

Research Statement

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My interests are algebraic geometry, representation theory, and geometric invariant theory. My Ph.D. thesis concerns: 1) the relation between conformal blocks and the moduli of G -bundles on a (possibly singular, reducible) curve, and 2) the tensor decomposition problem (how does a tensor product of group representations decompose into irreducibles).

In the first problem, I used conformal blocks to give a compactification of the moduli of G -bundles over Deligne-Mumford's stack of stable curves $\overline{\mathcal{M}}_g$. I showed that the sheaf of conformal blocks algebras is finitely generated and that, at sufficiently high level, conformal blocks can be identified with sections of an ample line bundle on a certain compactified moduli space of G -bundles. This generalizes a result of Belkale-Gibney for $G = \mathrm{SL}(r)$. This is discussed further in §1.

In the second, I studied certain divisors in flag varieties which arise from the tensor decomposition problem. By Borel-Weil theory, the tensor problem is related to describing the G -effective cone of line bundles on the flag variety $(G/B)^n$ that have a nonempty semistable locus. Unstable loci in $(G/B)^n$ can be described using Schubert calculus of Grassmannians G/P , and, in the case $G = \mathrm{SL}(r)$, I gave a simple numerical description of their possible codimension one components. This has interesting consequences for studying GIT quotients of $(\mathrm{SL}(r)/B)^n$. More details are given in §2.

Generalizations and related questions are described briefly in §3.

1. Conformal blocks and G -bundles on curves.

For a simple Lie algebra \mathfrak{g} , integer $l \geq 0$, and stable curve C , let $\mathbb{V}_{\mathfrak{g},l,C}$ be the associated space of conformal blocks. These are vector spaces which originated from physics and can be defined in terms of representation theory of the affine Lie algebra of \mathfrak{g} . Conformal blocks form vector bundles $\mathbb{V}_{\mathfrak{g},l}$ on the stack of genus g stable curves $\overline{\mathcal{M}}_g$, which have been of great interest to algebraic geometers at least since the '90s, when the search for a proof of the Verlinde formula initiated an intense study of conformal blocks by Beauville-Laszlo [2], Faltings [10], Kumar-Narasimhan-Ramanathan [12], Laszlo-Sorger [14], and many others.

Conformal blocks are deeply connected to moduli of G -bundles on C . For example, for a sufficiently high level l and *smooth* curve C of genus $g \geq 2$, $\mathbb{V}_{\mathfrak{g},l,C}^*$ is isomorphic to sections of an ample line bundle on Ramanathan's moduli space of G -bundles $\mathfrak{M}_G(C)$. In particular, $\mathrm{Proj} \mathcal{A}(C) \cong \mathfrak{M}_G(C)$, where $\mathcal{A}(C) := \bigoplus_{l \geq 0} \mathbb{V}_{\mathfrak{g},l,C}^*$ (see [15] for the algebra structure).

But, the moduli space of G -bundles is no longer compact for a singular curve $C \in \overline{\mathcal{M}}_g$, just as in the vector bundle case where one needs to allow vector bundles to degenerate into torsion-free sheaves to obtain compactifications. While $\mathrm{Proj} \mathcal{A}(C)$ is a natural candidate for a compactification, it has no obvious modular interpretation, and we need to show that $\mathcal{A}(C)$ is a finitely generated algebra to ensure that $\mathrm{Proj} \mathcal{A}(C)$ is actually a projective variety.

In [19], I prove finite generation of $\mathcal{A}(C)$ for any stable curve $C \in \overline{\mathcal{M}}_g$ and give a modular interpretation of $\mathrm{Proj} \mathcal{A}(C)$. Finite generation was previously only known for $\mathfrak{g} = \mathfrak{sl}(r)$ (due to Belkale-Gibney [5] and Moon-Yoo [16]), and no modular interpretation was known. I show

that, for sufficiently large m , the subalgebra $\mathcal{A}^m(C) = \bigoplus_{l \geq 0} \mathbb{V}_{g,lm,C}^*$ is isomorphic to the section ring of an ample line bundle on a certain compactification of the moduli space of G -bundles, called the the moduli space of “singular G -bundles,” which was introduced by Schmitt and Muñoz-Castañeda ([18]).

The proof of finite generation of $\mathcal{A}(C)$ for every $C \in \overline{\mathcal{M}}_g$ implies finite generation of the sheaf of algebras $\mathcal{A} = \bigoplus_{l \geq 0} \mathbb{V}_{g,l}^*$ on $\overline{\mathcal{M}}_g$, thus $\mathfrak{X} = \text{Proj } \mathcal{A}$ gives a flat, projective family $\mathfrak{X} \rightarrow \overline{\mathcal{M}}_g$ whose fiber over a smooth curve is Ramanathan’s moduli space of G -bundles and whose fiber over an arbitrary stable curve is a moduli space of singular G -bundles. This interpretation only applies to closed fibers of \mathfrak{X} over $\overline{\mathcal{M}}_g$, and it still remains to find a modular interpretation of \mathfrak{X} that works for the relative setting (if possible). There are many other possible approaches, for example Balaji has constructed, using Bruhat-Tits theory, flat moduli spaces over $\text{Spec } \mathbb{C}[[t]]$ that give degenerations of the moduli of G -bundles in the setting of a smooth curve degenerating into an irreducible curve with one node [1]. It would be interesting to see how Balaji’s moduli spaces relate to the conformal blocks compactification.

2. Tensor decomposition problem.

Let G be a connected semi-simple algebraic group over \mathbb{C} . The tensor decomposition problem asks, “Given two irreducible representations V_λ and V_μ of G , which irreducible representations V_ν appear in $V_\lambda \otimes V_\mu$?” By the Borel-Weil theorem, a triple of dominant integral weights (λ, μ, ν) satisfies this condition if and only if a certain line bundle on $(G/B)^3$ has a nonzero G -invariant section, where $B \subset G$ is a Borel subgroup. Thus, if the problem is loosened to looking for triples (λ, μ, ν) such that $V_{n\nu} \subseteq V_{n\lambda} \otimes V_{n\mu}$ for some $n > 0$ (this is called the “saturated tensor problem”), then the tensor problem becomes equivalent to the GIT problem of describing the G -effective cone $C \subseteq \text{Pic}(G/B)^3$ of line bundles with a nonempty semistable locus (here G is acting diagonally on $(G/B)^3$). For more detailed background on the tensor problem, see Kumar’s article [13].

In [3], Belkale-Kumar gave a set of inequalities defining the cone C , and this was proved to be a minimal set of inequalities by Ressayre in [17]. In [20], I generalize this result by finding inequalities for the sequence of subcones

$$C_k = \{\mathcal{L} \in \text{Pic}(G/B)^n : \text{unstable locus of } \mathcal{L} \text{ has codimension } \geq k\}.$$

The proof uses a natural relationship between the Hesselink stratification of $(G/B)^n$ and intersection theory of Grassmannians G/P . The inequalities are indexed by Schubert positions $v_1, \dots, v_n \in W$ (Weyl group of G) for G/P , ranging over all maximal parabolics P . Following Belkale-Kumar, I showed that one only needs to check the inequalities indexed by “Levi-movable” Schubert positions, which are Schubert positions such that translates of Schubert varieties by general elements of the Levi subgroup meet transversally at $eP \in G/P$ (if the intersection has negative expected dimension – in which case “transversality” isn’t really defined – then we just require that the tangent space of the intersection at eP is zero for general Levi translates). In the case $k = 1, 2$ (expected dimension zero or -1), this Levi movability condition can be tested using Belkale-Kumar’s deformed product of cohomology ([3]).

The extremal rays of the cone C were determined in [6]. A large class of extremal rays, “F-rays,” arise from GIT in the following way. Let $D \subset (G/B)^n$ be a G -invariant irreducible divisor such that $\dim H^0(\mathcal{O}(mD))^G = 1$ for all $m \geq 0$. Then $\mathcal{O}(D)$ spans an extremal ray called an F-ray, and D is called an F -divisor. A (reduced, irreducible) divisor is an F -divisor if and only if it is a component of the unstable locus of some line bundle on $(G/B)^n$, and the possible F -divisors which may appear in an unstable locus can be checked by Belkale-Kumar inequalities.

For $G = \mathrm{SL}(r)$, I give a description of all F -divisors as intersection loci

$$N_{\bar{v}} = \{(g_1 B, \dots, g_n B) \in (\mathrm{SL}(r)/B)^n : \bigcap_{i=1}^n g_i X_{v_i} \neq \emptyset\}$$

satisfying a simple numerical condition, where $X_v \subset \mathrm{SL}(r)/P$ is the Schubert variety corresponding to $v \in W$. As a consequence of the numerical condition, I show that one can always assume a line bundle on $X = (\mathrm{SL}(r)/B)^n$ has unstable locus of codimension ≥ 2 for the purposes of moduli problems. In particular there is a well-defined injective map

$$\mathrm{Pic}(X // G) \hookrightarrow \mathrm{Pic} X$$

for any GIT quotient of X . More can be said about the image of this map using Kempf's descent lemma as in [8], and it would be interesting to use this to further study the Picard groups of GIT quotients of $(G/B)^n$.

3. Future work

3.1 Conformal blocks and G -bundles on curves

- (i) Prove finite generation and explore modular interpretations for conformal blocks algebras on the stack of n -pointed curves $\overline{\mathcal{M}}_{g,n}$. Finite generation was proven for $\mathfrak{g} = \mathfrak{sl}(r)$ by Moon-Yoo ([16]), and one would like to generalize this to arbitrary simple Lie algebras.
- (ii) Spaces of conformal blocks are isomorphic to sections of determinant of cohomology line bundles on the stack of G -bundles $\mathrm{Bun}_G(C)$ (or possibly fractional powers of determinant of cohomology). Replacing curves with an appropriate class of higher dimensional varieties Z , is the section ring of determinant of cohomology on $\mathrm{Bun}_G(Z)$ still finitely generated?
- (iii) Recently it has been shown that certain modules over vertex operator algebras give rise to generalized sheaves of conformal blocks on $\overline{\mathcal{M}}_{g,n}$ (see for example [9]). How can one define an algebra structure on the sum of these sheaves, and when does this algebra have nice properties like finite generation or normality? Do these sheaves have a geometric interpretation that generalizes the relationship between traditional conformal blocks and moduli of G -bundles?

3.1 Tensor decomposition problem

- (i) First I would like to extend my results on unstable divisors in $(\mathrm{SL}(r)/B)^n$ to arbitrary semisimple groups, if possible. Then there are a number of generalizations of the tensor problem that one could consider, which I will describe in (ii), (iii).
- (ii) The first generalization is the “branching problem,” where one studies restrictions of representations for an embedding of groups $G \subset \widehat{G}$ (the tensor problem corresponds to the diagonal embedding $G \subset G^n$). The associated GIT problem takes place in the G -variety \widehat{G}/\widehat{B} for a Borel subgroup $\widehat{B} \subset \widehat{G}$. This version has been studied extensively by Ressayre, Kiers, and others, and inequalities and rays for the associated cone have been determined in [11], [17]. The unstable loci are again controlled by Schubert calculus of Grassmannians, and it would be interesting to study their divisorial components more closely.

- (iii) There is also a “quantum version” of the tensor problem, where the flag variety $(G/B)^n$ is replaced by the stack of parabolic G -bundles on a curve, and we want to find the cone of effective line bundles on this stack. The cone’s defining inequalities were determined by Belkale-Kumar in [4], and its rays were only recently determined for $G = \mathrm{SL}(r)$ by Belkale [7]. Finding the rays for arbitrary groups remains open. This version of the problem also has a rich connection to rigid local systems on a punctured sphere (see [7]).

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