Relation of Dot Product with Projection and some Pythagoream Theorem Proofs

Prerequisites

High School level Geometry, Trigonometry, and Linear Algebra

Introduction

My goal is to explore the relation between the dot product of two vectors and the projection of the vectors onto each other. I will work with a heavily constrained case, the dot product of vectors that: are 2 dimensional (in $\mathbb{R}^2$) and are unit vectors, that is, they have a magnitude of 1.

Problem Statement and Motivation

Let's say you are standing on an origin of a 2D coordinate plane. You may move a distance of 1, that is, you may move to any point P with coordinates (x, y) on the unit circle. That is, $x^2 + y^2 = 1$ by the pythagorean theorem (which I will derive in multiple ways in this note)! That also is, $x = \cos(\theta)$ and $y = \sin(\theta)$ where θ is the angle the vector from the origin to the point makes with the x-axis. And you get a reward R where $R(P) = R(x, y) = a * x + b * y$. For convenience, let's also have $a^2 + b^2 = 1$. And so what I am interested in is the dot-product of the two unit vectors: $\binom{a}{b} \cdot \binom{x}{y}$. Additionally, I am also interested in finding out what $\binom{x}{y}$ maximizes this dot-product. The answer turns out to always be $\binom{x}{y} = \binom{a}{b}$ and I'll show this geometrically by projection.

So I "know" or rather, I want to prove:

$$a \cdot b = \mid a \mid * \mid b \mid * \cos(\theta) \tag{1}$$

But since I am dealing with unit vectors:

$$a \cdot b = \cos(\theta) \tag{2}$$

For two unit vectors, $\mid \cos(\theta) \mid$ is the magnitude of the projection of either vector onto the other. For this note, I'll focus on the $\binom{a}{b}$ as the vector I am projecting the vector $\binom{x}{y}$ onto. And I know that the maximum value of $\cos(\theta)$ is 1 when $\theta = 0$ or when $\binom{x}{y} = \binom{a}{b}$.

To reiterate, my goal is to prove Equation 2, or equivalently, that $|a \cdot b|$ is the magnitude of the projection of $\binom{x}{y}$ onto $\binom{a}{b}$.