Eigen Vector & Eigen Value:

Almost all vectors change direction, when they are multiplied by **A**. Certain exceptional vectors ***x*** are in the same direction as ***Ax***. Those are the "**eigenvectors**". Multiply an eigenvector by **A**, and the vector ***Ax*** is a number λ times the original ***x***.

The basic equation is **Ax = λ x.** The number λ is an **eigenvalue of A.**

The eigenvalue λ tells whether the special vector ***x*** is stretched or shrunk or reversed or left unchanged—when it is multiplied by **A**. We may find λ = 2 or ½ or -1 or 1. The eigenvalue λ could be zero! Then *Ax* = *0x* means that this eigenvector x is in the null-space.

If A is the identity matrix, every vector has Ax = x. All vectors are eigenvectors of I. All eigenvalues are **λ=1.**

Det(A-λI) = 0

From the basic equation,

Aν=cν; where c is eigen value, ν is eigen vector

Aν=cν; Aν-cν = 0;

ν(A-cI) = 0; since ν = Iν

Since ν≠0, A-cI=0

A-cI is a matrix, and must have linearly dependent columns, which are non-invertible

Determinant of A-cI is zero, det (A-cI) = 0, | A-cI| = 0

c – Eigen value of A if and only if det (A-cI) = 0

Let A = $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; Estimate Eigen value and Eigen vector of A.

1. Let's find the Eigen values of matrix A

Let λ be the eigen value of A

| A – λI | = 0

$$\left| \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(2-\lambda) - 1 = 0$

$4-2\lambda-2\lambda+\lambda^2 -1 = 0$

$\lambda^2-4\lambda+3 = 0$ → Characteristic Polynomial

$(\lambda - 3)(\lambda-1)=0$

$\lambda=3$ or $\lambda=1$

The two eigen values of A are: $\lambda=3$ or $\lambda=1$

2. <u>Let's find the Eigen vectors of matrix A</u>

We know,

$A\nu = \lambda\nu$

$\lambda\nu - A\nu = 0$

$\lambda I\nu - A\nu = 0$; 'I' being Identity matrix

$(\lambda I - A)\nu = 0$;

For any Eigen value (λ), the Eigen space (E_λ) = $N(\lambda I - A)$

For $\lambda = 3$, $E_3 = N(3*I - A)$

$= N(3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix})$

$= N(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix})$

$= N(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix})$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} V = 0$

By applying Row echelon form:

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v1 - v2 = 0$

$v1 = v2$

Let $v1 = t$, then $v2 = t$

E3 = $\begin{bmatrix} v1 \\ v2 \end{bmatrix}$ = $t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ = span $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For λ = 1, E_1 = N(1*I – A)

= N($\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$-$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$)

= N($\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$)

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}V$ = 0

By applying Row echelon form:

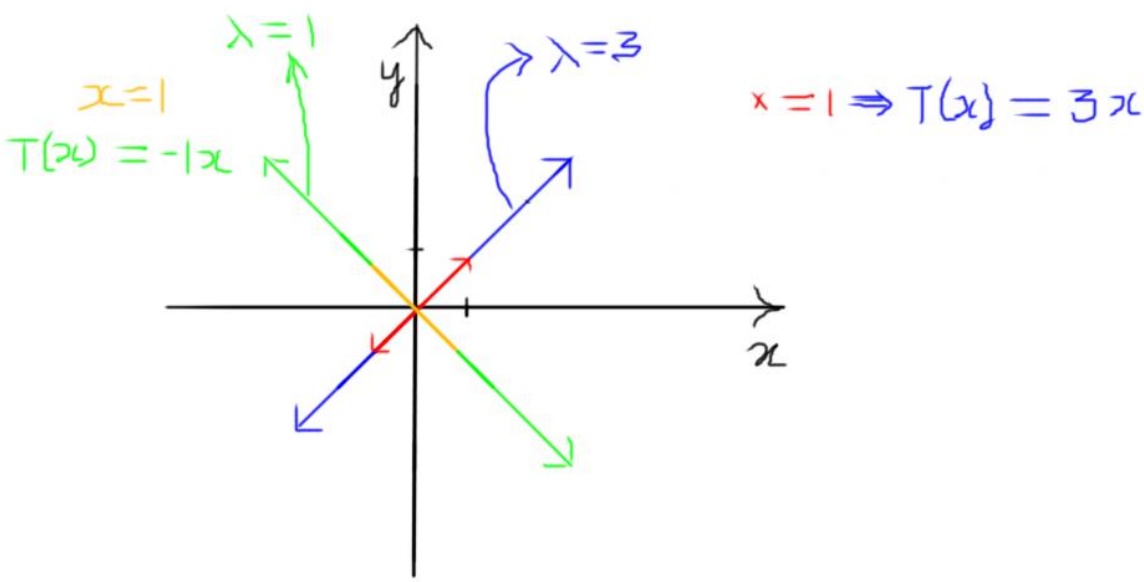
$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v1 \\ v2 \end{bmatrix}$ = $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-v1$ -$v2$ = 0

$v1 = -v2$

Let $v1 = t$, then $v2 = -t$

E_1 = $\begin{bmatrix} v1 \\ v2 \end{bmatrix}$ = $t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ = span $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$



Let A = $\begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix}$ **Find Eigen vector and Eigen value**

Let c be the eigen value of A

If and only if, A v = c v for some non-zero vector

$$\left| \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix} - c \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 5-c & 12 & -6 \\ -3 & -10-c & 6 \\ -3 & -12 & 8-c \end{vmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

(5-c)x – 12y – 6z = 0 ----------(1)

-3x –(10+c)y +6z = 0 ---------(2)

-3x -12y + (8-c)z = 0 ---------(3)

On Row (3) → (3) - (2) → (10+c-12)y+ (2-c)z= 0

(-2+c)y –(-2+c)z = 0

(-2+c)(y-z) = 0

(c-2)(y-z) = 0

On solving the equations, we get:

$-(\lambda - 2)^2 (\lambda + 1) = 0$

So $(\lambda - 2)^2 = 0$ or $\lambda + 1 = 0$

$(\lambda - 2)(\lambda - 2) = 0$ $\lambda = -1$

$\lambda = 2$ and $\lambda = 2$ $\lambda = -1$

So there are three values for λ: -1, +2, +2

We will get 2 vectors as solutions.

Defective matrix: A defective matrix is a square matrix that does not have a complete basis of eigenvectors, and is therefore not diagonalizable.

An n × n matrix is defective if and only if it does not have n linearly independent eigenvectors