We Need to Talk About Spectra

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In this talk, I will introduce the stable homotopy category from two complementary points of view. First, I will describe how attempts to stabilize the homotopy category of spaces naturally lead to spectra, using the sphere spectrum as a guiding example. Second, I will explain how spectra arise more importantly as representing objects for generalized cohomology theories, beginning with singular cohomology and K-theory. I hope to place emphasis on building intuition from concrete examples rather than formal foundations. I will also briefly indicate why one is led to consider structured models of spectra (like S-modules) and symmetric monoidal categories of spectra—and, if time permits, say a few words about highly structured ring spectra and $\mathbb{E}_\infty$-rings.

"There is no linear evolution; there is only a circumambulation of the self."
–Carl Jung

Figure 1: *Light Coming on the Plains No. I*, Georgia O'Keeffe 1917

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§1 A Brief History of Stable Homotopy Theory

We begin with a note on the history of stable homotopy theory, and provide a glimpse into where we will be headed with this talk[1]. Stable homotopy theory began with Freudenthal's suspension theorem around 1937, the statement of which will be essential to the subject of the following section in this talk [Fre38]. But to put it succinctly yet vaguely, it notes that for q relatively small, and for sufficiently connected X, the natural map $\pi_q(X) \longrightarrow \pi_q(\Omega\Sigma X) \cong \pi_{q+1}(\Sigma X)$ is an isomorphism[2]. Earlier examples of stable phenomena can be traced back to reduced homology and cohomology for example, where suspension induces an isomorphism with no connectivity issues. In the late 1950s, several stunning instances of *stable* phenomena providing deep geometric insights lent much more credence to the stable perspective. The first of these breakthroughs was Thom's monumental work on classifying cobordism classes of manifolds, by reducing it to a solvable problem in understanding stable homotopy types [Tho54]. Subsequently one must recognize Adams use of *stable* secondary cohomology operations to solve the Hopf invariant one problem [Ada60], as yet another impetus for the breakthrough of stable homotopy theory as a central subject in topology. Following the timeline to the early 1960s, Atiyah and Hirzebruch's introduction of complex topological K-theory, and their eponymous spectral sequence, set the foundations for generalized cohomology theories [AH61]. From classifying manifolds, to understanding higher algebraic K-theory, to pursuing chromatic homotopy theory, since then stable homotopy theory as been essential to mathematics.

§2 Prologue

Our work today will be to establish a concrete setting to handle such stable phenomena. As one sees spaces as the objects of study in unstable homotopy theory, we will see **spectra** as the objects of study in stable homotopy theory. One important interpretation of spectra is that they are objects that represent cohomology theories, and this is a perspective we will keep coming back to for a grounded intuition. To reiterate, just as homotopy theory sets itself in the homotopy category of spaces, we will situate stable homotopy theory in a **stable homotopy category**. Up to equivalence of categories, there has long

[1] Much of the history presented here was adapted from [EKMM95].

[2] Applied to S^n this provides the statement that for relatively small q, $\pi_{n+q}(S^n)$ is independent of n.

been consensus that "a" in the previous sentence may be replaced by "the"—and the consensus is that *the* appropriate stable homotopy category to work with is one first constructed by Boardman in the 1960s [Boa65]. An important note here however is that to do stable homotopy theory carefully, one must pay serious attention to the point-set level category before passing to homotopy types.

There is an analogy with the derived category of R-modules that is often helpful to think about. We will construct an object called the **sphere spectrum** that we will think of as our base ring R. In terms of (co)homology theories, the sphere spectrum should be thought of as the spectrum representing the homology theory given by stable homotopy groups. Spectra in the algebraic analogy will then be thought of as chain complexes (or more precisely a first approximation to the notion, which we will later find to be S-modules). This will define a category of spectra we will call $\mathcal{S}$. There is an appropriate notion of homotopy, of homotopy groups, and therefore of weak equivalences of spectra. By formally inverting the weak equivalences of spectra, we obtain the stable homotopy category $\mathrm{Ho}\mathcal{S}$.

Continuing with this algebraic analogy, the category of chain complexes $\mathbf{Ch}(R)$ has a commutative associative unital product given by the tensor product over R (with unit given by the chain complex representing R). We can also speak of a (differential) R-algebra in $\mathbf{Ch}(R)$ as a chain complex A with maps $R \longrightarrow A$ (the unit) and $A \otimes_R A \longrightarrow A$ (the product) satisfying associativity and unitality[3]. These notions exist before passing to the derived category $\mathcal{D}(R)$. It would be awkward to have an algebra in $\mathcal{D}(R)$ with unit and product defined only in the derived category. And yet this awkward predicament is what we are led to in $\mathcal{S}$. The category $\mathcal{S}$ indeed has a product called the **smash product** $\wedge$. However, this product is neither commutative nor associative nor unital. The induced product on the stable homotopy category however is commutative, associative, and unital with unit the sphere spectrum S. In the stable homotopy category, we can also make sense of **ring spectra**, which are the analogs of S-algebras in $\mathrm{Ho}\mathcal{S}$. In the cohomology perspective, ring spectra correspond precisely to multiplicative cohomology theories. But the awkward predicament is that this algebraic structure does not make sense at the level of $\mathcal{S}$.

The discovery in 1997, due to [EKMM97], was that there exists a precise analog of chain complexes in stable homotopy theory called **S-modules**, and that the category $\mathbf{Mod}_S$ has an associative, unital, commutative smash product $\wedge_S$. We will think of an S-module as a spectrum with additional structure, and we will say that a map of S-modules is a weak equivalence if it is a weak equivalence of spectra. The homotopy category of $\mathbf{Mod}_S$ denoted $\mathcal{D}(S)$ obtained by formally inverting the weak equivalences of S-modules was shown in [EKMM97] to be equivalent to the stable homotopy category $\mathrm{Ho}\mathcal{S}$.

Now in $\mathbf{Mod}_S$ (before passing to the homotopy category), there exists a point-set level notion of an S-algebra R with unit map $\eta\colon S \longrightarrow R$ and product map $\phi\colon R \wedge_S R \longrightarrow R$ such that the usual unitality and associativity diagrams commute. We will call such an algebra commutative if the commutativity diagram also commutes. The former of these notions is what is now known as an A_∞-ring spectrum, and the latter is what is commonly called an $\mathbb{E}_\infty$-ring spectrum (or for short, A_∞-rings and $\mathbb{E}_\infty$-rings). The latter are often referred to as highly structured ring spectra. In terms of cohomology theories, these are multiplicative cohomology theories with sophisticated associative and commutative structures. The modern perspective on $\mathbb{E}_\infty$-rings is that they are ho-

[3] And if we speak of commutative differential R-algebras, one also checks commutativity.

motopy theoretic generalizations of commutative rings. The algebraic theory obtained by studying $\mathbb{E}_\infty$-rings under this analogy has come to be known as **brave new algebra** (a reference to the Huxley novel, "Brave New World").

The upshot of having such an algebraic handle at the point-set level, particularly with S-modules, is that a large part of the ad-hoc constructions in stable homotopical algebra from the 1960s could now be unified under this algebraic framework. The details are omitted from this draft but we refer the reader to [EKMM97] for a treatment of (among other things): the Atiyah-Hirzebruch spectral sequence, topological K-theory, higher algebraic K-theory, topological Hochschild homology, and complex cobordism.

§3 Homotopy Groups

In this section we investigate the stabilization of the category $\mathcal{C}$ of pointed topological spaces (compactly generated and Hausdorff, say)[4]. There are two important functors on $\mathcal{C}$ that we will be dealing with today. The first of which is the suspension functor $\Sigma\colon \mathcal{C} \longrightarrow \mathcal{C}$ defined as $\Sigma X := S^1 \wedge X$. The second is the loop space functor $\Omega\colon \mathcal{C} \longrightarrow \mathcal{C}$ defined as $\Omega X := \underline{\mathrm{Hom}}_{\mathcal{C}}(S^1, X)$. We note that Σ and Ω are dual to each other in the sense that (Σ, Ω) is an adjoint pair[5]. What we will investigate in this section is the failure of (Σ, Ω) to be an equivalence of categories (even at the homotopy category level). More simply, given any space X, there is a natural map $\eta\colon X \longrightarrow \Omega\Sigma X$ (given by the adjoint to $\mathrm{id}_{\Sigma X}$). We will concern ourselves with understanding when this map is a weak equivalence. Recall that a weak equivalence is a map that induces isomorphisms on all homotopy groups.

The following theorem of Freudenthal offers the best starting point for this pursuit. Crudely, it states that the more X is connected, the closer the natural map $\eta\colon X \longrightarrow \Omega\Sigma X$ is to a weak equivalence.

Theorem 3.1 (Freudenthal, 1937) — If X is n-connected, then the natural map $\pi_q(X) \longrightarrow \pi_q(\Omega\Sigma X) \cong \pi_{q+1}(\Sigma X)$ is an isomorphism for $q \leq 2n$.

Now suppose X was some ordinary old space. Then by suspending it enough, we can increase its connectivity. Namely, $\Sigma^k X$ is at least $(k-1)$-connected. Therefore Freudenthal tells us that $\pi_q(\Sigma^k X) \longrightarrow \pi_{q+1}(\Sigma^{k+1} X)$ is an isomorphism for $q \leq 2k-2$. In other words, for a fixed q, there's an m large enough so that $\pi_{q+m}(\Sigma^m X) \cong \pi_{q+n}(\Sigma^n X)$ for all $n > m$. This stability defines a directed colimit, $\mathrm{colim}_n \pi_{q+n}(\Sigma^n X)$. We will call this the q-th **stable homotopy group** of X and denote it $\pi_q^{\mathrm{st}}(X)$.

Example 3.2 (Stable Homotopy Groups of Spheres)
Let $X = S^0$, then $\Sigma^k X = S^k$. The stable homotopy groups of S^0 are called the stable homotopy groups of the spheres. Their computation is an extremely challenging pursuit that many have undertaken. We present some low degree examples here. It is well known that for each k, $\pi_k(S^k) \cong \mathbb{Z}$, therefore $\pi_0^{\mathrm{st}}(S^0) \cong \mathbb{Z}$. It is a more involved result, due to Pontryagin, that $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 2$, yielding $\pi_1^{\mathrm{st}}(S^0) \cong \mathbb{Z}/2\mathbb{Z}$ where the generator roughly comes from the Hopf fibration.

We now go on a seemingly tangential question and ask ourselves, given a space X,

[4] And at times, I just might get down to CW complexes.

[5] So that for every $X, Y \in \mathcal{C}$ there is a natural isomorphism $\mathrm{Hom}(\Sigma X, Y) \cong \mathrm{Hom}(X, \Omega Y)$.

is there a space whose homotopy groups correspond to the stable homotopy groups of X? The answer remarkably is in the positive, and will be the beginning for our development of spectra. We first note that $\pi_{q+n}(\Sigma^n X) \cong \pi_q(\Omega^n\Sigma^n X)$, and therefore $\pi_q^{st}(X) \cong \operatorname{colim}_n \pi_q(\Omega^n\Sigma^n X)$. I will elide over this technical detail, but π_q commutes with filtered colimits of inclusion diagrams[6]. Going forward, I will find it customary to elide over any inclusion business—acknowledging that my purpose here is not to be precise by any means. This allows us to rewrite, $\pi_q^{st}(X) \cong \pi_q(\operatorname{colim}_n \Omega^n\Sigma^n X)$. Therefore, $\operatorname{colim}_n(\Omega^n\Sigma^n X)$ is the space we were after. We will call this space QX, and soon enough we will call it $\Omega^\infty\Sigma^\infty X$. The space QX has several desirable properties.

First, the reader may verify that the natural map $QX \longrightarrow \Omega\Sigma QX$ is a weak equivalence. Second, QX is an **infinite loop space**. Define $Q_k X := \operatorname{colim}_n \Omega^n\Sigma^{n+k} X$. Then $QX = Q_0 X$. For each k, we have a natural map $Q_k X \longrightarrow \Omega Q_{k+1} X$ induced by the unit map $\eta\colon X \longrightarrow \Omega\Sigma X$. This map moreover is a weak equivalence[7]! It follows that for each k, $QX \simeq \Omega^k Q_k X$. It is in this sense that QX is the space of loops in the space of loops in the space of loops in ... A quick calculation will also tell you that $\pi_q^{st}(X) \cong \pi_q(QX) \cong \operatorname{colim}_n \pi_{q+n}(Q_k X)$.

Example 3.3 (The Sphere Spectrum $\mathbb{S}$)

Let $X = S^0$. Our Q-construction in this case is given by $QS^0 = \operatorname{colim}_n \Omega^n S^n$. Define $\mathbb{S}_k = Q_k S^0 = \operatorname{colim}_n \Omega^n S^{n+k}$. We will call the collection $\{\mathbb{S}_k\}$ the sphere spectrum. We have seen that the stable homotopy groups of the spheres are given by $\pi_q^{st}(S^0) \cong \pi_q(\mathbb{S}_0) \cong \operatorname{colim}_n \pi_{q+n}(\mathbb{S}_k)$. We will soon define what it means to say $\pi_q(\mathbb{S})$, and at that point, identify $\pi_q(\mathbb{S})$ with $\pi_q(\mathbb{S}_0)$ and therefore with $\pi_q^{st}(S^0)$.

We now shift perspectives from the input X to the functor π_q^{st} itself. First we note that by construction, for any X, $\pi_q^{st}(X) \cong \pi_{q+1}^{st}(\Sigma X)$ with no restrictions on X. Putting this together with our compactness trick on S^1 (eliding over the inclusion mess again), we see that the natural map $X \longrightarrow \Omega\Sigma X$ induces isomorphisms $\pi_q^{st}(X) \longrightarrow \pi_q^{st}(\Omega\Sigma X)$ for all X. What we are seeing here is twofold, the stable homotopy groups are *stable* under suspension (so the name is befitting), moreover, while the unit map $X \longrightarrow \Omega\Sigma X$ is not a weak equivalence, it induces what we might want to call now a *stable* weak equivalence (i.e. a map inducing isomorphisms upon passage to stable homotopy groups). It is in this sense that in the stable world, suspension is an equivalence, with inverse the loop space functor. We will make this more precise when we discuss spectra. Before we conclude the section, we make a remark on how the sphere spectrum represents the stable homotopy groups. We note without proof that for each q, there is a natural isomorphism $\pi_q^{st}(X) \cong \pi_q(\mathbb{S}_q \wedge X)$ (the verification of this claim is a valuable exercise).

We conclude this section with some conceptual remarks. We began with a stability phenomenon, from Freudenthal's suspension theorem. In an attempt to treat this stability systematically, we found ourselves keeping track of a sequence of spaces, each of which encoded information about loop in the next one. The zeroth space in this sequence was an infinite loop space, with many desirable properties. In our brief exploration, we also saw that the stable homotopy groups look like a homology theory—in the sense that they form a functor from (pointed) spaces to graded abelian groups satisfying homotopy

[6]The keyphrase to look up here is *compactness* of objects in **Top**.

[7]I'm being modest here to be careful, but this map is actually a homotopy equivalence.

invariance, some exactness condition and a coproduct condition, but most importantly having a suspension isomorphism. In the following section, we will elaborate on this line of inquiry, and understand how to represent such stable (co)homology theories (just as we were able to do with stable homotopy groups, using the sphere spectrum).

§4 Generalized Cohomology Theories

In the previous section, we met a first kind of stability phenomena, in homotopy groups. We will now switch gears into another broad class of stability phenomena, in various reduced cohomology theories. The axiomatic study of generalized cohomology theories was pioneered by [ES45; AH61; Whi62; Bro62]. We provide a definition first, and then record examples.

Definition 4.1 (Generalized Cohomology Theory)**.** A (reduced) cohomology theory is a functor $\tilde{E}^\bullet\colon \mathcal{C}^{op} \longrightarrow \mathbf{Ab}^{\mathbb{Z}}$, equipped with a natural isomorphism of degree $+1$ called the **suspension isomorphism**, of the form: $\sigma\colon \tilde{E}^{\bullet+1}(\Sigma -) \cong \tilde{E}^\bullet(-)$ such that the following axioms hold.

1. Homotopy Invariance. If f_1, f_2 are two morphisms of pointed spaces with a (pointed) homotopy $f_1 \simeq f_2$ then, the induced pullback hom of graded abelian groups are equal $f_1^* = f_2^*$.
2. Exactness. For $i\colon A \longrightarrow X$ an inclusion of pointed spaces and $j\colon X \longrightarrow \mathrm{Cone}(i)$, there is an exact sequence of graded abelian groups:
$$\tilde{E}^\bullet(\mathrm{Cone}(i)) \xrightarrow{j^*} \tilde{E}^\bullet(X) \xrightarrow{i^*} \tilde{E}^\bullet(A).$$
3. Wedge Axiom. For $\{X_i\}_{i\in I}$ any set of pointed spaces, the canonical morphism $\tilde{E}^\bullet\left(\bigvee_{i\in I} X_i\right) \longrightarrow \prod_{i\in I} \tilde{E}^\bullet(X_i)$.

> **Remark 4.2.** Note that the stability phenomenon, namely the suspension isomorphism, is explicitly prescribed in the definition of a generalized cohomology theory.

Now for a (*not comprehensive*) list of examples out there in the real world.

Example 4.3 (Singular Cohomology)

Reduced singular cohomology with coefficients in an abelian group A, denoted $\tilde{H}^\bullet_{\mathrm{sing}}(-, A)\colon \mathcal{C}^{op} \longrightarrow \mathbf{Ab}^{\mathbb{Z}}$ (with trivial negative grading, for nonempty spaces I suppose) is the quintessential example of a cohomology theory. It is actually much better than a *generalized* cohomology theory, it is an *ordinary* cohomology theory (meaning the singular cohomology of S^0 is concentrated in degree 0).

Example 4.4 (Complex K-theory)

Topological (complex, reduced) K-theory, $\tilde{K}^\bullet \colon \mathcal{C}^{op} \longrightarrow \mathbf{Ab}^{\mathbb{Z}}$ on the other hand is the quintessential example of an *extraordinary* cohomology theory. The stability phenomenon is core to the definition of K-theory. Recall that for a space X, $K^0(X)$ is defined as the Grothendieck group of the commutative monoid of (finite rank) vector bundles on X (under Whitney sum). We can also define the *reduced* K^0 group as the kernel under the pullback map: $\ker(K^0(X) \longrightarrow K^0(*))$. The *higher* K-groups are defined by $\tilde{K}^{-i}(X) := \tilde{K}^0(\Sigma^i X)$ for all $i \geq 0$. Bott periodicity is an essential phenomenon of K-theory which says that $\tilde{K}^{-i}(X) = \tilde{K}^{2-i}(X)$ for all $i \geq 2$ (allowing us to extend K-theory for all i). The chain of isomorphisms (for $i > 0$): $\tilde{K}^{-i}(X) = \tilde{K}^0(\Sigma^i X) \cong \tilde{K}^{-i+1}(\Sigma X)$ provides the suspension isomorphism. K-theory is the canonical example of what is known as a *periodic* cohomology theory. It is also extraordinary because $\tilde{K}^0(S^0) \cong \mathbb{Z}$ and $\tilde{K}^1(S^0) \cong 0$, but Bott periodicity ensures that $\tilde{K}^{-2i}(S^0) \cong \mathbb{Z}$ for all i.

Remark 4.5. There is also real K-theory, denoted KO, where the Bott element has period 8. We won't say much about this cohomology theory.

Example 4.6 (Complex Cobordism)

For a smooth manifold X, define $MU^q(X)$ to be the set of cobordism classes of proper complex-oriented maps $f \colon Z \longrightarrow X$ of codimension q. To parse this definition, let $f \colon Z \longrightarrow X$ be a smooth map. If the relative codimension of f at each point of Z is even, then a complex orientation is an equivalence class of factorizations (and I'll leave it to the reader to identify the appropriate equivalence relation) of f of the form $Z \xrightarrow{i} E \xrightarrow{p} X$, where p is a complex vector bundle and i is an embedding equipped with a complex structure on its normal bundle. Two proper complex-oriented maps $f, f' \colon Z, Z' \longrightarrow X$ are called cobordant if there is a proper complex-oriented map $b \colon W \longrightarrow X \times [0,1]$ such that $X \times \{0\}$ and $X \times \{1\}$ are transversal to b and pulling back to b, we get f, f' respesctively. It is a result, dating back to the 1960s (for work along this line, see [Tho54; Ati61; Qui71]), that this is a generalized cohomology theory in the sense of our definitions.

Complex cobordism is a wonderfully fascinating cohomology theory. Thom's work in [Tho54] relates the classification of manifolds up to cobordism to $MU^\bullet$. There is a notion of a *complex oriented* cohomology theory, which roughly speaking is a multiplicative cohomology theory that is oriented on all (complex) vector bundles (for example, ordinary cohomology and topological K-theory are both complex-oriented). $MU^\bullet$ is one such theory, and is in fact the *universal* complex oriented cohomology theory. Generally speaking, to a complex-oriented cohomology theory A, one can associate a formal group law over its coefficient ring $A(*)$ (this roughly comes from, in a vague analogy, being able to express the first Chern class of a tensor product of line bundles in terms of the individual chern classes). The formal group law associated to $MU^\bullet$ (over the coefficient ring $MU^\bullet(*)$) is the *universal* formal group law. That is, every formal group law over any commutative ring R appears as the pullback of the formal group law over $MU(*)$ under a homomorphism $MU(*) \longrightarrow R$. This is a result of [Mil60; Qui69].

Example 4.7 (The Generalized Homology Theory of Stable Homotopy Groups)
Dual to a generalized cohomology theory, we may define a generalized homology theory. The functor $\pi_\bullet^{st} \colon \mathcal{C}^{op} \longrightarrow \mathbf{Ab}^{\mathbb{Z}}$ is one example of a generalized homology theory that we have already encountered. We exhibited the suspension isomorphism $\pi_q^{st}(X) \longrightarrow \pi_{q+1}^{st}(\Sigma X)$ explicitly. We now see this suspension isomorphism as a natural requirement from the perspective of homology.

For completeness, we will record a spectral sequence of Atiyah and Hirzebruch [AH61] that allows us to compute any generalized cohomology theory from its coefficient ring. We haven't discussed multiplicative cohomology theories entirely yet, but we remark that the spectral sequence furnished by the following theorem is actually *multiplicative*, for multiplicative cohomology theories.

Theorem 4.8 (Atiyah-Hirzebruch Spectral Sequence) — Let A be a generalized cohomology theory, and X a space. There is a spectral sequence, whose second page reads $E_2^{p,q} := H^p(X, A^q(*))$ that converges to $A^{p+q}(X)$.

We will pick up our conceptual treatment of generalized cohomology theories in the next section, where we will identify ways to represent these theories using stable objects in homotopy theory.

§5 Spectra

We begin with this remarkable theorem of Brown's [Bro62] on the representability of cohomology theories[8].

Theorem 5.1 (Brown Representability) — Let $E^\bullet$ be a generalized cohomology theory. There is a sequence of spaces $\{E_n\}$ with homotopy equivalences $\tilde{\sigma}_n \colon E_n \longrightarrow \Omega E_{n+1}$ such that

$$E^n(X) \cong [X, E_n]$$

for every X. Here $[X, E_n]$ denotes the homotopy classes of maps from X to E_n.

Note that by the prescription $E^{-n}(X) = [\Sigma^n X, E_0] \cong E^0(\Sigma^n X)$, we might as well assume that the sequence from the above theorem is indexed by $\mathbb{N}$. We quickly record a definition of what we have been building up to so far.

Definition 5.2 (Spectrum)**.** A **prespectrum** D is a sequence of spaces $\{D_n\}_{n\in\mathbb{N}}$ equipped with maps $\sigma_n \colon \Sigma D_n \longrightarrow D_{n+1}$ (and hence maps $\tilde{\sigma}_n \colon D_n \longrightarrow \Omega D_{n+1}$). A **spectrum**[9] E is a prespectrum $\{E_n\}_{n\in\mathbb{N}}$ such that the adjoint maps $\tilde{\sigma}_n \colon E_n \longrightarrow \Omega E_{n+1}$ are weak equivalences. A map of prespectra is a sequence of maps $f_n \colon D_n \longrightarrow D'_n$ compatible with the structure maps σ_n (or equivalently with the structure maps $\tilde{\sigma}_n$). A map of spectra is a map of prespectra.

Definition 5.3 (The Category of (Sequential) Ω-Spectra)**.** Let $\mathcal{S}$ denote the category whose objects are spectra, and whose morphisms are maps of spectra. The composition is defined in the obvious manner. We will call this the category of Ω-spectra, or when there's no ambiguity, just the category of spectra.

[8] We drop the tilde for reducedness from now
[9] Or what you might read of elsewhere as a sequential Ω-spectrum

We now return to Brown's representability result to make some remarks. We note first that every spectrum E produces a generalized cohomology theory $E^\bullet$ by simply defining $E^\bullet(X) := [X, E_n]$. In light of this result, the correspondence between the suspension isomorphism in cohomology theories, and the structure homotopy equivalences $\tilde{\sigma}$ is clear. We have already seen a spectrum before, in Section 3. We will explicitly lay out this example again in our improved language. We now present some examples:

Example 5.4 (The Eilenberg-Maclane Spectrum HA)

Let A be an abelian group. There is a spectrum HA whose n-th spaces are defined by Eilenberg-Maclane spaces $(HA)_n := K(A, n)$. The structure homotopy equivalences are given by the homotopy equivalences $K(A, n) \simeq \Omega K(A, n+1)$. The Eilenberg-Maclane spectrum represents singular cohomology with coefficients in A. Namely, $\tilde{H}^\bullet_{\text{sing}}(X; A) \cong [X, K(A, n)]$. There are two possible ways to show this. The first relies on showing that $[-, K(A, \bullet)]$ satisfies the Eilenberg-Steenrod axioms (and therefore must be identified with the ordinary reduced singular cohomology), and the second (and more geometric) proof relies on a bit of cellular obstruction theory. There was an analogy mentioned in the prologue between spectra and chain complexes. Under this analogy, one may want to imagine the functor $H\colon \mathbf{Ab} \longrightarrow \mathcal{S}$ as similar to the Eilenberg-Maclane object functor $\mathbf{Ab} \longrightarrow \mathbf{Ch}(\mathbf{Ab})$.

Example 5.5 (The Complex K-theory Spectrum KU)

The spectrum representing complex K-theory is called KU. Its zeroth space is $KU_0 = BU \times \mathbb{Z}$, and its first space is $KU_1 = U$ (here U is the infinite unitary group $U(\infty) = \operatorname{colim}_n U(n)$). Bott periodicity amounts to the statement that $\Omega^2(BU) \simeq BU \times \mathbb{Z}$ (or equivalently that $\Omega^2 U \simeq U$). This defines every KU_n. The intuition for the BU appearing in the zeroth space is that BU represents the classifying space for stable complex vector bundle (as $BU(n)$ is the classifying space for complex rank n vector bundles), and the $\mathbb{Z}$ keeps track of the dimensions of the fibers. We also see the periodicity of the cohomology theory at the level of the spectrum. In fact Bott periodicity for K-theory was first proved by showing that $\Omega^2(BU)$ is weak equivalent to $BU \times \mathbb{Z}$.

Example 5.6 (The Complex Cobordism Spectrum MU)

We won't say too much more about complex cobordism, but we will define its spectrum at the least. For each $2n$, define MU_{2n} to be the Thom space of the underlying real vector bundle of the universal rank n complex vector bundle $EU(n) \longrightarrow BU(n)$. We define MU_{2n+1} to be ΣMU_{2n}.

Example 5.7 (The Sphere Spectrum $\mathbb{S}$)

We finally define the sphere spectrum $\mathbb{S}$, the example we've been circling around throughout this talk. For each n, define $\mathbb{S}_k := \operatorname{colim}_n \Omega^n S^{n+k}$. Perhaps its zeroth space, $\mathbb{S}_0 = \operatorname{colim}_n \Omega^n S^n$ may be familiar to the reader from Section 3. $\mathbb{S}_0$ is namely our QS^0, its homotopy groups are the stable homotopy groups of the sphere. We also noted earlier in Section 3, that $\pi^{st}_q(X) \cong \pi_q(\mathbb{S}_q \wedge X)$. It is in this (dual) sense that $\mathbb{S}$ represents stable homotopy groups.

At this point it would probably be appropriate to inform the reader that our definition of spectra is only provisional. The point-set level structure we have defined does not allow for a good theory of cofibrations that one needs to do homotopy theory. This has to be done in a coordinate-free setting. For this, one typically resorts to indexing over finite-dimensional linear subspaces of $U \subset \mathbb{R}^\infty$ as opposed to the natural numbers. Defining an appropriate choice of Ω^W and Σ^W for finite-dimensional subspaces W (namely as the function space $F(S^W, -)$ and the smash product $S^W \wedge -$, where S^W is the one-point compactification of W), much of the structure we have described above carries over without effort. We now upgrade our category of sequential Ω-spectra to the category of spectra indexed on $\mathcal{U} \cong \mathbb{R}^\infty$ denoted $\mathcal{S}_\mathcal{U}$.

We make an important claim now.[10]

> **Claim 5.8.** Let $p\mathcal{S}$ denote the category of prespectra. The forgetful functor $\mathcal{S} \longrightarrow p\mathcal{S}$ is a right adjoint with left adjoint given by $L\colon p\mathcal{S} \longrightarrow \mathcal{S}$ defined as $LD_n := \mathrm{colim}_n \Omega^n D_{n+k}$.

This spectrification functor carries over to the coordinate-free setting. Without saying much more, I will going forth try to work with sequential spectra instead of coordinate-free spectra (for concreteness). And any time anything goes wrong, or could go wrong, I will revert back to coordinate-free spectra.

Proposition 5.9 (The Stable Loop-Suspension Adjunction)

There is a functor $\Omega^\infty\colon \mathcal{S} \longrightarrow \mathcal{C}$ given by taking a spectrum E and sending it to its zeroth space E_0. There is also a functor $p\Sigma^\infty\colon \mathcal{C} \longrightarrow p\mathcal{S}$ given by $(p\Sigma^\infty X)_k = \Sigma^k X$. The functor Ω^∞ is right adjoint to $\Sigma^\infty := Lp\Sigma^\infty\colon \mathcal{C} \longrightarrow \mathcal{S}$.

If a spectrum E is of the form $\Sigma^\infty X$ for some $X \in \mathcal{C}$, we say that it is a suspension spectrum. More generally, given any $n > 0$, we can define the **shift desuspension spectrum** of a space X denoted $\Sigma^\infty_n X$ as the spectrification of the prespectrum D given by $D_k = \Sigma^{k-n} X$ if $k \geq n$ and $D_k = *$ if $k < n$.

Example 5.10 (The Sphere Spectrum $\Sigma^\infty S^0$)

We see that the sphere spectrum $\mathbb{S}$ is simply $\Sigma^\infty S^0$. For $n > 0$, we see that $\Omega^\infty_n S^0$ is the spectrum whose j-th space is $(\Sigma^\infty_n S^0)_j = \mathrm{colim}_k \Omega^k S^{k-n+j}$ (here S^{-q} is interpreted to be a point for $q > 0$).

Writing out the suspension spectrum more explicitly we see: $\Sigma^\infty X_k = \mathrm{colim}_n \Omega^n \Sigma^{n+k} X$. Particularly, $\Omega^\infty \Sigma^\infty X = \mathrm{colim}_n \Omega^n \Sigma^n X$. We have seen this before in Section 3, as the Q-construction QX. We are now ready to define a notion of homotopy on $\mathcal{S}$. We say that maps $f_0, f_1\colon E \longrightarrow E'$ are homotopic if there exists a map $h\colon E \wedge I_+ \longrightarrow E'$ such that $h \circ \iota_j = f_j$ for $j = 0, 1$ where $\iota_j\colon E \longrightarrow E \wedge I_+$ is inclusion at 0 and 1 respectively.

Definition 5.11 (Homotopy Groups of Spectra)**.** For $q \geq 0$, the qth homotopy group of a spectrum $\pi_q(E)$ is defined as the homotopy class of maps $[\Sigma^\infty S^q, E]$. For $q < 0$ define $\pi_q(E)$ to be the class of maps $[\Sigma^\infty_{-q} S^0, E]$.

In the case we are dealing with, namely for sequential Ω-spectra, this notion is much

[10] Note, the formula furnished by the claim is technically false. One has to pass through inclusion spectra first, and then apply our prescribed spectrification formula. But as promised, I have chosen to elide over this fact.

simpler.

Theorem 5.12 (Homotopy Groups of Sequential Ω-Spectra) **—** Let E be a spectrum. Then we have the identification:

$$\pi_q(E) \cong \operatorname{colim}_k \pi_{q+k}(E_k)$$

We also have the identifications:

$$\pi_q(E) \cong \begin{cases} \pi_q(E_0) & q \geq 0 \\ \pi_{q+n}(E_n) & q+n \geq 0 \\ \pi_0(E_{-q}) & q < 0 \end{cases}$$

We now see that the stable homotopy groups of a space X we had defined in Section 3 are actually just the honest homotopy groups of the suspension spectrum $\Sigma^\infty X$. That is, $\pi_q^{st}(X) \cong \pi_q(\Sigma^\infty X)$.

Definition 5.13 (Weak Equivalences of Spectra)**.** We say that a map of spectra is a weak equivalence if it induces isomorphisms on all spectra.

Remark 5.14. We say that a spectrum is connective if all of its negative homotopy groups vanish. Note then that a map between connective spectra is a weak equivalence iff the induced map on zeroth spaces is a weak equivalence.

We are finally ready to define the stable homotopy category, as the homotopy category of spectra (and maybe now would be a great time to switch back to coordinate-free spectra).

Definition 5.15 (The Stable Homotopy Category)**.** Define the **stable homotopy category** to be the homotopy category $\mathrm{Ho}\mathcal{S}$ of $\mathcal{S}$. That is to say, it is the category obtained from $\mathcal{S}$ by formally inverting all weak equivalences.

We are now ready to systematically understand stable machinery, situating ourselves in the stable homotopy category. We pick up in the next section with ring spectra, S-modules, and a smash product of spectra.

§6 Further Topics

In progress. But refer back to the prologue meanwhile.

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