The Thing (+ Structure)

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A heuristic and informal introduction to operads in homotopy theory.

This note concerns itself with structure. Let us place ourselves in a nice category of topological spaces $\mathscr{T}$—like the category of compactly generated (nondegenerately) based weak Hausdorff spaces[1]. Recall the **loop functor** $\Omega\colon \mathscr{T} \longrightarrow \mathscr{T}$ given by $\Omega Y = \underline{\mathrm{Hom}}_{\mathscr{T}}(S^1, Y)$ (which is right adjoint to the **suspension functor** $\Sigma\colon \mathscr{T} \longrightarrow \mathscr{T}$ given by $\Sigma Y = Y \wedge S^1$). Here is a seemingly simple question, what is the structure of (singlefold) **loop spaces**, namely those of the form $X = \Omega Y$? Let us take a brief digression here and discuss the subtle but important distinction between structure and property. Consider the definition of a monoid for example. A monoid can be defined as a pair $(M, \otimes)$, where M is a set and $\otimes\colon M \times M \longrightarrow M$ – such that $\otimes$ is associative on M, and there exists an $e \in M$ such that $e \otimes m = m \otimes e = m$ for all $m \in M$. This definition is of the form "**thing satisfying property**." Such definitions are nice and easy when there is little data that one has to keep track of. Eventually, when we are working with other objects (like singlefold, iterated, and infinite loop spaces), we will find that there is too much data to handle, so that "thing satisfying property" becomes a calculationally inaccessible definition. We would instead like to define these objects as "**thing equipped with structure**," or more formally as *algebras* over something we will define as *operads*. The operad specific to that construction, along with what it means to be an algebra over an operad, will completely encapsulate the structure of the object we try to define. We will soon see, as a first example of algebras over operads, that monoids and commutative monoids readily admit such descriptions. What we will present today is an introduction to the ideas central to the theory with a strong emphasis on the visual story that was the precursor to the operad. We will then complete the discussion by formally defining what operads are, and presenting some key examples in stable homotopy theory.

The first part of this note is based on a talk I delivered at USC based on the work of Stasheff—and inspired by a talk Matthew Niemiro gave at the University of Chicago in the summer of 2024. The second part of these notes are based on classical literature on operads including [May97] and [May72]. The final discussion on examples of operads in stable homotopy theory was written in the summer of 2025 while I spent time at Chicago thinking about operads in my own work on motivic homotopy theory.

§1 The Structure of Loop Spaces

We will attempt to investigate the algebraic structure of $X = \Omega Y$. The first thing to note is that X admits a binary operation by "one loop then the other." This is however

[1] Hereforth, by *spaces* we will mean either this category or its objects.

imprecise, since there are several distinct ways of concatenating two loops. Although any two such ways are homotopic to each other, we must make a particular choice to be precise. One can for example take $\cdot : X \times X \longrightarrow X$ to be the product given by $(a, b) \mapsto a \cdot b$ where $(a \cdot b)(t) = a(2t)$ for $0 \leq t \leq 1/2$ and $b(2t-1)$ for $1/2 \leq t \leq 1$. Often we will write ab to mean $a \cdot b$.

This product on X is certainly neither unital nor associative. What we will see (in some precise sense) is the following: the product on X is unital and associative upto all higher coherent homotopies. Indeed given a triple of loops $(a, b, c) \in X^3$, the loops $(ab)c$ and $a(bc)$ are generically distinct as the following schematic diagrams indicate.

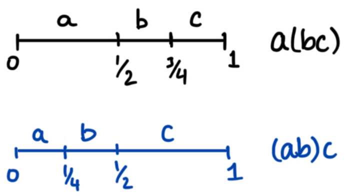


Figure 1: A schematic emphasizing the difference between the loops $a(bc)$ and $(ab)c$.

While the loops $a(bc)$ and $(ab)c$ are generically distinct, they are homotopic to each other. In other words, there is a path in X connecting the points $a(bc)$ and $(ab)c$. A priori of course, there are several paths connecting the two points in X. A prescription of such a path is a prescription of a homotopy between the loops $a(bc)$ and $(ab)c$. Therefore, to establish three-term associativity up to homotopy, we need to pick one such path for each triple $(a, b, c) \in X^3$. In other words, we need a map $I \times X^3 \longrightarrow X$ (satisfying some requirements, including those about the endpoints of the path, to be outlined later). We refer to the restriction of this map to $I \times \{(a, b, c)\}$ as the **associator** for (a, b, c) (and it is denoted $A_{a,b,c}$). Note that the associator $A_{a,b,c}$ will prescribe a path from $a(bc)$ to $(ab)c$. The homotopy between loops prescribed by the associators (or concatenations of) might sometimes be referred to as **1-homotopies**. We have so far established that the product on X is associative up to 1-homotopy.

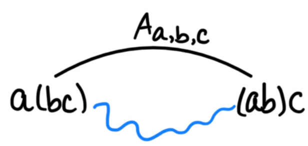
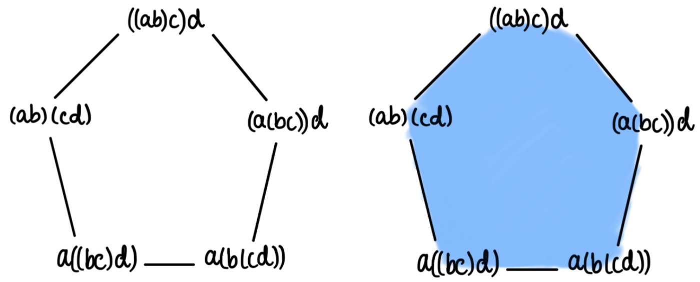


Figure 2: The chosen associator path connecting $a(bc)$ to $(ab)c$ and another arbitrary homotopy connecting the two loops.

Now to four-term associativity. Let $(a, b, c, d) \in X^4$ be a chosen quadruple. The diagram in Fig. 3 depicts all possible paranthesizations of $abcd$, and the associators connecting them. Again as before, each paranthesization of $abcd$ is distinct. The associators (under concatenation) connect each paranthesization through prescribed homotopies (or one might say 1-homotopies). However, there is a priori no unambiguous choice for a homotopy that connects two paranthesizations. For example, we see that there are two distinct homotopies connecting $(ab)(cd)$ and $(a(bc))d$. These distinct homotopies (think paths) however are themselves homotopic to each other. The prescription of a homotopy between any two 1-homotopies obtained by concatenation of associators from Fig. 3a is a 2D filling of the pentagon formed by the different paranthesizations. Again, there are a priori several fillings of the pentagon in Fig. 3a. Therefore, to establish four-term associativity up to homotopy, we need to pick one such filling for each quadruple $(a, b, c, d) \in X^4$. For reasons that may be mysterious at the moment, let $\mathscr{K}_4$ denote the filled polygon (a convex 2-polyhedron) with 5 vertices and 5 edges. We can then rephrase four-term associativity as a map $\mathscr{K}_4 \times X^4 \longrightarrow X$ (satisfying some requirements, including some about the boundary of the fillings). The homotopy between 1-homotopies prescribed by the $\mathscr{K}_4$ fillings (or concatenations of) might sometimes be referred to as **2-homotopies**. We have now established that the product on X is associative up to 2-homotopy.



(a) Different paranthesizations of a four-term product and connecting associators

(b) A 2D filling indicating a chosen higher homotopy between the associators

Figure 3: A diagram depicting four-term homotopy associativity

As you might have figured by now, we play the same game for five-term associativity. Let $(a, b, c, d, e) \in X^5$ be a chosen quintuple. This diagram is harder to draw by hand, so I am attaching a computer formatted version. Given two paranthesizations of $abcde$, there

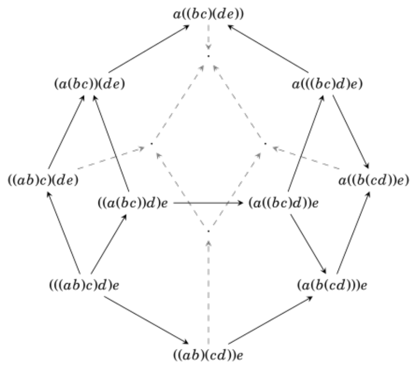


Figure 4: The five-term associativity diagram, where each pentagonal face is shaded in using the chosen 2-homotopies

is a priori no unambiguous choice for a 2-homotopy that connects the 1-homotopies that connect them. For example, we see that there are at least two 2-homotopies connecting the top and the middle 1-homotopies connecting $((a(bc))d)e$ and $(a((bc)d))e$ in Fig. 4. These distinct 2-homotopies however are themselves homotopic to each other. The prescription of a homotopy between any two 2-homotopies obtained by concatenation of the pentagonal fillings from Fig. 4 is a 3D filling of the polyhedron formed by the different paranthesizations. Again, there are a priori several fillings of the polyhedron in Fig. 4. Therefore, to establish five-term associativity up to homotopy, we need to pick one such fillinf for each quintuple $(a, b, c, d, e) \in X^5$. As before, let $\mathscr{K}_5$ denote the filled polyhedron (a 3-polyhedron to be specific) with 14 vertices and 9 faces. We can now rephrase five-term associativity as a map $\mathscr{K}_5 \times X^5 \longrightarrow X$ (satisfying some requirements, some again about the boundary of the fillings). The homotopy between 2-homotopies prescribed by the $\mathscr{K}_5$ fillings (or concatenations of) might sometimes be referred to as **3-homotopies**. We have now established that the product on X is associative up to 3-homotopy.

One repeats this story ad nauseum to see that at each level (for $n \geq 2$), to ensure n-term associativity we need a map $\mathscr{K}_n \times X^n \longrightarrow X$ (satisfying some additional structure requirements that we will outline later) where $\mathscr{K}_n$ is an $(n-2)$-polyhedron with i-cells indexed by all insertions of $(n-i-2)$ paranthesis pairs between n variables. The polyhedra $\mathscr{K}_n$ are called the **associahedra** (or Stasheff's associahedra). We will say that a space X is a **$\mathscr{K}$-space** if it admits structure maps $\mathscr{K}_n \times X^n \longrightarrow X$ as above. In [Sta63a] and [Sta63b], Stasheff proved that a connected space X has the weak homotopy type of a (singlefold) loop space iff X is a $\mathscr{K}$-space. Stasheff's associahedra $\mathscr{K}_n$ were historically one of the first examples of what are now known as **operads**, and his result was one of the first recognition principles for (singlefold) loop spaces. Using this story as motivation, we will formalize the notion of operads and concretely state Stasheff's result in the following section.

§2 Operads

This section is devoted to the study of operads and the structures they encode. We have already seen one example of an operadic structure in the previous section. In subsection 2.1, we will encounter the basic definitions one needs to get their hands dirty with the operadic machinery. Subsection 2.2 will introduce examples of E_n and E_∞-operads, tools that have now become essential in stable homotopy theory.

§2.1 Definitions

Operads are typically defined in two flavors: planar[2] (non-symmetric) and symmetric. Let us work in a symmetric monoidal category $\mathcal{C}$ with product $\otimes$ and unit object κ.

Definition 2.1 (Planar Operad)**.** A **planar operad** $\mathscr{C}$ consists of objects[3] $\mathscr{C}(j)$ for $j \geq 0$ (with $\mathscr{C}(0) = \kappa$ usually, and $\gamma\colon \mathscr{C}(0) \longrightarrow \mathscr{C}(0)$ seen as id_κ), a unit map $\eta\colon \kappa \longrightarrow \mathscr{C}(1)$ and composition maps

$$\gamma\colon \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1 + \cdots + j_k)$$

that are

1. unital in the sense that the following commute.

$$\begin{tikzcd} \mathscr{C}(j) \otimes \kappa^k \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_{\mathscr{C}(j)} \otimes \eta^k"'] & \mathscr{C}(j) \\ \mathscr{C}(j) \otimes \mathscr{C}(1)^k \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \begin{tikzcd} \kappa \otimes \mathscr{C}(j) \arrow[r, "\cong"] \arrow[d, "\eta \otimes \mathrm{id}_{\mathscr{C}(j)}"'] & \mathscr{C}(j) \\ \mathscr{C}(1) \otimes \mathscr{C}(j) \arrow[ur, "\gamma"'] & \end{tikzcd}$$

[2] The name planar comes from the combinatorial point of view of planar trees

[3] When $\mathcal{C}$ is concrete, it is common to interpret the elements of $\mathscr{C}(j)$ as j-ary operations. Hence the *oper* in operad.

2. associative in the sense that the following commutes for $k \geq 0$, $j_+ := \sum_{r=1}^{k} j_r$, $h_r = \sum_{q=1}^{j_r} i_{r,q}$, and $\sum_{t=1}^{j_+} i_t = i_+ = \sum_{r=1}^{k} h_r$.

$$\begin{tikzcd}
\mathscr{C}(k) \otimes (\otimes_{r=1}^{k} \mathscr{C}(j_r)) \otimes (\otimes_{t=1}^{j_+} \mathscr{C}(i_t)) \arrow[r, "\gamma \otimes \mathrm{id}"] \arrow[dd, "\mathrm{id} \otimes \mathrm{shuffle}"] & \mathscr{C}(j) \otimes (\otimes_{t=1}^{j_+} \mathscr{C}(i_t)) \arrow[d, "\gamma"] \\
& \mathscr{C}(i_+) \\
\mathscr{C}(k) \otimes (\otimes_{r=1}^{k} (\mathscr{C}(j_r) \otimes (\otimes_{q=1}^{j_r} \mathscr{C}(i_{r,q})))) \arrow[r, "\mathrm{id} \otimes \gamma^k"'] & \mathscr{C}(k) \otimes (\otimes_{r=1}^{k} \mathscr{C}(h_r)) \arrow[u, "\gamma"']
\end{tikzcd}$$

For now, by *operad* we will mean a planar operad.

Example 2.2 (The Planar Associative Operad)
The most canonical example of a planar operad (and as we will see later, a planar A_∞-operad) is the associative operad defined by $\mathscr{M}(j) = \kappa$ for all $j \geq 0$ (with η the identity, and γ the obvious identifications). Sometimes $\mathscr{M}$ is also written as Ass, Assoc, or **ASSO**.

Remark 2.3. We see that the Stasheff associahedra form an operad according to Defn. 2.1. We will denote this operad by $\mathscr{K}$ with $\mathscr{K}(j) = \mathscr{K}_j$.

We now define what it means to be an algebra over a (planar) operad. Hereforth, we restrict ourselves to **unital** operads, i.e. operads $\mathscr{C}$ for which $\mathscr{C}(0) = \kappa$

Definition 2.4 (Algebra over a Planar Operad)**.** Let $\mathscr{C}$ be an operad. A $\mathscr{C}$-algebra is an object $X \in \mathcal{C}$ equipped with maps

$$\theta \colon \mathscr{C}(j) \otimes X^j \longrightarrow X$$

for $j \geq 0$ that are unital and associative, as outlined below. The map $\theta \colon \mathscr{C}(0) \otimes \kappa \longrightarrow X$ gives X a base element $\kappa \longrightarrow X$ (we will later think of this as the unit for the product on X). If X is already based, we ask that this base element prescription coincides with the base element of X. The map θ is referred to as an **action** of $\mathscr{C}$ on X.

1. When we say θ is unital, we mean that the following commutes.

$$\begin{tikzcd}
\kappa \otimes X \arrow[r, "\cong"] \arrow[d, "\eta \otimes \mathrm{id}_X"'] & X \\
\mathscr{C}(1) \otimes X \arrow[ur, "\theta"'] &
\end{tikzcd}$$

2. When we say θ is associative, we mean that the following commutes.

$$\begin{tikzcd}
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes X^{j_+} \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[dd, "\mathrm{id}\otimes\mathrm{shuffle}"'] & \mathscr{C}(j_+) \otimes X^{j_+} \arrow[d, "\theta"] \\
& X \\
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes X^{j_1} \otimes \cdots \otimes \mathscr{C}(j_k) \otimes X^{j_k} \arrow[r, "\mathrm{id}\otimes\theta^k"'] & \mathscr{C}(k) \otimes X^k \arrow[u, "\theta"']
\end{tikzcd}$$

Remark 2.5. In the particular case that $\mathcal{C}$ is the category of spaces, with the cartesian monoidal structure, we call a $\mathscr{C}$-algebra a $\mathscr{C}$-space.

Example 2.6 (Monoids)

Note that algebras over the planar associative operad are precisely the monoidal objects in $\mathcal{C}$! In particular, for the case of $\mathcal{C} = \mathbf{Set}$, a monoid is a set M equipped with an $\mathscr{M}$-algebra structure.

Before we present Stasheff's result, we define planar A_∞-operads.

Definition 2.7 (Planar A_∞-operad)**.** Let $\mathcal{C}$ be our category of spaces. We say that an operad $\mathscr{C}$ is a (planar) **A_∞-operad** if each $\mathscr{C}(n)$ is contractible[4]. An space[5] $X \in \mathcal{C}$ is said to be an **A_∞-space** if it is a $\mathscr{C}$-algebra for some A_∞-operad $\mathscr{C}$.

The intuition is that A_∞-operads encapsulate the structure of homotopy coherent associativity.

Example 2.8 (Stasheff Associahedra as an A_∞-operad)

It is trivial that $\mathscr{M}$ itself is an A_∞-operad – it describes the structure of strict (on-the-nose) associativity. Stasheff's associahedra operad $\mathscr{K}$ is a very important non-trivial example of an A_∞-operad. Indeed by construction, each $\mathscr{K}_n$ is contractible.

Theorem 2.9 (Stasheff's Recognition Principle)

A connected space X is weak homotopy equivalent to ΩY for some space Y iff X is an A_∞-space.

Note that allowing arbitrary A_∞-operads above is not a problem since (by pulling back) any A_∞-space is also a $\mathscr{K}$-space. In more generality, we can say the following about spaces that are not necessarily connected. We choose to not define what *grouplike* or *group completion* means here – we just state the theorem for intrigue.

[4] In categories where the notion of homotopy equivalences is not readily available, we resort to using homology equivalences and chain complexes. In a broader sense, an A_∞-operad enriched in $\mathcal{C}$ is a free resolution of the associative operad enriched in $\mathcal{C}$.

[5] In the more general categorical context, algebras over an A_∞-operad are called A_∞-objects or A_∞-algebras.

Theorem 2.10 (Stasheff's Recognition Principle Contd.)

For each A_∞-space X, there exists a space Y and a group completion $X \longrightarrow \Omega Y$. It follows that if X is grouplike, then X has the weak homotopy type of a loop space.

Now we introduce the symmetric operads. Going forward, these are the only kinds of operads we will discuss – and hence we simply refer to them as operads. Let us also go back to $\mathcal{C}$ being an arbitrary symmetric monoidal category.

Definition 2.11 (Operad)**.** Let $\mathscr{C}$ be a planar operad as in Defn. 2.1. We say that $\mathscr{C}$ is an **operad** if each $\mathscr{C}(j)$ is equipped with a right Σ_j action, and the composition map γ is equivariant in the following sense. Let $\sigma \in \Sigma_k$ and $\tau_r \in \Sigma_{j_r}$. We write $\sigma(j_1, \cdots, j_k) \in \Sigma_j$ to mean the permutation of k blocks of letters induced by σ (considered as a permutation of $j = \sum_{r=1}^k j_r$ letters), and $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_j$ to mean the block sum. Equivariance refers to the commutativity of the following diagrams.

$$\begin{tikzcd}
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\gamma"'] & \mathscr{C}(k) \otimes \mathscr{C}(j_{\sigma(1)}) \otimes \cdots \otimes \mathscr{C}(j_{\sigma(k)}) \arrow[d, "\gamma"] \\
\mathscr{C}(j) \arrow[r, "\sigma(j_{\sigma(1)},\ldots,j_{\sigma(k)})"'] & \mathscr{C}(j)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \arrow[r, "\mathrm{id}\otimes\tau_1\otimes\cdots\otimes\tau_k"] \arrow[d, "\gamma"'] & \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \arrow[d, "\gamma"] \\
\mathscr{C}(j) \arrow[r, "\tau_1\oplus\cdots\oplus\tau_k"'] & \mathscr{C}(j)
\end{tikzcd}$$

Example 2.12 (The Associative and Commutative Operads)

Suppose $\mathcal{C}$ is closed under finite colimits. For a finite set S, denote by $\kappa[S] \in \mathcal{C}$ the coproduct of a copy of κ for each element of S. The **associative operad** enriched in $\mathcal{C}$ is defined to be $\mathscr{M}(j) = \kappa[\Sigma_j]$, and the **commutative operad** enriched in $\mathcal{C}$ is defined by $\mathscr{N}(j) = \kappa$. As noted before, $\mathscr{M}$ might sometimes be denoted by Ass, Assoc, or **ASSO**.

Example 2.13 (The Endomorphism Operad)

Let X be a space. We can define an operad enriched over spaces $\mathscr{E}_X$ called the **endomorphism operad** of X as follows. We define $\mathscr{E}_X(j)$ be the space of based maps $X^j \longrightarrow X$. Define the unit map $\eta \colon * \longrightarrow \mathscr{E}_X(1)$ to have image id_X. Define the composition map by $\gamma(f; g_1, \ldots, g_k) = f(g_1 \times \cdots \times g_k)$. Finally, for $\sigma \in \Sigma_j$ and $f \in \mathscr{E}_X(j)$ we can write $(f\sigma)(y) = f(\sigma y)$ for all $y \in X^j$. Here $\sigma(x_1, \ldots, x_k) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)})$. The endomorphism operad was one of the first motivating examples of operads.

Just as before, we say an operad $\mathscr{C}$ is **unital** if $\mathscr{C}(0) = \kappa$ – and we restrict ourselves to unital operads.

Definition 2.14 (Algebra over an Operad)**.** Let $\mathscr{C}$ be an operad. A **$\mathscr{C}$-algebra** is an algebra over $\mathscr{C}$ considered as a planar operad, with the restriction that the action maps $\theta\colon \mathscr{C}(j) \otimes X^j \longrightarrow X$ are all equivariant with respect to the symmetric group action. By equivariance we mean the commutativity of the following diagram.

$$\begin{tikzcd}
\mathscr{C}(j) \otimes X^j \arrow[rr, "\sigma\otimes\sigma^{-1}"] \arrow[dr, "\theta"'] & & \mathscr{C}(j) \otimes X^j \arrow[dl, "\theta"] \\
& X &
\end{tikzcd}$$

Example 2.15 (Monoids and Commutative Monoids)

The $\mathscr{M}$-algebras in $\mathcal{C}$ are precisely the monoidal objects in $\mathcal{C}$. Similarly, the $\mathscr{N}$-algebras in $\mathcal{C}$ are the commutative monoidal objects in $\mathcal{C}$. Returning to the examples in the introduction, for $\mathcal{C} = \mathbf{Set}$, a monoid (commutative monoid) is a set M equipped with an $\mathscr{M}$-algebra ($\mathscr{N}$-algebra) structure.

There is an analogous definition of an A_∞-operad, but there is some vocabulary to introduce first.

Definition 2.16 (Morphism of Operads)**.** A **morphism of operads** $\varphi\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a sequence of Σ_j-equivariant maps $\varphi_j\colon \mathscr{C}(j) \longrightarrow \mathscr{D}(j)$ that is compatible with the unit and composition maps of $\mathscr{C}$ and $\mathscr{D}$. By compatibility, we mean that the following diagrams must commute.

$$\begin{tikzcd}
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \arrow[r, "\gamma"] \arrow[d, "\varphi_k\otimes\varphi_{j_1}\otimes\ldots\otimes\varphi_{j_k}"'] & \mathscr{C}(j_+) \arrow[d, "\varphi_{j_+}"] \\
\mathscr{D}(k) \otimes \mathscr{D}(j_1) \otimes \cdots \otimes \mathscr{D}(j_k) \arrow[r, "\gamma"'] & \mathscr{D}(j_+)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C}(1) \arrow[rr, "\varphi_1"] & & \mathscr{D}(1) \\
& \kappa \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] &
\end{tikzcd}$$

The following in some sense is a cofibrancy condition for operads.

Definition 2.17 (Σ-free Operads)**.** We say that an operad $\mathscr{C}$ is **Σ-free** if Σ_j freely acts on $\mathscr{C}(j)$ for each j.

We now define what it means to have an operad $\mathscr{C}$ over a discrete operad $\mathscr{D}$.

Definition 2.18 (Path Component Operad)**.** Let $\mathscr{C}$ be an operad enriched over spaces. Denote by $\pi_0\mathscr{C}$ the **Set**-enriched operad given by the sequence of sets $(\pi_0\mathscr{C})(j) = \pi_0(\mathscr{C}(j))$.

Definition 2.19 (Augmentation)**.** We say that an operad $\mathscr{C}$ enriched over spaces is an operad over a discrete **Set**-enriched operad $\mathscr{D}$ if it is equipped with a morphism of operads $\epsilon\colon \mathscr{C} \longrightarrow \mathscr{D}$ such that $\pi_0\epsilon\colon \pi_0\mathscr{C} \longrightarrow \mathscr{D}$ is an isomorphism. In this case, we refer to ϵ as the **augmentation** of $\mathscr{C}$.

Finally, we have the notion of a local equivalence between two operads.

Definition 2.20 (Local Equivalence)**.** We say that a map of operads (enriched over spaces) $\varphi\colon \mathscr{C} \longrightarrow \mathscr{C}'$ is a **local equivalence** if each $\varphi_j\colon \mathscr{C}(j) \longrightarrow \mathscr{C}'(j)$ is a homotopy equivalence. We say that φ is a **local Σ-equivalence** if each φ_j is also Σ_j-equivariant.

We are now ready to define an A_∞-operad.

Definition 2.21 (A_∞-operad)**.** An **A_∞-operad** is a Σ-free operad over $\mathscr{M}$ such that $\epsilon\colon \mathscr{C} \longrightarrow \mathscr{M}$ is a local Σ-equivalence. We say that a space X is an **A_∞-space** if it is a $\mathscr{C}$-space over some A_∞ operad $\mathscr{C}$.

It is in this sense that A_∞-operads are resolutions of the associative operad. A_∞-operads are also sometimes referred to as E_1-operads. Recall that A_∞-operads record algebraic structures that are "associative up to higher coherent homotopy." There is a similar notion of an E_∞-operad, built as a resolution of $\mathscr{N}$, that records algebraic structures that are "associative and commutative up to higher coherent homotopy."[6]

Definition 2.22 (E_∞-operad)**.** An **E_∞-operad** is a Σ-free operad over $\mathscr{N}$ such that $\epsilon\colon \mathscr{C} \longrightarrow \mathscr{N}$ is a local equivalence. We say that a space X is an **E_∞-space** if it is a $\mathscr{C}$-space over some E_∞ operad $\mathscr{C}$.

There is in fact a simpler and equivalent definition of an E_∞-operad. An operad $\mathscr{C}$ enriched over spaces is an E_∞-operad iff it is Σ-free and each $\mathscr{C}_n$ is contractible. In practice, this is how one checks that an operad is an E_∞-operad[7]. We will see classical examples of E_∞-operads and E_n-operads in the next section.

There is a crucial aspect to the theory of operads that we have kept hidden all along – the -*ad* in operad. Recall the definition of a monad.

Definition 2.23 (Monad)**.** Let $\mathcal{C}$ be a category. We say that a functor $\mathbb{C}\colon \mathcal{C} \longrightarrow \mathcal{C}$ is a **monad** if it is equipped with natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \longrightarrow \mathbb{C}$ and $\mu\colon \mathbb{C}\mathbb{C} \longrightarrow \mathbb{C}$ such that the following diagrams commute.

$$\begin{tikzcd}
\mathbb{C}\mathbb{C}\mathbb{C} \arrow[r, "\mathbb{C}\mu"] \arrow[d, "\mu"'] & \mathbb{C}\mathbb{C} \arrow[d, "\mu"] \\
\mathbb{C}\mathbb{C} \arrow[r, "\mu"'] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathbb{C}\eta"] \arrow[d, "\eta"'] \arrow[dr, equal] & \mathbb{C}\mathbb{C} \arrow[d, "\mu"] \\
\mathbb{C}\mathbb{C} \arrow[r, "\mu"'] & \mathbb{C}
\end{tikzcd}$$

We say that η is the **unit** of $\mathbb{C}$, and μ is the **product** of $\mathbb{C}$.

> **Remark 2.24.** Note that a monad on $\mathcal{C}$ is just a monoidal object in the endofunctor category $[\mathcal{C}, \mathcal{C}]$.

We can also talk about morphisms between monads on a category $\mathcal{C}$.

Definition 2.25 (Morphism of Monads)**.** Let $\mathbb{C}$ and $\mathbb{C}'$ be monads on $\mathcal{C}$ (with unit η, η' and product μ, μ'). We say that a natural transformation $\psi\colon \mathbb{C} \longrightarrow \mathbb{C}'$ is a **morphism of monads** if the following diagrams commute for each $X \in \mathcal{C}$.

$$\begin{tikzcd}
\mathbb{C}\mathbb{C}X \arrow[r, "\psi^2"] \arrow[d, "\mu"'] & \mathbb{C}'\mathbb{C}'X \arrow[d, "\mu'"] \\
\mathbb{C}X \arrow[r, "\psi"'] & \mathbb{C}'X
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, "\eta_X"'] \arrow[dr, "\eta'_X"] & \\
\mathbb{C}X \arrow[rr, "\psi_X"'] & & \mathbb{C}'X
\end{tikzcd}$$

In light of the above definition, we can talk about $\mathrm{Mnd}(\mathcal{C})$ the category of monads on $\mathcal{C}$.

[6] The A in A_∞ presumably stands for *associativity* – since the structure it describes is *homotopy associative*. The E in E_∞ presumably stands for *everything* – since the structure it describes is *homotopy everything*.

[7] See [Ste79] for example.

Example 2.26 (Adjunction Monads)

Consider an adjunction $\mathcal{C} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathcal{D}$. The composite $\Gamma = \Omega\Sigma$ is actually a monad, with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \longrightarrow \Gamma$ (the unit of the adjunction), and product $\mu\colon \Gamma\Gamma \longrightarrow \Gamma$ given by $\Omega\varepsilon$ (where ε is the counit of the adjunction, $\varepsilon\colon \Sigma\Omega \longrightarrow \mathrm{id}_{\mathcal{D}}$). We refer to Γ as the **adjunction monad** for the adjunction (Σ, Ω).

Just as monoids and groups act on sets, monads act on objects of $\mathcal{C}$.

Definition 2.27 (Algebra over a Monad)**.** Let $\mathbb{C}$ be a monad on $\mathcal{C}$. We say that $X \in \mathcal{C}$ is a **$\mathbb{C}$-algebra** if it is equipped with a map $\theta\colon \mathbb{C}X \longrightarrow X$ such that the following diagrams commute.

$$\begin{tikzcd} \mathbb{C}\mathbb{C}X \arrow[r, "\mathbb{C}\theta"] \arrow[d, "\mu_X"'] & \mathbb{C}X \arrow[d, "\theta"] \\ \mathbb{C}X \arrow[r, "\theta"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & \mathbb{C}X \arrow[d, "\theta"] \\ & X \end{tikzcd}$$

We refer to θ as the **$\mathbb{C}$-action** on X (or simply the **action**). Let X and Y be $\mathbb{C}$-algebras (with actions θ and ϕ). We say that a map $f\colon X \longrightarrow Y$ in $\mathcal{C}$ is a **map of $\mathbb{C}$-algebras** if the following diagram commutes.

$$\begin{tikzcd} \mathbb{C}X \arrow[r, "\mathbb{C}f"] \arrow[d, "\theta"'] & \mathbb{C}Y \arrow[d, "\phi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We will denote by $\mathbb{C}[\mathcal{C}]$ the category of $\mathbb{C}$-algebras.

Example 2.28 (The Free $\mathbb{C}$-Algebra)

Note that for every $X \in \mathcal{C}$, $\mathbb{C}X$ is a $\mathbb{C}$-algebra with action $\mu_X\colon \mathbb{C}\mathbb{C}X \longrightarrow \mathbb{C}X$. We say that $\mathbb{C}X$ is the **free $\mathbb{C}$-algebra** on X.

Example 2.29 (Adjunction Monads contd.)

Let us go back to the adjunction monad Γ for an adjunction (Σ, Ω) between $\mathcal{C}$ and $\mathcal{D}$. It is formal that ΩY has a natural Γ-algebra structure for every $Y \in \mathcal{D}$. The Γ-action on ΩY is simply provided by $\Omega\varepsilon_Y\colon \Gamma\Omega Y = \Omega\Sigma\Omega Y \longrightarrow \Omega Y$. We have just shown that the right adjoint Ω actually takes values in $\Gamma[\mathcal{C}]$. Seen as a functor into this subcategory $\Gamma[\mathcal{C}]$, Ω is written as $\Omega_\Gamma\colon \mathcal{D} \longrightarrow \Gamma[\mathcal{C}]$. This functor is actually also a right adjoint! Beck's monadicity theorem provides necessary and sufficient conditions for Ω_Γ to be an equivalence.

A key aspect of the theory of operads is that each operad $\mathscr{C}$ gives rise to a monad $\mathbb{C}$ such that $\mathscr{C}$-algebras are precisely $\mathbb{C}$-algebras.

Definition 2.30. Let $\mathcal{C}$ be a symmetric monoidal category. Let $\mathscr{C}$ be an operad enriched in $\mathcal{C}$. Define a functor $\mathbb{C}\colon \mathcal{C} \longrightarrow \mathcal{C}$ by $X \mapsto \mathbb{C}X$ where $\mathbb{C}X$ is the following coequalizer.

$$\coprod_{n\in\mathbb{N}} \mathscr{C}(n) \otimes \kappa[\Sigma_n] \otimes X^n \rightrightarrows \coprod_{n\in\mathbb{N}} \mathscr{C}(n) \otimes X^n \longrightarrow \mathbb{C}X$$

The maps involved in the coequalizer utilize the right action of Σ_n on $\mathscr{C}(n)$ and the left action of Σ_n on n-tuples in X. There are several equivalent definitions of $\mathbb{C}$. For example, a neater definition using the tensor product of functors reads $\mathbb{C}X = \mathscr{C}(*) \otimes_\Lambda X^* = \int^{n\in\Lambda} \mathscr{C}(n) \otimes X^n$. Here Λ is the $\mathcal{C}$-enriched permutation category with $\mathrm{Ob}(\Lambda) = \mathbb{N}$ and $\Lambda(m,n) = \emptyset$ if $m \neq n$, $\kappa[\Sigma_n]$ otherwise. We view $\mathscr{C}$ as a functor $\mathscr{C}(*)\colon \Lambda^{op} \longrightarrow \mathcal{C}$ and X^* as a functor $X^*\colon \Lambda \longrightarrow \mathcal{C}$.

Claim 2.31 — $\mathbb{C}$ admits a monad structure so that its algebras are precisely the algebras over $\mathscr{C}$.

Proof. The unit map $\eta\colon \mathrm{id}_{\mathcal{C}} \longrightarrow \mathbb{C}$ of $\mathbb{C}$ is the one induced by the unit map of the operad $\eta \otimes X\colon \kappa \otimes X \longrightarrow \mathscr{C}(1) \otimes X$. The product map $\mu\colon \mathbb{C}\mathbb{C} \longrightarrow \mathbb{C}$ is induced by the Σ-equivariant composition map of $\mathscr{C}$ as follows.

$$\begin{array}{c} \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes X^{j_1} \otimes \cdots \otimes \mathscr{C}(j_k) \otimes X^{j_k} \\ \Big\downarrow \text{shuffle} \\ \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes X^{j_+} \\ \Big\downarrow \gamma \\ \mathscr{C}(j_+) \otimes X^{j_+} \end{array}$$

The associativity and unitality diagrams for $\mathscr{C}$ translate precisely into those for $\mathbb{C}$. As for algebras, note that the map $\theta\colon \mathbb{C}X \longrightarrow X$ that defines a $\mathbb{C}$-algebra structure on X provides maps $\theta_j\colon \mathscr{C}(j) \otimes_{\kappa[\Sigma_j]} X^j \longrightarrow X$. The maps θ_j indeed define a $\mathscr{C}$-algebra structure on X – the required commutative diagrams following from the diagrams in the definition of a $\mathbb{C}$-algebra. Conversely, each $\mathscr{C}$-algebra comes with maps $\theta_j\colon \mathscr{C}(j) \otimes_{\kappa[\Sigma_j]} X^j \longrightarrow X$. These maps define a $\mathbb{C}$-action $\theta\colon \mathbb{C}X \longrightarrow X$ on X – with the required diagrams again following from the ones for the operad action. $\square$

Remark 2.32. We have shown that the identity endofunctor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{C}$ restricts to an equivalence of subcategories between $\mathbb{C}[\mathcal{C}]$ and the category of $\mathscr{C}$-algebras (one may denote this subcategory by $\mathscr{C}[\mathcal{C}]$). We have technically not defined a morphism of $\mathscr{C}$-algebras, but it is clear what such a definition would look like. A map $f\colon X \longrightarrow Y$ between $\mathscr{C}$-algebras is called a morphism of $\mathscr{C}$-algebras if the following diagram commutes.

$$\begin{tikzcd} \mathscr{C}(j) \otimes X^j \arrow[r, "\mathrm{id} \otimes f^j"] \arrow[d, "\theta"'] & \mathscr{C}(j) \otimes Y^j \arrow[d, "\theta'"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Example 2.33 (The Free Monoid)

Consider the associative operad $\mathscr{M}$ and the commutative operad $\mathscr{N}$. The monad $\mathbb{M}$ associated to $\mathscr{M}$ is just the free monoid endofunctor! Namely, $\mathbb{M}X$ is the free monoidal object on $X \in \mathcal{C}$. For $\mathcal{C} = \mathbf{Set}$, this is just the free monoid on X. Similarly, the monad $\mathbb{N}$ associated to $\mathscr{N}$ is just the free commutative monoid endofunctor. $\mathbb{N}X$ is the free commutative monoidal object on $X \in \mathcal{C}$. For $\mathcal{C} = \mathbf{Set}$, this is the free commutative monoid on X. An $\mathbb{M}$-algebrais an object X equipped with a map $\mathbb{M}X \longrightarrow X$ that is compatible with the product $\mathbb{M}\mathbb{M} \longrightarrow \mathbb{M}$. This provides X with an associative unital product $X^2 \longrightarrow X$, making X monoidal. The same story applies for $\mathbb{N}$-algebras. With the converse stories also holding true—we see that $\mathbb{M}$-algebras in $\mathcal{C}$ are the monoidal objects, and the $\mathbb{N}$-algebras in $\mathcal{C}$ are the commutative monoidal objects—as expected.

We must note that not all monads arise from operads. For example, it is generally the case that adjunction monads never come from operads. The operadic monads form a calculationally convenient subcollection of monads, and for this reason they are prevalent in various areas of mathematics.

Perhaps regrettably, my biases shear this note in the direction of stable homotopy theory—but what is writing if not biased. In that spirit, the next subsection provides some important examples from the robust operadic machinery in stable homotopy theory.

§2.2 Examples

Will talk about the little n-cubes, the little n-disks, the Barratt-Eccles, and the Steiner operads.

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