$$\sum_{k=1}^{n} \left\lfloor k\sqrt{2} \right\rfloor$$

The problem is a sum of a Beatty Sequence, which is a sequence with a positive, irrational r of the form

$$\lfloor r \rfloor, \lfloor 2r \rfloor, \lfloor 3r \rfloor, \ldots$$

Let $S(\alpha, n)$ be the sum of the first n terms of a Beatty Sequence with $r = \alpha$ (in our case, we are concerned with $S(\sqrt{2}, n)$, but we'll get to that). There are a few simplifications based on α:

Case 0: $\alpha = 1$
Technically, this is not a Beatty Sequence, but I'm defining it here to simplify notation later. Obviously, with $\alpha = 1$ this is just a simple sum of consecutive integers, and so

$$S(1, n) = \frac{n(n+1)}{2}$$

Case 1: $\alpha \geq 2$
Let $\beta = \alpha - 1$. Then

$$\begin{aligned}
S(\alpha, n) &= \sum_{k=1}^{n} \lfloor k\alpha \rfloor \\
&= \sum_{k=1}^{n} \lfloor k(\beta + 1) \rfloor \\
&= \sum_{k=1}^{n} \lfloor k\beta + k \rfloor \\
&= \sum_{k=1}^{n} \lfloor k\beta \rfloor + \sum_{k=1}^{n} k \\
&= S(\beta, n) + S(1, n)
\end{aligned}$$

Case 2: $1 < \alpha < 2$
There is a useful property about Beatty Sequences, known as Raleigh's Theorem. Given an irrational $r > 1$, if s satisfies

$$r^{-1} + s^{-1} = 1$$

then the Beatty Sequences of r and of s partition $\mathbb{N}$, that is all positive integers are either in the r-sequence or the s-sequence. Thus define

$$\beta = \frac{1}{1 - \alpha^{-1}}$$

And $m = \lfloor \alpha n \rfloor$, so that the β sum ends at the same point as the α sum. Then

$$S(\alpha, n) + S(\beta, \lfloor m/\beta \rfloor) = S(1, m)$$

Also, $\lfloor m/\beta \rfloor = \lfloor m\beta^{-1} \rfloor = \lfloor m(1-\alpha^{-1}) \rfloor = \lfloor m - m/\alpha \rfloor = m - \lceil m/\alpha \rceil = m - n = \lfloor (\alpha - 1)n \rfloor$
Thus

$$S(\alpha, n) = S(1, \lfloor \alpha n \rfloor) - S(\beta, \lfloor (\alpha - 1)n \rfloor)$$

In the case where $\alpha = \sqrt{2}$, as $1 < \sqrt{2} < 2$, $\beta = 2 + \sqrt{2}$. The solution can thus be simplified as follows:

$$\begin{aligned} S(\sqrt{2}, n) &= S(1, \left\lfloor \sqrt{2}n \right\rfloor) - S(2 + \sqrt{2}, \left\lfloor (\sqrt{2} - 1)n \right\rfloor) \\ &= S(1, \left\lfloor \sqrt{2}n \right\rfloor) - 2S(1, \left\lfloor (\sqrt{2} - 1)n \right\rfloor) - S(\sqrt{2}, \left\lfloor (\sqrt{2} - 1)n \right\rfloor) \end{aligned}$$

Complexity Analysis
(Just for fun)
At each level of recursion, there is 1 subproblem and an $O(1)$ additional computation. The size of the subproblem decreases by a factor of

$$b = \frac{1}{\sqrt{2} - 1} \approx 2.41421356$$

Thus the time complexity for each level of recursion is:

$$T(n) = T\left(\frac{n}{2.41421356}\right) + O(1)$$

By application of the master theorem, the overall time complexity is $O(\log n)$, a significant improvement over the $O(n)$ naive formula.