Introduction

In this note(s), I will work with a matrix A with m rows and n columns. A can be viewed as a function that maps a vector, $\vec{v}$, in $\mathbb{R}^n$ to $A\vec{v}$ in $\mathbb{R}^m$. I have two, closely related, goals for this note concerning this matrix A. The first is to prove the Rank Nullity Theorem, $N = \mathrm{Dim}(\mathrm{Range}(A)) + \mathrm{Dim}(\mathrm{Nullspace}(A))$. And the second is to discus decomposing the domain of A (viewed as a function), $\mathbb{R}^n$, as the direct sum of the nullspace of A and any complementary subspace to it in $\mathbb{R}^n$. Note that for the second goal, one such complementary subspace to the nullspace is the rowspace, its orthogonal complement in $\mathbb{R}^n$.

Appendix: Linear Independence, Dependence, Redundancy, Nullspace

Let me start by reviewing the definition(s) of linear independence and dependence.

A set of vectors $\{a_1, a_2, ..., a_n\}$ is linearly independent if the only way to form the zero vector, $\vec{0}$, by taking a linear combination of the them is when all the weights are 0. That is, $x_1\vec{a_1} + x_2\vec{a_2} + ... + x_n\overrightarrow{a_n} = \vec{0}$ only when all the x_i are themselves 0. All the x_i being 0 is known as the trivial solution to this equation. Note, the above equation can be written in matrix form as $A\vec{x} = \vec{0}$ where $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. Then, if the only solution to this equation is $x = \vec{0}$, $\{a_1, a_2, ..., a_n\}$ is linearly independent. And matrix A has a trivial nullspace only containing the zero vector.

And if a set of vectors is not linearly independent, then it is linearly dependent. This means that there exists a nontrivial solution to $x_1\vec{a_1} + x_2\vec{a_2} + ... + x_n\overrightarrow{a_n} = \vec{0}$.

An upshot is of this is that at least one of the a_i can be expressed as a linear combination of the remaining vectors. To see this for a set of linearly independent vectors, consider the nontrivial solution, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. At least one of the x_i is not 0 so from $x_1\vec{a_1} + x_2\vec{a_2} + ... + x_n\overrightarrow{a_n} = \vec{0}$,

$$x_i\vec{a_i} = \sum_{j \neq i} -x_j\vec{a_j}$$

$$\vec{a_i} = \sum_{j \neq i} -\left(\frac{x_j}{x_i}\right)\vec{a_j}$$

Another upshot is that the matrix A, again formed by concatenating the a_i, has a nontrivial nullspace. Since $A\vec{x} = \vec{0}$ for a non-trivial $\vec{x}$, the nullspace of A contains that non-zero $\vec{x}$ at the very least. Moreover, the nullspace contains all scalar multiples of $\vec{x}$ as well, $\mathrm{span}(\vec{x})$ or $k\vec{x}$. I can show this by taking the equation, $x_1\vec{a_1} + x_2\vec{a_2} + ... + x_n\overrightarrow{a_n} = \vec{0}$, and multiplying both sides by k to get $kx_1\vec{a_1} + kx_2\vec{a_2} + ... + kx_n\overrightarrow{a_n} = \vec{0}$. $k\vec{0} = \vec{0}$ and thus $k\vec{x}$ also satisfies the equation $A(k\vec{x}) = \vec{0}$.

Linear Independence And Uniqueness

Say I have 2 vectors $\vec{a_1}$ and $\vec{a_2}$ that are linearly independent. Then I will show that $A\vec{x} = \vec{b}$ has exactly one solution. If $\vec{b} = \vec{0}$, this is evident by the definition of linear independence, as the solution is $\vec{x} = \vec{0}$.

Suppose for contradiction that $A\vec{x} = \vec{b}$ has two solutions, $\vec{u}$ and $\vec{v}$ where $\vec{u} \neq \vec{v}$, then:

$$A\vec{u} = A\vec{v} \Rightarrow A\vec{u} - A\vec{v} = \vec{0} \Rightarrow A(\vec{u} - \vec{v}) = 0$$

But this means

Axler proof

Key idea: Redundancy and NullSpace Where last implication follows from linearity. So if $\vec{u} \neq \vec{v}$, I have found a non-trivial vector in the nullspace of A, $\vec{u} - \vec{v}$.