_nA$ is the associated morphism of monads to $\tau^-: \mathcal{C}_n \to \mathcal{C}_n$. Since $1 \simeq \tau^-$ via the homotopy induced from $f \to (g_t^- \times 1^{n-1})f$ on little n-cubes f, where $g_t^-(s) = (s - \frac{1}{2}st)$, $(\pi_n \times q)w$ is fibre homotopic to the identity map. On points $y \in \pi_n^{-1}(V)$ written as in (2), we have

$$w(\pi_n \times q)(y) = [<\sigma^+(c''), \tau^-(d)>, x, a]$$

Construct a fibre-wise homotopy $1 \simeq w(\pi_n \times q)$ by deforming d into $\tau^-(d)$ as above (without changing c, x, or a) during the first half of the homotopy and then deforming c into $\sigma^+(c'')$ by deforming each c_r' linearly to g^+ (without changing $\tau^-(d), x$, or a) during the second half of the homotopy. It is easily verified that the disjoint images and empty intersections requirements on the little cubes of points of $\pi_n^{-1}(V)$ are preserved throughout the homotopy. Thus $\pi_n \times q$ is a fibre homotopy equivalence and V is distinguished. It remains to construct a neighborhood U of $F_{j-1}C_{n-1}(X/A)$ in $F_jC_{n-1}(X/A)$ and deformations of U and of $\pi_n^{-1}(U)$ which satisfy the conditions of Lemma 7.2(ii). Let (ℓ, v) represent (X, A) as a strong NDR-pair, and let $B = v^{-1}[0, 1)$; by definition, $\ell(I \times B) \subset B$. Define U to be the union of $F_{j-1}C_{n-1}(X/A)$ with

$$\{[c'', \pi(x_1), \ldots, \pi(x_j)] \mid x_r \in B \text{ for at least one index } r\}.$$

Let (h, u) be the representation of $(X/A, *)$ as an NDR-pair induced from (ℓ, v) by π, and let (h_j, u_j) and (ℓ_j, v_j) be the representations of $(X/A, *)^j$ and $(X, A)^j$ as NDR-pairs given by Lemma A. 4. Let $(\tilde{h}_j, \tilde{u}_j)$ be the representation of $(F_jC_{n-1}X/A, F_{j-1}C_{n-1}X/A)$ as an NDR-pair given by Proposition 2.6; then $\tilde{u}_j(x) < 1$ if and only if $x \in U$, and $\tilde{h}_j$ restricts to a strong deformation retraction $\tilde{h}_j: I \times U \to U$ of U onto $F_{j-1}C_{n-1}X/A$. Define $\tilde{\ell}^j: I \times \pi_n^{-1}(U) \to \pi_n^{-1}(U)$ by $\tilde{\ell}^j(t, y) = y$ for $y \in F^{j-1}E_n(X, A)$, where $F^{j-1}E_n(X, A) = \pi_n^{-1}(F_{j-1}C_{n-1}X/A)$, and by the following formula on points $y \in \pi_n^{-1}(U) - F^{j-1}E_n(X, A)$ written in the form (2)

$$\tilde{\ell}^j(t, y) = [\langle c, d\rangle, \ell_j(t, x), a] \tag{6}$$

$\tilde{\ell}^j$ is well-defined since $\ell(t, a) = a$ for $a \in A$, and clearly $\tilde{\ell}^j$ covers $\tilde{h}_j$ and is a strong deformation retraction of $\pi_n^{-1}(U)$ onto $F^{j-1}E_n(X, A)$. By Lemma 7. 2, it suffices to prove that if $x \in U$ and $x' = \tilde{h}_{j1}(x)$, then $\tilde{\ell}^j_1: \pi_n^{-1}(x) \to \pi_n^{-1}(x')$ is a homotopy equivalence. Since $\tilde{\ell}^j$ is constant on $F^{j-1}E_n(X, A)$, this is trivial for $x \in F_{j-1}C_{n-1}(X/A)$. Thus consider a typical element $x \in U - F_{j-1}C_{n-1}(X/A)$, say

$$x = [c'', \pi(x_1), \ldots, \pi(x_j)], \text{ where } c'' = \langle c''_1, \ldots, c''_j\rangle \text{ and } x_r \in X\text{-}A.$$

Let $\ell_{j1}(x_1, \ldots, x_j) = (x'_1, \ldots, x'_j)$. Some of the x'_r lie in A. By use of permutations and the equivalence relation, we may assume that $x'_r \notin A$ for $r \leq i$ and $x'_r \in A$ for $i < r \leq j$ (i may be zero), and then

$$x' = \tilde{h}_{j1}(x) = [\langle c''_1, \ldots, c''_i\rangle, \pi(x'_1), \ldots, \pi(x'_i)].$$

Consider the following diagram:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\pi_n^{-1}(x) \arrow[r, "\tilde{\ell}_1^j"] \arrow[d, shift right, "\pi_n \times q"'] & \pi_n^{-1}(x') \arrow[d, shift right, "\pi_n \times q"'] \\
x \times C_n A \arrow[r, "\tilde{h}_{j1} \times 1"] \arrow[u, shift right, "w"'] & x' \times C_n A \arrow[u, shift right, "w"']
\end{tikzcd}$$

Here the q and w are defined precisely as in the first part of the proof, and $\pi_n \times q$ and w are inverse homotopy equivalences. We shall construct a homotopy $H: I \times (x \times C_n A) \to \pi_n^{-1}(x')$ from $\tilde{\ell}_1^j \circ w$ to $w \circ (\tilde{h}_{j1} \times 1)$. This will imply that $\tilde{\ell}_1^j$ is homotopic to the composite of homotopy equivalences $w \circ (\tilde{h}_{j1} \times 1) \circ (\pi_n \times q)$. Since A is connected, we can choose paths $p_r: I \to A$ connecting x_r' to $*$ for $i < r \leq j$. Define H by the formula

(7) $\quad H(t, x, [d, a]) = [<\sigma^+(c''), \tau^-(d)>, x_1', \ldots, x_i', p_{i+1}(t), \ldots, p_j(t), a]$.

Clearly H is well-defined, and $H_o = \tilde{\ell}_1^j w$ and $H_1 = w\ (\tilde{h}_{j1} \times 1)$ are easily verified from (5) and (6). This completes the proof of (i).

(ii). Define a subspace $\mathcal{E}_n'(j; X, A)$ of $\mathcal{E}_n(j; X, A)$ by

$$\mathcal{E}_n'(j; X, A) = \{(<c_1, \ldots, c_j>, x_1, \ldots, x_j) \mid c_r' = g^+ \text{ if } x_r \notin A\},$$

where g^+ is defined in (3). Let $E_n'(X, A)$ denote the image of $\sum_{j \geq 0} \mathcal{E}_n'(j; X, A)$ in $E_n(X, A)$, and let $\pi_n': E_n'(X, A) \to C_{n-1}(X/A)$ be the restriction of π_n to $E_n'(X, A)$. With a few minor simplifications, the proof of (i) applies to show that π_n' is a quasi-fibration. We have been using $E_n(X, A)$ rather than

$E'_n(X, A)$ since the contractibility proof of Theorem 7.1 does not apply to $E'_n(X, A)$; a fortiori, these spaces are weakly homotopic equivalent and can be used interchangeably. We now have commutative diagrams

$$\begin{tikzcd} E'_n(X,A) \arrow[r, "\sigma_n"] \arrow[d, "\pi'_n"'] & E'_{n+1}(X,A) \arrow[d, "\pi'_{n+1}"] \\ C_{n-1}(X/A) \arrow[r, "\sigma_{n-1}"] & C_n(X/A) \end{tikzcd}$$

and

$$\begin{tikzcd} C_n X \arrow[r, "\sigma_n^+"] \arrow[d, "C_n\pi"'] & E'_{n+1}(X,A) \arrow[r, "i"] \arrow[d, "\pi'_{n+1}"] & C_{n+1}X \arrow[d, "C_{n+1}\pi"] \\ C_n(X/A) \arrow[r, "1"] & C_n(X/A) \arrow[r, "\sigma_n^+"] & C_{n+1}(X/A) \end{tikzcd}$$

where σ_n^+ is defined by $\sigma_n^+(f) = g^+ \times f$ on little n-cubes f, and i is the inclusion. $E'_{n+1}(X, A)$ was introduced in order to ensure that $C_{n+1}\pi \circ i = \sigma_n^+ \circ \pi'_{n+1}$. Lemma 4.9 implies that $\sigma_n \simeq \sigma'_n : C_nX \to C_{n+1}X$, naturally in X, and, since $\sigma'_n(c) = 1 \times c$ on little n-cubes c, we evidently have that $\sigma'_n \simeq \sigma_n^+ : C_nX \to C_{n+1}X$, naturally in X. Now pass these diagrams to limits with respect to the σ_n, observing that $\sigma_{n+1}\sigma_n^+ = \sigma_{n+1}^+\sigma_n$. For $x \in C_\infty(X/A)$ and $y \in (C_\infty\pi)^{-1}(x)$, we have a commutative diagram

$$\begin{array}{ccccc}
(C_\infty X, (C_\infty\pi)^{-1}(x), y) & \xrightarrow{\sigma^+_\infty} & (E'_\infty(X,A), \pi'^{-1}_\infty(x), \sigma^+_\infty y) & \xrightarrow{i} & (C_\infty X, (C_\infty\pi)^{-1}(\sigma^+_\infty x), \sigma^+_\infty y) \\
\Big\downarrow C_\infty\pi & & \Big\downarrow \pi'_\infty & & \Big\downarrow C_\infty\pi \\
(C_\infty(X/A), x) & \xrightarrow{1} & (C_\infty(X/A), x) & \xrightarrow{\sigma^+_\infty} & (C_\infty(X/A), \sigma^+_\infty x)
\end{array}$$

Clearly π'_∞ is still a quasi-fibration; since $\sigma_n \simeq \sigma^+_n$, naturally, both the top composite $i\sigma^+_\infty$ and the bottom map σ^+_∞, as well as π'_∞, induce isomorphisms on homotopy groups (or sets). Since $\pi'_{\infty*}\sigma^+_{\infty*}$ is a monomorphism, so is $(C_\infty\pi)_*$ on the left. Since $\sigma^+_{\infty*}\pi'_{\infty*}$ is an epimorphism, so is $(C_\infty\pi)_*$ on the right. It follows that

$$(C_\infty\pi)_*: \pi_*(C_\infty X, (C_\infty\pi)^{-1}(x), y) \to \pi_*(C_\infty(X/A), x)$$

is an isomorphism for all x and y, which verifies the defining property of a quasi-fibration.

The second part of the theorem has the following consequence.

Corollary 7.4. For any E_∞ operad $\mathcal{C}$, $\pi_*(CX)$ defines a homology theory on connected $X \in \mathcal{J}$ and $\pi_*(\Omega CSX)$ defines a homology theory on all $X \in \mathcal{J}$. These theories are isomorphic to stable homotopy theory, and the morphism of homology theories $\varepsilon_*: \pi_*(CX) \to \pi_*(NX)$ is precisely the stable Hurewicz homomorphism.

Proof. By Proposition 2.6 and the homotopy exact sequence of the quasi-fibrations $C_\infty X \to C_\infty M_f \to C_\infty T_f$, where $T_f = M_f/X$ is the mapping cone of $f: X \to Y$, $\pi_*(C_\infty X)$ satisfies the axioms for a homology theory on connected X. Since suspension preserves cofibrations and looping preserves fibrations, $\pi_*(\Omega C_\infty SX)$ satisfies the axioms for all X. The natural weak homotopy equivalences of Corollaries 6.3 and 6.4 clearly allow us to transfer the result to arbitrary E_∞ operads $\mathcal{C}$, and the maps $(\alpha_\infty)_*$ and $(\Omega\alpha_\infty S)_*$ define explicit isomorphisms with $\pi_*^s(X) = \pi_*(QX)$. The statement about $\mathcal{E}_*$ follows immediately from Corollary 6.13.

8. The smash and composition products

The purpose of this section is to record a number of observations relating the maps $\theta_n : C_n \Omega^n X \to \Omega^n X$ and $\alpha_n : C_n X \to \Omega^n S^n X$ to the smash and composition products, and to make a few remarks about non-connected spaces. The results of this section do not depend on the approximation theorem and are not required elsewhere in this paper; they are important in the applications and illustrate the geometric convenience of the use of the little cubes operads.

We identify $\Omega^n X$ with the space $\mathrm{Hom}_{\mathcal{J}}(S^n, X)$ of based maps $S^n \to X$, $S^n = I^n / \partial I^n$, and we write S for the inclusion $\Omega^n X \to \Omega^{n+1} SX$ given by suspension of maps.

For $X, Y \in \mathcal{J}$, the smash product defines a natural pairing $\Omega^m X \times \Omega^n Y \to \Omega^{m+n}(X \wedge Y)$; explicitly,

$$(f \wedge g)(s,t) = f(s) \wedge g(t)$$

for $f \in \Omega^m X$, $g \in \Omega^n Y$, $s \in I^m$, and $t \in I^n$. Observe that if $m \geq 1$ and if $\phi : \Omega^m X \times \Omega^m X \to \Omega^m X$ denotes the standard (first coordinate) loop product, then, for $f_1, f_2 \in \Omega^m X$ and $g \in \Omega^n Y$, we have the evident distributivity formula

$$\phi(f_1, f_2) \wedge g = \phi(f_1 \wedge g, f_2 \wedge g) .$$

Diagrammatically, this observation gives the following lemma.

Lemma 8.1. For $X, Y \in \mathcal{T}$, the following diagram is commutative:

$$\begin{tikzcd}
\Omega^m X \times \Omega^m X \times \Omega^n Y \arrow[rr, "\phi \times 1"] \arrow[d, "1 \times 1 \times \Delta"'] & & \Omega^m X \times \Omega^n Y \arrow[d, "\wedge"] \\
\Omega^m X \times \Omega^m X \times \Omega^n Y \times \Omega^n Y \arrow[d, "1 \times t \times 1"'] & & \Omega^{m+n}(X \wedge Y) \\
\Omega^m X \times \Omega^n Y \times \Omega^m X \times \Omega^n Y \arrow[rr, "\wedge \times \wedge"] & & \Omega^{m+n}(X \wedge Y) \times \Omega^{m+n}(X \wedge Y), \arrow[u, "\phi"']
\end{tikzcd}$$

where Δ is the diagonal and t is the switch map.

Now the loop products in this diagram are given by $\theta_{m,2}(c)$, where $c = \langle g^- \times 1^{m-1}, g^+ \times 1^{m-1} \rangle \in \mathcal{C}_m(2)$ with g^- and g^+ as defined in formulas (7.3) and (7.4), and the lemma generalizes to the following computationally important result.

Proposition 8.2. For $X, Y \in \mathcal{T}$ and all positive integers $m, n,$ and j, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{C}_m(j) \times (\Omega^m X)^j \times \Omega^n Y \arrow[rr, "\theta_{m,j} \times 1"] \arrow[d, "1 \times 1 \times \Delta"'] & & \Omega^m X \times \Omega^n Y \arrow[d, "\wedge"] \\
\mathcal{C}_m(j) \times (\Omega^m X)^j \times (\Omega^n Y)^j \arrow[d, "1 \times u"'] & & \Omega^{m+n}(X \wedge Y) \\
\mathcal{C}_m(j) \times (\Omega^m X \times \Omega^n Y)^j \arrow[rr, "1 \times \wedge^j"] & & \mathcal{C}_m(j) \times \Omega^{m+n}(X \wedge Y)^j \arrow[u, "\theta_{m,j}"']
\end{tikzcd}$$

where Δ is the iterated diagonal and u is the shuffle map.

Proof. We must verify the formula

$$\theta_{m,j}(c, x_1, \ldots, x_j) \wedge y = \theta_{m,j}(c, x_1 \wedge y, \ldots, x_j \wedge y)$$

for $x_i \in \Omega^m X$, $y \in \Omega^n Y$, and $c = \langle c_1, \ldots, c_j\rangle \in \mathcal{C}_m(j)$. By Theorem 5.1, if $s \in I^m$ and $t \in I^n$, then

$$\theta_{m,j}(c, x_1 \wedge y, \ldots, x_j \wedge y)(s,t) = \begin{cases} x_r(s') \wedge y(t) & \text{if } c_r(s') = s \\ * & \text{if } s \notin \operatorname{Im} c \end{cases} .$$

Visibly, this agrees with $\theta_{m,j}(c, x_1, \ldots, x_j)(s) \wedge y(t)$.

An equally trivial verification shows that we can pull back the smash product along the maps α_n in the sense of the following proposition.

Proposition 8.3. Define a map $\wedge : C_m X \times C_n Y \to C_{m+n}(X \wedge Y)$ by the following formula on points $[c, x] \in C_m X$ and $[d, y] \in C_n Y$, with $c = \langle c_1, \ldots, c_j\rangle \in \mathcal{C}_m(j)$, $x = (x_1, \ldots, x_j) \in X^j$, $d = \langle d_1, \ldots, d_k\rangle \in \mathcal{C}_n(k)$, and $y = (y_1, \ldots, y_k) \in Y^k$:

$$[c, x] \wedge [d, y] = [e, z],$$

where $$e = \langle c_1 \times d_1, \ldots, c_1 \times d_k, \ldots, c_j \times d_1, \ldots, c_j \times d_k\rangle$$

and $$z = (x_1 \wedge y_1, \ldots, x_1 \wedge y_k, \ldots, x_j \wedge y_1, \ldots, x_j \wedge y_k)$$

Then the following diagram is commutative:

$$\begin{tikzcd} C_m X \times C_n Y \arrow[r, "\wedge"] \arrow[d, "\alpha_m \times \alpha_n"'] & C_{m+n}(X \wedge Y) \arrow[d, "\alpha_{m+n}"] \\ \Omega^m S^m X \times \Omega^n S^n Y \arrow[r, "\wedge"] & \Omega^{m+n} S^{m+n}(X \wedge Y) \end{tikzcd}$$

(where we have identified $S^mX \wedge S^nY = X \wedge S^m \wedge Y \wedge S^n$ with $X \wedge Y \wedge S^m \wedge S^n = S^{m+n}(X \wedge Y)$ via the map $1 \wedge t \wedge 1$).

We can stabilize the smash products of the previous proposition, up to homotopy, by use of Lemma 4.9 and the following analogous result on change of coordinates.

Lemma 8.4. Let $X \in \mathcal{J}$. Define $S' : \Omega^{n-1}S^{n-1}X \to \Omega^n S^n X$, $n \geq 1$, by letting $S'f$, $f \in \Omega^{n-1}S^{n-1}X$, be the following composite:

$$S^n = S^1 \wedge S^{n-1} \xrightarrow{1 \wedge f} S^1 \wedge X \wedge S^{n-1} \xrightarrow{t \wedge 1} X \wedge S^1 \wedge S^{n-1} = S^n X.$$

Then S' is homotopic to S, where $Sf = f \wedge 1 : S^n \to S^n X$.

Proof. Let $\tau, \tau' : S^n \to S^n$ and $h : \tau \simeq \tau'$ be the maps and homotopy constructed in the proof of Lemma 4.9. For $f \in \Omega^{n-1}S^{n-1}X$ and $s \in I$, let $H_s(f) : S^n \to S^n X$ be the composite

$$S^n \xrightarrow{h_s^{-1}} S^n \xrightarrow{1 \wedge f} S^1 \wedge X \wedge S^{n-1} \xrightarrow{t \wedge 1} X \wedge S^n \xrightarrow{1 \wedge h_s} X \wedge S^n = S^n X.$$

Then $H_0(f) = f \wedge 1$ and $H_1(f) = (t \wedge 1) \circ (1 \wedge f)$, as required.

Of course, it is now clear that the n suspension maps $\Omega^{n-1}S^{n-1}X \to \Omega^n S^n X$ and $C_{n-1}X \to C_n X$ obtained by the n choices of privileged coordinate are all homotopic. It follows easily that the smash products of Proposition 8.3 are consistent under suspension, up to homotopy, as m and n vary.

We next discuss the composition product. Let $\tilde{F}(n)$ denote the space of based maps $S^n \to S^n$ regarded as a topological monoid under composition of maps. Let $\tilde{F}_i(n)$ denote the component of $\tilde{F}(n)$ consisting of the maps of degree i. As usual, we write

$$F(n) = \tilde{F}_1(n) \cup \tilde{F}_{-1}(n) \quad \text{and} \quad SF(n) = \tilde{F}_1(n).$$

$\tilde{F}(n)$ may be identified with $\Omega^n S^n$, and then, by (5.6), $S: \tilde{F}(n) \to \tilde{F}(n+1)$ agrees with $\sigma_n: \Omega^n S^n \to \Omega^{n+1} S^{n+1}$. We write $\tilde{F}$ for the monoid $\varinjlim \tilde{F}(n)$ and identify $\tilde{F}$ with QS^0 as a space. For $X \in \mathcal{J}$, define

$$c_n: \Omega^n X \times \tilde{F}(n) \to \Omega^n X$$

to be composition of maps. Then c_n is a right action of the monoid $\tilde{F}(n)$ on the space $\Omega^n X$. The diagram

$$\begin{tikzcd}
\Omega^n X \times \tilde{F}(n-1) \arrow[d, "1 \times S"'] \arrow[dr, "c_{n-1}"] & \\
\Omega^n X \times \tilde{F}(n) \arrow[r, "c_n"'] & \Omega^n X
\end{tikzcd}$$

is evidently commutative for all $n \geq 1$. Therefore, if $\{Y_i\} \in \mathcal{L}_\infty$, then the maps

$$c_n: Y_0 \times \tilde{F}(n) = \Omega^n Y_n \times \tilde{F}(n) \to \Omega^n Y_n = Y_0$$

induce a right action $c_\infty: Y_0 \times \tilde{F} \to Y_0$ of $\tilde{F}$ on Y_0. Of course, $c_\infty: QS^0 \times \tilde{F} \to QS^0$ coincides with the composition product on $\tilde{F}$. The composition product enjoys another stability property, which is quite analogous to the result of Lemma 5.6.

Lemma 8.5. For $X \in \mathcal{J}$ and $n \geq 1$, define $Pc_n : P\Omega^n X \times \tilde{F}(n) \to P\Omega^n X$ by $Pc_n(x, f)(t) = c_n(x(t), f)$ for $x \in P\Omega^n X$, $f \in F(n)$, and $t \in I$. Then the restriction Ωc_n of Pc_n to $\Omega^{n+1} X \times \tilde{F}(n)$ is the composite

$$\Omega^{n+1} X \times \tilde{F}(n) \xrightarrow{1 \times S'} \Omega^{n+1} X \times \tilde{F}(n+1) \xrightarrow{c_{n+1}} \Omega^{n+1} X$$

and the following diagram is commutative:

$$\begin{tikzcd}
\Omega^{n+1} X \times \tilde{F}(n) \arrow[r] \arrow[d, "\Omega c_n"'] & P\Omega^n X \times \tilde{F}(n) \arrow[r, "p \times 1"] \arrow[d, "Pc_n"] & \Omega^n X \times \tilde{F}(n) \arrow[d, "c_n"] \\
\Omega^{n+1} X \arrow[r] & P\Omega^n X \arrow[r, "p"] & \Omega^n X
\end{tikzcd}$$

The precise relationship between the smash and composition products is given by the following evident interchange formula.

Lemma 8.6. For $x \in \Omega^m X$, $y \in \Omega^n Y$, $f \in \tilde{F}(m)$, and $g \in \tilde{F}(n)$,

$$c_m(x, f) \wedge c_n(y, g) = c_{m+n}(x \wedge y, f \wedge g)$$

Lemma 8.7. The composition and smash products on $\tilde{F}$ are weakly homotopic, and both products are weakly homotopy commutative.

Proof. For $f \in \tilde{F}(m)$ and $g \in \tilde{F}(n)$, we have the formulas

$$(S')^m g \circ S^n f = f \wedge g = S^n f \circ (S')^m g$$

since $(S')^m g = 1^m \wedge g$ and $S^n f = f \wedge 1^n$. S and S' are homotopic by Lemma 8.4, and the result follows.

We shall obtain an enormous generalization of this lemma in the second paper of this series. There is an E_∞ operad $\mathcal{L}$ such that $\mathcal{L}$ acts on $\widetilde{F}$ (so as to induce the smash product) in such a manner that the composition product $\widetilde{F} \times \widetilde{F} \to \widetilde{F}$ is a morphism of $\mathcal{L}$-spaces.

Of course, there is a distributive law relating the loop product ϕ to the composition product, namely

$$\phi(f_1, f_2) \circ S'g = \phi(f_1 \circ S'g, f_2 \circ S'g)$$

for $f_1, f_2 \in \Omega^n X$ and $g \in \widetilde{F}(n-1)$. Diagrammatically, this gives

Lemma 8.8. For $X \in \mathcal{T}$, the following diagram is commutative

$$\begin{tikzcd}
\Omega^n X \times \Omega^n X \times \widetilde{F}(n-1) \arrow[r, "\phi \times S'"] \arrow[d, "1 \times 1 \times \Delta S'"'] & \Omega^n X \times F(n) \arrow[d, "c_n"] \\
\Omega^n X \times \Omega^n X \times \widetilde{F}(n) \times \widetilde{F}(n) \arrow[d, "1 \times t \times 1"'] & \Omega^n X \\
\Omega^n X \times \widetilde{F}(n) \times \Omega^n X \times \widetilde{F}(n) \arrow[r, "c_n \times c_n"] & \Omega^n X \times \Omega^n X \arrow[u, "\phi"']
\end{tikzcd}$$

The following generalized distributive law is proven, as was Proposition 8.2, simply by writing down the definitions.

Proposition 8.9. For $X \in \mathcal{T}$ and all positive integers $m, n,$ and j, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{C}_m(j) \times (\Omega^{m+n}X)^j \times \tilde{F}(n) \arrow[r, "\theta_{m,j} \times (S')^m"] \arrow[d, "1 \times 1 \times \Delta(S')^m"'] & \Omega^{m+n}X \times \tilde{F}(m+n) \arrow[d, "c_{m+n}"] \\
\mathcal{C}_m(j) \times (\Omega^{m+n}X)^j \times \tilde{F}(m+n)^j \arrow[d, "1 \times u"'] & \Omega^{m+n}X \\
\mathcal{C}_m(j) \times (\Omega^{m+n}X \times \tilde{F}(m+n))^j \arrow[r, "1 \times c^j_{m+n}"] & \mathcal{C}_m(j) \times (\Omega^{m+n}X)^j \arrow[u, "\theta_{m,j}"']
\end{tikzcd}$$

We can pull back the composition product along the approximation maps α_n, but this fact is slightly less obvious. The following reinterpretation of the definition of the maps $\theta_{n,j}$ will aid in the proof.

Lemma 8.10. For $X \in \mathcal{J}$, let ${}^jX$ denote the wedge of j copies of X and let $p: {}^jX \to X$ denote the folding map, the identity on each copy of X. Let $c = \langle c_1, \ldots, c_j \rangle \in \mathcal{C}_n(j)$ and $y = (y_1, \ldots, y_j)$, $y_r \in \Omega^n X$. Then $\theta_{n,j}(c, y): S^n \to X$ is the composite

$$S^n \xrightarrow{\bar{c}} {}^jS \xrightarrow{y_1 \vee \cdots \vee y_j} {}^jX \xrightarrow{p} X,$$

where $\bar{c}$ is the pinch map defined by $\bar{c}(v) = *$ unless $v = c_r(u)$ for some r and u, when $\bar{c}(v) = u$ in the r^{th} copy of S^n.

We next describe C_nS^o and $\alpha_n: C_nS^o \to \Omega^nS^n$; these maps play a central role in the homological applications of our theory.

Lemma 8.11. For any operad $\mathcal{C}$, CS^0 is homeomorphic to the disjoint union of the orbit spaces $\mathcal{C}(j)/\Sigma_j$ for $j \geq 0$.

Proof. If S^0 has points $*$ and 1, then any point of CS^0 other than $*$ can be written in the form $[c, 1^j]$, $c \in \mathcal{C}(j)$.

Lemma 8.12. Consider $\alpha_n : C_n S^0 \to \Omega^n S^n$. For $c \in \mathcal{C}_n(j)$, write $\alpha_n(c) = \alpha_n[c, \overset{j}{1}] \in \tilde{F}_j(n)$. Then $\alpha_n(c)$ is the composite

$$S^n \xrightarrow{\bar{c}} {}^jS^n \xrightarrow{p} S^n$$

Proposition 8.13. Define a map $c_n : C_n X \times C_n S^0 \to C_n X$ by

$$c_n([c, x], d) = [\gamma(d, c^k), x^k],$$

for $c \in \mathcal{C}_n(j)$, $x = (x_1, \ldots, x_j) \in X^j$, and $d \in \mathcal{C}_n(k)$. Then the following diagram is commutative for all n, $1 \leq n \leq \infty$:

$$\begin{array}{ccc} C_n X \times C_n S^0 & \xrightarrow{c_n} & C_n X \\ \Big\downarrow{\scriptstyle \alpha_n \times \alpha_n} & & \Big\downarrow{\scriptstyle \alpha_n} \\ \Omega^n S^n X \times \tilde{F}(n) & \xrightarrow{c_n} & \Omega^n S^n X \end{array}$$

Proof. Let $\eta_n(x) = \eta_n(x_1) \vee \ldots \vee \eta_n(x_j) : {}^jS^n \to {}^jS^n X$, where $\eta_n(x_r)(s) = [x_r, s]$ for $s \in S^n$. Since $\alpha_n = \theta_n \circ C_n \eta_n$, it suffices to verify the commutativity of the following diagram:

$$\begin{tikzcd}
S^n \arrow[r, "\overline{\gamma(d, c^k)}"] \arrow[d, "\bar{d}"'] & {}^{jk}S^n \arrow[r, "{}^k\eta_n(x)"] & {}^{jk}S^nX \arrow[r, "p"] & S^nX \\
{}^kS^n \arrow[r, "p"] & S^n \arrow[r, "\bar{c}"] & {}^jS^n \arrow[r, "\eta_n(x)"] & {}^jS^nX \arrow[u, "p"']
\end{tikzcd}$$

The result follows easily from the definition of γ, in 4.1.

Note that, in contrast to the smash product, the following diagrams are commutative for all n:

$$\begin{tikzcd}
C_nX \times C_nS^0 \arrow[r, "c_n"] \arrow[d, "\sigma_n \times \sigma_n"'] & C_nX \arrow[d, "\sigma_n"] \\
C_{n+1}X \times C_{n+1}S^0 \arrow[r, "c_{n+1}"] & C_{n+1}X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Omega^nS^nX \times F(n) \arrow[r, "c_n"] \arrow[d, "\sigma_n \times S"'] & \Omega^nS^nX \arrow[d, "\sigma_n"] \\
\Omega^{n+1}S^{n+1}X \times F(n+1) \arrow[r, "c_{n+1}"] & \Omega^{n+1}S^{n+1}X
\end{tikzcd}$$

Of course, $\alpha_n: C_nX \to \Omega^nS^nX$ fails to be a weak homotopy equivalence for non-connected spaces X, essentially because $\pi_0(\Omega^nS^nX)$ is a group and we have not built inverses into operads. Conceivably this could be done, but the advantages would be far outweighed by the resulting added complexity. It may be illuminating to compute $\pi_0(\alpha_n): \pi_0(C_0X) \to \pi_0(\Omega^nS^nX)$. Recall that if S is a based set (regarded as a discrete space), then MS (resp. NS) denotes the free monoid (resp., free commutative monoid) generated by S, subject to the relation $* = 1$. Let $\widetilde{M}S$ (resp., $\widetilde{N}S$) denote the free group (resp., free commutative group) generated by S, subject to the relation $* = 1$, and let $i: MS \to \widetilde{M}S$ (resp., $j: NS \to \widetilde{N}S$) denote the evident natural inclusions of monoids.

Proposition 8.14. For $X \in \mathcal{J}$, the horizontal arrows are all isomorphisms of monoids in the commutative diagrams

$$\begin{tikzcd} M\pi_o(X) \arrow[r] \arrow[d, "i"'] & \pi_o(C_1X) \arrow[d, "\pi_o(\alpha_1)"] \\ \widetilde{M}\pi_o(X) \arrow[r] & \pi_o(\Omega SX) \end{tikzcd} \quad \text{and, if } n > 1, \quad \begin{tikzcd} N\pi_o(X) \arrow[r] \arrow[d, "j"'] & \pi_o(C_nX) \arrow[d, "\pi_o(\alpha_n)"] \\ \widetilde{N}\pi_o(X) \arrow[r] & \pi_o(\Omega^n S^n X) \end{tikzcd}$$

Here the horizontal arrows are induced from the set maps

$$\pi_o(\eta_n): \pi_o(X) \to \pi_o(C_nX) \quad \text{and} \quad \pi_o(\eta_n): \pi_o(X) \to \pi_o(\Omega^n S^n X)$$

by the universal properties of the functors $M, N, \widetilde{M}$, and $\widetilde{N}$.

Proof. Fix $b \in \mathcal{C}_n(2)$ (with $b = \langle b_1, b_2 \rangle$ where $b_1(1) \leq b_2(0)$ if $n = 1$); then the product in C_nX may be taken to be

$$[c, x] \cdot [d, y] = [\gamma(b; c, d), x, y]$$

for $c \in \mathcal{C}_n(j)$, $x \in X^j$, $d \in \mathcal{C}_n(k)$, and $y \in X^k$. It follows easily that the image of $\pi_o(X)$ generates $\pi_o(C_nX)$ as a monoid. Thus the top horizontal arrows are epimorphisms and by the diagrams, it suffices to prove that the bottom horizontal arrows are isomorphisms. For $n > 1$, we have the evident chain of isomorphisms

$$\widetilde{N}\pi_o(X) \cong \widetilde{H}_o(X) \cong H_n(S^nX) \cong \pi_n(S^nX) \cong \pi_o(\Omega^n S^n X).$$

For $n = 1$, let X_g denote the component of g, where g runs through a set of points, one from each component of X. Define open subsets U_g of SX by

$$U_* = \{[x, s] \mid x \in X_* \text{ or } S < \tfrac{1}{4} \text{ or } s > \tfrac{3}{4}\}$$

and $$U_g = U_* \cup \{[x, s] \mid x \in X_g\} \qquad \text{for } g \neq *.$$

For $g \neq h$, $U_g \cap U_h = U_*$, and $\pi_1(U_*) = *$ since U_* is homotopy equivalent to SX_*. For $g \neq *$, $\pi_1(U_g)$ is free on one generator, since U_g is homotopy equivalent to $S(X_* \cup X_g)$, and therefore $\pi_1(SX) = \tilde{M}\pi_0(X)$ by the van Kampen theorem.

9. A categorical construction

We shall here introduce a very general categorical "two-sided bar construction". When we pass back to topology via geometric realization of simplicial spaces, this single construction will specialize to yield

(1) A topological monoid weakly homotopy equivalent to any given A_∞ space;

(2) The n-fold de-looping of a $\mathcal{C}_n$-space that is required for our recognition principle;

(3) Stasheff's generalization [28] of the Milgram classifying space of a topological monoid.

The construction also admits a variety of applications outside of topology; in particular, as we shall show in §10, it includes the usual two-sided bar constructions of homological algebra.

Throughout this section, we shall work in the category $\mathcal{S}\mathcal{J}$ of simplicial objects in an arbitrary category $\mathcal{J}$. Since verifications of simplicial identities are important, we recall the definition of simplicial objects and homotopies and then leave such verifications to the diligent reader.

Definition 9.1. An object $X \in \mathcal{S}\mathcal{J}$ is a sequence of objects $X_q \in \mathcal{J}$, $q \geq 0$, together with maps $\partial_i : X_q \to X_{q-1}$ and $s_i : X_q \to X_{q+1}$ in $\mathcal{J}$, $0 \leq i \leq q$, such that

$$\partial_i \partial_j = \partial_{j-1} \partial_i \quad \text{if } i < j$$

$$\partial_i s_j = \begin{cases} s_{j-1} \partial_i & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ s_j \partial_{i-1} & \text{if } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \quad \text{if } i \leq j$$

A map $f: X \to Y$ in $\mathcal{S}\mathcal{T}$ is a sequence $f_q: X_q \to Y_q$ of maps in $\mathcal{T}$ such that $\partial_i f_q = f_{q-1} \partial_i$ and $s_i f_q = f_{q+1} s_i$. A homotopy $h: f \simeq g$ in $\mathcal{S}\mathcal{T}$ between maps $f, g: X \to Y$ consists of maps $h_i: X_q \to Y_{q+1}$, $0 \leq i \leq q$, such that

$$\partial_o h_o = f_q \quad \text{and} \quad \partial_{q+1} h_q = g_q$$

$$\partial_i h_j = \begin{cases} h_{j-1} \partial_i & \text{if } i < j \\ \partial_j h_{j-1} & \text{if } i = j > 0 \\ h_j \partial_{i-1} & \text{if } i > j+1 \end{cases}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j \\ h_j s_{i-1} & \text{if } i > j \end{cases}$$

Thus a purely formal homotopy theory exists in $\mathcal{S}\mathcal{T}$, regardless of the choice of $\mathcal{T}$, and we can meaningfully speak of homotopy equivalences, deformation retracts, etc. When $\mathcal{T}$ is our category of spaces, these notions will translate back to ordinary homotopy theory via geometric realization.

We shall need a few very elementary observations about the relationship between $\mathcal{T}$ and $\mathcal{S}\mathcal{T}$. For $X \in \mathcal{T}$, define $X_* \in \mathcal{S}\mathcal{T}$ by letting $X_q = X$ and letting each ∂_i and s_i be the identity map. For a map $f: X \to X'$ in $\mathcal{T}$, define $f_*: X_* \to X'_*$ in $\mathcal{S}\mathcal{T}$ by $f_q = f$. The following lemma characterizes maps in and out of X_* in $\mathcal{S}\mathcal{T}$.

Lemma 9.2. Let $X \in \mathcal{T}$ and let $Y \in \mathcal{S}\mathcal{T}$. Then

(i) A map $\rho: X \to Y_o$ in $\mathcal{T}$ determines and is determined by the map $\tau_*(\rho): X_* \to Y$ in $\mathcal{S}\mathcal{T}$ defined by $\tau_q(\rho) = s_o{}^q\rho$; if

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "f"'] & Y_o \arrow[d, "g_o"] \\ X' \arrow[r, "\rho'"] & Y'_o \end{tikzcd}$$

is a commutative diagram in $\mathcal{T}$, where $g \in \mathcal{S}\mathcal{T}$, then

$$\begin{tikzcd} X_* \arrow[r, "\tau_*(\rho)"] \arrow[d, "f_*"'] & Y \arrow[d, "g"] \\ X'_* \arrow[r, "\tau_*(\rho')"] & Y' \end{tikzcd}$$

is a commutative diagram in $\mathcal{S}\mathcal{T}$.

(ii) A map $\lambda: Y_o \to X$ in $\mathcal{T}$ such that $\lambda\partial_o = \lambda\partial_1: Y_1 \to X$ determines and is determined by the map $\varepsilon_*(\lambda): Y \to X_*$ in $\mathcal{S}\mathcal{T}$ defined by $\varepsilon_q(\lambda) = \lambda \circ \partial_o{}^q$; if

$$\begin{tikzcd} Y_o \arrow[r, "\lambda"] \arrow[d, "g_o"'] & X \arrow[d, "f"] \\ Y'_o \arrow[r, "\lambda'"] & X' \end{tikzcd}$$

is a commutative diagram in $\mathcal{T}$, where $g \in \mathcal{S}\mathcal{T}$ and both $\lambda\partial_0 = \lambda\partial_1$ and $\lambda'\partial_0 = \lambda'\partial_1$, then

$$\begin{array}{ccc} Y & \xrightarrow{\varepsilon_*(\lambda)} & X_* \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_*} \\ Y' & \xrightarrow{\varepsilon_*(\lambda')} & X'_* \end{array}$$

is a commutative diagram in $\mathcal{S}\mathcal{T}$.

If $F: \mathcal{T} \to \mathcal{T}'$ is a functor, let $F_*: \mathcal{S}\mathcal{T} \to \mathcal{S}\mathcal{T}'$ denote the functor defined on objects $Y \in \mathcal{S}\mathcal{T}$ by $F_qY = F(Y_q)$, with face and degeneracy operators $F(\partial_i)$ and $F(s_i)$. If $\mu: F \to G$ is a natural transformation between functors $\mathcal{T} \to \mathcal{T}'$, let $\mu_*: F_* \to G_*$ denote the natural transformation defined by $\mu_q = \mu$.

Lemma 9.3. Let (C, μ, η) be a monad in $\mathcal{T}$. Then (C_*, μ_*, η_*) is a monad in $\mathcal{S}\mathcal{T}$, and the category $\mathcal{S}C[\mathcal{T}]$ of simplicial C-algebras is isomorphic to the category $C_*[\mathcal{S}\mathcal{T}]$ of C_*-algebras.

Proof. The first part is evident from Definition 2.1. For the second part, an object of either $\mathcal{S}C[\mathcal{T}]$ or $C_*[\mathcal{S}\mathcal{T}]$ consists of an object $X \in \mathcal{S}\mathcal{T}$ together with maps $\xi_q: CX_q \to X_q$ in $\mathcal{T}$ such that $(X_q, \xi_q) \in C[\mathcal{T}]$ and the following diagrams commute:

$$\begin{array}{ccc} CX_q & \xrightarrow{\xi_q} & X_q \\ {\scriptstyle C\partial_i}\downarrow & & \downarrow{\scriptstyle \partial_i} \\ CX_{q-1} & \xrightarrow{\xi_{q-1}} & X_{q-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} CX_q & \xrightarrow{\xi_q} & X_q \\ {\scriptstyle Cs_i}\downarrow & & \downarrow{\scriptstyle s_i} \\ CX_{q+1} & \xrightarrow{\xi_{q+1}} & X_{q+1} \end{array}$$

The point is that the diagrams which state that $\xi: C_*X \to X$ is a map in $\mathcal{S}\mathcal{T}$ are the same as the diagrams which state that each ∂_i and s_i on X is a morphism in $C[\mathcal{T}]$.

We need a new concept in order to make our basic construction.

Definition 9.4. Let (C, μ, η) be a monad in $\mathcal{T}$. A C-functor (F, λ) in a category $\mathcal{V}$ is a functor $F: \mathcal{T} \to \mathcal{V}$ together with a natural transformation of functors $\lambda: FC \to F$ such that the following diagrams are commutative:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FC \arrow[d, "\lambda"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FCC \arrow[r, "F\mu"] \arrow[d, "\lambda"] & FC \arrow[d, "\lambda"] \\ FC \arrow[r, "\lambda"] & C \end{tikzcd}$$

A morphism $\pi: (F, \lambda) \to (F', \lambda')$ of C-functors in $\mathcal{V}$ is a natural transformation $\pi: F \to F'$ such that the following diagram is commutative:

$$\begin{tikzcd} FC \arrow[r, "\pi"] \arrow[d, "\lambda"] & F'C \arrow[d, "\lambda'"] \\ F \arrow[r, "\pi"] & F' \end{tikzcd}$$

This definition should be compared with the defintion of a C-algebra: a monad in $\mathcal{T}$ can act from the left on an object of $\mathcal{T}$ and from the right on a functor with domain $\mathcal{T}$. The following elementary examples will play a central role in all of our remaining work.

<u>Examples 9.5.</u> (i) Let (C, μ, η) be a monad in $\mathcal{T}$. Then (C, μ) is itself a C-functor in $\mathcal{T}$. Since $(CX, \mu) \in C[\mathcal{T}]$ and $\mu: C^2X \to CX$ is a morphism in $C[\mathcal{T}]$ for any $X \in \mathcal{T}$, (C, μ) can also be regarded as a C-functor in $C[\mathcal{T}]$, by abuse of language.

(ii) Let $\alpha: (C, \mu, \eta) \to (D, \nu, \zeta)$ be a morphism of monads in $\mathcal{T}$. Recall that if (X, ξ) is a D-algebra, then $\alpha^*(X, \xi) = (X, \xi \cdot \alpha)$ is a C-algebra. Analogously, if (F, λ) is a D-functor in $\mathcal{T}$, then $\alpha^*(F, \lambda) = (F, \lambda \cdot F\alpha)$ is a C-functor in $\mathcal{T}$, in view of the following commutative diagrams:

$$\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[dr, "F\zeta"] \arrow[ddr, equal] & FC \arrow[d, "F\alpha"] \\
& FD \arrow[d, "\lambda"] \\
& F
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
FCC \arrow[rr, "F\mu"] \arrow[d, "F\alpha"'] & & FC \arrow[d, "F\alpha"] \\
FDC \arrow[r, "FD\alpha"] \arrow[d, "\lambda"'] & FDD \arrow[r, "F\nu"] \arrow[d, "\lambda"] & FD \arrow[d, "\lambda"] \\
FC \arrow[r, "F\alpha"] & FD \arrow[r, "\lambda"] & F
\end{tikzcd}$$

In particular, by (i), $(D, \nu \cdot D\alpha)$ is a C-functor in $D[\mathcal{T}]$; composing $D: \mathcal{T} \to D[\mathcal{T}]$ with $\alpha^*: D[\mathcal{T}] \to C[\mathcal{T}]$, we can also regard $(D, \nu \cdot D\alpha)$ as a C-functor in $C[\mathcal{T}]$. Clearly $\alpha: (C, \mu) \to (D, \nu \cdot D\alpha)$ is then a morphism of C-functors in $C[\mathcal{T}]$.

(iii) Let $\phi: \mathrm{Hom}_{\mathcal{T}}(X, \Lambda Y) \to \mathrm{Hom}_{\mathcal{V}}(\Sigma X, Y)$ be an adjunction between functors $\Lambda: \mathcal{V} \to \mathcal{T}$ and $\Sigma: \mathcal{T} \to \mathcal{V}$. Let $(\Lambda\Sigma, \nu, \zeta)$ be the monad in $\mathcal{T}$ which results by Lemma 2.10; thus $\zeta = \phi^{-1}(1_{\Sigma})$ and $\nu = \Lambda\phi(1_{\Lambda\Sigma})$. Clearly $(\Sigma, \phi(1_{\Lambda\Sigma}))$ is a $\Lambda\Sigma$-functor in $\mathcal{V}$.

(iv) Let $\alpha: (C, \mu, \eta) \to (\Lambda\Sigma, \nu, \zeta)$ be a morphism of monads in $\mathcal{J}$, with $\Lambda\Sigma$ as in (iii). Obviously $\phi(\alpha) = \phi(1) \cdot \Sigma\alpha: \Sigma C \to \Sigma$. Thus, by (ii) and (iii), $(\Sigma, \phi(\alpha))$ is a C-functor in $\mathcal{V}$ and

$$\alpha: (C, \mu) \to (\Lambda\Sigma, \Lambda\phi(\alpha))$$

is a morphism of C-functors in $C[\mathcal{J}]$.

Construction 9.6. Construct a category $\mathcal{B}(\mathcal{J}, \mathcal{V})$ and a functor $B_*: \mathcal{B}(\mathcal{J},\mathcal{V}) \to \mathcal{S}\mathcal{V}$ as follows. The objects of $\mathcal{B}(\mathcal{J}, \mathcal{V})$ are triples

$$((F, \lambda), (C, \mu, \eta), (X, \xi)),$$

abbreviated (F, C, X), where C is a monad in $\mathcal{J}$, F is a C-functor in $\mathcal{V}$ and X is a C-algebra. Define $B_*(F, C, X)$ by

$$B_q(F, C, X) = FC^qX,$$

with face and degeneracy operators given by

$$\begin{aligned}
\partial_0 &= \lambda & &, \quad \lambda: FC^qX \to FC^{q-1}X \\
\partial_i &= FC^{i-1}\mu & &, \quad \mu: C^{q-i+1}X \to C^{q-i}X, \quad 0 < i < q, \\
\partial_q &= FC^{q-1}\xi & &, \quad \xi: CX \to X \\
s_i &= FC^i\eta & &, \quad \eta: C^{q-i}X \to C^{q-i+1}X, \quad 0 \le i \le q
\end{aligned}$$

A morphism $(\pi, \psi, f): (F, C, X) \to (F', C', X')$ in $\mathcal{B}(\mathcal{J}, \mathcal{V})$ is a triple consisting of a morphism $\psi: C \to C'$ of monads in $\mathcal{J}$, a morphism $\pi: F \to \psi^*F'$ of C-functors in $\mathcal{V}$, and a morphism $f: X \to \psi^*X'$ of C-algebras, where ψ^*F' and ψ^*X' are as defined in Example 9.5 (iii). Define

$B_*(\pi, \psi, f)$ by

$$B_q(\pi, \psi, f) = \pi\psi^q f\colon FC^q X \to F'(C')^q X' ;$$

here $\pi\psi^q\colon FC^q \to F'(C')^q$ is a natural transformation of functors $\mathcal{J} \to \mathcal{V}$, and $\pi\psi^q f$ is defined by commutativity of the diagram

$$\begin{tikzcd} FC^q X \arrow[rr, "FC^q f"] \arrow[d, "\pi\psi^q"'] \arrow[drr, "\pi\psi^q f"] & & FC^q X' \arrow[d, "\pi\psi^q"] \\ F'(C')^q X \arrow[rr, "F'(C')^q f"] & & F'(C')^q X' \end{tikzcd}$$

The following observation will be useful in our applications.

Lemma 9.7. Let (F, λ) be a C-functor in $\mathcal{V}$ and let $G\colon \mathcal{V} \to \mathcal{V}'$ be any functor. Then $(GF, G\lambda)$ is a C-functor in $\mathcal{V}'$ and

$$B_*(GF, C, X) = G_* B_*(F, CX)$$

in $\mathcal{S}\mathcal{V}'$ for any C-algebra X.

We next show that, as one would expect, $B_*(C, C, X)$ can be regarded as a "simplicial resolution of X". This special case of our construction was known to Beck [5] and others. The proofs of the following two propositions consist solely of applications of Lemma 9.2 and formal verifications of simplicial identities.

<u>Proposition 9.8.</u> Let (C, μ, η) be a monad in $\mathcal{J}$ and let $(X, \xi) \in C[\mathcal{J}]$. Then $\varepsilon_*(\xi): B_*(C, C, X) \to X_*$ is a morphism in $\mathcal{S}C[\mathcal{J}]$ and $\tau_*(\eta): X_* \to B_*(C, C, X)$ is a morphism in $\mathcal{S}\mathcal{J}$ such that $\varepsilon_*(\xi) \circ \tau_*(\eta) = 1$ on X_*. Define $h_i: B_q(C, C, X) \to B_{q+1}(C, C, X)$, $0 \le i \le q$, by

$$h_i = s_o^{\,i}\eta\partial_o^{\,i}: C^{q+1}X \to C^{q+2}X\ ,\ \eta: C^{q+1-i}X \to C^{q+2-i}X\ .$$

Then h is a homotopy in $\mathcal{S}\mathcal{J}$ from the identity map of $B_*(C, C, X)$ to $\tau_*(\eta)\varepsilon_*(\xi)$, and $h_i \circ \tau_q(\eta) = \tau_{q+1}(\eta)$ for all i. Thus X_* is a strong deformation retract of $B_*(C, C, X)$ in $\mathcal{S}\mathcal{J}$.

Analogously, if for fixed F and C we regard $B_*(F, C, CY)$ as a functor of Y, then this functor can be regarded as a "simplicial resolution of F".

<u>Proposition 9.9.</u> Let (C, μ, η) be a monad in $\mathcal{J}$, let (F, λ) be a C-functor in $\mathcal{V}$, and let $Y \in \mathcal{J}$. Note that $(FY)_* = F_*Y_*$. Then $\varepsilon_*(\lambda): B_*(F, C, CY) \to F_*Y_*$ and $\tau_*(F\eta): F_*Y_* \to B_*(F, C, CY)$ are morphisms in $\mathcal{S}\mathcal{V}$ such that $\varepsilon_*(\lambda) \circ \tau_*(F\eta) = 1$ on F_*Y_*. Define $h_i: B_q(F, C, CY) \to B_{q+1}(F, C, CY)$, $0 \le i \le q$, by

$$h_i = s_q \cdots s_{i+1} \cdot FC^{i+1}\eta \cdot \partial_{i+1} \cdots \partial_q: FC^{q+1}Y \to FC^{q+2}Y,\quad \eta: Y \to CY\ .$$

Then h is a homotopy in $\mathcal{S}\mathcal{V}$ from $\tau_*(F\eta) \circ \varepsilon_*(\lambda)$ to the identity map of $B_*(F, C, CY)$, and $h_i \circ \tau_q(F\eta) = \tau_{q+1}(F\eta)$ for all i. Thus F_*Y_* is a strong deformation retract of $B_*(F, C, CY)$ in $\mathcal{S}\mathcal{V}$.

The following two theorems result by specialization of our previous results to Examples 9.5. In these theorems, we shall be given a morphism of monads $\alpha: C \to D$, and the functors α^* which assign C-algebras and C-functors to D-algebras and D-functors will be omitted from the notations.

The reader should think of α as the augmentation $\varepsilon: C \to M$ of the monad associated to an A_∞ operad, or as one of the morphisms of monads $\alpha_n: C_n \to \Omega^n S^n$, or as the composite of α_n and $\pi_n: C \times C_n \to C_n$, where C is the monad associated to an E_∞ operad.

<u>Theorem 9.10.</u> Let $\alpha: (C, \mu, \eta) \to (D, \nu, \zeta)$ be a morphism of monads in $\mathcal{T}$.

(i) For $(X, \xi) \in C[\mathcal{T}]$, $B_*(D, C, X)$ is a simplicial D-algebra and there are natural morphisms of simplicial C-algebras:

$$X_* \xleftarrow{\varepsilon_*(\xi)} B_*(C, C, X) \xrightarrow{B_*(\alpha, 1, 1)} B_*(D, C, X);$$

$\varepsilon_*(\xi)$ is a strong deformation retraction in $\mathcal{S}\mathcal{T}$ with right inverse $\tau_*(\eta)$ such that $B_*(\alpha, 1, 1) \circ \tau_*(\eta) = \tau_*(\zeta): X_* \to B_*(D, C, X)$.

(ii) For $(X, \xi') \in D[\mathcal{T}]$, there is a natural morphism

$$\varepsilon_*(\xi'): B_*(D, C, X) \to X_*$$

of simplicial D-algebras such that $\varepsilon_*(\xi') \circ \tau_*(\zeta) = 1$ on X_* and such that $\varepsilon_*(\xi') \circ B_*(\alpha, 1, 1) = \varepsilon_*(\xi'\alpha): B_*(C, C, X) \to X_*$.

(iii) For $Y \in \mathcal{J}$, there is a natural strong deformation retraction

$$\varepsilon_*(\nu \circ D\alpha): B_*(D, C, CY) \to D_*Y_*$$

of simplicial D-algebras with right inverse $\tau_*(D\eta)$.

When $D = \Lambda\Sigma$, as in example 9.5, we can "de-lambda" all parts of the theorem above; applied to $\alpha_n: C_n \to \Omega^n S^n$, this fact will lead to the n-fold "de-looping" of C_n-spaces.

Theorem 9.11. Let $\alpha: C \to \Lambda\Sigma$ be a morphism of monads in $\mathcal{J}$, where $\Lambda\Sigma$ results from an adjunction $\emptyset: \mathrm{Hom}_{\mathcal{J}}(X, \Lambda Y) \to \mathrm{Hom}_{\mathcal{V}}(\Sigma X, Y)$.

(i) For $(X, \xi) \in C[\mathcal{J}]$, $B_*(\Lambda\Sigma, C, X) = \Lambda_* B_*(\Sigma, C, X)$.

(ii) For $Y \in \mathcal{V}$, $(\Lambda Y, \Lambda\emptyset(1)) \in \Lambda\Sigma[\mathcal{J}]$ and there is a natural morphism

$$\varepsilon_*\emptyset(1): B_*(\Sigma, C, \Lambda Y) \to Y_*$$

in $\mathcal{S}\mathcal{V}$; $\varepsilon_*(\Lambda\emptyset(1)) = \Lambda_*\varepsilon_*\emptyset(1): \Lambda_* B_*(\Sigma, C, \Lambda Y) \to \Lambda_* Y_*$.

(iii) For $Y \in \mathcal{J}$, there is a natural strong deformation retraction

$$\varepsilon_*\emptyset(\alpha): B_*(\Sigma, C, CY) \to \Sigma_* Y_*$$

in $\mathcal{S}\mathcal{V}$ with right inverse $\tau_*(\Sigma\eta)$, $\eta: Y \to CY$.

Remark 9.12. We have described our basic construction in the form most suitable for the applications. However, as pointed out to me by MacLane, the construction admits a more aesthetically satisfactory symmetric generalization. If C is a monad in $\mathcal{J}$, then a left C-functor (E, ξ) from a category $\mathcal{U}$ is a functor $E: \mathcal{U} \to \mathcal{J}$ together with a natural transformation $\xi: CE \to E$ such that $\xi \circ \mu = \xi \circ C\xi$ and $\xi\eta = 1$; thus it is required that EX admit a natural structure of C-algebra for $X \in \mathcal{U}$. Now we can define $B_*(F, C, E)$, a functor from $\mathcal{U}$ to $\mathcal{V}$ where (F, λ) is a (right) C-functor in $\mathcal{V}$, by

$$B_*(F, C, E)(X) = B_*(F, C, EX)$$

on objects $X \in \mathcal{U}$. Since an object of $\mathcal{J}$ is equivalent to a functor from the unit category (one object, one morphism) to $\mathcal{J}$, our original construction is a special case. In the general context, $B_*(F, C, C)$ is a simplicial resolution of the functor F and $B_*(C, C, E)$ is a simplicial resolution of the functor E.

10. Monoidal categories

The construction of the previous section takes on a more familiar form when specialized to monoids in monoidal categories. We discuss this specialization here in preparation for the study of topological monoids and groups in [21] and for use in section 15.

A (symmetric) monoidal category $(\mathcal{U}, \otimes, *)$ is a category $\mathcal{U}$ together with a bifunctor $\otimes : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ and an object $* \in \mathcal{U}$ such that $\otimes$ is associative (and commutative) and $*$ is a two-sided identity object for $\otimes$, both up to coherent natural isomorphism; a detailed definition may be found in MacLane's paper [17]. For example, a category $\mathcal{U}$ with finite products (and therefore a terminal object $*$, the product of zero objects) is a symmetric monoidal category with its product as $\otimes$; we shall call such a category Cartesian monoidal. Observe that if $\mathcal{U}$ is a (symmetric or Cartesian) monoidal category, then so is $\mathcal{S}\mathcal{U}$, with $\otimes$ defined on objects $X, Y \in \mathcal{S}\mathcal{U}$ by $(X \otimes Y)_q = X_q \otimes Y_q$, $\partial_i = \partial_i \otimes \partial_i$ and $s_i = s_i \otimes s_i$, and with $* = (*)_*$.

A monoid (G, μ, η) in a monoidal category $\mathcal{U}$ is an object $G \in \mathcal{U}$ together with morphisms $\mu: G \otimes G \to G$ and $\eta: * \to G$ such that the following diagrams are commutative:

$$\begin{tikzcd} G \otimes G \otimes G \arrow[r, "1 \otimes \mu"] \arrow[d, "\mu \otimes 1"'] & G \otimes G \arrow[d, "\mu"] \\ G \otimes G \arrow[r, "\mu"] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} * \otimes G \arrow[r, "\eta \otimes 1"] \arrow[dr, "\cong"'] & G \otimes G \arrow[d, "\mu"] & G \otimes * \arrow[l, "1 \otimes \eta"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd}$$

These diagrams show that (G, μ, η) determines a monad in $\mathcal{U}$, which we shall still denote (G, μ, η), by

$$GX = G \otimes X$$

$$\mu(X) = \mu \otimes 1 : \ G \otimes G \otimes X \to G \otimes X$$

$$\eta(X) = \eta \otimes 1 : \ X \cong * \otimes X \to G \otimes X$$

A left G-object (X, ξ) is an object $X \in \mathcal{U}$ together with a map $\xi : G \otimes X \to X$ in $\mathcal{U}$ such that $\xi \eta = 1$ and $\xi(\mu \otimes 1) = \xi(1 \otimes \xi)$. Thus a left G-object is precisely a G-algebra. On the other hand, a right G-object (Y, λ) determines a G-functor in $\mathcal{U}$, which we shall still denote (Y, λ), by

$$YX = Y \otimes X \ \text{ and } \ \lambda(X) = \lambda \otimes 1 : Y \otimes G \otimes X \to Y \otimes X.$$

Thus a triple (Y, G, X) consisting of a monoid G in $\mathcal{U}$ and right and left G objects Y and X naturally determines an object (Y, G, X) of $\mathcal{B}(\mathcal{U}, \mathcal{U})$, and $B_*(Y, G, X)$ is a well-defined simplicial object in $\mathcal{U}$. Of course,

$$B_q(Y, G, X) = YG^qX = Y \otimes G \otimes \ldots \otimes G \otimes X, \ \ q \text{ factors } G,$$

with the familiar face and degeneracy operators

$$\partial_0 = \lambda \otimes 1^q \ , \ \ \partial_i = 1^i \otimes \mu \otimes 1^{q-i} \ \text{ if } \ 0 < i < q \, ,$$

$$\partial_q = 1^q \otimes \xi \ , \ \ s_i = 1^{i+1} \otimes \eta \otimes 1^{q+1-i} \ \text{ if } \ 0 \leq i \leq q \ .$$

Let us write $\mathcal{A}(\mathcal{U})$ for the evident category with objects (Y, G, X), as above. If $\mathcal{U}$ is symmetric and if (Y, G, X) and (Y', G', X') are objects of $\mathcal{A}(\mathcal{U})$, then, with the obvious structural maps, $(Y \otimes Y', G \otimes G', X \otimes X')$ is also an object of $\mathcal{A}(\mathcal{U})$, and we have the following lemma.

Lemma 10.1. Let $\mathcal{U}$ be a symmetric monoidal category and let (Y, G, X) and (Y', G', X') be objects of $\mathcal{A}(\mathcal{U})$. Then there is a commutative and associative natural isomorphism

$$B_*(Y, G, X) \otimes B_*(Y', G', X') \cong B_*(Y \otimes Y',\ G \otimes G',\ X \otimes X')$$

of simplicial objects in $\mathcal{U}$.

Proof. Since $\mathcal{U}$ is symmetric, we have shuffle isomorphisms

$$(X \otimes G^q \otimes Y) \otimes (X' \otimes G'^q \otimes Y') \cong X \otimes X' \otimes (G \otimes G')^q \otimes Y \otimes Y' ,$$

and these are trivially seen to commute with the ∂_i and s_i .

Now suppose that $\mathcal{U}$ is a monoidal category which is also Abelian. Then objects of $\mathcal{S}\mathcal{U}$ determine underlying chain complexes in $\mathcal{U}$ with differential $d = \sum (-1)^i \partial_i$; moreover, if $h: f \simeq g$ is a homotopy in $\mathcal{S}\mathcal{U}$ in the categorical sense of Definition 9.1, then $s = \sum (-1)^i h_i$ is a chain homotopy from f to g in the usual sense, $ds + sd = f - g$, by direct calculation. Therefore, regarding $B_*(Y, G, X)$ as a chain complex in $\mathcal{U}$, we recover the usual unnormalized two-sided bar constructions, together with their contracting homotopies when $X = G$ or $Y = G$. To normalize, we quotient out the sub-complex generated by the images of the degeneracies. Of course, if $\mathcal{U}$ is the category of (graded) modules over a commutative ring R, with $\otimes$ the usual tensor product over R and $* = R$, then a monoid G in $\mathcal{J}$ is an R-algebra and left and right G-objects are just left and right G-modules

When $\mathcal{U}$ is our Cartesian monoidal category of (unbased) topological spaces, geometric realization applied to the simplicial spaces $B_*(Y, G, X)$ will yield a complete theory of associated fibrations to principal G-fibrations for topological monoids G. The following auxiliary categorical observations, which mimic the comparison in [9 , p. 189] between "homogeneous" and "inhomogeneous" resolutions, will be useful in the specialization of this theory to topological groups and will be needed in section 15.

For the remainder of this section, we assume given a fixed Cartesian monoidal category $\mathcal{U}$. For $X \in \mathcal{U}$, let ε denote the unique map $X \to *$ and let $\Delta: X \to X \times X$ denote the diagonal map. A group (G, μ, η, χ) in $\mathcal{U}$ is a monoid (G, μ, η) in $\mathcal{U}$ together with a map $\chi: G \to G$ in $\mathcal{U}$ such that the following diagram commutes:

$$\begin{array}{ccccc} G \times G & & \xrightarrow{1 \times \chi} & & G \times G \\ \uparrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle \mu} \\ G & \xrightarrow{\varepsilon} & G & \xrightarrow{\eta} & G \end{array}$$

<u>Construction 10.2.</u> Define a functor $D_*: \mathcal{U} \to \mathcal{S}\mathcal{U}$ by letting

$$D_q X = X^{q+1} \ ,$$

with face and degeneracy operators given by

$$\partial_i = 1^i \times \varepsilon \times 1^{q-i}: X^{q+1} \to X^i \times * \times X^{q-i} \cong X^q$$

and $$s_i = 1^i \times \Delta \times 1^{q-i}: X^{q+1} \to X^{q+2} \ .$$

For a map $f: X \to Y$ in $\mathcal{U}$, define $D_q f = f^{q+1}$. Observe that D_* preserves products in the sense that the shuffle isomorphisms between $X^{q+1} \times Y^{q+1}$ and $(X \times Y)^{q+1}$ define an associative and commutative natural isomorphism between $D_*X \times D_*Y$ and $D_*(X \times Y)$ in $\mathcal{SU}$. Therefore, if (G, μ, η, χ) is a group in $\mathcal{U}$, then $(D_*G, D_*\mu, D_*\eta, D_*\chi)$ is a group in $\mathcal{SU}$ and if (X, ξ) is a left G-object, then $(D_*X, D_*\xi)$ is a left D_*G-object. By Lemma 9.2, if $\tau_q: X \to X^{q+1}$ is the iterated diagonal, then $\tau_*: X_* \to D_*X$ is a map in $\mathcal{SU}$. If G is a group in $\mathcal{U}$, then $\tau_*: G_* \to D_*G$ is a morphism of groups in $\mathcal{SU}$. In particular, left and right D_*G-objects determine left and right G_*-objects (that is, simplicial G-objects) via τ_*.

<u>Proposition 10.3.</u> Let (G, μ, η, χ) be a group in $\mathcal{U}$. Define

$$\alpha_*: B_*(*, G, G) \to D_*G$$

by letting $\alpha_q: G^{q+1} \to G^{q+1}$ be the map whose i-th coordinate is $\varepsilon^{i-1} \times \mu_{q+2-i}$, $1 \leq i \leq q+1$, where $\mu_j: G^j \to G$ is the iterated product ($\mu_1 = 1$, $\mu_2 = \mu$, $\mu_j = \mu(1 \times \mu_{j-1})$ if $j > 2$). Then α_* is an isomorphism of simplicial right G-objects; α_q^{-1} is the map whose i-th coordinate is $\varepsilon^{i-1} \times \mu(1 \times \chi) \times \varepsilon^{q-i}$ if $1 \leq i \leq q$ and is $\varepsilon^q \times 1$ if $i = q+1$.

<u>Proof.</u> Of course, the proof consists of easy diagram chases, but some readers may prefer to see formulas. Thus suppose that objects of $\mathcal{U}$ have underlying sets and write elements of $B_q(*, G, G)$ and of D_qG in the respective forms

$$[g_1, \ldots, g_q]g_{q+1} \quad \text{and} \quad (g_1, \ldots, g_{q+1}), \quad g_i \in G.$$

Write $\mu(g, g') = gg'$ and $\chi(g) = g^{-1}$. Then we have

$$\alpha_q([g_1, \ldots, g_q]g_{q+1}) = (g_1 g_2 \cdots g_{q+1}, g_2 \cdots g_{q+1}, \ldots, g_q g_{q+1}, g_{q+1})$$

and

$$\alpha_q^{-1}(g_1, \ldots, g_{q+1}) = [g_1 g_2^{-1}, g_2 g_3^{-1}, \ldots, g_q g_{q+1}^{-1}]g_{q+1} \ .$$

Visibly these are inverse functions. For $g \in G$, we have

$([g_1, \ldots, g_q]g_{q+1})g = [g_1, \ldots, g_q]g_{q+1}g$ and $(g_1, \ldots, g_{q+1})g = (g_1 g, \ldots, g_{q+1}g)$,

and α and α^{-1} are thus visibly G-equivariant; they commute with the face and degeneracy operators by similar inspections.

In line with Proposition 9.9 and the previous result we have the following observation.

<u>Proposition 10.4.</u> Let $X \in \mathcal{U}$ and let $\eta : * \to X$ be any map in $\mathcal{U}$. Define $h_i : D_q(X) \to D_{q+1}(X)$, $0 \leq i \leq q$, by the formula

$$h_i = s_0^i(\eta \times 1^{q-i})\partial_0^i : X^q \to X^{q+1} \ .$$

Then h is a strong deformation retraction of $D_*(X)$ onto $(*)_*$.

<u>Proof.</u> Since $*$ is a terminal object in $\mathcal{U}$, $\varepsilon\eta = 1$ on $*$ and $\varepsilon_* \circ \tau_*(\eta) = 1$ on $(*)_*$. It is trivial to verify that h is a homotopy from 1 to $\tau_*(\eta) \circ \varepsilon_*$ such that $h_i \circ \tau_q(\eta) = \tau_{q+1}(\eta)$ for all i.

11. Geometric realization of simplicial spaces

We shall use the technique of geometric realization of simplicial spaces to transfer the categorical constructions of the previous sections into constructions of topological spaces. This technique is an exceedingly natural one and has long been implicitly used in classifying space constructions. Segal [26] appears to have been the first to make the use of this procedure explicit.

In this section and the next, we shall prove a variety of statements to the effect that geometric realization preserves structure; thus we prove here that realization preserves cell structure, products (hence homotopies, groups, etc.), connectivity, and weak homotopy equivalences. Base-points are irrelevant in this section, hence we shall work in the category $\mathcal{U}$ of compactly generated Hausdorff spaces.

Let Δ_q denote the standard topological q-simplex,

$$\Delta_q = \{(t_o, \ldots, t_q) \mid 0 \le t_i \le 1 , \sum t_i = 1\} \subset R^{q+1} .$$

Define $\delta_i : \Delta_{q-1} \to \Delta_q$ and $\sigma_i : \Delta_{q+1} \to \Delta_q$ for $0 \le i \le q$ by

$$\delta_i(t_o, \ldots, t_{q-1}) = (t_o, \ldots, t_{i-1}, 0, t_i, \ldots, t_{q-1})$$

and

$$\sigma_i(t_o, \ldots, t_{q+1}) = (t_o, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{q+1})$$

Definition 11.1. Let $X \in \mathcal{SU}$. Define the geometric realization of X, denoted $|X|$, as follows. Let $\overline{X} = \sum_{q \geq 0} X_q \times \Delta_q$, where $X_q \times \Delta_q$ has the product topology (in $\mathcal{U}$) and $\sum$ denotes disjoint union. Define an equivalence relation $\approx$ on $\overline{X}$ by

$$(\partial_i x, u) \approx (x, \delta_i u) \quad \text{for } x \in X_q,\ u \in \Delta_{q-1},$$

$$(s_i x, u) \approx (x, \sigma_i u) \quad \text{for } x \in X_q,\ u \in \Delta_{q+1}.$$

As a set, $|X| = \overline{X}/(\approx)$. Let $F_q|X|$ denote the image of $\sum_{i=0}^{q} X_i \times \Delta_i$ in $|X|$ and give $F_q|X|$ the quotient topology. Then $F_q|X|$ is a closed subset of $F_{q+1}|X|$, and $|X|$ is given the topology of the union of the $F_q|X|$. The class of $(x, u) \in \overline{X}$ in $|X|$ will be denoted by $|x, u|$. If $f: X \to X'$ is a map in $\mathcal{SU}$, define $|f|: |X| \to |X'|$ by $|f||x, u| = |f(x), u|$. Observe that if each f_q is an inclusion (resp., surjection), then $|f|$ is an inclusion (resp., surjection).

Of course, if X is a simplicial set, then the classical geometric realization of X, due to Milnor, coincides with the geometric realization of X regarded as a discrete simplicial space. Further, if $\widetilde{X}$ denotes the underlying simplicial set of a simplicial space X, then $|X| = |\widetilde{X}|$ as sets and therefore any argument concerning the set theoretical nature of $|\widetilde{X}|$ applies automatically to $|X|$. The following definition will aid in the analysis of the topological properties of $|X|$.

Definition 11.2. Let $X \in \mathcal{SU}$. Define $sX_q = \bigcup_{j=0}^{q} s_j X_q \subset X_{q+1}$. We say that X is proper if each (X_{q+1}, sX_q) is a strong NDR-pair and that X is strictly proper if, in addition, each $(X_{q+1}, s_k X_q)$, $0 \leq k \leq q$, is an NDR-pair via a homotopy $h: I \times X_{q+1} \to X_{q+1}$ such that

$$h(I \times \bigcup_{j=0}^{k-1} s_j X_q) \subset \bigcup_{j=0}^{k-1} s_j X_q .$$

A point $(x, u) \in X_q \times \Delta_q$ is said to be non-degenerate if x is non-degenerate and u is interior (or if $q=0$).

Lemma 11.3. Let $X \in \mathcal{SU}$. Then each point of $\overline{X}$ is equivalent to a unique non-degenerate point. If X is proper, then each $(F_q|X|, F_{q-1}|X|)$ is an NDR-pair, $|X| \in \mathcal{U}$, and $F_q|X|/F_{q-1}|X|$ is homeomorphic to $S^q(X_q/sX_{q-1})$.

Proof. Define $\lambda: \overline{X} \to \overline{X}$ and $\rho: \overline{X} \to \overline{X}$ by the formulas

(1) $\lambda(x, u) = (y, \sigma_{j_1} \ldots \sigma_{j_p} u)$ if $x = s_{j_p} \ldots s_{j_1} y$ where y is non-degenerate and $0 \leq j_1 < \ldots < j_p$; and

(2) $\rho(x, u) = (\partial_{i_1} \ldots \partial_{i_q} x, v)$ if $u = \delta_{i_q} \ldots \delta_{i_1} v$ where v is interior and $0 \leq i_1 < \ldots < i_q$.

By [18, 14.2], the composite $\lambda \circ \rho$ carries each point of $\overline{X}$ into the unique equivalent non-degenerate point. Now

$$F_q|X| - F_{q-1}|X| = (X_q - sX_{q-1}) \times (\Delta_q - \partial\Delta_q).$$

If X is proper, then $(X_q \times \Delta_q, X_q \times \partial\Delta_q \cup sX_{q-1} \times \Delta_q)$ is an NDR-pair by Lemma A.3 and $|X| \in \mathcal{U}$ by [30, 9.2 and 9.4]. There is an evident one-to-one continuous map

$$F_q|X|/F_{q-1}|X| = (X_q \times \Delta_q)/(X_q \times \partial\Delta_q \cup sX_{q-1} \times \Delta_q) \to S^q(X_q/sX_{q-1})$$

determined by $X_q \to X_q/sX_{q-1}$ and any homeomorphism of pairs $(\Delta_q, \partial\Delta_q) \to (I^q, \partial I^q)$; the continuity of the inverse map follows easily from [30, 4.4].

As an immediate consequence of the lemma, we have the following proposition.

Proposition 11.4. Let X be a cellular object of $\mathcal{S}\mathcal{U}$, in the sense that each X_q is a CW-complex and each ∂_i and s_i is a cellular map. Then $|X|$ is a CW-complex with one $(n+q)$-cell for each n-cell of $X_q - sX_{q-1}$. Moreover, if $f: X \to X'$ is a cellular map between cellular objects of $\mathcal{S}\mathcal{U}$ (each f_q is cellular), then $|f|$ is cellular.

As in the case of simplicial sets, geometric realization is a product-preserving functor since we are working in $\mathcal{U}$.

Theorem 11.5. For $X, Y \in \mathcal{U}$, the map $|\pi_1| \times |\pi_2|: |X \times Y| \to |X| \times |Y|$ is a natural homeomorphism. Its inverse ζ is commutative and associative, and is cellular if X and Y are cellular.

Proof. We recall the definition of ζ, which is based on the standard triangulation of $\Delta_p \times \Delta_q$. Consider points

$$u = (t_o, \ldots, t_p) \in \Delta_p \quad \text{and} \quad v = (t_o', \ldots, t_q') \in \Delta_q .$$

Define $u^m = \sum_{i=0}^{m} t_i$, $0 \leq m < p$, and $v^n = \sum_{j=0}^{n} t_j'$, $0 \leq n < q$. Let $w^o \leq \ldots \leq w^{p+q-1}$ be the sequence obtained by ordering the elements of $\{u^m\} \cup \{v^n\}$ and define $w \in \Delta_{p+q}$ by

$$w = (t_o'', \ldots, t_{p+q}''), \text{ where } t_k'' = w^k - w^{k-1},\ w^{-1} = 0 \text{ and } w^{p+q} = 1.$$

Let $i_1 < \ldots < i_q$ and $j_1 < \ldots < j_p$ be disjoint sequences (not uniquely determined) such that $w^{j_s} \in \{u^m\}$ and $w^{i_r} \in \{v^n\}$. Then

$$u = \sigma_{i_1} \ldots \sigma_{i_q} w \quad \text{and} \quad v = \sigma_{j_1} \ldots \sigma_{j_p} w .$$

If $x \in X_p$ and $y \in Y_q$, define

$$\zeta(|x,u|, |y,v|) = |(s_{i_q} \ldots s_{i_1} x,\ s_{j_p} \ldots s_{j_1} y), w| .$$

It is easy to verify that ζ is well-defined and inverse to $|p_1| \times |p_2|$ by use of Lemma 11.3 (compare [18, 14.3]), and the commutativity and associativity of ζ follow formally from the commutativity and associativity of ζ^{-1}. The continuity of ζ, and the cellularity statement, follow from the commutative diagrams:

$$\begin{tikzcd}
X_p \times Y_q \times \Delta_p \times \Delta_q \arrow[r, "1\times t\times 1"] & X_p \times \Delta_p \times Y_q \times \Delta_q \arrow[r, "\pi\times\pi"] & F_p|X| \times F_q|Y| \arrow[d, "\zeta"] \\
X_p \times Y_q \times K(i,j) \arrow[u, "\subset"] \arrow[r, "s^i\times s^j\times\alpha(i,j)"] & X_{p+q} \times Y_{p+q} \times \Delta_{p+q} \arrow[r, "\pi"] & F_{p+q}|X\times Y|
\end{tikzcd}$$

Here $K(i,j)$ denotes the set of points of $\Delta_p \times \Delta_q$ which can determine given sequences $i = \{i_r\}$ and $j = \{j_s\}$ as above, $s^i = s_{i_q} \ldots s_{i_1}$ and $s^j = s_{j_p} \ldots s_{j_1}$, $\alpha(i,j)(u,v) = w$, and the π are quotient maps.

Corollary 11.6. Let $f: X \to B$ and $p: Y \to B$ be maps in $\mathcal{SU}$. Then $|X \times^B Y|$ is naturally homeomorphic to $|X| \times^{|B|} |Y|$, where $(X \times^B Y)_q = \{(x,y) \mid f_q(x) = p_q(y)\} \subset X_q \times Y_q$ gives the fibre product in $\mathcal{SU}$.

Proof. An easy verification shows that the restriction of ζ to $|X| \times^{|B|} |Y|$ takes values in $|X \times^B Y|$ and is inverse to

$$|p_1| \times |p_q| : |X \times^B Y| \to |X| \times^{|B|} |Y| .$$

Corollary 11.7. The geometric realization of a simplicial topological monoid (or group) G is a topological monoid (or group) and is Abelian if G is Abelian.

There are two obvious notions of homotopy in the category $\mathcal{SU}$, namely that of a simplicial map $I_* \times X \to Y$ and that given categorically in Definition 9.1. We now show that geometric realization preserves both types of homotopy.

Lemma 11.8. Let $X \in \mathcal{U}$. Then $|X_*|$ may be identified with X.

Proof. $X = F_o|X_*| = |X_*|$ since all simplices of $X_q = X$ are degenerate for $q > 0$.

Corollary 11.9. If $h: I_* \times X \to Y$ is a map in $\mathcal{S}\mathcal{U}$ and if $h_i: X \to Y$ is defined by $h_{i,q}(x) = h(i,x)$ for $x \in X_q$ and $i = 0$ or $i = 1$, then the composite $I \times |X| \xrightarrow{\zeta} |I_* \times X| \xrightarrow{|h|} |Y|$ is a homotopy between $|h_o|$ and $|h_1|$.

Proof. For $t \in I$, $|h|\zeta(t, |x,u|) = |h(t,x),u|$ by the definition of ζ.

Corollary 11.10. Let $h: f \simeq g$ be a homotopy between maps $f, g: X \to Y$ in $\mathcal{S}\mathcal{U}$, as defined in Definition 9.1. Then h determines a homotopy $\tilde{h}: I \times |X| \to |Y|$ between $|f|$ and $|g|$.

Proof. Let $\Delta[1]$ denote the standard simplicial 1-simplex [18, p. 14], regarded as a discrete simplicial space. By [18, Proposition 6.2, p. 16], if i_1 is the fundamental 1-simplex in $\Delta[1]$ and we define $H: \Delta[1] \times X \to Y$ by

$$H_q(s_{q-1} \cdots s_{i+1} s_{i-1} \cdots s_o i_1, x) = \partial_{i+1} h_i(x) \quad , \quad x \in X_q \, ,$$

then H is a map of simplicial sets, and therefore also of simplicial spaces (since the h_i and ∂_i are continuous). Now $|\Delta[1]|$ is homeomorphic to I and the composite

$$I \times |X| \to |\Delta[1]| \times |X| \xrightarrow{\zeta} |\Delta[1] \times X| \xrightarrow{|H|} |Y|$$

gives the desired homotopy $\tilde{h}$ between $|f|$ and $|g|$.

We next relate the connectivity of the spaces X_q to the connectivity of $|X|$.

Lemma 11.11. For $X \in s\mathcal{U}$, $\pi_o|X| = \pi_o(X_o)/(\sim)$, where $\sim$ is the equivalence relation generated by $[\partial_o x] \sim [\partial_1 x]$ for $x \in X_1$; here $[y]$ denotes the path component of a point $y \in X_o$.

Proof. X determines a simplicial set $\pi_o(X)$ with q-simplices the components of X_q and, by [18, p. 29 and p. 65], our assertion is that $\pi_o|X| = \pi_o|\pi_o(X)|$. If $(x, u) \in X_q \times \Delta_q$, $q > 0$, and if $f: I \to \Delta_q$ is a path connecting u to the point $\delta_o^q \Delta_o$, then the path $\tilde{f}(t) = |x, f(t)|$ in $|X|$ connects $|x, u|$ to a point of $X_o = F_o|X|$. If $x \in X_1$, then $g(t) = |x, (t, 1-t)|$ is a path in $|X|$ connecting $\partial_o x$ to $\partial_1 x$. The result follows easily.

Theorem 11.12. Fix $n \geq 0$. If X is a strictly proper simplicial space such that X_q is $(n-q)$-connected for all $q \leq n$, then $|X|$ is n-connected.

Proof. For $n = 0$, this follows from the lemma. For $n = 1$, we may assume that X_q is connected for $q \geq 2$, since otherwise we can throw away those components of X_q whose intersection with the simplicial subspace of X generated by X_o and X_1 is empty without changing the fundamental group of $|X|$. Then $|\Omega_* X|$ is weakly homotopy equivalent to $\Omega|X|$ by Theorem 12.3 below and therefore $|X|$ is simply connected since $|\Omega_* X|$ is connected. (For technical reasons, this argument does not iterate.) Now assume that $n \geq 2$. By the Hurewicz theorem, it suffices to prove that $\tilde{H}_i|X| = 0$ for $i \leq n$. We claim that $\tilde{H}_i F_q|X| = 0$ for $i \leq n$ and all $q \geq 0$.

$F_o|X| = X_o$ is n-connected, and we assume inductively that $\tilde{H}_i(F_{q-1}|X|) = 0$ for $i \leq n$. Since $(F_q|X|, F_{q-1}|X|)$ is an NDR-pair, we will have that $\tilde{H}_i(F_q|X|) = 0$ for $i \leq n$ provided that $\tilde{H}_i(F_q|X|/F_{q-1}|X|) = 0$ for $i \leq n$. Since $F_q|X|/F_{q-1}|X|$ is homeomorphic to $S^q(X_q/sX_{q-1})$, it suffices to prove that $\tilde{H}_i(X_q/sX_{q-1}) = 0$ for $i \leq n-q$; since X_q is $(n-q)$-connected and (X_q, sX_{q-1}) is an NDR-pair, this in turn will follow if we can prove that $\tilde{H}_i(sX_{q-1}) = 0$ for $i < n-q$. We shall in fact show that

$$\tilde{H}_i\Big(\bigcup_{j=0}^{k} s_jX_{q-1}\Big) = 0 \quad \text{for } i \leq n+1-q \;, \quad 0 \leq k \leq q.$$

We may assume, as part of our induction hypothesis on q, that

$$\tilde{H}_i\Big(\bigcup_{j=0}^{k} s_jX_{q-2}\Big) = 0 \quad \text{for } i \leq n+2-q \text{ and } 0 \leq k \leq q-1.$$

Observe that $s_j : X_{q-1} \to s_jX_{q-1}$ and $\partial_j : s_jX_{j-1} \to X_{q-1}$ are inverse homeomorphisms, $0 \leq j < q$. Thus $\tilde{H}_i(s_jX_{q-1}) = 0$ for $i \leq n+1-q$. Assume inductively that $\tilde{H}_i\Big(\bigcup_{j=0}^{k-1} s_jX_{q-1}\Big) = 0$ for $i \leq n+1-q$. Since X is strictly proper, the excision map

$$\Big(\bigcup_{j=0}^{k-1} s_jX_{q-1}, s_kX_{q-1} \cap \bigcup_{j=0}^{k-1} s_jX_{q-1}\Big) \to \Big(\bigcup_{j=0}^{k} s_jX_{q-1}, s_kX_{q-1}\Big)$$

is a map between NDR-pairs, and we therefore have the M yer-Vietoris exact sequence

$$\cdots \to H_i\Big(\bigcup_{j=0}^{k-1} s_j X_{q-1}\Big) \oplus H_i(s_k X_{q-1}) \to H_i\Big(\bigcup_{j=0}^{k} s_j X_{q-1}\Big)$$
$$\to H_{i-1}\Big(s_k X_{q-1} \cap \bigcup_{j=0}^{k-1} s_j X_{q-1}\Big) \to \cdots$$

If $s_k y = s_j z$ for $j < k$, then $y = \partial_{k+1} s_j z = s_j \partial_k z$; since $s_k s_j = s_j s_{k-1}$, it follows that

$$s_k X_{q-1} \cap \bigcup_{j=0}^{k-1} s_j X_{q-1} = \bigcup_{j=0}^{k-1} s_k s_j X_{q-2}$$

Now $s_k\colon \bigcup_{j=0}^{k-1} s_j X_{q-2} \to \bigcup_{j=0}^{k-1} s_k s_j X_{q-2}$ is a homeomorphism, with inverse ∂_k. By the induction hypothesis and the above exact sequence, $\tilde{H}_i\Big(\bigcup_{j=0}^{k} s_j X_{q-1}\Big) = 0$ for $i \leq n+1-q$, as required.

Theorem 11.13. Let $f\colon X \to Y$ be a simplicial map between strictly proper simplicial spaces. Assume that each f_q is a weak homotopy equivalence and that either $|X|$ and $|Y|$ are simply connected or that $|f|$ is an H-map between connected H-spaces. Then $|f|$ is a weak homotopy equivalence.

Proof. By the Whitehead theorem, it suffices to prove that $|f|$ induces an isomorphism on integral homology. In outline, the proof is the same as that of the previous theorem. One shows that $F_q|f|$ is a homology isomorphism by induction on q and the same sequence of reductions as was used in the previous proof, together with the naturality of Mayer-Vietoris sequences and the five lemma.

We complete this section by recalling a result due to Segal [26] on the spectral sequence obtained from the homology exact couple with respect to an arbitrary homology theory k_* of the filtered space $|X|$, where X is a proper simplicial space. Observe that $k_q(X)$ is a simplicial Abelian group for each fixed q; thus, regarding $k_q(X)$ as a chain complex with $d = \sum (-1)^i (\partial_i)_*$, there is a well-defined homology functor $H_*k_*(X)$ such that $H_pk_q(X)$ is the homology of $k_q(X)$ in degree p. By [18, 22.3], $H_*k_q(X)$ is equal to the homology of the normalized chain complex of $k_q(X)$, and the p-chains of the latter chain complex are easily seen to be isomorphic to $k_q(X_p, sX_{p-1})$.

Theorem 11.14. Let X be a proper simplicial space and let k_* be a homology theory. Then $E^2_{pq}X = H_pk_q(X)$ in the spectral sequence $\{E^rX\}$ derived from the k_* exact couple of the filtered space $|X|$.

Proof. $E^1_{pq}X = k_{p+q}(F_p|X|, F_{p-1}|X|)$, and d^1 is the boundary operator of the triple $(F_p|X|, F_{p-1}|X|, F_{p-2}|X|)$. The result follows from Lemma 11.3 and the following commutative diagram:

$$\begin{tikzcd}
k_q(X_p) \arrow[r, "S^p_*"] \arrow[ddd, "\sum(-1)^i"'] & k_{p+q}(X_p \times \Delta_p, X_p \times \dot{\Delta}_p) \arrow[r, "\pi_*"] \arrow[d, "\partial"] & k_{p+q}(F_p|X|, F_{p-1}|X|) \arrow[d, "\partial"] \\
 & k_{p+q-1}(X_p \times \dot{\Delta}_p, X_{p-1} \times \ddot{\Delta}_p) \arrow[r, "\pi_*"] \arrow[d, "\sum(-1)^i(1\times\alpha_i)_*"', "\cong"] & k_{p+q-1}(F_{p-1}|X|, F_{p-2}|X|) \\
 & \bigoplus_{i=0}^{p} k_{p+q-1}(X_p \times \dot{\Delta}_p, X_p \times \Delta^i_p) \arrow[d, "\oplus(-1)^i(1\times\delta_i)^{-1}_*"', "\cong"] & \\
\bigoplus_{i=0}^{p} k_q(X_p) \arrow[r, "\oplus S^{p-1}_*"] \arrow[d, "\nabla"'] & \bigoplus_{i=0}^{p} k_{p+q-1}(X_p \times \Delta_{p-1}, X_p \times \dot{\Delta}_{p-1}) \arrow[d, "\nabla"'] & \\
k_q(X_{p-1}) \arrow[r, "S^{p-1}_*"] & k_{p+q-1}(X_{p-1} \times \Delta_{p-1}, X_{p-1} \times \dot{\Delta}_{p-1}) \arrow[uuur, "\pi_*"'] &
\end{tikzcd}$$

Here $\dot{\Delta}_p = \partial\Delta_p$ is the (p-1)-skeleton of Δ_p and $\ddot{\Delta}_p$ is the (p-2)-skeleton. $\Delta^i_p = \bigcup_{j\neq i} \delta_j\Delta_{p-1}$, α_i is the inclusion $(\dot{\Delta}_p, \ddot{\Delta}_p) \to (\dot{\Delta}_p, \Delta^i_p)$, and $\sum(-1)^i(1\times\alpha_i)_*$ is an isomorphism by the Mayer-Vietoris sequence of the p+1 pairs $(X_p \times \dot{\Delta}_p, X_p \times \Delta^i_p)$. The maps $\delta_i : (\Delta_{p-1}, \dot{\Delta}_{p-1}) \to (\dot{\Delta}_p, \Delta^i_p)$ are clearly relative homeomorphisms. On the left, the maps are

$$\left(\sum(-1)^i\right)(x) = (x, -x, \ldots, (-1)^p x) \quad , \quad x \in k_q(X_p),$$

and

$$\nabla(x_o, \ldots, x_p) = \sum_{i=0}^{p} \partial_{i*}(x_i) \quad , \quad x_i \in k_q(X_p) \ .$$

(The other map ∇ is defined similarly, from $(\partial_i \times 1)_*$, and the maps π are the evident quotient maps.) Now the upper left rectangle commutes by a check of signs, the upper right and lower left rectangles commute by the naturality of ∂ and of S_*, and the triangle commutes by the face identifications used in the definition of the realization functor.

Of course, $\{E^r X\}$ is a right half-plane spectral sequence ($E^r_{pq} X = 0$ if $p < 0$). The convergence of such spectral sequences is discussed in [6]. The following observation is useful inthe study of products and coproducts in $\{E^r X\}$.

Lemma 11.15. For $X, Y \in \mathcal{SU}$, $\zeta : |X| \times |Y| \to |X \times Y|$ is filtration preserving, and the diagonal map $\Delta : |X| \to |X| \times |X|$ is naturally homotopic to a filtration preserving map.

Proof. $\zeta(F_p|X| \times F_q|X|) \subset F_{p+q}|X \times Y|$ by the definition of ζ in Theorem 11.5. For the second statement, define $g_i : \Delta_n \to \Delta_n$ for $i = 0$ and 1 and all $n \geq 0$ as follows. Let $u = (t_o, \ldots, t_n) \in \Delta_n$. Let p be the least integer such that $t_o + \ldots + t_p > 1/2$ and define

$$g_o(u) = \delta_n \ldots \delta_{p+1}(2t_o, \ldots, 2t_{p-1}, 1 - \sum_{i=0}^{p-1} 2t_i)$$

and

$$g_1(u) = \delta_o^{\,p}(1 - \sum_{i=p+1}^{n} 2t_i, 2t_{p+1}, \ldots, 2t_n)$$

Then g_i induces $G_i : |X| \to |X|$ such that G_i is homotopic to the identity map; thus Δ is homotopic to the filtration preserving map $(G_o \times G_1) \circ \Delta$.

12. Geometric realization and S_*, C_*, and Ω_*

In this section, we investigate the behavior of geometric realization with respect to the functors $S_*, C_*,$ and Ω_* defined on $\mathcal{S}\mathcal{T}$, where $\mathcal{T}$ is our category of based spaces. For $X \in \mathcal{S}\mathcal{T}$, we give $|X|$ the base-point $* \in X_0 = F_0|X|$; if X is proper, then it follows from Lemma 11.3 that $*$ is non-degenerate and that $|X| \in \mathcal{T}$.

Proposition 12.1. Realization commutes with suspension in the sense that there is a natural homeomorphism $\tau: |S_*X| \to S|X|$ for $X \in \mathcal{S}\mathcal{T}$.

Proof. Define $\tau|[x,s],u| = [|x,u|,s]$ for $x \in X_q$, $s \in I$, and $u \in \Delta_q$. It is trivial to verify that τ is well-defined and continuous, with continuous inverse.

The following pleasant result is more surprising. Its validity is what makes the use of simplicial spaces a sensible technique for the study of $\mathcal{C}$-spaces.

Theorem 12.2. Let $\mathcal{C}$ be any operad and let C be its associated monad in $\mathcal{T}$. Then there is a natural homeomorphism $\nu: |C_*X| \to C|X|$ for $X \in \mathcal{S}\mathcal{T}$ such that the following diagrams are commutative:

$$\begin{tikzcd} & {|C_*X|} \arrow[d, "\nu"] \\ {|X|} \arrow[ur, "|\eta_*|"] \arrow[r, "\eta"'] & C|X| \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {|C_*^2X|} \arrow[r, "\nu^2 = C\nu \circ \nu"] \arrow[d, "|\mu_*|"'] & C^2|X| \arrow[d, "\mu"] \\ {|C_*X|} \arrow[r, "\nu"] & C|X| \end{tikzcd}$$

If $(X, \xi) \in \mathcal{S}C[\mathcal{J}]$, then $(|X|, |\xi|\nu^{-1}) \in C[\mathcal{J}]$ and geometric realization therefore defines a functor $\mathcal{S}C[\mathcal{J}] \to C[\mathcal{J}]$.

Proof. Consider a point $|[c, x_1, \ldots, x_j], u| \in |C_* X|$, where $c \in \mathcal{C}(j)$, $x_i \in X_q$, and $u \in \Delta_q$. Define ν by the formula

(2) $\quad \nu|[c, x_1, \ldots, x_j], u| \;=\; [c, |x_1, u|, \ldots, |x_j, u|]$.

Clearly ν is compatible with the equivariance and base-point identifications used to define CX_q and with the face and degeneracy identifications used in the definition of the realization functor. For the latter, observe that

$$C\partial_i[c, x_1, \ldots, x_j] = [c, \partial_i x_1, \ldots, \partial_i x_j]$$

and similarly for the Cs_i. In view of this relationship between the iterated products X^j and CX, we can define ν^{-1} by

(3) $\quad \nu^{-1}[c, |x_1, u_1|, \ldots, |x_j, u_j|] = |[c, y_1, \ldots, y_j], v|$, where the iteration

$\zeta_j : |X|^j \to |X^j|$ of ζ is given by

$$\zeta_j(|x_1, u_1|, \ldots, |x_j, u_j|) \;=\; |(y_1, \ldots, y_j), v|.$$

By the associativity of ζ, ζ_j is unambiguous. By the commutativity of ζ, ν^{-1} is compatible with the equivariance identifications, and its compatibility with the remaining identifications is evident. The continuity of ν^{-1} follows from that of ζ_j, and it is clear from Theorem 11.5 that ν and ν^{-1} are ideed inverse functions. The commutativity of the stated diagrams is verified by an easy direct calculation from (2) and the formulas in Construction 2.4, and these diagrams, together with trivial formal diagram chases, imply that $(|X|, |\xi|\nu^{-1}) \in C[\mathcal{J}]$ if $(X, \xi) \in \mathscr{S}C[\mathcal{J}]$.

The relationship between $|\Omega_* X|$ and $\Omega|X|$ is more delicate. Indeed, if X is a discrete simplicial space, then each $\Omega X_q = *$ and therefore $|\Omega_* X| = *$, whereas $\Omega|X|$ is obviously non-trivial in general.

Theorem 12.3. For $X \in \mathscr{S}\mathcal{J}$, $|P_* X|$ is contractible and there are natural maps $\tilde{\gamma}$ and γ such that the following diagram commutes:

$$\begin{tikzcd}
|\Omega_* X| \arrow[r, "\subset"] \arrow[d, "\gamma"'] & |P_* X| \arrow[r, "|p_*|"] \arrow[d, "\tilde{\gamma}"'] & |X| \arrow[d, equal] \\
\Omega|X| \arrow[r, "\subset"] & P|X| \arrow[r, "p"] & |X|
\end{tikzcd}$$

Moreover, if X is proper and each X_q is connected, then $|p_*|$ is a quasi-fibration with fibre $|\Omega_* X|$ and therefore $\gamma\colon |\Omega_* X| \to \Omega|X|$ is a weak homotopy equivalence.

Proof. The standard contracting homotopy on PY, $Y \in \mathcal{J}$, is natural in Y; therefore, when applied to each PX_q, this homotopy defines a simplicial contracting homotopy $I_* \times P_* X \to P_* X$. Thus $|P_* X|$ is contractible by Corollary 11.9. For $f \in PX_q$, $u \in \Delta_q$, and $t \in I$, define $\tilde{\gamma}$ by the formula

(4) $$\tilde{\gamma}|f, u|(t) = |f(t), u| .$$

It is trivial to verify that $\tilde{\gamma}$ is a well-defined continuous map which restricts to an inclusion $\gamma\colon |\Omega_* X| \to \Omega|X|$ and satisfies $p\tilde{\gamma} = |p_*|$. The last statement will follow from Lemma 12.6 and Theorem 12.7 below.

Before completing the proof of the theorem above, we obtain an important consistency statement which interrelates our previous three results.

Theorem 12.4. For $X \in \mathcal{SJ}$, the iteration $\gamma^n : |\Omega_*^n X| \to \Omega^n |X|$ of γ is a morphism of C_n-algebras, and the following diagram is commutative:

$$\begin{tikzcd} {|C_{n*}X|} \arrow[r, "\nu"] \arrow[d, "|\alpha_{n*}|"'] & C_n|X| \arrow[d, "\alpha_n"] \\ {|\Omega_*^n S_*^n X|} \arrow[r, "\Omega^n\tau^n \circ \gamma^n"] & \Omega^n S^n |X| \end{tikzcd}$$

Proof. We must prove that the following diagram commutes:

$$\begin{tikzcd} C_n|\Omega_*^n X| \arrow[rr, "C_n\gamma^n"] \arrow[d, "\nu^{-1}"'] & & C_n\Omega^n|X| \arrow[d, "\theta_n"] \\ {|C_{n*}\Omega_*^n X|} \arrow[r, "|\theta_{n*}|"] & {|\Omega_*^n X|} \arrow[r, "\gamma^n"] & \Omega^n|X| \end{tikzcd}$$

and it clearly suffices to prove the commutativity of the diagram obtained by replacing ν^{-1} by ν. Thus consider

$$y = |[c, f_1, \ldots, f_j], u| \in |C_{n*}\Omega_*^n X|, \quad \text{where}$$

$c = \langle c_1, \ldots, c_j \rangle \in \mathcal{C}_n(j)$, $f_i \in \Omega^n X_q$, and $u \in \Delta_q$. Let $v \in I^n$. If $v \notin \bigcup c_i(J^n)$, then $\gamma^n |\theta_{n*}|(y)(v) = * = \theta_n \circ C_n\gamma^n \circ \nu(y)(v)$, and if $v = c_i(v')$, then, by Theorem 5.1 and the definitions of ν and γ,

$$\begin{aligned} \theta_n \circ C_n\gamma^n \circ \nu(y)(v) &= \theta_n[c, \gamma^n|f_1, u|, \ldots, \gamma^n|f_j, u|](v) \\ &= \gamma^n|f_i, u|(v') = |f_i(v'), u| \\ &= |\theta_n[c, f_1, \ldots, f_j](v), u| = \gamma^n|\theta_{n*}|(y)(v) . \end{aligned}$$

Thus γ^n is indeed a morphism of C_n-algebras. Since α_n is defined to be the composite $\theta_n \circ C_n\eta_n$, the commutativity of the following diagram gives that $\alpha_n \circ \nu = \Omega^n\tau^n \circ \gamma^n \circ |\alpha_{n*}|$:

$$\begin{tikzcd}
{|C_{n*}X|} \arrow[r, "\nu"] \arrow[d, "{|C_{n*}\eta_{n*}|}"'] & C_n|X| \arrow[d, "{C_n|\eta_{n*}|}"] \arrow[rrd, "C_n\eta_n"] & & \\
{|C_{n*}\Omega^n_*S^n_*X|} \arrow[r, "\nu"] \arrow[d, "{|\theta_{n*}|}"'] & C_n|\Omega^n_*S^n_*X| \arrow[r, "C_n\gamma^n"] & C_n\Omega^n|S^n_*X| \arrow[r, "C_n\Omega^n\tau^n"] \arrow[d, "\theta_n"] & C_n\Omega^nS^n|X| \arrow[d, "\theta_n"] \\
{|\Omega^n_*S^n_*X|} \arrow[rr, "\gamma^n"] & & \Omega^n|S^n_*X| \arrow[r, "\Omega^n\tau^n"] & \Omega^nS^nX
\end{tikzcd}$$

Here $\Omega^n\tau^n \circ \gamma^n \circ |\eta_{n*}| = \eta_n : |X| \to \Omega^nS^n|X|$ by an easy explicit calculation.

In order to complete the proof of Theorem 12.3, we shall prove a general result relating geometric realization to fibrations. We require some notations and a definition.

For $B \in \mathcal{U}$, let ΠB denote the space of all paths $I \to B$. For a map $p: E \to B$ in $\mathcal{U}$, define

$$\Gamma(p) = \{(e,f) \mid p(e) = f(0)\} \subset E \times \Pi B.$$

Define $\pi : \Pi E \to \Gamma(p)$ by $\pi(g) = (g(0), pg)$. Recall that p is a Hurewicz fibration if and only if there exists a lifting function $\lambda : \Gamma(p) \to \Pi E$ such that $\pi\lambda = 1$. In the applications, λ is usually "homotopy associative" in the sense that if $f, g \in \Pi B$ satisfy $f(1) = g(0)$, then the two maps $p^{-1}f(0) \to p^{-1}g(1)$

defined respectively by sending e to $\lambda(\lambda(e,f)(1),g)(1)$ or to $\lambda(e,gf)(1)$ are homotopic.

Definition 12.5. Let $p: E \to B$ be a map in $\mathcal{SU}$. Observe that if $\pi_q = \pi: \Pi E_q \to \Gamma(p_q)$, then $\pi_*: \Pi_* E \to \Gamma_*(p)$ is a map in $\mathcal{SU}$. We say that p is a simplicial Hurewicz fibration if there exists a map $\lambda_*: \Gamma_*(p) \to \Pi_* E$ such that $\pi_* \lambda_* = 1$ and such that the following associativity condition is satisfied.

(i) If $f, g \in \Pi B_q$ satisfy $f(1) = g(0)$ and if x_* and y_* denote the discrete simplicial subspaces of B generated by the q-simplices $x = f(0)$ and $y = g(1)$, then there exists a simplicial homotopy $H: I_* \times p^{-1}(x_*) \to p^{-1}(y_*)$ such that for any i-simplex e of $p^{-1}(x_*)$, with $p_i(e) = \gamma x$ for a composite γ of face and degeneracy operators (γ exists by the definition of x_*),

$$H_i(0,e) = \lambda_i(\lambda_i(e, \gamma f)(1), \gamma g)(1)$$

and

$$H_i(1,e) = \lambda_i(e, \gamma(gf))(1) .$$

We observe that the following statements, which shall be used in conjunction with (i), are valid in any simplicial Hurewicz fibration; in (ii) and (iii), e denotes an i-simplex of $p^{-1}(x_*)$ with $p(e) = \gamma x$, as in (i).

(ii) If $h: I \to \Pi B_q$ satisfies $h(t)(0) = x$ and $h(t)(1) = y$ for all $t \in I$, then the formula $H_i(t,e) = \lambda_i(e, \gamma h(t))(1)$ defines a simplicial homotopy $H: I_* \times p^{-1}(x_*) \to p^{-1}(y_*)$.

(iii) If $c(x): I \to B_q$ is the constant path at $x \in B_q$, then the formula $H_i(t, e) = \lambda_i(e, \gamma c(x))(t)$ defines a simplicial homotopy $H: I_* \times p^{-1}(x_*) \to p^{-1}(x_*)$ which starts at the identity map of $p^{-1}(x_*)$.

The standard natural constructions of Hurewicz fibrations apply simplicially; the only example that we shall need is the path space fibration.

Lemma 12.6. For $X \in \mathcal{S}\mathcal{T}$, $p_*: P_*X \to X$ is a simplicial Hurewicz fibration.

Proof. Choose a retraction $r: I \times I \to I \times 1 \cup 0 \times I$ such that

$$r(s, 0) = (0, 0) \quad \text{and} \quad r(1, t) = \begin{cases} (0, 2t) & , \quad 0 \leq t \leq 1/2 \\ (2t-1, 1) & , \quad 1/2 \leq t \leq 1 \end{cases}$$

For $Y \in \mathcal{T}$ and $p: PY \to Y$, define $\lambda: \Gamma(p) = \Pi PY$ by the formula

$$\lambda(e, f)(s)(t) = \begin{cases} e(u) & \text{if } r(s, t) = (0, u) \\ f(v) & \text{if } r(s, t) = (v, 1) \end{cases}$$

where $e \in PY$, $f \in \Pi Y$, and $e(1) = f(0)$. Clearly λ is a lifting function and $\lambda(e, f)(1) = fe$ is the standard product of paths. Thus if $f, g \in \Pi Y$ and $f(1) = g(0)$, then

$$\lambda(\lambda(e, f)(1), g)(1) = g(fe) \quad \text{and} \quad \lambda(e, gf)(1) = (gf)e$$

Now define $\lambda_q = \lambda: \Gamma(p_q) \to \Pi PX_q$. By the naturality of λ, λ_* is simplicial, and clearly $\pi_* \lambda_* = 1$. Condition (i) of Definition 12.5 is satisfied since the evident homotopies defined for each fixed γ are easily verified to fit together to define a simplicial homotopy.

Theorem 12.7. Let $p: E \to B$ be a simplicial Hurewicz fibration, in $\mathcal{S}\mathcal{T}$, and let $F = p^{-1}(*)$. Assume that B is proper and each B_q is connected. Then $|p|: |E| \to |B|$ is a quasi-fibration with fibre $|F|$.

Proof. We first define explicit lifting functions for the restrictions of $\sigma_{j_1} \cdots \sigma_{j_r}: \Delta_{q+r} \to \Delta_q$ to the inverse image of $\Delta_q - \partial\Delta_q$. We shall define

$$\gamma_{j_r \cdots j_1}: \Gamma(\sigma_{j_1} \cdots \sigma_{j_r}) \to \Pi(\Delta_{q+r})$$

by the inductive formula ($u \in \Delta_{q+r}$, $f \in \Pi(\Delta_q - \partial\Delta_q)$, $\sigma_{j_1} \cdots \sigma_{j_r} u = f(0)$):

$$\gamma_{j_r \cdots j_1}(u,f) = \gamma_{j_r}(u, \gamma_{j_{r-1} \cdots j_1}(\sigma_{j_r} u, f)),$$

and it remains to define $\gamma_j: \Gamma(\sigma_j) \to \Pi(\Delta_{q+1})$. Thus let $(u,f) \in \Gamma(\sigma_j)$. Let $f(s) = (t_o(s), \ldots, t_q(s)) \in \Delta_q$. Since $\sigma_j(u) = f(0)$,

$$u = (t_o(0), \ldots, t_{j-1}(0), at_j(0), (1-a)t_j(0), t_{j+1}(0), \ldots, t_q(0))$$

for some a, $0 \leq a \leq 1$ (a is well-defined since $t_j(0) > 0$). Define γ_j by

$$\gamma_j(u,f)(s) = (t_o(s), \ldots, t_{j-1}(s), at_j(s), (1-a)t_j(s), t_{j+1}(s), \ldots, t_q(s)).$$

Visibly, $\gamma_j(u,f)(0) = u$ and $\sigma_j \gamma_j(u,f) = f$, hence $\pi\gamma_j = 1$. Corresponding to the relation $\sigma_i \sigma_j = \sigma_{j-1}\sigma_i$ for $i < j$, we have $\gamma_{j,i} = \gamma_{i,j-1}$, by an easy verification. This implies that

(1) $\quad \sigma_j \gamma_{j_r \cdots j_1}(u,f) = \gamma_{i_{r-1} \cdots i_1}(\sigma_j u, f)$ if $s_j s_{i_{r-1}} \cdots s_{i_1} = s_{j_r} \cdots s_{j_1}$.

If $a = 0$ or $a = 1$ above, then $\gamma_j(u, f)(s) \in \operatorname{Im} \delta_j$ or $\operatorname{Im} \delta_{j+1}$, and it is an easy matter to verify the formula

(2) $\quad \delta_j \gamma_{j_r \dots j_1}(u, f) = \gamma_{i_{r+1} \dots i_1}(\delta_j u, f)$ if $\partial_j s_{i_{r+1}} \dots s_{i_1} = s_{j_r} \dots s_{j_1}$.

We can now show that $|p|: |p|^{-1}(V) \to V$ is a Hurewicz fibration for any open subset V of $F_q|B| - F_{q-1}|B|$, where, if $q = 0$, $F_{-1}|B| = \emptyset$. We must define a lifting function $\tilde{\lambda}_q: \Gamma(|p|) \to \Pi|p|^{-1}(V)$. Of course, by Lemma 11.3, we have that

$$\Pi V \subset \Pi(B_q - sB_{q-1}) \times \Pi(\Delta_q - \partial\Delta_q).$$

Let $(|e, w|, (f', f'')) \in \Gamma|p|$, where $(e, w) \in E_{q+r} \times \Delta_{q+r}$ is non-degenerate, $f': I \to B_q - sB_{q-1}$, and $f'': I \to \Delta_q - \partial\Delta_q$. Necessarily, we have

$$p_{q+r}(e) = s_{j_r} \dots s_{j_1} f'(0), \quad \text{where } \sigma_{j_1} \dots \sigma_{j_r} w = f''(0)$$

(as in the proof of Lemma 11.3). Define $\tilde{\lambda}_q$ by the formula

(3) $\quad \tilde{\lambda}_q(|e, w|, (f', f''))(t) = |\lambda_{q+r}(e, s_{j_r} \dots s_{j_1} f')(t), \gamma_{j_r \dots j_1}(w, f'')(t)|$.

Since λ_* is simplicial, formulas (1) and (2) show that $\tilde{\lambda}_q$ respects the equivalence relation used to define $|E|$, and it follows easily that $\tilde{\lambda}_q$ is continuous. Clearly $\pi\tilde{\lambda}_q = 1$, as required. We have now verified (i) of Lemma 7.2, and it remains to verify (ii) of that lemma. Fix $q > 0$. Let (k, v) be the representation of

$$(B_q \times \Delta_q, \ sB_{q-1} \times \Delta_q \cup B_q \times \partial\Delta_q)$$

as a strong NDR-pair obtained by use of Lemma A.3 from any given representations of (B_q, sB_{q-1}) and $(\Delta_q, \partial\Delta_q)$ as strong NDR-pairs. Define $U \subset F_q|B|$ to be the union of $F_{q-1}|B|$ and the image of $v^{-1}[0,1)$ under the evident map $B_q \times \Delta_q \longrightarrow F_q|B|$. Define $h_t: U \longrightarrow U$ by $h_t(x) = x$ for $x \in F_{q-1}|B|$ and by

(6) $\quad h_t|b,u| = |k_t(b,u)|$ for $(b,u) \in B_q \times \Delta_q$ with $v(b,u) < 1$.

Then h is a strong deformation retraction of U onto $F_{q-1}|B|$. To lift h, let $(e,w) \in E_{m+r} \times \Delta_{m+r}$ be a typical non-degenerate point such that $|e,w| \in |p|^{-1}(U)$ where, as in Lemma 11.3,

$$p_{m+r}(e) = s_{j_r} \dots s_{j_1} b \text{ and } u = \sigma_{j_1} \dots \sigma_{j_r} w$$

determines the non-degenerate representative (b,u) of $|p|(|e,w|)$. Here $m \leq q$ and we define H by the formulas

(7) $\quad H(t, |e,w|) = |\lambda_{m+r}(e, s_{j_r} \dots s_{j_1} c(b))(t), w|$ if $m < q$, where

$c(b): I \longrightarrow B_m$ is the constant path at b; and

(8) $\quad H(t, |e,w|) = |\lambda_{q+r}(e, s_{j_r} \dots s_{j_1} f')(t), \gamma_{j_r \dots j_1}(w, f'')(t)|$ if $m = q$,

where $f': I \longrightarrow B_q$ and $f'': I \longrightarrow \Delta_q$ are the paths defined by $f'(t) = \pi_1 k_t(b,u)$ and $f''(t) = \pi_2 k_t(b,u)$ (here π_1 and π_2 are the projections of $B_q \times \Delta_q$ onto its factors).

Here the $\gamma_{j_r \dots j_1}$ can be applied to the paths f'' in Δ_q (even though f'' does not have image in $\Delta_q - \partial\Delta_q$) because if $f''(0) \in \partial\Delta_q$, then f'' is the constant path at $f''(0)$ and the definition above of $\gamma_{j_r \dots j_1}(w, f'')$ is therefore unambiguous.

It is straightforward to verify that H is well-defined and continuous (note that $\gamma_{j_r \cdots j_1}(w, c(u)) = c(w))$, and that H covers h and deforms $|p|^{-1}(U)$ into $|p|^{-1}F_{q-1}|B|$. It remains to verify that $H_1: |p|^{-1}(x) \to |p|^{-1}h_1(x)$ is a weak homotopy equivalence for each $x \in U$. If $x \in F_{q-1}|B|$, then $h_1(x) = x$ and H is itself a homotopy $1 \simeq H_1: |p|^{-1}(x) \to |p|^{-1}(x)$. Thus assume that $x \notin F_{q-1}|B|$. In the notation of (8), let $x = |b, u| = |f'(0), f''(0)|$, so that $h_1(x) = |f'(1), f''(1)|$. Let $g: I \to B_q$ be any path connecting $g(0) = f'(0)$ to $g(1) = *$, and let $g' = g \circ f'^{-1}: I \to B_q$, where $f'^{-1}(t) = f'(1 - t)$; g' is then a path connecting $f'(1)$ to $*$. We shall first construct a homotopy equivalence $\tilde{f}(u): |p|^{-1}|b, u| \to |F|$ for any path $f: I \to B_q$ such that $f(0) = b$ and $f(1) = *$ and for any $u \in \Delta_q$. We shall then complete the proof by showing that the following diagram is homotopy commutative.

(9)
$$\begin{tikzcd} |p|^{-1}(x) = |p|^{-1}|f'(0), f''(0)| \arrow[r, "H_1"] \arrow[dr, "\tilde{g}(f''(0))"'] & |p|^{-1}|f'(1), f''(1)| = |p|^{-1}h_1(x) \arrow[d, "\tilde{g}'(f''(1))"] \\ & |F| \end{tikzcd}$$

Thus fix $f: I \to B_q$ with $f(0) = b$ and $f(1) = *$. Let $\Delta[q]$ denote the standard simplicial q-simplex [18, p. 14] regarded as a discrete simplicial space, and let $\overline{b}: \Delta[q] \to B$ be the unique simplicial map such that $\overline{b}(i_q) = b$, where

i_q (Δ_q in [18]) is the fundamental q-simplex in $\Delta[q]$. Let $E(b)$ denote the simplicial fibre product $E \times^B \Delta[q]$ of p and $\overline{b}$. Define $f_*: E(b) \to F \times \Delta[q]$ by

$$f_i(e, \gamma i_q) = (\lambda_i(e, \gamma f)(1), \gamma i_q),$$

where $e \in E_i$ satisfies $p_i(e) = \gamma b = \gamma \overline{b}(i_q)$ for some composite γ of face and degeneracy operators. Define $f_*^{-1}: F \times \Delta[q] \to E(b)$ by

$$f_i^{-1}(e, \gamma i_q) = (\lambda_i(e, \gamma f^{-1})(1), \gamma i_q), \quad e \in F_i \text{ and } \gamma i_q \in \Delta_i[q].$$

By (i), (ii), and (iii) of Definition 11.5, f_* and f_*^{-1} are inverse fibre homotopy equivalences over $\Delta[q]$. Therefore, by Corollary 11.6, the following composite is a fibre homotopy equivalence over $|\Delta[q]| = \Delta_q$:

$$|E| \times^{|B|} \Delta_q \xrightarrow{\zeta} |E(b)| \xrightarrow{|f_*|} |F \times \Delta[q]| \xrightarrow{|p_1| \times |p_2|} |F| \times \Delta_q .$$

Fix $u \in \Delta_q$, $u = |i_q, u|$. In $|E| \times^{|B|} \Delta_q$, $p_2^{-1}(u)$ may be identified with $|p|^{-1}|b, u|$, and the above composite restricts to give the desired homotopy equivalence $\tilde{f}(u): |p|^{-1}|b, u| \to |F|$. Finally, consider the diagram (9). Let $|e, w| \in |p|^{-1}(x)$ be as described above formula (7). We then have

(10) $$\tilde{g}(f''(0))|e, w| = |\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} g)(1), w| , \text{ and}$$

(11) $$\tilde{g}'(f''(1)) \circ H_1 |e, w|$$

$$= |\lambda_{q+r}(\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} f')(1), s_{j_r} \cdots s_{j_1} g')(1), \gamma_{j_r \cdots j_1}(w, f'')(1)| .$$

Definition 12.5 and $g' = gf'^{-1}$ imply that $g'(f''(1)) \cdot H_1$ is homotopic to the map $\ell : |p|^{-1}(x) \to |F|$ defined by

$$(12) \qquad \ell|e,w| = |\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} g)(1),\ \gamma_{j_r \cdots j_1}(w, f'')(1)| \ .$$

Finally, define $L: I \times |p|^{-1}(x) \to |F|$ by the formula

$$(13) \qquad L(t, |e,w|) = |\lambda_{q+r}(e, s_{j_r} \cdots s_{j_1} g)(1),\ \gamma_{j_r \cdots j_1}(w, f'')(t)| .$$

Then L is a homotopy from $\tilde{g}(f''(0))$ to the map ℓ .

13. The recognition principle and A_∞ spaces

We now have at our disposal all of the information required for the proof of the recognition principle. We prove our basic recognition theorem for n-fold loop spaces, $n < \infty$, and discuss A_∞ spaces here; E_∞ spaces will be studied in the next section. We first fix notations for our geometric constructions.

Let (C, μ, η) be a monad in $\mathcal{J}$, let (X, ξ) be a C-algebra, and let (F, λ) be a C-functor in $\mathcal{J}$; these notions are defined in Definitions 2.1, 2.2, and 9.4. Then Construction 9.6 yields a simplicial topological space $B_*(F, C, X)$, and we agree to write $B(F, C, X)$ for its geometric realization $|B_*(F, C, X)|$, as constructed in Definition 11.1; B defines a functor $\mathcal{B}(\mathcal{J}, \mathcal{J}) \to \mathcal{J}$, and we write $B(\pi, \psi, f) = |B_*(\pi, \psi, f)|$ for a morphism (π, ψ, f) in $\mathcal{B}(\mathcal{J}, \mathcal{J})$. Many of our C-functors F in $\mathcal{J}$ will be obtained by neglect of structure from C-functors (also denoted F) in the category $D[\mathcal{J}]$ of D-algebras, for some monad D in $\mathcal{J}$. Then $B_*(F, C, X)$ is a simplicial D-algebra, but this need not imply that $B(F, C, X)$ is itself a D-algebra. For example, this implication is not valid for $D = \Omega^n S^n$. However, by Theorem 12.2, if D is the monad in $\mathcal{J}$ associated to an operad $\mathcal{D}$, as obtained in Construction 2.4, then realization does define a functor $\mathcal{L}\, D[\mathcal{J}] \to D[\mathcal{J}]$ and B therefore defines a functor $\mathcal{B}(\mathcal{J}, D[\mathcal{J}]) \to D[\mathcal{J}]$.

We shall write $\tau(\zeta) = |\tau_*(\zeta)| : Y \to B(F, C, X)$ for any map $\zeta : Y \to FX$ in $\mathcal{J}$ and we shall write $\varepsilon(\pi) = |\varepsilon_*(\pi)| : B(F, C, X) \to Y$ for any map $\pi: FX \to Y$ in $\mathcal{J}$ such that the following diagram is commutative:

$$\begin{array}{ccc} FCX & \xrightarrow{\partial_0 = \lambda} & FX \\ {\scriptstyle \partial_1 = F\xi}\downarrow & & \downarrow{\scriptstyle \pi} \\ FX & \xrightarrow{\pi} & Y \end{array}$$

Here $\tau_*(\zeta)$ and $\varepsilon_*(\pi)$ are defined in Lemma 9.2, and $|Y_*| = Y$ by Lemma 11.8; τ and ε are natural, in the evident sense.

We must dispose of one minor technical point before proceeding to the theorems. Since we wish to apply the results of the previous two sections, we shall always tacitly assume that $B_*(F, C, X)$ is a strictly proper simplicial space, in the sense of Definition 11.2. This is in fact a harmless assumption, at least when C is the monad associated to an operad $\mathcal{C}$, in view of the results of the appendix. In Proposition A.10, we show that $\mathcal{C}$ can, if necessary, be replaced functorially by a very slightly altered operad $\mathcal{C}'$ which maps onto $\mathcal{C}$ and is such that $B_*(F, C', X)$ is strictly proper for reasonable functors (such as Ω, S, C, C' and their composites) and for $\mathcal{C}$-spaces (X, θ) such that $(X, *)$ is a strong NDR-pair. If $(X, *)$ is not well-behaved, for example if $*$ is degenerate, then Lemma A.11 shows that (X, θ) can be replaced by $(X', \theta') \in \mathcal{C}[\mathcal{J}]$ where $(X', *)$ is a strong NDR-pair.

In our basic theorem, we shall assume given a morphism of operads $\pi: \mathcal{D} \to \mathcal{C}_n$, where $\mathcal{C}_n$ is the n-th little cubes operad of Definition 4.1 and $\mathcal{D}$ is some other operad; as in Construction 2.4, we shall also write π for the associated morphism of monads $D \to C_n$. Observe that if $Y \in \mathcal{T}$ then $(\Omega^n Y, \theta_n \pi) \in D[\mathcal{T}]$, where θ_n is as defined in Theorem 5.1, and, by Theorem 5.2, θ_n coincides with the composite

$$C_n \Omega^n Y \xrightarrow{\alpha_n} \Omega^n S^n \Omega^n Y \xrightarrow{\xi_n = \Omega^n \phi^n(1)} \Omega^n Y .$$

Here $\phi: \mathrm{Hom}_{\mathcal{T}}(X, \Omega Y) \to \mathrm{Hom}_{\mathcal{T}}(SX, Y)$ is the standard adjunction homeomorphism of (5.1) and $\alpha_n: C_n \to \Omega^n S^n$ is the morphism of monads constructed in Theorem 5.2. Of course, we are identifying the notions of $\mathcal{C}_n$-space and of C_n-algebra via Proposition 2.8, and similarly for $\mathcal{D}$. Since $\alpha_n \pi: D \to \Omega^n S^n$ is a morphism of monads in $\mathcal{T}$, $(S^n, \phi^n(\alpha_n \pi))$ is a D-functor in $\mathcal{T}$ by Examples 9.5. Thus, if $(X, \xi) \in D[\mathcal{T}]$, then $B(S^n, D, X)$ is defined. With these notations, we have the following theorem, which implies the recognition principle stated in Theorem 1.3.

<u>Theorem 13.1.</u> Let $\pi: D \to C_n$ denote the morphism of monads associated to a local equivalence $\pi: \mathcal{D} \to \mathcal{C}_n$ of Σ-free operads. Let (X, ξ) be a D-algebra and consider the following morphisms of D-algebras:

$$X \xleftarrow{\varepsilon(\xi)} B(D, D, X) \xrightarrow{B(\alpha_n \pi, 1, 1)} B(\Omega^n S^n, D, X) \xrightarrow{\gamma^n} \Omega^n B(S^n, D, X).$$

(i) $\varepsilon(\xi)$ is a strong deformation retraction with right inverse $\tau(\zeta)$, where $\zeta : X \to DX$ is given by the unit ζ of D.

(ii) $B(\alpha_n\pi, 1, 1)$ is a weak homotopy equivalence if X is connected.

(iii) γ^n is a weak homotopy equivalence for all X.

(iv) The composite $\gamma^n \circ B(\alpha_n\pi, 1, 1) \circ \tau(\zeta) : X \to \Omega^n B(S^n, D, X)$ coincides with the adjoint of $\tau(1) : S^n X \to B(S^n, D, X)$.

(v) $B(S^n, D, X)$ is $(m+n)$-connected if X is m-connected.

Moreover, the following conclusions hold for $Y \in \mathcal{J}$.

(vi) $\varepsilon\phi^n(1) : B(S^n, D, \Omega^n Y) \to Y$ is a weak homotopy equivalence if Y is n-connected; for all Y, the following diagram is commutative and $\Omega^n\varepsilon\phi^n(1)$ is a retraction with right inverse $\phi^{-n}\tau(1)$:

$$\begin{tikzcd}
B(D, D, \Omega^n Y) \arrow[r, "B(\alpha_n\pi, 1, 1)"] \arrow[d, "\varepsilon(\theta_n\pi)"'] & B(\Omega^n S^n, D, \Omega^n Y) \arrow[dl, "\varepsilon(\xi_n)"'] \arrow[d, "\gamma^n"] \\
\Omega^n Y & \Omega^n B(S^n, D, \Omega^n Y) \arrow[l, "\Omega^n \varepsilon \phi^n(1)"']
\end{tikzcd}$$

(vii) $\varepsilon\phi^n(\alpha_n\pi) : B(S^n, D, DY) \to S^n Y$ is a strong deformation retraction with right inverse $\tau(S^n\zeta)$.

Proof. $\varepsilon(\xi)$ and $B(\alpha_n\pi, 1, 1)$ are morphisms of D-algebras since $\varepsilon_*(\xi)$ and $B_*(\alpha_n\pi, 1, 1)$ are morphisms of simplicial D-algebras by Theorem 9.10. By Theorem 9.11, we have

$$B_*(\Omega^n S^n, D, X) = \Omega^n_* B_*(S^n, D, X) .$$

Thus γ^n is a well-defined morphism of D-algebras by Theorem 12.4. Now (i) and (vii) hold on the level of simplicial spaces by Theorems 9.10 and 9.11 and therefore hold after realization by Corollary 11.10. By the approximation theorem (Theorem 6.1) and Proposition 3.4, each composite $\alpha_n\pi: D^{q+1}X \to \Omega^n S^n D^q X$ is a weak homotopy equivalence if X is connected, and (ii) follows from Theorem 11.13. Part (iii) follows from Theorem 12.3; here X need not be connected since each $\Omega^i S^n D^q X$ for $i<n$ is certainly connected. Part (iv) is trivial (from a glance at the explicit definitions) and (v) follows from Theorem 11.12. Finally, the upper triangle in the diagram of (vi) commutes by the naturality of ε, since $\xi_n \alpha_n = \theta_n$, and the lower triangle commutes by the naturality of γ^n, since $\xi_n = \Omega^n \phi^n(1)$ and $\varepsilon_* \Omega^n \phi^n(1) = \Omega^n_* \varepsilon_* \phi^n(1)$ by Theorem 9.11 and since γ^n reduces to the identity on $\Omega^n Y = |\Omega^n_* Y_*|$; the fact that $\varepsilon\phi^n(1)$ is a weak homotopy equivalence for n-connected spaces Y follows from the diagram.

$B(S^n, D, X)$ should be thought of as an n-fold de-looping of X. As such for $Y \in \mathcal{T}$, $B(S^n, D, \Omega^n Y)$ should give back Y but with its bottom homotopy groups killed. This is the content of part (vi). Similarly, DY approximates $\Omega^n S^n X$, hence $B(S^n, D, DY)$ should give back $S^n Y$. This is the content of part (vii), but with a curious twist: the proof of (vii) in no way depends on the approximation theorem and the result is valid even when Y is not connected, in which case DY fails to approximate $\Omega^n S^n Y$.

For $(X,\xi) \in D[\mathcal{J}]$, the diagram

$$X \xleftarrow{\varepsilon(\xi)} B(D,D,X) \xrightarrow{\gamma^n B(\alpha_n \pi,1,1)} \Omega^n B(S^n, D, X)$$

is to be thought of as displaying an explicit natural weak homotopy equivalence between X and $\Omega^n B(S^n, D, X)$ <u>in the category of D-algebras.</u> The use of weak homotopy equivalence in this sense is essential: it is not possible, in general, to find a morphism $f: X \to \Omega^n Y$ of D-algebras which is a (weak) homotopy equivalence. For example, if $D = C_n$ and if X is a connected N-algebra (that is, a connected commutative monoid) regarded as a C_n-algebra by pull-back along the augmentation $\varepsilon: C_n \to N$, then, for any space Y, the only morphism of C_n-algebras from X to $\Omega^n Y$ is the trivial map! Indeed, for any such f, commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{C}_n(1) \times X & \xrightarrow{1 \times f} & \mathcal{C}_n(1) \times \Omega^n Y \\
\Big\downarrow{\scriptstyle \varepsilon \times 1} & & \Big\downarrow{\scriptstyle \theta_{n,1}} \\
\mathcal{N}(1) \times X = X & \xrightarrow{f} & \Omega^n Y
\end{array}$$

implies $\theta_{n,1}(c, f(x)) = f(x)$ for $x \in X$ and all $c \in \mathcal{C}_n(1)$, and a glance at the definition of θ_n in Theorem 5.1 shows that this implies $f(x)(s) = *$ for all $s \in S^n$.

Thus we cannot do better than to obtain a weak homotopy equivalence of D-algebras between a given D-algebra X and an n-fold loop space, and it is clearly reasonable to demand that an n-fold de-looping of X be (n-1)-connected (hence n-connected if X is connected). Subject to these two desiderata, the n-fold de-looping of X is unique up to weak homotopy equivalence.

Corollary 13.3. Under the hypotheses of Theorem 13.1, if

$$(X, \xi) \xleftarrow{f} (X', \xi') \xrightarrow{g} (\Omega^n Y, \theta_n \pi)$$

is a weak homotopy equivalence of connected D-algebras, where Y is n-connected, then the diagram

$$B(S^n, D, X) \xleftarrow{B(1,1,f)} B(S^n, D, X') \xrightarrow{\varepsilon\phi^n(g)} Y$$

displays a weak homotopy equivalence between Y and $B(S^n, D, X)$.

Proof. $\varepsilon\phi^n(g) = \varepsilon\phi^n(1) \circ B(1,1,g)$ by the naturality of ε ; $\varepsilon\phi^n(1)$ is a weak homotopy equivalence by the theorem and $B(1,1,f)$ and $B(1,1,g)$ are weak homotopy equivalences by Theorem 11.13 since $S^n D^q f$ and $S^n D^q g$ are weak homotopy equivalences for all q (as follows readily from the approximation theorem: $S^n (\Omega^n S^n)^q$ is certainly a functor which preserves weak homotopy equivalences).

Remarks 13.3. The idea of proving a recognition principle by geometrically realizing simplicial constructions based on monads is due to Beck [5]. In that paper, Beck sketched a proof of the fact that (in our terminology) if (X, ξ) is a $\Omega^n S^n$-algebra, then the diagram

$$X \xleftarrow{\varepsilon(\xi)} B(\Omega^n S^n, \Omega^n S^n, X) \xrightarrow{\gamma^n} \Omega^n B(S^n, \Omega^n S^n, X)$$

displays a weak homotopy equivalence between X and $\Omega^n B(S^n, \Omega^n S^n, X)$. Of course, our results prove this and add that $\varepsilon(\xi)$ and γ^n are morphisms of C_n-algebras (not of $\Omega^n S^n$-algebras) and that

$$B(1, \alpha_n, 1): B(S^n, C_n, X) \rightarrow B(S^n, \Omega^n S^n, X)$$

is a weak homotopy equivalence if X is connected. Unfortunately, the only $\Omega^n S^n$-algebras that seem to occur "in nature" are n-fold loop spaces, and Beck's recognition theorem is thus of little practical value.

The little cubes operads are of interest because their geometry so closely approximates the geometry of iterated loop spaces; for precisely this reason, a recognition principle based solely on these operads would also be of little practical value. We have therefore allowed more general operads in Theorem 13.1. We next exploit this generality to obtain our recognition principle for A_∞ spaces, as defined in Definition 3.5. Recall that the category of operads over $\mathcal{M}$ of Definition 3.3 has the product ∇ described in Definition 3.9. In view of Proposition 3.10, the following theorem is an immediate consequence of Theorem 13.1 and Corollary 13.2.

Theorem 13.4. Let $\mathcal{C}$ be any A_∞ operad, let $\mathcal{D} = \mathcal{C} \nabla \mathcal{C}_1$ and let $\psi: \mathcal{D} \to \mathcal{C}$ and $\pi: \mathcal{D} \to \mathcal{C}_1$ be the projections. Then π is a local Σ-equivalence of operad. Therefore, if (X, θ) is a connected $\mathcal{C}$-space, then there exists one and, up to weak homotopy equivalence, only one connected space Y such that $(X, \theta\psi)$ is weakly homotopy equivalent as a $\mathcal{D}$-space to $(\Omega Y, \theta_1 \pi)$, namely $Y = B(S, D, X)$.

Of course, Theorem 13.4 implies that a connected A_∞ space X is weakly homotopy equivalent to a topological monoid, namely the Moore loop space $\Lambda B(S, D, X)$. As was first proven by Adams (unpublished), a more direct construction is possible. Recall that, by Proposition 3.2, the notions of topological monoid and of M-algebra are equivalent.

Theorem 13.5. Let $\mathcal{C}$ be any A_∞ operad and let $\delta: C \to M$ be the morphism of monads associated to the augmentation $\mathcal{C} \to \mathcal{M}$. Let (X, θ) be a C-algebra and consider the following morphisms of C-algebras:

$$X \xleftarrow{\varepsilon(\theta)} B(C, C, X) \xrightarrow{B(\delta, 1, 1)} B(M, C, X).$$

(i) $\varepsilon(\theta)$ is a strong deformation retraction with right inverse $\tau(\eta)$, where $\eta: X \to CX$ is given by the unit η of C.

(ii) $B(\delta, 1, 1)$ is a weak homotopy equivalence if X is connected.

(iii) $B(M, C, X)$ has a natural structure of topological monoid.

(iv) If (G, ϕ) is an M-algebra (that is, a topological monoid) then

$\varepsilon(\phi): B(M, C, G) \to G$ is a morphism of monoids and the following diagram is commutative (hence $\varepsilon(\phi)$ is a weak homotopy equivalence if G is connected):

$$\begin{tikzcd}
B(C, C, G) \arrow[r, "B(\delta, 1, 1)"] \arrow[d, "\varepsilon(\phi)"'] & B(M, C, G) \arrow[d, "B(1, \delta, 1)"] \arrow[dl, "\varepsilon(\phi)"'] \\
G & B(M, M, G) \arrow[l, "\varepsilon(\phi)"']
\end{tikzcd}$$

(v) For $Y \in \mathcal{J}$, $\varepsilon(\nu \cdot M\delta): B(M, C, CY) \to MY$, $\nu: M^2 \to M$, is a strong deformation rectraction of topological monoids (that is, the required deformation is given by morphisms of monoids h_t) with right inverse $\tau(M\eta)$.

<u>Proof.</u> In view of Theorem 9.10 and the fact that, by Proposition 3.4, $\delta: CY \to MY$ is a weak homotopy equivalence if Y is a connected space, the theorem follows from the facts that geometric realization preserves homotopies (Corollary 11.10), weak homotopy equivalences (Theorem 11.13), monoids (Corollary 11.7), and C-algebras (Theorem 12.2).

Like Theorem 13.1, the result above implies its own uniqueness statement.

Corollary 13.6. Under the hypotheses of Theorem 13.5, if

$$(X,\theta) \xleftarrow{f} (X',\theta') \xrightarrow{g} (G,\phi\delta)$$

is a weak homotopy equivalence of connected D-algebras, where (G,ϕ) is an M-algebra, then the diagram

$$B(M,C,X) \xleftarrow{B(1,1,f)} B(M,C,X') \xrightarrow{B(1,1,g)} B(M,C,G) \xrightarrow{\varepsilon(\phi)} G$$

displays a weak homotopy equivalence of topological monoids between G and $B(M,C,X)$.

Remarks 13.7. By Corollary 3.11, any E_∞ space is an A_∞ space ; by the previous theorem any connected A_∞ space is weakly homotopy equivalent to a topological monoid. These two facts are the starting point of Boardman and Vogt's proof [7 , 8] of the recognition principle for E_∞ spaces. Given an E_∞ space, they construct a homotopy equivalent monoid and show that the monoid can be given a structure of E_∞ space such that the (monoid) product commutes with the (operad) action. Then, as we shall see in [21], the classifying space of the monoid inherits a structure of E_∞ space and the argument can be iterated. While conceptually very natural, this line of argument leads to formidable technical complications; a glance at Lemma 1.9 will reveal one major source of difficulty, and another source of difficulty will be discussed in section 15.

14. E_∞ spaces and infinite loop sequences

Our recognition principle for E_∞- spaces, as defined in Definition 3.5, will follow from Theorem 13.1 by use of the product (Definition 3.8) in the category of operads and passage to limits. Throughout this section, $\mathcal{C}$ will denote a fixed E_∞ operad, $\mathcal{D}_n$ will denote the product operad $\mathcal{C} \times \mathcal{C}_n$ for $n \geq 1$ or $n = \infty$, and $\pi_n : \mathcal{D}_n \to \mathcal{C}_n$ and $\psi_n : \mathcal{D}_n \to \mathcal{C}$ will denote the projections. By Proposition 3.10, the π_n are local equivalences, and Theorem 13.1 thus applies to the study of $\mathcal{D}_n$-spaces. The inclusions $\sigma_n : \mathcal{C}_n \to \mathcal{C}_{n+1}$ of Definition 4.1 (e) give rise to inclusions $\tau_n = 1 \times \sigma_n : \mathcal{D}_n \to \mathcal{D}_{n+1}$, and $\mathcal{D}_\infty$ is the limit of the $\mathcal{D}_n$ for finite n. As in Construction 2.4, we write C, C_n, and D_n for the monads in $\mathcal{J}$ associated to $\mathcal{C}$, $\mathcal{C}_n$, and $\mathcal{D}_n$, and we use the same letter for morphisms of operads and for their associated morphisms of monads in $\mathcal{J}$. We let $\nu_n : D_n^2 \to D_n$ and $\zeta_n : 1 \to D_n$ denote the product and unit of D_n.

A connected C-algebra (X, θ) determines a D_n-algebra $(X, \theta\psi_n)$ for all $n \geq 1$ and thus has an n-fold de-looping $B(S^n, D_n, X)$ by Theorem 13.1. By the definition of the functor B_* in Construction 9.6, the following lemma will imply that the $B(S^n, D_n, X)$ fit together to form a (weak) Ω-spectrum.

Lemma 14.1. Let $\eta = \phi^{-1}(1):1 \to \Omega S$. Then, for all $n \geq 1$,

$$\eta S^n:(S^n, \phi^n(\alpha_n \pi_n)) \to (\Omega S^{n+1}, \Omega\phi^{n+1}(\alpha_{n+1}\pi_{n+1}\tau_n))$$

is a morphism of D_n-functors in $\mathcal{T}$. Therefore, for all $i \geq 0$, the functor $QS^i = \varinjlim \Omega^j S^{i+j}$ inherits a structure of D_∞-functor in $D_\infty[\mathcal{T}]$ by passage to limits from the actions $\Omega^j\phi^{i+j}(\alpha_{i+j}\pi_{i+j})$ of D_{i+j} on $\Omega^j S^{i+j}$.

Proof. The first statement holds since the following diagram is commutative:

$$\begin{tikzcd}
S^n D_n \arrow[r, "\eta S^n D_n"] \arrow[d, "S^n(\alpha_n \pi_n)"] & \Omega S^{n+1} D_n \arrow[r, "\Omega S^{n+1}\tau_n"] \arrow[d, "\Omega S^{n+1}(\alpha_n \pi_n)"] & \Omega S^{n+1} D_{n+1} \arrow[d, "\Omega S^{n+1}(\alpha_{n+1}\pi_{n+1})"] \\
S^n \Omega^n S^n \arrow[r, "\eta S^n \Omega^n S^n"] \arrow[d, "\phi^n(1)"] & \Omega S^{n+1}\Omega^n S^n \arrow[r, "\Omega S^{n+1}\sigma_n"] \arrow[d, "\Omega S\phi^n(1)"] & \Omega S^{n+1}\Omega^{n+1}S^{n+1} \arrow[d, "\Omega\phi^{n+1}(1)"] \\
S^n \arrow[r, "\eta S^n"] & \Omega S^{n+1} \arrow[r, "1"] & \Omega S^{n+1}
\end{tikzcd}$$

Here $\sigma_n = \Omega^n\phi^{-1}(1):\Omega^n S^n \to \Omega^{n+1}S^{n+1}$, as in formula (5.5), and $\sigma_n\alpha_n\pi_n = \alpha_{n+1}\sigma_n\pi_n = \alpha_{n+1}\pi_{n+1}\tau_n$ by Theorem 5.2 and the definitions of the π_n and τ_n. Since QS^i is defined by passage to limits from the inclusions

$$\sigma_j S^i = \Omega^j\eta S^{i+j}:\Omega^j S^{i+j} \to \Omega^{j+1}S^{i+j+1},$$

the second statement does follow from the first.

We precede our recognition theorem for E_∞ spaces with two further lemmas. These will lead to a structural description of the homotopy type of the n-fold de-looping of a D_∞-algebra which is based on D_∞ itself, rather than on D_n. Recall that, by Proposition 5.4, there are morphisms of monads $\beta_n: C_n \to \Omega C_{n-1}S$ such that $\alpha_n = (\Omega\alpha_{n-1}S)\beta_n$. We require the analogous result for the D_n.

Lemma 14.2. There exist morphisms of monads $\delta_n: D_n \to \Omega D_{n-1}S$ for $n > 1$ such that the following diagrams are commutative:

$$\begin{array}{ccc} D_n & \xrightarrow{\delta_n} & \Omega D_{n-1}S \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \Omega\pi_{n-1}S} \\ C_n & \xrightarrow{\beta_n} & \Omega C_{n-1}S \end{array} \qquad \text{and} \qquad \begin{array}{ccc} D_n & \xrightarrow{\delta_n} & \Omega D_{n-1}S \\ \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \Omega\tau_{n-1}S} \\ D_{n+1} & \xrightarrow{\delta_{n+1}} & \Omega D_n S \end{array}$$

Proof. Recall that $\Omega D_{n-1}S$ is a monad in $\mathcal{J}$ by Lemma 5.3. Let $X \in \mathcal{J}$. By Definition 3.8 and Construction 2.4, a typical point of D_nX has the form $[(d, c), y]$, where $d \in \mathcal{C}(j)$, $c = \langle c_1, \ldots, c_j \rangle \in \mathcal{C}_n(j)$, and $y \in X^j$. For $t \in I$, write

$$\beta_n[c, y](t) = [\langle c''_{r_1}, \ldots, c''_{r_i} \rangle, z],$$

where $c_r = c'_r \times c''_r$ with $c'_r: J \to J$, the r_k are those indices r such that $t \in c'_r(J)$, and $z \in (SX)^i$ is as determined in the proof of Proposition 5.4. By Notations 2.3 and Definition 4.1(d), we can choose degeneracy operators

$\sigma_{k_1}, \ldots, \sigma_{k_{j-i}}$ such that

$$\sigma_{k_1} \cdots \sigma_{k_{j-i}} c = \langle c_{r_1}, \ldots, c_{r_i} \rangle .$$

We define δ_n by the formula

$$\delta_n[(d,c),y](t) = [(\sigma_{k_1} \cdots \sigma_{k_{j-i}} d, \langle c''_{r_1}, \ldots, c''_{r_i} \rangle), z] .$$

It is then easy to verify that δ_n is a well-defined morphism of monads such that the stated diagrams commute.

Let $\delta_{ij}: D_{i+j} \to \Omega^i D_j S^i$ denote the composite morphism of monads

$$D_{i+j} \xrightarrow{\delta_{i+j}} \Omega D_{i+j-1} S \xrightarrow{\Omega \delta_{i+j-1} S} \Omega^2 D_{i+j-2} S^2 \longrightarrow \cdots \longrightarrow \Omega^i D_j S^i ,$$

and define $\beta_{ij}: C_{i+j} \to \Omega^i C_j S^i$ similarly. Define $\delta_{i\infty}: D_\infty \to \Omega^i D_\infty S^i$ by passage to limits over j.

<u>Lemma 14.3.</u> Let $\lambda_{ij}: D_j S^i D_{i+j} \to D_j S^i$ be the composite

$$D_j S^i D_{i+j} \xrightarrow{D_j \phi^i(\delta_{ij})} D_j D_j S^i \xrightarrow{\nu_j S^i} D_j S^i .$$

Then $(D_j S^i, \lambda_{ij})$ is a D_{i+j}-functor in $D_j[\mathcal{T}]$, and

$$\tau_j S^i : (D_j S^i, \lambda_{ij}) \longrightarrow (D_{j+1} S^i, \lambda_{i,j+1} \circ D_{j+1} S^i \tau_{i+j})$$

and

$$\alpha_j \pi_j S^i : (D_j S^i, \lambda_{ij}) \longrightarrow (\Omega^j S^{i+j}, \Omega^j \phi^{i+j}(\alpha_{i+j} \pi_{i+j}))$$

are morphisms of D_{i+j}-functors in $D_j[\mathcal{T}]$. By passage to limits over j,

$D_\infty S^i$ inherits a structure of D_∞-functor in $D_\infty[\mathcal{J}]$, with action

$$\lambda_{i\infty}: D_\infty S^i D_\infty \xrightarrow{D_\infty \phi^i(\delta_{i\infty})} D_\infty D_\infty S^i \xrightarrow{\nu_\infty S^i} D_\infty S^i ,$$

and $\alpha_\infty \pi_\infty S^i : D_\infty S^i \to QS^i$ is a morphism of D_∞-functors in $D_\infty[\mathcal{J}]$.

Proof. $D_j\phi^i(\delta_{ij}) = D_j\phi^i(1)\cdot D_jS^i\delta_{ij}$, and it is trivial to verify that $\nu_jS^i\cdot D_j\phi^i(1)$ gives D_jS^i a structure of $\Omega^i D_jS^i$-functor in $D_j[\mathcal{J}]$ by use of Lemma 5.3. Thus (D_jS^i, λ_{ij}) is a D_{i+j}-functor in $D_j[\mathcal{J}]$ by Example 9.5(ii). The following two commutative diagrams show that τ_jS^i and $\alpha_j\pi_jS^i$ are morphisms of D_{i+j}-functors in $D_j[\mathcal{J}]$, and thus complete the proof:

$$\begin{tikzcd}
D_jS^iD_{i+j} \arrow[r, "\tau_jS^iD_{i+j}"] \arrow[d, "D_j\phi^i(\delta_{ij})"'] & D_{j+1}S^iD_{i+j} \arrow[r, "D_{j+1}S^i\tau_{i+j}"] \arrow[d, "D_{j+1}\phi^i(\delta_{ij})"] & D_{j+1}S^iD_{i+j+1} \arrow[d, "D_{j+1}\phi^i(\delta_{i,j+1})"] \\
D_jD_jS^i \arrow[r, "\tau_jD_jS^i"] \arrow[d, "\nu_jS^i"'] & D_{j+1}D_jS^i \arrow[r, "D_{j+1}\tau_jS^i"] & D_{j+1}D_{j+1}S^i \arrow[d, "\nu_{j+1}S^i"] \\
D_jS^i \arrow[rr, "\tau_jS^i"] & & D_{j+1}S^i
\end{tikzcd}$$

$$\begin{tikzcd}
D_j S^i D_{i+j} \arrow[r, "\alpha_j \pi_j S^i"] \arrow[d, "D_j \phi^i(\delta_{ij})"'] & \Omega^j S^{i+j} D_{i+j} \arrow[r, "\Omega^i S^{i+j} \alpha_{i+j} \pi_{i+j}"] \arrow[d, "\Omega^j S^j \phi^i(\delta_{ij})"] & \Omega^j S^{i+j} \Omega^{i+j} S^{i+j} \arrow[d, "\Omega^j S^j \phi^i(1)"] \\
D_j D_j S^i \arrow[r, "\alpha_j \pi_j S^i"] \arrow[d, "\nu_j S^i"'] & \Omega^j S^j D_j S^i \arrow[r, "\Omega^j S^j \alpha_j \pi_j S^i"] & \Omega^j S^j \Omega^j S^{i+j} \arrow[d, "\Omega^j \phi^j(1)"] \\
D_j S^i \arrow[rr, "\alpha_j \pi_j S^i"] & & \Omega^j S^{i+j}
\end{tikzcd}$$

The upper left and bottom rectangles commute since τ_j and $\alpha_j \pi_j$ are natural and are morphisms of monads. The upper right rectangles commute by Lemma 14.2 and Proposition 5.4, which imply that

$$\delta_{i,j+1} \cdot \tau_{i+j} = \Omega^i \tau_j S^i \cdot \delta_{ij}$$

and

$$\alpha_{i+j} \cdot \tau_{i+j} = \Omega^i \alpha_j S^i \cdot \beta_{ij} \cdot \pi_{i+j} = \Omega^i \alpha_j \pi_j S^i \cdot \delta_{ij} \ .$$

Recall that by Theorems 5.1 and 5.2, if $Y = \{Y_i\} \in \mathcal{L}_\infty$, so that $Y_i = \Omega Y_{i+1}$, then $(Y_o, \theta_\infty \pi_\infty)$ is a D_∞-algebra and $\theta_\infty : C_\infty Y_o \to Y_o$ factors as the composite

$$C_\infty Y_o \xrightarrow{\alpha_\infty = \varinjlim \alpha_n} \Omega^\infty S^\infty Y_o \xrightarrow{\xi_\infty = \varinjlim \Omega^n \phi^n(1)} Y_o \ .$$

We shall write $W: \mathcal{L}_\infty \to D_\infty[\mathcal{T}]$ for the functor given on objects by $WY = (Y_o, \theta_\infty \pi_\infty)$. Recall also that if $Z \in \mathcal{T}$, then $Q_\infty Z$ denotes the free infinite loop sequence $\{QS^iZ\}$ generated by Z, as described in formulas (5.7), (5.8), and (5.9).

We retain the notations of the previous section for our geometric constructions, and we have the following recognition theorem for E_∞-spaces.

<u>Theorem 14.4.</u> Let (X,ξ) be a D_∞-algebra, and regard X as a D_n-algebra via the restriction of ξ to $D_nX \subset D_\infty X$. Then the following is a commutative diagram of morphisms of D_j-algebras for all $i \geq 0$ and $j \geq 1$:

$$\begin{tikzcd}
 & & \Omega^j B(S^{i+j}, D_{i+j}, X) \arrow[dd, "\Omega^j B(\eta S^{i+j}, \tau_{i+j}, 1)"] \\
B(\Omega^j S^{i+j}, D_{i+j}, X) \arrow[urr, "\gamma^j"] \arrow[dd, "B(\sigma_j S^i, \tau_{i+j}, 1)"'] & & \\
 & & \Omega^j B(\Omega S^{i+j+1}, D_{i+j+1}, X) \arrow[dd, "\Omega^j \gamma"] \\
B(\Omega^{j+1} S^{i+j+1}, D_{i+j+1}, X) \arrow[urr, "\gamma^j"] \arrow[drr, "\gamma^{j+1}"'] & & \\
 & & \Omega^{j+1} B(S^{i+j+1}, D_{i+j+1}, X)
\end{tikzcd}$$

Define an infinite loop sequence $B_\infty X = \{B_i X\}$ by

$$B_i X = \varinjlim \Omega^j B(S^{i+j}, D_{i+j}, X)$$

and, for $i \geq 0$, define a morphism γ^∞ of D_∞-algebras by

$$\gamma^\infty = \varinjlim \gamma^j : B(QS^i, D_\infty, X) \to B_i X .$$

Consider the further morphisms of D_∞-algebras

$$B(\alpha_\infty \pi_\infty S^i, 1, 1): B(D_\infty S^i, D_\infty, X) \to B(QS^i, D_\infty, X)$$

and

$$\varepsilon(\xi) : B(D_\infty, D_\infty, X) \to X$$

(i) $\varepsilon(\xi)$ is a strong deformation retraction with right inverse $\tau(\zeta_\infty)$, where $\zeta_\infty : X \to D_\infty X$ is given by the unit ζ_∞ of D_∞.

(ii) $B(\alpha_\infty \pi_\infty S^i, 1, 1)$ is a weak homotopy equivalence if $i > 0$ or if $i = 0$ and X is connected.

(iii) γ^∞ is a weak homotopy equivalence for all i and X.

(iv) The composite $\gamma^\infty B(\alpha_\infty \pi_\infty, 1, 1)\tau(\zeta_\infty): X \to B_0 X$ coincides with $\iota = \varinjlim \phi^{-1}\tau(1), \tau(1): S^j X \to B(S^j, D_j, X)$.

(v) $B_i X$ is $(m+i)$-connected if X is m-connected.

(vi) Let $Y = \{Y_i\} \in \mathcal{L}_\infty$ and define $\omega : B_\infty WY \to Y$ by

$$\omega_i = \varinjlim \Omega^j \varepsilon \phi^{i+j}(1): B_i WY \to Y_i$$

(where $\Omega^j \varepsilon \phi^{i+j}(1): \Omega^j B(S^{i+j}, D_{i+j}, \Omega^{i+j} Y_{i+j}) \to \Omega^j Y_{i+j}$).

Then ω_i is a weak homotopy equivalence if Y_i is i-connected and, for all Y, the following diagram is commutative and ω_0 is a retraction with right inverse ι:

$$\begin{tikzcd}
B(D_\infty, D_\infty, WY) \arrow[rr, "B(\alpha_\infty \pi_\infty, 1, 1)"] \arrow[d, "\varepsilon(\theta_\infty \pi_\infty)"'] & & B(Q, D_\infty, WY) \arrow[d, "\gamma^\infty"] \arrow[dll, "\varepsilon(\xi_\infty)"'] \\
Y_0 & & B_0 WY \arrow[ll, "\omega_0"']
\end{tikzcd}$$

(vii) Let $Z \in \mathcal{J}$. Then the composite

$$B_\infty D_\infty Z \xrightarrow{B_\infty(\alpha_\infty \pi_\infty)} B_\infty QZ \xrightarrow{\omega} Q_\infty Z$$

is a strong deformation retraction in $\mathcal{L}_\infty$ with right inverse the adjoint $\phi_\infty(\iota\zeta_\infty)$ of $\iota\zeta_\infty : Z \to B_0 D_\infty Z$.

Proof. In view of the definitions of Construction 9.6, the specified spaces and maps are well-defined by Lemmas 14.1 and 14.3. The diagram commutes by the naturality of γ^j (since $\sigma_j = \Omega^j \eta$) and by the definition of γ^{j+1} Of course, $\varepsilon(\xi)$, $B(\alpha_\infty \pi_\infty S^i, 1, 1)$, and γ^∞ are morphisms of D_∞-algebras by Theorems 12.2 and 12.4. Now (i) follows from Proposition 9.8 and Corollary 11.10, (ii) follows from the approximation theorem (Theorem 6.1), Propositions 3.4 and 3.10, and Theorem 11.13, and (iii) follows from Theorem 12.3. Parts (iv) and (v) follow from the corresponding parts of Theorem 13.1 by passage to limits. For (vi), ω is well-defined since the following diagram commutes by the naturality of ε and of γ and by the fact that $\gamma = 1$ on $\Omega Z = |\Omega_* Z_*|$, $Z \in \mathcal{J}$:

$$\begin{tikzcd}
B(S^n, D_n, \Omega^n Y_n) \arrow[r, "B(\eta S^n, \tau_n, 1)"] \arrow[d, "\varepsilon\phi^n(1)"'] & B(\Omega S^{n+1}, D_{n+1}, \Omega^{n+1} Y_{n+1}) \arrow[d, "\gamma"] \arrow[dl, "\varepsilon\Omega\phi^n(1)"'] \\
Y_n = \Omega Y_{n+1} & \Omega B(S^{n+1}, D_{n+1}, \Omega^{n+1} Y_{n+1}) \arrow[l, "\Omega\varepsilon\phi^{n+1}(1)"']
\end{tikzcd}$$

The commutativity of the diagram in (vi) follows by passage to limits from Theorem 13.1(vi). If Y_o is connected, then $\omega_o = \Omega^i \omega_i$ is a weak homotopy equivalence by parts (i), (ii), and (iii) and the diagram; it follows that ω_i is a weak homotopy equivalence if Y_i is i-connected. For (vii), the explicit deformations of $B(S^n, D_n, D_n Z)$ given by Proposition 9.9 and Corollary 11.10, and the loops of these homotopies, are easily verified to yield deformations $h_{i,t}$ of $B_i D_\infty Z$ in the limit such that $\Omega h_{i+1,t} = h_{i,t}$. The fact that $\phi_\infty(\iota\zeta_\infty)$ is the right inverse to $\omega B_\infty(\alpha_\infty \pi_\infty)$ follows by passage to limits from Theorem 13.1(vii) and the definition (5.9) of ϕ_∞.

Up to weak homotopy equivalence in $\mathcal{L}_\infty$, there is only one connective $Y \in \mathcal{L}_\infty$ such that WY is weakly homotopy equivalent as a D_∞-algebra to a given connected D_∞-algebra X.

Corollary 14.5. If $(X, \xi) \xleftarrow{f} (X', \xi') \xrightarrow{g} (Y_0, \theta_\infty \pi_\infty)$ is a weak homotopy equivalence of connected D_∞-algebras, where $Y = \{Y_i\} \in \mathcal{L}_\infty$ and each Y_i is connected, then the diagram of infinite loop sequences

$$B_\infty X \xleftarrow{B_\infty f} B_\infty X' \xrightarrow{B_\infty g} B_\infty WY \xrightarrow{\omega} Y$$

displays a weak homotopy equivalence in $\mathcal{L}_\infty$ between Y and $B_\infty X$.

Proof. By Theorem 11.13 and passage to limits, each functor B_i preserves weak homotopy equivalences between connected D_∞-algebras; since $Y_i = \Omega Y_{i+1}$, each Y_i is i-connected, and therefore each ω_i is a weak homotopy equivalence by the theorem.

Since our de-loopings B_i are not constructed iteratively, we should verify that $B_{i+j}X$ is indeed weakly homotopy equivalent to $B_i B_j X$. To see this, define functors $\Omega^j\colon \mathcal{L}_\infty \to \mathcal{L}_\infty$ for all integers j by letting the i-th space $\Omega^j_i Y$ of $\Omega^j Y$, $i \geq 0$, be

$$\Omega^j_i Y = \begin{cases} Y_{i-j} & \text{if } i \geq j \\ \Omega^j Y_i & \text{if } i < j \end{cases} .$$

Observe that if $j \geq 0$, then the zero-th space of $\Omega^{-j}Y$ is Y_j. Clearly $\Omega^j \Omega^k Y = \Omega^{j+k} Y$ for all j and k, and $\Omega^0 = 1$. We have the following addendum to part (vi) of the theorem.

<u>Corollary 14.6.</u> If $Y \in \mathcal{I}_\infty$ and $\Omega_i^j Y$ is connected for all i, then $\omega: B_\infty W\Omega^j Y \to \Omega^j Y$ is a weak homotopy equivalence in $\mathcal{I}_\infty$. In particular, if (X, ξ) is a D_∞-algebra and if $j \geq 0$ or if $j = 0$ and X is connected, then, for $i \geq 0$,

$$\omega_i: B_i B_j X = B_i W\Omega^{-j} B_\infty X \to \Omega_i^{-j} B_\infty X = B_{i+j} X$$

is a weak homotopy equivalence.

We require one further, and considerably less obvious, consistency result. Recall that if an operad acts on a space X, then, by iterative use of Lemma 1.5, the same operad acts on each $\Omega^i X$, $i > 0$. We thus obtain functors $\Omega^i: D_\infty[\mathcal{T}] \to D_\infty[\mathcal{T}]$, and we wish to compare the infinite loop sequences $\Omega^i B_\infty X$ and $B_\infty \Omega^i X$, at least for D_∞-algebras which arise from C-algebras. To this end, let $\tau_n' = 1 \times \sigma_n': \mathcal{D}_n \to \mathcal{D}_{n+1}$, where $\sigma_n': \mathcal{C}_n \to \mathcal{C}_{n+1}$ is the inclusion of Lemma 4.9 (which gives the first coordinate the privileged role). Let $\tau_{ij}': \mathcal{D}_j \to \mathcal{D}_{i+j}$ denote the composite morphism of operads

$$\mathcal{D}_j \xrightarrow{\tau_j'} \mathcal{D}_{j+1} \xrightarrow{\tau_{j+1}'} \mathcal{D}_{j+2} \longrightarrow \cdots \longrightarrow \mathcal{D}_{i+j},$$

and define $\tau_{i\infty}': \mathcal{D}_\infty \to \mathcal{D}_\infty$ by passage to limits over j; this makes sense since $\tau_{i+j} \cdot \tau_{ij}' = \tau_{i,j+1}' \cdot \tau_j$. It follows easily from Lemma 4.9 that $\tau_{i\infty}'$ is a local Σ-equivalence of operads.

Proposition 14.7. For $i > 0$, let $D'_\infty S^i$ denote the functor $D_\infty S^i$ regarded as a D_∞-functor in $D_\infty[\mathcal{J}]$ via the action

$$D_\infty S^i D_\infty \xrightarrow{D_\infty S^i \tau'_{i\infty}} D_\infty S^i D_\infty \xrightarrow{\lambda_{i\infty}} D_\infty S^i \ .$$

Then $\varepsilon(\xi \circ D_\infty \phi^i(1)) : B(D'_\infty S^i, D_\infty, \Omega^i X) \to X$ is a well-defined morphism of D_∞-algebras for any D_∞-algebra (X, ξ), and $\varepsilon(\xi \circ D_\infty \phi^i(1))$ is a weak homotopy equivalence if X is i-connected.

Proof. Let $\xi_i : D_\infty \Omega^i X \to \Omega^i X$ denote the D_∞-algebra structure map determined from ξ by Lemma 1.5 (the previous notation $\Omega^i \xi$ would be confusing here). We claim that ξ_i factors as the following composite:

$$D_\infty \Omega^i X \xrightarrow{\tau'_{i\infty}} D_\infty \Omega^i X \xrightarrow{\delta_{i\infty}} \Omega^i D_\infty S^i \Omega^i X \xrightarrow{\Omega^i D_\infty \phi^i(1)} \Omega^i D_\infty X \xrightarrow{\Omega^i \xi} \Omega^i X.$$

Since $\tau'_{i\infty}$ results by replacing each little ∞-cube c by the ∞-cube $1^i \times c$, the proofs of Lemma 14.2 and of Proposition 5.4 imply that

$$\delta_{i\infty} \tau'_{i\infty}[(d, c), y_1, \ldots, y_j](s) = [(d, c), [y_1, s], \ldots, [y_j, s]]$$

for $d \in \mathcal{C}(j)$, $c \in \mathcal{C}_\infty(j)$, $y_r \in \Omega^i X$, and $s \in I^i$.

Since $\phi^i(1)$ is the evaluation map, $\phi^i[y, s] = y(s)$, ξ_i is indeed equal to the stated composite by Lemma 1.5. Therefore the following diagram is commutative, and this implies that $\varepsilon(\xi \circ D_\infty \phi^i(1))$ is well-defined by Lemma 9.2, Construction 9.6, and the definition of $\lambda_{i\infty}$ in Lemma 14.3:

$$\begin{tikzcd}
D_\infty S^i D_\infty \Omega^i X \arrow[r, "D_\infty \phi^i(\delta_{i\infty}\tau'_{i\infty})"] \arrow[dd, "D_\infty S^i \xi_i"'] \arrow[ddr, "D_\infty \phi^i(\xi_i)"] & D_\infty D_\infty S^i \Omega^i X \arrow[r, "\nu_\infty S^i"] \arrow[d, "D_\infty D_\infty \phi^i(1)"] & D_\infty S^i \Omega^i X \arrow[d, "D_\infty \phi^i(1)"] \\
 & D_\infty D_\infty X \arrow[r, "\nu_\infty"] \arrow[d, "D_\infty \xi"] & D_\infty X \arrow[d, "\xi"] \\
D_\infty S^i \Omega^i X \arrow[r, "D_\infty \phi^i(1)"] & D_\infty X \arrow[r, "\xi"] & X
\end{tikzcd}$$

Moreover, by the naturality of ε and of γ^i, the following diagram is also commutative:

$$\begin{tikzcd}
B(D_\infty, D_\infty, \Omega^i X) \arrow[r, "B(\delta_{i\infty}\tau'_{i\infty}, 1, 1)"] \arrow[d, "\varepsilon(\xi_i)"'] & B(\Omega^i D'_\infty S^i, D_\infty, \Omega^i X) \arrow[d, "\gamma^i"] \arrow[dl, "\varepsilon(\Omega^i\xi \circ \Omega^i D_\infty \phi^i(1))"] \\
\Omega^i X & \Omega^i B(D'_\infty S^i, D_\infty, \Omega^i X) \arrow[l, "\Omega^i \varepsilon(\xi \circ D_\infty \phi^i(1))"]
\end{tikzcd}$$

Here $\delta_{i\infty}\tau'_{i\infty} : D_\infty \to \Omega^i D'_\infty S^i$ is a morphism of D_∞-functors in $D_\infty[\mathcal{J}]$ by a simple diagram chase from Lemma 5.3. By Theorem 14.4(i) and Theorem 12.3, $\varepsilon(\xi_i)$ and γ^i are weak homotopy equivalences. For connected spaces Z, $\tau'_{i\infty} : D_\infty Z \to D_\infty Z$ and $\delta_{i\infty} : D_\infty Z \to \Omega^i D_\infty S^i Z$ are weak homotopy equivalences by Proposition 3.4 and by the approximation theorem (since $\alpha_\infty \pi_\infty = \Omega^i \alpha_\infty \pi_\infty S^i \circ \delta_{i\infty}$). By Theorem 11.13, $B(\delta_{i\infty}\tau'_{i\infty}, 1, 1)$ is thus a weak homotopy equivalence if X is i-connected and, by the diagram, $\varepsilon(\xi \circ D_\infty \phi^i(1))$ is then also a weak homotopy equivalence.

Lemma 14.8. Let $(X, \theta\psi_\infty)$ be the D_∞-algebra determined by a C-algebra (X, θ). Then

$$B(1, \tau'_{i\infty}, 1) \colon B(D'_\infty S^i, D_\infty, X) \to B(D_\infty S^i, D_\infty, X)$$

is a well-defined morphism of D_∞-algebras and is a weak homotopy equivalence if X is connected.

Proof. Since $\psi_\infty \colon D_\infty \to C$ is the projection, we obviously have $\psi_\infty = \psi_\infty \tau'_{i\infty}$. In view of the definition of $D'_\infty S^i$, $(1, \tau'_{i\infty}, 1)$ is thus a morphism in the category $\mathcal{B}(\mathcal{J}, D[\mathcal{J}])$ of Construction 9.6 and $B(1, \tau'_{i\infty}, 1)$ is well-defined. The last part follows from Proposition 3.4 and Theorem 11.13.

By combining the previous lemma (applied to $\Omega^i X$ instead of to X) and proposition with Theorem 14.4 and Corollaries 14.5 and 14.6, we obtain the desired comparison between $\Omega^i B_\infty X$ and $B_\infty \Omega^i X$ for C-algebras (X, θ).

Theorem 14.9. Let (X, θ) be a C-algebra and let $\xi = \theta\psi_\infty$. Assume that X is i-connected. Then the diagram

$$X \xleftarrow{\varepsilon(\xi \circ D_\infty \phi^i(1))} B(D'_\infty S^i, D_\infty, \Omega^i X) \xrightarrow{\gamma^\infty \circ B(\alpha_\infty \pi_\infty S^i, \tau'_{i\infty}, 1)} B_i \Omega^i X$$

displays a weak homotopy equivalence of D_∞-algebras between (X, ξ) and $W\Omega^{-i}B_\infty\Omega^i X$. Therefore the infinite loop sequences $B_\infty X$ and $\Omega^{-i}B_\infty\Omega^i X$ are weakly homotopy equivalent.

Remark 14.10. Observe that if $Y \in \mathcal{L}_\infty$, then $\Omega^i WY$ and $W\Omega^i Y$ have the same underlying space, namely $\Omega^i Y_o$, and respective actions $\Omega^i\theta_\infty \circ \pi_\infty$ and $\theta_\infty \circ \pi_\infty$. By passage to limits, Lemma 5.6 implies that the action $\Omega^i\theta_\infty$ of C_∞ on $\Omega^i Y_o$ derived in Lemma 1.5 satisfies $\Omega^i\theta_\infty = \theta_\infty \circ \sigma'_{i\infty}$ (where $\sigma'_{i\infty}$ is defined from the σ'_n as $\tau'_{i\infty}$ was defined from the $\tau'_n = 1 \times \sigma'_n$). Therefore $\Omega^i\theta_\infty \circ \pi_\infty = \theta_\infty \circ \pi_\infty \circ \tau'_{i\infty}$, and, by Proposition 3.4, the action maps $D_\infty\Omega^i Y_o \to \Omega^i Y_o$ of $\Omega^i WY$ and $W\Omega^i Y$ are weakly homotopic (at least if $\Omega^i Y_o$ is connected).

15. Remarks concerning the recognition principle

The purpose of this section is to indicate the intent of our recognition theorem for E_∞ spaces in pragmatic terms, to describe some spectral sequences which are implicit in our goemetric constructions, to discuss the connectivity hypotheses in the theorems of the previous section, and to indicate a few directions for possible generalizations of our theory. We shall also construct a rather curious functor from Σ-free operads to E_∞ operads.

Of course, Theorem 14.4 implies that a connected E_∞ space X determines a connective cohomology theory. Pragmatically, this is not the importance of our results. A cohomology theory cannot be expected to be of very much use without an explicit hold on the representing spaces. Ideally, one would like to know their homotopy groups, and one surely wants at least to know their ordinary homology and cohomology groups. Our results are geared toward such computations via homology operations derived directly from the E_∞ structure, and it is crucial for these applications that the homology operations derived from a given C-algebra structure map $\theta: CX \longrightarrow X$, where $\mathcal{C}$ is any E_∞ operad, necessarily agree with the homology operations derived on the equivalent infinite loop space B_0X from the canonical C_∞-algebra structure map $\theta_\infty: C_\infty B_0 X \longrightarrow B_0 X$. In the notations of Theorem 14.4, our theory yields the following commutative diagram, in which the indicated maps are all (weak) homotopy equivalences:

$$\begin{tikzcd}
D_\infty X \arrow[d, "\psi_\infty"', "\simeq"] & D_\infty B(D_\infty, D_\infty, X) \arrow[l, "D_\infty \varepsilon(\theta\psi_\infty)"', "\simeq"] \arrow[r, "D_\infty \gamma^\infty B(\alpha_\infty \pi_\infty, 1, 1)", "\simeq"'] \arrow[d, equal] & D_\infty B_0 X \arrow[d, "\pi_\infty", "\simeq"'] \\
CX \arrow[d, "\theta"'] & B(D_\infty D_\infty, D_\infty, X) \arrow[d, "B(\nu_\infty, 1, 1)"] & C_\infty B_0 X \arrow[d, "\theta_\infty"] \\
X & B(D_\infty, D_\infty, X) \arrow[l, "\varepsilon(\theta\psi_\infty)"', "\simeq"] \arrow[r, "\gamma^\infty B(\alpha_\infty \pi_\infty, 1, 1)", "\simeq"'] & B_0 X
\end{tikzcd}$$

Thus the given geometry $\theta: CX \to X$ is automatically transformed into the little cubes geometry $\theta_\infty: C_\infty B_0 X \to B_0 X$. The force of this statement will become apparent in our subsequent applications of the theory to such spaces as F and BTop, where there will be no direct geometric connection between the relevant E_∞ operad $\mathcal{C}$ and the operad $\mathcal{C}_\infty$.

We indicate one particularly interesting way in which this statement can be applied. With (X, θ) as above, let $f: Z \to X$ be any map of spaces. By use of the adjunction ϕ_∞ of (5.9), we obtain a map of infinite loop sequences $g = \phi_\infty(\iota f): Q_\infty Z \to B_\infty X$ such that the following diagram is commutative:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "\eta_\infty"'] & X \arrow[d, "\iota"] \\
QZ \arrow[r, "g_0"] & B_0 X
\end{tikzcd}$$

Obviously g_0 is a map of C_∞-algebras, by Theorem 5.1. On mod p homology then, identifying $H_*(X)$ with $H_*(B_0 X)$ via ι_* and using Theorem 14.4(iv), we are guaranteed that $(g_0)_*$ transforms the homology operations on QZ coming from $\theta_\infty : C_\infty QZ \to QZ$ into the homology operations on X coming from $\theta: CX \to X$. Since $H_*(QZ)$ is freely generated by $H_*(Z)$ under homology operations (see [20 , Theorem 2.5] for a precise statement), it follows that $(g_o)_*$ is completely determined by f_* and the homology operations on $H_*(X)$.

Theorem 14.9 will have several important concrete applications. For example, the spaces occuring in Bott periodicity are all $\mathcal{C}$-spaces for an appropriate E_∞-operad $\mathcal{C}$ and the various Bott maps $X \to \Omega X'$ (e.g., $X = BU$ and $X' = SU$) are all $\mathcal{C}$-morphisms, where $\Omega X'$ has the $\mathcal{C}$-space structure determined by Lemma 1.5 from that on X'. Via Theorem 14.9, it follows that our spectra $B_\infty X$ are weakly homotopy equivalent to the connective spectra obtained fromthe periodic Bott spectra by killing the bottom homotopy groups. Less obvious examples will arise in the study of submonoids of F.

We should observe that our constructions produce a variety of new spectral sequences, in view of Theorem 11.14. Probably the most interesting of these are the spectral sequences $\{{}^iE^rX\}$ derived by use of ordinary mod p homology in Theorem 11.14 from the simplicial spaces $B_*(D_\infty S^i, D_\infty, X)$ of Theorem 14.4, where X is a connected D_∞-algebra and $i > 0$. Of course, $B(D_\infty S^i, D_\infty, X)$ is weakly homotopy equivalent to the i-th de-looping B_iX of X. For each j and q, the homology $H_q(D_\infty S^i D_\infty^j X)$ is a known functor of $H_*(X)$, determined by [20 , Theorem 2.5], since D_∞ may be replaced by Q. The differentials

$$d = \sum_{i=0}^{j} (-1)^i H_q(\partial_i): H_q(D_\infty S^i D_\infty^j X) \rightarrow H_q(D_\infty S^i D_\infty^{j-1} X)$$

are in principle computable from knowledge of the homology operations on $H_*(X)$; these operations determine $H_q(\partial_j)$, and the $H_q(\partial_i)$ for $i < j$ depend only on the additive structure of $H_*(X)$ as they are derived from natural transformations of functors on $\mathcal{J}$ (with known behavior on homology). Therefore ${}^iE^2$ is a well-defined computable functor of the R-algebra $H_*(X)$, where R is the Dyer-Lashof algebra (see [20]), and $\{{}^iE^rX\}$ converges to $H_*(B_iX)$. It appears unlikely that these spectral sequences will be of direct computational value, but they are curious and deserve further study. In particular, one would like to have a more precise description of ${}^iE^2X$, perhaps as some homological functor of $H_*(X)$, and, in the case $i = 1$, one would like to know the relationship between $\{{}^1E^rX\}$ and the Eilenberg-Moore spectral sequence (derived by use of the Moore loop space on B_1X) converging from $\mathrm{Tor}^{H_*(X)}(Z_p, Z_p)$ to $H_*(B_1X)$.

Although all of our constructions of spaces and maps are perfectly general, the validity of our recognition principle is restricted to connected E_∞ spaces since its proof is based on the approximation theorem. A necessary condition for an H-space X to be homotopy equivalent to a loop space is that X be group-like, in the sense that $\pi_o(X)$ is a group under the induced product. It is trivial to verify that a homotopy associative group-like H-space

X is homotopy equivalent to $X_o \times \pi_o(X)$, where X_o is the component of the identity element. It follows that a group-like E_∞ space X is weakly homotopy equivalent to an infinite loop space since both X_o and the Abelian group $\pi_o(X)$ are. Such a statement is of no pragmatic value since the equivalence does not preserve the E_∞ space structures: there are many examples (such as $\Omega^2 BU$ and QS^o) of E_∞ spaces with non-trivial homology operations on zero-dimensional classes but, as a product, $X_o \times \pi_o(X)$ has only trivial homology operations on such classes (see [20, Theorem 1.1]).

A more satisfactory result can be obtained by reworking everything in the previous section with C, C_j, and D_j replaced by the monads ΩCS, $\Omega C_j S$, and $\Omega D_j S$. Of course, any $\Omega D_\infty S$-algebra is a D_∞-algebra by pull-back along $\delta_{1\infty}: D_\infty \to \Omega D_\infty S$, and therefore any $\Omega D_\infty S$-algebra is a group-like E_∞ space Given a $\Omega D_\infty S$-algebra (X, ξ), define $\tilde{B}_\infty X = \{\tilde{B}_i X\}$ by

$$\tilde{B}_i X = \varinjlim \Omega^j B(S^{i+j}, \Omega D_{i+j-1} S, X)$$

and consider the following spaces and maps:

$$\begin{array}{ccccccc} X & \xleftarrow{\varepsilon(\xi)} & B(\Omega D_\infty S, \Omega D_\infty S, X) & \xrightarrow{B(\Omega\alpha_\infty \pi_\infty S, 1, 1)} & B(\Omega QS, \Omega D_\infty S, X) & \xrightarrow{\gamma^\infty} & \tilde{B}_o X \\ & & \Big\downarrow \gamma & & \Big\downarrow \gamma & & \\ & & \Omega B(D_\infty S, \Omega D_\infty S, X) & \xrightarrow{\Omega B(\alpha_\infty \pi_\infty S, 1, 1)} & \Omega B(QS, \Omega D_\infty S, X) & & \end{array}$$

By Theorems 12.2 and 12.4, $\varepsilon(\xi)$, $B(\Omega\alpha_\infty \pi_\infty S, 1, 1)$, and γ^∞ are morphisms of D_∞-algebras (not of $\Omega D_\infty S$-algebras). $\varepsilon(\xi)$ is a homotopy equivalence by Proposition 9.8 and Corollary 11.10, and (granting the appendix to have been generalized so as to show that the various simplicial spaces are strictly proper) the maps γ^{∞}, γ, and $B(\alpha_{\infty}\pi_{\infty}S, 1, 1)$ are weak homotopy equivalences by Theorem 12.3, the approximation theorem, and Theorem 11.13. It follows from the commutative square that the map $B(\Omega\alpha_{\infty}\pi_{\infty}S, 1, 1)$ is also a weak homotopy equivalence. Thus (X, ξ) is weakly homotopy equivalent as a D_{∞}-algebra to $W\widetilde{B}_{\infty}X$. The remaining results of the previous section can be similarly reproven for $\Omega D_{\infty}S$-algebras, with all connectivity hypotheses lowered by one (e.g., Y_i need only be $(i-1)$-connected in the analog of Theorem 14.4(vi)). We omit the details since no applications are presently in view.

Finally, we mention several possible generalizations of our theory. There are various places where it should be possible to replace strictly commuting diagrams by diagrams which only commute up to appropriate homotopies. The technical cost of weakening the notion of operad surely cannot be justified by results, but the notion of $\mathcal{C}$-space might profitably be weakened. It would be useful for applications to BO and BU with the tensor product H-space structure if all reference to base-points could be omitted, but this appears to be awkward within our context. A change in a different direction, suggested by Stasheff, is to define the notion of a homotopy $\mathcal{C}$-space by retaining the commutativity with permutations, degeneracies, and unit that we have required of an action θ of $\mathcal{C}$ on X, but only requiring the resulting map $\theta: CX \to X$ to be such that the various ways of composing θ and $\mu: C^2 \to C$ to obtain maps $C^q X \to X$ agree up to appropriately coherent homotopies.

This possible refinement to our theory is related to an objection that might be raised. We have not proven, nor have we needed, that a space X which is homotopy equivalent to an E_∞ space Y is itself an E_∞ space. This was proven by Boardman and Vogt [7 , 8] (and was essential to their proof of the recognition theorem)by means of a change of operads. With a recognition theorem based on the notion of a homotopy $\mathcal{C}$-space,such an argument might be unnecessary. Alternatively, their argument may generalize to replace a homotopy $\mathcal{C}$-space by a $\mathcal{Q}$-space, for a related operad $\mathcal{Q}$. Of course, one would expect the notion of a homotopy $\mathcal{C}$-space to be homotopy invariant. Indeed, let $f: X \to Y$ be a homotopy equivalence with homotopy inverse g, where $(Y,\theta) \in \mathcal{C}[\mathcal{T}]$. Define $\theta': CX \to X$ to be the composite

$$CX \xrightarrow{Cf} CY \xrightarrow{\theta} Y \xrightarrow{g} X$$

By Corollary A.13, we may replace f by its mapping cylinder (at the price of growing a whisker on $\mathcal{C}$) and thus assume that f is an inclusion, and we may then assume that X is a strong deformation retraction of Y with retraction g. Now $gf = 1$ trivially implies that $\theta'\eta = 1$ on X, but θ' fails to define a C-algebra structure map since the third square in the following diagram only homotopy commutes:

$$\begin{tikzcd} CCX \arrow[r, "CCf"] \arrow[d, "\mu"'] & CCY \arrow[r, "C\theta"] \arrow[d, "\mu"'] & CY \arrow[r, "Cg"] \arrow[d, "\theta"] & CX \arrow[r, "Cf"] \arrow[d, "\theta'"] & CY \arrow[d, "\theta"] \\ CX \arrow[r, "Cf"] & CY \arrow[r, "\theta"] & Y \arrow[r, "g"] & X & Y \arrow[l, "g"'] \end{tikzcd}$$

Intuitively, this is a minor deficiency which should evaporate with the study of the notion of homotopy $\mathcal{C}$ -spaces.

Similarly, the notion of a morphism of $\mathcal{C}$ -spaces can certainly be weakened to an appropriate notion of homotopy $\mathcal{C}$ -morphism (most simply between actual $\mathcal{C}$ -spaces but also between homotopy $\mathcal{C}$ -spaces). The maps f and g above ought then to be homotopy $\mathcal{C}$ -morphisms. As further examples, one would expect the product on an E_∞ space to be a homotopy morphism (see Lemma 1.9) and one would expect the homotopy inverse of a $\mathcal{C}$ -morphism which is a homotopy equivalence to be a homotopy $\mathcal{C}$-morphism. Our theory avoids such a notion at the negligible cost of reversing the direction of certain arrows. We have not pursued these ideas since they are not required for any of the immediately visible applications.

Finally, we point out the following procedure for constructing new operads from old ones.

Construction 15.1. Let $\mathcal{C}$ be an operad. Define $\mathcal{D}\mathcal{C}(j) = |D_*\mathcal{C}(j)|$ where $D_*: \mathcal{U} \to \mathcal{S}\mathcal{U}$ is the functor defined in Construction 10.2. Then $\mathcal{D}\mathcal{C}$ is an operad with respect to the data specified by

(a) $\mathfrak{Q}(\gamma) = |D_*\gamma| : \mathfrak{Q}\mathcal{C}(k) \times \mathfrak{Q}\mathcal{C}(j_1) \times \dots \times \mathfrak{Q}\mathcal{C}(j_k) \longrightarrow \mathfrak{Q}\mathcal{C}(j)$, $j = \Sigma\, j_s$, where we have used the fact that D_* and realization preserve products to identify the left-hand side with $|D_*(\mathcal{C}(k) \times \mathcal{C}(j_1) \times \dots \times \mathcal{C}(j_k))|$.

(b) The identity of $\mathfrak{Q}\mathcal{C}$ is $1 \in \mathcal{C}(1) = F_0|D_*\mathcal{C}(1)|$.

(c) The right action of Σ_j on $\mathfrak{Q}\mathcal{C}(j)$ is the composite

$$\mathfrak{Q}\mathcal{C}(j) \times \Sigma_j \xrightarrow{1 \times |\mathcal{T}_*|} |D_*\mathcal{C}(j)| \times |D_*\Sigma_j| = |D_*(\mathcal{C}(j) \times \Sigma_j)| \xrightarrow{|D_*\alpha|} \mathfrak{Q}\mathcal{C}(j),$$

where $\mathcal{T}_*$ is defined in Construction 10.2 and α is the action of Σ_j on $\mathcal{C}(j)$.

By Proposition 10.4 and Corollary 11.10, each $\mathfrak{Q}\mathcal{C}(j)$ is contractible hence, by (c), $\mathfrak{Q}\mathcal{C}$ is an E_∞ operad if $\mathcal{C}$ is a Σ-free operad.

The E_∞ operad $\mathfrak{Q} = \mathfrak{Q}\mathcal{M}$ has been implicitly exploited by Barratt [4] (see Remark 6.5). This operad is technically convenient because DX is a topological monoid for any $X \in \mathcal{J}$; indeed, the product is induced from the evident pairings

$$\oplus : \mathfrak{Q}(j) \times \mathfrak{Q}(k) = |D_*(\Sigma_j \times \Sigma_k)| \longrightarrow |D_* \Sigma_{j+k}| = \mathfrak{Q}(j+k)$$

by the formula $[d, y]\ [d', y'] = [d \oplus d', y, y']$.

APPENDIX

We prove the technical lemmas on NDR-pairs that we have used and discuss whiskered spaces, monoids, and operads here.

Definition A.1. A pair (X, A) of spaces in $\mathcal{U}$ is an NDR-pair if there exists a map $u: X \to I$ such that $A = u^{-1}(0)$ and a homotopy $h: I \times X \to X$ such that $h(0,x) = x$ for all $x \in X$, $h(t,a) = a$ for all $(t,a) \in I \times A$, and $h(1,x) \in A$ for all $x \in u^{-1}[0,1)$; the pair (h,u) is said to be a representation of (X,A) as an NDR-pair. If, further, $ux < 1$ for all x, so that $h(1,x) \in A$ for all $x \in X$, then (X,A) is a DR-pair. An NDR-pair (X,A) is a strong NDR-pair if $uh(t,x) < 1$ whenever $ux < 1$; thus, if $B = u^{-1}[0,1)$, it is required that (h,u) restrict to a representation of (B,A) as a DR-pair.

By [30, 7.1], (X,A) is an NDR-pair if and only if the inclusion $A \subset X$ is a cofibration. There is little practical difference between the notions of NDR-pair and strong NDR-pair in view of the following example and the discussion below of whiskered spaces.

Example A.2. Define the (reduced) mapping cylinder M_f of a map $f: X \to Y$ in $\mathcal{J}$ to be the quotient space of $X \times I + Y$ obtained by identifying $(x, 0)$ with $f(x)$ and $(*, t)$ with $* \in Y$. Embed X in M_f by $x \to (x, 1)$. It is trivial that $(\overline{M}_f, X)$ is an NDR-pair, where $\overline{M}_f$ is the unreduced mapping cyclinder, but f must be well-behaved near the base-points to ensure that (M_f, X) is an

NDR-pair. Thus let (h,u) and (j,v) represent $(X,*)$ and $(Y,*)$ as NDR-pairs and assume that $vf(x) = u(x)$ and $j(t,f(x)) = fh(t,x)$ for $x \in X$ and $t \in I$. Then (k,w) represents (M_f, X) as an NDR-pair, where

$$w(y) = v(y) \quad \text{and} \quad w(x,s) = \begin{cases} u(x) & 0 \leq s \leq 1/2 \\ \min(u(x), 2-2s) & 1/2 \leq s \leq 1 \end{cases}$$

$$k(t,y) = j(t,y) \quad \text{and} \quad k(t,(x,s)) = \begin{cases} (h(t,x), s+st) & 0 \leq s \leq 1/2 \\ (h(2t-2st,x), s+t-st) & 1/2 \leq s \leq 1 \end{cases}$$

If (h,u) and (j,v) represent $(X,*)$ and $(Y,*)$ as strong NDR-pairs, then (k,w) represents (M_f, X) as a strong NDR-pair. Of course, (M_f, Y) is represented as a DR-pair by (u', h'), where $u'(y) = 0$, $u'(x,s) = \frac{1}{2} s \cdot u(x)$, and

$$h'(t,y) = y \quad \text{and} \quad h'(t,(x,s)) = (x, s(1-t)) .$$

We have frequently used the following result of Steenrod [30, 6.3].

<u>Lemma A.3.</u> Let (h,u) and (j,v) represent (X,A) and (Y,B) as NDR-pairs. Then (k,w) represents the product pair

$$(X,A) \times (Y,B) = (X \times Y, X \times B \cup A \times Y)$$

as an NDR-pair, where $w(x,y) = \min(ux, vy)$ and

$$k(t,x,y) = \begin{cases} \left(h(t,x), j\left(\frac{ux}{vy} t, y\right)\right) & \text{if } vy \geq ux \\ \left(h\left(\frac{vy}{ux} t, x\right), j(t,y)\right) & \text{if } ux \geq vy \end{cases}$$

Further, if (Y, B) is a DR-pair, then so is $(X, A) \times (Y, B)$, since $vy < 1$ for all y implies $w(x, y) < 1$ for all (x, y).

The proof of the following addendum to this lemma is virtually the same as Steenrod's proof of [30, 6.3].

Lemma A.4. Let (h, u) represent (X, A) as an NDR-pair. Then (h_j, u_j) represents $(X, A)^j = (X^j, \bigcup_{i=1}^{j} X^{i-1} \times A \times X^{j-i})$ as a Σ_j-equivariant NDR-pair, where $u_j(x_1, \ldots, x_j) = \min(ux_1, \ldots, ux_j)$ and

$$h_j(t, x_1, \ldots, x_j) = (h(t_1, x_1), \ldots, h(t_j, x_j))$$

with

$$t_i = \begin{cases} t \min\limits_{j \neq i} (ux_j/ux_i) & \text{if some } ux_j < ux_i\,,\ j \neq i \\ t & \text{if all } ux_j \geqslant ux_i\,,\ j \neq i \end{cases}$$

The following sharpening of [30, 7.2] is slightly less obvious.

Lemma A.5. Let (B, A) and (X, B) be NDR-pairs. Then there is a representation (h, u) of (X, A) as an NDR-pair such that $h(I \times B) \subset B$.

Proof. Let (j, v) and (k, w) represent (B, A) and (X, B) as NDR-pairs. Define $f: I \times B \to I$ by $f(t, b) = (1-t)w(b) + tv(b)$. Since $B \to X$ is a cofibration, there exist maps $\tilde{j}: I \times X \to X$ and $\tilde{f}: I \times X \to I$ which make the following diagrams commutative:

$$\begin{tikzcd} 0 \times B \arrow[rr] \arrow[dd] & & I \times B \arrow[dl, "j"'] \arrow[dd] \\ & X & \\ 0 \times X \arrow[ur, "1"] \arrow[rr] & & I \times X \arrow[ul, "\tilde{j}"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \times B \arrow[rr] \arrow[dd] & & I \times B \arrow[dl, "f"'] \arrow[dd] \\ & I & \\ 0 \times X \arrow[ur, "w"] \arrow[rr] & & I \times X \arrow[ul, "\tilde{f}"'] \end{tikzcd}$$

Define u by $u(x) = \max(\tilde{f}(1, k(1,x)), w(x))$ and define h by

$$h(t,x) = \begin{cases} k(2t, x) & 0 \leq t \leq 1/2 \\ \tilde{j}(2t-1, k(1,x)) & 1/2 \leq t \leq 1 \end{cases}$$

It is easy to verify that the pair (h, u) has the desired properties.

We shall shortly need the following lemma on unions, in which the requisite verifications and the continuity proof are again simple and omitted.

Lemma A.6. Let A_i, $1 \leq i \leq n$, be subspaces of X, and let (h_i, u_i) represent (X, A_i) as an NDR-pair. Assume that

(a) $h_j(I \times A_i) \subset A_i$ for $i < j$ and

(b). $u_j x < 1$ implies $u_j h_i(t,x) < 1$ for $i < j$, $t \in I$ and $x \in X$.

Then (j, v) represent $(X, A_1 \cup \ldots \cup A_n)$ as an NDR-pair, where $vx = \min(u_1 x, \ldots, u_n x)$ and

$$j(t,x) = h_n(t_n, h_{n-1}(t_{n-1}, \ldots, h_1(t_1, x) \cdots)),$$

with

$$t_i = \begin{cases} t \min\limits_{j \neq i} u_j x / u_i x & \text{if some } u_j x < u_i x \\ t & \text{if all } u_j x \geq u_i x \ . \end{cases}$$

The functors we have been studying preserve NDR-pairs and strong NDR-pairs in a functorial way; the following ad hoc definition will conveniently express this for us.

Definition A.7. A functor $F: \mathcal{J} \to \mathcal{J}$ is admissible if any representation (h,u) of (X,A) as an NDR-pair determines a representation (Fh, Fu) of (FX, FA) as an NDR-pair such that $(Fh)_t = F(h_t)$ on X and such that, for any map $g: X \to X$ with $ug(x) < 1$ whenever $u(x) < 1$, the map $Fu: FX \to I$ satisfies $(Fu)(Fg)(y) < 1$ whenever $Fu(y) < 1$, $y \in FX$. As examples, S, C, and Ω are admissible (where C is the monad associated to any operad $\mathcal{C}$), with

$$(Su)[x,s] = u(x) \quad , x \in X \text{ and } s \in I;$$

$$(Cu)[c, x_1, \ldots, x_j] = \max_i u(x_i), \quad c \in \mathcal{C}(j) \text{ and } x_i \in X;$$

$$(\Omega u)(f) = \max_{s \in I} uf(s) \quad , f \in \Omega X.$$

Clearly any composite of admissible functors is admissible.

We now discuss whiskered spaces, monoids, and operads. Growing a whisker is a standard procedure for replacing a given base-point by a non-degenerate base-point. For our purposes, what is more important is that the new base-point is strongly and functorially non-degenerate.

Definition A.8 (i) Let $(X,*)$ be a pair in $\mathcal{U}$, $* \in X$. Define $X' = X \vee I$, where I is given the base-point 0 in forming the wedge, and give X' the base-point $1 \in I$. $(X', 1)$ is represented as an NDR-pair by (h,u), where $u(x) = 1$ and $h(t,x) = x$ for $x \in X$ and, for $s \in I$,

$$u(s) = \begin{cases} 1 & \text{if } s \leq 1/2 \\ 2-2s & \text{if } s \geq 1/2 \end{cases} \quad \text{and} \quad h(t,s) = \begin{cases} s+st & \text{if } s \leq 1/2 \\ s+t-st & \text{if } s \geq 1/2 \end{cases}$$

Let $\iota: X \to X'$ and $\rho = h_1: X' \to X$ denote the evident inclusion and retraction. If $f:(X,*) \to (Y,*)$ is a map of pairs, let $f' = f \vee 1: X' \to Y'$; then, by Example A.2, $(M_{f'}, X')$ is a strong NDR- pair (since $uf' = u$ and $h_t f' = f' h_t$), and $(M_{f'}, Y')$ is a DR-pair.

(ii) Let G be a topological monoid with identity e. Then G' is a topological monoid with identity 1 under the product specified by the formula

$$gs = g = sg \quad \text{for } g \in G \text{ and } s \in I$$

and the requirement that the product on G' restrict to the given product on G and the usual multiplication on I. The retraction $\rho: G' \to G$ is clearly a morphism of monoids.

(iii) Let $\mathcal{C}$ be an operad; to avoid confusion, let e denote the identity element in $\mathcal{C}(1)$. Define a new operad $\mathcal{C}'$ and a morphism $\rho: \mathcal{C}' \to \mathcal{C}$ of operads by $\mathcal{C}'(j) = \mathcal{C}(j)$ as a Σ_j-space, with $\rho_j = 1$, for $j > 1$ and by $\mathcal{C}'(1) = \mathcal{C}(1)'$ as a monoid under γ', with ρ_1 the retraction; the maps γ' are defined by commutativity of the diagrams

$$\begin{tikzcd}
\mathcal{C}'(k) \times \mathcal{C}'(j_1) \times \dots \times \mathcal{C}'(j_k) \arrow[r, "\gamma'"] \arrow[d, "\rho_k \times \rho_{j_1} \times \dots \times \rho_{j_k}"'] & \mathcal{C}'(j) \\
\mathcal{C}(k) \times \mathcal{C}(j_1) \times \dots \times \mathcal{C}(j_k) \arrow[r, "\gamma"] & \mathcal{C}(j) \arrow[u, "\subset"']
\end{tikzcd}$$

for $j = j_1 + \dots + j_k \neq 1$ or $k \neq 1$. Of course, $\mathcal{C}'(0) = * = \mathcal{C}(0)$.

Lemma A.9. Let C and C' denote the monads in $\mathcal{J}$ associated to an operad $\mathcal{C}$ and its whiskered operad $\mathcal{C}'$. Let $X \in \mathcal{J}$. Then there is a natural homeomorphism χ from the mapping cylinder M_η of $\eta: X \to CX$ to $C'X$ such that the following diagram commutes:

$$\begin{tikzcd}
 & M_\eta \arrow[dd, "\chi"] \arrow[dr, "r"] & \\
X \arrow[ur, "i"] \arrow[dr, "\eta'"'] & & CX \\
 & C'X \arrow[ur, "\rho"'] &
\end{tikzcd}$$

(where i and r are the standard inclusion and retraction)

Proof. On $CX \subset M_\eta$, let $\chi: CX \to C'X$ be the evident inclusion, and define $\chi(x, s) = [s, x]$ for $(x, s) \in X \times I$, where $s \in I \subset \mathcal{C}(1)'$ on the right. Since

$$(x, 0) = \eta(x) = [e, x] \in M_\eta \text{ and } [0, x] = [e, x] \in C'X,$$

χ is well-defined, and the remaining verifications are easy.

Proposition A.10. Let $\mathcal{C}$ be an operad and let C' be the monad in $\mathcal{T}$ associated to $\mathcal{C}'$. Let X be a C'-algebra and F a C'-functor in $\mathcal{T}$ (e.g., X a $\mathcal{C}$-space and F a C-functor). Assume that F is an admissible functor and that $(X,*)$ is a strong NDR-pair. Then $B_*(F,C',X)$ is a strictly proper simplicial space.

Proof. Let (h,u) represent $(X,*)$ as a strong NDR-pair. As shown in Definition A.7, (h,u) determines a representation (Ch,Cu) of $(CX,C*) = (CX,*)$ as a strong NDR-pair. Clearly $Ch_t \circ \eta = \eta \circ h_t$ and $Cu \circ \eta = u$, hence, by Example A.2, (M_η, X) is a strong NDR-pair. By the lemma above, (M_η, X) is homeomorphic to $(C'X, \eta'X)$ and (h,u) thus explicitly determines a representation of $(C'X, \eta'X)$ as a strong NDR-pair. Write $D = C'$ to simplify notation, and let

$$Y = B_{q+1}(F,D,X) = FD^{q+1}X \quad \text{and} \quad A_i = \operatorname{Im} s_i \subset Y,$$

$$\text{where } s_i = FD^i\eta', \ \eta': D^{q-i}X \to D^{q+1-i}X .$$

Now (h,u) determines a representation $(D^{q-i}h, D^{q-i}u)$ of $(D^{q-i}X,*)$ as a strong NDR-pair and, with X replaced by $D^{q-i}X$, we have just shown that this representation explicitly determines a representation, (k_i, w_i) say, of $(D^{q+1-i}X, \eta'D^{q-i}X)$ as a strong NDR-pair. Since FD^i is admissible, $(h_i, u_i) = (FD^ik_i, FD^iw_i)$ is then a representation of (Y, A_i) as a strong NDR-pair. Since $FD^i\eta'$ is a natural transformation, the following diagram commutes for $i < j$ and $t \in I$:

$$\begin{tikzcd} FD^{q+1}X \arrow[r, "FD^j k_{jt}"] & FD^{q+1}X \\ FD^q X \arrow[u, "FD^i\eta'"] \arrow[r, "FD^{j-1}k_{jt}"] & FD^q X \arrow[u, "FD^i\eta'"'] \end{tikzcd}$$

Therefore $h_j(I \times A_i) \subset A_i$ for $i < j$. By Definition 11.2, $B_*(F,D,X)$ will be strictly proper if it is proper and, by Lemma A.6, $B_*(F,D,X)$ will be proper provided that $u_j y < 1$ implies $u_j h_i(t,y) < 1$ for $i < j$, $t \in I$ and $y \in Y$. By our definition of an admissible functor, this will hold provided that

$$(D^{j-i}w_j)k_i(t,x) < 1 \quad \text{whenever} \quad (D^{j-i}w_j)(x) < 1,$$

for $i < j$, $t \in I$ and $x \in D^{q+1-i}X$. Here k_i and $D^{j-i}w_j$ are explicitly determined by the original representation (h,u) of $(X,*)$ as a strong NDR-pair, and the result is easily verified by inspection of the definitions.

The requirement that $(X,*)$ be a stong NDR-pair is no real restriction in the proposition above in view of the following lemma.

Lemma A.11. Let θ be an action of an operad $\mathcal{C}$ on a based space $X \in \mathcal{U}$. Then there is an action θ' of $\mathcal{C}$ on X' such that $\rho: X' \to X$ is a morphism of $\mathcal{C}$-spaces.

Proof: Let $\iota : X \subset X'$ and define $\theta'_j : \mathcal{C}(j) \times (X')^j \to X'$ by

$$\theta'_j(c, x_1, \dots, x_j) = \begin{cases} \iota\theta_j(c, \rho x_1, \dots, \rho x_j) & \text{if some } x_i \notin (I - 0) \\ x_1 \dots x_j & \text{if all } x_i \in I \end{cases}$$

Here $x_1 \dots x_j \in I \subset X'$; both parts of the domain are closed, and both definitions yield $0 = *$ on the intersection. The requisite verifications are all straightforward.

The following lemma is relevant to the remarks at the end of § 15.

Lemma A.12. Let $(Y, \theta) \in \mathcal{C}[\mathcal{J}]$, let $Y \subset Z$, and let $h: I \times Z \to Z$ be a homotopy such that

$$h(1, z) = z, \quad h(t, y) = y, \quad h(0, z) \in Y, \text{ and } h(tt', z) = h(t, h(t', z))$$

for $z \in Z$, $y \in Y$, and $t, t' \in I$. Then there is an action $\tilde{\theta}$ of $\mathcal{C}'$ on Z such that the retraction $r = h_0 : Z \to Y$ is a morphism of $\mathcal{C}'$-spaces.

Proof. Define $\tilde{\theta}_j$ on $\mathcal{C}(j) \times Z^j$ by commutativity of the diagram

$$\begin{tikzcd} \mathcal{C}(j) \times Z^j \arrow[r, "1 \times r^j"] \arrow[d, "\tilde{\theta}_j"'] & \mathcal{C}(j) \times Y^j \arrow[d, "\theta_j"] \\ Z & Y \arrow[l, "\supset"'] \end{tikzcd}$$

and define $\tilde{\theta}_1 = h$ on $I \times Z \subset \mathcal{C}'(1) \times Z$. The requisite verifications are again completely straightforward.

<u>Corollary A.13.</u> If $(Y,\theta) \in \mathcal{C}[\mathcal{J}]$ and $f: X \to Y$ is any map in $\mathcal{J}$ then there is an action $\tilde{\theta}$ of $\mathcal{C}'$ on M_f such that the retraction $r: M_f \to Y$ is a morphism of $\mathcal{C}'$-spaces.

<u>Proof.</u> Define $h: I \times M_f \to M_f$ by $h_t = h'_{1-t}$, where h' is as defined in Example A.2, and apply the lemma.

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