Sums of Powers Of Two

Introduction

My goal is to explore the following relation:

$$1 + 2 + 4 + 8 + ... + 2^n = 2^{n+1} - 1 \tag{1}$$

I'll start by providing the standard proof by induction. Then I'll discuss how I personally reason about it and find it plausible. This article will include tree visualizations, focus on the recursive nature of this sum, discuss how I use approximations to simplify my reasoning, and finally, I'll conclude with 2 bonus sections where I walk through another sum of terms recurrence relation and where I discuss both problems from a Computer Science and programming perspective.

Prerequisites

Knowledge of proof by induction. Which I believe people learn in high school when they take Precalculus or in their second Algebra course.

For the bonus CS section at the end, the main prerequisite is knowing binary arithmetic and generally having some introductory level experience.

Proof By Induction

I'll focus on the left-hand side (LHS) and right-hand side (RHS) of Equation 1 separately and express them as functions.

For the LHS, I'll define the function:

$$S(n) = 2^0 + 2^1 + ... + 2^n \tag{2}$$

This function has $n + 1$ terms. And here it is expressed as a sum:

$$S(n) = \sum_{i=0}^{n} 2^i \tag{3}$$

For the RHS, I'll define the function:

$$F(n) = 2^{n+1} - 1 \tag{4}$$

Let P_n be the claim that $S(n) = F(n)$. I want to prove P_n is true for all $n = 0, 1, 2, ...$ To this end, I'll use induction. The first step is to check the base case, P_0. $S(0) = 2^0 = 1$ and $F(0) = 2^{0+1} - 1 = 1$. $S(0) = F(0)$. Done.

Next, if I can show $P_k \Rightarrow P_{k+1}$, I'll be done. By induction, I would have proved Equation 1.

Here is the claim P_k:

$$2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1 \tag{5}$$

And here is the claim P_{k+1}:

$$2^0 + 2^1 + ... + 2^n + 2^{n+1} = 2^{n+2} - 1 \tag{6}$$

Let me prove that $P_k \Rightarrow P_{k+1}$ in 3 ways:

1. the textbook way
2. essentially the textbook way, but with a focus on the recursive natures of $S(n)$ and $F(n)$.
3. a visual approach, using tree diagrams

Approach 1: Textbook Induction

Assuming Equation 5 is true, I want to show that Equation 6 is true as well. Observe that I can transform the LHS of Equation 6 by plugging in the RHS of Equation 5. I'll do that:

$$2^{n+1} - 1 + 2^{n+1} = 2^{n+2} - 1 \tag{7}$$

Write both sides in terms of 2^n using exponent rules:

$$2^n * 2 - 1 + 2^n * 2 = 2^n * 4 - 1 \tag{8}$$

Factor 2^n in the LHS:

$$2^n * 4 - 1 = 2^n * 4 - 1 \tag{9}$$

And we are done. We have successfully proved that $P_k \Rightarrow P_{k+1}$. And since we already verified the base case, our proof of Equation 1 is complete.

Approach 2: Induction emphasizing recursive definitions

So the previous proof felt slightly unsatisfactory. What I really want to know, besides simply proving the correctness of Equation 1 is more insight as to *why* it's true. If someone looks at $S(n)$ with fresh eyes, defined in Equation 2, would they be able to come up with $F(n)$, defined in Equation 4? Why is $F(n)$ plausible? Well, the prior proof, at least to me, did not seem to help me too much answer these questions. So the proofs in this subsection and the next attempt to answer my questions

Ok, so actually I slightly lied. The last proof actually did help me, namely one key step in it. And that key step was the substitution of the RHS of Equation 5 into Equation 6. This exploited, and more importantly, displayed, the recursive structure of $S(n)$. That is, we noticed that $S(n+1)$ expanded out contained $S(n)$.

Approach 3: "Approximate" Visual Induction

Draw the tree and table. Possibly to do so side-by-side? As figuring out the pattern behind sums of powers of 2, it's likely an observer would simply notice the pattern by looking at this table. Maybe expanding out a few more levels to convince themselves of the increasingly promising pattern they've formulated that is $F(n)$.

This section, I'd like to introduce a way I reasoned about $F(n)$ being plausible. Again, from the previous section, the key idea is the exponential growth of $S(n)$. Visually at each level, 2^d more nodes are introduced. So my candidate function to match or approximate $S(n)$ could grow exponentially. And the base is 2. So why not simply try the function 2^n? Indeed $S(n) \cong 2^n$. And I'll present a visual "proof" of this.