Phys 242 HW1

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1 Some analytic results

Recovering the free particle Schrodinger equation

We're first asked to show

$$i\hbar\frac{\partial K(b,a)}{\partial t_b} = -\frac{\hbar^2}{2m}\frac{\partial^2 K(b,a)}{\partial {x_b}^2}$$

given that the form of K for a free particle is

$$K(b,a) = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}}\exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right)$$

This can be done by direct computation:

$$\begin{aligned} i\hbar\frac{\partial K(b,a)}{\partial t_b} &= \frac{m(i\hbar(t_a - t_b) + m(x_b - x_a)^2)}{2\hbar(t_b - t_a)^3\sqrt{\frac{-2\pi m}{h(t_a - t_b)}}} \\ &= -\frac{\hbar^2}{2m}\frac{\partial^2 K(b,a)}{\partial {x_b}^2} \end{aligned}$$

Next, we can use this to recover the free particle Schrodinger equation. Recall

$$\psi(x,t) = \int_{-\infty}^{\infty} \mathrm{dx}' K(x,t;x',t')\psi(x',t')$$

Differentiating with respect to x or t commutes with the integral over x' so that

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \int_{-\infty}^{\infty} \mathrm{dx}' i\hbar\frac{\partial}{\partial t}K(x,t;x',t')\psi(x',t')$$

$$-\frac{h}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) = \int_{-\infty}^{\infty} \mathrm{dx}' \left(-\frac{h}{2m}\frac{\partial^2}{\partial x^2}K(x,t;x',t')\right)\psi(x',t')$$

Having shown previously that

$$-\frac{h}{2m}\frac{\partial^2}{\partial x^2}K(x,t;x',t') = i\hbar\frac{\partial}{\partial t}K(x,t;x',t')$$

the above operations on ψ are equivalent and we obtain the free particle schrodinger equation.

Analytic form of the harmonic oscillator propagator

Let's first consider the integral

$$I_1 = \int_{-\infty}^{\infty} \mathrm{dx}_1 \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{2}{2}} \exp\left(\frac{im}{2\hbar\epsilon}\left((x_2-x_1)^2+(x_1-x_0)^2\right) - \frac{\omega}{2}(x_0^2+x_1^2+x_2^2)\right)$$

We can expand the exponent to

$$\frac{imx_0^2}{2\epsilon\hbar} - \frac{imx_1x_0}{\epsilon\hbar} + \frac{imx_1^2}{\epsilon\hbar} + \frac{imx_2^2}{2\epsilon\hbar} - \frac{imx_1x_2}{\epsilon\hbar} - \frac{1}{2}x_0^2\omega - \frac{x_1^2\omega}{2} - \frac{x_3^2\omega}{2}$$

Collecting terms in powers of x_1 this becomes

$$x_1^2\left(\frac{im}{\epsilon\hbar} - \frac{\omega}{2}\right) + x_1\left(-\frac{imx_0}{\epsilon\hbar} - \frac{imx_2}{\epsilon\hbar}\right) - \frac{1}{2}x_0^2\omega - \frac{x_2^2\omega}{2} + \frac{imx_0^2}{2\epsilon\hbar} + \frac{imx_2^2}{2\epsilon\hbar}$$
$$\equiv -a_1x_1^2 + b_1x_1 + c_1$$

Notice that we always have $\mathrm{Re}(a_1) < 0$. I_1 is therefore just a gaussian integral:

$$I_1 = \frac{m}{2\pi i\hbar\epsilon}\sqrt{\frac{\pi}{a}}e^{\frac{b_1^2}{4a_1}+c_1}$$

Now consider

$$I_2 = \int_{-\infty}^{\infty} \mathrm{dx}_2 \sqrt{\frac{m}{2\pi i\hbar\epsilon}} \exp\left(\frac{im}{2\hbar\epsilon}(x_3 - x_2)^2 - \frac{\omega}{2}x_3^2\right) I_1$$

Clearly I_2 can be put into the same form as I_1, and we can know without doing any calculation at all that $\mathrm{Re}(a_2) < 0$ or else I_2 (and our propagator) will diverge. Then I_2 can be computed exactly as I_1. We can repeat this reasoning for all I_N, and presumably pick out a pattern in the form of I_N and next show inductively that I_N converges to the form given in the lecture notes.

It's also obvious that this procedure is going to painfully messy, and probably beyond my algebraic abilities.

So I'm going to abandon this problem and invoke ambiguity of the problem statement (see page 7 of lecture 4) to say that we're actually meant to solve the much more tractable problem of finding the classical action.

S_{cl} for the harmonic oscillator

The classical system is described by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

which yields an Euler-Lagrange equation of motion given by

$$-m\omega^2 x_{cl} - m\ddot{x}_{cl} = 0$$

with boundary conditions $x_{cl}(t_i) = x_i$ and $x_{cl}(t_f) = x_f$. The general solution is a linear combination of sinusoids of angular frequency ω. The particular solution that satisfies our boundary conditions is

$$x_{cl} = \frac{\sin\omega(t_f - t)}{\sin\omega(t_f - t_i)}x_i + \frac{\sin\omega(t - t_i)}{\sin\omega(t_f - t_i)}x_f$$

where the sign of the $t_f - t$ and $t - t_i$ terms has been chosen so that the angular frequency of the sines remains $+\omega$.

The classical action is straightforward to compute from here:

$$S_{cl} = \int_{t_i}^{t_f} \mathrm{d}t L(x_{cl}, \dot{x}_{cl}) = \frac{1}{2} m\omega \frac{(x_i + x_f)^2 \cos\omega(t_f - t_i) - 2x_i x_f}{\sin\omega(t_f - t_i)}$$

2 Harmonic Oscillator Simulation