Whitehead's Theorems

Ajay Srinivasan

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An illustration of the power of Eckmann-Hilton duality.

This talk concerns the two following theorems, each of which has historically been referred to as **Whitehead's theorem** and is an essential result in the homotopical foundations of algebraic topology.

Theorem 0.1 (Homotopical Whitehead's Theorem)
A weak homotopy equivalence $f\colon Y \longrightarrow Z$ between CW complexes is a homotopy equivalence.

Theorem 0.2 (Homological Whitehead's Theorem)
An integral homology isomorphism $f\colon Y \longrightarrow Z$ between simple spaces is a weak homotopy equivalence.

We might as well assume that we are working in the category $\mathbf{Top}^{*/}_{\mathrm{CW}}$ of based spaces homotopy equivalent to a CW complex. The hypothesis of Thm. 0.1 and conclusion of Thm. 0.2 assert that $f_*\colon \pi_*(Y) \longrightarrow \pi_*(Z)$ is an isomorphism. The hypothesis of Thm. 0.2 asserts that $f_*\colon H_*(Y) \longrightarrow H_*(Z)$ is an isomorphism. A **simple space** is one whose fundamental group is Abelian and acts trivially on the higher homotopy groups.

Classically, Thm. 0.1 was quite easy to prove. However, Thm. 0.2 was harder to prove, with a subtle Serre spectral sequence argument required. In 1983, [May83] realized that there is a proof of Thm. 0.2 which is precisely Eckmann-Hilton dual to that of Thm. 0.1.

§I The Homotopical Whitehead's Theorem

By $\mathscr{T}$ we will mean the nice category of compactly generated based weak Hausdorff spaces. Let $X \wedge Y$ be the smash product $X \times Y / X \vee Y$ and let $F(X,Y)$ be the function space of based maps $X \longrightarrow Y$. The source of duality is the adjunction $F(X \wedge Y, Z) = F(X, F(Y,Z))$. Let $CX = X \wedge I$, $\Sigma X = X \wedge S^1$ the reduced suspension of X, $PX = F(I,X)$ the path space of X, and $\Omega X = F(S^1, X)$ the loop space of X, where I is based at 1 in forming CX and 0 in forming PX. For based map $f\colon X \longrightarrow Y$, let $Cf = Y \sqcup_f CX$ be the mapping cone of f and let $Mf = X \times_f PY$ be the mapping fiber of f. Let $\pi(X,Y)$ denote the pointed set of homotopy classes of based maps $X \longrightarrow Y$. For spaces J and K, we have the long exact sequences of pointed sets (or whatever category you're working in).

$$\cdots \longrightarrow \pi(\Sigma^n Cf, K) \longrightarrow \pi(\Sigma^n Y, K) \longrightarrow \pi(\Sigma^n X, K) \longrightarrow \pi(\Sigma^{n-1} Cf, K) \longrightarrow \cdots \quad (1)$$

$$\cdots \longrightarrow \pi(J, \Omega^n Mf) \longrightarrow \pi(J, \Omega^n X) \longrightarrow \pi(J, \Omega^n Y) \longrightarrow \pi(J, \Omega^{n-1} Mf) \longrightarrow \cdots \quad (2)$$

Claim I.1 — Let $e\colon Y \longrightarrow Z$ be a map such that $\pi(J, Me) \cong 0$. If $hi_1 = eg$ and $hi_0 = fi$ in the following diagram, where i_0, i_1, and i are the evident inclusions, then there exist $\tilde{g}$ and $\tilde{h}$ which make the diagram commute.

$$\begin{tikzcd}
J \arrow[rr, "i_0"] \arrow[dd, "i"'] & & J \wedge I^+ \arrow[dl, "h"'] \arrow[dd] & & J \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll, "e"'] & \\
CJ \arrow[rr, "i_0"] \arrow[ur, "f"] & & CJ \wedge I^+ \arrow[ul, dashed, "\tilde{h}"'] & & CJ \arrow[ll, "i_1"'] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd}$$

Proof. Define $k_0\colon J \longrightarrow Me$ by $k_0(j) = (g(j), \omega_0(j))$, where $\omega_0(j) \in PZ$ is specified by

$$\omega_0(j)(s) = \begin{cases} f(j, 1-2s) & \text{if } s \leq 1/2 \\ h(j, 2s-1) & \text{if } s \geq 1/2. \end{cases}$$

Pick a homotopy $k\colon J \wedge I^+ \longrightarrow Me$ from k_0 to the trivial map and define $\tilde{g}\colon CJ \longrightarrow Y$ and $\omega\colon J \wedge I^+ \longrightarrow PZ$ by $k(j,t) = (\tilde{g}(j,t), \omega(j,t))$. We may define $\tilde{h}\colon CJ \wedge I^+ \longrightarrow Z$ by $\tilde{h}(j,s,t) = \omega(j, u(s,t))(v(s,t))$, where $u(s,t) = \min(s, 2t)$ and $v(s,t) = \max\left(\frac{1}{2}(1+t)(1-s), 2t-1\right)$. □

Now we move on to the cellular aspects of the theory, a generalization of the theory of CW complexes.

Definition I.2. Let $\mathscr{J}$ be any collection of spaces such that $\Sigma J \in \mathscr{J}$ if $J \in \mathscr{J}$. A map $e\colon Y \longrightarrow Z$ is said to be a **weak $\mathscr{J}$-equivalence** if $e\colon \pi(J,Y) \longrightarrow \pi(J,Z)$ is a bijection for all $J \in \mathscr{J}$. A **$\mathscr{J}$-complex** is a space X together with subspaces X_n and maps $j_n\colon J_n \longrightarrow X_n$, $n \geq 0$, such that $X_0 = \{*\}$, J_n is a wedge of spaces in $\mathscr{J}$, $X_{n+1} = Cj_n$, and X is the union of the X_n. The evident map from the cone on a wedge summand of J_{n-1} into X is called an **n-cell**. The restriction of j_n to a wedge summand is called an **attaching map**. A subspace A of a $\mathscr{J}$-complex X is said to be a **subcomplex** if A is a $\mathscr{J}$-complex such that each $A_n \subset X_n$ and the composite of each n-cell $CJ \longrightarrow A_n \subset A$ and the inclusion $i\colon A \longrightarrow X$ is an n-cell of X.

Example I.3 (Cell complexes)

Consider $\mathscr{J} = \{S^n : n \geq 0\}$, where $S^n = \Sigma S^{n-1}$. A weak $\mathscr{J}$-equivalence between connected spaces is a weak homotopy equivalence. We call a $\mathscr{J}$-complex X a **cell complex**. If J_n is a wedge of n-spheres, then X is a CW complex with a single vertex and based attaching maps. It is easy to see that any connected CW complex is homotopy equivalent to one of such form.

Lemma I.4 (Homotopy Extension Lifting Property (HELP))

Let A be a subcomplex of a $\mathscr{J}$-complex X and let $e\colon Y \longrightarrow Z$ be a weak $\mathscr{J}$-equivalence. If $hi_1 = eg$ and $hi_0 = fi$ in the following diagram, then there exist $\tilde{g}$ and $\tilde{h}$ which make the following diagram commute.

$$
\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \wedge I_+ \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"] & & X \wedge I^+ \arrow[ul, dashed, "\tilde{h}"'] & & X \arrow[ll, "i_1"'] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}
$$

Proof. By the hom-tensor adjunction, (1), and the fact that $\mathscr{J}$ is closed under suspensions, the hypothesis implies $\pi(J, Me) \cong 0$ for all $J \in \mathscr{J}$. We construct compatible maps $\tilde{g}_n\colon X_n \longrightarrow Y$ and homotopies $\tilde{h}_n\colon X_n \wedge I^+ \longrightarrow Z$ from $f|_{X_n}$ to $e\tilde{g}_n$ by inducting on n, starting with the trivial maps $\tilde{g}_0$ and $\tilde{h}_0$ and extending given maps $\tilde{g}_{n-1}$ and $\tilde{h}_{n-1}$ over cells in A_n by useing the maps g and h and over cells of X_n not in A_n by using the case of (CJ, J) already handled in Claim I.1. $\square$

Remark I.5. Taking $e = \mathrm{id}_Y$, we see that the inclusion $i\colon A \longrightarrow X$ is actually a cofibration.

Theorem I.6

For every weak $\mathscr{J}$-equivalence $e\colon Y \longrightarrow Z$ and every $\mathscr{J}$-complex X, $e_*\colon \pi(X, Y) \longrightarrow \pi(X, Z)$ is a bijection.

Proof. By applying HELP to the pair $(X, *)$, we know that e_* is a surjection. We also see that $X \wedge I^+$ is a $\mathscr{J}$-complex which contains $X \wedge (\partial I)^+$ as a subcomplex, and we see that e_* is an injection by applying HELP to this pair. $\square$

Theorem I.7 (Cellular Whitehead's Theorem)

Every weak $\mathscr{J}$-equivalence between $\mathscr{J}$-complexes is a homotopy equivalence

Proof. This follows immediately from Thm. I.6. Let $f\colon Y \longrightarrow Z$ be a weak $\mathscr{J}$-equivalence. Then we know that $f_*\colon \pi(X, Y) \longrightarrow \pi(X, Z)$ is a bijection for every $\mathscr{J}$-complex X. Let $X = Z$. Then we know that there is a map $g\colon Z \longrightarrow Y$ (unique up to homotopy) such that $f \circ g$ is homotopic to id_Z. Let $X = Y$. Then $g \circ f\colon Y \longrightarrow Y$ is a map such that $f_*([g \circ f]) = [f]$. However, $f_*([\mathrm{id}_Y]) = [f]$ as well. By injectivity, we must have that $g \circ f$ is homotopic to id_Y. $\square$

Thm. 0.1 is just an application of Thm. I.7 to Example I.3.

§I* The Homological Whitehead's Theorem

We now dualize everything in Section I. The dual of Claim I.1 admits a dual proof.

Claim I*.1 — Let $e\colon Y \longrightarrow Z$ be a map such that $\pi(Ce, K) = 0$. If $p_1 h = ge$ and $p_0 h = pf$ in the following diagram, where p_0, p_1, and p are the evident projections, then there exist $\tilde{g}$ and $\tilde{h}$ which make the diagram commute.

$$
\begin{tikzcd}
PK \arrow[dd] & & F(I^+, PK) \arrow[ll] \arrow[rr] \arrow[dd] & & PK \arrow[dd] \\
& Y \arrow[ul] \arrow[ur, dashed] \arrow[rr] \arrow[dr] & & Z \arrow[ur, dashed] \arrow[dr] & \\
K & & F(I^+, K) \arrow[ll] \arrow[rr] & & K
\end{tikzcd}
$$

Now comes the dual co-cellular theory.

Definition I*.2. Let $\mathscr{K}$ be any collection of spaces such that $\Omega K \in \mathscr{K}$ if $K \in \mathscr{K}$. A map $e\colon Y \longrightarrow Z$ is said to be a **weak $\mathscr{K}$-equivalence** if $e\colon \pi(Z, K) \longrightarrow \pi(Y, K)$ is a bijection for all $K \in \mathscr{K}$. A **$\mathscr{K}$-tower** is a space X together with maps $X \longrightarrow X_n$ and $k_n\colon X_n \longrightarrow K_n$, for $n \geq 0$, such that $X_0 = \{*\}$, K_n is a product of spaces in $\mathscr{K}$, $X_{n+1} = Mk_n$, and X is the inverse limit of the X_n. The evident map from X to the paths on a factor of K_{n-1} is called an **n-cocell**. The projection of k_n to a factor is called a **coattaching map**. A map $p\colon X \longrightarrow A$ is said to be a **projection onto a quotient tower** if A is a $\mathscr{K}$-tower, p is the inverse limit of maps $X_n \longrightarrow A_n$, and the composite of p and each n-cocell $A \longrightarrow A_n \longrightarrow PK$ is an n-cocell of X.

Example I*.3 (Postnikov towers)

Let $\mathscr{A}$ be any collection of Abelian groups which contains 0, for example the collection **Ab** of all Abelian groups. Let $\mathscr{K}\mathscr{A}$ be the collection of all Eilenberg-Maclane spaces $K(A, n)$ for $A \in \mathscr{A}$ and $n \geq 0$. (We ask that the Eilenberg-Maclane spaces have homotopy types of CW complexes – by a theorem due to Milnor, this won't affect closure under loops.) A $\mathscr{K}\mathbf{Ab}$-tower X such that K_n is a $K(\pi_{n+1}, n+2)$ for $n \geq 0$ is called a **simple Postnikov tower** and satisfies $\pi_n(X) = \pi_n$. Its coattaching map k_n is usually denoted by k^{n+2} and called a **k-invariant**.

Theorem I*.4 (coHELP)

Let A be a quotient tower of a $\mathscr{K}$-tower X and let $e\colon Y \longrightarrow Z$ be a weak $\mathscr{K}$-equivalence. If $p_1 h = ge$ and $p_0 h = pf$ in the following diagram, then there exist $\tilde{g}$ and $\tilde{h}$ which make the diagram commute.

$$\begin{tikzcd}
X \arrow[ddd, "p"'] & & F(I^+, X) \arrow[ll, "p_0"'] \arrow[rr, "p_1"] \arrow[ddd] & & X \arrow[ddd, "p"] \\
& & & & \\
& Y \arrow[uul, "f"] \arrow[uur, dashed, "\tilde{h}"] \arrow[rr, "e"] \arrow[dr, "h"] & & Z \arrow[uur, dashed, "\tilde{g}"] \arrow[dr, "g"'] & \\
A & & F(I^+, A) \arrow[ll, "p_0"] \arrow[rr, "p_1"'] & & A
\end{tikzcd}$$

Proof. By (2) and (1) and the fact that $\mathscr{K}$ is closed under loops, the hypothesis implies that $\pi(Me, K) = 0$ for all $K \in \mathscr{K}$. The conclusion follows inductively by a cocell by cocell application of Claim I*.1. $\square$

In particular, the projection $p\colon X \longrightarrow A$ is a fibration.

Theorem I*.5

For every weak $\mathscr{K}$-equivalence $e\colon Y \longrightarrow Z$ and every $\mathscr{K}$-tower X, $e^*\colon \pi(Z, X) \longrightarrow \pi(Y, X)$ is a bijection.

Proof. The surjectivity and injectivity of e^* result by application of coHELP to the quotient towers $X \longrightarrow *$ and $F(I^+, X) \longrightarrow F(\partial I^+, X)$, respectively. $\square$

The cocellular Whitehead theorem can be seen formally from above.

Theorem I*.6 (Whitehead)

Every weak $\mathscr{K}$-equivalence between $\mathscr{K}$-towers is a homotopy equivalence.

Now we approximate our spaces by CW-complexes and by Postnikov towers. For a space X with the homotopy type of a CW-complex, we have

$$\tilde{H}^n(X; A) = \pi(X, K(A, n))$$

However, $\mathscr{K}$-towers hardly ever have the homotopy type of CW-complexes. The way around this is to pass to the true homotopy category $\mathrm{ho}\mathscr{T}$. For any space X, there is a CW-complex ΓX and a weak homotopy equivalence $\gamma\colon \Gamma X \longrightarrow X$. The morphisms of $\mathrm{ho}\mathscr{T}$ can be specified by

$$[X, Y] = \pi(\Gamma X, \Gamma Y),$$

with the evident composition. By Thm. I*.5, we have $[X, Y] = \pi(X, Y)$ if X has the homotopy type of a CW-complex. Either as a matter of definition or as a consequence of the fact that cohomology is an invariant of weak homotopy type, we have

$$\tilde{H}^n(X; A) = [X, K(A, n)]$$

for any space X.

Now we come back to Example I*.3. Say that a map $f\colon Y \longrightarrow Z$ is an **$\mathscr{A}$-cohomology isomorphism** if $f^*\colon H^*(Z;A) \longrightarrow H^*(Y;A)$ is an isomorphism for every $A \in \mathscr{A}$. If Y and Z are CW-complexes, then f is an $\mathscr{A}$-cohomology isomorphism iff it is a weak $\mathscr{K}\mathscr{A}$-equivalence.

Claim I*.7 — For every $\mathscr{A}$-cohomology isomorphism $f\colon Y \longrightarrow Z$ and every $\mathscr{K}\mathscr{A}$ tower X, $f^*\colon [Z,X] \longrightarrow [Y,X]$ is a bijection.

Proof. By assuming that Y and Z are CW-complexes WLOG, the result is a special case of Thm. I*.5. □

Now the big theorem.

Theorem I*.8 (Whitehead)
The following statements are equivalent for a map $f\colon Y \longrightarrow Z$ in $\mathrm{ho}\mathscr{T}$ between connected spaces Y and Z of the weak homotopy type of $\mathscr{K}\mathscr{A}$-towers.

1. f is an isomorphism in $\mathrm{ho}\mathscr{T}$.
2. $f_*\colon \pi_*(Y) \longrightarrow \pi_*(Z)$ is an isomorphism.
3. $f^*\colon H^*(Z;A) \longrightarrow H^*(Y;A)$ is an isomorphism for all $A \in \mathscr{A}$.
4. $f^*\colon [Z,X] \longrightarrow [Y,X]$ is a bijection for all $\mathscr{K}\mathscr{A}$-towers X.

If $\mathscr{A} = \mathbf{Mod}_R$, the collection of R-modules for R commutative, then we can also add the following.

5. $f_*\colon H_*(Y;R) \longrightarrow H_*(Z;R)$ is an isomorphism.

Proof. Claim I*.7 provides (3) $\implies$ (4), (4) $\implies$ (1) is formal as in the proof of Thm. I.7, and (1) $\iff$ (2) by the definition of $\mathrm{ho}\mathscr{T}$; (2) $\implies$ (3) and (2) $\implies$ (5) since homology and cohomology are invariants of weak homotopy type, and (5) $\implies$ (3) by the universal coefficients spectral sequence. □

When $\mathscr{A} = \mathbf{Mod}_{\mathbb{Z}} = \mathbf{Ab}$, the implication (5) $\implies$ (2) is a grand generalization of the homological Whitehead's theorem. The standard theory of Postnikov towers shows us that if X is simple or nilpotent, it can be approximated by a simple Postnikov tower.

References

[May83] May, J. Peter. *The Dual Whitehead's Theorems*. 1983. URL: `http://www.math.uchicago.edu/~may/PAPERS/47.pdf`.