Derived Fun: Towards Homotopical Algebra

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This two-part series presents the historical motivation for model categories through the evolving need to define derived functors in general settings. The first talk begins with the classical example of Tor in homological algebra, motivating what a derived functor is in abelian settings; the second turns to the cotangent complex, presenting a necessity for a derived functor in a nonabelian setting. Along the way, we'll see how model categories emerged as the right framework to make sense of derived constructions beyond abelian categories—and why they remain essential tools in modern homotopy theory. For those newer to the subject, this is an introduction to a powerful framework in homotopy theory; for those already steeped in it, a chance to revisit the foundations with fresh eyes.

"We rise by lifting others"
–*Robert Ingersoll*

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These are notes for a two-part talk series delivered at the University of Chicago in the summer of 2025, leading up to a discussion of the historical motivation for model categories in homotopy theory. The goal of the talks in part was to demonstrate the continued utility/value in model categories as a framework for homotopy theory at the face of modern developments in ∞-category theory—or at the least persuade the ∞-category enthusiast to value a firm grasp on model categories. For the seasoned members of the audience, these talks offer a less-than-frequently discussed story that historically motivated the advent of model categories—in homotopical algebra.

§1 Homological Algebra

In the spirit of problem-oriented thinking, we identify a motivating problem for our story—**derived functors**. Derived functors are central objects of study in homological algebra, where one aims to study so-called pure algebra using homology theory, or study homology using pure algebra. Here, we present the motivating example of **Tor** and some generalities.

Before we depart on our journey, we introduce some terminology that'll appear often in these talks. In a sense the most fundamental objects of study in algebra are chain complexes. A **chain complex** of R-modules[1] is a sequence of R-modules

$$\cdots \longrightarrow M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \longrightarrow \cdots$$

such that for each $i \geq 0$, $f_i \circ f_{i+1} = 0$. We will use the shorthand $M_\bullet$ to denote a chain complex as above. A chain complex, depicted as above, is furthermore called an **exact sequence** if for each $i \geq 0$, $\ker f_i = \mathrm{im} f_{i+1}$. We will call every truncated exact sequence of the form $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ a **short exact sequence**, and every exact sequence where there is (at least a priori) no truncation in sight, a **long exact sequence**. Denote by $M'_\bullet$, $M_\bullet$, $M''_\bullet$ three chain complexes of R-modules. A **short exact sequence of chain complexes** (of R-modules, say) refers to a short exact sequence $0 \longrightarrow M'_\bullet \longrightarrow M_\bullet \longrightarrow M''_\bullet \longrightarrow 0$, by which we mean a commuting diagram of the form:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & M'_{i+1} \arrow[r] \arrow[d] & M_{i+1} \arrow[r] \arrow[d] & M''_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_i \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d] & M''_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_{i-1} \arrow[r] \arrow[d] & M_{i-1} \arrow[r] \arrow[d] & M_{i-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with exact rows and columns chain complexes. A commutative diagram of the form obtained by erasing the rightmost column and all the zeros in the above diagram, is called a **chain map**—i.e., a map of chain complexes. Before we begin, it is best to recall these central motifs from homology theory.

(i) Homology measures the failure of exactness.

(ii) Homology (with coefficients in R) takes short exact sequences of chain complexes to long exact sequences of R-modules.

The story of Tor begins with Künneth's work on the homology of a product of spaces. His work established a relation between the Betti numbers and the torsion coefficients of the product and those of the individual factors. His results were later strengthened by successors, like Cartan and Eilenberg to provide what is now known as the **Künneth theorem**. In the case of homology over a field, say $\mathbb{R}$, it takes quite the simple form,

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X;\mathbb{R}) \otimes H_j(Y;\mathbb{R}) \xrightarrow{\sim} H_n(X \times Y;\mathbb{R}) \longrightarrow 0$$

where the middle arrow denotes an isomorphism (we phrase it in this manner for a specific reason). This simple form is due to the fact that vector spaces have no torsion, so one need not account for the torsion subgroups of the individual homologies or of the product homology. That changes when we begin working with coefficients in a ring, when

[1]Here, we only consider non-negatively graded chain complexes, meaning $i \geq 0$.

homology takes values in R-modules, and we need a correction factor for the torsion subgroups.

$$
\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i+j=n} H_i(X;R) \otimes H_j(Y;R) \arrow[r] & H_n(X \times Y;R) \arrow[dl] \\
& \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i(X;R), H_j(Y;R)) \arrow[r] & 0
\end{tikzcd}
$$

The correction factor $\mathrm{Tor}_1^R(-,-)$ is called the **torsion product**. But where does this come from?

In the spirit of motivation, at least initially, let us take $R = \mathbb{Z}$ so that R-modules are just abelian groups. Let B be an abelian group. Let us look at the congruence classes modulo l in B. These congruence classes form an abelian group B/lB. One succinct way to describe this group is as the tensor product $\mathbb{Z}/l\mathbb{Z} \otimes B$. Recall the short exact sequence associated to $\mathbb{Z}/l\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{l\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow 0.$$

Seeing as $B/lB \cong \mathbb{Z}/l\mathbb{Z} \otimes B$, we can ask ourselves, to what extent can we recover an exact sequence describing B/lB by tensoring the above exact sequence with B. Noting the isomorphism $\mathbb{Z} \otimes B \cong B$, by tensoring with B on the right, we obtain a sequence:

$$0 \longrightarrow B \xrightarrow{l\cdot} B \longrightarrow B/lB \longrightarrow 0$$

But note most importantly that this is NOT an exact sequence! The map $B \longrightarrow B/lB$ is indeed surjective, and the kernel of this map is indeed the image of $l\cdot$. However, $l\cdot$ can be far from injective! Granted, if B has no l-torsion elements (like when B is free), the map $l\cdot$ is injective and the above sequence certainly is exact. But when B has non-trivial l-torsion, the kernel of $l\cdot$ is precisely the subgroup of l-torsion elements in B (denoted $B[l]$). We can now append the above sequence to the left to obtain a true exact sequence.

$$0 \longrightarrow B[l] \longrightarrow B \xrightarrow{l\cdot} B \longrightarrow B/lB \longrightarrow 0$$

With this intuition in sight, let A be a finitely generated abelian group. As an abelian group, A has a **free resolution** (i.e. is the quotient of a free abelian group by another free abelian group) as the exact sequence: $0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ (where F_i are free). Note that by the fundamental theorem, the map $F_1 \longrightarrow F_0$ captures all of the torsion information in A (namely if A was free, this map would be 0). The tensor product $A \otimes B$ can be thought of as a direct sum of the groups of congruence classes in B modulo those prime powers that appear in the torsion group decomposition of A. Once again, in the spirit of obtaining an exact sequence describing $A \otimes B$, we can tensor the free resolution of A by B to get the exact sequence $F_1 \otimes B \longrightarrow F_0 \otimes B \longrightarrow A \otimes B \longrightarrow 0$. As before, the map on the left can be far from injective. Its kernel is a direct sum of the torsion subgroups $B[p^k]$ for those prime powers p^k that appear in the torsion group decomposition of A. We denote this direct sum by $\mathrm{Tor}_1^{\mathbb{Z}}(A,B)$. This definition extends the tensor product sequence to an exact sequence: $0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A,B) \longrightarrow F_1 \otimes B \longrightarrow F_0 \otimes B \longrightarrow A \otimes B \longrightarrow 0$. We have committed a grave oversight so far in our construction. A priori, A can have two distinct free resolutions–so what is the guarantee that $\mathrm{Tor}_1^{\mathbb{Z}}(A,B)$ defined over either of these resolutions will yield isomorphic results? We will handle this oversight conceptually in the next paragraph—for intuition at the moment, take this for granted. Looking beyond our oversight, this definition of $\mathrm{Tor}_1^{\mathbb{Z}}(-,B)$ allows us to do much better than the defining exact sequence. Suppose $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence. By the snake lemma, there is an exact sequence $0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A', B) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A, B) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A'', B) \longrightarrow A' \otimes B \longrightarrow A \otimes B \longrightarrow A'' \otimes B \longrightarrow 0$. Notice what we have done here, we have taken a short exact sequence of abelian groups $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, which we can really think of as a short exact sequence of the free resolutions (i.e. chain complexes), and obtained a longer exact sequence on passage to $\mathrm{Tor}_1(-, B)$ and $- \otimes B$. Or more succinctly, if I were to say $- \otimes B$ was $\mathrm{Tor}_0(-, B)$, then passing to $\mathrm{Tor}_i(-, B)$ takes a short exact sequence of chain complexes to a long exact sequence of abelian groups. In presenting the story this way, I hope the reader can sense a homological presence here.

Now rather than pass to some kind of limit to extend the definition of Tor to all abelian groups A, we jump right into the conceptually clearer definition of $\mathrm{Tor}_i^R(M, N)$ for a general commutative ring R and R-modules M and N, using homological methods. Let us fix an R-module N. We will define $\mathrm{Tor}_i^R(-, N)$ as follows. Given an R-module M, we pick a free resolution of M that we denote $F_\bullet \longrightarrow M \longrightarrow 0$ (this is shorthand for the exact sequence $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ where each F_i is free). Note that unlike in the abelian group case, this free resolution can be of more than length one (whereas for abelian group, we can only have free resolutions of length one —in a sense, this is exactly why there is no higher Tor for abelian groups). For any R-module K, we can speak of the tensor product of K with N over R, $K \otimes_R N$ defined by the products $k \otimes n$ such that $rk \otimes n = k \otimes rn$ for every $r \in R$ and $k \in K$ and $n \in N$. Over $R = \mathbb{Z}$, this is just the regular tensor product. Tensoring $F_\bullet$ (by which we implicitly mean $F_\bullet \longrightarrow 0$) from the free resolution of M, with N over R, we obtain a chain complex $F_\bullet \otimes_R N$ given by $\cdots \longrightarrow F_1 \otimes_R N \longrightarrow F_0 \otimes_R N \longrightarrow 0$. The i-th homology of this chain complex, as an R-module, is called $\mathrm{Tor}_i^R(M, N)$. Now we handle the ambiguity in picking a free resolution of M. We deduce the independence of $\mathrm{Tor}_i^R(M, N)$ from the choice of free resolution of M using the following lemma.

Claim 1.1 — Given a map $f\colon M \longrightarrow M'$ of R-modules, and free resolutions $F_\bullet \longrightarrow M \longrightarrow 0$ and $F'_\bullet \longrightarrow M' \longrightarrow 0$, there is an extension (unique up to chain homotopy equivalence) of f to a map $\tilde{f}$ of chain complexes:

$$\begin{tikzcd}
F_\bullet \arrow[r] \arrow[d, "\tilde{f}_\bullet"'] & M \arrow[r] \arrow[d, "f"] & 0 \\
F'_\bullet \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

Proof Sketch. The fundamental fact we will use is that free R-modules are **projective**. This property can be stated as follows. For any free R-module F, $M \twoheadrightarrow N$ any epimorphism of R-modules, and $F \longrightarrow N$ any R-module hom, there is a lift $F \longrightarrow M$ such that the following diagram commutes.

$$\begin{tikzcd}
 & M \arrow[d, two heads] \\
F \arrow[r] \arrow[ur, dashed] & N
\end{tikzcd}$$

A succinct way to state this is that projective modules are exactly those modules that have the **left lifting property** with respect to epimorphisms. In this sketch we will briefly outline the construction of the lift $\tilde{f}$ and argue that any lift of the 0 map is nullhomotopic.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] & F_1 \arrow[r, two heads, "\partial_1"] \arrow[d, dashed, "\tilde{f}_1"] \arrow[dr, dotted] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r, "\partial_2'"] & F_1' \arrow[r, two heads, "\partial_1'"] & M' \arrow[r] & 0
\end{tikzcd}$$

We start with the above picture, where the dotted line indicates the composite $f \circ \partial_1$. Applying projectivity of F_1, we get a lift $F_1 \longrightarrow F_1'$ denoted by the dashed line. We will call this lift $\tilde{f}_1$. Now noticing that that $0 = f\partial_1\partial_2 = \partial_1'\tilde{f}_1\partial_2$, we see that the composite $\tilde{f}_1\partial_2$ maps into the image of ∂_2'. Applying projectivity at this composite, yields the lift $\tilde{f}_2$.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] \arrow[d, dashed, "\tilde{f}_2"] \arrow[dr, dotted] & F_1 \arrow[r, two heads, "\partial_1"] \arrow[d, dashed, "\tilde{f}_1"] \arrow[dr, dotted] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r, "\partial_2'"] & F_1' \arrow[r, two heads, "\partial_1'"] & M' \arrow[r] & 0
\end{tikzcd}$$

Extending this procedure inductively provides the lift $\tilde{f}_\bullet$. Of course one has to check that this is a chain map, but that is built into the construction.

Now we briefly argue that every lift of the zero map is nullhomotopic. That this is enough to show that every map of R-modules has a unique lift up to homotopy boils down to the abelianness of the setting of R-modules (namely that the hom-sets of R-modules are all abelian groups). Say $f = 0$, and we have a lift $\tilde{f}_\bullet$ as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & F_3 \arrow[r, "\partial_3"] \arrow[d, "\tilde{f}_3"] & F_2 \arrow[r, "\partial_2"] \arrow[d, "\tilde{f}_2"] \arrow[dl, dashed] & F_1 \arrow[r, two heads, "\partial_1"] \arrow[d, "\tilde{f}_1"] \arrow[dl, dashed] & M \arrow[r] \arrow[d, "f"] \arrow[dl, dashed] & 0 \\
\cdots \arrow[r] & F_3' \arrow[r, "\partial_3'"] & F_2' \arrow[r, "\partial_2'"] & F_1' \arrow[r, two heads, "\partial_1'"] & M' \arrow[r] & 0
\end{tikzcd}$$

To define a nullhomotopy of $\tilde{f}_\bullet$ (treating f as $\tilde{f}_0$ and M, M' as F_0, F_0'), is to define a sequence of maps $\psi_n \colon F_n \longrightarrow F_{n+1}'$ such that $\tilde{f}_n = \partial_{n+1}'\psi_n + \psi_{n-1}\partial_n$. We won't say much here about why this is the right notion of a nullhomotopy for chain complexes—but we will say this much: note that a nullhomotopic $\tilde{f}_\bullet$ induces the 0 map on homology (passing an n-cycle in F_n to $\tilde{f} = \partial'\psi + \psi\partial$ shows that the image is a boundary) [2]. Set $\psi_0 = 0$ and $\psi_{-1} = 0$. Inductively, defining $h_n := \tilde{f}_n - \psi_{n-1}\partial_n$, we see that $\partial_n' h_n = 0$. This means that h_n maps into $\ker \partial_n' = \mathrm{im}\partial_{n+1}'$. On the other hand, there is an epimorphism $\partial_{n+1}' \colon F_{n+1}' \twoheadrightarrow \mathrm{im}\partial_{n+1}'$. Applying projectivity of F_n gives us a map $\psi_n \colon F_n \longrightarrow F_{n+1}'$. The chain homotopy condition at this level is easy to check. □

We make it excruciatingly clear as to why this guarantees that any two free resolutions $F_\bullet \longrightarrow M \longrightarrow 0$ and $F_\bullet' \longrightarrow M \longrightarrow 0$ will produce the same homology. Consider the following diagrams which denote the applications of Claim 1.1 to $\mathrm{id}_M \colon M \longrightarrow M$ and the relevant resolutions. We use $f_\bullet$ to denote a lift of id_M to $F_\bullet \longrightarrow F_\bullet'$, and $g_\bullet$ to denote a lift of id_M to $F_\bullet' \longrightarrow F_\bullet$. Noticing that the composite $g_\bullet \circ f_\bullet$ is a lift of id_M to $F_\bullet \longrightarrow F_\bullet$, we know by Claim 1.1 again that $g_\bullet \circ f_\bullet \simeq \mathrm{id}_{F_\bullet}$. Similarly we also know that $f_\bullet \circ g_\bullet \simeq \mathrm{id}_{F_\bullet'}$. In other terms, $f_\bullet$ represents a homotopy equivalence from $F_\bullet$ to $F_\bullet'$.

[2] One can show treating $\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{(\mathrm{id}, -\mathrm{id})} R^2 \longrightarrow 0$ as the analog of an interval for chain complexes, and defining a nullhomotopy using this interval, will yield the same notion of a nullhomotopy as above.

$$\begin{tikzcd}
F_\bullet \arrow[r] \arrow[d, dashed, "f_\bullet"] \arrow[dd, bend right=60, "\mathrm{id}_{F_\bullet}"'] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
F'_\bullet \arrow[r] \arrow[d, dashed, "g_\bullet"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
F_\bullet \arrow[r] & M \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
F'_\bullet \arrow[r] \arrow[d, dashed, "g_\bullet"] \arrow[dd, bend right=60, "\mathrm{id}_{F'_\bullet}"'] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
F_\bullet \arrow[r] \arrow[d, dashed, "f_\bullet"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
F'_\bullet \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

In the above process, note that we only used projectivity of the resolution! And using only this, we have established that $\mathrm{Tor}_i^R(M, N)$ is independent of the chosen resolution. This means that if instead of a free resolution of M, I had begun with a **projective resolution** of M (i.e. a resolution of M by projective instead of free R-modules), I would have gotten the same answer! A posteriori then, it suffices to begin with the more general notion of a projective resolution, and then follow through with the program. The constructions $\mathrm{Tor}_i^R(-,-)$ defined this way are called the **higher Tor functors**, sometimes called the higher torsion product or the derived tensor product. It is easy to note that right-exactness of $- \otimes_R N$ tells us $\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N$. What's also quite easy to check is that Tor is symmetric in both arguments and that if either M or N is projective (and therefore if either is free), then $\mathrm{Tor}_i^R(M, N)$ vanishes for all M, N.

Perhaps the most valuable insight gained from defining the higher Tor functors this way is that $\mathrm{Tor}_i^R(-, N)$ sends short exact sequences of R-modules (thought of as short exact sequences of the associated free resolutions) to long exact sequences of R-modules. Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of R-modules. There is a long exact sequence, by the snake lemma, given by:

$$\cdots \longrightarrow \mathrm{Tor}_i^R(M', N) \longrightarrow \mathrm{Tor}_i^R(M, N) \longrightarrow \mathrm{Tor}_i^R(M'', N) \longrightarrow \mathrm{Tor}_{i-1}^R(M', N) \longrightarrow \cdots$$

Also recalling that every projective abelian group is free, and noting that every abelian group has a length one free resolution, we can see that $\mathrm{Tor}_i^{\mathbb{Z}}(A, B) \cong 0$ for $i \geq 2$, $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is as defined before, and $\mathrm{Tor}_0^{\mathbb{Z}}(A, B) \cong A \otimes B$.

This concludes our discussion on Tor—we now move on to some generalities. R-modules are particularly nice for computational purposes since the collection of R-modules and that of R-module homomorphisms possess several favorable properties. For one, hom-sets between R-modules have an abelian group structure (i.e. you can add and subtract R-module homomorphisms). There is also a zero R-module, and every R-module homomorphism has a kernel and a cokernel. In fact, every monomorphism of R-modules represents the kernel of some R-module hom, and every epimorphism of R-modules represents the cokernel of some R-module hom. These properties put together are precisely what allow us to talk about exact sequences, chain complexes, and projective and injective objects for example. We call a collection of objects and morphisms a **category**, and one satisfying these properties an **abelian category**. By a functor $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ between categories, we mean a map from the collection of objects in $\mathcal{A}$ to the collection of objects in $\mathcal{B}$ that also sends morphisms in $\mathcal{A}$ from A to A' to morphisms in $\mathcal{B}$ from $F(A) \to F(A')$ (such that F parses over composition and preserves the identity morphisms).

Let $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor of abelian categories. Suppose $\mathcal{F}$ is right-exact, meaning it takes exact sequences in $\mathcal{A}$ to right-exact sequences in $\mathcal{B}$. A projective object in $\mathcal{A}$ is defined to be an object that satisfies the left lifting property with respect to epimorphisms in $\mathcal{A}$. We assume that $\mathcal{A}$ has enough projectives, meaning that every object in $\mathcal{A}$ admits an epimorphism from a projective object. The right-exactness of $\mathcal{F}$ says that to every exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, $\mathcal{F}$ assigns an exact sequence $\mathcal{F}(A') \longrightarrow \mathcal{F}(A) \longrightarrow \mathcal{F}(A'') \longrightarrow 0$—we are looking for the appropriate extension of this short exact sequence to a long exact sequence. As before, given an object $A \in \mathcal{A}$, we take a projective resolution of A denoted by $P_\bullet \longrightarrow A \longrightarrow 0$. Applying $\mathcal{F}$ to the chain complex $P_\bullet$ in $\mathcal{A}$, we get another chain complex $\mathcal{F}(P_\bullet)$ in $\mathcal{B}$. We define the i-th **left derived functor** of $\mathcal{F}$ evaluated at A to be $L^i\mathcal{F}(A) := H_i(\mathcal{F}(P_\bullet))$, the i-th homology of this chain complex. By the lifting argument for projective resolutions presented in Claim 1.1, this prescribes a sequence of functors $L^i\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$. Among other properties already noted above (like the natural isomorphism $L^0\mathcal{F} \cong \mathcal{F}$, or the vanishing of higher derived functors at projective objects), the most notable one is the following. For every short exact sequence in $\mathcal{A}$ given by $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, we obtain a long exact sequence: $\cdots \longrightarrow L^i\mathcal{F}(A') \longrightarrow L^i\mathcal{F}(A) \longrightarrow L^i\mathcal{F}(A'') \longrightarrow L^{i-1}\mathcal{F}(A') \longrightarrow \cdots \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$. We see that Tor is just the left derived functor of $- \otimes_R N$.

The construction is summarized as follows. We can embed $\mathcal{A}$ into the category of chain complexes (up to quasi-isomorphism) in $\mathcal{A}$ by taking projective resolutions. Then we apply $\mathcal{F}$ to land in the category of chain complexes in $\mathcal{B}$ (up to quasi-isomorphism). Taking i-th homology lands you in $\mathcal{B}$. One might notice that I have been biased towards right-exact functors throughout this talk—indeed $- \otimes_R N$ is one such functor. But you can run this whole story dually! Beginning from $\mathrm{Hom}_R(M, -)$, which is a **left-exact functor**, one can extend the left-exact sequences to the right using **injective resolutions** instead of projective resolutions. This defines what is called the **Ext functor**. Generalizing this to the abelian category setting, we have a dual program. Consider a left-exact functor $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ (and assuming $\mathcal{A}$ has enough injectives, meaning every object admits a monomorphism to an injective object). We can again embed $\mathcal{A}$ into the category of cochain complexes in $\mathcal{A}$ (up to quasi-isomorphism, by a dual to Claim 1.1), by taking injective resolutions. Passing through $\mathcal{F}$, we land in the category of cochain complexes (up to quasi-isomorphism) in $\mathcal{B}$. Taking the i-th cohomology of the resulting cochain complex, we get the i-th **right derived functor** of $\mathcal{F}$. One important application of right derived functors is in the case of the left-exact functor of taking global sections of an abelian sheaf—the associated right derived functor is called sheaf cohomology. The moral of the story roughly is this, every right-exact functor deserves a left derived functor, and every left-exact functor deserves a right derived functor.

Next time, we will see how to generalize derived functors beyond abelian categories, why we would want to do this, how homotopy theoretic lifting properties are key to understanding the derived functor—and a historical application of this effort, in building the cotangent complex for commutative rings (which form a nonabelian category).

§2 Homotopical Algebra

Let's pick up where we left off last time. Concisely, what we have achieved can be described as follows. Let $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor of abelian categories, say right-exact. We define the i-th left derived functor of $\mathcal{F}$ as the composite:

$$\mathcal{A} \xrightarrow{\mathbf{B}^0(-)} \mathcal{K}(\mathcal{A}) \xrightarrow{\mathbb{L}\mathrm{Ch}(\mathcal{F})} \mathcal{K}(\mathcal{B}) \xrightarrow{H_i(-)} \mathcal{B}$$

Here $\mathcal{K}(\mathcal{A})$ denotes the category of chain complexes in $\mathcal{A}$ up to homotopy (same for $\mathcal{K}(\mathcal{B})$). Given an object $A \in \mathcal{A}$, $\mathbf{B}^0(-)$ sends it to the chain complex concentrated at 0 with value A. The functor $\mathbb{L}\mathrm{Ch}(\mathcal{F})$ can then best be described as the composite

$\mathcal{K}(\mathcal{A}) \xrightarrow{P} \mathcal{K}(\mathcal{A}) \xrightarrow{\mathcal{K}(\mathcal{F})} \mathcal{K}(\mathcal{B})$, where P is the functorial projective replacement, and $\mathcal{K}(\mathcal{F})$ is the prolongation of $\mathcal{F}$ to chain complexes. So the concentrated complex $\mathbf{B}^0(A) = \cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0$, in the next step, is replaced functorially by a *nicer* chain complex $P\mathbf{B}^0(A)$. First of course one must note that our replacement P has the property that $PA_\bullet$ is quasi-isomorphic to $A_\bullet$ for every $A_\bullet \in \mathcal{K}(\mathcal{A})$ (so that the word replacement is appropriate). Next, we must note that after replacement, we get a projective resolution, say $P_\bullet$. Projective resolutions are particularly simple kinds of objects in $\mathrm{Ch}(\mathcal{A})$, in that they satisfy the following left lifting property over any map of chain complexes $M_\bullet \longrightarrow N_\bullet$ that's a degreewise epimorphism.

$$\begin{tikzcd}
 & \cdots \arrow[r] & M_2 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & M_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[ur, dashed] \arrow[dr] & P_1 \arrow[r] \arrow[ur, dashed] \arrow[dr] & P_0 \arrow[r] \arrow[ur, dashed] \arrow[dr] & 0 & \\
 & \cdots \arrow[r] & N_2 \arrow[r] & N_1 \arrow[r] & N_0 \arrow[r] & 0
\end{tikzcd}$$

In a sense, it is precisely this lifting property that makes the whole theory work. In particular, this allows us to pass $P\mathbf{B}^0(A)$ to $\mathcal{F}$ to get $\mathbb{L}\mathrm{Ch}(\mathcal{F})\mathbf{B}^0(A)$, a chain complex up to homotopy in $\mathcal{B}$. Finally, taking i-th homology lands us back in $\mathcal{B}$. In this sense, we can perhaps see the middle arrow $\mathbb{L}\mathrm{Ch}(\mathcal{F})$ as being the meat of the construction from last time—and this is what we will explore today, homotopy theoretically.

Homotopical algebra stems from the necessity to construct derived functors, and talk about (co)homology, beyond abelian categories. The fundamental insight in going to homotopical algebra from homological algebra is one due to Dold and Kan—that simplicial objects should be seen as a generalization of chain complexes.

Theorem 2.1 (Dold-Kan Correspondence)

For $\mathcal{A}$ an abelian category, there is an equivalence between simplicial objects in $\mathcal{A}$ and chain complexes in $\mathcal{A}$.

To see where this insight comes into play in our theory, one must remark that every object $A \in \mathcal{A}$ can be considered as a discrete simplicial object in $s\mathcal{A}$ (here $s\mathcal{A}$ denotes the simplicial objects in $\mathcal{A}$). In this sense, we have an embedding $\mathcal{A} \hookrightarrow s\mathcal{A} \simeq \mathrm{Ch}(\mathcal{A})$. Now we can ask ourselves, what does the procedure $\mathbb{L}$ look like in the context of $s\mathcal{A}$, particularly when $\mathscr{A}$ is not abelian? In 1958, Kan had the insight that the homotopy theory of simplicial groups was equivalent to the homotopy theory of connected pointed spaces. There was also significant work at the time on defining **derived categories** of an abelian category (what one can think of as chain complexes up to quasi-isomorphisms, which are chain maps that induce isomorphism on all homologies). Quillen was able to put together these insights to note that it is enticing to treat the procedure $\mathbb{L}$ homotopy theoretically—and that it is worthwhile to talk about a homotopy theory of simplicial objects in $\mathcal{A}$.

Before we get too excited (or rather, I get to excited) by the theory, we must return to our principle of problem-oriented thinking. We begin with a theory of differentials on a commutative ring—and the hope is to build a conceptually complete cohomology theory of commutative rings. This is what André and Quillen set out to do in the late 1960s. The starting point for this theory is the module of Kähler differentials on a commutative ring R. The idea is to obtain an analog of the theory of differential forms on a commutative ring or a scheme. We can think of an affine variety as being represented by the coordinate ring $S = R[t_1, \ldots, t_n]$. We know from calculus that the derivative of a polynomial is once again a polynomial. In other terms, given a polynomial $f \in S$, we can expect a good theory of differentials to express df as $f'(t_1)dt_1 + f'(t_2)dt_2 + \cdots f'(t_n)dt_n$, where each $f'(t_i)$ is a polynomial in S. What we have just conceptually motivated is an expectation that the universal theory of differentials on S over R should read $\Omega^1_{S/R} = \bigoplus_{i=1}^n Sdt_i$. Taking this to be definition, the map $d\colon S \longrightarrow \Omega^1_{S/R}$ called the universal **derivation**, takes f to df. The map d possesses several desirable properties. First, $dr = 0$ for every $r \in R$. Next, d satisfies the addition and the Leibniz product rules: $d(f+g) = df + dg$ and $d(fg) = fdg + gdf$. A map $d\colon S \longrightarrow M$ for an S-module M is called a derivation of S. In a sense, $\Omega^1_{S/R}$ is the universal S-module admitting a derivation of S—in that every derivation of S factors through the universal derivation $d\colon S \longrightarrow \Omega^1_{S/R}$.

It is obvious how one can generalize this definition to any ring S admitting a homomorphism $\varphi\colon R \longrightarrow S$. We will call an R-module homomorphism $d\colon S \longrightarrow M$ into an S-module an R-linear derivation of S if d satisfies the Leibniz rule. The module of **Kähler differentials** $\Omega_{S/R}$ is defined as the S-module that admits a universal derivation $d\colon S \longrightarrow \Omega_{S/R}$. As before, we can construct $\Omega_{S/R}$ by building a free S-module with generators ds for each $s \in S$, and quotienting by relations $dr = 0$, $d(f+g) = df + dg$, and $d(fg) = fdg + gdf$. What does this construction imply geometrically? One can show that the module of Kähler differentials is compatible with scalar extensions and therefore compatible with localizations. This means that we can glue together the definition above on affine schemes/varieties, and get a sheaf of Kähler differentials, sometimes called the **cotangent sheaf**. In sum, this allows us to generalize differential form theory to the realm of algebraic geometry. What we have produced thus far is called the functor of Kähler differentials, it is a functor that takes a commutative ring R over T to a T-module $\Omega_{T/R}$.

Consulting with differential geometry for a second—say $f\colon X \longrightarrow Y$ is a surjective equidimensional map between connected manifolds with connected fibers. Then the tangent bundle of X relative to Y, $T_{X/Y}$ is defined as the kernel of the map $T_X \longrightarrow\!\!\!\!\rightarrow f^*T_Y$ (intuitively all this is saying is that the fibers of $T_{X/Y}$ are just the tangent spaces of the fibers of f). In other terms, there is a short exact sequence $0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow f^*T_Y \longrightarrow 0$. Dualizing, we get a short exact sequence in cotangent bundles, $0 \longrightarrow f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$. We can see this as one way of defining relative differentials in differential geometry. Let us go back to the case of commutative rings now. Given a pair of ring homomorphisms $R \xrightarrow{f} S \xrightarrow{g} T$, there is actually a priori only a right exact sequence of T-modules $\Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R} \longrightarrow \Omega_{T/S} \longrightarrow 0$. Seeing $\Omega_{S/R} \otimes_S T$ as the pullback $f^*\Omega_{S/R}$, this reads as the exact sequence $f^*\Omega_{S/T} \longrightarrow \Omega_{T/R} \longrightarrow \Omega_{T/S} \longrightarrow 0$. This is sometimes called the conormal sequence. What we are seeing here, is something that looks like it should be called a right exact functor (but can't quite be called that since **CRng** is not an abelian category). How do we understand the failure of this sequence to be exact? Well, remember, right exact-ish functors morally deserve a left derived functor. In a sense, this is what André and Quillen set out to do.

Let $\mathcal{C}$ denote the category of commutative rings **CRng** from now. Note that as alluded to earlier, there is a functor $\mathcal{C} \longrightarrow s\mathcal{C}$ that takes an object $R \in s\mathcal{C}$ to the discrete simplicial object at R, denoted $R_* \in s\mathcal{C}$. If we were to take the analogy between simplicial objects and chain complexes seriously, then we must develop a way to replace R_* up-to "quasi-isomorphism" by a simple "projective" object (here the quotation marks indicate, up-to appropriate analog). The idea now is to appeal to Kan's intuition, and treat $s\mathcal{C}$ homotopy theoretically to find this sort of replacement procedure—in a sense, this is what

Quillen achieved. He was able to define a **model** for a homotopy theory on $s\mathcal{C}$. In a sense, his model on $s\mathcal{C}$ draws from the homotopy theory of simplicial sets. Three fundamental classes of maps describe the homotopy theory of simplicial sets: weak equivalences, Kan fibrations, and cofibrations. Recall that a map $f\colon X_* \longrightarrow Y_*$ of simplicial sets is called a weak equivalence if the map obtained on passage to geometric realization is a weak equivalence of topological spaces. In any category, given a commuting diagram as follows, we will say that $\pi'\colon X' \longrightarrow Y'$ has the left lifting property (LLP) with respect to $\pi\colon X \longrightarrow Y$, and that $\pi\colon X \longrightarrow Y$ has the right lifting property (RLP) with respect to $\pi'\colon X' \longrightarrow Y'$.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Recall now that a Kan fibration $p_*\colon X_* \longrightarrow Y_*$ is a map that has the RLP with respect to all horn inclusions, as in the following diagram.

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & X_* \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y_* \end{tikzcd}$$

The cofibrations of simplicial sets are just the degreewise monomorphisms. We will say that an object $P \in \mathcal{C}$ is projective if $\mathbb{Z} \longrightarrow P$ has the LLP with respect to every ring epimorphism (in other terms, this is saying that $\mathrm{Hom}(P, R) \longrightarrow \mathrm{Hom}(P, R')$ is surjective whenever there is an epimorphism $R \twoheadrightarrow R'$). Note that by taking the tensor algebra of the free abelian group describing R as a quotient, every ring R admits an epimorphism $P \twoheadrightarrow R$ from a projective ring. In other words, $\mathcal{C}$ has enough projectives. We are now ready to define the model on $s\mathcal{C}$, inspired by that on **sSet**.

We will call a map $f_*\colon X_* \longrightarrow Y_*$ a **fibration** in $s\mathcal{C}$ if for every projective P in $\mathcal{C}$, the induced map on simplicial sets $\mathrm{Hom}(P, f_*)\colon \mathrm{Hom}(P, X_*) \longrightarrow \mathrm{Hom}(P, Y_*)$ is a Kan fibration. Similarly, we say that $f_*\colon X_* \longrightarrow Y_*$ is a **weak equivalence** in $\mathcal{C}$ if for every projective P in $\mathcal{C}$, the map on hom-simplicial-sets $\mathrm{Hom}(P, f_*)$ is a weak equivalence of simplicial sets. Finally, a **cofibration** in $s\mathcal{C}$ is a map that has the LLP with respect to every fibration that is also a weak equivalence (or rather, what's called an **acyclic fibration**). To see the analogy with simplicial sets here, recall that the cofibrations of simplicial sets are exactly the degreewise monomorphisms, and the acyclic fibrations are the Kan fibrations with contractible fibers. There is a folklore result that the degreewise monomorphisms, namely the cofibrations, are exactly the class of maps with the LLP with respect to acyclic fibrations in **sSet**.[3] We will also need an essential property of the homotopy theory on simplicial sets, which is phrased in the following theorem.

Theorem 2.2 (Functorial Factorization in **sSet**)
Every map of simplicial sets $f_*\colon X_* \longrightarrow Y_*$ can be functorially (in f_*) factorized as $p \circ i$ where i is a cofibration and p is an acyclic fibration

Without saying much about how this is proved, we will just note that the argument here is a simplicial analog of killing homotopy groups by attaching cells in algebraic topology.

[3] In fact the lift in the LLP diagram is unique up to homotopy, and this can be shown to be true in $s\mathcal{C}$ as well.

And in the same vein, Quillen proved that functorial factorization actually holds for $s\mathcal{C}$ too.

Theorem 2.3 (Functorial Factorization in $s\mathcal{C}$)

Every map $f_*\colon X_* \longrightarrow Y_*$ in $s\mathcal{C}$ can be functorially (in f_*) factorized as $p \circ i$ where i is a cofibration and p is an acyclic fibration

Taking these results to be encased within their black boxes, our replacement procedure is now in sight. We will say that an object $X_* \in s\mathcal{C}$ is **cofibrant** if the unique map $\mathbb{Z}_* \longrightarrow X_*$ is a cofibration. In other words, if for every acyclic fibration $Y_* \longrightarrow Z_*$, there is a commuting diagram.

$$\begin{tikzcd} & Y_* \arrow[d] \\ X_* \arrow[r] \arrow[ur, dashed] & Z_* \end{tikzcd}$$

Thm. 2.3 tells us that any map $\mathbb{Z}_* \longrightarrow X_*$ can be factorized (functorially in X_*) in a unique up to homotopy fashion as a composite $\mathbb{Z}_* \xrightarrow{\iota} \tilde{X}_* \xrightarrow{p} X_*$, where ι is a cofibration (meaning $\tilde{X}_*$ is cofibrant), and p is a weak equivalence. In other terms, we see that every $X_* \in s\mathcal{C}$ is weak equivalent to a cofibrant object in a functorial manner, and moreover this cofibrant object is unique up to homotopy. So we may define a **cofibrant resolution** of X_* to be an acyclic fibration $Q_* \longrightarrow X_*$ where Q_* is cofibrant. By the assumptions of the context, we know that every X_* has a (up to homotopy) unique cofibrant resolution. This looks very much like what we did in the world of homological algebra. In Appendix A, we will see that there is actually a homotopy theory of chain complexes that reconciles the two replacement procedures we have seen so far.

To keep track of what we have done so far, we took $\mathcal{C}$, embedded into $s\mathcal{C}$ as the discrete simplicial ring, and now we have passed through to the category of simplicial rings up to weak equivalence by applying a cofibrant replacement functor. At this moment, we remain in the category of simplicial rings up to weak equivalence. As before, now we can apply the functor at hand, $\Omega_{T/-}$, to the cofibrant replacement Q_* by the formula $\Omega_{R/Q_*} \otimes_R T$ (which you should really see as the pullback by the cofibrant resolution, $\iota^*\Omega_{T/R}$) to get a simplicial T-module. Appealing to Dold-Kan now, Thm. 2.1, we can see simplicial T-modules as chain complexes of T-modules. The associated chain complex (up to quasi-isomorphism) to $\Omega_{R/Q_*} \otimes_R T$ is called the **cotangent complex** of T over R, denoted $\mathbb{L}^\bullet_{T/R}$.

Applications of the cotangent complex in algebraic geometry are plentiful. Passing this construction to the scheme-theoretic/variety world, given any morphism of schemes $f\colon X \longrightarrow Y$, we can talk about the cotangent complex of relative differentials $\mathbb{L}^\bullet_{Y/X}$. One important fact is that the cotangent complex, in this sense, vanishes at higher degrees if f is smooth—and reduces on passage to homology to $\Omega_{Y/X}$ at degree 0. In the case where f is a closed embedding of smooth schemes/varieties, the cotangent complex reduces to the conormal bundle from differential geometry, which is the dual to the normal bundle. In this sense, we can see that the cotangent complex represents some sort of universal version of the module of Kähler differentials that unifies geometric constructions like the cotangent bundle and the normal bundle in algebraic geometry. Cotangent complexes are the starting points of modern study in **deformation theory** and **moduli problems** since in some sense they control the deformations of morphisms of schemes/varieties. This for example, was the starting point for Lurie in DAG IV and X.

We finish by defining what a model category is.

Definition 2.4 (Model Category)**.** A category $\mathcal{C}$ equipped with three classes of morphisms respectively called weak equivalences, cofibrations, and fibrations is called a model category if the following hold:

1. $\mathcal{C}$ has all small limits and colimits.
2. If f and g are two composable morphisms and two of f, g, or $g \circ f$ are weak equivalences, then so is the third.
3. Weak equivalences, cofibrations, and fibrations remain so under taking retracts.
4. Fibrations are exactly those maps that have the RLP with respect to acyclic cofibrations and acyclic cofibrations are exactly those maps that have the LLP with respect to fibrations.
5. Cofibrations are exactly those maps that have the LLP with respect to acyclic fibrations and acyclic fibrations are exactly those maps that have the RLP with respect to cofibrations.
6. Any morphism f can functorially (in f) be factorized as a composite $p \circ i$ where p is a fibration and i is an acyclic cofibration and also as a composite $q \circ j$ where q is an acyclic fibration and j is a cofibration.

§A Appendix: Quillen Adjunctions and Model Structures on Chain Complexes

We offer some general theory on derived functors of model categories here, and argue that the theory of derived functors presented in Section 1 falls out as a special instance.

We first note that the category of (co)chain complexes in an abelian category form a model category.

Theorem A.1 (Projective Model Structure on $\mathrm{Ch}(-)$, [Qui67])
Let $\mathcal{A}$ be an abelian category with enough projectives, and $\mathrm{Ch}(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. There is a model structure on $\mathrm{Ch}(\mathcal{A})$ whose:

1. weak equivalences are quasi-isomorphisms.
2. fibrations are chain maps that are degreewise epimorphisms.
3. cofibrations are chain maps that are degreewise monomorphisms with projective cokernels.

Theorem A.2 (Injective Model Structure on $\mathrm{coCh}(-)$, [Qui67])

Let $\mathcal{A}$ be an abelian category with enough injectives, and $\mathrm{coCh}(\mathcal{A})$ the category of co-chain complexes in $\mathcal{A}$. There is a model structure on $\mathrm{coCh}(\mathcal{A})$ whose:

1. weak equivalences are quasi-isomorphisms.
2. fibrations are cochain maps that are degreewise epimorphisms with injective kernels.
3. cofibrations are cochain maps that are degreewise monomorphisms.

Remark A.3. We note that the cofibrant objects in $\mathrm{Ch}(\mathcal{A})$ are exactly the chain complexes that are degreewise projective, and that the fibrant objects in $\mathrm{coCh}(\mathcal{A})$ are exactly the cochain complexes that are degreewise injective.

Before we begin, we introduce the homotopy category of a model category. In a sense, this is the category obtained by inverting weak equivalences—sometimes this construction is also referred to as the localization of a model category at the weak equivalences.

Definition A.4. Given a model category $\mathcal{C}$, its **homotopy category** denoted by $\mathrm{Ho}\mathcal{C}$ is defined as the category obtained by formally inverting the weak equivalences in $\mathcal{C}$. There is a canonical functor $\gamma\colon \mathcal{C} \longrightarrow \mathrm{Ho}\mathcal{C}$ that acts as the identity on objects and sends weak equivalences to corresponding isomorphisms.

Remark A.5. In [Qui67], Quillen showed that $\mathrm{Ho}\mathcal{C}$ is actually equivalent to the subcategory of $\mathcal{C}$ whose objects are the objects in $\mathcal{C}$ that are both fibrant and cofibrant, and whose morphisms are homotopy classes of maps.

Remark A.6. The homotopy category of $\mathrm{Ch}(\mathcal{A})$ is what is referred to in literature as the **derived category** of $\mathcal{A}$, denoted $\mathcal{D}(\mathcal{A})$.

We will only discuss the left derived functor going forth, but the story for the right derived functor is dual.

Definition A.7. Let $\mathcal{F}\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between model categories that preserves weak equivalences between cofibrant objects in $\mathcal{C}$ (we call such a functor a **left Quillen functor**[4]). Note that $\mathcal{F}$ induces a functor $\gamma \circ \mathcal{F}\colon \mathcal{C} \longrightarrow \mathrm{Ho}\mathcal{D}$. This functor, by assumption, takes weak equivalences of cofibrant objects in $\mathcal{C}$ to isomorphisms in $\mathrm{Ho}\mathcal{D}$. By functorial factorization, there is a well-defined cofibrant replacement functor[5] $\overline{Q}\colon \mathcal{C} \longrightarrow \pi\mathcal{C}_C$ that takes X to a cofibrant $\overline{Q}(X)$. The functor $\gamma\mathcal{F}\overline{Q}\colon \mathcal{C} \longrightarrow \mathrm{Ho}\mathcal{D}$ takes an object $X \in \mathcal{C}$, produces a functorial cofibrant replacement $\overline{Q}X$, and applies $\mathcal{F}$ to this replacement. Recalling that $\overline{Q}$ takes weak equivalences in $\mathcal{C}$ to weak equivalences between cofibrant objects, and that by assumption, $\gamma\mathcal{F}$ takes weak equivalences of cofibrant objects to isomorphisms—we see that $\gamma\mathcal{F}\overline{Q}$ must factor through $\mathrm{Ho}\mathcal{C}$ to produce a functor $\mathbb{L}\mathcal{F}\colon \mathrm{Ho}\mathcal{C} \longrightarrow \mathrm{Ho}\mathcal{D}$ such that $\mathbb{L}\mathcal{F} \circ \gamma = \gamma \circ (\mathcal{F} \circ \overline{Q})$. The functor $\mathbb{L}\mathcal{F}$ is called the **left derived functor** of $\mathcal{F}$.

[4] Although in literature, this actually means something much stronger—for our purposes, call it abuse of terminology, but this suffices.

[5] Here $\mathcal{C}_C$ denotes the subcategory of cofibrant objects in $\mathcal{C}$, and $\pi\mathcal{C}_C$ denotes the category whose objects are cofibrant objects in $\mathcal{C}$ and whose morphisms are homotopy classes of morphisms between cofibrant objects in $\mathcal{C}$.

Remark A.8. By virtue of construction, for any left Quillen functor $\mathcal{F}\colon \mathcal{C} \longrightarrow \mathcal{D}$ and any $X \in \mathcal{C}$, $\mathbb{L}\mathcal{F}(X)$ is isomorphic to $\mathcal{F}(X)$ if X is cofibrant.

Let $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right-exact functor between abelian categories, and let $\mathrm{Ch}(\mathcal{F})$ denote the induced map on chain complexes. We note that since the weak equivalences of chain complexes are quasi-isomorphisms, and since the quasi-isomorphisms of levelwise projective chain complexes are exactly chain homotopy equivalences, $\mathrm{Ch}(\mathcal{F})$ preserves weak equivalences of cofibrant objects by additivity. We may therefore call it left Quillen in our sense—and we can talk about its left derived functor $\mathbb{L}\mathrm{Ch}(\mathcal{F})$. The cofibrant replacement functor that goes into the definition of the derived functor here is exactly the projective replacement functor from Section 1. The construction of $L_i\mathcal{F}$ from Section 1 can now be homotopically summarized as the composite:

$$\mathcal{A} \xrightarrow{\mathbf{B}^0(-)} \mathcal{D}(\mathcal{A}) \xrightarrow{\mathbb{L}\mathrm{Ch}(\mathcal{F})} \mathcal{D}(\mathcal{B}) \xrightarrow{H_i(-)} \mathcal{B}$$

Where $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ are precisely the homotopy categories of $\mathrm{Ch}(\mathcal{A})$ and $\mathrm{Ch}(\mathcal{B})$ respectively.

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