ON THE IRRATIONALITY OF A CUBIC THREEFOLD

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1. Introduction

In this note, we concern ourselves with a vital result in the birational geometry of threefolds. Recall that a *rational map* f from a variety X to a variety Y (understood to be irreducible and defined over an algebraically closed field $\Bbbk$, say $\mathbb{C}$) is a morphism from a non-empty Zariski open (hereafter referred to as Z-open) $U \subset X$ to Y (denoted $f : X \dashrightarrow Y$). Locally, a rational map is given by a family of rational functions. A rational map $f : X \dashrightarrow Y$ is said to be *birational* if it has a rational inverse $Y \dashrightarrow X$, and if such an f exists, we say X and Y are *birationally equivalent*. Finally, we say that X is *rational* if it is birationally equivalent to $\mathbb{P}^n$ for some n. One particularly illuminating example of a rational variety is that of a smooth degree 2 hypersurface (i.e. a quadric) of dimension n. A smooth quadric hypersurface X is always rational, since stereographic projection provides a birational map from X to $\mathbb{P}^n$.

Our goal in particular is to study the rationality of the (non-singular) cubic threefold, that is to say, the hypersurface of degree three in complex projective four-space. Particularly, we will develop the notion of the *intermediate Jacobian* of a threefold V – and using its properties, develop a birational invariant for threefolds as we discuss the proof sketch of the following theorem regarding V (first presented in [CG72]).

Theorem 1.1. *Let V be a non-singular cubic threefold. V is not rational.*

The use of the intermediate Jacobian to study algebraic curves on the threefold is in some sense analogous to the use of Jacobian varieties to study divisors on an algebraic curve, introduced in 2. This note is heavily inspired by a lecture delivered by Claire Voisin at the Collège de France in 2019 (see 1:06:12 and onward in [Voi19]) as part of her series on birational invariants. We also refer to [CG72] and [Voi02] for various elements of this exposition.

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2. Intermediate Jacobian of Threefolds

Let X be a smooth projective 3 dimensional variety on $\mathbb{C}$ such that $H^{3,0}(X) \cong 0$ and $H^1_B(X,\mathbb{Z}) \cong 0$ (here $H^1_B(X,\mathbb{Z})$ refers to the first integral Betti/singular cohomology group of X)[1]. Note in particular that a (nonsingular) cubic threefold, and $\mathbb{P}^3$ are both examples of such a variety. See subsection 17.3 in [Ara12] for details on the Hodge structure of a projective hypersurface. That $H^{3,0}(X) \cong 0$ for a cubic threefold actually follows from a quick computation with the adjunction formula. This is because the adjunction formula tells us that the canonical line bundle of a cubic threefold has no sections. Having access to the full Hodge decomposition of $H^3_B(X,\mathbb{C})$ for a cubic threefold turns out to be helpful soon. Also, by the Lefschetz hyperplane theorem (see p.156 in [GH94]), we get $H^1(X,\mathbb{Z}) \cong 0$. Hereforth, we use *threefold* to mean an X satisfying the hypotheses introduced above. For such an X, note that the Hodge decomposition of the middle cohomology group reads as follows: $H^3(X,\mathbb{C}) \cong H^{1,2}(X) \oplus H^{2,1}(X)$, where $H^{1,2}(X) = \overline{H^{2,1}(X)}$. Now note that since $H^3_B(X,\mathbb{R}) \cong H^{1,2}(X)$ as $\mathbb{R}$-vector spaces, ($H^3(X,\mathbb{Z})$ (modulo torsion)) $\subset H^3(X,\mathbb{R})$. We may then define the *intermediate Jacobian* of X to be the complex torus obtained by quotienting out this lattice from $H^{1,2}(X)$, namely $J(X) := H^{1,2}(X)/(H^3(X,\mathbb{Z})$ (mod. torsion)).

We now note that $J(X)$ is an abelian variety since it admits a non-degenerate intersection form (more particularly what's known as a *Riemann form*) $\langle\, , \,\rangle \in \Lambda^2 H_1(J(X),\mathbb{Z})^\vee$ satisfying the *Riemann conditions* (see [GH94] pp. 200-207 for a review of these conditions). First note that since $J(X)$ was defined by quotienting out a $\mathbb{C}$-vector space (isomorphic to $\mathbb{C}^g$ for some g) by the lattice $H^3(X,\mathbb{Z})$, we have $H_1(J(X),\mathbb{Z}) \cong H^3(X,\mathbb{Z})$. Therefore to obtain an intersection form on the lattice $H_1(J(X),\mathbb{Z})$ it suffices to obtain one on $H^3(X,\mathbb{Z})$. But there is a natural non-degenerate intersection form (by Poincaré duality) on $H^3(X,\mathbb{Z})$ given by $\langle\alpha,\beta\rangle = \int_{X_{\mathrm{an}}} \alpha \smile \beta$ (where X_{an} is the analytification of X). In this way, we obtain a *polarization*[2] on the complex torus $J(X)$, making it an abelian variety. This bilinear form on $H_1(J(X),\mathbb{Z})$ is also *unimodular* (i.e., considered as an integer matrix, it has determinant ± 1, or equivalently it is invertible) by Poincaré duality, making $J(X)$ a *principally polarized abelian variety* (sometimes referred to as a *ppav*). Namely, a principally polarized abelian variety is a pair (A,θ) (where A is an abelian variety with polarization $\theta \in \mathrm{Pic}(A)$) such that the bilinear intersection form $\langle\, , \,\rangle_\theta$

[1] We drop the B subscript on all Betti (co)homology groups since the (co)homology we are working with will be clear from context. Unless otherwise mentioned, we will be working with Betti (co)homology throughout this work.

[2] Note that in [GH94] a polarization is defined as a class in $H^2(J(X),\mathbb{Z})$ satisfying the Riemann conditions, this is effectively the same thing as what we've done here since $H^2(J(X),\mathbb{Z}) \cong \Lambda^2 H_1(J(X),\mathbb{Z})^\vee$ by the cup product.

defined by θ is unimodular. Note that this somewhat new notation is justifiable by the following argument. Previously we had defined a polarization on $J(X)$ to be (equivalently) a bilinear form in $\Lambda^2 H_1(J(X),\mathbb{Z})^\vee$ or a class in $H^2(J(X),\mathbb{Z})$. The first Chern class gives us an isomorphism $H^2(J(X),\mathbb{Z}) \cong \mathrm{Pic}^0(J(X))$. So a (principally) polarized abelian variety can really be thought of as a pair (A,θ) with A a complex torus, and $\theta \in \mathrm{Pic}^0(A)$ a (principal) polarization on A.

As mentioned in 1, the intermediate Jacobian is a generalization of the *Jacobian variety* of an algebraic curve which is defined as follows. Let C be a non-singular algebraic curve of genus g. Each element $[\gamma] \in H_1(C,\mathbb{Z})$ defines a map on the space of global sections of holomorphic 1-forms on C by integration $\omega \mapsto \int_\gamma \omega$, and in this way $H_1(C,\mathbb{Z})$ may be considered as a lattice in $H^{0,1}(C) \cong H^0(C,\Omega^1_C)^\vee$. We can therefore define the Jacobian of C to be the complex torus defined as $J(C) = H^{0,1}(C)/(H_1(C,\mathbb{Z})$ (modulo torsion)) (note that this torus is manifestly of the form $\mathbb{C}^g/\Lambda$ as seen from the Hodge diamond of C). As before, we obtain a unimodular intersection pairing (satisfying the Riemann conditions) on $H^1(J(C),\mathbb{Z})$ from the unimodular intersection pairing on $H_1(C,\mathbb{Z})$ thanks to Poincaré duality. In this way, $J(C)$ is a principally polarized abelian variety[3]. To see the vital role played by the Jacobian variety in the birational geometry of curves, the reader might wish to refer to Torelli's theorem which states that the Jacobian variety of a nonsingular algebraic curve (as a ppav) uniquely characterizes the curve up to isomorphism (see [GH94] p.359 for details and an exposition on its significance).

Unfortunately, the intermediate Jacobian $J(X)$ of a threefold X with the principal polarization discussed above is not a birational invariant. This can be seen by considering the blow-up $\hat{X}_C$ of X along a curve $C \subset X$. In this case, the intermediate Jacobian of $\hat{X}_C$ is given by the direct sum of the intermediate Jacobian of X and the Jacobian of C (see the proof of Thm. 2.2 for an explanation obtained by reading off the Hodge diamond of the blow-up). Nonetheless, this hints at a birational invariant for the class of threefolds considered in this section. Thm. 2.2 shows that one can construct a birational invariant of threefolds by first decomposing the intermediate Jacobian into a product of irreducible principally polarized abelian varieties, and removing those abelian varieties that are Jacobians of curves. To go through with this procedure, we use the following result, a corollary of Poincaré's complete reducibility theorem (not proven here, but see [Swi74] Thm. 34. Cor. 3 for details).

Theorem 2.1. *Let (A,θ) be a principally polarized abelian variety. Then (A,θ) decomposes in a unique way (up to permutation of factors) as:*

$$A = \prod_i A_i$$

$$\theta = \sum_i \mathrm{pr}_i^* \theta_i$$

where each (A_i,θ_i) is an irreducible principally polarized abelian variety, and $\mathrm{pr}_i : A \to A_i$ are the usual projection maps onto the i-th factor.

[3] In fact, by the Abel-Jacobi theorem, it is identified with the Picard variety of degree 0 divisors on C modulo linear equivalence (see [GH94] pp. 227 for details).

Thm. 2.1 allows us to introduce the following birational invariant for threefolds.

Theorem 2.2 (the Clemens-Griffith invariant)**.** *Given a threefold X, let $J(X)$ denote its intermediate Jacobian discussed in this section. Let $J(X) = \prod_i A_i$ denote the decomposition of $J(X)$ obtained from Thm. 2.1. Then*

$$J(X)_{\mathrm{CG}} := \prod_{i \in \mathcal{J}} A_i$$

where $\mathcal{J}$ contains those indices i such that the corresponding A_i is not isomorphic to the Jacobian of a (nonsingular algebraic) curve, is a birational invariant.

Proof (Sketch). The proof proceeds in two parts. The first argument shows that every birational map $X \dashrightarrow Y$ can be factored through a sequence of blow-ups and subsequently a sequence of blow-downs along smooth subvarieties (a result due to Hironaka in [Hir64]). Consequently, it suffices to check invariance of $J(X)_{\mathrm{CG}}$ under blow-ups. The second argument is an explicit computation of the intermediate Jacobian of a blow-up along a smooth subvariety $Z \subset X$ as a consequence of Thm. 7.31 in [Voi02].

While the first argument is a nontrivial result due to [Hir64] (also see [Wło03]), we sketch out the intuition in the case of surfaces (and correspondingly, blowups at points) here. This sketch is due to a proof presented in a set of Ravi Vakil's lecture notes on blow-ups of surfaces (see section 3.1 [Vak02]). While this is not what we need in particular, we hope it will act as a toy example for how singularities are resolved in practice.

In Progress (but may be omitted).

By the above reasoning, it suffices to verify the invariance of $J(X)_{\mathrm{CG}}$ under blow-ups of X along smooth subvarieties. Since $\dim X = 3$, we have two possibilities for blow-ups – we may consider the blow-up of X along a point $p \in X$, or a curve $C \subset X$. But note that blowing up X along a point p does not change its third cohomology (recall that a blow-up along a point only adds ones to the central entry of the Hodge diamond, see [Voi02] Thm. 7.31 for the Hodge structure of a blowup), and therefore leaves the intermediate Jacobian invariant (from which it follows that $J(X)_{\mathrm{CG}}$ is invariant). Now suppose we blow up X along a nonsingular curve $Z \subset X$ to obtain the blow-up $\mathrm{Bl}_Z(X)$ (or sometimes written $\tilde{X}^Z$). We know from Thm. 7.31 in [Voi02] that $H^3(\mathrm{Bl}_Z(X), \mathbb{Z}) \cong H^3(X, \mathbb{Z}) \oplus H^1(Z, \mathbb{Z})$. In particular, the isomorphism of Hodge structures from Thm. 7.31 in [Voi02] (namely that the Hodge structure of the blow-up decomposes as above), also induces a decomposition of the complex torus $J(X)$ and tells us that $J(\mathrm{Bl}_Z(X)) \cong J(X) \oplus J(Z)$. This decomposition also passes to the intersection form (or rather the polarization) on $J(\mathrm{Bl}_Z(X))$ as $\langle , \rangle_{\mathrm{Bl}_Z(X)} = \langle , \rangle_X - \langle , \rangle_Z$. Therefore, $J(\mathrm{Bl}_Z(X)) \cong J(X) \oplus J(Z)$ as principally polarized abelian varieties. It is now clear that $J(\mathrm{Bl}_Z(X))_{\mathrm{CG}} \cong J(X)_{\mathrm{CG}}$ (as ppav's). ■

3. The Irrationality Theorem

An immediate corollary of Thm 2.2 is the following.

Theorem 3.1. *Let X be a rational threefold. Then the intermediate Jacobian of X can be decomposed as a product of Jacobians of irreducible curves. Namely, $J(X) \cong \prod_i J(C_i)$ where each C_i is a nonsingular algebraic curve.*

Proof. Let X be a rational threefold. Then there exists a birational map $X \dashrightarrow \mathbb{P}^3$. By Thm. 2.2, $J(X)_{\mathrm{CG}} = J(\mathbb{P}^3)_{\mathrm{CG}}$. But recall that $\mathbb{P}^3$ has trivial middle cohomology, and therefore $J(\mathbb{P}^3)_{\mathrm{CG}} = 0$. So $J(X)_{\mathrm{CG}} = 0$ as well. Then by definition, we obtain $J(X) = \prod_i J(C_i)$ where each C_i is a nonsingular algebraic curve. ■

Now we come to our final goal, the irrationality of a cubic threefold.

Theorem 3.2. *Let V be a cubic threefold. Then the intermediate Jacobian of V is irreducible, but not isomorphic to the Jacobian of a curve. A cubic threefold is therefore irrational.*

Proof (Very Crude Sketch). For the most part, we refer the reader to Thm. 13.12 in [CG72] for the proof. Here, we point out several key steps in an arguably simpler proof first presented by Beauville in [Bea82] (who in turn credits the keypoint in the argument to Mumford) which aims to prove a disagreement between the dimensions of the singular loci of the θ-divisors[4] of $J(V)$ and the Jacobian of a curve (in particular, we also get irreducibility along the way)[5].

Let Θ denote a θ-divisor of $J(V)$. The punchline in [Bea82] is that Θ has exactly one singular point (with multiplicity 3). From which we conclude that Θ (and hence)[6] $J(V)$ is irreducible. Furthermore, we obtain $\dim(\Theta_{\mathrm{sing}}) = 0$. Now for a genus g non-singular algebraic curve C with θ-divisor of $J(C)$ chosen to be Ξ, we have $\dim(\Xi_{\mathrm{sing}}) = g - 4$ (see Cor. 4.5 and Lemma 3.3 in [ACGH13] for the Brill-Noether theory description of the singular locus of a divisor). So if $J(V) \cong J(C)$ for some curve C, we must have that its genus is such that $g \leq 4$. However, since $\dim(J(C)) = g$, and from the Hodge diamond of V (see [Ara12] subsection 17.3) $\dim(J(V)) = 5$, we must have $g = 5$. This shows that $J(V) \ncong J(C)$ for any curve C. The irrationality of V now follows from Thm. 3.1 as in the claim. ■

References

[ACGH13] E. Arbarello, M. Cornalba, P. Griffiths, and J.D. Harris. *Geometry of Algebraic Curves: Volume I.* Grundlehren der mathematischen Wissenschaften. Springer New York, 2013. ISBN: 9781475753233. URL: https://books.google.com/books?id=LanxBwAAQBAJ.

[Ara12] D. Arapura. *Algebraic Geometry over the Complex Numbers.* Universitext. Springer New York, 2012. ISBN: 9781461418092. URL: https://books.google.com/books?id=FQslb7pH8EgC.

[Bea82] Arnaud Beauville. “Les singularites du diviseur Θ de la jacobienne intermediaire de l'hypersurface cubique dans IP4”. In: *Algebraic Threefolds.* Ed. by Alberto Conte. Berlin, Heidelberg: Springer Berlin Heidelberg, 1982, pp. 190–208. ISBN: 978-3-540-39342-9.

[4]Given a principally polarized abelian variety (A, θ), we call the divisor corresponding to $\theta \in \mathrm{Pic}^0(A)$ the θ-divisor of A

[5]See Cor. 4.11 in [Huy23] for a condensed version of the same argument, outsourcing some results to classical Brill-Noether theory, but presented in English.

[6]Although not mentioned before, a routine verification, as in [CG72] Lemma 3.20, shows that $J(V)$ is irreducible iff Θ is irreducible as a divisor

[CG72] C. Herbert Clemens and Phillip A. Griffiths. "The Intermediate Jacobian of the Cubic Threefold". In: *Annals of Mathematics* 95.2 (1972), pp. 281–356. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970801.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, Ltd, 1994. ISBN: 9781118032527. DOI: https://doi.org/10.1002/9781118032527.ch2. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/9781118032527.

[Hir64] Heisuke Hironaka. "Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero: I". In: *Annals of Mathematics* 79.1 (1964), pp. 109–203. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970486.

[Huy23] Daniel Huybrechts. *The Geometry of Cubic Hypersurfaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2023.

[Swi74] H.P.F. Swinnerton-Dyer. *Analytic Theory of Abelian Varieties*. Lecture note series. Cambridge University Press, 1974. ISBN: 9780521205269. URL: https://books.google.com/books?id=SZm7SdqVyWEC.

[Vak02] Ravi Vakil. *Complex Algebraic Surfaces Class 7*. Oct. 2002. URL: https://math.stanford.edu/~vakil/02-245/sclass7B.pdf.

[Voi02] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I*. Ed. by Translator Leila Schneps. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002. URL: https://doi.org/10.1017/CBO9780511615344.

[Voi19] Claire Voisin. *Dimension 3: les invariants de Clemens-Griffiths et Artin-Mumford*. 2019. URL: https://www.college-de-france.fr/fr/agenda/cours/invariants-birationnels/dimension-3-les-invariants-de-clemens-griffiths-et-artin-mumford.

[Wło03] Jarosław Włodarczyk. "Toroidal varieties and the weak factorization theorem". In: *Inventiones mathematicae* 154.2 (2003), pp. 223–331. ISSN: 1432-1297. DOI: https://doi.org/10.1007/s00222-003-0305-8. URL: https://link.springer.com/article/10.1007/s00222-003-0305-8.

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