Étale Cohomology

Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$

P. Deligne

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Translator's Preface

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Introduction

This volume aims to facilitate the use of ℓ-adic cohomology for the non-expert. I hope that it will often allow them to avoid resorting to the terse exposition in SGA 4 and SGA 5. It also contains some new results.

The first exposition, written by J.F. Boutot, surveys SGA 4. It provides the principal results - with a minimal generality, often insufficient for applications - and an idea of their proof. For complete results, or detailed proofs, SGA 4 is available.

The "Report on the trace formula" contains a complete proof of the trace formula for the Frobenius endomorphism. The proof is that given by Grothendieck in SGA 5, stripped of the unnecessary details. This Report should allow the user to forget SGA 5, which we may consider as a series of digressions, some very interesting. Its existence allows the publishing of SGA 5 as is soon. It is completed by the exposition "Applications of the trace formula to trigonometric sums" which explains how the trace formula allows for the study of trigonometric sums, and provides examples.

The target audience of the other expositions is more limited, and their writing style reflects this. The exposition "L-functions modulo ℓ^n and modulo p" is a modular generalization of the Report, based on the study SGA 4 XVII 5.5 of symmetric powers. The exposition "The cohomology class associated to a cycle" defines this class in various contexts, and provides the compatibility between intersections and cup-products. In "Duality" are gathered some known results that lack a reference, and some compatibilities. The exposition "Theorems of finiteness in ℓ-adic cohomology" is new. It notably provides, in cohomology without support, theorems of finiteness analogous to those known in cohomology with compact support.

For more details on the expositions, I refer to their respective introductions.

I finally thank J.L. Verdier for allowing me to reproduce here his notes "Derived Categories (État 0)." I believe they remain very useful, and had become untraceable.

Bures-sur-Yvette, 20 September 1976
Pierre Deligne

CHAPTER 1

Ariadne's thread for SGA 4, SGA $4\frac{1}{2}$, and SGA 5

The expositions I to VI of SGA 4 give the general theory of Grothendieck topologies. Very detailed, they can be valuable when studying exotic topologies, such as the one that gave birth to crystalline cohomology. For the étale topology, so close to classical intuition, such an imposing guardrail is not necessary : it suffices to be familiar with (for example) the book of Godement [God58], and to have a bit of faith. Other possible references: the chapters I to III of Artin's notes [Art62], the Bourbaki exposition of Giraud [Gir64] or 2.1 in this volume. The expositions VII and VIII start the study of étale topology. They are more detailed than chapter III of Artin and than 2.2 here.

The study of curves is the key to étale cohomology. It is started by Artin in IX of SGA 4; the consequences, as to the cohomological dimension, are given in X. The essential points are repeated in 2.3 here. The exposition XI is not necessary for the following; it contains an elegant proof of the theorem of comparison between étale cohomology and classical cohomology in the particular case of smooth complex algebraic varieties.

The sections XII and XIII prove the fundamental base-change theorem for a proper morphism. As Artin noted in [Art69], his approximation theorem allows one to simplify the proof. This path is followed in 2.4 where multiple dévissages of the proof are only sketched, and where the applications of XIV are only very briefly indicated. The theorem allows one to define the "cohomology with compact support," and the "superior direct image with proper support" functors $R^i f_!$ (where the properties are reduced to those of superior direct image functors for a proper morphism). This is the subject of the verbose XVII; the essential point is said in 2.4, but the pieces of XVII can be of independent utility.

The expositions XV and XVI are centered on the fundamental theorem of local acyclicity for smooth morphisms. The influence of SGA 7 shows through in the less-detailed parallel exposition in 2.5, which also contains the proof of required results of SGA 2.

The exposition 2.6 is dedicated to Poincaré duality; it is clearer than SGA 4 XVIII but does not pretend to give a complete proof. For those who care about a purely algebraic proof, it is simplified by a reference to 6.3; it does not contain the proof of the formalism that the trace morphism fits into (XVIII §2).

There exists in étale cohomology a formalism of duality analogous to that of coherent cohomology. To establish this, Grothendieck used the resolution of singularities and the purity conjecture (for the statement, see 5 2.1.4), established in a relative framework in SGA 4 XVI, and - modulo the resolution - in equal characteristic in SGA 4 XIX. The key-points are established by another method, in 8, for schemes of finite type over a regular scheme of dimension 0 or 1. Various developments are given in SGA 5 I. In SGA 5 III, we show how this formalism implies the very general trace formula of Lefschetz-Verdier.

We see that in the original version of SGA 5, the Lefschetz-Verdier formula was only established conjecturally. Moreover, the local terms were not calculated there. For an application to L-functions, this seminar contains another proof, it is complete, in the particular case of the Frobenius morphism. This is the one in 2. Other references: for the statement and the schema of dévissages: the Bourbaki exposition of Grothendieck [Gro95]; for a brief description of the reduction (due to Grothendieck) of the crucial case to a case already treated by Weil, [Del76] §10; for an ℓ-adic treatment of this last case, 5.3.

In SGA 5, we will again find a detailed treatment of the passing to the $\mathbb{Z}/\ell$-cohomology of the $\mathbb{Z}/\ell^n$-cohomology (V and VI, by J.P. Jouanolou), and of the formalism of the Frobenius morphism (XV, by C. Houzel). The expositions X and XII, written by I. Bucur, give the calculation of the Euler-Poincaré characteristic of a sheaf on a curve, and that of the trace of the endomorphism of its cohomology defined by certain correspondances. In VII, Jouanolou gives the calculation of the cohomology of classical schemes, and applications. Otherwise, Serre's presentation in "Introduction à la théorie de Brauer" was recalled in his book [Ser78] (3rd part).

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CHAPTER 2

Étale cohomology : the starting points

by P. Deligne, written by J.F. Boutot

This work resumes 6 presentations given by P. Deligne at Arcata in August 1974 (AMS Summer School on algebraic geometry), under the title: “Inputs of etale cohomology." A 7^{th}-presentation became the “report on the trace formula,” in this same volume. The goal of these presentations was to give proofs of fundamental theorems in étale cohomology, freed from the gangue of nonsense that surrounds them in SGA 4. We have not sought to state the theorems under their most general form, nor followed the dévissages, sometimes clever, that their proof required. We have on the contrary placed the emphasis on the “irreducible” cases, which, all dévissages done, remained to be treated.

We hope that this text, which does not claim any originality, aids the reader to consult with profit the 3 volumes of SGA 4.

Convention 0.0.1. We only consider quasi-compact schemes (= union of finite affine open subschemes) and quasi-separated (= such that the intersection of any two open affine subschemes is quasi-compact), and we simply call them schemes.

1. Grothendieck topologies

At the origin, Grothendieck topologies appeared as underlying his theory of descent (cf SGA 1 VI, VIII); the usage of correspondant cohomology theories is later. The same approach is taken here in formalizing the classical notions of localization, of the local property of gluing, we bring out the general concept of the Grothendieck topology; to justify its introduction into algebraic geometry, we show a theorem of faithfully flat descent, a generalization of the classical Hilbert's theorem 90.

1.1. Sieves. Let X be a topological space, and $f : X \to \mathbb{R}$ a real-valued function on X. The continuity of f is a property of local nature; said otherwise, if f is continuous on every sufficiently small open subset of X, f is continuous on all of X. To formalize the notion of a "property of local nature," we introduce some definitions.

We say that a set $\mathcal{U}$ of open subsets of X is a *sieve* if for every $U \in \mathcal{U}$ and $V \subset U$, we have $V \in \mathcal{U}$. We say that a sieve is *covering* if the union of all of the open subsets belonging to this sieve is equal to X.

Given a family $\{U_i\}_{i\in\mathcal{I}}$ of open subsets of X, the sieve generated by $\{U_i\}_{i\in\mathcal{I}}$ is by definition the set of open subsets U of X such that U is contained in some U_i.

We say that a property $P(U)$, defined for every open subset U of X, is *local* if, for every covering sieve $\mathcal{U}$ of every open subset U of X, $P(U)$ is true if and only if $P(V)$ is true for every $V \in \mathcal{U}$. For example, given $f : X \to \mathbb{R}$, the property "f is continuous on U" is local.

1.2. Sheaves. We make precise the notion of locally given functions on X.

1.2.1. **Point of view of sieves**: Let $\mathcal{U}$ be a sieve of open subsets of X. We mean bya $\mathcal{U}$-locally given function on X the data for every $U \in \mathcal{U}$ of a function f_U on U such that, if $V \subset U$, we have $f_V = f_U|_V$.

1.2.2. **Point of view of Čech**: If the sieve $\mathcal{U}$ is generated by a family of open subsets U_i of X, to provide a function $\mathcal{U}$-locally is to provide a function f_i on each U_i such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$.

Said differently, if $Z = \coprod U_i$, giving a function $\mathcal{U}$-locally is to provide a function on Z which is constant on the fibers of the natural projection $Z \to X$.

1.2.3. The continuous functions form a sheaf; this means that for every covering sieve $\mathcal{U}$ of an open subset V of X and every $\mathcal{U}$-locally given function $\{f_U\}$ such that every f_U is continuous on U, there exists a unique continuous function f on V such that $f|_U = f_U$ for every $U \in \mathcal{U}$.

1.3. Fields. We will make precise now the notion of a locally given vector bundle on X.

1.3.1. **Point of view of sieves:** Let $\mathcal{U}$ be a sieve of open subsets of X. We mean bya $\mathcal{U}$-locally given vector bundle on X the data of

1.3.1.1. a vector bundle E_U over each $U \in \mathcal{U}$.
1.3.1.2. if $V \subset U$, an isomorphism $\rho_{U,V} \colon E_V \xrightarrow{\sim} E_U|_V$, such that the following is true.
1.3.1.3. if $W \subset V \subset U$, the diagram

$$\begin{tikzcd} E_W \arrow[rr, "\rho_{U,W}"] \arrow[dr, "\rho_{V,W}"'] & & E_U|_W \\ & E_V|_W \arrow[ur, "\rho_{U,V}|_W"'] & \end{tikzcd}$$

commutes, i.e. $\rho_{U,W} = (\rho_{U,V}|_W) \circ \rho_{V,W}$.

1.3.2. **Point of view of Čech:** If the sieve $\mathcal{U}$ is generated by a family of open subsets U_i of X, providing a vector bundle $\mathcal{U}$-locally is to provide:
1.3.2.1. a vector bundle E_i on each U_i,
1.3.2.2. if $U_{ij} = U_i \cap U_j = U_i \times_X U_j$, an isomorphism $\rho_{ji} : E_i|_{U_{ij}} \xrightarrow{\sim} E_j|_{U_{ij}}$, so that
1.3.2.3. if $U_{ijk} = U_i \times_X U_j \times_X U_k$, the diagram

$$\begin{tikzcd} E_i|_{U_{ijk}} \arrow[rr, "\rho_{ki}|_{U_{ijk}}"] \arrow[dr, "\rho_{ji}|_{U_{ijk}}"'] & & E_k|_{U_{ijk}} \\ & E_j|_{U_{ijk}} \arrow[ur, "\rho_{kj}|_{U_{ijk}}"'] & \end{tikzcd}$$

commutes, i.e. $\rho_{ki} = \rho_{kj} \circ \rho_{ji}$ on U_{ijk}.

Said differently, if $Z = \coprod U_i$ and if $\pi : Z \to X$ is the natural projection, to provide a vector bundle $\mathcal{U}$-locally is to provide:
1.3.2.1. a vector bundle E over Z,
1.3.2.2. if x and y are two points in Z such that $\pi(x) = \pi(y)$, an isomorphism $\rho_{yx} \colon E_x \xrightarrow{\sim} E_y$ between the fibers of E at x and at y, depending continuously on (x, y) and such that,
1.3.2.3. if x, y and z are three points of Z such that $\pi(x) = \pi(y) = \pi(z)$, we have $\rho_{zx} = \rho_{zy} \circ \rho_{yx}$.

1.3.3. A vector bundle E over X defines a vector bundle given $\mathcal{U}$-locally $E_{\mathcal{U}}$: the system of restrictions E_U of E to objects of $\mathcal{U}$. The fact that the notion of a vector bundle is of local nature may be expressed so: for every covering sieve $\mathcal{U}$ of X, the functor $E \mapsto E_{\mathcal{U}}$, is an equivalence of categories.

1.3.4. If in 2.1.1, we replace "open subset of X" by "subspace of X," we obtain the notion of a *sieve of subspaces of* X. In this framework too we have gluing theorems. For example: let X be a normal space and $\mathcal{C}$ a sieve of subspaces of X generated by a locally finite closed cover of X, then the functor $E \mapsto E_{\mathcal{C}}$, from vector bundles over X to $\mathcal{C}$-locally given vector bundles is an equivalence of categories.

In algebraic geometry, it is useful to also consider "sieves of spaces over X." This is what we will see in the following paragraph.

1.4. Faithfully flat descent.

1.4.1. In the framework of schemes, the Zariski topology is not fine enough for the study of non-linear problems and we are brought to replace in the preceding definitions, open immersion by more general morphisms. From this point of view, the techniques of descent appear as localization techniques. So the following statement of descent can be expressed by saying that the considered properties are of local nature for the faithfully flat topology (we say that a morphism of schemes is faithfully flat if it is flat and surjective).

PROPOSITION 1.4.2. *Let A be a ring and B a faithfully flat A-algebra. Then:*

(i) A sequence $\Sigma = (M' \longrightarrow M \longrightarrow M'')$ of A-modules is exact when the sequence $\Sigma_{(B)}$ deduced by extension of scalars to B is exact.
(ii) An A-module M is of finite type (resp. of finite presentation, flat, locally free of finite rank, invertible (i.e. locally free of rank 1)) when the B-module $M_{(B)}$ is.

PROOF. The functor $M \mapsto M_{(B)}$ being exact (flatness of B), it suffices to show that, if an A-module N is nonzero, $N_{(B)}$ is nonzero. If N is nonzero, N contains a nonzero cyclic submodule A/a; so $N_{(B)}$ contains a cyclic submodule $(A/a)_{(B)} = B/a\,B$, nonzero by surjectivity of the structural morphism $\varphi : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ (if $V(a)$ is nonempty, $\varphi^{-1}(V(a)) = V(a\,B)$ is nonempty).

For every family (x_i) of elements of $M_{(B)}$, there exists a submodule of finite type M' of M such that $M'_{(B)}$ contains the x_i's. If $M_{(B)}$ is of finite type and if the x_i's generate $M_{(B)}$, we have $M'_{(B)} = M_{(B)}$, so $M' = M$ and M is of finite type.

If $M_{(B)}$ is of finite presentation, we can, from above, find a surjection $A^n \longrightarrow M$. If N is the kernel of this surjection, the B-module $N_{(B)}$ is of finite type, so N is, and M is of finite presentation. The assertion for "flat" results also from (i); "locally free of finite rank" means "flat and of finite presentation" and the rank is tested by extension of scalars to fields. ■

1.4.3. Let X be a scheme and $\mathscr{S}$ a class of X-schemes stable under fiber product over X. A class $\mathcal{U} \subset \mathscr{S}$ is a *sieve* on X (relative to $\mathscr{S}$) if, for every morphism $\varphi\colon V \to U$ of X-schemes, with $U, V \in \mathscr{S}$ and $U \in \mathcal{U}$, we have $V \in \mathcal{U}$. The sieve *generated* by a familly $\{U_i\}_{i\in\mathcal{I}}$ of X-schemes in $\mathscr{S}$ is the class of $V \in \mathscr{S}$ such that there exists a morphism of X-schemes from V into some U_i.

1.4.4. Let $\mathcal{U}$ be a sieve on X. We mean by a $\mathcal{U}$-locally given quasi-coherent module on X, the data of:
(a) a quasi-coherent module E_U on each $U \in \mathcal{U}$,
(b) for every $U \in \mathcal{U}$ and for every morphism $\varphi\colon V \to U$ of X-schemes in $\mathscr{S}$, an isomorphism $\rho_\varphi : E_V \xrightarrow{\sim} \varphi^* E_U$, such that

(c) if $\psi\colon W \longrightarrow V$ is a morphism of X-schemes in $\mathscr{S}$, the diagram

$$\begin{tikzcd} E_W \arrow[rr, "\rho_{\varphi\circ\psi}"] \arrow[dr, "\rho_\psi"'] & & \psi^*\varphi^* E_U \\ & \psi^* E_V \arrow[ur, "\psi^*\rho_\varphi"'] & \end{tikzcd}$$

commutes, i.e. $\rho_{\varphi\circ\psi} = (\psi^*\rho_\varphi)\circ\rho_\psi$.

If E is a quasi-coherent module on X, we denote by $E_{\mathcal{U}}$ the $\mathcal{U}$-locally given module having value $\varphi_U^* E$ on $\varphi_U : U \longrightarrow X$ and such that, for every morphism $\psi\colon V \longrightarrow U$ is the restriction isomorphism ρ_ψ is the canonical isomorphism $E_V = (\varphi_U \circ \psi)^* E \xrightarrow{\sim} \psi^*\varphi_U^* E = \psi^* E_U$.

Theorem 1.4.5. *Let $\{U_i\}_{i\in\mathcal{I}} \in \mathscr{S}$ a finite family of flat X-schemes on X such that X is the union of the images of U_i, and let $\mathcal{U}$ be the sieve generated by $\{U_i\}_{i\in\mathcal{I}}$. Then the functor $E \mapsto E_{\mathcal{U}}$ is an equivalence from the category of quasi-coherent modules on X to the category of $\mathcal{U}$-locally given quasi-coherent modules*

Proof. We only treat the case where X is affine and where $\mathcal{U}$ is generated by an affine X-scheme U, faithfully flat on X. The reduction to this case is formal. We let $X = \operatorname{Spec}(A)$ and $U = \operatorname{Spec}(B)$.

If the morphism $U \longrightarrow X$ admits a section, X belongs to the sieve $\mathcal{U}$ and the assertion is evident. We reduce to this case.

A $\mathcal{U}$-locally given quasi-coherent module defines modules M', M'', M''' on $U, U\times_X U$ and $U\times_X U\times_X U$, and isomorphisms $\rho\colon p^*M^\bullet \simeq M^\bullet$ for every projection morphism p between these spaces; this is a cartesian diagram

$$\begin{tikzcd} M^*\colon M' \arrow[r, shift left] \arrow[r, shift right] & M'' \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & M''' \end{tikzcd}$$

over

$$\begin{tikzcd} U_*\colon U & U\times_X U \arrow[l, shift left] \arrow[l, shift right] & U\times_X U\times_X U \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

Conversely, M^* determines the $\mathcal{U}$-locally given module: for $V \in \mathcal{U}$, there exists $\varphi : V \longrightarrow U$ and we write $M_U = \varphi^* M'$; for $\varphi_1, \varphi_2\colon V \longrightarrow U$, we have a natural identification $\varphi_1^* M' \simeq (\varphi_1\times\varphi_2)^* M'' \simeq \varphi_2^* M'$, and we see using M''' that these identifications are compatible, so that the definition is legitimate. In short, it comes down to the same thing, to $\mathcal{U}$-locally give a module or a cartesian diagram M^* on U_*.

Translating to algebraic terms: to give M^* is to give a cartesian diagram of modules

$$\begin{tikzcd} M' \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & M'' \arrow[r, shift left=3, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=3, "\partial_2"'] & M''' \end{tikzcd}$$

over the diagram of rings

$$\begin{tikzcd} B \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & B\otimes_A B \arrow[r, shift left=3, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=3, "\partial_2"'] & B\otimes_A B\otimes_A B \end{tikzcd}$$

[to make that precise: we have $\partial_i(bm) = \partial_i(b)\cdot\partial_i(m)$, the usual identities such that $\partial_0\partial_1 = \partial_0\partial_0$ are true, and "cartesian" means that the morphisms $\partial_i\colon M'\otimes_{B,\partial_1}(B\otimes_A B)\longrightarrow M''$ and $M''\otimes_{B\otimes_A B,\partial_i}(B\otimes_A B\otimes_A B)\longrightarrow M'''$ are isomorphisms].

The functor $E\mapsto E_{\mathcal{U}}$ becomes the functor which, to an A-module M, associates

$$M^* = (\, M\otimes_A B \rightrightarrows M\otimes_A B\otimes_A B \, \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \, M\otimes_A B\otimes_A B\otimes_A B\,).$$

It admits for a right adjoint, the functor

$$(\ M' \rightrightarrows M'' \substack{\rightarrow\\ \rightarrow\\ \rightarrow} M'''\ ) \mapsto \ker(\ M' \rightrightarrows M''\ ).$$

We have to prove that the adjunction arrows:

$$M \longrightarrow \ker(\ M\otimes_A B \rightrightarrows M\otimes_A B\otimes_A B\ )$$

and

$$\ker(\ M' \rightrightarrows M''\ )\otimes_A B \longrightarrow M'$$

are isomorphisms. According to Prop. 2.1.4.2(i), it suffices to just prove it after a faithfully flat base change $A\longrightarrow A'$ (B becomes $B' = B\otimes_A A'$). Taking $A' = B$, this brings us back to the case where $U\longrightarrow X$ admits a section. ■

1.5. A particular case : Hilbert's theorem 90.

1.5.1. Let K be a field, K' a Galois extension of K and $G = \mathrm{Gal}(K'/K)$. Then the homomorphism

$$K'\otimes_K K' \longrightarrow \bigoplus_{\sigma\in G} K'$$
$$x\otimes y \mapsto \{x\cdot\sigma(y)\}_{\sigma\in G}$$

is bijective.

We deduce from this that to locally give a module for a sieve generated by $\mathrm{Spec}(K')$ over $\mathrm{Spec}(K)$ is to give a vector K'-space equipped with a semi-linear G-action, i.e.:

a) a vector K'-space V',
b) for every $\sigma\in G$, an endomorphism φ_σ of the group structure of V' such that $\varphi_\sigma(\lambda v) = \sigma(\lambda)\varphi_\sigma(v)$, for every $\lambda\in K'$ and $v\in V'$, verifying the condition
c) for every $\sigma,\tau\in G$, we have $\varphi_{\tau\sigma} = \varphi_\tau\circ\varphi_\sigma$.

Let $V = V'^G$ be the group of invariants by this G-action; this is a vector K-space, according to Thm. 2.1.4.5, we have:

Proposition 1.5.2. *The inclusion of V in V' defines an isomorphism $V\otimes_K K' \xrightarrow{\sim} V'$.*

In particular, if V' is of dimension l and if $v'\in V$ is nonzero, φ_σ and determined by the constant $c(\sigma)\in K'^*$ such that $\varphi_\sigma(v') = c(\sigma)v'$ and the condition c) is written as

$$c(\tau\sigma) = c(\tau)\cdot\tau(c(\sigma)).$$

according to the proposition there exists a nonzero invariant vector $v = \mu v'$, $\mu\in K'^*$. We therefore have for every $\sigma\in G$,

$$c(\sigma) = \mu \cdot \sigma(\mu^{-1}).$$

Said otherwise, every 1-cocycle of G with values in K'^* is a cobordism.

COROLLARY 1.5.3. *We have that* $H^1(G, K'^*) \cong 0$.

1.6. Grothendieck topologies. We transcribe now the definitions of the previous paragraphs in a global abstract framework for both the case of topological spaces and that of schemes.

1.6.1. Let $\mathscr{S}$ be a category and U an object of $\mathscr{S}$. We mean by a *sieve* on U a subset $\mathcal{U}$ of $\mathrm{Ob}(\mathscr{S}/U)$ such that if $\varphi\colon V \longrightarrow U$ belongs to $\mathscr{U}$ and if $\psi\colon W \longrightarrow V$ is a morphism in $\mathscr{S}$, so $\varphi \circ \psi\colon W \longrightarrow U$ belongs to $\mathcal{U}$.

If $\{\varphi_i\colon U_i \longrightarrow U\}$ is a family of morphisms, the sieve generated by the U_i is by definition the set of morphisms $\varphi\colon V \longrightarrow U$ that factorize through one of the φ_i.

If $\mathcal{U}$ is a sieve on U and if $\varphi\colon V \longrightarrow U$ is a morphism, the restriction $\mathcal{U}_V$ of $\mathcal{U}$ to V is by definition the sieve on V made up of the morphisms $\psi\colon W \longrightarrow V$ such that $\varphi \circ \psi\colon W \longrightarrow U$ belongs to $\mathcal{U}$.

1.6.2. The data of a *Grothendieck topology* on $\mathscr{S}$ consists of the data for every object U of $\mathscr{S}$, a set $C(U)$ of sieves on U, called covering sieves, such that the following axioms are satisfied:

a) The sieve generated by the identity of U is covering.
b) If $\mathcal{U}$ is a covering sieve on U and if $V \longrightarrow U$ is a morphism, the sieve $\mathcal{U}_V$ is covering.
c) A locally covering sieve is covering. Said otherwise, if $\mathcal{U}$ is a covering sieve on U and if $\mathcal{U}'$ is a sieve on U such that, for every $V \longrightarrow U$ belongs to $\mathcal{U}$, the sieve $\mathcal{U}'_V$ is covering, then $\mathcal{U}'$ is covering.

We mean by a *site* the data of a category equipped with a Grothendieck topology.

1.6.3. Given a site $\mathscr{S}$, we call a contravariant functor $\mathcal{F}$ from $\mathscr{S}$ to the category of sets a *presheaf* on $\mathscr{S}$. For every object U of $\mathscr{S}$, we call the elements of $\mathcal{F}(U)$ the sections of $\mathcal{F}$ over U. For every morphism $V \longrightarrow U$ and for every $s \in \mathcal{F}(U)$, we denote by $s|_V$ (s restricted to V) the image of s in $\mathcal{F}(V)$.

If $\mathcal{U}$ is a sieve on U, we mean by a $\mathcal{U}$-locally given section the data, for every $V \longrightarrow U$ in $\mathcal{U}$, of a section $s_V \in \mathcal{F}(V)$ such that, for every morphism $W \longrightarrow V$, we have $s_V|_W = s_W$. We say that $\mathcal{F}$ is a *sheaf* if, for every object U of $\mathscr{S}$, for every covering sieve $\mathcal{U}$ on U and for every $\mathcal{U}$-locally given section $\{s_V\}$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_V = s_V$, for every $V \longrightarrow U$ in $\mathcal{U}$.

We define abelian sheaves in an analogous manner by replacing the category of sets by that of abelian groups. We can show that the category of abelian sheaves on $\mathscr{S}$ is an abelian category with sufficiently many injectives. A sequence $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ of sheaves is exact if, for every object U of $\mathscr{S}$ and for every $s \in \mathcal{G}(U)$ such that $g(s) = 0$, there locally exists t such that $f(t) = s$; i.e. if there exists a covering sieve $\mathcal{U}$ on U and for every $V \in \mathcal{U}$, a section t_V of $\mathcal{F}$ over V such that $f(t_V) = s|_V$.

1.6.4. **Examples**: We saw two above.

a) Let X be a topological space and $\mathscr{S}$ the category whose objects are open subsets of X and whose morphisms are natural inclusions. The Grothendieck topology on $\mathscr{S}$ corresponding to the usual topology of X is that for which a sieve $\mathcal{U}$ on an open subset U of X is covering if the union of the open subsets in this is sieve is equal to U. It is clear that the category of sheaves on $\mathscr{S}$ is equivalent to the category of sheaves on X in the usual sense.

b) Let X be a scheme and $\mathscr{S}$ the category of schemes over X. We mean by the qcff (quasi-compact faithfully flat) topology on $\mathscr{S}$ the Grothendieck topology for which a sieve on an X-scheme U is covering if it is generated by a finite family of flat morphisms whose images cover U.

1.6.5. **Cohomology:** We always suppose that the category $\mathscr{S}$ has a final object X. So we mean by the global sections of an abelian sheaf $\mathcal{F}$, and we denote by $\Gamma\mathcal{F}$ or $H^0(X, \mathcal{F})$, the group $\mathcal{F}(X)$. The functor $\mathcal{F} \mapsto \Gamma\mathcal{F}$ is a left exact functor from the category of abelian sheaves on $\mathscr{S}$ to the category of abelian groups, and we denote by $H^i(X, \bullet)$ its derived functors (or satellites). These cohomology groups represent the obstructions to passing from the local to the global. By definition, if $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is an exact sequence of abelian sheaves, we have a long exact sequence in cohomology:

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \cdots \\ \cdots \longrightarrow H^n(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{G}) \longrightarrow H^n(X, \mathcal{H}) \longrightarrow H^{n+1}(X, \mathcal{F}) \longrightarrow \cdots \end{aligned}$$

1.6.6. Given an abelian sheaf $\mathcal{F}$ on $\mathscr{S}$, we mean by an $\mathcal{F}$-torsor a sheaf $\mathcal{G}$ equipped with an action $\mathcal{F} \times \mathcal{G} \longrightarrow \mathcal{G}$ of $\mathcal{F}$ such that locally (after restriction to all objects of a sieve covering the final object X) $\mathcal{G}$ equipped with the action of $\mathcal{F}$ is isomorphic to $\mathcal{F}$ equipped with the canonical action $\mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$ by translations.

We can show that $H^i(X, \mathcal{F})$ is interpreted as the set of isomorphism classes of $\mathcal{F}$-torsors.

2. Étale topology

We specialize the definitions of the preceding chapter to the case of the étale topology of a scheme X (2.2.1, 2.2.2, 2.2.3). The correspondent cohomology coincides in the case where X is the spectrum of a field K with the Galois cohomology of K (2.2.4).

2.1. Étale topology. We start with some reminders on the notion of étale morphisms.

DEFINITION 2.1.1. Let A be a (commutative) ring. We say that an A-algebra B is étale if B is an A-algebra of finite presentation and if the following equivalent conditions are satisfied:

a) For every A-algebra C and for every ideal of square zero J of C, the canonical map

$$\mathrm{Hom}_{A-\mathrm{alg}}(B, C) \longrightarrow \mathrm{Hom}_{A-\mathrm{alg}}(B, C/J)$$

is a bijection.

b) B is a flat A-module and $\Omega_{B/A} = 0$ (we denote by $\Omega_{B/A}$ the module of relative differentials).

c) Let $B = A[x_1, \cdots, x_n]/I$ be a presentation of B. Then for every prime ideal r of $A[x_1, \cdots, x_n]$ containing I, there exists polynomials $p_1, \cdots, p_n \in I$ such that I_r is generated by the images of $p_1, \cdots, p_n$ and $\det(\partial p_i/\partial x_i) \notin r$.

[see SGA I, exposition I or [Ray70], chapter V].

We say that a morphism of schemes $f\colon X \longrightarrow S$ is étale if for every $x \in X$ there exists an affine open neighborhood $U = \mathrm{Spec}(A)$ of $f(x)$ and an affine open neighborhood $V = \mathrm{Spec}(B)$ of x in $X \times_S U$ such that B is an étale A-algebra.

2.1.2. **Examples:** a) If A is a field, an A-algebra B is étale if and only if it is a finite product of separable extensions of A.

b) If X and S are schemes of finite type over $\mathbb{C}$, a morphism $f\colon X \longrightarrow S$ is étale if and only if its analytification $f^{\mathrm{an}}\colon X^{\mathrm{an}} \longrightarrow S^{\mathrm{an}}$ is a local isomorphism.

SORITE 2.1.3. a) (base-change) If $f\colon X \longrightarrow S$ is an étale morphism, the same is true of $f_{S'}\colon X \times_S S' \longrightarrow S'$ for every morphism $S' \longrightarrow S$.

b) (composition) The composite of two étale morphisms is an étale morphism.

c) If $f\colon X \longrightarrow S$ and $g\colon Y \longrightarrow S$ are two étale morphisms, every S-morphism from X to Y is étale.

d) (descent) Let $f\colon X \longrightarrow S$ be a morphism. If there exists a faithfully flat morphism $S' \longrightarrow S$, such that $f_{S'}\colon X \times_S S' \longrightarrow S'$ is étale, then f is étale.

2.1.1. Let X be a scheme. Let $\mathscr{S}$ the category of étale X-schemes; by 2.2.1.3 c) every morphism of $\mathscr{S}$ is an étale morphism. We mean by the *étale topology* on $\mathscr{S}$ the topology for which a sieve on U is covering if it is generated by a finite family of morphisms $\varphi_i\colon U_i \longrightarrow U$ such that the union of the images of φ_i covers U. We mean by the *étale site* of X, and we denote by $X_{\text{ét}}$, the site defined by $\mathscr{S}$ equipped with the étale topology.

2.2. Examples of sheaves.

2.2.1. **Constant sheaf:** Let C be an abelian group and suppose for simplicity that X is Noetherian. We will note by C_X (or even C if there is no ambiguity) the sheaf defined by $U \mapsto C^{\pi_0(U)}$, where $\pi_0(U)$ is the (finite) set of connected components of U. The most important case will be $C = \mathbb{Z}/n$. We thus have by definition

$$H^0(X, \mathbb{Z}/n) = \mathbb{Z}/n^{\pi_0(X)}.$$

Moreover $H^1(X, \mathbb{Z}/n)$ is the set of isomorphism classes of $\mathbb{Z}/n$-torsors (2.1.6.6.), said otherwise the Galois étale coverings of X of group $\mathbb{Z}/n$. Particularly, if X is connected and if $\pi_1(X)$ is its fundamental group for a chosen basepoint, we have

$$H^1(X, \mathbb{Z}/n) = \mathrm{Hom}(\pi_1(X), \mathbb{Z}/n).$$

2.2.2. **Multiplicative group:** We will note by $\mathbb{G}_{m,X}$ (or $\mathbb{G}_m$ if there is no ambiguity) the sheaf defined by $U \mapsto \Gamma(U, \mathcal{O}_U^*)$; it is indeed a sheaf thanks to the theorem of faithfully flat descent (2.1.4.5). We have by definition

$$H^0(X, \mathbb{G}_m) = H^0(X, \mathcal{O}_X)^*;$$

particularly if X is reduced, connected and proper over an algebraically closed K, we have:

$$H^0(X, \mathbb{G}_m) = K^*$$

Proposition 2.2.3. *We have an isomorphism:*

$$H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X),$$

where $\mathrm{Pic}(X)$ *is the group of classes of invertible sheaves on* X*.*

Proof. Let $*$ be the functor that, to an invertible sheaf $\mathcal{L}$ on X, associates the following presheaf $\mathcal{L}^*$ on $X_{\text{ét}}$: for $\varphi \colon U \longrightarrow X$ étale,

$$\mathcal{L}^*(U) = \mathrm{Isom}_U(\mathcal{O}_U, \varphi^*\mathcal{L}).$$

By Prop. 2.1.4.2 (i) and Thm. 2.1.4.5 (fully faithful), this presheaf is a sheaf; it's also a $\mathbb{G}_m$-torsor. We also verify that

a) the functor $*$ is compatible with the localization (étale);
b) it induces an equivalence from the category of trivial invertible sheaves (i.e. isomorphic to $\mathcal{O}_X$) with the category of trivial $\mathbb{G}_m$-torsors: $\mathcal{L}$ is trivial iff $\mathcal{L}^*$ is.

Moreover, by Prop. 2.1.4.2 (ii) and Thm. 2.1.4.5,

c) the notion of an invertible sheaf is local for the étale topology.

It formally follows from a), b), c) that $*$ is an equivalence between the category of invertible sheaves on X and that of $\mathbb{G}_m$-torsors on $X_{\text{ét}}$; it induces the needed isomorphism. We construct the inverse equivalence as follows: if T is a $\mathbb{G}_m$-torsor, there exists a finite étale cover $\{U_i\}$ of X such that the torsors T/U_i are trivial; so T is trivial over each étale V on X in the sieve $\mathcal{U} \subset X_{\text{ét}}$ generated by $\{U_i\}$. On each $V \in \mathcal{U}$, $T|_V$ corresponds to an invertible sheaf $\mathcal{L}_V$ (by b)) and the $\mathcal{L}_V$ constitute a $\mathcal{L}_U$ (by a)). By c), the latter provides an invertible sheaf $\mathcal{L}(T)$ on X, and $T \mapsto \mathcal{L}(T)$ is the required inverse of $*$. ∎

2.2.4. **Roots of unity:** For every integer $n > 0$, we mean by the sheaf of nth roots of unity, and we denote by μ_n, the kernel of raising to the nth power in $\mathbb{G}_m$. If X is a scheme on a separable algebraically closed K and if n is invertible in K, the choice of a primitive nth root of unity $\xi \in K$ defines an isomorphism $i \mapsto \xi^i$ of $\mathbb{Z}/n$ with μ_n.

The relation between cohomology with coefficients in μ_n and cohomology with coefficients in $\mathbb{G}_m$ is given by the exact sequence in cohomology deduced from the

THEOREM 2.2.5 (Kummer Theory). *If n is invertible on X, raising to the nth power in $\mathbb{G}_m$ is an epimorphism of sheaves. We then have an exact sequence*

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0.$$

PROOF. Let $U \longrightarrow X$ be an étale morphism and $a \in \mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. Since n is invertible on U, the equation $T^n - a = 0$ is separable; said otherwise $U' = \operatorname{Spec} \mathcal{O}_U[T]/(T^n - a)$ is étale over U. Moreover $U' \longrightarrow U$ is surjective and admits an nth root on U', and the result follows. ■

2.3. Fibers, direct images.

2.3.1. We call a *geometric point* of X a morphism $\overline{x} \longrightarrow X$, where $\overline{x}$ is the spectrum of a separably closed field $K(\overline{x})$. We will denote by $\overline{x}$ (by abuse of notation), implying the morphism $\overline{x} \longrightarrow X$. If x is the image of $\overline{x}$ in X, we say that $\overline{x}$ is centered at x. If the field $K(\overline{x})$ is an algebraic extension of the residual field $K(x)$, we say that $\overline{x}$ is an *algebraic* geometric point of X.

We mean by an étale neighborhood of $\overline{x}$ a commutative diagram

$$\begin{tikzcd} & U \arrow[d] \\ \overline{x} \arrow[ur] \arrow[r] & X \end{tikzcd}$$

where $U \longrightarrow X$ is an étale morphism.

The *strict localization* of X at $\overline{x}$ is the ring $\mathcal{O}_{X,\overline{x}} = \varinjlim \Gamma(U, \mathcal{O}_U)$, the inductive limit being taken over the étale neighborhoods of $\overline{x}$. This is a strictly henselian local ring whose residual field is the separable closure of the residual field $K(x)$ of X in x in $K(\overline{x})$. It plays the role of a local ring for the étale topology.

2.3.2. Given a sheaf $\mathcal{F}$ on $X_{\text{ét}}$, we mean by the *fiber* of $\mathcal{F}$ at $\overline{x}$ the set (resp. the group, $\cdots$) $\mathcal{F}_{\overline{x}} = \varinjlim \mathcal{F}(U)$, the inductive limit is always taken over the étale neighborhoods of $\overline{x}$.

For a homomorphism of sheaves $\mathcal{F} \longrightarrow \mathcal{G}$ to be a mono/epi/isomorphism it is neccessary and sufficient that this be the case for the homomorphisms $\mathcal{F}_{\overline{x}} \longrightarrow \mathcal{G}_{\overline{x}}$ induces on the fibers at every geometric point of X. If X is of finite type on an algebraically closed field, it suffices that it is so on the rational points of X.

2.3.3. If $f\colon X \longrightarrow Y$ is a morphism of schemes and F a sheaf on $X_{\text{ét}}$, the *direct image* $f_*\mathcal{F}$ of $\mathcal{F}$ by f is the sheaf on $Y_{\text{ét}}$ defined by $f_*\mathcal{F}(V) = F(X \times_Y V)$ for every V étale on Y.

The functor $f_*\colon \mathbf{Ab}(\mathbf{Sh}(X_{\text{ét}})) \longrightarrow \mathbf{Ab}(\mathbf{Sh}(Y_{\text{ét}}))$ is left exact. Its right derived functors $R^q f_*$ are called the superior direct images. If $\overline{y}$ is a geometric point of Y, we have

$$(R^q f_* \mathcal{F})_{\overline{y}} = \varinjlim H^q(V \times_Y X, \mathcal{F}),$$

inductive limit taken over the étale neighborhoods V of $\overline{y}$.

Let $\mathcal{O}_{Y,\overline{y}}$ be the strict localization of Y at $\overline{y}$, $\tilde{Y} = \operatorname{Spec}(\mathcal{O}_{Y,\overline{y}})$ and $\tilde{X} = X \times_Y \tilde{Y}$. We can extend $\mathcal{F}$ to $\tilde{X}_{\text{ét}}$ (this is a particular case of the general notion of an inverse image) in the following manner: let $\tilde{U}$ be an étale scheme on $\tilde{X}$, then there exists an étale neighborhood V of $\overline{y}$ and an étale scheme U over $X \times_Y V$ such that $\tilde{U} = U \times_V \tilde{Y}$; we write

$$\mathcal{F}(\tilde{U}) = \varinjlim F(U \times_V V'),$$

the inductive limit is taken over the étale neighborhoods V' of $\overline{y}$ that contain V. With this definition, we have

$$(R^q f_* F)_{\overline{y}} = H^q(\tilde{X}, \mathcal{F}).$$

The functor f_* has a left adjoint f^*, the "inverse image" functor. If $\overline{x}$ is a geometric point of X and $f(\overline{x})$ its image in Y, we have $(f^*\mathcal{F})_{\overline{x}} = \mathcal{F}_{f(\overline{x})}$. This formula shows that f^* is as an exact functor. The functor f_* thus sends injective sheaves to injective sheaves, and the spectral sequence of the composite functor $\Gamma \circ f_*$ (resp. $g_* f_*$) furnishes the

THEOREM 2.3.4 (Leray spectral sequence). *Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{ét}}$ and $f\colon X \longrightarrow Y$ a morphism of schemes (resp. morphisms of schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$). We have a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_* F) \implies H^{p+q}(X, F)$$

$$(\textit{resp. } E_2^{p,q} = R^p g_* R^q f_* F \implies R^{p+q}(gf)_* F).$$

COROLLARY 2.3.5. *If $R^q f_* F = 0$ for every $q > 0$, we have $H^p(Y, f_*\mathcal{F}) = H^p(X, \mathcal{F})$ (resp. $R^p g_*(f_*\mathcal{F}) = R^p(gf)_* F$) for every $p \geq 0$.*

This applies particularly in the following case:

PROPOSITION 2.3.6. *Let $f\colon X \longrightarrow Y$ be a finite morphism (or even, by passing to the limit, an integral morphism) and $\mathcal{F}$ an abelian sheaf on X. Then $R^q f_* \mathcal{F} = 0$, for every $q > 0$.*

In effect, let $\overline{y}$ be a geometric point of Y, $\tilde{Y}$ the spectrum of the strict localization of Y at y and $\tilde{X} = X \times_Y \tilde{Y}$; by what precedes, it suffices to show that $H^q(\tilde{X}, \mathcal{F}) = 0$ for every $q > 0$. However $\tilde{X}$ is the spectrum of a product of strictly henselian local rings [see [Ray70], chapter I], the functor $\Gamma(\tilde{X}, \bullet)$ is exact because every surjective étale $\tilde{X}$-scheme admits a section, from which follows the assertion.

2.4. Galois cohomology. For $X = \mathrm{Spec}(K)$ the spectrum of a field, we will see that the étale cohomology is identified with the Galois cohomology.

2.4.1. We will start with a topological analogue. If K is the field of functions of an affine integral algebraic variety $Y = \mathrm{Spec}(A)$ over $\mathbb{C}$, we have $K = \varinjlim_{f\in A} A[1/f]$. Said differently $X = \varprojlim U$, U ranging over the set of open subsets of Y. We know that there exist arbitrarily small Zariski open subsets which for the classical topology are $K(\pi, 1)$. We will therefore not be surprised if we consider $\mathrm{Spec}(K)$ itself as a $K(\pi, 1)$, π being the fundamental group (in the algebraic sense) of X, said differently the Galois group of $\overline{K/K}$, where $\overline{K}$ is a separable closure of K.

2.4.2. More precisely, let K be a field, $\overline{K}$ be a separable closure of K and $G = \mathrm{Gal}(\overline{K}/K)$ the topological Galois group. To every finite étale K-algebra A (a finite product of separable extensions of K), we will associate the finite set $\mathrm{Hom}_K(A, \overline{K})$. The Galois group G acts on this set through a discrete quotient (so finite). If $A = K[T]/(F)$, it is identified with the set of roots in $\overline{K}$ of the polynomial F. Galois theory, under the form given to it by Grothendieck, says that:

Proposition 2.4.2. *The functor:*

(finite étale K-algebras) $\longrightarrow$ (finite sets on which G acts continuously)

associating to an étale algebra A, the set $\mathrm{Hom}_K(A, \overline{K})$ *is an anti-equivalence of categories.*

We deduce from this an analogous description of sheaves for the étale topology on $\mathrm{Spec}(K)$:

Proposition 2.4.3. *The functor:*

(étale sheaves on $\mathrm{Spec}(K)$*)* $\longrightarrow$ *(sets on which G acts continuously)*

that to a sheaf $\mathcal{F}$ associates its fiber $\mathcal{F}_{\overline{K}}$ at the geometric point $\mathrm{Spec}(\overline{K})$ *is an equivalence of categories.*

We say that G acts continuously on a set E if the stabilizer of each element of E is an open subgroup of G. The functor in the inverse sense is written in hte evident manner: let A be a finite étale K-algebra, $U = \mathrm{Spec}(A)$ and $U(\overline{K}) = \mathrm{Hom}_K(A, \overline{K})$ the G-set corresponding to A; so we have $\mathcal{F}(U) = \mathrm{Hom}_{G-\mathrm{set}}(U(\overline{K}, \mathcal{F}_{\overline{k}})$.

Particularly, if $X = \mathrm{Spec}(K)$, we have $\mathcal{F}(X) = \mathcal{F}_{\overline{K}}^{G}$. If we restrict ourselves to abelian sheaves, we obtain on passing to derived functors, canonical isomorphisms

$$H^q(X_{\text{ét}}, \mathcal{F}) = H^q(G, \mathcal{F}_{\overline{K}}).$$

2.4.5. a) To the constant sheaf $\mathbb{Z}/n$ corresponds $\mathbb{Z}/n$ with trivial G-action.
b) To the sheaf of n-th roots of unity μ_n corresponds the group $\mu_n(\overline{k})$ of n-th roots of unity in $\overline{K}$, with the natural G-action.
c) To the sheaf $\mathbb{G}_m$ corresponds the group $\overline{K}^*$ with the natural G-action.

3. Cohomology of curves

In the case of topological spaces, dévissages using the Künneth formula and simplicial decompositions permit us to come back to calculate the cohomology of the interval $I = [0, 1]$ for which we have $H^0(I, \mathbb{Z}) = \mathbb{Z}$ and $H^q(I, \mathbb{Z}) = 0$ for $q > 0$.

In our case, the dévissages will result in more complicated objects, namely curves on an algebraically closed field; we will calculate their cohomology in this chapter. The situation is more complex than in the topological case since the cohomology groups are trivial only for $q > 2$. The essential ingredient of the calculations is the trivialy of the Brauer group of the field of functions on such a curve (Tsen's theorem, 2.3.2).

3.1. Brauer group. Let us first recall the classical definition:

Definition 3.1.1. Let K be a field and A a K-algebra of finite dimension. We say that A is a central simple algebra on K if the following equivalent conditions are satisfied:

a) A has no nontrivial bilateral ideal and its center is K.
b) There exists a finite Galois extension K'/K such that $A_{K'} = A \otimes_K K'$ is isomorphic to an algebra of square matrices on K'.
c) A is K-isomorphic to an algebra of square matrices on a division ring with center K.

Two such algebras are said to be equivalent if the division rings that are associated to them by c) are K-isomorphic. If these algebras are of the same dimension, this amounts to saying that they are K-isomorphic. The tensor product defines by passing to the quotient, an abelian group structure on the set of equivalence classes. This is the group that we classically call the *Brauer group* of K and we denote it by $\mathrm{Br}(K)$.

3.1.2. We will denote by $\mathrm{Br}(n, K)$ the set of K-isomorphism classes of K-algebras A such that there exists a finite Galois extension K' of K for which A_K is isomorphic to the algebra $M_n(K')$ of $n \times n$ square matrices on K'. By definition $\mathrm{Br}(K)$ is the union of subsets $\mathrm{Br}(n, K)$ for $n \in \mathbb{N}$. Let $\overline{K}$ be an algebraic closure of K and $G = \mathrm{Gal}(\overline{K}/K)$. The set $\mathrm{Br}(n, K)$ is the set of "forms" of $M_n(\overline{K})$, it is therefore canonically isomorphic to $H^1(G, \mathrm{Aut}(M_n(\tilde{K})))$.

We know that every automorphism of $M_n(\overline{K})$ is interior. Consequently the group $\mathrm{Aut}(M_n(\overline{K}))$ is identified with the linear projective group $\mathrm{PGL}(n, \overline{K})$ and we have a canonical bijection:

$$\theta_n \colon \mathrm{Br}(n, K) \xrightarrow{\sim} H^1(G, \mathrm{PGL}(n, \overline{K}).$$

On the other hand, the exact sequence:

$$1 \longrightarrow \overline{K}^* \longrightarrow \mathrm{GL}(n, \overline{K}) \longrightarrow \mathrm{PGL}(n, \overline{K}) \longrightarrow 1, \qquad (*)$$

allows us to define a boundary operator

$$\Delta_n \colon H^1(G, \mathrm{PGL}(n, \overline{K})) \longrightarrow H^2(G, \overline{K}^*).$$

Composition θ_n and Δ_n, we obtain a map:

$$\delta_n\colon \operatorname{Br}(n,K) \longrightarrow H^2(G,\overline{K}^*).$$

We easily verify that the maps δ_n are compatible with each other and define a group homomorphism:

$$\delta\colon \operatorname{Br}(K) \longrightarrow H^2(G,\overline{K}^*).$$

PROPOSITION 3.1.3. *The homomorphism* $\delta\colon \operatorname{Br}(K) \longrightarrow H^2(G,\overline{K}^*)$ *is bijective.*

This results in the following two lemmas:

LEMMA 3.1.4. *The map* $\Delta_n\colon H^1(G,\operatorname{PGL}(n,\overline{K})) \longrightarrow H^2(G,\overline{K}^*)$ *is injective.*

By [Ser65], Cor. to Prop. I-44, it suffices to check that every time we twist the exact sequence (*) by an element of $H^1(G,\operatorname{PGL}(n,\overline{K}))$, the H^1 of the middle group is trivial. This middle group is the group of $\overline{K}$-points of the multiplicative group of a simple central algebra A of rank n^2 over K. To prove that $H^1(G, A^*_{\overline{K}}$, we interpret A^* as the group of automorphisms of the free A-module L of rank 1, and H^1 as the set of "forms" of L – of A-modules of rank n^2 over K, automatically free.

LEMMA 3.1.5. *Let* $\alpha \in H^2(G,\overline{K}^*)$*,* K' *a finite extension of* K *contained in* $\overline{K}$*,* $n = [K' : K]$*, and* $G' = \operatorname{Gal}(\overline{K}/K')$*. If the image of* α *in* $H^2(G',\overline{K}^*)$ *is zero, then,* α *is in the image of* Δ_n*.*

Note first of all that we have:

$$H^2(G',\overline{K}^*) \simeq H^2(G,(\overline{K}\otimes_K K')^*).$$

[From a geometric point of view if we write $X = \operatorname{Spec}(K)$, $X' = \operatorname{Spec}(K')$ and $\pi\colon X' \longrightarrow X$ the canonical morphism, we have $R^q\pi_*(\mathbb{G}_{m,X}) = 0$ for $q > 0$ and consequently $H^q(X',\mathbb{G}_{m,X'}) \simeq H^q(X,\pi_*\mathbb{G}_{m,X'}$ for $q \geq 0$].

Moreover choosing a base of K' as a vector space over K allows us to define a homomorphism

$$(\overline{K}\otimes_K K')^* \longrightarrow \operatorname{GL}(n,\overline{K})$$

which, to an element x, assigns the endomorphism of multiplication by x of $\overline{K}\otimes_K K'$. We thus have a commutative diagram with rows exact:

$$\begin{tikzcd}
1 \arrow[r] & \overline{K}^* \arrow[r] \arrow[d, equal] & (\overline{K}\otimes_K K')^* \arrow[r] \arrow[d] & (\overline{K}\otimes_K K')^*/\overline{K}^* \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \overline{K}^* \arrow[r] & \operatorname{GL}(n,\overline{K}) \arrow[r] & \operatorname{PGL}(n,\overline{K}) \arrow[r] & 1
\end{tikzcd}$$

The lemma results in the following commutative diagram that we deduce from above by passing to cohomology:

$$\begin{tikzcd}
H^1(G,(\overline{K}\otimes_K K')^*/\overline{K}^*) \arrow[r] \arrow[d] & H^2(G,\overline{K}^*) \arrow[r] \arrow[d, equal] & H^2(G,(\overline{K}\otimes_K K')^*) \\
H^1(G,\operatorname{PGL}(n,\overline{K})) \arrow[r, "\Delta_n"] & H^2(G,\overline{K}^*) &
\end{tikzcd}$$

The knowledge of the Brauer group, particularly its triviality, is extremely important in Galois cohomology as the following proposition shows.

PROPOSITION 3.1.6. *Let K be a field, $\overline{K}$ an algebraic closure of K and $G = \mathrm{Gal}(\overline{K}/K)$. Suppose that, for every finite extension K' of K, we have $\mathrm{Br}(K') = 0$. Then we have:*

(i) $H^q(G, \overline{K}^*) = 0$ for every $q > 0$.
(ii) $H^q(G, F) = 0$ for every torsion G-module F and for every $q \geq 2$.

[For the proof, see [Ser62] or [Ser65]].

3.2. Tsen's theorem.

DEFINITION 3.2.1. We say that a field K is C_1 if every nonconstant homogenous polynomial $f(x_1, \cdots, x_n)$ of degree $d < n$ has a nontrivial zero.

PROPOSITION 3.2.2. *If a field K is C_1, we have $\mathrm{Br}(K) = 0$.*

It amounts to showing that every division ring D with center K is finite over K and equal to K. Let r^2 be the degree of D over K and $Nrd\colon D \longrightarrow K$ be the reduced norm. [locally for the étale topology on K, D is isomorphic - noncanonically - to an algebra of matrices M_r and the reduced norm coincides with the determinant map. This is well-defined, independent of the chosen isomorphism between D and M_r because every automorphism of M_r is interior and two similar matrices have the same determinant. This locally defined map for the étale topology descends, because of its local uniqueness, to a map $\mathrm{Nrd}\colon D \longrightarrow K$].

The only zero of Nrd is the zero element of D, because, if $x \neq 0$, we have $\mathrm{Nrd}(x) \cdot \mathrm{Nrd}(x^{-1}) = 1$. On the other hand, if $\{e_1, \cdots, e_{r^2}\}$ is a basis of D over K and if $x = \sum_i x_i e_i$, the function $\mathrm{Nrd}(x)$ is written as a homogenous polynomial $\mathrm{Nrd}(x_1, \cdots, x_{r^2})$ of degree r [this is clear locally for the étale topology]. Since K is C_1, we have $r^2 \leq r$, that is $r = 1$ and $D = K$.

THEOREM 3.2.3 (Tsen). *Let $\Bbbk$ be an algebraically closed field and K an extension of transcendance degree 1 of $\Bbbk$. Then K is C_1.*

Suppose first that $K = \Bbbk(X)$. Then

$$f(T) = \sum a_{i_1, \cdots, i_n} T_1^{i_1} \cdots T_n^{i_n}$$

a homogenous polynomial of degree $d < n$ with coefficients in $\Bbbk(X)$. By multiplying the coefficients by a common denominator, we can suppose that they are in $\Bbbk[X]$. So let $\delta = \sup \deg(a_{i_1 \cdots i_n})$. We search for a nontrivial zero in $\Bbbk[X]$ by the method of undetermined coefficients by writing each T_i $(i = 1, \cdots, n)$ as a polynomial of degree N in X. So the equation $f(T) = 0$ becomes a system of homogenous equations in the $n \times (N+1)$ coefficients of polynomials $T_i(X)$ expressing the nullity of the coefficients of the polynomial in X obtained by replacing T_i by $T_i(X)$. This polynomial is of degree $\delta + Nd$ at most, so there are $\delta + Nd + 1$ equations in $n \times (N+1)$ variables. As $\Bbbk$ is algebraically closed, this system has a nontrivial solution if $n(N+1) > Nd + \delta + 1$, which is the case for large enough N if $d < n$.

It is clear that, to show the theorem in the general case, it suffices to show it when K is a finite extension of a pure transcendental extension $\Bbbk(X)$ of $\Bbbk$. Let

$f(T) = f(T_1, \cdots, T_n)$ be a homogenous polynomial of degree $d < n$ with coefficients in K. Let $s = [K : \Bbbk(X)]$ and $e_1, \cdots, e_s$ a basis for K over $\Bbbk(X)$. Let us introduce new variables U_{ij}, sn many, such that $T_i = \sum U_{ij} e_j$. For the polynomial $f(T)$ to have a nontrivial zero in K, it suffices that the polynomial $g(X_{ij}) = N_{K/\Bbbk}(f(T))$ have a nontrivial zero in $\Bbbk(X)$. However g is a homogenous polynomial of degree sd in sn variables, and the result follows.

COROLLARY 3.2.4. *Let $\Bbbk$ an algebraically closed field and K an extension of transcendence degree 1 of $\Bbbk$. Then the étale cohomology groups $H^q(\mathrm{Spec}(K), \mathbb{G}_m)$ are trivial for every $q > 0$.*

3.3. Cohomology of smooth curves. From now on, without explicit mention of the contrary, the cohomology groups considered are the étale cohomology groups.

PROPOSITION 3.3.1. *Let $\Bbbk$ be an algebraically closed field and X a connected projective nonsingular curve over $\Bbbk$. Then we have:*

$$H^0(X, \mathbb{G}_m) = \Bbbk^*,$$
$$H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$$

,

$$H^q(X, \mathbb{G}_m) = 0 \ \textit{for} \ q \geq 2.$$

Let η be the generic point of X, $j : \eta \longrightarrow X$ be the canonical morphism and $\mathbb{G}_{m,\eta}$ the multiplicative group of the field of fractions $K(X)$ of X. For every closed point x of X, let $i_x : x \longrightarrow X$ be the canonical immersion and $\mathbb{Z}_x$ the constant sheaf with value $\mathbb{Z}$ on x. So $j_* \mathbb{G}_{m,\eta}$ is the sheaf of nonzero meromorphic functions on X and $\bigoplus_{x \in X} i_{x*} \mathbb{Z}_x$ the sheaf of divisors, we thus have an exact sequence of sheaves:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow j_* \mathbb{G}_{m,\eta} \xrightarrow{\mathrm{div}} \bigoplus_{x \in X} i_{x*} \mathbb{Z}_x \longrightarrow 0. \qquad (*)$$

LEMMA 3.3.2. *We have $R^q j_* \mathbb{G}_{m,\eta} = 0$ for every $q > 0$.*

It suffices to show that the fiber of this sheaf at every closed point x of X is trivial. If $\tilde{\mathcal{O}}_{X,x}$ is the henselization of X at x and K the field of fractions of $\tilde{\mathcal{O}}_{X,x}$, we have

$$\mathrm{Spec}(K) = \eta \times_X \mathrm{Spec}(\tilde{\mathcal{O}}_{X,x}),$$

so $(R^q j_* \mathbb{G}_{m,\eta})_x = H^q(\mathrm{Spec}(K), \mathbb{G}_m)$. However K is an algebraic extension of $\Bbbk(X)$, so an extension of transcendence degree 1 of $\Bbbk$: the lemma results from Cor. 2.3.2.4.

LEMMA 3.3.3. *We have $H^q(X, j_* \mathbb{G}_{m,\eta}) = 0$ for every $q > 0$.*

Indeed from Lem. 2.3.3.2 and from the Leray spectral sequence for j, we deduce:

$$H^q(X, j_* \mathbb{G}_{m,\eta}) = H^q(\eta, \mathbb{G}_{m,\eta})$$

for every $q \geq 0$ and the right hand side is trivial for $q > 0$ by Cor. 2.3.2.4.

LEMMA 3.3.4. *We have $H^q\left(X, \bigoplus_{x \in X} i_{X*} \mathbb{Z}_x\right) = 0$ for every $q > 0$.*

Indeed for every closed point x of X, we have $R^q i_{x*} \mathbb{Z}_x = 0$ for $q > 0$, because i_x is a finite morphism (see Prop. 2.2.3.6), and

$$H^q(X, i_{x*} \mathbb{Z}_x) = H^q(x, \mathbb{Z}_x).$$

The right hand side is trivial for very $q > 0$, because x is the spectrum of an algebraically closed field [We see that the lemma is true more generally for every "skyscraper" sheaf on X].

We deduce from the preceding lemmas and the exact sequence (*) the equalities:

$$H^q(X, \mathbb{G}_m) = 0 \text{ for } q \geq 2,$$

and an exact sequence of cohomology in low degrees:

$$1 \longrightarrow H^0(X, \mathbb{G}_m) \longrightarrow H^0(X, j_*\mathbb{G}_{m,\eta}) \longrightarrow H^0(X, \bigoplus_{x \in X} i_{x*}\mathbb{Z}_x) \longrightarrow H^1(X, \mathbb{G}_m) \longrightarrow 1$$

which is no different than the exact sequence:

$$1 \longrightarrow \Bbbk^* \longrightarrow \Bbbk(X)^* \longrightarrow \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 1.$$

From Prop. 2.3.3.1 we deduce that the cohomology groups of X with coefficients in $\mathbb{Z}/n$, n coprime to the characteristic of $\Bbbk$, have a reasonable form:

Corollary 3.3.5. *If X is of genus g and if n is invertible in $\Bbbk$, $H^q(X, \mathbb{Z}/n) = 0$ for $q > 2$, and free on $\mathbb{Z}/n$ of rank $1, 2g, 1$ for $q = 0, 1, 2$. Replacing $\mathbb{Z}/n$ by the isomorphic group μ_n, we have canonical isomorphisms:*

$$H^0(X, \mu_n) = \mu_n$$
$$H^1(X, \mu_n) = \mathrm{Pic}^0(X)_n$$
$$H^2(X, \mu_n) = \mathbb{Z}/n.$$

As the field $\Bbbk$ is algebraically closed, $\mathbb{Z}/n$ is isomorphic (noncanonically) to μ_n. From the Kummer exact sequence:

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0,$$

and from Prop. 2.3.3.1, we deduce the equalities:

$$H^q(X, \mathbb{Z}/n) = 0 \text{ for } q > 2,$$

and, in low degrees, the exact sequences:

$$0 \longrightarrow H^0(X, \mu_n) \longrightarrow \Bbbk^* \xrightarrow{n} \Bbbk^* \longrightarrow 0$$
$$0 \longrightarrow H^1(X, \mu_n) \longrightarrow \mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X) \longrightarrow H^2(X, \mu_n) \longrightarrow 0$$

Moreover we have an exact sequence:

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

and $\mathrm{Pic}^0(X)$ is identified with the group of rational points over $\Bbbk$ of an abelian variety of dimension g, the Jacobian of X. In such a group, multiplication by n is surjective and its kernel is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2g$ (since n is invertible in $\Bbbk$); from which follows the corollary.

A clever dévissage, using the "trace method," permits us to obtain the following corollary.

Proposition 3.3.6 (SGA4 IX 5.7). *Let $\Bbbk$ be an algebraically closed field, X an algebraic curve over $\Bbbk$ and F a torsion sheaf on X. Then:*

(i) We have $H^q(X, F) = 0$ for $q > 2$.

(ii) If X is affine, we also have $H^q(X, F) = 0$ for $q > 1$.

For the proof, and for an exposition on the "trace method," we refer the reader to SGA4 IX 5.

3.4. Dévissages. To calculate the cohomology of varieties of dimension > 1 we employ fibrations by curves, which allows us to study the morphisms whose fibers are of dimension ≤ 1. This principle possesses several variants, let us indicate some of them.

3.4.1. Let A be $\mathbb{k}$-algebra of finite type and $a_1, \cdots, a_n$ generators of A. If we write $X_0 = \mathrm{Spec}(\mathbb{k})$, $X_i = \mathrm{Spec}(\mathbb{k}[a_1, \cdots, a_i])$, $X_n = \mathrm{Spec}(A)$, the canonical inclusions $\mathbb{k}[a_1, \cdots, a_i] \longrightarrow \mathbb{k}[a_1, \cdots, a_i, a_{i+1}]$ define morphisms $X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$ whose fibers are of dimension ≤ 1.

3.4.2. In the case of a smooth morphism, we can be more precise. We mean by an *elementary fibration* a morphism of schemes $f\colon X \longrightarrow S$ which can be placed in a commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Y \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

satisfying the following conditions:

(i) j is a dense open immersion into each fiber and $X = \overline{X} - Y$.
(ii) $\overline{f}$ is projective and smooth, with fibers irreducible, geometric, and of dimension 1.
(iii) g is an étale covering and no fiber of g is empty.

We mean by a *good neighborhood* relative to S an S-scheme X such that there exist S-schemes $X = X_n, \cdots, X_0 = S$ and elementary fibrations $f_i\colon X_i \longrightarrow X_{i-1}$, $i = 1, \cdots, n$. We can show [SGA 4, XI, 3.3] that if X is a smooth scheme on an algebraically closed $\mathbb{k}$ every rational point of X has an open neighborhood that is a good neighborhood (relative to $\mathrm{Spec}(\mathbb{k})$).

3.4.3. We can dévisse a proper morphism $f\colon X \longrightarrow S$ in the following manner. By Chow's lemma, there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & & \overline{X} \arrow[ll, "\pi"'] \arrow[dl, "\overline{f}"] \\ & S & \end{tikzcd}$$

where π and $\overline{f}$ are projective morphisms, π is moreover an isomorphism over a dense open subset of X. Locally on S, $\overline{X}$ is a closed subscheme of a projective space of the type $\mathbb{P}^n_S$.

We dévisse the latter by considering the projection $\varphi\colon \mathbb{P}^n_S \longrightarrow \mathbb{P}^1_S$ which sends the point with homogenous coordinates $(x_0, x_1, \cdots, x_n)$ on (x_0, x_1). This is a rational map defined outside the closed subset $Y \simeq \mathbb{P}^{n-2}_S$ of $\mathbb{P}^n_S$ defined by the homogenous equations $x_0 = x_1 = 0$. Let $u\colon P \longrightarrow \mathbb{P}^n_S$

be the blow-up at Y; the fibers of u are of dimension ≤ 1. Moreover there exists a natural morphism $v\colon P \longrightarrow \mathbb{P}^1_S$ that extends the rational map φ and v makes P a $\mathbb{P}^1_S$-scheme locally isomorphic to the projective space of type $\mathbb{P}^{n-1}$ which can in turn be projected onto a $\mathbb{P}^1$, etc.

3.4.4. We can sweep a smooth projective variety X by a *Lefschetz pencil*. The blowup $\tilde{X}$ of the intersection of the axis of the pencil with X projects onto $\mathbb{P}^1$ and the fibers of this projection are the hyperplane sections of X by the hyperplanes of the pencil.

4. Base-change theorem for a proper morphism

4.1. Introduction. This section is devoted to the proof and applications of:

THEOREM 4.1.1. *Let $f\colon X \longrightarrow S$ be a proper morphism of schemes and $\mathcal{F}$ an abelian torsion sheaf on X. Then, for any $q \geq 0$, the fiber of $R^q f_*\mathcal{F}$ at a geometric point s of S is isomorphic to the cohomology $H^q(X_s, \mathcal{F})$ of the fiber $X_s = X \otimes_S \operatorname{Spec} \Bbbk(s)$ of f at s.*

For $f\colon X \longrightarrow S$ a proper and separated continuous map (separated meaning that the diagonal of $X \times_S X$ is closed) between topological spaces, and F an abelian sheaf on X, the analogous result is well-known, and elementary: as f is closed, the $f^{-1}(V)$'s for neighborhoods V of s form a fundamental system of neighborhoods of X_s, and we verify that $H^*(X_s, \mathcal{F}) = \varinjlim_U H^*(U, \mathcal{F})$, for U ranging over the neighborhoods of X_s. In practice, X_s also has a fundamental system $\mathcal{U}$ of neighborhoods U from which it is deformation retracted and, for F constant, we thus have $H^*(X_s, \mathcal{F}) = H^*(U, \mathcal{F})$. In figurative terms: the special fiber swallows the general fiber.

In the case of schemes the proof is more delicate and it is indispensable to suppose that $\mathcal{F}$ is of torsion (SGA4 XII 2). Considering the description of the fibers of $R^q f_*\mathcal{F}$ (see 2.2.3.3.), Thm. 2.4.1.1 is essentially equivalent to

THEOREM 4.1.2. *Let A be a strictly henselian local ring and $S = \operatorname{Spec}(A)$. Let $f\colon X \longrightarrow S$ a proper morphism and X_0 the closed fiber of f. Then, for every abelian torsion sheaf $\mathcal{F}$ on X and for every $q \geq 0$, we have $H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X_0, \mathcal{F})$.*

By passing to the limit we see that it suffices to show the theorem when A is the strict henselization of a $\mathbb{Z}$-algebra of finite type at a prime ideal. We first treat the case $q = 0$ or 1 and $\mathcal{F} = \mathbb{Z}/n$ (see 2.4.2). An argument based on the notion of constructible sheaves (see 2.4.3) shows that it actually suffices to consider the case where $\mathcal{F}$ is constant. On the other hand the dévissage (see 2.3.4.3.) allows us to suppose that X_0 is a curve; in this case it only remains to show the theorem for $q = 2$ (see 2.4.4).

Among other applications (see 2.4.6), the theorem allows us to define the notion of cohomology with proper support (see 2.4.5).

4.2. Proof for $q = 0$ or 1 and $\mathcal{F} = \mathbb{Z}/n$. The result for $q = 0$ and $\mathcal{F}$ constant is equivalent to the following proposition [the Zariski connection theorem]:

PROPOSITION 4.2.1. *Let A be a noetherian and henselian local ring and $S = \operatorname{Spec}(A)$. Let $f\colon X \longrightarrow S$ be a proper morphism and X_0 the closed fiber of f. Then the sets of connected components $\pi_0(X)$ and $\pi_0(X_0)$ are in bijection.*

It boils down to the same thing to show that the sets of clopen subspaces $\operatorname{Clo}(X)$ and $\operatorname{Clo}(X_0)$ are in bijection. We know that the set $\operatorname{Clo}(X)$ corresponds bijectively to the set of idempotents of $\Gamma(X, \mathcal{O}_X)$, similarly $\operatorname{Clo}(X_0)$ corresponds bijectively to the set of idempotents of $\Gamma(X_0, \mathcal{O}_{X_0})$. It is therefore a question of showing that the canonical map

$$\operatorname{Idem}\Gamma(X, \mathcal{O}_X) \longrightarrow \operatorname{Idem}\Gamma(X_0, \mathcal{O}_{X_0})$$

is bijective.

We will denote by $\mathfrak{m}$ the maximal ideal of A, $\Gamma(X,\mathcal{O}_X)^\wedge$ the completion of $\Gamma(X,\mathcal{O}_X)$ for the $\mathfrak{m}$-adic topology and, for every integer $n \geq 0$, $X_n = X \otimes_A A/\mathfrak{m}^{n+1}$. By the theorem of finiteness for proper morphisms (see [EGA3], 3.2), $\Gamma(X,\mathcal{O}_X)$ is a finite A-algebra; since A is henselian, it follows that the canonical map

$$\mathrm{Idem}\Gamma(X,\mathcal{O}_X) \longrightarrow \mathrm{Idem}\Gamma(X,\mathcal{O}_X)^\wedge$$

is bijective.

By the comparison theorem for proper morphisms (see [EGA3] 4.1), the canonical map

$$\Gamma(X,\mathcal{O}_X)^\wedge \longrightarrow \varprojlim \Gamma(X_n,\mathcal{O}_{X_n})$$

is bijective. In particular the canonical map

$$\mathrm{Idem}\Gamma(X,\mathcal{O}_X)^\wedge \longrightarrow \varprojlim \mathrm{Idem}\Gamma(X_n,\mathcal{O}_{X_n})$$

is bijective. But, since X_n and X_0 have the same underlying topological space, the canonical map

$$\mathrm{Idem}\Gamma(X_n,\mathcal{O}_{X_n}) \longrightarrow \mathrm{Idem}\Gamma(X_0,\mathcal{O}_{X_0})$$

is bijective for every n, which completes the proof.

Since $H^1(X,\mathbb{Z}/n)$ is in bijection with the set of isomorphism classes of galois étale coverings of X of group $\mathbb{Z}/n$, the theorem for $q = 1$ and $\mathcal{F} = \mathbb{Z}/n$ results in the following proposition.

Proposition 4.2.2. *Let A be a noetherian and henselian local ring and $S = \mathrm{Spec}(A)$. Let $f\colon X \longrightarrow S$ be a proper morphism and X_0 the closed fiber of f. Then the restriction functor*

$$\mathrm{Et.cov.(X)} \longrightarrow \mathrm{Et.cov.(X_0)}$$

is an equivalence of categories.

[If X_0 is connected and if we choose a geometric point of X_0 as the basepoint, that is to say that the canonical map $\pi_1(X_0) \longrightarrow \pi_1(X)$ on the (profinite) fundamental groups is bijective].

Prop. 2.4.2.1 shows that this functor is fully faithful. Indeed, if X' and X'' are two étale covers of X, an X-morphism from X' to X'' is determined by its graph which is a clopen subspace of $X \times_X X''$.

It is therefore a question of showing that every étale cover X_0' of X_0 extends to an étale cover of X. We know that the étale covers are independent of nilpotent elements (see [SGA1], Ch. I), consequently X_0' lifts uniquely to an étale cover X_n' of X_n for every $n \geq 0$, said otherwise to an étale cover $\mathcal{X}'$ of the formal scheme $\mathcal{X}$ the completion of X along X_0. By Grothendieck's theorem of algebrization of formal coherent sheaves (see the existence theorem of [EGA3] Ch. 5), $\mathcal{X}'$ is the formal completion of an étale cover $\overline{X}'$ of $\overline{X} = X \otimes_A \hat{A}$.

By passing to the limit, it suffices to show the proposition in the case where A is the henselization of a $\mathbb{Z}$-algebra of finite type. We can then apply Artin's approximation theorem to the functor $F\colon \mathbf{Alg}_A \longrightarrow \mathbf{Set}$ which, to an A-algebra B, assigns the set of isomorphism classes of étale covers of $X \otimes_A B$. Indeed this functor is locally of finite presentation: if B_i is an inductive filtered system of A-algebras and if $B = \varinjlim B_i$, we have $F(B) = \varinjlim F(B_i)$. By Artin's theorem, given an element $\overline{\xi} \in F(\hat{A})$, in this case the isomorphism class of $\overline{X}'$, there exists $\xi \in F(A)$ having the same image of $\overline{\xi}$ in $F(A/\mathfrak{m})$. Said otherwise there exists an étale cover X' of X whose restriction to X_0 is isomorphic to X_0'.

4.3. Constructible sheaves. In this paragraph, we consider a *noetherian* scheme X and we call a sheaf on X an *abelian* sheaf on X_{et}.

DEFINITION 4.3.1. We say that a sheaf $\mathcal{F}$ on X is locally constant constructible (in short, l.c.c.) if it is represented by an étale cover of X.

DEFINITION 4.3.2. We say that a sheaf $\mathcal{F}$ on X is constructible if it satisfies the following equivalent conditions:

(i) There exists a finite surjective family of subschemes X_i of X such that the restriction of $\mathcal{F}$ to X_i is l.c.c..
(ii) There exists a finite family of finite morphisms $p_i\colon X_i' \longrightarrow X$, for eachc i a constant constructible sheaf (= defined by a finite abelian group) $\mathcal{C}_i$ on X_i', and a monomorphism $\mathcal{F} \longrightarrow \prod_i p_{i*}\mathcal{C}_i$.

We easily check that the category of constructible sheaves on X is an abelian category. Moreover, if $u\colon \mathcal{F} \longrightarrow \mathcal{G}$ is a homomorphism of sheaves and if $\mathcal{F}$ is constructible, the sheaf $\mathrm{Im}(u)$ is constructible.

LEMMA 4.3.3. *Every torsion sheaf $\mathcal{F}$ is an inductive filtered limit of constructible sheaves.*

Indeed, if $j\colon U \longrightarrow X$ is an étale scheme of finite type on X, an element $\xi \in \mathcal{F}(U)$ such that $n\xi = 0$ defines a homomorphism of sheaves $j_!\underline{\mathbb{Z}/n}_U \longrightarrow \mathcal{F}$ whose image (the smalles subsheaf of $\mathcal{F}$ whose ξ is a local section) is a constructible subsheaf of $\mathcal{F}$. It is clear that $\mathcal{F}$ is an inductive limit of such subsheaves.

DEFINITION 4.3.4. Let $\mathcal{C}$ be an abelian category and T a functor defined on $\mathcal{C}$ taking values in the category of abelian groups. We will say that T is effaceable in $\mathcal{C}$ if, for every object A of $\mathcal{C}$ and every $\alpha \in T(A)$, there exists a monomorphism $u\colon A \longrightarrow M$ in $\mathcal{C}$ such that $T(u)\cdot\alpha = 0$.

LEMMA 4.3.5. *The functors $H^q(X,\cdot)$ for $q > 0$ are effaceable in the category of constructible sheaves on X.*

It suffices to remark that, if $\mathcal{F}$ is a constructible sheaf, there necessarily exists an integer $n > 0$ such that $\mathcal{F}$ is a sheaf of $\mathbb{Z}/n$-modules. So there exists a monomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$, where $\mathcal{G}$ is a sheaf of $\mathbb{Z}/n$-modules and $H^q(X,G) = 0$ for every $q > 0$. We can for example take for G to be the Godement resolution $\prod_{x\in X} i_{x^*}\mathcal{F}_{\overline{x}}$, where x ranges over the points of X and $i_x\colon \overline{x} \longrightarrow X$ is a geometric point centered at x. By Lem. 2.4.3.3 $\mathcal{G}$ is an inductive limit of constructible sheaves, from which the lemma follows, since the functors $H^q(X,\cdot)$ commute with inductive limits.

LEMMA 4.3.6. *Let $\varphi^\bullet\colon T^\bullet \longrightarrow T'^\bullet$ be a morphism of cohomology functors defined on an abelian category $\mathcal{C}$ and taking values in the category of abelian groups. Suppose that T^q is effaceable for $q > 0$ and let $\mathcal{E}$ be a subset of objects of $\mathcal{C}$ such* *that every object of $\mathcal{C}$ is contained in an object of $\mathcal{E}$. Then the following conditions are equivalent:*

(i) $\varphi^q(A)$ *is bijective for every* $q \geq 0$ *and every* $A \in \mathrm{Ob}\mathcal{C}$.
(ii) $\varphi^0(M)$ *is bijective and* $\varphi^q(M)$ *surjective for every* $q > 0$ *and every* $M \in \mathcal{E}$.
(iii) $\varphi^0(A)$ *is bijective for every* $A \in \mathrm{Ob}\mathcal{C}$ *and* $(T')^q$ *is effaceable for every* $q > 0$.

The proof is by recurrence on q and does not present difficulties.

PROPOSITION 4.3.7. *Let X_0 be a subscheme of X. Suppose that, for every $n \geq 0$ and for every scheme X' finite over X, the canonical map*

$$H^q(X', \mathbb{Z}/n) \longrightarrow H^q(X'_0, \mathbb{Z}/n),$$

where $X'_0 = X' \times_X X_0$, is bijective for $q = 0$ and surjective for $q > 0$. Then, for every torsion sheaf $\mathcal{F}$ on X and for every $q \geq 0$, the canonical map

$$H^q(X, \mathcal{F}) \longrightarrow H^q(X_0, \mathcal{F})$$

is bijective.

By passing to the limit, it suffices to the assertion for $\mathcal{F}$ constructible. We apply Lem. 2.4.3.6 by taking $\mathcal{C}$ to be the category of constructible sheaves on X, $T^q = H^q(X, \cdot)$, $(T')^q = H^q(X_0, \cdot)$ and $\mathcal{E}$ the set of constructible sheaves of the form $\pi_{p_i^*}\mathcal{C}_i$, where $p_i \colon X'_i \longrightarrow X$ is a finite morphism and $\mathcal{C}_i$ is a constant sheaf finite on X'_i.

4.4. End of the proof. By the method of fibration by curves (2.3.4.3.), we come back to proving the theorem in relative dimension ≤ 1. By the preceding paragraph, it will suffice to prove that, if S is the spectrum of a local strictly henselian noetherian ring, $f \colon X \longrightarrow S$ a proper morphism whose closed fiber X_0 is of dimension ≤ 1 and n an integer ≥ 0, the canonical homomorphism

$$H^q(X, \mathbb{Z}/n) \longrightarrow H^q(X_0, \mathbb{Z}/n)$$

is bijective for $q = 0$ and surjective for $q > 0$.

The cases $q = 0$ and 1 were seen above and we have $H^q(X_0, \mathbb{Z}/n) = 0$ for $q \geq 3$; it suffices then to treat the case $q = 2$. Evidently we can suppose that n is a power of a prime number. If $n = p^r$, where p is the characteristic of the residual field of S, the theory of Artin-Schreier shows that we have $H^2(X_0, \mathbb{Z}/p^r) = 0$. If $n = \ell^r$, $\ell \neq p$, we deduce from Kummer's theory a commutative diagram

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "\alpha"] \arrow[d] & H^2(X, \mathbb{Z}/\ell^r) \arrow[d] \\ \mathrm{Pic}(X_0) \arrow[r, "\beta"] & H^2(X_0, \mathbb{Z}/\ell^r) \end{tikzcd}$$

where the map β is surjective [We have seen this in section 2.3 for a smooth curve on an algebraically closed field, but similar arguments apply to any curve on a separably closed field].

To conclude, it suffices to show:

PROPOSITION 4.4.1. *Let S be the spectrum of a henselian noetherian local ring and $f\colon X \longrightarrow S$ a proper morphism whose closed fiber X_0 is of dimension ≤ 1. Then the canonical restriction map*

$$\operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(X_0)$$

is surjective [It is sufficient that the morphism f is separated *of finite type].*

To simplify the proof, we suppose that X is integral, even though this is not necessary. Every invertible sheaf on X_0 is associated to a Cartier divisor (because X_0 is a curve, hence quasi-projective), it suffices then to show that the canonical map $\operatorname{Div}(X) \longrightarrow \operatorname{Div}(X_0)$ is surjective.

Every divisor on X_0 is a linear combination of divisors whose support is concentrated at a single closed non-isolated point of X_0. Let x be one such point, $t_0 \in \mathcal{O}_{X_0,x}$ be a regular non-invertible element of $\mathcal{O}_{X_0,x}$ and D_0 be the divisor concentrated at x of local equation t_0. Let U be an open neighborhood of x in X such that there exists a section $t \in \Gamma(U, \mathcal{O}_U)$ relevant to t_0. Let Y be the closure of U of equation $t = 0$; even if it means taking U to be small enough, we can suppose that x is the only point of $Y \cap X_0$. Then Y is quasi-finite over S at x; since S is the spectrum of a henselian local ring, we can deduce that $Y = Y_1 \coprod Y_2$, where Y_1 is finite over S and where Y_2 does not meet X_0. Moreover, as X is separated over S, Y_1 is closed in X.

Even if it means replacing U by a smaller open neighborhood of x, we can suppose that $Y = Y_1$, said otherwise, that Y is closed in X. We can then define a divisor D on X associated to D_0 by supposing $D|_{X-Y} = 0$ and $D|_U = \operatorname{div}(t)$ which makes sense because t is invertible on $U - Y$.

REMARK 4.4.2. In the case where f is proper, we can conduct a proof similar to that of Prop. 2.4.2.2. Indeed, as X_0 is a curve, there exists no obstruction to lift an invertible sheaf on X_0 to infinitesimal neighborhoods X_n of X_0, so to the formal completion $\mathcal{X}$ of X along X_0. We conclude then by successively applying Grothendieck's existence theorem and Artin's approximation theorem.

4.5. Cohomology with proper support.

DEFINITION 4.5.1. Let X be a separated scheme of finite type on a field k. By a theorem of Nagata, there exists a proper scheme $\overline{X}$ over k and an open immersion $j\colon X \longrightarrow \overline{X}$. For every torsion sheaf $\mathcal{F}$ on X we note $j_!\mathcal{F}$ the extension by 0 of $\mathcal{F}$ to $\overline{X}$ and we define the cohomology groups of proper support $H^q_c(X, \mathcal{F})$ by writing

$$H^q_c(X, \mathcal{F}) = H^q(\overline{X}, j_!\mathcal{F}).$$

We will show that this definition is independent of the chosen compactifiction $j\colon X \longrightarrow \overline{X}$. Let $j_1\colon X \longrightarrow \overline{X}_1$ and $j_2\colon X \longrightarrow \overline{X}_2$ be two compactifications. Then X sends itself into $\overline{X}_1 \times \overline{X}_2$ by $x \mapsto (j_1(x), j_2(x))$ and the closed image $\overline{X}_3$ of X

by this map is a compactification of X. We thus have a commutative diagram:

$$\begin{tikzcd} & \overline{X}_1 & \\ X \arrow[ur, "j_1"] \arrow[rr, "(j_1,j_2)"] \arrow[dr, "j_2"'] & & \overline{X}_3 \arrow[ul, "p_1"'] \arrow[dl, "p_2"] \\ & \overline{X}_2 & \end{tikzcd}$$

p_1 and p_2, the restrictions of natural projections to $\overline{X}_3$, are proper morphisms.

It suffices then to treat the case where we have a commutative diagram

$$\begin{tikzcd} & \overline{X}_2 \arrow[d, "p"] \\ X \arrow[ur, "j_2"] \arrow[r, "j_1"'] & \overline{X}_1 \end{tikzcd}$$

Lemma 4.5.2. *We have $p_*(j_{2!}\mathcal{F}) = j_{1!}\mathcal{F}$ and $R^q p_*(j_{2!}\mathcal{F}) = 0$, for $q > 0$.*

We note right away that the lemma suffices to conclude our claim. Using the Leray spectral sequence of the morphism p, we deduce that we have, for every $q \geq 0$,

$$H^q(\overline{X}_2, j_{2!}\mathcal{F}) = H^q(\overline{X}_1, j_{1!}\mathcal{F}).$$

To show the lemma, we reason fiber by fiber using the base-change theorem (see 2.4.1.1) for p. The result is immediate, since, over a point of X, p is an isomorphism at, over a point of $\overline{X}_1 - X$, $j_{2!}\mathcal{F}$ is trivial over the fiber of p. with p a proper morphism.

4.5.3. Likewise, if $f\colon X \longrightarrow S$ is a separated morphism of finite type between noetherian schemes, there exists a proper morphism $\overline{f}\colon \overline{X} \longrightarrow S$ and an open immersion $j\colon X \longrightarrow \overline{X}$. We define then the superior direct images with proper support $R^q f_!$ by writing for every torsion sheaf $\mathcal{F}$ on X

$$R^q f_! = R^q f_*(j_!\mathcal{F}).$$

We verify as before that this definition is independent of the chosen compactification.

Theorem 4.5.4. *Let $f\colon X \longrightarrow S$ be a separated morphism of finite type between noetherian schemes and $\mathcal{F}$ a torsion sheaf on X. Then the fiber of $R^q f_!\mathcal{F}$ at a geometric point s of S is isomorphic to the cohomology with proper support $H^q_c(X_s, \mathcal{F})$ of the fiber X_s of f at s.*

This is a simple variant of the base-change theorem for a proper morphism (see Thm. 2.4.1.1). More generally, if

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\ S & S' \arrow[l, "g"] \end{tikzcd}$$

is a cartesian diagram, we have a canonical isomorphism.

$$g^*(R^q f_! \mathcal{F}) \cong R^q f'_!(g'^* \mathcal{F}). \tag{4.5.4.1}$$

4.6. Applications.

THEOREM 4.6.1. *[Annihilation theorem] Let $f\colon X \longrightarrow S$ be a separated morphism of finite type whose fibers are of dimension $\leq n$ and $\mathcal{F}$ a torsion sheaf on X. Then we have $R^q f_! \mathcal{F} = 0$ for $q > 2n$.*

By the base-change theorem, we can suppose that S is the spectrum of a separably closed field. If $\dim X = n$, there exists an affine open subset U of X such that $\dim(X - U) < n$; we then have an exact sequence $0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{X-U} \longrightarrow 0$ and, by recurrence on n, it suffices to prove the theorem for $X = U$ affine. Then the method of fibration by curves (see 2.3.4.1.) and the base-change theorem allows us to reduce to a curve on a separably closed field for which we deduce the required result from Tsen's theorem (see Prop. 2.3.3.6).

THEOREM 4.6.2. *Let $f\colon X \longrightarrow S$ be a separated morphism of finite type and $\mathcal{F}$ a constructible sheaf on X. Then the sheaves $R^q f_! \mathcal{F}$ are constructibles.*

We will only consider the case where $\mathcal{F}$ is annihilated by an invertible integer over X.

To prove the theorem we reduce to the case where $\mathcal{F}$ is a constant sheaf $\underline{\mathbb{Z}/n}$ and where $f\colon X \longrightarrow S$ is a proper smooth morphism whose fibers are geometrically connected curves of genus g. For n invertible over X, the sheaves $R^q f_* \mathcal{F}$ are then locally free of finite rank, trivial for $q > 2$ (Thm. 2.4.6.1). Replacing $\mathbb{Z}/n$ by the sheaf locally isomorphic (over S) to μ_n, we have canonically

$$\begin{aligned} R^0 f_* \mu_n &= \mu_n \\ R^1 f_* \mu_n &= \underline{\mathrm{Pic}}(X/S)_n \\ R^2 f_* \mu_n &= \mathbb{Z}/n. \end{aligned} \tag{6.2.1}$$

THEOREM 4.6.3 (Comparison with classical cohomology). *Let $f\colon X \longrightarrow S$ be a separated morphism between schemes of finite type over $\mathbb{C}$, and $\mathcal{F}$ a torsion sheaf over X. Let us denote by an exponent ^{an} the functor of passing to usual topological spaces, and by $R^q f_!^{an}$ the derived functors of the direct image functor with proper support by f^{an}. We have*

$$(R^q f_! \mathcal{F})^{an} \cong R^q f_!^{an} \mathcal{F}^{an}.$$

Particularly, for $S =$ a point and $\mathcal{F}$ the constant sheaf $\mathbb{Z}/n$,

$$H_c^q(X, \mathbb{Z}/n) \cong H_c^q(X^{an}, \mathbb{Z}/n).$$

Dévissages using the base-change theorem allow us to reduce to the case where X is a proper smooth curve, where $S =$ a point, and where $\mathcal{F} = \mathbb{Z}/n$. The cohomology groups considered are then trivial for $q \neq 0, 1, 2$ and we invoke GAGA: indeed, if X is proper over $\mathbb{C}$, we have $\pi_0(X) = \pi_0(X^{an})$ and $\pi_1(X) =$ profinite completion of $\pi_1(X^{an})$, from which the assertion for $q = 0, 1$. For $q = 2$, we use the Kummer exact sequence and the fact that, by GAGA again, $\mathrm{Pic}(X) = \mathrm{Pic}(X^{an})$.

THEOREM 4.6.4 (Cohomological dimension of affine schemes). *Let X be an affine scheme of finite type over a separably closed field and $\mathcal{F}$ a torsion sheaf on X. We then have $H^q(X,\mathcal{F}) = 0$ for $q > \dim(X)$.*

For a very pretty proof we refer the reader to [SGA4V3], XIV § 2 and 3.

REMARK 4.6.5. This theorem is in some way a substitute for Morse theory. We will indeed consider the classical case where X is smooth and affine over $\mathbb{C}$ immersed in an affine space of type $\mathbb{C}^N$. Then, for almost every point $p \in \mathbb{C}^N$, the function "distance to p" on X is a Morse function and the indices of its critical points are smaller than $\dim(X)$. So X is obtained by reattaching the handles of index smaller than $\dim(X)$, from which the classical analogue of Thm. 2.4.6.4 follows.

5. Local acyclicity of smooth morphisms

Let X be a complex analytic variety and $f\colon X \longrightarrow D$ a morphism from X to the unit disk. We denote by $[0,t]$ the right closed segment with endpoints 0 and t in D and $]0,t]$ the semi-open segment. If f is *smooth*, the inclusion

$$j\colon f^{-1}(]0,t]) \hookrightarrow f^{-1}([0,t])$$

is a homotopy equivalence; we can push the special fiber $X_0 = f^{-1}(0)$ into $f^{-1}([0,t])$.

In practice, for small enough t, $f^{-1}(]0,t])$ will be fibered over $]0,t]$ such that the inclusion

$$X_t = f^{-1}(t) \hookrightarrow f^{-1}(]0,t])$$

will also be a homotopy equivalence. We then call the following homotopy class of maps the *cospecialization morphism*:

$$\mathrm{cosp}\colon X_0 \hookrightarrow f^{-1}([0,t]) \xleftarrow{\simeq} f^{-1}(]0,t]) \xleftarrow{\simeq} X_t$$

We can express this construction in terms of images by saying that, for a smooth morphism, the general fiber contains the special fiber.

Let us no longer suppose that f is necessarily smooth (but let us suppose that $f^{-1}(]0,t])$ is fibered over $]0,t]$). We can define again a morphism cosp^* in cohomology when $j_*\mathbb{Z} = \mathbb{Z}$ and $R^q j_*\mathbb{Z} = 0$ for $q > 0$. Under these hypotheses, the Leray spectral sequence for j shows that we have

$$H^*(f^{-1}([0,t]),\mathbb{Z}) \xrightarrow{\sim} H^*(f^{-1}(]0,t]),\mathbb{Z})$$

and cosp^* is the composite morphism:

$$\mathrm{cosp}^*\colon H^*(X_t,\mathbb{Z}) \xleftarrow{\sim} H^*(f^{-1}(]0,t]),\mathbb{Z}) \xleftarrow{\sim} H^*(f^{-1}([0,t]),\mathbb{Z}) \longrightarrow H^*(X_0,\mathbb{Z})$$

The fiber of $R^q j_*\mathbb{Z}$ at a point $x \in X_0$ can be calculated as follows. We take a ball B_ε centered at x with small enough radius ε in an ambient space, and for η small enough, we suppose $E = X \cap B_\varepsilon \cap f^{-1}(\eta t)$; this is the *variety of evanescent cycles* at x. We have:

$$(R^q j_*\mathbb{Z})_x \xleftarrow{\sim} H^q(X \cap B_\varepsilon \cap f^{-1}(]0,\eta t]),\mathbb{Z}) \xrightarrow{\sim} H^q(E,\mathbb{Z})$$

and the cospecialization morphism is defined in cohomology as long as the varieties of evanescent cycles are acyclic [$H^0(E,\mathbb{Z}) = \mathbb{Z}$ and $H^q(E,Z) = 0$ for $q > 0$], which can be expressed by saying that f is *locally acyclic*.

This chapter is devoted to the analogue of this situation for a smooth morphism of schemes and for the étale cohomology. However it is essential in this body of work to limit ourselves to torsion coefficients and to *first order residual characteristics*. Paragraph 1 is devoted to generalities on locally acyclic morphisms and cospecialization arrows. In paragraph 2, we demonstrate that a smooth morphism is locally acyclic. In paragraph 3, we reconcile this result with those from the preceding chapter to deduce two applications: a theorem of specialization of cohomology groups (the cohomology of geometric fibers of a proper smooth morphism is locally constant) and a theorem for base-change by a smooth morphism.

In all that follows, we fix an integer n and the word "scheme" will be used to mean "scheme on which n is invertible." "Geometric point" will always mean an "algebraic geometric point" (see 2.2.3) $x\colon \operatorname{Spec}(k) \longrightarrow X$, with k algebraically closed.

5.1. Locally acyclic morphisms.

Notation 5.1.1. Given a scheme S and a geometric point s of S, we denote by $\tilde{S}^s$ the strictly localized spectrum of S at s.

Definition 5.1.2. We say that a geometric point t of S is a generalization of s if it is defined by an algebraic closure of the residual field of a point in $\tilde{S}^s$. We also say that s is a specialization of t and we call the S-morphism $t \longrightarrow \tilde{S}^s$ the specialization arrow.

Definition 5.1.3. Let $f\colon X \longrightarrow S$ be a morphism of schemes. Let s be a geometric point of S, t a generalization of s, x a geometric point of X over s and $\tilde{X}^x_t = \tilde{X}^x \times_{\tilde{S}^s} t$. Then we say that $\tilde{X}^x_t$ is a variety of evanescent cycles of f at the point x.

We say that f is locally acyclic if, the reduced cohomology of every variety of evanescent cycles $\tilde{X}^x_t$ is trivial:

$$\tilde{H}^*(\tilde{X}^x_t, \mathbb{Z}/n) = 0, \tag{5.1.3.1}$$

i.e. $H^0(X^x_t, \mathbb{Z}/n) = \mathbb{Z}/n$ and $H^q(\tilde{X}^x_t, \mathbb{Z}/n) = 0$ for $q > 0$.

Lemma 5.1.4. *Let $f\colon X \longrightarrow S$ be a locally acyclic morphism and $g\colon S' \longrightarrow S$ a quasi-finite mophism (or the projective limit of quasi-finite morphisms). Then the morphism $f'\colon X' \longrightarrow S'$ derived from f by base-change is locally acyclic.*

We end up verifying that every variety of evanescent cycles of f' is a variety of evanescent cycles of f.

Lemma 5.1.5. *Let $f\colon X \longrightarrow S$ be a locally acyclic morphism. For every geometric point t of S, giving rise to a Cartesian diagram*

$$\begin{tikzcd} X_t \arrow[r, "\varepsilon'"] \arrow[d] & X \arrow[d, "f"] \\ t \arrow[r, "\varepsilon"] & S \end{tikzcd}$$

We have $\varepsilon'_*\mathbb{Z}/n = f^*\varepsilon_*\mathbb{Z}/n$ and $R^q\varepsilon'_*\mathbb{Z}/n = 0$ for $q > 0$.

Let $\overline{S}$ be the closure of $\varepsilon(t)$, S' the normalization of $\tilde{S}$ in $k(t)$, and consider the Cartesian diagram:

$$\begin{tikzcd} X_t \arrow[r, "i'"] \arrow[d] & X' \arrow[r, "\alpha'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ t \arrow[r, "i"] & S' \arrow[r, "\alpha"] & S \end{tikzcd}$$

The local rings of S' are normal to separably closed fraction fields. They are thus strictly Henselian, and the local acyclicity of f' (Lemma 2.5.1.4) provides $f'_*\mathbb{Z}/n = \mathbb{Z}/n$, $R^q f'_*\mathbb{Z}/n = 0$ for $q > 0$. Since α is an integer, we then have $R^q\varepsilon'_*\mathbb{Z}/n = \alpha'_* R^q i'_*\mathbb{Z}/n = \alpha'_* f'^* R^q i_*\mathbb{Z}/n = f^*\alpha_* R^q i_*\mathbb{Z}/n = f^* R^q \varepsilon_*\mathbb{Z}/n$; and hence the lemma.

5.1.6. Given a locally acyclic morphism $f\colon X \longrightarrow S$ and a specialization arrow $t \longrightarrow \tilde{S}^s$, we will define canonical homomorphisms, called cospecialization arrows

$$\operatorname{cosp}^*\colon H^*(X_t, \mathbb{Z}/n) \longrightarrow H^*(X_s, \mathbb{Z}/n),$$

connecting the cohomology of the general fiber $X_t = X \times_S t$ to that of the special fiber $X_s = X \times_S s$.

Consider the Cartesan diagram

$$\begin{tikzcd} X_t \arrow[r, "\varepsilon'"] \arrow[d] & \tilde{X} \arrow[d, "f'"] & X_s \arrow[l] \arrow[d] \\ t \arrow[r, "\varepsilon"] & \tilde{S}^s & s \arrow[l] \end{tikzcd}$$

derived from f from base-change. By Lemma 2.5.1.4, f' is again locally acyclic. From the definition of local acyclicity, we immediately get that the restriction to X_s of the sheaves $R^q\varepsilon'_*\mathbb{Z}/n$ is $\mathbb{Z}/n$ for $q = 0$, and 0 for $q > 0$. By Lemma 2.5.1.5, we even know that $R^q\varepsilon'_*\mathbb{Z}/n = 0$ for $q > 0$. We define cosp^* as the composite arrow

$$H^*(X_t, \mathbb{Z}/n) \cong H^*(X, \varepsilon'_*\mathbb{Z}/n) \longrightarrow H^*(X_s, \mathbb{Z}/n) \tag{5.1.6.1}$$

Variant: Let $\overline{S}$ be the closure of $\varepsilon(t)$ in $\overline{S}^s$, S' the normalization of $\overline{S}$ in $k(t)$ and X'/S' derived from X/S by base-change. The diagram 2.5.1.6. can again be written as:

$$H^*(X_t, \mathbb{Z}/n) \cong H^*(X', \mathbb{Z}/n) \longrightarrow H^*(X_s, \mathbb{Z}/n).$$

THEOREM 5.1.7. *Let S be a locally Noetherian scheme, s a geometric point of S and $f\colon X \longrightarrow S$ a morphism. Suppose*

(a) the morphism f is locally acyclic,
(b) for every specialization morphism $t \longrightarrow \tilde{S}^s$ and for every $q \geq 0$, the cospecialization arrows $H^q(X_t, \mathbb{Z}/n) \longrightarrow H^q(X_s, \mathbb{Z}/n)$ are bijections.

Then the canonical homomorphism $(R^q f_*\mathbb{Z}/n)_s \longrightarrow H^q(X_s, \mathbb{Z}/n)$ is bijective for every $q \geq 0$.

To prove the theorem, it is clear that we can suppose $S = \tilde{S}^s$. We will in fact show that, for every sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules F on S, the canonical homomorphism $\varphi^q(F)\colon (R^q f_* f^* F)_s \longrightarrow H^q(X_s, f^*F)$ is bijective.

Every sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules is the fitered inductive limit of constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules (Lemma 2.4.3.3). Moreover every constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules injects into a sheaf of the form $\prod i_{\lambda *} C_\lambda$, where $i_\lambda \colon t_\lambda \longrightarrow S$ is a finite family of generalizations of s and C_λ a free $\mathbb{Z}/n\mathbb{Z}$-module of finite rank over t_λ. By the definition of cospecialization arrows, condition (b) means that the homomorphisms $\varphi^q(F)$ are bijections if F is of this form.

We conclude using a variant of Lemma 2.4.3.6:

Lemma 5.1.6. *Let $\mathcal{C}$ be an abelian category in which filtered inductive limits exist. Let $\varphi^\bullet \colon T^\bullet \longrightarrow T'^\bullet$ be a morphism of cohomological functors commuting with filtered inductive limits, defined over $\mathcal{C}$ and with values in the category of abelian groups. Suppose that there exist two subsets $\mathcal{D}$ and $\mathcal{E}$ of objects of $\mathcal{C}$ such that:*

(a) every object of $\mathcal{C}$ is a filtered inductive limit of objects in $\mathcal{D}$,

(b) every object of $\mathcal{D}$ is contained in an object of $\mathcal{E}$.

Then the following conditions are equivalent.

(i) $\varphi^q(A)$ is bijective for every $q \geq 0$ and every $A \in \operatorname{Ob} \mathcal{C}$.

(ii) $\varphi^q(M)$ is bijective for every $q \geq 0$ and every $M \in \mathcal{E}$.

The proof of the lemma is accomplished by passing to the inductive limit, recurrence on q and repeated application of the five lemma of exact sequences in cohomology derived from an exact sequence $0 \longrightarrow A \longrightarrow M \longrightarrow A' \longrightarrow 0$, with $A \in \mathcal{D}$, $M \in \mathcal{E}$, $A' \in \operatorname{Ob} \mathcal{C}$.

Corollary 5.1.9. *Let S be the spectrum of a strictly henselian locally noetherian ring and $f \colon X \longrightarrow S$ a locally acyclic morphism. Suppose that, for every geometric point t of S we have $H^0(X_t, \mathbb{Z}/n) = \mathbb{Z}/n$ and $H^q(X_t, \mathbb{Z}/n) = 0$ for $q > 0$ [said otherwise the geometric fibers of f are acyclic]. Then we have $f_* \mathbb{Z}/n = \mathbb{Z}/n$ and $R^q f_* \mathbb{Z}/n = 0$ for $q > 0$.*

Corollary 5.1.10. *Let $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$ be morphisms of locally noetherian schemes. Then, if f and g are locally acyclic, then so is $g \circ f$.*

We can suppose that X, Y, and Z are strictly local and that f and g are local morphisms. It suffices then to show that, if z is an algebraic geometric point of Z, we have $H^0(X_z, \mathbb{Z}/n) = \mathbb{Z}/n$ and $H^q(X_z, \mathbb{Z}/n) = 0$ for $q > 0$.

Since g is locally acyclic, we have $H^0(Y_z, \mathbb{Z}/n) = \mathbb{Z}/n$, and $H^q(Y_z, \mathbb{Z}/n) = 0$ for $q > 0$. On the other hand, the morphism $f_z \colon X_z \longrightarrow Y_z$ is locally acyclic (Lemma 2.5.1.4) and its geometric fibers are acyclic, since these are the varieties of evanescent cycles of f. By Corollary 2.5.1.9, we then have $R^q f_{z_*} \mathbb{Z}/n = 0$ for $q > 0$. Moreover $f_{z^*} \mathbb{Z}/n$ is constant on the fiber $\mathbb{Z}/n$ over Y_z. We are done with the help of the Leray spectral sequence for f_z.

5.2. Local acyclicity of a smooth morphism.

Theorem 5.2.1. *A smooth morphism is locally acyclic.*

Let $f \colon X \longrightarrow S$ be a smooth morphism. The assertion is local for the étale topology on X and S, we can then suppose that X is an affine space of dimension d over S. Passing to the limit, we can suppose that S is Noetherian and the transitivity of local acyclicity (2.5.1.10) shows that it suffices to treat the case $d = 1$.

Let s be a geometric point of S and x a geometric point of X centered at a closed point of X_s. It suffices to show that the geometric fibers of the morphism

$\tilde{X}^x \longrightarrow \tilde{S}^s$ are acyclic. We can write hereforth that $S = \tilde{S}^s = \mathrm{Spec}(A)$ and $X = \tilde{X}^x$. We have that $X \cong \mathrm{Spec}\{T\}$, from which we know that $A\{T\}$ is the Henselization of $A[T]$ at the point $T = 0$ over s.

If t is a geometric point of S, the fiber X_t is the projective limit of affine smooth curves over t. We then have $H^q(X_t, \mathbb{Z}/n) = 0$ for $q \geq 2$ and it suffices to show that $H^0(X_t, \mathbb{Z}/n) = \mathbb{Z}/n$ and $H^1(X_t, \mathbb{Z}/n) = 0$ for n coprime to the residual characteristic of S. This results in the following two propositions.

Proposition 5.2.2. *Let A be a strictly Henselian local ring, $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}\, A\{T\}$. Then the geometric fibers of $X \longrightarrow S$ are connected.*

We can get here by passing to the limit to the case where A is a strict Henselization of a finite type $\mathbb{Z}$-algebra.

Let $\bar{t}$ be a geometric point of S, localized at t, and k' a finite separable extension of $k(t)$ in $k(\bar{t})$. We write $t' = \mathrm{Spec}(k')$ and $X_{t'} = X \times_S \mathrm{Spec}(k')$. It remains to verify that, for every $\bar{t}$ and t', $X_{t'}$ is connected (by which we mean connected and nonempty). Let A' be the normalization of A in k', i.e. the ring of elements of k' integers over the image of A in $k(t)$. We have $A\{T\} \times_A A' \xrightarrow{\sim} A'\{T\}$: the object on the left is in fact Henselian local (since A' is finite over A, and local) and the limit of étale local algebras over $A'[T] = A[T] \times_A A'$. The scheme $X_{t'}$ is then again the fiber at t' of $X' = \mathrm{Spec}(A'\{T\})$ over $S' = \mathrm{Spec}(A')$. The local scheme X' is normal, thus integral; its localization $X_{t'}$ is still integral, a fortiori connected.

Proposition 5.2.3. *Let A be a strict Henselian local ring, $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A\{T\})$. Let $\bar{t}$ be a geometric point of S and $X_{\bar{t}}$ the corresponding geometric fiber. Then every étale galois cover of $X_{\bar{t}}$ of order coprime to the characteristic of the residual field of A is trivial.*

Lemma 5.2.3.1 (Zariski-Nagata theorem of purity in dimension 2). Let C be a regular local ring of dimension 2 and C' an étale normal finite C-algebra over the open set complementary to the closed point of $\mathrm{Spec}(C)$. Then C' is étale over C.

In fact C' is normal of dimension 2, so $\mathrm{depth}(C') = 2$. Since $\mathrm{depth}(C') + \mathrm{pd}(C') = \dim(C) = 2$, we conclude that C' is free over C. So the set of points of C where C' is ramified is defined by an equation, the discriminant; since it doesn't contain a point of height 1, it is empty.

Lemma 5.2.3.2 (the particular case of Abhyankar's lemma). Let $S = \mathrm{Spec}(V)$ be a line, π a uniformization, η the generic point of S, X smooth over S, irreducible, of relative dimension 1, $\tilde{X}_\eta$ a Galois étale covering of X_η, of degree n invertible over S, and $S_1 = \mathrm{Spec}(V[\pi^{1/n}])$. Note by an index $_1$ the base-change of S to S_1. Then, $\tilde{X}_{1\eta}$ extends to an étale covering of X_1.

Let $\tilde{X}_1$ be the normalization of X_1 in $\tilde{X}_{1\eta}$. Given the structure of the groups of moderate inertia of the localized valuation rings of X at the generic points of the special fiber X_s, $\tilde{X}_1$ is étale over X_1 over the general fiber, and at the generic points of the special fiber.

5.3. Applications.

6. Poincaré Duality

6.1. Introduction.

6.2. The case of curves.

6.3. The general case.

6.4. Variants and Applications.

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CHAPTER 3

Report on the trace formula

CHAPTER 4

L-functions modulo ℓ^n and modulo p

CHAPTER 5

The cohomology class associated to a cycle

1. Cohomology with support and cup-products

1.1. H^1 and torsors.

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2. The cohomology class associated to a cycle

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CHAPTER 6

Duality

CHAPTER 7

Applications of the trace formula to trigonometric sums

CHAPTER 8

Theorems of finiteness in ℓ-adic cohomology

CHAPTER 9

Derived categories, état 0

CHAPTER 10

Erratum for SGA 4