Sums of Powers Of Two

Introduction

My goal is to explore the following relation:

$$1 + 2 + 4 + 8 + ... + 2^n = 2^{n+1} - 1 \tag{1}$$

I'll start by providing the standard proof by induction. Then I'll discuss how I personally reason about it and find it plausible. This article will include tree visualizations, focus on the recursive nature of this sum, discuss how I use approximations to simplify my reasoning, and finally, I'll conclude with 2 bonus sections where I walk through another sum of terms recurrence relation and where I discuss both problems from a Computer Science and programming perspective.

Prerequisites

Knowledge of proof by induction. Which I believe people learn in high school when they take Precalculus or in their second Algebra course.

For the bonus CS section at the end, the main prerequisite is knowing binary arithmetic and generally having some introductory level experience.

Proof By Induction

I'll focus on the left-hand side (LHS) and right-hand side (RHS) seperately and express them as functions.

For the LHS, I'll define the function:

$$S(n) = 2^0 + 2^1 + ... + 2^n \tag{2}$$

This function has $n + 1$ terms. And here it is expressed as a sum:

$$S(n) = \sum_{i=0}^{n} 2^i \tag{3}$$

For the RHS, I'll define the function:

$$F(n) = 2^{n+1} - 1 \tag{4}$$

Let P_n be the claim that $S(n) = F(n)$. I want to prove P_n is true for all $n = 0, 1, 2, ...$ To this end, I'll use induction. The first step is to check the base case, P_0. $S(0) = 2^0 = 1$ and $F(0) = 2^{0+1} - 1 = 1$. $S(0) = F(0)$. Done.

Next, if I can show $P_k \Rightarrow P_{k+1}$, I'll be done.

Here is the claim P_k:

$$2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1 \tag{5}$$

And here is the claim P_{k+1}:

$$2^0 + 2^1 + ... + 2^n + 2^{n+1} = 2^{n+2} - 1 \tag{6}$$

Let me prove this the textbook way and then repeat it but with a focus on the recursive natures of $S(n)$ and $F(n)$.

Approach 1: Textbook Induction

Assuming Equation 5 is true, I want to show that Equation 6 is true as well. Observe that I can transform the LHS of Equation 6 by plugging in the RHS of Equation 5. I'll do that:

$$2^{n+1} - 1 + 2^{n+1} = 2^{n+2} - 1 \quad (7)$$

Write both sides in terms of 2^n using exponent rules:

$$2^n * 2 - 1 + 2^n * 2 = 2^n * 4 - 1 \quad (8)$$

Factor 2^n in the LHS:

$$2^n * 4 - 1 = 2^n * 4 - 1 \quad (9)$$

And we are done. We have successfully proved that $P_k \Rightarrow P_{k+1}$.

Approach 2: Induction emphasizing recursive definitions

So the previous proof felt slightly unsatisfactory. The key step was the substitution of the RHS of Equation 5 into Equation 6. This exploited the recursive structure of $S(n)$. That is, we noticed that $S(n + 1)$ expanded out contained $S(n)$.