AMATH 751: Final Project
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1 Introduction

The purpose of this paper is to examine a novel form of the 2 dimensional rocket equation presented in Kang,Y., Bae S. (2006) [1]. The equation is presented without proving any mathematical properties of the solution, so all properties presented here are derived by the author of this paper. In particular, this paper proves existence and uniqueness of the solution, stability of equilibrium points, provides periodic solutions and proves the system of ODEs is Lipschitz continuous. Accompanied with these proofs are some plots of orbits of the solution which are obtained numerically using the 4th order Runge-Kutta discretization method. The structure of the paper is as follows: section 1 provides a derivation of the speed-flight angle form of the rocket equation which was not fully provided by [1]. Section 2 provides some analytical properties of the system of ODEs given in defining the rocket equation in [1]. Lastly, section 3 explores the some numerical solutions.

2 Formulation of Rocket Equation

The form of the rocket equation, which will be considered here, is in terms of the speed and flight angle of the rocket relative to the horizontal x-axis. To arrive at the speed-angle form of the rocket equation, first consider the usual formulation given in (1).

$$M\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{dM}{\mathrm{d}t}\vec{u} + \vec{F}_{tot} \tag{1}$$

Here M $\vec{v}$, $\vec{u}$ and $\mathrm{F_{tot}}$ are the rocket mass, velocity, exhaust velocity and the sum of the total forces acting on the rocket, respectively. Begin by defining θ to be the angle between the velocity vector $\vec{v}$ and the horizontal x-axis and $v = |\vec{v}|$ the speed of the rocket. We identify the x and y components of the velocity and total forces as v_1, v_2 and F_{tot_1}, F_{tot_2}, respectively. Since $\cos\theta = v_1/v$ and $\sin\theta = v_2/v$, the velocity of the rocket can be expressed in terms of speed and flight angle:

$$\vec{v} = v\cos(\theta)\hat{i} + v\sin(\theta)\hat{j} \tag{2}$$

Next, taking the exhaust velocity to be in the opposite direction of the rocket velocity velocity:

$$\vec{u} = -u\frac{\vec{v}}{|\vec{v}|} = -u\frac{\vec{v}}{v} \tag{3}$$

where $u = |\vec{u}|$ is a constant. Decomposing the forces into the force of gravity $\vec{F}_g$, air resistance $\vec{F}_{air}$ and inserting (2) and (3) into (1) the system has the component form of:

$$M\left[\dot{v}\cos(\theta) - v\sin(\theta)\dot{\theta}\right] = -u\dot{M}\cos(\theta) - F_{air_1} \tag{4}$$

$$M\left[\dot{v}\sin(\theta) + v\cos(\theta)\dot{\theta}\right] = -u\dot{M}\sin(\theta) - F_{g_2} - F_{air_2} \tag{5}$$

Multiplying equations (4) and (5) by $\sin(\theta)$ and $\cos(\theta)$ respectively, the equation (6) = (4) + (5) and equation (7) = (4) − (5).

$$M\dot{v} = -u\dot{M} - (F_{air_1}\cos(\theta) + F_{air_2}\sin(\theta) + \sin(\theta)F_{g2}) \tag{6}$$

$$Mv\dot{\theta} = F_{air_1}\sin(\theta) - F_{air_2}\cos(\theta) - F_{g_2}\cos(\theta) \tag{7}$$

Representing the components of the vector $\vec{F}_{air} = (F_{air_1}, F_{air_2})$ in polar coordinates: $F_{air_1} = |\vec{F}_{air}|\cos(\theta)$ and $F_{air_2} = |\vec{F}_{air}|\sin(\theta)$, and upon plugging this in to (6) and (7) and rearranging to obtain the final system:

$$\dot{v} = -u\frac{\dot{M}}{M} - \frac{F_g}{M}\sin(\theta) - \frac{F_{air}}{M} \tag{8}$$

$$\dot{\theta} = -\frac{F_g}{Mv}\cos(\theta) \tag{9}$$

where $F_g = |\vec{F}_g| = |F_{g_2}|$ and $F_{air} = |\vec{F}_{air}|$. It is immediately clear that (9) is discontinuous at $v = 0$. To restrict the problem to the case where the RHS of (8) and (9) are continuous functions, redefine (9) as:

$$\begin{aligned} \dot{\theta} &= -\frac{F_g}{Mv}\cos(\theta), \quad v \geq 1 \\ \dot{\theta} &= -\frac{F_g}{M}\cos(\theta), \quad v < 1 \end{aligned} \tag{10}$$

Note that a shortcoming of this modification is that if $v < 1$ the solution will be unrealistic, but for most reasonable sets of parameters $v \geq 1$ for all t where it is defined. Also, (10) has discontinuous partial derivatives at $v = 1$, hence we will have to use Lipschitz continuity to prove uniqueness of the solution. Note that this paper will assumed that the system given by (8) and (10) is such that $M(t) > 0$, $F_g \geq 0$ and $\dot{M}(t) \leq 0$ and are globally continuous and bounded functions of time. Also, F_{air} is of the form $F_{air} = c(t)v(t)$ where $c(t)$ is a strictly positive, globally continuous and bounded function of time. Hence the system given by (8) and (10) is continuous over the domain $\mathbb{R} \times \mathbb{R}^2$. Lastly, denote the RHS of (8) and (10) as $f(t, v, \theta)$.

3 Properties of the System

Employing the assumptions stated above, a few analytically properties of the solution are proved in this section. Immediately we can deduce from the Existence Theorem for ODEs, since f is continuous, for any set of initial conditions $v(t_o) = v_o \geq 0$ and $\theta(t_o) = \theta_o$, there exists at least one solution on some interval containing t_o. On the other hand f not C^1, hence for uniqueness, we will show that the system is satisfies the Lipschitz condition.

3.1 Lipschitz Condition

For the system given by (8) and (10), if F_g and M are such that $\sup_{t \in \mathbb{R}} F_g/M$ exists, then f is globally Lipschitz in v and θ.

Proof The proof will be to show that each component of f is Lipschitz, hence f is Lipschitz itself. First, (8) clearly has continuous partial derivatives in v and θ, hence is Lipschitz. For the system given in (10) let $\mathbf{x} = (v_1, \theta_1)^T$ and $\mathbf{y} = (v_2, \theta_2)^T$ in $\mathbb{R}^2$.
Case 1: $v_1,\ v_2 \geq 1$

$$|f_2(t,\mathbf{x}) - f_2(t,\mathbf{y})| = \frac{F_g}{M}\left|\frac{cos(\theta_1)}{v_1} - \frac{cos(\theta_2)}{v_2}\right| \leq \frac{F_g}{M}|cos(\theta_1) - cos(\theta_2)| \qquad (11)$$

Numerically it can be shown that $(cos(x_1) - cos(x_2))^2 \leq (x_1 - x_2)^2$, hence we have the inequality:

$$(11) \leq \frac{F_g}{M}\left[(\theta_1-\theta_2)^2\right]^{1/2} \leq \frac{F_g}{M}\left[(\theta_1-\theta_2)^2 + (v_1-v_2)^2\right]^{1/2} \leq L_1|\mathbf{x}-\mathbf{y}|$$

where $L_1 = \sup\limits_{t\in\mathbb{R}} \dfrac{F_g}{M}$.

The other cases where where $v_1 < 1$ and/or $v_2 < 1$ follow a similar proof and are omitted here. Hence, the components of the system given by (8) and (10) are globally Lipschitz, hence f is globally Lipschitz itself. $\square$

As an immediate consequence of the system being continuous and satisfying the Lipschitz condition, we can state some properties of the solution. For the system of ODEs given by (8) and (10) with set of initial conditions $v(t_o) = v_o \geq 0$, $\theta(t_o) = \theta_o$ and chosen parameter u, the following two properties hold:

1) There exists a unique solution $\varphi(t, t_o, \theta_o, u)$ defined on some maximal interval $J \subset \mathbb{R}$ containing t_o.

2) $\varphi(t, t_o, \theta_o, u)$ is continuous in (t, t_o, θ_o, u)

Next we show that the maximal interval of existence given by J can be extended to the entire real line.

3.2 Existence of solution for all Time

For the system of ODEs defined by (8) and (10) with a set of initial conditions $v(t_o) = v_o > 0$ and $\theta(t_o) = \theta_o$, the unique solution $\varphi(t)$ existing on a maximal interval J, is defined for all $t \in (-\infty, \infty)$.
Proof Integrating (8) from t_o to t:

$$v = v_o + \int_{t_o}^{t}\left[-u\frac{\dot{M}}{M} - \frac{F_{grav}}{M}\sin(\theta)\right]ds - \int_{t_o}^{t}\frac{cv}{M}ds$$
$$|v| \leq v_o + \left|\int_{t_o}^{t}\left[u\frac{\dot{M}}{M} + \frac{F_{grav}}{M}\right]ds\right| + \left|\int_{t_o}^{t}\frac{c|v|}{M}ds\right|$$

and setting

$$h(t) := v_o + \left| \int\limits_{t_o}^{t} \left[u\frac{\dot{M}}{M} + \frac{F_{grav}}{M} \right] ds \right| \quad \text{and} \quad m(t) = \frac{c}{M}$$

the inequality is expressed explicitly in terms of its variables

$$|v(t)| \leq h(t) + \left| \int\limits_{t_o}^{t} m(s)|v(s)|ds \right|$$

so by the Gronwall Inequality:

$$|v(t)| \leq h(t) + \left| \int\limits_{t_o}^{t} m(s)h(s) \right| e^{|\int\limits_{t_o}^{t} m(\xi)d\xi|}$$

Hence $v(t)$ is bounded on any finite interval of time and from (10) it is clear θ is also bounded on any finite interval of time. So for any constant $c > 0$, $\varphi(t)$ is bounded on $J \cap (-c, c)$, hence the solution $\varphi(t)$ exists for all $t \in (-\infty, \infty)$.

3.3 Equilibrium point

In order to asses the equilibrium points and their stability for the system given in (8) and (10), we will make some physically justified assumptions about the asymptotic behavior of $c(t)$ and $M(t)$ and its first derivative. First we will assume that the drag coefficient converges to a constant, i.e. $c(t) \to c_o \in \mathbb{R}_+$ as $t \to \infty$. In other words, the rocket eventually returns to a layer of relatively constant atmospheric density. Also, the mass of the rocket should converge to a constant when the fuel is spent, $M(t) \to M_o \in \mathbb{R}_+$, and hence $\dot{M}(t) \to 0$ as $t \to \infty$.

With the above assumptions, we check the equilibrium points of the system where $\dot{v}(t) = 0$ and $\dot{\theta}(t) = 0$. It is clear from (10) that for the restriction $3\pi/2 \leq \theta_o < -\pi/2$, $\theta_o \neq \pi/2$, that $\dot{\theta} = 0$ if and only if $\theta = 3\pi/2$ or $\theta = -\pi/2$. The *only if* holds since for $3\pi/2 \leq \theta_o < \pi/2$ we have $\theta(t) \to 3\pi/2$ and for $\pi/2 < \theta_o \leq -\pi/2$ we have $\theta(t) \to -\pi/2$ as $t \to \infty$. So these equilibrium points for $\theta(t)$ either occur at t_o when $\theta_o = 3\pi/2$ or $\theta_o = -\pi/2$, or asymptotically as $t \to \infty$. For $\dot{v} = 0$ from equation (8) we have:

$$v = [-u\dot{M} - F_g sin(\theta)]/c$$

Which is a constant when $t \to \infty$ since $\dot{M} \to 0$, $F_g \to gM_o$ and $sin(\theta) \to -1$. This is the terminal speed of the rocket given by $v = M_o g/c$. More generally, we list the domain of attraction D for a set of initial conditions $v(t_o) = v_o$ and $\theta(t_o) = \theta_o$.

For the equilibrium point $(M_o g/c, 2n\pi + 3\pi/2)$, where $n \in \mathbb{N}$ is fixed, D is given by:

$$D = \{(v_o, \theta_o) \in \mathbb{R}^2 : v_o > 0, \quad 2n\pi + 3\pi/2 \leq \theta_o < \pi/2 + 2n\pi\}$$

For the equilibrium point $(M_o g/c, 2n\pi - \pi/2)$, D is given by:

$$D = \{(v_o, \theta_o) \in \mathbb{R}^2 : v_o > 0, \quad 2n\pi + \pi/2 < \theta_o \leq -\pi/2 + 2n\pi\}$$

The other equilibrium point occurs at $(\pi/2 + 2n\pi, -F_g/c)$. This situation is clearly non-physical as the speed is negative. Furthermore the equilibrium is unstable since if the initial angle is $\theta_o < \pi/2$ then $\theta(t) \to -\pi/2$ and if $\theta_o > \pi/2$ then $\theta(t) \to 3\pi/2$.

3.4 Periodic Solution

Since $\theta(t)$ is strictly decreasing to zero, for the system to be in equilibrium it is required that $\theta(t) \equiv \pi/2 + n\pi$, where $n \in \mathbb{N}$ is fixed. First consider the case where system given in (8) and (10) is autonomous by setting M, F_g and c to be positive constants. Not this corresponds to the projectile case. Let $\Omega = [\frac{\pi}{2} + n\pi] \times [1, \infty) \subset \mathbb{R}^2$ for a fixed $n \in \mathbb{N}$, then we have $f \in C^1[\Omega, \mathbb{R}^2]$ and the Bendixon criterion for no periodic orbits lying entirely in Ω is:

$$div f = \frac{\partial f_1}{\partial v} + \frac{\partial f_2}{\partial \theta} \neq 0 \quad \forall (v, \theta)^T \in \Omega \tag{11}$$

which gives:

$$v \neq \frac{F_g}{c}$$

So in order for a periodic solution to possibly exist, we require $v = F_g/c$. But in this case, v must be a constant to satisfy (8) since:

$$\dot{v} = \frac{F_g}{M} - \frac{c}{M}\frac{F_g}{c} = 0$$

Hence the only periodic solution for the autonomous system is the trivial one where $\theta(t)$ and $v(t)$ are constants. For the non autonomous case, a non trivial periodic solution exists for some functions M, F_g and c. For example, $v(t) = sin^2(t) + 1$ is a periodic solution to the system given in (12).

$$\begin{aligned} \dot{v} &= 9.8 - \frac{cv}{M_o} \\ \dot{\theta} &= -\frac{9.8}{v}\cos(\theta) \qquad v(0) = 2, \quad \theta(0) = 3\pi/2 \end{aligned} \tag{12}$$

where $c(t) = -\dfrac{2M_o sin(t) cos(t) - 9.8M_o}{sin^2(t) + 1}$, $M = M_o \in \mathbb{R}_+$ and $F_g = 9.8M_o$. The function $c(t)$ is strictly positive for any positive M_o.

4 Numerical Results

Here some numerical solutions are provided for different scenarios solved using a 4^{th} order Runge-Kutta discretization. The first solutions in Figure 1 are given for the parameter values in Table 1 where T is the final simulated time. The mass function is taken to be $M(t) = M_E + M_F e^{-t/\tau}$ where τ is used to control the burn rate of the fuel, M_E is the empty rocket mass (without fuel) and M_F is the initial mass of the fuel. The difference is the initial angle with values on either side of $\pi/2$. We can see that the left and right systems asymptotically approach terminal speed and $-\pi/2$ and $3\pi/2$ respectively.

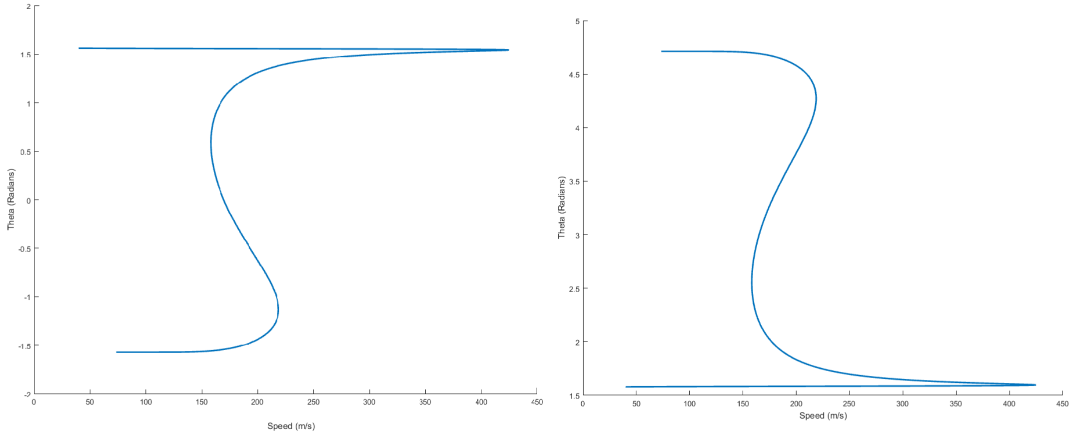

Figure 1 Solutions in phase space for Left: $\theta_o = \pi/2.01$ and Right: $\theta_o = \pi/1.99$, for the parameters given in Table 1.

Table 1: Parameter Values for Figure 1

v_o	T	u	τ	M_E	M_F	c
40	500	10,000	100	75	75	10

For these parameters, the terminal speed is about $73.5m/s$, hence the equilibrium point approached by the solutions are $(73.5, -\pi/2)$ and $(73.5, 3\pi/2)$. Here τ is small compared to u, this scenario depicts a quick burst of efficient fuel accelerating the rocket to about $424.5m/s$. The magnitudes of the force of gravity plus the air resistance is greater than the magnitude of the thrust which first occurs at about $30s$. For the given value of τ, over 25% of fuel is burned within these $30s$ and 50% of the fuel burned within about $71s$.

Another scenario is where the rocket fails because of an inefficient fuel source. Figure 2 represents the flight of a poorly built rocket whose empty weight is heavier by $25kg$, the fuel is more massive by $325kg$ and the magnitude of the thrust provided by the fuel is lowered by a factor of 100.

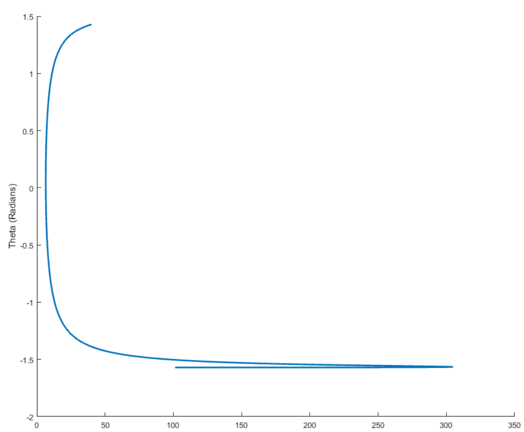

Figure 2 Solutions in phase space for $\theta_o = \pi/2.01$ and parameters given in Table 2.

Table 2: Parameter Values for Figure 2

v_o	T	u	τ	M_E	M_F	c
40	500	100	10	100	400	10

Even though the fuel is spent at a higher rate (by decreasing τ) the angle of the velocity vector immediately begins to decrease, i.e. downward velocity is generated. Since the rocket is now pointing downward, the fuel now produces downward thrust to a maximum of about $304.6 m/s$ occurring at $73.3 s$. After this the rocket begins to approach the equilibrium point given by the terminal speed and flight angle at $(98, \pi/2)$.

5 Conclusion

The non autonomous speed-flight angle form of the rocket equation provided in Kang,Y., Bae S. (2006) admits a unique and globally defined solution for realistic choices of parameters and functions $M(t)$ and $c(t)$. The coordinate singularity produced at $v = 0$ does not impose a large restriction on the possible formulation of the problem. Referring to the second numerical example (Figure 2), this is an extreme case in terms of the chosen parameters, yet the minimum speed was $3.45 m/s$. In most cases $v \geq 1$, hence the results obtained in this paper can be applied to the majority of formulations of the of the rocket equation presented Kang,Y., Bae S. (2006).

References

[1] Kang, Y., & Bae, S. (2006). Two-dimensional motions of rockets. European Journal of Physics, 28(1), 135-144. doi:10.1088/0143-0807/28/1/015