Operads and Holomorphic 17-forms on $\overline{\mathcal{M}}_{g,n}$

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A biased survey on the operational structure of $\overline{\mathcal{M}}_{g,n}$, including a particularly specific example of what we can learn from this structure. Written for *Math 731: Moduli Spaces of Curves* at the University of Michigan.

"We shall not cease from exploration,
and the end of all our exploring
will be to arrive where we started
and know the place for the first time."
–*T.S. Eliot*

Figure 1: *Wanderer above the Sea of Fog*, Caspar David Friedrich 1818

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Throughout the semester we've explored various perspectives to study the geometry of moduli spaces of curves. With this talk, we take one step further in that direction, using the *structure* on moduli spaces to look at their geometry. Allow me to be vague in what I mean by *structure* here for a moment, when I say that the particular kind of structure we will be looking at today is an *operational* one. There are objects, originating from homotopy theory that encapsulate exactly this notion of operational structure called *operads*. In today's talk, we will explore the presence of these operadic structures in the geometry of moduli spaces—leading to a characterization of the space of holomorphic 17-forms on $\overline{\mathcal{M}}_{g,n}$. I'll preface this talk by saying that we will prove very few things in great detail because that is emphatically not the point of the talk. The point is to present a story—and hopefully the level of details in this talk allows for good storytelling. Nonetheless, if you feel like some detail could add to the way the story is told here, please reach out.

§1 Operads

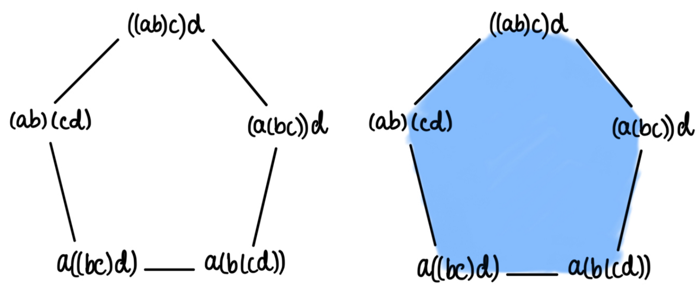
Let us elaborate slightly here on the kind of thing we mean by operational structure. Consider the definition of a monoid for example, the simplest operational structure. A monoid can be defined as a pair $(M, \otimes)$ where M is a set and $\otimes\colon M \times M \longrightarrow M$ is a function such that $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ for all $a, b, c \in M$, and there exists an $e \in M$ such that $e \otimes m = m \otimes e = m$ for all $m \in M$. This definition is of the form I like to call *thing satisfying property*. The utility of such a definition is grand when there is little data to keep track of—as in the example of a monoid. However, there are situations where we will find that there is too much data to handle, so that *thing satisfying property* becomes calculationally inaccessible—as in the example of loop spaces. We would instead like to separate out this structure into an object that wholly encapsulates it, so that our definitions can be of the form *thing that is an algebra over structure object*. An operad is exactly this kind of structure object. We will soon see, as a first example of operads, that there are operads that wholly encapsulate the monoid and commutative monoid structures.

§1.1 Loop Spaces and the Stasheff Associahedra

Let us return very briefly to the example of loop spaces. The question is this, what is the operational structure of singlefold loop spaces? Recall that these are the spaces of the form $X = \Omega Y$. It is well known that ΩY admits a product by concatenation. Perhaps it is important to also remember that there is no canonical choice for this product, so that a noncanonical choice is necessary. However, all choices are homotopic to each other. In other terms, there is a map $X^2 \longrightarrow X$, given by say concatenating halfway through. This product however lacks the most desirable operational properties, it is neither associative, nor unital. We will however find that there is concrete meaning to the statement:

> **Claim 1.1 —** The product on X given by concatenation is associative up to higher coherent homotopy.

We only discuss associativity here, but unitality is also encapsulated in the structures we introduce. Let us first look at three term associativity. Given a triple $(a, b, c) \in X^3$, the loops $(ab)c$ and $a(bc)$ are homotopic to each other. In other words, there is a path in X connecting the two points. As before, there is no canonical choice for this path—but all choices are homotopic to each other. Therefore to establish three-term associativity up to homotopy, we need to pick one such path for each triple $(a, b, c) \in X^3$. Using the notation $\mathscr{K}_3 := I$, this is a map $I \times X^3 \longrightarrow X$ (although I am eliding over some essential properties of this map that allow it to capture any structure). As for four-term associativity, let $(a, b, c, d) \in X^4$. There are a priori five distinct paranthesizations of $abcd$. The three-term associators connecting them form a pentagon (as shown in Fig. 2). We run into a problem again. Although any two vertices on this pentagon are



(a) Different paranthesizations of a four-term product and connecting associators

(b) A 2D filling indicating a chosen higher homotopy between the associators

Figure 2: A diagram depicting four-term homotopy associativity

connected by paths/homotopies, there are a priori multiple choices for a homotopy that connects two paranthesizations. The upshot however is that any two such homotopies are themselves homotopic to each other via a 2-homotopy. The prescription of such a 2-homotopy between any two 1-homotopies is a 2D filling of the pentagon. Again there are a priori several fillings of the pentagon—so four-term associativity up to homotopy is the choice of one such filling for each quadruple $(a, b, c, d) \in X^4$. Writing $\mathscr{K}_4$ for the pentagon, this is a map $\mathscr{K}_4 \times X^4 \longrightarrow X$.

As we move to five-term associativity, let $(a, b, c, d, e) \in X^5$. The paranthesization diagram is harder to draw by hand, so I am attaching a computer formatted version. Given two paranthesizations of $abcde$, we have 1-homotopies connecting them, and a 2-homotopy

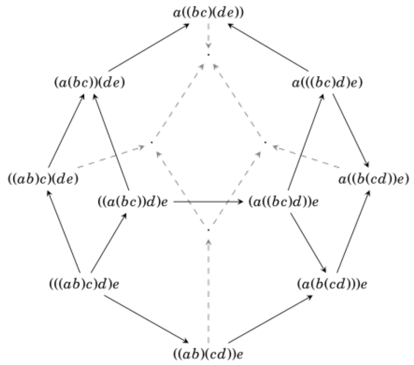


Figure 3: The five-term associativity diagram, where each pentagonal face is shaded in using the chosen 2-homotopies

connecting any two such 1-homotopies—so that up to 2-homotopy there is no problem. However, there are a priori two such 2-homotopies connecting two given 1-homotopies. The upshot once more is that there is a noncanonical choice of a 3-homotopy connecting any two 2-homotopies—provided by a filling of this polyhedron. So five-term associativity is a choice of a map $\mathscr{K}_5 \times X^5 \longrightarrow X$ where $\mathscr{K}_5$ is the underlying polyhedron in Figure 3.

Repeating this story ad nauseum, we see that associativity up to all higher coherent homotopy is a list of maps $\mathscr{K}_n \times X^n \longrightarrow X$ (satisfying additional structure requirements to be outline later) where $\mathscr{K}_n$ is an (n-2)-polyhedron with i-cells indexed by all insertions of $(n-i-2)$ paranthesis pairs between n variables. The polyhedra $\mathscr{K}_n$ aare called the *Stasheff associahedra*. We say that a space X is a $\mathscr{K}$-algebra if it admits structure maps $\mathscr{K}_n \times X^n \longrightarrow X$ as above.

Theorem 1.2 (Stasheff recognition principle [Sta63])
A connected space X of CW type is weak equivalent to a singlefold loop space ΩY iff X is a $\mathscr{K}$-algebra.

It is in this precise sense that $\mathscr{K}$ wholly encapsulates the structure of coherent homotopy associativity[1]. The Stasheff associahedra were the first examples of what now are known as **operads**.

§1.2 Definitions: Operads, Algebras, and Modules

In the example of the Stasheff associahedra, we saw that the structure in sight was encapsulated by a sequence of objects $\mathscr{K}_n$ that at each level encoded information about n-ary operations. This motivates the following definition, of an operad.

Definition 1.3 (Operad)**.** An operad $\mathscr{C}$ (with values in a symmetric monoidal category $(V, \otimes, \kappa)$) is a sequence of objects $\mathscr{C}(j)$ for $j \geq 0$ each equipped with a right $\mathbb{S}_j$ action (these objects are thought of as encapsulating j-ary operations), along with a unit map $\eta\colon \kappa \longrightarrow \mathscr{C}(1)$, and a *composition* product: $\gamma\colon \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1 + \cdots + j_k)$ that are suitably $\mathbb{S}_j$-equivariant, unital, and associative.

We will also say that an operad $\mathscr{C}$ is **unital** if $\mathscr{C}(0) = \kappa$ (in which case we can think

[1] There is in fact a much stronger version of Theorem 1.2 for a general X of CW type, phrased in terms of group completions.

of η as a unit map). Intuitively operads, whose j-th levels are thought of as encoding j-ary operations, encapsulate wholly myriads of operational structures found in nature. In this sense, perhaps the quintessential (unital) operad one must keep in mind is the *endomorphism operad*, End_X defined for any $X \in V$ by $\mathrm{End}_X(j) = \mathrm{Hom}_V(X^j, X)$[2]. The simplest unital operad however is given by the commutative operad Com with $\mathrm{Com}(j) = \kappa$. We will see why this is called the commutative operad in just a second (and what kinds of operational structure it encodes).[3]. As a visual aid, you may want to think of an operad (passing to a concrete category) as follows: each element of $\mathscr{C}(j)$ corresponds to a rooted tree with one vertex, j inputs numbered from 1 to j and one output. The composition maps then tell us how to graft such compatible trees together. I must also add that the *unital associative product* verbage in the definition is incredibly suggestive of an operad itself being some kind of monoidal object. And this is true in a very precise sense, an operad can equivalently be thought of as a monoid in the symmetric monoidal category of symmetric sequences. The latter is the presheaf category on the permutation groupoid.

Now that we have defined our structure-encapsulating objects, we can define what it means to slap this structure onto something.

Definition 1.4 (Algebra over an operad)**.** Let $\mathscr{C}$ be an operad. A $\mathscr{C}$-algebra is an object $X \in V$ with maps $\theta\colon \mathscr{C}(j) \otimes X^j \longrightarrow X$ for $j \geq 0$ that are suitably unital, associative, and $\mathbb{S}_j$-equivariant. The map θ is referred to as an action of $\mathscr{C}$ on X.

The algebras over the commutative operad Com in V are exactly the commutative monoidal objects in V—hence the name[4]. An algebra over an operad is a special case of a left module over the operad, and turns out to be equivalent to the notion.[5] With these definitions, we see that the Stasheff associahedra were space-valued (unital) operads—and particularly the operads whose (connected) algebras were singlefold loop spaces. An equivalent way to define an algebra over an operad (or a left operadic module) is to define it as an operad morphism $\varphi\colon \mathscr{C} \longrightarrow \mathrm{End}_X$ (and we see this to be in complete analogy of a left G-module, say). There is also a notion of right modules over operads, defined by considering right actions by $\mathscr{C}$.

Definition 1.5 (Right module over an operad)**.** Let $\mathscr{C}$ be an operad. A right $\mathscr{C}$-module is a sequence of objects $M(j)$ for $j \geq 0$ each equipped with a right $\mathbb{S}_j$ action and right action maps: $\varphi\colon M(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \longrightarrow M(j_1 + j_2 + \cdots + j_k)$.

Again intuitively, these are the right modules over the operad seen as a monoid in the category of symmetric sequences. Dualizing everything at sight, we also make sense of cooperads, coalgebras, and right comodules over cooperads. We will see later that the FA-modules considered in [CLPW25] are exactly right comodules over the commutative $\mathbf{Vect}_{\mathbb{C}}$-cooperad $\mathscr{C}(j) = \mathbb{C}$.

§1.3 $\overline{\mathcal{M}}_{g,n}$, and the Associated (co)Operads on (co)Homology

We return to moduli spaces now, elaborating on their operational structure in the sense of the previous sections. More precisely, we will see $\overline{\mathcal{M}}_{g,n}$ as forming an operad now.

[2] A generalized version of Hilbert's thirteenth problem can be equivalently phrased as: in some category V of spaces, is the endomorphism operad $\mathrm{End}_{\mathbb{R}}$ generated by $\mathrm{End}_{\mathbb{R}}(2)$ and $\mathrm{End}_{\mathbb{R}}(3)$.

[3] There is also the associative operad given by $\mathrm{Ass}(j) = \kappa[\mathbb{S}_j]$.

[4] Similarly, the Ass-algebras in V are exactly the monoidal objects.

[5] Technically speaking, a left module over an operad should be thought of as a left module over the monoid in the category of symmetric sequences. However, there is an equivalence of categories between left modules and algebras over an operad.

We will soon see that this moduli space has a much more desirable structure—it has the structure of what is known as a modular operad. We will elaborate on this note extensively in Section 2.

Throughout, by $\mathcal{M}_{g,n}$ we mean the Deligne-Mumford moduli stack of smooth genus g curves with n marked points, and by $\overline{\mathcal{M}}_{g,n}$ its compactification given by the Deligne-Mumford moduli stack of stable genus g curves with n marked points. Let $\overline{\mathcal{M}}_0$ denote the **Var**-valued symmetric sequence [6] defined by $\overline{\mathcal{M}}_0(n) = \overline{\mathcal{M}}_{0,n+1}$. The symmetric group action on $\overline{\mathcal{M}}_0$ is by permutation of the (last n) marked points. The zeroth marked point is thought of as an output, while the last n marked points are thought of as the inputs. Similarly, let $\overline{\mathcal{M}}$ be the **DM**-valued symmetric sequence $\overline{\mathcal{M}}(n) = \coprod_g \overline{\mathcal{M}}_{g,n+1}$. $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_0$ actually form operads with composition products defined as follows. In the case of $\overline{\mathcal{M}}_0$, an element of $\overline{\mathcal{M}}_0(n) \times \overline{\mathcal{M}}(j_1) \times \cdots \times \overline{\mathcal{M}}(j_n)$ is a tuple of a (genus 0) curve C with $n+1$ marked points, and curves D_i with $1 + j_i$ marked points. We send this tuple to the reducible curve obtained by gluing the zeroth marked point of D_i with the i-th marked point of C. The composition product on $\overline{\mathcal{M}}$ is defined similarly.

Claim 1.6 — $\overline{\mathcal{M}}_0$ and $\overline{\mathcal{M}}$ (with the prescribed composition products) form **Var**-valued and **DM**-valued operads respectively.

By taking homology, we get operads $H_\bullet(\overline{\mathcal{M}}_0)$ and $H^\bullet(\overline{\mathcal{M}})$ with values in $\mathbf{dgVect}_\mathbb{C}$. These are what we call dg-operads. Similarly by taking cohomology, we get dg-cooperads $H^\bullet(\overline{\mathcal{M}}_0)$ and $H^\bullet(\overline{\mathcal{M}})$. For now, we record results about the algebras over these operads. Let $q_n \in H_{2(n-2)}(\overline{\mathcal{M}}_0(n))$ be the fundamental class of $\overline{\mathcal{M}}_0(n) = \overline{\mathcal{M}}_{0,n+1}$. Then q_2 generates the suboperad $H_0(\overline{\mathcal{M}}_0) \subseteq H_\bullet(\overline{\mathcal{M}}_0)$. This suboperad is just the commutative operad. Intuitively then, the algebras over $H_\bullet(\overline{\mathcal{M}}_0)$ are going to be commutative algebras with some additional graded structure.

Theorem 1.7 (Kontsevich–Manin '94 [KM94])

The $H^\bullet(\overline{\mathcal{M}}_0)$-algebras are graded vector spaces with multilinear (graded) totally symmetric operations $(x_1, \ldots, x_n)$ of degree $n-2$ (for $n \geq 2$) satisfying the WDVV relations. In particular, the operation (x_1, x_2) is a commutative associative multiplication.

Kapranov attributes the following result on $H_\bullet(\overline{\mathcal{M}})$ to developments in Gromov-Witten theory and [KM94].

Theorem 1.8

For any smooth projective variety V (over $\mathbb{C}$), $H_\bullet(V, \mathbb{C})$ has a canonical algebra structure over $H_\bullet(\overline{\mathcal{M}})$. In particular, $H_\bullet(V, \mathbb{C})$ satisfies the WDVV relations.

We will make improvements on these statements in Section 2.1 using the language of cyclic operads.

§2 Modular Operads, $\overline{\mathcal{M}}_{g,n}$, and $H^\bullet(\overline{\mathcal{M}}_{g,n})$

In this section we specialize to the modular setting and conceptually bring out the structure of the operad $\overline{\mathcal{M}}$.

[6] What we call symmetric sequences here, [GK98] call $\mathbb{S}$-modules.

§2.1 Cyclic Operads and Stable $\mathbb{S}$-modules

The first piece of intuition regarding the operads $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_0$ is that the distinction between one output and n input in the n-ary information of $\overline{\mathcal{M}}(n)$ and $\overline{\mathcal{M}}_0$ is artificial in consideration of the $\mathbb{S}$-action. Indeed, each $\overline{\mathcal{M}}_0(n)$ and $\overline{\mathcal{M}}(n)$ actually admit $\mathbb{S}_{n+1}$ actions by permuting all marked points. The authors of [GK98] begin with this intuition. Just as an $\mathbb{S}$-module[7] (with values in $\mathbf{Vect}_{\mathbb{C}}$ say) was a sequence of spaces $\mathcal{V}(n)$ equipped with an $\mathbb{S}_n$-action, we have the following definition of a cyclic $\mathbb{S}$-module.

Definition 2.1 (Cyclic $\mathbb{S}$-module)**.** A cyclic $\mathbb{S}$-module $\mathcal{V}$ is a sequence of objects $\mathcal{V}(n)$ each equipped with a $\mathbb{S}_{n+1}$-action.

The nomenclature comes from the fact that these are exactly the sequences of objects $\mathcal{V}(n)$ that are at the same time $\mathbb{S}_n$-modules and modules over the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ generated by $(01\ldots n)$. For a cyclic $\mathbb{S}$-module $\mathcal{V}$ and I a $(k+1)$-element set, we define:

$$\mathcal{V}((I)) := \left(\bigoplus_{f:\{0,1,\ldots,k\}\cong I} \mathcal{V}(k) \right)_{\mathbb{S}_{k+1}}$$

When $I = \{0, 1, \ldots, n-1\}$, we write $\mathcal{V}((n)) := \mathcal{V}((I)) = \mathcal{V}(n-1)$. For each n, let τ_n denote the cycle $(01\ldots n)$.

Definition 2.2 (Cyclic operads)**.** We say that a cyclic $\mathbb{S}$-module $\mathcal{V}$ is a cyclic operad if its underlying $\mathbb{S}$-module is an operad and the composition maps are equivariant with respect to the $\mathbb{Z}/(n+1)\mathbb{Z}$ action generated by τ_n.

The $\mathbb{S}_{n+1}$-action on $\overline{\mathcal{M}}_0(n)$ described at the start of this section makes $\overline{\mathcal{M}}_0$ a cyclic operad with values in **Var**. Note that $\overline{\mathcal{M}}_0((n)) = \overline{\mathcal{M}}_{0,n}$. The same is true for $\overline{\mathcal{M}}$ (however now replacing **Var** with **St**). On taking homology, the cyclic structure descends to provide cyclic dg-operads $H_\bullet(\overline{\mathcal{M}}_0)$ and $H_\bullet(\overline{\mathcal{M}})$. Now just as operads have algebras, cyclic operads can have cyclic algebras.

Definition 2.3 (Cyclic algebras)**.** A cyclic algebra over a cyclic operad $\mathcal{V}$ is a pair (A, g) where A is an algebra over the underlying operad of $\mathcal{V}$ and g is a scalar product on A invariant under the $\mathcal{V}$-action.

Returning to the canonical example of Com, we see that Com is a cyclic operad whose cyclic algebras are exactly commutative algebras with an invariant scalar product in the usual sense. In the cyclic language, the algebras over $H_\bullet(\overline{\mathcal{M}}_0)$ and $H_\bullet(\overline{\mathcal{M}})$ admit a much neater description.

Theorem 2.4 (Manin '99 [Man99] and Results from Gromov-Witten Theory)

1. A finite dimensional (graded) cyclic algebra over $H_\bullet\overline{\mathcal{M}}_0$ is the same as a formal germ over a potential Frobenius manifold.

2. For a smooth projective variety V, the intersection pairing g on $H_\bullet(V, \mathbb{C})$ makes it a cyclic $H_\bullet(\overline{\mathcal{M}})$-algebra.

Note that the (formal) germ of a Frobenius manifold can be thought of as a deformation of the Frobenius algebra given by the tangent space at the point, and hence can be

[7] These are exactly what we have been calling symmetric species, but we make the shift in terminology here.

thought of as a graded Frobenius algebra with some additional structure coming from the WDVV relations that define the Frobenius structure on the manifold.

We have, for the purpose of our talk, somewhat exhausted the operadic structure of $\overline{\mathcal{M}}_0$. The operad $\overline{\mathcal{M}}$ however still possesses a rich theory that we are yet to exploit. The first thing we must notice is that $\overline{\mathcal{M}}$ actually comes from a collection of spaces $\overline{\mathcal{M}}_{g,n}$ for $n, g \geq 0$ with an $\mathbb{S}_n$-action on each $\overline{\mathcal{M}}_{g,n}$ by permuting marked points. But this sequence of spaces is concentrated in the regime where the stability condition $2g + n - 2 > 0$ is satisfied. We can convert this observation to a definition in the operadic setting.

Definition 2.5 (Stable $\mathbb{S}$-Modules)**.** A stable $\mathbb{S}$-module $\mathcal{V}$ is a collection of objects $\mathcal{V}((g,n))$ for $n, g \geq 0$ equipped with a right $\mathbb{S}_n$-action on $\mathcal{V}((g,n))$ such that $\mathcal{V}((g,n)) = 0$ if $2g + n - 2 \leq 0$.

Any cyclic $\mathbb{S}$-module $\mathcal{V}$ such that $\mathcal{V}((n))$ for $n \geq 2$ may be seen as a stable $\mathbb{S}$-module by setting $\mathcal{V}((0,n)) = \mathcal{V}((n))$ and $\mathcal{V}((g,n)) = 0$ for $g > 0$. Conversely, the genus 0 piece of any stable $\mathbb{S}$-module is itself a cyclic $\mathbb{S}$-module. As notation, for a finite set I, we write:

$$\mathcal{V}((g,I)) := \left(\bigoplus_{f\colon [n] \cong I} \mathcal{V}((g,n)) \right)_{\mathbb{S}_n}$$

Stable $\mathbb{S}$-modules form a category (with morphisms given by $\mathbb{S}$-equivariant maps at each level). We conclude this section by defining a monad $\mathbb{M}$ on this category. Recall that a connected labelled graph G is called stable if $2g(v) - 2 + n(v) > 0$ for each vertex v of G. For a stable $\mathbb{S}$-module $\mathcal{V}$ and a stable graph G, we let $\mathcal{V}((G))$ be the tensor product

$$\mathcal{V}((G)) := \bigotimes_{v \in \mathrm{Vert}(G)} \mathcal{V}((g(v), \mathrm{Leg}(v))).$$

Let $\Gamma((g,n))$ be the category whose objects are pairs (G, ρ) where G is a stable genus g graph and ρ is a bijection between $\mathrm{Leg}(G)$ and $[n]$ (and whose morphisms are morphisms of stable graphs preserving the labelling ρ). It is a routine exercise to show that $\Gamma((g,n))$ has a finite number of isomorphism classes of objects (this is essentially because there are a finite number of stable graphs of genus g with a given number of flags). We will denote such an isomorphism class $[\Gamma((g,n))]$. Finally, let $\mathrm{Iso}\Gamma((g,n))$ denote the maximal subgroupoid of $\Gamma((g,n))$.

Claim 2.6 (Getzler-Kapranov '98 [GK98]) — There is a monad $\mathbb{M}\colon \mathbb{S}\mathbf{Mod} \longrightarrow \mathbb{S}\mathbf{Mod}$ on the category of $\mathbb{S}$-modules defined by:

$$\mathbb{M}\mathcal{V}((g,n)) = \mathrm{colim}_{G \in \mathrm{Iso}\Gamma((g,n))} \mathcal{V}((G)) \cong \bigoplus_{G \in [\Gamma((g,n))]} \mathcal{V}((G))_{\mathrm{Aut}(G)}.$$

Where the unit map $\eta\colon \mathcal{V} \longrightarrow \mathbb{M}\mathcal{V}$ is given by the inclusion of the summand $\mathcal{V}((*_{g,n})) \cong \mathcal{V}((g,n))$ associated to the terminal stable graph with genus g, n legs, and no edges. Noting that $\mathbb{M}^2\mathcal{V}$ is the colimit of $\mathcal{V}((G_0))$ over the maximal groupoid of diagrams $G_0 \longrightarrow G_1$, the multiplication maps $\mu\colon \mathbb{M}\mathbb{M}\mathcal{V} \longrightarrow \mathbb{M}\mathcal{V}$ are induced by sending the contractions $G \longrightarrow G/I$ to G.

The algebras over this monad are exactly the modular operads—and $\overline{\mathcal{M}}$ is the quintessential example thereof.

§2.2 Modular Operads

We begin with the definition of a modular operad, the kind of thing that $\overline{\mathcal{M}}$ is.

Definition 2.7 (Modular Operads)**.** A modular operad $\mathcal{A}$ is an algebra over the monad $\mathbb{M}$.

However conceptually satisfying this definition maybe, it is unclear what the practical ramifications are. The first claim I will elide over is that given a stable $\mathbb{S}$-module $\mathcal{A}$ and a structure map $\mu\colon \mathbb{M}\mathcal{A} \longrightarrow \mathcal{A}$, and a morphism of stable graphs $G_0 \longrightarrow G_1$, there is a morphism $\mathcal{A}((f))\colon \mathcal{A}((G_0)) \longrightarrow \mathcal{A}((G_1))$ (this is in some sense given by noting that $\mathcal{A}((g(u), \mathrm{Leg}(u))) = \mathcal{A}((f^{-1}(v)))$, and constructing maps $\mathcal{A}((f^{-1}(v))) \longrightarrow \mathcal{A}((g(v), \mathrm{Leg}(v)))$ by precomposing the structure map μ with the universal map $\mathcal{A}((G)) \longrightarrow \mathbb{M}\mathcal{A}((g, n)))$. The next claim I will elide over is the following, which provides an equivalent characterization of modular operads.

Claim 2.8 (Getzler-Kapranov '98 [GK98]) — A stable $\mathbb{S}$-module $\mathcal{A}$ with a structure map $\mu\colon \mathbb{M}A \longrightarrow A$ is a modular operad iff the prescription above defines a functor on the category $\Gamma((g, n))$ of stable graphs.

There is a still more practically utile way to define modular operads. Note that given any modular operad $\mathcal{A}$, there are contraction maps $\xi_{ij} = \mu_{G^{ij}_{g,I}} : \mathcal{A}((G^{ij}_{g,I})) \cong \mathcal{A}((g, I)) \longrightarrow \mathcal{A}((g+1, I \setminus \{i, j\}))$, where $G^{ij}_{g,I}$ is the stable graph whose flags are elements of I (a finite set), equipped with a single vertex of genus g, and a single edge joining the flags i and j. These contraction maps are equivariant with respect to bijections of the set I.

Theorem 2.9 (Getzler-Kapranov '98 [GK98])

Let $\mathcal{A}$ be a stable graded cyclic operad with contraction maps

$$\xi_{ij}\colon \mathcal{A}((g, I)) \longrightarrow \mathcal{A}((g+1, I \setminus \{i, j\}))$$

equivariant with respect to bijections of the set I, that are suitably coherent. Then $\mathcal{A}$ is a modular operad.[a]

[a] To maybe give a hint as to what these coherence conditions look like, the first of them is to ask that if one contracts the vertices i and j then k and l, it is the same as contracting k and l then i and j. The subsequent conditions are compatibility conditions between the contraction maps and the operadic composition maps of $\mathcal{A}$.

We add that a graded stable cyclic operad is a cyclic operad $\mathcal{P}$ such that each $\mathcal{P}((n))$ has an $\mathbb{S}_n$-invariant decomposition into a direct sum over its different genera pieces such that the composition maps are additive in genera. We see in hindsight that the structure provided by Theorem 2.9 is exactly the kind of structure that the stable graded cyclic operad $\overline{\mathcal{M}}$ has. The contraction maps are capturing the fact that given a stable genus g curve with n marekd points, gluing together two distinct marked points yields a stable genus $g+1$ curve with $n-2$ marked points. Therefore $\overline{\mathcal{M}}$ has the structure of a modular operad. Taking homology, we get a modular dg-operad $H_\bullet(\overline{\mathcal{M}})$. We add here that the homology of $\overline{\mathcal{M}}_0$ in turn has the structure of a cyclic dg-operad (now from the point of view that the genus 0 piece of a modular operad is a cyclic operad). This cyclic operad is called the *hypercommutative operad*. We have seen before that the algebras over the hypercommutative operad are just the hypercommutative algebras in the sense of Theorem 1.7. We also add that, dualizing, we get the modular dg-cooperad $H^\bullet(\overline{\mathcal{M}})$, and the hypercommutative cooperad $H^\bullet(\overline{\mathcal{M}}_0)$. We will soon find it helpful to also note that the fixed genus piece of a modular operad is a right module over the cyclic operad given by its genus 0 part. Dually, the genus g part of a modular cooperad is a right comodule over the genus 0 cyclic cooperad. We will talk about what this means specifically in the context of the cohomology of moduli spaces when discussing FA-modules in Section 3.

We close this section by adding that while we do not touch on these aspects, there are myriads of applications for the theory of operads we have presented here in algebraic geometry. See [GK94; Get95; Get98; Kap98] for just a peek.

§3 Holomorphic p-forms on $\overline{\mathcal{M}}_{g,n}$

We are now ready to step into the realm we originally set out to study, the space of holomorphic p-forms on $\overline{\mathcal{M}}_{g,n}$. We will denote this space by $H^{p,0}(\overline{\mathcal{M}}_{g,n})$. Before we begin, we state what is known about $H^{p,0}(\overline{\mathcal{M}}_{g,n})$. For $p = 1, 3, 5$, we know from [AC98] that the space of holomorphic p-forms vanish. In fact for $p \leq 9$ odd, we know from [BFP24] that there exist no nontrivial holomorphic p-forms (since the corresponding cohomology groups vanish).

For even $p \leq 12$, we know from [CLP23] that $H^p(\overline{\mathcal{M}}_{g,n})$ is pure Hodge-Tate, so there exist no non-trivial holomorphic p-forms in these cases either. For $p = 11$, [CLP23] shows that $H^{11,0}(\overline{\mathcal{M}}_{1,n})$ is generated by pullbacks of the distinguished generator of $H^{11,0}(\overline{\mathcal{M}}_{1,11})$ (which itself corresponds to the weight 12 cusp form Δ of $\mathrm{SL}_2(\mathbb{Z})$), and there are no non-trivial holomorphic 11-forms for $g \neq 1$ or $n < 11$. The study of the motivic structure of the thirteenth and fourteenth cohomology groups of $\overline{\mathcal{M}}_{g,n}$ shows that there exist no nontrivial p-forms when $p = 13, 14$. Also, [Fon24] shows that there are no non-trivial holomorphic p-forms when $p = 14, 16, 18$. Conditional upon the conjectural vanishing of $H^{20,0}(\overline{\mathcal{M}}_{3,15})$ and $H^{20,0}(\overline{\mathcal{M}}_{3,16})$, there are also no non-trivial holomorphic 20-forms on $\overline{\mathcal{M}}_{g,n}$ (the Langlands program predicts that $H^k(\overline{\mathcal{M}}_{g,n})$ is pure Hodge-Tate for all even $k \leq 20$.). The case of $p = 15$ comes from [CLPW24] (where we learn that there are no non-trivial holomorphic 15-forms if $g \neq 1$, and in this case, the holomorphic 15-forms are generated by pullbacks of the weight 15 cusp form Δ_{15} on $\mathrm{SL}_2(\mathbb{Z})$ for $n \geq 15$).

The cases $p = 17$ and conjecturally $p = 19$ were settled in [CLPW25] using the operadic machinery we outlined the development of. We will say at the out set that if $g > 2$, then they show that there exist no non-trivial holomorphic 17-forms, and conjecturally the same is true for holomorphic 19-forms (the conjecture being the vanishing of $H^{19}(\overline{\mathcal{M}}_{3,15})$). This result is the main goal of the final section of this talk—we will however only touch upon the vanishing of the space of holomorphic 17-forms (restricting ourselves to the case of $g > 2$).

§3.1 The Arbarello-Cornalba Inductive Method

In this section, we elaborate on the inductive method of [AC98] used to demonstrate the vanishing of $H^p(\overline{\mathcal{M}}_{g,n})$ for $p = 1, 3, 5$ (and later $p = 7, 9$). This is in fact the argument used to demonstrate that the even cohomology (up to degree 12) of the stable moduli space is pure Hodge-Tate.

The inductive method begins with the excision long exact sequence associated to the boundary $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$:

$$\cdots \longrightarrow H^k_c(\mathcal{M}_{g,n}) \longrightarrow H^k(\overline{\mathcal{M}}_{g,n}) \longrightarrow H^k(\partial\mathcal{M}_{g,n}) \longrightarrow H^{k+1}_c(\mathcal{M}_{g,n}) \longrightarrow \cdots .$$

Therefore, whenever we know that $H^k_c(\mathcal{M}_{g,n})$ vanishes, we know that the map $H^k(\overline{\mathcal{M}}_{g,n}) \longrightarrow H^k(\partial\mathcal{M}_{g,n})$ is injective. Writing $\widetilde{\partial\mathcal{M}}_{g,n}$ for the normalization of the boundary $\partial\mathcal{M}_{g,n}$, [AC98] proved the following improvement using Hodge-theoretic arguments.

Theorem 3.1 (Arbarello-Cornalba '98 [AC98])
If $H^k_c(\mathcal{M}_{g,n}) = 0$, then the pullback:

$$H^k(\overline{\mathcal{M}}_{g,n}) \longrightarrow H^k(\widetilde{\partial\mathcal{M}}_{g,n})$$

is injective.

The utility of the above theorem comes from knowing the vanishing of the compact support cohomology of $\mathcal{M}_{g,n}$ for certain regimes. The first result of this kind was proved in [AC98], and then improved upon by [BFP24].

Lemma 3.2 (Bergström-Faber-Payne '24 [BFP24])
Let $g \geq 1$. Then $H^k_c(\mathcal{M}_{g,n}) = 0$ for $k < 2g$ and $n = 0, 1$, and also for $k < 2g - 2 + n$ and $n \geq 2$.

The inductive aspect of the method lies from knowing that each irreducible component of the normalization of the boundary $\widetilde{\partial\mathcal{M}}_{g,n}$ is a quotient of a product of "smaller" stable moduli spaces by a finite group action. Therefore, if we had enough base cases, using the Künneth formula and the Lemma above, we can conclude something about the cohomology of the stable moduli space that we started out with. This is for example what was done in [CLP23] to show that the even cohomology of $\overline{\mathcal{M}}_{g,n}$ (up to degree 12) is pure Hodge-Tate, and what was done in [AC98] to show that the first, third, and fifth cohomology of $\overline{\mathcal{M}}_{g,n}$ vanish, while the second cohomology is tautological.

An essential observation of [CLP23] was that the pullback arrow to the normalization: $H^k(\overline{\mathcal{M}}_{g,n}) \longrightarrow H^k(\widetilde{\partial\mathcal{M}}_{g,n})$ is the first arrow in a chain complex, whose j-th term can be identified with the cohomology of the normalization of the closure of the codimension j boundary strata with coefficients in a local system (the determinant of the permutation representation on the branches of the boundary divisor). We have seen this chain complex before. It is exactly the weight k row in the E_1-page of the spectral sequence obtained via Poincaré duality from the Deligne weight spectral sequence for the pair $(\overline{\mathcal{M}}_{g,n}, \partial\mathcal{M}_{g,n})$.

§3.2 The Deligne Weight Spectral Sequence

Recall that the (Poincaré dual of) Deligne weight spectral sequence is a natural starting point for the study of the weight-graded compact support cohomology of a DM-stack with a given normal crossings compactification. In the specific case of the compactification of $\mathcal{M}_{g,n}$ by $\overline{\mathcal{M}}_{g,n}$, the construction is summarized as follows. Recall the stratification of $\overline{\mathcal{M}}_{g,n}$ by dual graphs, where each stratum corresponds to the moduli space of curves with a prescribed dual graph of genus g with n legs. Let D_Γ denote the locus of stable curves in $\overline{\mathcal{M}}_{g,n}$ with dual graph Γ. The normalization of D_Γ is the smooth and proper DM-stack $\tilde{D}_\Gamma = \overline{\mathcal{M}}_\Gamma/\mathrm{Aut}(\Gamma)$. We collect the codimension j pieces of such normalizations to get $\tilde{D}^j = \coprod_{|E(\Gamma)|=j} \tilde{D}_\Gamma$.

The image of a small neighborhood of a point in $\tilde{D}^j$ is contained in exactly j analytic branches of the boundary divisor of $\overline{\mathcal{M}}_{g,n}$, and the monodromy action on these branches defines a local system of rank j on $\tilde{D}^j$ whose determinant is denoted ε^j. The E_1-page of the Deligne weight spectral sequence for $H_c^\bullet(\mathcal{M}_{g,n})$ is given by $E_1^{j,k} \cong H^k(\tilde{D}^j, \varepsilon^j)$. However, note that the branches of D containing the image of a small neighborhood of a point in $\tilde{D}_\Gamma$ are naturally identified with the edges of Γ, and ε^j is trivialized pulling back to $\overline{\mathcal{M}}_\Gamma$. Thus, the system ε^j is the determinant of the permutation action of $\mathrm{Aut}(\Gamma)$ on $E(\Gamma)$. And thus we have:

$$E_1^{j,k} \cong \bigoplus_{|E(\Gamma)|=j} (H^k(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma))^{\mathrm{Aut}(\Gamma)}$$

The spectral sequence degenerates at E_2 to $\mathrm{gr}_k H_c^{j+k}(\mathcal{M})_{g,n}$, which is thus canonically identified with H^j of the complex described by the E_1 page, graded cohomologically by the number of edges of Γ. The data in this spectral sequence is neatly swept up by the language of modular operads. We will see that there is a procedure called the Feynman transform, that for one takes the modular cooperad $H^\bullet(\overline{\mathcal{M}})$ to a modular operad whose value at (g, n) is the direct sum over all weights of the complex above. This is also sometimes called the Getzler-Kapranov complex $\mathrm{GK}_{g,n}^k$.

Note that the first map in this complex can be written as $H^k(\overline{\mathcal{M}}_{g,n}) \longrightarrow \bigoplus_{|E(\Gamma)|=1} H^k(\overline{\mathcal{M}}_\Gamma)^{\mathrm{Aut}(\Gamma)}$, and can be identified with the pullback of the map into H^k of the normalization of the boundary.

§3.3 The Feynman Transform of a Modular Cooperad and the GK-Complex

We now outline the procedure prescribed by the Feynman transform. We only define the Feynman transform of a dg-modular cooperad $P^\bullet$ of $\mathbb{Q}$-vector spaces here. Let $\mathcal{P}^\bullet$ be a dg-modular cooperad. We write: $\mathcal{P}^k((\Gamma)) := \bigoplus_{\sum k_v = k} \mathcal{P}^{k_v}((g_v, n_v))$. The Feynman transform of a modular operad prescribes what is known as a $\mathfrak{K}$-modular operad, that we write $F\mathcal{P}$. A $\mathfrak{K}$-modular operad, crudely is a modular operad whose structure maps (i.e. composition and contraction maps) have cohomological degree $+1$. They are in this sense, also known as twisted modular operads. The underlying $\mathbb{S}$-module of $F\mathcal{P}$ is given by $F\mathcal{P}(g, n) = \bigoplus_{[\Gamma]} \left[\bigotimes_\Gamma \mathcal{P} \otimes \mathbb{F}[-1]^{\otimes |E(\Gamma)|}\right]_{\mathrm{Aut}(\Gamma)}$. The direct sum here is taken over isomorphism classes of stable graphs Γ with genus g and n legs (so an element of $[\Gamma((g, n))]$). The differential is defined using the cooperadic structure maps of $\mathcal{P}$. The case of interest to us is the modular dg-cooperad $H^\bullet(\overline{\mathcal{M}})$ where the additional grading is cohomological or the weight grading (and these gradings descend down to the Feynman transform). The weight k Getzler-Kapranov complex is defined to be the k-th associated graded of the Feynman transform $FH^\bullet(\overline{\mathcal{M}})$ evaluated at (g, n). In other words, $\mathrm{GK}_{g,n}^k := \mathrm{gr}_k FH^\bullet(\overline{\mathcal{M}})((g, n))$. As we saw before, this complex is canonically identified with the k-th row of the weight spectral sequence, so we have $H^\bullet(\mathrm{GK}_{g,n}^k) = \mathrm{gr}_k H_c^\bullet(\mathcal{M}_{g,n})$.

Similarly, we also have the modular cooperad $H^{\bullet,0}(\overline{\mathcal{M}})$ of holomorphic forms, which is graded by the degree. The Hodge weight $(k, 0)$ Getzler-Kapranov complex is $\mathrm{GK}_{g,n}^{k,0} := \mathrm{gr}_k FH^{\bullet,0}(\overline{\mathcal{M}})((g, n))$. The generators of this complex are dual graphs of stable curves of genus g with n numbered legs, each of whose vertices is decorated by a copy of $H^{k_v,0}(\overline{\mathcal{M}}_{g_v,n_v})$ so that $\sum_{v\in\Gamma} k_v = k$. We also have the identification $H_c^\bullet(\mathcal{M}_{g,n})^{k,0} \cong H^\bullet(\mathrm{GK}_{g,n}^{k,0})$. The reader may want to pause here to see that on one side of the duality, we have the cohomology of the stable moduli space—and on the other side of the duality, we have the compact support cohomology of the open moduli space.

§3.4 FA-Modules and $H^{p,0}(\overline{\mathcal{M}}_{g,n})$

We say something very briefly here about the FA-modules considered in [CLPW25]. The FA-modules as considered in that paper[8], are exactly the right comodules over the cocommutative operad Com^c in our sense (this is a routine exercise to work out[9]). In our context, $\overline{\mathcal{M}}$ forms a modular operad, and $H^\bullet(\overline{\mathcal{M}})$ thus is a modular cooperad. The genus 0 piece, $H^\bullet(\overline{\mathcal{M}}_0)$ is the cyclic hypercommutative cooperad. And any fixed genus piece $H^\bullet(\overline{\mathcal{M}}_{g,*})$ is an operadic right comodule over the hypercommutative operad. The degree 0 piece of the hypercommutative cooperad is the commutative cooperad.[10]. Therefore, by restriction, $H^\bullet(\overline{\mathcal{M}}_{g,*})$ is an operadic right comodule over the commutative cooperad. It is also therefore an FA-module. We also get an induced FA-module structure on the holomorphic forms $H^{\bullet,0}(\overline{\mathcal{M}}_{g,*})$.

Given a partition $m = \lambda_1 + \cdots + \lambda_l$, let V_λ be the associated irreducible representation of $\mathbb{S}_m$ (the so-called Specht module). For each such λ, there is an associated FA-module C_λ, whose definition we omit. For $\lambda_1 \geq 2$, the FA-module C_λ is simple. If $\lambda_1 = 1$, then C_λ is not simple, but the FA-module $\tilde{C}_{1^m} := \mathrm{coker}(C_{1^{m+1}} \longrightarrow C_{1^m})$. The results of [CLPW25] are summarized as follows.

Theorem 3.3 (Canning-Larson-Payne-Willwacher '25 [CLPW25])
There is an equality of FA-modules, for $g \geq 1$ and $k \leq 18$.

$$H^{k,0}(\overline{\mathcal{M}}_{g,*}) \cong \begin{cases} \tilde{C}_{1^k} & k = 11, 15, 17 \wedge g = 1 \\ C_{2^7} & k = 17 \wedge g = 2 \\ 0 & \text{otherwise.} \end{cases}$$

§3.5 Operadic Arbarello-Cornalba Induction

We exhibit the main method of the [CLPW25] result, an operadic version of the Arbarello-Cornalba induction. Let $\mathcal{P}^\bullet$ be a graded modular cooperad, and let $\mathcal{T}^\bullet \subset \mathcal{P}^\bullet$ be a graded modular subcooperad. Then $F\mathcal{T}^\bullet \subset F\mathcal{P}^\bullet$ is a graded $\mathfrak{K}$-modular suboperad.

Theorem 3.4 (Canning-Larson-Payne-Willwacher '25 [CLPW25])
Let $\mathcal{T}^\bullet \subset \mathcal{P}^\bullet$ be an inclusion of graded modular cooperads. Suppose

1. the inclusion induces isomorphisms $\mathcal{T}^k((\Gamma)) \cong \mathcal{P}^k((\Gamma))$ for all stable graphs Γ of genus g with n legs and one or two edges, and
2. the induced maps $H^{k+e}(F\mathcal{T}^k((g,n))) \longrightarrow H^{k+e}(F\mathcal{P}^k((g,n)))$ are isomorphisms for $e = 0, 1$. Then $\mathcal{T}^k((g,n)) = \mathcal{P}^k((g,n))$

Proof. The inclusion $F\mathcal{T}^k \subset F\mathcal{P}^k$, together with the assumption that the inclusion induces isomorphisms $\mathcal{T}^k((\Gamma)) \cong \mathcal{P}^k((\Gamma))$, provides the following injection.

$$H^k(F\mathcal{T}^k((g,n))) \hookrightarrow H^k(F\mathcal{P}^k((g,n))),$$

[8] Which are functors $FA \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ on the category of finite sets and all maps.
[9] It may help to note that every map of sets factors as an injection followed by a surjection on the image
[10] Technically speaking here, to have a unital hypercommutative cooperad, we need to allow for unstable operations in genus 0. We elide over this here.

and a surjection:

$$H^{k+1}(F\mathcal{T}^k((g,n))) \twoheadrightarrow H^{k+1}(F\mathcal{P}^k((g,n))).$$

If both of these are isomorphisms (i.e. if assumption (2)) holds, then $\mathcal{T}^k((g,n)) = \mathcal{P}^k((g,n))$ by the Five Lemma. $\square$

In particular, if (1) is satisfied, if $H^k(F\mathcal{P}^k((g,n))) = 0$, and if $H^{k+1}(F\mathcal{T}^k((g,n))) = 0$, then $\mathcal{T}^k((g,n)) = \mathcal{P}^k((g,n))$.

Suppose $\mathcal{P} \subset H^\bullet(\overline{\mathcal{M}})$ is a subcooperad. Then $H^k(F\mathcal{P}^k((g,n))) = 0$ when g and n are sufficiently large compared to k (this was the Arbarello-Cornalba insight [AC98]). The primary new observation is the vanishing of $H^{k+1}(F\mathcal{T}^k)$ for an appropriate choice of $\mathcal{T}^\bullet$. This vanishing is due to a vanishing result of the cohomology of graph complexes in [CLPW25] that we omit here.

§3.6 Holomorphic 17-forms

Let $\mathcal{T}^\bullet$ be the modular dg-cooperad whose underlying stable $\mathbb{S}$-module is:

$$\mathcal{T}^k((g,n)) = \begin{cases} \mathbb{C} & k = 0 \\ V_{n-16,1^{16}} & g = 1 \wedge k = 17 \\ \operatorname{Ind}_{\mathbb{S}_{14}\times\mathbb{S}_{n-14}}^{\mathbb{S}_n}(V_{2^7} \boxtimes 1) & g = 2 \wedge k = 17 \\ 0 & \text{otherwise} \end{cases}$$

For $g \leq 2$, [CLPW25] show that $\mathcal{T}^{17}((g,n)) = H^{17,0}((\overline{\mathcal{M}}_{g,n}))$. Thus $\mathcal{T}^\bullet$ is a modular subcooperad of $H^{\bullet,0}(\overline{\mathcal{M}})$. Also, for all $g \geq 3$ and $2g-2+n \leq 17$ or $(g,n) = (3,14)$, $\overline{\mathcal{M}}_{g,n}$ has no holomorphic forms since it is rationally connected. So on these pairs, $\mathcal{T}^{17}((g,n)) \cong H^{17,0}(\overline{\mathcal{M}}_{g,n})$ too. Now if (g,n) lie outside these cases, we argue using the operadic Arbarello-Cornalba induction. Note that $H^{17,0}(\overline{\mathcal{M}}_\Gamma) \cong \bigotimes_{v\in V(\Gamma)} H^{17,0}(\overline{\mathcal{M}}_{g(v),n(v)})$ by the Hodge-Künneth decomposition, since $H^{p,0}(\overline{\mathcal{M}}_{g(v),n(v)})$ is vanishing for $1 \leq p \leq 8$. Therefore, from the base cases, we may assume by induction on g and n that the inclusion $\mathcal{T}^\bullet \subset H^{\bullet,0}(\overline{\mathcal{M}})$ induces isomorphisms $\mathcal{T}^17((\Gamma)) \cong H^{17,0}(\overline{\mathcal{M}}((\Gamma)))$ for all stable graphs Γ of genus g with n legs and one or two edges.

Because of the assumption $2g-2+n > 17$, we can use the computation of the compact support cohomology of open moduli spaces to see that:

$$H^17(\mathrm{GK}_{g,n}^{17,0}) = H_c^{17}(\mathcal{M}_{g,n})^{17,0} \subset H_c^{17}(\mathcal{M}_{g,n}, \mathbb{C}) = 0.$$

The authors of [CLPW25] use a graph complex cohomology computation to show that $H^{18}(F\mathcal{T}^{17}) = 0$. Thus the natural maps from (2) in the operadic Arbarello-Cornalba induction lemma are trivially isomorphisms. We conclude that $\mathcal{T}^{17}((g,n)) \cong H^{17,0}(\overline{\mathcal{M}}_{g,n})$ for all (g,n).

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