Q1. Verify that Eigen value is 4 and Eigen Vector v = (1, 1) for given matrix A.

A = $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

Let's find the Eigen value of Matrix A.

$| A - \lambda I | = 0$

$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda) - 6 = 0$

$2 - 3\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

$\lambda = 4$ or $\lambda = -1$

Hence we have verified that **Eigen value = 4**

Let's find the Eigen vectors of matrix A
$(\lambda I - A) v = 0$

For any Eigen value (λ), the Eigen space ($E\lambda$) = $N(\lambda I - A)$

For $\lambda = 4$, $E4 = N(4*I - A)$
$= N(4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}) = N(\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix})$

$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} v = O$

$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = O$

$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = O$

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = O$

$v1 - v2 = 0$

$v1 = v2$

E4 = $\begin{bmatrix} v1 \\ v2 \end{bmatrix}$ **=** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence we have verified that **<u>Eigen Vector, v = (1, 1)</u>**

Q2. Verify that Eigen value is 3 and Eigen Vector v = (2, 1, -1) for given matrix A.

A = $\begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$

Let's find the Eigen value of Matrix A.

| λI – A | = 0

$\begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} - \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix} = 0$

$\begin{vmatrix} \lambda - 1 & 2 & 6 \\ 2 & \lambda - 2 & 5 \\ -2 & -1 & \lambda - 8 \end{vmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$

$$\left[\begin{array}{ccc} \lambda-1 & 2 & 6 \\ 2 & \lambda-2 & 5 \\ -2 & -1 & \lambda-8 \end{array}\right] \begin{array}{cc} \lambda-1 & 2 \\ 2 & \lambda-2 \\ -2 & -1 \end{array}$$

$$(\lambda-1)(\lambda-2)(\lambda-8) + 2\cdot5(-2) + 6\cdot2(-1)$$
$$- (-2)(\lambda-2)\cdot6 \quad - (-1)5(\lambda-1) - (\lambda-8)\cdot2\cdot2$$

$$= (\lambda^2-3\lambda+2)(\lambda-8) - 20 - 12$$
$$+ 12(\lambda-2) \quad + 5(\lambda-1) - 4(\lambda-8)$$

$$= \lambda^3 - 3\lambda^2 + 2\lambda - 8\lambda^2 + 24\lambda - 16 - \cancel{32}$$
$$+ 12\lambda - 24 \quad + 5\lambda - 5 - 4\lambda + \cancel{32}$$

$$= \lambda^3 - 11\lambda^2 + 39\lambda - 45$$

$\downarrow$

$1, 3, 5, 9, 15, 45$

$\lambda = 1 \Rightarrow 1 - 11 + 39 - 45 \neq 0$

$\lambda = 3 \Rightarrow 27 - 99 + 117 - 45 = 4 \neq 0$

$\lambda = 5 \Rightarrow 125 - 275 + 195 - 45 = 0$ ✓

$$\begin{array}{r} 39 \times 3 \\ \hline 117 \\ 27 \\ \hline 144 \end{array} \qquad \begin{array}{r} 95 \\ 45 \\ \hline 140 \end{array}$$

$\lambda = 9 \Rightarrow 729 - 891 + 351 - 45$

$= 1080 - 936 = 144 \neq 0$

$\lambda = 15 \Rightarrow 3375 - 2475 + 585 - 45 \neq 0$

$\lambda = 45 \Rightarrow 91125 - 22275 + 1755 - 45 \neq 0$

$$\begin{array}{r|l} & \lambda^2 - 6\lambda + 9 \\ \hline \lambda - 5 & \lambda^3 - 11\lambda^2 + 39\lambda - 45 \\ & \lambda^3 - 5\lambda^2 \\ \hline & -6\lambda^2 + 39\lambda \\ & -6\lambda^2 + 30\lambda \\ \hline & 9\lambda - 45 \\ & 9\lambda - 45 \\ \hline & 0 \end{array}$$

$(\lambda - 5)(\lambda^2 - 6\lambda + 9) = 0$

$\boxed{\lambda = 5}$ $\lambda^2 - 3\lambda - 3\lambda + 9 = 0$

$(\lambda - 3)(\lambda - 3) = 0$

$\lambda = 5$ & $\lambda = 3$

Hence, verified that **<u>Eigen value is 3</u>**

Let's find the Eigen vectors of matrix A

$(\lambda I - A)\, v = 0$

For λ = 3, E3 = $\left(3\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}1 & -2 & -6\\-2 & 2 & -5\\2 & 1 & 8\end{bmatrix}\right)v = \begin{bmatrix}2 & 2 & 6\\2 & 1 & 5\\-2 & -1 & -5\end{bmatrix}v = 0$

⇨ $\begin{bmatrix}2 & 2 & 6\\0 & -1 & -1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v1\\v2\\v3\end{bmatrix} = 0$

⇨ $\begin{bmatrix}1 & 1 & 3\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v1\\v2\\v3\end{bmatrix} = 0$

$v_1+v_2+3v_3 = 0$ -----(1)
$v_2 + v_3 = 0$ ---------(2)
Given that, Eigen Vector v = (2, 1, -1)

Substituting the given eigen vector values in above (1) and (2) equations

$v_1+v_2+3v_3 = 2 + 1 - 3 = 0$
$v2 + v3 = 1 - 1 = 0$

<u>Given Eigen vector values satisfies the equations (1) and (2). Hence proved.</u>

Q3. Given that v1 = (1, -2) and v2 = (1, 1) are eigen vectors of A, determine the eigen values of A.

A = $\begin{bmatrix}4 & 1\\2 & 3\end{bmatrix}$

Let's find the Eigen value of Matrix A.

$(A - \lambda I)v = 0$

$\begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$v = 0 ----------(1)

Given that v1 = (1, -2) and v2 = (1, 1) are Eigen vectors of A

i) Let's take v1 = (1, -2)
Substituting the Eigenvector values in (1),
$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0$

(4-λ)-2=0 ➔ 2-λ = 0 ➔ **λ = 2**

ii) Let's take v2 = (1, 1)
Substituting the Eigenvector values in (1),
$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

(4-λ) + 1 = 0 ➔ 5 - λ = 0➔ **λ = 5**

Eigenvalues of A are **λ = 2** and **λ = 5**

Q4. Determine all eigen values and corresponding eigen vectors of the given matrix A. If,

(a) $A = \begin{bmatrix} 1 & 6 \\ 2 & -3 \end{bmatrix}$

Let's find the Eigen value of Matrix A.

$| A - \lambda I | = 0$

$\begin{vmatrix} 1-\lambda & 6 \\ 2 & -3-\lambda \end{vmatrix} = 0$

(1 - λ)(-3 - λ) - 12 = 0
-3 -λ +3 λ + λ^2 – 12 = 0
λ^2 + 2 λ - 15 = 0
λ^2 + 5 λ - 3 λ – 15 = 0
(λ+5)(λ-3) = 0
Hence the Eigenvalues are λ =3 or λ=-5

Let's find the Eigen vectors of matrix A
(λ I – A) v = 0

$$\begin{bmatrix} \lambda-1 & -6 \\ -2 & \lambda+3 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = 0 \text{ ------(1)}$$

Let's put the value of λ =3 in the above equation (1)

$$\begin{bmatrix} 3-1 & -6 \\ -2 & 3+3 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -6 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = 0$$

2v1 – 6v2 = 0

v1 – 3v2 = 0

v1 = 3v2

Let v2=t, then v1=3t

Eigen Vector for λ=3 is $E_3 = \begin{bmatrix} v1 \\ v2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Let's put the value of λ =-5 in the above equation (1)

$$\begin{bmatrix} -5-1 & -6 \\ -2 & -5+3 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -6 & -6 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = 0$$

v1 = -v2

Let v_1=t, then v_2=-t

Eigen Vector for λ=-5 is $E_{-5} = \begin{bmatrix} v1 \\ v2 \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) A = $\begin{bmatrix} \mathbf{3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$

Let's find the Eigen value of Matrix A.

$|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-3 & 0 & 0 \\ 0 & \lambda-2 & 1 \\ -1 & 1 & \lambda-2 \end{vmatrix} = 0$$

(λ - 3)[(λ-2)(λ-2) – 1*1] - 0 + 0 = 0
(λ - 3)[λ^2 - 4λ +3] = 0
(λ - 3)(λ -3)(λ-1) = 0
λ=3 and λ=1

Let's find the Eigen vectors of matrix A

(λ I – A) v = 0

$$\begin{bmatrix} \lambda-3 & 0 & 0 \\ 0 & \lambda-2 & 1 \\ -1 & 1 & \lambda-2 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0 \text{ ------(1)}$$

i) Let's put the value of **<u>λ =3</u>** in the above equation (1)

$$\begin{bmatrix} 3-3 & 0 & 0 \\ 0 & 3-2 & 1 \\ -1 & 1 & 3-2 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

0 + 0 + v1 = 0 ➔ v1=0
v2 + v3 = 0 ➔ v2 = -v3
-v1 + v2 + v3 = 0 ➔ v1 = v2 + v3 ➔-v1 –v3 + v3 = 0 ➔ v1=0

Let v2 = t, then v3 = -t

$$\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

We have 2 equations and 3 variables, hence we have infinite number of solutions.

ii) Let's put the value of **<u>λ =1</u>** in the above equation (1)

$$\begin{bmatrix} 1-3 & 0 & 0 \\ 0 & 1-2 & 1 \\ -1 & 1 & 1-2 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

-2v1 = 0 ➔v1 = 0
-v2+v3 = 0 ➔ v2 = v3
-v1+v2-v3 = 0 ➔ 0 +v2 – v2 = 0

Let v2 = t, then v3=t

$$\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(c) A = $\begin{bmatrix} \mathbf{6} & \mathbf{3} & \mathbf{-4} \\ \mathbf{-5} & \mathbf{-2} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{bmatrix}$

Let's find the Eigen value of Matrix A.

| λI – A| = 0

$$\begin{vmatrix} \lambda-6 & -3 & 4 \\ 5 & \lambda+2 & -2 \\ 0 & 0 & \lambda+1 \end{vmatrix} = 0$$

(λ-6)[(λ+2)(λ+1)-0*(-2)] - (-3)[5(λ+1) – 0*(-2)]+ 4 [5*0 – 0(λ+2)] = 0

(λ-6)(λ+2)(λ+1) + 3(5λ+5)+4* 0 = 0

(λ-6)(λ+2)(λ+1) + 3*5(λ+1) = 0

(λ+1) [(λ-6)(λ+2) + 15)] = 0

(λ+1) [λ^2 -6λ+2λ-12 + 15] = 0

(λ+1) [λ^2 -4λ+ 3] = 0

(λ+1) [λ^2 - 3λ - λ+ 3] = 0

(λ+1) [(λ - 3)(λ - 1)] = 0

λ = -1, λ = 1 and λ = 3 are the Eigen values of matrix X

Let's find the Eigen vectors of matrix A

$(\lambda I - A)\, v = 0$

$$\begin{bmatrix} \lambda-6 & -3 & 4 \\ 5 & \lambda+2 & -2 \\ 0 & 0 & \lambda+1 \end{bmatrix}\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0 \text{ ------(1)}$$

i) Let's find the Eigenvector when $\lambda = -1$

Put $\lambda = -1$ in equation (1)

$$\begin{bmatrix} -1-6 & -3 & 4 \\ 5 & -1+2 & -2 \\ 0 & 0 & -1+1 \end{bmatrix}\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -7 & -3 & 4 \\ 5 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

We will get 2 equations with 3 variables, hence we will have infinite number of solutions

$R_1 \rightarrow 2R_2 + R_1$

$$\begin{bmatrix} 3 & -1 & 0 \\ 5 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

$3v1 - v2 = 0 \Rightarrow v2=3v1$

$5v1+v2-2v3 = 0 \Rightarrow 5v1 + 3v1 - 2v3 = 0 \Rightarrow 8v1= 2v3 \Rightarrow 4v1 = v3$

Let v1=t, then

$$\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

ii) Let's find the Eigenvector when $\lambda = 1$

Put $\lambda = 1$ in equation (1)

$$\begin{bmatrix} 1-6 & -3 & 4 \\ 5 & 1+2 & -2 \\ 0 & 0 & 1+1 \end{bmatrix}\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -5 & -3 & 4 \\ 5 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

(R2) → R1+R2

$$\begin{bmatrix} -5 & -3 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

We will get 2 equations with 3 variables, hence we will have infinite number of solutions

(R3) → R3-R2

$$\begin{bmatrix} -5 & -3 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

-5v1-3v2+v3 = 0
2V3=0 → v3 = 0

-5v1-3v2 = 0→ 5v1 = -3v2 → v1 = -3/5 v2

Let v2=t, then

$$\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = t \begin{bmatrix} -3/5 \\ 1 \\ 0 \end{bmatrix}$$

iii) Let's find the Eigenvector when $\lambda = 3$

Put $\lambda = 3$ in equation (1)

$$\begin{bmatrix} 3-6 & -3 & 4 \\ 5 & 3+2 & -2 \\ 0 & 0 & 3+1 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & -3 & 4 \\ 5 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = 0$$

-3v1 -3v2+4v3 = 0

5v1+5v2-2v3 = 0

4v3 = 0 → v3 = 0

-3v1 -3v2 = 0 → v1 = -v2

$5v1 + 5v2 = 0 \rightarrow v1 = -v2$

Let v1 = t, then

$$\begin{bmatrix} v1 \\ v2 \\ v3 \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$