Inverses

I am going to review a few fundamental properties of inverses of functions. First, a way to think about them. Next, a way to compute them. A geometric interpretation of them as a reflection about the line $y = x$ and derive this.

And wrap it off with the computing derivative of inverse function.

Audience

This is targeted at, or at least assumes, basic precalculus understanding and the optional ending section differential

Motivation, Definition, Computation

Say I have a function, f, of x. f takes in x and produces $f(x)$. So for the function, f, I have this pair $(x, f(x))$. Todo draw picture of x and an arrow over a box, f, that points to $f(x)$. Now the inverse reverses the direction of the arrow and completes a loop. So for the inverse function, f^-1, I have this pair $(f(x), x)$. Given $f(x)$ as the input, it outputs x. Note the swapping. I have heard the terms 'image' and 'preimage': $f(x)$ is the image, and x is the preimage. And functions take in preimages and return their images. Inverses then take in images and return the preimages whose image is in the input image.

There is some relation between x and $f(x)$ that the function f defines. Now the inverse is not some magical, and especially some new relation I pulled out of thin air. Rather, it is but the exact same relation f at play except with this idea of swapping: f applied to the output of the inverse function f^-1 yields the input of the inverse function. Whereas for the original function, f, f applied to its input yields its output.

Computation: Swapping Simply switch x and y to compute the inverse function.

Derivative of inverses

I will motivate this identity:

$$\frac{\partial(f^{-1})}{\partial x} = \frac{1}{\frac{\partial f}{\partial x} \mid x = f^{-1}(x)}$$

Let me start by considering the functions $f(x) = x^n$ and its inverse $f^{-1}(x) = x^{\frac{1}{n}}$. For f, assuming the input is significantly greater than 1, a small change to the input is associated with a large change to the output. The exact relation quantifying the ratio of the changes in output to input is given by the power rule of differentiation.

And I want to say something like "a small change to the output of f^{-1} is associated with a large change of the input of f^{-1}".

Let me actually define two points on f: $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$. $\Delta y = f(x + \Delta x) - f(x)$ and $(\Delta y) = k * \Delta x$ where k is some large scaling factor that, for a very small Δx, is the derivative at x, $k \approx n * x^{n-1}$. $\frac{\Delta y}{\Delta x} = k \approx \frac{\partial y}{\partial x} \mid x = x$. Again, Δx and Δy are both very small, in fact as $\Delta x \to 0$, $\Delta y \to 0$. So both these points are very close together on the graph of f.

Thus now I have, for free, information on two close together points on the graph of f^{-1}, the points $(f(x), x)$ and $(f(x + \Delta x), (x + \Delta x))$. The difference quotient that these two points specify, which approxmiates the derivative of f^{-1} at $f(x)$, is then $\frac{1}{k}$ as the "rise" and "run" get swapped. $\frac{\Delta \text{ rise}}{\Delta \text{ run}} = \frac{\Delta x}{f(x+\Delta x)-f(x)} = \frac{1}{k}$. Thus I have demonstrated the reciprical nature of the derivatives of f and f^{-1}.

The remaining detail is, for the derivative of f^{-1}, where the involved derivatives are specified at. Consider the pair that is a point on f^{-1}, (x, y). The roles of 'x' and 'y', are swapped as they are traditionally used in the context of f: $f^{-1}(x) = y$ where $f(y) = x$. So $f^{(-1)'}(x) = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$.

Just for fun, let me also get this result via implicit differentiation.

TODO and implicit differentiation same idea y = x^n. x = y^n. Again seeing swapping at play. The input x, has corresponding y such that y^n = x. And in this relation, the derivative nature of the variables are swapped. Small x delta induces large y delta in original fn. But in the inverse, small y associated with large x change as x = y^n shows. Already notions of their derivaties being recriprocals of each other are suggestive to me.

Bonus Bonus: Derivatives of inverses of functions whose derivatives 'resemble' said functions

Two examples, $f(x) = e^x$ and $f(\theta) = \sin(\theta)$.

Let me start with the more involved example first, $f(\theta) = \sin(\theta)$. Then $f'(\theta) = \cos(\theta)$. And I view cos and sin as complements, two different legs of the same right triangle, united by the pythagorean relation: $(\sin(\theta))^2 + (\cos(\theta))^2 = 1$. So I have: $\cos(\theta) = \sqrt{1 - (\sin(\theta))^2}$, so the derivative of $\sin(\theta)$, $\cos(\theta)$, can be simply expressed as a function of $\sin(\theta)$.

So what about $f^{-1'}(x)$? First step back and view the action of f^{-1} as $f^{-1}(x) = \theta \mid \sin(\theta) = x$. So f contains the coordinate pair (θ, x) while f^{-1} contains the coordinate pair (x, θ). $\frac{\partial f}{\partial \theta} = \cos(\theta)$. So $\frac{\partial f^{-1}}{\partial x} = \frac{1}{\cos(\theta)}$. But as $\cos(\theta)$ resembles $f(\theta) = \sin(\theta)$ by the Pythagorean relation $\frac{\partial f^{-1}}{\partial x} = \frac{1}{\sqrt{1-(\sin(\theta))^2}} = \frac{1}{\sqrt{1-x^2}}$. The last step, substituting x for $\sin(\theta)$, defines my idea.

(The way I learned this in high school was drawing a right triangle with hypoteneous 1 which is equivalent) (1/cos(arcsin(x)) and use a triangle to compute the denominator arcsin(x) takes you from a ratio scalar, x, to an angle, but the deriviitive of sin(theta) is cos(theta), which takes in an angle, and sends it back to a ratio scalar. So it's like you start in the "ratio-scalar/X" world, go to "angle/theta" world via the inverse but then the derivative sends you back from "theta" to "X" world (cos and sin are both ratio-scalars). It all hinges on the derivative having this world mapping direction action. TODO make last idea more clear, literally just pasting my mental dialogue here lol)

Outcome versus Expectation I would expect derivative of arcsin to involve some trig function(s). And I don't have any clue, no guesses, about the derivative of ln, but 1/x seems quite surprising and well, arbitrary to me. If you consider for $f(x) = y = e^x$, $f'(x) = e^x = y$ so $f^{-1'}(y) = \frac{1}{y}$ (idea is to use 'y' as the canoical output variable of f but input variable of f^{-1}). Well if you squint for the trig one, you see the pythagorean relation. But it's clean and involves no trig. And the "cleanliness" of both stems from the same resemblance of the f and f' And this resemblance shows up in $f^{-1'}$ as well, which I find neat, maybe obvious, but still neat.