Theory of Linear Algebra: Math 217 Theory Companion for the Textbook[1]

Math 217 is a course about both the computational side and the theoretical side of linear algebra. People who love math or who are good at math—like most Math 217 students—usually learn and enjoy computation best when they deeply understand the theory behind it.

The theory and computation of linear algebra complement each other. After mastering some computational skill (like row reducing a matrix), it is easier to understand powerful theorems (like the rank-nullity theorem). Conversely, knowing the theory well helps us see straight to the point of what needs to be done— or to take intellectual leaps in a productive new direction— in solving difficult technical problems. This is a key reason employers are so eager to hire math majors.

Our textbook, by Otto Bretscher, is an excellent introduction to the computational side of linear algebra. Bretscher's book is so well-written that Michigan students can usually more-or-less teach themselves the computational techniques by studying the book, supplemented with practice through webwork and Part A Homework problems. And we are counting on you to do this! Of course, you should ask for help when you need it–that's what office hours are for. The ability to teach yourself technical material is a valuable skill—a skill you will hone in Math 217—and part of what makes your mathematical training so valuable.

The **theoretical side of linear algebra** is mostly not developed in Bretscher's textbook. In Math 217, students develop the theory on the worksheets, supplemented by their work on Part B Homework Problems. The key points of the theory can be found in the framed boxes on the worksheets; the exercises surrounding those boxes will walk you through examples and proofs to solidify your understanding.

In this handout, we organize the theory of linear algebra scattered throughout the worksheets in one place, with precisely stated definitions, theorems, and proofs. The presentation follows Bretscher's computational story, so Chapter and Section numbers here correspond to Chapters and Sections in the textbook. In doing proofs in Math 217, you should refer to this document (or the worksheets) for the definitions you will need.

To succeed in Math 217, you should memorize and understand the *precise* statements of definitions and theorems here. Some are *different* than in the textbook, which does not treat all concepts in the generality you will need.

This document **does not replace the textbook**. In particular, the textbook contains many important examples and computational techniques that are crucial to succeeding in

[1]Please email Professor Smith if you find typos or errors. Thanks to Jack Moody, Katie Maddon, Jordan Katz, Dominic Russell, Mark Spencer, Aarush Garg, Zhengxi Tan, Shrikar Thodlar, Miranda Riggs, Suki Dasher, Margaret Ehinger, Betty Qi, Ben Li, Eris Llangos, Jacob Block for useful remarks.

Math 217. For your convenience, we also provide a list of the vocabulary and techniques from the book you should be sure to understand, but we only list them briefly. **Please refer to the book to make sure you fully understand all the listed words and techniques.** The book's **Summaries** at the end of each section are especially useful.

1. Chapter 1

Section 1.2. A **vector,** loosely speaking, is any kind of mathematical object that can be *added* to another vector or scaled by a real number. The collection of all vectors of a particular type is called a **vector space.** Put differently, a **vector space** is *set with extra structure* whose elements will be called **vectors.** By "extra structure," we mean the vector addition and scalar multiplication, which must satisfy certain natural axioms. The precise definition[2] is a full page long:

Definition 1.1. A **vector space**[3] is a set V, equipped with a rule for **addition** of any two vectors and for **scalar multiplication** of a vector by a scalar. The addition $+$ must satisfy the following axioms

(1) The set V is closed under addition: For any two vectors v and w of V, the sum $v+w$ is also in V.
(2) Addition is commutative: For all $v, w \in V$, $v + w = w + v$.
(3) Addition is associative: For all $v, w, y \in V$, $(v + w) + y = v + (w + y)$.
(4) There is an additive identity: There exists $0_V \in V$ such that $v + 0_V = 0_V + v = v$ for all $v \in V$. [We often write 0 for 0_V, remembering that the meaning of 0 depends on the context.]
(5) Every element has an additive inverse: for every $v \in V$, there exists a vector $y \in V$ such that $v + y = y + v = 0_V$.

The scalar multiplication must satisfy the following axioms

(6) The set V is closed under scalar multiplication: For any vector v in V and any scalar $\lambda \in \mathbb{R}$, the scalar multiple λv is also in V.
(7) For two scalars $a, b \in \mathbb{R}$, we have $a(bv) = (ab)v$ for all vectors $v \in V$.
(8) For $1 \in \mathbb{R}$, we have $1v = v$ for all $v \in V$.

And finally, scalar multiplication distributes over addition:

(9) $\lambda(v + w) = \lambda\, v + \lambda\, w$ for all $v, w \in V$ and all $\lambda \in \mathbb{R}$.
(10) $(a + b)v = av + bv$ for all vectors $v \in V$ and all scalars $a, b \in \mathbb{R}$.

[2] The book mentions vector spaces in 1.2 but does not give a careful definition until Chapter 4. The book also uses the non-standard term "linear space" instead of vector space—these are completely synonymous.

[3] Technically speaking, what we define here is a **vector space over the real numbers**, meaning we always consider the scalars to be real numbers in Math 217. However, vector spaces can be defined over any "field" of scalars. Important examples include vector spaces over $\mathbb{C}$ or vector spaces over finite fields. We will only briefly hint at vector spaces over $\mathbb{C}$ at the end of Math 217, but you can learn more in Math 420. Vector spaces over finite fields are an important tool in error correcting codes; take Math 567 to learn more.

The familiar Cartesian **coordinate space** $\mathbb{R}^n$ is the main example of a vector space, and the only one the book introduces in Chapter 1. We write its elements as *column vectors* (of size n). In some previous courses, you may have written (x, y) instead of $\begin{bmatrix} x \\ y \end{bmatrix}$ for a point in $\mathbb{R}^2$. The addition and scalar multiplication are the usual (coordinate-wise) addition and scalar multiplication you know from Math 215. This is a very important example, but there are many others.

Example 1.2. More examples of vector spaces:

(1) In physics and multi-variable calculus, you learned a vector is a "directed magnitude" represented by an arrow. The set of all such vectors (say, in 3-space) forms a vector space with the usual notion of vector addition (placing arrows "head-to-tail") and scalar multiplication (scaling the magnitude). We call this vector space the "coordinate-free space" of dimension three, and denote it $\mathbb{E}^3$. The zero vector has magnitude zero (so no particular direction), represented by an "arrow" of length zero.

(2) Let $\mathcal{F}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Since high school, you have added such functions (by simply adding the outputs) and multiplied them by scalars $cf(x)$. The set $\mathcal{F}$ forms a vector space with these familiar notions of addition and scalar multiplication. The constant function $f(x) = 0$ is the zero element in $\mathcal{F}$.

(3) The set $\mathbb{R}^{m \times n}$ of all $m \times n$ matrices is a vector space with the usual notion of addition and scalar multiplication of matrices ("entry-by-entry").

(4) The set $\mathcal{P}$ of all polynomials is a vector space with the usual notions of polynomial addition and scalar multiplication. Even though it is possible to multiply polynomials, this multiplication is *not part of the vector space structure* of $\mathcal{P}$.

(5) The set of all solutions $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the equation $5x_1 + 3x_2 - x_3 = 0$ is vector space. This set can be written $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid 5x_1 + 3x_2 - x_3 = 0 \right\}$. Notice that two solutions to this equation can be added in an obvious way (componentwise, just like in $\mathbb{R}^3$). Also, we can multiply any solution by a scalar multiple to get another solution. You should run through the axioms of a vector space to convince yourself that the solutions to the equation form a vector space. Geometrically, this vector space can be represented as the points on a *plane* in $\mathbb{R}^3$. More generally, the solutions to any *homogeneous* system of linear equations—that is, one of the form $A\vec{x} = \vec{0}$— is a vector space.

Section 1.3.

Definition 1.3. A **linear combination** of vectors $v_1, \dots, v_n$ is any vector v of the form

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

where $a_1, a_2, \dots, a_n$ are scalars.

In Section 1.3, the book defines linear combinations only in the context that the v_i are (column) vectors in coordinate space $\mathbb{R}^n$. However, linear combinations are defined in any context where "addition and scalar multiplication" make sense—that is, in any *vector space.*

Example 1.4. Here are some examples of linear combinations:

(1) Every vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$ is a linear combination of the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Indeed,
$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\vec{e}_1 + b\vec{e}_2.$$
(2) More generally, every vector in $\mathbb{R}^n$ is a linear combination of the standard unit column vectors $\vec{e}_1, \dots, \vec{e}_n$.
(3) The polynomials x and x^2 are both "vectors" in the vector space $\mathcal{P}$ of polynomials in one variable x. The polynomials $5x + x^2$ and $\pi x^2 - 17x$ are two different linear combinations of the polynomials x and x^2. The polynomial $x^2 + x + 1$ is *not* a linear combination of x and x^2.
(4) Consider the functions $f(x) = sin^2(x)$ and $g(x) = cos^2(x)$. These are vectors in the vector space of all functions $\mathcal{F}$ from $\mathbb{R}$ to $\mathbb{R}$. The constant function $h(x) = 1$ is a linear combination of f and g. Indeed: for all x, we have
$$sin^2(x) + cos^2(x) = 1$$
so $h(x) = f(x) + g(x)$.

Book Concepts you must master. VOCABULARY: system of linear equations, consistent, inconsistent, matrix, augmented matrix, row vector, column vector, elementary row operation, reduced row echelon form (rref), leading one (or pivot) in rref, rank, free variables, leading variables, linear combination, matrix addition and multiplication.

IMPORTANT SKILLS: Solving linear systems of equations using the techniques described on page 15 of textbook (see example on page 14-15). Using the book theorems 1.3.1, 1.3.3 and 1.3.4 on interpreting whether or not a linear system has no solutions, exactly one solution, or infinitely many solutions. Computing the rank of a matrix and the reduced row echelon form. Being able to determine when some vector is a linear combination of some given vectors. These are not just computational techniques you can forget after Chapter 1: all the theory later will be built upon these technique.

2. Chapter 2

The main idea is the concept of linear transformation, and matrices as a tool to understand them.

Section 2.1: Linear Transformations.

Definition 2.1. Let V and W be vector spaces. A **linear transformation** is a mapping $V \xrightarrow{T} W$ that satisfies both of the following two conditions:

(1) $T(x+y) = T(x) + T(y)$ for all vectors $x, y \in V$; and
(2) $T(kx) = kT(x)$ for all vectors $x \in V$ and all scalars k.

The vector space V is called the **source** of T, whereas W is the **target** of T. Another word for source is **domain**; another word for target is **codomain**.

Example 2.2. Let A be an $n \times m$ matrix. The map

$$\mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n$$

defined by left multiplication by A

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \mapsto A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

is a linear transformation. We call this the *transformation given by multiplication by the matrix A.*

Proof of Example: To prove T_A is linear, we need to show that for all column vectors $x, \vec{y} \in \mathbb{R}^m$ and all scalars k, that

$$T_A(\vec{x} + \vec{y}) = T_A(\vec{x}) + T_A(\vec{y}) \quad \text{and} \quad T_A(k\vec{x}) = kT_A(\vec{x}),$$

that is, that T_A respects addition and scalar multiplication. This follows from the definition of T_A and basic properties of matrix multiplication:

$$T_A(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = T_A(\vec{x}) + T_A(\vec{y})$$

(the second equality comes from the distributive property of matrix multiplication) and

$$T_A(k\vec{x}) = Ak\vec{x} = kA\vec{x} = kT_A(\vec{x}).$$

(See also Theorem 1.3.10 in the textbook). $\square$

Remark 2.2.1. Somewhat misleadingly, the book gives *only* this example of a linear transformation (and calls it the "definition") in Chapter 2. However, there are many important examples of vector spaces and linear transformations throughout mathematics, science and engineering, many of which are already familiar to you. In Math 217, you must use the definition of linear transformation in this document and be able to apply it to examples beyond coordinate space.

Example 2.3. Here are a few familiar examples of linear transformations. Identify the SOURCE and TARGET, and verify each is a linear transformation.

(1) The *rotation map* of the Cartesian plane rotating vectors counterclockwise by $\frac{\pi}{2}$;
(2) The *evaluation map* $f(x) \mapsto f(0)$ from the space of all functions to $\mathbb{R}$;
(3) The *differentiation map* on the space $\mathcal{C}^\infty$ of infinitely differentiable functions.
(4) Let $\mathbb{R}^{m\times n}$ be the vector space of $m \times n$ matrices. Fix any $n \times p$ matrix A. Then the map
$$\mathbb{R}^{m\times n} \xrightarrow{R_A} \mathbb{R}^{m\times p} \quad \text{defined by } \; B \mapsto BA$$
is a linear transformation. Similarly, if C is an $p \times m$ matrix, then the map
$$\mathbb{R}^{m\times n} \xrightarrow{L_C} \mathbb{R}^{p\times n} \quad \text{defined by } \; B \mapsto CB$$
is a linear transformation. So both left and right multiplication by a fixed matrix is linear. Be sure you can prove this, using the basic properties of matrix multiplication.

IMPORTANT SPECIAL CASE: The **Cartesian coordinate space** $\mathbb{R}^n$ is the most important case of a vector space because as we will soon see, many vector spaces can be modelled by $\mathbb{R}^n$. There is a useful concrete description of all linear transformations $\mathbb{R}^m \to \mathbb{R}^n$:

Key Theorem 2.4. *Let $\mathbb{R}^m \xrightarrow{T} \mathbb{R}^n$ be a linear transformation. There exists a* **unique** *matrix A such that for all $\vec{x} \in \mathbb{R}^m$, we have*
$$T(\vec{x}) = A\vec{x}.$$
Morever, the matrix A is the $n \times m$ matrix
$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \end{bmatrix},$$
whose columns are the images of the standard unit column vectors $\vec{e}_j$ under T.

The matrix A is called the **standard matrix** *of the transformation T.*

The **Key Theorem** guarantees that *every linear transformation*
$$\mathbb{R}^m \xrightarrow{T} \mathbb{R}^n$$
of coordinate spaces is a left multiplication by some matrix A, and gives us a recipe to find A. This partially justifies why the book *defines* a linear transformation as a matrix multiplication (see Definition 2.1.1 in the book). However, since many important vector spaces *are not just*

coordinate space, the book's definition is inadequate for the more sophisticated mathematical treatment you need to understand in Math 217.

The proof of the Key Theorem uses the following **Unreasonably Useful Lemma**:

Lemma 2.5. *Let A be an $n \times m$ matrix. The j-th column of A is the matrix product*
$$A\vec{e}_j$$
where $\vec{e}_j$ is the j-th standard unit column vector.

Proof of Lemma: Exercise. □

Proof of the Key Theorem. Take $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ from the source $\mathbb{R}^m$. Write
$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_m\vec{e}_m.$$
Applying T, and using the fact that T is linear, we have
$$\begin{aligned} T(\vec{x}) =& T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_m) \\ =& x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_mT(\vec{e}_m) \\ =& \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}. \end{aligned}$$
Thus for any vector $\vec{x}$, we have $T(\vec{x}) = A\vec{x}$, where A is the matrix as defined in the theorem.

It remains to be shown that A is the unique matrix with this property. Suppose, on the contrary, that there is some other matrix B such that
$$T(\vec{x}) = B\vec{x}$$
for all vectors $\vec{x} \in \mathbb{R}^m$. Then for all vectors $\vec{x} \in \mathbb{R}^m$, we have
$$A\vec{x} = B\vec{x}.$$
In particular, taking $\vec{x}$ to be the standard unit column vector $\vec{e}_j$, we have
$$A\vec{e}_j = B\vec{e}_j$$
for each $\vec{e}_j \in \mathbb{R}^m$. By the Unreasonably Useful Lemma, we conclude that the j-th columns and A and B are the same. Since this holds for each of the n columns of the matrices A and B, we conclude that $A = B$. □

Definition 2.6. An **isomorphism** of vector spaces is a *bijective* linear transformation. Vector spaces V and W are **isomorphic** if there exists an isomorphism $V \xrightarrow{T} W$. We write $V \cong W$ to mean "V is isomorphic to W."

The **inverse** of an isomorphism $V \xrightarrow{T} W$ is the unique map $W \xrightarrow{T^{-1}} V$ assigning to each $\vec{w} \in W$ the unique vector v in V such that $T(v) = w$.

We can think of an isomorphism T as a "renaming" of elements. The inverse mapping T^{-1} "undoes the renaming."

Example 2.7. The *transpose map* sending each row vector $\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ to the corresponding column vector $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ defines an isomorphism

$$\mathbb{R}^{1\times n} \to \mathbb{R}^{n\times 1} (= \mathbb{R}^n) \qquad \alpha \mapsto \alpha^\top.$$

Of course, the space of row vectors is not really any different from the space of column vectors, since we can write any row as a column and vice-versa, and addition and scalar multiplication works the same way in either case. This intuitive idea of "essentially the same after renaming" is exactly what we mean by *isomorphism.* In symbols, $\mathbb{R}^{1\times n} \cong \mathbb{R}^{n\times 1}$.

Note that **isomorphism is not the same as equality**: the statement "$\mathbb{R}^{1\times n} = \mathbb{R}^{n\times 1}$" *false*, unless $n = 1$, as this would mean that row vectors and column vectors are literally the *same* without any renaming.

Example 2.8. Consider the map $\mathbb{C} \to \mathbb{R}^2$ sending a complex number $x + iy$ to the point $\begin{bmatrix} x \\ y \end{bmatrix}$ in the coordinate plane. This is an isomorphism, as you should check. Don't forget that verifying isomorphism is *more* than just checking that the map is a bijection. The meaning of addition and scalar multiplication must be preserved by the map.

Example 2.9. The **coordinate free space** $\mathbb{E}^2$ of "directed magnitudes" in the plane (see Example 1.2 (1)) is isomorphic to the Cartesian coordinate space $\mathbb{R}^2$. Indeed, you have probably been using this fact without formally acknowledging it since Calc III or physics. The isomorphism $\mathbb{E}^2 \to \mathbb{R}^2$ is defined as follows: given a directed magnitude (arrow) $\vec{v}$, slide it so that its tail is at the origin of the Cartesian plane. Then the coordinates of its head give the corresponding point in $\mathbb{R}^2$. Importantly, *adding the "arrows" in the usual head-to-tail fashion agrees with the usual addition of the corresponding column vectors,* and similarly for scalar multiplication. This "sameness" of the addition and scalar multiplication (ie, the vector space structure) is precisely what we mean by *isomorphism.*

We will use this isomorphism between $\mathbb{R}^2$ and $\mathbb{E}^2$ frequently in Math 217 without explicit mention, sometime abusing notation by thinking of an element $\vec{v} \in \mathbb{R}^2$ as an arrow in 2-space.[4] The same is true in any dimension, of course. That is, $\mathbb{E}^n \cong \mathbb{R}^n$ in an obvious way, and we often make this identification without explicit mention.

Example 2.10. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most two. The map $\mathcal{P}_2 \to \mathbb{R}^3$ sending the polynomial $a + bx + cx^2$ to the column vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an isomorphism. It is not the only isomorphism between $\mathcal{P}_2$ and $\mathbb{R}^3$. Can you find some others?

Proposition 2.11. *Let $V \xrightarrow{T} W$ be an isomorphism of vector spaces. The inverse map $W \xrightarrow{T^{-1}} V$ is also linear, hence also an isomorphism.*

Proof. [5] We need to check that

$$T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2) \text{ and } T^{-1}(kw_1) = kT^{-1}(w_1)$$

for all $w_1, w_2 \in W$ and all scalars k.

Suppose $T^{-1}(w_1) = v_1$ and $T^{-1}(w_2) = v_2$. By definition of T^{-1}, this means that $T(v_1) = w_1$ and $T(v_2) = w_2$. So because T is linear, $T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$, which shows that $T^{-1}(w_1 + w_2) = v_1 + v_2$ (again, by the definition of the inverse map T^{-1}). Thus T^{-1} respects addition.

Likewise, linearity of T guarantees that $T(kv_1) = kT(v_1) = kw_1$. This means that $T^{-1}(kw_1) = kv_1$. So $T^{-1}(kw_1) = kT^{-1}(w_1)$, showing that T^{-1} respects scalar multiplication as well. □

[4]You may wonder why we even bother with such formality, since each "directed magnitude" vector $\vec{v}$ can be obviously identified with a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, and you have always done so in previous courses. Professional mathematicians basically agree. In this case, where the isomorphism is natural and obvious, mathematicians say that there is a **canonical** isomorphism. Mathematicians often identity "canonically isomorphic" vector spaces as we have done here with $\mathbb{R}^2$ and $\mathbb{E}^2$. However, there can be many isomorphisms, even from a vector space to itself, that are not so obvious. See Example 2.17.1.

[5]You will notice that some proofs in this document are in smaller type. In these cases, we recommend trying to absorb the **statement** of the proposition or theorem only on your first read-through. As your understanding develops, you will eventually want to read and understand the small-type proofs as well, especially if you are interested in earning an A in Math 217.

Section 2.3: Composition of linear transformations and matrix products.

Theorem 2.12. *A composition of linear transformations $V \xrightarrow{T} W \xrightarrow{S} Q$, where V, W and Q are vector spaces, is linear.*

Proof. We need to verify that $S \circ T$ satisfies the two conditions of Definition 2.1:

(1) $(S \circ T)(x + y) = (S \circ T)(x) + (S \circ T)(y)$ for all vectors x, y in V; and
(2) $(S \circ T)(kx) = k(S \circ T)(x)$ for all vectors $x \in V$ and all scalars k.

First,
$(S \circ T)(x+y) = S(T(x+y)) = S(T(x)+T(y)) = S(T(x))+S(T(y)) = (S\circ T)(x)+(S\circ T)(y)$,
where the linearity of T justifies the second equality and the linearity of S justifies the third. So $S \circ T$ respects addition.

Similarly,

$$(S \circ T)(kx) = S(T(kx)) = S(kT(x)) = kS(Tx)) = k(S \circ T)(x).$$

So $S \circ T$ respects scalar multiplication. Thus $S \circ T$ is linear. □

Example 2.13. Recall the differentiation mapping $\mathcal{C}^\infty \xrightarrow{\frac{d}{dx}} \mathcal{C}^\infty$ sending a function f to its derivative $\frac{df}{dx}$. We have observed that differentiation is a **linear** transformation. Theorem 2.14 says that the second derivative $\frac{d^2}{dx^2}$ is linear too, since it is the composition $\frac{d}{dx} \circ \frac{d}{dx}$. Indeed, the n-th derivative $\frac{d^n}{dx^n}$ is linear for all n. Do you see why?

What about composing maps of coordinate spaces? In this case, the Key Theorem says that linear maps are given by matrix multiplication. What is the matrix of a composition?

Theorem 2.14. *Consider a composition of linear transformations $\mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n \xrightarrow{T_B} \mathbb{R}^p$, where T_A is left multiplication by A and T_B is left multiplication by B. Then the composition $T_B \circ T_A$ is left multiplication by BA. That is,*

$$T_B \circ T_A = T_{BA}.$$

Put differently, the standard matrix of a composition $T_B \circ T_A$ is the product of the corresponding standard matrices BA, in the same order.

Proof. We compute

$$(T_B \circ T_A)(\vec{x}) = T_B(A\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$$

where the third equality comes from the associative property of matrix multiplication.[6]

[6]The associative property is proved in the textbook (Theorem 2.3.6). The book has a slightly different, but equivalent, presentation of matrix multiplication. It *defines* the matrix product BA as the standard matrix of the composition $T_B \circ T_A$, whereas we have defined it as $BA = B[\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_m] = [B\vec{a}_1\ B\vec{a}_2\ \dots\ B\vec{a}_m]$, which

So the map $T_B \circ T_A$ is the same as left multiplication by the matrix BA. $\square$

Example 2.15. Consider the two linear transformations R and T, both with source and target $\mathbb{R}^2$: R is rotation by $\frac{\pi}{2}$ counterclockwise and T is reflection over the x-axis. The standard matrix of $R \circ T$ can be computed in two ways.

First, we can use the Key theorem to directly analyze where $R \circ T$ sends $\vec{e}_1$ and $\vec{e}_2$: this gives $R(T(\vec{e}_1)) = R(\vec{e}_1) = \vec{e}_2$ and $R(T(\vec{e}_2)) = R(-\vec{e}_2) = \vec{e}_1$, so the standard matrix of $R \circ T$ is

$$A_{R \circ T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Or, we can compute the standard matrices for R and T and multiply them:

$$A_R A_T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Theorem 2.14 says that these matrices are the same: $A_{R \circ T} = A_R A_T$. You should check this by computing the matrix product. Interestingly, our computation shows that reflection over the x-axis followed by rotation 90 degrees is the same as reflection over the line $y = x$. Do you see why?

Section 2.4: Invertibility of Linear Transformations and Matrices.

Definition 2.16. A matrix A is **invertible** if it is *square* (of dimension $n \times n$, say) and there exists a matrix B such that $AB = BA = I_n$, an $n \times n$ identity matrix.

The matrix B is called the **inverse** of A. Some basic facts you should make sure you can prove are:

(1) The inverse of A—if it has one— is *unique*[7];
(2) B is the inverse of A if and only if A is the inverse of B.

We write A^{-1} for the inverse of A, when it exists.

Remark 2.16.1. In the definition of invertible matrix we have been deliberately redundant by emphasizing that *invertible matrices are always square.* If we assumed A is any size and that $AB = BA = I_n$, it follows from thinking about the dimensions of A and B that both must be $n \times n$.

Building on the idea that linear transformations of coordinate spaces are modelled by matrices, we see that invertible matrices correspond to invertible linear transformations:

Theorem 2.17. *A linear transformation $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ is invertible if and only if its standard matrix A is invertible. In this case, the standard matrix of T^{-1} is A^{-1}.*

the book calls Theorem 2.3.2. These two definitions of matrix multiplication can be seen to be equivalent using the Key Theorem.

[7] Basic proof technique: assume there are two, B and B', then show $B = B'$.

Proof. By definition of standard matrix, we know that $T(\vec{x}) = A\vec{x}$ for all vectors $\vec{x} \in \mathbb{R}^n$.

First assume that $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ is invertible. Because T is invertible, its inverse T^{-1} is also linear (by Proposition 2.11). By the Key Theorem, T^{-1} is represented by a matrix—call it B. By the definition of invertible mapping, both compositions $T \circ T^{-1}$ and $T^{-1} \circ T$ are the identity map. Thus the standard matrix of both $T \circ T^{-1}$ and $T^{-1} \circ T$ is the identity matrix I_n. Using Theorem 2.14, we conclude that $AB = I_n$ and also $BA = I_n$. By definition, we conclude that A is an invertible matrix.

Conversely, suppose that A is invertible with inverse A^{-1}. The standard matrix of the composition

$$T_{A^{-1}} \circ T_A \;:\; \mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^n \xrightarrow{T_{A^{-1}}} \mathbb{R}^n$$

is $A^{-1}A = I_n$ by Theorem 2.14. This means that $T_{A^{-1}} \circ T_A = T_{I_n}$, so is the identity map. Similarly, $T_A \circ T_{A^{-1}}$ is the identity map. So T_A has inverse mapping $T_{A^{-1}}$. That is T_A is invertible.

This proof shows also that the inverse of T is given by multiplication by the inverse matrix A^{-1}, so that A^{-1} is the standard matrix of T^{-1}.

□

Remark 2.17.1. Let A be any invertible $n \times n$ matrix. Theorem 2.17 tells us that the linear transformation $\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^n$ given by left multiplication by A is an invertible linear transformation, hence an **isomorphism.** There are infinitely many different invertible $n \times n$ matrices, hence infinitely many different self-isomorphisms[8] $\mathbb{R}^n \to \mathbb{R}^n$. Such a self-isomorphism is also called a *change of coordinates.* Changes of coordinates will play an important role in the theory in Chapter 4.

Section 2.4 of the textbook has many different and **important** ways to think about invertibility of matrices, many of which build on our computational understanding of reduced row echelon form. For example, the following is essentially a re-phrasing of Theorem 2.17 in terms of systems of equations:

Theorem 2.18. *Let A be an $n \times n$ matrix. Thinking of A as the coefficient matrix of a system of n linear equations in n unknowns, the following are equivalent:*

(1) *The matrix A is invertible.*
(2) *The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_A(\vec{x}) = A\vec{x}$ is an isomorphism.*
(3) *The system $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$.*
(4) *The reduced row echelon form $rref(A)$ is the identity matrix.*
(5) *The rank of A is n.*

[8]Another word for self-isomorphism is *automorphisms*, with the prefix "auto" meaning "self."

Moreover, if there is a unique solution to $A\vec{x} = \vec{b}$ for one $\vec{b} \in \mathbb{R}^n$, then there is a unique solution for every $\vec{b} \in \mathbb{R}^n$, namely $A^{-1}\vec{b}$.

Proof. The equivalence of (1) and (2) is Theorem 2.17, but we give a different proof that (1) implies (2) in the spirit of solving systems of linear equations. Assume (1). To prove that T_A is bijective, take arbitrary $\vec{y}$ in the target $\mathbb{R}^n$. We need to show that there is a **unique** $\vec{x}$ in the source $\mathbb{R}^n$ such that $T_A(\vec{x}) = \vec{y}$. That is, we must show there is a unique solution to the system of equations $A\vec{x} = \vec{y}$. Multiplying both sides of this equation by A^{-1}, this is clear: $\vec{x} = A^{-1}\vec{y}$ is the unique solution. So (1) implies (2).

To see (2) implies (3) is easy: assuming T_A is an isomorphism, we know it is bijective. By definition, this means that for every $\vec{b}$ in the target, there is a unique $\vec{v}$ in the source such that $T_A(\vec{v}) = \vec{b}$. But such a $\vec{v}$ is precisely a solution to $A\vec{x} = \vec{b}$! That is, the bijectivity of T_A means that for all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution. So (2) implies (3). But the converse is clear as well: $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$ exactly says that T_A is bijective. So (2) and (3) are equivalent to each other, and hence to (1) by Theorem 2.17.

Finally, to see that (1), (2) and (3) are equivalent to (4) and (5), suppose that we know that $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$. Think about the process of row-reducing the augmented matrix $[A \,|\, \vec{b}]$ to find it. There is a unique solution if and only if $rref(A)$ is the identity matrix, regardless of what $\vec{b}$ is. This says (4) is equivalent to (3), and clearly also (5), by definition of rank, since the matrix is square. The proof is complete. $\square$

Theorem 2.18 tells us that a matrix A has an inverse if and only if $rref(A)$ is the identity matrix. Even more is true: the process of putting A into reduced row echelon form gives a useful process for constructing A^{-1}, as explained in the textbook, Theorem 2.4.5.

If you *already know A is square*, the next fact saves effort in checking invertibility. For the proof, we refer to the textbook, Theorem 2.4.8.

Corollary 2.19. *Let A be an $n \times n$ matrix. Then A is invertible if and only if there exists a matrix B such that $AB = I_n$* **or** *such that $BA = I_n$.*

Corollary 2.19 says that, for a *square matrix A*, we only need to check *one* of the products *AB or BA* is the identity in order to conclude A is invertible with inverse B. The other will follow automatically. If A is not square, this statement is false:

Example 2.20. It is important in Theorem 2.19 to remember the hypothesis that A is *square*. The statement *"If $AB = I_n$, then A is invertible"* is false in general. Here is a counterexample:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Book Concepts you must Master from Chapter 2. VOCABULARY: linear transformation, matrix of a linear transformation, coordinate space $\mathbb{R}^n$, standard unit vectors $\vec{e}_j$, domain (or source), target, image, injective, surjective, bijective, invertible map, invertible matrix, inverse map, inverse matrix, names for basic algebraic properties (commutative, associative, distributive, additive or multiplicative identity, additive or multiplicative inverse, etc).

IMPORTANT SKILLS: Verifying given maps are linear transformations using Definition 2.1, recognizing common linear transformation (including rotations, projections, reflections), finding the standard matrix of a linear transformation (using the Key Theorem). You must be able to use book Theorems 2.1.2 and 2.1.3.

You should be fast and accurate at multiplying matrices, and be adept at thinking of matrix multiplication in many ways (eg, in Thm 2.3.2 for AB you could write B as a row of columns $[C_1 \;\; C_2 \;\; \cdots \;\; C_n]$ and then AB is the matrix $[AC_1 \;\; AC_2 \;\; \cdots \;\; AC_n]$). You should know how to find the matrix of a composition of linear transformations. You should have an arsenal of counterexamples ready: matrices A and B that don't commute, non-zero matrices A and B such that $AB = 0$, etc.

You must know how to find the INVERSE of a given matrix. Make sure you can use the technique explained in book Theorem 2.4.5 (and demonstrated just prior, starting with Example 1 on page 90 but summarized succintly near the top of page 91). You should be able to immediately write down the inverse of a 2×2 matrix; see Theorem 2.4.9. You should be able to tell if a matrix is invertible (Thms 2.4.3, 2.4.7, 2.4.8, 2.4.9) and understand how inverse matrices come up in solving systems (Thm 2.4.4). You should also know all the equivalent characterizations of invertible matrices in Summary 3.1.8 on page 118.

3. CHAPTER 3

THE MAIN IDEA IS THE CONCEPT OF A **BASIS** FOR A VECTOR SPACE. THIS GIVES US THE IMPORTANT NOTION OF **DIMENSION** OF VECTOR SPACE.

A deep idea is that bases allow us introduce **COORDINATES** for any vector space, so we can model any (finite dimensional) vector space on the coordinate space $\mathbb{R}^n$ and any linear transformation by matrix multiplication. Coordinates are introduced in the book in Section 3.4. but we postpone this important until the next section of these notes, where we develop this theory in detail.

Section 3.1. Span, Kernel and Image.

Definition 3.1. The **span** of a set $\{v_1, \ldots, v_n\}$ of vectors is the set *all* linear combinations. That is,

$$Span\{v_1, \ldots, v_n\} = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n \mid c_i \in \mathbb{R}\}.$$

A set $\mathcal{S}$ of vectors **spans** W if $W = \text{Span } \mathcal{S}$.

Example 3.2. (1) Let $\vec{e}_1$ and $\vec{e}_2$ be the standard unit vectors in $\mathbb{R}^3$. Their span in $\mathbb{R}^3$ is the xy-plane, or in set-notation, $\{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R}\}$. Similarly, the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

also span a plane through the origin in $\mathbb{R}^3$. It is the plane $\{\begin{bmatrix} x \\ y \\ x \end{bmatrix} \mid x, y \in \mathbb{R}\}$, which can also be described as the plane in $\mathbb{R}^3$ defined by the equation $x = z$.

(2) Consider the subset $\{1, x, x^2, x^3\}$ of the vector space of all polynomials. Its span is the set of all linear combinations of $1, x, x^2, x^3$—that is, all polynomials of the form $c_1 + c_2x + c_3x^2 + c_4x^3$. So Span $\{1, x, x^2, x^3\}$ is the vector space $\mathcal{P}_3$ of all polynomials of degree at most three.

(3) The most obvious spanning set for the coordinate space $\mathbb{R}^3$ is the set of the three standard unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. However, this is definitely not the only spanning set. The space $\mathbb{R}^3$ is also spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, although this is less obvious (prove it!). It is also spanned by the four vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, although one might reasonably argue that the fourth vector is redundant.

(4) A line through the origin in $\mathbb{R}^3$ is a vector space. It is spanned by any non-zero vector in it. Do you see why?

(5) The vector space $\mathcal{P}$ of all polynomials is spanned by the polynomials $\{1, x, x^2, x^3, \dots\}$. No finite subset spans $\mathcal{P}$. Do you see why?

In all the examples above, the Span of the given set of vectors forms a vector space in its own right. Do you see why? We prove this formally in Proposition 3.12.

Remark 3.2.1. We can define the span of an infinite set $\mathcal{S}$ as the collection of all linear combinations of elements in $\mathcal{S}$. That is, Span $\mathcal{S} = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n \mid v_i \in \mathcal{S}\ c_i \in \mathbb{R}\}$.

Definition 3.3. The **kernel** of a linear transformation $V \xrightarrow{T} W$ of vector spaces is the set of all vectors v in the source such that $T(v) = 0$. That is,

$$\ker T = \{v \in V \mid T(v) = 0\}.$$

Definition 3.4. The **image** of a linear transformation $V \xrightarrow{T} W$ of vector spaces is the set of all vectors w in the target such that there exists v in the source such that $T(v) = w$. That is,

$$\operatorname{im} T = \{w \in W \mid \text{there exists some } \ v \in V \ \text{ with } T(v) = w\}.$$

The kernel and image are two important spaces associated with a linear transformation $T : V \to W$. The kernel is a subset of the *source* and the image is a subset of the *target.*

Example 3.5. Let $p : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the projection onto the xy-plane

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

Then

$$\operatorname{im} p = \{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R}\} \quad \text{and} \quad \ker p = \{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \mid z \in \mathbb{R}\}.$$

That is, the image of p is the xy-plane and the kernel of p is the z-axis.

Proposition 3.6. *A linear transformation $V \xrightarrow{T} W$ is injective if and only if its kernel is trivial.*

Proof. Suppose T is injective. Take v in the kernel of T. Then $T(v) = 0$. But also $T(0) = 0$. So v and 0 have the same image under T. By definition of injective, $v = 0$. So the kernel of T can contain only the zero element.

Conversely, suppose $\ker T$ is zero. If $T(v) = T(w)$, then because T is linear, $T(v-w) = 0$. So $v - w$ is in the kernel, making $v = w$. Thus T is injective. $\square$

An important special case is when our source and target are coordinate spaces $\mathbb{R}^n$. This is the case from which you should draw your intuition, and the only case the book discusses in Chapter 3. The next two propositions describe how to think about the kernel and image in this case.

Theorem 3.7. *Let $\mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n$ be defined by $T_A(\vec{x}) = A\vec{x}$ for some $n \times m$ matrix A. Then*

(1) *The kernel of T_A is the space of solutions to the linear system $A\vec{x} = \vec{0}$.*
(2) *The image of T_A is the span of the columns of A.*

Proof. For (1): The kernel of T_A is the set of vectors $\vec{x}$ such that $T_A\vec{x} = \vec{0}$. By definition of T_A, this is the set of all $\vec{x}$ such that $A\vec{x} = \vec{0}$, or the solutions to the linear system $A\vec{x} = \vec{0}$.

For (2): The vectors in the image of T_A are those of the form

$$T_A(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}) = T_A(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_m\vec{e}_m).$$

Using the linearity of T_A, this is the same as

$$x_1 T_A(\vec{e}_1) + x_2 T_A(\vec{e}_2) + \cdots + x_m T_A(\vec{e}_m).$$

But the vectors $T_A(\vec{e}_j) = A\vec{e}_j$ are the columns of the matrix A (by the Unreasonably Useful Lemma). So the vectors in the image of T_A are those that can be written as linear combinations of the columns of A. That is, the image of T_A is the span of the columns of A. $\square$

Remark 3.7.1. A common abuse of language is to say "kernel A" and "image A" instead of "kernel T_A" and "image T_A." We always interpret the kernel and image "of a matrix" to mean the kernel and image of the corresponding linear transformation given by (left) multiplication by this matrix.

Example 3.8. Consider the map $\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^3$ defined by $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y+z \\ y \\ x+z \end{bmatrix}$. This map can also be described as $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. According to the previous proposition, the kernel of T is the solution space of the system of linear equations

$$A\vec{x} = \vec{0},$$

which is the line spanned by $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. And the image is the span of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Of course, this image is thus the plane spanned by the two vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$, since the third column gives no additional information.

Section 3.2: Subspaces and Bases.

Definition 3.9. A **subspace** of a vector space V is a non-empty subset W which is closed under addition and scalar multiplication. That is, a **subspace** is a subset W of V such that

(1) The zero vector (of V) is in W;
(2) If $x, y \in W$, then also $x + y \in W$;
(3) If $x \in W$ and k is any scalar, then also $kx \in W$.

In Chapter 3 of the book, the only subspaces considered are subspaces of $\mathbb{R}^n$.

Example 3.10. Any line or plane *through the origin* in $\mathbb{R}^3$ is a subspace. In fact, these are the *only* subspaces of $\mathbb{R}^3$, besides the zero vector space $\{\vec{0}\}$ and the whole space $\mathbb{R}^3$.

Proof of Example 3.10. A line through the origin consists of all vectors of the form $\{t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid t \in \mathbb{R}\}$. Adding two such we have

$$t_1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + t_2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (t_1 + t_2) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

so such a line is closed under addition. Scalar multiplying we have

$$k(t_1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}) = (kt_1) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

so it is also closed under scalar multiplication.

Similarly, a plane through the origin will consist of all points satisfying the equation $ax + by + cz = 0$. If $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ are two points in this plane, then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$, so also

$$a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = 0,$$

so $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$ lies on on the plane as well. Thus the plane is closed under addition. Likewise, if $ax_1 + by_1 + cz_1 = 0$, then also $a(kx_1) + b(ky_1) + c(kz_1) = 0$. So if $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is on the plane, then any scalar multiple is also on the plane. $\square$

An important point is that **every subspace is a vector space in its own right.** Do you see why? Thus lines and planes through the origin in $\mathbb{R}^3$ are vector spaces. In your reading, when you see the words "Let V be a vector space," a plane in $\mathbb{R}^3$ is a pretty good picture to have in mind: it is concrete enough to visualize but not so special as to trivialize the discussion.

Proposition 3.11. *Let $T : V \to W$ be a linear transformation of vector spaces.*

(1) *The kernel of T is a subspace of the source V.*
(2) *The image of T is a subspace of the target W.*

Proof. (1) To show the kernel of T is a subspace of V, we must check the three conditions of the definition of subspace.

Obviously $0_V \in \ker T$, since $T(0_V) = 0_W$. It remains to show that the ker T is closed under addition and scalar multiplication.

Take v_1 and v_2 in ker T. By definition of kernel, $T(v_1) = 0$ and $T(v_2) = 0$. Because T is linear, we know $T(v_1) + T(v_2) = T(v_1 + v_2) = 0$. So $v_1 + v_2$ is in the kernel of T.

Take $v \in$ ker T. By definition, this means that $T(v) = 0$. So for any scalar, $kT(v) = k\ 0 = 0$. By linearity of T, we have $T(kv) = 0$. This shows that kv is in the kernel of T. We conclude that the kernel is a subspace.

(2) To show the image of T is a subspace of W, we must check the three conditions of the definition of subspace. Clearly $0_W \in$ im T since $0_W = T(0_V)$. It remains to show that im T is closed under addition and scalar multiplication.

Take w_1 and w_2 in the image of T. By definition, this means $T(v_1) = w_1$ and $T(v_2) = w_2$ for some $v_1, v_2 \in V$. Because T is linear, we have

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2,$$

so that also $w_1 + w_2$ is in the image of T.

Similarly, if w is in the image, we can write $T(v) = w$ for some $v \in V$. So for any scalar k,

$$kw = kT(v) = T(kv),$$

showing that kw is in the image. We conclude that im T is a subspace. □

Proposition 3.12. *Let V be a vector space and $\mathcal{S}$ any subset. The Span of $\mathcal{S}$ is a subspace of V.*

Proof. We need to show:

(1) 0 is the span of $\mathcal{S}$.
(2) If v_1 and v_2 are in the span of $\mathcal{S}$, then also $v_1 + v_2$ is in the span of $\mathcal{S}$;
(3) If v is in the span of $\mathcal{S}$ and k is any scalar, then kv is in the span of $\mathcal{S}$.

The details are left for you: use the definition of span, write out what it means that a vector in the span of $\mathcal{S}$ and check these three properties. This will complete the proof. □

Linear Independence.

Definition 3.13. A **relation** on a set of vectors $\{v_1, \dots, v_n\}$ is any expression of the form

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0,$$

where the c_i are scalars. That is, a relation is a linear combination that equals the **zero** vector.

Every set $\{v_1, v_2, \ldots, v_n\}$ has the **trivial relation:**

$$0\,v_1 + 0\,v_2 + \cdots + 0\,v_n = 0.$$

We say a relation

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

is **non-trivial** if at least one of the coefficients c_i is non-zero.

Definition 3.14. A set of vectors $\{v_1, \ldots, v_n\}$ is **linearly independent** if the only relation on it is the trivial relation. That is, $\{v_1, \ldots, v_n\}$ is **linearly independent** if whenever $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ for some scalars c_i, then $c_1 = c_2 = \cdots = c_n = 0$.

Definition 3.15. A set of vectors $\{v_1, \ldots, v_n\}$ is **linearly dependent** if it is not linearly independent. That is, $\{v_1, \ldots, v_n\}$ is **linearly dependent** if there exists a relation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ with at least one $c_i \neq 0$.

Example 3.16.

(1) The vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ in $\mathbb{R}^3$ are linearly independent. To prove this, consider a relation

$$c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = 0.$$

Expanding out, this says that $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}$, which is possible only if the c_i are all zero. So every relation on the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is trivial. By definition, the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are linearly independent.

(2) The vectors $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are linearly dependent because

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 0$$

is a non-trivial relation on them.

(3) Any set of vectors containing the zero vector is linearly dependent. Indeed, a non-trivial relation on $\{0_V, v_1, \ldots, v_n\}$ is

$$1\ 0_V + 0\ v_1 + \cdots + 0\ v_n.$$

(4) A one-element set $\{v\}$ is linearly independent, provided $v \neq 0$.

Proposition 3.17. *A two-element set $\{v_1, v_2\}$ is linearly independent if and only if neither vector is a multiple of the other.*

Proof. Suppose that $\{v_1, v_2\}$ is linearly independent. We use "proof by contradiction" to show that neither vector is a multiple of the other. Suppose on the contrary that $cv_1 = v_2$. Then we have a relation

$$cv_1 + (-1)v_2 = 0.$$

Since the relation is non-trivial ($-1 \neq 0$), this contradicts linear independence. A similar argument (reversing the roles of v_1 and v_2) shows that if $cv_2 = v_1$, we also get a contradiction.

Conversely, suppose neither vector is a multiple of the other. We need to show that $\{v_1, v_2\}$ is linearly independent. If not, then there there is a non-trivial relation

$$c_1v_1 + c_2v_2 = 0.$$

At least one of the c_i is not zero, say $c_1 \neq 0$. Then rearranging, we have $v_1 = \frac{-c_2}{c_1}v_2$, contrary to assumption. We can argue similarly if $c_2 \neq 0$. □

Remark 3.17.1. You can also define relations and linear independence for an infinite set $\mathcal{S}$. A relation on an arbitrary set $\mathcal{S}$ is some (finite) sum $c_1v_2 + \cdots + c_nv_n = 0$ where the v_i are vectors in $\mathcal{S}$ and the c_i are scalars. An arbitrary set $\mathcal{S}$ is linearly dependent if there is some non-trivial relation on $\mathcal{S}$. So an arbitrary set $\mathcal{S}$ of vectors is linearly independent if for *any* relation, $c_1v_2+\cdots+c_nv_n = 0$ where $v_i \in \mathcal{S}$ and $c_i \in \mathbb{R}$, it follows that $c_1 = c_2 = \cdots = c_n = 0$. That is, a (possibly infinite) set $\mathcal{S}$ is linearly independent it has no non-trivial relations.

Example 3.18. The infinite set $\mathcal{S} = \{1, x, x^2, x^3, x^4, \dots\}$ is a linearly independent subset of the vector space $\mathcal{P}$ of all polynomials.

The definition of **linear independence** is probably the trickiest so far. Please memorize it as stated here; doing so will help you with proofs. For vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$, the book gives many useful ways to think about whether or not they are linearly independent. Please study Summary 3.2.9 carefully on page 129.

Bases.

Definition 3.19. A **basis** for a vector space V is a set of vectors in V which is BOTH *linearly independent* and *spans* V.

Example 3.20. Some natural bases for familiar vector spaces:

(1) A basis for $\mathbb{R}^n$ is the set $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$. This is called the **standard basis.**
(2) A basis for the space $\mathbb{R}^{2\times 2}$ of 2×2 matrices is

$$\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}.$$

(3) A basis for the vector space $\mathbb{C}$ of complex numbers is $\{1, i\}$.
(4) A basis for the vector space $\mathcal{P}_4$ of polynomials of degree at most four is

$$\{1, x, x^2, x^3, x^4\}.$$

(5) A basis for the vector space $\mathcal{P}$ of *all* polynomials is $\{1, x, x^2, \dots\}$.

(6) The plane W in $\mathbb{R}^3$ defined by $x+y+z=0$ is a vector space with basis $\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\}$.

This example shows that not every vector space has an obvious or natural basis. For example, another basis for W is $\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\}$.

Theorem 3.21. *If a subset $\mathcal{B}$ is a basis for a vector space V, then every element v in V can be written uniquely as a linear combination vectors in $\mathcal{B}$.*

Proof. Take $v \in V$. Because $\mathcal{B}$ spans V, we know that v can be written as a linear combination of the elements of $\mathcal{B}$. To show this is unique, suppose we can write v as a linear combination in two ways. Write

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

where the v_i are in $\mathcal{B}$ and the a_i and c_i are scalars (some could be zero). Subtract one expression from the other to get

$$0 = (a_1 - c_1)v_1 + (a_2 - c_2)v_2 + \cdots + (a_n - c_n)v_n.$$

Because the v_i are elements of a basis, they are linearly independent. So by definition, this relation on the v_i must be trivial. This means that $a_i = c_i$ for all i. Thus the expression for v as a linear combination of the basis elements is unique. $\square$

The next two theorems are helpful for our intuition about bases:

Theorem 3.22. (1) *A set of vectors is a basis if and only if it is a minimal spanning set. ("Minimal spanning set" means that that removing any vector will produce a set that fails to span V.)*

(2) *A set of vectors is a basis if and only if it is a maximal linearly independent set. ("Maximal linearly independent set" means that adding in any vector to the set will produce a set that fails to be linearly independent.)*

We write down the proof of Theorem 3.22 for finite sets of vectors only. The proof is the same for infinite sets but the notation is somewhat more clumsy.

Proof of Theorem 3.22. (1) Let $\{v_1, \ldots, v_n\}$ be a minimal spanning set of vectors in V. This means that removing any vector from this set will produce a set that fails to span V. To show this set is a basis, we only need to show it is linearly independent, since we already know it spans.

Suppose, on the contrary, that we have a non-trivial relation

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.$$

Some coefficient, say c_i, is not zero. Rearranging, we have

$$v_i = -\frac{c_1}{c_i}\, v_1 - \frac{c_2}{c_i}\, v_2 - \cdots - \hat{i} - \cdots - \frac{c_n}{c_i}\, v_n,$$

where the notation $\hat{i}$ means the v_i term is omitted. This shows that v_i is in the span of the smaller set $\{v_1, \ldots, \hat{i}, \ldots, v_n\}$. So the smaller set $\{v_1, \ldots, \hat{i}, \ldots, v_n\}$ in fact spans all of V, contrary to the minimality of the original spanning set. This contradiction establishes that the set $\{v_1, \ldots, v_n\}$ is linearly independent. So it is a basis.

For the converse, assume $\{v_1, \ldots, v_n\}$ is a basis. To prove that it is a minimal spanning set, suppose on the contrary, that removing (say, after relabeling the vectors) v_n is also a spanning set for V. This means that $v_n = c_1 v_1 + \cdots + c_{n-1} v_{n-1}$. But then we have the relation $c_1 v_1 + \cdots + c_{n-1} v_{n-1} - v_n = 0$, contrary to the assumption that the set is a basis.

(2) Let $\{v_1, \ldots, v_n\}$ be a maximal set of linearly independent vectors in V. This means that adding any vector to this set will make it linearly *dependent.* We only need to check that the set *spans* V, since we know already the set is linearly independent.

Take an arbitrary $w \in V$. Since $\{v_1, \ldots, v_n, w\}$ is linearly dependent, there is a non-trivial relation

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n + aw = 0.$$

Note that $a \neq 0$, because otherwise we would have a non-trivial relation on the set $\{v_1, \ldots, v_n\}$, contrary to assumption. Hence, rearranging, we have

$$w = -\frac{c_1}{a}\, v_1 - \frac{c_2}{a}\, v_2 - \cdots - \frac{c_n}{a}\, v_n.$$

This says that w is in the span of $\{v_1, \ldots, v_n\}$. So the set $\{v_1, \ldots, v_n\}$ spans V and hence is a basis.

Conversely, assume $\{v_1, \ldots, v_n\}$ is a basis. Suppose it is not a maximal linearly independent set. Then we can add some vector w so that $\{v_1, \ldots, v_n, w\}$ is linearly independent. Because $\{v_1, \ldots, v_n\}$ spans V, we can write $w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ for some scalars c_i. But then $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n - w = 0$ is a non-trivial relation, contrary to the linear independence of $\{v_1, \ldots, v_n, w\}$. This contradiction establishes that $\{v_1, \ldots, v_n\}$ is a maximal linearly independent set. □

Theorem 3.23. *Every vector space has a basis.*

Proof. This is actually a hard theorem to prove in the infinite dimensional case. It is more straightforward in the finite dimensional case. We omit this proof for now. □

Section 3.3. Dimension. Vector spaces typically have many bases, but the number of elements in any basis is always the same:

Theorem 3.24. *All bases of a vector space have the same number (possibly infinite) of elements.*

Proof. Fix a vector space V. If all bases for V are infinite, the theorem holds. So assume V has a finite basis $\{v_1, \ldots, v_n\}$. We need to show every basis for V has n elements. This is proved in the book, Theorem 3.3.2. We will give a different proof in Section 3.4. □

Definition 3.25. The **dimension** of a vector space is the number of vectors in a basis.

Note that the dimension can be infinite.

Example 3.26. Referring to the bases we found in Example 3.20, we see $\mathbb{R}^n$ has dimension n, the space $\mathbb{R}^{2\times 2}$ has dimension 4, the space of complex numbers has dimension 2, the space $\mathcal{P}_4$ (of polynomials of degree four or less) has dimension 5, the space $\mathcal{P}$ (of all polynomials) has infinite dimension, and the plane W has dimension 2.

Theorem 3.27. *Let V be a vector space. The dimension of V is the maximal number of linearly independent vectors in V. Alternatively, the dimension of V is the minimal number of vectors needed to span V.*

Proof. This is really a corollary of Theorem 3.22. Since a maximal set of linearly independent elements in V is a basis (by Theorem 3.22), we know the number of elements in such a set is the dimension. Likewise, since a minimal spanning set is a basis (by Theorem 3.22), we know the number of elements in such a set is the dimension. $\square$

Corollary 3.28. *Let V be a vector space of dimension n and let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be any set of n vectors. Then $\mathcal{B}$ spans V if and only if $\mathcal{B}$ is linearly independent.*

Proof. Suppose $\mathcal{B}$ spans V. Since V is n-dimensional, the minimal number of spanning vectors is n by Theorem 3.27. So $\mathcal{B}$ is a minimal spanning set—if we remove any vectors we have a smaller set so it can't span. This means $\mathcal{B}$ is a basis by Theorem 3.22, so the elements are linearly independent.

Conversely, suppose the n vectors of $\mathcal{B}$ are linearly independent. Since V is n- dimensional, this means $\mathcal{B}$ is a maximal linearly independent set—adding any extra vector to it would produce a linearly dependent set. So $\mathcal{B}$ is a basis, and so must also span V. $\square$

Proof Tip: These theorems imply that if V has dimension n **and we have n vectors** $\{v_1, \ldots, v_n\}$, then to check they are a basis we can check **either** they are linearly independent **or** they span V. This can significantly shorten your struggle in many proofs.

The book restates these results in the following useful form:

Book Theorem 3.3.4: Let V be a vector space of dimension m. Then

(1) We can find at most m linearly independent vectors in V.
(2) We need at least m vectors to span V.
(3) Any set of m linearly independent vectors in V is a basis.
(4) Any set of m vectors which spans V is a basis.

Actually, the theorem in the book states this only in the special case that V is a subspace of $\mathbb{R}^n$. The proof is exactly the same.

The Rank-Nullity Theorem. This is many mathematicians' number one favorite linear algebra theorem, and definitely one of the most useful theorems of Math 217.

Theorem 3.29. Rank-Nullity Theorem. *Let $V \xrightarrow{T} W$ be a linear transformation, where V is finite dimensional. Then*

$$\dim V = \dim \mathit{ker}\, T + \dim \mathit{im}\, T.$$

That is: "dimension of kernel plus dimension of image = dimension of source."

The reason for the name "Rank-Nullity theorem" comes from some older terminology in linear algebra: The **nullity** of T is the dimension of the kernel. The **rank** of a linear transformation T is the dimension of the image. The next result ensures this terminology is consistent with our previous definition of rank.

Theorem 3.30. *If $\mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n$ is the transformation given by left multiplication by the matrix A, the dimension of the image of T_A is the rank of the corresponding matrix A.*

Proof. This is Theorem 3.3.6 in the book. This was also proved on Worksheet 10, Problem 4.

□

Example 3.31. Consider the linear transformation $\mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n$ given by left multiplication by the $n \times m$ matrix A. The kernel is the solution space of the linear system

$$A\vec{x} = 0.$$

As you know from chapter one, this solution space will have d free variables, where d is the total number of variables m minus the rank of A (number of leading ones in $rref(A)$). The d free variables means that the kernel is d-dimensional, where $d = m - \operatorname{rank}(A)$. So we recover

$$\dim(\mathit{source}) = \dim(\mathit{kernel}) + \dim(\mathit{image}).$$

Example 3.32. Consider the projection $\mathbb{R}^3 \to \mathbb{R}^2$ sending $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$. The kernel is the z-axis, which has basis $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; hence the kernel has dimension one. The image is all of $\mathbb{R}^2$, hence has dimension two. This confirms that $2 + 1 = 3$, the dimension of the source $\mathbb{R}^3$.

The next theorem is useful in practice to check whether column vectors are linearly independent.

Theorem 3.33. *Column vectors $\vec{v}_1, \dots, \vec{v}_d$ in $\mathbb{R}^n$ are linearly independent if and only if the $n \times d$ matrix $[\vec{v}_1 \ \vec{v}_2 \ \ \dots \vec{v}_d]$ has rank d. In particular, vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$ if the $n \times n$ matrix formed by its columns is invertible.*

Proof. This is a corollary of the rank-nullity theorem. Let A be the matrix $[\vec{v}_1 \ \vec{v}_2 \ \ \dots \vec{v}_d]$ and consider the transformation

$$\mathbb{R}^d \xrightarrow{T_A} \mathbb{R}^n \qquad \vec{x} \mapsto A\vec{x}.$$

We know that image is spanned by the columns of A, so the image has dimension d if and only if the columns $\vec{v}_1, \dots, \vec{v}_d$ are linearly independent. But the dimension of the image is the rank of A by Theorem 3.30, so the rank is d if and only if $\{\vec{v}_1, \dots, \vec{v}_d\}$ is linear independent. $\square$

Important Skills from Book Chapter 3:

- recognizing invertible matrices (Summary 3.1.8 on p118; Summary 3.3.10 on p142).
- finding a basis for the image of T_A (technique in book theorem 3.2.4).
- checking vectors are linearly independent, in particular, you should know the characterizations in Summary 3.2.9 of the book.
- finding the kernel and image of a linear transformation (Example 1 on p 136—be sure to know Theorem 3.3.5 from the book, as well as Theorem 3.3.8),
- computing the dimension of a subspace of $\mathbb{R}^n$
- using the rank-nullity theorem.

Exam 1 will cover the book Chapters 1, 2, 3 (except 3.4) and Chapter 4 (except 4.3). Sections 3.4 and 4.3 of the book will be on Exam 2.

3.4 Coordinatization.

The book's Section 3.4 (and 4.3, which is the same material in the setting of abstract vector spaces) is so important, it should really be its own chapter called Modeling Vectors by Columns and Transformations by Matrices.

This material is the heart of Math 217–the deepest and hardest and most important. You will need to reread the material many times and in different presentations.[9]

3.34. **Coordinates.** Let V be a vector space with ordered basis $\mathcal{B} = (v_1, v_2, \ldots, v_n)$. Recall that each vector v in V can be written uniquely as

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

for some scalars a_i (by Theorem 3.21).

Definition 3.35. The **$\mathcal{B}$-coordinates** of a vector v in V are the unique scalars $a_1, \ldots, a_n$ such that

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n.$$

We usually gather the coordinates into a column vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

called the **$\mathcal{B}$-coordinate column vector** of v. We also write $[v]_{\mathcal{B}}$ for this column vector.

By *ordered basis* here, we are emphasizing that we consider the elements of the basis in a particular order. For example, the ordered basis (v_1, v_2) is a *different* ordered basis than (v_2, v_1), even though the sets $\{v_1, v_2\}$ and $\{v_2, v_1\}$ are identical.

Example 3.36. (1) Let $\mathcal{E}$ be the standard ordered basis $(\vec{e}_1, \ldots, \vec{e}_n)$ for $\mathbb{R}^n$. Then the $\mathcal{E}$-coordinates of a vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ are just the standard coordinates, since

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n.$$

That is,

$$[\vec{x}]_{\mathcal{E}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

[9]The material in §3.4 will not be on Exam 1. That being said, because it is so difficult, reading this as soon as you are ready will give you more time to soak it in. If you feel confident on the main ideas in the earlier parts of this document, reading this may help you understand what is "really going on" so could help on Exam 1 as well.

So the coordinates with respect to the standard basis are simply the usual coordinates you already know.

(2) Let $V \subset \mathbb{R}^3$ be plane spanned by two vectors $\vec{v}_1$ and $\vec{v}_2$, as in Example 1 of page 147 of the textbook. Each point of the plane is a *unique* combination $a\vec{v}_1 + b\vec{v}_2$. The coordinates with respect to the ordered basis $(\vec{v}_1, \vec{v}_2)$ are thus $\begin{bmatrix} a \\ b \end{bmatrix}$. Please study this example in the book, which continues through page 148 up to the top of page 149, since I can't draw as nice picture as they have.

(3) Consider the vector space $\mathbb{C}$, with ordered basis $(1, i)$. The coordinates of $z = x + yi$ with respect to this basis are $\begin{bmatrix} x \\ y \end{bmatrix}$.

(4) Consider the vector space $\mathbb{R}^{2\times 2}$ of two-by-two matrices with ordered basis $\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$ where

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the $\mathcal{B}$-coordinates of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

If we instead use the basis $\mathfrak{C} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right)$, then the $\mathfrak{C}$-coordinates of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\begin{bmatrix} a - c + d \\ b \\ d \\ c - d \end{bmatrix}$.

A CRUCIAL IDEA IS THAT COORDINATES LET US IDENTIFY A VECTOR SPACE WITH $\mathbb{R}^n$:

Theorem 3.37. *Let V be a vector space with ordered basis $\mathcal{B} = (v_1, v_2, \ldots, v_n)$. The map*

$$V \xrightarrow{L_{\mathcal{B}}} \mathbb{R}^n$$

sending each v to its $\mathcal{B}$-coordinates

$$v \mapsto [v]_{\mathcal{B}}$$

is an isomorphism of vector spaces, called the **coordinate isomorphism** *with respect to $\mathcal{B}$.*

Remember that an *isomorphism* is a *bijective linear transformation*— a way of saying two vector spaces are "essentially the same, just with different names." In this theorem, **the target $\mathbb{R}^n$ is the space of $\mathcal{B}$-coordinates for V.** Basically, the coordinate isomorphism is a "labelling" of all the vectors in V by their coordinates (with respect to the basis $\mathcal{B}$).

Now you know why the spaces $\mathbb{R}^n$ are called **coordinate spaces!**

Example 3.38. (1) Consider the vector space $\mathbb{C}$, with ordered basis $(1, i)$. The "obvious map"

$$\mathbb{C} \to \mathbb{R}^2 \quad \text{sending} \quad x + yi \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$$

is an isomorphism of vector spaces—it is the coordinate isomorphism determined by the basis $(1, i)$.

(2) The "obvious map"

$$\mathbb{R}^{2\times 2} \to \mathbb{R}^4 \quad \text{sending} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

is the precisely the coordinate isomorphism induced by the basis $(E_{11}, E_{12}, E_{21}, E_{22})$ described in Example 3.36 above.

(3) If we had picked more exotic bases for either of the example in (1) and (2), we would get different isomorphisms. For example, the basis $\mathfrak{C}$ for $\mathbb{R}^{2\times 2}$ discussed in Example 3.36 gives the isomorphism

$$\mathbb{R}^{2\times 2} \to \mathbb{R}^4 \quad \text{sending} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a - c + d \\ b \\ d \\ c - d \end{bmatrix}.$$

Not all bases are created equal—part of the art of being a good user of linear algebra is choosing convenient bases in which to study your problem.

3.39. **An Important idea.** When we use a basis $\mathcal{B}$ to identify V with $\mathbb{R}^n$, the basis vectors $v \in \mathcal{B}$ play the role of the standard unit vectors $\vec{e}_i$ in $\mathbb{R}^n$ in a sense.

To see this, take any ordered basis $\mathcal{B} = (v_1, \ldots, v_n)$ for V. The $\mathcal{B}$ coordinate of the v_i are

$$[v_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad [v_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad [v_n]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

So the vector $x_1 v_1 + \cdots + x_n v_n$ is expressed in $\mathcal{B}$-coordinates as

$$[x_1 v_1 + \cdots + x_n v_n]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Under the coordinate isomorphism $L_{\mathcal{B}} : V \to \mathbb{R}^n$, the vector v_j corresponds to $\vec{e}_j$. So, when we identify V with $\mathbb{R}^n$ using $\mathcal{B}$-coordinates, the basis $\mathcal{B}$ gets identified with the standard basis for $\mathbb{R}^n$.

We can now easily prove that every basis has the same number of elements:

Proof of Theorem 3.24. We prove this only for finite bases. It is true in general but would require a long digression about different kinds of "infinities"—a worthy topic but better left for another course.

Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ and $\mathcal{A} = (w_1, w_2, \ldots, w_m)$ be two different ordered bases for V. We need to show $m = n$. The coordinate isomorphisms

$$V \xrightarrow{L_{\mathcal{B}}} \mathbb{R}^n \quad \text{and} \quad V \xrightarrow{L_{\mathcal{A}}} \mathbb{R}^m$$

give two different isomorphisms of V with coordinate spaces. The composition

$$\mathbb{R}^n \xrightarrow{L_{\mathcal{B}}^{-1}} V \xrightarrow{L_{\mathcal{A}}} \mathbb{R}^m$$

is an isomorphism as well. The corresponding $m \times n$ matrix of the composition, therefore, is invertible. So $m = n$. □

Remark 3.39.1. Of course, $\mathbb{R}^2$ has many different bases. So there are many different ways to "coordinatize" $\mathbb{R}^2$. The standard Cartesian coordinates (or "x-y coordinates") you've known since middle school is the coordinate system given by the standard basis. Non-standard coordinates on $\mathbb{R}^n$ can be confusing because we are so brainwashed to think in standard coordinates.

You might wonder why one would want to use a *non-standard basis* for $\mathbb{R}^n$. It turns out that for many problems, a clever choice of basis will be very helpful. This will be a major theme both in Chapters 5 (where we will choose *orthonormal bases* for subspaces of $\mathbb{R}^n$) and in Chapter 7 (where we will choose *eigenbases* for $\mathbb{R}^n$). We'll see some simple examples in this document, too, in Example 3.53 and Example 3.52.

Remark 3.39.2. Some vector spaces, like $\mathbb{R}^n$, come with a *canonical* (meaning, "natural" or "obvious") choice of basis. For $\mathbb{R}^n$, we probably all agree that the standard basis is a canonical basis. For polynomials, we have the obvious basis $\{1, x, x^2, \ldots\}$, and for matrices, the basis in Example 3.36. Of course, all these vector spaces also have non-standard bases, which depending on the problem you are trying to solve, might turn out to be more convenient.

Caution 3.39.3. Many vector spaces don't have a canonical basis, which means they don't have a "natural choice" of coordinates. Think of the plane W defined by $x + y + z = 0$ in $\mathbb{R}^3$. If we fix a basis for W, say

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right),$$

we can "coordinatize" W which allows us to think of this 2-dimensional vector space as a copy of $\mathbb{R}^2$. But an equally reasonable choice of basis is

$$\mathcal{A} = (\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}).$$

The coordinates of a vector, say $\vec{v} = \begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}$ are different in these bases! You should check that

$$\vec{v} = 3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

so

$$[\vec{v}]_{\mathcal{A}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \text{whereas} \qquad [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}.$$

So, if you are a scientist or engineer or mathematician trying to communicate with another, you need to have a systematic way to understand and compare coordinates in different bases.

Comparing different Coordinates. AN IMPORTANT ISSUE TO UNDERSTAND IS THIS: If we have two different bases for a vector space V, say $\mathcal{B}$ and $\mathcal{A}$, how do the $\mathcal{B}$-coordinates and $\mathcal{A}$-coordinates compare?

Theorem 3.40. *Let $\mathcal{B}$ and $\mathcal{A}$ be two different ordered bases for an n-dimensional vector space V. The map from the space of $\mathcal{B}$-coordinates to the space of $\mathcal{A}$-coordinates*

$$\mathbb{R}^n \to \mathbb{R}^n \qquad [v]_{\mathcal{B}} \mapsto [v]_{\mathcal{A}} \tag{3.40.0.1}$$

is a bijective linear map— that is, an isomorphism. In particular, this map is given by left multiplication by the $n \times n$ matrix

$$S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} [v_1]_{\mathcal{A}} & [v_2]_{\mathcal{A}} & \cdots & [v_n]_{\mathcal{A}} \end{bmatrix},$$

whose columns are the elements of the basis $\mathcal{B}$ expressed in $\mathcal{A}$-coordinates.

Proof of Theorem 3.40. This map is the composition of the coordinate isomorphisms from Theorem 3.37:

$$\mathbb{R}^n \xrightarrow{L_{\mathcal{B}}^{-1}} V \xrightarrow{L_{\mathcal{A}}} \mathbb{R}^n \qquad [v]_{\mathcal{B}} \mapsto v \mapsto [v]_{\mathcal{A}},$$

hence it is also an isomorphism.

Because it is a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$, the Key Theorem tells us it is given by multiplication by *some* matrix S. We can find S by finding each of its columns. We know that the j-th column of S is the image of $\vec{e}_j$ under the transformation.[10] So let's follow $\vec{e}_j$ through the composition:

$$\mathbb{R}^n \xrightarrow{L_{\mathcal{B}}^{-1}} V \xrightarrow{L_{\mathcal{A}}} \mathbb{R}^n$$

[10]These three sentences are all coming from the Key Theorem 2.4. If those lines don't make sense, you should go reread about it in Section 2.1.

$$\vec{e}_j \mapsto \vec{v}_j \mapsto [\vec{v}_j]_{\mathcal{A}}.$$

Note here that $\vec{e}_j$ maps to v_j since $[v_j]_{\mathcal{B}} = \vec{e}_j$. Thus the j-th column of the matrix S is $[v_j]_{\mathcal{A}}$, as claimed.

□

Definition 3.41. The matrix

$$S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} [v_1]_{\mathcal{A}} & [v_2]_{\mathcal{A}} & \cdots & [v_n]_{\mathcal{A}} \end{bmatrix}.$$

of Theorem 3.40 is called the **change of basis matrix from $\mathcal{B}$ to $\mathcal{A}$.** It can be characterized as the unique matrix such that

$$S_{\mathcal{B}\to\mathcal{A}}[v]_{\mathcal{B}} = [v]_{\mathcal{A}}$$

for all vectors v in V.

The matrix $S_{\mathcal{B}\to\mathcal{A}}$ transforms the column of $\mathcal{B}$-coordinates of each v into its column of $\mathcal{A}$-coordinates, so the change of basis matrix is often called the "change of coordinates matrix."

Example 3.42. Consider the vector space $\mathbb{R}^n$. Let us compare coordinates in the standard basis $\mathcal{E} = (\vec{e}_1, \ldots, \vec{e}_n)$ to coordinates in some non-standard basis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ for $\mathbb{R}^n$. What is the matrix that will change the $\mathcal{B}$-coordinates of a vector into the standard coordinates? To figure this out, we need to see where the standard unit vectors $\vec{e}_i$ are taken. We have

$$\vec{e}_i = [\vec{v}_i]_{\mathcal{B}} \mapsto [\vec{v}_i]_{\mathcal{E}}$$

which is simply the column vector $\vec{v}_i$. Hence we have verified that the change of coordinate matrix is the matrix

$$S_{\mathcal{B}\to\mathcal{E}} = [\vec{v}_1 \;\; \vec{v}_2 \;\; \ldots \;\; \vec{v}_n].$$

This is the only case the book considers in Chapter 3 (Theorem 3.4.4). The general case is in Chapter 4 (Definition 4.3.3).

Example 3.43. The vector space $\mathbb{R}^2$ has nonstandard basis $\mathcal{B} = (\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix})$. How do we convert coordinates in this basis to coordinates in the standard basis $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$—that is, to standard coordinates?

Theorem 3.40 tells us we must multiply the $\mathcal{B}$-coordinate column by

$$S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$$

to transform to standard coordinates. Let's check this for the vector $\vec{v} = \begin{bmatrix}3\\1\end{bmatrix}$. Since $\vec{v} = \vec{v}_1 + \vec{v}_2$, its $\mathcal{B}$-coordinates are $\begin{bmatrix}1\\1\end{bmatrix}$. That is, $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}1\\1\end{bmatrix}$. We convert to standard coordinates by multiplying by $S_{\mathcal{B}\to\mathcal{E}}$:

$$S_{\mathcal{B}\to\mathcal{E}}[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = [\vec{v}]_{\mathcal{E}},$$

which is, of course, $\vec{v}$ expressed in standard coordinates!

Caution: Do not confuse $S_{\mathcal{B}\to\mathcal{A}}$ with $S_{\mathcal{A}\to\mathcal{B}}$, which is the matrix transforming $\mathcal{A}$-coordinates into $\mathcal{B}$-coordinates! Of course, these transformations are inverse to each other (do you see why?) Thus their matrices are as well. We state and prove this formally in Proposition 3.45 below.

Computation Tip: It is almost always easier to find the change of basis matrix **from** a less standard basis **to** a more standard one. Go ahead and directly try to compute the matrix $S_{\mathcal{E}\to\mathcal{B}}$ in Example 3.43 by writing the vectors $\vec{e}_i$ as a linear combination of the $(\vec{v}_1, \vec{v}_2)$. You will see what I mean. By contrast, notice how simple it was to find the change of basis matrix $S_{\mathcal{B}\to\mathcal{E}}$ **to** the standard basis.

Example 3.44. Let $\mathcal{P}_1$ be the vector space of polynomials of degree 1 or less. Its elements are the functions[11] of the form $f(x) = mx + b$. It has an "obvious" basis $\mathcal{A} = (1, x)$. Another basis is $\mathcal{B} = (1, x-1)$. The change of basis matrix **from** $\mathcal{B}$ to $\mathcal{A}$ is easy to find:

$$S_{\mathcal{B}\to\mathcal{A}} = [[1]_{\mathcal{A}} \ \ [x-1]_{\mathcal{A}}] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Given $g \in \mathcal{P}_1$, we can write g as a linear combination of the elements in the basis $\mathcal{B}$, so that $g = c + d(x-1)$ for some $c, d \in \mathbb{R}$. This means that, in $\mathcal{B}$-coordinates, we can write g as $[g]_{\mathcal{B}} = \begin{bmatrix} c \\ d \end{bmatrix}$. To convert to $\mathcal{A}$-coordinates, we apply the change of basis transformation:

$$S_{\mathcal{B}\to\mathcal{A}} \ [g]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c-d \\ d \end{bmatrix} = [g]_{\mathcal{A}}.$$

So the $\mathcal{A}$-coordinates of g are $[g]_{\mathcal{A}} = \begin{bmatrix} c-d \\ d \end{bmatrix}$. This reflects the fact that we can write the polynomial g either as $c + d(x-1)$ or as $(c-d) + dx$.

To find the matrix $S_{\mathcal{A}\to\mathcal{B}}$, instead of directly computing from Definition 3.41, we can just invert $S_{\mathcal{B}\to\mathcal{A}}$. So

$$S_{\mathcal{A}\to\mathcal{B}} = [S_{\mathcal{B}\to\mathcal{A}}]^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

You should confirm that

$$S_{\mathcal{A}\to\mathcal{B}} \ [g]_{\mathcal{A}} = [g]_{\mathcal{B}}$$

for an arbitrary $g \in \mathcal{P}_1$.

[11] which are usually called "linear functions" though they are not linear transformations if $b \neq 0$.

Proposition 3.45. *With notation as in Definition 3.41, we have* $S_{\mathcal{A}\to\mathcal{B}} = [S_{\mathcal{B}\to\mathcal{A}}]^{-1}$.

Proof. For any vector $v \in V$, we have matrix multiplications

$$S_{\mathcal{A}\to\mathcal{B}}\, S_{\mathcal{B}\to\mathcal{A}}[v]_{\mathcal{B}} = S_{\mathcal{A}\to\mathcal{B}}\,[v]_{\mathcal{A}} = [v]_{\mathcal{B}},$$

by the definition of the change of basis matrix. So

$$(S_{\mathcal{A}\to\mathcal{B}}\, S_{\mathcal{B}\to\mathcal{A}})\,[v]_{\mathcal{B}} = [v]_{\mathcal{B}},$$

which means that $S_{\mathcal{A}\to\mathcal{B}}\, S_{\mathcal{B}\to\mathcal{A}}$ represents the identity map, hence must be I_n. So

$$S_{\mathcal{A}\to\mathcal{B}}\, S_{\mathcal{B}\to\mathcal{A}} = I_n.$$

Since both matrices are $n \times n$, we conclude that they are inverse to each other (Proposition 2.19). That is, $S_{\mathcal{A}\to\mathcal{B}} = S_{\mathcal{B}\to\mathcal{A}}^{-1}$. ☐

Modelling linear transformations by matrix multiplication. Suppose we have a linear transformation

$$V \xrightarrow{T} V.$$

Fix an ordered basis for V, say $\mathcal{B} = (v_1, \ldots, v_n)$. Then we can think of V as "modelled by" $\mathbb{R}^n$ by identifying each vector with its coordinate column. Does this mean we can think of the linear transformation T as "modelled by" a linear transformation

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n$$

of the $\mathcal{B}$-coordinate space? The answer is YES!

Generalized Key Theorem 3.46. *Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an ordered basis for the vector space V. Let $V \xrightarrow{T} V$ be a linear transformation. Then the corresponding map of $\mathcal{B}$-coordinate columns*

$$[v]_{\mathcal{B}} \mapsto [T(v)]_{\mathcal{B}}$$

is a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. Moreover, the matrix of this transformation is the $n\times n$ matrix

$$[T]_{\mathcal{B}} = \begin{bmatrix}[T(v_1)]_{\mathcal{B}} & [T(v_2)]_{\mathcal{B}} & \ldots & [T(v_n)]_{\mathcal{B}}\end{bmatrix},$$

whose j-th column is $T(v_j)$ expressed in the basis $\mathcal{B}$. That is,

$$[T(v)]_{\mathcal{B}} = [T]_{\mathcal{B}}[v]_{\mathcal{B}},$$

for every vector $v \in V$.

Definition 3.47. The matrix $[T]_{\mathcal{B}}$ in Theorem 3.46 is called the **matrix of T with respect to the basis** $\mathcal{B}$, or simply the **$\mathcal{B}$-matrix** of T. That is, the $\mathcal{B}$ matrix of T is the $n \times n$ matrix

$$[T]_{\mathcal{B}} = \begin{bmatrix}[T(v_1)]_{\mathcal{B}} & [T(v_2)]_{\mathcal{B}} & \ldots & [T(v_n)]_{\mathcal{B}}\end{bmatrix}.$$

We should think of this matrix as the one which tells us what T does to a vector in V by describing the effect on the $\mathcal{B}$-coordinates of v.

Example 3.48. Consider the map $T : \mathbb{C} \to \mathbb{C}$ sending $z \mapsto iz$. It is easy to check that this is linear. The standard identification of $\mathbb{C}$ with $\mathbb{R}^2$ is the coordinate isomorphism defined by the basis $(1, i)$ for $\mathbb{C}$. This map identifies $z = x + iy$ with the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. Note that $T(x + iy) = -y + ix$, so the corresponding linear map of the coordinate-spaces is

$$\mathbb{R}^2 \to \mathbb{R}^2 \quad \text{sending} \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$$

which has matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Thus the matrix of T with respect to the basis $(1, i)$ is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. You should also check that its columns are the images of the basis elements 1 and i under T, expressed again in $\mathcal{B}$-coordinates.

Proof of Theorem 3.46. The map $[v]_{\mathcal{B}} \mapsto [T(v)]_{\mathcal{B}}$ is the composition

$$\mathbb{R}^n \xrightarrow{L_{\mathcal{B}}^{-1}} V \xrightarrow{T} V \xrightarrow{L_{\mathcal{B}}} \mathbb{R}^n.$$

Since a composition of linear transformations is linear, this map is linear.

To find the matrix of the composition, we use the Key Theorem 2.4. We know the map is given by some matrix, and we just need to figure out which one. We find it by finding each column: we know the j-th column should be the image of $\vec{e}_j$ under this composition. To figure out the image of $\vec{e}_j$ for each j, follow $\vec{e}_j$ through the composition map:

$$\vec{e}_j \mapsto v_j \mapsto T(v_j) \mapsto [T(v_j)]_{\mathcal{B}},$$

where the last vector is the column of $\mathcal{B}$-coordinates for $T(v_j)$. [Here, the first arrow $\vec{e}_j \mapsto v_j$ is because $[v_j]_{\mathcal{B}} = \vec{e}_j$.] So the j-th column of $[T]_{\mathcal{B}}$ is $[T(v_j)]_{\mathcal{B}}$, as claimed. □

Theorem 3.46 says that any linear transformation $V \longrightarrow V$ can be treated like a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ simply by identifying V with the $\mathcal{B}$-coordinate space $\mathbb{R}^n$. Thus, we have a way of thinking of any linear transformation as a matrix multiplication!

Example 3.49. Let $\mathcal{P}_4$ be the vector space of polynomials of degree four or less. Consider the map $d : \mathcal{P}_4 \to \mathcal{P}_4$ sending $f \mapsto f'$. A basis for $\mathcal{P}_4$ is $\mathcal{B} = \{1, x, x^2, x^3, x^4\}$. Under d, we compute the image of each basis element, expressed in $\mathcal{B}$-coordinates, to find the matrix. For example, we compute the third column of the $\mathcal{B}$-matrix in detail as follows:

The third element in the $\mathcal{B}$-basis is x^2.
We apply the transformation d to obtain $2x$.
We then rewrite this result as a linear combination of the basis $\mathcal{B}$ to find the $\mathcal{B}$-coordinates:
$01 + 2x + 0x^2 + 0x^3 + 0x^4$.

Thus the third column of the $\mathcal{B}$-matrix is $[d(x^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Doing this for each column, we see that the matrix of d with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 3.50. Let $\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m$ be multiplication by a matrix A. What is the matrix of $[T_A]_{\mathfrak{B}}$ where $\mathfrak{B}$ is the standard basis? Using Definition 3.47, we recover precisely A, as you should check! So you have already mastered the process of finding $\mathcal{B}$-matrices in the important case where $V = \mathbb{R}^n$ and $\mathcal{B}$ is the standard basis.

One more time, we rephrase the **Key Theorem** and give another proof:

Generalized Key Theorem 3.51. Alternate Version. *Let V be a vector space with ordered basis $\mathcal{B} = (v_1, \dots, v_n)$. Let $V \xrightarrow{T} V$ be a linear transformation, and let*

$$[T]_{\mathcal{B}} = \big[[T(v_1)]_{\mathcal{B}} \quad [T(v_2)]_{\mathcal{B}} \quad \dots \quad [T(v_n)]_{\mathcal{B}}\big]$$

be the $\mathcal{B}$-matrix of T. Then for any vector $v \in V$,

$$[T(v)]_{\mathcal{B}} = [T]_{\mathcal{B}}\,[v]_{\mathcal{B}}.$$

That is, to compute $T(v)$, we can compute the $\mathcal{B}$-coordinates of the vector $T(v)$ by multiplying the $\mathcal{B}$-coordinates column vector $[v]_{\mathcal{B}}$ of v by the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of T.

Think through each bit of notation in the Generalized Key Theorem, asking yourself "what kind of thing" each symbol denotes. Which symbols represent objects in the unspecified abstract setting, and which are columns or matrices? Notice how the subscripts $\mathcal{B}$ represent concrete columns and matrices; these *model* (abstract) vectors and transformation by using the particular coordinate system $\mathcal{B}$. Read it through keeping in mind the case where $V = \mathbb{R}^n$ and $\mathcal{B} = (\vec{e}_1, \dots, \vec{e}_n)$. Do you see how it recovers the Key Theorem from Chapter 2?

Proof. Since $v_1, \dots, v_n$ is a basis for V, we can write an arbitrary $v \in V$ as

$$v = x_1 v_1 + x_2 v_2 + \dots x_n v_n.$$

So by linearity,

$$T(v) = x_1 T(v_1) + x_2 T(v_2) + \dots + x_n T(v_n).$$

So also for coordinate columns:

$$[T(v)]_{\mathcal{B}} = x_1 [T(v_1)]_{\mathcal{B}} + x_2 [T(v_2)]_{\mathcal{B}} + \dots + x_n [T(v_n)]_{\mathcal{B}}.$$

That is

$$[T(v)]_{\mathcal{B}} = [[T(v_1)]_{\mathcal{B}} \ \ [T(v_2)]_{\mathcal{B}} \ \ \cdots \ \ [T(v_n)]_{\mathcal{B}}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

In more compact notation, this says

$$[T(v)]_{\mathcal{B}} = [T]_{\mathcal{B}}\,[v]_{\mathcal{B}}$$

for every vector $v \in V$. $\square$

Example 3.52. Consider the map $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ given by projection onto the line spanned by $\vec{u}_1$. Let $\vec{u}_2$ be any vector perpendicular to $\vec{u}_1$. Note that $\mathcal{B} = (\vec{u}_1, \vec{u}_2)$ is a basis for $\mathbb{R}^2$. What is the $\mathcal{B}$-matrix of π? We compute

$$\pi(\vec{u}_1) = \vec{u}_1 = 1\vec{u}_1 + 0\vec{u}_2, \quad \pi(\vec{u}_2) = 0 = 0\vec{u}_1 + 0\vec{u}_2.$$

So the coordinates of the images of the basis elements are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\end{bmatrix}$. So the $\mathcal{B}$-matrix is

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$

As you might imagine, the linear transformation in these coordinates is easier to understand than the one we worked out in Chapter 2, Section 2.

Example 3.53. Consider the map $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ given by multiplication by $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$. How can we understand it geometrically? If we instead use the basis $\mathcal{B} = (\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix})$, we compute

$$[T]_{\mathcal{B}} = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}.$$

This means that T stretches all vectors in the $\begin{bmatrix}1\\1\end{bmatrix}$ direction by 3, while fixing all vectors in the direction of $\begin{bmatrix}-1\\1\end{bmatrix}$. Sketch the square in the source $\mathbb{R}^2$ determined by the basis $\mathcal{B}$ and its image in the target $\mathcal{B}$.

Comparing matrices of T in Different Bases. An important issue to understand is this: If I have two different bases for V, say $\mathcal{B}$ and $\mathcal{A}$, how do the $\mathcal{B}$-matrix and $\mathcal{A}$ matrix of a linear transformation $V \xrightarrow{T} V$ compare?

Theorem 3.54. *Let V be an n-dimensional vector space, and let $V \xrightarrow{T} V$ be a linear transformation. Suppose that $\mathcal{B}$ and $\mathcal{A}$ are two different bases for V. Then*

$$[T]_{\mathcal{B}} = S_{\mathcal{A}\to\mathcal{B}}\,[T]_{\mathcal{A}}\,S_{\mathcal{B}\to\mathcal{A}},$$

which we can also write as

$$[T]_{\mathcal{B}} = S^{-1}\,[T]_{\mathcal{A}}\,S,$$

where $S = S_{\mathcal{B}\to\mathcal{A}}$ is the change of basis matrix from $\mathcal{B}$ to $\mathcal{A}$ coordinates.

Think about the source and target of each matrix in the statement of Theorem 3.54. The matrix $S_{\mathcal{B}\to\mathcal{A}}$ converts $\mathcal{B}$-coordinates into $\mathcal{A}$-coordinates: its source is the $\mathcal{A}$-coordinate space and its target is the $\mathcal{B}$-coordinate space (of course, both of which are just copies of $\mathbb{R}^n$). Likewise, $S_{\mathcal{A}\to\mathcal{B}}$ takes in $\mathcal{A}$-coordinates and spits out $\mathcal{B}$-coordinates—this map is the inverse of $S_{\mathcal{B}\to\mathcal{A}}$. On the other hand, the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{A}}$ live entirely in the world of $\mathcal{B}$-coordinates and $\mathcal{A}$-coordinates, respectively: The source and target of $[T]_{\mathcal{B}}$ are both the $\mathcal{B}$-coordinate space, whereas the both the source and target of $[T]_{\mathcal{A}}$ is $\mathcal{A}$-coordinate space. The composition $S_{\mathcal{A}\to\mathcal{B}}\,[T]_{\mathcal{A}}\,S_{\mathcal{B}\to\mathcal{A}}$, should be viewed as follows:

$$\{\mathcal{B}-\text{coord space}\} \xrightarrow{S_{\mathcal{B}\to\mathcal{A}}} \{\mathcal{A}-\text{coord space}\} \xrightarrow{[T]_{\mathcal{A}}} \{\mathcal{A}-\text{coord space}\} \xrightarrow{S_{\mathcal{B}\to\mathcal{A}}} \{\mathcal{B}-\text{coord space}\}$$

although technical of course all these coordinate spaces are just $\mathbb{R}^n$:

$$\mathbb{R}^n \xrightarrow{S_{\mathcal{B}\to\mathcal{A}}} \mathbb{R}^n \xrightarrow{[T]_{\mathcal{A}}} \mathbb{R}^n \xrightarrow{S_{\mathcal{B}\to\mathcal{A}}} \mathbb{R}^n.$$

Follow an element though the composition. An element in the source is a $\mathcal{B}$-coordinate column vector, so has the form $[v]_{\mathcal{B}}$ for a unique $v \in V$. Then the composition takes

$$[v]_{\mathcal{B}} \mapsto [v]_{\mathcal{A}} \mapsto [T(v)]_{\mathcal{A}} \mapsto [T(v)]_{\mathcal{B}}.$$

This is exactly what the matrix $[T]_{\mathcal{B}}$ accomplishes! So

$$[T]_{\mathcal{B}} = S_{\mathcal{A}\to\mathcal{B}}\,[T]_{\mathcal{A}}\,S_{\mathcal{B}\to\mathcal{A}}.$$

Finally, since $S_{\mathcal{A}\to\mathcal{B}}$ and $S_{\mathcal{B}\to\mathcal{A}}$ are inverse to eachother, we see also that

$$[T]_{\mathcal{B}} = S^{-1}\,[T]_{\mathcal{A}}\,S,$$

where $S = S_{\mathcal{B}\to\mathcal{A}}$. This proves Theorem 3.54, showing it is really just common sense once you unravel all the notation.

Definition 3.55. Two $n \times n$ matrices A and B are similar if there exists an invertible $n \times n$ matrix S such that $B = S^{-1}AS$.

Proposition 3.56. *Fix a linear transformation $V \xrightarrow{T} V$. The $\mathcal{B}$-matrices of T in different bases are all similar to each other.*

Proof. If B is the matrix of T with respect to $\mathcal{B}$ and A is the matrix of T with respect to $\mathcal{A}$, then $B = S^{-1}AS$ where $S = S_{\mathcal{B}\to\mathcal{A}}$ is the change of basis matrix from $\mathcal{B}$ to $\mathcal{A}$. □

Alternate Proof of Theorem 3.54. We need to check that the two matrices $[T]_{\mathcal{B}}$ and $S^{-1}\,[T]_{\mathcal{A}}S$ are the same matrix. To do this, we can check that they have the same jth column for each $j = 1, 2, \dots n$. Using the Unreasonably Useful Lemma, we can get at the j-th column of each by multiplying by $\vec{e}_j$.

To compute $S^{-1}\,[T]_{\mathcal{A}}\,S$, recall that the j-th column of S is v_j expressed in the basis $\mathcal{A}$. So

$$(S^{-1}\,[T]_{\mathcal{A}}\,S)\vec{e}_j = (S^{-1}\,[T]_{\mathcal{A}})\,(S\vec{e}_j) = (S^{-1}\,[T]_{\mathcal{A}})\,[v_j]_{\mathcal{A}} = S^{-1}\,([T]_{\mathcal{A}}[v_j]_{\mathcal{A}}).$$

But by definition of $[T]_{\mathcal{A}}$, we have $[T]_{\mathcal{A}}[v]_{\mathcal{A}} = [T(v)]_{\mathcal{A}}$ for all vectors v, so in particular,

$$(S^{-1}\,[T]_{\mathcal{A}}\,S)\vec{e}_j = S^{-1}\,([T]_{\mathcal{A}}\,[v_j]_{\mathcal{A}}) = S^{-1}[T(v)]_{\mathcal{A}}.$$

But of course S^{-1} is the matrix which transforms $\mathcal{A}$-coordinates into $\mathcal{B}$-coordinates, so this is

$$[T(v_j)]_{\mathcal{B}}.$$

We have just shown that the j-th column of $S^{-1}\,[T]_{\mathcal{A}}\,S$ is precisely $[T(v_j)]_{\mathcal{B}}$. So we have an equality of matrices

$$S^{-1}\,[T]_{\mathcal{A}}\,S = [T]_{\mathcal{B}}.$$

□

Non-Standard Coordinates on $\mathbb{R}^n$. Of course, $\mathbb{R}^n$ has many basis. How does the $\mathcal{B}$-matrix of a fixed transformation compare to the standard matrix? The preceding discussion specializes as follows:

Theorem 3.57. *Fix a linear transformation $\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^n$ given by multiplication by the $n \times n$ matrix A. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$. Then the $\mathcal{B}$ matrix of T and the standard matrix A of T are related by*

$$[T]_{\mathcal{B}} = S^{-1}AS,$$

where S is the matrix formed from the basis $\mathcal{B}$, that is

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}.$$

This Theorem is really just a special case of Theorem 3.54. Do you see why? The book treats all of the preceding material in Chapter 3 only in the case $V = \mathbb{R}^n$ and compares $\mathcal{B}$-matrices only to standard matrices. They treat the general case in Chapter 4. I personally think it is actually *easier* to understand the general case, where we are less “brainwashed" to rely on standard coordinates. (It might be easier to memorized the special case in Theorem 3.57 above, which you should do, but *only* memorizing without understanding will not take you to the A-level in Math 217.)