Notes On Advanced Quantum Field Theory
The Theory of Elementary Interactions
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April 4, 2019

Lecture 1: Introduction

The goal for this course is to explain the current "standard model" for particle physics. This is too lofty of a goal for this course, so what we focus on is the textitbuilding blocks of the standard model, such that we understand the origin and purpose of each term of the Lagrangian.

Topics covered include

1. Path integral quantization

   Specifically, and luckily, we'll work with Gaussian integrals

2. Review perturbation theory via path integrals

   Includes familiar tools such as Wick's theorem, Feynman rules, et cetera

3. Renormalization

   Allows us to discuss effective QFT (e.g., eliminating infinities) in more detail

4. (Non-)Abelian gauge theories (classical)

   Full exploit of path integrals to deduce quantizations from classical field theories

5. Quantization of non-abelian gauge theories

   Using path integrals and lattices (non-perturbative calculations)

6. Spontaneous symmetry breaking mechanisms (classical)

Path Integrals

We begin by supposing that the quantization is already done, and we have a quantum system with a Hilbert space $\mathcal{H}$, a Hamiltonian $\hat{H}$, and a propagator, from integrating the Schroedinger equation, $U(t) = e^{-i\hat{H}t}$.

Now work out a representation for the propagator by Taylor expanding

$$U(t) = e^{-i\hat{H}t} = \left(e^{-\frac{it}{N}\hat{H}}\right)^N = \lim_{N\to\infty}\left(\mathbb{I} - \frac{it}{N}\hat{H}\right)^N \tag{1}$$

Let $\{|j\rangle\}$ be a basis for the Hilbert space $\mathcal{H}$ and consider the transition amplitude of evolving from an eigenstate $|\phi_i\rangle$ to another eigenstate $|\phi_f\rangle$

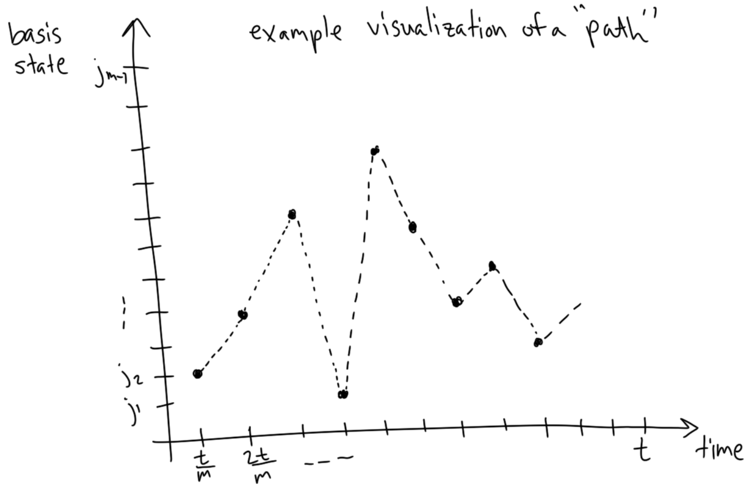
$$\langle\phi_f|\, U(t)\, |\phi_i\rangle = \langle\phi_f| \left(e^{-\frac{it}{N}\hat{H}}\right)^N |\phi_i\rangle \tag{2}$$

Now, insert the completeness relation of our HIlbert space in between each of the N exponentials

$$\langle\phi_f| U(t) |\phi_i\rangle = \sum_{j_1,\ldots,j_{N-1}} \langle\phi_f| e^{-\frac{it}{N}\hat{H}} |j_{N-1}\rangle \langle j_{N-1}| e^{-\frac{it}{N}\hat{H}} \ldots |j_1\rangle \langle j_1| e^{-\frac{it}{N}\hat{H}} |\phi_i\rangle \tag{3}$$

$$\equiv \sum_{\text{paths}} f(j_1,\ldots,j_{N-1}). \tag{4}$$

So, the transition amplitude of this state evolution is a sum over all of the paths through the basis states $j_1,\ldots,j_{N-1}$. An example schematic of a path is visualized below.



To work out the function of the path $f(j_1,\ldots,j_{N-1})$, we need to calculate the transition amplitude in the path integral setting, and seek to write it as an exponential of some function of the states $\mathcal{L}(j,k)$, which ends up being the Lagrangian

$$\langle j| (\mathbb{I} - \frac{it}{N}\hat{H}) |k\rangle \simeq e^{\frac{it}{N}\mathcal{L}(j,k)}. \tag{5}$$

So, the transition amplitude will end up being a product of a bunch of exponentials with arguments dependent of the Lagrangian

$$\langle\phi_f| U(t) |\phi_i\rangle = \sum_{\text{paths}} f(j_1,\ldots,j_{N-1}) \tag{6}$$

$$= \sum_{j_1,\ldots,j_{N-1}} e^{\frac{it}{N}\sum_{k=2}^{N-1}\mathcal{L}(j_k,j_{k-1})} \tag{7}$$

Before moving forward, what makes the path integral so interesting?

1. It allows the calculation of quantum quantities (e.g., transition amplitudes) via well-understood classical solutions, via the saddle point method (e.g., for handling highly oscillatory integrals).

2. It can be used to build (nonperturbative) approximation schemes, such as Monte Carlo sampling over paths.

Example: General Nonrelativistic Quantum Mechanical System

Now we assume a little bit more about our quantum system. Suppose our quantum system is inspired by a classical system with pairs of canonical coordinates and momenta and the Hamiltonian $H(\{q^j\},\{p^j\}) = H(q,p)$.

Now, we turn around the path integral sum over paths to guess a quantum Hamiltonian and Hilbert space from this classical Hamiltonian via

$$U(q_i, q_f, T) = \langle q_f | U(T) | q_i \rangle = \langle q_f | e^{-iT\hat{H}} | q_i \rangle \tag{8}$$

Proceed as before, mulitplying the exponentials and inserting $N-1$ completeness relations in between the N copies of the exponential. In the continuous basis of canonical coordinates, the completeness relation is

$$\mathbb{I} = \left(\prod_j \int dq_k^j \right) |q_k\rangle \langle q_k| \tag{9}$$

So, the transition amplitude in this case is, with $\epsilon \equiv \delta t = \frac{T}{N}$

$$\langle q_f | U(T) | q_i \rangle = \sum_{k_1,\ldots,k_{N-1}} \langle q_f | e^{-i\epsilon\hat{H}} |q_{k_{N-1}}\rangle \langle q_{k_{N-1}}| \ldots |q_{k_1}\rangle \langle q_{k_1}| e^{-i\epsilon\hat{H}} |q_i\rangle \,. \tag{10}$$

There are three cases for the dependence of the quantum Hamiltonian on the canonical coordinates in the expression for the propagator. It can depend purely on position, purely on momenta, or most realistically, it can depend on both.

In the case that the Hamiltonian is a function purely dependent on canonical position, such that $\hat{H} = g(\hat{q})$, we easily calculate the transition amplitude, which relates the quantum and classical canonical positions, since the $|q_k\rangle$ are energy eigenstates of the position-dependent Hamiltonian

$$\langle q_{k+1} | g(\hat{q}) | q_k \rangle = g(q_k) \prod_j \delta(q_k^j - q_{k+1}^j) \tag{11}$$

$$= g\left(\frac{q_{k+1} + q_k}{2}\right) \left(\prod_j \int \frac{dp_k^j}{2\pi} \right) e^{i \sum_j p_k^j (q_{k+1}^j - q_k^j)}. \tag{12}$$

Where we used the Dirac delta distribution identity $\delta(q) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip\cdot q}$ to introduce the canonical momenta into the transition amplitude. Also note that the Dirac delta function forces $q_{k+1} = q_k$, such that $f(q_k) = f(\frac{q_{k+1}+q_k}{2})$, and we write it in this fashion for later use.

Next, in the case that the Hamiltonian is a function purely dependent on canonical momenta, such that $\hat{H} = h(\hat{p})$, the transition amplitude is calculated by inserting the completeness relation for the momentum eigenbasis.

$$\langle q_{k+1}| h(\hat{p}) |q_k\rangle = \langle q_{k+1}| h(\hat{p}) \cdot \prod_j \int dp_k^j \, |p_k\rangle \langle p_k|q_k\rangle \tag{13}$$

$$= \prod_j \int \frac{dp_k^j}{2\pi} h(p_k) e^{i\sum_j p_k^j (q_{k+1}^j - q_k^j)} \tag{14}$$

Where the inner product of the position and momentum eigenstates is a Fourier phase element $\langle p|q\rangle = \frac{1}{2\pi} e^{ip\cdot q}$, and we get the sum, since the subscript k denotes N total canonical coordinate pairs.

The more realistic situation is when the Hamiltonian is dependent on both position and momenta $\hat{H} = \hat{H}(\hat{q}, \hat{p}) = g(\hat{q}) + h(\hat{p})$. Suppose the dependencies are linearly separable in the quantum Hamiltonian. Then we may translate between classical position and momenta via the Taylor expansion to first order

$$e^{-i\epsilon\hat{H}} = \mathbb{I} - i\epsilon\hat{H} = \mathbb{I} - i\epsilon(g(\hat{q}) + h(\hat{p})). \tag{15}$$

Using tihs linearity, we can write this dependence in the derived formula as

$$\langle q_{k+1}| e^{-i\epsilon\hat{H}(\hat{q},\hat{p})} |q_k\rangle = \prod_j \int \frac{dp_k^j}{2\pi} e^{-i\epsilon H(\frac{q_{k+1}+q_k}{2}, p_k)} e^{i\sum_j p_k^j (q_{k+1}^j - q_k^j)} \tag{16}$$

Putting all this together into the propagator, which is really the transition amplitude for a nonrelativistic quantum system,

$$U(q_I, q_f; T) = \left(\prod_{jk} \int dq_k^j \int \frac{p_k^j}{2\pi} \right) e^{i\sum_k \left(\sum_j p_k^j (q_{k+1}^j - q_k^j) - \epsilon H(\frac{q_{k+1}}{2}, p_k)\right)}. \tag{17}$$

Take note that there is nothing quantum on the RHS: no hats! We have used purely classical data to define the quantum propagator, or, transition amplitude, on the LHS, such that $U(q_i, q_f; T) \propto e^{-i\epsilon H(q_i, q_f; T)}$.

A few other remarkable points:
Using the saddle point method, we can build an approximation scheme for U. This is useful for solving highly oscillatory integrals, as we see in the transition

amplitude above (e.g., $e^{i\cdots}$), since such integrals can be approximated by its saddle points (or critical points), which correspond to classical paths of the system.

Monte Carlo sampling of the system can also be used to approximate the transition amplitude by building an estimator for the RHS, sampling over classical configurations, and summing up the estimator.

Now, the expression for U was not-so-pretty, but imagine continuous time variables and integrals when you see $\sum_k$ and ϵ above. In the limit as $N \to \infty$ (the number of completeness relations inserted), the quantum propagator is expressed in a continuous form with strange new "integrals".

$$U(q_i, q_f; t) = \left(\int \mathcal{D}q \int \mathcal{D}p \right) e^{i \int_0^T dt\, (\sum_j p^j \dot{q}^j - H(q,p))} \tag{18}$$

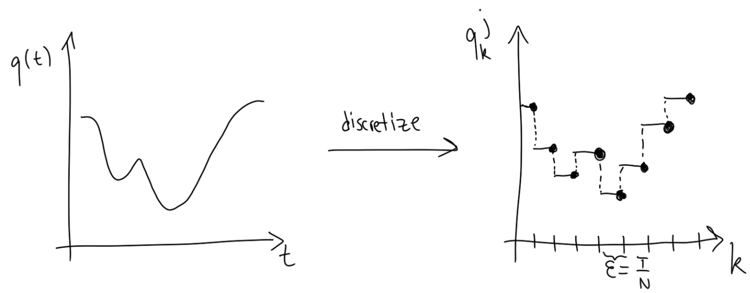
Do not think of these as literal integrals, as we do not have a proper measure space to integrate over. Think of them as algorithms for now, something totally new that will be applied to solve this expression above.

Lecture 2: Gaussian Path Integrals

Recall the propagator, or transition amplitude, for a nonrelativistic quantum system

$$U(q_i, q_f; T) = \left(\prod_j \int \mathcal{D}q^j(t) \int \mathcal{D}p^j(t) \right) e^{i \int_0^T dt\, \mathcal{L}(q^j, \dot{q}^j)}. \tag{19}$$

To work with this, we often discretize $q(t) \to q_k^j$



$$U(q_i, q_f; T) = \left(\prod_{j,k} \int dq_k^j \int \frac{dp_k^j}{2\pi} \right) e^{i \sum_k (\sum_j p_k^j (q_{k+1}^j - q_k^j) - \epsilon H)} \tag{20}$$

Evaluate these very many integrals to get an answer dependent on $\epsilon = \frac{T}{N}$, since we discretized, take the limit as $\epsilon \to 0$ and deal with any encountered infinities.

Key Example

Consider the classical Hamiltonian

$$H = \frac{p^2}{2m} + V(q). \tag{21}$$

Calculate the transition amplitude (**Exercise**)

$$\begin{aligned} U(q_i, q_f; T) &= \left(\prod_{j,k} \int dq_k^j \int \frac{dp_k^j}{2\pi} \right) e^{i \sum_k (\sum_j p_k^j (q_{k+1}^j - q_k^j) - \epsilon H)} && (22) \\ &= \left(\prod_k \int dq_k \int \frac{dp_k}{2\pi} \right) e^{i \sum_k (p_k (q_{k+1} - q_k) - \epsilon (\frac{p_k^2}{2m} + V(q)))} && (23) \\ &= \left(\prod_k \int dq_k \right) \sqrt{\frac{-im}{2\pi\epsilon}} e^{i \sum_k \frac{m}{2\epsilon} (q_{k+1} - q_k)^2 - \epsilon V(\frac{q_{k+1} + q_k}{2})}. && (24) \end{aligned}$$

We may also write this in the following notation, using the fact that the argument of the exponential is the discretized version of the action, now without the p-integral

$$\lim_{\epsilon\to 0} U(q_i, q_f; T) = \int \mathcal{D}q(t) e^{\mathcal{S}[q(t)]} \tag{25}$$

Where the action is

$$\mathcal{S}[q(t)] = \int_0^T dt \; (\frac{m}{2}\sum_j (\dot{q}^j)^2 - V(q)). \tag{26}$$

Note that if our system is a harmonic oscillator $V(q) = \frac{1}{2}m\omega q^2$, we can do the full integral.

Path Integrals for Scalar Fields

Recall the classical scalar field with Lagrangian density and Hamiltonian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \tag{27}$$

$$H = \int d^3x \; (\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)). \tag{28}$$

The path integral prescription for quantum scalar fields gives the transition amplitude, by blind application of the above, we conjecture that

$$\langle \phi_b | \, e^{-i\hat{H}T} \, |\phi_a\rangle = \left(\int \mathcal{D}\phi \int \mathcal{D}\pi\right) e^{i\int_0^T d^4x\, (\pi\dot{\phi} - H(\phi)} \tag{29}$$

Where the boundary terms are $\phi(t=0,x) = \phi_a(x)$ and $\phi(t=T,x) = \phi_b(x)$.

As explained above, to make sense of this quantity, we must discretize, evaluate, and take the continuum limit as $\epsilon \to \infty$. When we discretize, note that we only discretize space, as discretizing time in this way will cause trouble with the conjugate momenta.

The field operators are discretized over a "grid" of points x_j each of width ϵ, such that

$$\phi(t,x) \;\; \to \;\; \phi(t,x_j) \equiv q^j(t). \tag{30}$$

Then discretize the integral by turning it into a sum over the grid

$$\int d^3x \;\; \to \;\; \epsilon^3 \Sigma_{j\in\mathbb{Z}^3}. \tag{31}$$

Next the derivative can be discretized via a finite difference. Note that much better choices of a symmetric difference can be used, which are more computationally nice.

$$\nabla_\mu \phi(x) \quad \rightarrow \quad \frac{(\phi(x_j + \epsilon_\mu) - \phi(x_j))}{|\epsilon_\mu|}. \tag{32}$$

Where μ chooses one of four directions to calculate the derivative, and

$$\epsilon_\mu = \epsilon\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\} \tag{33}$$

And, lastly, the potential just becomes evaluated at each x_j

$$V(\phi(x)) \quad \rightarrow \quad V(\phi(x_j)). \tag{34}$$

Then the Lagrangian is discretized to a sum over a bunch of terms (**Exercise**), but the only relevant term to the construction of the Hamiltonian is the time derivative of the field operator $\dot{\phi}$

$$L = \int d^3x \, \mathcal{L} \rightarrow \epsilon^3 \sum_j \frac{1}{2}(\dot{\phi}_j)^2 \tag{35}$$

And the discretized conjugate momentum becomes

$$\pi^j = \frac{\partial L}{\partial \dot{q}^j} = \frac{\partial L}{\partial \dot{\phi}^j} \quad \rightarrow \quad \epsilon^3 \dot{q}^j \tag{36}$$

Finally, we have the discretized Hamiltonian, where we display the ϵ terms to show that if we did not add the ϵ^3 term to the discretized Lagrangian, we would be stuck with an extra ϵ^{-3} on the discretized Hamiltonian

$$H = \epsilon^3 \sum_j \epsilon^{-3} \pi_j^2 + \frac{1}{2}\left(\frac{q_{j+\epsilon^\mu} - q_j}{\epsilon}\right)^2 + V(q). \tag{37}$$

In summary, the discretization of the scalar field gives us a nonrelativistic lattice system such that the discretized Hamiltonian is the sum of a kinetic energy term and a potential energy term. The second step is to evaluate the (nonrelativistic) path integral, and the third step is to take the continuum limit as $\epsilon \rightarrow 0$, which will later be re-branded as renormalization.

The most important case of the scalar field is the quadratic potential, which corresponds to the Klein-Gordon field (e.g., discretizing Klein-Gordon theory yields the quadratic potential below), is

$$V(q) = \frac{1}{2} q^T \mathbf{A} q. \tag{38}$$

Gaussian Integrals

Consider the following integral

$$I = \int_{-\infty}^{\infty} dx\ e^{-x^2} = \sqrt{\pi}. \tag{39}$$

Proof:

$$I = \int_{-\infty}^{\infty} dx\ e^{-x^2} \tag{40}$$

$$I^2 = \left(\int_{-\infty}^{\infty} dx\ e^{-x^2}\right)\left(\int_{-\infty}^{\infty} dy\ e^{-y^2}\right) \tag{41}$$

$$= \int_{0}^{\infty} rdr \int_{0}^{2\pi} d\theta\, e^{-r^2} \tag{42}$$

$$I^2 = 2\pi \int_{0}^{\infty} \frac{d}{dr}\left(-\frac{1}{2}e^{-r^2}\right) dr = \pi \tag{43}$$

This is actually a special case of the more general form

$$\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}ax^2+bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}} \tag{44}$$

$$\int_{-\infty}^{\infty} dx\ e^{iax^2+ibx} = \sqrt{\frac{2\pi i}{a}} e^{\frac{-ib^2}{2a}} \tag{45}$$

$$\tag{46}$$

We will extensively use the moments generated by the Gaussian integrand

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} dx\ x^n e^{-\frac{1}{2}ax^2}}{\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}ax^2}}. \tag{47}$$

Note that if n is odd, then the moment is zero. So, rewrite the exponent as $2m$. Then (**Exercise**)

$$\langle x^{2m} \rangle = \frac{(2m-1)!!}{a^m}. \tag{48}$$

Note that the double factorial $(2m-1)!!$ represents the number of ways to join $2m$ points in pairs. – "All science should in linear algebra or combinatorics." –

Another closed form of this integral is in terms of derivatives

$$\langle x^{2m}\rangle = \left(\frac{d}{db}\right)^{2m}\left(\frac{\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}ax^2+bx}}{\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}ax^2}}\right)\Big|_{b=0} \tag{49}$$

$$= \left(\frac{d}{db}\right)^{2m} e^{\frac{b^2}{2a}}\Big|_{b=0}. \tag{50}$$

To evaluate Gaussian integrals of many variables, where $x \in \mathbb{R}^n$, consider

$$I(\mathbf{A}, B) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n\ e^{-x^T\mathbf{A}x+B^Tx} \tag{51}$$

Where $\mathbf{A}$ is an $n\times n$ symmetric real matrix and B is an $n\times 1$ real vector. Since $\mathbf{A}$ is real, symmetric, it contains orthogonal $\mathbf{O}$ and diagonal matrices $\mathbf{D}$, such that $\mathbf{O}^T\mathbf{O} = \mathbb{I}$ and $\mathbf{D}$ is diagonalized with the eignevalues of $\mathbf{A}$.

$$\mathbf{O}^T\mathbf{D}\mathbf{O} = \mathbf{A} \tag{52}$$

Assume that $B = 0$ and define $y = \mathbf{O}x$. Then

$$I(\mathbf{A}, B=0) = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n\ e^{-y^T\mathbf{D}y} \tag{53}$$

$$= \prod_{j=1}^{n} \int_{-\infty}^{\infty} dy_j\ e^{-y_j^2\lambda_j} \tag{54}$$

$$= \prod_{j=1}^{n} \sqrt{\frac{\pi}{\lambda_j}} \tag{55}$$

$$I(\mathbf{A}, B=0) = \sqrt{\frac{\pi^n}{det(\mathbf{A})}}. \tag{56}$$

The $B \neq 0$ case (**Exercise**) results in the following

$$I(\mathbf{A}, B) = \sqrt{\frac{\pi^n}{det(\mathbf{A})}} e^{B^T\mathbf{A}^{-1}B}. \tag{57}$$

Lecture 3: Correlation Functions and Path Integrals

Recall the generating function for a single-variable Gaussian probability distribution $e^{\frac{1}{2}ax^2}$ and the moment-generating integral

$$I = \int_{-\infty}^{\infty} dx \; x^{2n} e^{\frac{1}{2}ax^2} = \frac{(2n-1)!!}{a^n}. \tag{58}$$

We also derived the identity with the generating function for the multivariable Gaussian probability distribution

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \; e^{-\frac{1}{2}x^T \mathbf{A} x + J^T x} = \sqrt{\frac{\pi^n}{det(\mathbf{A})}} e^{J^T \mathbf{A}^{-1} J}. \tag{59}$$

The 2-point correlation function for the n-variable Gaussian is (**Exercise**), for $j \neq k$

$$\langle x_j x_k \rangle \equiv \frac{\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \; x_j x_k e^{-\frac{1}{2}x^T \mathbf{A} x}}{\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \; e^{-\frac{1}{2}x^T \mathbf{A} x}} \equiv [\mathbf{A}^{-1}]_{jk}. \tag{60}$$

Note that this is also equal to the second derivative with respect to the vector J, evaluated at $J = 0$

$$\langle x_j x_k \rangle \equiv \frac{\frac{\partial^2}{\partial J_j \partial J_k} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \; e^{-\frac{1}{2}x^T \mathbf{A} x + J^T x}}{\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \; e^{-\frac{1}{2}x^T \mathbf{A} x + J^T x}} \Bigg| J = 0. \tag{61}$$

Higher Order Moments: l-point Correlation Functions

The l-point correlationn function has similar form

$$\langle x_{j_1} \dots x_{j_l} \rangle \equiv \frac{\int_{-\infty}^{\infty} dx_{j_1} \cdots \int_{-\infty}^{\infty} dx_{j_l} \; x_{j_1} \dots x_{j_l} \; e^{-\frac{1}{2}x^T \mathbf{A} x}}{\int_{-\infty}^{\infty} dx_{j_1} \cdots \int_{-\infty}^{\infty} dx_{j_l} \; e^{-\frac{1}{2}x^T \mathbf{A} x}}. \tag{62}$$

By Wick's theorem (proof by induction), for even l, the l-point correlation function is proportional to the sum of the products over the permutation group on l symbols, the "Wick sum". We write "proportional to" for reasons of symmetry and soon eliminating redundant terms.

$$\langle x_{j_1} \dots x_{j_l} \rangle \propto \sum_{\pi \in S_l} [\mathbf{A}^{-1}]_{j_{\pi^{-1}(1)} j_{\pi^{-1}(2)}} \dots [\mathbf{A}^{-1}]_{j_{\pi^{-1}(l-1)} j_{\pi^{-1}(l)}}. \tag{63}$$

Example: 4-point Correlation

To calculate the 4-point correlation function, we begin by considering the $4! = 24$ total permutations on 4 symbols. Since $\mathbf{A}^{-1}$ is symmetric

$$[\mathbf{A}^{-1}]_{jk} = [\mathbf{A}^{-1}]_{kj} \tag{64}$$

And there are only $\frac{24}{2!2!2!} = 3$ unique terms (products of two matrix elements), which are (**Exercise**)

$$\begin{aligned}\langle x_{j_1} x_{j_2} x_{j_3} x_{j_4} \rangle &= [\mathbf{A}^{-1}]_{j_1 j_2} [\mathbf{A}^{-1}]_{j_3 j_4} \\ &+ [\mathbf{A}^{-1}]_{j_1 j_3} [\mathbf{A}^{-1}]_{j_2 j_4} \\ &+ [\mathbf{A}^{-1}]_{j_1 j_4} [\mathbf{A}^{-1}]_{j_2 j_3} \\ &= \langle x_{j_1} x_{j_2} \rangle \langle x_{j_3} x_{j_4} \rangle + \langle x_{j_1} x_{j_3} \rangle \langle x_{j_2} x_{j_4} \rangle + \langle x_{j_1} x_{j_4} \rangle \langle x_{j_2} x_{j_3} \rangle\end{aligned}$$

With the approporiate choice of **A**, in the context of path integrals and perturbative field theory, these products of correlations functions are exactly correspondent to Feynman propagator, and, in turn, the Feynman diagrams, just as we studied in *Lecture 9* of the last lecture series (Quantum Field Theory) for the 4-particle Wick contraction.

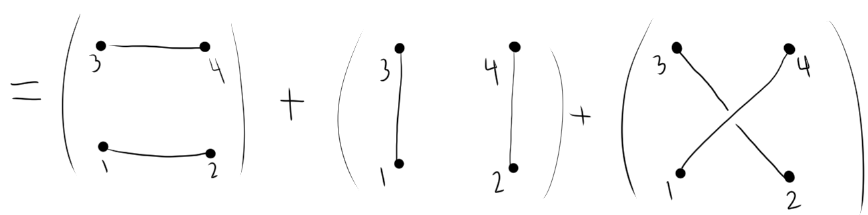


Figure 1: Feynman diagram correspondence of the 4-point correlation function

Keeping only unique terms, the proprotionality relation becomes an equivalence

$$\langle x_{j_1} \dots x_{j_l} \rangle = \sum_{\text{unique } \pi^{-1}} [\mathbf{A}^{-1}]_{j_{\pi^{-1}(1)} j_{\pi^{-1}(2)}} \dots [\mathbf{A}^{-1}]_{j_{\pi^{-1}(l-1)} j_{\pi^{-1}(l)}} \tag{65}$$

(**Exercise**) Calculate the 6-point correlation function with $\frac{6!}{2!2!2!} = 90$ unique terms

$$\langle x_{j_1} x_{j_2} x_{j_3} x_{j_4} x_{j_5} x_{j_6} \rangle = [\mathbf{A}^{-1}]_{j_1 j_2} [\mathbf{A}^{-1}]_{j_3 j_4} [\mathbf{A}^{-1}]_{j_5 j_6} + \dots \tag{66}$$

In short summary,

- We can calculate *all* moments for the Gaussian probability distribution.
- We have a diagrammatic calculus to calculate the l-point correlation functions, which end up being exactly the Feynman propagators/diagrams, with appropriate choice of **A**, and is the direct connection of quantum field theory and Gaussian integrals.

The Matrix A for Path Integrals

Let the potential V be quadratic in the canonical position coordinate per particle q_k (e.g., a one-dimensional chain of oscillators), such that the transition amplitude, which will be discretized, evaluated, and limited $\epsilon \to 0$, from some state q_a to another q_b is

$$U(q_a, q_b; T) = \left(\prod_k \int \frac{dq_k}{c(\epsilon)} \right) e^{\frac{1}{2} i q^T \mathbf{A} q} \tag{67}$$

Where we know the quadratic form contains a kinetic energy term plus a potential energy term

$$q^T \mathbf{A} q = \sum_k \left(m \frac{(q_{k+1} - q_k}{\epsilon}^2 - \epsilon V(\frac{q_{k+1} + q_k}{2}) \right). \tag{68}$$

This results is $\mathbf{A}$ as a *tridiagonal* matrix for the kinetic energy term and a potential energy term which is a matrix with elements quadratic in q

$$\mathbf{A} = \begin{bmatrix} \frac{2m}{\epsilon} & -\frac{m}{\epsilon} & 0 & \cdots & & & \\ -\frac{m}{\epsilon} & \frac{2m}{\epsilon} & -\frac{m}{\epsilon} & 0 & \cdots & & \\ 0 & -\frac{m}{\epsilon} & \frac{2m}{\epsilon} & -\frac{m}{\epsilon} & 0 & \cdots & \\ \vdots & 0 & \ddots & \ddots & \ddots & & \end{bmatrix} + [V(q^2)] \tag{69}$$

The transition amplitude is then calculated similarly to last lecture as

$$U(q_a, q_b; T) = \frac{\infty \text{ const.}}{\sqrt{\det(\mathbf{A})}} \tag{70}$$

The infinite constant will not be a problem since the l-point correlation is normalized, and the same exact infinite constant will appear in the denominator and cancel the constant. So, in terms of q_k, the l-point correlation reads

$$\langle q_{j_1} \dots q_{j_l} \rangle \equiv \frac{\prod_k \int \frac{dq_k}{c(\epsilon)} q_{j_1} \dots q_{j_l} \; e^{-\frac{1}{2} i q^T \mathbf{A} x}}{\prod_k \int \frac{dq_k}{c(\epsilon)} \; e^{-\frac{1}{2} i q^T \mathbf{A} x}}. \tag{71}$$

Note that with periodic boundary conditions, the elements of follow a modulo relation $\mathbf{A}_{jk} = f((j-k) \bmod n)$, where n is the number of sites/oscillators in the chain.

Assuming that $\mathbf{A}$ is invertible, there exists a unitary matrix Q, such that $Q^T Q = \mathbb{I}$ and $Q^T \mathbf{A} Q = D$, with diagonal matrix D with the eigenvalues of $\mathbf{A}$ along the diagonal. Then the determinant of $\mathbf{A}$ is easy to calculate, since

$$\det(\mathbf{A}) = \prod_{j=1}^{n} \lambda_j(A). \tag{72}$$

Correlations Functions and Quantum Observables

Consider the transition amplitude over 2-point spatial correlations

$$U(q_a, q_b; T) \propto \int \mathcal{D}\phi(x)\ \phi(x_1)\phi(x_2)e^{i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi)} \tag{73}$$

With an expression like this, always discretize by sending $\phi(x_j) \to q_j$, evaluate the integral, and enter the contiuum limit with the boundary conditions

$$\phi(-T, x) = \phi_a(x) \tag{74}$$

$$\phi(T, x) = \phi_b(x). \tag{75}$$

Apply the following condition, exploiting the boundary conditions, to factor the full field "integral" over the individual fields and the boundary of the field

$$\int \mathcal{D}\phi(x) = \int \mathcal{D}\phi_1(x) \int \mathcal{D}\phi_2(x) \int_{\partial\phi} \mathcal{D}\phi(x) \tag{76}$$

Where the boundary $\partial\phi = \partial\phi_1 + \partial\phi_2$ is defined by

$$\phi_1(x) = \phi(x_1^0, x_1) \tag{77}$$

$$\phi_2(x) = \phi(x_2^0, x_2) \tag{78}$$

So, after performing the boundary integral, we introduce quantum stuff to the expression from the classical 2-point function above, for $x_2^0 > x_1^0$

$$U(q_a, q_b; T) \propto \int \mathcal{D}\phi_1(x) \int \mathcal{D}\phi_2(x)\ \phi(x_1)\phi(x_2) \langle\phi_b| e^{-i\hat{H}(T-x_2^0)} |\phi_2\rangle \\ \times \langle\phi_2| e^{-i\hat{H}(x_2^0-x_1^0)} |\phi_1\rangle \langle\phi_1| e^{-i\hat{H}(x_1^0+T)} |\phi_a\rangle$$

Now, apply the Schroedinger-picture field operator to write the classical field operators in terms of quantum field operators. The formula is

$$\hat{\phi}_S(x) |\phi_1\rangle = \phi(x_1) |\phi_1\rangle \tag{79}$$

So, the purely quantum expression for the 2-point function is

$$U(q_a, q_b; T) \propto \int \mathcal{D}\phi_1(x) \int \mathcal{D}\phi_2(x)\ \ \langle\phi_b| e^{-i\hat{H}(T-x_2^0)} \hat{\phi}_S(x) |\phi_2\rangle \\ \times \langle\phi_2| e^{-i\hat{H}(x_2^0-x_1^0)} \hat{\phi}_S(x) |\phi_1\rangle \langle\phi_1| e^{-i\hat{H}(x_1^0+T)} |\phi_a\rangle$$

This is called the time-ordered expectation value of the field operators in the Heisenberg picture. The equation holds for $x_2^0 < x_1^0$ as well, and we can write it as

$$U(q_a, q_b; T) \propto \langle\phi_b| e^{-i\hat{H}T}\mathcal{T}[\hat{\phi}_H(x_1)\hat{\phi}_H(x_2)]e^{-i\hat{H}T} |\phi_a\rangle \tag{80}$$

Now, enter the limit as $T \to \infty$, where we bring the interacting vacuum state and the normalization for the full transition amplitude (**Exercise**), and introduce the *most important formula* for this course

$$\langle\Omega| \mathcal{T}[\hat{\phi}_H(x_1)\hat{\phi}_H(x_2)] |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \; \phi(x_1)\phi(x_2)e^{i\int_{-T}^{T} d^4x\,\mathcal{L}(\phi)}}{\int \mathcal{D}\phi \; e^{i\int_{-T}^{T} d^4x\,\mathcal{L}(\phi)}}. \tag{81}$$

So, the LHS is built of purely quantum observables equal to the classical expression of path integrals!

This expression will end up to be the propagator, which is also the inverse of the Klein-Gordon operator, which is what we call $\mathbf{A}$ in the scalar quantum field theory.

The solution to this is well-known for the case when $\mathcal{L}$ is quadratic in the field operators, and one can easily discretize, evaluate the Gaussian integral, and take the limit as $\epsilon \to 0$.

(Exercise) Calculate the l-point formula for the time-ordered expectation value of the field operators in the Heisenberg picture.

Lecture 4: Functional Quantization of the Scalar Field

The path integral formalism for quantization of fields is an incredibly efficient tool, but one must learn when, and when not, ot use it. Through the lectures and many examples, we'll develop an intuition for when to trust quantization via path integrals.

Recall the action of the scalar field S with classical field operators ϕ

$$S_0 = \int d^4x \; \mathcal{L}_0 = \int d^4x \; \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \right). \tag{82}$$

We will (1) discretize, tantamount to imposing a cutoff Λ, (2) evaluate the integrals, and (3) enter the continuum limit where $\epsilon \to 0$. Start the discretization by putting the field on a lattice (a Lorentz manifold) with spacing ϵ and then compactify the space onto a torus for periodic boundary conditions.

Mathematically, we are transforming from a four-dimensional Minkowski space $\mathcal{M}^4$ to a four-domensional torus $(\mathbb{Z}/N\mathbb{Z})^4$, where $N = \frac{L}{\epsilon}$ is the number of sites, and L is the total size of grid.
Continue discretization with the field operators

$$\phi(x) \to \phi(x_j) \equiv q_j, \tag{83}$$

$$x_j = \epsilon j \in \frac{L}{N}(\mathbb{Z}/N\mathbb{Z})^4. \tag{84}$$

And the partial derivatives are replaced for the forward difference, which is not the best method, but it's "good enough for government work"

$$\partial_\mu \phi(x) \to \frac{\phi(x_j + \epsilon e^\mu) - \phi(x_j)}{\epsilon}. \tag{85}$$

And the space-time integral becomes a sum over the sites on the torus

$$\int d^4x \; \to \epsilon^4 \sum_{j \in (\mathbb{Z}/N\mathbb{Z})^4} \tag{86}$$

Now following the path integral quanitzation recipe, consider the transition amplitude in terms of the discretized action

$$\langle \phi_f | \, U(q_i, q_f; T) \, | \phi_i \rangle = \int \mathcal{D}\phi \; e^{iS_0} \tag{87}$$

Where we follwo the "algorithm" of the integral-differential operator and discretize it to a product, over the torus sites, of integrals (N^4 total integrals) over the field operators, the canonical position variables

$$\int \mathcal{D}\phi \to \prod_j \int d\phi(x_j) \equiv \prod_j \int dq_j. \tag{88}$$

Discretization in Momentum Space

Thus far we have worked entirely in real (position) space. Let's Fourier transform over into momentum space to continue discretization. The Fourier transform is a unitary transformation with Jacobian equal to 1 (**Exercise**). First, the field operators transform as

$$\phi(x_j) = \frac{1}{V} \sum_n e^{-ik_n \cdot x_j} \phi(k_n) \tag{89}$$

Where $V = L^4$ is the volume of the 4D torus. Notationally, the k argument to the field operator in momentum space $\phi(k)$ will denote the Fourier transform of the field operator in real space $\phi(x)$. The wavenumber k_n is discretized over the torus as

$$k_n = \frac{2\pi n^\mu}{L},\ n^\mu \in \mathbb{Z}/N\mathbb{Z} \text{ and } |k^\mu| < \frac{\pi}{\epsilon} \tag{90}$$

Note that the Fourier space field operator is complex, such that $\phi(-k) = \phi^*(k)$, and we therefore have two independent variables per field operator in momentum space: the real part $\Re\phi(k_n)$ and the imaginary part $\Im\phi(k_n)$ for positive time-component $k_n^0 > 0$.
So, in momentum space, the discretized integral-differential operator is (**Exercise**)

$$\int \mathcal{D}\phi = \prod_{n:k_n^0>0} \int d\,\Re\phi(k_n) \int d\,\Im\phi(k_n). \tag{91}$$

And the discretized action for the scalar field in momentum space is (**Exercise**)

$$S_0 = -\frac{1}{V} \sum_{k_n^0>0} (m^2 - k_n^2)((\Re\phi_n)^2 + (\Im\phi_n)^2) \tag{92}$$

Where $\phi_n \equiv \phi(k_n)$, and the following relation for the Kronecker delta is used to obtain this expression

$$\delta_{k,0} = \frac{1}{n} \sum_{j=0}^{n-1} e^{\frac{2\pi ijk}{n}}. \tag{93}$$

Our expression for the path integral for the Klein-Gordon field discretized to a lattice (four-dimensional with periodic boundary conditions) is comprised of Gaussian integarls over a finite number of degrees of freedom

$$I_0 = \int \mathcal{D}\phi\, e^{iS_0} = \left(\prod_{k_n^0>0} \int d\,\Re\phi_n \int d\,\Im\phi_n \right) e^{-i\frac{1}{V}\sum_{k_n^0>0}(m^2-k_n^2)|\phi_n|^2}. \tag{94}$$

Now, onto evaluating this integral, it's just a bunch of Gaussian integrals, and we know how to solve those. We get the following, and unrestrict k_n to get the second line (**Exercise**)

$$I_0 = \prod_{k_n^0 > 0} \sqrt{\frac{-i\pi V}{m^2 - k_n^2}} \cdot \sqrt{\frac{-i\pi V}{m^2 - k_n^2}} \tag{95}$$

$$I_0 = \prod_{k_n} \sqrt{\frac{-i\pi V}{m^2 - k_n^2}} \tag{96}$$

Note that k_n is bounded, but we have an infinity when $V \to \infty$ (continuum limit), but this integral does not yet have full operational meaning and is proportional to the transition amplitude $I_0 \propto \langle \phi_f | U(q_i, q_f; T) | \phi_i \rangle$, and the infinities will cancel and drop out in the full expression.

Heuristic Argument for I_0

As the "surface area of knowledge" we need to remember the path integral formalism is small, there is a heuristic way to obtain this result without formal discretization, etc., using the aforementioned intuition.

Recall the Gaussian integral whose argument is quadratic in its independent variable

$$\int dx \; e^{-x^T \mathbf{A} x} = \sqrt{\frac{\pi^n}{\det(\mathbf{A})}} \propto (\det(\mathbf{A}))^{-\frac{1}{2}} \tag{97}$$

For the Klein-Gordon field, consider the path integral with the Klein-Gordon operator and field operators substituted

$$\int \mathcal{D}\phi \; e^{iS} \;\sim\; \int \mathcal{D}\phi \; e^{\frac{1}{2}\int d^4x \; \phi(x)(-\partial^2 - m^2)\phi(x)} \tag{98}$$

So, we are boldly extrapolating to say that $\mathbf{A}$ is like the Klein-Gordon operator and the x is like the field operator

$$\int d^4x \; \phi(x)(-\partial^2 - m^2)\phi(x) \sim x^T \mathbf{A} x. \tag{99}$$

Furthermore, we say that the path integral is proportional to the determinant of the Klein-Gordon operator

$$\int \mathcal{D}\phi \; e^{iS} \;\propto\; (\det(-\partial^2 - m^2))^{-\frac{1}{2}}. \tag{100}$$

Operationally Well-Defined Quantities

As mentioned, I_0 cancels for operationally well-defined quanities, such as the 2-point correlation function, a time-ordered expectation value of products of the field operators. For example, using the path integral formalism

$$\langle\Omega|\,\mathcal{T}[\phi(x_1)\phi(x_2)]\,|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{iS}}{\int \mathcal{D}\phi\ e^{iS}} \tag{101}$$

To check our bold extrapolations, calculate the discretized field operator product

$$\phi(x_1)\phi(x_2) = \frac{1}{V^2}\sum_m e^{-ik_m\cdot x_1}\phi_m \sum_l e^{-ik_l\cdot x_2}\phi_l \tag{102}$$

So, the discretized RHS numerator of the time-ordered expectation value above is just a bunch of independent Gaussian integrals, quadratic in its independent variables (**Exercise**)

$$\begin{aligned}
\text{numerator} &= \frac{1}{V^2}\sum_{l,m} e^{-i(k_m\cdot x_1 + k_l\cdot x_2)}\left(\prod_{k_n^0>0}\int d\,\Re\phi_n \int d\,\Im\phi_n\right) \\
&\quad\times (\Re\phi_m + i\Im\phi_m)(\Re\phi_l + i\Im\phi_l)e^{-i\frac{1}{V}\sum_{k_n^0>0}(m^2-k_n^2)((\Re\phi_n)^2+(\Im\phi_n)^2)} \\
&= \frac{1}{V^2}\sum_m e^{-ik_m\cdot(x_1-x_2)}\left(\prod_{k_n^0>0}\frac{-i\pi V}{m^2-k_n^2}\right)\frac{-iV}{m^2-k_n^2-i\epsilon} \\
&= \frac{1}{V^2}\sum_m e^{-ik_m\cdot(x_1-x_2)}\cdot I_0\cdot\frac{-iV}{m^2-k_n^2-i\epsilon}
\end{aligned}$$

Where we drastically cut down the number of integrals to evaluate, since any integrals involving products like $\Re\phi_m\cdot\Im\phi_l$ or $\Im\phi_m\cdot\Re\phi_l$ form odd integrands and evaluate to zero. The integeral will also be zero for terms where $m \neq l$ and for terms where $k_m = k_l$. Integrals where $k_m = -k_l$ will *not* be zero.

Bringing this together, the RHS of the time-ordered expectation value has boiled down to

$$\begin{aligned}
\langle\Omega|\,\mathcal{T}[\phi(x_1)\phi(x_2)]\,|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{iS}}{\int \mathcal{D}\phi\ e^{iS}} && (103)\\
&= \lim_{V\to\infty} -i\frac{1}{V}\sum_n \frac{e^{-ik_n\cdot(x_1-x_2)}}{m^2-k_n^2-i\epsilon} && (104)\\
&= \int \frac{d^4k}{(2\pi)^4}\ \frac{ie^{-ik\cdot(x_1-x_2)}}{-m^2+k^2+i\epsilon} && (105)\\
\langle\Omega|\,\mathcal{T}[\phi(x_1)\phi(x_2)]\,|\Omega\rangle &= D(x_1-x_2) && (106)
\end{aligned}$$

So, the path integral formalism gives us exactly the propagator we wish to see. Note that is we were to just boldy extrapolate, without discretization, etc., we would get the same result!

For example,

$$\frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{iS}}{\int \mathcal{D}\phi\ e^{iS}} = \frac{(\partial^2 - m^2)^{-\frac{1}{2}} D(x_1 - x_2)}{(\partial^2 - m^2)^{-\frac{1}{2}}} \tag{107}$$

Since if $\mathbf{A} \sim (-\partial^2 - m^2)$
Then $[\mathbf{A}^{-1}]_{jk} \sim \frac{1}{(-\partial^2 - m^2)_{x_1 x_2}} = D(x_1 - x_2)$
And $\delta^{(4)}(x - y) = (-\partial^2 - m^2)D(x - y)$.

Example: 4-point Correlation Function

Note that all 3-point correlations are zero, since they all have odd integrands. The 4-point correlation function starts off as

$$\langle\Omega|\, \mathcal{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\, |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)e^{iS}}{\int \mathcal{D}\phi\ e^{iS}}. \tag{108}$$

The numerator contains the quantities of $(\Re\phi_m + i\Im\phi_m)\ldots(\Re\phi_l + i\Im\phi_l)$, and most terms will vanish as before, leaving us with terms where $k_l = -k_m$ and $k_q = -k_p$, and we end up with, after applying Wick's theorem and sending $V \to \infty$ (**Exercise**), something like

$$\sum_{k_l, k_q} e^{-i\ldots} \int \ldots \phi_{k_l}\phi_{-k_l}\phi_{k_q}\phi_{-k_q} e^{\cdots}$$
$$= D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3)$$

Interacting QFT via Path Integrals

Consider the action with a free part and an interacting part, namely the phi-fourth interaction,

$$S = S_0 + S_{int} = S_0 + \frac{i\lambda}{4!}\int d^4x\ \phi^4(x). \tag{109}$$

Then the time-ordered expectation value for 2-point correlations can be Taylor expanded, since λ is small,

$$\langle\Omega| \mathcal{T}[\phi(x_1)\phi(x_2)] |\Omega\rangle = lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{i(S_0+S_{int})}}{\int \mathcal{D}\phi\ e^{i(S_0+S_{int})}} \tag{110}$$

$$= lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{iS_0}(1+S_{int}+\frac{1}{2}S_{int}^2+\dots)}{\int \mathcal{D}\phi\ e^{iS_0}(1+S_{int}+\frac{1}{2}S_{int}^2+\dots)} \tag{111}$$

Where $S_{int} = \frac{i\lambda}{4!}\int d^4z\ \phi^4(z)$, and each term above is an integral of powers of time-ordered quantum field operators which end up as Feynman diagrams, for example, of the form

$$\frac{\lambda^m}{4!^m}\int d^4z_1 \cdots \int d^4z_m \int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\phi^4(z_1)\dots\phi^4(z_m)e^{iS_0} \tag{112}$$

$$= \frac{\lambda^m}{4!^m}\int d^4z_1 \cdots \int d^4z_m \langle\Omega| \mathcal{T}[\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}^4(z_1)\dots\hat{\phi}^4(z_m)] |\Omega\rangle \tag{113}$$

$$= \text{Sum of Feynman diagrams} \tag{114}$$

Lecture 5: Functional Derivatives and Generating Functionals

Here we will finish the functional quantization of the scalar field.

Recall that we can compute time-ordered correlation functions for the quantum scalar field entirely in terms of classical quantities, which is equivalent to a sum over all diagrams,

$$\langle\Omega| T[\hat{\phi}(x_1)\dots\hat{\phi}(x_n)] |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\ \phi(x_1)\dots\phi(x_n)e^{iS[\phi(x_1),\dots,\phi(x_n)]}}{\int \mathcal{D}\phi\ e^{iS[\phi(x_1),\dots,\phi(x_n)]}} \tag{115}$$

For example, the 2-point correlation function for the Klein-Gordon field is the Feynman propagator

$$\langle\Omega| T[\hat{\phi}(x_1)\hat{\phi}(x_2)] |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{iS}}{\int \mathcal{D}\phi\ e^{iS}} = D_F(x_1 - x_2) \tag{116}$$

More elegantly, and analogous to multivariate Gaussian integrals, we found the Feynman propagator $D_F(x-y)$, which is the inverse operator of the Klein-Gordon operator $-\partial^2 - m^2$, to be similiar to the inverse of a matrix $\mathbf{A}$, making the Klein-Gordon operator the matrix $\mathbf{A}$.

$$D_F(x_j - x_k) \sim [\mathbf{A}^{-1}]_{jk} = \frac{\int dx_1\dots dx_n\ x_j x_k e^{-\frac{1}{2}x^T\mathbf{A}x}}{\int dx_1\dots dx_n\ e^{-\frac{1}{2}x^T\mathbf{A}x}} \tag{117}$$

To compute these n-point correlation functions, or elements of this "inverse matrix", we used derivatives of the generating functional, which is what we generalize in this lecture. Recall the multivariate Gaussian generating functional

$$Z[J] = \int dx_1\dots dx_n e^{-\frac{1}{2}x^T\mathbf{A}x - J^Tx} = e^{\frac{1}{2}J^T\mathbf{A}^{-1}J}. \tag{118}$$

Functional Derivatives

The functional derivative is a tool from the calculus of variations that we now define by an example that is the continuum analog of the standard partial derivative

$$\frac{\delta}{\delta J(x)}J(y) = \delta^{(4)}(x-y). \tag{119}$$

There is a *one-to-one* mapping from the discrete representation to the continuous with correspondences

$$
\begin{array}{ccc}
x \in \mathbb{R} & \to & j \in \mathbb{Z} \\
J(x) \in C(\mathbb{R}) & \to & J_j \in L_2(\mathbb{Z}) \\
\frac{\delta}{\delta J(x)} F[J(y)] & \to & \frac{\partial}{\partial J_j} F[J_1, J_2, \dots]
\end{array} \tag{120}
$$

Where $C(\mathbb{R})$ is a continuous function space and $L_2(\mathbb{Z})$ is ...

Example 1

$$
\begin{aligned}
\frac{\delta}{\delta J(x)} e^{i \int d^4y\, J(y)\phi(y)} &= i e^{i \int d^4y\, J(y)\phi(y)} \frac{\delta}{\delta J(x)} \left(\int d^4y\, J(y)\phi(y) \right) \\
&= i e^{i \int d^4y\, J(y)\phi(y)} \int d^4y\, \frac{\delta J(y)}{\delta J(x)} \phi(y) \\
&= i e^{i \int d^4y\, J(y)\phi(y)} \int d^4y\, \delta^{(4)}(x-y)\phi(y) \\
\frac{\delta}{\delta J(x)} e^{i \int d^4y\, J(y)\phi(y)} &= i\phi(x) e^{i \int d^4y\, J(y)\phi(y)}
\end{aligned}
$$

Example 2: Derivatives of Delta functions

$$
\begin{aligned}
\frac{\delta}{\delta J(x)} \int d^4y\, (\partial_\mu J(y)) v^\mu(y) &= \frac{\delta}{\delta J(x)} \left(\text{boundary term} - \int d^4y\, J(y) \partial_\mu v^\mu(y) \right) \\
&= -\partial_\mu v^\mu(x)
\end{aligned}
$$

Note that the boundary term is almost always zero, except for topologically interesting theories.

The Generating Functional

Define the generating functional as

$$
Z[J] = \lim_{T \to \infty(1-i\epsilon)} \int \mathcal{D}\phi\; e^{iS + iJ(x)\phi(x)}. \tag{121}
$$

This expression is obviously useful, since correlation functions are directly related to derivatives of $Z[J]$

$$
\langle \Omega | \, T[\hat{\phi}(x)\hat{\phi}(y)] \, | \Omega \rangle = \frac{-\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} Z[J] \big|_{J=0}}{Z[J] \big|_{J=0}} \tag{122}
$$

So, if you can compute the generating functional $Z[J]$, you have *all* of the n-point correlation functiosn via derivatives for your field theory.

In free field theories, such as the Klein-Gordon field, the action is quadratic in the field operators, and the argument of the exponential is $Z[J]$ is

$$i(S_0 + J(x)\phi(x) = i \int d^4x \, \left(\frac{1}{2}\phi(x)(-\partial^2 - m^2 + i\epsilon)\phi(x) + J(x)\phi(x) \right). \quad (123)$$

To homogenize quadraticity, complete the square by introducing the shift (with Jacobian = 1

$$\phi'(x) = \phi(x) - i \int d^4y \; D_F(x-y)J(y). \quad (124)$$

This is analogous to the positional shift $x' = x - \mathbf{A}^{-1}J$, and works becuase the Feynman propagator is the inverse of the Klein-Gordon operator, such that

$$(-\partial^2 - m^2)D_F(x-y) = i\delta^{(4)}(x-y) \quad (125)$$

With the variable change, the exponential argument becomes

$$\begin{aligned} i(S_0 + J(x)\phi(x) =& i \int d^4x \, \left(\frac{1}{2}\phi'(x)(-\partial^2 - m^2 + i\epsilon)\phi'(x) \right) \\ & - i \int d^4x \int d^4y \, \left(\frac{1}{2} J(x)(-iD_F(x-y))J(y) \right). \end{aligned}$$

So, the generating functional is then

$$Z[J] = Z_0 e^{-\frac{1}{2}\int d^4x d^4y \, (J(x)D_F(x-y)J(y))} \quad (126)$$

Where the free field contribution, independent of J, is

$$Z_0 = \int \mathcal{D}\phi' \; e^{i\int d^4x \, \left(\frac{1}{2}\phi'(x)(-\partial^2 - m^2 + i\epsilon)\phi'(x)\right)}. \quad (127)$$

Examples of Free Theory Correlations Functions

Example 1: The 2-point correlation function, with the Z_0 cancelled out,

$$\langle\Omega| \, T[\hat{\phi}(x)\hat{\phi}(y)] \, |\Omega\rangle = -\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} e^{-\frac{1}{2}\int d^4x d^4y \, (J(x)D_F(x-y)J(y))} \Big|_{J=0}. \quad (128)$$

Example: The 4-point correlation function, with notation $\hat{\phi}_i = \hat{\phi}(x_i)$, $J_i = J(x_i)$, and $D_{xi} = D(x - x_i)$

$$\langle 0| T[\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4] |0\rangle = \frac{\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4} Z[J]\big|_{J=0}}{Z[J=0]} \tag{129}$$

$$= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\left(-\int d^4x' \ J_{x'} D_{x'4} e^{-\frac{1}{2}\int d^4x \int d^4y J_x D_{xy} J_y}\right)\big|_{J=0} \tag{130}$$

$$= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\left(-D_{34} + \int d^4x' \int d^4y' \, J_{x'} D_{x'3} J_{y'} D_{y'4}\right) \times e^{\cdots}\big|_{J=0} \tag{131}$$

$$= \frac{\delta}{\delta J_1}\left(D_{34}\int d^4x' \, J_{x'} D_{x'2} + D_{24}\int d^4y' \, J_{y'} D_{y'3} + D_{23}\int d^4z' \, J_{z'} D_{z'4} + \mathcal{O}(J^2)\right) e^{\cdots}\big|_{J=0} \tag{132}$$

$$= D_{34}D_{12} + D_{24}D_{13} + D_{23}D_{14} \tag{133}$$

Interacting Fields

The time-ordered expectation value, which contains the generating functions by Taylor expansion, for the (classical) phi-fourth interacting theory is

$$\langle \Omega| T[\phi_1 \dots \phi_n] |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\,\phi e^{i(S_0+S_{int})}\phi(x_1)\dots\phi(x_n)}{\int \mathcal{D}\phi \, e^{i(S_0+S_{int})}} \tag{134}$$

Where $S_{int} = -\frac{i\lambda}{4!}\int d^4x \, \phi^4(x)$.

Fermionic Fields

For the (classical) *fermionic field* $\hat{\psi}$, the 2-point correlation function, vacuum expectation value, is

$$\langle \Omega| T[\psi(x)\psi(y)] |\Omega\rangle = \frac{\int \mathcal{D}\,\psi e^{iS}\psi(x)\psi(y)}{\int \mathcal{D}\psi \, e^{iS}} \tag{135}$$

Rule number one for this expression (1) is to not think abou this operationally, and rule number two (2) is to think in analogy to complex numbers, which can provide a more clear representation and make things easier.

The Fermi fields obey the relations

$$\psi^2(x) = 0 \tag{136}$$

$$\psi(x)\psi(y) = -\psi(y)\psi(x) \tag{137}$$

Vignette: Anticommuting Numbers (Grassman Numbers)

Let V be an n-dimensional vector space with basis $\theta_a \in V$, $a = 1, \ldots, n$. Thus, elements of the vector space $v \in V$ have the form

$$v = \sum_{a=1}^{n} v_a \theta_a. \tag{138}$$

To build a bigger vector space $\mathcal{G}(V)$ from V, we first endow V with the product operation denoted by concatenation (e.g., $\theta_a \cdot \theta_b \cdot \theta_c = \theta_a \theta_b \theta_c$).

This gives us an infinite dimensional vector space with span

$$\mathcal{S}^{\infty}(V) = \mathrm{span}\{\theta_a, \theta_a\theta_b, \theta_a\theta_b\theta_c, \ldots\} \tag{139}$$

Now restrict the basis to obey the following suggestive relations

$$\theta_a \theta_b = -\theta_b \theta_a \tag{140}$$

$$\theta_a^2 = 0. \tag{141}$$

Then the new vector space has dimension $\dim(\mathcal{G}(V)) = 2^n$, and is the infinite dimensional span modulo the elements of the underlying vector space

$$\mathcal{G}(V) = \mathcal{S}^{\infty}(V)/v. \tag{142}$$

(**Exercise**) Check that this structure is well-defined. Note that this is exactly the space of differential forms for a tangent space V (also known as the space of "classical fermions").

The basis of $\mathcal{G}(V)$ is now

$$\{1,\, \theta_a,\, \theta_a\theta_b,\, \theta_a\theta_b\theta_c, \ldots\} \tag{143}$$

With $a = 1, \ldots, n$, followed by $1 \le a < b \le n$, $a < b < c$, etc.

Then a general element of the new vector space $f \in \mathcal{G}(V)$ is

$$f = \alpha + \sum_{p=1}^{n} \sum_{1 \le j_1 < \cdots < j_p \le n} \alpha_{j_1 j_2 \ldots j_p}\, \theta_{j_1}\theta_{j_2} \ldots \theta_{j_p} \tag{144}$$

$$\text{, where } \alpha_{j_1 j_2 \ldots j_p} \in \mathbb{C}. \tag{145}$$

Lecture 6: Grassmann Numbers

We left off with an object meant to be the classical version of a fermion: a Grassman number. It is the object of a vector space $\mathcal{G}_n(V)$, generated by basis $\{1, \theta_1, \ldots, \theta_n \in V\}$. Imposing the anticommutation relation

$$\{\theta_j, \theta_k\} = 0, \, \forall j, k = 1, \ldots, n \tag{146}$$

This is a 2^n-dimensional vector space with monomial basis

$$\{1, \theta_1, \theta_2, \ldots, \theta_n, \theta_1\theta_2, \ldots, \theta_{n-1}\theta_n, \ldots, \theta_1\theta_2 \ldots \theta_n\}. \tag{147}$$

An arbitrary element $f \in \mathcal{G}_n(V)$

$$f = f_0 + \sum_{j_1 < \cdots < j_p} f_p(j_1, \ldots, j_p)\, \theta_{j_1} \ldots \theta_{j_p} \tag{148}$$

We now define functions, linear and nonlinear, representations, complex numbers, calculus, derivatives, and integrals of Grassman numbers.

Functions of Grassman Numbers

Think of θ_j as anticommuting numbers/variables.

Note: A "wrong" definition is to let f be an infinitely differentiable function $f \in C^\infty(\mathbb{R})$, adjoin the symbol $f(\theta_j)$, and impose the anticommutation relation $\{f(\theta_j), \theta_k\} = 0, \, \forall \, j, k$. Then each f will produce a new anticommuting object, which leads to an uncountably infinite number of objects, and is not what we expect from a function.

Linear functions should be linear maps from the vector space to itself, $F \in \mathcal{M}_{2^n}(\mathbb{C})$, the space of $2^n \times 2^n$ matrices

$$F : \; \mathcal{G}_n(V) \to \mathcal{G}_n(V) \tag{149}$$

Nonlinear functions will be defined by analogy to the functional calculus of matrices; consider a matrix $M \in \mathcal{M}_m(\mathbb{C})$. As long as M is diagonalizable, define the nonlinear function $f(M) = S^{-1}f(D)S$, where M is diagonalized by S, such that $M = S^{-1}DS$. Since a diagonal matrix occupies a commutative algebra, we know how to define function for D and then rotate from D to M via S.

In summary so far, one strategy to define functions of Grassman numbers is to represent $\mathcal{G}_n(V)$ as matrices and use functional calculus. Working with representations, we now need matrices to represent the Grassman numbers.

Representations of $\mathcal{G}_n(V)$

Consider a map from our vector space to a concrete space of matrices

$$\pi : \ \mathcal{G}_n(V) \to M_d(\mathbb{C} \tag{150}$$

Which must obey the anticommutation relation

$$\{\pi(\theta_j), \pi(\theta_k)\} = 0, \ \forall j, k. \tag{151}$$

Construct the "Jordan-Wigner representation", which will look very familiar. Begin with a Hilbert space $\mathcal{H} = \mathbb{C}^{2^n}$, with dimension 2^n, and Pauli operators

$$\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{152}$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{153}$$

Recall that these Pauli operators obey the relations

$$\{\sigma^+, \sigma^z\} = 0 \tag{154}$$

$$(\sigma^+)^2 = 0. \tag{155}$$

Construct the representation of $\mathcal{G}_n(V)$

$$\pi(\theta_1) = \sigma_1^+ \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_{n-1} \otimes \mathbb{I}_n \tag{156}$$

$$\pi(\theta_2) = \sigma_1^z \otimes \sigma_2^+ \otimes \cdots \otimes \mathbb{I}_{n-1} \otimes \mathbb{I}_n \tag{157}$$

$$\cdots \tag{158}$$

$$\pi(\theta_n) = \sigma_1^z \otimes \sigma_2^z \otimes \cdots \otimes \sigma_{n-1}^z \otimes \sigma_n^+ \tag{159}$$

Single variable representation ($n = 1$)

Let $F \in C^\infty(\mathbb{R}, \mathbb{R})$ and $\mathcal{G}_1(V) \simeq \{a + b\theta : \ a, b \in \mathbb{C}\}$. This should be consistent with the functional calculus of the representation $\pi(\cdot)$, such that $F(\pi(\cdot)) = S^{-1}F(D)S$.
To evaluate the function F, we write out the Taylor series, evaluate at $x = \theta$, and impose the defined relations (e.g., $\theta_j^2 = 0$, $\forall j$)

$$F(x) = \sum_{j=0}^{\infty} \frac{F^{(j)}(x=0)x^j}{j!} \tag{160}$$

$$F(\theta) = \sum_{j=0}^{\infty} \frac{F^{(j)}(\theta=0)\theta^j}{j!} \tag{161}$$

$$F(\theta) = F^{(0)}(\theta = 0) + F^{(1)}(\theta = 0) \cdot \theta \tag{162}$$

Example 1:

$$F(x) = \sin(x) \tag{163}$$
$$F(\theta) = (x - \frac{x^3}{3!} + \dots)\big|_{x=\theta} \tag{164}$$
$$F(\theta) = \theta \tag{165}$$

Example 2:

$$F(x) = x + x^3 \to F(\theta) = \theta \tag{166}$$

Multiple variable representation

Let $F \in C^\infty(\mathbb{R}^n, \mathbb{R})$, with the Taylor expansion

$$F(\theta_1, \dots, \theta_n) = F^{(0)}(\theta_1 = 0, \dots, \theta_n = 0) + \sum_{j=1}^{n} \theta_j \frac{\partial F(0, \dots, 0)}{\partial \theta_j} + \mathcal{O}(\theta^2) \tag{167}$$

Example 1: $n = 2$

$$F(x, y) = e^{-\lambda xy} \tag{168}$$
$$F(\theta_1, \theta_2) = (1 - \lambda xy + \lambda^2 x^2 y^2 + \dots)\big|_{x=\theta_1, y=\theta_2} \tag{169}$$
$$F(\theta_1, \theta_2) = 1 - \lambda\theta_1\theta_2 \tag{170}$$

Example 2: $n = 2$

$$F(x, y) = e^{-\lambda_1 x - \lambda_2 y} \tag{171}$$
$$F(\theta_1, \theta_2) = (1 - \lambda_1\theta_1)(1 - \lambda_2\theta_2) \tag{172}$$
$$F(\theta_1, \theta_2) = 1 - \lambda_1\theta_1 - \lambda_2\theta_2 + \lambda_1\lambda_2\theta_1\theta_2 \tag{173}$$

Note that even though we lose higher order terms such as θ_j^2, nonlinear features are preserved in the multivariable case.

Complex Grassman Numbers

Let θ_1, $\theta_2 \in \mathcal{G}_2(V)$, a 4-dimensional vector space with basis $\{1, \theta_1, \theta_2, \theta_1\theta_2\}$, and define the quantities θ and $\theta^* \in \mathcal{G}_2(V)$ as

$$\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}} \tag{174}$$
$$\theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}}. \tag{175}$$

To extend to the multivariable case, define

$$\theta_j^* = \frac{\theta_{j_1} - i\theta_{j_2}}{\sqrt{2}} \tag{176}$$

Where $\theta_{j_1}, \theta_{j_2} \in \mathcal{G}_{2n}(V)$.

Grassman Derivatives

Define the derivative with respect to Grassman variables as the map

$$\partial_{\theta_j} : \mathcal{G}_n(V) \to \mathcal{G}_n(V) \tag{177}$$

Which obeys the relations

1. $\partial_{\theta_j}(\theta_k) = \delta_{jk}$
2. $\partial_{\theta_j}(\theta_{k_1} \ldots \theta_{k_p}) = \delta_{jk_1}\theta_{k_2} \ldots \theta_{k_p} - \delta_{jk_2}\theta_{k_1}\theta_{k_3} \ldots \theta_{k_p} + \cdots + (-1)^{p-1}\delta_{jk_p}\theta_{k_1} \ldots \theta_{k_{p-1}}$.

The second relation follows from the anticommutation relation, and can be thought of as bringing the corresponding θ_j to the front via anticommutations and then differentiating.
For example,

$$\partial_{\theta_2}(\theta_1\theta_2) = -\partial_{\theta_2}(\theta_2\theta_1) = -\theta_1. \tag{178}$$

(**Exercise**) These relations can be extended by linearity for any Grassman number. This definition of ∂_{θ_j} obeys the product rule and the chain rule.

Grassman Integrals

Following the analogy of the common definite integral, the Grassman integral should be a linear map which obeys shift invariance, such that $\theta \to \theta + \eta$,

$$\int d\theta : \mathcal{G}_n(V) \to \mathbb{C}. \tag{179}$$

The only consistent definition to satisfy these two constraints for a single Grassman variable is

$$\int d\theta \ (a + b\theta) = b, \ a, b \in \mathbb{C} \tag{180}$$

Note the very interesting property here that, by this definition, the integral and the derivative are the exact same thing

$$\int d\theta \, (a + b\theta) = \partial_\theta(a + b\theta). \tag{181}$$

This also holds for the multivariable case, where the highest order term is picked off from the Taylor series

$$\int d\theta_n \ldots d\theta_1 \, (f_0 + \sum_{j_1 < \cdots < j_p} f_p(j_1, \ldots, j_p)\theta_{j_1} \ldots \theta_{j_p}) = f_n(1, \ldots, n). \tag{182}$$

This is a weird definition, but it works, behaves correctly under change of variables, and does what we need it to do.

Example: $n = 2$ independent Grassman numbers

$$\int d\theta^* d\theta \, e^{-\lambda\theta^*\theta} = \int d\theta^* d\theta \, (1 - \lambda\theta^*\theta) \tag{183}$$

$$= \partial_{\theta^*}\partial_\theta \, (1 - \lambda\theta^*\theta) \tag{184}$$

$$\int d\theta^* d\theta \, e^{-\lambda\theta^*\theta} = \lambda \tag{185}$$

Note that the order of integration (θ first, θ^* second) is by convention.

Multivariable Gaussian Integrals

Consider the multivariable Gaussian integral

$$I = \int d\theta_1^* d\theta_1 \ldots d\theta_n^* d\theta_n \; e^{-\sum_{j,k} \theta_j^* B_{jk} \theta_k} \tag{186}$$

Where $B^\dagger = B$ and we diagonalize via $U^\dagger B U = D$.

Make a change of variables $\theta'_j = U_{jk}\theta_k$, where U is a unitary matrix, such that the product of these new Grassman variables is (**Exercise**)

$$\theta'_1\theta'_2 \ldots \theta'_n = \sum_{k_1, \ldots, k_n} U_{1k_1} U_{2k_2} \ldots U_{nk_n} \theta_{k_1}\theta_{k_2} \ldots \theta_{k_n} \tag{187}$$

$$= \sum_{\pi \in S_n} U_{1\pi(1)} \ldots U_{n\pi(n)} \mathrm{sgn}(\pi)\theta_1\theta_2 \ldots \theta_n \tag{188}$$

$$\theta'_1\theta'_2 \ldots \theta'_n = \det(U)\theta_1\theta_2 \ldots \theta_n \tag{189}$$

So the Gaussian integral becomes

$$I = \int d{\theta'}_1^* d\theta'_1 \ldots d{\theta'}_n^* d\theta'_n \; e^{-\sum_{j,k} {\theta'}_j^* (UBU^\dagger)_{jk} \theta'_k} \det(U)\det(U^\dagger) \tag{190}$$

$$= \int d{\theta'}_1^* d\theta'_1 \ldots d{\theta'}_n^* d\theta'_n \; e^{-\lambda_1 {\theta'}_1^* \theta'_1} \ldots e^{-\lambda_n {\theta'}_n^* \theta'_n} \cdot (1) \cdot (1) \tag{191}$$

$$= \lambda_1\lambda_2 \ldots \lambda_n \tag{192}$$

$$I = \det(B) \tag{193}$$

Recall that for the normal Gaussian integral case, we got $I = (\det(B))^{-1}$.

(**Exercise**) The generating functional for the Grassman calculus is, where J is a vector of Grassman numbers,

$$Z[J] = \int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n \; e^{-\theta^\dagger B\theta + J^\dagger \theta + \theta^\dagger J} \tag{194}$$

$$= e^{J^\dagger B^{-1} J}. \tag{195}$$

The moments are (**Exercise**)

$$\int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n \; \theta_j \theta_k^* e^{-\theta^\dagger B\theta} = \det(B) \cdot [B^{-1}]_{jk} \tag{196}$$

Side: The mixing of regular and Grassman numbers provides the basis for the supersymmetric method. Consider the Gaussian integral

$$\int d\Phi e^{-\Phi^\dagger M\Phi} = \int dx_1 \dots dx_n dx_1^* \dots dx_n^* \int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n \, e^{-\Phi^\dagger M\Phi} \tag{197}$$

$$= (\det A)(\det A)^{-1} \tag{198}$$

$$\int d\Phi e^{-\Phi^\dagger M\Phi} = 1 \tag{199}$$

Where $\Phi = (x_1, \dots, x_n, \theta_1, \dots, \theta_n)$ and M is a $2n \times 2n$ matrix

$$M = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \tag{200}$$

Lecture 7: Functional Quantization of the Dirac Field

We now employ Grassmann numbers/variables to build a path integral-like object that provides the n-point correlation functions for the Dirac (spinor) field.

Consider the Grassmann integral of the complex Grassmann variables θ and θ^*, the Grassmann Gaussian generating function

$$Z[J] = \left(\prod_{j=1}^{n} \int d\theta_j^* d\theta_j \right) e^{-\sum_{j,k} \theta_j^* B_{jk} \theta_k + \sum_j (J_j^* \theta_j + \theta_j^* J_j)} \tag{201}$$

Where the Grassmann variables and auxiliary fields J and J^* obey the anticommutation relations

$$\{\theta_j, \theta_k^*\} = \{\theta_j, \theta_k\} = \{\theta_j^*, \theta_k^*\} = 0 \tag{202}$$
$$\{J_j, \theta_k\} = \{J_j^*, \theta_k\} = \{J_j, J_k^*\} = 0. \tag{203}$$

Calculating these Gaussian integrals, the generating functional becomes

$$Z[J] = e^{-\sum_{j,k} J_j^* [B^{-1}]_{jk} J_k}. \tag{204}$$

Since the matrix B is unitary, such that $B^\dagger = B$, the generating functional $Z[J]$ is Hermitian.

Note that the generating functional takes a vector of Grassmann numbers, anticommuting objects, as input and yields an expression quadratic in the Grassmann numbers which evaluates to a real number, a commuting object, since observables correspond to Hermitian operators and real numbers as their eigenvalues; the expectation value must always be a real number, not a Grassmann number.