ROUTLEDGE ADVANCES IN RISK MANAGEMENT

Emerging Financial Derivatives

Understanding exotic options and structured products

Jerome Yen and Kin Keung Lai

ROUTLEDGE

Emerging Financial Derivatives

Exotic options and structured products have been two of the most popular financial products over the past ten years and will soon become very important to the emerging markets, especially China. This book first discusses the products' recent development in the world and provides a comprehensive overview of the major products. The book also discusses the risks of issuing and buying such products as well as the techniques to price them and to assess the risks. Volatility is the most important factor in determining the return and risk. Therefore, a significant part of the book's content discusses how we can measure the volatility by using local and stochastic volatility models — the Heston Model and Dupire Model, the volatility surface, the term structure of volatility, variance swaps, and breakeven volatility.

The book introduces a set of dimensions which can be used to describe structured products to help readers to classify them. It also describes the more commonly traded exotic options with details. The book discusses key features of each exotic option which can be used to develop structured products and covers their pricing models and when to issue such products that contain such exotic options. This book contains several case studies about how to use the models or techniques to price and hedge risks. These case analyses are illuminating.

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Preface

Over the past decade, structured products experienced dramatic changes in their nature and in the appetite of investors. Although the demand for sophisticated products decreased in the Western world, the conditions in emerging markets such as China and India were different. The size of the investment product markets, which includes structured products, in China increased by over 30 percent in 2012, and we expect a similar number in 2014. Therefore, there is still an increasing need for structure developers who know how to develop structured products with underlying assets that cover equities/stocks, foreign exchange, and commodities. Also, for the emerging market, a great escalation in the sophistication of such structured products has taken place to meet the different requirements of investors to cope with the markets under different conditions.

Structured products can be divided into two categories: (1) flow products and (2) theme-based or ad hoc products. Flow products are standard products such as equity-linked notes (ELNs) that are always available on the product shelf. Theme-based or ad hoc products are designed to take advantage of special market conditions, for example, gold price bullish movement, so that investors can earn higher returns. Therefore, we feel that the structured products business is not just technical driven; two very important components are forecasting the potential movement of the underlying asset to write the story of the product as well as understanding the sentiments of the investors. Only when the story of the structured product matches the sentiments of the investors will there be business or transactions.

Most of the reference books on structured products either focus too much on technical issues, such as pricing models, or just provide an introduction about the nature of the products. It is difficult to find a book that provides a balanced discussion on technical issues as well as business or managerial issues. This is the motivation for our writing this book: our objective was to write a book that allows readers to understand a wider spectrum of issues faced by practitioners in the domain of structured products.

Topics covered in this book include the following:

- movements of various assets, trends, volatility, correlation, and behavior of assets under extreme market conditions;

- estimation of volatility, skewness, kurtosis, correlation, mean reversion, and the formation of market views;
- volatility smile, volatility surface, and local volatility models;
- volatility index and variance swap;
- mathematical or numerical methods for product pricing;
- survey of equity and foreign exchange structured products.

The book is designed for practitioners as a reference or as a textbook for students at the master's level who feel that plain-vanilla options or the Black—Scholes model based on the normal distribution is no longer adequate and that they need to learn more sophisticated pricing models, for example, stochastic volatility models such as Heston and local volatility models such as Dupire. This book could be used for a short semester course of 7 or 8 weeks if students have already finished one course in derivatives or a long semester of 12 or 13 weeks, so that a few lectures can be used to cover fundamental topics such as options and their pricing. Several research assistants provided significant support in finishing this book. Without their support, this book would not have been completed. They are: Yuen Kwan (Claire) Chan, Ying Lun (John) Cheung, Cheuk Hang (William) Leung, Xinlu Tang, Boyang Zhao, Yilin Yang, and Pinzhi (Michael) Zhou.

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1 Survey and classification of structured products

1.1 Background

Structured products are designed as part of prepackaged investment strategies by financial engineers, often combining elementary financial instruments such as bonds, stocks, futures, and options that are traded independently in spot and futures markets, to offer tailor-made risk return profiles to the investors. These products are designed to help investors pursue a broad range of portfolio and risk management objectives, such as protection, optimization, enhancement of returns, and leverage. They can help diversify portfolios, address various market conditions, or manage risks and taxes.

There are various ways in which structured products can be classified, and we are going to discuss them in detail.

1.2 Literature review

1.2.1 History and product development

In the early 1990s, many investment banks thought up new solutions to attract more investors to equity markets. The idea was to create innovative options (products?) with sophisticated payoffs that would he based on all types of assets such as stocks indices, commodities, foreign exchange, and all kinds of funds. Also, banks were looking for intelligent ways to provide investors with easy access to these innovations by issuing wrappers (medium-term notes, insurance life contracts, and collective funds) in a tax-efficient manner. Moreover, it was important to structure a business that was capable of following an issued financial asset throughout its life. Therefore, structured roles were created to compose complex over the counter products, while quantitative analysts developed pricing models to enable traders to hedge the products until maturity. Banks were also conscious of the importance of providing secondary markets that introduced the liquidity the business needed to expand. (See *Exotic Options and Hybrids: A Guide to Structuring, Pricing and Trading* by Mohamed Bouzoubaa and Adel Osseiran.)

Another school of thought considers that structured investment products came into being because companies wanted to issue debt, implying fewer opportunity costs to the investors. Traditionally, one of the ways to do this was to issue a convertible bond, i.e., debt that under certain circumstances can be converted into equity. In exchange for the potential for a higher return (if the equity value would increase and the bond could be converted at a profit), investors were willing to accept lower interest rates. However this trade-off and its actual worth are debatable because the movement of the equity value of the company was unpredictable. Investment banks then decided to add features to the basic convertible bond, such as increased income in exchange for limits on convertibility of the stock, or principal protection. These extra features were all based on strategies investors themselves could formulate, using options and other derivatives, but these were prepackaged as one product. The goal was again to give investors more reasons to accept a lower interest rate on debt in exchange for certain features. On the other hand, the goal for the investment banks was to increase profit margins because the newer products with added features were harder to value, making it harder for banks' clients to see how much profit the banks were making from it.

1.3 History and market development

Structured products gained popularity in the United States during the 1980s and were introduced in Europe in the mid-1990s, during years of low interest rates. Though the trend came into being because companies wanted to issue debt more cheaply, first, convertible bonds were issued that under certain circumstances could be converted into equity. In exchange for potential higher returns, investors were ready to accept lower interest rates. Investment banks and financial engineers kept innovating different financial instruments to provide more and more reasons for investors to accept lower interest rates, and features such as increased income in exchange for limits on convertibility of the stock or principal protection were added.

The past two decades have seen a huge growth in structured products globally. The 2000–2003 large drops in stock markets were one of the reasons that motivated investors to look at structured product as an influential alternative investment strategy; the other reason was increased market volatility.

When structured products were first introduced to high net-worth individuals in the early 1990s, the value proposition was focused on innovation, access to capital markets, and potential for enhanced performance. Since then, the structured products market has undergone major changes. The number of private investors using them has increased significantly, and a wider range of standardized products has been introduced.

According to the definition given by Roberto Knop, "a structured product is a financial instrument, and its return depends on the composition of other, simpler products. It consists of a loan, and one or more derivative products. The special feature here will be the conversion of the original risks of each of its components."

Sajit Das insists that structured products represent a special class of fixed-income instruments, and their principal appeal is the capacity to generate highly customized exposures for investors, consistent with their investment objectives with predetermined risk return parameters.

Many other similar definitions of structured financial products have been put forth that in turn focus special attention on the fact that an instrument's basis is the combination of a security (or a monetary asset) and one or several derivatives.

However we cannot fully claim that a structured product is a "security packaged together with derivative(s)." The concept of a structured product is so wide that all components of the instrument are not always clear. The usage of derivatives is not necessary. Thus, the Finnish Association of Structured Products asserts that "banks do not include a derivative component and banks are not necessarily using derivatives to hedge the underlying risk, even though there are features in structured products that resemble derivative like behavior."

The following definition of a structured product seems to be the most appropriate for understanding its essence: a structured product is a complex financial instrument with predetermined conditions of payoff and initial capital return, linked to a certain underlying asset. The product payoff depends on the underlying asset's dynamics, and the type of payoff and periodicity are defined by components of the structured product, i.e., securities (mainly fixed income instruments) and derivative instruments.

1.4 The goals and purposes of structured products

The uses of structured products are very broad, including the following:

- **Arbitrage:** Both investors and issuers can carry out arbitrage trades with derivatives and underlying assets by means of structured products.
- **Investment restrictions:** Such groups of investors as pension and mutual funds and insurance companies can access derivatives transactions via structured products.
- **Taxation and accounting:** Structured products are simple from the perspective of accounting and taxation as they are considered as separate security, and the value of derivatives is already included in the product price.
- **Creation of products "a la carte":** The freedom to create products is almost unbounded. Products are customized to fit the unique requirements of investors.
- **Hedging:** Structured products can be used not only for investments but also to hedge positions against market risks.
- **Access to new markets:** With the help of structured products, investors can access exotic instruments and new markets, for example, assets and instruments of developing markets that would otherwise be difficult for investors to access directly.

- **Cheap funding source:** A part of the funds intended for fixed income investment can be used by the issuer for its own financing at rates cheaper than the market rates.

1.5 The classification of structured products

The goals and purposes of structured products define their classification. In modern practice, there is no uniform and universally accepted system of product classification. There are a number of reasons for this: newness of the market, access restrictions among private investors, constant creation of new instruments, and proper interpretation of products by investment banks.

The lack of a standardized classification of structured products has tough implications for all market stakeholders. The consequences are as follows:

1. Difficulties due to the lack of definition from the legal perspective
2. Issue and information disclosure problems
3. Complex risk and yield calculations
4. The placement of restrictions and limits on transactions with products
5. Difficult product distribution
6. Low liquidity.

Of course, in order to solve the problem of classification, general coordination across the entire market of structured products is required. Only then can clear and uniform categories that define the types and classes of assets be determined.

After the products currently available in the market were studied, the following classification was prepared (see Appendix A).

1.5.1 Classification by levels of principal protection

On the basis of degree of protection of the capital, structured products can be divided into the following categories:

- **Principal protected products:** Those that provide full protection of the initial capital, independent on the underlying asset's price movements
- **Partially protected products:** Those that guarantee the return of the initial capital only at a certain level in the form of a percentage of the originally invested sum.

1.5.2 Classification by quantity of payments

- **Coupon products:** Products that provide more than one payment, similar to usual bonds, throughout the lifespan
- **Non-coupon products:** Products that offer only one payment at maturity, which includes both the return of the initial capital and the profit and loss amount.

1.5.3 Classification by type of underlying asset

The following underlying assets can be linked to the product:

- Security
- Interest rate
- Currency
- Index
- Basket of assets (currencies, securities, commodities, etc.)
- Commodities
- Credit quality
- Volatility
- Spread
- The consumer price index and other macroeconomic indicators
- Property price index.

1.5.4 Classification by form of structured product

Structured products can be issued in the following forms:

- Security
- Deposit
- Fund
- Private banking service.

1.5.5 Classification by type of investor

Each structured product is designed for a predetermined group of investors and customers. It is possible to outline three basic groups of investors:

- **Retail group:** Mass consumers
- **Group of institutional investors:** Large investment banks, mutual and pension funds, state funds, etc.
- **Individual investors:** Wealthy consumers.

1.5.6 Classification by behavior of underlying assets

Structured products' payoffs depend on the dynamics of the underlying assets to which they are linked. The behavior models can be defined as follows:

- Growth/falling
- Lateral movement
- Occurrence/non-occurrence of an event
- High/low volatility.

1.5.7 Classification by degree to which payoff depends on price path of underlying asset

Payoffs of structured products can be either defined by the value of a variable at the maturity date or by the value of a variable through the lifespan of a product. Thus, the payoff can be *independent* of and/or *dependent* upon the price path of the underlying asset.

1.5.8 Classification by payoff functions

As noted earlier, the basic peculiarity of structured products is the core element: derivative financial instruments. Almost all derivatives can be used for the creation of structured products. The type of derivatives and their combinations define the payoffs' functions that differentiate one product from another. Having investigated the products offered on the market, the following types of payoff functions can be segregated:

- **Tracking functions:** Payoffs are fully defined by the movement of the underlying asset and a change of 1% provides 1 percent change in the price of the product. Product example: protected tracker.
- **Leveraged functions:** Financial leverage is used. These products bear the risk of a partial loss of the initial capital. Product example: leverage long with sop loss note.
- **Basket functions:** Payoffs are defined by the dynamics of one asset versus a basket of underlying assets. Product example: Altiplano note.
- **Barrier functions:** Payoffs based on reaching or not reaching the underlying asset of a certain barrier level. Product example: knock-in, knock-out note.
- **Functions with floating parameters:** The main parameters of options can be changed (for example, a strike) when the underlying asset has overcome a certain level. Product example: Cliquet note.
- **Fixed payoff functions:** Payments in this case are fixed. Product example: reverse convertible.
- **Swap functions:** Within those functions, the payoffs are defined by spreads between prices (values) of certain underlying assets or by their volatility. Product example: dispersion note.

The disclosure of mentioned indicators and their detailed descriptions will allow all market participants to outline more accurately the limits and possibilities of the market's functioning and further development.

1.6 Case analysis

We end this chapter by giving some examples of how we can classify a product using the foregoing criteria. The examples chosen are real financial products. Before introducing the classification, let us introduce the mechanism of the products first.

1.6.1 Non-deliverable swap

A non-deliverable swap is also called a single-/double-touch quanto swap. This product has the following characteristics:

- **Currency:** There are two types of currency, domestic currency and foreign currency, and they are agreed to by both counterparties.
- **Underlying asset:** In our example, the underlying asset is an FX rate. Such an FX rate may not be the domestic currency and the foreign currency agreed at the beginning.
- **Payment of the swap:** There are two payment legs of the swap: *funding leg* and *coupon leg*. The buyer of the swap pays a contractual agreed fixed rate to the seller of the swap. On the other hand, the seller of the swap pays a contractual agreed coupon rate to the buyer of the swap. Hence, the buyer of the swap is said to pay a funding leg and receive a coupon leg, while the seller of the swap is said to pay a coupon leg and receive a funding leg.
- **Barriers:** There are two pairs of barrier levels. These barriers are used as criteria for determining the amount of coupon payments received by the buyer of the swap.
- **FX rate:** Rate at which the underlying currency is converted to foreign currency.

At the fixing date, the buyer of the contract pays a fixed amount on receiving different coupon payments according to different pre-agreed situations that are settled in foreign currency.

1.7 Mechanism of non-deliverable swap

In this section, two situations are investigated: double touch and single touch.

1.7.1 Case 1 (double touch)

To be specific, the following notations are defined:

1. Two pairs of barriers: UB1 and LB1 (pair 1 barrier) and UB2 and LB2 (pair 2 barrier).
2. Three different coupon rates: R_{C1}, R_{C2}, and R_{C3}.
3. Funding rates: R_C.
4. Two foreign exchange rates: FX_1 and FX_2.
5. Notional: N.

The following rules must also be obeyed:

1. FX_1 and FX_2 are the USDCNY and EURUSD FX rates, respectively. However, the roles of FX_1 and FX_2 are different. The FX rate FX_1 is treated as the exchange rate for settlement, whereas the FX rate FX_2 is treated as the underlying asset.

2 Initially, UB1 > UB2 > *Spot* FX_2 > LB2 > LB1.
3 Domestic currency is treated as CNY, and foreign currency is treated as USD.

If $FX_2 > UB1$ and $FX_2 < LB1$ during the index barrier observation period, then the coupon rate will be R_{C1}. However, if $FX_2 < UB1$ and $FX_2 > LB1$, and at the same time it turns out out that $FX_2 > UB2$ or $FX_2 < UB2$, then the coupon rate will be R_{C2}. Otherwise, the coupon rate will be R_{C3}.

The following figures illustrate the mechanism of how the product works. Figure 1.1 illustrates the scenario in which the coupon rate R_{C1} will be received, whereas Figure 1.2 illustrates the scenario in which the coupon rate R_{C3} will be employed. Figure 1.3 gives one interpretation of when R_{C2} will be given.

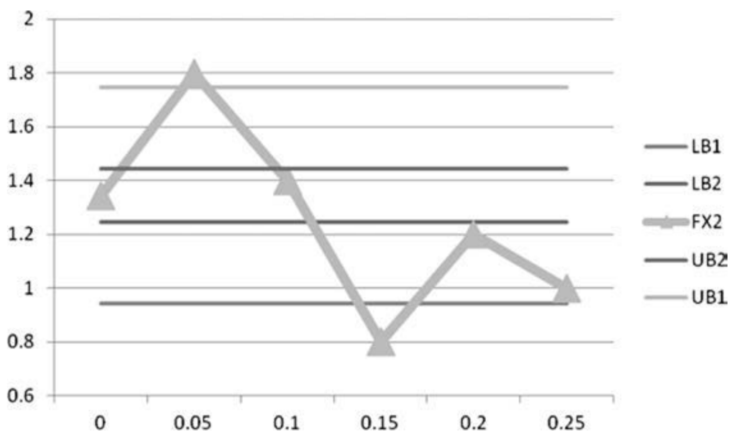


Figure 1.1 Situation in which coupon rate R_{C1} is given.

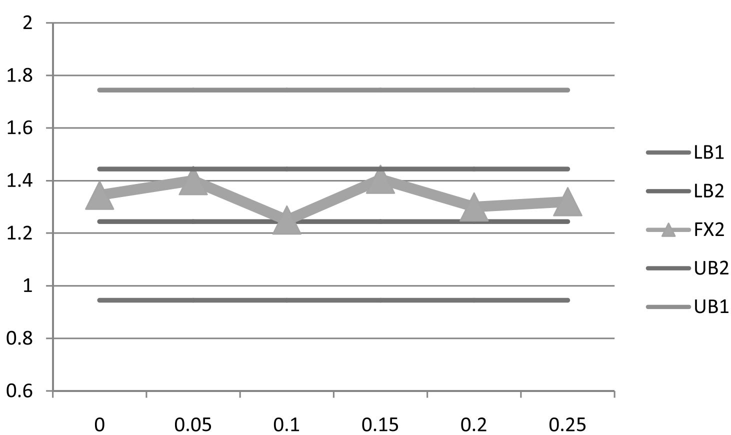


Figure 1.2 Situation in which coupon rate R_{C3} is given.

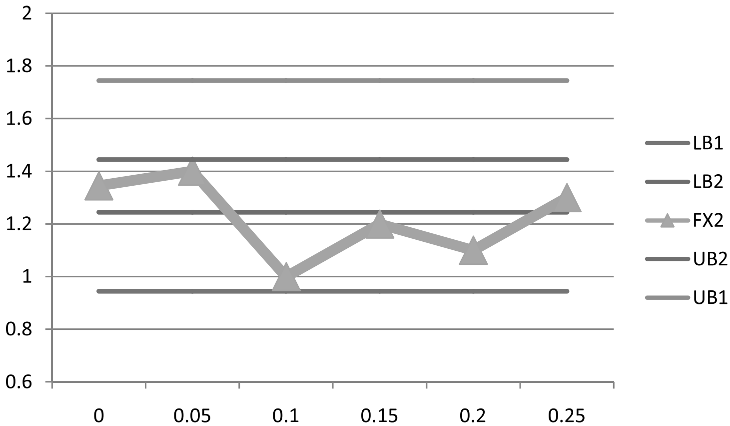


Figure 1.3 One interpretation in which coupon rate R_{C2} is given.

1.7.2 Case 2 (single touch)

All notations and conditions defined in Case 1 can be used for this case also. However, coupon payments are completely different.

If $FX_2 > UB1$ or $FX_2 < LB1$ during the index barrier observation period, then the coupon rate will be R_{C1}. However, if $FX_2 < UB1$ and $FX_2 > LB1$, and at the same time $FX_2 > UB2$ or $FX_2 < UB2$, then the coupon rate will be R_{C2}. Otherwise, the coupon rate will be R_{C3}.

The following figures illustrate the mechanism of how the product works. Figures 1.4 and 1.5 give two independent interpretations in which the coupon rate R_{C1} is given. Figure 1.6 gives an interpretation of when R_{C2} will be given, whereas Figure 1.7 illustrates the scenario in which the coupon rate R_{C3} will be employed.

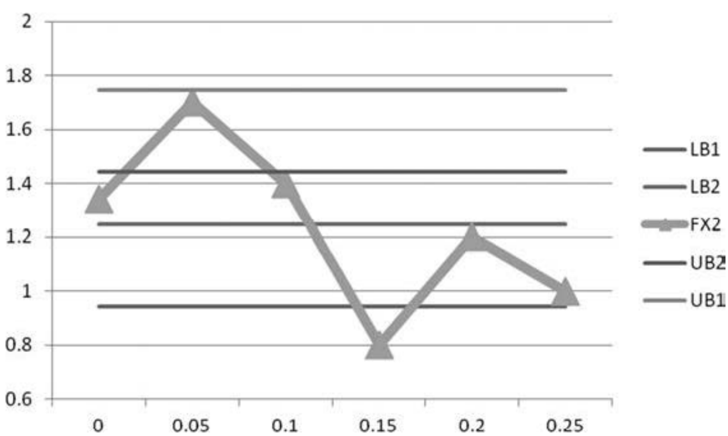


Figure 1.4 An interpretation in which R_{C1} is given.

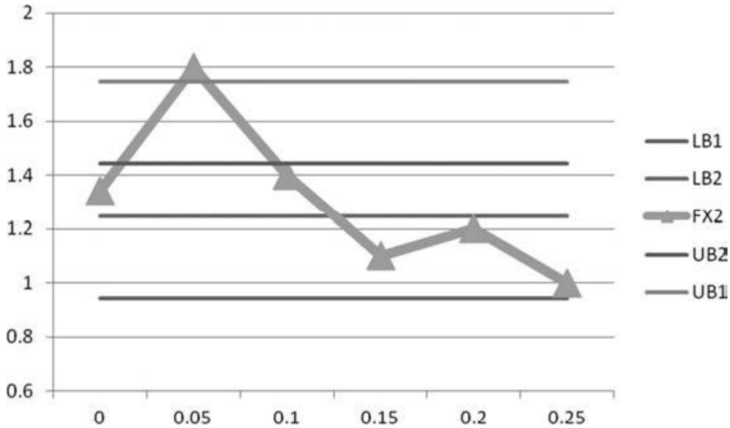


Figure 1.5 Another interpretation in which R_{C1} is given.

1.8 Basic analysis for buyer

The product mentioned promises a positive and CNY-based cash flow to the investor. It also offers a tool to bet on the volatility of the underlying FX. To be specific, assuming that $R_{C1} > R_{C2} > R_{C3}$, the buyer of the contract bets for high volatility because the probability of triggering the “pair 1 barrier” becomes larger if the volatility is high, which results in getting the high coupon rate R_{C1}. Hence, the payoff of the buyers becomes larger if the volatility of the underlying asset is high. On the other hand, if R_{C3} is the largest, the buyer of the contract bets for low volatility because the probability of not triggering both “pair 1 barrier” and “pair 2 barrier” becomes smaller if the volatility is high. Thus, the payoff of the buyers becomes larger if the volatility of the underlying asset is low.

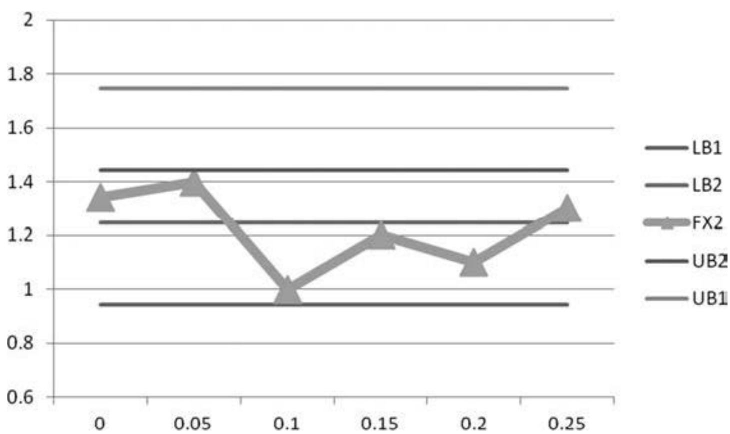


Figure 1.6 Another interpretation in which R_{C1} is given.

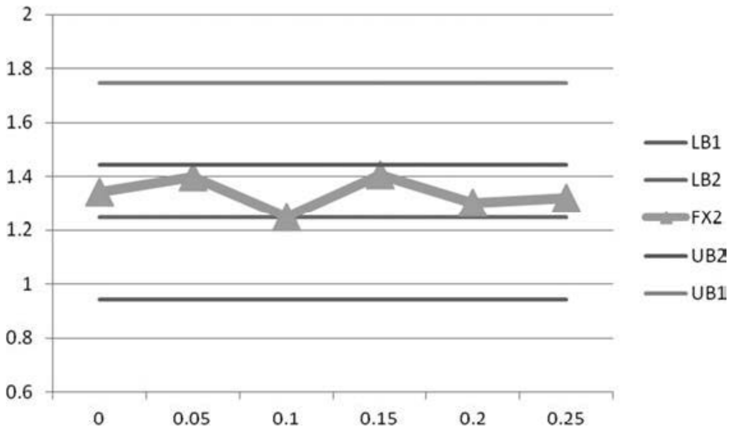


Figure 1.7 The scenario in which R_{C3} is given.

Although there are two cases of the non-deliverable swap, both lead to the same conclusion under the aforementioned classification criteria, which can be summarized in Table 1.1.

1.9 Auto-callable ratio par forward

The auto-callable ratio par forward is associated with the underlying asset, the FX rate. Such a contract is a synthetic forward consisting of a call and put that have the same strike and expiry date but different notionals.

In addition, the product has three different variations:

1 Multiple knock-out type (or American knock-out type)
2 Guarantee type (based on American knock-out type)
3 Bonus type.

Table 1.1

	Non-deliverable swap
By levels of principal protection	Partially protected product
By quantity (periodicity) of payments	Coupon product
By the type of underlying asset	FX rate
By the form of a structured product	Private banking service
By the type of investor	Individual investor/group of institutional investor
By behavior of underlying asset	Occurrence/non-occurrence of triggered event
By the degree to which the payoff depends on the price path of the underlying asset	Independent of the price path
By the payoff functions	Barrier function

1.9.1 Mechanism of auto-callable ratio par forward

To be specific, it is assumed that there are 22 fixing dates, Party A is the seller, and Party B is the investor (buyer). The entire period is divided into two periods: the first period includes the first 11 periods, and the second period includes the last 11 periods. Moreover, the following terms are defined:

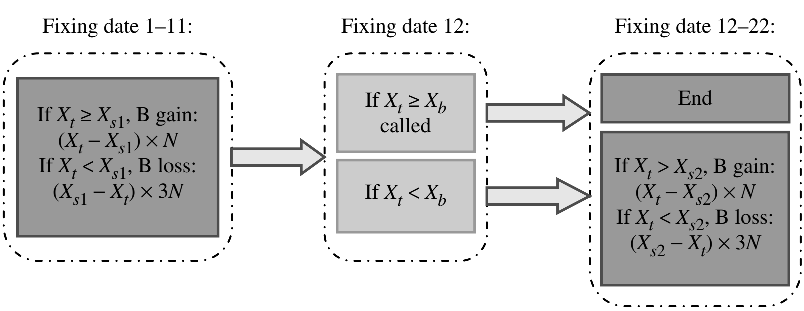
1 X_t is the FX rate.
2 N is the notional in domestic currency.
3 X_{s1} is the strike rate for the first period.
4 X_{s2} is the strike rate for the second period.
5 X_b is the barrier rate.

There are different types of auto-callable ratio par forward, which are now described.

1.9.2 The original (one-time knock-out) type

For this type, the product can be called at the 12th fixing date. On each fixing date in the first period, if the FX rate X_t is greater than or equal to X_{s1}, then the buyer earns $\left(X_t - X_{s1}\right) \times N$. However, if the FX rate X_t is less than X_{s1}, then the buyer loses an amount equal to $\left(X_{s1} - X_t\right) \times 3N$. On the 12th fixing date, if the FX rate is greater than or equal to the barrier rate, the product is called by the seller, and the contract is terminated. On the other hand, if the FX rate is smaller than the barrier rate, the contract can continue, and on each fixing date in the second period, the buyer gains $(X_t - X_{s2}) \times N$ if $X_t > X_{s2}$ and the buyer loses $\left(X_{s2} - X_t\right) \times 3N$ if $X_t < X_{s2}$.

The cash flow on each fixing date can be summarized in the following flow chart:

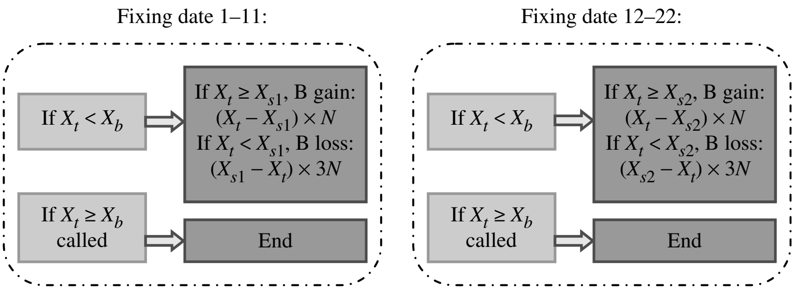


1.9.3 Multiple knock-out (or American knock-out) type

This type can be called by the seller on each fixing date, which is determined by whether the FX rate X_t is greater than or equal to the barrier rate X_b.

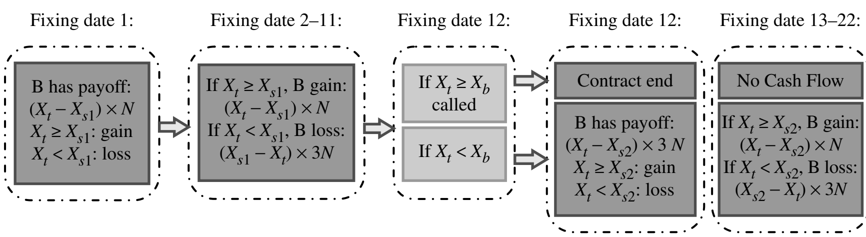
If the product is not called, then in the first period, the buyer can gain $\left(X_t - X_{s1}\right)\times N$ when $X_t \geq X_{s1}$ and loses $\left(X_{s1} - X_t\right)\times 3N$ when $X_t < X_{s1}$. On the other hand, in the second period, the buyer can gain $(X_t - X_{s2})\times N$ if $X_t \geq X_{s2}$ and loses $\left(X_{s2} - X_t\right)\times 3N$ if $X_t < X_{s2}$.

The cash flow on each fixing date can be summarized in the following flow chart



1.9.4 The guarantee type (based on American knock-out)

Properties and cash flows of this type are similar to the American knock-out (multiple knock-out) type, except that on the first fixing date, an auto-call is not available, and the buyer gains $\left(X_t - X_{s1}\right)\times N - X_t \geq X_{s1}$ and loses $\left(X_t - X_{s1}\right)\times N \_ X_t < X_{s1}$.

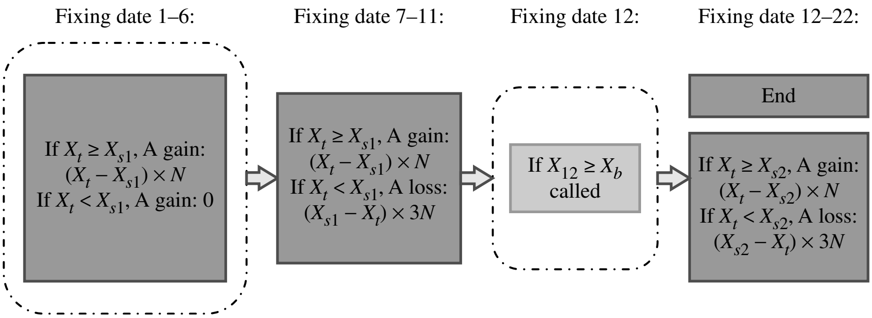


1.9.5 The bonus type

For this type, the callable feature is the same as the original (or one-time knock-out) type. However, cash flows are completely different on each fixing date. Instead of considering two periods, there are three periods: 1st period from fixing day 1 to fixing day 6, 2nd period from fixing day 7 to fixing day 11, and the last period from fixing day 12 to fixing day 22.

In the first period, the buyer gains $\left(X_t - X_{s1}\right)\times N$ if $X_t \geq X_{s1}$ and gets nothing if $X_t < X_{s1}$. In the second period, the buyer gains $\left(X_t - X_{s1}\right)\times N$ if $X_t \geq X_{s1}$ and loses $\left(X_{s1} - X_t\right)\times 3N - X_t < X_{s1}$. In the last period, the buyer gains $(X_t - X_{s2})\times N$ _ $X_t \geq X_{s2}$ and loses an amount equal to $\left(X_{s2} - X_t\right)\times 3N - X_t < X_{s2}$.

The cash flow on each fixing date can be summarized as follows:



1.9.6 Basic analysis of one-time knock-out type

Because of the difference in the notional, the auto-callable ratio par forward indeed has a larger leverage of loss rather than gain. So the investor should have a strong market view that the FX rate will stay in the benefit range (above strike if the investor calls foreign currency, below strike if the investor puts foreign currency) during the target period. Otherwise, the investor will lose more money. And if the FX rate goes out of the benefit range at the first fixing date of the second period (like the 12th fixing date in our sample), it will not be called after the first fixing date of the second period, and what is worse, it will continue to lose money until the last fixing date of the second period. The auto-callable barrier is beneficial to the seller as it guarantees that the seller of the RPF will not lose too much. Once the FX rate is deep in the money at the callable date (like over the barrier in our sample), the product will be knocked out. So the features of leverage and auto-call are both beneficial to the seller only.

If the FX rate stays in the benefit range in the target period, which means the FX rate behaves well and does not fluctuate much, the buyer will enjoy a good return.

Again, we focus on the classification of the product. With the classification criteria, we obtain Table 1.2.

Table 1.2

	Auto-callable ratio par forward
By levels of principal protection	Partially protected product
By quantity (periodicity) of payments	Coupon product
By the type of the underlying asset	FX rate
By the form of the structured product	Private banking service
By the type of investor	Individual investor
By the behavior of the underlying asset	Occurrence/non-occurrence of triggered event
By the degree that the payoff depends on the price path of the underlying asset	Independent of the price path
By the payoff functions	Barrier function

Appendix A

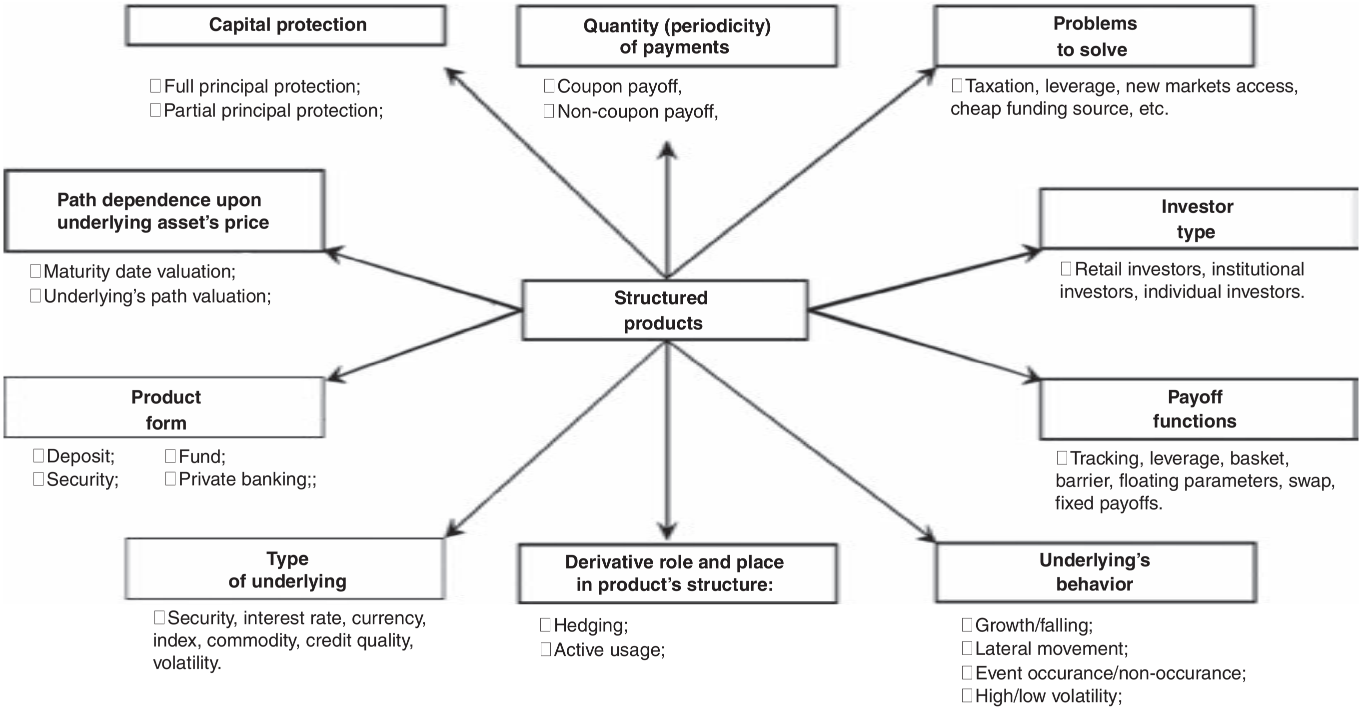


Figure A.1 Classification of structured products.

Further Reading

1 V. Omelchenko. 2009. Definition and classification of structured financial products. http://en.wikipedia.org/wiki/Structured_product#Origin.
The classification of a structured product in economics: http://www.ukessays.com/essays/finance/the-classification-of-a-structured-product-in-economics-finance-essay.php.

2 Tools and methods for pricing exotic options

2.1 Background

Options trading became popular since, 1973, after the Chicago Board Options Exchange (CBOE) standardized and integrated options transactions. In the following 20 years or so, the options market expanded significantly, and many different exotic options were developed. In general, for a particular exotic option, a corresponding closed formula might not exist there, too. Tools/methods need to be explored to price exotic options.

Fischer Black and Myron Scholes are most likely the two economists who first made a breakthrough in derivatives pricing. They developed an analytical model now popularly known as the *Black–Scholes model (BS model)*, which opened the area of research on option pricing.

In this chapter, unless otherwise specified, the following assumptions are made.

2.1.1 Assumptions for BS model

1 Efficient markets (market movements cannot be predicted).
2 Commissions are non-existent.
3 Interest rates do not change over the life of the option (and are known).

To be specific, the following notations are used in this chapter:

S = underlying stock price
X = strike price
r = risk-free interest rate
V = volatility
τ = time to maturity
D = dividend
T = maturity.

Moreover, the stock price is assumed to follow the lognormal distribution:

$$\frac{dS}{S} = \mu dt + \sigma dZ_t^Q,$$

where Z_t^Q is a risk-neutral standard Weiner process, μ is the drift rate, which in the risk-neutral world is given as.

With the above assumptions, it is now possible to explore different tools or models for pricing options. How to select the appropriate models or methodologies? Such selection can be based on the duration of the contract, the purpose of such pricing, and how accurate the pricing reflects the market condition. In addition, if constant volatility needs to be used, then how can the volatility be estimated so that its pricing would not be too far from the one obtained from the market? Normally for products with a short duration, such as 1 week or 10 days, constant volatility can be the best choice, and such volatility can be estimated from the combination of historical volatility and implied volatility that derived from options that traded on the market. For longer duration products, it is too risky or inaccurate to use those models that are based on constant volatility. Any information that can be used to capture the views of the option contract holders, which include their expected asset price and volatility movement, such as volatility term structure, interest term structure, and so on, are welcome to be included in the pricing. Therefore, the volatility is no longer a constant but a stochastic one.

2.2 European option

According to Wikipedia, an option is a contract that gives the buyer (the owner) the right, but not the obligation, to buy or sell an underlying asset at a specified strike price on or before a specified date. The seller has the corresponding obligation to fulfill the transaction, which is to sell or buy the underlying asset, if the buyer decides to "exercise" the option. The buyer pays a *premium* to the seller for this right. An option that conveys to the owner the right to buy an underlying asset is referred to as a call; an option that conveys the right of the owner to *sell* an asset is referred to as a put.

Options pricing is a research area with a long history. In general, the value of an option commonly consists of two parts:

- The first part is the intrinsic value, which is defined as the difference between the market value of the underlying asset and the strike price of the given option.
- The second part is the time value, which depends on a set of other factors that, through a multi-variable, non-linear interrelationship, reflect the discounted expected value of that difference at expiration.

The formula for a European call option on a non-dividend stock is

$$c_t = Se^{-\mathrm{D}\tau}N(d_1) - Xe^{-\mathrm{r}\tau}N(d_2),$$

whereas for a European put option on a non-dividend stock the formula is

$$p_t = Xe^{-\mathrm{r}\tau}N(-d_2) - Se^{-\mathrm{D}\tau}N(-d_1),$$

where $N(d_1)$ and $N(d_2)$ are the cumulative normal distribution functions for d_1 and d_2, which are defined as

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r - D + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r - D - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}};$$

d_2 can be further simplified as

$$d_2 = d_1 - \sigma\sqrt{\tau}.$$

In addition, the volatility of the stock price is an important input, which can be estimated from historical data, which is known as historical volatility; however, it only reflects the past history and consists of no view of the future. A more practical method that utilizes the additional information about the view of the option buyers is to use implied volatility that is extracted from the market traded options, especially those put options based on the Black-Scholes Formula, and use numerical methods like the Newton-Rhapson or simplified methods like approximation (Brenner and Subrahmanyam, 1988).

Having outlined the foundations of option pricing via the BS framework, we can describe numerous other methods to price European options and the underlying theories.

Here we look at some other models to price standard vanilla European options.

2.2.1 Pricing

2.2.1.1 Binomial method

The binomial method is a common method to price all types of options, both vanilla and the more exotic types owing to its flexibility (Cox, Ross, Rubinstein, 1979). European options can be related to the BS model, for reasons discussed later.

First, we consider the fundamental properties of the binomial method. It essentially models the movement of the stock price over time and hence the option price by considering the movements at each time node where the up (u) and down (d) "jumps" are given as

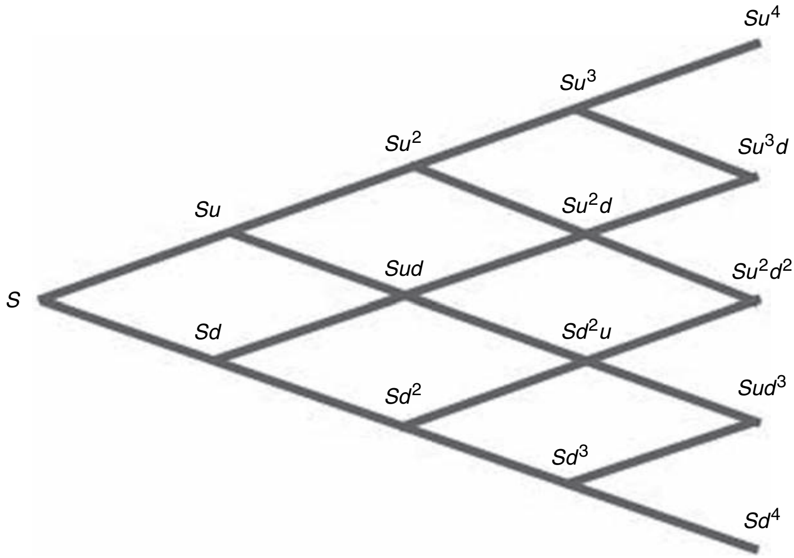
$$u = e^{\sigma\sqrt{\tau}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\tau}}.$$

From the formula u and d, it can be seen that ud=1. The probability of an upward stock movement (i.e., increasing) is

$$p = \frac{e^{(\mathrm{r}-\mathrm{D})\tau} - d}{u - d}.$$

For the probability of a downward stock movement, note that the combined probabilities must equal 1; hence, the probability of the downward movement is $1 - p$.

An illustration of how the binomial tree works is shown in the following example:



Having found the value of stock price at the end of the nodes, we can calculate the option value by means of backward induction, that is, working from the far right of the lattice, back to the origin. A simple computer algorithm is able to solve the option values with a large number of nodes. For European call and put options, the binomial model is given as

$$c = e^{\sigma\sqrt{\tau}} \sum_{i=j}^{n} \left(\frac{n!}{i!(n-i)!} \right) p^{\mathrm{i}} (1-p)^{\mathrm{n-i}} \left(Su^{\mathrm{i}} d^{\mathrm{n-i}} - X \right)$$

and

$$p = e^{\sigma\sqrt{\tau}} \sum_{i=0}^{\mathrm{j-1}} \left(\frac{n!}{i!(n-i)!} \right) p^{\mathrm{i}} (1-p)^{\mathrm{n-i}} \left(X - Su^{\mathrm{i}} d^{\mathrm{n-i}} \right),$$

where j is given by the following condition:

$$j = \Phi \left(\frac{\ln \left(X / Sd^{\mathrm{n}} \right)}{\ln (u / d)} \right).$$

Here ϕ represents the next non-negative integer greater than the term in brackets.

There is a correlation between the binomial model and the BS pricing model in the context of the valuation of options. With a significant number of time nodes, the binomial method begins to converge, and the convergence of this value ultimately becomes the value obtained from the closed-form formula.

2.2.1.2 Trinomial method (Boyle)

The trinomial method of pricing options was introduced by Boyle (1986); it is an attempt to model stock price movements better than the binomial method. As one can guess by its name, the trinomial method is similar to the binomial lattice in that the stock price is modeled by a tree, but instead of two possible paths, the trinomial tree has three paths: up, down, and a stable path. The probabilities of each are

$$p_d = \left(\frac{e^{\sigma\sqrt{\tau/2}} - e^{(r-D)\tau/2}}{e^{\sigma\sqrt{\tau/2}} - e^{-\sigma\sqrt{\tau/2}}} \right)^2, \quad p_u = \left(\frac{e^{(r-D)\tau/2} - e^{-\sigma\sqrt{\tau/2}}}{e^{\sigma\sqrt{\tau/2}} - e^{-\sigma\sqrt{\tau/2}}} \right)^2, \quad p_m = 1 - p_d - p_u.$$

The probabilities can also be represented as

$$p_d = -\sqrt{\frac{t}{12\sigma^2}}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{6}, \quad p_u = \sqrt{\frac{t}{12\sigma^2}}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{6} \quad \text{and} \quad p_d = \frac{2}{3}.$$

Note also that the trinomial method converges much faster than the binomial method.

2.2.1.3 Adaptive mesh model

Normally, constant volatility is used in such pricing with trees. However, for pricing options with a longer duration, information of volatility term structure is needed to make the pricing closer to the market, or to reflect the views of the investors that reflected in the prices (volatility) that they were willing to offer for options with different durations. The Boot-trap approach can be used to extract the volatility to be used in a different time-interval. If such volatility fluctuated significantly, then constant mesh size would not provide sufficient accuracy, and an adaptive mesh model can be a good solution for such a situation. Also, for pricing exotic options like a barrier option, when the asset size is getting close to the barrier, the price changes very significantly. Therefore, using the smaller mesh size can more accurately capture the price change in such an area.

Adaptive Mesh Model (AMM) is a very flexible approach that helps in enhancing the efficiency in trinomial trees to a great extent. It uses coarse time and price steps in most of the tree when market or data points are smooth, but small sections of finer mesh are used in more volatile regions to improve resolution in such critical areas. In other words, the model is based on a lattice-like numerical process in which regions of high resolution are embedded in critical regions of a lower resolution mesh.

The AMM is a variant of the standard lattice models in that the mesh size can be adjusted based on the market condition; for example, the volatility of stock prices.

Compared with the standard lattice approach, AMM provides a faster convergence and more accurate results. However, an indicator needs to be determined and calculated to guide the adjustment of the mesh size. Although the adaptive mesh is more common in pricing path-dependent options such as barrier options, it can be used to price European options as well. Detailed descriptions of the model can be found by referring to Figlewski and Gao (1999).

2.2.1.4 Monte Carlo simulation

One of the most commonly used techniques today to price options is what is known as Monte Carlo simulation (MCS). Recall that the stock price is assumed to follow the lognormal distribution, i.e.,

$$\frac{dS}{S} = \mu dt + \sigma dZ_t^Q.$$

Employing the techniques in solving stochastic differential equation, the stock price can be written as follows:

$$S_T = S_t \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t\varepsilon}\right\}, \varepsilon \sim N(0,1).$$

To obtain the value of a European option, we divide the interval $T - t$ into n different time points, say, $t = t_0, t_1, t_2, \ldots, t_n = T$. To be specific, assume we generate a total of N paths. Then, for each path generated, the value of $S_{t_{i+1}}$ is obtained using the formula above, that is, $S_{t_{i+1}} = S_{t_i} \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\varepsilon_i\right\}$ for each ε_i until we get S_T. After getting S_T, we can obtain the present values of the payoffs for this path. To obtain the price of the option, we simply take average of the N paths generated.

2.3 American option

American options give the holder the right to exercise the option at or before the expiry date. This characteristic of American options renders pricing somewhat more cumbersome in particular cases.

The only case in which a closed-form solution to pricing an American option exists is where we are valuing an American *call* option with no dividends throughout its life, because it is never beneficial to exercise the option prior to expiry. The detailed reasons behind this will not be considered here, but two primary reasons exist:

1 Holding the call option instead of exercising it and holding the stock is an insurance of sorts. An adverse stock movement (fall) can result in losses for the stock holder, but holding the call enables the holder of the call to insure against any falls.
2 There is the concept of time value of money. Paying the strike price earlier rather than later means that the holder of the option loses out on the time value the money can achieve for the remainder of the option.

Hence, we can price American calls with no dividends via the standard BS European call option formula and force dividends to 0.

Another American call option that can be priced using an exact formula is an American call option with a single known dividend via a *pseudo-American formula*. The formula was derived by Roll, Geske, and Whaley, see Roll (1977). Derivations of the closed-form solution have been omitted; only the formula is presented here:

$$c = \left(S - D_1 e^{-r(t_1 - t)}\right) N(b_1) - (X - D_1) e^{-r(t_1 - t)} N(b_2) + \left(S - D_1 e^{-r(t_1 - t)}\right)$$
$$N_2\left(a_1; -b_1; -\sqrt{\frac{t_1 - t}{T - t}}\right) - X e^{-r(t_1 - t)} N_2\left(a_2; -b_2; -\sqrt{\frac{t_1 - t}{T - t}}\right).$$

The variables are defined as follows:

$$a_1 = \frac{\ln\left(\frac{S - D_1 e^{-r(t_1 - t)}}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \; a_2 = a_1 - \sigma\sqrt{\tau},$$
$$b_1 = \frac{\ln\left(\frac{S - D_1 e^{-r(t_1 - t)}}{S^*}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \; b_2 = b_1 - \sigma\sqrt{\tau}.$$

$N_2(a; b; rho)$ is the bivariate cumulative normal distribution function, and S^* is the *critical stock price* for which the following equation is satisfied: $S^* + D_1 - X = \text{call price of critical stock price}$.

Making use of the above, the critical stock price can be solved iteratively via the *bisectional method*. Several approaches used in pricing European options can be used in pricing American options, some of which include the Binomial Tree, Trinomial Tree, Monte Carlo, and so on. However, because the exercise time is different, the pricing formula can be different.

The rest of this section looks at some of the tools/methods to price standard vanilla American options.

2.3.1 Pricing

Pricing of American options can be done by Binomial Tree, Trinomial Tree, Adaptive Mesh Model, and Monte Carlo, which were all discussed in the previous subsection. We will not repeat explanations of the materials that have already been discussed, but instead we will just highlight the differences between European options and American options.

2.3.1.1 Binomial trees (Cox, Ross, Rubinstein)

Binomial trees are widely used for pricing American-type options as they are computationally inexpensive and handle American options well. The binomial method constructs a tree lattice that represents the movements of the stock under GBM and prices the option relative to the stock price through backward induction. Again, the binomial version used follows from Cox, Ross, and Rubinstein (1979).

The following step is a major difference between pricing European options and pricing American options when using a binomial tree. Because the exercise times are different, through backwards induction it is possible to determine the price of an American call or put through the following formula:

$$c_{\mathrm{i,j}} = \max\left\{Su^{\mathrm{j}}d^{\mathrm{i-j}} - X, e^{-\mathrm{r(T-t)}}\left[pc_{\mathrm{i+1,j+1}} + (1-p)c_{\mathrm{i+1,j}}\right]\right\}.$$

$$p_{\mathrm{i,j}} = \max\left\{X - Su^{\mathrm{j}}d^{\mathrm{i-j}}, e^{-\mathrm{r(T-t)}}\left[pp_{\mathrm{i+1,j+1}} + (1-p)p_{\mathrm{i+1,j}}\right]\right\}.$$

2.3.1.2 Trinomial trees (Boyle)

The trinomial tree is similar to the binomial method in that it employs a lattice-type method for pricing options. The difference is that the trinomial method provides an accurate value faster than its binomial counterpart owing to the use of a three-pronged path compared to the two-pronged path seen with binomial trees.

The probabilities of the price going up during the next time period are given as

$$p_d = \left(\frac{e^{\sigma\sqrt{\tau/2}} - e^{(r-D)\tau/2}}{e^{\sigma\sqrt{\tau/2}} - e^{-\sigma\sqrt{\tau/2}}}\right)^2, \quad p_u = \left(\frac{e^{(r-D)\tau/2-} e^{-\sigma\sqrt{\tau/2}}}{e^{\sigma\sqrt{\tau/2}} - e^{-\sigma\sqrt{\tau/2}}}\right)^2, \quad p_m = 1 - p_d - p_u.$$

The respective American call and put options can now be priced via backward induction.

Call:

$$c_{i,j} = \max\left\{Su^{\max(0,j-1)}d^{\max(0,2i-i-j)} - X, e^{-\mathrm{r(T-t)}} \times\left[p_u c_{i+1,\mathrm{j+2}} + p_m c_{i+1.\mathrm{j+1}} + p_d c_{i+1,\mathrm{j}}\right]\right\}.$$

Put:

$$p_{i,j} = \max\left\{X - Su^{\max(0,j-1)}d^{\max(0,2i-i-j)}, e^{-r(T-t)} \times\left[p_u p_{i+1,j+2} + p_m p_{i+1.j+1} + p_d p_{i+1,j}\right]\right\}.$$

For further studies of the trinomial model, see Boyle (1986).

2.3.1.3 Monte Carlo simulation

The Monte Carlo method does not easily handle the pricing of American options because of the early exercise characteristic; early research deemed the pricing of American options to be not even possible, using MCS. Simulation of option prices tends to employ a backward induction technique, which tends to overestimate the price of an option.

Various algorithms have been put forward to price American options using backward induction, but many algorithms are computationally expensive because they do not converge readily.

A number of authors, including Broadie, Glasserman, and Kou (1997) and Fu et al. (2000), have suggested that the most flexible and easily implemented procedure is the simulated tree algorithm, but it has its own drawbacks, the primary one being exponential growth in computation time with the number of exercise opportunities.

In the rest of this chapter, we are going to discuss several exotic options and the corresponding tools for pricing them.

2.4 Asian options

Asian options are options in which the underlying variable is the average price over a period of time. Because of this fact, Asian options have lower volatility, which makes them cheaper relative to their European counterparts. They are commonly traded on currencies and commodity products that have low trading volumes. Asian options have been used a lot in helping firms, like airline companies, trading firms, etc., to hedge their balance sheet or cash flow risk. Normally, a temporal spike in commodity prices or fluctuation in exchange rates would not create financial difficulty for such firms, but sustained ones do. They were originally used in 1987 when Banker's Trust Tokyo office used them for pricing average options on crude oil contracts—hence the name "Asian" option. Another advantage of using Asian options is that they reduce the risk of market manipulation when a financial instrument, such as commodity or index futures, is about to mature.

They are broadly segregated into three categories: (1) arithmetic average Asians, (2) geometric average Asians, and (3) the averaging of these forms on a *weighted average* basis, where a given weight is applied to each stock being averaged. This can be useful for obtaining an average for a sample with a highly skewed sample population.

To date, there are no known closed-form analytical solutions for options, because the lognormal assumptions generally collapse. A further breakdown of these options suggests that Asians are either based on the *average price* of the underlying asset, or alternatively, they are the *average strike* type. The corresponding payoff functions are as follows:
Average price Asian:

$$V = \max\left(0, \eta\left(S_A - X\right)\right).$$

Average strike Asian:

$$V = \max\left(0, \eta\left(S_T - S_A\right)\right),$$

where η is a binary variable set to 1 for a call and -1 for a put.

Arithmetic averaging is the sum of the sampled asset prices divided by the number of samples:

$$Avg_A = \frac{S_1 + S_2, \ldots + S_n}{n}.$$

For geometric averaging, the average value is taken as

$$Avg_G = \sqrt[n]{S_1 S_2 \ldots S_n},$$

where the *n*th root of the sample values multiplied together is taken. The payoff functions for Asian options are given as follows:
For an average price Asian:

$$V = \max\left(0, \eta\left(S_A - X\right)\right).$$

For an average strike Asian:

$$V = \max\left(0, \eta\left(S_T - S_A\right)\right),$$

where η is a binary variable that is set to 1 for a call and −1 for a put. Asians can be either a European- or American-style exercise.

Next we discuss some methods to price standard Asian options (both arithmetic and geometric) under a variety of methods.

2.4.1 Pricing

2.4.1.1 Geometric closed form (Kemna and Vorst, 1990)

Kemna and Vorst (1990) developed a closed-form pricing solution to geometric averaging options by altering the volatility and cost of the carry term. Geometric averaging options can be priced via a closed-form analytic solution because the geometric average of the underlying price follows a lognormal distribution, whereas under average rate options, this condition collapses.

The solutions for geometric averaging Asian call and puts are given as

$$c = Se^{(b-r)\tau} N\left(d_1\right) - Xe^{-r\tau} N\left(d_2\right) \quad \text{and} \quad p = Xe^{-r\tau} N\left(-d_2\right) - Se^{(b-r)\tau} N\left(d_1\right),$$

where $N(x)$ is the cumulative normal distribution function, and the formulas for different parameters are as follows:

$$d_1 = \frac{In\left(\frac{S}{X}\right) + \left(b + \frac{1}{2}\sigma_A^2\right)T}{\sigma_A \sqrt{T}},\ d_2 = \frac{In\left(\frac{S}{X}\right) + \left(b - \frac{1}{2}\sigma_A^2\right)T}{\sigma_A \sqrt{T}} \text{ or}$$

$$d_2 = d_1 - \sigma_A \sqrt{T}, \sigma_A = \frac{\sigma}{\sqrt{3}}, b = \frac{1}{2}\left(r - D - \frac{\sigma^2}{6}\right).$$

Here σ is the observed volatility, *r* is the risk-free interest rate, *D* is the dividend yield, σ_A is adjusted volatility, and *b* is adjusted dividend yield.

2.4.1.2 Arithmetic rate approximation

As there are no closed-form solutions to arithmetic averages owing to inappropriate use of the lognormal assumption under this form of averaging, a number of approximations have emerged in the literature. There are at least two different arithmetic rate approximations available for pricing Asian options. They include the *Turnbull and Wakeman approximation* and the *Levy approximation*.

2.4.1.3 Turnbull and Wakeman approximation

Turnbull and Wakeman (TW) (1991) used the fact that the distribution under arithmetic averaging is *approximately* lognormal, and they put forward the first and second moments of the average in order to price the option.

The analytical approximations for a call and a put under TW are given as

$$c \approx Se^{(\mathrm{b}-\mathrm{r})\mathrm{T}_2} N(d_1) - Xe^{-\mathrm{r}\mathrm{T}_2} N(-d_2)$$

and

$$p \approx Xe^{-\mathrm{r}\mathrm{T}_2} N(-d_2) - Se^{(\mathrm{b}-\mathrm{r})\mathrm{T}_2} N(d_1),$$

where

$$d_1 = \frac{In\left(\frac{S}{X}\right) + \left(b + \frac{1}{2}\sigma_A^2\right)T_2}{\sigma_A\sqrt{T_2}} \quad \text{and} \quad d_2 = d_1 - \sigma_A\sqrt{T_2}.$$

Here, T_2 is the time to maturity. For averaging options that have already begun their averaging period, T_2 is simply T (the original time to maturity); if the averaging period has not yet begun, then T_2 is $T - \tau$.

The adjusted volatility and dividend yield are given as $\sigma_A = \sqrt{\frac{\ln(M_2)}{T} - 2b}$ and $b = \frac{\ln(M_1)}{T}$, respectively.

To generalize the equations, we assume that the averaging period has not yet begun and give the first and second moments as

$$M_1 = \frac{e^{(\mathrm{r}-\mathrm{D})\mathrm{T}} - e^{(\mathrm{r}-\mathrm{D})\tau}}{(r-D)(T-\tau)}$$

and

$$M_2 = \frac{2e^{\left(2(\mathrm{r}-\mathrm{D})+\sigma^2\right)\mathrm{T}} S^2}{\left(r-D+\sigma^2\right)\left(2r-2q+\sigma^2\right)T^2} + \frac{2S^2}{(r-D)T^2}\left(\frac{1}{2(r-D)+\sigma^2} - \frac{e^{(\mathrm{r}-\mathrm{D})\mathrm{T}}}{r-D+\sigma^2}\right).$$

If the averaging period has already begun, we must adjust the strike price as

$$X_A = \frac{T}{T_2} X - \frac{(T - T_2)}{T_2} S_{Avg},$$

where T is the original time to maturity, T_2 the remaining time to maturity, X is the original strike price, and S_{Avg} is the average asset price. Haug (1998) noted that if $r = D$, the formula will not generate a solution.

2.4.1.4 Levy approximation

Levy created another analytical approximation to give more accurate results than the TW approximation. The approximation of value of a call is given as

$$c \approx S_Z N(d_1) - X_Z e^{-rT_2} N(d_2).$$

By the formula of put-call parity, we get the price of a put as

$$p \approx c - S_Z + X_Z e^{-rT_2},$$

where

$$d_1 = \frac{1}{\sqrt{K}} \left[\frac{\ln(L)}{2} \ln(X_Z) \right],\ d_2 = d_1 - \sqrt{K},\ S_Z = \frac{s}{(r-D)T} \left(e^{-DT_2} e^{-rT_2} \right),$$

$$X_Z = X - S_{Avg} \frac{T - T_2}{T}, K = \ln(L) - 2\left[rT_2 + \ln(S_Z) \right],\ L = \frac{M}{T^2}, \text{and}$$

$$M = \frac{2S^2}{r - D + \sigma^2} \left\{ \frac{2e^{(2(r-D)+\sigma^2)T_2 - 1}}{2(r-D) + \sigma^2} \right\} + \frac{e^{(r-D)T_2 - 1}}{r - D}.$$

The variables are the same as defined under the TW approximation.

2.4.1.5 Monte Carlo Simulation

Various methods using MCS have been developed to price arithmetic Asian options. The aforementioned analytical approximations by TW, Levy, and Curran can all be computed using simulation. MCS can give relatively accurate prices for options, and in the case of Asian options, which are highly path dependent, this method is particularly useful.

In the beginning, we gave a geometric closed-form solution to Asian options originally presented by Kemna and Vorst (1990). The authors show that the geometric solution can be used as a *control variate* within a MCS framework.

The control variate technique can be used to find more accurate analytical solutions to a derivative's price if there is a similar derivative with a known analytical solution. With this in mind, MCS is then performed on the two derivatives in parallel.

Given the price of the geometric Asian, we can price the arithmetic Asian by considering the equation

$$V_A = V_A^* - V_B^* + V_B,$$

where V_A^* is the estimated value of the arithmetic Asian through simulation, V_B^* is the simulated value of the geometric Asian, and V_B is the exact value of the aforementioned geometric Asian.

2.5 Barrier options

Barrier options are the most important of exotic options; so important that a significant amount of structured products contain such a barrier nature even though, compared with vanilla options, the barrier (H) provides one additional hurdle. An "In" barrier means that a barrier option becomes active once crossing a particular barrier level. An "Out" barrier means that a barrier option becomes inactive once crossing such a barrier level. For example, an Up & In barrier becomes active when the underlying price hits a barrier from below. For a simple Up & In European call, only maturity above the strike (X) is inadequate. During the life of the option, the underlying price must cross the barrier level to knock in (activate) the option contract. Also, for such a barrier option, the barrier level must be higher than the strike (H > X), or the barrier would become meaningless.

Barrier options are path-dependent and have many formats and styles for investors or product developers who have different views about the market and purposes for using such exotic options (for example, for reducing the cost). The key characteristic is that these types of options are either activated (knock-in) or terminated (knock-out) upon reaching a certain barrier (H). Barrier options can also be cross-asset; for example, when the Hang Seng Index (HSI) reaches H = 28000, a knock in call can be made on HSBC at X = 92. However, stability of correlation between HSI and HSBC is a key issue in pricing such cross-asset barrier options. In the following, we will discuss the most basic type of barrier options—the single barrier.

Theoretically, barrier options come in 16 flavors—but some of them are meaningless as mentioned before or depending on whether the Barrier option started as "activated" or "non-activated"—each with its own characteristics, which are outlined below:

1 Up & In & (H>X) Put or Call
2 Up & Out & (H>X) Put or Call
3 Down & In & (H>X) Put or Call (meaningless)
4 Down & Out & (H>X) Put or Call (meaningless)
5 Up & In & (H<X) Put or Call (meaningless)
6 Up & Out & (H<X) Put or Call (meaningless)
7 Down & In & (H<X) Put or Call
8 Down & Out & (H<X) Put or Call

Each of these types can be a call or a put, giving a total of eight single barrier types. An "in" barrier means the barrier becomes active after crossing a particular barrier level; for example, an up-and-in barrier becomes active when the underlying's price reaches a barrier from below.

If we add one more dimension, European or American, then it would become very complicated. Also, innovations in exotic options created more complicated forms of barrier options, including double barriers (similar to Double Touch), soft barriers, time-dependent barriers, CPI-linked barriers, Parisians, and partial time barriers, which all came to exist mainly because of the needs of the market. In the following, we will discuss the various methods of pricing single barriers.

Other forms of barriers include double barriers, Parisians, and partial time barriers. Here, we discuss the various methods of pricing single barriers.

Barriers can take either American or European form, and despite the seemingly complex payoffs, they are widely used in the markets and are generally cheaper than plain vanilla-type options.

2.5.1 Pricing

2.5.1.1 Analytical closed form (Merton, Reiner, and Rubinstein)

Barrier options were traded on the OTC way back in the late 1960s. They have been used extensively to manage risks related to commodities, FX, and interest rate exposures.

In 1973, Merton created the first analytical formula for a down and out call option. Since then, pricing of exotic options became a major research area in financial engineering. Merton's work was followed by the more detailed and comprehensive research of Reiner and Rubinstein (RR) (1991), who developed the formulas for all 8 types of barrier options. Haug (1998) organized and restructured the set of formulas provided by Reiner & Rubinstein to create a recipe like pricing templates. The major concept of pricing barrier options is that the value of the contract is determined by two parts: value of the vanilla option and the possibility of crossing or not crossing the barrier. Simply the value of a barrier option equals the value of a plain vanilla option multiplied by the possibility of crossing (knock-in) or not crossing (knock-out) the barrier.

We consider the 16 single barrier options as follows, given by Reiner and Rubinstein under a BS framework.

When the barrier level is lower than or equal to the strike price, the following values are given.

Call values:

$$c_{\text{down-in}} \left| H \le X = Se^{-\text{DT}} \left(\frac{H}{S}\right)^{2\text{m}} N(y) - Xe^{-\text{rT}} \left(\frac{H}{S}\right)^{2\text{m}-2} N\left(y - \sigma\sqrt{T}\right) \right.$$

$$c_{\text{down-out}} \left| H \le X = C_{\text{BS}} - \left(c_{\text{down-in}} \left| H \le X\right.\right) \right.$$

$$c_{\text{up-in}} \left| H \le X = C_{\text{BS}} \right.$$

$$c_{\text{up-out}} \left| H \le X = C_{\text{BS}} - \left(c_{\text{up-in}} \left| H > X\right.\right) \right.$$

Put values:

$$p_{\text{up-out}}\Big| H \le X = -Se^{-DT}N(-x_1) + Xe^{-rT}N\left(-x_1 - \sigma\sqrt{T}\right)$$
$$+ Se^{-DT}\left(\frac{H}{S}\right)^{2m} N(-y_1) - Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N\left(-y_1 - \sigma\sqrt{T}\right)$$
$$p_{\text{up-in}}\left| H \le X = p_{\text{BS}} - \left(p_{\text{up-out}}\right| H \le X\right)$$

$$p_{\text{down-in}}\Bigg| H \le X = -Se^{-DT}N(-x_1) + Xe^{-rT}N\left(-x_1 - \sigma\sqrt{T}\right) + Se^{-DT}\left(\frac{H}{S}\right)^{2m}$$
$$\times\left[N(y) - N(y_1)\right] - Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2}\left[N\left(y - \sigma\sqrt{T}\right) - N\left(-y_1 - \sigma\sqrt{T}\right)\right]$$
$$p_{\text{down-out}}\left| H \le X = p_{\text{BS}} - \left(p_{\text{down-in}}\right| H \le X\right)$$

When the barrier is greater than the strike price, the following formulas are used:

Call values:

$$c_{\text{down-out}}\,| H > X$$
$$= Se^{-DT}N(x_1) - Xe^{-rT}N\left(x_1 - \sigma\sqrt{T}\right) - Se^{-DT}\left(\frac{H}{S}\right)^{2m} N(y_1)$$
$$+ Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N\left(y_1 - \sigma\sqrt{T}\right)$$
$$c_{\text{down-in}}\left| H > X = c_{\text{BS}} - \left(c_{\text{down-out}}\right| H > X\right)$$
$$c_{\text{up-out}}\,| H > X = c_{\text{BS}}$$
$$c_{\text{up-out}}\left| H \le X = c_{\text{BS}} - \left(c_{\text{up-in}}\right| H > X\right)$$
$$p_{\text{up-out}}\,| H \le X = -Se^{-DT}N(-x_1) + Xe^{-rT}N\left(-x_1 - \sigma\sqrt{T}\right)$$
$$\left.+ Se^{-DT}\left(\frac{H}{S}\right)^{2m} N(-y_1) - Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N\left(-y_1 + \sigma\sqrt{T}\right)\right)$$
$$p_{\text{up-in}}\left| H \le X = p_{\text{BS}} - \left(p_{\text{up-out}}\right| H \le X\right)$$
$$c_{\text{up-in}}\,| H > X = Se^{-DT}N(x_1) - Xe^{-rT}N\left(x_1 - \sigma\sqrt{T}\right) - Se^{-DT}\left(\frac{H}{S}\right)^{2m}[N(-y)$$
$$-N(-y_1)] + Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2}[N\left(-y + \sigma\sqrt{T}\right) - N\left(-y_1 + \sigma\sqrt{T}\right)]$$

Put values:

$$p_{\text{up-in}} \Big| H > X = Se^{-DT}\left(\frac{H}{S}\right)^{2m} N(-y) + Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N\left(-y+\sigma\sqrt{T}\right)$$

$$p_{\text{up-out}} \,|H > X = p_{\text{BS}} - \left(p_{\text{up-in}} \,|H > X\right)$$

$$p_{\text{down-out}} \,|H > X = 0$$

$$p_{\text{down-in}} \,|H > X = p_{\text{BS}},$$

where

$$m = \frac{r - D + \frac{1}{2}\sigma^2}{\sigma^2}$$

$$y = \frac{In\left(\frac{H^2 / SX}{\sigma\sqrt{T}}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T}$$

$$x_1 = \frac{In(\text{S/ H})}{\sigma\sqrt{T}} + m\sigma\sqrt{T}$$

$$y_1 = \frac{In(\text{H/ S})}{\sigma\sqrt{T}} + m\sigma\sqrt{T},$$

where $N(x)$ is the cumulative normal distribution of x, H is the barrier value, S is the asset price, X is the strike price, c_{BS} and p_{BS} and are the values of a European call-and-put option, respectively, under the BS framework.

Haug handles the equations differently, using a set of standard equations and binary variables in order to generalize the formula for implementation. Both Haug and Reiner and Rubinstein formulas are the same, but are merely presented differently; the programmer can choose which to use, but we use Haug's method for more versatile programming.

Haug gives six standard formulas. We use similar notation as Haug's text for generalization:

$$A = \phi Se^{-DT} N(\phi x_1) - \phi Xe^{-rT} N\left(\phi x_1 - \phi\sigma\sqrt{T}\right)$$

$$B = \phi Se^{-DT} N(\phi x_2) - \phi Xe^{-rT} N\left(\phi x_2 - \phi\sigma\sqrt{T}\right)$$

$$C = \phi Se^{-DT}\left(\frac{H}{S}\right)^{2(m+1)} N(\eta y_1) - \phi Xe^{-rT}\left(\frac{H}{S}\right)^{2m} N\left(\eta y_1 - \eta\sigma\sqrt{T}\right)$$

$$D = \phi Se^{-DT}\left(\frac{H}{S}\right)^{2(m+1)} N(\eta y_2) - \phi Xe^{-rT}\left(\frac{H}{S}\right)^{2m} N\left(\eta y_2 - \eta\sigma\sqrt{T}\right)$$

$$E = Ke^{-rT}\left[N\left(\eta y_2 - \eta\sigma\sqrt{T}\right) - \left(\frac{H}{S}\right)^{2m} N\left(\eta y_2 - \eta\sigma\sqrt{T}\right)\right]$$

$$F = K\left[\left(\frac{H}{S}\right)^{m-\lambda} N(\eta Z) + \left(\frac{H}{S}\right)^{m-\lambda} N\left(\eta y_2 - \eta\sigma\sqrt{T}\right)\right],$$

where

$$x_1 = \frac{\ln\left(S/X\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T} \qquad x_2 = \frac{\ln\left(S/X\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}$$

$$y_1 = \frac{\ln\left(H^2/SX\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T} \quad y_2 = \frac{\ln\left(H/S\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}$$

$$z = \frac{\ln\left(H/S\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \qquad \mu = \frac{r - D - \frac{1}{2}\sigma^2}{\sigma^2} \qquad \lambda = \sqrt{\mu^2 + \frac{2(r-D)}{\sigma^2}}.$$

K is the rebate paid if the barrier has not been knocked during its lifespan. The reason for this is if the barrier for a certain barrier option has not been touched, then the issuer of such a contract could save a lot in hedging cost, and the issuer would be willing to share part of such earnings with the customer. Haug then gives the values of each of the barrier options and their respective binary values.

If $X > H$	Value	η	ϕ	**If $X < H$**	Value	η	ϕ
Down-in call	C+E	1	1	**Down-in call**	A−B+D+E	1	1
Down-out call	A−C+F	1	1	**Down-out call**	B−D+F	1	1
Up-in call	A+E	−1	1	**Up-in call**	B−C+D+E	−1	1
Up-out call	F	−1	1	**Up-out call**	A−B+C−D+F	−1	1
Down-in put	B−C+D+E	1	−1	**Down-in put**	A+E	1	−1
Down-out put	A−B+C−D+F	1	−1	**Down-out put**	F	1	−1
Up-in put	A−B+D+E	−1	−1	**Up-in put**	C+E	−1	−1
Up-out put	B−D+F	−1	−1	**Up-out put**	A−C+F	−1	−1

Note that the aforementioned analytical formulas present a method to price barrier options in continuous time, but often in industry, the asset price is sampled at discrete times, where periodic measurements rather than a continuous lognormal distribution of the asset prices is assumed. Many pricing platforms used in the industry followed the above framework in their implementation. However, if the duration of such a product is long, then the use of constant volatility might not generate satisfactory accuracy, and so more flexible pricing methods are needed.

2.5.1.2 Binomial method

Like most other path-dependent options, barrier options can be priced via lattice trees, such as binomial, trinomial, or adaptive mesh models by solving the PDE using a generalized finite difference method. The classic Cox et al. (1979) paper introduced the binomial method, which has since been adapted to various other option types, including barrier options.

First, we consider the fundamental properties of the binomial method. It essentially models the movement of the stock price over time and hence the option price by considering the movements at each time node where the up and down "jumps" are given as

$$u = e^{\alpha\sqrt{T}} \quad \text{and} \quad d = e^{-\alpha\sqrt{T}}.$$

From the formulas for u and d, it can be seen that $ud = 1$. The probability of the stock movement up (i.e., increasing) is

$$p = \frac{e^{(r-D)\tau} - d}{u - d}.$$

For the probability of a down movement, because the combined probabilities must equal 1, the probability of the down movement is $1 - p$.

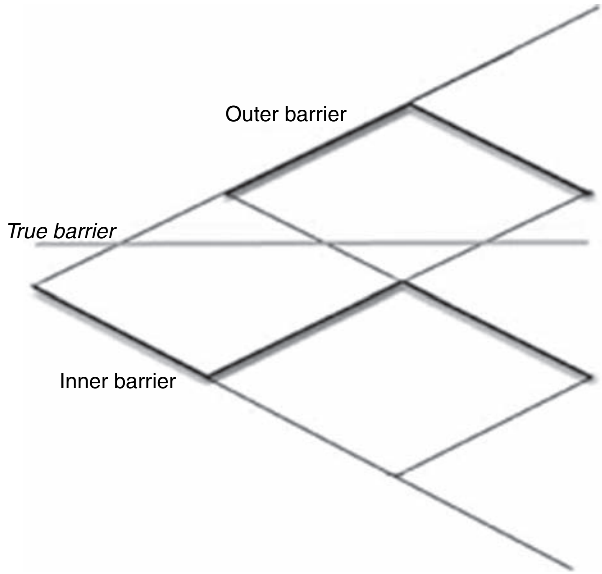
An important aspect to note is that with barriers, the zigzag movements of a binomial model undoubtedly creates problems, as the true barrier of the barrier in question is often not the same as that implied by the tree. Derman et al. (1995) and Hull (2002) illustrate this by considering an *inner barrier* and an *outer barrier*, between which lies the true barrier.

The existence of an inner and outer barrier is due to the implicit assumption of the binomial tree that places the barriers at node points, whereas, in fact, the true barrier usually lies in between.

Hull (2002) and Derman et al. (1995) present several alternatives to adjust for the discrepancy by using lattice methods:

1 Position the nodes where the barriers are. This approach is effective unless the initial asset price is close to the barrier.
2 Adjust for the nodes by assuming the barrier calculated by the tree is incorrect. By calculating the inner and outer barriers and working backward through the lattice, the first value that is encountered (i.e., the inner barrier) is given as correct, followed by the second value encountered (the outer barrier), and so forth. Once we reach the initial node, we can interpolate between the values (see Hull, 2002).
3 We can use an adaptive mesh model, which is introduced in the following text.

Alternatively, Boyle and Lau (1994) suggested a way to determine which nodes can approximate price of a barrier option (where the inner and outer nodes are the closest to each other). The number of time steps that will give the most accurate prices when using the binomial lattice is given as



$$Node(i) = \frac{i^2\sigma^2 T}{\left[\ln\left(S/H\right)\right]^2},$$

Derman et al. (1995) show that the convergence using a binomial tree to price a barrier option is slow due to two sources of errors that come from *Stock Price Quantization and Option Specification Error*. These create even more difficult problems and even slower convergence. It is also possible to extend the binomial tree to an implied binomial tree to incorporate the volatility smile that would generate results that are closer to the market (Derman and Kani, 1994a and 1994b).

2.5.1.3 Trinomial method

The trinomial method is essentially a simple and elegant extension of the binomial method, and in pricing plain-vanilla options, converges quickly compared to the binomial method. In the case of barrier options, the standard trinomial method suggested by Boyle (1986) can be applied, but again, lattice errors significantly slow down convergence.

Ritchken and Kamrad (RK) (1991) proposed a modified technique to value barrier options under a trinomial framework, which gives better results than the standard lattice.

They give the up, down, and middle magnitudes as:

$$u = \lambda\sigma\sqrt{T} \qquad d = -\lambda\sigma\sqrt{T} \qquad m = 0$$

With the respective probabilities as

$$p_u = \frac{1}{2\lambda^2} + \frac{\mu\sqrt{T}}{2\lambda\sigma} \qquad p_u = \frac{1}{2\lambda^2} - \frac{\mu\sqrt{T}}{2\lambda\sigma} \qquad p_m = 1 - \frac{1}{\lambda^2}.$$

μ is the drift term given as

$$\mu = r - D - \frac{1}{2}\sigma^2,$$

and λ is the control term that governs and determines the space between the price layers on the trinomial tree. When λ is equal to 1, the trinomial tree becomes the well-known binomial tree, and when λ is equal to $\sqrt{3}$, the RK trinomial tree becomes the standard trinomial tree given by Boyle. To determine the value of λ, Levitan (2001) used a method that makes use of the number of consecutive down movements leading to the lowest lattice on the tree. The formula for λ is given as

$$\lambda = \frac{\ln\left(S/H\right)}{\eta_0 \sigma \sqrt{T}},$$

where η_0 is the number of consecutive down movements leading to the lowest lattice on the tree. This is shown to converge rapidly toward the actual value. Illustrations of this follow. Note that the trinomial method is essentially pricing under a finite difference method (see Rubinstein, 2000).

2.5.1.4 Adaptive mesh model

Figlewski and Gao (1999) proposed a lattice method that builds on the existing binomial or trinomial lattice trees to price American style options. In their subsequent paper, see Figlewski, Gao, and Ahn (1999), they extended this approach to price discrete barrier options. In the case of barrier options, the adaptive mesh model can be used even when the asset price is closed to the barrier, an attribute that creates difficulties in standard lattice models. A mesh is constructed similarly to the one shown above, but with the nodes placed along the barriers to give more accurate simulated paths. The barrier option value can then be solved via backwards induction on the tree, and this can be implemented within a Monte Carlo framework.

Figlewski, Gao, and Ahn also pointed out that because of the numerous paths that can be constructed in both binomial and trinomial adaptive mashes, American style barrier options can also be priced in a similar fashion.

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3 Stochastic and local volatility models, volatility surface, term structure, and break-even volatility

3.1 Implied volatility, volatility surface, and term structure

3.1.1 Implied volatility

Implied volatility of an option contract is the volatility of that implied (calculated) from the price of the option with an equation, like Black and Scholes (BS), which contains the market view of the contract holders about the future asset movement. For a put option, such implied volatility indicates the level of the down risk that reflects in the cost of purchasing such hedging instruments. For a call option, it indicates the upside opportunity or how bullish the underlying asset is.

Implied volatilities of options vary with different strikes (X) and time to maturities (T), and a volatility surface can be formed along these two dimensions. For a given time to maturity, the implied volatility varies with strikes, which is called volatility smile or volatility smirk. For a given strike, the implied volatility varies with different maturities, which is called volatility term structure. When pricing European options, Black and Scholes (1973) proposed the pricing model of underlying assets under the following assumptions:

1 No dividend before the option maturity
2 No arbitrage
3 A constant risk-free interest rate
4 No transaction cost or taxes
5 Divisible securities
6 Continuous trading
7 Constant volatility.

What is more, Black and Scholes assumed that the asset price follows a geometric Brownian process (GBP):

$$\frac{dS}{S} = \mu dt + \sigma d,$$

where μ is the mean historical price, σ is the constant variance of the underlying asset's price, and W represents a standard Wiener process. From Ito's lemma, the logarithmic of the underlying asset price should follow the formula

$$d\ln S = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW.$$

However, such a Geometric Brownian process that follows a normal distribution has one big drawback—it cannot model underlying asset movement with fat tails, jumps, unstable skewness, or any correlation between asset movement and its volatility. Therefore, more complicated models are needed to capture the dynamics of the assets through calibration to obtain those parameters to describe the process.

In the Black–Scholes (BS) model, the theoretical value of a plain-vanilla option is a monotonic increasing function of the volatility of the underlying asset. Furthermore, except in the case of American options with dividends whose early exercise could be optimal, the price is a strictly increasing function of volatility. This means it is usually possible to compute a unique implied volatility from the market price of an option. This implied volatility is best regarded as a rescaling of option prices, which makes comparisons between different strikes, expirations, and underlying easier and more intuitive.

3.1.2 Volatility surface

Suppose also that at a given time during the trading day, the market releases the option price. The implied volatility can be calculated by inverting the BS formula. When the implied volatility is plotted against the strike price, the resulting graph is typically downward sloping for equity markets, or valley-shaped for currency markets. For markets where the graph is downward sloping, such as for equity options, the term *volatility skew* is often used. For other markets, such as FX options or equity index options, where the typical graph turns up at either end, the more familiar term *volatility smile* is used. For example, the implied volatility for upside (i.e., high strike) equity options is typically lower than for at-the-money equity options. However, the implied volatilities of options on foreign exchange contracts tend to rise in both the downside and upside directions. In equity markets, a small tilted smile is often observed when the option is near-the-money, as a kink in the general downward sloping implicit volatility graph (Figure 3.1). Sometimes the term smirk is used to describe a skewed smile.

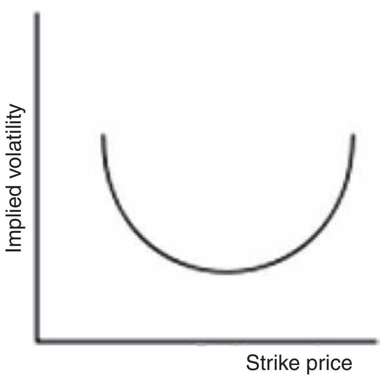

Figure 3.1 A simple plot of the implied volatility against the strike price.

Zhang and Xiang (2008) defined the concept of moneyness as the logarithm of the strike price over the forward price, normalized by the standard deviation of the expected asset return, as follows:

$$\xi \equiv \frac{\ln\left(K/F_0\right)}{\bar{\sigma}\sqrt{\tau}},$$

where $\bar{\sigma}$ is the historical volatility of the underlying asset price; τ is the time to maturity; K is the strike price; and F_0 is the forward index level. Then, the implied volatility smirk can be defined by employing the moneyness as an independent variable. The implied volatility changes according to moneyness.

On the other hand, if implied volatilities with different strikes and a given maturity are combined together under a certain weighting scheme, then the implied volatility term structure can be defined as a curve where weighted implied volatilities change with different maturities (Luo and Zhang, 2012). For example, weighted average implied that volatility for six months and for one month in the future can be different, and that indicates that the risk or opportunity is different from now until these two time spots. When the weighted implied volatilities of call options and put options are calculated, we obtain a *call implied volatility curve* and a *put implied volatility curve*, respectively. The spread between call and put curve is called the *call-put term structure spread*. This spread contains certain market information.

When the implied volatility of the put term structure is larger than that of the call term structure, market participants worry about the market on the maturity date. If these two curves cross, there are two conditions. First, when the implied volatility of the put curve is larger than that of the call curve before the crossing point and smaller after that, it means the market trend may reverse in a short term and go up. Second, when the implied volatility of call curve is larger than the put curve before the crossing point and smaller after that, it means the market trend may reverse in a short term and go down. Finally, when the implied volatility of the call term structure is larger than that of the put term structure, the view from investors is that the market will still go up. By using this functionality of the term structure, we can employ the term structure to predict the underlying asset price efficiently, which is discussed in the following sections.

3.1.3 Volatility term structure

The volatility term structure reflects the fact that the implied volatility varies with different (time to) maturities. While analyzing the implied volatility term structure, the key point is to figure out how the implied volatility is calculated from the options market data. There are mainly two types of methods used for implied volatility valuation: *the model-based implied volatility valuation method* and *the model-free implied volatility valuation method*.

The most widely used model for implied volatility valuation is the BS model. Researchers reverse the BS model and obtain a deterministic volatility function.

However, this method suffers from a number of constraints. Heston (1993) proposed a model that assumes volatility is a stochastic process. Second, in the case of model-free volatility, Carr and Madan (1998), Demeterfi et al. (1999), Britten-Jones and Neuberger (2000), and Carr and Wu (2009) have presented a volatility expectation based on variance swap contracts. Britten-Jones and Neuberger (2000) further proposed an integrated volatility defined as the integral of call and put options prices for all strikes at a given expiry date. The extant literature has showed that the BS model has a few shortcomings. First, for a given maturity, options with different strikes have different implied volatilities. This is the reason why the BS model cannot explain the volatility smile curve. Second, it assumes the implied volatility at a given maturity to be constant, which is unsuitable for predicting the underlying asset's price trend. Hence, in this chapter, we prefer the model-free integrated implied volatility model but use a discrete form of the integral and consider the process to be risk-neutral.

3.2 Local volatility model

Most derivatives exhibit persistent patterns of volatility that vary by strike. In some markets, the patterns form a smile curve. In others, such as equity index options markets, they form more of a skewed curve. This has motivated the name "volatility skew." In practice, either the term "volatility smile" or "volatility skew" (or simply skew) may be used to refer to the general phenomenon of volatilities varying by strike (Derman and Kani, 1994a and 1994b). In many cases, it is important to capture this phenomenon when pricing financial derivatives. In this section and in the following section, the local volatility model and the stochastic volatility model are introduced.

3.2.1 Local volatility model

Local volatility is an instantaneous volatility that is a function of time t and the underlying asset price S_t. Typically, Dupire (1994) showed that if the spot price follows a risk-neutral random walk and if no-arbitrage market prices for European vanilla options are available for all strikes K and expiries T, then, under the assumption that all call options with different strikes and maturities should be priced in a consistent manner, the local volatility can be extracted analytically from these option prices. A detailed mathematical proof is not presented here. The final formula derived by Dupire for calculating the local volatility is given by

$$\sigma_{\text{loc}}^2(K,T) = 2\frac{\frac{\partial C_{\text{market}}}{\partial T} + \left(r(T) - d(T)\right)K\frac{\partial C_{\text{market}}}{\partial K} d(T)C_{\text{market}}}{K^2\frac{\partial^2 C_{\text{market}}}{\partial K^2}}. \tag{3.1}$$

This local volatility model formula was later called *Dupire equation*. Note also that the local volatility is the deterministic function of the underlying asset price and time.

In 2002, Sepp used market-implied volatility to express the Dupire local volatility formula as follows:

$$\sigma_{\text{loc}}^2(K,T) = 2\frac{\dfrac{\sigma_{\text{imp}}}{T-t} + 2\dfrac{\partial\sigma_{\text{imp}}}{\partial T} + 2\big(r(T)-d(T)\big)K\dfrac{\partial\sigma_{\text{imp}}}{\partial K}}{K^2\left(\dfrac{\partial^2\sigma_{\text{imp}}}{\partial K^2} - d_1\sqrt{T-t}\left(\dfrac{\partial\sigma_{\text{imp}}}{\partial K}\right)^2 + \dfrac{1}{\sigma_{\text{imp}}}\left(\dfrac{1}{K\sqrt{T-t}} + d_1\dfrac{\partial\sigma_{\text{imp}}}{\partial K}\right)^2\right)}, \tag{3.2}$$

where

$$d_1 = \frac{\ln\left(S/K\right) + \left(r(T)-d(T)+\frac{1}{2}\sigma_{\text{imp}}^2\right)(T-t)}{\sigma_{\text{imp}}\sqrt{T-t}}.$$

Nevertheless, this deterministic function suffers from two weaknesses. First, because local volatility is a function of both strike and time to maturity, and it is possible that not all strikes are available at each time to maturity, the number of local volatilities is finite and is usually not enough for further calculation and applications. As a result, researchers are inclined to use interpolation to obtain series data of local volatility for further calculations. In this way, the algorithm of interpolation becomes very important because a weak algorithm results in inadequate precision. Second, there is the intrinsic problem of Dupire's equation. The indeterminacy of the equation may cause local volatility to be extremely large or very small.

We take a long-term expiry of options from the Hang Seng Index (HSI) as an example. The local volatility fluctuates greatly and suffers greatly from errors caused by implied volatility calculated from long-term options. As shown in Figure 3.2, the local volatility has reached 800% when using the long-term maturity option and a large underlying asset price. Besides, there also exist many negative values of local volatility calculated from these two formulas. In practice, neither of these equations is stable.

In order to solve this problem, we propose to add a *mean-reversion term* to Sepp's equation when deriving local volatility from implied volatility in order to correct computational errors. The mean-reversion phenomenon of stock prices was first proposed by Keynes (1936): "all sorts of considerations enter into market valuation which are in no way relevant to the prospective yield" (p. 152). When the stock prices diverge from the long-run mean level, they eventually revert to the previous level. The larger the deviation, the greater the probability that stock prices will revert to the mean. Cecchetti et al. (1990) found that the asset price usually vibrates within the 60% confidence

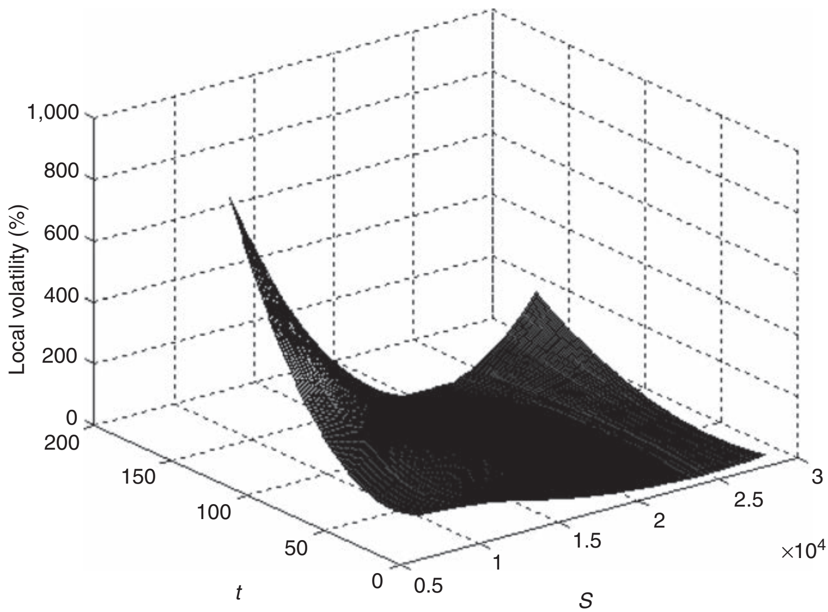

Figure 3.2 Local volatility surface.

interval of its long-run mean level, and the asset price finally returns to its mean level.

Hence, the mean-reversion process is a common event in stock markets. Bessembinder et al. (1995) studied the term structure of commodity futures prices and concluded that mean reversion exists in the equilibrium asset price. Engle and Patton (2001) found that the mean-reversion phenomenon also existed in the volatility of asset price when they sought a model to predict volatility. That is, high volatility of underlying asset price eventually evolves into its long run mean level.

3.2.2 Mean-reversion process

In this section we focus on the evolution of the local volatility of the underlying asset price and propose a novel local volatility model with a mean-reversion process.

As local volatility is a function of time and the underlying asset price, we fix the underlying asset price[1] and analyze the mean-reversion process of the local volatility against time. Implied volatility is a function of maturity and strike price. However, the series of maturities of different options is discrete. In order to analyze the relationship between the underlying asset price and implied volatility, Sepp (2002) proposed a local volatility calculated from implied volatility based on Dupire's works. In his model, local volatility is a function of continuous time. Hence, to improve this model, we propose a mean-reversion term and apply this term to the local volatility model.

Assume that X_t is the long-run mean of the local volatility, and θ is the reversion rate. The farther local volatility moves away from its mean, the greater the reversion rate will be. Based on this assumption, we formulate our novel local volatility with mean reversion as follows:

$$Y_t' = X_t + \left(Y_t - X_t\right)e^{-\theta|Y_t - X_t|}, \tag{3.3}$$

where $X_t = \frac{1}{t}\sum_{i=1}^{t}\sigma_{loc}\left(S,i\right)$ and $\theta \geq 1$. θ can be determined from real market data, Y_t is the local volatility calculated by Sepp's formula, and Y_t' is the volatility computed by our novel local volatility model. We view the distance between the local volatility and its mean as the distance, which increases dramatically, and the right-hand side of Equation 3.3 decreases sharply. Hence, we assert that if the observed horizon is long enough, Y_t' will finally converge to its long-run mean level.

Hypothesis 1: If a local volatility is given with a mean-reversion process as

$$Y_t' = X_t + \left(Y_t - X_t\right)e^{-\theta|Y_t - X_t|},$$

then, Y_t' *will finally converge to its long-run mean level.*

Proof:

$$\begin{aligned} E\left[Y_t'\right] &= E\left[X_t\right] + E\left[\left(Y_t - X_t\right)e^{-\theta|Y_t - X_t|}\right] \\ &= E\left[X_t\right] + E\left[Z_t e^{-\theta|Z_t|}\right]\left(where, Z_t = Y_t - X_t\right) \\ &= E\left[X_t\right] + \int_{-\infty}^{+\infty} Z_t^2 e^{-\theta|Z_t|} dZ_t \\ &= E\left[X_t\right] + \int_{0}^{+\infty} Z_t^2 e^{-\theta Z_t} dZ_t + \int_{-\infty}^{0} Z_t^2 e^{\theta Z_t} dZ_t \otimes. \end{aligned}$$

In $\otimes$*, we calculate two indefinite integrals as follows:*

$$\begin{aligned} \int_{0}^{+\infty} Z_t^2 e^{-\theta Z_t} dZ_t &= e^{-\theta Z_t}\int_{0}^{+\infty} Z_t^2 dZ_t + Z_t^2 \int_{0}^{+\infty} e^{-\theta Z_t} dZ_t \\ &= \frac{1}{3}Z_t^3 e^{-\theta Z_t}\Big|_0^{+\infty} - \frac{1}{\theta}Z_t^2 e^{-\theta Z_t}\Big|_0^{+\infty} \\ &= \lim_{Z_t \to +\infty}\frac{1}{3}Z_t^3 e^{-\theta Z_t} - 0 - \left(\lim_{Z_t \to +\infty}\frac{1}{\theta}Z_t^2 e^{-\theta Z_t} - 0\right) = 0, \end{aligned}$$

and

$$\int_{-\infty}^{0} Z_t^2 e^{\theta Z_t} dZ_t = e^{\theta Z_t} \int_{-\infty}^{0} Z_t^2 dZ_t + Z_t^2 \int_{-\infty}^{0} e^{\theta Z_t} dZ_t = \frac{1}{3} Z_t^3 e^{\theta Z_t} \Big|_{-\infty}^{0} + \frac{1}{\theta} Z_t^2 e^{\theta Z_t} \Big|_{-\infty}^{0}$$

$$= 0 - \lim_{Z_t \to -\infty} \frac{1}{3} Z_t^3 e^{\theta Z_t} + 0 - \lim_{Z_t \to -\infty} \frac{1}{\theta} Z_t^2 e^{\theta Z_t} = 0.$$

Then, ⊗ can be modified as follows:

$$\therefore E\left[Y_t'\right] = E\left[X_t\right].$$

This means that for a long horizon of observations, the expected new local volatility will equal that of the long-run mean level. Hence, the new local volatility will finally revert to its long-run mean level.

Therefore, combining Equations 3.2 and 3.3, we propose our novel mean-reversion local volatility model as follows:

$$\sigma_{loc}'(S,t)\Big|_{S=S_0}$$
$$= \frac{1}{t}\sum_{i=1}^{t} \sigma_{loc}(S,i) + \left(\sigma_{loc}(S,i) - \frac{1}{t}\sum_{i=1}^{t} \sigma_{loc}(S,i)\right) e^{-\theta\left|\sigma_{loc}(S,t) - \frac{1}{t}\sum_{i=1}^{t} \sigma_{loc}(S,i)\right|}\Bigg|_{S=S_0}, \tag{3.4}$$

where S_0 is a given underlying asset price.

After the local volatility is precisely modeled, we use *Monte Carlo simulation* to estimate the future underlying asset price trend. We apply the normal distribution with the mean of the risk-free interest rate and the variance of the local volatility. Finally, the path-dependent trend of the underlying asset price is simulated from the local volatility.

3.2.3 Local volatility surface

Dupire (1994) presented a deterministic equation to calculate the local volatility from the option price based on the assumption that all call options with different strikes and maturities should be priced in a consistent manner.

Recall that such a deterministic function suffers from two weaknesses. The first shortcoming is an intrinsic problem of Dupire's equation. The indeterminacy of the equation may cause the local volatility on the left to be extremely large or very small. Second, because local volatility is a function of both strike and time to maturity and it is possible that strikes are not available for each time to some. As a result, researchers are inclined to use interpolation to obtain a series data of local volatility for further calculations.

To address the first problem, we propose a novel local volatility model with the mean-reversion process described in the previous section. In this section, we focus on the second problem by using the *least squares method*. We define a surface function $\varphi(x)$ as follows. The left-hand side of the function represents the fitted local volatility:

$$\varphi(S,t) = \sum_{i=0}^{M-1} \sum_{j=0}^{T-1} A_{ij} S^i t^j , \tag{3.5}$$

where A is an $M \times T$ matrix, S is the underlying asset price, and t is time (in years). Our aim is to find an optimal parameters matrix A such that the difference between the fitted local volatility and the real local volatility is minimal. The errors function is defined as follows:

$$E_A = \sum_{K=1}^{N} \omega(S_i,t_j) \left[\sum_{i=0}^{M-1} \sum_{j=0}^{T-1} A_{ij} S_i t_j - \sigma(S_i,t_j) \right], \tag{3.6}$$

where $k = 1,\ldots,N$, N denotes the total number of local volatilities, $\sigma(S_i,t_j)$ is the real local volatility with the underlying asset price S_i and time t_j, $\omega(S_i,t_j)$ is the weighted matrix at different points. Obviously, if we minimize the summation of errors, we will obtain an optimal parameters matrix for the surface function. Because a parabolic function can reach an optimal value within a given variable range, we differentiate Equation 3.6 and get a partial differential equation of the summation error function to a given parameter.

$$\frac{\partial E_A}{\partial A_{ij}} = 0. \tag{3.7}$$

We let the partial differential result be zero. In this way, the optimal value of the parabolic function is derived. After moving the real local volatility to the right-hand side, Equation 3.7 is transformed into Equation 3.8.

$$\begin{pmatrix} (\varphi_{00},\varphi_{00})(\varphi_{00},\varphi_{01}) & \cdots & (\varphi_{00},\varphi_{M-1T-1}) \\ \vdots & \ddots & \vdots \\ (\varphi_{M-1T-1},\varphi_{00})(\varphi_{MT},\varphi_{12}) & \cdots & (\varphi_{M-1T-1},\varphi_{M-1T-1}) \end{pmatrix} \begin{pmatrix} A_{00} \\ \vdots \\ A_{M-1T-1} \end{pmatrix} = \begin{pmatrix} (\varphi_{00},\sigma) \\ \vdots \\ \varphi_{M-1T-1},\sigma \end{pmatrix}, \tag{3.8}$$

where $\varphi_{00} = S^0 t^0$, $\varphi_{01} = S^0 t^1$, $\ldots$, $\varphi_{M-1T-1} = S^{M-1} t^{T-1}$ and $(\varphi_{00},\varphi_{00}),= \sum_{k=1}^{N} S_k^0 t_k^0 \bullet S_k^0 t_k^0$, $(\varphi_{M-1T-1},\varphi_{M-1T-1}) = \sum_{k=1}^{N} S_k^{M-1} t_k^{T-1} \bullet S_k^{M-1} t_k^{T-1}$, and $(\varphi_{M-1T-1},\sigma) = \sum_{k=1}^{N} S_k^{M-1} t_k^{T-1} \bullet \sigma_k$. The optimal parameters matrix A can be obtained by solving Equation 3.8.

3.3 Stochastic volatility

In the BS model, volatility is assumed to be constant over the entire time to maturity. However, this cannot explain variations of volatility smile and skew with different strikes. The stochastic volatility model is developed to solve the problem. In fact, the stochastic volatility model is designed to incorporate the empirical observation that volatility appears not to be constant and indeed varies, at least in part, randomly.

Many researchers have investigated stochastic volatility models, for example:

- Hull and White ($\rho = 0$, 1987) and Wiggins ($\rho \neq 0$, 1987)

$$\frac{d\sigma}{\sigma} = \mu dt + \xi dW \tag{3.9}$$

- Scott ($\rho \neq 0$, 1989)

$$d\ln\left(\sigma^2\right) = \left(w - \zeta\ln\left(\sigma^2\right)\right)dt + \xi dW \tag{3.10}$$

- Stein and Stein ($\rho = 0$, 1991)

$$d\sigma = \left(w - \zeta\sigma\right)dt + \xi dW \tag{3.11}$$

The first model (Equation 3.9) was introduced by Hull and White (1987), who took ($\rho = 0$), and Wiggins (1987), who considered the general case ($\rho \neq 0$). Here the volatility is an exponential Brownian motion, and it can grow indefinitely (or equivalently the logarithm of the volatility is a drifting Brownian motion). Scott (1989) considered the case (Equation 3.10) in which the logarithm of the volatility is an Ornstein–Uhlenbeck (OU) process or a Gauss–Markov process. The models represented by Equations 3.9 and 3.10 have the advantage in that the volatility is always strictly positive. The third model (Equation 3.11) was proposed by Scott (1987) and further investigated by Stein and Stein (1991). These authors specialized in the case $\rho = 0$. In this model, the volatility process itself is an OU process with a mean reversion level " ω ." However, the disadvantage of the foregoing models is that volatility σ can easily become negative. In order to ensure that the volatility σ is positive, we use the Heston model, discussed in the following section.

3.3.1 Heston model

In 1993, Heston proposed the following model (the Heston model):

- Heston ($\rho \neq 0$, 1993)

$$d\sigma^2 = \left(w - \zeta\sigma^2\right)dt + \xi\sigma dW_2 \tag{3.12}$$

From the above model it can be seen that the volatility is related to a square root process. Further, the volatility can be interpreted as the radial distance from the origin of a multidimensional OU process. For small *dt*, this model keeps the volatility positive and is the most popular model because it allows for a correlation between asset returns and volatility at the same time. Together with the assumption that the underlying asset price follows a GBP, the models are formulated as follows:

$$dS_t = \mu(t) S_t dt + \sqrt{V_t} S_t dW_1$$
$$dV_t = \kappa(\theta - V_t) dt + \sigma\sqrt{V_t} dW_2 ,$$

where θ is the long-run mean of volatility, κ is the speed of the instant volatility returning to the long-run mean level, and σ is the volatility of the volatility. These three parameters satisfy the condition $2\kappa_t\theta > \sigma^2$ and ensure that the process of V_t is strictly positive. Besides, W_1 and W_2 are two standard Wiener processes and have a correlation ρ (or $\rho = < W_1, W_2 >$).

In the basic Heston model, the underlying asset price is determined by a stochastic process with a constant drift rate and stochastic volatility. Stochastic volatility is another stochastic process but has a correlation with the first process. Therefore, it is very similar to the GBP, where the underlying asset price is a stochastic process with a constant drift rate and constant volatility. The stochastic differential equation (SDE) of the GBP is as follows:

$$dS_t = \mu S_t dt + \sigma S_t dW_t , \tag{3.13}$$

where $\propto$ is the drift rate in per cent. Basically, the drift rate is equal to the risk-free interest rate minus the stock dividend. σ is the percent volatility. They are both assumed to be constant in the assumption of geometric Brownian movement (GBM). W_t is a standard Wiener process.

In our model, we have already fitted out the implied volatility term structure. The term structure of the interest rate is also available from HIBOR.[2] For a given time *t* and an initial stock price S_0 , an analytical solution can be derived from the SDE of (3.13) by applying the Ito's interpretation[3] (Ito, 1951).

$$S_{t+1} = S_t \exp\left(\left(\mu_t - \frac{V_t}{2}\right)t + \sqrt{V_t} W_t^S\right), \tag{3.14}$$

where S_t is the stock price at time *t*. The right-hand side of (3.14) represents the calculation of the future stock price based on the initial value. For a given time *t*, the implied volatility is obtained from the volatility term structure described above. μ is the drift rate, a function of time *t*. According to Heston (1993), the drift rate of the stochastic process is equal to the interest rate minus the stock dividend, namely, $\mu_t = r_t - D_t$. W_t^S is a Wiener process generated by the normal distribution with a mean of 0 and a variance of 1. However, because W_t^S has an optimal correlation with W_t^V, they are calculated as follows:

$$W_t^V = \sqrt{dt}\varepsilon$$
$$W_t^S = \rho W_t^V + \sqrt{1-\rho^2}\sqrt{dt}\varepsilon. \tag{3.15}$$

Similarly, the stochastic process of implied volatility based on the term structure is generated as follows:

$$V_{t+1} = V_t e^{\left(\kappa(\theta - V_t) - \frac{\sigma^2}{2}\right)dt / V_t + \frac{\sigma W_t^V}{\sqrt{V_t}}}, \tag{3.16}$$

where κ is the mean reversion speed, θ is the long-run mean level of implied volatility obtained from the term structure, and σ is the volatility of the volatility.

The Heston model has been widely applied in equities, gold, and foreign exchange markets. Besides, many extended models based on the Heston model have been proposed. For example, Christoffersen et al. (2006) proposed a two-factor stochastic volatility model based on the Heston model to control the level and slope of the volatility smirk. Andersen et al. (2002) and Chernov and Ghysels (2000) employed an efficient method of moments approach to estimate the structural parameters of the Heston model. Bates (2000) used an iterative two-procedure model to measure the structural parameters and spot volatilities.

The following section introduces another stochastic volatility called the constant elasticity of variance (CEV) model. In Section 3.3.3, we will use both models to simulate the asset price for prediction by considering the correlation of the underlying asset and the volatility process as dynamic variables. The underlying asset price is simulated by the analytical solution of GBM.

3.3.2 CEV model

The CEV option price model was first proposed by Cox (1975). It contains the BS model as a member. Cox assumes the underlying price process to be stochastic, driven by a differential equation:

$$dS_t = \mu S_t dt + \sigma S_t^\gamma dW_t, \tag{3.17}$$

where μ, σ, and γ are constant parameters and μ is the drift rate. Here, we assume it to be the risk-free interest rate. σ is the volatility and γ is the constant parameter. These variables satisfy the conditions $\sigma \geq 0$ and $\gamma \geq 0$. At any given time *t*, the standard deviation of the returns is defined as follows:

$$V_t = \sigma S_t^{(\gamma-2)/2}. \tag{3.18}$$

Therefore, the CEV exponent (Emanuel and MacBeth, 1982) γ determines the relationship between volatility and the underlying price. When γ equals 2, the volatility equals the constant variable σ. This is the standard BS model, and the option price in the CEV model is the same as in the BS model. When γ is less than 2, we can see the leverage effect from Equation 3.18, which means the underlying asset price increases when the volatility decreases. In contrast, when γ is greater than 2, this is the so-called inverse leverage effect, where the volatility rises with an increase in the underlying price.

Based on the stochastic differential equation (Equation 3.17), we can derive the distribution of the underlying price from the stochastic volatility as follows (according to the GBM):

$$S_{t+1} = S_t e^{\left(r_t - \frac{1}{2}\left(\frac{V_t}{S_t^{1-\gamma}}\right)^2\right)t + \frac{V_t \sqrt{t} W_t^S}{S_t^{1-\gamma}}}, \tag{3.19}$$

where r_t is the risk-free interest rate.

In the next section, we combine Equations 3.18 and 3.19 and use Monte Carlo simulation to calculate the distribution of the underlying asset price. Given the initial underlying price S_0, we can calculate the initial volatility by applying Equation 3.17. Iteratively, the next day price is computed from the volatility of the previous day, by using Equation 3.19.

3.3.3 Empirical analysis

3.3.3.1 Data description

For empirical tests, we used data from the Hang Seng Index (HSI) options. Although the original file contains transaction data of every minute, only the closing price of every day is treated as the option price. The options data span the period from January 2, 2007, to December 31, 2009, covering 739 trading days and 321,932 trading records. Each record contains contract information of the option, the trading time (precise to the second), the bid and ask price at the trading time, the risk-free interest rate, and the closing value of the HSI. At the same time, we also compute simple averages of the bid and ask price as the closing price of the option. Because the strike price and time to maturity are contained in the option contract, we can use the BS model to calculate the implied volatility from each entry data. To make the comparison convenient, we append the implied volatility at the end of each entry.

As the implied volatility of options with different maturities and different strike prices can generate a surface, we calculate the implied volatility term structure from it by averaging the implied volatility along the strike axle. Besides, we only consider options with maturities of less than six months. Figure 3.3 shows the average volatility over the next 4 months. The stars represent the implied

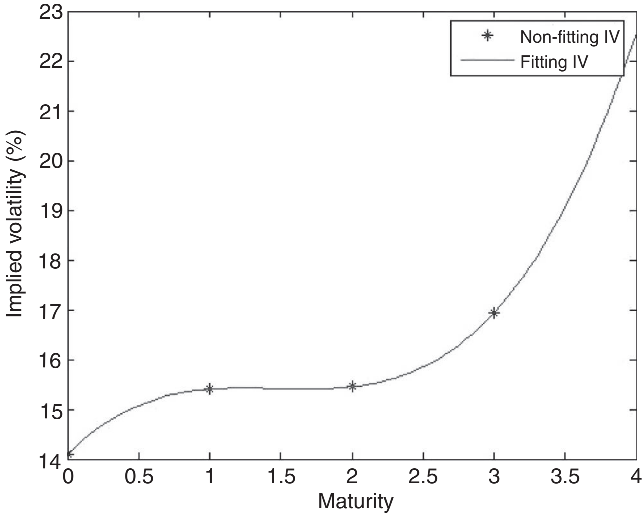

Figure 3.3 Model-free term structure fitting curve.

volatility data before fitting. The line is the model-free term structure curve after fitting. This fitting curve functions as an average implied volatility for Heston model simulation. There are 21 trading days in each month. Then, there are 21 points between each maturity.

3.3.3.2 Heston and CEV simulation

As mentioned in Section 3.3.2, the CEV model includes μ, σ, and γ constant parameters, in which μ is equal to the interest rate, as we assume there are no dividends during the testing period. σ is the volatility of the volatility. Therefore, we calculate σ from the time series and consider σ as the standard deviation of the term structure. Then, there is a unique sigma each day. According to Ncube (2009), the value of γ is important because γ determines the relationship between the volatility and the underlying price. Similarly, we set γ equal to 0.5.

We estimate the parameters of the Heston model using the historical market options data. The optimal parameters are set as follows: $\hat{K} = 1.476$, $\hat{\theta} = 0.306$, $\hat{\sigma} = 0.499$, and $\hat{\rho} = -0.899$.

We simulate the Heston volatility process by using the term structure as the long-run mean level of the volatility. The above optimal parameters are also used in the simulation of the volatility process. When simulating the underlying price stochastic process, we fix the random seed of the GBM solution to be 11. Then, the random motion term can be calculated from these random numbers. The random motion term of the volatility process is calculated from that of the underlying asset price process by using the aforementioned correlation coefficient. Based on all these parameters, we generate the process as shown in Figure 3.4.

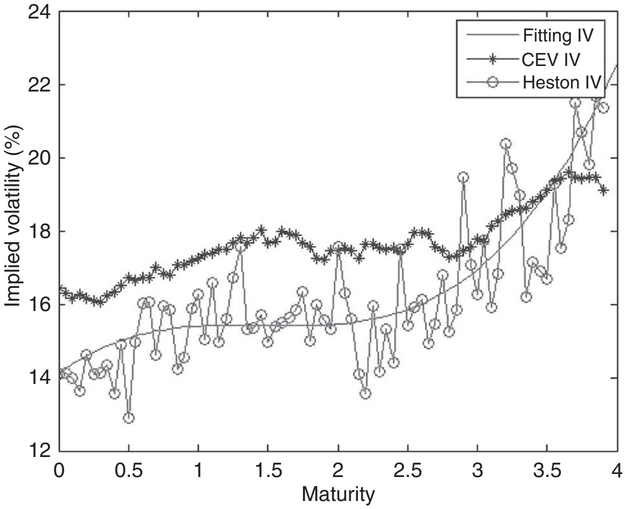

Figure 3.4 Heston and CEV IV process.

The cycle line represents the mode-free fitting term structure. The star line is the stochastic process of the CEV volatility. Owing to the small value of γ , the implied volatility deviates to a small extent with different maturities. The cycle line is the Heston volatility process, generated by using Monte Carlo simulation. After applying the implied volatility term structure as the long-run mean level, the volatility vibrates along the fitting term structure curve.

As described in Section 3.3.1, the underlying price process is also a stochastic process and has a correlation with the volatility process. The optimal correlation coefficient is −0.899. By using the result of Figure 3.4, we simulate the process of the underlying price as shown in Figure 3.5. The cycle line in this figure is the time series of the HSI from December 2, 2009 to March 26, 2010. The star line is the CEV underlying process of the HSI. The cycle line is the Heston underlying process of the HSI. Because the process of the underlying price has a high correlation with the volatility process, the underlying series also vibrates along the time series of the HSI.

To evaluate the prediction ability, we compare the one-day-ahead prediction errors of the CEV and Heston models, measured by the mean absolute error (MAE) in Equation 3.20.

$$E = \frac{1}{N}\sum_{i=1}^{N}\left|\hat{y}_l - y_i\right|, \tag{3.20}$$

where $\hat{y}_\iota$ is the estimated i^{th} underlying price; y_i is the ith market HSI.

As shown in Table 3.1, average errors of the Heston model are smaller than those of the CEV model, which means the Heston model with the adaptive correlation coefficient scheme works better than the CEV model in respect of one-day-ahead

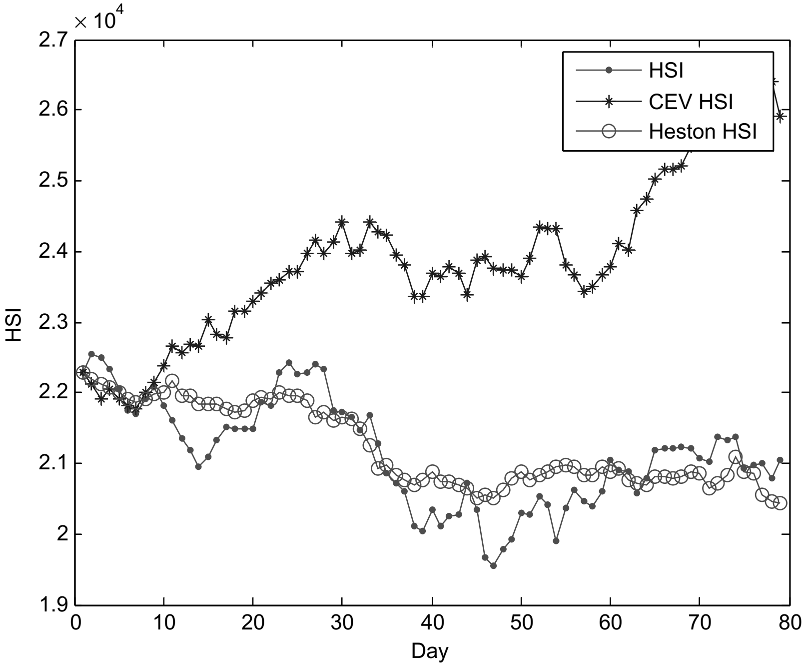

Figure 3.5 Heston and CEV HSI process vs. HSI.

prediction of the underlying price. Besides, we also estimate the distribution performance of our model. The underlying price distribution is a measure of the future price based on the term structure. In our test, we measure it over a horizon of 79 days of the price distribution. The MAE of our model is nearly three times less than that of the CEV model. Except the comparative position of the underlying price, we also compare the precision of the trend prediction of our model with that of the CEV model. As shown in Table 3.1, our model has 62.82% trend prediction precision over the 79-day period, whereas the CEV model has only 56.41%.

3.4 An adaptive correlation Heston model for stock prediction

Heston (1993) proposed a stochastic model for options pricing, the Heston model, which considers the process of the underlying asset price as a GMB with

Table 3.1 One-day ahead comparison

Model	*Distribution errors*	*Trend prediction (%)*	*One-day-ahead errors*
CEV	2810.8	56.41	1393.35
Heston	375.90	62.82	211.92

stochastic volatility. The process of volatility takes the long-run mean of the volatility into consideration, which ensures that the volatility varies within a reasonable range. This overcomes the disadvantages of the BS model, which is constrained by several assumptions. Besides, Heston assumed a constant correlation between the underlying price process and the stochastic volatility process. This ensures that the process of the underlying price evolves reasonably.

However, after observation of market data of the HSI, we find that the correlation coefficient is not necessarily a constant, but varies with time. Coefficients of the volatility process, including the rate of reverting to the mean level, the long-run mean volatility level, and the volatility of the volatility vary, which ensures that the process of volatility evolves within a corrective process. Therefore, we apply the Heston model with a dynamic correlation coefficient to establish a deterministic relationship between the underlying price and the volatility.

We find that the constant assumption does not hold in the HSI market. Figure 3.6 shows a comparison of the VHSI and the HSI from June 2008 to July 2011. The HSI line represents the time series data of the HSI, and the VHSI line represents the time series of the VHSI. The two indexes are obviously highly correlated. When the HSI went down during the period between June 2008 and November 2008, the trend of the VHSI was obviously upward. When the HSI climbed from March 2009 to Nov 2009, VHSI declined gradually. When the price fluctuated from January 2010 to July 2011, the VHSI also experienced high fluctuations.

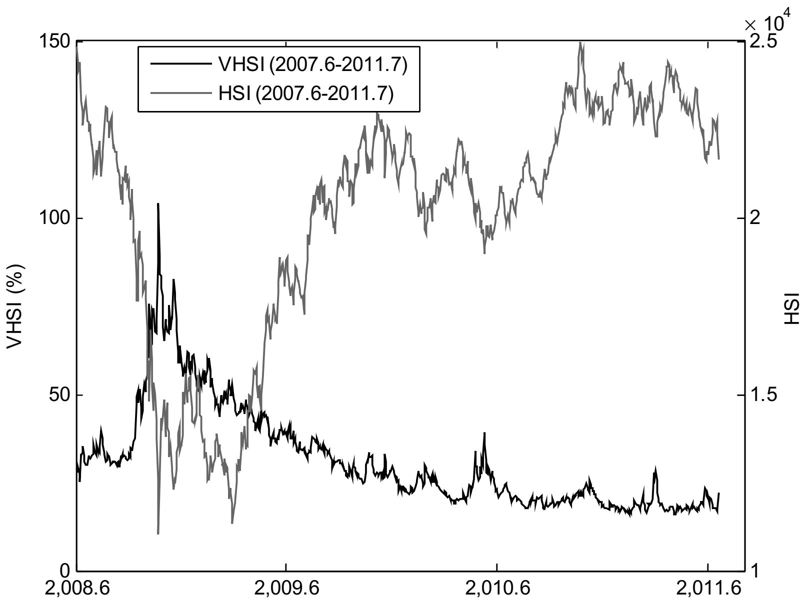

Figure 3.6 Time series of the VHSI and the HSI.

To numerically analyze the correlation between the VHSI and the HSI, we use the Pearson's correlation coefficient, which defines the correlation as the covariance of two variables divided by the product of their standard deviation:

$$\rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E\left[(X-\mu_X)(Y-\mu_Y)\right]}{\sigma_X \sigma_Y}. \tag{3.21}$$

Here, in our analysis, X represents the VHSI series, and Y represents the HSI series. We divide the time series data into four sections and calculate their respective correlation coefficients. Table 3.2 shows that the different sections have different time periods, and the correlation coefficients are also different from each other. Besides, the number of observations affects the correlation coefficients.

The correlation coefficients of sections 1 and 3 are relatively high. Their numbers of observations are also low. Section 4 also has a correlation of −0.5937, but its number of observations is high. As is known, observations over a long period of time may be unable to capture the trend of time series. Nevertheless, Section 2 does have observations over a short period, but its correlation is also low because the trend of this section is fluctuating and unstable. This analysis suggests that the correlation coefficient may be affected by two factors:

1 The number of observations affects the correlation coefficient. If the observation period is too large, the correlation coefficient is affected severely.
2 The trend of the underlying asset price should be monotonous. If the process of the underlying asset price fluctuates during the observation period, the coefficient also changes dramatically.

Therefore, a suitably long period of observations and a stable time series trend are critical for obtaining high correlation coefficients. In the next section, we propose an adaptive correlation coefficient scheme to capture the correlation between the underlying price process and the volatility process.

3.4.1 Adaptive correlation Heston model

Compared with the BS model, Heston (1993) assumes a European option with strike price K and time to maturity T can be priced by using the following equation. This is a closed-form solution for the Heston model (Moodley, 2005):

Table 3.2 Correlation coefficients between the VHSI and HSI

Section No.	*Time period*	*Correlation coefficient*	*Number of observations*
1	June to November 2008	−0.9629	120
2	December 2008 to March 2009	0.0974	80
3	April to November 2009	−0.8858	160
4	December 2009 to July 2011	−0.5937	409

$$C(S_t, V_t, t, T) = S_t P_1 - Ke^{-r(T-t)} P_2, \tag{3.22}$$

where

$$P_j(x, V_t, T, K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\varphi\ln(K)} f_j(x, V_t, T, \varphi)}{i\varphi}\right) d\varphi$$

$$x = \ln(S_t)$$

$$f_j(x, b_j, T, \varphi) = \exp\{C(T-t, \varphi) + D(T-t, \varphi)V_t + i\varphi x\}$$

$$C(T-t, \varphi) = r\varphi i T + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma\varphi i + d)\tau - 2\ln\left(\frac{1 - ge^{dr}}{1-g}\right)\right]$$

$$D(T-t, \varphi) = \frac{b_j - \rho\sigma\varphi i + d}{\sigma^2}\left(\frac{1 - e^{dr}}{1 - ge^{dr}}\right)$$

$$g = \frac{b_j - \rho\sigma\varphi i + d}{b_j - \rho\sigma\varphi i - d}$$

$$d = \sqrt{(\rho\sigma\varphi i - b_j)^2 - \sigma^2(2u_j\varphi i - \varphi^2)}.$$

For $j = 1, 2$, we have

$$u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}, \quad a = \kappa\theta, \quad b_1 = \kappa + \lambda - \rho\sigma, \quad b_2 = \kappa + \lambda.$$

Thus, Heston's method seems to be suitable for pricing European options. However, as mentioned earlier, we observe that the coefficient of the correlation between the underlying asset price and the volatility is not constant but varies with the time and shape of the stochastic process. Therefore, we propose to use the principle of the least squares method to estimate the correlation coefficient.

From the market option data, we already know the option prices. Then, we can use Equation 3.22 to estimate the option price and compare the results with real market data. We rewrite the function of Equation 3.22 as follows:

$$C(S_t, V_t, t, T) = F(S_t, V_t, t, T, K; \kappa, \theta, \sigma, \rho), \tag{3.23}$$

where the parameters of S_t, V_t, t, T, K are already known or can be calculated from real options market data. Here, we define a vector $\alpha = [S_t, V_t, t, T, K]$. The parameters κ, θ, σ, ρ need to be estimated. We define a parameter vector $\beta = [\kappa, \theta, \sigma, \rho]$. The difference between the estimated option price and the real option price is $r_i = y_i - F(\alpha_i, \beta)$, where y_i is the i^{th} real option price. The squared summation of the difference is Φ.

$$\Phi = \sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} \left(y_i - F\left(\alpha_i, \beta\right)\right)^2, \tag{3.24}$$

where n is the total number of options data. According to the spirit of the least squares method, if the first differential of Equation 3.24 is equal to 0, Φ will arrive at an optimal value. That is,

$$\frac{\partial \Phi}{\partial \beta_j} = 2\sum_{i=1}^{n} r_i \frac{\partial r_i}{\partial \beta_j} = -2\sum_{i=1}^{n} r_i \frac{F\left(r_i, \beta\right)}{\partial \beta_j} = 0, \tag{3.25}$$

where $j = 1,2,3,4$ corresponds to the parameter index of β. By solving Equation 3.25, we finally obtain an optimal parameter group, including the optimal correlation coefficient.

3.4.2 Empirical Analysis

As mentioned in Section 3.4.1, the CEV model includes the constant parameters μ, σ, and γ in which μ is equal to the interest rate because we assume there are no dividends during the period. σ is the volatility of the volatility. Therefore, we calculate σ from the time series and consider σ as the standard deviation of the term structure. Then, there is a unique sigma each day. According to Ncube (2009), the value of γ is important because γ determines the relationship between the volatility and the underlying asset price. Similarly, we set γ equal to 0.5.

Before we simulate the volatility process of the CEV model, it is important to discretize it. In this simulation, we consider time steps of one day in the CEV model. There are 21 trading days in a month, which means there are 21 volatilities in one month. By using Monte Carlo simulation, we generate the volatility process. Besides, when generating random numbers for simulation, we set the random seed of normal distribution as 11 and calculate the random term of the underlying asset price process from these random numbers. At each evaluation day, once we use the underlying price to generate the volatility, according to Equation 3.19, we can use the volatility to estimate the underlying price of the next day. The first underlying price is from market data.

The Heston model is a stochastic volatility model. The parameters of the Heston model affect the volatility process greatly. Hence, in order to obtain an optimal group of parameters, we train the Heston model by using historical options data from January 2, 2007, to November 30, 2009. Data for the last month of maturity of the option are left for testing. Section 3.3.3 discussed the four parameters of $\beta = \left[\kappa, \theta, \sigma, \rho\right]$ to be estimated. In our empirical running results, we obtain the optimal group of parameters as follows, using the algorithm described in Section 3.3.3:

$$\hat{\kappa} = 1.746, \quad \hat{\theta} = 0.306, \quad \hat{\sigma} = 0.499, \quad \hat{\rho} = -0.899.$$

The implied volatility is a measure of the future average volatility, which reflects the degree of anxiety on the part of investors about future prices of

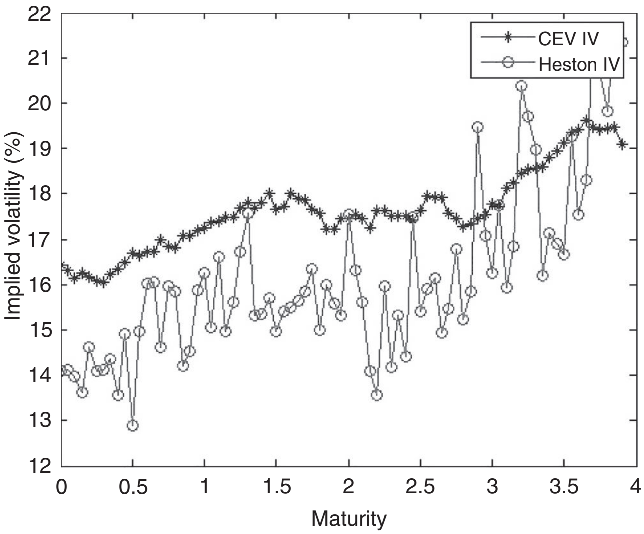

Figure 3.7 CEV implied volatility vs. Heston implied volatility.

options and stocks. The volatility term structure, therefore, contains significant information about the market view. The Heston simulation based on the term structure, therefore, becomes significant. To make use of the historical options market data, we simulate the Heston volatility process by using the term structure as the long-run mean of volatility. The aforementioned optimal parameters are also used in simulation of the volatility process.

As described in Section 3.4.3, the underlying price process is also a stochastic process and has a correlation with the volatility process. The optimal correlation coefficient is −0.899, which is dynamically estimated by the least squares method. By using this parameter, we simulate the Heston model and the CEV model. In Figure 3.7, the blue star line is the CEV implied volatility. The cycle line is the process of Heston volatility.

As shown is Table 3.3, the average error of the Heston model is smaller than that of the CEV model, which means the Heston model with the adaptive correlation coefficient scheme works better than the CEV model for prediction of the one-day-ahead underlying price. Besides, we also estimate the distribution performance of our model. The underlying price distribution is a measure of the future price based

Table 3.3 One-day ahead comparison

Model	*Seventy-nine-day errors*	*One-day ahead errors*
CEV Model	2810.8	1393.35
Heston Model	375.90	211.92

on the term structure, which we measure over a horizon of 79 days in the test. The MAE of our model is nearly three times less than that of the CEV model.

Notes

1 Because the local volatility is a function of the underlying asset's price and time and the time is assumed to be fixed, we go through the proof of Y_S' with different prices S converging to its long-run mean level. This is the same process as when the underlying asset price is fixed as mentioned above. Therefore, we only discuss one case in this chapter.

2 HIBOR is Hong Kong Interbank Offered Rate, the annualized offer rate that banks in Hong Kong offer for a specified period ranging from overnight to one year, normally including overnight, 1 week, 2 weeks, 1 month, 2 months, 3 months, 6 months, and 1 year.

3 In mathematics, a particular type of stochastic process applies the Ito's Lemma to calculate the differential of a function. It is named after Kiyoshi Ito, who discovered the lemma. It is the counterpart of stochastic calculus in ordinary calculus. It uses the Taylor series expansion by conserving the second-order term in calculus. This lemma is widely used in finance and related fields. It is best known for its application for deriving the Black–Scholes option pricing model.

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4 Market view formation

Market view is investors' expectation or forecast of future performance of a specific asset or market, given a set of quantitative or qualitative clues. The formation of the market view is of utmost importance to both investors (*buy side*) and financial institutions (*sell side*) because it provides the basis of the development of trading strategies. In this chapter, we will describe some quantitative methods used in the prediction of market movement and also some examples of trading strategies under different market scenarios.

Market view can be defined in different dimensions, for example, directional movement of asset price, growth or decline of volatility, correlation between asset price and volatility, and so on. To begin with, let us consider the nine-grid diagram (Figure 4.1) that can serve as a good indicator of the expected market performance.

In this chapter, some techniques for establishing the market view in the equity market and the foreign exchange market are demonstrated with examples.

4.1 Equity market view formation

4.1.1 Volatility forecast

According to the *Black–Scholes (BS) equation* (Black & Scholes, 1973):

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$
$$p = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1),$$

where

$$d_1 = \frac{\ln(S_0 / K) + (r - q + \sigma^2 / 2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

From the pricing formula, the option price depends on the following parameters:

- Underlying asset price
- Underlying asset volatility
- Underlying asset dividend yield
- Strike price

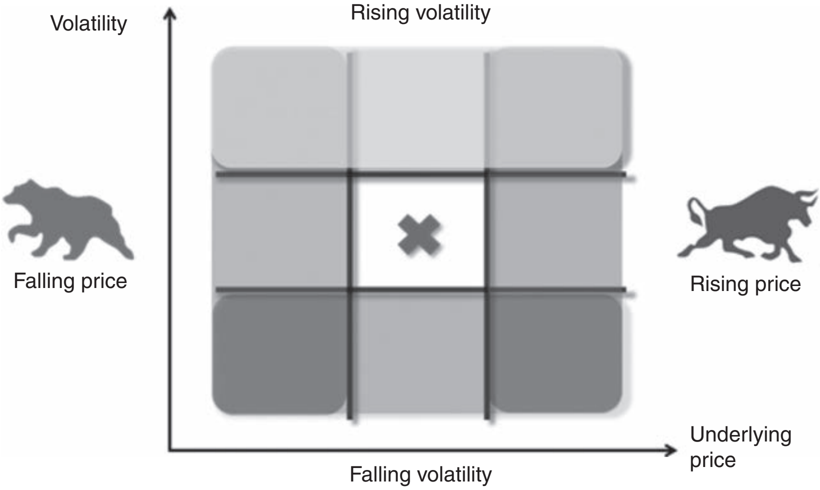


Figure 4.1 Market view grid.

- Risk-free interest rate
- Time to maturity.

Among all the required parameters, asset volatility is the only one that is unobservable. Therefore, given all the required parameters and the market price of the option contract, we can deduce the *implied volatility* of the option. More generally, the implied volatility is the volatility for which theoretical price of an option equals the market price. Under the BS model option pricing model,

$$\sigma_{\text{BSM}}^{\text{iv}} = c_{\text{BSM}}^{-1}\left(S_0, r, K, T, q, c^{\text{mrk}}\right),$$

where c_{BSM}^{-1} is the inverse function of the option pricing formula and c^{mrk} is the market option price. As a hedging tool, an option's price can reflect the market expectation of the future performance of the underlying asset. Therefore, given a set of options of the same underlying asset with a different time to maturity and all other parameters identical, we can obtain a set of implied volatilities of different maturities. We can then construct the volatility term structure and observe the trend of the expected volatility of the underlying asset (Figure 4.2).

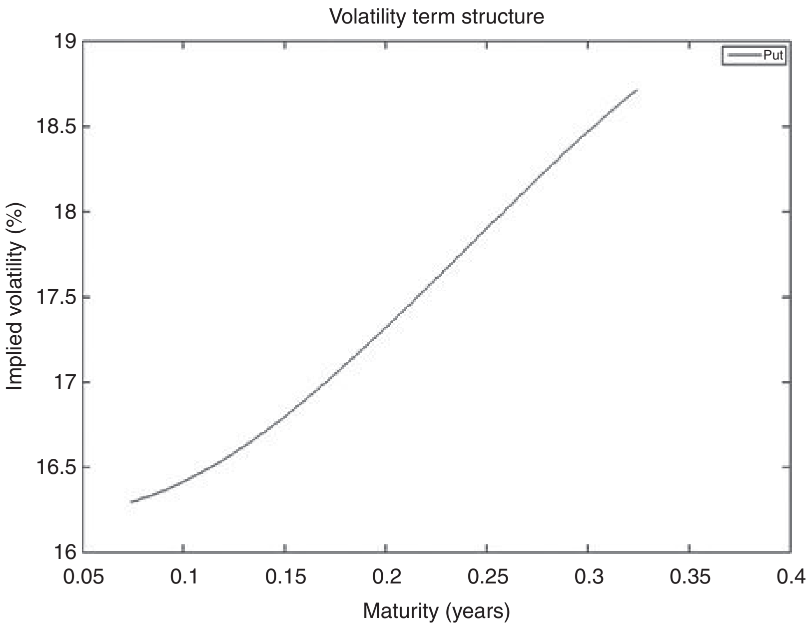


Figure 4.2 Implied volatility term structure.

Forward Implied Volatility

From the volatility term structure, we can further deduce the forward volatility (Taleb, 1997). If we assume that returns of the underlying asset in different non-overlapping periods are independent and thus the variance is additive, we can have

$$\sigma_{0,j}^2 = \frac{1}{j}\left(\sigma_{0,1}^2 + \sigma_{1,2}^2 + \ldots + \sigma_{j-1,j}^2\right) = \frac{j-1}{j}\sigma_{0,j-1}^2 + \frac{1}{j}\sigma_{j-1,j}^2.$$

Thus, forward volatility can be obtained by

$$\sigma_{\mathrm{j-1,j}} = \sqrt{j\sigma_{\mathrm{0,j}}^2 - (j-1)\sigma_{\mathrm{0,j-1}}^2}.$$

Compared to the implied volatility term structure, the forward volatility can better represent the implied volatility of an asset in a specific period of time in the future. From the plot of the forward volatility against time, we can again observe the trend of the expected volatility of the asset.

4.2 Volatility modeling

Although the foregoing method can extract the market expectation of asset volatility, it is only feasible for assets with frequently traded options. We can make use of the techniques in econometrics, e.g., the autoregressive conditional heteroskedasticity (ARCH) models. This section describes a popular member of the ARCH model that is often used to model stock volatility: the nonlinear GARCH (NGARCH) model (Engle & Ng, 1993). The model setup is

$$\sigma_{t+1}^2 = \omega + \alpha\sigma_t^2 (z_t - \theta)^2 + \beta\sigma_t^2, R_t = \sigma_t z_t,$$

where z_t is a random variable with mean zero and variance one. The first step of volatility modeling is to calibrate the variables. Assuming asset return follows the normal distribution, we can estimate the variables by the maximum likelihood estimation (MLE). The log likelihood function is

$$\sum_{t=1}^{T} \left[-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\left(\sigma_t^2\right) - \frac{1}{2}\frac{R_t^2}{\sigma_t^2} \right],$$

where σ_t^2 can be obtained recursively. By changing the set of variables $\Theta = (\omega, \alpha, \beta, \theta)$, we find the set that maximizes the log-likelihood function. After calibration of the variables, the long-run variance can be obtained by taking the unconditional expectation of both sides of the equation, yielding

$$\sigma^2 = \frac{\omega}{1 - \alpha\left(1 + \theta^2\right) - \beta}.$$

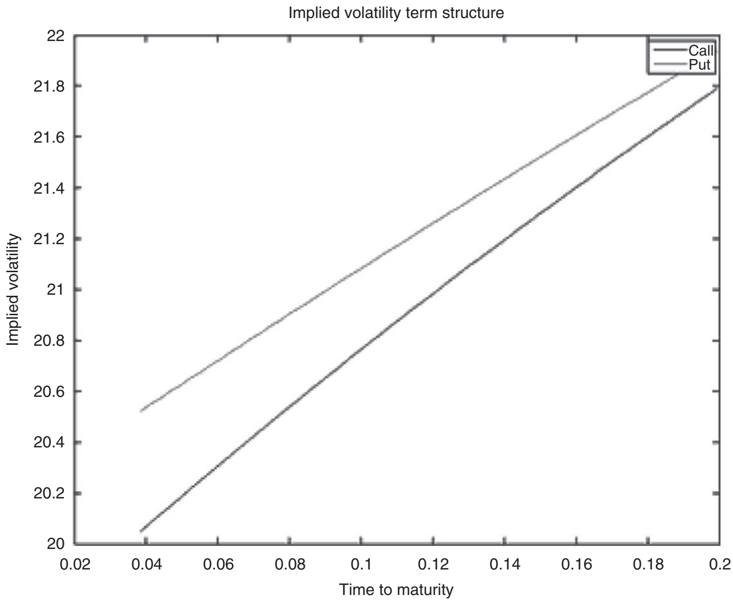
Because the volatility of stock usually shows a mean-reversal behavior (Bali & Demirtas, 2008), stock volatility has a tendency to be pulled back to the long-run mean. Therefore, as a preliminary check, we can generate a brief view of the future volatility behavior by comparing the current volatility with the long-run mean.

4.2.1 Price forecast

4.2.1.1 Volatility term structure

Options are one of the most commonly used tools used for hedging against a long/short position in the underlying asset. Because the option price is a monotonically increasing function of volatility, an option's higher implied volatility implicitly suggests that such protection is more costly. Similar to insurance, an option is more expensive when the probability or the magnitude of loss in the underlying asset is larger. Therefore, we can deduce that other things being constant, a higher implied volatility in a call (put) option implies the market expects the underlying asset value to fall (rise). When we plot the volatility term structure of call and put options in the same graph, we can generate a brief overview of the future asset price movement.

Example



In this example, implied volatility term structures of the call option and the put option are drawn in the same graph. We can observe that both plots show a trend of increasing volatility. Moreover, the plot for the put option is above that for the call option for the entire period. That may imply that investors are willing to pay more (in terms of implied volatility) for the downside risk of the underlying asset; i.e., there is a falling trend of the underlying asset.

4.2.1.2 Monte Carlo simulation

As stated in the previous section, the volatility term structure cannot be used if the asset does not have a corresponding option that is frequently traded. In this case, we can adopt Monte Carlo simulation (MCS) to find the theoretical distribution of the cumulative return of the underlying asset, from which we can obtain the distribution of the future asset price. With MCS, we can choose any models that can generate a series of future returns or price process, e.g., *GARCH option pricing model*, *Heston model*, etc. In the following, we will use the most basic model, the BS model option pricing model for demonstration purposes.

Under the assumptions of the BS model (for details, see Chapter 7), asset price is assumed to follow a GBP, i.e.,

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ is the percentage drift, σ is the percentage volatility, and W_t is a Wiener process. By Itō's lemma,

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), W_t - W_s \sim N(0, t-s).$$

From this relation, we can generate a large number of random W_t and obtain the distribution of the future asset price (Figure 4.3).

4.2.2 Simple trading strategies under various scenarios

4.2.2.1 Increasing volatility: long strangle

If an investor believes that the volatility of an asset is going up, a trading strategy that guarantees a high payoff when the asset moves far away from the current price (or a specific price) is suitable. Assuming the investor does not have a strong view on the directional movement of the asset, but predicts that its volatility is going up, he or she can go into a *long* position of a *strangle*. A strangle is a combination of a call option and a put option with the strike price of the call option higher than the put option. The payoff function is thus

$$\begin{aligned}\text{Payoff} &= \max(S_t - K_C, 0) + \max(K_P - S_t, 0)\\ &= \max(0, S_t - K_C, K_P - S_t),\end{aligned}$$

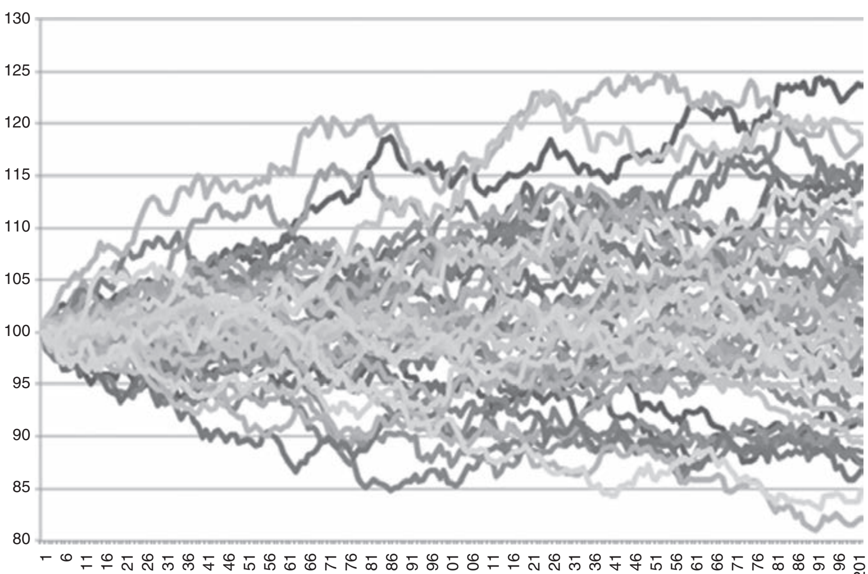


Figure 4.3 Fifty simulated paths under the BS model.

where K_C is the strike price of the call option, K_P is the strike price of the put option, and $K_C > K_P$. From the payoff diagram, we can observe that the payoff is the largest when the asset price goes far away from the two strike prices. Therefore, the value of this product increases when the volatility goes up. This can be easily seen from the components of the trading strategy. As a *long strangle* can be viewed as a *portfolio of two options*, we can calculate the vega of the product easily by using the BS model option pricing model

$$\begin{aligned}\nu_{\text{Strangle}} &= \nu_C + \nu_P \\ &= S_t\sqrt{T}\left[N'\left(d_1^C\right) + N'\left(d_1^P\right)\right],\end{aligned}$$

where

$$d_1^C = \frac{\ln\left(S_0 / K_C\right) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}, \quad d_1^P = \frac{\ln\left(S_0 / K_P\right) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}.$$

The vega of a long position is the sum of the vegas of the two embedded options. A relatively large and positive vega makes the strangle a good product when the volatility is going up (Figure 4.4).

4.2.2.2 Decreasing Volatility: Range Accrual

When the volatility of an asset is going down, investors may demand a product that provides a payoff when the asset price is moving within a small interval.

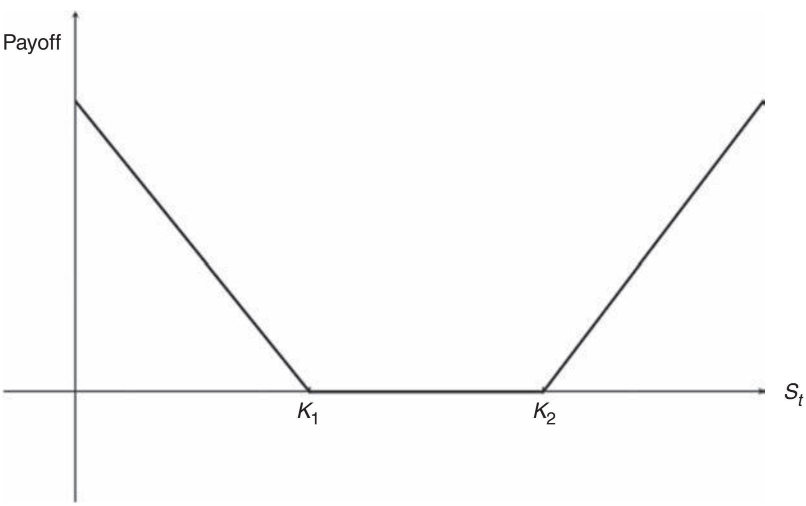


Figure 4.4 Payoff diagram of a long strangle.

The *range accrual* can provide a payoff pattern that matches investors' demand. A range accrual is a structured note that provides a fixed payoff if the indexed asset price is within a predetermined range I on each observation date. The general payoff function is

$$\text{payoff} = P \times \sum_{i=1}^{N} 1_{S(i)\in I} \times \frac{1}{N}, 1_{S(i)\in I} = \begin{cases} 1, S(i) \in I \\ 0, S(i) \notin I \end{cases}.$$

The range accrual can be viewed as a series of *binary options*. Because there is a payoff only when the underlying asset price falls in a specific range, the chance of a positive payoff and thus value for the product is higher when the volatility is low.

4.2.2.3 Upward trend

When investors expect the asset price to go up, they can buy any product with a positive *delta*, e.g., long positions in forward/futures, call options, or even the asset itself. In addition, if investors believe the upward movement will be small in magnitude, they can add an up-and-out barrier or a callable feature to reduce the cost of the product. Conversely, if investors believe the upward movement will be very strong, they can add an up-and-in barrier to make the product less expensive.

4.2.2.4 Downward trend

Investors can buy any product with a negative delta if they believe the asset price is going to drop. Similar to the trading strategies used in *upward trend*, investors can reduce the costs by adding a down-and-out barrier for a small downward trend and a down-and-in barrier for a strong downward trend.

4.3 Foreign exchange market view formation

4.3.1 Volatility and rate forecast

There are three sources of risk that affect FX market behavior:

1 Stochastic FX rate
2 Stochastic volatility
3 Stochastic skewness.

Different models have taken into account different types of risks listed. In the following, we discuss some of the popular models used to incorporate these types of risk and replicate the FX rate process.

4.3.1.1 BS model and the Garman–Kohlhagen extensions

Similar to the BS model for stock options, the FX rate is assumed to follow a lognormal process (Garman & Kohlhagen, 1983). Therefore, the FX option price is calculated in the same way as for equity options. To include the interest rates in the two countries involved in the FX rate, the option pricing function is modified to

$$c = S_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2)$$
$$p = K e^{-r_d T} N(-d_2) - S_0 e^{-r_f T} N(-d_1),$$

where

$$d_1 = \frac{\ln(S_0 / K) + \left(r_d - r_f + \sigma^2 / 2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Given a closed-form solution of the FX option price, the implied volatility can be obtained. By constructing the volatility term structure, we can get the expected trend of volatility of the FX rate.

4.3.1.2 Bates' jumps and the stochastic volatility model

Bates combined Merton's and Heston's approaches and modeled exchange rate processes with jumps and stochastic volatility (Bates, 1996). The model is built on three assumptions:

1 Markets are frictionless.
2 Both domestic and foreign risk-free interest rates are known and are constant.
3 The exchange rate follows a geometric jump diffusion with the instantaneous conditional variance that follows a mean-reverting square root process.

The model can be described with the following system of equations:

$$dS_t = (r - q - \lambda\mu_J) S_t dt + \sqrt{v_t} S_t dz_t^{(1)} + J_t S_t dN_t$$
$$dv_t = \kappa(\theta - v_t) dt + \sigma_v \sqrt{v_t} dz_t^{(2)}$$
$$cov\left(dz_t^{(1)}, dz_t^{(2)}\right) = \rho dt$$
$$prob(dN_t = 1) = \lambda dt, \ln(1 + J_t) \sim N\left(\ln(1 + \mu_J) - \frac{\sigma_J^2}{2}, \sigma_J^2\right),$$

where $r - q$ is the instantaneous expected rate of appreciation of the foreign currency, λ is the annual frequency of jumps, J is the random percentage conditional on a jump occurring, and N is a Poisson counter with intensity λ. In order to calibrate the unknown parameters, we consider the pricing function of a European option (Bakshi & Madan, 2000):

$$C = e^{-qt} S_0 \Pi_1 - e^{-rt} K \Pi_2,$$

where

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(\frac{e^{-i\omega \ln(K)}\phi(\omega - i)}{i\omega\phi(-i)}\right) d\omega$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(\frac{e^{-i\omega \ln(K)}\phi(\omega)}{i\omega}\right) d\omega.$$

The characteristic function is

$$\phi = e^{A+B+C+D},$$

with

$$A = i\omega s_0 + i\omega(r-q)\tau$$

$$B = \frac{\theta\kappa}{\sigma^2}\left((\kappa - \rho\sigma i\omega - d)\tau - 2\ln\left(\frac{1-ge^{-d\tau}}{1-g}\right)\right)$$

$$C = \frac{\frac{v_0}{\sigma^2}(\kappa - \rho\sigma i\omega - d)\left(1-e^{-d\tau}\right)}{1-ge^{-d\tau}}$$

$$D = -\lambda\mu_J i\omega\tau + \lambda\tau\left((1+\mu_J)^{i\omega} e^{\frac{1}{2}\sigma_J^2 i\omega(i\omega-1)} - 1\right)$$

$$d = \sqrt{(\rho\sigma i\omega - \kappa)^2 + \sigma^2\left(i\omega + \omega^2\right)}$$

$$g = \frac{\kappa - \rho\sigma i\omega - d}{\kappa - \rho\sigma i\omega + d}.$$

By employing numerical integration methods such as *composite Simpson's rule*, option prices can be easily calculated, given a set of parameters with the foregoing equations. Therefore, together with the market prices of FX options, the unknown parameters can be calibrated using optimization methods, e.g., the *least square method*:

$$\min_{\Omega} S(\Omega) = \min_{\Omega} \sum_{i=1}^{N}\left[C_i^{\Omega}(K_i,T_i) - C_i^{M}(K_i,T_i)\right]^2,$$

where Ω is a vector of parameters, and M refers to the market value. After calibration of the parameters, we can generate sample paths of the FX rate movement by the simple Euler method. With millions of sample terminal FX rates, we can obtain the distribution of the rate in a future period of time and thereby form the market view.

Composite Simpson's Rule

The Composite Simpson's Rule (Burden & Faires, 2011) is a numerical method used for computing the integration of a function. Given $f \in C^4[a, b]$, $h = \frac{b-a}{2m}$, $x_k = a + kh$, we have

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a) + 4\sum_{k=0}^{m-1} f(x_{2k+1}) + 2\sum_{k=1}^{m-1} f(x_{2k}) + f(b)\right].$$

The error term is written as

$$error = -\frac{b-a}{180} h^4 f^{(4)}(\xi), \xi \in (a,b).$$

4.3.1.3 Stochastic skew model

The stochastic skew model (SSM) is built based on time-changed Lévy processes (Carr & Wu, 2005):

$$\ln S_t / S_0 = (r_d - r_f)t + \left(L^R_{T^R_t} - \xi^R T^R_t\right) + \left(L^L_{T^L_t} - \xi^L T^L_t\right),$$

where L^R_t is a Lévy process that generates positive skewness, L^L_t and is a Lévy process that generates negative skewness. The model specified in Carr & Wu's paper is

$$\begin{aligned}
\ln S_t / S_0 &= (r_d - r_f)t \\
&\quad + \sigma\sqrt{v^R_t}\,dW^R_t + \int_0^\infty (e^x - 1)\left[\mu^R(dx,dt) - k^R(x)\,dx v^R_t\,dt\right] \\
&\quad + \sigma\sqrt{v^L_t}\,dW^L_t + \int_{-\infty}^0 (e^x - 1)\left[\mu^L(dx,dt) - k^L(x)\,dx v^L_t\,dt\right] \\
dv^j_t &= \kappa(1 - v_t)\,dt + \sigma_v\sqrt{v_t}\,dZ^j_t, \rho^j dt = E\left[dW^j_t dZ^j_t\right], j = R, L.
\end{aligned}$$

4.4 Simple trading strategies under various scenarios

With the formation of a two-dimensional (*directional and volatility*) market view, the development of trading strategies is similar to that for equity products. In this section, a few interesting FX-linked structured products are introduced.

4.4.1 Appreciation with small volatility: FX range bet digital option

The payoff diagram of the FX range but digital option is as shown in Figure 4.5 (the rate in the diagram is the value of USD expressed in CNY).

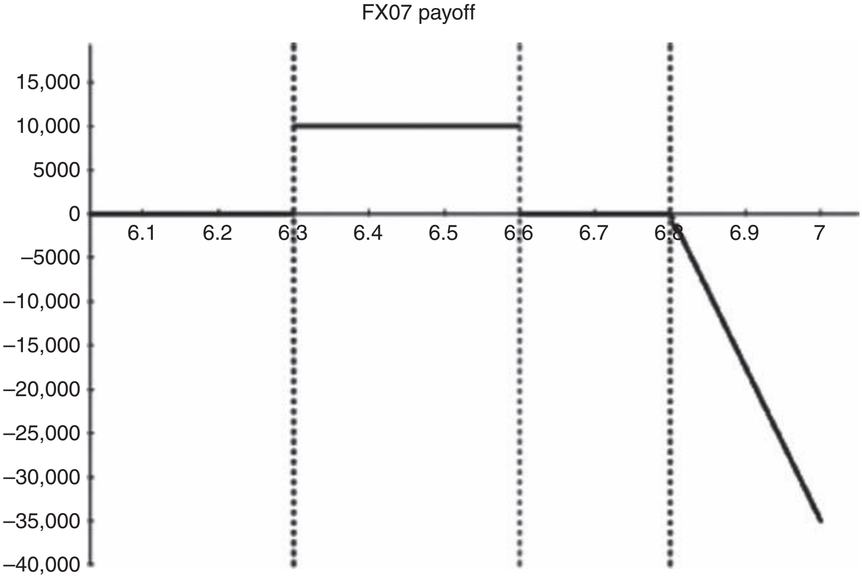


Figure 4.5 Payoff diagram of the FX range but digital option.

From the payoff diagram, we can observe that if the FX rate is higher than the strike rate (i.e., the depreciation of CNY), the payoff is negative because the investor is committed to selling the notional amount of USD against CNY at the strike price. The payoff is positive only when the FX rate falls between the first two barriers. There is no payment between the two parties otherwise.

Obviously, FX range but digital option is the combination of a short position of a call option of CNY (or a put option of USD) and a digital option. Because the loss incurred from the short position of a call (or put) option can be unlimited, this product is only suitable for investors who strongly believe CNY (or the underlying currency) will not depreciate during the observation period.

4.4.2 Sharp depreciation: Bullish G7

The payoff diagram of a Bullish G7 is shown in Figure 4.6. The FX rate is the amount of JPY per USD.

A Bullish G7 is the combination of a short position in a put option on JPY (or a call option on USD) and a long position on a digital option. From the payoff diagram, it can be seen that investors can gain when JPY depreciates and falls below the strike of the digital option. The payoff becomes negative when the FX rate is below the strike rate of the put option. There is no payment between the two parties otherwise. This product is suitable for investors with a strong view that the underlying currency is depreciating sharply.

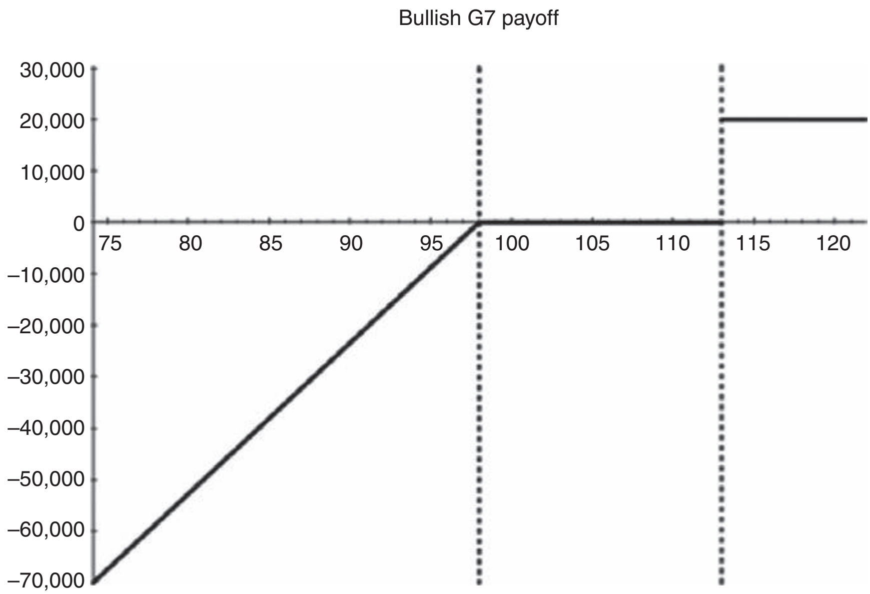


Figure 4.6 Payoff diagram of Bullish G7.

4.5 Conclusion

For both investors and issuers, the very first step of their trading actions is the formation of a market view, which can be a simple directional view, or a more complex view combining directional forecast, changes in volatility, and even the various correlations. In order to match with the complex and specific market view and demand, many financial institutions have specially designed different payoff patterns and trading strategies for different needs. These strategies are packaged in the form of a single product called a *structured product*. In the following chapters, a classification of structured products and some popular tools for their pricing are introduced. More examples for different asset classes are also presented.

Trading strategies in general

In general, the 3 × 3 market view grid can be transformed into a *trading strategy grid* with the x-axis being delta and the y-axis being vega: given a rising (falling) trend in the asset price, we can buy products with a positive (negative) delta; given a rising (falling) volatility, we can buy products with a positive (negative) vega.

Some simple trading strategies under different market scenarios are summarized in the following table:

		Trend		
		Falling	*Ambiguous*	*Rising*
Volatility	*Falling*	Short-call options	Short straddle Short strangle	Short put options
	Ambiguous	Short asset Short futures		Long Asset Long Futures
	Rising	Long put options	Long straddle Long strangle	Long call options

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5 Structured equity products

Structured equity products refer to financial instruments that offer multiple payoffs and comprise a variety of underlying equity components, from benchmark or customized indices to baskets or single stocks. They are designed to meet specific investor requirements and incorporate features such as capital protection and leverage. In this chapter, a few structured equity products are highlighted. For each product introduced, a brief analysis, together with some pricing and hedging solutions, is provided.

5.1 Equity accumulator with honeymoon

This product was fairly popular among investors in Hong Kong in 2007 considering the market conditions at that time. It is an accumulator of the underlying stock with a contract period of 12 months. During the life of the contract, there are 12 settlement dates and 12 observation periods. The first observation period is the so-called "honeymoon," as the product is not to be knocked out even if a knock-out event occurs.

5.1.1 Basic analysis

For the buyer of this product (i.e., long position), two scenarios are possible at any time in the contract period except the first observation periods:

1. A knock-out event does not occur if the underlying spot price is below the out price. The buyer of the product is obligated to buy a number of shares of the underlying at the forward price on each relevant settlement date.
2. A knock-out event is triggered in any of the 12 observation periods if the underlying spot price is at or above the out price. The contract will be ended immediately, and the knock-out day will be treated as the last settlement date of the contract.

Table 5.1 Example

Spot reference	HKD50.6
Underlying	China Life
Type	Equity accumulator
Forward price	HKD40.1582
Out price	HKD52.9725
Number of shares	DNS × [DA + (M × DB)]
Daily number shares	1,700
DA	DAYS above forward price in relevant observation periods
DB	DAYS below forward price in relevant observation periods
Knock-out	Spot ≥ out price; knock-out mechanism invalid during first observation period
M	Multiplier = 2
Observation Periods	**Settlement Date**
Oct 15, 2007 to Nov 15, 2007	Nov 15, 2007 (for transaction 1)
Nov 15, 2007 to Dec 17, 2007	Dec 17, 2007 (for transaction 2)
Sep 15, 2008 to Oct 15, 2008	Oct 15, 2008 (for transaction 12)

From the payoff structure, the buyer of the product should hold a market view that the underlying stock price will stay between the forward price and out price until the expiration date. The payoff diagram from each share of the underlying share is shown in Figure 5.1.

According to our example, the buyer can earn a limited but huge profit from this product if the price of China Life falls below HKD10 during the contract period (Figure 5.2; scenarios 1, 2, and 4). However, he may suffer a huge loss if the market condition reverses (Figure 5.2; scenarios 3 and 5). This will mean the buyer has an obligation to buy double the shares (1,700 at the forward price 40.1582), which is higher than the market price on the settlement date. This product term enhances the investors' loss in the bear market, and the loss is very likely to surpass the early profit.

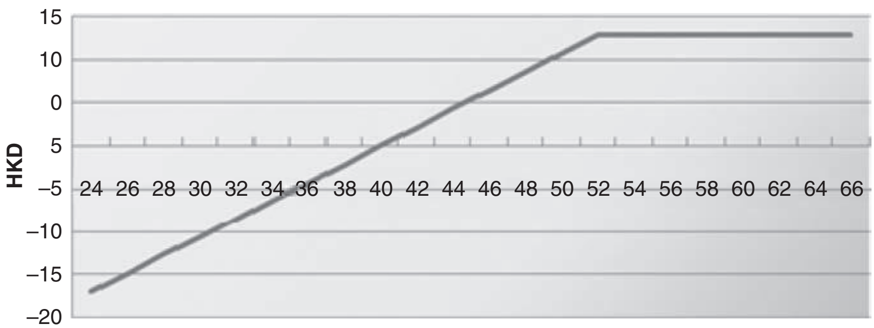


Figure 5.1 Payoff diagram showing gain from each share of China Life.

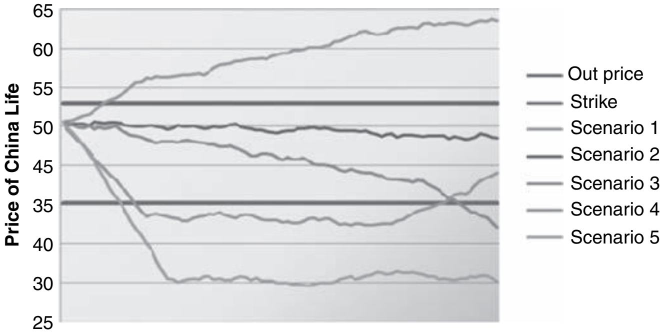


Figure 5.2 Scenario simulation of China Life price.

Real market scenario

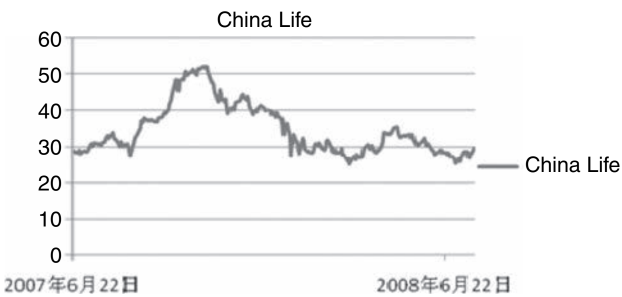


Figure 5.3 China Life spot rate between June 22, 2007 and June 22, 2008.

This product was first traded on October 15, 2007, when the price of China Life reached a high level, which made the product term appear favorable to investors. However, in the next one year the price fell and stayed at about 30 HKD. The lower price with the forward price at 40.1582 HKD resulted in EAHIT3's investors suffering large losses. This decline of the price had nothing to do with the fundamentals of the company. It was a crash of the whole market and was primarily caused by the breaking out of subprime crisis together with cessation of the "through train program." The two pieces of bad news resulted in capital outflow from the HK market and exaggerated panic in the market indirectly.

5.1.2 Pricing

In this section three different methods are used to calculate the price. First, we employ the BS model with the implied volatility regressed on the implied volatility surface and Monte Carlo to get the prices. Second, we employ the Heston model and Monte Carlo to get prices. Third, we try a different method to get an approximate analytical solution.

5.1.2.1 BS model with regression on implied volatility surface

Sigma is derived by regressions on the implied volatility surface. The regression model is given as

$$\sigma = \beta_0 + \beta_1 \ln(1+\tau) + \beta_2 M + \beta_3 M^2 + \beta_4 \ln(1+\tau) M + \beta_5 \ln(1+\tau) M^2.$$

In the regression, τ denotes the time to maturity, which is trading days/252. M denotes the moneyness of an option calculated in the form of forward price, $\ln(K/Se^{rt})$, where r is the approximation of the interest rate. If the stock is highly volatile, this approximation makes little difference to the price. In this model, the shape of the implied volatility surface is captured by the slope and curvature of the moneyness and the time to maturity. The non-flat volatility surface represents different market views toward different moneyness and times to maturity.

Statistical results of regressions

Settlement date	*June 18*	*June 25*	*July 2*	*July 9*
β_0	0.4193	0.3970	0.4421	0.4142
β_1	0.2719	0.2439	0.1283	0.3013
β_2	0.3090	0.1739	0.5491	0.5262
β_3	−0.7668	0.9238	−0.9446	−1.0699
β_4	−1.6548	−1.0167	−1.8146	−2.1906
β_5	3.7999	−0.1774	3.1966	3.9755
P value of the model	0.2846	0.0036	0.1609	0.0001
R square	0.1611	0.3258	0.1330	0.3654

As a result of the insufficient explanation for the regression conveyed by R square, the regression result can only be a reference. However, it is still more credible than the original BS model using the volatility of ATM options. From statistical results of regressions, we find that the result is favorable on valuations for June 25 and July 9.

Price with MC simulation

Valuation date	*June 18*	*June 25*	*July 2*	*July 9*
Price	−3,737,700	−3,329,700	−3,628,400	−3,114,900

The foregoing prices are calculated by Monte Carlo simulation (MCS) with 100,000 paths. Owing to the low spot price of China Life, especially compared to the forward price, the value of this product is negative.

Greek letters

Valuation date	*June 18*	*June 25*	*July 2*	*July 9*
Delta	293,700	271,580	262,230	246,750
Gamma	1873.3	1,845.6	−5365.1	−2,450.3

The table above shows the Greek letters calculated by MCS. The price of this product is very sensitive to the spot price; moreover, the delta is also sensitive to the spot price according to the gamma's value.

5.2 Heston model

A more realistic and complicated model for pricing is the Heston model, which models the volatility process as a stochastic process correlated with the stock price process. The model is given by

$$dS = rSdt + \sqrt{v}SdW_1$$
$$dv = \kappa(\theta - v)dt + \eta\sqrt{v}dW_2$$
$$dW_1 dW_2 = pdt,$$

where all the parameters are given for a risk-neutral world.

5.2.1 Calibration

As the parameters of the Heston model cannot be observed directly, we have to calibrate the model.

Calibration is a tedious and challenging task due to the noisy nature of the market. To overcome this problem, we chose to set the initial volatility as given. As proved in some previous papers, v0 should be near the implied volatility of a short-term ATM option, so we set it as 0.1 (about 33% to the volatility). Then we calibrate the model to the observed implied volatility surface once per valuation day, and the risk-neutral parameters of each day are shown in the following table.

Valuation date	*June 18*	*June 25*	*July 2*	*July 9*
κ	46.3897	15.9101	47.4402	21.5913
θ	0.2327	0.2532	0.2321	0.2604
η	0.6216	0.7612	0.4100	0.3820
ρ	0.6857	0.6348	−0.1448	0.6284
v_0	0.1000	0.1000	0.1000	0.1000

The value of a European vanilla option can have a closed-form solution and may be evaluated by numerical integration or a fast Fourier transform (FFT) (Carr & Madan, 1999).

A simpler method is MCS. A simple Euler discretization of the variance process is

$$\nu_{i+1} = \nu_i + \kappa(\theta - \nu_i)\Delta t + \eta\sqrt{\nu_i}\sqrt{\Delta t}W_2, W_2 \sim N(0,1).$$

However, because there is no restriction on the range of the process, it may give rise to a negative variance. To alleviate this problem, we may implement a Milstein discretization scheme (Gatheral, 2006):

$$\nu_{i+1} = \nu_i + \kappa(\theta - \nu_i)\Delta t + \eta\sqrt{\nu_i}\sqrt{\Delta t}W + \frac{\eta^2}{4}\Delta t\left(W_2^2 - 1\right)$$

or

$$\nu_{i+1} = \left(\sqrt{\nu_i} + \frac{\eta}{2}\sqrt{\Delta t}W\right)^2 + \kappa(\theta - \nu_i)\Delta t - \frac{\eta^2}{4}\Delta t.$$

On the other hand, the stock process can be discretized as

$$x_{i+1} = x_i - \frac{v_i}{2}\Delta t + \sqrt{v_i \Delta t}W_1,$$

where $x_i = \ln S_i / S_0, W_1 \sim N(0,1)$. Because the parameters are assumed to be risk-neutral, the expected payoff of the option should be discounted by the risk-free rate to the present time so as to obtain the current value of the options. The following table shows the pricing results.

Valuation date	*June 18*	*June 25*	*July 2*	*July 9*
Price	−3,746,600	−3,314,500	−3,624,300	−3,116,400

These prices are almost the same as in the BS model. The spot price on each day was much lower than the forward price, the put options contained in this product were deeply in-the-money, and call options were significantly out-of-the-money. So their sums are dominated by the value of put options, and the prices of the product were not very sensitive to the parameters' input.

5.3 Approximate analytical solution

Assuming short sales are allowed, one can buy this product and then short the stock on every trading date. In this way, we can replicate the portfolio with a strip of plain-vanilla barrier options and forwards. Owing to the special term in the first month, this product together with short sales can be decomposed into a forward and a barrier out option every day.

The trading strategy of replication is as follows (refer to Table 5.2): In the first month, we use one position of EAHIT3 and daily short sales of stocks (which are all closed at the settlement date) to replicate a strip of barrier put options (expired one per day) with barrier levels at 52.9725, at a strike price of $40.1582e^{-r\tau_i}$ (τ_i represent the trading days between every settlement date and every observation date) and N1 long positions on forwards (all expired at the first settlement date with the strike price equal to 40.1582). The numbers N1, N2, …, N12 denote the trading days in the following 12 months, while N = sum (N1 + N2 + …. + N12). And the daily amounts of short sales on China Life depend on whether the spot price is higher or lower than the forward price of 40.1582. If the price of China Life closes above 40.1582, then we short sell 1 × 1,700 = 1700 shares of China Life, else we short sell 2 × 1,700 = 3400 shares of China Life. The replication trading strategy in the first month is different from other months as a result of "honeymoon."

On the left 11 months, replication portfolios are constructed in a similar manner. But we cannot construct all the replications if the knock-out event occurs. However, we show that there exists only a small difference between them, which can almost be neglected. As opposed to the first month, we do not need the forward contracts to replicate. We use EAHIT3 to replicate a strip of barrier put

Table 5.2

First month	*Spot price*	*EAHIT3*	*Daily short sales of stock**	*Short position on barrier put (Open)**	*Long position on barrier call (Open)**	*Forward (Open)**
Initial position		1	0	2N−N1	N−N1	N1
1st day	42.6	1	−1	2N−N1−1	N−N1	N1
2nd day	43.5	1	−1	2N−N1−2	N−N1	N1
3rd day	40.1	1	−2	2N−N1−3	N−N1	N1
4th day	39.5	1	−2	2N−N1−4	N−N1	N1
5th day	39.7	1	−2	2N−N1−5	N−N1	N1
6th day	41.5	1	−1	2N−N1−6	N−N1	N1
—	—	—	—	—	—	—
N1−1 day	43.6	1	−1	2N−2N1+1	N−N1	N1
N1 day	42.9	1	−1	2N−2N1	N−N1	N1
Total amounts of trade		1	−(N1+DB)	N1 expired	0	All closed at the settlement date
Left position in the first month		1	0	2N−2N1	N−N1	0

*Amounts times 1,700.

options (expired two per day) with the strike at $40.1582e^{-r\tau_i}$ and a strip of barrier call options (expired one per day) with the strike equal to that of the put at the same expiration date. The detailed strategies are shown in Table 5.3.

Table 5.3 takes the replication strategies in the second month as an example; strategies in the remaining months are the same as in the second month.

An unfavorable event occurs when the closing price of China Life exceeds 52.9725. In this situation, we cannot do the replication completely as there is a difference between the discount of *K*. Assuming there is a knock-out event in the second month, our strategies in fact replicate the options with *K* set at $40.1582e^{-r\tau_i}$, which is only slightly different from our desired τ_i. But this difference has little impact on the price. We have approximately estimated that the extreme impact of this difference can be no more than 0.2% of the price. See Table 5.4.

5.3.1 Price with analytical solution

Valuation date	*June 18*	*June 25*	*July 2*	*July 9*
Pricing	−4,067,600	−3,615,800	−3,861,000	−3,349,200

These prices give the cost of our replication, which can be divided into a portfolio of vanilla options. Because the rest of the duration of this product is only 4 months as on the valuation date, we do not need to consider the "honeymoon" in the first month. We can find that the price given by this method is about 10% higher than that given by MCS. The 10% gap between these two methods may be due to the assumption not being realistic, because short sales are not allowed,

Table 5.3

Second Month	*Spot price*	*EAHIT3*	*Daily short sales on stock**	*Short position on barrier put (Open)**	*Long position on barrier call (Open)**	*Forward (Open)**
Initial position		1	0	2N−2N1	N−N1	0
1st day	42.3	1	1	2N−2N1−2	N−N1−1	0
2nd day	43.7	1	1	2N−2N1−4	N−N1−2	0
3rd day	41.5	1	1	2N−2N1−6	N−N1−3	0
4th day	38.5	1	2	2N−2N1−8	N−N1−4	0
5th day	39.6	1	2	2N−2N1−10	N−N1−5	0
—	—	—	—	—	—	0
N1–1 day	44.6	1	1	2N−2N1−2N2+2	N−N1−N2+1	0
N1 day	45.9	1	1	2N−2N1−2N2	N−N1−N2	0
Total		1	N2+DB	2N−2N1	N−N1−N2	0

*Amounts below 1,700.

Table 5.4

Second month	*Spot price*	*EAHIT3*	*Daily short sales on stock**	*Short position on barrier put (Open)**	*Long position on barrier call (Open)**	*Forward (Open)**
Initial position		1	0	2N–2N1	N–N1	0
1st day	42.3	1	1	2N–2N1–2	N–N1–1	0
2nd day	44.7	1	1	2N–2N1–4	N–N1–2	0
3rd day	46.5	1	1	2N–2N1–6	N–N1–3	0
4th day	48.5	1	1	2N–2N1–8	N–N1–4	0
5th day (knock-out)	53.1	1	1	2N–2N1–10	N–N1–5	0
—	—	—	—	—	—	0

*Amounts below 1,700.

which provides liquidity to the investors. Because of the incompleteness of replication and fees on short sales, these prices can only be regarded as a reference.

5.4 Risk and hedging

5.4.1 Use vanilla option for hedge

In the previous section we introduced replication strategies that generate a payoff that is little different from this product. We can also use these strategies for hedging. The strategies can be employed to hedge partial risk according to investors' needs. For example, a buyer of this product can hedge the short positions on barrier put options and leave the long call options exposed.

Hedging downward risk is very expensive and troublesome. Other hedge strategies using vanilla options include buying a strip of deeply out-of-the-money put options. The advantage of this strategy is that investors can choose the strike price of the put options consistent with their risk tolerances. And the lower the strike price the investors use, the lower is the cost they incur for the hedge.

5.4.2 Use VIX for hedge

The favorite scenario for investors is where the price of China Life remains above 40.1582. Therefore, investors may expect the volatility to stay low. For these reasons, they will have motivations to avoid the risk of volatility. Hence, VIX is a good product for investors to hedge against the volatility risk. What is more, long positions on VIX can result in benefits from both sides of the price movement. If the volatility becomes larger and the price of China Life rises quickly, then the investors can benefit both from the VIX and this product. If the volatility becomes larger and the price of China Life falls quickly, investors can benefit from VIX to counter the loss on this product. If the volatility stays low or even declines, then investors will suffer a loss on VIX but gain on the product. However, VIX cannot hedge the slowly downward trend of the stock, and its main advantage is that it can conveniently hedge the price movements caused by quick changes in the entire market.

5.4.3 Dynamic hedge with greek letters

Similar to most derivatives, we can hedge this product with dynamic Greek letters, which mainly include delta, gamma, and vega hedges. By analyzing the behavior of the Greeks of this product, we can assess at a quick glance how the value of this product changes, which corresponds to the risk for investors.

Figure 5.4 shows how the delta changes along with the spot price. From this graph, we can dig out some characteristics of the delta:

1 The delta is invariant and has a large value when the spot price is less than 28 (or another value near 28). However, when the spot price is greater than the critical value, the delta decreases speedily when the spot price increases.

2 In general, the delta is positive When the spot price is about 48, the delta becomes negative.

Figure 5.5 shows how the gamma changes along with the spot price. From this graph, we can dig out some characteristics of the gamma:

1 The gamma is stable and hovers near zero when the spot price is less than 28 (or other values close to 28).

2 When the spot price is between 28 and 49, the gamma decreases rapidly. Then it rebounds quickly when the spot price is greater than 49.

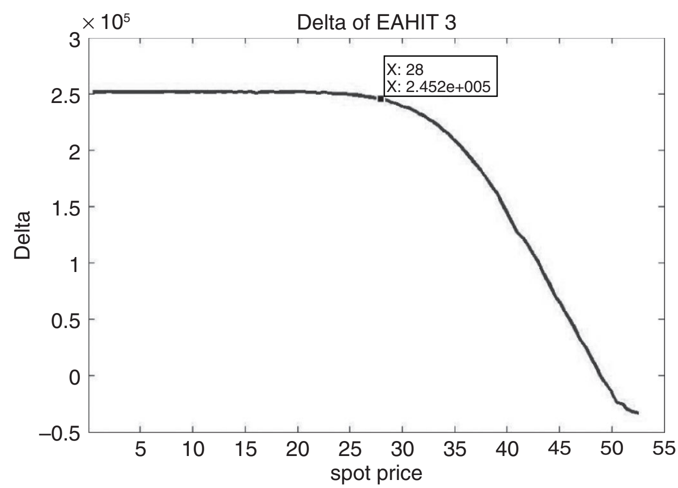


Figure 5.4 Delta.

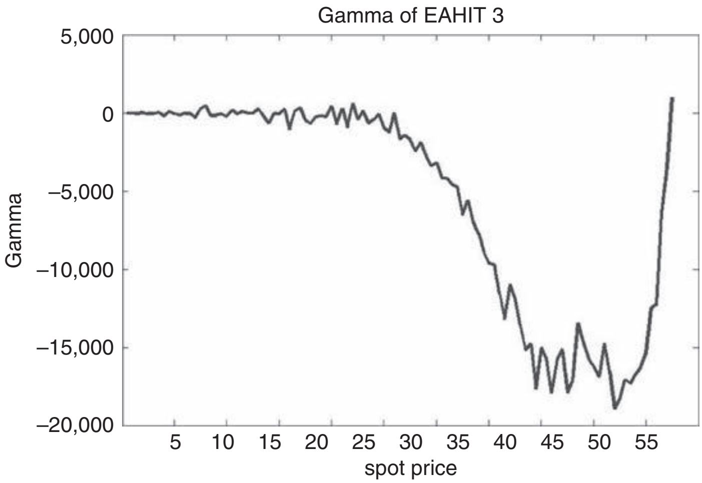


Figure 5.5 Gamma.

Figure 5.6 shows how the vega moves according to the spot price and volatility. From this graph, some characteristics of the vega are apparent:

1 The vega is smooth when the spot price is below 30 and the volatility is below 60%.
2 When the spot price stays near the forward price 40.1582, the vega decreases and becomes negative.
3 The vega becomes more volatile along with the increase of volatility.

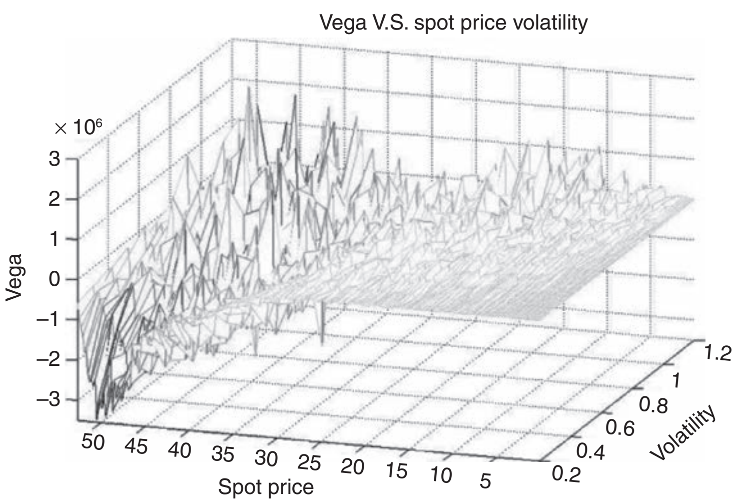


Figure 5.6 Vega.

5.5 Equity accumulator with advance delivery

This is a forward transaction for stock with advance delivery. Through accumulating stocks according to daily closing prices of the underlying asset and making a settlement at the end of each observation period, the contract attracts investors on account of its high potential profit. This product seems attractive at its issuance for its low knock price, low forward price, and the terms of advance delivery, which seem to provide the buyers with a powerful protection mechanism. Sellers of the contract, however, are faced with the probability of large returns while bearing limited risk.

5.5.1 Basic analysis

At the time of issuance, its underlying asset, China Telecom, has just been at a historical high, and its future movement is hard to predict. If the price goes up a little, investors would gain a decent profit. On the other hand, if the price reverses, investors face the possibility of a huge loss. Optimists and pessimists entered on the buyers' side and sellers' side, respectively.

When the contract is initially sold, the seller is obliged to pay the buyer a cash amount of $(S_T - F) \times$ number of shares on the scheduled settlement date. S_T, the

Table 5.5 Example

Spot reference	HKD6.6
Underlying	China Telecom (0728.HK)
Type	Equity accumulator
Trade date	Jan 11, 2008
Forward price (F)	HKD5.3599
Out price	HKD6.8495
Monthly number of shares	$20000 \times [DA + (2 \times DB)]^*$
DA	DAYS above forward price in relevant observation periods
DB	DAYS below forward price in relevant observation periods
Knock-out	Spot ≥ out price; knock-out mechanism invalid during first month
Settlement	Cash, but share delivery is applicable upon buyer's instruction; *advance delivery* within first month
Settlement price (S_T)	Closing at the end of each observation period or on the knock-out day (advance delivery excluded)
Settlement amount	$(S_T - F) \times$ monthly number of shares
Observation Periods	**Settlement Date**
Jan 14, 2008 to Feb 11, 2008	Jan 15, 2007 (for transaction 1)
Feb 12, 2008 to Mar 11, 2008	Mar 13, 2008 (for transaction 2)
Mar 12, 2008 to Apr 11, 2008	Apr 15, 2008 (for transaction 3)
Dec 12, 2008 to Jan 12, 2009	Jan 14, 2009 (for transaction 12)

*The formula is valid for the 2nd to the 12th observation period.

settlement price, denotes the closing price at the end of each day of the corresponding observation period. However, if a knock-out event occurs during observation periods, the settlement price equals the specific closing price on the knock-out day.

For the buyer of this product (i.e., long position), at any point in time during the contract period except the first observation periods, two scenarios are possible:

1 A knock-out event does not occur. If the price of China Telecom anytime in the observation period is lower than the out price, one share is cumulated, and at the end day of this observation period, the seller should pay his counterparty an extra amount of 20,000 × 1 × (settlement price − 5.3599). On the contrary, if the price of China Telecom anytime in the observation period goes below the forward price, two shares are cumulated, and the extra amount goes up to 20,000 ×2 × (settlement price − 5.3599).
2 A knock-out event is triggered in any of the 12 observation periods if the underlying spot price is at or above the out price. Cumulating of stocks ceases, and days above (days below) until the very day when a knock-out event occurs.

To sum up, the month end payoff depends on two factors: the path of the stock price and the closing price on the settlement day or the knock-out day. Neither of the factors can solely determine the final cash settlement amount. The path of the stock price determines the number of shares, whereas the end day closing determines the unit payment amount.

According to our example, if the contract has been knocked out, the coming movements of stock prices are irrelevant for the payoff of this month (Figure 5.7; path A). On the other hand, if the contract has not been knocked out, the buyer can earn a profit from this product if the price of China Telecom falls below HKD 1.2401 (= 6.6 − 5.3599) during the contract period (Figure 5.7; path B). However, the buyer may suffer a huge loss if the market condition reverses such that the final settlement price is below the forward price (Figure 5.7; path C). This will mean that the buyer has an obligation to buy double the shares of 20,000 at the forward

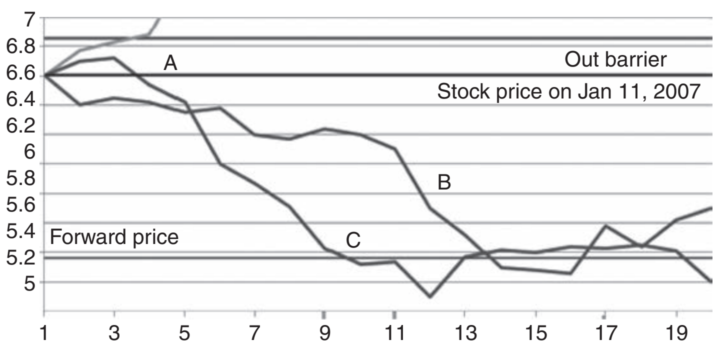


Figure 5.7 Relationship between simulated price paths and payoffs.

price of 5.3599, which is higher than the market price on the settlement date. Though at most points of time in this month, the stock price is above the forward price, buyers still lose money. This product term enhances the investors' loss in the bear market, and the loss is very likely to surpass the early profit.

5.6 Advance delivery

During the first month the situation differs both in knock-out terms and the settlement cash amount owing to "advance delivery terms." In the first month, the number of shares is fixed at 20,000 × 19, where 19 is the number of scheduled valuation dates in the first month. What is more, the settlement price is determined as the closing price on the trading day of the contract, say, January 11, 2008. Thus, the cash settlement amount is fixed at 471,238 HKD (= [6.6 − 5.3599] × 20,000 × 19), which is positive for buyers of the contract.

In addition, the knock-out term would not be valid until the end of the first observation period. If, for example, the stock price drops to 5.2 on January 15, 2008, the pre-agreed payment of 471,238 HKD would still be effective. In such circumstances, the contract would be terminated on February 11, 2008, indicating invalidation of the terms with respect to all months thenceforth until the end of 2008.

5.7 Factors that affect the value of the contract

1 **Current stock price and the trend of stock price movements:** The current stock price directly affects the first-month-fixed-amount payment and further, possibly affects the future stock prices that determine the future pay off. The trend of price movements affects the value of the contract in a similar way.

2 **Risk-free rates and volatilities:** Risk-free rates and volatilities determine the future movements of stock prices, thus affecting the contract value.

3 **Knock-out event:** A lower knock-out price causes early knock-out of the contract with both positive and negative effects for investors. On the one hand, an early knock-out of the contract deprives the investor of more opportunities to benefit from the contract. But on the other hand, the early knock-out ensures that the investor gets a certain amount of acquired profit, thus lowering the risk investors have to bear.

   In Figure 5.8, the dots depict the possible distributions of possible values of the contract. The later the contract is knocked out, the more uncertain the value of the contract would be. An extreme case is when knock-out event occurs on the very last day of the contract, when the value may vary over incredibly very wide range. Another notable feature is that the first-month knockout event occurrence would not lead to any uncertainty in the value of the contract owing to the protection of advance delivery.

4 **Multiplier:** The multiplier influences the value of the contract through the adjustment of cumulated stocks. If stock price goes below the forward price, a lager multiplier implies a larger cumulation of stocks. Thus, if the final

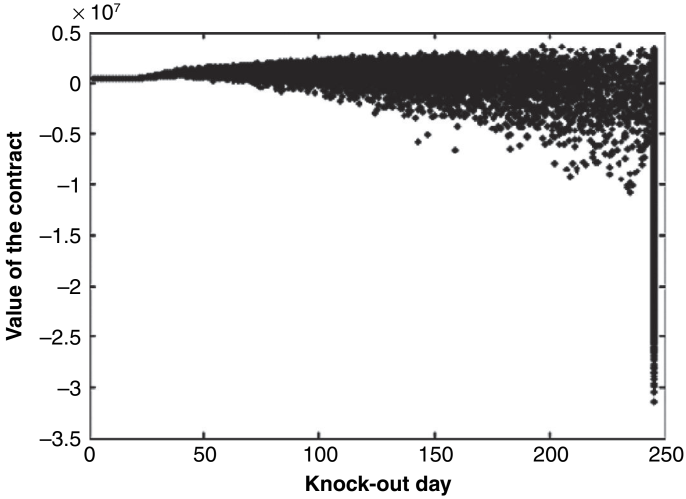


Figure 5.8 The relationship between value of the contract and the knock-out day.

settlement price is above the forward price, buyers receive larger payments from the sellers of this accumulator, and vice versa. Therefore, a larger multiplier suggests a larger variance for comparable accumulator contracts. In addition, considering the usual and obvious trend, or "momentum effect" in stock price movements, which states that low prices are always followed by continuous low prices, the negative impact of the multiplier is larger than its positive impact. To conclude, the multiplier usually decreases the value of the contract, as shown in Figure 5.9.

5.8 Product highlights and risks

From the *buyers'* perspectives, this product is attractive for the following reasons:

1 **Low knock-out price and low forward price:** Compared with the stock price of China Telecom at the issuance of contract, the knock-out price is only 103.78% (= 6.8495 ÷ 6.6) of the stock price as on January 11, 2008, while the forward price is 81.21% (= 5.3599 ÷ 6.6) of the stock price; both are relatively low. This fact implies that if the stock price goes up by 3.8%, buyers can gain a large sum of money, which seems almost risk free because China Telecom has gone through a period of rapid growth. Moreover, even if the stock price goes down in the near future, things would not get that bad for there would still be a profit of about 20%. As long as the present level of the stock price is maintained in the mid and long term, there will be considerable profits for buyers of the contract.

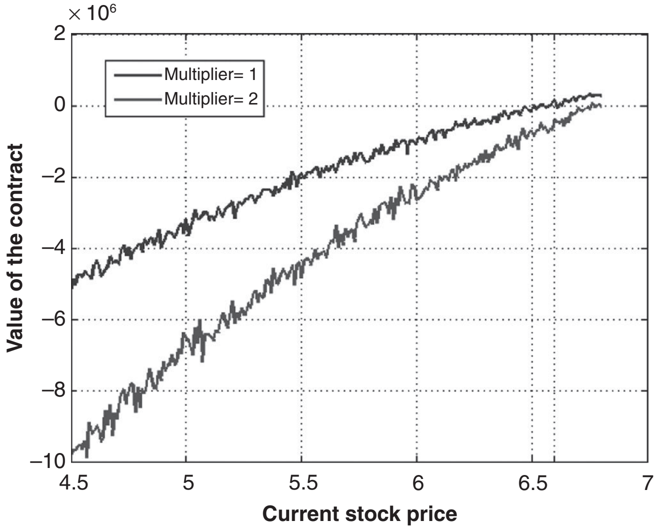


Figure 5.9 The relationship between the value of the contract and the multiplier.

2 **Advance delivery:** The fixed amount of about 480,000 HKD ensures that the buyer has a positive cash flow. On the other, the disconnect between the stock price and the first-month payoff helps the buyers avoid possible losses brought about by short-term fluctuations in prices. For an optimistic investor who firmly believes that China Telecom would continue rising in the mid-to-long run, accumulator with advance delivery matches his market view.

From the *sellers*' perspectives, this product is attractive for the following reasons:

1 **Limited risk:** Owing to the protection of knock-out terms, sellers of the contract are actually facing a limited risk with this maximum payment per share of 1.4896. Moreover, they could adopt appropriate hedging strategies to keep the risk within an acceptable range.
2 **Huge potential profit**: If the stock price goes below the forward price, sellers of the contract would profit from this transaction. For sellers who are pessimistic about the future market or just about China Telecom, this profit might motivate them to enter this contract.
3 **Risk:** The associated risk is somewhat difficult to gauge and may often be neglected by over-optimistic investors. Because China Telecom is at its highest point in history, the probability of the market turning down is relatively large, and if that should happen, buyers of the contract would suffer great losses. On the contrary, the risk for sellers with respect to this transaction is obvious: if the stock price stays above the forward price, sellers would have to complete their payments until the contract is knocked out or expired.

Table 5.6 Maximum profit and loss for both parties

	Buyer		*Seller*	
	Maximum profit	*Maximum loss*	*Maximum profit*	*Maximum loss*
Per unit	1.4896	−5.3599	5.3599	−1.4896
Total	13937222	−47982258	47982258	−13937222

Note that these calculations are not rigorous and are only used for intuitive analyses.

5.9 Summary

In general, this product is suited for market conditions with varied views about the future market movements. When the asset prices are relatively high (low) and market participants are uncertain about the coming movements in the mid and long term, accumulator with advance delivery perfectly meets the needs for both sides. An optimist enters the buy side and expects a profit, while things are the opposite for a pessimist. Both sides face risks and need appropriate hedging strategies.

Real market scenario

Figure 5.10 China Telecom spot rate between November 15, 2002, and January 10, 2008.

As demonstrated in Figure 5.10, since mid-2007, China Telecom has entered a volatile period with sharp ups and downs. In January 2008, when it was issued, the stock price was almost at the highest point in its history. The market was faced with an ambiguous future. From our point of view, however, this stock was at a high point without sufficient backup power. A strong support would be at 5 HKD, and the other support would be at 4 HKD. China Telecom would face huge pressure if it should move upward. But if it moves downward, the price would decrease sharply.

In such circumstances, when the market is volatile and further downward movement is possible, the implied volatilities of the market would be much larger than the historical volatilities. That is why later on we adopt implied volatilities from market data to price this product.

On the other hand, being one of the largest telecommunication service providers in China, China Telecom is in good financial health. However, with competition in this market turning out to be fiercer than ever before, its operating profit ratio has declined slowly from 1999 to 2007.

Therefore, when it was first issued, the future market was unclear. For one thing, the China Telecom share had been rising for a long period, and one could naturally expect an increase in the near future. But on the other hand, negative news from macroeconomics and the declining Hong Kong stock market were putting great pressure on future movements of China Telecom. Besides, high volatility in the market indicated a different market sentiment. As we have stated, it is this discrepancy in the market view that gives birth to this product, the accumulator with advance delivery.

Figure 5.11 Closing for China Telecom between January 11, 2008, and July 25, 2008.

Figure 5.11 depicts the results of the actual stock price movement during the specified period. As can be seen, the knock-out barrier has never been touched, indicating the buyers have to hold the contract. The price of China Telecom went below the forward price in March. After that, buyers of this contract had to bear a big loss. Since June, the price fell sharply, enlarging buyers' losses dramatically.

5.9.1 Pricing

Because the payoff of each observation simultaneously depends on the path of the stock price (which determines the monthly number of shares) and the closing price on the last day of the period (which determines the settlement price), it is hard to split this accumulator into combinations of plain-vanilla barrier options. Therefore, traditional analytical solutions and binominal trees have difficulties in pricing this accumulator. MCS is a more suitable pricing method.

The MCS method is convenient to program and easy to understand, but it suffers from the following two flaws:

1 **Computationally time consuming:** This makes it difficult to calculate Greek letters with a high degree of accuracy.
2 **Lack of sufficient stability:** Unlike other types of options whose paths do not affect the final payoff or affect it in a limited way, when pricing accumulators, the path of stock prices is used to cumulate the number of shares, and the latter variable, the number of shares, is multiplied by 20,000 to calculate the final payoff. This indicates that possibly, some tiny discrepancies in random numbers would be cumulated and amplified 20,000 times, thus restricting the stability of the MC method. For more stable price outputs, more sample paths are needed for simulations.

To illustrate, here we assume the movement of stock price follows geometric Brownian motion (GBM), i.e.,

$$d \ln S = \left(r - \frac{\sigma^2}{2} \right) dt + \sigma dz.$$

Before simulating the process with the given equation, we need to calibrate the parameters in the model. In the BS model, the unknown parameters are constant drift (usually represented by the difference between the risk-free rate and the dividend yield) and volatility. Besides, we also need to calculate the number of trading days to decide the length of simulation.

5.10 Parameters

5.10.1 Trading days

In the calculation of trading days, we have adopted the Hong Kong public calendar and eliminated public holidays and weekends. We have taken the differences between ending days of observation periods and settlement days into consideration. We have also considered the differences between trading days (245 days in total) and days used for discounting (368 days).

5.10.2 Interest rates

Interest rates are used to calculate the drift rates of the stocks as well as to discount in our pricing model. When interest rates are used to calculate the drift, we adopt the forward rates calculated from the term structure of spot rates, because we have chosen an iterative measure to calculate the stock prices:

$$S(i+1) = S(i)\exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}\right].$$

When interest rates are used to discount, we choose the spot rates directly. They have been changed into continuous compounding and have been properly annualized as listed in Table 5.7.

5.10.3 Volatilities

The volatilities we use are the implied volatilities calculated from options traded at the Hong Kong Exchange (HKEx). We have controlled the strike price of the options and the time to maturity as well, so as to match our product. We eliminated those *abnormalities* whose implied volatilities are either 0 or much larger than other options. The calculated implied volatilities are listed in Table 5.8.

5.10.4 Stock prices

Although not stated in the contract, we assume an implied dividend protection mechanism in this contract. Thus, all the stock prices we adopted are adjusted with dividends with a backward exemption right. The stock prices are listed in Table 5.9.

5.11 Algorithm

The steps of the MC method are as follows:

1 Simulate the path of the stock price according to the assumed model.
2 Identify whether the product is knocked out.
   a Calculate the corresponding number of shares, settlement prices, and payoffs for each month if not knocked out.
   b Record the specific location of the knock-out event, and calculate the number of shares, settlement prices, and related payoffs if knocked out.
3 Discount all the payoffs and add them together to obtain the value of the contract for one simulation.
4 Repeat the above steps, and calculate the mean of every simulation.

Table 5.7 Interest rate term structure*

	Jan 11		*Jun 18*		*Jun 25*		*Jul 2*		*Jul 9*	
	SPOT	Forward	SPOT	Forward	SPOT	Forward	SPOT	Forward	SPOT	Forward
o/n	1.832%	1.832%	1.198%	1.198%	1.095%	1.095%	1.331%	1.331%	1.386%	1.386%
1w	2.499%	2.602%	1.350%	1.373%	1.646%	1.731%	1.550%	1.583%	1.503%	1.521%
2w	2.844%	3.189%	1.751%	2.152%	1.649%	1.653%	1.646%	1.742%	1.599%	1.696%
1m	3.132%	3.419%	1.908%	2.065%	1.797%	1.944%	1.827%	2.008%	1.803%	2.006%
2m	3.320%	3.509%	2.138%	2.367%	2.071%	2.346%	2.118%	2.408%	2.103%	2.404%
3m	3.375%	3.485%	2.341%	2.748%	2.318%	2.812%	2.291%	2.636%	2.301%	2.697%
4m	3.391%	3.441%	2.466%	2.842%	2.449%	2.842%	2.390%	2.690%	2.400%	2.698%
5m	3.391%	3.389%	2.591%	3.090%	2.578%	3.094%	2.487%	2.874%	2.497%	2.883%
6m	3.389%	3.379%	2.716%	3.340%	2.703%	3.327%	2.585%	3.073%	2.594%	3.078%
7m	3.355%	3.150%	2.794%	3.260%	2.810%	3.454%	2.663%	3.132%	2.671%	3.136%
8m	3.322%	3.094%	2.872%	3.424%	2.916%	3.653%	2.741%	3.285%	2.748%	3.282%
9m	3.288%	3.011%	2.954%	3.609%	3.025%	3.903%	2.821%	3.468%	2.830%	3.487%
10m	3.267%	3.083%	3.049%	3.907%	3.118%	3.955%	2.902%	3.632%	2.890%	3.431%
11m	3.246%	3.028%	3.144%	4.084%	3.212%	4.143%	2.986%	3.820%	2.954%	3.594%
12m	3.227%	3.027%	3.236%	4.257%	3.305%	4.330%	3.063%	3.915%	3.015%	3.684%

*All rates acquired from the website of the Hong Kong Association of Banks.

Table 5.8 Volatility term structure

	Jun 18		*Jun 25*		*Jul 2*		*Jul 9*	
	SPOT	*Forward*	*SPOT*	*Forward*	*SPOT*	*Forward*	*SPOT*	*Forward*
1m	50.000%	50.000%	62.390%	62.390%	61.780%	61.780%	59.276%	59.276%
2m	49.610%	49.217%	59.880%	57.260%	60.490%	59.172%	60.341%	61.388%
3m	53.950%	61.721%	55.840%	46.724%	60.720%	61.177%	61.956%	65.066%
4m	55.930%	61.489%	55.570%	54.752%	58.150%	49.648%	61.012%	58.088%
5m	51.450%	26.886%	53.640%	45.102%	57.880%	56.787%	59.858%	55.001%
6m	48.830%	32.722%	54.190%	56.860%	58.940%	63.977%	59.499%	57.669%
7m	55.740%	86.269%	57.300%	73.237%	not needed for marking to market			

Table 5.9 Stock Prices*

Jan 11	*Jun 18*	*Jun 25*	*Jul 2*	*Jul 9*
6.60	4.72	4.48	4.17	4.22

*Raw data are exported from the Wind Database.

To tackle the advance delivery term, we first add the fixed amount of HK$ 471,238 and discount it to the proper pricing days during every simulation.

5.11.1 Pricing results

The pricing results are shown in Table 5.10.
Our prices have basically reflected the key features of the contract:

1 The prices are negative, which indicates that buyers of the contract have suffered great losses owing to the sharp decline of China Telecom.
2 Among the results of four pricing days, the price on July 2 is the lowest, in accordance with the lowest price of the underlying asset on July 2.
3 We also priced the contract on the initial day, say, January 11, 2008. The pricing result, −445310 (with 100,000 simulations) suggests a negative initial value for the contract. This may be owing to the parameters used for implied volatilities. The data shows that on January 11, the overall market view was pessimistic about the future of China Telecom.

Table 5.10 Pricing results

Pricing Dates		*Jun 18*	*Jun 25*	*Jul 2*	*Jul 9*
Ours	10^4	−646930.00	−719320.00	−877020.00	−820550.00
	10^5	−652180.00	−723430.00	−879750.00	−825360.00
	10^6	−650540.00	−722040.00	−881290.00	−822490.00
Interbank		−604480.03	−739803.43	−933307.66	−817508.58
NX		−687157.00	−807595.00	−989065.00	−918121.00

*To make them comparable with vendors, our pricing results are measured in USD.

Table 5.11 Greek letters*

Pricing Dates	*Jun 18*	*Jun 25*	*Jul 2*	*Jul 9*
Delta (HKD)	5209100.00	3575100.00	2757400	4981600.00
Delta (USD)	667010.00	457880.00	353580	638570.00
Gamma (HKD)	411500000.00	1661400000.00	−3154700000	594010000.00
Gamma (USD)	52691000.00	212780000.00	−404530000	76144000.00
Vega (HKD)	2106800.00	−184600.00	514360	−5746600.00
Vega (USD)	269770.00	−23643.00	65958	−736640.00

*Because the price is sensitive to Monte Carlo simulation, the Greek letters might be problematic for mixing together actual value changes and value discrepancies brought about by the pricing method.

5.11.2 Greeks

The Greek letters are also calculated for the four pricing days and are listed in Table 5.11.

5.12 SW05—Daily callable fixed coupon swap (Equity)

SW05 is a daily callable fixed coupon swap, and the knock-out event is monitored continuously in each period. This product can be treated as a swap associated with a basket of equities as the underlying. The prices of the underlying equities are used to determine whether an early knock-out is triggered. The swap payments of each period are settled on the corresponding settlement day, which is usually some business days (e.g., 3 days) after the day of early termination or some business days after the end of each period if there is no early termination. The investor receives a fixed coupon rate and pays the floating rate that is determined by the reference rate (e.g., HIBOR) while the cash flows of the counterparty are reversed.

5.12.1 Basic analysis

Assume the underlying of our product is a basket of two equities and the percentage knock-out barrier (B) for each equity is the same. Then on any day during each period, if the closing prices of both equity 1 and 2 are higher or equal to the barrier, the contract will be terminated. A knock-out event is said to have occurred.

Figure 5.12 illustrates the knock-out mechanism. Let there be four swap periods, and let the percentage knock-out barrier B be set to 105%. It can be seen that in the first three periods, the prices of Equity 1 and Equity 2 are smaller than 105%, implying that the contract is not knocked out. However, in the fourth period, both equities are greater than or equal to 105%, meaning that the contract is knocked out.

To be specific in describing the payoff, the following are assumed for the product:

1 The notional amount is in HKD and is denoted by N.
2 The reference rate used is 1 month HIBOR and is denoted as r_t.

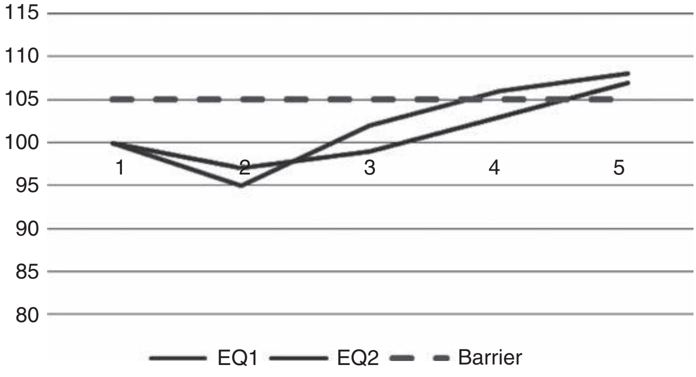
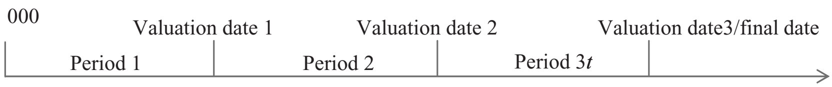


Figure 5.12 Equity price movement.

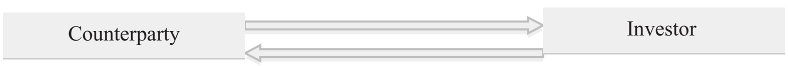
3 The fixed accrual rate used is denoted as R (e.g., 10%).
4 The strike rate is defined as $R_{s,t}$, and the knock-out rate is denoted as $R_{b,t}$ ..
5 There are three swap periods with three settlement days in total as follows.



Within each period $i\, t_i^s \left(t_i < t_i^{i+1} < t_{i+1}\right)$, then early termination of the contract occurs. The counterparty pays an amount that equals $N \times R \times \frac{t_i^{i+1} - t_i}{365}$ to the investor and receives an amount that equals $N \times r_{t_i^{i+1}} \times \frac{t_i^{i+1} - t_i}{365}$ from the counter-party.

The following flowchart summarizes the cash flows:

$$N \times R \times \frac{t_i^{i+1} - t_i}{365}$$



$$N \times r_{t_i^{i+1}} \times \frac{t_i^{i+1} - t_i}{365}.$$

If the contract is held until period 3 and there is no early termination, and if all equities prices are greater than or equal to the strike price at the final date, then counterparty pays an amount that equals $N \times R \times \frac{t_3 - t_2}{365}$ to investor and receives an amount that equals $N \times r_{t_3} \times \frac{t_3 - t_2}{365}$ from the investor.

On the other hand, if the price(s) of one or more equities is/are smaller than or equal to the strike price on the final date, then in addition to the above cash flow, the investor will buy the worst-performing equity at the strike price. The investor will lose (strike price—the equity spot price) × aggregate number.

Real market scenario

Figure 5.13 One-month HIBOR rate.

We assume this product is a 3-Month HKD daily callable fixed coupon swap linked to a basket of equities. As the accrual rate is much larger than the HIBOR, without early termination, payments on the equity payment dates are profitable to the investor.

- If the equities' prices are all very stable and stay between the strike price and exit price, the product is profitable to the investor because three exchange payments will be made.
- If the equities' prices all go up very high and early termination occurs, the product will still be profitable to the investor although the profit is limited. The later the product is knocked out, the more beneficial it is to the investor.
- If any of the equities' price goes below its strike price on the final valuation date, the investor may suffer a loss on buying the worst-performing equity. The loss can be unlimited.

5.12.2 Pricing

5.12.2.1 Interest rate equity basket model

Because a basket of equities is involved in pricing, we should consider the correlation between equities in simulation. Also, no Quanto feature (only one currency would be considered) is involved in this product. Therefore, the IR equity basket model is used in pricing.

In order to obtain the random variables with the desired correlations from independent random variables for simulation, *Cholesky decomposition* is used. The derivation is as follows:

Consider a correlation matrix Σ with dimension $n \times n$ that is symmetric and positive-definite (i.e., $X^T \Sigma X > 0$ for any non-zero X with dimension $n \times 1$). Then the correlation matrix can be decomposed as a $n \times n$ lower triangular matrix A such that $AA^T = \Sigma$. Once the matrix A is obtained, the random variables with desired correlation can be expressed as AZ, where Z is a column vector of independent standard normal random variables. Here is the derivation of the Cholesky decomposition for a 2 × 2 correlation matrix Σ:

Consider a 2 × 2 covariance matrix Σ, represented as

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where ρ is the correlation between W_1 and W_2. The matrix A satisfying the relation $AA^T = \Sigma$ is as follows:

$$A = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}.$$

Let Z be a 1 × 2 column vector of independent standard normal random variables:

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}.$$

As such, we can sample from bivariate independent normal random variables (1,) by setting $W = AZ$, implying that

$$\begin{aligned} W_1 &= Z_1 \\ W_1 &= \rho Z_1 + \sqrt{1-\rho^2 Z_2} \end{aligned}$$

Because no Quanto feature exists in this template, we just need to simulate two correlated stock prices under the same measure:

$$\frac{dS_{t,\mathrm{HKD}}}{S_{\mathrm{t,HKD}}} = \left(r_{\mathrm{t,HKD}} - Div_{\mathrm{t,HKD}}\right)dt + \sigma_t dW_{\mathrm{t,HED}}.$$

5.12.2.2 Implied volatility

To capture the volatility from the implied volatility surface we may simply use the Black–Scholes (BS) model, which assumes a constant or time-dependent volatility given specific moneyness. However, because the volatility surface is not always flat for all moneyness values, this model cannot be used to calibrate volatilities with several degrees of moneyness simultaneously. We may turn to another model that takes into account the behavior of the volatility smile, e.g., the Dupire model. The model is given by

$$\sigma_{\text{loc}}^2(K,T) = \frac{2\left(\frac{\partial C_{\text{market}}}{\partial T} + \left(r(T) - d(T)\right)K\frac{\partial C_{\text{market}}}{\partial K} + d(T)C_{\text{market}}\right)}{K^2 \frac{\partial C^2_{\text{market}}}{\partial^2 K}}.$$

Note that one potential problem of using the Dupire formulas is that, for some financial instruments, option prices for different strikes and maturities are not available or are not sufficient to calculate the right local volatility. Another problem is that for strikes far in- or out-of-the-money, the numerator and denominator of this equation may become very small, which could lead to numerical inaccuracies.

5.13 Monte Carlo simulation

Owing to the path-dependent nature of the product, MCS is a good simulation technique. To start with, the following conditions are assumed:

1 The underlying equities follow GBM with time-dependent volatilities, $\sigma_{1,t}$ and $\sigma_{2,t}$.
2 The interest rates are deterministic.
3 The market is frictionless, complete, and without arbitrage opportunities.

Under a risk-neutral pricing measure:

$$\frac{dS1_{t,HKD}}{S1_{t,HKD}} = \left(r_{t,HKD} - Div1_{t,HKD}\right)dt + \sigma_{1,t}dW_1$$

$$\frac{dS2_{t,HKD}}{S2_{t,HKD}} = \left(r_{t,HKD} - Div2_{t,HKD}\right)dt + \sigma_{2,t}dW_2,$$

with $dW_1 dW_2 = pdt$. The simulation of equities is as follows:

$$S1_{\mathrm{t_{i+1}},\mathrm{HKD}} = S1_{\mathrm{t_i},\mathrm{HKD}} \cdot \exp\left(\left(r_{\mathrm{t_i},\mathrm{HKD}} - Div1_{\mathrm{t_i},\mathrm{HKD}} - 0.5\sigma_{1,\mathrm{t}}^2\right)\left(t_{i+\mathrm{t}} - t_{\mathrm{i}}\right) + \sigma_{1,\mathrm{t_i},}\sqrt{t_{i+1} - t_i}W_1\right),$$

$$S2_{t_{i+1},HKD} = S1_{2,HKD}\exp\left(\left(r_{t_i,HKD} - Div2_{t_i,HKD} - 0.5\sigma_{2,t}^2\right)\left(t_{i+t} - t_i\right) + \sigma_{2,t_i,}\sqrt{t_{i+1} - t_i}W_2\right).$$

Based on the simulated underlying equities and the resultant payoff under the corresponding simulation, the theoretical instrument value is calculated by

$$NPV = \frac{1}{N}\sum_{k=1}^{N} payoff\left(S1^k, S^k\right)P^k\left(0,T\right),$$

where $S1^k$, $S2^k$ are the k^{th} simulated paths, $payoff(\cdot)$ is the final payoff, and $P^k(0,T)$ is the corresponding discounted factor.

5.13.1 Pricing result and greeks

In this section the product is assumed to have the following specifications:

1. Underlying interest rate: HKD
2. Effective date: Jan 18, 2012
3. Termination Date: Apr 18, 2014
4. Barrier = 100%
5. Strike = 95%
6. Notional per fixing = HKD 1,500,000
7. Yield curve interpolation = Linear LogDF
8. Implied volatility interpolation = Linear vol × vol × T
9. Implied volatility extrapolation = flat
10. Quality = 5
11. Simulation paths = 100,000
12. Payment schedule

Period	*Pay Date*
1	Feb 20, 2012
2	Mar 30, 2012
3	Apr 18, 2012

The pricing results are given by

NPV	*Equity Leg*	*Fund Leg*
−27,833.79	477,051.21	−504,885.00

In the calculation of the Greeks, because there are no analytical solutions available, simple numerical differentiation methods, e.g., difference equations, can be used instead.

$$\text{NPV.Delta} = \frac{\begin{bmatrix}\text{[NPV(+1\% changed in} & - & \text{NPV(1\% changed in} \\ \text{All EQ spot prices)} & & \text{All EQ spot prices)]}\end{bmatrix}}{2}$$

NPV.Gamma =

$$\left[\begin{matrix}\text{NPV(+1\% changed in} \\ \text{All EQ spot prices)}\end{matrix} + \begin{matrix}\text{NPV(1\% changed in} \\ \text{All EQ spot prices)}\end{matrix} - 2 \quad \text{NPV(Original)}\right]$$

NPV.Vega =

$$\left[\frac{\text{NPV(+1\% shift in All EQ implied vol)} \;\; \text{NPV(1\% shift in All EQ implied vol)}}{2}\right]$$

The results are as follows:

NPV.Delta	5,868.13
NPV.Gamma	−747.26
NPV.Vega	−3,120.70

5.14 Conclusion

In this chapter a few newly developed equity-linked structured products were introduced, with some descriptions of pricing and Greeks' calculations. It may be noticed that although the payoff structures of different products can greatly vary, there are some standard methods for pricing the product:

1 Decompose the product into a portfolio of some standard products (e.g., vanilla options) and apply the BS formula to each of the embedded options.
2 Apply the simulation method under the assumption of different models.

In the following chapter some FX-linked structured products will be introduced. However, the methods used will be somewhat similar.

6 Foreign exchange-linked structured products

Foreign exchange (FX)-linked structured products refer to any combination of FX options and other FX products, including FX-linked forwards and swaps. As the world is becoming more interconnected under globalization, the demand for financial products that serve to hedge or speculate against foreign exchange-related risk has been increasing. In this chapter, a few newly designed FX-linked structured products are introduced. For each product introduced, a brief analysis, together with some pricing and hedging solutions, is provided.

6.1 Bullish G7

There are two types of Bullish G7 products: Bullish G7A and Bullish G7B. The former is a strip of 12 European call digital options on the underlying FX rate, whereas the latter is a strip of 12 ordinary European put options on the FX rate. Options on the same type of products usually have the same strike, with maturities ranging from 1 to 12 months.

6.1.1 Basic analysis

Bullish G7A

Table 6.1 Example of a Bullish G7A

Spot reference	105.94 USD-JPY
Underlying	USD/JPY FX Rate
Type	Digital USD CALL/JPY PUT
Style	European
Strike rate	112.80
Settlement amount	JPY 1,800,000.00
Premium	JPY 10,750,000

For the buyer of this product (i.e., long position), on each expiry date, there can be two scenarios:

1 USD/JPY trades below 112.80. There is no payment between the buyer and seller of the product.
2 USD/JPY trades at or above 112.80. The seller of the product is obliged to pay JPY 1,8000,000 to the buyer.

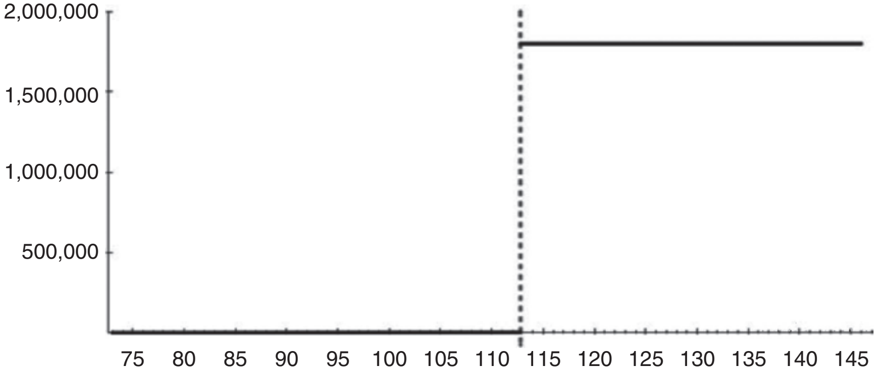


Figure 6.1 Bullish G7A payoff diagram.

From the payoff structure, the buyer of the product should hold a market view that USD is appreciating against JPY and will stay at a high level for at least a year. The payoff diagram for one expiration date is shown in Figure 6.1.

When the product matures, the buyer can get at the most a total of JPY 21,600,000 (= 1,800,000 × 12) (see Figure 6.2; scenario 1). However, in the worst case, the buyer may get nothing if the spot rate is below 112.80 on each settlement date (see Figure 6.2; scenario 2).

Bullish G7B

Table 6.2 Example of a Bullish G7B

Spot reference	105.94 USD-JPY
Underlying	USD/JPY FX Rate
Type	JPY CALL/USD PUT
Notional amount	USD 800,000
Style	European
Strike rate	98.65
Premium	JPY 1,027,200

The buyer of this product is faced with the following two scenarios on each expiry date:

1 USD/JPY trades below 98.65. The buyer (i.e., long position) of the product can fetch an amount of JPY 800,000 × (98.65 − USD/JPY spot rate at expiry time).
2 USD/JPY trades at or above 98.65. There is no payment between the two parties.

This payoff structure suggests that the buyer of the product should hold a market view that USD is likely to weaken against JPY. The payoff diagram on each expiry date is the same as a vanilla put option (see Figure 6.3).

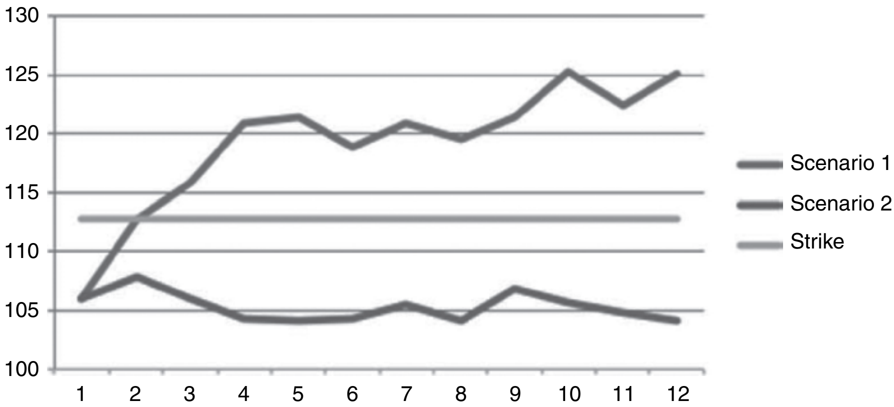


Figure 6.2 Simulation of USD/JPY spot rate.

6.1.2 Bullish G7—a combination

For an investor with a clear directional view of the FX rate, one can enter into the *opposite* direction of these two products. For instance, if an investor believes that USD is strengthening against JPY, he or she can enter into a long position of Bullish G7A and a short position of Bullish G7B (see Figure 6.4).

Although the payoff structure looks complicated, the whole package is actually a linear combination of one Bullish G7A and one (short) Bullish G7B, which are linear combinations of 12 digital options and 12 (short) put options, respectively.

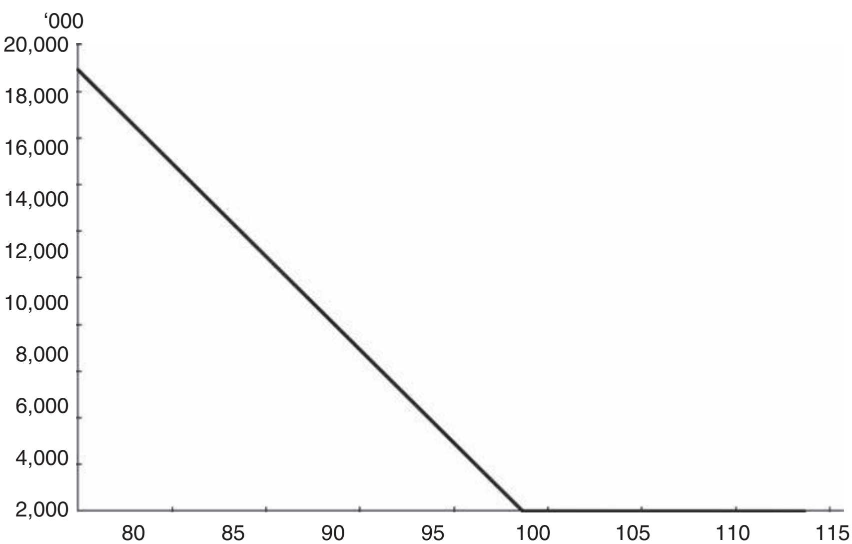


Figure 6.3 Bullish G7B payoff diagram.

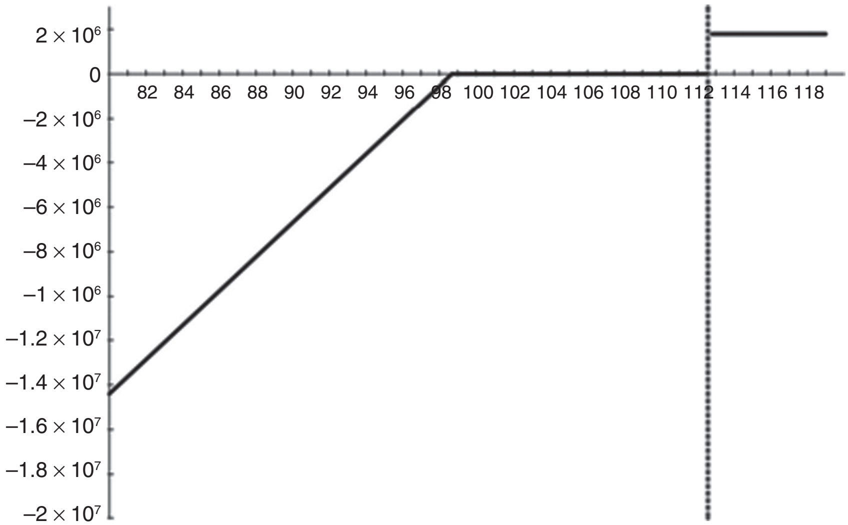


Figure 6.4 Bullish G7 payoff diagram.

Therefore, the value and Greeks of the whole product can be calculated as a combination of those simpler products.

Real market scenario

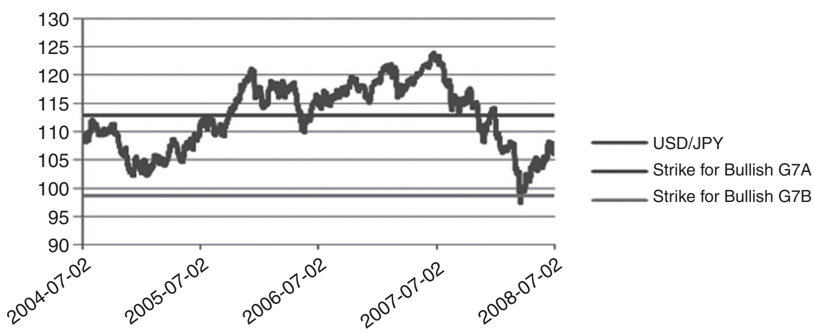


Figure 6.5 USD/JPY spot rate between January 2, 2004, and July 2, 2008.

From the historical data shown in Figure 6.5, it can be seen that most of the time the USD/JPY FX rate stayed above 112.80 (strike for Bullish G7A) before 2007. However, owing to the shock of the subprime crisis, many people began to lose confidence in the US, and the FX rate began to drop rapidly. The rate once even fell below 98.65 (strike for Bullish G7B). Since then, the effect of the subprime crisis began fading, and the market regained its confidence in the US. At that time, it was unclear whether the rate would go above 112.80, but the risk of the rate falling again below 98.65 was small. The package of going long on Bullish G7A and shorting Bullish G7B was thus a good avenue for speculation.

6.1.3 Pricing

As mentioned earlier, both Bullish G7A and Bullish G7B can be decomposed into 12 options with the same strike but different maturities. In this and the following section, the product is priced from the buyer's perspective. Thus, the value of a short position is just the negative of the result.

6.1.4 Closed-form BS model

Under the BS model, the FX rate is assumed to follow GBM with constant drift and volatility. An analytical solution for the option prices can then be obtained as follows.

Bullish G7A

$$P = Q_A \sum_{i=1}^{12} \left(N\left(d_{2,i}\right) \times \exp\left(-r_i^J \times T_i\right)\right)$$

$$d_{2,i} = \frac{\ln\left(S_0/K_A\right) + \left(r_i^J - r_i^U - \sigma_i^2/2\right)T_i'}{\sigma_i\sqrt{T_i}}, i = 1, 2, \ldots, 12,$$

Bullish G7B

$$P = Q_B \sum_{i=1}^{12} \left(K_B N\left(-d_{2,i}\right) - S_0 \exp\left[\left(r_i^J - r_i^U\right)T_i'\right] N\left(-d_{1,i}\right)\right) \times \exp\left(r_i^J T_i\right)$$

$$d_{1,i} = \frac{\ln\left(S_0/K_A\right) + \left(r_i^J - r_i^U - \sigma_i^2/2\right)T_i'}{\sigma_i\sqrt{T_i}}$$

$$d_{2,i} = \frac{\ln\left(S_0/K_A\right) + \left(r_i^J - r_i^U - \sigma_i^2/2\right)T_i'}{\sigma_i\sqrt{T_i}}, i = 1, 2, \ldots, 12,$$

where

P = the price of this product
Q_A = the settlement amount of Bullish G7A
Q_B = the nominal amount of Bullish G7B
S_0 = the spot rate on the valuation date

K_A = the strike price of Bullish G7A
K_B = the strike price of Bullish G7B
T_i' = the maturity of the ith option decomposed from Bullish G7 (calculated using the number of working days between the valuation date and the expiration date)
T_i = the maturity of the ith option decomposed from Bullish G7 (calculated using the number of calendar days between the valuation date and the settlement date)
r_i^J, r_i^U = the continuously compounded interest rate in Japan and the US, respectively, whose duration is T_i
σ_i = the volatility of the ith option decomposed from Bullish G7
$N(\cdot)$ = the cumulative probability of the normal distribution.

6.2 Monte Carlo simulation

With this method, the product can be priced easily, under any given model in the risk-neutral world. In order to verify the foregoing analytical solution, sample paths of the FX rate process can be generated under the assumptions of the BS model. The result should be close to the result obtained earlier.

Besides a simple model like the BS model, Monte Carlo simulation (MCS) is also capable of tackling more complicated models, e.g., the Heston model. The model is given as follows:

$$dS = rSdt + \sqrt{v}SdW_1$$
$$dv = \kappa(\theta - v)dt + \eta\sqrt{v}dW_2$$
$$dW_1 dW_2 = pdt.$$

As the parameters cannot be observed directly, we need calibrate the model.

Problems in model calibration

Values obtained in the calibration of parameters may be unreasonable. For instance, in our previous example, the initial value of the volatility on July 2 was found to be lower than 3%, which is counterintuitive. From the data, it is found that the volatility from the put option is the main reason for this result. Thus, to overcome this problem, we need to (1) kick off some singular points and (2) choose to set the initial volatility as a given one.

The value of a European vanilla option can have a closed-form solution and may be evaluated by the numerical integration method or fast Fourier transform (FFT) (Carr & Madan, 1999).

A simpler method is MCS. A simple Euler discretization of the variance process is

$$v_{i+1} = v_i + \kappa(\theta - v_i)\Delta t + \eta\sqrt{v_i}\sqrt{\Delta t}W_2, W_2 \sim \mathrm{N}(0,1).$$

However, because there is no restriction on the range of the process, a negative variance may result. To alleviate this problem, we may implement a Milstein discretization scheme (Gatheral, 2006):

$$v_{i+1} = v_i + \kappa(\theta - v_i)\Delta t + \eta\sqrt{v_i}\sqrt{\Delta t}W + \frac{\eta^2}{4}\Delta t\left(W_2^2 - 1\right)$$

or

$$v_{i+1} = \left(\sqrt{v_i} + \frac{\eta}{2}\sqrt{\Delta t}W\right)^2 + \kappa(\theta - v_i)\Delta t - \frac{\eta^2}{4}\Delta t.$$

On the other hand, the stock process can be discretized as

$$x_{i+1} = x_i - \frac{v_i}{2}\Delta t + \sqrt{v_i \Delta t}\,\mathrm{W}_1,$$

where $x_i = \ln S_t / S_0, W_1 \sim N(0,1)$. Because the parameters are assumed to be risk-neutral parameters, the expected payoff of the option should be discounted by the risk-free rate to the present time so as to obtain the current value of the options.

6.2.1 Implied volatility surface

In all of the foregoing models, volatility plays a very important role. Thus, to obtain a more reliable and accurate implied value, volatility can be regressed on the implied volatility surface as follows:

$$\sigma = \beta_0 + \beta_1 \ln(1+\tau) + \beta_2 M + \beta_3 M^2 + \beta_4 \ln(1+\tau)M + \beta_5 \ln(1+\tau)M^2,$$

where τ denotes the time to maturity, and M denotes the moneyness of an option, which is obtained by $\ln\left(K/Se^{rt}\right)$. In this model, the shape of the implied volatility surface can be captured by the slope and curvature of the moneyness and the time to maturity.

6.2.2 Analysis of the greeks

As mentioned in the previous sections, Bullish G7A and Bullish G7B are strips of 12 ordinary digital options and put options, respectively. Thus, they can be separated, and the Greek letters can be calculated as the sum of the 12 options. In the following sections, we show the sketch map of the general dynamic behavior of the Greeks of one ordinary option and then analyze the Greeks of options decomposed from the two products, assuming the current date is September 7, 2008. Note that only two digital options have not expired on September 7, 2008.

6.2.2.1 Bullish G7A

Delta

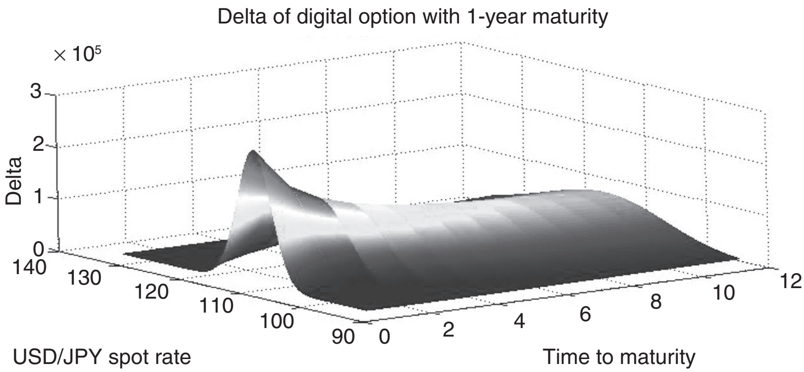


Figure 6.6 General dynamic behavior of the delta of one digital option.

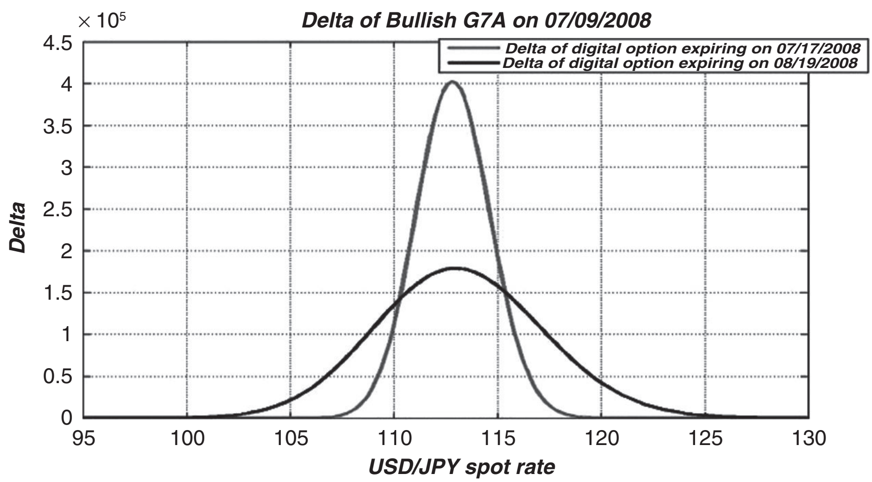


Figure 6.7 The delta of each digital option decomposed from Bullish G7A.

From Figures 6.6 and 6.7, three points are noteworthy:

1 The delta of a digital option increases when the time to maturity decreases. It tells us that the value of the Bullish G7A is much more sensitive to the price of the underlying when the expiration date arrives.
2 Unlike the ordinary call option, the delta of a digital call option reaches its maximum value when the spot rate is near the strike and converges to 0 when the USD/JPY spot rate is high/low enough. (*Note: The delta of an ordinary call option converges to 1 when the underlying asset price goes to infinity*.)
3 The delta of Bullish G7A is always positive.

Gamma

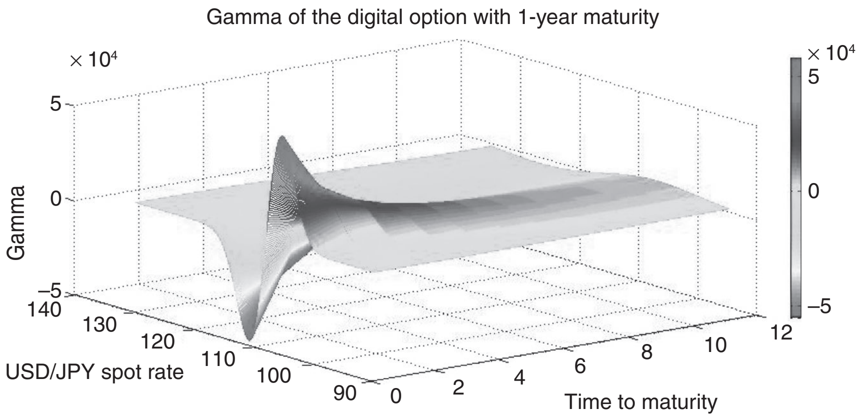


Figure 6.8 General dynamic behavior of the gamma of one digital option.

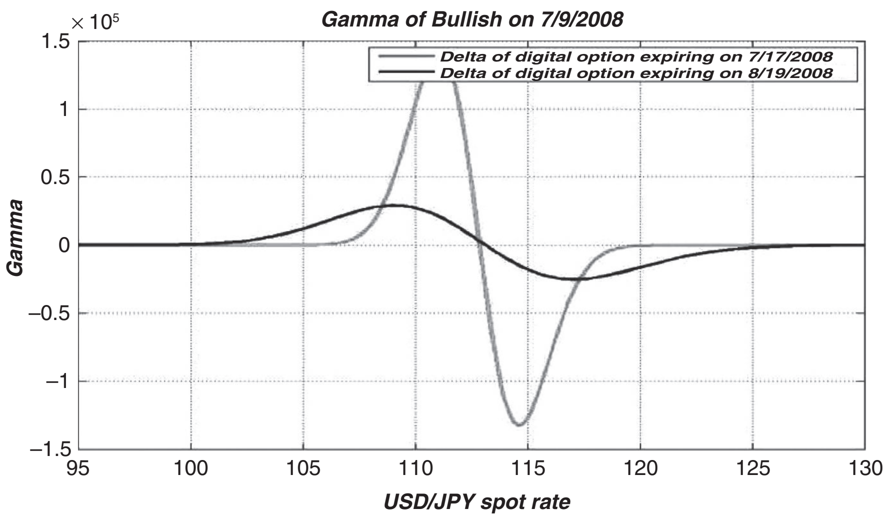


Figure 6.9 The gamma of each digital option decomposed from Bullish G7A.

Several points about Figures 6.9 and 6.10 are noteworthy:

1 The absolute value of gamma increases and fluctuates more sharply when the time to maturity decreases; the reason is the same as the explanation given for delta. This means the sensitivity of Bullish G7A to the USD/JPY spot rate fluctuates more as the expiration date approaches. And if one hedges the Bullish G7A with a dynamic delta hedging strategy, one's position should be frequently adjusted when it is near the expiration date.

2 Unlike delta, the absolute value of gamma is small when the spot rate is near the strike. And gamma is positive when the USD/JPY spot rate is low, but negative when it is high. This means delta increases to a maximum value and then decreases.
3 Gamma also converges to 0 when the USD/JPY spot rate is much higher/lower than the strike.

Vega

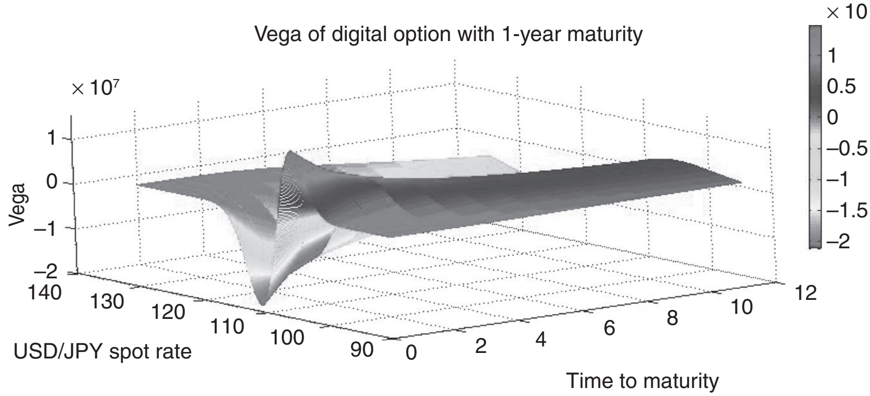


Figure 6.10 General dynamic behavior of the vega of one digital option.

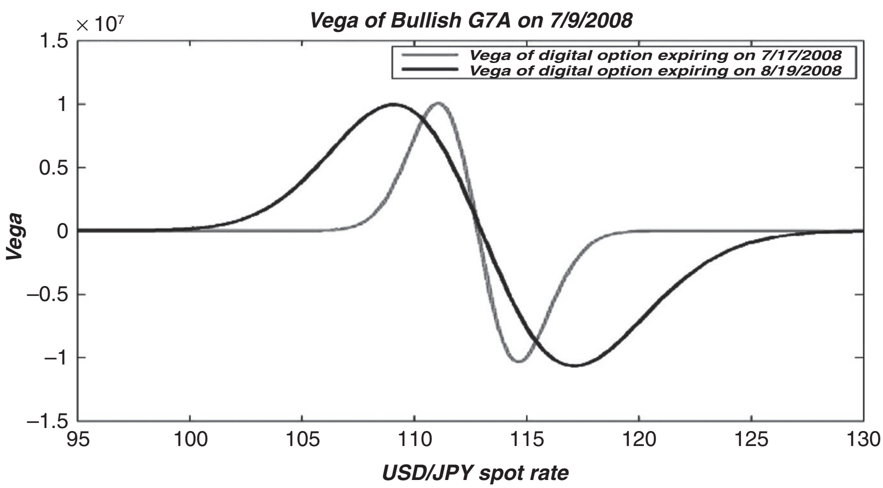


Figure 6.11 The vega of each digital option decomposed from Bullish G7A.

Several points about Figures 6.10 and 6.11 are noteworthy:

1. Vega increases and fluctuates much more when the time to maturity decreases. This means that the value of Bullish G7A becomes much more sensitive to the volatility of the USD/JPY FX rate when the expiration date approaches.
2. And like gamma, vega is relatively small when the USD/JPY FX rate is near the strike. This means that the value of the Bullish G7A is not sensitive to the volatility when Bullish G7A is at-the-money.
3. The vega of Bullish G7A is positive when the USD/JPY spot rate is below the strike (not exactly) and negative when the USD/JPY spot rate is above the strike (not exactly). Note that this is different from the vega of the ordinary call option, where vega is always positive. This is because the digital option gives a constant cash amount to option holders when the underlying asset price is higher than the strike, but the ordinary call option gives an amount that is proportional to the increase in the underlying asset price.
4. It also converges to 0 when the USD/JPY spot rate is high/low enough.
5. Among all the aforementioned points, we should pay special attention to the third one: it tells us that an increase in volatility is not always good for the buyer as in the case of the ordinary call option.

6.2.2.2 Bullish G7B

Delta

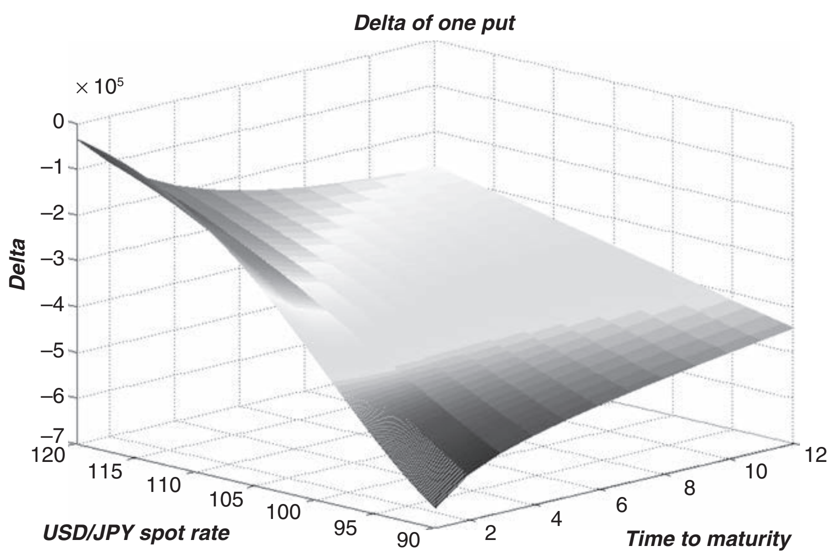


Figure 6.12 General dynamic behavior of the delta of one put option.

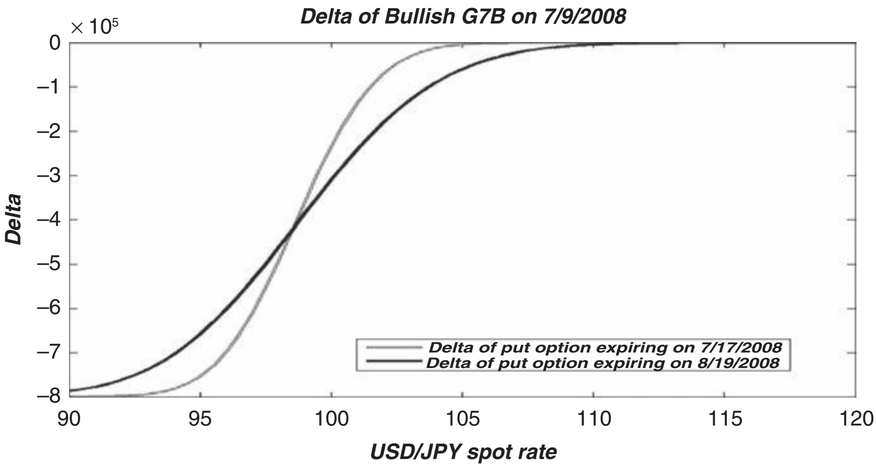


Figure 6.13 The delta of each put option decomposed from Bullish G7B.

Two points about Figures 6.12 and 6.13 are noteworthy:

1. Delta becomes more volatile when the expiration day approaches.
2. The delta of Bullish G7B is always negative. The absolute value of the delta of Bullish G7B converges to −1,600,000 when the USD/JPY spot rate is low enough and converges to 0 when the USD/JPY spot rate is high enough.

Gamma

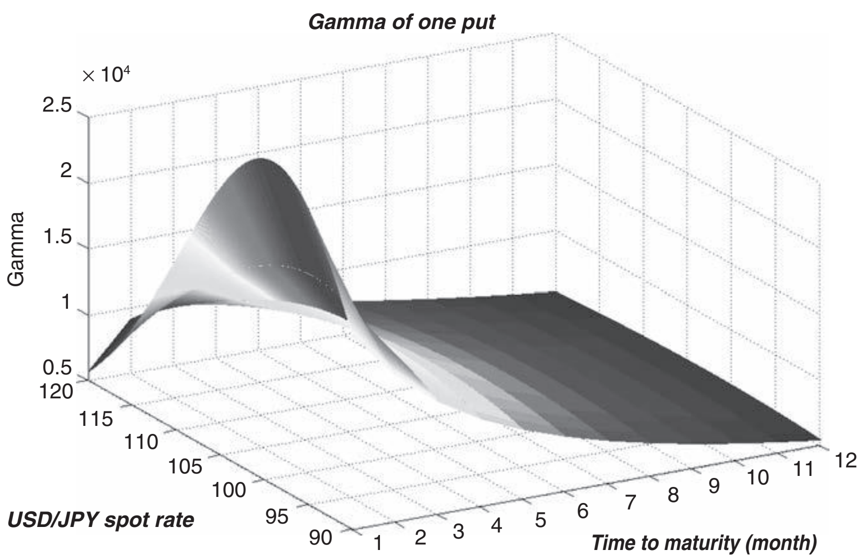


Figure 6.14 The general dynamic behavior of the gamma of one put option.

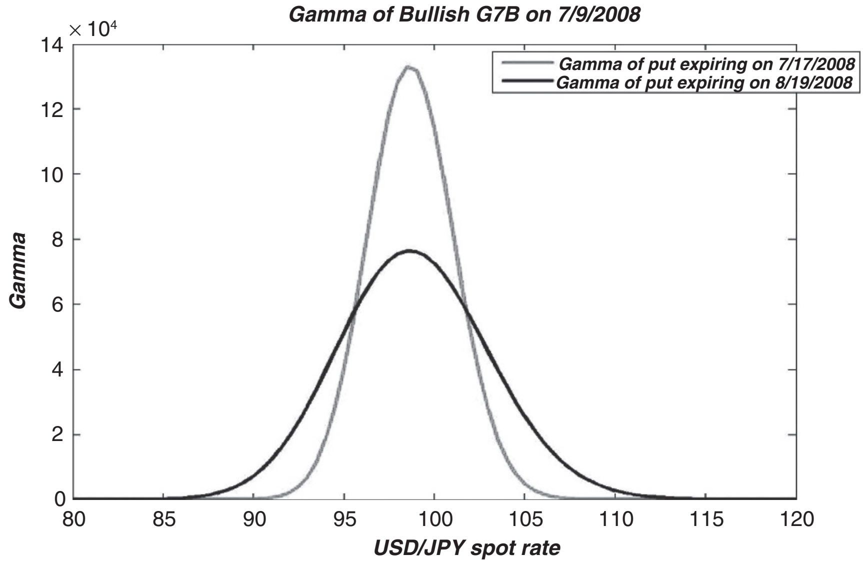


Figure 6.15 The gamma of each put option decomposed from Bullish G7B.

Two points about Figures 6.14 and 6.15 are noteworthy:

1 Gamma becomes larger and more volatile when the expiration day approaches. This means that if one assumes the delta hedging strategy, the delta position should be adjusted frequently when the expiration day approaches.
2 Gamma is always positive. It gets its maximum value when the spot rate is near the strike and converges to 0 when the spot rate is high/low enough. This means that if one assumes a delta hedging strategy to hedge the Bullish G7B, the delta should be adjusted frequently when the spot rate is near the strike.

Vega

Several points about Figures 6.16 and 6.17 are noteworthy:

1 Vega decreases when the expiration date approaches.
2 Vega gets its maximum value when the USD/JPY FX rate is near the strike. This means that the value of the Bullish G7B is quite sensitive to volatility when Bullish G7B is at-the-money.
3 Unlike the vega of the Bullish G7A, the vega of Bullish G7B is always positive, meaning that volatility is always beneficial to the long side of the Bullish G7B. What should be noted is that the issuer plays a short side of the Bullish G7B.
4 Vega also converges to zero when the USD/JPY spot rate is high/low enough.

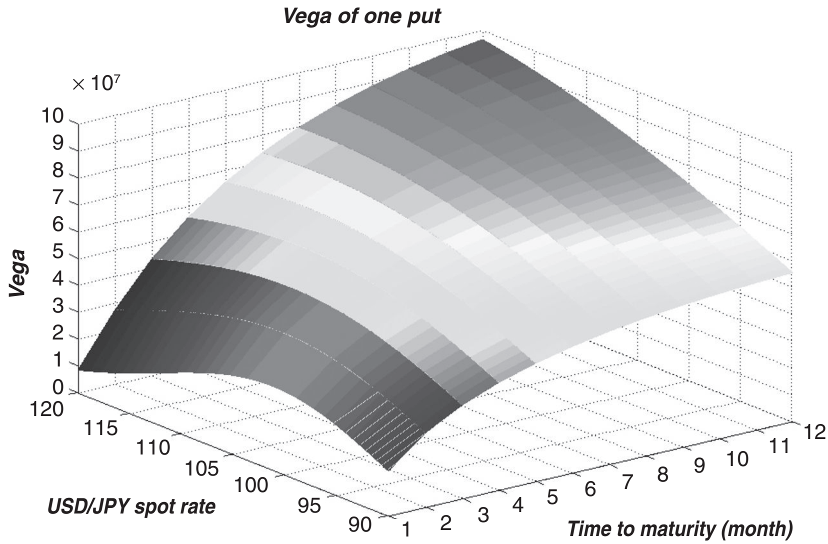


Figure 6.16 General dynamic behavior of the vega of one put option.

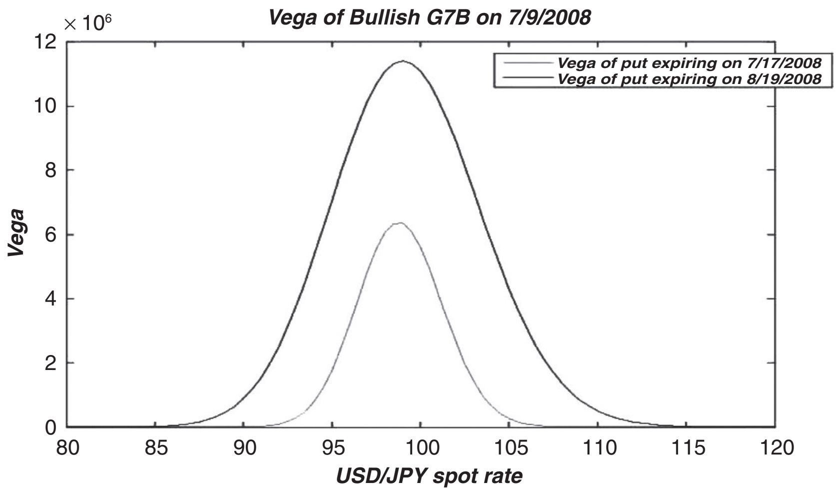


Figure 6.17 The vega of each put option decomposed from Bullish G7B.

It should be noted that in the example of the Bullish G7 package, the component of Bullish G7B is short positions. Thus, the Greek of Bullish G7B should be multiplied by −1 when calculating the Greeks of the entire package.

6.3 Hedging strategies

This section analyzes hedging strategies for the package Bullish G7 (i.e., *long* in Bullish G7A and *short* in Bullish G7B). In Bullish G7A, the payoff is always

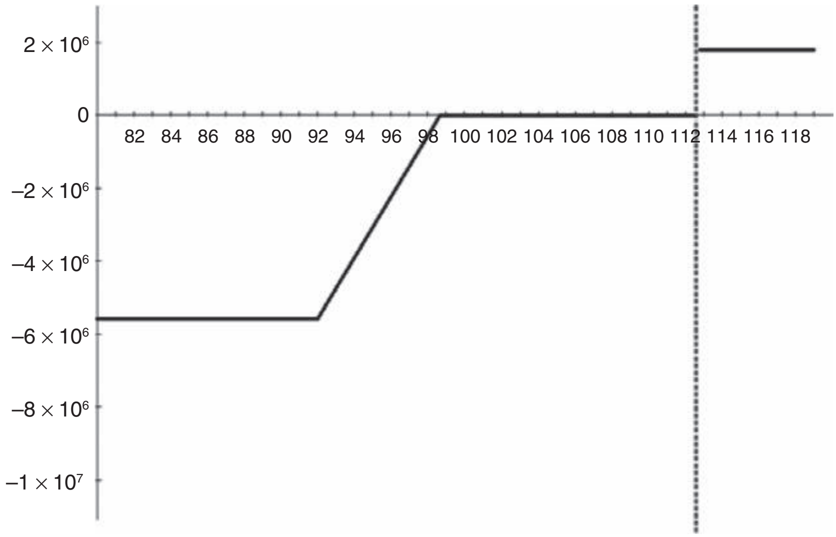


Figure 6.18 Payoff diagram of Bullish G7 hedged with a plain-vanilla option.

non-negative for the long position, and so it can be neglected in hedging. Here, two simple hedging methods are introduced:

- **Hedging with plain vanilla:** In order to constrain the loss amount, investors can long a strip of put options to hedge the short position of Bullish G7B with a strike below that of Bullish G7Bs (i.e., 98.65 in the previous example) so that the hedging cost is smaller.

  The hedging strategy can protect investors from suffering a huge loss in the unlikely event that USD depreciates substantially against JPY.
- **Dynamic hedging:** Instead of looking at the payoff diagram and undertaking *static hedging*, investors can track the Greek of the entire portfolio and adjust it dynamically. With this method, the form of the hedging instruments does not matter. The only concern of the investors is the structure and value of the Greek letters of the instruments. Because the Greek of a portfolio is just the linear combination of those of each component, the amount of hedging instruments resulting in a desired amount of risk can be easily calculated.

6.3.1 EKIKO 1

This product contains two products, EKIKO 1A and EKIKO 1B, with similar characteristics. EKIKO 1A is a strip of 12 exotic barrier options. Each embedded option is a European-style knock-in (KI) barrier option with an American-style knock-out (KO) barrier and has the same contract details except the time to maturity. EKIKO 1B is a strip of 12 American-style KO options with different maturities.

6.3.2 Basic analysis

6.3.2.1 EKIKO 1A

Table 6.3 Example of an EKIKO 1A

Spot reference	1.3638 EUR-USD
Underlying	EUR/USD FX Rate
Type	USD PUT / EUR CALL
Style	European
Strike rate	1.3950
Out strike rate	1.3000
European in strike rate	1.4200
Call currency and amount	USD 837,000.00
Put currency and amount	EUR 600,000.00

For the buyer of this product, on each expiration date:

1 If the EUR/USD FX rate is above 1.42, the buyer can get an amount equal to USD 600,000 × (S_t − 1.395).
2 If the EUR/USD FX rate is below 1.42, there is no transaction between the two parties.

However, at any time before the expiration date, if the spot EUR/USD FX rate is at or below the out strike rate 1.3000, the product will be knocked out and no further transaction will be carried out.

The payoff diagram of Figure 6.19 is valid only when the knock-out event does not occur before the expiration date. Owing to the knock-in and knock-out

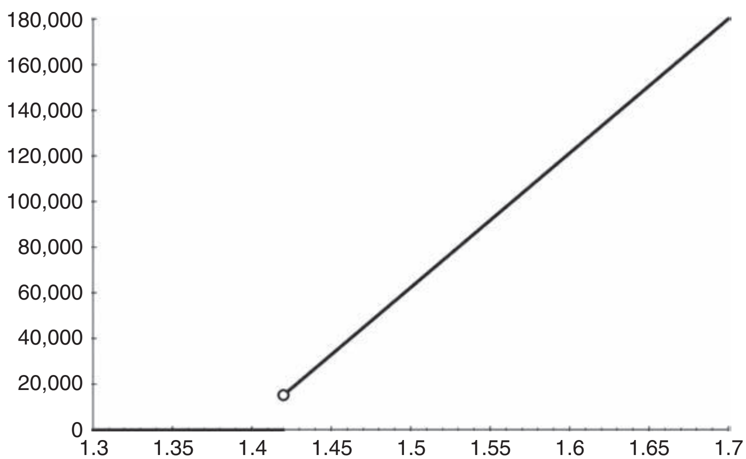


Figure 6.19 Payoff diagram of EKIKO 1A.

features, the price of each embedded option is lower than that of a vanilla European call option. This product is thus suitable for the following target clients:

1 Corporations whose domestic currency is USD but have short positions on EUR assets (e.g., expected costs dominated in EUR).
2 Individual investors and financial institutions with an optimistic view of EUR

EKIKO 1B.

Table 6.4 Example of an EKIKO 1B

Spot reference	1.3638 EUR-USD
Underlying	EUR/USD FX Rate
Type	USD CALL / EUR PUT
Style	American
Strike rate	1.3950
Out strike rate	1.3000
European in strike rate	1.4200
Call currency and amount	USD 418,500.00
Put currency and amount	EUR 300,000.00

At any time before the expiration date, the buyer can exercise the option and get an amount equal to USD $300,000 \times (1.395 - S_t)^+$. However, if the spot EUR/USD FX rate is at or below the out strike rate 1.3000, the product will be knocked out.

Note that because this product is an American-style option, in theory investors exercise the option when the FX rate approaches the knock-out rate. Because it is possible for the investor to avoid the knock-out event, a more realistic way to view the payoff structure is to treat the knock-out rate as a cap. Owing to this cap, or barrier, the price of each embedded option in EKIKO 1B is lower than an ordinary American put option. This product is thus suitable for the following target clients:

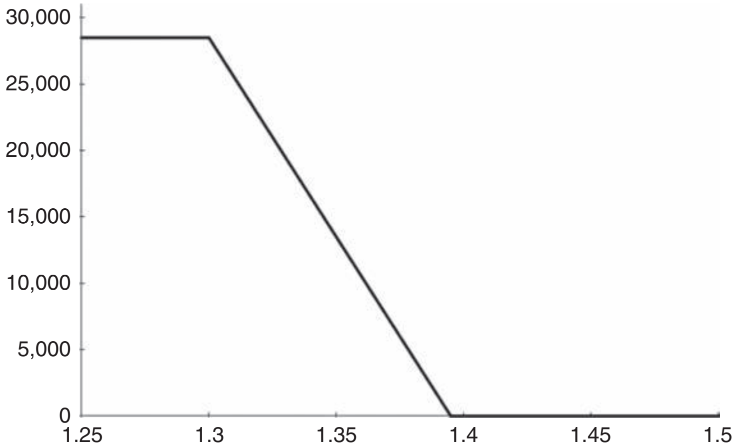


Figure 6.20 Payoff diagram of EKIKO 1B.

1 Corporations with domestic currency in USD but who own EUR assets (e.g., future EUR cash flows).
2 Individual investors and financial institutions who have the market view that EUR/USD will fall in a small range.

Real market scenario

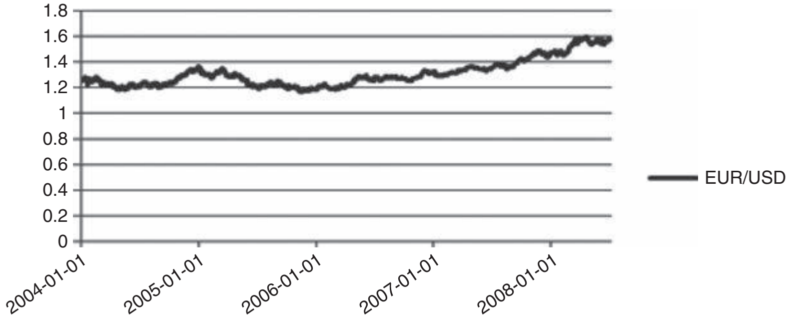


Figure 6.21 EUR/USD rate from 2004 to 2008.

As a result of the subprime crisis and the recession in the US economy, the EUR/USD rate rose by about 17% from June 2007 to June 2008. However, thanks to the demand from emerging nations and the deflation of USD, exports from America were maintained during those few years, and the trade deficit tended to decrease for a long time. The demand for USD from international trade was strong and durative support to USD. Moreover, the interest rate in the United States was expected to reach the bottom and was brewing a rebound. The decreasing gap between the interest rates in the European Union and United States pulled the EUR/USD rate down. By shorting EKIKO 1A or longing EKIKI 1B, investors can benefit from the appreciation of USD if the market forecast was accurate.

6.3.3 Pricing

6.3.3.1 The BS model

In the BS model, the dynamic process of the price of the underlying (i.e., the EUR/USD rate) follows the geometric Brownian motion (GBM) with constant drift and volatility. Because the embedded options in the product are American style, there is no closed-form solution. Thus, several numerical methods, namely, MCS, binomial tree, and finite difference are adapted for pricing of the product.

- **MC simulation:** Millions of paths of the FX rate are simulated, and the values of the options in each path are calculated. The product price is taken to be the average value of the product values in the different paths.
- **Binomial tree:** A tree of the FX rate is constructed under the assumption of GBM. The value of the option is obtained by backing up from nodes at the expiration date to the node at the current time.
- **Finite difference:** Derivatives in the partial differential equation under the BS model are replaced by numerical differences. The product's value can then be obtained by solving the PDE.

However, these methods may be unstable and lead to numerical errors when calculating the deeply out-of-the-money options. Moreover, owing to the sensitivity of pricing, there is a big difference between pricing EKIKI 1B with working days and calendar days.

6.4 Heston model

In the Heston model, because both the price (FX rate) and the volatility of the underlying are stochastic, it is hard to price the product with a closed-form solution, binomial tree, and finite difference methods. Thus, MCS is used to price the product. The procedures are the same as were previously described.

6.4.1 Dupire model and implied volatility surface

In the Dupire model, the local volatility is given by (Dupire, 1994):

$$\sigma_{loc}^2(K,T) = \frac{2\left(\frac{\partial C_{\text{market}}}{\partial T} + \left(r(T) - d(T)\right)K\frac{\partial C_{\text{market}}}{\partial K} + d(T)C_{\text{market}}\right)}{K^2\frac{\partial C^2_{\text{market}}}{\partial^2 K}}.$$

The advantage of the local volatility model is that it can capture the instantaneous volatility structure very well. Moreover, the local volatility can be calculated from the market data from the above equation. Thus, the result is model-free.

However, because only discrete times to maturity and strikes exist in the market, the partial derivatives in the equation can only be approximated. The results may be unacceptable when the data are not dense enough. Another flaw of the local volatility model is that these types of models fail to capture the dynamics of the volatility smile; the smile predicted by local volatility is opposite to the observed market behavior: when the price of the underlying decreases, the local volatility models predict that the smile shifts to higher prices; when the price increases, these models predict that the smile shifts to lower prices. Owing to these limitations, we may use the regression method described in the previous product to get the implied volatility surface instead.

6.4.2 Risk and hedging

From the foregoing *real market scenario*, USD is predicted to be falling. Thus, in this section the investor is assumed to take a short position on EKIKO 1A and a long position on EKIKO 1B. The corresponding payoff diagram on each expiration date, assuming that no early exercise or knock-out occurs during the period, is shown in Figure 6.22.

Assuming that no early exercise or knock-out event occurs, the investor will get a maximum profit of $28,500 if EUR/USD falls below 1.3000 at each expiration date, and this amount decreases along with the increase of EUR/USD above 1.3000. The investor will get a total payoff of zero if EUR/USD lies between 1.395 and 1.42. But if EUR/USD rises above 1.42, the issuer must pay an amount of $\$\,600{,}000 \times (S_t - 1.395)$, which might be illimitable.

On the other hand, if EUR/USD touches or falls below 1.3000, both EKIKO1A and EKIKO1B will be knocked out. Thus, the investor will pay nothing. But because EKIKO1B contains 12 American barrier options, for every American option the investor may exercise early and get a payoff of less than 0.095 × 300,000 = $28,500.

From the payoff structure, it is clear that potentially this product contains high risk because the downside can be unlimited. Thus, a hedging strategy may be needed if the investor cannot ensure that EUR/USD will not rise substantially.

6.4.3 Analysis of greeks—EKIKO 1A

Delta

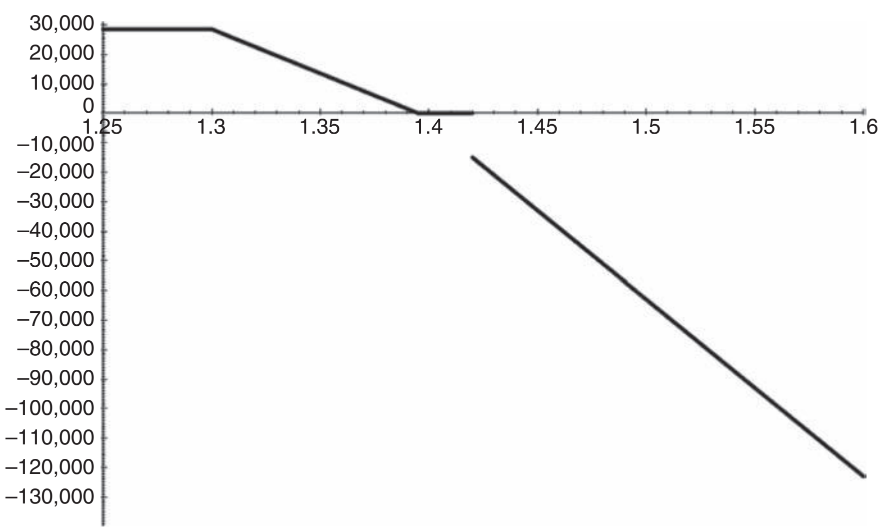


Figure 6.22 Payoff diagram EKIKO.

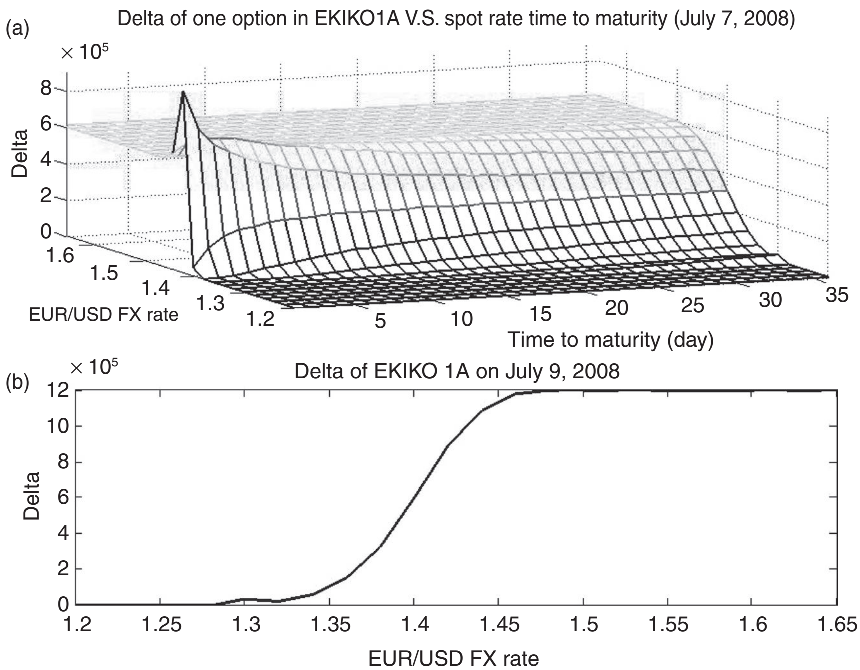


Figure 6.23 (a) General dynamic behavior of the delta of each option. (b) The delta of EKIKO 1A.

Some key characteristics of the behavior of delta are apparent from Figure 6.23a and b:

1 The delta of EKIKO 1A or the delta of each option embedded in EKIKO 1A increases with the EUR/USD FX rate. It increases from zero when the spot rate is below 1.3 (or a spot rate near 1.3) and converges finally when the option is deep in-the-money.
2 The delta of each option embedded in EKIKO 1A becomes more volatile when the expiration date is very close. Otherwise, the delta is not sensitive to the change in the duration.
3 The delta of EKIKO 1A or the delta of each option embedded in EKIKO 1A is always positive. This means that the issuer should buy the underlying asset for hedging because of its short position on EKIKO 1A.

Gamma

Some key characteristics of the behavior of gamma are apparent from Figure 6.24a and b:

1 The gamma of EKIKO 1A or the gamma of each option embedded in EKIKO 1A fluctuates sharply when the option is near-the-money, and the gamma converges to zero when the option is deep-in-the money.

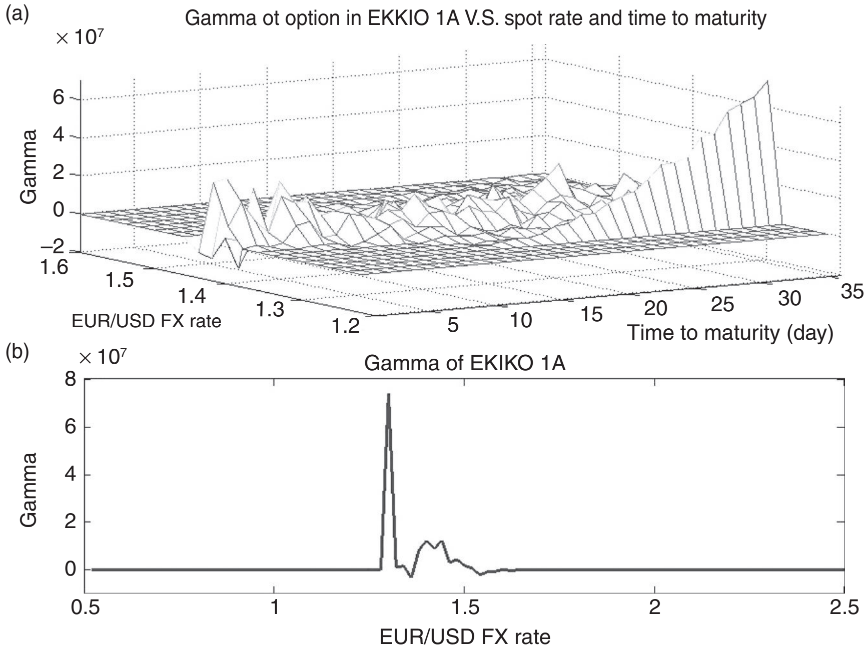


Figure 6.24 (a) The general dynamic behavior of the gamma of each option. (b) The gamma of EKIKO 1A.

2 The gamma is large when the spot rate is just a little higher than the knock-out barrier. This makes Greeks-hedging strategies more difficult.

Vega

Some key characteristics of the behavior of vega are apparent from Figure 6.25a and b:

1 The vega of EKIKO 1A or the vega of each option embedded in EKIKO 1A fluctuates sharply when the option is near-the-money, and it reaches a minimum when the spot rate is near the knock-in barrier. Then the vega converges to zero when the option is deep-in-the-money.
2 When the volatility increases, the vega fluctuates more fiercely.
3 The vega is small, between −1 and 1, which means we need not pay much attention to the vega risk for hedging.

6.4.4 Analysis of greeks—EKIKO 1B

Delta

Some key characteristics of the behavior of delta are apparent from Figure 6.26a and b:

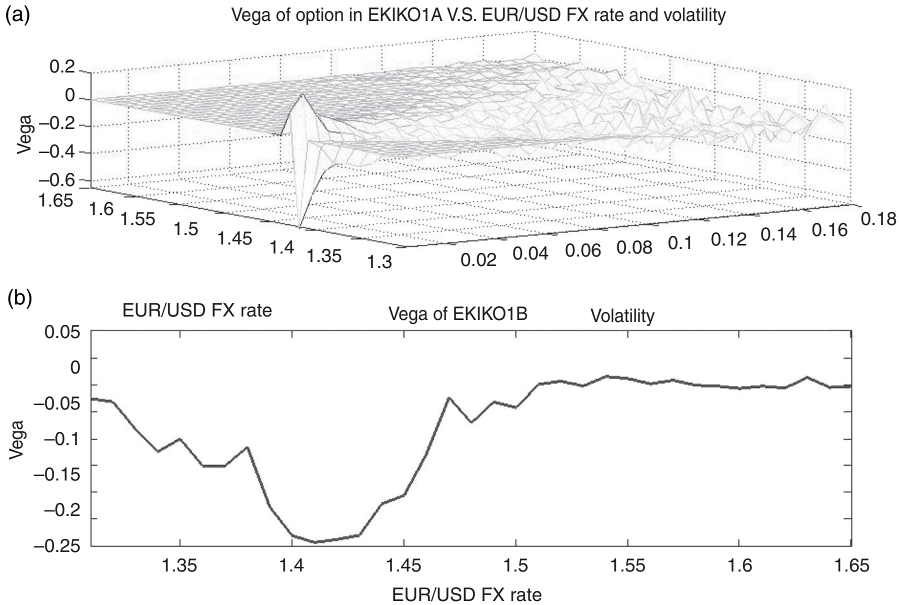


Figure 6.25 (a) The general dynamic behavior of the vega of each option. (b) The vega of EKIKO 1A.

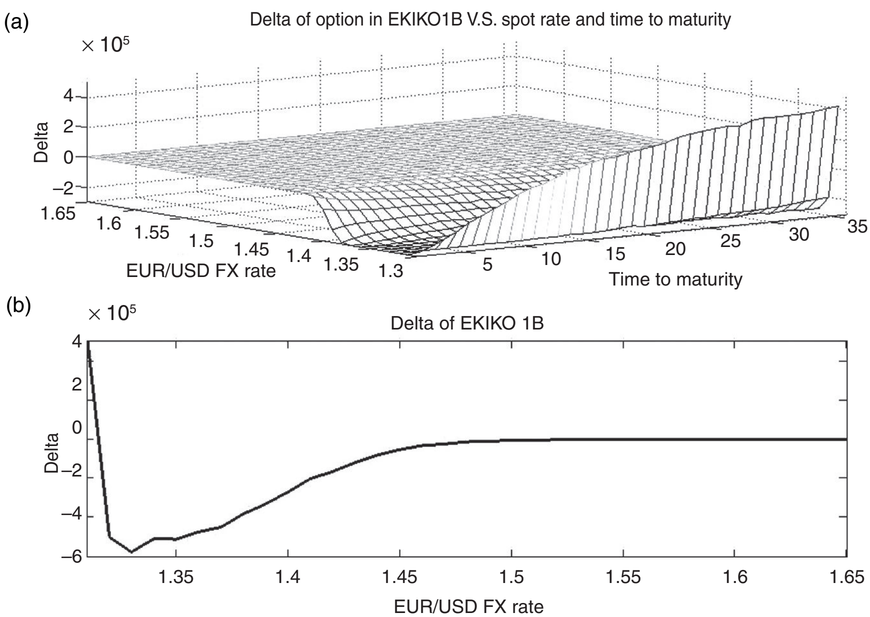


Figure 6.26 (a) The general dynamic behavior of the delta of each option. (b) The delta of EKIKO 1B.

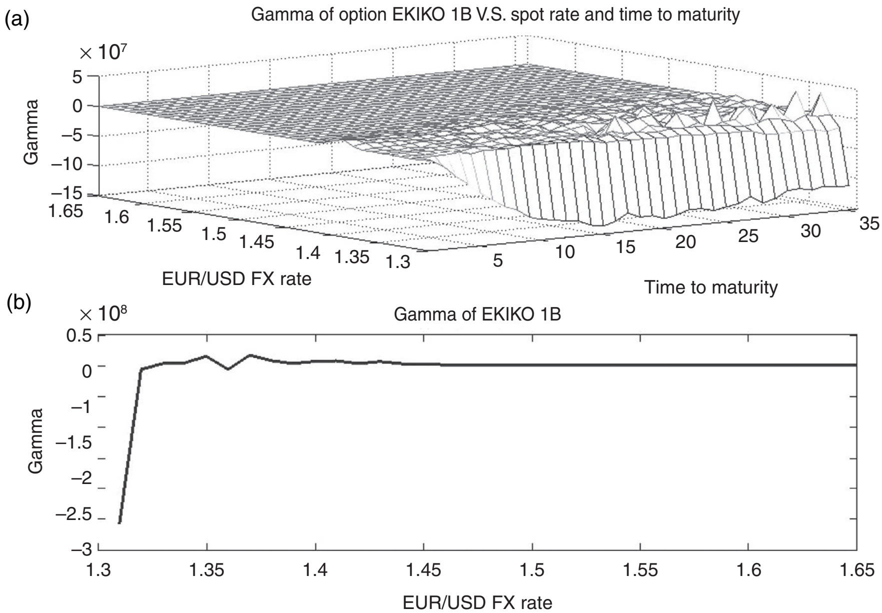


Figure 6.27 (a) The general dynamic behavior of the gamma of each option. (b) The gamma of EKIKO 1B.

1 The delta is slightly larger when the spot rate is greater than 1.30. Then the delta decreases rapidly and after reaching its minimum, it increases with the EUR/USD FX rate until it converges to zero when the option is deep in-the-money.
2 The delta of each option embedded in EKIKO 1B becomes less volatile when the expiration date is very close. Otherwise, the delta is not sensitive to the change in the duration.
3 The delta of EKIKO 1B or the delta of each option embedded in EKIKO 1B is negative except when the spot rate is only slightly higher than 1.30.

Gamma

Some key characteristics of the behavior of gamma are apparent from Figure 6.27a and b:

1 The gamma reaches its minimum when the spot rate is 1.30, and then increases rapidly when the spot rate increases from 1.3 to 1.32 (or other points near 1.32). When the option is near-the-money, the gamma just hovers near zero, and ultimately converges to zero.
2 The gamma becomes steadier when the duration is shorter. This means that we do not need to consider the gamma risk when we use Greeks-hedging strategies in the last period of the option.
3 The gamma is large at the knock-out barrier.

Vega

Some key characteristics of the behavior of vega are apparent from Figure 6.28a and b:

1 The vega fluctuates sharply when the option is near-the-money. Then the vega converges to zero when the option is deep in-the-money.
2 When the volatility increases, the vega fluctuates more sharply.
3 The vega is small and is between −1 and 1, which means we do not need to pay much attention to the vega risk for hedging.

6.5 Hedging strategy

In this section we consider hedging strategies for the package EKIKO 1 discussed in regard to the Greeks. Because the payoff for long position is always non-negative, it can be neglected in hedging. Here, two methods are introduced.

1 **Hedging with plain vanilla:** To hedge a short position on EKIKO 1A, the investor can buy a strip of vanilla options with the same characteristics as the options embedded in EKIKO 1A except the OUT-Strike and Knot-In-Strike. Of course, the strike of these vanilla options can be even greater than 1.42 if the investor can bear some risk; then the cost of hedging decreases.

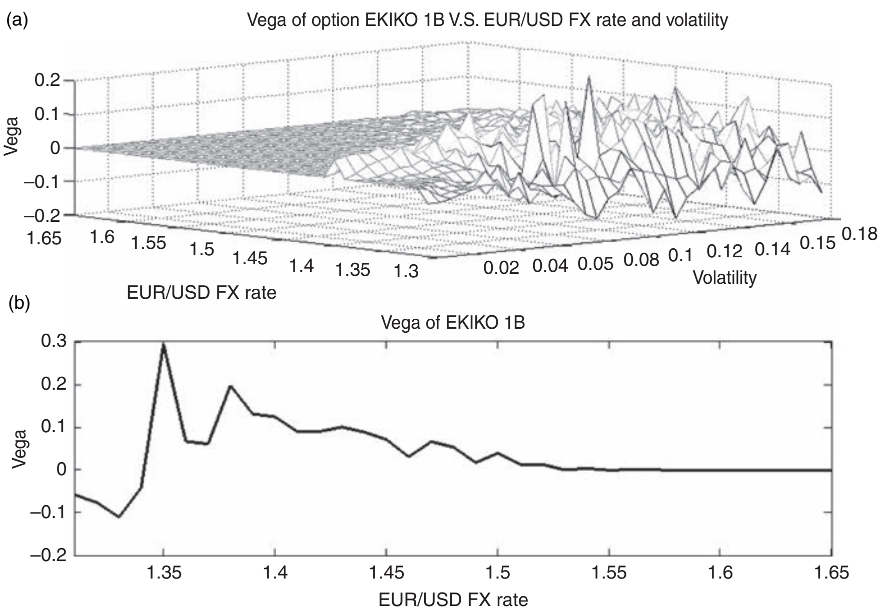


Figure 6.28 (a) The general dynamic behavior of the vega of each option. (b) The vega of EKIKO 1B.

2 **Dynamic hedging:** Considering the coherence of the trading book, most traders may employ delta and vega hedging. The most important issues are recalibrating and revising the dynamic hedging parameters on the trading floor.

6.5.1 FX ratio par forward

In this product the buyer of a long position is committed to buying one currency against another currency on a pre-specified date at a contracted forward (strike) rate, if the spot FX rate lies above the contracted forward rate. If the spot FX rate lies below the forward rate, the buyer buys the currency for a larger pre-specified amount. Because the difference is notional, this product allows the buyer to get a better forward rate than in a normal forward transaction. On each valuation date, a knock-out event occurs if the spot FX rate is above the barrier rate. The knock-out event only affects that period—i.e., there is no payment for that period—while the remaining contract continues. This product is also embedded with a bonus feature that guarantees the buyer will not suffer a downside loss in the first several valuation periods.

6.5.2 Basic analysis

Assume in an FX ratio par forward contract, a buyer is committed to buying notional USD against CNY. Then on each valuation date, there can be two scenarios:

1 Knock-out is triggered if the FX spot rate is at or above the barrier rate. There is no payment between the two parties.
2 A knock-out event does not occur if the FX spot rate is below the barrier rate. The counterparty must buy the notional amount of USD against CNY at the strike rate (sometimes, there is a leverage). Also, if the spot rate is higher than the strike rate, the counterparty's gain will be limited. However, if the

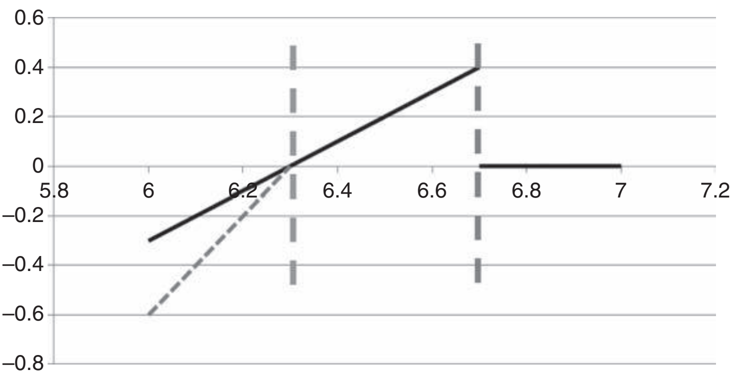


Figure 6.29 FX ratio par forward payoff with respect to the FX rate.

spot rate is below the strike rate, the counterparty will incur a loss that could potentially be unlimited.

According to our example, this product is suitable for investors who want to buy USD against CNY and take the view that the FX spot rate will be higher than the strike rate and less than the barrier rate, and are willing to accept a limited gain if the FX spot rate is less than the barrier rate.

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