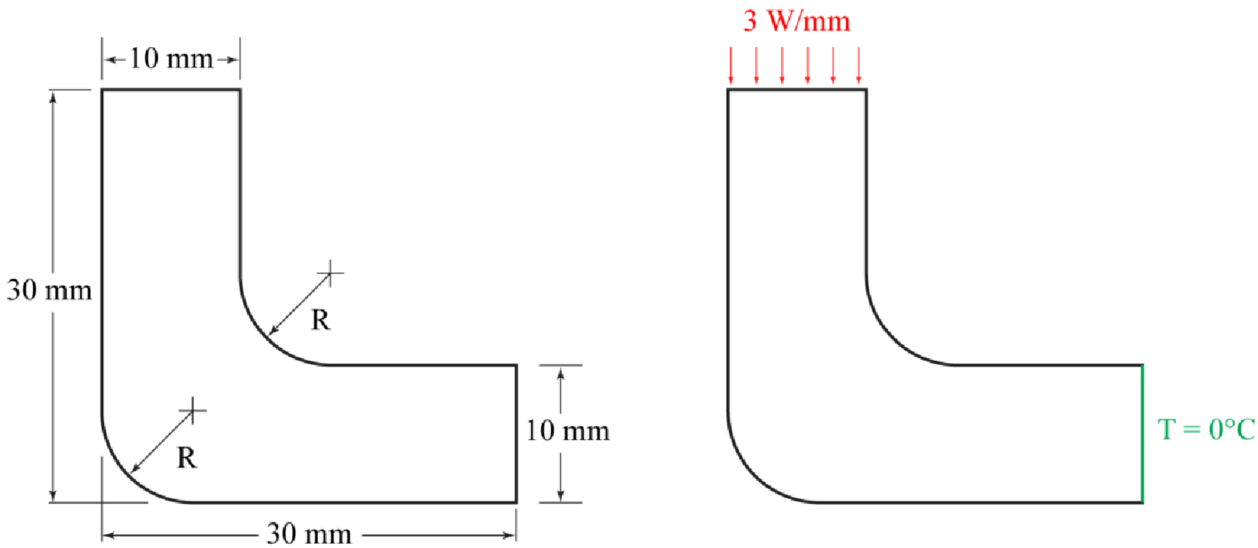
Problem Statement:

A thick copper plate of 2 mm thickness is subjected to a uniform heat flux of 3 W/m along its top surface and a prescribed temperature of 0° C at the rightmost edge. Two of the corners of the plate have a blend radius R (a design parameter). Over the surface of the plate, heat is conducted to the surrounding air with a heat transfer coefficient of 0.001 W/mm^2-K and an ambient air temperature of $T_\infty = 20^\circ C$. The thermal conductivity of copper is k = 0.40 W/mm-K.



Task 1:

Heat Equation for the given project is taken from the Law of Conversation of Energy. That is:

$$\dot{E}_{in} + \dot{E}_{gen} - \dot{E}_{out} = \dot{E}_{stored}$$

From the above details, we consider that there is no Energy storage and Energy generation within the copper plate. We also know that heat flux from the Fourier's Law is taken as:

$$\bar{q} = -K\,\Delta T$$

So we can simplify the equation by considering the conductivity of the object is constant. Using the Divergence Theorem we can say that :

$$S(x) - \frac{d}{dx}(q) = 0$$

$$\nabla(K\,\nabla T + S(x)) = 0$$

Here the boundary conditions are

1. Natural Boundary Conditions: $q = \bar{q}\ on\ \Gamma_q$

2. Essential Boundary Conditions: $T = \bar{T} on\, \Gamma_T$

The Strong from for the problem is given by: $\bar{q} = h\,(T - T\infty)$

$$\nabla(K\,\nabla T)\,\Delta_z = h(T - T_\infty)$$

$$\Rightarrow \nabla K \nabla T\, \Delta z = h\,(T - T\infty)$$

Weak Form:

Derivation of the weak form is done by multiplying the strong form with a test function "w" and then integrating the whole equation.

$$\int_\Omega w\Big(\nabla\big(-\,k\nabla T\,\Delta_z\big) - h\big(T - T_\infty\big)\Big)d\Omega = 0$$

$$\int_\Omega \nabla(wk\nabla T)\Delta_z d\Omega + \int_\Omega \nabla w\, k\, \nabla T\, \Delta_z d\Omega - \int_\Omega w\Big(h\big(T - T_\infty\big)\Big)d\Omega = 0$$

$$\int_\Omega \nabla(wk\nabla T)\Delta_z d\Omega = \int_\Gamma \nabla w\, k\, \nabla T\, \hat{n}\, \Delta_z d\Gamma + \int_\Omega w\Big(h\big(T - T_\infty\big)\Big)d\Omega$$

We know that $k\,\nabla T = \bar{q}$ on Γ_q

Now the equation becomes:

$$\int_\Omega \nabla(wk\nabla T)\Delta_z d\Omega = \int_\Gamma \nabla w\, \bar{q}\, \Delta_z d\Gamma + \int_\Omega w\Big(h\big(T - T_\infty\big)\Big)d\Omega$$

Here $\nabla(wk\nabla T)\Delta_z$ is the conduction in the domain.

$\nabla w\, \bar{q}\, \Delta_z$ is the heat flow out of the domain.

$w\Big(h\big(T - T_\infty\big)\Big)$ is the heat source term.

The final weak form obtained from the given problem is:

$$\int_\Omega \nabla(wk\nabla T)\Delta_z d\Omega + \int_\Omega whTd\Omega = \int_\Gamma \nabla w\, \bar{q}\, \Delta_z d\Gamma - \int_\Omega whT_\infty d\Omega$$

The boundary conditions are the heat flux q and the temperature T at the top surface and the rightmost end respectively.

$$w = 0\, on\, \Gamma_T \text{ and } \quad -\,K\,\nabla T\,\hat{n} = \bar{q}\, on\, \Gamma_q$$

Discretization of the weak form:

We know that

$$T = d^e N^e$$

$$w = w^e N^e$$

And also

$$\nabla T = d_I^e \nabla N_I^e$$

$$\nabla w = w_I^e \nabla N_I^e$$

The gather matrices are given by:

$$d^e = L^e d$$

$$w^e = L^e w$$

Here the elemental temperatures and the weights are obtained by using the gathering matrices d^e and w^e respectively whereas the temperature and weight gradients are given using ∇T and ∇w .

Now, by substituting the above equations in the weak form, we get the discretized equation as below:

$$\Rightarrow \int_{\Omega} \nabla w K \nabla T \, \Delta_z d\Omega + \int_{\Omega} wh\, T\, d\,\Omega = -\int_{\Gamma} w\bar{q}\Delta_z d\Gamma - \int_{\Omega} wh\, T_{\infty} d\Omega$$

$$w^e N_I^e d_I^e \nabla N_i^e \int_{\Omega_e} \left(B^e\right)^T D^e B^e \, d\Omega \, d\Omega + w^T N^T h \, d^e N^e d\Omega = w^T N^T \Delta_z [\int_{\Omega_e} \left(B^e\right)^T D^e B^e d\Omega - h d_{\infty}^e N_{\infty}^e] \, d\Omega$$

$$w^T \Delta_z \sum_e [\left(L^e\right)^T \int_{\Omega_e} \left(B^e\right)^T D^e B^e d\,\Omega\, L^e d\Omega + N^T h N^e \, d\Omega\,] = -w^T \sum_e [\left(L^e\right)^T \int_{\Omega_e} \left(N^e\right)^T \bar{q}\, d\Gamma - \int_{\Omega_e} \left(N^e\right)^T h \left(d_{\infty}^e N_{\infty}^e\right) d\Omega]$$

Here we know that the above equations get into the form $k\, d = f$.

$$K = \sum_e \left(L^e\right)^T K^e \left(L^e\right)$$

$$K^e = \int_{\Omega_e} (\left(B^e\right)^T D^e B^e + N^T h\, N^e) d\Omega$$

$$f = \sum_e \left(L^e\right)^T f^e$$

$$f^e = - \int_{\Omega} \left(N^e\right)^T q \, d\bar{\Gamma} - h(d^e_{\infty} N^e_{\infty}) d\Omega$$

$$k \, d = f$$

Convergence Testing:

As computing the exact solutions for 2D and 3D problems is difficult, we use the Method of Manufactured Solutions to test the convergence rate of the code by creating an arbitrary solution using a continuous equation. We can see that the shape of the copper plate is in a way that the origin is not in the domain. So we can take any continuous equation which is not contained within the approximation function space to create a solution for this problem. Now we are taking the temperature function with a continuous equation and then calculating the heat flux using the gradients of the temperature. These equations are used to obtain the strong form, which is later used to solve for the heat source term.

Now we can take any continuous function for Method of Manufacturing Solutions. As sinusoidal equations are smooth periodic functions, let us consider:

$T^*(x, y) = sin\, x \,.\, cos\, y$

Let's consider $r = sin\, x \,.\, cos\, y$

Now the gradients of temperature are given by,

$$\frac{\partial r}{\partial x} = cos\, x \,.\, cos\, y$$

$$\frac{\partial r}{\partial y} = sin\, x \,.\, sin\, y$$

Now the Heat Flux for the Manufactured Solution is calculated as follows:

$$\bar{q}^* = - k \, \nabla T^*$$

$$\Rightarrow \bar{q}^* = - k \, [\frac{\partial r}{\partial x} \hat{i} + \frac{\partial r}{\partial y} \hat{j}]$$

$$\Rightarrow \bar{q}^* = - k \, [(cos\, x \,.\, cos\, y) \, \hat{i} + (sin.\, x \,.\, sin\, y) \, \hat{j}\,]$$

Strong Form:

From the above equation, we need to find the divergence of the heat flux for the strong form. That is as follows:

$$\nabla \bar{q}^* - s = 0$$

$$\Rightarrow \nabla \bar{q}^{*} = s$$

$$\Rightarrow s = -2k \cos y . \sin x$$

Now we have got:

$$\nabla k \nabla T - 2k \cos y . \sin x = 0$$

Boundary Conditions:

Now we have the temperature, heat flux and hear source terms for the manufactured solution. We have to apply the boundary conditions now. The temperature is considered to be flowing out of the right most edge of the given copper plate. So it is one of the boundary conditions and then the heat flux is flowing out of the object and is computed by dot product of outward normal vector $\hat{n}$.

1. Natural Boundary Conditions: $q = q^{*} \hat{n}$ on Γ_q
2. Essential Boundary Conditions: $T = T^{*}$ on Γ_T

Now we have the outward normal vector

$$\hat{n} = k\,[(\cos x . \cos y)\,\hat{i} + (\sin . x . \sin y)\,\hat{j}]$$

And finally the heat flux of the domain is given as:

$$\bar{q} = \bar{q}^{*} . \hat{n}$$

$$\bar{q} = k\,[(\cos x . \cos y)\,\hat{i} + (\sin . x . \sin y)\,\hat{j}] . (-2k \cos y . \sin x)$$

$$\bar{q} = -2k \cos y . \sin x . \cos(x - y)$$