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HILBERT TRANSFORMS

Volume 1

Frederick W. King

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Hilbert Transforms

The Hilbert transform arises widely in a variety of applications, including problems in aerodynamics, condensed matter physics, optics, fluids, and engineering.
This work, written in an easy-to-use style, is destined to become the definitive reference on the subject. It contains a thorough discussion of all the common Hilbert transforms, mathematical techniques for evaluating them, and a detailed discussion of their application. Especially valuable features are the tabulation of analytically evaluated Hilbert transforms, and an atlas that immediately illustrates how the Hilbert transform alters a function. These will provide useful and convenient resources for researchers.

A collection of exercises is provided for the reader to test comprehension of the material in each chapter. The bibliography is an extensive collection of references to both the classical mathematical papers, and to a diverse array of applications.

Frederick W. King is a Professor in the Department of Chemistry at the University of Wisconsin-Eau Claire.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Hilbert transforms

Volume 1

FREDERICK W. KING

University of Wisconsin-Eau Claire

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To the memory of my mother

Contents

Volume II

Preface

My objective in this book is to present an elementary introduction to the theory of the Hilbert transform and a selection of applications where this transform is applied. The treatment is directed primarily at mathematically well prepared upper division undergraduates in physics and related sciences, as well as engineering, and first-year graduate students in these areas. Undergraduate students with a major in applied mathematics will find material of interest in this work.

I have attempted to make the treatment self-contained. To that end, I have collected a number of topics for review in Chapter 2. A reader with a good undergraduate mathematics background could possibly skip over much of this chapter. For others, it might serve as a highly condensed review of material used later in the text. The principal background mathematics assumed of the reader is a solid foundation in basic calculus, including introductory differential equations, a course in linear and abstract algebra, some exposure to operator theory basics, and an introductory knowledge of complex variables. Readers with a few deficiencies in these areas will find a number of recommendations for further reading at the end of Chapter 2. Some of the applications discussed require the reader to be familiar with basic electrodynamics.

A focus of the book is on problem solving rather than on proving theorems. Theorems are, for the most part, not stated or proved in the most general form possible. The end-notes will typically provide additional reference sources of more detailed discussions about the various theorems presented. I have not attempted to sketch the proof of every theorem stated, but for the key results connected to the Hilbert transform, at minimum an outline of the essential elements is usually presented. Consistent with the problem-solving emphasis is that all the different techniques that I know for evaluating Hilbert transforms are displayed in the book.

I take the opportunity to introduce special functions in a number of settings. I do this for two reasons. Special functions occur widely in problems of great importance in many areas of physics and engineering, and, accordingly, it is essential that students gain exposure to this important area of mathematics. Since many Hilbert transforms evaluate to special functions, it is imperative that the reader know when to stop doing algebraic manipulations. I have incorporated several mathematical topics for which few or no applications are known to the writer. The selection process was governed in part by the potential that I thought a particular area might have in problem solving, and I have done this with the full knowledge that crystal-ball reading is an art rather than a science!

The exercises are intended as a means for the reader to test his/her comprehension of the material in each chapter. The vast majority of the problems are by design routine applications of ideas discussed in the text. A small percentage of the problems are likely to be fairly challenging for an undergraduate reader, and a few problems could be labeled rather difficult. Most readers will have no trouble deciding when they have encountered an example of this latter group.

I have compiled an extensive table of Hilbert transforms of common mathematical functions. I hope this table will be useful in three ways. First, it serves as the answer key for a number of exercises that are placed throughout the text. Since many additional Hilbert transform pairs can be established by differentiation, or by appropriate multiplicative operations, etc., this table can be used to generate a great number of exercises, much to the delight of the reader. Second, I hope it will provide a useful reference source for those looking for the Hilbert transform of a particular function. Finally, for those searching for a particular Hilbert transform not present in the table, finding related transforms may give an idea on how to approach the evaluation, and give some clues as to whether a closed form expression in terms of standard functions is likely to be possible. In several sections the table includes a few specific cases followed by the general formula. This has been done to allow the reader to access the Hilbert transform of some of the simpler special cases as quickly as possible, rather than reducing a more complicated general formula.

The mini atlas of functions and the associated Hilbert transforms given in Appendix 2 is intended to provide a visual representation for a selection of Hilbert transform pairs. I hope this will be valuable for students in the applied sciences and engineering.

The reference list is rather extensive, but is not intended to be exhaustive. There are far too many published articles on Hilbert transforms to provide a complete set of references. I have attempted to give a generous number of references to applications. Many citations are given to the classical mathematical papers on the topics of the book, and for the serious student these works can be read with great profit. The Notes section at the end of each chapter gives a guide as to where to start reading for further information on topics discussed in the chapter. Elaborations and further details on the proofs of different theorems will often be located in the references cited in the end-notes.

My final task is to thank those who have helped. Logan Ausman, Dr. Matt Feldmann, Geir Helleloid, Dr. Kai-Erik Peiponen, Dr. Ignacio Porras, Dr. Jarkko Saarinen, and Corey Schuster read various chapters and made a number of useful suggestions to improve the presentation. Dr. Walter Reid and Dr. Jim Walker gave me some helpful comments on a preliminary draft of the first three chapters. Julia Boryskina and Hristina Ninova assisted with the translation of a number of technical papers. Several other students did translations and I offer a collective thanks to them.

Julia also provided assistance in the construction of the atlas of Hilbert transforms and with a number of the figures. Ali Elgindi did some numerical checking on the table of Hilbert transforms and Julia also did a few preliminary tests. Thanks are extended to Irene Pizzie for her efforts to improve the presentation.

The author would greatly appreciate if readers would bring to his attention any errors that escaped detection. The URL http://www.chem.uwec.edu/king/ is the web address where corrections will be posted. It is the author's intention to maintain this site actively.

Symbols

The first occurrence or a definition is indicated by a section reference or an equation number. HT is an abbreviation for the tables of Hilbert transforms given in the Appendixes (Table 1.1).

$\lvert a \rvert$	sum of the components of the multi-index a; §15.5
$\arg z$	argument of a complex number; Eq. (2.69)
A_p	the A_p condition for $1 \leq p \leq \infty$; Eq. (7.377), §7.12
$b\Omega$	boundary of a bounded domain Ω; §3.1
B	Boas transform operator; §16.4
B_n	generalization of the Boas transform operator; Eq. (16.84)
$\mathbf{B}(t)$	magnetic induction; §17.9
$\mathcal{B}$	generalization of the Boas transform operator; Eq. (16.80)
$B(a,b)$	beta function (Euler's integral of the first kind); Eq. (5.112), HT-01
$BV([a,b])$	class of functions that have bounded variation on the interval $[a,b]$; §4.25
C	designation for a contour (usually closed); §2.8.1.
C	a positive (often unspecified) constant; in derivations such a constant need not be the same at each occurrence, even though the same symbol is employed.
C	SI unit for charge, the coulomb, §19.1
$\mathbb{C}$	the set of complex numbers; §2.10
C_n	symmetry operation such that rotation by $2\pi/n$ leaves the system invariant; §21.3
C^∞	infinitely differentiable function for all points of $\mathbb{R}$; §2.15.2
C_0^∞	class of functions that are infinitely differentiable with compact support; §2.15.2
C^k	class of functions that are continuously differentiable up to order k; §2.15.2

C_0^k	class of functions that are continuously differentiable up to order k and have compact support; §2.15.2.
C_p	positive constant depending on the parameter p; often not the same at each occurrence in the sequence of steps of a proof
cas	Hartley cas function; Eq. (5.59)
$\mathscr{C}f$	Cauchy transform of the function f; Eq. (3.19)
$C(z)$	Fresnel cosine integral; Eq. (14.171), HT-01
chirp(x)	chirp function, Exercise 18.13
$\mathrm{Ci}(x)$	cosine integral; Eq. (8.78), HT-01
$\mathrm{ci}(x)$	cosine integral; HT-01
$\mathrm{cie}(\alpha, \beta)$	cosine-exponential integral; HT-01
$\mathrm{Cie}(\alpha, \beta)$	cosine-exponential integral; HT-01
$\mathrm{Cl}_2(x)$	Clausen function; HT-01
$C_n^\lambda(x)$	Gegenbauer polynomials (ultraspherical polynomials); §9.1, Eq. (11.298), HT-01

$\mathbf{D}$	electric displacement; §19.1
$\mathcal{D}$	space of all C^∞ functions with compact support; §2.15.2, §10.2
$\mathcal{D}'$	space of all distributions on $\mathcal{D}$; §10.2
$\mathcal{D}'_+$	space of distributions with support on the right of some point; §10.2
$\mathcal{D}_{L^p}$	space of test functions; §10.2
$\mathcal{D}'_{L^q}$	space of distributions; §10.2
$\mathcal{D}'_{\mathscr{L}}$	space of distributions; Eq. (17.240)
$D(x)$	Dawson's integral; Eq. (5.32)
$D_n(\theta)$	Dirichlet kernel; Eq. (6.56)
$D_n^\lambda(x)$	ultraspherical function of the second kind; Eq. (11.299)

$-e$	electronic charge
E	identity element; §2.10, Eq. (2.150)
E	energy of a signal; Eq. (18.1)
E	one-dimensional Euclidean space
E^1	one-dimensional Euclidean space; §2.11.1
E^n	n-dimensional Euclidean space; §2.11.1
E^σ	class of entire functions of exponential type; §2.8.7, §7.4
$\mathcal{E}$	space of all C^∞ functions with arbitrary support on $\mathbb{R}$; §10.2
$\mathcal{E}(t)$	envelope function; Eq. (18.76)
$\mathcal{E}'$	space of distributions having compact support; §10.2
$E_1(x)$	exponential integral; Eq. (5.98)
E_n	eigenvalues of the unperturbed Hamiltonian; §22.4, Eq. (22.57)
$E_n(z)$	exponential integral; Eq. (14.200), HT-01
$\mathbf{E}(t)$	electric field; §17.9
$\mathbf{E_i}(\omega)$	incident electric field; Eq. (20.3)
$\mathbf{E_r}(\omega)$	reflected electric field; Eq. (20.3)

$\mathbf{E_L}$	left circularly polarized electric wave; Eq. (21.16)
$\mathbf{E_R}$	right circularly polarized electric wave; Eq. (21.17)
$\mathrm{Ei}(x)$	exponential integral function; Eq. (5.101), HT-01
$\mathrm{erf}(z)$	error function; Eq. (5.27), HT-01
$\mathrm{erfc}(z)$	complementary error function; Eq. (5.141)
$\mathbf{E}_\upsilon(z)$	Weber's function; HT-01
$\mathbf{F}$	Lorentz force; Eq. (21.50).
$f()$ or f	function (at no particular specified point); §1.2
$f(x)$	function f evaluated at the point x; §1.2
f_j	oscillator strength; §19.2
$f[n]$	element of a discrete sequence; §13.2, §13.6
$\{f[n]\}$	discrete sequence; §13.6
$f_{\mathrm{e}}(x)$	even function; Eq. (4.8)
$f_{\mathrm{o}}(x)$	odd function; Eq. (4.9)
$f_{\downarrow}(c)$	limit approaching c from $c+0$; Eq. (2.22)
$f_{\uparrow}(c)$	limit approaching c from $c-0$; Eq. (2.23)
$\mathcal{F}f$	fourier transform of the function f; §2.6, Eq. (2.46)
$\mathcal{F}_n f$	n-dimensional Fourier transform of the function f; §15.6
$\mathcal{F}^{-1}f$	inverse Fourier transform of the function f; §2.6, Eq. (2.47)
$\mathcal{F}_{\mathrm{c}}f$	Fourier cosine transform of the function f; Eq. (5.41)
$\mathcal{F}_{\mathrm{s}}f$	Fourier sine transform of the function f; Eq. (5.40)
$\mathcal{F}_N$	N-point DFT operator; §13.4
$\mathcal{F}_Q$	fractional Fourier transform; §18.10, Eq. (18.147)
$\mathfrak{F}_\alpha$	discrete fractional Fourier transform; §18.13, Eq. (18.240)
$\hat{f}$	Fourier transform of the function f; §2.6
$\tilde{f}$	conjugate series of f; Eq. (6.118); alternative notation for $\mathcal{H}f$; §6.1
f'	derivative of the function f
$f^+(z)$	function f evaluated at an interior point to a contour; Eq. (3.152)
$f^-(z)$	function f evaluated at an exterior point to a contour; Eq. (3.153)
$F_n(\theta)$	Fejér kernel; Eq. (6.63)
${}_1F_1(\alpha;\beta;x)$	Kummer's confluent hypergeometric function; Eq. (5.30), HT-01
${}_2F_1(a,b;c;z)$	hypergeometric function (or Gauss' hypergeometric function); HT-01
$\boldsymbol{F}_{\mathrm{ext}}$	external force; §17.9
$\boldsymbol{F}_{\mathrm{rad}}$	radiative reaction force; §17.9
$f(\omega,\,0)$	scattering amplitude at $\theta=0$; Eq. (19.309)
$F_{\mathbf{h}}$	scattering factor; §23.4
$\mathrm{floor}[x]$	the greatest integer $\leq x$
$G(a,x)$	Hilbert transform of the Gaussian function; §4.7. The abbreviation $G(1,x)\equiv G(x)$ is employed; §9.3
$G^{(n)}_{kl_1\cdots l_2}(t_1,t_2,\ldots,t_n)$	tensor components of the nth-order response function; §22.1

$\hbar$	Planck's constant divided by 2π
H	Hilbert transform operator on $\mathbb{R}$; Eqs. (1.2) and (1.4)
$\mathcal{H}$	Hilbert transform operator for the disc; Eq. (3.202)
$\mathbf{H}$	magnetic field; Eq. (19.7)
$\mathcal{H}_\tau$	Hilbert transform operator for period 2τ; Eq. (3.286)
Hf	Hilbert transform of the function f; §1.2
$(Hf)(x)$	Hilbert transform of the function f on the real line evaluated at the point x; Eq. (1.2)
$H_\mathrm{e} f$	Hilbert transform of the even function f on $\mathbb{R}^+$; Eq. (4.11)
$H_\mathrm{o} f$	Hilbert transform of the odd function f on $\mathbb{R}^+$; Eq. (4.12)
$H_1 f$	one-sided Hilbert transform of the function f; Eq. (8.18)
$H_\mathrm{I} f$	Hilbert's integral of the function f; Eq. (7.33)
$H_n f$	n-dimensional Hilbert transform of the function f; Eq. (15.26)
$\mathscr{H}_n f$	general n-dimensional Hilbert transform of the function f in E^n; Eq. (15.2)
$\mathscr{H}_{n,\varepsilon} f$	general n-dimensional truncated Hilbert transform of the function f in E^n; Eq. (15.7)
$H_{(k)} f$	Hilbert transform of the function $f(x_1, x_2, \ldots, x_k, \ldots, x_n)$ in the variable x_k; Eq. (15.36)
H^{-1}	inverse Hilbert transform operator; Eq. (4.26)
H^+	adjoint of the Hilbert transform operator; Eq. (4.194)
H_α	fractional Hilbert transform operator; Eqs. (18.209) and (18.216)
$\mathscr{H}$	Hamiltonian for an electronic system; §22.4, Eq. (22.54)
$\mathscr{H}_0$	unperturbed Hamiltonian for an electronic system; §22.4, Eq. (22.57)
$\mathscr{H}$	space of test functions; §10.14
$\mathcal{H}$	inner product space; §2.10.1
$\mathcal{H}$	Hilbert space; §2.10
$H_n f$	n-dimensional Hilbert transform of the function f, for $n \geq 2$; Eq. (15.26)
$H_1(f, g)(x)$	bilinear Hilbert transform; §16.5
$H_a(f, g)(x)$	bilinear singular integral operator; Eq. (16.85)
$H_n(x)$	Hermite polynomials; §9.3, Eq. (9.39), HT-01
$\mathrm{H}(x)$	Heaviside step function; Eqs. (10.54) and (18.116)
$\mathsf{H}(\omega)$	response function for a linear system; Eq. (13.1), §18.2
$\mathsf{H}_p(\omega)$	fractional Hilbert filter; §18.9, Eq. (18.142)
$H^p(D)$	Hardy space for the unit disc; §2.10.2
H^p	Hardy space for the upper half complex plane; §2.10.2
H_ν	response function at the frequency ν; Eq. (13.3)
$H_\varepsilon f$	truncated Hilbert transform; Eq. (3.3)
$H_E f$	truncated Hilbert transform; Eq. (4.507)

$H_{\mathrm{M}}f$	maximal Hilbert transform function; Eq. (7.280)
$\mathcal{H}_{\mathrm{M}}f$	maximal Hilbert transform function; Eq. (7.282)
$H_{\mathrm{S}}F$	Hilbert–Stieltjes transform of the function F; Eq. (4.551)
H_{K}	Kober's extension of the Hilbert transform operator; §16.3
H_{R_m}	Redheffer's extension of the Hilbert transform operator; §16.2
$\mathbf{H}_j$	vectorial Hilbert transform operator; Eq. (16.100)
$H_{\theta,\varepsilon}$	truncated directional Hilbert transform operator; Eq. (16.103)
H_{θ}	directional Hilbert transform operator; Eq. (16.104)
H_{θ}	helical Hilbert transform operator; Eq. (16.131)
H_{M_θ}	directional maximal Hilbert transform operator; Eq. (16.105)
H_{M_θ}	maximal helical Hilbert transform operator; Eq. (16.132)
$H_{\mathrm{M}_{n\theta}}$	double maximal helical Hilbert transform operator; Eq. (16.136)
$H_{\Gamma}f$	Hilbert transform of f along the curve Γ; Eq. (16.109)
$\bar{H}_{\Gamma}f$	modified Hilbert transform of f along the curve Γ; Eq. (16.113)
$\mathrm{H}_{\mathrm{A}}f$	Hartley transform of a function f; Eq. (5.58)
$\mathrm{H}_{\mathrm{A}}^{-1}$	inverse Hartley transform operator; Eq. (5.60)
$H_{\pm v}^{(1)}(z),\ H_{\pm v}^{(2)}(z)$	Bessel functions of the third kind (Hankel functions of the first kind and second kind, respectively); §9.9
$h_n^{(1)}(z)$	spherical Bessel functions of the third kind; Eq. (9.131) (spherical Hankel functions of the first kind)
$h_n^{(2)}(z)$	spherical Bessel functions of the third kind; Eq. (9.132) (spherical Hankel functions of the second kind)
$h_n(x)$	Hermite–Gaussian functions; Eq. (18.179)
$\mathbf{h}_k$	discrete Hermite–Gaussian vector functions; Eqs. (18.254) and (18.255)
$\mathbf{H}_v(z)$	Struve's function; Eq. (9.77), HT-01
$H_{\mathrm{D}}\{f[n]\}$	discrete Hilbert transform of the sequence $\{f[n]\}$; Eq. (13.127)
$\{H_{\mathrm{sD}}f\}(x)$	semi-discrete Hilbert transform of the sequence $\{f[]\}$; Eq. (13.133)
$\mathcal{H}_{\mathrm{D}}\{f[n]\}$	alternative definition of the discrete Hilbert transform; Eq. (13.158)
$(\mathcal{H}_{\mathrm{D}_{pq}}x)[n]$	discrete fractional Hilbert transform; Eq. (18.269)

i	imaginary unit (engineers typically use j); §2.8
I	identity operator; §4.4
I	interval; §7.9
$\lvert I\rvert$	length of an interval; §7.9
$I_n(x)$	modified Bessel function of the first kind; HT-01
$i(t)$	input (time-dependent in general) to a system; §17.1–17.2
iff	if and only if
Im	imaginary part of a complex function
inf	infimum, the greatest lower bound of a set; §2.8

Ind f	index of a function; Eq. (11.179)
J	SI unit of energy, the joule; §19.1
$J_{\pm v}(z)$	Bessel function of the first kind; §9.6, HT-01
$\mathbf{J}_v(z)$	Anger's function; §9.12, HT-01
$j_n(z)$	spherical Bessel function of the first kind; Eq. (9.115)
k	wave number; Eq. (19.87)
$\mathbf{k}$	wave vector; §20.7
$k(x, y)$	Kernel function; §1.2, Eq. (1.3)
$K(x)$	Calderón–Zygmund kernel function; §15.1
$K_n(x)$	modified Bessel function of the third kind; HT-01
$l(I)$	length of an interval I; §2.11.1
$\mathscr{L}f$	Laplace transform of the function f; Eq. (5.91)
$\mathscr{L}_2 f$	bilateral (or two-sided) Laplace transform of the function f; Eq. (5.92)
L	class of functions that are Lebesgue integrable on a given interval; §2.11.1
$L(a, b)$	class of functions that are Lebesgue integrable on the interval (a, b); 2.11.1
L^1_{loc}	class of functions that are Lebesgue integrable on every subinterval of a given interval; Eq. (4.121)
L^2	class of functions that are Lebesgue square integrable on a given interval; §2.11.1
L^p	class of functions f such that $\lvert f\rvert^p$ is Lebesgue integrable on a given interval; §2.11.1
$L^p(\mathbb{R})$	class of functions f such that $\lvert f\rvert^p$ is Lebesgue integrable on the real line; §2.11.1
l^p	§13.11
$l^p(\mathbb{Z})$	§13.11
L^∞	class of essentially bounded functions; §2.11.1
$L^p_{2\pi}$	class of periodic functions f such that $\lvert f\rvert^p$ is Lebesgue integrable on the interval $(0, 2\pi)$. $L^p_{2\tau}$ has a similar meaning for periodic functions with period 2τ.
$L^{\alpha,p}$	class of functions f such that $\lvert x\rvert^\alpha \lvert f(x)\rvert^p$ is Lebesgue integrable on a particular interval; Eq. (7.186)
$L^p(\mu)$	class of μ-measurable functions; §7.12
$L_n(x)$	Laguerre polynomials; §9.4, Eq. (9.60)
$\mathbf{L}_v(z)$	modified Struve function; HT-01
$\mathrm{Li}_n(z)$	polylogarithm function; HT-01
$\mathrm{Li}_2(z)$	dilogarithm function; HT-01
Lip m	Lipschitz condition of order m; §2.3
log	logarithm to the base e; the alternative notation ln is also common usage
$\log^+ f$	maximum of $\{\log\lvert f\rvert, 0\}$; Eq. (7.74)

$\mathbf{M}$	magnetization; Eq. (20.111)
$\operatorname{mod} z$	modulus of a complex number; Eq. (2.68)
$m(E)$	measure of the set E; §2.11.1
Mf	Hardy–Littlewood maximal function; §7.9
$\mathcal{M}f$	Mellin transform of f; Eq. (5.102)
$\mathcal{M}^{-1}$	inverse Mellin transform operator; Eq. (5.107)
m	SI unit for length, the meter; §19.1
$m\{g(\lambda)\}$	distribution function of g; §4.25, §7.2, Eqs. (4.556), (7.55)
$m_{X,Y}(\omega)$	relative multiplier connecting $X(\omega)$ and $Y(\omega)$; Eq. (18.60)
$\mathbb{N}$	set of positive integers; $1, 2, 3, \ldots$
N	complex refractive index; Eq. (19.90)
N^{NL}	nonlinear complex refractive index; §22.13
$\mathcal{N}$	number of molecules per unit volume; §19.2
$n(\omega)$	angular frequency-dependent refractive index; Eq. (19.91)
$n^{\mathrm{NL}}(\omega, E)$	nonlinear refractive index; §22.13, Eq. (22.238)
$N_{\pm}(\omega)$	complex refractive indices for circularly polarized modes; §21.3, Eq. (21.47)
$n_{\pm}(\omega)$	real parts of $N_{\pm}(\omega)$; Eqs. (21.79) and (21.80)
$\mathcal{O}$	linear operator on a vector space; §2.10
$\mathcal{O}^{+}$	*adjoint* operator to $\mathcal{O}$; §2.10
$\mathcal{O}^{-1}$	inverse of an operator $\mathcal{O}$; §2.10
$O(\ )$	Bachmann order notation, of the order of; Eq. (2.1)
$o(\ )$	Landau order notation, of the order of; Eq. (2.6)
$\mathcal{O}'_C$	space of distributions that decrease rapidly at infinity; §10.2
$P\int$	Cauchy principal value; §2.4, Eq. (2.18)
$P(r, \theta)$	Poisson kernel for the disc; Eq. (3.49)
$P(x, y)$	Poisson kernel for the half plane; Eq. (3.31)
P_{ε}	Poisson operator; §7.10, Eq. (7.290)
P_{+}	projection operator; Eq. (4.352)
P_{-}	projection operator; Eq. (4.353)
$Pf(x^{-1})$	pseudofunction; §10.1
$\mathbf{P}(\mathbf{x})$	electric polarization of a medium; Eq. (19.1)
$\mathrm{P}_n(x)$	one of the orthogonal polynomials; §9.1
$P_n(x)$	Legendre polynomials; §9.2, Eqs. (9.10) and (9.27)
$P_{\nu}^{m}(x)$	associated Legendre function of the first kind; HT-01
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomials; §9.1
$\mathcal{P}_{\tau}$	space of periodic testing functions of period τ; §10.2
$\mathcal{P}'_{\tau}$	space of periodic distributions of period T; §10.2
$p.v.\dfrac{1}{x}$	distribution; §10.1

$\mathcal{P}\frac{1}{x}$	distribution; §10.1
$Q(r,\theta)$	conjugate Poisson kernel for the disc; Eq. (3.50)
$Q(x,y)$	conjugate Poisson kernel for the half plane; Eq. (3.32)
Q_ε	conjugate Poisson operator; §7.10, Eq. (7.291)
$Q_n(x)$	Legendre function of the second kind; Eq. (11.263)
$Q_\nu^m(x)$	associated Legendre function of the second kind; HT-01
$Q_n^{(\alpha,\beta)}(x)$	Jacobi function of the second kind; HT-01
R	reflection operator; Eq. (4.73)
R	radius for a semicircular contour
A	Radon transform; §5.10, Eqs. (5.152) and (5.155)
$R_i(z_i)$	residue corresponding to the pole at $z = z_i$; §2.8.5
$R_j f$	Riesz transform of the function f; §15.12
$\mathbb{R}$	real line; the set of real numbers
$\mathbb{R}^+$	positive real axis interval; §3.4
$\mathbb{R} \times \mathbb{R}$	Euclidean plane
$\mathbb{R}^n$	n-dimensional Euclidean space; §2.15.2
$\mathcal{R}$	simply connected region; §2.8.1
$\mathcal{R}$	radius for a semicircular contour
$\Re_p$	Riesz constant; §4.20, Eqs. (4.382) and (4.384)
$\tilde{r}(\omega)$	generalized or complex reflectivity; Eq. (20.1)
$\tilde{r}_\pm(\omega)$	generalized reflectivity for circularly polarized modes; Eq. (21.132)
$r(\omega)$	reflectivity amplitude; Eq. (20.1)
$r(t)$	response (time-dependent) from a system; §17.1, §17.2
R_{n0}	rotational strength; Eq. (21.233)
$R(\omega)$	reflectivity; Eq. (20.2)
$re^{i\theta}$	polar form of the complex number z
Re	real part of a complex number
$\mathrm{rect}(x)$	reactangular pulse function; §18.7.3
$\mathrm{Res}\{g(z)\}_{z=z_0}$	residue at the pole $z = z_0$ of the function g; §2.8.5, Eq. (2.93)
$\mathsf{S}f$	Stieltjes transform of the function f; Eqs. (5.77), and (8.6)
S_a	dilation operator (homothetic operator); Eqs. (4.70) and (15.68)
sgn	signum function (sign function); Eqs. (1.14) and (18.120)
S^{n-1}	locus of points $x \in \mathbb{R}^n$ for which $\lvert x\rvert = 1$; §16.6
$S(z)$	Fresnel sine integral; Eq. (14.172), HT-01
$S(E)$	S-function (S-matrix); §17.12
$S(\omega)$	Fourier transform of a signal $s(t)$ in the frequency domain; §18.1, Eq. (18.2)
$s(t)$	signal in the time domain; §18.1
$\mathrm{Shi}(z)$	hyperbolic sine integral function; Eq. (14.201), HT-01
$\mathrm{Si}(x)$	sine integral; Eq. (8.79), HT-01

$\mathrm{si}(x)$	integral; Eq. (9.170), HT-01
$\mathrm{sie}\,(\alpha,\ \beta)$	sine-exponential integral; HT-01
$\mathrm{sinc}\,x$	sinc function; Eq. (4.260), HT-01
sup	supremum, the least upper bound
supp	support of the function; §2.15.2
T	finite Hilbert transform operator; chap. 11, Eq. (11.2)
T	used to denote a distribution; §2.15.2
T_{ab}	finite Hilbert transform operator on the interval (a, b); Eq. (12.98)
$T_n(x)$	Chebyshev polynomials of the first kind; §9.1, HT-01
$\mathbb{T}$	circle group; §3.10
Tr	trace; §22.4, Eq. (22.63)
$U_n(x)$	Chebyshev polynomials of the second kind; §9.1, HT-01
$u[n]$	unit step sequence; Eq. (13.91)
V	total variation of a function; Eq. (4.554)
V	SI unit for potential, the volt; §19.1
$w(x)$	weight function; §9.1
$W(x)$	weight function; §14.4
w_i	weight points in a quadrature scheme; Eq. (14.15)
$W^{p,m}$	Sobolev space; §10.2
$\bar{x}_j$	any value in the interval $[x_{j-1}, x_j]$; §2.11
x_i	sampling points in a quadrature scheme; Eq. (14.15)
$\lvert x\rvert$	norm of x in E^n; §15.1
$\mathbf{\times}$	vector cross product
$\times$	direct product; §10.6. Also used for Cartesian product of Euclidean spaces; §15.13
$\boldsymbol{x}(t)$	time-dependent particle displacement; §17.2, §17.9
$X(z)$	Z transform (one-sided or two-sided); Eqs. (13.38) and (13.39)
$Y_v(z)$	Bessel function of the second kind (Weber's function, Neumann's function); §9.6, 9.8, HT-01
$y_n(z)$	spherical Bessel function of the second kind; Eq. (9.116)
z	complex variable, $z = x + \mathrm{i}y$; Eq. (2.67)
$\bar{z}$	complex conjugate of z
z^*	complex conjugate of z
z_{I}	inverse point (or image point) of z; Eq. (3.35)
$\mathbb{Z}$	set of integers $0, \pm 1, \pm 2, \ldots$
$\mathbb{Z}^+$	set of non-negative integers $0, 1, 2, \ldots$
$Z\{x_n\}$	Z transform of the sequence $\{x_n\}$; §13.6, Eq. (13.38)

$\mathcal{Z}$	space of test functions whose Fourier transforms belong to $\mathcal{D}$; §10.2
$\mathcal{Z}'$	space of ultradistributions; §10.2
$\mathcal{Z}_1$	space of Fourier transforms of test functions belonging to $\mathscr{S}_1$; §10.14
$Z_s(\omega)$	surface impedance function; Eq. (20.129)

Greek letters

α	polarizability § 19.1, Eq. (19.6)
$\alpha(\omega)$	absorption coefficient of a medium; Eq. (19.92)
$\beta(2)$	Catalan's constant (0.915 965 594 177 219 015 1 . . .); HT-01
γ	Euler's constant (0.577 215 664 9 . . .)
Γ	contour in the complex plane (frequently used to signify a non-closed contour)
$\Gamma(z)$	gamma function; Eq. (4.118), HT-01
$\Gamma(a, z)$	incomplete gamma function; Eq. (8.38), HT-01
Γ_{mn}	damping constant for a transition between the levels m and n; §22.4, Eq. (22.71)
$\delta(x)$	Dirac delta distribution; §2.15, §10.3, Eq. (10.1)
$\delta[n]$	unit sample sequence; Eq. (13.92)
$\delta^+(x)$	Heisenberg delta function; Eq. (10.9)
$\delta^-(x)$	Heisenberg delta function; Eq. (10.10)
δ_{nm}	Kronecker delta; Eq. (2.38). When one of the subscript indices appears with a negative sign, the notation $\delta_{n,-m}$ is employed
Δ	difference operator; Eq. (2.305)
Δ	Laplacian operator; Eq. (15.161)
Δ_τ	Dirac comb distribution; Eq. (10.212)
$\Delta R(\omega)$	magnetoreflection; Eq. (21.125)
Δx	length of an interval; §2.11, Eq. (2.166)
ε	permittivity of the medium; Eq. (19.5)
$\varepsilon^{\mathrm{NL}}$	nonlinear dielectric permittivity; §22.13
ε_0	vacuum permittivity; §19.1
ϵ_{ijk}	Levi-Civita pseudotensor; Eq. (21.212)
$\boldsymbol{\varepsilon}(\mathbf{k}, \omega)$	spatial-dependent electric permittivity; Eq. (20.177)
$\varsigma(n)$	Riemann zeta function; Eq. (2.288)
$\theta(\omega)$	phase; Eq. (20.1)
$\theta(\omega)$	ellipticity per unit length; Eq. (21.195)
$\theta_{\mathrm{F}}(\omega)$	ellipticity function; Eq. (21.95)
$\kappa(\omega)$	measure of the absorption of a propagating wave in a medium; Eq. (19.92)
$\kappa_\pm(\omega)$	imaginary parts of $N_\pm(\omega)$; Eqs. (21.79) and (21.80)
$\kappa^{\mathrm{NL}}(\omega, E)$	imaginary part of the nonlinear complex refractive index; §22.13, Eq. (22.238)
$\Lambda(x)$	unit triangular function; Eq. (4.265)

Λ_α	space of Lipschitz continuous functions; §6.16, §15.1
μ	continuous Borel measure; §7.12
μ	magnetic permeability; §19.1
$\mu_o(A)$	Lebesgue outer measure of a set A; Eq. (2.183)
μ_0	permeability of the vacuum; §19.1
$\boldsymbol{\mu}$	electric dipole operator; Eq. (22.56)
$\Pi_{2a}(x)$	unit rectangular step function; Eqs. (9.19) and (18.122). For $a = 1/2$, the abbreviation $\Pi(x) \equiv \Pi_1(x)$ is employed
ρ	density operator; §22.4, Eq. (22.51)
$\rho(\mathbf{r})$	electronic density; §23.4, Eq. (23.75)
ρ_{mn}	matrix element of the density operator; §22.4
$\rho_s(t)$	auto-convolution function for a signal; Eq. (18.16)
$\rho_f(t)$	auto-correlation function; Eq. (18.23)
$\rho_{fg}(t)$	cross-correlation function; Eq. (18.21)
σ	type of an entire function; Eq. (2.110)
$\sigma(H)$	symbol of H; Eq. (5.37)
$\sigma(0)$	conductivity at $\omega = 0$; §19.8, Eq. (19.176)
$\sigma(\omega)$	complex conductivity; Eq. (20.84)
$\sigma_t(\omega)$	total scattering cross-section; Eq. (19.316)
Σ	unit sphere; §15.1
τ_a	translation operator; Eqs. (4.64) and Eq. (15.63)
φ_n	eigenfunctions of the unperturbed Hamiltonian; §22.4, Eq. 22.57
ϕ	test function in a particular space; §2.15.2, §10.1
$\phi(\omega)$	optical rotatory dispersion; Eq. (21.192)
$\phi(t)$	instantaneous phase; Eq. (18.78)
$\phi_F(\omega)$	magneto-rotatory dispersion function; Eq. (21.94)
$\Phi(z, s, v)$	Lerch function; HT-01
$\Phi(\omega)$	complex optical rotation function; Eq. (21.197)
χ	(linear) electric susceptibility; §19.1
$\boldsymbol{\chi}^{(n)}$	nth-order electric susceptibility tensor; §22.1
χ_m	magnetic susceptibility; Eq. (20.112)
$\chi_S(x)$	characteristic function associated with the set S; Eq. (2.191)
$\chi_{[x_1,x_2]}$	characteristic function with the interval where the function is non-zero specified by a subscript; §2.11.1
$\{\psi_n\}$	sequence of step functions; §2.14
$\psi(x)$	step function; Eq. (2.190)
$\psi(z)$	psi (or digamma) function; Eq. (4.222), HT-01
$\psi^{(n)}(z)$	Polygamma function; HT-01
$\Psi(\omega)$	ellipticity function, Eq. (21.193)
ω	angular frequency
$\omega(t)$	instantaneous frequency; Eq. (18.79)
ω_z	complex angular frequency; §17.7, Eq. (17.53)
ω_r	real part of a complex angular frequency; §17.7, Eq. (17.53)

$\omega_{\rm i}$	imaginary part of a complex angular frequency; §17.7, Eq. (17.53)
$\omega_{\rm p}$	plasma frequency of the medium; Eq. (19.13)
$\omega_{\rm c}$	cyclotron frequency; Eq. (21.56)
ω_{mn}	energy separation (in frequency units) between the levels m and n; §22.4
Ω	part of the Calderón–Zygmund kernel; §15.1

Miscellaneous notations

$\sum_{k=-\infty}^{\infty\prime}$	summation with a particular value of k excluded (usually $k = 0$)
$\dfrac{\partial}{\partial x}$	partial derivative operator (with respect to x)
∂^m	shorthand for the mth derivative (with respect to the variable under discussion), for $m \geq 1$.
$(a)_k$	Pochhammer symbol; Eq. (5.31). The notation a_k is also employed
$\forall$	for all
∇^2	Laplacian (del-squared) operator; Eq. (7.3)
$\boldsymbol{\nabla}$	gradient (del) operator
$\Rightarrow$	implies;
$\sim$	same order as; §2.2
$\sim$	correspondence; Eq. (3.269)
$\sim$	twiddle sign, employed to indicate asymptotic equivalence between functions in a particular limit; §8.1
$f[\,]$	functional notation, e.g. $f[g(x)]$; §2.15.2, Eq. (2.283)
$\mathcal{O}^+$	adjoint of a linear operator $\mathcal{O}$; §2.10
$[\alpha, \beta]$	closed interval, that is $\alpha \leq x \leq \beta$; §2.3
$[\mathcal{O}_1, \mathcal{O}_2]$	commutator of two operators; §2.10, Eq. (2.146)
$\{\mathcal{O}_1, \mathcal{O}_2\}$	anticommutator of two operators; Eq. (4.67)
(α, β)	open interval, that is $\alpha < x < \beta$; §2.3
(f, g)	scalar product for two functions $f(x)$ and $g(x)$; Eq. (14.52)
$\langle f, g\rangle$	inner product for two functions f and g in Dirac bra–ket notation; §2.10
$\begin{pmatrix} m \\ n \end{pmatrix}$	binomial coefficient; Eq. (2.309)
$*$	convolution operator, e.g. $f * g$; Eq. (2.53)
*	complex conjugate of a function (e.g. z^*)
$\star$	pentagram symbol for the cross-correlation operation; Eq. (18.21)
$\int_a^b f(x)\mathrm{d}x$	Riemann or Lebegue integral (depending on context); §2.11
$\int_C$	integral along the specified contour C; §2.8.1. $\int_\Gamma$ is sometimes used when the contour is not closed

$\int_{\mathbb{R}} f(x)\mathrm{d}x$	integral over the real line; §2.11.1
$\int_{\mathbb{R}^2} f(x,y)\mathrm{d}x\,\mathrm{d}y$	integral over the xy-plane; Eq. (2.213)
$\int_{\mathbb{T}} f(\theta)\mathrm{d}\theta$	integral over a 2π period
$\oint_C$	integral along the closed contour C taken in a specified orientation; §2.8.1
$\int_E f(x)\mathrm{d}x$	Lebesgue integral of f on E; Eq. (2.198)
$\int\limits_{\lvert x-t\rvert>\varepsilon} f(x,t)\mathrm{d}t$	integral for which a segment $(x-\varepsilon, x+\varepsilon)$ is excluded; Eq. (3.3)
$\exists$	there exists
$\in$	belongs to
$\notin$	does not belong to
$\mathscr{S}$	space of test functions that have rapid decay; §10.2
$\mathscr{S}_1$	space of test functions that belong to $\mathscr{S}$ and vanish on the interval function $(-a,\ a)$ for some $a>0$; §10.14
$\mathscr{S}'$	space of all tempered distributions; §10.2
$\langle\ \rvert$	Dirac bra; §2.10
$\lvert\ \rangle$	Dirac ket; §2.10
$\lvert f\rvert$	absolute value of the function f
$\lvert x\rvert$	norm of x in E^n; §15.1
$\lvert a\rvert$	sum of the components of the multi-index a; §15.5
$\lvert\theta\in[-\pi,\pi]:\lvert g(\theta)\rvert\geq\lambda\rvert$	distribution function of g, §7.2, Eq. (7.55)
$\lVert\phi\rVert$	norm of a vector ϕ; §2.10, Eq. (2.136)
$\lVert f\rVert_p$	pth-power norm of f; Eq. (2.202)
$\lVert f\rVert_\infty$	essential supremum of $\lvert f\rvert$; Eq. (2.203)
$\lVert f(\theta)\rVert_{\alpha,p}$	weighted norm; Eq. (7.186)
$\lVert f(\theta)\rVert_{\alpha,\infty}$	weighted norm; Eq. (7.187)
$\lVert f\rVert_{p,\mu}$	norm $(\int\lvert f\rvert^p\,\mathrm{d}\mu)^{1/p}<\infty$, $1<p<\infty$; Eq. (7.376)
$\lVert f\rVert_{W^{p,m}}$	Sobolev norm; §10.2, Eq. (10.40)
$\lVert p\rVert$	longest subinterval; Eq. (2.167)
$\{x_i : a\leq x\leq b\}$	set of points $\{x_i\}$ such that $a\leq x\leq b$; §2.10
$\varnothing$	empty set; §2.10
$\subset$	subset of, as in $A\subset B$, A is a (proper) subset of B; §2.10
$\subseteq$	included in, as in $A\subseteq B$, A is included in B; §2.10
$\cap$	intersection of sets, as in $A\cap B$; §2.10
$\cup$	union of sets, as in $A\cup B$; §2.10
$\bigcup\limits_k A_k$	union of the collection of sets A_k
$\backslash$	relative complement of a set, that is, the relative complement of B with respect to A (the difference of A and B) is denoted by $A\backslash B$; §2.10
$(m)!!$	double factorial; Eqs. (4.119) and (4.120)
$\lfloor m\rfloor$	floor function, the greatest integer less than or equal to m; Eq. (9.28)

$[m/2]$	value $m/2$ if m is an even integer or $(m-1)/2$ if m is an odd integer
$[\arg f(z)]_C$	change in $\arg f(z)$ as the contour is traversed; Eq. (11.174)
$\{x_n\}$	sequence of numbers
$\otimes$	tensor product (direct product); §10.6, Eq. (10.88)

Abbreviations

a.e.	almost everywhere: §2.11.1
CD	circular dichroism; §21.1
DFT	discrete Fourier transform; §13.2
DFHT	discrete fractional Hilbert transform; §18.14
DFRFT	discrete fractional Fourier transform; §18.13
EMD	empirical mode decomposition; §18.16
FFT	fast Fourier transform; §14.9, §14.10
FHT	fractional Hilbert transform; §18.9
FRFT	fractional Fourier transform; §18.9
FTNMR	Fourier transform nuclear magnetic resonance (spectroscopy); §1.1
FTIR	Fourier transform infrared (spectroscopy); §1.1
IDFT	inverse DFT; §13.4
IMF	intrinsic mode function; §18.16
MCD	magnetic circular dichroism; §21.1
MOR	magnetic optical rotation; §21.3
MRS	magnetic rotation spectra; §21.3
ORD	optical rotatory dispersion; §21.1
SHG	second-harmonic generation; §22.2
THG	third-harmonic generation; §22.2

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1

Introduction

1.1 Some common integral transforms

Transform techniques have become familiar to recent generations of undergraduates in various areas of mathematics, science, and engineering. The principal integral transform that is perhaps best known is the Fourier transform. The jump from the time domain to the frequency domain is a characteristic feature of a number of important instrumental methods that are routinely employed in many university science departments and industrial laboratories. Fourier transform nuclear magnetic resonance spectroscopy (acronym FTNMR) and Fourier transform infrared spectroscopy (FTIR) are two extremely significant techniques where the Fourier transform methodology finds important application. Two transforms derived from the Fourier transform, the Fourier sine and Fourier cosine transforms, also find wide application. The Laplace transform is often encountered fairly early in the undergraduate mathematics curriculum, because of its utility in aiding the solution of certain types of elementary differential equations. The transforms that bear the names of Abel, Cauchy, Mellin, Hankel, Hartley, Hilbert, Radon, Stieltjes, and some more modern inventions, such as the wavelet transform, are much less well known, tending to be the working tools of specialists in various areas. The focus of this work is about the Hilbert transform. In the course of discussing the Hilbert transform, connections with some of the other transforms will be encountered, including the Fourier transform, the Fourier sine and Fourier cosine offspring, and the Hartley, Laplace, Stieltjes, Mellin, and Cauchy transforms. The Z-transform is studied as a prelude to a discussion of the discrete Hilbert transform.

In this chapter the principal objective is to provide a non-rigorous introduction to the Hilbert transform, and to establish the idea of the Hilbert transform operator. Some brief historical comments are presented on the emergence of the Hilbert transform. Finally, some areas are given where the Hilbert transform finds application.

1.2 Definition of the Hilbert transform

Many of the common integral transforms can be written in the following form:

$$g(x) = \int_a^b k(x, y) f(y) \mathrm{d}y, \tag{1.1}$$

where $k(x,y)$ is called the *kernel function*, or just the kernel of the equation. Equation (1.1) can also be thought of as an example of an *integral equation*, if one desires to determine the function f in terms of g. More specifically, it is termed a Fredholm equation of the first kind. The limits on the integral can be finite or infinite. When the kernel function has a singularity in the integration range, it is possible in a number of cases to extend the definition of the integral in Eq. (1.1) to accommodate these cases. Such equations are referred to as singular integral equations.

The Hilbert transform on $\mathbb{R}$, the real line, is defined by

$$Hf(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x-y}, \quad \text{for } x \in \mathbb{R}. \tag{1.2}$$

The kernel function in this definition is given by

$$k(x,y) = \frac{1}{\pi(x-y)}, \tag{1.3}$$

which is singular when $y = x$. The symbol $P\int$ denotes an extension of the normal definition of the integral called the *Cauchy principal value*. This is discussed in detail in Chapter 2. The integral becomes well behaved for many common functions if an infinitesimally small section of the integration interval centered at the singularity $y = x$ is deleted, as part of the definition of the integral. This is the essential effect of evaluating the integral as a principal value integral.

A word on notation may be useful at this juncture. Commonly, f is used to denote a function of a single variable and $f(x)$ is the value of the function evaluated at the point x. It is prevalent in the sciences to use the notation $f(x)$ to denote the function and also the value of the function evaluated at the point x. Usually the context makes it clear which of the two meanings is intended, although the use of f or $f(\ )$ for the function, and $f(x)$ for the value of the function evaluated at the point x, makes the meaning much clearer. The interpretation of Eq. (1.2) is that Hf signifies a new function and $Hf(x)$ is the value of this function evaluated at the point x. The notation Hf is used when there is no need to specify the point at which the transform is evaluated, which is convenient in a number of cases, particularly where additional operators such as the Fourier or inverse Fourier transform operator are also being applied to the function f. Occasionally the notation $H[\ ]$ or $H\{\ \}$ is employed; this is expedient when the Hilbert transform of a product of functions is taken, but the notation is not used exclusively for this purpose. In this book the notation $H[f(x)]$ or $H\{f(x)\}$ is employed with some frequency as a shorthand for $H[f(t)](x)$. In the latter form, t is the dummy integration variable for the Hilbert transform, and the function Hf is evaluated at the point x. Occasionally the notation $H[f,x]$ is used in the literature to denote the Hilbert transform of the function f evaluated at the point x. Sometimes, mostly by mathematicians, the Hilbert transform of the function

f is denoted by $\tilde{f}$. In the literature, the symbol T is also employed to denote the Hilbert transform. In this book, T is used to denote the finite Hilbert transform. When no confusion is likely, operator identities involving H are written with no function specified, and it is assumed that functions can be found for which the operator equality holds.

Historically, Eq. (1.2) was not the definition given by David Hilbert. Working in the area of integral equations, he arrived at a pair of integral equations connecting the real and imaginary parts of a function analytic in the unit disc, leading to the definition of the Hilbert transform for the circle (Hilbert, 1904, 1912). The transform appearing in Eq. (1.2) seems to have been first discussed with some level of rigor by the cricket loving English mathematician G. H. Hardy (1902, 1908), and named by him in 1924 the Hilbert transform, in honor of Hilbert's contribution. It is perhaps interesting to speculate how this transform might have been named by later workers had Hardy not graciously named the transform as he did. In a sense, Alfred Tauber's contribution (Tauber, 1891) appears to have been overlooked. In hindsight, perhaps the transform should bear the names of the three aforementioned authors. Most of the early developments on the Hilbert transform were not performed by David Hilbert, but by Hardy (1924a, 1924b, 1932) and Titchmarsh (1925a, 1929, 1930a, 1930b). A related form was given by Young (1912). Variants of the Hilbert transform on $\mathbb{R}$ are presented in later chapters; these include the Hilbert transform for the circle, the finite Hilbert transform, the multi-dimensional Hilbert transform, the discrete Hilbert transform, and others.

The reader is alerted to the existence of an alternative definition of the Hilbert transform for the real line, one where the kernel $k(x,y) = \{\pi(y-x)\}^{-1}$ is employed. Unfortunately, a consensus agreement on the definition has not been reached, and both forms occur rather commonly in the literature, though the definition given in Eq. (1.2) appears to be increasingly favored. For a number of purposes this difference in sign is not important, but obviously is significant for the evaluation of the Hilbert transform of a particular function, which means that the reader needs to be alert to the sign choice when pulling entries from tables of Hilbert transforms. Occasionally the Hilbert transform is defined with the factor π^{-1} omitted. Employing the definition given in Eq. (1.2) does have the advantage that factors of π that would frequently appear are incorporated into the definition of the Hilbert transform. A few authors define the Hilbert transform with the imaginary unit factor included, that is, π^{-1} is replaced by $(\pi \mathrm{i})^{-1}$.

Note that nothing has been said about what conditions must be specified for the function f in order that the integral in Eq. (1.2) exists. Different levels of rigor can be brought to bear on this question. For almost all applications in the physical sciences, the existence of the Riemann integral of the function $|f|^2$ over the interval $(-\infty, \infty)$ is all that is required to guarantee that the Hilbert transform of f is bounded. The Hilbert transform can be defined for a wider class of functions than the aforementioned, and this is addressed in Chapter 3.

1.3 The Hilbert transform as an operator

The key idea in the application of any of the simple integral transforms is that the function f is acted on by an "integral operator," to yield a new function, g, which is referred to as the "name" transform of f. In the case of the Hilbert transform, the integral operation is given by

$$H \equiv \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(\,) - s}, \tag{1.4}$$

where the identity of the function and the point at which the Hilbert transform is evaluated are left unspecified. The Hilbert transform of f can be thought of as the application of the integral operator in Eq. (1.4) on the function $f(\,)$, to yield

$$Hf(\,) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(\,) - s} f(s), \tag{1.5}$$

and the left-hand side of Eq. (1.5) is frequently denoted by the function $g(\,)$. Clearly, the function g depends on the entire shape of f. In other words, g at some point x, $g(x)$, is not determined simply by the value of the function f evaluated at the same point. That is, g has a *non-local* dependence on f. The situation where $g(x)$ is determined directly by the value $f(x)$ arises when there is a simple functional connection between f and g; for example, suppose $g(x) = \sin[f(x)]$, then the value of g at the point x depends only on the value of f evaluated at x. This notion has important consequences. A function f could be zero over a large region of the real axis and finite for a small region, but its Hilbert transform could be everywhere non-zero. Applications will be encountered later that reflect this type of behavior.

To visualize the changes that take place when the Hilbert transform of a function is evaluated, consider the following choice:

$$f(x) = \frac{a}{a^2 + x^2}, \tag{1.6}$$

where a is a real positive constant. This functional form appears in several diverse applications, and is sometimes referred to as a Cauchy pulse, and in other applications is closely related to the Lorentzian profile. The Hilbert transform of this function is given by

$$g(x) = Hf(x) = \frac{x}{a^2 + x^2}. \tag{1.7}$$

Figure 1.1 shows a plot of $f(x)$ and its Hilbert transform for the value $a = 1$.

The particular methods that are most effective for evaluating this relatively straightforward Hilbert transform are discussed in Chapter 2 and illustrated with examples in Chapters 3 and 4.

The function f of the preceding example can be recovered from g using the expression $f(x) = -Hg(x)$. In fact, this is a rather general result. The two formulas

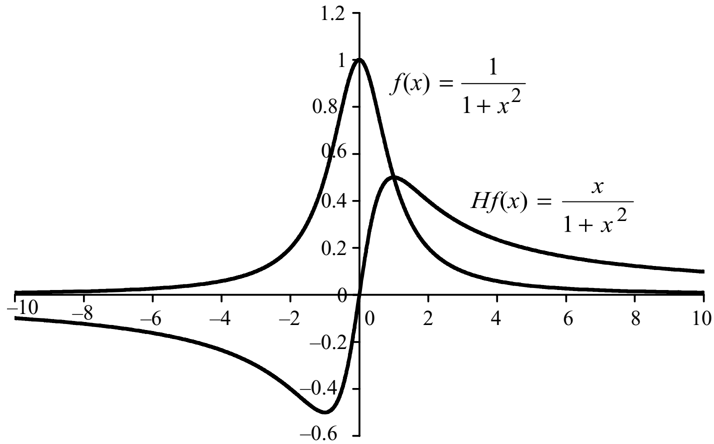

Figure 1.1. Plot of the Cauchy pulse and its Hilbert transform.

$g(x) = Hf(x)$ and $f(x) = -Hg(x)$ constitute a *Hilbert transform pair*. This Hilbert transform pair is explored in detail in later chapters, and it is shown that there is a very close connection with the theory of analytic functions. Pairs of functions that satisfy this type of skew-reciprocal character have been known for a considerable time. For example, the results (for $a > 0$)

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\sin as\,\mathrm{d}s}{x-s} = -\cos ax \tag{1.8}$$

and

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\cos as\,\mathrm{d}s}{x-s} = \sin ax \tag{1.9}$$

were given well over one hundred years ago (Schlömilch, 1848 p. 153; Bierens de Haan, 1867). The sine and cosine functions thus form a Hilbert transform pair.

Hardy (1908, 1924a, 1924b, 1928a, 1932) was one of those who pioneered the study of the mathematical foundations of the Hilbert transform. Prior to Hilbert's publications, Hardy (1902) had investigated the properties of Cauchy principal value integrals, and, in particular, he derived the preceding two formulas. Let $I(x, a)$ denote the following integral:

$$I(x,a) = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\sin as\,\mathrm{d}s}{x-s}, \tag{1.10}$$

where a is a constant. From the preceding formula, Hardy obtained the following differential equation:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}x^2} + a^2 I = 0. \tag{1.11}$$

The topic of differentiation of the Hilbert transform is discussed in detail later. The solution of Eq. (1.11) is

$$I(x, a) = \alpha \cos ax + \beta \sin ax, \tag{1.12}$$

where α and β are arbitrary constants. Setting $x = 0$ and using the result

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as \, \mathrm{d}s}{s} = \operatorname{sgn} a, \tag{1.13}$$

where

$$\operatorname{sgn} a = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -1, & a < 0 \end{cases} \tag{1.14}$$

gives $\alpha = -\operatorname{sgn} a$. It is straightforward to show that $I(x, a) = -I(-x, -a)$, from which it follows that $\beta = 0$, and hence

$$I(x, a) = -\operatorname{sgn} a \cos ax. \tag{1.15}$$

Hardy (1902) gave this result for the case $a > 0$, and he also gave a result equivalent to Eq. (1.9).

1.4 Diversity of applications of the Hilbert transform

Historically, work on Hilbert transforms developed on three main fronts. Mathematicians made the seminal developments in the first quarter of the twentieth century by putting the Hilbert transform into various useful forms, and established a number of key theorems that turned out to be of critical importance for future developments in the physical sciences. Hilbert transforms arose first in potential theory. Around the time of the dawn of modern quantum theory, Kramers (1926, 1927) and, working independently, Kronig (1926) obtained the reciprocal relations between the frequency dependent refractive index and the absorption coefficient of a medium. The resulting equations involved principal value integrals over the frequency interval $[0, \infty)$, which can be recast as a pair of standard Hilbert transforms. These equations became known in the physics and chemistry literature as the Kramers–Kronig relations. In parallel with this development, electrical engineers applied the same and some closely related mathematical ideas in circuit analysis (Carson, 1926). The real and imaginary parts of the general complex impedance were found to be connected to each other by Hilbert transforms. These relations are sometimes referred to as the Bode relations (Bode, 1945). In branches of engineering the Hilbert transforms are sometimes referred to as Wiener–Lee transforms (Papoulis, 1962, p. 192). In modern signal processing the terms 90° phase shift filter or quadrature filter are also employed to describe a Hilbert transform. The former of these two designations comes from the fact that the Hilbert transform of a sine function yields a cosine function, and this can be recast as a sine function with a shift of the argument by 90°. Somewhat later, with activity rising significantly in the early 1960s, Hilbert transforms found important applications in the study of various scattering processes in elementary particle physics and some other branches of physics. The key equations developed to describe the scattering processes are called *dispersion relations*, which are, in many cases, Hilbert transform relations or relatively minor extensions of the Hilbert transform concept. The Hilbert transform technique has clearly acquired multiple names as it has been employed in different applications. This multiplicity of names makes it more difficult to assess the true impact of Hilbert's contribution to transform calculus in the physical sciences. In addition to Hilbert, perhaps it is not inappropriate to give due credit to the nineteenth century mathematicians Poisson and, in particular, Cauchy, whose contributions laid the foundations for the work of Hilbert and others on the transform that finds such a diverse number of applications.

The question of why one should be interested in studying the theory of Hilbert transforms can be best answered in the following manner. There are numerous practical applications of Hilbert transforms, such as those mentioned in the preceding paragraph. To that list of applications can be added problems in aerofoil theory, crack formation in materials, aspects of the theory of elasticity, applications in wave propagation theory, problems in potential theory, and the study of dispersion forces. Further applications arise in certain areas in digital signal processing, and problems in the reconstruction of images. Readers with an interest in the stock market might be fascinated to see how a discrete version of the Hilbert transform has been used as a modeling tool (Ehlers, 2001). For some of these topics, the Hilbert transform or some variant of the standard form occurs as part of an integral equation or of an integro-differential equation. An example that is discussed later is the study of solitary waves. Because of the rich and diverse array of applications, the study of Hilbert transform theory can be a rewarding exercise.

Hilbert transform theory of course finds a number of applications in pure mathematics. The theory of the conventional Hilbert transform can be viewed as a paradigm for the mathematical investigation of singular integrals in general. This opens up a whole area of study in singular integral equations. Hilbert transform theory has served as a springboard to the study of singular integrals in n-dimensional Euclidean space. The Hilbert transform has played an important role in addressing some fundamental questions in the theory of Fourier series. This transform has a very close connection to some areas of complex analysis, and it plays an essential role in the theory of Fourier transforms of causal functions. The Hilbert transform is the key ingredient in characterizing operators that commute with the translation and dilation operators. Parts of all of the aforementioned topics are discussed in an introductory fashion in the following chapters.

Notes

The end-notes for each chapter provide sources, both books and journal articles, where additional reading on various topics may be pursued. The books that are recommended on standard topics reflect in large part the contents of the author's personal library. On many standard topics, particularly the background material covered in Chapter 2, the reader should be able to find a large number of additional reference texts beyond the ones cited. For a delightful account of the life and times of David Hilbert, intended for a general audience, see Reid (1996).

§1.1 For further reading on integral equations, consult Gakhov (1966), Hochstadt (1973), Tricomi (1985), Mikhlin and Prössdorf (1986), Pipkin (1991), Muskhelishvili (1992), and Kress (1999). Good sources on integral transforms with an applied emphasis include Sneddon (1972) and Davies (1978). The books by Zayed (1996) and Debnath and Bhatta (2007), and the individual accounts in Poularikas (1996a), are highly recommended reading.

§1.2 Hardy's work referenced in this book can be found in the seven volumes of his collected papers, Hardy (1966).

§1.3 Additional Hilbert transform pairs can be found in the nineteenth century literature; see, for example, Schlömilch (1848) or Bierens de Haan (1867). For some more recent collections of Hilbert transforms, see the following: Erdélyi *et al*. (1954, Vol. II), MacDonald and Brachman (1956), Smith and Lyness (1969), Alavi-Sereshki and Prabhakar (1972), and Hahn (1996a, 1996b). Hilbert transform relations of the type given in Eqs. (1.8) and (1.9) are due to the great French mathematician Augustin-Louis Cauchy.

§1.4 Some further reading on various applications of the Hilbert transform can be found in: Tricomi (1985, p. 173) and Zayed (1996, p. 287) for aerofoil theory; Wright and Hutchinson (1999) for the determination of oscillator phases for atomic motions; Ferry (1970), Booij and Thoone (1982), Madych (1990), and Herdman and Turi (1991), for elasticity theory; Aki and Richards (1980, p. 852) for crack propagation; Hinojosa and Mickus (2002) for the study of gravity gradient profiles; Červený and Zahradník (1975) for a review of geophysical applications; Weaver and Pao (1981), Beltzer (1983), and Bampi and Zordan (1992), for wave propagation theory; Duffin (1972), Nabighian (1984), and Sugiyama (1992), for aspects of potential theory; Sakai and Vanasse (1966) for an application in Fourier spectroscopy; Karl (1989), Hahn (1996a, 1996b), Oppenheim, Schafer, and Buck, (1999), for signal processing; and Lowenthal and Belvaux (1967), Herman (1980), Kohlmann (1996), Arnison *et al*. (2000), Davis, McNamara, and Cottrell (2000), and Shaik and Iftekharuddin (2003), for image reconstruction theory. A study of dispersion forces using dispersion theoretic techniques can be found in Feinberg, Sucher, and Au (1989). A number of applications have been made in Raman spectroscopy; see Chinsky *et al*. (1982), Stallard *et al*. (1983), Patapoff, Turpin, and Peticolis (1986), and Lee and Yeo (1994). For the development of a dispersion-type relation for the ground-state energy of two-electron atomic systems as a function of nuclear charge, see Ivanov and Dubau (1998). For further general reading on matters mathematical, see Butzer and Trebels (1968), Butzer and Nessel (1971), and Pandey (1996). A concise introductory account on the Hilbert transform can be found in Peters (1995).

Exercises

The table of Hilbert transforms in Appendix 1 should prove to be of value to you, both for checking the answers to a number of exercises throughout the book, and for solving some of the exercises.

1.1 Given

$$I(x,a) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\cos as \, \mathrm{d}s}{x - s},$$

where a is a constant, set up a differential equation by differentiation with respect to x, and hence determine the value for $I(x,a)$. Justify the differentiation step.

1.2 Given

$$P \int_{-\infty}^{\infty} f(s)\mathrm{d}s = \lim_{\varepsilon \to 0+} \left\{ \int_{-\infty}^{x-\varepsilon} f(s)\mathrm{d}s + \int_{x+\varepsilon}^{\infty} f(s)\mathrm{d}s \right\},$$

show for

$$f(s) = (x - s)^{-1} \text{ that } P \int_{-\infty}^{\infty} f(s)\mathrm{d}s = 0.$$

1.3 What is the value of $\int_{-\infty}^{\infty} \mathrm{d}s/(x - s)$?

1.4 Show that $Hf(x)$ equals $-a/(x^2 + a^2)$ for $f(s) = s/(a^2 + s^2)$, where a is a positive constant. Hint: The identity

$$\frac{s}{(s^2 + a^2)(x - s)} = \frac{1}{x^2 + a^2} \left\{ \frac{x}{x - s} + \frac{xs}{s^2 + a^2} - \frac{a^2}{s^2 + a^2} \right\}$$

leads to a straightforward calculation.

1.5 Show that $Hf(x)$ equals $x/a(x^2 + a^2)$ for $f(s) = 1/a^2 + s^2$, where a is a positive constant. Hint: The identity

$$\frac{1}{(s^2 + a^2)(x - s)} = \frac{1}{x^2 + a^2} \left\{ \frac{x}{s^2 + a^2} + \frac{1}{x - s} + \frac{s}{s^2 + a^2} \right\}$$

simplifies the calculation.

1.6 If c is a constant, evaluate $H[c]$.

1.7 Evaluate $H[\sin(ax + b)]$, where a and b are real constants.

1.8 Evaluate $H[\cos(ax + b)]$, where a and b are real constants.

1.9 Evaluate $H[\sin^2(\alpha x)]$, where α is a real constant.

1.10 If α is a real constant, does $H[x^{-1} \sin(\alpha x)]$ converge?

1.11 For α a real constant, determine whether $H[x^{-1} \cos(\alpha x)]$ converges.

1.12 Prove Eq. (1.13).

1.13 For $f(x) = x(x^2 + \alpha^2)^{-1}$ with α a real constant greater than zero, how does Hf behave as $\alpha \to 0+$?

1.14 If

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-\alpha x}, & x \geq 0 \end{cases} \quad \text{with } \alpha > 0,$$

evaluate $Hf(x)$.

1.15 If $f(x) = \begin{cases} 0, & |x| > 1 \\ 1, & |x| \leq 1 \end{cases}$, evaluate $Hf(x)$.

1.16 For $a > 0$, is the statement $H[x^2(a^2 + x^2)^{-1}] = -ax\,(a^2 + x^2)^{-1}$, true or false?

2

Review of some background mathematics

2.1 Introduction

The principal objective of this chapter is to present some of the essential basic mathematical background that is employed in later sections. A good deal of this material should be straightforward for a well trained undergraduate mathematics or physics major; however, there are a few slightly more advanced topics that are treated concisely. For these topics, collateral reading in a standard text is highly recommended. Some suggestions of where to start additional reading are provided in the end-notes. The mathematically talented reader could bypass most of this chapter and skip to the derivations in Chapter 3.

Almost all the mathematical notation employed can be found in the List of symbols; please consult that list for the definition or for the first use of a particular symbol. Some common notational devices are reviewed first, and this is followed by a concise description of some of the more important mathematical tools, such as Fourier analysis, complex variable theory, and the basics of integration theory, i.e. topics that are central to later developments. Further extensions of some of these tools are given later as needed.

2.2 Order symbols $O(\,)$ and $o(\,)$

There are a number of situations in later sections where it is necessary to address the asymptotic behavior of $f(x)$, for x approaching some limit, which may be finite or infinite. In a number of circumstances, $f(x)$ may be so complicated that it may be an advantage to replace $f(x)$ by a simpler choice, $g(x)$, as x approaches the limit under consideration. Three different notations are employed to express the asymptotic behavior of $f(x)$.

If the value of the ratio $f(x)/g(x)$ is bounded by a constant as $x \to \alpha$, then $f(x)$ is at most *of the order of magnitude* of $g(x)$ as $x \to \alpha$. Symbolically this is written using the Bachmann order notation:

$$f(x) = O(g(x)). \tag{2.1}$$

Commonly, this notation is also used to imply, for a positive constant C, that $|f(x)| \leq C\,|g(x)|$ as x approaches some given limit. Some examples of the use of this notation are as follows. If $P_k(x)$ denotes a polynomial of degree k, then

$$P_k(x) = O(x^k), \quad \text{as } x \to \infty. \tag{2.2}$$

A second example is

$$\sin x = O(x), \quad \text{as } x \to 0. \tag{2.3}$$

If $f(x)$ is bounded for all x as $x \to$ some limit, then

$$f(x) = O(1), \quad \text{as } x \to \text{some limit}. \tag{2.4}$$

For example,

$$\exp(-|x|) = O(1), \quad \text{as } x \to \infty. \tag{2.5}$$

If $f(x)/g(x) \to 1$ as $x \to$ some limit, then $f(x)$ has the same order as $g(x)$, which is expressed symbolically as $f(x) \sim g(x)$ as x tends to its limit. This is clearly a more restrictive condition than Eq. (2.1).

If $f(x)/g(x) \to 0$ as x tends to its limit, then $f(x)$ has a smaller order of magnitude than $g(x)$, which is denoted symbolically by

$$f(x) = o(g(x)). \tag{2.6}$$

Some examples are as follows:

$$x^m = o(\mathrm{e}^x), \quad \text{as } x \to \infty, \quad \text{for } m > 0; \tag{2.7}$$

$$x^{-m} = o(1), \quad \text{as } x \to \infty, \quad \text{for } m > 0; \tag{2.8}$$

$$P_k(x) = o(x^{k+1}), \quad \text{as } x \to \infty; \tag{2.9}$$

and

$$\cos x - 1 = o(x), \quad \text{as } x \to 0. \tag{2.10}$$

If $f(x) = o(g(x))$ as x tends to its limit, this implies $f(x) = O(g(x))$.

2.3 Lipschitz and Hölder conditions

A function f satisfies the Lipschitz condition at a point x_0 if there exists a positive constant C such that

$$|f(x) - f(x_0)| \leq C|x - x_0|, \tag{2.11}$$

for all values of x in some neighborhood of x_0.

A function f satisfies the Hölder condition at a point x_0 if

$$|f(x) - f(x_0)| \leq C|x - x_0|^m, \tag{2.12}$$

for all values of x in some neighborhood of x_0, and C and m are positive constants. The parameter C is called the Hölder constant, and m is termed the order of the Hölder condition. The designations *index* and *exponent* are also employed in place of order. Equation (2.12) is also referred to as a Lipschitz condition of order m. This is written as

$$f \in \mathrm{Lip}\, m. \tag{2.13}$$

A function f satisfies a Hölder condition on an interval $[\alpha, \beta]$ if

$$|f(x_2) - f(x_1)| \leq C|x_2 - x_1|^m, \tag{2.14}$$

for all x_1 and x_2 on $[\alpha, \beta]$. A function satisfying a Hölder condition on an interval is continuous on that interval. The reader is reminded that square brackets, as in $[\alpha, \beta]$, denote the closed interval $\alpha \leq x \leq \beta$, and that parentheses (,) as in (α, β) designate the open interval $\alpha < x < \beta$.

2.4 Cauchy principal value

Consider an integral with a singularity present in the integration interval. As an example, suppose

$$f(t) = \int_\alpha^\beta \frac{\mathrm{d}x}{(t - x)}, \tag{2.15}$$

where t lies in the interval (α, β). If the evaluation of this integral is approached as an improper Riemann integral, then

$$\begin{aligned} f(t) &= \lim_{\varepsilon \to 0} \int_\alpha^{t-\varepsilon} \frac{\mathrm{d}x}{(t - x)} + \lim_{\rho \to 0} \int_{t+\rho}^\beta \frac{\mathrm{d}x}{(t - x)} \\ &= -\lim_{\varepsilon \to 0} \{\log|t - x|\}_\alpha^{t-\varepsilon} - \lim_{\rho \to 0} \{\log|x - t|\}_{t+\rho}^\beta \\ &= \lim_{\varepsilon \to 0} \lim_{\rho \to 0} \log\left|\frac{t - \alpha}{\beta - t}\frac{\rho}{\varepsilon}\right|. \end{aligned} \tag{2.16}$$

In this book the notation log always refers to a logarithm to the base e. The result in Eq. (2.16) can take on any value, depending on the value of the ratio ρ/ε. Since the integral in Eq. (2.15) is ill defined as it stands, a simple modification of the approach indicated in Eq. (2.16), first utilized by Cauchy in 1822 (Smithies, 1997), leads to the following:

$$f(t) = \lim_{\varepsilon \to 0}\left[\int_{\alpha}^{t-\varepsilon} \frac{\mathrm{d}x}{(t-x)} + \int_{t+\varepsilon}^{\beta} \frac{\mathrm{d}x}{(t-x)}\right]$$
$$= \log\left|\frac{t-\alpha}{\beta-t}\right|, \tag{2.17}$$

which is clearly well defined for $t \neq \alpha$ and $t \neq \beta$. The limiting operation given in Eq. (2.17) is called the principal value of the integral, or, more appropriately, the Cauchy principal value of the integral. The common notations employed to symbolize the limiting process displayed in Eq. (2.17) are

$$P\int f(x)\mathrm{d}x, \quad PV\int f(x)\mathrm{d}x, \quad VP\int f(x)\mathrm{d}x, \quad \int^{*} f(x)\mathrm{d}x, \quad ⨍ f(x)\mathrm{d}x,$$

and $f(x)$ has a singularity in the interval over which the integral is evaluated. That is,

$$P\int_{\alpha}^{\beta} f(x)\mathrm{d}x = \lim_{\varepsilon \to 0}\left[\int_{\alpha}^{t-\varepsilon} f(x)\mathrm{d}x + \int_{t+\varepsilon}^{\beta} f(x)\mathrm{d}x\right], \tag{2.18}$$

where $f(x)$ has a singularity at $x = t$. The third of the five notational devices given, *VP*, (valeur principale) is seen in European writings. In a number of works, the standard integral sign is used to denote a principal value integral, with an explicit statement given to the reader that a principal value integral is under discussion. In the remainder of this work, the first of the five symbols just listed is employed.

It is not difficult to find examples of functions with singularities such that the limit in Eq. (2.18) is not finite. For the case $f(x) = (x - t)^{-2}$ with $t \in (\alpha, \beta)$, the Cauchy principal value integral in Eq. (2.18) diverges.

2.5 Fourier series

Some of the basics of Fourier series that find application in other chapters are reviewed in this section. It is possible in a number of cases to represent various experimental data sets by a Fourier series, from which important information can be extracted on taking the Hilbert transform. Fourier series arise in a natural way when the Hilbert transforms of periodic functions are considered.

2.5.1 Periodic property

Let the function f satisfy

$$f(x \pm P) = f(x), \tag{2.19}$$

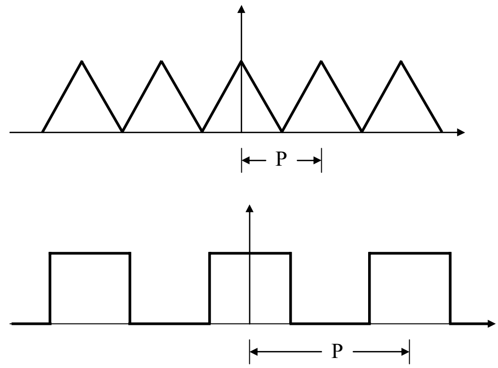

Figure 2.1. Two examples of periodic functions. The period for each example is P.

for all values of x, and $P > 0$. If Eq. (2.19) holds, f is called a *periodic* function, and the smallest P for which the equation holds is termed the period of f. Figure 2.1 illustrates two examples.

Some elementary examples of periodic functions are $\cos x$ and $\sin x$, since

$$\cos x = \cos(x + 2\pi) = \cos(x + 4\pi) = \cos(x + 6\pi) = \cdots, \tag{2.20}$$

$$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \cdots. \tag{2.21}$$

The period in both examples is 2π. For the functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ with n a positive integer, the period is $2L$.

2.5.2 Piecewise continuous functions

A function f is *piecewise continuous* in an interval (a, b) if it satisfies the following requirements: the interval (a, b) can be dissected into a finite number of subintervals in which f is continuous, and that the limits as x approaches the endpoints of each subinterval are finite. The term *jump discontinuity* is employed in the following way. If the limits

$$f_{\downarrow}(c) \equiv f(c + 0) = \lim_{x \to c+0} f(x) \tag{2.22}$$

and

$$f_{\uparrow}(c) \equiv f(c - 0) = \lim_{x \to c-0} f(x) \tag{2.23}$$

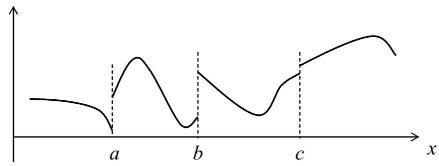

Figure 2.2. The discontinuities in the function occur at $x = a$, $x = b$, and $x = c$. The discontinuities are non-infinite.

exist, but are different, then the function has a jump discontinuity at the point c. A piecewise continuous function in an interval has a finite number of jump discontinuities in that interval. An example of a piecewise continuous function is shown in Figure 2.2.

2.5.3 Definition of Fourier series

Suppose that the function f is defined on an interval $(-L, L)$ and is periodic outside the interval with period $2L$, that is $f(x) = f(x + 2L)$, then the Fourier series expansion of f is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right). \tag{2.24}$$

The coefficients a_n and b_n are called the Fourier coefficients. The coefficients can be determined in the following manner. Multiply both sides of Eq. (2.24) by $\cos(m\pi x/L)$ and integrate over the interval $(-L, L)$ to obtain

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{2.25}$$

Multiplying both sides of Eq. (2.24) by $\sin(m\pi x/L)$ and integrating over the interval $(-L, L)$ yields

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{2.26}$$

To obtain these expressions, the following elementary integrals are employed:

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \begin{cases} 0, m \neq n \\ L, m = n \end{cases}, \tag{2.27}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 0, \tag{2.28}$$

where m and n take on integer values.

The conditions that are necessary for f to be expanded in the form of a Fourier series are examined next. If the following requirements (due to Dirichlet) hold:

(1) f is defined and single-valued on the interval $(-L, L)$, with the possible exception of a finite number of points;
(2) f is periodic with period $2L$;
(3) f and its derivative are piecewise continuous in $(-L; L)$,

then it is possible to write the following:

$$\frac{1}{2}\{f(x+0)+f(x-0)\} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right). \tag{2.29}$$

If x is not a point of discontinuity, then

$$f(x) = \frac{1}{2}\{f(x+0)+f(x-0)\}. \tag{2.30}$$

Two examples of well known Fourier series expansions are:

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\sin nx, \quad \text{for } -\pi < x < \pi, \tag{2.31}$$

and

$$\log\left\{2\cos\left(\frac{x}{2}\right)\right\} = \sum_{n=1}^{\infty}(-1)^{n+1}\cos nx, \quad \text{for } -\pi < x < \pi. \tag{2.32}$$

The three conditions on f are known to be sufficient conditions to guarantee convergence. The conditions are not necessary; that is, if the three conditions as stated are not satisfied, then the series in Eq. (2.29) may or may not converge. From a practical standpoint, with a view to the type of applications to be discussed later, the Dirichlet conditions will almost always be satisfied.

An alternative Fourier series expansion is employed in later sections. With the same conditions stated previously, it is possible to write the Fourier series for f in complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}. \tag{2.33}$$

The coefficients c_n can be determined by multiplying both sides of Eq. (2.33) by $e^{im\pi x/L}$ and integrating over the interval $(-L, L)$, to obtain

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx. \tag{2.34}$$

If f is not continuous at the point x, then $1/2\{f(x+0)+f(x-0)\}$ replaces $f(x)$ in Eq. (2.33).

In later chapters it will be necessary to expand a function in terms of a complete set of orthonormal functions on a particular interval $[a,b]$. A set of real-valued functions $\phi_n(x), n=0,1,2,\ldots$, is *orthogonal* on the interval $[a,b]$ if

$$\int_a^b \phi_n(x)\phi_m(x)\mathrm{d}x = 0, \quad \text{for } n \neq m. \tag{2.35}$$

The preceding formula can be generalized to include a weight function $w(x)$ in the integrand, for which case the $\phi_n(x)$ are termed orthogonal on the interval $[a,b]$ with weight $w(x)$. The set of functions is termed *orthonormal* if Eq. (2.35) is satisfied and

$$\int_a^b \phi_n^2(x)\mathrm{d}x = 1 \tag{2.36}$$

holds for all n. The last two results are often combined into the following single result:

$$\int_a^b \phi_n(x)\phi_m(x)\mathrm{d}x = \delta_{nm}. \tag{2.37}$$

The symbol δ_{nm} is called the Kronecker delta, and is defined by

$$\delta_{mn} = \begin{cases} 1, & \text{for } n = m \\ 0, & \text{for } n \neq m \end{cases} \quad \text{with } n, m \in \mathbb{Z}, \tag{2.38}$$

where $\mathbb{Z}$ denotes the set of integers, $0, \pm 1, \pm 2, \ldots$. On the interval $[0,\pi]$, the set of functions $\{(\sqrt{(2/\pi)}) \sin nx\}_{n=1}^{\infty}$ forms a complete orthonormal set. If a function f and its derivative f' are piecewise continuous in an interval $[a,b]$, then the function can be expanded in terms of a complete orthonormal set:

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad \text{for } a \leq x \leq b, \tag{2.39}$$

where the c_n are termed generalized Fourier coefficients. The partial sum of the first N terms of the series is defined as follows:

$$S_N(x) = \sum_{n=1}^{N} c_n\phi_n(x). \tag{2.40}$$

The set of functions ϕ_n is termed *complete* if

$$\lim_{N\to\infty} \int_a^b \{f(x) - S_N(x)\}^2 \,\mathrm{d}x = 0. \tag{2.41}$$

The generalized Fourier series for f converges in the mean square sense, also called converges in norm, if Eq. (2.41) holds.

2.5.4 Bessel's inequality

If $\{\phi_n\}$ form an orthonormal set on the interval $[a,b]$, then the coefficients in the expansion in Eq. (2.39) can be determined from

$$c_n = \int_a^b f(x)\phi_n(x)\mathrm{d}x. \tag{2.42}$$

By a straightforward calculation, it follows that

$$\begin{aligned}\int_a^b \{f(x) - S_N(x)\}^2\,\mathrm{d}x &= \int_a^b f^2(x)\mathrm{d}x - 2\int_a^b f(x)S_N(x)\mathrm{d}x + \int_a^b S_N^2(x)\mathrm{d}x \\ &= \int_a^b f^2(x)\mathrm{d}x - 2\sum_{n=0}^{\infty} c_n \sum_{m=0}^{N} c_m \int_a^b \phi_n(x)\phi_m(x)\mathrm{d}x \\ &\quad + \sum_{n=0}^{N} c_n \sum_{m=0}^{N} c_m \int_a^b \phi_n(x)\phi_m(x)\mathrm{d}x \\ &= \int_a^b f^2(x)\mathrm{d}x - \sum_{n=0}^{N} c_n^2. \end{aligned} \tag{2.43}$$

It follows on taking the limit $N \to \infty$ that

$$\sum_{n=0}^{\infty} c_n^2 \leq \int_a^b f^2(x)\mathrm{d}x, \tag{2.44}$$

which is Bessel's inequality.

2.6 Fourier transforms

In this section a few properties of Fourier transforms that are employed later are concisely reviewed. Ideas developed from Fourier analysis provide one route to the derivation of the conjugate function relationships, and these ideas have an important bearing on issues connected with the numerical evaluation of Hilbert transforms. They also play a central role in the connection of causal arguments with analytic behavior, and hence to the Hilbert transform relations.

2.6.1 Definition of the Fourier transform

Suppose f is an absolutely integrable function on $\mathbb{R}$, that is

$$\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x < \infty; \tag{2.45}$$

then the Fourier transform of f, denoted by $\mathcal{F}f$, is defined by

$$\mathcal{F}f(x) = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}s. \tag{2.46}$$

The notation $\hat{f}$ is also widely used in the mathematics literature to denote $\mathcal{F}f$, and this is employed occasionally in this book when the use of $\mathcal{F}$ would make the expression rather cluttered. In the engineering disciplines, the notation $\hat{f}$ is sometimes used to denote the Hilbert transform, a convention not employed in this book. Condition (2.45) is sufficient, but not necessary, for the existence of the Fourier transform. There are functions which do not satisfy Eq. (2.45), but still have a well defined Fourier transform. An example is the function $x^{-1}\sin x$. Weaker conditions on f can be given to ensure that $\mathcal{F}f$ is defined (Titchmarsh, 1948, chap. 3; Champeney, 1987, p. 47; Papoulis, 1962, p. 9).

Let $g(x) = \mathcal{F}f(x)$, then the inverse transform, written symbolically as $\mathcal{F}^{-1}g(x)$, is given by

$$f(s) = \mathcal{F}^{-1}g(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}x. \tag{2.47}$$

The Fourier transform is also commonly defined in the following ways:

$$\mathcal{F}f(x) = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{-\mathrm{i}2\pi xs}\,\mathrm{d}s, \quad \mathcal{F}^{-1}f(x) = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{\mathrm{i}2\pi xs}\,\mathrm{d}s, \tag{2.48}$$

$$\mathcal{F}f(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(s)\mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}s, \quad \mathcal{F}^{-1}f(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(s)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s, \tag{2.49}$$

$$\mathcal{F}f(x) = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s, \quad \mathcal{F}^{-1}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}s. \tag{2.50}$$

The alternative definitions given in Eqs. (2.48) and (2.49) reflect that the placement of the 2π factor is arbitrary. Equations (2.48) and (2.49) have a more symmetric appearance than the pair given by Eqs. (2.46) and (2.47). The definitions of the Fourier transform given in Eqs. (2.46) and (2.50) differ only in the sign of the exponent term. In formulas involving both the Fourier and Hilbert transforms, this difference in sign can lead to some formulas differing by a minus sign. This occurs widely in the scientific and engineering literature. Since the Hilbert transform can also be defined with one of two possible sign choices, the reader needs to be alert when reading literature sources, where unexpected signs may be traceable either to the choice of definition of the Hilbert transform or the Fourier transform employed. An important formula where this sign issue arises is given in Section 5.2 (see Eq. (5.3)).

Using the definition given in Eq. (2.46), two examples are as follows:

$$f(x) = \mathrm{e}^{-ax^2}, \text{ with } a > 0, \quad \mathcal{F}f(x) = \sqrt{\left(\frac{\pi}{a}\right)}\mathrm{e}^{-x^2/(4a)}. \tag{2.51}$$

For

$$f(x) = \frac{1}{a^2 + x^2}, \text{ with } a > 0, \quad \mathcal{F}f(x) = \pi a^{-1} e^{-a|x|}. \tag{2.52}$$

2.6.2 Convolution theorem

The convolution of two functions f and g defined on $\mathbb{R}$ is given by the following definition:

$$\{f * g\}(x) = \int_{-\infty}^{\infty} f(s)g(x - s)\, ds, \tag{2.53}$$

provided the integral exists. The notation $f(x) * g(x)$ is also commonly employed in place of $\{f * g\}(x)$. A result that will turn out to be extremely useful at a later juncture is the convolution theorem for Fourier transforms. This result states that the Fourier transform of the convolution of f and g is the product of the Fourier transforms of f and g; thus,

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}. \tag{2.54}$$

The convolution operation defined in Eq. (2.53) obeys the commutative and distributive laws. The associative property is obeyed for functions in certain classes; see, for example, Howell (2001, p. 376).

2.6.3 The Parseval and Plancherel formulas

If the functions f and $g \in L^2(\mathbb{R})$ and $\hat{f}$ and $\hat{g}$ denote the corresponding Fourier transforms, then

$$\int_{-\infty}^{\infty} \left|\hat{f}(x)\right|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2\, dx \tag{2.55}$$

and

$$\int_{-\infty}^{\infty} \hat{f}(x)\hat{g}^*(x)dx = \int_{-\infty}^{\infty} f(x)g^*(x)dx. \tag{2.56}$$

The notation L denotes the class of Lebesgue integrable functions, a topic that is discussed in some detail in Section 2.11, and the superscript $*$ denotes the complex conjugate. Mathematicians call the first of these two relations Parseval's formula (sometimes the Parseval–Plancherel formula), and physicists associate the formula with Rayleigh. The second equation is designated as Plancherel's formula, although a number of authors refer to both equations as Parseval's formula. These results have a number of practical applications, and there are analogs of these formulas involving the Hilbert transform. Each formula can be obtained rather quickly from the other. Equation (2.56) can be established as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}(x)\hat{g}^*(x)\mathrm{d}x &= \int_{-\infty}^{\infty} \hat{f}(x)\left(\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} g^*(y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y\right)\mathrm{d}x\\ &= \int_{-\infty}^{\infty} g^*(y)\left(\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \hat{f}(x)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}x\right)\mathrm{d}y\\ &= \int_{-\infty}^{\infty} g^*(y)f(y)\mathrm{d}y. \end{aligned}\tag{2.57}$$

The symmetric distribution of the 2π term has been used to avoid these factors in the final formula. If the definition in Eq. (2.46) is employed, then the right-hand side of Eq. (2.57) must be multiplied by a factor of 2π. The interchange of order of integration can be justified using Fubini's theorem (see Section 2.12). Setting $g = f$ in Eq. (2.57) recovers Eq. (2.55).

2.7 The Fourier integral

Combining Eqs. (2.24), (2.25), and (2.26) leads to

$$\begin{aligned}f(x) =&\frac{1}{2L}\int_{-L}^{L} f(s)\,\mathrm{d}s + \frac{1}{L}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(s)\cos\left(\frac{n\pi s}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}s\right.\\ &\left.+\int_{-L}^{L} f(s)\sin\left(\frac{n\pi s}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}s\right)\\ =&\frac{1}{2L}\int_{-L}^{L} f(s)\mathrm{d}s + \frac{1}{L}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(s)\cos\left[\frac{n\pi(s-x)}{L}\right]\mathrm{d}s\right),\end{aligned}\tag{2.58}$$

where the trigonometric identity

$$\cos(A - B) = \sin A\sin B + \cos A\cos B \tag{2.59}$$

has been used. If the substitution $n\pi/L = t_n$ is employed, and $\delta t = t_{n+1} - t_n$, then

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(s)\mathrm{d}s + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(s)\cos[t_n(s-x)]\mathrm{d}s\right)\delta t. \tag{2.60}$$

If f is integrable on $\mathbb{R}$, then in the limit as $L \to \infty$ the first term on the right-hand side of Eq. (2.60) vanishes, and the sum in Eq. (2.60) can be replaced by an integral, so that

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty} f(s)\cos[t(s-x)]\mathrm{d}s, \tag{2.61}$$

which can be rewritten as follows:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} f(s)\mathrm{e}^{\mathrm{i}t(s-x)}\,\mathrm{d}s. \tag{2.62}$$

Both of the preceding two results are referred to as the Fourier integral formula. Equation (2.61) is occasionally useful for the situation where $f(x)$ has particular symmetry properties. The reader is reminded of the following definitions of even and odd functions. An even function satisfies the condition $f(-x) = f(x)$, and an odd function satisfies $f(-x) = -f(x)$. An immediate consequence is

$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x, \quad \text{for } f(x) \text{ even}, \tag{2.63}$$

and

$$\int_{-a}^{a} f(x)\mathrm{d}x = 0, \quad \text{for } f(x) \text{ odd}. \tag{2.64}$$

For $f(x)$ even, Eq. (2.61) simplifies, with the aid of Eq. (2.59), to

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \cos xt\,\mathrm{d}t \int_{0}^{\infty} \cos ts\, f(s)\mathrm{d}s; \tag{2.65}$$

and for $f(x)$ an odd function,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin xt\,\mathrm{d}t \int_{0}^{\infty} \sin ts\, f(s)\mathrm{d}s. \tag{2.66}$$

Equations (2.65) and (2.66) are the Fourier cosine and Fourier sine integral formulas, respectively.

2.8 Some basic results from complex variable theory

In this section, some of the essential basic ideas on functions of a single complex variable are summarized. More involved results are developed elsewhere as required. The theory of analytic functions plays a central role in several developments in Hilbert transform theory. This branch of mathematics is the key link between the experimental notion of causality and the reciprocal relations that bear the names of Kramers, Kronig, and Bode, and also dispersion theoretic methods. On the mathematical side, analytic function theory provides a connecting link between conjugate functions expressed as Fourier series, Fourier integrals, or as Hilbert transforms. Analytic function theory represents a powerful set of techniques for solving problems in many branches of pure and applied mathematics.

A complex variable z is defined by

$$z = x + \mathrm{i}y, \tag{2.67}$$

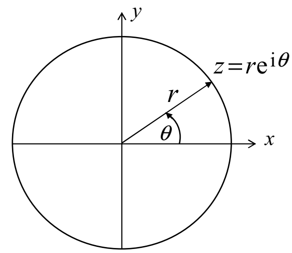

Figure 2.3. Argand diagram displaying the point z in terms of the polar variables r and θ.

where i is the imaginary unit, with the property that $\mathrm{i}^2 = -1$, and x and y are real numbers. Parenthetically it is noted that the symbol j is used in various branches of engineering in place of i. Since z is determined by an ordered pair (x, y), it can be conveniently represented diagrammatically by points in the xy-plane, called the complex plane, or, as it is also termed, an Argand diagram. In many applications, a polar representation of a complex number is of considerable value. The point z in Figure 2.3 can be characterized by (x, y) or by (r, θ), where r, the modulus, and θ, the argument, are respectively given by

$$\operatorname{mod} z = |z| = |x + \mathrm{i}y| = r = \sqrt{(x^2 + y^2)} \tag{2.68}$$

and

$$\arg z = \theta = \tan^{-1}\left(\frac{x}{y}\right). \tag{2.69}$$

An idea that will find later use is the ability to express z in a polar representation about some point z_0, which is not the origin of the x–y coordinates. In this case,

$$z = z_0 + R\mathrm{e}^{\mathrm{i}\theta}, \tag{2.70}$$

where the circle on which z lies has a radius R and origin z_0, and the angle θ is measured from the new origin.

A function f is continuous at a point z_0 if, for a given $\varepsilon > 0$, there exists a number δ such that

$$|f(z) - f(z_0)| < \varepsilon, \tag{2.71}$$

for all points z in a domain satisfying the condition $|z - z_0| < \delta$. If f is a single-valued function in a domain of the complex plane, then f is differentiable at a point z_0 of that domain, if

$$\frac{f(z) - f(z_0)}{z - z_0}$$

tends to a unique limit as $z \to z_0$, and this limit is denoted $f'(z_0)$, that is

$$f'(z_0) = \left.\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right|_{z=z_0}. \tag{2.72}$$

If a function f is single-valued and differentiable at every point z in a domain of the complex plane, then the function is called *analytic* in that domain. The terms *holomorphic* and *regular* are often used synonymously with analytic. If a function f is differentiable in a domain, except for a finite number of points, those points are called the *singular* points (or singularities) of the function.

There are several types of singular points that will be encountered later. The key types are as follows. A singular point z_0 is termed *isolated* if there is a neighborhood of sufficiently small radius ε surrounding the point containing no other singular points. If the following condition holds for positive integer m,

$$\lim_{z\to z_0} (z - z_0)^m f(z) = a \neq 0, \tag{2.73}$$

then the point z_0 of $f(z)$ is termed a *pole of order m*. The case where $m = 1$ is called a *simple pole*. If a function f can be defined at a singular point z_0, that is a meaning can be attached to $\lim_{z\to z_0} f(z)$, then the singularity is called a *removable* singularity. A singularity that is neither an isolated singularity nor a pole, is termed an *essential* singularity. The latter is defined more rigorously later in Section 2.8.5. The preceding discussion of singularities can be recast in terms of the Laurent series expansion for an analytic function, and that topic is also treated in Section 2.8.5. Here are some examples of the types of singularities that have just been discussed. The function $f(z) = (z^2 - \alpha^2)^{-1}$ has simple poles at $z = \alpha$ and $z = -\alpha$, while the function $f(z) = (z - \beta)^{-4}$ has a pole of order four at $z = \beta$; in both cases, the singularities are isolated. A function with a removable singularity is $f(z) = z^{-2}(1 - \cos z)$, which is finite in the limit $z \to 0$. The function $f(z) = \sin z^{-1}$ has an essential singularity at $z = 0$.

Let $f(z)$ be written in the form

$$f(z) = u(x,y) + \mathrm{i}v(x,y), \tag{2.74}$$

where $u(x,y)$ and $v(x,y)$ are real-valued functions. Then the necessary and sufficient conditions for f to be analytic in a domain of the complex plane is that the four partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ exist and satisfy in that domain the following:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2.75}$$

These are called the Cauchy–Riemann equations.

Functions that are not single-valued will be encountered in later chapters. Consider the case $f(z) = z^{1/n}$ with $n = 2, 3, 4, \ldots$. This is an example of a multiple-valued function. The case $n = 2$ is examined in the sequel in some detail. If z is expressed in polar notation, that is $z = re^{\mathrm{i}\theta}$, then

$$f(z) = \sqrt{(r)}\, e^{\mathrm{i}\theta/2}, \tag{2.76}$$

where θ denotes the angle shown in Figure 2.4. If one complete cycle around the contour labeled Γ is made, starting and returning to the point P, then $\theta \to \theta + 2\pi$, and

$$f(z) = \sqrt{(r)}\, e^{\mathrm{i}(\theta+2\pi)/2} = -\sqrt{(r)}\, e^{\mathrm{i}\theta/2}, \tag{2.77}$$

which results in a different value for the function. By making another complete circuit of the contour, the original value is obtained. A point in the complex plane is called a *branch point* if the value of a function is altered when one complete cycle around the point is completed. The point $z = 0$ is a branch point of the function $\sqrt{z}$. The function $f(z) = \sqrt{z}$ can be converted to a single-valued function by agreeing to restrict θ, so that

$$\begin{aligned} f(z) &= \sqrt{(r)}\, e^{\mathrm{i}\theta/2}, \quad \text{for } 0 \le \theta < 2\pi, \\ f(z) &= \sqrt{(r)}\, e^{\mathrm{i}(\theta+2\pi)/2}, \quad \text{for } 0 \le \theta < 2\pi. \end{aligned} \tag{2.78}$$

The function f can be regarded as being defined on two separate sheets, the collection of which is called the Riemann surface for the function, as in Figure 2.5.

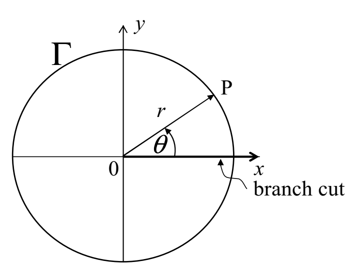

Figure 2.4. Branch cut and polar notation for the function $f(z) = \sqrt{z}$.

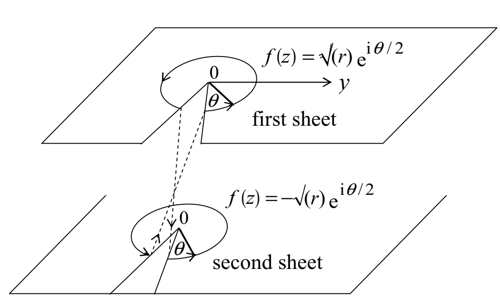

Figure 2.5. Riemann surfaces for the function $\sqrt{z}$.

To achieve a connection between the two sheets a *branch line* (or *branch cut*) is made from $z = 0$ to $z = \infty$, often taken along the positive real axis, although it can be set at any angle from the real axis, and all points that lie on this line are removed from the definition of the function. The outcome is that, on each sheet, the function is single-valued and continuous.

2.8.1 Integration of analytic functions

To carry out an integration of a function of a complex variable, it is necessary to specify the path of integration. Consider the curve shown in Figure 2.6. If the integral from (z_1, z_2) is to be evaluated, it is necessary to specify the path of the contour, that is

$$\int_{z_1}^{z_2} f(z)\mathrm{d}z \to \int_C f(z)\mathrm{d}z, \tag{2.79}$$

where the notation $\int_C$ symbolizes not only the endpoints, but also the pathway between them. A contour, or piecewise smooth arc, is said to be closed if the starting and ending points are the same. A contour described parametrically by $z(t)$ is called *smooth* if $z(t)$ has a continuous derivative along the entire contour; it is called *piecewise smooth* if the contour can be represented by a finite sum of smooth arcs. Figure 2.7 indicates a *simple* closed contour and Figure 2.8 illustrates a non-simple closed contour.

An integral taken around a closed contour C is often indicated by $\oint_C$. The sense of direction for the integral needs to be specified. Unless stated to the contrary, all contour integrals will assume a *counter-clockwise* sense. This is sometimes indicated with an arrow pointing in the counter-clockwise sense on the circle for the integral symbol just given.

In the following discussion, and for the rest of this book, the focus is on regions (designated $\mathcal{R}$) that are *simply connected*. A simply connected region $\mathcal{R}$ is one where

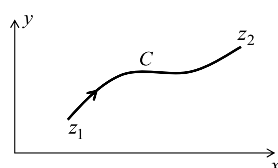

Figure 2.6. Contour showing the path between z_1 and z_2.

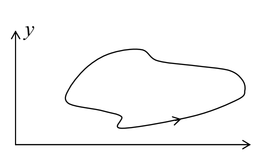

Figure 2.7. Simple closed contour.

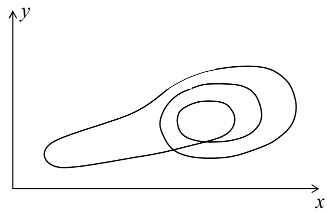

Figure 2.8. Non-simple closed contour.

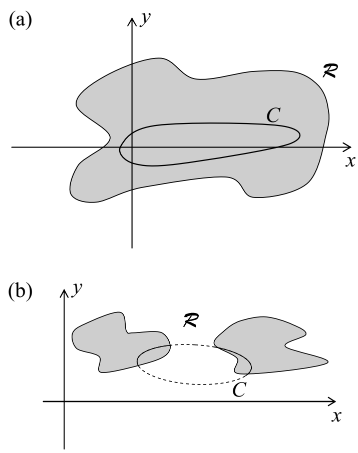

Figure 2.9. (a) A simply connected region. (b) A region that is not simply connected.

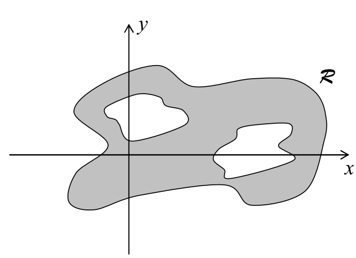

Figure 2.10. A multiply connected region.

any closed contour in $\mathcal{R}$ contains only points belonging to $\mathcal{R}$. Figure 2.9(a) shows a simply connected region, and Figure 2.9(b) illustrates a region that is not simply connected, that is, it is disconnected.

The dashed portion of the curve C in Figure 2.9(b) does not lie entirely in $\mathcal{R}$. Figure 2.10 illustrates a multiply connected region. In this case, it is possible to connect points within $\mathcal{R}$ by different contours which cannot be continuously deformed into one another without passing through points not in the region $\mathcal{R}$.

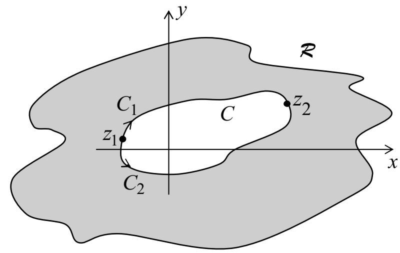

Figure 2.11. C_1 is the contour from z_1 to z_2 in the clockwise sense and C_2 is the contour from z_1 to z_2 in the counter-clockwise sense. C denotes the entire contour.

2.8.2 Cauchy integral theorem

One of the two results that will prove to be central in later developments is the Cauchy integral theorem (the other is the Cauchy integral formula, which is discussed in Section 2.8.3). The key result of this section is also referred to as the Cauchy–Goursat theorem. If $f(z)$ is analytic and if $f'(z)$ is continuous at each point of a domain, including the boundary C, then

$$\oint_C f(z)\mathrm{d}z = 0. \tag{2.80}$$

As a simple example, consider the choice $f(z) = z^2$, with the contour C taken to be a circle of radius r and center at the origin, then it is straightforward calculation (using $z = r\mathrm{e}^{\mathrm{i}\theta}$) to show that Eq. (2.80) holds true. There are a number of proofs of this theorem available (see the end-notes), with different starting assumptions. Equation (2.80) with the conditions specified is due to Cauchy. The hypothesis requiring $f'(z)$ to be continuous was discovered by Goursat to be unnecessary. That leads to the statement of the Cauchy–Goursat theorem as: let $f(z)$ be analytic in a domain and on its boundary C. Then Eq. (2.80) holds.

A result that has considerable practical value is based on the type of closed contour shown in Figure 2.11. For $f(z)$ analytic in the region $\mathcal{R}$, it follows that

$$\int_C f(z)\mathrm{d}z = \int_{-C_1} f(z)\mathrm{d}z + \int_{C_2} f(z)\mathrm{d}z = 0 \tag{2.81}$$

and hence

$$\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z. \tag{2.82}$$

In writing Eq. (2.81), it is assumed that the contour is traversed in the counter-clockwise orientation, which is taken to be the positive sense along the curve. In later sections, a number of contours are examined that include segments that, in some appropriate limit, have the same form as in Eq. (2.81). When this occurs, the evaluation of the overall contour integral is simplified. Equation (2.80) has a number of important practical applications in applied fields, and in particular to topics to be discussed later.

There is a converse theorem due to Morera. Let $f(z)$ be continuous throughout a region $\mathcal{R}$; if

$$\oint_C f(z)\mathrm{d}z = 0 \tag{2.83}$$

for every contour C in $\mathcal{R}$, then $f(z)$ is analytic throughout $\mathcal{R}$.

2.8.3 Cauchy integral formula

The Cauchy integral formula is one of the most powerful results in analytic function theory. The formula finds extensive application in the discussion of Hilbert transforms. Let $f(z)$ be analytic within and on a closed contour C, then, if z_0 is an interior point of C,

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{z - z_0}\,\mathrm{d}z. \tag{2.84}$$

This is a rather remarkable mathematical result: the value of $f(z)$ at an interior point z_0 is determined by the values taken by the function on a prescribed contour encircling the point. There is no analog of this result in the theory of functions of a real variable.

As an elementary example, consider the integral

$$\frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{z-2}\,\mathrm{d}z,$$

where C is a circular contour of radius R centered at $z = 2$ and $f(z) = z^2$. Then a straightforward calculation, on writing $z - 2 = R\mathrm{e}^{\mathrm{i}\theta}$, yields the following:

$$\begin{aligned}
\frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{z-2}\,\mathrm{d}z &= \frac{1}{2\pi\mathrm{i}} \oint_C \frac{z^2}{z-2}\,\mathrm{d}z \\
&= \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{(2 + R\mathrm{e}^{\mathrm{i}\theta})^2 \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{R\mathrm{e}^{\mathrm{i}\theta}} \\
&= \frac{1}{2\pi} \int_0^{2\pi} (2 + R\mathrm{e}^{\mathrm{i}\theta})^2\,\mathrm{d}\theta \\
&= 4.
\end{aligned} \tag{2.85}$$

2.8.4 Jordan's lemma

A function that is analytic everywhere in the finite part of the complex plane, except at a finite number of poles, is said to be *meromorphic*. Jordan's lemma is as follows. If C denotes a semicircular contour with radius R and center the origin, and $f(z)$ is

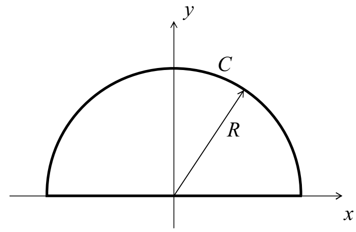

Figure 2.12. Semicircular contour in the upper half plane.

meromorphic in the upper half plane and satisfies the condition $|f(z)| \to 0$ uniformly as $|z| \to \infty$ for $0 \leq \arg z \leq \pi$, then, for $m > 0$,

$$\lim_{R\to\infty} \int_C e^{imz} f(z) dz = 0. \tag{2.86}$$

The first part of the stated condition means that there exists a radius R_0 chosen sufficiently large to enclose all the singularities of $f(z)$. The uniformity constraint implies that a constant $\varepsilon > 0$ can be selected so that, for some radius $R > R_0$, and for z on C it follows that

$$|f(z)| \leq \varepsilon. \tag{2.87}$$

The contour being employed is shown in Figure 2.12. Semicircular contours of this type find routine application in many problems of scientific interest. They are most frequently converted into closed semicircular contours by incorporating a segment of the real axis from $-R$ to R. The conditions given in the statement of Jordan's lemma for Eq. (2.86) to hold are sufficient. The result actually applies under weaker conditions. These conditions are encountered in later sections.

2.8.5 The Laurent expansion

Let $f(z)$ be analytic within and on the boundary of a circle C, centered at the point $z = z_0$, then the value of the function f at any point within C can be represented by the power series given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{2.88}$$

where

$$a_n = \frac{1}{n!} \frac{d^n f(z)}{dz^n}\bigg|_{z=z_0} = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}. \tag{2.89}$$

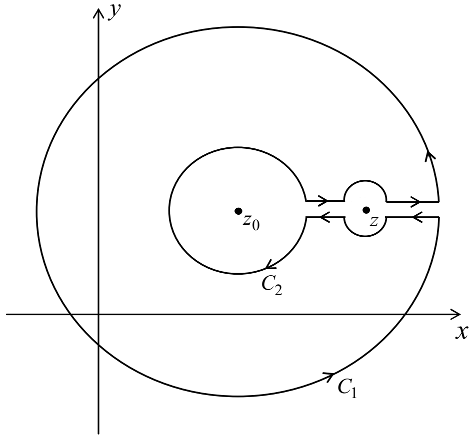

Figure 2.13. The circular contours C_1 and C_2 have a common origin at z_0.

This is the generalization of the well known Taylor series expansion to cover analytic functions. Its proof, which starts with the Cauchy integral formula, is straightforward, and is left as an exercise for the reader.

If the function $f(z)$ is not analytic in C, then the following approach can be employed. Suppose $f(z)$ is analytic in the annular region and on the concentric circular contours C_1 and C_2 (see Figure 2.13), then $f(z)$ can be expanded in the power series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{2.90}$$

where

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{f(z)\mathrm{d}z}{(z - z_0)^{n+1}} \tag{2.91}$$

and Γ is any contour contained within the annular region and encircling the point z_0. The contributions from the horizontal sections of the contour adjacent to the point z cancel as the radius of the semicircular arcs at this point shrink to zero. The directional orientation of these straight line segments is obviously important to obtain the cancellation. The proof of Eq. (2.90) follows directly upon integrating $(z' - z)^{-1} f(z')$ (with respect to z') around the contour shown in Figure 2.13 and making appropriate expansions of the denominators for the contour integrals on C_1 and C_2. The expansion in Eq. (2.90) is the Laurent series for an analytic function. The contribution to the Laurent series from the terms with $n \geq 0$ is called the *analytic part*, and the remaining terms are called the *principal part*.

The Laurent series facilitates the discussion of singular points. Suppose that $f(z)$ has an isolated singularity at the point $z = z_0$ and is analytic everywhere else within and on the boundary of a circle C centered at $z = z_0$. Then the Laurent series converges for all points z in the circle C except at the point z_0. If

$$f(z) = \cdots \frac{a_{-3}}{(z - z_0)^3} + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots, \tag{2.92}$$

then $f(z)$ has a *pole of order m*$(m > 0)$ at $z = z_0$ if $a_n = 0$ for $n = -(m+1), -(m+2), \ldots$. If the principal part is *non-terminating*, that is there is no value m satisfying the aforementioned condition, then $f(z)$ has an *essential singularity* at $z = z_0$. When $f(z)$ has a pole of order m, the coefficient a_{-1} is called the residue of $f(z)$ at $z = z_0$. For a pole of order one, the residue, which is denoted by $\mathrm{Res}(z_0)$ (the notation $R_0(z_0)$ is also sometimes seen in the literature), can be determined from the following formula:

$$a_{-1} \equiv \mathrm{Res}(z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{2.93}$$

When the context does not make it clear that the function f is under consideration, then the notation $\mathrm{Res}\{f(z)\}_{z=z_0}$ is employed to designate the residue at $z = z_0$. For a pole of order m, it is easy to show from the Laurent series that

$$a_{-1} \equiv \mathrm{Res}(z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \{(z - z_0)^m f(z)\}. \tag{2.94}$$

2.8.6 The Cauchy residue theorem

The Cauchy residue theorem, or, as it is frequently called, the residue theorem, plays a central role in a number of places in this book. It is an essential bit of mathematical machinery required for the evaluation of many Hilbert transforms. Suppose $f(z)$ is analytic within and on a closed contour C except for the point $z = z_0$, where the function has a pole of order m. With reference to Figure 2.14, application of the

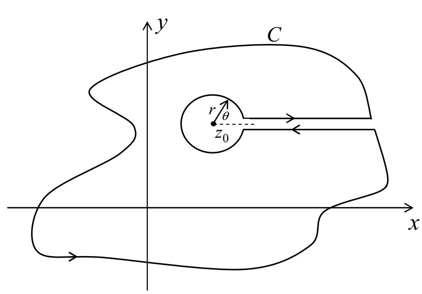

Figure 2.14. Standard modification of a contour to avoid a singularity.

Cauchy–Goursat theorem leads to

$$\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &= \oint_C \sum_{n=-m}^{\infty} a_n (z-z_0)^n\,\mathrm{d}z \\
&= \sum_{n=-m}^{\infty} a_n \oint_C (z-z_0)^n\,\mathrm{d}z \\
&= \sum_{n=-m}^{\infty} a_n \int_0^{2\pi} r^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}\,\mathrm{i}\,\mathrm{d}\theta \\
&= 2\pi\mathrm{i} \sum_{n=-m}^{\infty} a_n \delta_{n,-1} \\
&= 2\pi\mathrm{i}a_{-1}.
\end{aligned} \tag{2.95}$$

The preceding argument can be generalized in a straightforward fashion to include a finite number of poles. The residue theorem is stated thus: suppose $f(z)$ is analytic within and on the boundary of a closed contour C, except for a finite number of poles, then

$$\oint_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{j=1} \mathrm{Res}(z_j), \tag{2.96}$$

where the sum runs over the finite number of poles at the points z_j located within C, and $\mathrm{Res}(z_j)$ denotes the residue corresponding to the pole at $z = z_j$. Cauchy's theorem (Eq. (2.80)) obviously corresponds to the special case of Eq. (2.96) when there are no poles in the region enclosed by C.

2.8.7 Entire functions

A function analytic everywhere in the finite complex plane is called an *entire* function. The synonym sometimes employed is *integral* function. Two examples of entire functions are $\sin z$ and e^z. Note that the point at infinity, where the function may have an isolated singularity, is excluded from the definition of an entire function. If an entire function is bounded at infinity, then it can be shown that the function is a constant in the whole complex plane (Liouville's theorem).

Let $f(z)$ be a meromorphic function with poles, assumed for simplicity to be simple, at $z_k, k = 1, \ldots, n$, and arranged in order of increasing absolute value. Denote by $r_k, k = 1, \ldots, n$, the residues of the poles at z_k. Let C be a circular contour of radius R centered at the origin, passing through no singularity of $f(z)$ and enclosing n poles. Suppose that $f(z)$ is bounded above by the constant M assumed independent of n, $|f(z)| < M$, then, in the limit $R \to \infty$, a short calculation using the residue theorem leads to

$$f(z) = f(0) + \sum_{k=1}^{n} r_k \left\{ \frac{1}{z - z_k} + \frac{1}{z_k} \right\}. \tag{2.97}$$

This is referred to as the Mittag–Leffler expansion of $f(z)$.

Suppose $g(z)$ is a meromorphic function whose poles are all simple (of order one) and are located at $\alpha_1, \alpha_2, \alpha_3, \ldots$, then an entire function $f(z)$ can be constructed from

$$f(z) = g(z) - \sum_{n=1}^{\infty} \frac{a_n}{z - \alpha_n}, \tag{2.98}$$

where a_n is the residue of $g(z)$ at $z = \alpha_n$. If it is assumed that $g(z)$ is bounded at infinity, then so is $f(z)$, and hence $f(z)$ is a constant. Therefore from Eq. (2.98) it follows that

$$g(z) = g(0) + \sum_{n=1}^{\infty} \left\{ \frac{a_n}{z - \alpha_n} + \frac{a_n}{\alpha_n} \right\}. \tag{2.99}$$

This result is a particular case of the Mittag–Leffler expansion of a meromorphic function.

Suppose that $f(z)$ is entire and has first-order zeros at $\beta_1, \beta_2, \beta_3, \ldots$, none of which are located at the origin, then from this function the following meromorphic function can be constructed, which has simple poles at $\beta_1, \beta_2, \beta_3, \ldots$:

$$g(z) = \frac{\mathrm{d} \log f(z)}{\mathrm{d}z} = \frac{1}{f(z)} \frac{\mathrm{d}f(z)}{\mathrm{d}z}. \tag{2.100}$$

The logarithm of a complex number being understood in the sense that if $z = \mathrm{e}^w$, then $w = \log z$. The function $\log z$ is multi-valued. If $g(z)$ satisfies the conditions necessary to carry out a Mittag–Leffler expansion, then

$$\frac{\mathrm{d} \log f(z)}{\mathrm{d}z} = \left. \frac{\mathrm{d} \log f(z)}{\mathrm{d}z} \right|_{z=0} + \sum_{n=1}^{\infty} \left\{ \frac{1}{z - \beta_n} + \frac{1}{\beta_n} \right\}. \tag{2.101}$$

From Eq. (2.101) it follows that

$$\int_0^z \frac{\mathrm{d} \log f(z')}{\mathrm{d}z'} \mathrm{d}z' = \int_0^z \left[\left. \frac{\mathrm{d} \log f(z')}{\mathrm{d}z'} \right|_{z'=0} + \sum_{n=1}^{\infty} \left\{ \frac{1}{z' - \beta_n} + \frac{1}{\beta_n} \right\} \right] \mathrm{d}z'; \tag{2.102}$$

that is,

$$\log f(z) = \log f(0) + z \left[\frac{\mathrm{d} \log f(z)}{\mathrm{d}z} \right]_{z=0} + \sum_{n=1}^{\infty} \left[\log \left(\frac{\beta_n - z}{\beta_n} \right) + \frac{z}{\beta_n} \right], \tag{2.103}$$

which can be rewritten as follows:

$$f(z) = f(0) \exp\left\{z\left[\frac{\mathrm{d}\log f(z)}{\mathrm{d}z}\right]_{z=0}\right\}\prod_{n=1}^{\infty}\left(1-\frac{z}{\beta_n}\right)\mathrm{e}^{\beta_n^{-1}z}. \tag{2.104}$$

If $f(z)$ is an even function, the summation limits in Eq. (2.101) extend from $-\infty$ to ∞ with $n = 0$ omitted, and the condition $\beta_{-n} = -\beta_n$ applies, so, with the appropriate change to the lower summation limit, Eq. (2.103) simplifies to

$$f(z) = f(0)\prod_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\left(1-\frac{z}{\beta_n}\right)\mathrm{e}^{\beta_n^{-1}z} = f(0)\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\beta_n^2}\right). \tag{2.105}$$

Equations (2.104) and (2.105) are particular cases of the Weierstrass factorization formula. A simple example is $f(z) = z^{-1}\sin z$, for which the zeros are located at $\beta_n = -\beta_{-n} = n\pi$, for $n \in \mathbb{Z}$, and hence

$$\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right). \tag{2.106}$$

The rate of growth of an entire function is characterized by its *order*. An entire function f is of order ρ if, for $\theta_1 < \theta < \theta_2$,

$$\lim_{r\to\infty}\sup\frac{\log\left|f(r\mathrm{e}^{\mathrm{i}\theta})\right|}{r^{\rho+\varepsilon}} = 0, \tag{2.107}$$

for each $\varepsilon > 0$, and uniformly for θ in the given range. The reader is reminded of the following set theory notation. A set S that is bounded above has a least upper bound that is called the *supremum*, denoted $\sup S$, and if the set S is bounded below it has a greatest lower bound, called the *infimum*, which is denoted $\inf S$. Equation (2.107) is frequently stated in terms of the maximum modulus, denoted $M(r)$,

$$M(r) = \max_{0\leq\theta\leq 2\pi}\left|f(r\mathrm{e}^{\mathrm{i}\theta})\right| \tag{2.108}$$

so that

$$\rho = \lim_{r\to\infty}\sup\frac{\log\log M(r)}{\log r}. \tag{2.109}$$

By convention, the order of a constant is zero. Another factor used in the characterization of entire functions is the *type*. An entire function $f(z)$ of order ρ has type σ given by

$$\sigma = \lim_{r\to\infty}\sup\frac{\log M(r)}{r^{\rho}}, \tag{2.110}$$

with $0 \le \sigma \le \infty$. A function $f(z)$ of order one and type σ ($\sigma < \infty$) is said to be a function of exponential type σ. If

$$f(z) = O(\mathrm{e}^{a|z|}), \quad z \to \infty, \tag{2.111}$$

where a is a positive constant, then σ is the lower bound of a.

2.9 Conformal mapping

Conformal mapping is a powerful problem-solving technique that finds widespread use in complex analysis. A few of the key aspects of conformal mapping that are used in later applications are summarized in this section.

The equations

$$u = u(x,y) \quad \text{and} \quad v = v(x,y) \tag{2.112}$$

set up a correspondence between a domain in the xy-plane and a domain of the uv-plane. If each point in the xy-plane corresponds to only one point in the uv-plane, and conversely, then this is called a one-to-one transformation or mapping. It can be shown that the transformation is one-to-one if u and v are continuously differentiable in a domain D and if the Jacobian of the transformation, $\partial(u,v)/\partial(x,y)$, where

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}, \tag{2.113}$$

does not vanish in D. If a mapping preserves the sense and the magnitude of angles, as in Figure 2.15, then the transformation is called *conformal*. If only the angles are preserved, the transformation is called *isogonal*.

Suppose C denotes a simple closed curve enclosing the region $\mathcal{R}$ in the complex z-plane, as in Figure 2.16, then the Riemann mapping theorem states that there exists a function $w = f(z)$ that is analytic in $\mathcal{R}$ (which is not the entire complex plane), which maps each point of $\mathcal{R}$ one-to-one into a corresponding point in the unit disc.

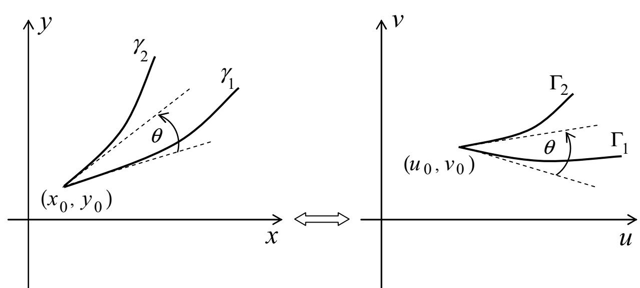

Figure 2.15. Preservation of the angle θ on mapping from the xy-plane to the uv-plane.

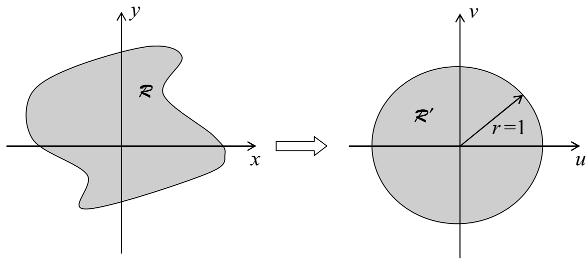

Figure 2.16. Mapping a region $\mathcal{R}$ in the xy-plane to the unit disc in the uv-plane.

Some of the simple and useful transformations are as follows:

$$w = z + \alpha \quad \text{(translation)}; \tag{2.114}$$

$$w = az \quad (\text{contraction for } 0 < a < 1, \text{ stretching for } a > 1); \tag{2.115}$$

$$w = ze^{i\theta} \quad (\text{rotation by } \theta); \tag{2.116}$$

$$w = z^{-1} \quad \text{(inversion)}. \tag{2.117}$$

The *linear* transformation takes the form

$$w = \alpha z + \beta, \tag{2.118}$$

where α and β are complex constants. The *bilinear* transformation is given by

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \text{for } \alpha\delta - \beta\gamma \neq 0. \tag{2.119}$$

In later sections, there will be interest in mapping the upper half complex z-plane into the interior of the unit circle in the w-plane. This allows a connection to be made between functions analytic in the upper half plane and conjugate series. The conversion is accomplished using the Cayley transformation,

$$w = -\frac{z - i}{z + i}, \tag{2.120}$$

and is illustrated in Figure 2.17.

It is possible to give a generalization of the bilinear transformation given in Eq. (2.119) (see, for example, Miller, 1970, p. 218 and Nehari, 1975, p. 164). Tables of various conformal transformations can be found in a number of sources (see, for example, Kober (1957) or Krantz (1999b)).

A number of more intricate issues involving analytic functions that build off the theorems and definitions given in Sections 2.8 and 2.9 are discussed in subsequent chapters. It will quickly become apparent to the reader that analytic function theory is one of the principal underlying mathematical structures of much of the original work in the area of Hilbert transform theory.

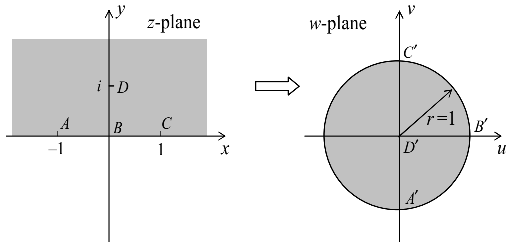

Figure 2.17. Mapping of the upper half of the xy-plane into the unit disc in the uv-plane.

2.10 Some functional analysis basics

The standard notation from set theory that is employed throughout is as follows. The elements x of a set having a property P are written $\{x : P\}$ or as $\{x|P\}$, for example, $A = \{x : a \leq x \leq b\}$ is read as: A is the set of x for which the condition $a \leq x \leq b$ holds. The empty set containing no members is written as $\varnothing$. If α is an element of A, this is written as $\alpha \in A$, and if α is not an element of A, this is denoted by $\alpha \notin A$. If the set A is a subset of the set B, then $A \subset B$ or $B \supset A$. If $A \subset B$ and $A \neq B$ then A is termed a proper subset of B, and the symbol $\subset$ is often used to indicate a proper subset. If the set A is a subset of the set B with the possibility that $A = B$, then $A \subseteq B$. The intersection of sets is written as $A \cap B$, which is the set of members belonging to both A and B, or $A \cap B = \{x : x \in A \text{ and } x \in B\}$. The union of sets is written as $A \cup B$, which is the set of members belonging to either A or B, that is $A \cup B = \{x : x \in A \text{ or } x \in B\}$. The difference (or complement) of two sets, $A - B$, is denoted by $A \backslash B$, and $A \backslash B = \{x : x \in A \text{ and } x \notin B\}$. If every point in a set S has a neighborhood lying in the set, then S is called an *open set*. An open interval (a, b) is an example of an open set. A set S is called a *closed set* if its complement is open. The closed interval $[a, b]$ is a closed set. The complement of $[a, b]$ is $(-\infty, a) \cup (b, \infty)$, which is an open set. If a set S of real numbers x is such that for all members of the set there exists numbers M and m such that $m \leq x \leq M$, then the set is called a *bounded set*. The set is bounded above by the upper bound M and bounded below by the lower bound m.

Let $S \subset \mathbb{R}$. A point p is a limit point (also termed an accumulation point) of a set of points S iff every neighborhood of p contains at least one point of S distinct from p. The abbreviation iff stands for *if and only if*. That is, if A is an open set containing p, $S \cap (A \backslash \{p\}) \neq \varnothing$. Let $S \subset \mathbb{R}$, then the closure of the set S denoted by $\bar{S}$ is the set S together with all the limits points of S.

A few definitions from elementary linear algebra that are employed in later chapters are recounted. The set of complex numbers is denoted by $\mathbb{C}$. Consider the set (designated V_n) of elements $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$ ($\equiv \{\phi_i\}$) for which the following operations hold:

addition,

$$\phi_i = \phi_j + \phi_k; \tag{2.121}$$

multiplication by a scalar α,

$$\phi_i = \alpha\phi_j. \tag{2.122}$$

Suppose that the following conditions apply:

(i) if $\phi_i \in V_n$ and $\phi_j \in V_n$, then $(\phi_i + \phi_j) \in V_n$; (2.123)
(ii) if $\phi_i \in V_n$ and α is a (complex) constant, then $\alpha\phi_i \in V_n$; (2.124)
(iii) $\exists$ a null element 0, such that $\phi_i + 0 = \phi_i$, for any $\phi_i \in V_n$; (2.125)
(iv) for any $\phi_i \in V_n, \exists$ a $\phi_i' \in V_n$, such that $\phi_i' + \phi_i = 0$; (2.126)
(v) $1 \cdot \phi_i = \phi_i$, for all $\phi_i \in V_n$. (2.127)

Suppose also that the commutative and associative laws of addition apply:

(vi) $\phi_i + \phi_j = \phi_j + \phi_i$; (2.128)
(vii) $(\phi_i + \phi_j) + \phi_k = \phi_i + (\phi_j + \phi_k)$; (2.129)

and that for $\alpha, \beta \in \mathbb{C}$:

(viii) associative law of multiplication, $\alpha(\beta\phi_i) = (\alpha\beta)\phi_i$; (2.130)
(ix) distributive law, $(\alpha + \beta)\phi_i = \alpha\phi_i + \beta\phi_i$; (2.131)
(x) distributive law, $\alpha(\phi_i + \phi_j) = \alpha\phi_i + \alpha\phi_j$. (2.132)

Then the set $\{\phi_i\}$ having the properties (i)–(x) is called a *linear vector space* and ϕ_i are called vectors.

A group is a set of elements with a binary (usually multiplication) operation such that the following apply. (i) If a and b are elements of the group, then so is ab. (ii) For all a, b, and c in the group, then $a(bc) = (ab)c$. (iii) The group contains an identity element e such that $ae = ea = a$ for each element a of the group. (iv) For each element a of the group there is a unique inverse element a^{-1} in the group satisfying $aa^{-1} = a^{-1}a = e$. With the additional condition $ab = ba$ for elements a and b in the group, it is called a commutative or Abelian group. A set of elements for which a binary multiplication operation is defined that is associative, and in which the domain of the set includes all ordered pairs of the set, is termed a semigroup. There is no requirement for an identity or inverse element in a semigroup. A field is a set of elements for which the operations of addition and multiplication are defined and the following conditions hold. With addition as the group operation, the set is a commutative group. With multiplication as the group operation, with 0 omitted, the set is a group, with multiplication commutative. The distributive property holds for all elements in the set.

Two linear vector spaces X and Y over the same field are isomorphic if there exists a one-to-one mapping of the vectors of X onto the vectors of Y such that if $T : X \to Y$, then, for $x, y \in X$ and a scalar c, (i) $T(x+y) = Tx+Ty$; (ii) $T(cx) = cTx$.

The *inner product* of two vectors is denoted by (ϕ_i, ϕ_j). The inner product (ϕ_i, ϕ_j) for continuous functions on a suitable space is given by $\int \phi_i^*(t)\phi_j(t)\mathrm{d}t$, where the integral is over the domain for which $\phi_n(t)$ is defined. A notational device that is widely used, particularly in the physics community, is the so-called bra–ket symbolism of Dirac. In this convention, a vector is represented by the ket symbol $|\phi\rangle$ and the bra is $\langle\phi|$. The inner product is written in Dirac notation as $\langle\phi_i|\phi_j\rangle$. The inner product has the following properties:

$$(\phi_i, \phi_j) = (\phi_j, \phi_i)^*, \tag{2.133}$$

$$(\alpha\phi_i + \beta\phi_j, \phi_k) = \alpha(\phi_i, \phi_k) + \beta(\phi_j, \phi_k), \tag{2.134}$$

and

$$(\phi_i, \phi_i) \geq 0, \text{ and } (\phi_i, \phi_i) = 0, \quad \text{iff } \phi_i = 0. \tag{2.135}$$

In Eq. (2.133) the $*$ denotes complex conjugation. The *norm* (length) of a vector ϕ is written as follows:

$$\|\phi\| = \surd(\phi, \phi). \tag{2.136}$$

The norm $\|\phi\|$ satisfies the following three properties:

$$\|\phi\| \geq 0, \text{ and } \|\phi\| = 0, \quad \text{iff } \phi = 0, \tag{2.137}$$

$$\|\alpha\phi\| = |\alpha| \, \|\phi\|, \tag{2.138}$$

and

$$\left\|\phi_i + \phi_j\right\| \leq \|\phi_i\| + \left\|\phi_j\right\|. \tag{2.139}$$

Equation (2.139) is termed the *triangle inequality*. A space equipped with a norm is termed a normed space.

A linear operator $\mathcal{O}$ on a vector space V_n is a procedure for determining, for each $\phi_i \in V_n$, a unique ϕ_j, also in V_n, that is

$$\phi_j = \mathcal{O}\phi_i. \tag{2.140}$$

For a scalar α, and linear operators $\mathcal{O}_1$ and $\mathcal{O}_2$, then the following must hold:

$$\mathcal{O}_1(\phi_i + \phi_j) = \mathcal{O}_1\phi_i + \mathcal{O}_1\phi_j, \tag{2.141}$$

$$(\mathcal{O}_1 + \mathcal{O}_2)\phi_i = \mathcal{O}_1\phi_i + \mathcal{O}_2\phi_i, \tag{2.142}$$

$$(\mathcal{O}_1\mathcal{O}_2)\phi_i = \mathcal{O}_1(\mathcal{O}_2\phi_i), \tag{2.143}$$

and

$$\mathcal{O}_1\alpha\phi_i = \alpha\mathcal{O}_1\phi_i. \tag{2.144}$$

In general, the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ do not commute, that is

$$\mathcal{O}_1\mathcal{O}_2\phi_i \neq \mathcal{O}_2\mathcal{O}_1\phi_i. \tag{2.145}$$

The commutator of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ is defined by

$$[\mathcal{O}_1, \mathcal{O}_2] = \mathcal{O}_1\mathcal{O}_2 - \mathcal{O}_2\mathcal{O}_1. \tag{2.146}$$

This is a useful notational device for representing the difference in the order of application of a product of operators on some element of a vector space.

A particularly important class of operator equations is of the form

$$\mathcal{O}\phi_i = \alpha_i\phi_i, \tag{2.147}$$

which are termed *eigenvalue* problems. The function ϕ_i in Eq. (2.147) is called an *eigenfunction* (or, less commonly, a characteristic function), and the constant α_i is termed the *eigenvalue* (less frequently, the characteristic value).

The *adjoint* operator of $\mathcal{O}$ is denoted by $\mathcal{O}^+$ (the notation $\mathcal{O}^\dagger$ is also widely employed), and is defined by the requirement that

$$(\phi_i, \mathcal{O}^+\phi_j) = (\phi_j, \mathcal{O}\phi_i)^*, \quad \text{for any } \phi_i, \phi_j \in V_n. \tag{2.148}$$

The inverse of an operator is denoted by $\mathcal{O}^{-1}$, and is defined by the following relationship:

$$\mathcal{O}^{-1}\mathcal{O} = \mathcal{O}\mathcal{O}^{-1} = E, \tag{2.149}$$

where E is the identity element defined by

$$E\phi_i = \phi_i, \quad \text{for any } \phi_i \in V_n. \tag{2.150}$$

Note that the symbol E is also employed to designate the one-dimensional Euclidean space; however, the context should make it completely clear to the reader which meaning is intended. An operator that is *self-adjoint* satisfies the following condition:

$$\mathcal{O}^+ = \mathcal{O}. \tag{2.151}$$

Such an operator is called Hermitian. This group of operators plays a central role in quantum theory.

2.10.1 Hilbert space

An inner product on a linear vector space V assigns to every pair f and g in V a number (in general complex) denoted by (f, g). A space having an inner product defined on it is termed an inner product space, and is denoted by $\mathcal{H}$. A *Cauchy sequence* in $\mathcal{H}$, $\{\phi_n\}$, is defined by the following property: for every $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that

$$\|\phi_n - \phi_m\| < \varepsilon, \text{ for } n, m > N(\varepsilon). \tag{2.152}$$

The sequence $\{\phi_n\}$ is convergent if there exists an element $\phi \in \mathcal{H}$ such that $\lim_{n\to\infty} \|\phi_n - \phi\| = 0$. A vector space is complete if each Cauchy sequence contained therein converges to an element contained in the space:

$$\lim_{n,m\to\infty} \|\phi_n - \phi_m\| = 0. \tag{2.153}$$

An inner product space for which every Cauchy sequence converges to an element of that space is called a *Hilbert space*.

An operator $\mathcal{O}$ acting on a Hilbert space $\mathcal{H}$ is called *isometric* if

$$\|\mathcal{O}\phi\| = \|\phi\|, \text{ for each } \phi \in \mathcal{H}. \tag{2.154}$$

If in addition to Eq. (2.154) the operator satisfies on $\mathcal{H}$ the condition

$$\mathcal{O}^+ = \mathcal{O}^{-1}, \tag{2.155}$$

it is called a *unitary* operator.

2.10.2 The Hardy space H^p

Analytic functions will prove to be particularly important in the developments of the next chapter. The Hardy spaces deal with these functions and the two cases that arise are the unit disc $|z| < 1$ and the upper half plane $\operatorname{Im} z > 0$. Let f be analytic on the unit disc D and let $0 < p < \infty$, then $f \in H^p(D)$ if

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right) = \|f\|_{H^p}^p < \infty. \tag{2.156}$$

If $p = \infty$, then

$$\|f\|_{H^\infty} = \sup_{z\in D} |f(z)|. \tag{2.157}$$

For the upper half plane, $f \in H^p$ if $f(z)$ is analytic in the upper half plane complex plane and

$$\int_{-\infty}^{\infty} |f(x + iy)|^p \, dx < \infty, \tag{2.158}$$

for all $y > 0$.

2.10.3 Topological space

Topological notions enter into several aspects of an introductory account of the theory of Hilbert transforms. Some of the basic ideas are given in this subsection. Let X denote a nonempty set and let $\mathscr{T}$ be a collection of subsets of X. The collection $\mathscr{T}$ is called a *topology* on X iff the following three axioms hold:

(i) $\varnothing$ and X belong to $\mathscr{T}$;
(ii) the union of any number of sets in $\mathscr{T}$ belongs to $\mathscr{T}$;
(iii) The intersection of any two sets in $\mathscr{T}$ belongs to $\mathscr{T}$.

Consider the example where X is given by

$$X = \{a, b, c, d, e\}, \tag{2.159}$$

and let

$$\mathscr{T} = \{\varnothing, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}, \tag{2.160}$$

then the conditions (i)–(iii) hold, and $\mathscr{T}$ is a topology on X. However if

$$\mathscr{T} = \{\varnothing, X, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}, \tag{2.161}$$

then $\mathscr{T}$ is a not a topology on X, since $\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\} \notin \mathscr{T}$, and so axiom (iii) fails in this example.

If X is a nonempty set and $\mathscr{T}$ is a topology on X, then the pair $(X, \mathscr{T})$ is called a topological space. The sets $\mathscr{T}_i$ contained in $\mathscr{T}$ are called open sets.

Let $(X, \mathscr{T}_1)$ and $(Y, \mathscr{T}_2)$ denote two topological spaces. The topological space $(X, \mathscr{T}_1)$ is a subspace of $(Y, \mathscr{T}_2)$, written $(X, \mathscr{T}_1) \subset (Y, \mathscr{T}_2)$, or, more simply, $\mathscr{T}_1 \subset \mathscr{T}_2$, when $X \subset Y$ and the open sets of $\mathscr{T}_1$ equal the intersections of X with the open sets of $\mathscr{T}_2$.

Given two topological spaces $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$, then $\mathscr{T}_1 \subset \mathscr{T}_2$ if all the open sets of $\mathscr{T}_1$ are in $\mathscr{T}_2$. The topology $\mathscr{T}_1$ is called a *weaker topology* of X than $\mathscr{T}_2$ and $\mathscr{T}_2$ is termed a stronger topology of X than $\mathscr{T}_1$. Retaining the same X as in the preceding example, let

$$\mathscr{T}_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}; \tag{2.162}$$

and

$$\mathscr{T}_2 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\{a, b, c\}, \{a, b, c, d\}\}, \tag{2.163}$$

then clearly $\mathscr{T}_1 \subset \mathscr{T}_2$ and $\mathscr{T}_2$ is a stronger topology on X than $\mathscr{T}_1$.

Two topological spaces $(X, \mathscr{T}_1)$ and $(Y, \mathscr{T}_1)$ are termed topologically equivalent, or homeomorphic, iff there exists a map $f : X \to Y$ such that f is bijective and both f and f^{-1} are continuous. A bijective function is injective (*one-to-one*) and surjective (*onto*). The function f is called a homeomorphism. A function $f : X \to Y$ is invertible iff it is bijective. A continuous linear bijection $f : X \to Y$ is an isomorphism if the inverse f^{-1} is a continuous linear mapping $f^{-1} : Y \to X$. If $X = Y$, then f is called an automorphism on X.

2.10.4 Compact operators

The notion of a compact set is considered first. A set S is called a *compact set* if every infinite sequence of elements in S, $\{X_n\}$ say, has a subsequence that converges to an element of S. Compact sets are bounded. Every closed and bounded interval $[a, b]$ on the real line is an example of a compact set.

Let $\mathcal{O}$ denote an operator mapping the normed space X into the normed space Y. The operator $\mathcal{O}$ is called a *compact operator* if it maps every bounded set in X into a set in Y whose closure is compact. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two compact operators on $\mathcal{H}$, then $\mathcal{O}_1 + \mathcal{O}_2$ is a compact operator, and for a scalar α the operator $\alpha\mathcal{O}_1$ is also compact. Compact operators have important applications in the theory of integral equations. As an example, suppose the kernel function $K(x, y)$ is continuous on the square $0 \le x,\ y \le 1$, then the operator defined by

$$\mathcal{K}f(x) = \int_0^1 K(x, y)f(y)\mathrm{d}y, \tag{2.164}$$

is a compact operator on the space of square integrable functions on $[0, 1]$.

2.11 Lebesgue measure and integration

For most of the applications of the Hilbert transform technique that have been made in the physical sciences the functions of interest represent physically realizable systems and are invariably Hölder continuous over the domain of study. When this is not the case, the function can usually be modified in a straightforward manner so that this condition is true. In such cases, the standard Riemann interpretation of the integral is sufficient. However, it is extremely useful to generalize beyond Riemann integration. The first and obvious reason is that the Hilbert transform can be defined for a wider class of functions. In addition, the jump to reading original papers becomes a lot easier if the reader has had an exposure to Lebesgue integration. This is a vast topic, and only a few of the essential elements of the theory are outlined in this section.

It will help if the reader first recalls the notion of the Riemann integral. Let f be a continuous function defined on the interval $[a, b]$. Suppose the interval $[a, b]$ is divided into a set of subintervals by choosing partition points x_i such that

$$a = x_0 < x_1 < x_2 < x_3 \cdots < x_{m-1} < x_m = b, \tag{2.165}$$

as shown in Figure 2.18.

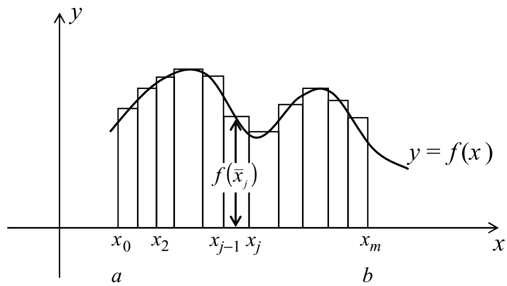

Figure 2.18. Riemann sum used to evaluate an integral.

The length of the jth subinterval is defined by

$$\Delta x_j = x_j - x_{j-1}, \tag{2.166}$$

and $\bar{x}_j$ is used to denote any value in the subinterval $[x_{j-1}, x_j]$. Let $\|p\|$ designate the longest subinterval, that is

$$\|p\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_m\}. \tag{2.167}$$

The Riemann integral of f over the interval $[a, b]$ is given by

$$\int_a^b f(x)\mathrm{d}x = \lim_{\|p\|\to 0} \sum_j f(\bar{x}_j)\Delta x_j, \tag{2.168}$$

assuming the limit exists. If f is positive in the interval $[a, b]$, a useful interpretation is that the integral corresponds to the area under the graph of f from a to b.

Before proceeding, a bit of terminology on types of convergence is reviewed. If the series $\sum_n |a_n|$ converges, then so does the series $\sum_n a_n$, and the series is said to be *absolutely convergent* or the series *converges absolutely*. The converse of the preceding statement is, of course, not true in general. The terminology *square summable* is used for a series that satisfies $\sum_n |a_n|^2 < \infty$. For integrals, $f(x)$ is absolutely integrable on (a, b) if $\int_a^b |f(x)|\mathrm{d}x$ converges. If this integral converges then the integral of $f(x)$ on (a, b) is called absolutely convergent. If $\int_a^b |f(x)|\mathrm{d}x$ is divergent but $\int_a^b f(x)\mathrm{d}x$ converges, then the latter integral is termed *conditionally convergent*. The convergence of $\int_a^b f(x)\mathrm{d}x$ does not, of course, imply that $\int_a^b |f(x)|\mathrm{d}x$ is convergent.

A sequence of functions $f_1(x) + f_2(x) + f_3(x) + \cdots$ converges to a sum $S(x)$ if, for a given $\varepsilon > 0$, there exists a number N (which in general will depend on both ε and x) such that $|S(x) - S_n(x)| < \varepsilon$ for $n > N$, where $S_n(x) = \sum_{k=1}^n f_k(x)$ denotes the nth partial sum. If there exists an N independent of x and depending only on ε, the sequence of functions *converges uniformly* to $S(x)$. In other words, for sufficiently large n the graph of $S_n(x)$ can be bounded above by $S(x) + \varepsilon$ and bounded below by $S(x) - \varepsilon$. If the $f_k(x)$ are continuous in an interval $x \in [a, b]$ and if $\sum_k f_k(x)$ converges uniformly to $S(x)$ for $x \in [a, b]$, then $S(x)$ is continuous for $x \in [a, b]$. Consider the example $f_k(x) = x^2/(1+x^2)^k, k = 1, 2, \ldots,$ for $-1/2 \leq x \leq 1/2$. Then

$$S_n(x) = 1 - \frac{1}{(1 + x^2)^n} \tag{2.169}$$

and hence, in the limit $n \to \infty$,

$$S(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0. \end{cases} \tag{2.170}$$

Since $S(x)$ is not continuous in the interval $-1/2 \leq x \leq 1/2$, the sequence is not uniformly convergent for the stated range of x.

A useful test for determining uniform convergence is the Weierstrass M-test. If $|f_k(x)| \leq M_k$ for $k = 1, 2, \ldots, x \in (a, b)$, and $\sum_k M_k$ converges, then $\sum_k f_k(x)$ converges uniformly for $x \in (a, b)$. For example, the sequence $x, x^2, x^3, \ldots$ for x in the interval $x \in (0, 1/2)$ converges uniformly, since each term is bounded by $1/2$, and the sum $\sum_{k=1}^{\infty} 1/2^k$ converges.

A result that will find considerable application is the fact that a uniformly convergent series of continuous functions can be integrated term-by-term, that is

$$\int_a^b \sum_{k=1}^{\infty} f_k(x) \mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b f_k(x) \mathrm{d}x. \tag{2.171}$$

Note that uniform convergence of a series is not a necessary condition for the term-by-term integration of the series. If the sequence $\{f_k(x)\}$, $k = 1, 2, \ldots,$ is uniformly convergent in the interval $[a, b]$, then

$$\lim_{k \to \infty} \int_a^b f_k(x) \mathrm{d}x = \int_a^b \lim_{k \to \infty} f_k(x) \mathrm{d}x. \tag{2.172}$$

For an unbounded interval, for example $\mathbb{R}$, uniform convergence of the sequence $\{f_k(x)\}$ is not sufficient to establish the interchange in Eq. (2.178). If the sequence $\{f_k(x)\}, k = 1, 2, \ldots,$ is continuous and the $f_k(x)$ have continuous first derivatives in $[a, b]$, and if $\sum_{k=1}^{\infty} f_k'(x)$ is uniformly convergent in the interval $[a, b]$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} f_k'(x). \tag{2.173}$$

The integral $g(x) = \int_{\alpha}^{\infty} f(t, x) \mathrm{d}t$ is said to be uniformly convergent for $x \in [a, b]$ if, for each $\varepsilon > 0$, a number N can be found depending on $\varepsilon > 0$ but not on x such that

$$\left|g(x) - \int_{\alpha}^{\beta} f(t,x)\mathrm{d}t\right| < \varepsilon, \tag{2.174}$$

for all $\beta > N$ and all $x \in [a,b]$. The Weierstrass M-test can be used to determine uniform convergence of integrals. If a function $M(t) \geq 0$ can be found such that $|f(t,x)| \leq M(t)$ for $x \in [a,b]$, $t > \alpha$, and $\int_{\alpha}^{\infty} M(t)\mathrm{d}t$ converges, then $\int_{\alpha}^{\infty} f(t,x)\mathrm{d}t$ is uniformly convergent for $x \in [a,b]$. If $f(t,x)$ is continuous for $t > \alpha, x \in [a,b]$, and $g(x) = \int_{\alpha}^{\infty} f(t,x)\mathrm{d}t$ is uniformly convergent for $x \in [a,b]$, then $g(x)$ is continuous in the stated interval. For a point x_0,

$$\lim_{x\to x_0} \int_{\alpha}^{\infty} f(t,x)\mathrm{d}t = \int_{\alpha}^{\infty} \lim_{x\to x_0} f(t,x)\mathrm{d}t. \tag{2.175}$$

Also,

$$\int_{a}^{b} \left\{\int_{\alpha}^{\infty} f(t,x)\mathrm{d}t\right\}\mathrm{d}x = \int_{\alpha}^{\infty} \left\{\int_{a}^{b} f(t,x)\mathrm{d}x\right\}\mathrm{d}t. \tag{2.176}$$

More care is needed if the function $g(x)$ is also integrated over an infinite interval, even when the convergence of $g(x)$ is uniform. Further discussion on interchanging integration order is given in Section 2.12. If the previously stated conditions on $f(t,x)$ apply, and if $\partial f(t,x)/\partial x$ is continuous for $t > \alpha, x \in [a,b]$, and $\int_{\alpha}^{\infty} \partial f(t,x)/\partial x\, \mathrm{d}t$ converges uniformly in $[a,b]$, then, assuming α is independent of x, it follows that

$$\int_{\alpha}^{\infty} \frac{\partial f(t,x)}{\partial x}\,\mathrm{d}t = \frac{\partial}{\partial x}\int_{\alpha}^{\infty} f(t,x)\mathrm{d}t. \tag{2.177}$$

2.11.1 The notion of measure

To proceed further, the notion of the *measure* of a set is introduced. This is a generalization of the notion of length in one dimension, of area in two dimensions, and so on. Consider the set E of points $\{x : a \leq x \leq b\}$, then the following geometric interpretation of measure for the one-dimensional case can be given. The measure of the subset E_i satisfies the following conditions:

(i) $m(E_i) \geq 0$, for $E_i \subset E$; (2.178)
(ii) if $E_i \subseteq E_j$, then $m(E_i) \leq m(E_j)$; (2.179)
(iii) for an empty set $\varnothing, m(\varnothing) = 0$; (2.180)
(iv) the measure of sets is additive. For two disjoint sets E_i and E_j,

$$m(E_i + E_j) = m(E_i) + m(E_j), \text{ with } E_i \cap E_j = \varnothing. \tag{2.181}$$

If E_i denotes the set of points in the interval $[a,\alpha]$ and E_j denotes the set of points in the interval $[\beta,b]$, then the measure of these two sets if they are disjoint is the sum

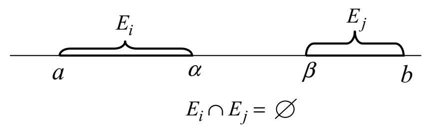

Figure 2.19. Disjoint intervals.

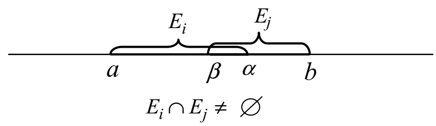

Figure 2.20. Overlapping intervals.

of the two line segments shown in Figure 2.19. If the sets E_i and E_j are not disjoint, as in Figure 2.20, then

$$m(E_i + E_j) = m(E_i) + m(E_j) - m(E_i \cap E_j). \tag{2.182}$$

In this manner the interval (α, β) is counted only once. The preceding appeals to the intuitive notion of the measure of a one-dimensional interval as the length of the interval. In a similar manner, an interpretation of the measure of sets in the Euclidean plane $\mathbb{R} \times \mathbb{R}$ can be given as the area of the associated domain over which the set is defined. The notation E^1 is used to denote the one-dimensional Euclidean space and E^n designates the n-dimensional Euclidean space.

Two sets A and B are termed *equivalent* if there is a one-to-one mapping from the set A onto B. A set S that is finite or equivalent to the set of all positive integers is called *countable*. A set S of real numbers is termed *open* if each point of S is at the center of an open interval totally contained in S. An *open cover* of the set A is the collection of open sets $\{S_\alpha\}$ such that $A \subset \bigcup S_\alpha$.

The Lebesgue outer measure of a set A, denoted $\mu_o(A)$, is defined by

$$\mu_o(A) = \inf \left\{ \sum_k m(A_k) \right\}, \quad \text{with } A \subset \bigcup_k A_k, \tag{2.183}$$

where the inf is taken over the class of all countable open coverings of A. Some properties of μ_o are as follows:

$$\mu_o(\varnothing) = 0; \tag{2.184}$$

$$0 \leq \mu_o(I) \leq \infty; \tag{2.185}$$

$$\mu_o(I) = l(I), \tag{2.186}$$

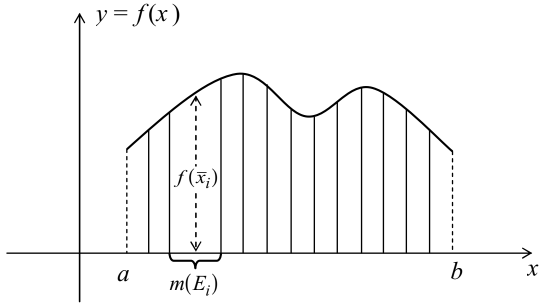

Figure 2.21. Riemann sum in terms of the measure of the intervals.

where $l(I)$ denotes the length of the interval I; and

$$\mu_o(\{\alpha\}) = 0, \tag{2.187}$$

where $\{\alpha\}$ denotes a particular point. This last result reflects the fact that points have zero length, that is, points are dimensionless.

The result for the Riemann integral given in Eq. (2.168) can be recast in terms of measure theoretic language. In Figure 2.21 the continuous function f is shown on the interval $[a, b]$, which is represented by the set of points E, and the interval $[a, b]$ is partitioned such that

$$E = \sum_{i=1}^{m} E_i, \quad \text{with } E_i \cap E_j = 0, \quad \text{for any } i, j \text{ pair}, \tag{2.188}$$

where the measure of a subdivision E_i is denoted by $m(E_i)$. If $\bar{x}_i$ is any point belonging to the set E_i, then the sum $\sum_{i=1}^{m} f(\bar{x}_i)m(E_i)$ can be formed. By increasing the number of subsets indefinitely, so that for any E_i, $m(E_i) \to 0$, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{m(E_i)\to 0} \sum_i f(\bar{x}_i)m(E_i) \tag{2.189}$$

provided the limit exists and is independent of the subdivision process. The connection with the definition of the Riemann integral given previously should be apparent.

For what follows, it is useful to introduce the idea of a step function, defined on the interval $[a, b]$ by

$$\psi(x) = \begin{cases} c_j, & \text{for } x_{j-1} < x < x_j,\ j = 1, 2, \ldots, n \\ d_j, & \text{for } x = x_j,\ j = 0, 1, 2, \ldots, n. \end{cases} \tag{2.190}$$

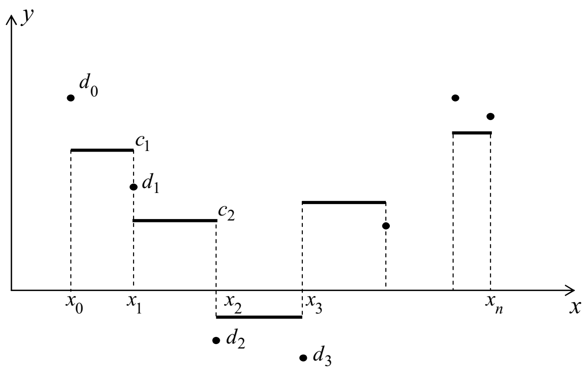

Figure 2.22. Step function on the interval $[x_0,\ x_n]$.

Given a set S, the characteristic function associated with this set is defined by

$$\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S, \end{cases} \tag{2.191}$$

so that, for a constant c, the function $c\chi_{[x_1,x_2]}$ is zero everywhere except on the interval $[x_1, x_2]$, where it takes the value c. Using the characteristic function allows Eq. (2.190) to be written in the form

$$\psi(x) = \sum_{j=1}^{n} c_j \chi_{(x_{j-1},\, x_j)} + \sum_{j=0}^{n} d_j \chi_{\{x_j\}}, \tag{2.192}$$

where χ serves to designate the subinterval lengths or the endpoint locations of the subintervals. An example of a step function is shown in Figure 2.22. The Riemann integral of $\psi(x)$ on the interval $[a, b]$ is given by

$$\int_a^b \psi(x)\mathrm{d}x = \sum_{j=1}^{n} c_j(x_j - x_{j-1}). \tag{2.193}$$

Note that the values at the endpoints of the subintervals do not enter this result. For a continuous function f defined on an interval $[a, b]$, the function can be bounded above and below by step functions. If the integrals of these two step functions are equal in the limit of an infinite sequence of partitions, the function is Riemann integrable on $[a, b]$. If the function has a finite number of discontinuities, a similar bounding process can be carried out in terms of step functions.

Using the notion of the characteristic function just defined, the idea of Lebesgue measurability can be introduced. If S is a subset of $\mathbb{R}$, then the set S is said to be Lebesgue measurable if and only if χ_S is a measurable function, and, if χ_S is integrable, then the measure of the set is written as follows:

$$m(S) = \int \chi_S(x)\mathrm{d}x. \tag{2.194}$$

An essential distinction between the Riemann definition of an integral and Lebesgue's approach, which is now considered, is that, in the former, the domain of the function is partitioned, whereas Lebesgue partitions the range of values taken by the function. Figure 2.23 will help clarify the situation. In this figure, a continuous bounded function is displayed on the interval $[a, b]$. The function f takes values in the interval y_{i-1} to y_i for the five ranges shown on the x-axis. The measure of these five intervals are collectively designated $m(E_i)$. If $\overline{x}_i$ denotes any point in E_i, the sum $\sum_{i=1}^{n} f(\overline{x}_i)m(E_i)$ can be formed. The sets $E_i = \{x : x \in [a, b], y_{i-1} < f(x) < y_i\}$ are taken to be Lebesgue measurable, and, when this is so, the function is termed Lebesgue measurable. By construction, the sum just formed satisfies the following:

$$\sum_{i=1}^{n} y_{i-1}m(E_i) \leq \sum_{i=1}^{n} f(\overline{x}_i)m(E_i) \leq \sum_{i=1}^{n} y_i m(E_i). \tag{2.195}$$

In the limit that the partitions $\delta y_i = y_i - y_{i-1} \to 0$, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{\max \delta y \to 0} \sum_i f(\overline{x}_i)m(E_i). \tag{2.196}$$

If this limit exists, then it is the Lebesgue integral of f. The reader can contrast this result with Eq. (2.189) and interpret the difference in meaning of the two similar looking results. The class of Lebesgue integrable functions includes the class of Riemann integrable functions, but the reverse is not true.

A function which can be written as

$$f(x) = \sum_{j=1}^{n} c_j \chi_{E_j}(x), \tag{2.197}$$

where the domain of f is the measurable set E, is called a *simple function* or a *generalized step function*. The E_j are disjoint measurable subsets of E, and $E = \bigcup_j E_j$. The Lebesgue integral of f on E is given by

$$\int_E f(x)\mathrm{d}x = \sum_{j=1}^{n} c_j \mu_j(E_j). \tag{2.198}$$

Some general remarks about the functions that are encountered later in applications are appropriate. The functions arising in practical applications are most often measurable. For measurable functions, sums, products, and limits of these functions are measurable. Functions that are equal everywhere, except on sets of measure

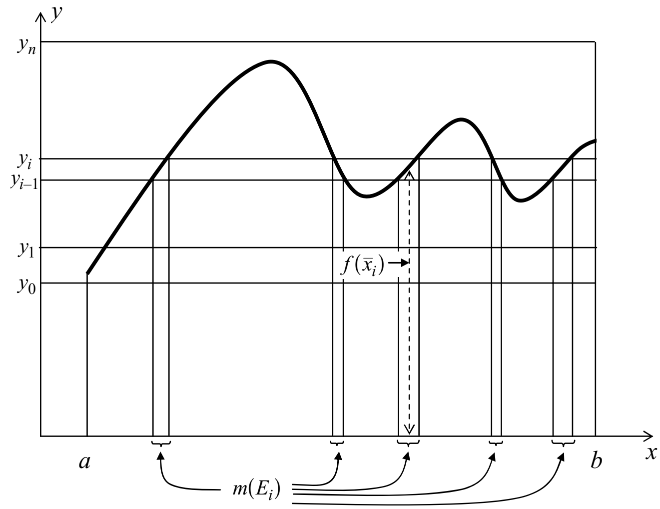

Figure 2.23. Partition of the values taken by the function $y = f(x)$.

zero, are termed equivalent functions for the domain under consideration. The often employed terminology is that the functions are equal *almost everywhere*. The commonly employed abbreviation for the latter is the obvious *a.e*. If the function f has some particular property for all values of x except for values of x of a set of measure zero, then the property is said to be true for *almost all* x.

The standard symbol employed to denote the class of Lebesgue integrable functions is L. A function $f \in L$ if

$$\int |f(x)|\mathrm{d}x < \infty, \tag{2.199}$$

where the integral is over a specified domain. The notation $f \in L(\mathbb{R})$ is employed when the integral is to be taken over the interval $(-\infty, \infty)$, and $f \in L(a, b)$ is used when the integral is taken over the interval (a, b). In this work the domain when left unspecified is taken to be $\mathbb{R}$, so a statement $f \in L^p$ for some specified p is shorthand for $f \in L^p(\mathbb{R})$.

Some properties that find application later are now reviewed. If $f_1(x) \in L$ and $|f_2(x)| < c\,|f_1(x)|$ for c some positive constant, then $f_2(x) \in L$. If $|f(x)|^p$ is Lebesgue integrable for $p > 0$, then $f \in L^p$; L^1 is often shortened to L. The term *integrable* is commonly used synonymously for functions $f \in L$. The integration interval is usually self-evident when such statements are made. When it is not, it is appropriate to write $f \in L^p(a, b)$, where a and b are specified; for example, $f \in L^p(\mathbb{R})$. On a finite integration interval, the class L^p includes L^q, where $q > p$, that is, if $f(x) \in L^q$, then also $f(x) \in L^p$. Additional care is needed for an infinite integration interval. On such an interval, $f(x)$ may $\in L^p$ for only one value of p. For example, consider

$f(x) = \{(a + |\log x|)\sqrt{x}\}^{-1}$ for $a > 0$, then, on the interval $(0, \infty)$, the integral

$$\int_0^\infty \frac{\mathrm{d}x}{\{(a + |\log x|)\sqrt{x}\}^p}$$

can be considered in two parts. At the upper limit, it follows, with the aid of the substitution $x = \mathrm{e}^y$, that

$$\int^\infty \frac{\mathrm{d}x}{\{(a + |\log x|)\sqrt{x}\}^p} \to \int^\infty \frac{\mathrm{e}^{(1-p/2)y}\,\mathrm{d}y}{(a+y)^p}, \tag{2.200}$$

which converges for $p \geq 2$. In a similar fashion, the lower limit behaves as

$$\int_0 \frac{\mathrm{d}x}{\{(a + |\log x|)\sqrt{x}\}^p} \to -\int_\infty \frac{\mathrm{e}^{(p/2-1)y}\mathrm{d}y}{(a + |y|)^p}, \tag{2.201}$$

which converges for $p \leq 2$. The combination of these two results indicates that $f \in L^2(0, \infty)$.

The notation $\|\,\|$ is used to designate the norm, so that

$$\|f\|_p = \left\{\int_{-\infty}^\infty |f(x)|^p\,\mathrm{d}x\right\}^{1/p}, \tag{2.202}$$

where the integral will normally be understood in the sense of Lebesgue. If this needs to be made explicit, the notation $\|f\|_{L^p(\mathbb{R})}$ is employed. For the particular case that $p = \infty$, then

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(x)| : x \in \mathbb{R}\}, \tag{2.203}$$

where the abbreviation ess sup stands for the *essential supremum*, which signifies the least upper bound of $|f(x)|$ except on sets of measure zero. Functions belonging to L^∞ are referred to as *essentially bounded* functions.

To deal with products of functions the following result due to Hölder will prove to be useful. If $f \in L^p$ for $1 \leq p \leq \infty$ and $g \in L^q$ for $1 \leq q \leq \infty$, then $fg \in L^r$, where $r^{-1} = p^{-1} + q^{-1}$. The particular case where $f \in L^2$ and $g \in L^2$, leading to $fg \in L^1$, occurs frequently in applications. Recall from what has been discussed previously that $f \in L^p$ does not imply $f \in L^q$ for any other value $1 \leq q \leq \infty$. However, a useful result is the following: if $f \in L^p \cap L^q$ for $1 \leq p < q \leq \infty$, then $f \in L^r$ with $p \leq r \leq q$.

Let $\{f_n\}$ denote a sequence of functions each of which belongs to the class L^p. If the sequence converges to $f(x)$ for each $x \in \mathbb{R}$, then the sequence is said to *converge pointwise* to $f(x)$. If the sequence converges to $f(x)$ for almost all $x \in \mathbb{R}$, the sequence is said to converge pointwise almost everywhere to $f(x)$. Examples can be constructed for each of these two situations where the resulting function $f \notin L^p$.

Consider the case

$$f_n(x) = n\chi_{(0,\,1/n)}. \tag{2.204}$$

Each term of the sequence $\{f_n\}$ is in L^p, and the sequence converges pointwise to $f(x) = 0$. A sequence $\{f_n\}$ converges to a function $f \in L^p$ if $f \in L^p$ and $\lim_{n\to\infty} \|f_n - f\|_p \to 0$. For the example in Eq. (2.204), it follows for $p > 1$ that $\lim_{n\to\infty} \|f_n\|_p \to \infty$.

A result that finds application in several areas in later chapters concerns the switching of the order of limit and integral in various settings. Let the sequence of functions $\{f_n\} \subseteq L^1(\mathbb{R})$ and $\lim_{n\to\infty} f_n = f$, *a.e.*, and suppose there exists a function $g \in L^1(\mathbb{R})$ such that *for all* (henceforth denoted by the symbol $\forall$) n

$$|f_n(x)| \le g(x), \quad a.e. \text{ on } \mathbb{R}, \tag{2.205}$$

then $f \in L^1(\mathbb{R})$, and, using $\int_{\mathbb{R}} \mathrm{d}x$ to denote the integral over $\mathbb{R}$,

$$\int_{\mathbb{R}} f(x)\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\mathrm{d}x = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\mathrm{d}x. \tag{2.206}$$

This is the Lebesgue dominated convergence theorem.

This section concludes with some comments on the Borel measure. A Borel field $\mathcal{B}$ is defined as the collection of subsets that satisfy the following: if $B_j \in \mathcal{B}$ then the finite unions $\bigcup_{j=1}^{n} B_j$ and the countable union $\bigcup_{j=1}^{\infty} B_j$ belong to $\mathcal{B}$. If $B \in \mathcal{B}$ then its complement, denoted by B^{c}, satisfies $B^{\mathrm{c}} \in \mathcal{B}$. The members B_j of a Borel field are the Borel sets. The most commonly occurring sets that arise in analysis are frequently Borel sets. The sets of open intervals (a, b) for $a < b$ on $\mathbb{R}$, the set of closed intervals $[a, b]$ on $\mathbb{R}$, and the set of half-open intervals $(a, b]$ or $[a, b)$ on $\mathbb{R}$, are examples. Every Borel set is measurable. A function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable if $f^{-1}(B)$ is a Borel set for each Borel set B in $\mathbb{R}$. A Borel measurable set is Lebesgue measurable; however, examples can be found that make it clear that the converse of this statement is not true.

2.12 Theorems due to Fubini and Tonelli

In later sections multi-dimensional integrals are often encountered, and there will be a need to reverse the integration order of the variables involved. Doing this in a cavalier fashion invites courting disaster. A well known example will serve as an illustration. Consider the function

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}. \tag{2.207}$$

Is it true that

$$\int_0^1 \mathrm{d}x \int_0^1 f(x,y)\mathrm{d}y = \int_0^1 \mathrm{d}y \int_0^1 f(x,y)\mathrm{d}x? \tag{2.208}$$

It is straightforward to show that the left-hand side of Eq. (2.208) equals $\pi/4$ and that the right-hand side evaluates to $-\pi/4$. This example is revisited shortly.

The applications of interest where there is a need to switch integration order fall into two main categories. These are: (i) functions continuous on the integration ranges involved, and (ii) principal value integrals. This section treats the former, and the latter topic is discussed in Section 2.13.

The following result is well known from introductory calculus. The integration of a function $f(x,y)$ continuous in the closed rectangle $a \le x \le b, c \le y \le d$, can be carried out by first integrating with respect to x, then y, or with respect to y, then x. That is

$$\int_c^d \mathrm{d}y \int_a^b f(x,y)\mathrm{d}x = \int_a^b \mathrm{d}x \int_c^d f(x,y)\mathrm{d}y, \tag{2.209}$$

and the value of the integral is independent of the order of integration. When an improper integral is considered, then the following result holds: if the integral

$$F(x) = \int_c^\infty f(x,y)\mathrm{d}y \tag{2.210}$$

converges uniformly for $a \le x \le b$, then

$$\int_c^\infty \mathrm{d}y \int_a^b f(x,y)\mathrm{d}x = \int_a^b \mathrm{d}x \int_c^\infty f(x,y)\mathrm{d}y. \tag{2.211}$$

Convergence of Eq. (2.210) is not sufficient to make the interchange in Eq. (2.211). If $f(x,y)$ is discontinuous on a finite number of curves in the y integration range, and $\int_c^d f(x,y)\mathrm{d}y$ converges uniformly for $a \le x \le b$, then Eq. (2.211) holds.

When both integration ranges are infinite, uniform convergence is not sufficient to guarantee the interchange of order on integration. If the integral $\iint f(x,y)\mathrm{d}x\,\mathrm{d}y$ taken over the entire first quadrant exists, then

$$\int_0^\infty \mathrm{d}y \int_0^\infty f(x,y)\mathrm{d}x = \int_0^\infty \mathrm{d}x \int_0^\infty f(x,y)\mathrm{d}y. \tag{2.212}$$

The modern statements indicating the conditions for the interchange of order of a double integral are the theorems of Fubini and Tonelli. Fubini's theorem for $\mathbb{R}^2$ is as follows: suppose $f(x,y)$ is a real-valued measurable function on $\mathbb{R} \times \mathbb{R}$, that is $f(x,y) \in L^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} f(x,y)\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(x,y)\mathrm{d}y = \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} f(x,y)\mathrm{d}x. \tag{2.213}$$

This means that a function $F(x)$, defined by

$$F(x) = \int_{\mathbb{R}} f(x,y)\mathrm{d}y, \tag{2.214}$$

exists for almost all x in $\mathbb{R}$, and that $\int_{\mathbb{R}} F(x)\,\mathrm{d}x$ exists. A related theorem of Tonelli is: if $f(x,y)$ is measurable and one of the integrals

$$\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} |f(x,y)|\mathrm{d}y, \quad \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} |f(x,y)|\mathrm{d}x,$$

exists, then

$$\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(x,y)\mathrm{d}y = \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} f(x,y)\mathrm{d}x. \tag{2.215}$$

For proofs of these results, consult, for example, Kestelman (1960, p. 206) or Weir (1973, p. 123).

For the example considered in Eq. (2.207), $f(x,y)$ is continuous and measurable in the integration interval $0 < x < 1, 0 < y < 1$, and Eq. (2.215) does not hold; therefore,

$$\int_0^1 \int_0^1 |f(x,y)|\mathrm{d}x\,\mathrm{d}y, \int_0^1 \mathrm{d}x \int_0^1 |f(x,y)|\mathrm{d}y, \quad \text{and} \quad \int_0^1 \mathrm{d}y \int_0^1 |f(x,y)|\mathrm{d}x$$

must diverge. Since

$$\frac{\partial g(x,y)}{\partial y} = f(x,y), \quad \text{with } g(x,y) = y(x^2+y^2)^{-1}, \tag{2.216}$$

it is straightforward to show that

$$\int_0^1 \mathrm{d}x \int_0^1 |f(x,y)|\mathrm{d}y = \int_0^1 \left(\frac{1}{x} - \frac{1}{x^2+1}\right)\mathrm{d}x = \infty. \tag{2.217}$$

2.13 The Hardy–Poincaré–Bertrand formula

Before proceeding to the key topic of this section, a few preliminary ideas are discussed. The most important general inequality employed in this book is Hölder's inequality, which, for the integration region Ω, takes the following form:

$$\int_{\Omega} |fg|\mathrm{d}\Omega \le \left[\int_{\Omega} |f|^p\,\mathrm{d}\Omega\right]^{1/p} \left[\int_{\Omega} |g|^q\,\mathrm{d}\Omega\right]^{1/q}, \tag{2.218}$$

where p and q (termed conjugate exponents) are connected by

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{2.219}$$

and the functions f and g satisfy $f \in L^p$, with $1 \leq p \leq \infty$, and $g \in L^q$, with $1 \leq q \leq \infty$. Hölder's inequality finds a number of applications in many proofs given in later chapters. The derivation of Eq. (2.218) is developed in a pair of exercises at the end of this chapter. The appearance of the condition given by Eq. (2.219) in any theorem is often linked to the application of Hölder's inequality in the proof. An important special case of Eq. (2.218) occurs for the case $p = q = 2$, and this is referred to as the Cauchy–Schwarz–Buniakowski inequality.

On occasion, it is necessary to switch the order of integration in a multiple integral, when one or more of the integrals are defined only as a Cauchy principal value integral. This need arises for several properties of the Hilbert transform. The discussion given in the preceding section does not handle the general cases where principal value integrals arise. If the functions f and g belong to the classes L^p and L^q, respectively, and if Eq. (2.219) is satisfied, then

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x\, P\int_{-\infty}^{\infty} \frac{g(y)\mathrm{d}y}{y-x} = \int_{-\infty}^{\infty} g(y)\mathrm{d}y\, P\int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{y-x}. \tag{2.220}$$

That is, the order of integration where one principal value integral is involved may be interchanged with an ordinary integral. To see how Eq. (2.219) arises, set

$$G(x) = P\int_{-\infty}^{\infty} \frac{g(y)\mathrm{d}y}{y-x}, \quad F(y) = P\int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{y-x}, \tag{2.221}$$

and hence

$$\int_{-\infty}^{\infty} f(x)G(x)\mathrm{d}x = \int_{-\infty}^{\infty} g(x)F(x)\mathrm{d}x. \tag{2.222}$$

By Hölder's inequality, both integrals in Eq. (2.222) are bounded if $f(x)$ and $g(x)$ belong to the classes L^p and L^q, respectively, and Eq. (2.219) holds. The preceding statement is not obvious. The reader should expect the additional requirements that $G \in L^q$ and $F \in L^p$. However, these latter two conditions need not be explicitly stated, since, by an important theorem due to M. Riesz, if $f \in L^p$ for $p > 1$, then $F \in L^p$. The proof of Riesz's theorem is relatively straightforward for particular values of p; however, a discussion and general proof of this result is postponed until Section 4.20. Since both integrals in Eq. (2.222) are bounded, applying the ideas of Section 2.12 establishes the validity of Eq. (2.220). The argument presented makes it clear why the condition given in Eq. (2.219) is required.

A similar result to Eq. (2.220) can be written when the integration interval is finite. When all four integration intervals are finite, the less stringent condition

$$\frac{1}{p} + \frac{1}{q} \leq 1 \tag{2.223}$$

replaces Eq. (2.219). For a finite interval, the class L^p belongs to $L^{p'}$ for $p' > p$, and hence in Eq. (2.219) p and q can be replaced by larger values, and Eq. (2.223) follows. The English savant Hardy (1908) gave a detailed analysis of results related to Eq. (2.220), though the conditions he imposed on the functions f and g were less general than those specified in the preceding paragraph.

If f and g are Hölder continuous on C, then, for $t \in C$, the singular integral equation

$$g(t) = \frac{1}{\pi \mathrm{i}} P \int_C \frac{f(s)\mathrm{d}s}{s-t} \tag{2.224}$$

has the solution

$$f(t) = \frac{1}{\pi \mathrm{i}} P \int_C \frac{g(s)\mathrm{d}s}{s-t}. \tag{2.225}$$

The details of this are explored later in this chapter. With the obvious change of integration variable, the two preceding equations yield

$$f(t) = -\frac{1}{\pi} P \int_C \frac{\mathrm{d}s}{s-t} \frac{1}{\pi} P \int_C \frac{f(s')\mathrm{d}s'}{s'-s}. \tag{2.226}$$

This result is sometimes referred to as the Poincaré–Bertrand formula (see, for example, Henrici (1986, p. 118) and Pandey (1996, p. 66).

The next issue to be considered is the situation where an interchange of integration order is made in a two-dimensional integral and both integrals involve a Cauchy principal value. Suppose the functions ϕ_1 and ϕ_2 belong to the classes L^p and L^q, respectively, and that Eq. (2.219) is satisfied, then

$$\begin{aligned} &\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_1(x)}{x-t} \mathrm{d}x \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_2(y)}{y-x} \mathrm{d}y \\ &\quad = \frac{1}{\pi} P \int_{-\infty}^{\infty} \phi_2(y)\mathrm{d}y \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_1(x)}{(x-t)(y-x)} \mathrm{d}x - \phi_1(t)\phi_2(t). \end{aligned} \tag{2.227}$$

This result is most commonly referred to as the Poincaré–Bertrand formula. Note that it is rather different from the result given the same name in Eq. (2.226). The formula was discussed by Poincaré (1910), and somewhat later by Bertrand (1922, 1923a, 1923b), but the result had already been published by Hardy (1908) under less general conditions than those given for Eq. (2.227). As a consequence of Hardy's seminal contribution, Eq. (2.227) is called in this book the Hardy–Poincaré–Bertrand formula, which is a departure from common usage. This formula has practical applications for developing some identities for the Hilbert transform operator, as well as applications in solving certain types of singular integral equations. A generalization of Eq. (2.227) involving contour integrals is discussed by Muskhelishvili (1992, p. 56).

The following simplified argument can be provided to establish Eq. (2.227). Start with

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_1(x)}{x-t} \mathrm{d}x \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_2(y)}{y-x} \mathrm{d}y = \frac{1}{\pi^2} P \int_{-\infty}^{\infty} \frac{\phi_1(x)G(x)\mathrm{d}x}{x-t}, \tag{2.228}$$

where

$$G(x) = \int_{-\infty}^{\infty} \frac{\phi_2(y) - \phi_2(x)}{y - x}\,\mathrm{d}y. \tag{2.229}$$

To write the last line, the following result has been employed:

$$P\int_{-\infty}^{\infty} \frac{\mathrm{d}y}{y - x} = 0. \tag{2.230}$$

This result is also employed in the following sequence of steps. From Eqs. (2.228) and (2.230), it follows that

$$\begin{aligned}
\frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\phi_1(x)}{x-t}\,\mathrm{d}x \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\phi_2(y)}{y-x}\,\mathrm{d}y &= \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\{\phi_1(x)G(x) - \phi_1(t)G(t)\}\mathrm{d}x}{x-t} \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \left[\frac{\{\phi_2(y) - \phi_2(x)\}\phi_1(x)}{(x-t)(y-x)}\right. \\
&\quad \left. - \frac{\{\phi_2(y) - \phi_2(t)\}\phi_1(t)}{(x-t)(y-t)}\right]\mathrm{d}y \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \left[\frac{\{\phi_2(y) - \phi_2(x)\}\phi_1(x)}{(x-t)(y-x)}\right. \\
&\quad \left. - \frac{\{\phi_2(y) - \phi_2(t)\}\phi_1(t)}{(x-t)(y-t)}\right]\mathrm{d}x,
\end{aligned} \tag{2.231}$$

and the interchange of integration order in the last step is justified by Fubini–Tonelli, assuming the functions ϕ_1 and ϕ_2 are Hölder continuous (with suitable exponent) and integrable on $\mathbb{R}$. The structure of the factor in the square braces is such that there is no non-integrable singularity in the integration interval. Using a partial fraction expansion,

$$\begin{aligned}
&\frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\phi_1(x)}{x-t}\,\mathrm{d}x \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\phi_2(y)}{y-x}\,\mathrm{d}y \\
&\quad = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \frac{\{\phi_2(y) - \phi_2(x)\}\phi_1(x) - \{\phi_2(y) - \phi_2(t)\}\phi_1(t)}{(x-t)(y-t)}\,\mathrm{d}x \\
&\qquad + \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \frac{\{\phi_2(y) - \phi_2(x)\}\phi_1(x)}{(y-x)(y-t)}\,\mathrm{d}x \\
&\quad = \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\phi_2(y)}{y-t}\,\mathrm{d}y \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\phi_1(x)}{x-t}\,\mathrm{d}x \\
&\qquad + \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\phi_2(y)}{y-t}\,\mathrm{d}y \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\phi_1(x)}{y-x}\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{y-t}\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\phi_1(x)\phi_2(x)}{y-x}\,\mathrm{d}x \\
&=\frac{1}{\pi}P\int_{-\infty}^{\infty}\phi_2(y)\mathrm{d}y\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\phi_1(x)\mathrm{d}x}{(y-x)(x-t)}-\phi_1(t)\phi_2(t),
\end{aligned}
\tag{2.232}
$$

where Eq. (2.226) has been employed to complete the last step. A more rigorous discussion of the Hardy–Poincaré–Bertrand formula, one making it clear how the condition given in Eq. (2.219) arises, requires some additional results. Further discussion of this topic is postponed to Section 4.23.

The Hardy–Poincaré–Bertrand formula can also be written in a form involving contour integrals. If C denotes a smooth arc or closed contour, and the function $\varphi(x,y)=\phi_1(x)\phi_2(y)$ is Hölder continuous on C, then

$$
\int_C\frac{\phi_1(x)}{x-t}\,\mathrm{d}x\int_C\frac{\phi_2(y)}{y-x}\mathrm{d}y=\int_C\phi_2(y)\,\mathrm{d}y\int_C\frac{\phi_1(x)}{(x-t)(y-x)}\,\mathrm{d}x-\pi^2\phi_1(t)\phi_2(t).
\tag{2.233}
$$

For an authoritative discussion of the Hardy–Poincaré–Bertrand formula along arcs, consult Muskhelishvili (1992), which is the standard reference.

2.14 Riemann–Lebesgue lemma

The following two integrals arise in several applications:

$$
\int_{-\infty}^{\infty}f(x)\cos\lambda x\,\mathrm{d}x \quad\text{and}\quad \int_{-\infty}^{\infty}f(x)\sin\lambda x\,\mathrm{d}x,
$$

and attention is focused on the limit $\lambda\to\infty$. These integrals are dealt with by the Riemann–Lebesgue lemma. If $f\in L(\mathbb{R})$, then

$$
\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}f(x)\sin\lambda x\,\mathrm{d}x=0
\tag{2.234}
$$

and

$$
\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}f(x)\cos\lambda x\,\mathrm{d}x=0.
\tag{2.235}
$$

The limit remains the same when the integration range is finite for each of the preceding two integrals. The case of a finite integration range $[a,b]$ is treated first. If $f(x)$ is a constant c over the integration range, then

$$
\int_a^b c\sin\lambda x\,\mathrm{d}x=\frac{c}{\lambda}\{\cos a\lambda-\cos b\lambda\},
\tag{2.236}
$$

and the right-hand side of this equation goes to zero as $\lambda\to\infty$. Suppose a sequence of step functions $\{\psi_n(x)\}$ is considered, such that the interval (a,b) is subdivided into a finite number of subintervals over which $\psi_n(x)$ is a constant. Then on each of the subintervals it follows that

$$\lim_{\lambda\to\infty} \int \psi_n(x) \sin \lambda x \, \mathrm{d}x = 0, \tag{2.237}$$

using the result given in Eq. (2.236). Hence, on the interval (a, b),

$$\lim_{\lambda\to\infty} \int_a^b \psi_n(x) \sin \lambda x \, \mathrm{d}x = 0. \tag{2.238}$$

If $f \in L$, then an increasing sequence of step functions $\{\psi_n\}$ can be found such that, for a given $\varepsilon > 0$,

$$\int_a^b |f(x) - \psi_n(x)| \mathrm{d}x < \varepsilon. \tag{2.239}$$

Hence,

$$\left| \int_a^b f(x) \cos \lambda x \, \mathrm{d}x - \int_a^b \psi_n(x) \cos \lambda x \, \mathrm{d}x \right| \leq \int_a^b |f(x) - \psi_n(x)| \mathrm{d}x, \tag{2.240}$$

which, together with Eq. (2.239), establishes the desired result. The argument for the integral involving $\sin \lambda x$ goes in a similar manner. For the infinite integration range the following approach can be employed. There exists an integer m and an $\varepsilon > 0$ such that

$$0 \leq \int_{-\infty}^{\infty} \{f(x) - \psi_m(x)\} \mathrm{d}x < \varepsilon, \tag{2.241}$$

and, for $\lambda \geq m$,

$$\left| \int_{-\infty}^{\infty} \psi_m(x) \cos \lambda x \, \mathrm{d}x \right| < \varepsilon. \tag{2.242}$$

The following inequality can be written:

$$\left| \int_{-\infty}^{\infty} f(x) \cos \lambda x \, \mathrm{d}x \right| \leq \left| \int_{-\infty}^{\infty} \{f(x) - \psi_m(x)\} \cos \lambda x \, \mathrm{d}x \right| + \left| \int_{-\infty}^{\infty} \psi_m(x) \cos \lambda x \, \mathrm{d}x \right|. \tag{2.243}$$

This result can be simplified on using Eq. (2.242), so that, for $\lambda \geq m$,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} f(x) \cos \lambda x \, \mathrm{d}x \right| &< \int_{-\infty}^{\infty} \{f(x) - \psi_m(x)\} \mathrm{d}x + \varepsilon \\ &< 2\varepsilon. \end{aligned} \tag{2.244}$$

As an example of Eq. (2.234), consider the case $f(x) = x(x^2 + a^2)^{-2}$ for $a > 0$, then

$$\lim_{\lambda\to\infty}\int_0^\infty \frac{x\sin\lambda x\,\mathrm{d}x}{(x^2+a^2)^2} = \lim_{\lambda\to\infty}\frac{\pi\lambda\mathrm{e}^{-\lambda a}}{4a} = 0. \tag{2.245}$$

2.15 Some elements of the theory of distributions

It has long been recognized in the physical sciences that there is considerable value to having a "function" with the following properties:

$$\delta(x) = 0, \quad \text{for } x \neq 0, \tag{2.246}$$

with

$$\int_\alpha^\beta \delta(x)\mathrm{d}x = 0, \quad \text{for} \quad 0 < \alpha < \beta, \quad \text{or } \alpha < \beta < 0, \tag{2.247}$$

$$\int_{-\infty}^\infty \delta(x)\mathrm{d}x = 1, \tag{2.248}$$

and, assuming that $f(x)$ is regular at $x = 0$,

$$\int_{-\infty}^\infty f(x)\delta(x)\mathrm{d}x = f(0). \tag{2.249}$$

The reader may recognize this "function" as the *delta function* or the *Dirac delta function*. In engineering it is frequently called the *unit impulse*. A number of books in physics supplement the two conditions given with the assignment $\delta(x) = \infty$ for $x = 0$. Shortly, it will be observed that this need not always be true. If you think about the normal definition of a function, then the combination of Eqs. (2.246) and (2.248) do not fit the description. If a function vanishes almost everywhere, then its Riemann integral and its Lebesgue integral (see Section 2.11) taken over the real line would vanish, which is clearly in contradiction with Eq. (2.248). Functions of the type typified by the delta function are referred to as *distributions* or as *generalized functions*. At least some comfort can be taken in the fact that they were not called super functions! The terms distribution and generalized function are often used synonymously, though some writers attach slightly different conceptual meanings to the two terms.

Consider the following function:

$$f(x) = \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}. \tag{2.250}$$

The factor of π is inserted so that

$$\int_{-\infty}^\infty f(x)\mathrm{d}x = 1. \tag{2.251}$$

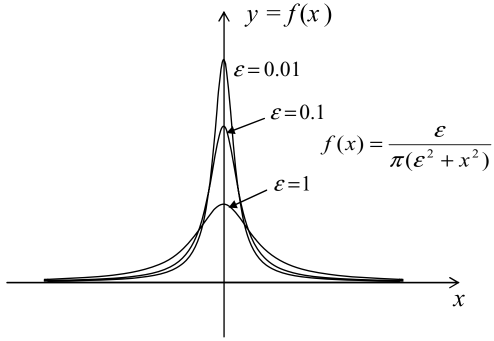

Figure 2.24. Plot of $f(x)$ in Eq. (2.250) for $\varepsilon = 0.01, 0.1$, and 1.

Figure 2.24 shows a plot of $f(x)$ for three values of ε. It is clearly evident that as ε becomes smaller, the function becomes increasingly sharply peaked, and approaches zero everywhere except around $x = 0$. This suggests a possible definition of the delta distribution as follows:

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}. \tag{2.252}$$

Alternative representations are possible for the delta distribution, for example

$$\delta(x) = \lim_{\lambda \to \infty} \sqrt{(\lambda/\pi)}\, \mathrm{e}^{-\lambda x^2} \tag{2.253}$$

and

$$\delta(x) = \lim_{\lambda \to \infty} \frac{\sin \lambda x}{\pi x}. \tag{2.254}$$

If you cannot visualize these two functions, plot them and examine the behavior as λ becomes progressively larger. Both these functions are normalized to unit area.

Before preceding with more formal issues, a list is provided of some of the properties of the delta distribution that find wide usage in a variety of physical applications, a number of which are encountered later in this book in various sections. If the delta distribution does not appear as part of an integrand, then the following equations are to be viewed as symbolic relationships. They become identities when applied to suitable test functions (which are described later in this section) and then integrated over $\mathbb{R}$:

$$\delta(x - a) = 0, \quad x \neq a; \tag{2.255}$$

$$\delta(-x) = \delta(x); \tag{2.256}$$

$$x\delta(x) = 0; \tag{2.257}$$

$$\delta(ax) = |a|^{-1}\,\delta(x), \quad a \neq 0; \tag{2.258}$$

$$\delta(ax - b) = |a|^{-1}\delta\left(x - \frac{b}{a}\right), \quad a \neq 0; \tag{2.259}$$

$$\delta'(-x) = -\delta'(x); \tag{2.260}$$

$$x\delta'(x) = -\delta(x); \tag{2.261}$$

$$x^2\delta'(x) = 0; \tag{2.262}$$

$$\delta(x^2 - a^2) = \frac{1}{2\,|a|}\{\delta(x-a) + \delta(x+a)\}, \quad a \neq 0; \tag{2.263}$$

$$\int_a^b \delta(x-c)\mathrm{d}x = \begin{cases} 1, & a \leq c \leq b \\ 0, & c < a, \text{ or } c > b \text{ with } a < b, \end{cases} \tag{2.264}$$

assuming $f(x)$ is regular at $x = 0$;

$$f(x)\delta'(x) = f(0)\delta'(x) - f'(x)\delta(x), \tag{2.265}$$

assuming $f(x)$ is regular at $x = a$;

$$f(x)\delta(x-a) = f(a)\delta(x-a); \tag{2.266}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a); \tag{2.267}$$

$$P\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{s^2 - x^2} = \frac{\pi^2}{4}\delta(x); \tag{2.268}$$

and

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}xs}\,\mathrm{d}s. \tag{2.269}$$

Equation (2.267) has considerable value in applications. It is called the *sifting property* of the delta function. Equation (2.269) has an important role in Fourier transform theory. The alert reader will note that no property has been specified for the square or higher powers of the Dirac delta function. There is no meaning attached to powers of the Dirac delta function. In general, it is a much more delicate issue to deal with the multiplication of distributions (that are a function of the same variable), and in many cases it is not possible to arrive at a definition.

2.15.1 Generalized functions as sequences of functions

One approach to the discussion of generalized functions, sometimes referred to as Temple's theory, is to define them in terms of a sequence of functions. Consider the sequence of functions defined by

$$f_n(x) = \frac{\sin nx}{\pi x}, \quad n = 1, 2, 3, \ldots. \tag{2.270}$$

These functions satisfy

$$\int_{-\infty}^{\infty} \frac{\sin nx}{\pi x} \mathrm{d}x = \operatorname{sgn} n = 1, \tag{2.271}$$

which can be established by contour integration. In order that the sequence in Eq. (2.270) might provide a representation for the delta distribution in the limit $n \to \infty$, it is necessary to establish that Eqs. (2.247)–(2.249) are satisfied. Taking note of Eq. (2.271), it is straightforward to see that Eq. (2.248) is satisfied. To show that Eq. (2.247) holds requires the following result:

$$\begin{aligned} \int_{\alpha}^{\beta} \lim_{n\to\infty} \frac{\sin nx}{\pi x} \, \mathrm{d}x &= \lim_{n\to\infty} \int_{\alpha}^{\beta} \frac{\sin nx \, \mathrm{d}x}{\pi x} \\ &= \frac{1}{\pi} \lim_{n\to\infty} \int_{n\alpha}^{n\beta} \frac{\sin t \, \mathrm{d}t}{t} \\ &= 0, \end{aligned} \tag{2.272}$$

for $0 < \alpha < \beta$ or $\alpha < \beta < 0$. If $f_1, f_2, \ldots,$ are equivalent to summable functions and $|f_n|$ is bounded above almost everywhere by a summable function in the region of integration, then the interchange of limit and integration can be made, as was done in Eq. (2.272). The term summable function is here being used synonymously with integrable function, as is the common practice. It is necessary to show that in the limit $n \to \infty$, Eq. (2.249) holds. Suppose $g(x)$ is continuous and $g'(x)$ is also continuous and bounded on $\mathbb{R}$, then

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x) \mathrm{d}x &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\sin nx}{\pi x} g(x) \mathrm{d}x \\ &= \lim_{n\to\infty} \left\{ \int_{-\infty}^{\infty} \frac{\sin nx}{\pi x} \{g(x) - g(0)\} \mathrm{d}x + g(0) \int_{-\infty}^{\infty} \frac{\sin nx}{\pi x} \, \mathrm{d}x \right\} \\ &= g(0) + \lim_{n\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \sin nx \frac{\{g(x) - g(0)\}}{x} \, \mathrm{d}x. \end{aligned} \tag{2.273}$$

Using the stated assumption on g, or by assuming $\{g(x) - g(0)\}/x$ is in $L(\mathbb{R})$, the Riemann–Lebesgue lemma (see Eq. (2.234)) allows the last integral in Eq. (2.273) to be set to zero, and so

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x) \mathrm{d}x = g(0), \tag{2.274}$$

which establishes Eq. (2.249).

A second sequence is as follows:

$$f_n(x) = \begin{cases} -n, & |x| < (2n)^{-1} \\ 2n, & (2n)^{-1} \le |x| \le n^{-1} \\ 0, & \text{otherwise.} \end{cases} \tag{2.275}$$

First note that

$$\lim_{n\to\infty} f_n(x)|_{x=0} = -\infty. \tag{2.276}$$

Also,

$$\begin{aligned}\int_{-\infty}^{\infty} f_n(x)\mathrm{d}x &= \left\{\int_{-\infty}^{-n^{-1}} + \int_{-n^{-1}}^{-(2n)^{-1}} + \int_{-(2n)^{-1}}^{(2n)^{-1}} + \int_{(2n)^{-1}}^{n^{-1}} + \int_{n^{-1}}^{\infty}\right\} f_n(x)\mathrm{d}x \\ &= \int_{-n^{-1}}^{-(2n)^{-1}} (2n)\mathrm{d}x + \int_{-(2n)^{-1}}^{(2n)^{-1}} (-n)\mathrm{d}x + \int_{(2n)^{-1}}^{n^{-1}} (2n)\mathrm{d}x \\ &= 1. \end{aligned} \tag{2.277}$$

If $g(x)$ has the same properties as stated previously, and $h(x) = g(x) - g(0)$, then

$$\begin{aligned}\lim_{n\to\infty} \int_{-\infty}^{\infty} g(x)f_n(x)\mathrm{d}x &= g(0) + \lim_{n\to\infty} \int_{-\infty}^{\infty} h(x)f_n(x)\mathrm{d}x \\ &= g(0) + \int_{-n^{-1}}^{-(2n)^{-1}} (2n)h(x)\mathrm{d}x + \int_{-(2n)^{-1}}^{(2n)^{-1}} (-n)h(x)\mathrm{d}x \\ &\quad + \int_{(2n)^{-1}}^{n^{-1}} (2n)h(x)\mathrm{d}x. \end{aligned} \tag{2.278}$$

Now, the integrals in Eq. (2.278) can be simplified in the following manner. Given an $\varepsilon > 0$, pick an integer N such that, for $n > N$ and $|x| < N^{-1}$, $|h(x)| < \varepsilon$. Thus

$$\begin{aligned}&\int_{-n^{-1}}^{-(2n)^{-1}} (2n)h(x)\mathrm{d}x + \int_{-(2n)^{-1}}^{(2n)^{-1}} (-n)h(x)\mathrm{d}x + \int_{(2n)^{-1}}^{n^{-1}} (2n)h(x)\mathrm{d}x \\ &\quad\le 2n\int_{-n^{-1}}^{-(2n)^{-1}} |h(x)|\mathrm{d}x + n\int_{-(2n)^{-1}}^{(2n)^{-1}} |h(x)|\mathrm{d}x + 2n\int_{(2n)^{-1}}^{n^{-1}} |h(x)|\mathrm{d}x \\ &\quad\le \varepsilon\left\{2n\int_{-n^{-1}}^{-(2n)^{-1}} \mathrm{d}x + n\int_{-(2n)^{-1}}^{(2n)^{-1}} \mathrm{d}x + 2n\int_{(2n)^{-1}}^{n^{-1}} \mathrm{d}x\right\} \\ &\quad= 3\varepsilon. \end{aligned} \tag{2.279}$$

This establishes the sifting property.

Other sequence representations for the delta function can be found. Two generalized functions are equivalent if the sequence representations are equivalent. That is, for some generalized function represented by the sequences $g_n(x)$ and $h_n(x)$, and some suitable function $f(x)$, it follows that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} g_n(x)f(x)\mathrm{d}x = \lim_{n\to\infty}\int_{-\infty}^{\infty} h_n(x)f(x)\mathrm{d}x. \tag{2.280}$$

The important requirement is that all the equivalent sequences in the class defining the delta distribution must lead to the following property:

$$\langle\delta,f\rangle \equiv \int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0), \tag{2.281}$$

where $f(x)$ is a sufficiently well behaved function. The notation $\langle f,g\rangle$ is used to define the integral over $\mathbb{R}$ of the scalar product for a pair of functions, including cases where one, but not both, are generalized functions. The notation (f,g) is also widely employed for the same purpose. The context, that is the appearance of two functions, will usually make it perfectly clear that an open interval, which uses the same symbolism, is not being discussed. The term *test function* is commonly used to denote the function f in Eq. (2.281). Test functions are frequently defined in a number of different ways. Test functions can be continuous, continuously differentiable to some finite degree, or infinitely differentiable. In addition to the smoothness condition, a more stringent specification of the asymptotic behavior for the function as the argument approaches infinity is usually provided.

2.15.2 Schwartz distributions

Some preliminary terminology is considered. The closure of the set of all points x for which $f(x) \neq 0$ is called the *support* of the function f. The support of f is denoted by supp f. The function f is said to have *compact support* if supp f is a bounded set. For example, the square-pulse function of width $2a$ is defined by

$$f(x) = \begin{cases} 0, & \text{for } -\infty < x < -a \\ 1, & \text{for } -a < x < a \\ 0, & \text{for } a < x < \infty. \end{cases} \tag{2.282}$$

The support of the square-pulse function is $(-a,a)$, and, since f is bounded in the interval $(-a,a)$, the function has compact support.

Recall the notion of the term function: a rule for going from some variable x to some value $f(x)$. By analogy, the term *functional* is a rule for assigning to some function a value. For example, for the integral

$$I[f] = \int_a^b |f(x)|^2 \, \mathrm{d}x, \tag{2.283}$$

the value of the integral I is determined once the function f is specified. The square bracket notation is a commonly used designation for a functional dependence. In what immediately follows, interest will focus on functionals, where the rule is: the functional operation of T on f is $\langle T, f \rangle$. A linear functional on a vector space satisfies the relationship

$$\langle T, \alpha f + \beta g \rangle = \alpha \langle T, f \rangle + \beta \langle T, g \rangle, \tag{2.284}$$

where α and β are constants and f and g are elements of the vector space.

If f is a suitable test function and T is a distribution that is locally integrable on $\mathbb{R}$, that is the integral $\int_a^b |T(x)| \, \mathrm{d}x$ is finite for every interval (a, b), and if T is a continuous linear functional that can be represented by

$$\langle T, f \rangle = \int_{-\infty}^{\infty} T(x) f(x) \mathrm{d}x, \tag{2.285}$$

T is called a *regular distribution*. Distributions that are not regular are called *singular distributions*. Equation (2.285) is also employed for singular distributions with a symbolic interpretation applied to the integral. An example of a singular distribution is the Dirac delta distribution, and another example is $p.v.\ x^{-1}$, which is interpreted as x^{-1} when $x \neq 0$, and in a limiting sense as $|x| \to 0$. The latter example cannot be expressed in the form of Eq. (2.285) for a general test function, but must be interpreted as a Cauchy principal value integral. This example is discussed in detail later in Chapter 10.

Laurent Schwartz was one of the principal architects of the development of the mathematical theory of distributions. He defined test functions in the following manner. All functions that are infinitely differentiable and that vanish outside some bounded set K of $\mathbb{R}^n$, the n-dimensional Euclidean space, belong to the class $\mathcal{D}(\mathbb{R}^n)$. When the meaning is obvious, $\mathcal{D}(\mathbb{R}^n)$ is abbreviated to $\mathcal{D}$. Functions that are infinitely differentiable for all points of $\mathbb{R}$ belong to the class C^∞. The notation C_0^∞ is used to denote the class of functions that are infinitely differentiable with compact support. The set of functions that are continuously differentiable up to order k is denoted by C^k and by C_0^k if they are continuously differentiable up to order k and have compact support. The context should make it clear to the reader that the kth power of C is not intended. The set K will in general be different for dissimilar functions.

What type of functions belong to $\mathcal{D}$? Elementary functions such as polynomials and trigonometric functions satisfy the differentiability requirement, but do not meet the restricted support condition. Furthermore, the elementary infinitely differentiable functions are analytic, and, consequently, if the function is zero on an interval, then it is identically zero. This follows directly from Liouville's theorem: if $f(z)$ is analytic in the entire complex plane and $|f(z)|$ is bounded, then $f(z)$ must be a constant. To circumvent this difficulty, the function can be defined appropriately on multiple intervals, with the proviso that the derivatives vanish outside the region where the function has its support. As an example, consider for the one-dimensional Euclidean space the function ϕ given by

$$\phi(x) = \begin{cases} c e^{\{-(1-x^2)^{-1}\}}, & \text{for } |x| < 1 \\ 0, & \text{for } |x| \geq 1, \end{cases} \tag{2.286}$$

where c is a constant. This function belongs to $\mathcal{D}$. Clearly ϕ has compact support, and the function is infinitely differentiable everywhere. It is straightforward to generalize this test function to an n-dimensional Euclidean space.

If ϕ_1 and ϕ_2 are any two arbitrary functions belonging to $\mathcal{D}$, then a continuous linear functional T satisfies Eq. (2.284) and, for the set $\{\phi_n\}$, the relationship

$$\lim_{n\to\infty} \langle T, \phi_n \rangle = \langle T, \lim_{n\to\infty} \phi_n \rangle. \tag{2.287}$$

Distributions in the sense of Schwartz (some authors use Schwartz–Sobolev, to reflect the early pioneering contributions of the Soviet mathematician S. L. Sobolev) are those functionals defined on $\mathcal{D}$ that are linear and continuous.

2.16 Summation of series: convergence accelerator techniques

A great many Hilbert transforms cannot be reduced to a simple closed form, but can be represented as an infinite series. In such cases, various summation techniques can be employed to accelerate the convergence of these series, thus providing a convenient route to numerical evaluation of the Hilbert transform. These techniques become increasingly important for series which converge very slowly.

Consider the Riemann zeta function $\varsigma(n)$, defined for integer argument by

$$\varsigma(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad \text{for } n = 2, 3, \ldots. \tag{2.288}$$

For $n = 2$ it is straightforward to show that

$$\varsigma(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{2.289}$$

If you were requested to evaluate this series to ten significant digits by direct termwise addition of the individual contributions, the number of terms requiring evaluation is obviously enormous. The reader should make a rough estimation of the summation limit required to obtain ten significant digits. Clearly this type of brute force strategy is not a viable approach to evaluate this or other series with similar rates of convergence.

Two of the many approaches that are available to accelerate the convergence of slowly converging series are outlined in this section. These techniques can generate more than ten digits of precision for series converging like Eq. (2.288) using typically fewer than thirty terms in the sum.

2.16.1 Richardson extrapolation

Let S denote the series

$$S = \sum_{k=0}^{\infty} a_k, \tag{2.290}$$

and let the nth partial sum be denoted by

$$A_n = \sum_{k=0}^{n} a_k. \tag{2.291}$$

Suppose that, for large n, A_n has the following series representation:

$$A_n = \sum_{k=0}^{n} a_k \approx b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + \frac{b_3}{n^3} + \cdots, \quad \text{as } n \to \infty. \tag{2.292}$$

To obtain an accurate numerical estimate for the series in Eq. (2.290), it suffices to determine the lim $n \to \infty$ in Eq. (2.292). Clearly, in this limit knowledge of the coefficients b_i, $i = 1, 2, \ldots$, is not required; it is only necessary to evaluate b_0. In a practical computation of the value of b_0, the series expansion in Eq. (2.292) is truncated at some term, say the Nth, and it is assumed that all the A_k for $k = 0, 1, \ldots, N$ are known by direct numerical evaluation. The approach just sketched is termed the Richardson extrapolation (Richardson, 1927). The sequence of partial sums are expanded in a power series of inverse powers of n, that is

$$\begin{aligned}
A_n &= b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots + \frac{b_N}{n^N} \\
A_{n+1} &= b_0 + \frac{b_1}{n+1} + \frac{b_2}{(n+1)^2} + \cdots + \frac{b_N}{(n+1)^N} \\
&\vdots \\
A_{n+N} &= b_0 + \frac{b_1}{n+N} + \frac{b_2}{(n+N)^2} + \cdots + \frac{b_N}{(n+N)^N}.
\end{aligned} \tag{2.293}$$

The solution of this set of simultaneous equations for b_0 is obtained using Cramer's rule, with the following result:

$$b_0 = \frac{\begin{vmatrix} A_n & A_{n+1} & A_{n+2} & \cdots & A_{n+N} \\ k_0 & k_1 & k_2 & \cdots & k_N \\ k_0^2 & k_1^2 & k_2^2 & \cdots & k_N^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_0^N & k_1^N & k_2^N & \cdots & k_N^N \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ k_0 & k_1 & k_2 & \cdots & k_N \\ k_0^2 & k_1^2 & k_2^2 & \cdots & k_N^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_0^N & k_1^N & k_2^N & \cdots & k_N^N \end{vmatrix}}, \tag{2.294}$$

where

$$k_j = \frac{1}{n+j}. \tag{2.295}$$

The expression for b_0 can be significantly simplified by recognizing that the denominator is a Vandermonde determinant and that the numerator can be expressed in terms of Vandermonde determinants. A Vandermonde determinant can be written in a simple closed form:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ k_1 & k_2 & k_3 & \cdots & k_n \\ k_1^2 & k_2^2 & k_3^2 & \cdots & k_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_1^{n-1} & k_2^{n-1} & k_3^{n-1} & \cdots & k_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (k_j - k_i). \tag{2.296}$$

If the numerator of Eq. (2.294) is expanded by the first row, then

$$b_0 = \frac{\sum_{j=0}^{N} (-1)^j A_{n+j} \left|K_{1j}\right|}{\prod_{0 \le m < l \le N} (k_l - k_m)}, \tag{2.297}$$

where $|K_{1j}|$ denotes the minor of the determinant of the numerator obtained by deleting the first row and the $(j+1)$th column. Each of these minors can be expressed as a

Vandermonde determinant by factoring the term $(k_0 k_1 k_2 \cdots k_N) k_j^{-1}$. It follows that

$$\left|K_{1j}\right| = (k_0 k_1 k_2 \cdots k_N) k_j^{-1} \prod_{\substack{0 \leq l < m < N \\ (l, m \neq j)}} (k_m - k_l). \tag{2.298}$$

Now,

$$\prod_{\substack{0 \leq l < m < N \\ (l, m \neq j)}} (k_m - k_l) = \frac{\prod\limits_{0 \leq l < m < N} (k_m - k_l)}{\prod\limits_{i=j+1}^{N} (k_i - k_j) \prod\limits_{i=0}^{j-1} (k_i - k_j)}, \tag{2.299}$$

and $\left|K_{1j}\right|$ simplifies on using

$$\prod_{i=0}^{j-1} (k_i - k_j) = \frac{(-1)^j j!(n-1)!}{(n+j)^j (n+j-1)!}, \tag{2.300}$$

$$\prod_{i=j+1}^{N} (k_i - k_j) = \frac{(-1)^{N-j} (N-j)!(n+j-1)!}{(n+j)^{N-j-1} (N+n)!}, \tag{2.301}$$

and

$$(k_0 k_1 k_2 \cdots k_N)\, k_j^{-1} = \frac{(n+j)(n-1)!}{(N+n)!}, \tag{2.302}$$

to give

$$\left|K_{1j}\right| = \frac{(-1)^N (n+j)^N}{(N-j)!j!} \prod_{0 \leq l < m < N} (k_m - k_l), \tag{2.303}$$

and a rather compact expression for b_0 is obtained:

$$b_0 = \sum_{j=0}^{N} \frac{(-1)^{j+N} A_{n+j} (n+j)^N}{(N-j)!j!}. \tag{2.304}$$

Increasing the size of n and N leads to improved estimates for b_0 and hence of S. The success of this technique clearly hinges on how closely the partial sums can be represented by a series expansion of the form given in Eq. (2.292) and on the size of n and N selected. The obvious drawback to the use of Eq. (2.304) is the high likelihood of numerical instabilities arising from the alternating sign structure of the sum and the rapidly increasing size of individual contributions to the sum (try Exercise 2.21). Issues associated with numerical instabilities can usually be circumvented by judicious selection of the values of n and N. Alternatively, working in higher precision arithmetic can delay the onset of numerical instability in the calculation, allowing the sum to be evaluated to the required level of precision. In a practical application, a grid of values of b_0 is constructed for increasing values of n and N. The onset of erratic convergence signals the arrival of the numerical instability problem.

2.16.2 The Levin sequence transformations

Work on sequence transformations to accelerate the convergence of summation of series has a very long history, dating back to work of the mathematicians Stirling and Euler. A robust group of sequence transformations were introduced by Levin (1973), and these have been found to be particularly useful in many applications. Over the past quarter of a century a number of interesting refinements and extensions of Levin's work have emerged. In this section, the basic approach is sketched. The original work of Levin was based on the properties of the Vandermonde determinant. A different approach is considered here, which is straightforward in application. Some preliminary results are required first.

The difference operator Δ is defined by

$$\Delta f(n) = f(n+1) - f(n), \quad \text{for } n \in \mathbb{Z}^+, \tag{2.305}$$

where $\mathbb{Z}^+$ denotes the set of non-negative integers $0, 1, 2, \ldots$. The notation Δ is also employed to designate the Laplacian operator (defined later in Section 15.10); however, the reader should have no difficulty in determining the intended meaning of this symbol from the context of the discussion. Repeated application of the difference operator is given by

$$\Delta^k f(n) = \Delta\{\Delta^{k-1} f(n)\}, \quad \text{for } k \in \mathbb{N}, \tag{2.306}$$

where $\mathbb{N}$ denotes the set of positive integers, $1, 2, \ldots$, and with

$$\Delta^0 f(n) = f(n). \tag{2.307}$$

Equation (2.306) can be written in the form

$$\Delta^k f(n) = \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} f(n+j), \tag{2.308}$$

where $\binom{k}{j}$ denotes a binomial coefficient, defined by

$$\binom{k}{j} = \frac{k!}{(k-j)!\,j\,!}. \tag{2.309}$$

The standard binomial expansion for $(1-x)^k$, with $k \in \mathbb{Z}^+$, is

$$(1-x)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^j, \tag{2.310}$$

which, on repeated differentiation with respect to x, leads to

$$(1-x)^{k-1} = -\frac{1}{kx} \sum_{j=0}^{k} (-1)^j j \binom{k}{j} x^j, \tag{2.311}$$

$$(1-x)^{k-2} = \frac{1}{k(k-1)x^2} \left\{ \sum_{j=0}^{k} (-1)^j j^2 \binom{k}{j} x^j - \sum_{j=0}^{k} (-1)^j j \binom{k}{j} x^j \right\}, \tag{2.312}$$

and so on. Letting $x = 1$ in these expansions leads to

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} = 0 \tag{2.313}$$

and

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n = 0, \quad n = 1, 2, \ldots, k-1. \tag{2.314}$$

A direct consequence of the preceding results is that the difference operator Δ^k for $k \in \mathbb{N}$, acting on a function f which is a polynomial in n of degree $k-1$, yields zero. That is, suppose

$$f(n) = \sum_{i=0}^{k-1} a_i n^i, \tag{2.315}$$

where the a_i are constants, then

$$\begin{aligned}
\Delta^k f(n) &= \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} \sum_{i=0}^{k-1} a_i (n+j)^i \\
&= (-1)^k \sum_{i=0}^{k-1} a_i \left\{ n^i + \sum_{j=1}^{k} (-1)^j \binom{k}{j} (n+j)^i \right\} \\
&= (-1)^k \sum_{i=0}^{k-1} a_i \left\{ n^i + \sum_{j=1}^{k} (-1)^j \binom{k}{j} \left\{ n^i + \sum_{m=1}^{i} \binom{i}{m} n^{i-m} j^m \right\} \right\}
\end{aligned}$$

$$= (-1)^k \sum_{i=0}^{k-1} a_i \sum_{m=1}^{i} \binom{i}{m} n^{i-m} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^m$$

$$= 0. \tag{2.316}$$

Suppose the sum in Eq. (2.290) is written as follows:

$$S = A_n + r_n, \tag{2.317}$$

where r_n denotes a remainder term, which clearly has the form $r_n = \sum_{k=n+1}^{\infty} a_k$. Levin derived acceleration transformations for sequences of the form

$$A_n = T_{kr} + R_{kn}, \quad \text{for } r \leq n \leq r+k, \tag{2.318}$$

where T_{kr} is an approximation to the limit $n \to \infty$ of the sequence $\{A_n\}$; hence, it is assumed that $S \approx T_{kr}$. The term R_{kn} incorporates the structure of the remainder terms, and the index k is used to denote the number of terms in an expansion of the remainder, that is it denotes an order parameter. For an assumed functional form for the remainder, a sequence of transformations can be determined, and the index r signifies the minimal index n employed to determine the partial sum. Levin considered a sequence satisfying

$$S - A_n = -\omega_n \sum_{j=0}^{k-1} \frac{c_j}{(n+\alpha)^j}, \quad \text{for } k, n \in \mathbb{Z}^+, \tag{2.319}$$

where α and c_j are constants, and ω_n denotes a remainder term depending on n. Multiply the preceding equation by $(n+\alpha)^{k-1}$ to obtain

$$\frac{(S - A_n)(n+\alpha)^{k-1}}{\omega_n} = -\sum_{j=0}^{k-1} c_j (n+\alpha)^{k-j-1}; \tag{2.320}$$

applying the difference operator Δ^k and making use of Eq. (2.316) yields

$$\Delta^k \left\{ \frac{(S - A_n)(n+\alpha)^{k-1}}{\omega_n} \right\} = 0 \tag{2.321}$$

and hence

$$S = \frac{\Delta^k \left\{ \dfrac{A_n (n+\alpha)^{k-1}}{\omega_n} \right\}}{\Delta^k \left\{ \dfrac{(n+\alpha)^{k-1}}{\omega_n} \right\}}. \tag{2.322}$$

Making use of Eq. (2.308) leads to

$$S = \frac{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{n+\alpha+j}{n+\alpha+k}\right)^{k-1} \frac{A_{n+j}}{\omega_{n+j}}}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{n+\alpha+j}{n+\alpha+k}\right)^{k-1} \frac{1}{\omega_{n+j}}}. \tag{2.323}$$

The factor $(n+\alpha+k)^{-k+1}$, which is inserted into both the numerator and denominator, is incorporated to improve the numerical stability of the formula. If the remainder factor ω_n is taken to be

$$\omega_n = a_n, \quad \text{for } n \in \mathbb{Z}^+, \tag{2.324}$$

then Eq. (2.323) can be written as follows:

$$S = \frac{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{n+\alpha+j}{n+\alpha+k}\right)^{k-1} \frac{A_{n+j}}{a_{n+j}}}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{n+\alpha+j}{n+\alpha+k}\right)^{k-1} \frac{1}{a_{n+j}}}, \tag{2.325}$$

which is referred to as Levin's t transformation. If

$$\omega_n = (n+\alpha)a_n, \quad \text{for } n \in \mathbb{Z}^+, \tag{2.326}$$

then

$$S = \frac{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{n+\alpha+j}{n+\alpha+k}\right)^{k-2} \frac{A_{n+j}}{a_{n+j}}}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{n+\alpha+j}{n+\alpha+k}\right)^{k-2} \frac{1}{a_{n+j}}}, \tag{2.327}$$

which is Levin's u transformation. Additional transformations can be obtained by an appropriate choice of ω_n. Further transformations can also be obtained by selecting different forms in place of Eq. (2.319). In the absence of any specific knowledge of α in Eq. (2.319), a common procedure is to start with the choice $\alpha = 0$.

Notes

A collection of many key papers in classical analysis pertaining to a number of topics in this chapter can be found in Birkhoff (1973). The seminal papers were often not written in English, and a strength of the Birkhoff collection is that these papers have been translated into English. There are also some useful annotations on various papers in this work. There are a number of excellent books on mathematical physics where many of the topics of this chapter are discussed in detail: see, for example, Hassani (1999) and McQuarrie (2003).

§2.2 The earliest appearance of the order symbols $O(\ )$ and $o(\ )$ known to the author are Bachmann (1894) for the former and Landau (1909) for the latter. The names of these authors are not commonly attached to the order symbols. For additional discussion, see, for example, Whittaker and Watson (1927), Titchmarsh (1939), or Meyer (1979).

§2.3 For further reading on Hölder continuous functions, see Gakhov (1966), Roos (1969, p. 220), Henrici (1986), and particularly Muskhelishvili (1992).

§2.4 The latter four books are also useful references for additional reading on Cauchy principal value integrals. Davies and Davies (1989) present a concise discussion on higher-order singularities.

§2.5 There are many sources on Fourier series. Walker (1988) is a good introduction to the topic; Bary (1964) and Edwards (1982) present more advanced discussions, and Zygmund (1968) is a masterly account of the subject. Katznelson (1976), Benedetto (1997), and Krantz (1999a) could be consulted for some background on harmonic analysis.

§2.6 Titchmarsh (1948) presents an authoritative account on the Fourier transform. Howell (1996) provides a highly readable discussion with a view to applications. Wiener (1933) and Champeney (1987) provide concise presentations of the important Fourier theorems, Walker (1988) gives an easy to read account of the basics; and Bochner and Chandrasekharan (1949) present a more advanced discussion. Zygmund (1968) is also a very useful resource, as is the book by Campbell and Foster (1948).

§2.7 For further reading on the Fourier integral, see Titchmarsh (1948) and Bochner (1959a).

§2.8 For readers interested in the historical development of the subject, Mitrinović and Kečkić (1984, chap. 10) provide a concise biographical sketch of Cauchy, with a focus on his contributions to the development of the calculus of residues. Smithies (1997) gave a more extended discussion of Cauchy's work, and this is highly recommended reading. For some well written accounts on the theory of functions of a complex variable, see Carrier, Krook, and Pearson (1983), Churchill and Brown (1984), Jeffrey (1992), Needham (1997), or Krantz (1999b), and, at the more advanced level, Greene and Krantz (1997). A good condensed account can be found in Morse and Feshbach (1953).

§2.8.7 An elementary account of entire functions can be found in Markushevich (1966). A good introduction can be found in Holland (1973), and more advanced treatments are provided by Boas (1954) and Cartwright (1962).

§2.10 For further reading on functional analysis, see Oden (1979) or Kantorovich and Akilov (1982). For additional discussion and proofs of the various results on compact operators, see Hochstadt (1973) or Hassani (1999). The books by Garnett (1981), Koosis (1998), and Duren (2000) can be consulted for additional information on H^p spaces. The bra-ket symbolism was introduced in quantum mechanics by Dirac (1939).

§2.11 For further reading on topics connected with this section the following introductory texts are highly recommended reading: Titchmarsh (1939), Weir (1973), Priestley (1997) and Burk (1998). For more advanced discussions, see Cohn (1980), Benedetto (1997), and Folland (1999). A good discussion of uniform convergence can be found in Titchmarsh (1939). This text also provides a very readable account of several other topics sketched in this chapter. For more details on the conditions for switching limits with integrals, see, for example, Kestelman (1960, p. 141).

§2.12 For further introductory reading on interchanging the order in repeated integrals, the books by Weir (1973) and particularly Priestley (1997) are highly recommended. Good discussions can be found in Courant (1936) and Titchmarsh (1939). For a more advanced treatment, see Folland (1999).

§2.13 Hardy (1908) discusses the inversion of integration order for double integrals where one or both integrals are taken in the principal value sense, and both cases of finite and infinite integration limits are treated. For further general reading see Poincaré (1910), Bertrand (1922, 1923a, 1923b), Titchmarsh (1948), Gakhov (1966), Love (1977), Tricomi (1985), Henrici (1986), and Muskhelishvili (1992). For a discussion related to Eq. (2.220) see Hardy (1908), Kober (1943b) or the Tricomi, Love, and Titchmarsh references just cited. The papers by Davies, Davies, and White (1990), Davies, Glasser, and Davies (1992), and Davies *et al.* (1996) are useful resources for the Hardy–Poincaré–Bertrand theorem. A straightforward derivation of the Hardy–Poincaré–Bertrand formula for Hölder continuous functions can be found in Baird, Sancaktar, and Zweifel (1977). For an extension to n dimensions, see Newell (1968).

§2.15 A good introduction to some of the material of this section is in Lighthill (1970). Gel′fand and Shilov (1964, 1968), Bremermann (1965a), Schwartz (1966a, 1966b), Roos (1969), and Kanwal (1998) can be consulted for additional information. For a concise account of some of the history of the Dirac delta function, including references to publications predating the seminal work of Paul Dirac, see Van Der Pol and Bremmer (1987, chap. 5). The key early paper by Dirac on the delta function is Dirac (1927).

§2.16 Discussion of convergence accelerator techniques can be found in the books by Wimp (1981) and Brezinski and Redivo Zaglia (1991). Highly readable accounts directed towards individuals in the physical sciences can be found in Weniger (1989, 1991, 1996). Smith and Ford (1979, 1982) and Fessler, Ford, and Smith (1983) are also useful reading. Some example applications to problems in atomic physics can be found in Porras and King (1994), Pelzl and King (1998), King (1999), and Pelzl, Smethells, and King (2002).

§Exercises Inequalities such as the Hölder inequality (Hölder, 1889; see also Rogers, 1888), Exercises 2.16 and 2.17, play an important role in the derivations in a number of places in this book. Some good sources on inequalities are: Hardy, Littlewood, and Pólya (1952), Mitrinović (1970), and Mitrinović, Pečarić, and Fink (1991). The review by Fink (2000) is also worth reading. A delightful introductory account can be found in Steele (2004).

Exercises

2.1 For the following functions, determine the $O(\ )$ and $o(\ )$ behavior as $x \to \infty$ and $x \to 0$.

(i) $x^{-1} \sin x$,

(ii) $x^{-1}(1 - \cos x)$,

(iii) $x^m e^{-ax}$ (for $m > 0$ and $a > 0$),

(iv) $\dfrac{\sum_{p=0}^{m} a_p x^p}{\sum_{p=0}^{n} b_p x^p}$, (for $m < n$ and $a_p \neq 0, b_p \neq 0$),

(v) $\operatorname{sech} x$.

2.2 Are the following functions Hölder continuous on the specified interval? Justify your answers:

(i) e^{-ax} for $a > 0$, with $x \in [-1, 1]$,

(ii) $\dfrac{\sin\theta}{\theta}$, $\theta \in [-\pi, \pi]$,

(iii) $\{(b - x)(x - a)\}^{-1}$, $x \in (a, b)$,

(iv) $(x^2 + x - 2)^{-1}$, $x \in [0, 2]$.

2.3 Determine whether the following integrals are convergent or divergent, where a and b are real constants. For the convergent cases, determine the value of the integral.

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax \, dx}{x}$;

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax \, dx}{x^2}$;

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x - a}$; for $\operatorname{Im} a > 0$;

(iv) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x - a)^2}$, for real a;

(v) $\displaystyle\int_{0}^{\infty} \frac{(e^{-ax} - e^{-bx})\, dx}{x}$, for $a > 0,\ b > 0$.

2.4 For the same integrals given in Exercise 2.3, which (if any) of the divergent integrals exist when the Cauchy principal value is employed.

2.5 Starting from the Parseval formula, deduce the Plancherel formula, Eq. (2.56). [Hint: Start by trying the replacement $f \to f + g$.]

2.6 Determine whether the following functions have a simple pole (find the order), a removable singularity, or an essential singularity: (i) $(z-3)^{-2}\{1-\cos(z-3)\}$; (ii) $\{(2 - z)^{-1} + (2 + z)^{-1}\}$; (iii) $e^{(z-1)^{-1}}$.

2.7 To verify that the endpoints of the subintervals are of no consequence when evaluating an integral involving step functions, determine the value of the integrals $\int_0^5 f_1(x)\mathrm{d}x$ and $\int_0^5 f_2(x)\mathrm{d}x$, where

$$f_1(x) = \begin{cases} 2, & 0 \le x < 1 \\ -2, & 1 \le x < 3 \\ 3, & 3 \le x \le 5 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 2, & 0 \le x < 1 \\ 1, & x = 1 \\ -2, & 1 < x < 3 \\ 1, & x = 3 \\ 3, & 3 < x < 5 \\ -1, & x = 5. \end{cases}$$

2.8 Determine if the function $x^{-1}\sin x$ is absolutely integrable on $(0, \infty)$.

2.9 Does the series $\sum_{n=0}^{\infty} (\sin(2n+1)\theta)/(2n+1)$ converge uniformly on the interval $-\pi \le \theta \le \pi$?

2.10 For the following functions, determine the value(s) of p such that $f \in L^p$ when the integration interval is $(0, \infty)$:

(i) $f(x) = \dfrac{\log x}{1+x}$;

(ii) $f(x) = \dfrac{x^2}{\cosh x}$;

(iii) $f(x) = \dfrac{\sin^m x}{x^n}$; for m, n positive integers,

(iv) $f(x) = \dfrac{\sqrt{(x)}\log x}{(1+x)}$.

2.11 For what values of p (if any) is $f \in L^p$, where

(i) $f(x) = (x\{a + |\log|x||\})^{-1}$, with $a > 0$ and $x \in \mathbb{R}$,

(ii) $f(x) = (|x|\{a + |\log|x||^2\})^{-1}$, with $a > 0$ and $x \in \mathbb{R}$?

2.12 If $f(x) = \mathrm{e}^{x^{-1}}$, for what values of p is $f \in L^p(0,1)$?

2.13 If $f(x,y) = (x-y)(x+y)^{-3}$, determine if

$$\int_0^1 \left\{ \int_0^1 f(x,y)\mathrm{d}x \right\} \mathrm{d}y = \int_0^1 \left\{ \int_0^1 f(x,y)\mathrm{d}y \right\} \mathrm{d}x.$$

2.14 For the choice $f(x,y) = \{(y-\alpha x)(y-\beta x)\}^{-1}$, for $\alpha \ne 0, \beta \ne 0, \alpha \ne \beta$, evaluate $g(\alpha,\beta) = \int_{-1}^{1} \mathrm{d}x\, P\int_{-1}^{1} f(x,y)\,\mathrm{d}y - \int_{-1}^{1} \mathrm{d}y\, P\int_{-1}^{1} f(x,y)\mathrm{d}x$, depending on whether (i) $\beta < 0 < \alpha$, (ii) $\alpha < 0 < \beta$, or (iii) α and β have the same sign. If the lower limits are replaced by zero, determine the value of $g(\alpha,\beta)$ and specify the conditions on α and β in order that the repeated integrals are convergent.

2.15 By first writing $\int_0^{\infty} (\sin x/x)\mathrm{d}x = \int_0^{\infty} x^{-1}(\mathrm{d}(1-\cos x)/\mathrm{d}x)\mathrm{d}x$, use integration by parts, the identity $x^{-2} = \int_0^{\infty} t\mathrm{e}^{-xt}\,\mathrm{d}t$, and a change of integration order to evaluate the integral in Eq. (2.271). Justify these steps.

2.16 This problem and the following one address the derivation of the Hölder inequality, which finds frequent application in analysis. For $0 < \tau < 1$ and $x > 0$, show by elementary calculus that $x^\tau \leq \tau x + (1-\tau)$ and that equality occurs if and only if $x = 1$. Hence, or otherwise, show that, for $\alpha \geq 0$ and $\beta \geq 0$, $\alpha^\tau \beta^{1-\tau} \leq \tau\alpha + (1-\tau)\beta$.

2.17 If $f \in L^p$ and $g \in L^q$ with $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, show that $fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$. [Hint: Use Exercise 2.16 with $\tau = p^{-1}, \alpha = \left(|f(x)|/\|f\|_p\right)^p$ and $\beta = \left(|g(x)|/\|g\|_q\right)^q$].

2.18 If $f \in L^p$ for $1 \leq p \leq \infty$ and $g \in L^q$ for $1 \leq q \leq \infty$ establish that $fg \in L^r$, where $r^{-1} = p^{-1} + q^{-1}$.

2.19 If $f \in L^p \cap L^q$ for $1 \leq p < q \leq \infty$, show that $f \in L^r$ with $p \leq r \leq q$.

2.20 Derive the result

$$\Delta^k f(n) = \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} f(n+j),$$

where Δ denotes the difference operator and $k \in \mathbb{Z}^+$.

2.21 Apply the Richardson extrapolation procedure, Eq. (2.304), to evaluate the series $\sum_{k=1}^{\infty} k^{-2}$ to a precision of ten digits. Also demonstrate the instability issue that arises as the index j increases.

2.22 Use the Levin u sequence transformation to evaluate $\sum_{k=1}^{\infty} k^{-2}$ to eight digits of precision. Determine the point at which the convergence of the calculation starts to become erratic for the working precision employed in the calculation.

2.23 Show that $\sum_{k=1}^{\infty} k^{-2} = 2\sum_{k=1}^{\infty} (-1)^{k-1} k^{-2}$ and evaluate the latter sum using the Levin u sequence transformation. Compare the convergence behavior for this example with that found in Exercise 2.22. Explain any differences that are apparent.

3

Derivation of the Hilbert transform relations

3.1 Hilbert transforms – basic forms

The principal objective of this chapter is to introduce the basic forms for the Hilbert transform on the real line and on the circle. The important connections with the theory of functions of a complex variable are also established. The Hilbert transform on the real line is defined in this book by the principal value integral

$$Hf(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\, \mathrm{d}t. \tag{3.1}$$

This integral is often written in the following form:

$$Hf(x) = \lim_{\varepsilon \to 0+} H_\varepsilon f(x), \tag{3.2}$$

where

$$H_\varepsilon f(x) = \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\, \mathrm{d}t. \tag{3.3}$$

The function $H_\varepsilon f$ is sometimes referred to as the *truncated* Hilbert transform of f. The designation "truncated" is also used to describe other variants of the standard Hilbert transform. Let

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\, \mathrm{d}t; \tag{3.4}$$

then the function f is connected to g by the following result:

$$f(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(t)}{x-t}\, \mathrm{d}t. \tag{3.5}$$

Equations (3.4) and (3.5) constitute a *Hilbert transform pair*. The reader is alerted to the fact that, in some sources, the Hilbert transform is defined with a sign change for the kernel of the integrand in Eq. (3.1). The definition given in Eq. (3.1) follows that employed in recent major works devoted to the Hilbert transform (Henrici, 1986;

Hahn, 1996a, 1996b; Pandey, 1996). The required conditions on the functions f and g are addressed in detail in later Sections of this chapter. At this point it will be assumed that f and g are integrable in any finite interval, with $|f|^2$ and $|g|^2$ integrable on the interval $(-\infty, \infty)$.

In the literature, the Hilbert transform pairs are presented in several different forms. They were given originally by Hilbert (1904) in the following manner:

$$u(\sigma) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}v(s)}{\mathrm{d}s} \log\left(2\left|\sin\frac{s-\sigma}{2}\right|\right)\mathrm{d}s, \tag{3.6}$$

$$v(\sigma) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}u(s)}{\mathrm{d}s} \log\left(2\left|\sin\frac{s-\sigma}{2}\right|\right)\mathrm{d}s, \tag{3.7}$$

and

$$u(\xi) = \frac{1}{\pi}\int_{-1}^{1} \frac{\mathrm{d}v(x)}{\mathrm{d}x} \log\left(2\left|\sin\pi\frac{x-\xi}{2}\right|\right)\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} v(x)\cot\left(\pi\frac{x-\xi}{2}\right)\mathrm{d}x, \tag{3.8}$$

$$v(\xi) = -\frac{1}{\pi}\int_{-1}^{1} \frac{\mathrm{d}u(x)}{\mathrm{d}x} \log\left(2\left|\sin\pi\frac{x-\xi}{2}\right|\right)\mathrm{d}x = -\frac{1}{2}\int_{-1}^{1} u(x)\cot\left(\pi\frac{x-\xi}{2}\right)\mathrm{d}x. \tag{3.9}$$

In Eqs. (3.6)–(3.9) the Cauchy principal value is understood. Equations (3.6) and (3.7) represent the Hilbert transform pair for a function defined on the unit circle. The derivation of this form will be discussed later in Section 3.9. Kellogg (1904) gave the results

$$u(s) = P\int_{0}^{1} v(x)\cot\pi(s-x)\mathrm{d}x + \int_{0}^{1} u(x)\mathrm{d}x \tag{3.10}$$

and

$$v(s) = -P\int_{0}^{1} u(x)\cot\pi(s-x)\mathrm{d}x + \int_{0}^{1} v(x)\mathrm{d}x, \tag{3.11}$$

which he acknowledges as being due to Hilbert. Young (1912) gave the following results:

$$g(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{f(x+t) - f(x-t)}{t}\,\mathrm{d}t, \tag{3.12}$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{g(x+t) - g(x-t)}{t}\,\mathrm{d}t, \tag{3.13}$$

and

$$g(x) = \frac{1}{2\pi}\int_0^\pi \{f(x+t) - f(x-t)\}\cot(t/2)\mathrm{d}t, \tag{3.14}$$

$$f(x) = \frac{1}{2\pi}\int_0^\pi \{g(x+t) - g(x-t)\}\cot(t/2)\mathrm{d}t, \tag{3.15}$$

which were motivated by results from Fourier series. Young used the opposite sign convention in Eq. (3.12) to that employed in Eq. (3.1). Hardy (1924a) gave the transform pair as

$$g(x) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} f(t)\log\left|1 - \frac{x}{t}\right|\mathrm{d}t \tag{3.16}$$

and

$$f(x) = -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} g(t)\log\left|1 - \frac{x}{t}\right|\mathrm{d}t. \tag{3.17}$$

Equations (3.4) – (3.17) constitute the classical transform pairs. A number of extensions to the preceding results have been developed, and some of these will be touched upon elsewhere. Several of the forms just given are obviously interrelated, and these connections are explored in this chapter.

In later sections the Cauchy integral will be encountered. In the complex plane this integral takes the form

$$g(s) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z)\mathrm{d}z}{z-s}, \tag{3.18}$$

where C is a closed contour. This obviously has a very close tie-in with the Hilbert transform when f is holomorphic in the region enclosed by the contour C. Closely related is the Cauchy transform, defined in the following manner. Let C^∞ denote the space of complex-valued functions that are infinitely differentiable on $\mathbb{R}$. If $b\Omega$ denotes the boundary (the notation $\partial\Omega$ is also commonly employed) of a bounded domain Ω, and f is a C^∞ function defined on $b\Omega$, then the Cauchy transform, denoted $(\mathscr{C}f)(z)$, is given by

$$\mathscr{C}f(z) = \frac{1}{2\pi\mathrm{i}}\oint_{b\Omega} \frac{f(s)\mathrm{d}s}{s-z}. \tag{3.19}$$

This result is also connected to the Hilbert transform.

3.2 The Poisson integral for the half plane

One of the key mathematical ideas connected to the Hilbert transform is examined in this Section. The derivation of the Poisson integral formula for the upper half plane is considered, and from this result it will be shown how the Hilbert transform on $\mathbb{R}$

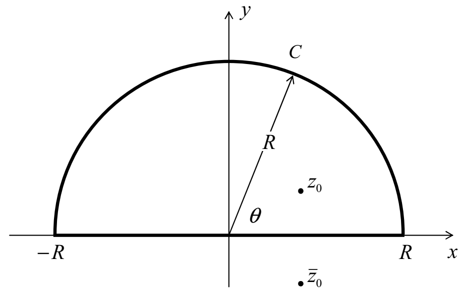

Figure 3.1. Semicircular contour with center at the origin and linear section along the real axis.

emerges. The Poisson integral formula allows the values of a function in the upper half plane to be determined from its values on the x-axis. Consider the contour C depicted in Figure 3.1. Let the point z_0 be inside the contour C; and denote the image point (the reflection across the x-axis) in the lower half plane by $\bar{z}_0$, where the bar denotes complex conjugate. Let $f(z)$ be analytic in the upper half plane, and suppose that $f(z) \to 0$ as $z \to \infty$. From the Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z - z_0}\, \mathrm{d}z, \tag{3.20}$$

and, from the Cauchy integral theorem,

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z - \bar{z}_0}\, \mathrm{d}z = 0. \tag{3.21}$$

Equation (3.21) follows since the point $\bar{z}_0$ is not enclosed by the contour C. On subtracting Eq. (3.21) from Eq. (3.20) it follows that

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi \mathrm{i}} \oint_C f(z) \left\{ \frac{1}{z - z_0} - \frac{1}{z - \bar{z}_0} \right\} \mathrm{d}z \\ &= \frac{(z_0 - \bar{z}_0)}{2\pi \mathrm{i}} \int_{-R}^{R} \frac{f(x)\mathrm{d}x}{(x - x_0)^2 + y_0^2} + \frac{(z_0 - \bar{z}_0)}{2\pi \mathrm{i}} \int_{C_R} \frac{f(z)\mathrm{d}z}{(z - z_0)(z - \bar{z}_0)}. \end{aligned} \tag{3.22}$$

The second integral in Eq. (3.22) vanishes in the limit as $R \to \infty$. This is most easily seen by converting to polar coordinates, with $z = R\mathrm{e}^{\mathrm{i}\theta}$. Suppose the maximum value of $|f(R\mathrm{e}^{\mathrm{i}\theta})|$ on C_R (the semicircular section of the contour C) is denoted by

M_R, then

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{f(z)\mathrm{d}z}{(z-z_0)(z-\bar{z}_0)}\right| \leq \lim_{R\to\infty}\int_0^{\pi}\left|\frac{f(R\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}}{(R\mathrm{e}^{\mathrm{i}\theta}-z_0)(R\mathrm{e}^{\mathrm{i}\theta}-\bar{z}_0)}\right|\mathrm{d}\theta$$
$$\leq \lim_{R\to\infty}\frac{M_R}{R}\int_0^{\pi}\left|\frac{\mathrm{e}^{\mathrm{i}\theta}}{\left(\mathrm{e}^{\mathrm{i}\theta}-\frac{z_0}{R}\right)\left(\mathrm{e}^{\mathrm{i}\theta}-\frac{\bar{z}_0}{R}\right)}\right|\mathrm{d}\theta$$
$$= \lim_{R\to\infty}\frac{M_R}{R}\int_0^{\pi}\left|\mathrm{e}^{-\mathrm{i}\theta}\right|\mathrm{d}\theta$$
$$= \lim_{R\to\infty}\frac{\pi M_R}{R} = 0. \tag{3.23}$$

The last limit in Eq. (3.23) is obtained by virtue of the restriction placed on $f(z)$. Therefore in the limit as $R \to \infty$, Eq. (3.22) simplifies to

$$f(z_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{(x-x_0)^2+y_0^2}. \tag{3.24}$$

Equation (3.24) is called the Poisson integral formula for the half plane. The function f can be written as

$$f(z) = u(x,y) + \mathrm{i}v(x,y), \tag{3.25}$$

where u and v are real-valued harmonic functions. A real function $u(x,y)$ with continuous second partial derivatives, a C^2 function, is called harmonic if it satisfies Laplace's equation, $\nabla^2 u = 0$. If the region of interest is simply connected, and if u is harmonic in this region, then there exists a harmonic function v, called the harmonic conjugate of u, such that $f = u + \mathrm{i}v$ is analytic in the same region. Equation (3.24) can be separated into its real and imaginary parts to yield

$$u(x_0,y_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{u(x)\mathrm{d}x}{(x-x_0)^2+y_0^2} \tag{3.26}$$

and

$$v(x_0,y_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{v(x)\mathrm{d}x}{(x-x_0)^2+y_0^2}, \tag{3.27}$$

where the notational abbreviations $u(x,0) \equiv u(x)$and $v(x,0) \equiv v(x)$ have been adopted. Equations (3.26) and (3.27) are also referred to as Poisson integrals for the half plane.

If the preceding process is repeated, but in place of taking the difference of Eqs. (3.20) and (3.21), the sum of these two equations is taken, then it follows that

$$f(z_0) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{(x - x_0) f(x) \mathrm{d}x}{(x - x_0)^2 + y_0^2}, \tag{3.28}$$

where the assumption required to obtain Eq. (3.28) is that $M_R \to 0$ as $z \to \infty$. If Eq. (3.25) is employed, then

$$u(x_0, y_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x - x_0) v(x) \mathrm{d}x}{(x - x_0)^2 + y_0^2} \tag{3.29}$$

and

$$v(x_0, y_0) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x - x_0) u(x) \mathrm{d}x}{(x - x_0)^2 + y_0^2}. \tag{3.30}$$

These results are called the *conjugate Poisson formulas*. The important feature of these formulas is that knowledge of the values of the real part of $f(x)$ on the real axis $(-\infty, \infty)$ determines the imaginary part of $f(z)$ in the upper half plane, and knowledge of the imaginary part of $f(x)$ on the real axis $(-\infty, \infty)$ determines the real part of $f(z)$ in the upper half plane.

The Poisson kernel for the half plane is defined by

$$P(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, \tag{3.31}$$

and the conjugate Poisson kernel for the half plane is given by

$$Q(x, y) = \frac{1}{\pi} \frac{x}{x^2 + y^2}. \tag{3.32}$$

From the form of Eqs. (3.29) and (3.30), it should be clear that these kernels will play a central role in the limit as $y_0 \to 0+$. The Poisson and conjugate Poisson kernels will be exploited in several sections later in the book, in particular in Section 7.10.

The situation $y_0 \to 0+$ in Eqs. (3.29) and (3.30) is now considered, and the integrals that arise are interpreted as Cauchy principal value integrals (see Section 2.4), so that

$$u(x_0) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x) \mathrm{d}x}{x_0 - x} \tag{3.33}$$

and

$$v(x_0) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x) \mathrm{d}x}{x_0 - x}. \tag{3.34}$$

Equations (3.33) and (3.34) represent a Hilbert transform pair. The important interpretation to give to these results is similar to that provided previously for the conjugate Poisson integrals. Knowledge of the real part of $f(x)$ on the interval $(-\infty, \infty)$ determines the imaginary part of $f(x)$, and knowledge of the imaginary part of $f(x)$ on $(-\infty, \infty)$ determines the real part of $f(x)$. The reader needs to keep in mind the restrictions placed on $f(z)$ at the start of this section. These restrictions can be weakened considerably, and this is considered shortly. One immediate loose end not yet covered concerns the details of looking at the limit $y_0 \to 0+$. What is the justification for interchanging the limit and the integral? Is there a singularity on the contour? Just how do the Cauchy principal value integrals arise? These issues are addressed later in this chapter. For additional discussion on the Poisson integral for the half plane, see the end of Section 4.22, and for some direct connections with the Hilbert transform, see Section 7.10.

3.3 The Poisson integral for the disc

The Poisson integral formula plays a key role in the development of the theory of Hilbert transforms. This section examines the Poisson integral for the disc. Suppose $f(z)$ is analytic inside and on the boundary of a disc of radius r centered at the origin. Let z_0 be a point inside the disc, the *inverse point* (also referred to as the image point) of z_0, denoted z_I, is defined by the following relationship:

$$|z_0||z_\mathrm{I}| = r^2, \tag{3.35}$$

and z_I lies outside the disc, as shown in Figure 3.2.

On the boundary of the disc $f(z)$ can be written as follows:

$$f(z) \equiv f(r, \theta) = u(r, \theta) + \mathrm{i}v(r, \theta), \tag{3.36}$$

where u and v are real-valued functions. At the point z_0 the function f can be written as follows:

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{z - z_0} + a\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{z - z_\mathrm{I}}, \tag{3.37}$$

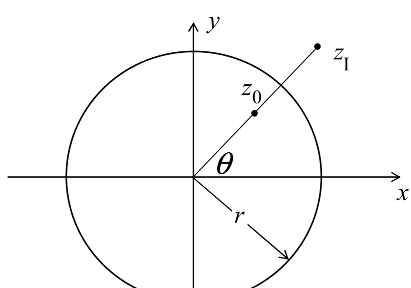

Figure 3.2. Circular contour, center the origin, with z_0 interior and the image point z_I exterior to the disc.

where the contour C is the perimeter of the disc in Figure 3.2 and a is a constant. The first part of Eq. (3.37) is the Cauchy integral theorem, and the second integral is zero (by the Cauchy–Goursat theorem), since the image point z_{I} is outside the contour. Consider first the case that $a = -1$, and, on converting to polar coordinates,

$$z = r\mathrm{e}^{\mathrm{i}\theta} \tag{3.38}$$

and

$$z_0 = r_0\mathrm{e}^{\mathrm{i}\theta_0}; \tag{3.39}$$

then Eq. (3.37) can be written, using Eq. (3.35), as follows:

$$f(r_0\mathrm{e}^{\mathrm{i}\theta_0}) = \frac{1}{2\pi}\int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta})\left[\frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta_0 - \theta)}\right]\mathrm{d}\theta, \tag{3.40}$$

which is Poisson's formula for the circle. It provides a means to find the real and imaginary parts of the function inside the disc, in terms of values of the function on the boundary of the disc. If $r_0 = 0$, then Eq. (3.40) simplifies to

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta. \tag{3.41}$$

Starting from Eq. (3.37) with the alternative choice $a = 1$, then

$$f(r_0\mathrm{e}^{\mathrm{i}\theta_0}) = \frac{1}{2\pi}\int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta})\left[1 + \frac{2\mathrm{i}rr_0\sin(\theta_0 - \theta)}{r^2 + r_0^2 - 2rr_0\cos(\theta_0 - \theta)}\right]\mathrm{d}\theta. \tag{3.42}$$

Substituting Eq. (3.36) into Eq. (3.42) and separating into real and imaginary parts leads to

$$u(r_0, \theta_0) = \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\mathrm{d}\theta - \frac{rr_0}{\pi}\int_0^{2\pi} \frac{v(r, \theta)\sin(\theta_0 - \theta)\mathrm{d}\theta}{r^2 + r_0^2 - 2rr_0\cos(\theta_0 - \theta)} \tag{3.43}$$

and

$$v(r_0, \theta_0) = \frac{1}{2\pi}\int_0^{2\pi} v(r, \theta)\mathrm{d}\theta + \frac{rr_0}{\pi}\int_0^{2\pi} \frac{u(r, \theta)\sin(\theta_0 - \theta)\mathrm{d}\theta}{r^2 + r_0^2 - 2rr_0\cos(\theta_0 - \theta)}. \tag{3.44}$$

If the point z_0 is selected on the boundary of the circle, then Eq. (3.43) simplifies, on taking the limit as $r_0 \to r$, to yield

$$u(r, \theta_0) = \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\mathrm{d}\theta - \frac{1}{2\pi}P\int_0^{2\pi} v(r, \theta)\cot\left(\frac{\theta_0 - \theta}{2}\right)\mathrm{d}\theta, \tag{3.45}$$

and Eq. (3.44) yields

$$v(r,\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} v(r,\theta)\mathrm{d}\theta + \frac{1}{2\pi}P\int_0^{2\pi} u(r,\theta)\cot\left(\frac{\theta_0-\theta}{2}\right)\mathrm{d}\theta. \tag{3.46}$$

Equations (3.45) and (3.46) are the Hilbert transform relations for the disc (Hilbert, 1904). Using Eq. (3.41) and the notational simplifications $u(0,\theta) \equiv u(0)$ and $v(0,\theta) \equiv v(0)$ yields

$$u(r,\theta_0) = u(0) - \frac{1}{2\pi}P\int_0^{2\pi} v(r,\theta)\cot\left(\frac{\theta_0-\theta}{2}\right)\mathrm{d}\theta, \tag{3.47}$$

and

$$v(r,\theta_0) = v(0) + \frac{1}{2\pi}P\int_0^{2\pi} u(r,\theta)\cot\left(\frac{\theta_0-\theta}{2}\right)\mathrm{d}\theta. \tag{3.48}$$

The details of taking the limit $r_0 \to r$ in Eqs. (3.43) and (3.44) are addressed in Section 3.3.1.

3.3.1 The Poisson kernel for the disc

This subsection explores some key properties of the Poisson kernel that play an important role in the discussion of the Hilbert transform of periodic functions. The Poisson kernel for the unit disc is defined by

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}, \quad \text{for} \quad 0 \le r < 1. \tag{3.49}$$

The function $Q(r,\theta)$ is termed the conjugate Poisson kernel, and is defined by

$$Q(r,\theta) = \frac{2r\sin\theta}{1-2r\cos\theta+r^2}, \quad \text{for} \quad 0 \le r < 1. \tag{3.50}$$

These kernels have a direct tie-in with complex variable theory. For $|z| < 1$, the following expansion holds:

$$\frac{1+z}{1-z} = 1 + 2\sum_{k=1}^{\infty} z^k = 1 + 2\sum_{k=1}^{\infty} r^k \mathrm{e}^{\mathrm{i}k\theta}. \tag{3.51}$$

The real part of this expansion is $P(r,\theta)$ and the imaginary part is $Q(r,\theta)$.

For $0 \le r < 1$, $P(r,\theta)$ satisfies the following three conditions:

(1) $\quad \dfrac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta)\mathrm{d}\theta = 1, \qquad (3.52)$

(2) $P(r,\theta) > 0,$ (3.53)

(3) for every $\delta > 0,\ \max\limits_{\delta\le|\theta|\le\pi} P(r,\theta) \to 0,$ as $r \to 1,$ (3.54)

or equivalently, for every $\varepsilon > 0$ and $\delta > 0$ then

$0 \le P(r,\theta) < \varepsilon,$ for $\delta \le |\theta| \le \pi$ as $r \to 1.$ (3.55)

Employing the identification

$$u(re^{i\theta}) \equiv u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)f(t)dt, \tag{3.56}$$

an important property of the Poisson kernel is that the radial limit $r \to 1$ is given by

$$\lim_{r\to 1} u(r,\theta) = f(\theta), \tag{3.57}$$

a result given by Poisson. Let the function f be periodic with period 2π and continuous at θ_0. Then as $re^{i\theta}$ tends to $e^{i\theta_0}$, $u(re^{i\theta})$ tends to $f(\theta_0)$. To establish this result, use the identity

$$\int_{-\pi}^{\pi} P(r,\theta - t)f(t)dt = \int_{-\pi}^{\pi} P(r,t)f(\theta - t)dt, \tag{3.58}$$

and select a δ such that $0 < \delta < \pi$; then

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)f(t)dt - f(\theta_0)\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t)f(\theta - t)dt - f(\theta_0)\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta - t) - f(\theta_0)|P(r,t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\delta} |f(\theta - t) - f(\theta_0)|P(r,t)dt \\ &\quad + \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(\theta - t) - f(\theta_0)|P(r,t)dt \\ &\quad + \frac{1}{2\pi}\int_{\delta}^{\pi} |f(\theta - t) - f(\theta_0)|P(r,t)dt,\end{aligned} \tag{3.59}$$

which, on using the notational abbreviation $\int_{|t|\ge\delta} = \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}$ and making use of the triangle inequality,

$$|f(\theta - t) - f(\theta_0)| \le |f(\theta - t)| + |f(\theta_0)|, \tag{3.60}$$

yields

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)f(t)\mathrm{d}t - f(\theta_0)\right| \leq \frac{1}{2\pi}\int_{|t|\geq\delta} |f(\theta-t)|P(r,t)\mathrm{d}t$$
$$+ \frac{1}{2\pi}\int_{|t|\geq\delta} |f(\theta_0)|P(r,t)\mathrm{d}t$$
$$+ \frac{1}{2\pi}\int_{|t|<\delta} |f(\theta-t) - f(\theta_0)|P(r,t)\mathrm{d}t. \tag{3.61}$$

For the third integral on the right-hand side of Eq. (3.61), a sufficiently small δ can be selected so that an $\varepsilon > 0$ can be found such that

$$|f(\theta-t) - f(\theta_0)| < \varepsilon, \tag{3.62}$$

and the integral simplifies as follows:

$$\frac{1}{2\pi}\int_{|t|<\delta} |f(\theta-t) - f(\theta_0)|P(r,t)\mathrm{d}t < \frac{\varepsilon}{2\pi}\int_{|t|<\delta} P(r,t)\mathrm{d}t$$
$$= C_\delta\varepsilon, \tag{3.63}$$

where C_δ is a positive constant depending on δ. The first and second integrals on the right-hand side of Eq. (3.61) simplify using Eq. (3.55) and the periodicity of f to yield

$$\frac{1}{2\pi}\int_{|t|\geq\delta} |f(\theta-t)|P(r,t)\mathrm{d}t + \frac{1}{2\pi}\int_{|t|\geq\delta} |f(\theta_0)|P(r,t)\mathrm{d}t$$
$$\leq \frac{\varepsilon}{2\pi}\int_{|t|\geq\delta} |f(\theta-t)|\mathrm{d}t + \frac{\varepsilon}{2\pi}\int_{|t|\geq\delta} |f(\theta_0)|\mathrm{d}t$$
$$\leq \frac{\varepsilon}{2\pi}\left\{\int_{-\pi}^{\pi} |f(\theta-t)|\mathrm{d}t + \int_{-\pi}^{\pi} |f(\theta_0)|\mathrm{d}t\right\}$$
$$= \varepsilon\left\{|f(\theta_0)| + \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\mathrm{d}t\right\}. \tag{3.64}$$

Therefore

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)f(t)\mathrm{d}t - f(\theta_0)\right| = \varepsilon\left\{C_\delta + |f(\theta_0)| + \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\mathrm{d}t\right\}$$
$$\to 0, \text{ as } \varepsilon \to 0, \tag{3.65}$$

and Eq. (3.57) is established.

In Eq. (3.40), set $r = 1$ and replace r_0 by r; then, by the preceding argument, Fatou's theorem is obtained: let f be a bounded analytic function in the interior of a unit disc, then the radial limit (sometimes termed the non-tangential limit) $\lim_{r\to 1} f(re^{i\theta})$ exists for almost all θ in $[0, 2\pi)$. That is, as $re^{i\theta} \to e^{i\theta_0}$ from within the sector of angle less than π, which is assumed to be symmetric about the radius vector from the origin to the vertex point $e^{i\theta_0}$, $f(re^{i\theta}) \to f(e^{i\theta_0})$. The term sector is understood in the asymptotic sense for points approaching closely the vertex point $e^{i\theta_0}$ on the boundary of the disc.

The function $Q(r, \theta)$ can be rewritten as follows:

$$Q(r, \theta) = \frac{4r \sin\theta/2 \cos\theta/2}{(1 - r)^2 + 4r \sin^2\theta/2}, \tag{3.66}$$

and, in the limit $r \to 1$,

$$\lim_{r\to 1} Q(r, \theta) = \cot\frac{\theta}{2}. \tag{3.67}$$

This is the kernel of the Hilbert transform on the circle.

The limiting process employed to obtain Eqs. (3.45) and (3.46) is now revisited. The following proof is based on Weiss (1965). The problem is simplified by considering the unit disc (putting $r = 1$ in Eqs. (3.43) and (3.44), and then replacing r_0 by r). The integrand of the second integral in Eqs. (3.43) and (3.44) is directly proportional to the conjugate Poisson kernel $Q(r, \theta_0 - \theta)$. Attention is now focused on the integral involving this kernel. Consider the function

$$F(z) = e^{-\{u_P(r,\theta_0) + iu_Q(r,\theta_0)\}}, \tag{3.68}$$

where $u_P(r, \theta_0)$ denotes the Poisson integral of $u(r, \theta_0)$ and $u_Q(r, \theta_0)$ designates the conjugate Poisson integral of $u(r, \theta_0)$. The function F is bounded and analytic in the interior of the unit disc, and, by Fatou's theorem, the radial limit as $r \to 1$ exists *a.e.* The argument given in Eqs. (3.58)–(3.65) establishes that the radial limit for $u_P(r, \theta_0)$ exists *a.e.* It follows that the radial limit of $u_Q(r, \theta_0)$ exists *a.e.*, which leads to the results given in Eqs. (3.45) and (3.46).

3.4 Hilbert transform on the real line

The development of the Hilbert transform from the point of view of complex analysis is addressed in this section. Let f be analytic in the upper half complex plane, and suppose $f(z) \to 0$ as $z \to \infty$. Consider the contour shown in Figure 3.3. From the Cauchy integral theorem,

$$\frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - x_0} = 0. \tag{3.69}$$

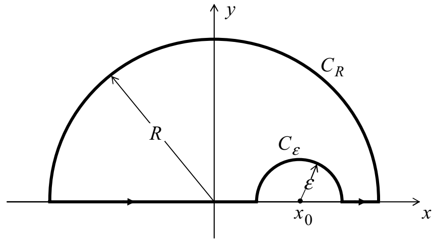

Figure 3.3. Semicircular contour in the upper half plane with half circle indentation at x_0.

Note that the singularity lies outside the closed contour and that the kernel is everywhere analytic in the upper half plane. The contour integral in Eq. (3.69) can be simplified as follows:

$$\oint_C \frac{f(z)\mathrm{d}z}{z-x_0} = \int_{-R}^{x_0-\varepsilon} \frac{f(x)\mathrm{d}x}{x-x_0} + \int_{C_\varepsilon} \frac{f(z)\mathrm{d}z}{z-x_0} + \int_{x_0+\varepsilon}^{R} \frac{f(x)\mathrm{d}x}{x-x_0} + \int_{C_R} \frac{f(z)\mathrm{d}z}{z-x_0}. \quad (3.70)$$

Attention is now directed to the evaluation of the right-hand side of Eq. (3.70) in the limits as $R \to \infty$ and $\varepsilon \to 0$. The first and third integrals on the right-hand side of Eq. (3.70) simplify to the Cauchy principal value integral:

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \left\{ \int_{-R}^{x_0-\varepsilon} \frac{f(x)\mathrm{d}x}{x-x_0} + \int_{x_0+\varepsilon}^{R} \frac{f(x)\mathrm{d}x}{x-x_0} \right\} = P\int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{x-x_0}. \quad (3.71)$$

To deal with the second integral on the right-hand side of Eq. (3.70), involving integration over the contour C_ε, set $z - x_0 = \varepsilon \mathrm{e}^{\mathrm{i}\theta}$, which leads to

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{f(z)\mathrm{d}z}{z-x_0} &= \lim_{\varepsilon\to 0} \int_{\pi}^{0} \frac{f(x_0+\varepsilon \mathrm{e}^{\mathrm{i}\theta})\varepsilon \mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\,\mathrm{d}\theta}{\varepsilon \mathrm{e}^{\mathrm{i}\theta}} \\ &= -\mathrm{i} \lim_{\varepsilon\to 0} \int_0^{\pi} f(x_0+\varepsilon \mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta \\ &= -\mathrm{i}\pi f(x_0). \end{aligned} \quad (3.72)$$

Note that the contour is traversed in the counter-clockwise direction. The last integral in Eq. (3.70), taken over the contour C_R, is simplified by setting $z = R\mathrm{e}^{\mathrm{i}\theta}$; hence

$$\begin{aligned} \lim_{R\to\infty} \int_{C_R} \frac{f(z)\mathrm{d}z}{z-x_0} &= \lim_{R\to\infty} \int_0^{\pi} \frac{f(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{R\mathrm{e}^{\mathrm{i}\theta} - x_0} \\ &= 0, \end{aligned} \quad (3.73)$$

provided $f(z) \to 0$ as $z \to \infty$. Therefore Eq. (3.69) simplifies to

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{z - x_0} = P \int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{x - x_0} - \mathrm{i}\pi f(x_0) = 0. \tag{3.74}$$

Now set $f(x) = u(x) + \mathrm{i}v(x)$, where u and v are real-valued functions; then Eq. (3.74) can be split into two conjugate relations:

$$u(x_0) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)\mathrm{d}x}{x_0 - x} \tag{3.75}$$

and

$$v(x_0) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)\mathrm{d}x}{x_0 - x}. \tag{3.76}$$

These two equations constitute the Hilbert transform pair for the real line.

From Eqs. (3.75) and (3.76), it follows that

$$u(x_0) = \frac{1}{\pi^2} P \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x - x_0} P \int_{-\infty}^{\infty} \frac{u(s)\mathrm{d}s}{x - s}, \tag{3.77}$$

which is *Hilbert's integral formula*. Hardy (1928a) proved this formula to be true when u belongs to $L^2(\mathbb{R})$ and is continuous at $x = x_0$. He also proved the result for functions u belonging to L^p for $p > 1$, and continuous at x_0. The analog of Eq. (3.77) on $\mathbb{R}^+$, the positive real axis, that is the interval $[0, \infty)$, has been known for well over one hundred years (Schlömilch, 1848, p. 158).

3.4.1 Conditions on the function f

This subsection elaborates on the conditions to be satisfied by the function f, in order that the Hilbert transform exists on $\mathbb{R}$. First recall Hölder's inequality,

$$\int_a^b |f(x)g(x)|\mathrm{d}x \leq \left[\int_a^b |f(x)|^p \,\mathrm{d}x\right]^{1/p} \left[\int_a^b |g(x)|^q \,\mathrm{d}x\right]^{1/q}, \tag{3.78}$$

where p and q satisfy

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{3.79}$$

and the functions f and g belong to the following spaces: $f \in L^p$, for $1 \leq p \leq \infty$, and $g \in L^q$, for $1 \leq q \leq \infty$. Making use of the identification

$$k(x, t) = \frac{1}{\pi(x - t)}, \tag{3.80}$$

and employing the definition given in Eq. (3.3) yields

$$\begin{aligned} H_\varepsilon f(x) &= \frac{1}{\pi}\int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\,\mathrm{d}t \\ &= \int_{|x-t|>\varepsilon} k(x,t)f(t)\mathrm{d}t \\ &\le \int_{|x-t|>\varepsilon} |f(t)k(x,t)|\mathrm{d}t \\ &\le \left[\int_{|x-t|>\varepsilon} |f(t)|^p\,\mathrm{d}t\right]^{1/p} \left[\int_{|x-t|>\varepsilon} |k(x,t)|^q\,\mathrm{d}t\right]^{1/q}. \end{aligned} \tag{3.81}$$

If a small ε-sized neighborhood about the origin is excluded, the kernel function $k(x,t)$ belongs to L^q for $q > 1$. Assume $f \in L^p$, for $1 < p < \infty$, then $H_\varepsilon f(x)$ exists. The last issue is to examine the $\lim \varepsilon \to 0+$. Let $f \in L^p$ for $1 < p < \infty$ and suppose $x \in [-\alpha, \alpha]$ for $\alpha > 0$. Set

$$f(x) = f_1(x) + f_2(x), \tag{3.82}$$

where

$$f_1(x) = \chi_{[-2\alpha,2\alpha]} f(x); \tag{3.83}$$

then

$$Hf(x) = \lim_{\varepsilon\to 0+} H_\varepsilon f(x) = \lim_{\varepsilon\to 0+}\int_{|x-t|\ge\varepsilon} \frac{f_1(t)\mathrm{d}t}{x-t} + \lim_{\varepsilon\to 0+}\int_{|x-t|\ge\varepsilon} \frac{f_2(t)\mathrm{d}t}{x-t}. \tag{3.84}$$

The second integral can be written as follows:

$$\lim_{\varepsilon\to 0+}\int_{|x-t|\ge\varepsilon} \frac{f_2(t)\mathrm{d}t}{x-t} = \int_{|t|\ge 2\alpha} \frac{f(t)\mathrm{d}t}{x-t}, \tag{3.85}$$

which is not a singular integral, and, by the condition on f, exists *a.e*. The first integral in Eq. (3.84) can be expressed as follows:

$$\lim_{\varepsilon\to 0+}\int_{|x-t|\ge\varepsilon} \frac{f_1(t)\mathrm{d}t}{x-t} = \lim_{\varepsilon\to 0+}\left\{\int_{-2\alpha}^{x-\varepsilon} \frac{f_1(t)\mathrm{d}t}{x-t} + \int_{x+\varepsilon}^{2\alpha} \frac{f_1(t)\mathrm{d}t}{x-t}\right\}, \tag{3.86}$$

and since $f_1 \in L$ this integral exists *a.e*. Because α can be selected arbitrarily, this gives the required result.

An alternative approach is based on functions that are Hölder continuous. Let

$$H_\varepsilon f(x) = -\frac{1}{\pi}\int_\varepsilon^\infty \frac{f(x+t)-f(x-t)}{t}\,\mathrm{d}t, \tag{3.87}$$

which is Young's form for the integral. Suppose that f vanishes sufficiently rapidly as $t \to \pm\infty$ so that the integrals $\int^\infty f(t)\mathrm{d}t$ and $\int^\infty f(-t)\mathrm{d}t$ do not diverge. When integrals are displayed with one limit, as in the preceding examples, the reader is requested to focus attention on the behavior of the integral in the vicinity of the given limit, and not on the behavior at the unstated other limit. If f satisfies a Hölder condition

$$|f(x)-f(y)| \le C|x-y|^\alpha, \text{ for } 0<\alpha<1 \text{ and } |x-y|\le\delta, \tag{3.88}$$

for some $\delta > 0$ and C a positive constant, then, using the form given in Eq. (3.87) and taking $\lim \varepsilon \to 0+$, leads to

$$|Hf(x)| \le \int_0^\delta \frac{|f(x+t)-f(x-t)|}{t}\,\mathrm{d}t + \int_\delta^\infty \frac{|f(x+t)-f(x-t)|}{t}\,\mathrm{d}t. \tag{3.89}$$

The second of these integrals is bounded using the asymptotic condition specified for $f(x)$. The first integral is also bounded if $f(x)$ satisfies the Hölder condition of Eq. (3.88). The functions likely to be encountered in practical applications are expected to satisfy a Hölder condition, or can be so modified in a straightforward manner so that this is the case.

The result of the preceding paragraph can be formalized as the following theorem (Wood, 1929). Suppose $\int^\infty t^{-1}f(t)\mathrm{d}t$ and $\int^\infty t^{-1}f(-t)\mathrm{d}t$ both exist. If f satisfies a Lipschitz condition of order α for $0<\alpha<1$, over any finite interval, then $Hf \in \mathrm{Lip}\,\alpha$. Titchmarsh (1925a) gave the theorem in a slightly different form. Suppose f is square integrable over $\mathbb{R}$ and satisfies a uniform Lipschitz condition, $f \in \mathrm{Lip}\,\alpha$ for $0<\alpha<1$, then $Hf \in \mathrm{Lip}\,\alpha$. The reader can pursue the proof of these theorems in the given references.

Attention is now focused on proving the existence of Hf for the case $f \in L(\mathbb{R})$. The following proof is based on that given in Titchmarsh (1948). First, a preliminary result: if f belongs to $L(0,1)$ and $x^{-1}f(x) \in L(1,\infty)$, then

$$\lim_{y\to 0+}\left\{v(x,y) - \frac{1}{\pi}\int_y^\infty \frac{\{f(x-t)-f(x+t)\}\mathrm{d}t}{t}\right\} = 0, \tag{3.90}$$

where

$$v(x,y) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{(x-t)f(t)\mathrm{d}t}{(x-t)^2+y^2}. \tag{3.91}$$

Equation (3.90) is established as follows:

$$\begin{aligned}
v(x,y) &- \frac{1}{\pi}\int_y^\infty \frac{\{f(x-t)-f(x+t)\}\mathrm{d}t}{t} \\
&= \frac{1}{\pi}\left\{\int_0^y \frac{\{f(x-t)-f(x+t)\}t\,\mathrm{d}t}{t^2+y^2} - y^2\int_y^\infty \frac{\{f(x-t)-f(x+t)\}\mathrm{d}t}{t(t^2+y^2)}\right\} \\
&\le \frac{1}{2\pi}\int_0^1 |f(x-yt)-f(x+yt)|\mathrm{d}t - \frac{1}{\pi}\int_1^\infty \frac{\{f(x-ty)-f(x+ty)\}\mathrm{d}t}{t(t^2+1)}.
\end{aligned} \tag{3.92}$$

In the limit $y \to 0+$, both the preceding integrals vanish, and hence Eq. (3.90) follows. Now for the key result: if $f \in L(\mathbb{R})$, then Hf exists for almost all x. An argument attributed by Titchmarsh to Littlewood is employed. Let $u(x,y)$ be defined by

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^\infty \frac{f(t)\mathrm{d}t}{(x-t)^2+y^2}, \tag{3.93}$$

and suppose for simplicity that $f(x) \ge 0$; then, for $y > 0$, a function ϕ can be defined by

$$\phi(z) \equiv u(x,y) + \mathrm{i}v(x,y) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^\infty \frac{f(t)\mathrm{d}t}{z-t}. \tag{3.94}$$

Define the function ψ by

$$\psi(z) = \mathrm{e}^{-\phi(z)} = \mathrm{e}^{-u(x,y)-\mathrm{i}v(x,y)}. \tag{3.95}$$

The function ψ satisfies $|\psi(z)| \le 1$ since $u(x,y) \ge 0$, and in the limit $y \to 0+$, $\psi(z)$ tends to a finite limit for almost all x since $\psi(z)$ is bounded and analytic for $y > 0$. From this it follows that $\phi(z)$ tends to a finite limit for almost all x as $y \to 0+$. In the limit $y \to 0+$, $u(x,y)$ tends to $f(x)$ *a.e.* The easiest way to verify this is to recall the definition of the Dirac delta distribution (see Eq. (2.252)). Hence, it follows that $v(x,y)$ tends to a finite limit for almost all x as $y \to 0+$. Making use of Eq. (3.90), it therefore follows that Hf exists for almost all x. To summarize the results of the last few sections: the Hilbert transform of a function $f \in L^p$ for $1 \le p < \infty$ exists almost everywhere. In Section 4.25, there is further discussion on the existence of the Hilbert transform in a more general setting.

The statement about the asymptotic behavior of $f(z)$ can be replaced by the requirement that the function f belongs to the Hardy space H^p; that is, $f \in H^p$ for $1 \le p < \infty$. When this is not the situation, it may still be possible to arrive at Hilbert transform pairs under certain circumstances. This issue is considered in Section 3.4.2.

3.4.2 The Phragmén–Lindelöf theorem

The derivation given at the start of Section 3.4 utilized the condition $f(z) \to 0$ as $z \to \infty$, from which it follows that the contribution given by the contour integral on the large semicircular arc is zero (see Eq. (3.73)). When $f(z)$ does not have this asymptotic behavior, it may be possible to make an alternative choice for the function, in order to meet the aforementioned condition. For example, suppose $f(z)$ behaves like

$$f(z) = f_\infty + az^{-2}, \text{ as } z \to \infty, \tag{3.96}$$

where f_∞ and a are constants. Then the choice $g(z) = f(z) - f_\infty$ would satisfy the necessary asymptotic requirements, and a Hilbert transform pair can be established for the real and imaginary parts of $g(z)$, assuming $f(z)$ satisfies at least the requirements given in Section 3.4.1. In this case, however, extra information about the function is needed, that is f_∞ must be explicitly known.

Another issue that arises when jumping from a physically realizable function defined (by the context of the problem) on the real axis to a function that is analytic in the upper half plane, is the asymptotic behavior of the function as $z \to \infty$. Suppose for the function of interest it is known that $f(x)$ vanishes as $x \to \pm\infty$, what can be said about the asymptotic behavior of the function in the upper half plane as $z \to \infty$? Clearly, bounded behavior on the real axis does not guarantee bounded behavior in the complex plane. For example, $f(x) = \mathrm{e}^{-x^2}$ is bounded on the real axis as $x \to \pm\infty$, but $f(z)$ is not bounded on the imaginary axis in the upper half plane as $z \to \infty$. This issue can be dealt with using a theorem due to Phragmén and Lindelöf. Nevanlinna's (1970, p. 43) statement of the Phragmén–Lindelöf theorem is employed. Suppose $f(z)$ is analytic in the upper half plane and bounded at every finite point of the real axis, such that

$$|f(x)| \leq 1, \tag{3.97}$$

and that the maximum modulus of $f(z)$ over the half circle with center the origin and radius R in the upper half plane is

$$M(R) = \max_{0 \leq \theta \leq \pi} \left| f(R\mathrm{e}^{\mathrm{i}\theta}) \right|. \tag{3.98}$$

Then either the function is bounded in the upper half plane,

$$|f(z)| \leq 1, \quad z \in \text{ upper half plane}, \tag{3.99}$$

or

$$\lim_{R \to \infty} \inf \left\{ \frac{\log M(R)}{R} \right\} = \alpha > 0, \tag{3.100}$$

where α is a constant.

Suppose $f(x) = (x + w)^{-1}$, with Im $w > 0$ and $|w| \geq 1$; then

$$M(R) \sim R^{-1} \tag{3.101}$$

and

$$\lim_{R\to\infty} \inf \left\{ \frac{\log M(R)}{R} \right\} = 0. \tag{3.102}$$

The function therefore satisfies Eq. (3.99). Consider the modification of the function such that $f(x) = (x+w)^{-1}\mathrm{e}^{-\mathrm{i}\alpha x}$ (for $\alpha > 0$), where the exponential term is interpreted as a phase factor. In this case,

$$M(R) \sim R^{-1}\mathrm{e}^{\alpha R}, \tag{3.103}$$

and

$$\lim_{R\to\infty} \inf \left\{ \frac{\log M(R)}{R} \right\} = \alpha. \tag{3.104}$$

That is, $|f(z)|$ increases rapidly as $z \to \infty$.

For the choice $f(x) = (x + w)^{-1}$, it is apparent that the function is bounded for z in the upper half plane. The second choice of function is not bounded for z in the upper half plane. The most direct way to see this is to examine the behavior along the imaginary axis (in the upper half plane). The implication is clearly that the Hilbert transform of the first function can be evaluated by resorting to a semicircular contour in the upper half plane, as in Figure 3.3. The Hilbert transform of the second of the two functions cannot be evaluated in the same manner, since the contribution from the integral along the semicircular segment of the contour is divergent when $R \to \infty$. In this case the integral on the semicircular contour resembles the integral that occurs in the statement of Jordan's lemma (see Section 2.8.4), but with the important distinction of the sign difference of the exponent of the phase factor. The practical consequence is that if it is desired to find the Hilbert transform connections between experimental data that involves a phase factor, then the sign choice of the phase exponent is important. If the function is analytic in the upper half plane and a phase factor is present, then the sign of the exponent must be positive. If the function is analytic in the lower half plane, then the exponent of the phase factor must be negative. If these conditions on the phase factor hold, then the Hilbert transform relations can be established by contour integration.

3.4.3 Some examples

A number of Hilbert transforms are evaluated throughout this book, and many are left as exercises for the reader to explore. At this juncture three simple examples of Hilbert transforms on $\mathbb{R}$, all of which have important application in later sections, are examined in detail. The first example considered is the Hilbert transform of a constant, c. The result is as follows:

$$
\begin{aligned}
Hc(x) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{c\,\mathrm{d}s}{x-s} \\
&= \frac{c}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{s} \\
&= \frac{c}{\pi} \lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{-\varepsilon} \frac{\mathrm{d}s}{s} + \int_{\varepsilon}^{\infty} \frac{\mathrm{d}s}{s} \right\} \\
&= \frac{c}{\pi} \lim_{\varepsilon \to 0} \left\{ -\int_{\varepsilon}^{\infty} \frac{\mathrm{d}s}{s} + \int_{\varepsilon}^{\infty} \frac{\mathrm{d}s}{s} \right\} \\
&= 0.
\end{aligned} \tag{3.105}
$$

Alternatively, $Hc(x)$ can be evaluated as follows:

$$
\begin{aligned}
Hc(x) &= \frac{c}{\pi} \lim_{T \to \infty} \lim_{\varepsilon \to 0} \left\{ \int_{-T}^{x-\varepsilon} \frac{\mathrm{d}s}{x-s} + \int_{x+\varepsilon}^{T} \frac{\mathrm{d}s}{x-s} \right\} \\
&= \frac{c}{\pi} \lim_{T \to \infty} \lim_{\varepsilon \to 0} \left\{ \log\left\{ \frac{T+x}{\varepsilon} \right\} + \log\left\{ \frac{\varepsilon}{T-x} \right\} \right\} \\
&= \frac{c}{\pi} \lim_{T \to \infty} \log\left\{ \frac{T+x}{T-x} \right\} \\
&= 0.
\end{aligned} \tag{3.106}
$$

The Hilbert transform of $\sin ax$ on the real line is given by

$$
\begin{aligned}
H(\sin ax) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\sin as\,\mathrm{d}s}{x-s} \\
&= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\sin(as+ax)\mathrm{d}s}{s} \\
&= -\frac{\cos ax}{\pi} P \int_{-\infty}^{\infty} \frac{\sin as\,\mathrm{d}s}{s} - \frac{\sin ax}{\pi} P \int_{-\infty}^{\infty} \frac{\cos as\,\mathrm{d}s}{s}.
\end{aligned} \tag{3.107}
$$

The notation $H(\sin ax)$ has been employed in place of the slightly more cumbersome form $H[\sin as](x)$, where the Hilbert transform integration is with respect to the variable s and the evaluation point for the transform is x. When no confusion is likely to occur, this notational simplification is employed in later sections. Equation (3.107) can be simplified on using

$$
P \int_{-\infty}^{\infty} \frac{\cos as\,\mathrm{d}s}{s} = 0, \tag{3.108}
$$

which follows from the odd character of the integrand, and

$$P\int_{-\infty}^{\infty}\frac{\sin as\,\mathrm{d}s}{s}=\int_{-\infty}^{\infty}\frac{\sin as\,\mathrm{d}s}{s}=\pi\,\mathrm{sgn}\,a, \tag{3.109}$$

where sgn denotes the signum function, which was defined in Eq. (1.14). Therefore,

$$H(\sin ax)=-\mathrm{sgn}\,a\cos ax, \tag{3.110}$$

which for $a>0$ simplifies to

$$H(\sin ax)=-\cos ax. \tag{3.111}$$

For $a<0$, put $a=-\alpha$, with $\alpha>0$, to yield

$$H\sin(-\alpha x)=\cos\alpha x, \tag{3.112}$$

which is of the form given in Eq. (3.111). In a similar manner, the Hilbert transform of $\cos ax$ is given by

$$\begin{aligned}
H(\cos ax)&=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\cos as\,\mathrm{d}s}{x-s}\\
&=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\cos(as+ax)\mathrm{d}s}{s}\\
&=-\frac{\cos ax}{\pi}P\int_{-\infty}^{\infty}\frac{\cos as\,\mathrm{d}s}{s}+\frac{\sin ax}{\pi}P\int_{-\infty}^{\infty}\frac{\sin as\,\mathrm{d}s}{s}\\
&=\mathrm{sgn}\,a\sin ax.
\end{aligned} \tag{3.113}$$

Therefore it follows that

$$H(\cos ax)=\sin|a|x. \tag{3.114}$$

This particular case is probably one of the oldest examples of a published Hilbert transform (Cauchy, 1822). As a final elementary example, consider the evaluation of $H((x+\mathrm{i}\alpha)^{-1})$, where α is real. A partial fraction expansion yields

$$\begin{aligned}
H\left(\frac{1}{x+\mathrm{i}\alpha}\right)&=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{(s+\mathrm{i}\alpha)(x-s)}\\
&=\frac{1}{\pi(x+\mathrm{i}\alpha)}P\int_{-\infty}^{\infty}\left\{\frac{s}{s^2+\alpha^2}-\frac{\mathrm{i}\alpha}{s^2+\alpha^2}+\frac{1}{x-s}\right\}\mathrm{d}s\\
&=-\frac{\mathrm{i}\alpha}{\pi(x+\mathrm{i}\alpha)}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{s^2+\alpha^2},
\end{aligned} \tag{3.115}$$

where the final result follows using Eq. (3.105) and the odd character of the first term in the integrand. Using a change of integration variable and paying attention to the sign of α gives

$$H\left(\frac{1}{x+\mathrm{i}\alpha}\right) = -\frac{\mathrm{i}\alpha}{\pi(x+\mathrm{i}\alpha)}\begin{cases} \alpha^{-1}\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{s^2+1}, & \text{for } \alpha > 0 \\ -\alpha^{-1}\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{s^2+1}, & \text{for } \alpha < 0, \end{cases} \tag{3.116}$$

and hence

$$H\left(\frac{1}{x+\mathrm{i}\alpha}\right) = \begin{cases} -\mathrm{i}(x+\mathrm{i}\alpha)^{-1}, & \text{for } \alpha > 0 \\ \mathrm{i}(x+\mathrm{i}\alpha)^{-1}, & \text{for } \alpha < 0. \end{cases} \tag{3.117}$$

The reader is requested to decide what happens for the case $\alpha = 0$ in this example.

3.5 Transformation to other limits

The Hilbert transform relations given in Eqs. (3.4) and (3.5) can be transformed into different forms with various trigonometric substitutions. Starting from the relationship

$$g(x) = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{x-s} \tag{3.118}$$

and introducing the following change of variables:

$$x = -\cot\alpha \tag{3.119}$$

and

$$s = -\cot\beta, \tag{3.120}$$

Eq. (3.118) transforms to

$$g(-\cot\alpha) = \frac{1}{\pi}P\int_0^{\pi}\frac{f(-\cot\beta)[1+\cot^2\beta]\mathrm{d}\beta}{\cot\beta-\cot\alpha}. \tag{3.121}$$

Recalling the standard trigonometric identity

$$\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}, \tag{3.122}$$

then

$$\frac{\cot^2\beta+1}{\cot\beta-\cot\alpha} = \cot\beta+\cot(\alpha-\beta), \tag{3.123}$$

and hence Eq. (3.121) can be rewritten as follows:

$$g(-\cot\alpha) = \frac{1}{\pi}P\int_0^{\pi} f(-\cot\beta)\cot(\alpha-\beta)\mathrm{d}\beta + \frac{1}{\pi}P\int_0^{\pi} f(-\cot\beta)\cot\beta\,\mathrm{d}\beta. \tag{3.124}$$

The conjugate relation is given by

$$f(-\cot\alpha) = \frac{1}{\pi}P\int_0^{\pi} g(-\cot\beta)\cot(\beta-\alpha)\mathrm{d}\beta - \frac{1}{\pi}P\int_0^{\pi} g(-\cot\beta)\cot\beta\,\mathrm{d}\beta. \tag{3.125}$$

With the change of variables $\alpha = \pi x$ and $\beta = \pi s$, the preceding relations can be written as follows:

$$g(-\cot\pi x) = P\int_0^{1} f(-\cot\pi s)\cot[\pi(x-s)]\mathrm{d}s + P\int_0^{1} f(-\cot\pi s)\cot\pi s\,\mathrm{d}s \tag{3.126}$$

and

$$f(-\cot\pi x) = P\int_0^{1} g(-\cot\pi s)\cot[\pi(s-x)]\mathrm{d}s \;-\; P\int_0^{1} g(-\cot\pi s)\cot\pi s\,\mathrm{d}s. \tag{3.127}$$

These relations have a form that is similar to the Hilbert transform pairs for functions that are periodic, as will be seen shortly in Section 3.11, though the preceding results make no assumptions on the periodic character of $f(x)$ and $g(x)$. If in place of Eqs. (3.119) and (3.120) the substitutions

$$x = \tan\alpha \tag{3.128}$$

and

$$s = \tan\beta \tag{3.129}$$

are employed, then Eq. (3.118) transforms to

$$g(\tan\alpha) = \frac{1}{\pi}P\int_{-\pi/2}^{\pi/2} \frac{f(\tan\beta)[1+\tan^2\beta]\mathrm{d}\beta}{\tan\alpha-\tan\beta}. \tag{3.130}$$

Using the standard trigonometric identity

$$\cot(\alpha-\beta) = \frac{\tan\alpha\tan\beta+1}{\tan\alpha-\tan\beta}, \tag{3.131}$$

then

$$\frac{\tan^2\beta + 1}{\tan\alpha - \tan\beta} = \cot(\alpha - \beta) - \tan\beta, \tag{3.132}$$

and so Eq. (3.130) becomes

$$g(\tan\alpha) = \frac{1}{\pi} P \int_{-\pi/2}^{\pi/2} f(\tan\beta)\cot(\alpha - \beta)\mathrm{d}\beta - \frac{1}{\pi} P \int_{-\pi/2}^{\pi/2} f(\tan\beta)\tan\beta\,\mathrm{d}\beta, \tag{3.133}$$

and the conjugate relationship is given by

$$f(\tan\alpha) = \frac{1}{\pi} P \int_{-\pi/2}^{\pi/2} g(\tan\beta)\cot(\beta - \alpha)\mathrm{d}\beta + \frac{1}{\pi} P \int_{-\pi/2}^{\pi/2} g(\tan\beta)\tan\beta\,\mathrm{d}\beta. \tag{3.134}$$

With the substitutions $\alpha = \pi x$ and $\beta = \pi s$, the preceding relationships can be recast as follows:

$$g(\tan\pi x) = P \int_{-1/2}^{1/2} f(\tan\pi s)\cot[\pi(x - s)]\mathrm{d}s - P \int_{-1/2}^{1/2} f(\tan\pi s)\tan\pi s\,\mathrm{d}s \tag{3.135}$$

and

$$f(\tan\pi x) = P \int_{-1/2}^{1/2} g(\tan\pi s)\cot[\pi(s - x)]\mathrm{d}s + P \int_{-1/2}^{1/2} g(\tan\pi s)\tan\pi s\,\mathrm{d}s. \tag{3.136}$$

Equation (3.118) can be cast into additional forms with appropriate transformations: this is left as an exercise for the curious reader.

If the function f is given by

$$f(\omega) = g(\omega) + \mathrm{i}h(\omega), \tag{3.137}$$

where g and h form a Hilbert transform pair, then the transformation

$$\omega = -\tan\frac{\delta}{2}, \tag{3.138}$$

along with the identifications $F(\delta) = f(-\tan\delta/2)$, $G(\delta) = g(\tan\delta/2)$, and $H(\delta) = h(\tan\delta/2)$ allows Eq. (3.137) to be written as follows:

$$F(\delta) = G(\delta) - \mathrm{i}H(\delta), \tag{3.139}$$

where g is taken to be an even function and h is an odd function. Equation (3.138) and the resulting expression for $F(\delta)$ are sometimes called the Wiener–Lee transform (see Papoulis, 1962, p. 201). A Fourier series expansion can be developed for the functions G and H and a simple connection exists between the coefficients of the two expansions. Further developments of this idea are given in Section 3.13, and applications to optical properties are discussed later in Sections 20.10 and 20.11.

3.6 Cauchy integrals

A simple integral equation of the Cauchy type is now considered. Let $g(z)$ be defined by

$$g(z) = \frac{1}{\pi\mathrm{i}} \oint_C \frac{f(s)\mathrm{d}s}{s-z}, \tag{3.140}$$

where C is a simple closed contour. The quantity f is sometimes referred to as the *density function*, or simply the *density* of the Cauchy integral, and it is assumed to be Hölder continuous on the contour. The function $(s-z)^{-1}$ is called the Cauchy kernel. The Cauchy integral is usually defined with a denominator factor of $2\pi\mathrm{i}$, and this form for Eq. (3.140) can be obtained with a trivial change of function. The solution of this integral equation is given by

$$f(z) = \frac{1}{\pi\mathrm{i}} \oint_C \frac{g(s)\mathrm{d}s}{s-z}. \tag{3.141}$$

So Eqs. (3.140) and (3.141) form a reciprocal pair. The demonstration that Eq. (3.141) is the solution of Eq. (3.140) can be performed in a few different ways. Perhaps the simplest employs the Hardy–Poincaré–Bertrand formula (see Section 2.13). On multiplying both sides of Eq. (3.140) by $[\pi\mathrm{i}(z-w)]^{-1}\,\mathrm{d}z$, and then integrating around the contour C, leads to the following:

$$\begin{aligned}\frac{1}{\pi\mathrm{i}} \oint_C \frac{g(z)\mathrm{d}z}{z-w} &= \frac{1}{\pi\mathrm{i}} \oint_C \frac{\mathrm{d}z}{z-w} \frac{1}{\pi\mathrm{i}} \oint_C \frac{f(s)\mathrm{d}s}{s-z} \\ &= f(w) - \frac{1}{\pi^2} \oint_C f(s)\mathrm{d}s \oint_C \frac{\mathrm{d}z}{(z-w)(s-z)},\end{aligned} \tag{3.142}$$

where the last line follows by direct application of the Hardy–Poincaré–Bertrand formula (Eq. (2.233)). The second integral in Eq. (3.142) can be evaluated by splitting the integrand via partial fractions to obtain

$$\begin{aligned}\oint_C \frac{\mathrm{d}z}{(z-w)(s-z)} &= \frac{1}{s-w} \oint_C \left\{\frac{1}{z-w} - \frac{1}{z-s}\right\} \mathrm{d}z. \\ &= 0.\end{aligned} \tag{3.143}$$

Assuming both w and s are interior to C, the last result follows using the Cauchy integral formula. If w and s are exterior to C, then Eq. (3.143) also follows using the Cauchy integral theorem. If w and s are on C, then a suitable modification of the integration path also leads to Eq. (3.143). Therefore, Eq. (3.142) represents the solution of Eq. (3.140) and is of the form given in Eq. (3.141).

The solution of Eq. (3.140) can be obtained without recourse to the Hardy–Poincaré–Bertrand formula. Before addressing an alternative method for the solution of the Cauchy integral equation given, some preliminary issues are first examined.

Consider the Cauchy integral, given by

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{g(s)\mathrm{d}s}{s - z}, \tag{3.144}$$

where z is not on the contour Γ. If $g(s) = f(s)$ and Γ is a simple closed contour, then this gives the Cauchy integral formula, provided $f(z)$ is analytic within and on the contour Γ and z is an interior point of Γ. The simplest example of Eq. (3.144) occurs when Γ is taken to be a closed contour and $g(s)$ satisfies $g(s) = 1$; then

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{\mathrm{d}s}{s - z}. \tag{3.145}$$

If the point z is *interior* to the closed contour Γ, then Eq. (3.145) is just the Cauchy integral formula with $f(z) = 1$, and so

$$\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{\mathrm{d}s}{s - z} = 1. \tag{3.146}$$

It may help to visualize the contour by inspecting Figure 3.4, examining the limit $\varepsilon \to 0$, and noting that the contributions from the horizontal contour segments cancel.

If the point z is *exterior* to Γ, then, from the Cauchy integral theorem,

$$\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{\mathrm{d}s}{s - z} = 0. \tag{3.147}$$

The final possibility is that the point z lies on Γ. In this case the contour Γ is taken to be of the form shown in Figure 3.5. Then,

$$\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{\mathrm{d}s}{s - z} = \lim_{\rho \to 0} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma - \Gamma_{ab}} \frac{\mathrm{d}s}{s - z}. \tag{3.148}$$

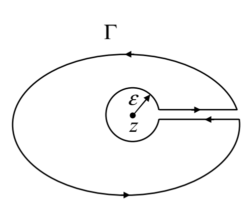

Figure 3.4. Contour Γ with the point z encircled by a suitable indentation of the main path of the contour.

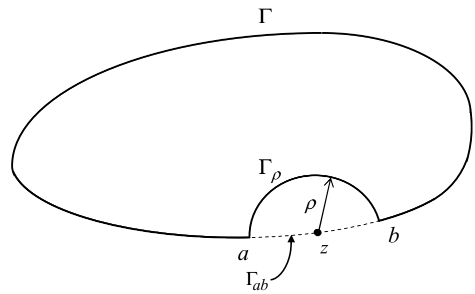

Figure 3.5. Contour Γ with a semicircular-like indentation at the point z, which lies on Γ.

This contour integral can be simplified by writing

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma-\Gamma_{ab}} \frac{\mathrm{d}s}{s-z} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma-\Gamma_{ab}+\Gamma_\rho} \frac{\mathrm{d}s}{s-z} - \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\rho} \frac{\mathrm{d}s}{s-z}. \tag{3.149}$$

In the limit $\rho \to 0$, the first integral on the right-hand side of Eq. (3.149) is zero, since the point z is exterior to $\Gamma - \Gamma_{ab} + \Gamma_\rho$. On setting $s - z = \rho \mathrm{e}^{\mathrm{i}\theta}$, the second integral simplifies in the limit $\rho \to 0$ to yield

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\rho} \frac{\mathrm{d}s}{s-z} = \frac{1}{2\pi \mathrm{i}} \int_{\pi}^{0} \frac{\rho \mathrm{i} \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{\rho \mathrm{e}^{\mathrm{i}\theta}} = -\frac{1}{2}. \tag{3.150}$$

Therefore,

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{\mathrm{d}s}{s-z} = \begin{cases} 0, & z \text{ exterior to } \Gamma \\ 1/2, & z \text{ on } \Gamma \\ 1, & z \text{ interior to } \Gamma. \end{cases} \tag{3.151}$$

The limit of the Cauchy integral is now examined as $z \to z_0$, a point on Γ, from the opposite sides of the contour. Let C be a closed contour in the complex plane, and set

$$f^+(z) = f(z), \text{ if } z \text{ is interior to } C, \tag{3.152}$$

and

$$f^-(z) = f(z), \text{ if } z \text{ is exterior to } C. \tag{3.153}$$

The plus and minus signs signify, respectively, the positive and negative sides of the contour. The terms left and right are also used. When the contour is not closed, f^+ will denote the function to the left of the contour and f^- will designate the function to the right of the contour, with the contour traversed in the counter-clockwise orientation. In the limit $z \to z_0$,

$$f^+(z_0) = \lim_{z \to z_0} f(z), \qquad \text{limit from the left,} \tag{3.154}$$

$$f^-(z_0) = \lim_{z \to z_0} f(z), \qquad \text{limit from the right.} \tag{3.155}$$

The function $f(z)$ is said to be continuous from the right at z_0 if $f^-(z_0)$ exists. In general, the boundary values $f^+(z_0)$ and $f^-(z_0)$ will not be equal, and the function $f(z)$ is therefore not everywhere continuous in the complex plane. If $f(z)$ is continuous everywhere in the neighborhood of C, and $f^+(z_0)$ and $f^-(z_0)$ exist, but are different, then the function is said to be *sectionally continuous* in the neighborhood of C. When the contour is not closed, no distinction is made between the limits from the left and right at the endpoints of the contour. That is, the function f is continuous from the left and right at the endpoints.

If the function $f(z)$ is analytic in each region of the complex plane except C, and the function is sectionally continuous in the neighborhood of C, the function is termed *sectionally analytic* in the plane cut by the contour C. From the example considered in Eqs. (3.145)–(3.151), it follows for the situation described in Figure 3.6 that

$$g(z_2) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(s)\mathrm{d}s}{s - z_2} = 0, \tag{3.156}$$

$$g(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(s)\mathrm{d}s}{s - z_0} = \frac{1}{2} f(z_0), \tag{3.157}$$

$$g(z_1) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(s)\mathrm{d}s}{s - z_1} = f(z_1). \tag{3.158}$$

Equation (3.158) is simply the statement of the Cauchy integral formula for the function $f(z)$, since the point z_1 is interior to the contour. Equation (3.156) follows directly from the Cauchy integral theorem, since the point z_2 is exterior to the contour, and $(s - z_2)^{-1} f(s)$ is analytic in the region interior to the contour. For the case where the point is on the contour, Eq. (3.157), the approach of Eq. (3.151) is employed. The integral on the semicircular contribution, where Γ_ρ is traversed in a clockwise

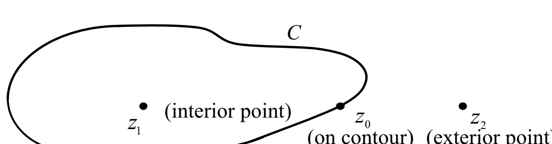

Figure 3.6. Arbitrary closed contour C with points z_0, z_1, and z_2 shown on, interior to, and exterior to C, respectively.

orientation, is given by

$$-\frac{1}{2\pi\mathrm{i}}\lim_{\rho\to 0}\int_{\Gamma_\rho}\frac{f(s)\mathrm{d}s}{s-z_0} = -\lim_{\rho\to 0}\frac{1}{2\pi\mathrm{i}}\int_{\pi}^{0}\frac{f(z_0+\rho\mathrm{e}^{\mathrm{i}\theta})\rho\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\,\mathrm{d}\theta}{\rho\mathrm{e}^{\mathrm{i}\theta}}$$
$$= \frac{1}{2}f(z_0), \tag{3.159}$$

and hence Eq. (3.157) follows.

3.7 The Plemelj formulas

The results derived in Section 3.6 are now used to develop some important results connected with the question: what are the boundary values $f^+(z_0)$ and $f^-(z_0)$ of a function f that can be represented in the form of a Cauchy integral,

$$f(z) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{g(s)\,\mathrm{d}s}{s-z}, \tag{3.160}$$

and is sectionally analytic? The key results obtained in this section are usually attributed to Plemelj (1908a, 1908b), but were apparently first given in 1873 by the Russian mathematician Yu.-K. V. Sokhotsky (the alternative spellings Sokhotskyi, Sokhotski, Sokhotskiĭ, Sohocki, and Sohockiĭ are also used) in his thesis work. It is assumed that $g(s)$ is Hölder continuous on C. Then it follows that

$$f(z) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{[g(s)-g(z)]\mathrm{d}s}{s-z} + \frac{1}{2\pi\mathrm{i}}\oint_C \frac{g(z)\mathrm{d}s}{s-z}. \tag{3.161}$$

If z is a point interior to the contour C, the second integral evaluates to $g(z)$ (see Eq. (3.151)), and when z is exterior to the contour, this term evaluates to zero. It then follows that

$$f^+(z) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{[g(s)-g(z)]\mathrm{d}s}{s-z} + g(z) \tag{3.162}$$

and

$$f^-(z) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{[g(s)-g(z)]\mathrm{d}s}{s-z}. \tag{3.163}$$

When the point z moves onto the contour, as depicted in Figure 3.7, it is designated as the point z_0. When the point z approaches z_0 from the positive direction, Eq. (3.162) becomes

$$f^+(z_0) = \frac{1}{2}g(z_0) + \frac{1}{2\pi\mathrm{i}}\oint_C \frac{g(s)\mathrm{d}s}{s-z_0}. \tag{3.164}$$

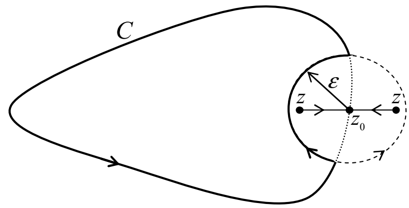

Figure 3.7. Closed contour, showing a point interior or exterior to C approaching the boundary, and the modification of C that is employed.

If the point z moves onto the contour from the negative side, the contour is reoriented along the dashed section as shown in Figure 3.7, and then Eq. (3.163) can be written as follows:

$$f^{-}(z_0) = -\frac{1}{2}g(z_0) + \frac{1}{2\pi \mathrm{i}} \oint_C \frac{g(s)\mathrm{d}s}{s - z_0}. \tag{3.165}$$

Equations (3.164) and (3.165) are most commonly called the Plemelj formulas, but they are also referred to as the Sokhotsky's formulas, and sometimes the Sokhotsky–Plemelj formulas, for reasons indicated at the start of this section (see for example, Gakhov (1966, p. 25) and Henrici (1986, p. 88)). From Eqs. (3.164) and (3.165), it follows immediately that

$$f^{+}(z_0) - f^{-}(z_0) = g(z_0) \tag{3.166}$$

and

$$f^{+}(z_0) + f^{-}(z_0) = \frac{1}{\pi \mathrm{i}} \oint_C \frac{g(s)\mathrm{d}s}{s - z_0}, \tag{3.167}$$

which are equivalent to the Plemelj formulas. Equations (3.166) and (3.167) are also called the Plemelj formulas. The following section illustrates one important application of these results.

3.8 Inversion formula for a Cauchy integral

The following integral equation is now revisited:

$$g(z_0) = \frac{1}{\pi \mathrm{i}} \oint_C \frac{f(s)\mathrm{d}s}{s - z_0}, \tag{3.168}$$

where z_0 is on the contour C. It has already been demonstrated that a solution of this equation can be found by employing the Hardy–Poincaré–Bertrand formula. A different approach based on the Plemelj formulas just developed is used to solve this integral equation.

Let $F(z)$ be a sectionally analytic function vanishing in the limit as $z \to \infty$ and defined by

$$F(z) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(s)\mathrm{d}s}{s - z}. \tag{3.169}$$

Let $G(z)$ be a sectionnally analytic function defined by the following conditions:

$$G(z) = \begin{cases} F(z), & z \text{ interior to C} \\ -F(z), & z \text{ exterior to C.} \end{cases} \tag{3.170}$$

From Eq. (3.168) and the Plemelj formulas Eqs. (3.166) and (3.167), it follows that

$$F^+(z_0) - F^-(z_0) = f(z_0) \tag{3.171}$$

and

$$F^+(z_0) + F^-(z_0) = \frac{1}{\pi \mathrm{i}} \oint_C \frac{f(s)\mathrm{d}s}{s - z_0} = g(z_0). \tag{3.172}$$

Using the definition in Eq. (3.170),

$$\begin{aligned} G^+(z_0) - G^-(z_0) &= F^+(z_0) + F^-(z_0) \\ &= g(z_0). \end{aligned} \tag{3.173}$$

Employing Eq. (3.173) and the Plemelj formula Eq. (3.166), $G(z)$ can be written as the Cauchy integral

$$G(z) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{g(s)\mathrm{d}s}{s - z}, \tag{3.174}$$

from which it follows, using the Plemelj formula Eq. (3.167), that

$$\begin{aligned} G^+(z_0) + G^-(z_0) &= \frac{1}{\pi \mathrm{i}} \oint_C \frac{g(s)\mathrm{d}s}{s - z_0} \\ &= F^+(z_0) - F^-(z_0). \end{aligned} \tag{3.175}$$

Using Eq. (3.175), the solution of Eq. (3.168) is obtained as

$$f(z_0) = \frac{1}{\pi \mathrm{i}} \oint_C \frac{g(s)\mathrm{d}s}{s - z_0}, \tag{3.176}$$

which represents the required inversion formula for the Cauchy integral. An immediate application of the ideas of this section is to establish the Hilbert transform relationships for the circle, a topic that is presented in Section 3.9.

3.9 Hilbert transform on the circle

It is a short jump from the preceding developments to find a transform pair for functions defined on the unit circle. If the contour in Eqs. (3.168) and (3.176) is taken to be the unit circle, centered at the origin, then the change of variables

$$s = \mathrm{e}^{\mathrm{i}\theta} \tag{3.177}$$

and

$$z_0 = \mathrm{e}^{\mathrm{i}\theta_0} \tag{3.178}$$

yields

$$\begin{aligned} \frac{\mathrm{d}s}{s - z_0} &= \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_0}} \\ &= \frac{1}{2}\left[\cot\left(\frac{\theta - \theta_0}{2}\right) + \mathrm{i}\right]\mathrm{d}\theta. \end{aligned} \tag{3.179}$$

Let the functions u and v be defined as follows:

$$u(\theta) = f(\mathrm{e}^{\mathrm{i}\theta}) \tag{3.180}$$

and

$$v(\theta) = \mathrm{i}g(\mathrm{e}^{\mathrm{i}\theta}), \tag{3.181}$$

where $u(\theta)$ and $v(\theta)$ are periodic with period 2π; then Eq. (3.168) simplifies to

$$v(\theta_0) = \frac{\mathrm{i}}{2\pi}\int_0^{2\pi} u(\theta)\mathrm{d}\theta + \frac{1}{2\pi}P\int_0^{2\pi} u(\theta)\cot\left(\frac{\theta - \theta_0}{2}\right)\mathrm{d}\theta, \tag{3.182}$$

and Eq. (3.176) becomes

$$u(\theta_0) = -\frac{\mathrm{i}}{2\pi}\int_0^{2\pi} v(\theta)\mathrm{d}\theta - \frac{1}{2\pi}P\int_0^{2\pi} v(\theta)\cot\left(\frac{\theta - \theta_0}{2}\right)\mathrm{d}\theta. \tag{3.183}$$

With the assumptions

$$\int_0^{2\pi} u(\theta)\mathrm{d}\theta = \int_0^{2\pi} v(\theta)\mathrm{d}\theta = 0, \tag{3.184}$$

Eqs. (3.182) and (3.183) reduce to

$$v(\theta_0) = -\frac{1}{2\pi}P\int_0^{2\pi} u(\theta)\cot\left(\frac{\theta_0 - \theta}{2}\right)\mathrm{d}\theta \tag{3.185}$$

and

$$u(\theta_0) = \frac{1}{2\pi} P \int_0^{2\pi} v(\theta) \cot\left(\frac{\theta_0 - \theta}{2}\right) d\theta, \tag{3.186}$$

which represent the Hilbert transform pair on the unit circle. The quantity v is termed the *conjugate function* of u. The reader should note that the minus sign in the preceding pair of equations can be shifted from Eq. (3.185) to (3.186) by starting with a denominator factor of $z_0 - s$ in place of $s - z_0$ in Eq. (3.168). Alternatively, a minus sign can be introduced into the definition in Eq. (3.180) or Eq. (3.181).

3.10 Alternative approach to the Hilbert transform on the circle

Let $\phi(z)$ be analytic inside and on the boundary of the unit circle centered at the origin, and on the boundary let

$$\phi(\theta) = u(\theta) + iv(\theta), \tag{3.187}$$

where $u(\theta)$ and $v(\theta)$ denote the real and imaginary parts of $\phi(\theta)$, respectively. The growth condition on $\phi(z)$ is fixed by the requirement that $\phi \in H^p(D)$ for $1 \leq p < \infty$. The functions u and v are periodic with period 2π. Consider the contour shown in Figure 3.8; then,

$$\begin{aligned} \oint_C \frac{\phi(z)dz}{z - z_0} &= \lim_{\varepsilon \to 0} \int_{C - \Gamma_{ab}} \frac{\phi(z)dz}{z - z_0} \\ &= \lim_{\varepsilon \to 0} \left\{ \int_{C - \Gamma_{ab} + \Gamma_\varepsilon} \frac{\phi(z)dz}{z - z_0} - \int_{\Gamma_\varepsilon} \frac{\phi(z)dz}{z - z_0} \right\} \\ &= \pi i \phi(z_0). \end{aligned} \tag{3.188}$$

To obtain this result, the Cauchy integral theorem has been used to show that the integral around the contour $C - \Gamma_{ab} + \Gamma_\varepsilon$ contributes zero in the lim $\varepsilon \to 0$. On

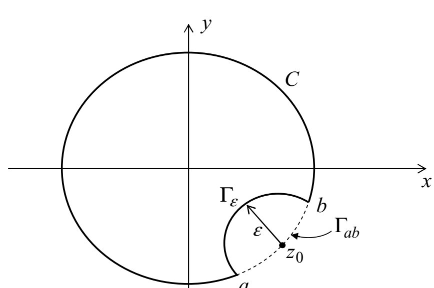

Figure 3.8. Circular contour with a semicircular-like indentation at the point z_0, which is located on C.

introducing polar coordinates into Eq. (3.188), $z = e^{i\theta}$ and $z_0 = e^{i\theta_0}$, and expressing the result in terms of the real and imaginary parts, $u(\theta)$ and $v(\theta)$,

$$u(\theta_0) + iv(\theta_0) = \frac{1}{2\pi i} P \int_0^{2\pi} \left[\cot\left(\frac{\theta - \theta_0}{2}\right) + i \right] [u(\theta) + iv(\theta)] d\theta. \tag{3.189}$$

Separation of the real and imaginary parts leads to

$$v(\theta_0) = \frac{1}{2\pi} P \int_0^{2\pi} u(\theta) \cot\left(\frac{\theta_0 - \theta}{2}\right) d\theta + \frac{1}{2\pi} \int_0^{2\pi} v(\theta) d\theta \tag{3.190}$$

and

$$u(\theta_0) = -\frac{1}{2\pi} P \int_0^{2\pi} v(\theta) \cot\left(\frac{\theta_0 - \theta}{2}\right) d\theta + \frac{1}{2\pi} \int_0^{2\pi} u(\theta) d\theta. \tag{3.191}$$

With the change of variables $\theta = 2\pi x$ and $\theta_0 = 2\pi s$, Eqs. (3.190) and (3.191) can be written as follows:

$$v(s) = P \int_0^1 u(x) \cot[\pi(s - x)] dx + \int_0^1 v(x) dx \tag{3.192}$$

and

$$u(s) = -P \int_0^1 v(x) \cot[\pi(s - x)] dx + \int_0^1 u(x) dx. \tag{3.193}$$

With the further conditions

$$\int_0^1 u(x) dx = 0 \tag{3.194}$$

and

$$\int_0^1 v(x) dx = 0 \tag{3.195}$$

satisfied, the reciprocal character between the real and imaginary parts becomes apparent:

$$v(s) = P \int_0^1 u(x) \cot[\pi(s - x)] dx \tag{3.196}$$

and

$$u(s) = -P \int_0^1 v(x) \cot[\pi(s - x)] dx. \tag{3.197}$$

Using the fact that $u(\theta)$ and $v(\theta)$ are periodic functions with period 2π, and employing Eqs. (3.194) and (3.195), allows Eqs. (3.193) and (3.192) to be written as follows:

$$u(s) = -\frac{1}{2} P \int_{-1}^{1} v(x) \cot\left[\frac{\pi(s-x)}{2}\right] \mathrm{d}x \tag{3.198}$$

and

$$v(s) = \frac{1}{2} P \int_{-1}^{1} u(x) \cot\left[\frac{\pi(s-x)}{2}\right] \mathrm{d}x, \tag{3.199}$$

which is one of the original two forms given by Hilbert (1904, 1912, p. 75). From Eqs. (3.198) and (3.199), it follows that

$$u(s) = -\frac{1}{4} P \int_{-1}^{1} \cot\left[\frac{\pi(s-x)}{2}\right] \mathrm{d}x \; P \int_{-1}^{1} u(y) \cot\left[\frac{\pi(x-y)}{2}\right] \mathrm{d}y, \tag{3.200}$$

which is called *Hilbert's integral formula* (on the circle).

The condition required for convergence of the integral in Eq. (3.198) is either that $v(x)$ is continuous for $x = s$ or the more general condition (see Fatou, 1906; Plessner, 1923; Hardy and Littlewood, 1929),

$$\int_{0}^{t} \{v(x+s) - v(x-s)\} \mathrm{d}s = o(t), \text{ as } t \to 0. \tag{3.201}$$

This condition is developed in Exercise 3.11.

Equation (3.199) provides the basis for selecting the definition of the Hilbert transform on the unit circle for a periodic function f with period 2π as follows:

$$\mathcal{H}f(x) = \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(s) \cot\left(\frac{x-s}{2}\right) \mathrm{d}s, \tag{3.202}$$

provided that the integral exists as a Cauchy principal value. A function $f \in L^p(\mathbb{T})$ for $1 < p < \infty$ if

$$\int_{0}^{2\pi} |f(s)|^p \, \mathrm{d}s < \infty, \tag{3.203}$$

where the symbol $\mathbb{T}$ denotes the circle group, and it is employed to designate the interval $[0, 2\pi)$. A key result is the following: if $f \in L^p(\mathbb{T})$ for $1 < p < \infty$, then $\mathcal{H}f \in L^p(\mathbb{T})$. This result will be discussed in detail in Section 6.17.

The definition given in Eq. (3.202) can be extended to an arbitrary interval $[-\tau, \tau]$ for periodic functions with period 2τ, and this topic is considered in Section 3.14. A comment on notation is appropriate at this point. It is common in the literature to employ the same notation for the Hilbert transform operator on the circle and the real line. The context is intended to make it clear to the reader as to which definition of H is to be employed. In this work a second symbol is adopted, as indicated in

Eq. (3.202). This has obvious advantages when both operators are being discussed at the same time.

3.11 Hardy's approach

Hardy (1908) derived the Hilbert transform inversion formulas on the circle from a study of the inversion of the order of integration of double integrals involving Cauchy principal values. Consider the trigonometric identity

$$\cot(w - v)\{\cot w - \cot v\} + \cot w \cot v + 1 = 0, \tag{3.204}$$

which yields, with the substitutions $w = \pi(s - t)$ and $v = \pi(s - x)$, the following result:

$$\cot\pi(s - t)\cot\pi(t - x) = \cot\pi(s - x)\{\cot\pi(s - t) + \cot\pi(t - x)\} + 1. \tag{3.205}$$

Let α denote a constant satisfying $0 < \alpha < 1$; then

$$P\int_0^1 \cot\pi(t - \alpha)\mathrm{d}t = 0, \tag{3.206}$$

and from Eq. (3.205) it follows, for $0 < x < 1$ and $0 < s < 1$, that

$$P\int_0^1 \cot\pi(s - t)\cot\pi(t - x)\mathrm{d}t = 1. \tag{3.207}$$

Let

$$h(s, t, x) = \cot\pi(s - t)\cot\pi(t - x)\phi(s), \tag{3.208}$$

where $\phi(s)$ is integrable on the interval $(0, 1)$; then

$$P\int_0^1 \mathrm{d}s\, P\int_0^1 h(s, t, x)\mathrm{d}t = P\int_0^1 \mathrm{d}t\, P\int_0^1 h(s, t, x)\mathrm{d}s + \phi(x). \tag{3.209}$$

To see how this result emerges, the following straightforward but slightly tedious approach is utilized. First note that

$$\cot x - x^{-1} = O(x),\ \text{as}\ x \to 0. \tag{3.210}$$

Writing $\cot\pi(s-t) = \cot\pi(s-t) - [\pi(s-t)]^{-1} + [\pi(s-t)]^{-1}$ and similarly for $\cot\pi(t-x)$ yields

$$\begin{aligned}
&P\int_0^1 \mathrm{d}s\, P\int_0^1 \cot\pi(s-t)\cot\pi(t-x)\phi(s)\mathrm{d}t \\
&\quad = \int_0^1 \mathrm{d}s \int_0^1 \left[\cot\pi(s-t) - \frac{1}{\pi(s-t)}\right]\left[\cot\pi(t-x) - \frac{1}{\pi(t-x)}\right]\phi(s)\mathrm{d}t \\
&\qquad + \int_0^1 \mathrm{d}s\, P\int_0^1 \left[\cot\pi(s-t) - \frac{1}{\pi(s-t)}\right]\frac{\phi(s)\mathrm{d}t}{\pi(t-x)} \\
&\qquad + \int_0^1 \mathrm{d}s\, P\int_0^1 \left[\cot\pi(t-x) - \frac{1}{\pi(t-x)}\right]\frac{\phi(s)\mathrm{d}t}{\pi(s-t)} \\
&\qquad + P\int_0^1 \mathrm{d}s\, P\int_0^1 \frac{\phi(s)\mathrm{d}t}{\pi^2(s-t)(t-x)}.
\end{aligned} \tag{3.211}$$

For the first integral on the right-hand side of the preceding equation, there is no singularity arising from the terms in square brackets, and so the integration order can be switched using the results of Fubini–Tonelli. The integration order in the second and third integrals can be switched making use of Eq. (2.220), adjusted for the finite interval $(0, 1)$. From the Hardy–Poincaré–Bertrand formula, Eq. (2.233), it follows for a finite interval $(0, 1)$ and with $\phi_1(x) = 1$ and $\phi_2(x) = \phi(x)$, and the change of variables $y \to s$ and $x \leftrightarrow t$, that

$$\frac{1}{\pi}P\int_0^1 \frac{\mathrm{d}t}{t-x}\frac{1}{\pi}P\int_0^1 \frac{\phi(s)}{s-t}\,\mathrm{d}s = \frac{1}{\pi}P\int_0^1 \phi(s)\mathrm{d}s\frac{1}{\pi}P\int_0^1 \frac{\mathrm{d}t}{(t-x)(s-t)} - \phi(x), \tag{3.212}$$

with $x \in (0, 1)$. Using Eq. (3.212) and adding the other three integrals, with the integration order reversed, leads to the desired outcome, Eq. (3.209). The result just proved is a particular case of a more general result given by Hardy (1908). It is one of several results given by him for interchanging the integration order of double integrals involving Cauchy principal values. In this book the outcome of Hardy's work is referred to as the Hardy–Poincaré–Bertrand formula, and this result was discussed in Section 2.13.

Equation (3.209) can be simplified as follows:

$$\begin{aligned}
&P\int_0^1 \phi(s)\mathrm{d}s\, P\int_0^1 \cot\pi(s-t)\cot\pi(t-x)\mathrm{d}t \\
&\quad = P\int_0^1 \cot\pi(t-x)\mathrm{d}t\, P\int_0^1 \cot\pi(s-t)\phi(s)\mathrm{d}s + \phi(x)
\end{aligned} \tag{3.213}$$

and hence

$$P\int_0^1 \cot\pi(x-t)\mathrm{d}t\, P\int_0^1 \cot\pi(t-s)\phi(s)\mathrm{d}s = \int_0^1 \phi(s)\mathrm{d}s - \phi(x). \tag{3.214}$$

Setting

$$P\int_0^1 \cot\pi(x-s)\phi(s)\mathrm{d}s = \chi(x) + \int_0^1 \chi(s)\mathrm{d}s \tag{3.215}$$

yields

$$P\int_0^1 \cot\pi(x-t)\chi(t)\mathrm{d}t = -\phi(x) + \int_0^1 \phi(s)\mathrm{d}s. \tag{3.216}$$

With the following conditions imposed:

$$\int_0^1 \chi(s)\mathrm{d}s = \int_0^1 \phi(s)\mathrm{d}s = 0, \tag{3.217}$$

then

$$P\int_0^1 \cot\pi(x-s)\phi(s)\mathrm{d}s = \chi(x) \tag{3.218}$$

and

$$P\int_0^1 \cot\pi(x-t)\chi(t)\mathrm{d}t = -\phi(x). \tag{3.219}$$

Equations (3.218) and (3.219) represent a Hilbert transform pair for the circle (see Eqs. (3.196) and (3.197)).

3.11.1 Hilbert transform on $\mathbb{R}$

A similar approach to the scheme just outlined was given by Hardy to arrive at the Hilbert transform pair on $\mathbb{R}$. From the trigonometric identity

$$\csc u \csc v = \csc(u+v)\{\cot u + \cot v\}, \tag{3.220}$$

and with the substitutions $u = \pi(s-t)$ and $v = \pi(t-x)$,

$$\csc\pi(s-t)\csc\pi(t-x) = \csc\pi(s-x)\{\cot\pi(s-t) + \cot\pi(t-x)\}. \tag{3.221}$$

Making use of Eq. (3.206) leads to

$$\int_0^1 \csc\pi(s-t)\csc\pi(t-x)\mathrm{d}t = 0. \tag{3.222}$$

Setting

$$h(s,t) = \csc \pi(s-t) \csc \pi(t-x)\phi(s), \tag{3.223}$$

and using Eq. (3.209),

$$\int_0^1 \phi(s)\mathrm{d}s\, P \int_0^1 \csc \pi(s-t) \csc \pi(t-x)\mathrm{d}t$$
$$= P \int_0^1 \csc \pi(t-x)\mathrm{d}t\, P \int_0^1 \csc \pi(s-t)\phi(s)\mathrm{d}s + \phi(x) \tag{3.224}$$

and hence

$$P \int_0^1 \csc \pi(t-x)\mathrm{d}t\, P \int_0^1 \csc \pi(s-t)\phi(s)\mathrm{d}s = -\phi(x). \tag{3.225}$$

Introducing the definition

$$\chi(t) = P \int_0^1 \csc \pi(s-t)\phi(s)\mathrm{d}s \tag{3.226}$$

leads to

$$\phi(t) = -P \int_0^1 \csc \pi(s-t)\chi(s)\mathrm{d}s. \tag{3.227}$$

Making the change of variables

$$\tan \pi s = u, \quad \tan \pi t = v, \tag{3.228}$$

and using the substitutions

$$f(u) = \frac{\phi(\pi^{-1}\tan^{-1} u)}{\sqrt{(1+u^2)}}, \quad g(u) = \frac{\chi(\pi^{-1}\tan^{-1} u)}{\sqrt{(1+u^2)}}, \tag{3.229}$$

allows Eqs. (3.226) and (3.227) to be written in the following form:

$$g(v) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(u)\mathrm{d}u}{v-u} \tag{3.230}$$

and

$$f(v) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(u)\mathrm{d}u}{v-u}, \tag{3.231}$$

which Hardy labeled "a most interesting pair of formulas." These are of course the Hilbert transform pair on $\mathbb{R}$.

Hardy (1908) also gave an alternative approach to the Hilbert transform on $\mathbb{R}$ that is more direct than the preceding approach. Suppose $f(x)$ satisfies $|f(x)| \leq c$ (some constant) for all x, and that $f(x)$ as well as its first two derivatives are continuous on $\mathbb{R}$, and that the integrals

$$\int^{\infty} \frac{f(x)\log x\,\mathrm{d}x}{x}, \quad \int_{-\infty} \frac{f(x)\log(-x)\mathrm{d}x}{x}$$

are convergent. Then

$$P\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{x-s} P\int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{s-t} = -\pi^2 f(x). \tag{3.232}$$

This is a specific case of the Hardy–Poincaré–Bertrand formula (see Section 2.13 and, in particular, the argument leading to Eq. (2.232)). With the identification

$$g(s) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{s-t}, \tag{3.233}$$

it follows that

$$f(x) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{g(s)\mathrm{d}s}{x-s}. \tag{3.234}$$

These results have since been derived under less stringent assumptions on the function f. Parenthetically it is noted that when Hardy gave the preceding derivation, he remarked, "The inversion formula itself has a familiar look, but I am not aware that I have ever seen it before." One might expect that Hardy would probably have been aware of Hilbert's contribution via the work of Kellogg (1904) or from Hilbert's early publications on the subject. So it is rather uncertain what the basis for his remark might be.

3.12 Fourier integral approach to the Hilbert transform on $\mathbb{R}$

Consider the analytic function w defined by

$$w(z) = \int_0^{\infty} [a(t) - \mathrm{i}b(t)]\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t, \tag{3.235}$$

and suppose that $w(z) = u(x,y) + \mathrm{i}v(x,y)$, where u and v are both real-valued functions. If the limit $y \to 0+$ is examined, and the following identifications employed:

$$f(x) = u(x,0) \tag{3.236}$$

and

$$g(x) = v(x,0), \tag{3.237}$$

then

$$f(x) = \int_0^\infty [a(t)\cos xt + b(t)\sin xt]\mathrm{d}t \tag{3.238}$$

and

$$g(x) = -\int_0^\infty [b(t)\cos xt - a(t)\sin xt]\mathrm{d}t. \tag{3.239}$$

The coefficients $a(t)$ and $b(t)$ can be determined as follows. Multiply Eq. (3.238) by $\cos s\tau$ and integrate over $(-\infty, \infty)$ to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \cos s\tau\, f(s)\mathrm{d}s &= \int_{-\infty}^{\infty} \cos s\tau \,\mathrm{d}s \int_0^\infty \{a(t)\cos st + b(t)\sin st\}\mathrm{d}t \\ &= \int_0^\infty a(t)\mathrm{d}t \int_{-\infty}^{\infty} \cos s\tau \cos\, st\,\mathrm{d}s \\ &\quad + \int_0^\infty b(t)\mathrm{d}t \int_{-\infty}^{\infty} \cos s\tau \,\sin st\,\mathrm{d}s. \end{aligned} \tag{3.240}$$

As an exercise, the reader is invited to produce an argument supporting the change of the order of integration. Is it possible to justify the change by introducing a convergence factor, say $\mathrm{e}^{-\varepsilon s^2}$, and then examining the limit $\varepsilon \to 0+$? The second integral is zero because of the odd character of the integrand. Hence,

$$\begin{aligned}\int_{-\infty}^{\infty} \cos s\tau\, f(s)\mathrm{d}s &= \frac{1}{2}\int_0^\infty a(t)\mathrm{d}t \int_{-\infty}^{\infty} \{\cos[(t-\tau)s] + \cos[(t+\tau)s]\}\mathrm{d}s \\ &= \frac{1}{2}\int_0^\infty a(t)\{2\pi\delta(t-\tau) + 2\pi\delta(t+\tau)\}\mathrm{d}t \\ &= \pi a(\tau), \end{aligned} \tag{3.241}$$

and so

$$a(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\cos st\,\mathrm{d}s. \tag{3.242}$$

On multiplying Eq. (3.238) by $\sin s\tau$, the same approach just outlined leads to

$$b(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\sin st\,\mathrm{d}s. \tag{3.243}$$

Using Eqs. (3.242) and (3.243), Eq. (3.238) can be expressed as follows:

$$f(x) = \frac{1}{\pi}\int_0^\infty \mathrm{d}t \int_{-\infty}^{\infty} f(s)\cos[t(s-x)]\mathrm{d}s, \tag{3.244}$$

which represents the Fourier integral formula for $f(x)$. In a similar manner, Eq. (3.239) can be expressed using the expressions for $a(t)$ and $b(t)$ just given as follows:

$$g(x) = -\frac{1}{\pi}\int_0^\infty \mathrm{d}t \int_{-\infty}^\infty f(s)\sin[t(s-x)]\mathrm{d}s. \tag{3.245}$$

The integral in Eq. (3.245), apart from the minus sign, is referred to as the *Fourier allied integral*. The integral connection between $f(x)$ and $g(x)$ can be established in the following way. From Eq. (3.239), using the approach just given,

$$a(t) = \frac{1}{\pi}\int_{-\infty}^\infty g(s)\sin st\,\mathrm{d}s \tag{3.246}$$

and

$$b(t) = -\frac{1}{\pi}\int_{-\infty}^\infty g(s)\cos st\,\mathrm{d}s. \tag{3.247}$$

Substituting these results into Eq. (3.238) yields

$$f(x) = \frac{1}{\pi}\int_0^\infty \mathrm{d}t \int_{-\infty}^\infty g(s)\sin[t(s-x)]\mathrm{d}s. \tag{3.248}$$

Equations (3.245) and (3.248) form a reciprocal pair. Frequently in applications the function f is even and g is odd. When this is the case, Eqs. (3.245) and (3.248) simplify to

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos xt\,\mathrm{d}t \int_0^\infty g(s)\sin st\,\mathrm{d}s,\ \text{for } g(s) \text{ odd}, \tag{3.249}$$

and

$$g(x) = \frac{2}{\pi}\int_0^\infty \sin xt\,\mathrm{d}t \int_0^\infty f(s)\cos st\,\mathrm{d}s,\ \text{for } f(s) \text{ even}. \tag{3.250}$$

These relations can serve as an alternative approach to calculating Hilbert transforms. For example, if $f(x) = x^{-1}\sin ax$ with $a > 0$, then

$$\begin{aligned} g(x) = Hf(x) &= \frac{2}{\pi}\int_0^\infty \sin xt\,\mathrm{d}t \int_0^\infty \frac{\sin as\cos st}{s}\mathrm{d}s \\ &= \frac{1}{\pi}\int_0^\infty \sin xt\,\mathrm{d}t \int_0^\infty \frac{\{\sin[(a-t)s] + \sin[(a+t)s]\}}{s}\mathrm{d}s \\ &= \frac{1}{2}\int_0^\infty \sin xt\{\mathrm{sgn}\,(a-t) + \mathrm{sgn}(a+t)\}\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \int_0^a \sin xt \mathrm{d}t \\
&= \frac{1 - \cos ax}{x}. \qquad (3.251)
\end{aligned}
$$

If the Hilbert transform of an odd function is evaluated using the allied integral approach, then Eq. (3.249) is employed with a minus sign inserted. This sign change is required since $f(x)$, which is conjugate to $g(x)$, equals $-Hg(x)$. For example, to determine $H(\sin ax)$ for $a > 0$:

$$
\begin{aligned}
H(\sin ax) &= -\frac{2}{\pi} \int_0^\infty \cos xt \, \mathrm{d}t \int_0^\infty \sin st \sin as \, \mathrm{d}s \\
&= -\frac{1}{2\pi} \int_0^\infty \cos xt \, \mathrm{d}t \int_{-\infty}^\infty \{\cos(t-a)s - \cos(t+a)s\} \mathrm{d}s \\
&= -\frac{1}{2\pi} \int_0^\infty \cos xt \, \mathrm{d}t \int_0^\infty \{\mathrm{e}^{\mathrm{i}(t-a)s} - \mathrm{e}^{\mathrm{i}(t+a)s}\} \mathrm{d}s \\
&= -\frac{1}{2\pi} \int_0^\infty \cos xt \{2\pi\delta(t-a) - 2\pi\delta(t+a)\} \mathrm{d}t \\
&= -\int_0^\infty \cos xt \, \delta(t-a) \mathrm{d}t \\
&= -\cos ax. \qquad (3.252)
\end{aligned}
$$

From Eq. (3.239) it follows that

$$
\begin{aligned}
\frac{g(x+s) - g(x-s)}{s} &= \frac{1}{s} \int_0^\infty [a(t)\{\sin[(x+s)t] - \sin[(x-s)t]\} \\
&\quad - b(t)\{\cos[(x+s)t] - \cos[(x-s)t]\}] \mathrm{d}t \\
&= \frac{2}{s} \int_0^\infty \sin st [a(t) \cos xt + b(t) \sin xt] \mathrm{d}t. \qquad (3.253)
\end{aligned}
$$

Integrating both sides of Eq. (3.253) over the interval (α, β), for $0 < \alpha < \beta$, leads to

$$
\begin{aligned}
\frac{1}{\pi} \int_\alpha^\beta \frac{[g(x+s) - g(x-s)] \mathrm{d}s}{s} &= \frac{2}{\pi} \int_\alpha^\beta \frac{\mathrm{d}s}{s} \int_0^\infty \sin st [a(t) \cos xt + b(t) \sin xt] \mathrm{d}t \\
&= \frac{2}{\pi} \int_0^\infty [a(t) \cos xt + b(t) \sin xt] \mathrm{d}t \int_\alpha^\beta \frac{\sin st \, \mathrm{d}s}{s}. \qquad (3.254)
\end{aligned}
$$

For all values of s, the integral over t converges uniformly, and so the integration order can be switched. On taking the limits $\alpha \to 0$ and $\beta \to \infty$, it follows that

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \frac{[g(x+s)-g(x-s)]\mathrm{d}s}{s} \\
&\quad= \lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \frac{2}{\pi}\int_0^\infty [a(t)\cos xt + b(t)\sin xt]\mathrm{d}t \int_\alpha^\beta \frac{\sin st\,\mathrm{d}s}{s} \\
&\quad= \frac{2}{\pi}\int_0^\infty [a(t)\cos xt + b(t)\sin xt]\mathrm{d}t \lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \int_\alpha^\beta \frac{\sin st\,\mathrm{d}s}{s} \\
&\quad= \int_0^\infty [a(t)\cos xt + b(t)\sin xt]\mathrm{d}t \\
&\quad= f(x).
\end{aligned} \tag{3.255}$$

The switch of the limit and the integral in Eq. (3.255) can be justified using Lebesgue's dominated convergence theorem.

In a similar manner, from Eq. (3.238),

$$\begin{aligned}
\frac{f(x+s)-f(x-s)}{s} &= \frac{1}{s}\int_0^\infty [a(t)\{\cos[(x+s)t]-\cos[(x-s)t]\} \\
&\qquad + b(t)\{\sin[(x+s)t]-\sin[(x-s)t]\}]\mathrm{d}t \\
&= \frac{2}{s}\int_0^\infty \sin st[b(t)\cos xt - a(t)\sin xt]\mathrm{d}t.
\end{aligned} \tag{3.256}$$

From Eq. (3.256), the following integral can be formed:

$$\begin{aligned}
&\lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \frac{1}{\pi}\int_\alpha^\beta \frac{[f(x+s)-f(x-s)]\mathrm{d}s}{s} \\
&\quad= \lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \frac{2}{\pi}\int_\alpha^\beta \frac{\mathrm{d}s}{s}\int_0^\infty \sin st[b(t)\cos xt - a(t)\sin xt]\mathrm{d}t
\end{aligned} \tag{3.257}$$

and hence

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \frac{[f(x+s)-f(x-s)]\mathrm{d}s}{s} \\
&\quad= \frac{2}{\pi}\int_0^\infty [b(t)\cos xt - a(t)\sin xt]\mathrm{d}t \lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \int_\alpha^\beta \frac{\sin st\,\mathrm{d}s}{s}
\end{aligned}$$

$$= \int_0^\infty [b(t)\cos xt - a(t)\sin xt]\mathrm{d}t$$
$$= -g(x). \tag{3.258}$$

Therefore, the conjugate relationships are as follows:

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{[g(x+s) - g(x-s)]\mathrm{d}s}{s} \tag{3.259}$$

and

$$g(x) = -\frac{1}{\pi}\int_0^\infty \frac{[f(x+s) - f(x-s)]\mathrm{d}s}{s}. \tag{3.260}$$

These two results correspond to the form given by Young (see Eqs. (3.12) and (3.13)), apart from the choice of sign convention employed. These integrals can be put in a different form using the following sequence:

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty \frac{[g(x+s) - g(x-s)]\mathrm{d}s}{s} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_\varepsilon^\infty \frac{[g(x+s) - g(x-s)]\mathrm{d}s}{s} \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\left\{\int_\varepsilon^\infty \frac{g(x+s)\mathrm{d}s}{s} - \int_\varepsilon^\infty \frac{g(x-s)\mathrm{d}s}{s}\right\} \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\left\{\int_{x+\varepsilon}^\infty \frac{g(t)\mathrm{d}t}{t-x} + \int_{-\infty}^{x-\varepsilon} \frac{g(t)\mathrm{d}t}{t-x}\right\}, \end{aligned} \tag{3.261}$$

and hence

$$f(x) = -\frac{1}{\pi}P\int_{-\infty}^\infty \frac{g(t)\mathrm{d}t}{x-t}. \tag{3.262}$$

In a similar manner,

$$g(x) = \frac{1}{\pi}P\int_{-\infty}^\infty \frac{f(t)\mathrm{d}t}{x-t}. \tag{3.263}$$

Equations (3.262) and (3.263) are recognized as the Hilbert transform pair derived previously.

An alternative approach to obtain Eq. (3.260), as outlined by Titchmarsh (1948, p. 120), proceeds as follows. Starting with Eq. (3.245), it follows that

$$\begin{aligned} g(x) &= -\frac{1}{\pi}\int_0^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} f(s)\sin[t(s-x)]\mathrm{d}s \\ &= \frac{1}{\pi}\lim_{\lambda\to\infty}\int_0^{\lambda} \mathrm{d}t \int_{-\infty}^{\infty} f(s)\sin[t(s-x)]\mathrm{d}s \\ &= -\frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} \frac{\{1-\cos[\lambda(s-x)]\}f(s)\mathrm{d}s}{s-x}. \end{aligned} \tag{3.264}$$

This final integral can be broken up into the separate intervals $(-\infty, 0]$ and $[0, \infty)$, and, with the appropriate change of variables, may be simplified to

$$g(x) = -\frac{1}{\pi}\lim_{\lambda\to\infty}\int_0^{\infty} \frac{\{1-\cos\lambda s\}}{s}\{f(x+s)-f(x-s)\}\mathrm{d}s \tag{3.265}$$

and hence

$$g(x) = -\frac{1}{\pi}\int_0^{\infty} \frac{\{f(x+s)-f(x-s)\}\mathrm{d}s}{s}, \tag{3.266}$$

which is Eq. (3.260) given previously. The contribution to the integral depending on λ vanishes as $\lambda \to \infty$, if f is integrable. A simple example should help clarify this limit: for $\alpha > 0$,

$$\lim_{\lambda\to\infty}\int_0^{\infty} \cos\lambda x\, \mathrm{e}^{-\alpha x}\, \mathrm{d}x = \lim_{\lambda\to\infty}\frac{\alpha}{\alpha^2+\lambda^2} = 0. \tag{3.267}$$

The integrand of the integral in Eq. (3.267) is shown in Figure 3.9. Some readers will recognize this as the form of the free induction decay (the "FID") curve for a single resonance from a Fourier transform magnetic resonance experiment; others will be familiar with this from a study of simple harmonic motion with less than critical damping. As $\lambda \to \infty$, the oscillations of the cosine term increase sharply, and the integral in Eq. (3.267) consists of a sum of contributions of alternating sign, with the neighboring contributions canceling in the limit, leading to a zero value for the integral. The result given in Eq. (3.266) follows directly by application of the Riemann–Lebesgue lemma (see Section 2.14). An alternative and simple way to think about the limiting process in Eq. (3.265) is to set $h(s,x) = f(x+s) - f(x-s)$; then

$$\lim_{\lambda\to\infty}\int_0^{\infty} \frac{\cos\lambda s}{s}\, h(s,x)\mathrm{d}s = \lim_{\lambda\to\infty}\int_0^{\infty} \frac{\cos u}{u}\, h\Big(\frac{u}{\lambda},x\Big)\mathrm{d}u = 0, \tag{3.268}$$

since $h(0,x) = 0$.

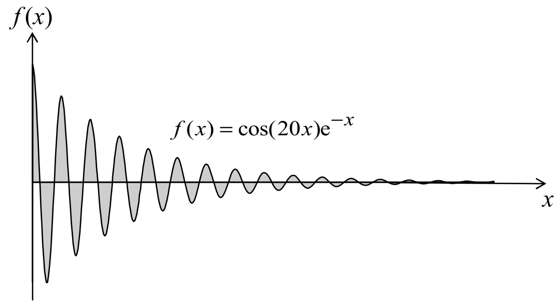

Figure 3.9. Free induction decay curve for a single resonance.

3.13 Fourier series approach

Suppose the function $f(x)$ is periodic and integrable, and can be represented by the Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{3.269}$$

The reader is reminded that the symbol $\sim$ denotes correspondence. From the combination $(1/2)\{f(x-t) - f(x+t)\}$,

$$\frac{1}{2}\{f(x-t) - f(x+t)\} = \sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx)\sin nt. \tag{3.270}$$

Using this construction, the left-hand side of the equation is invariant to the addition of an arbitrary constant in the definition of f, so the term in a_0 disappears from the following steps. Multiplying both sides of Eq. (3.270) by t^{-1}, integrating over t, and using the result

$$\int_0^{\infty}\frac{\sin nt\,\mathrm{d}t}{t} = \frac{\pi}{2}, \text{ for } n > 0, \tag{3.271}$$

leads to

$$\frac{1}{\pi}\int_0^{\infty}\frac{\{f(x-t) - f(x+t)\}\mathrm{d}t}{t} = \sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx). \tag{3.272}$$

The series in Eq. (3.272), which is termed the *conjugate series* to Eq. (3.269), is identified with the function g, so that

$$g(x) = \sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx), \tag{3.273}$$

and thus

$$\frac{1}{\pi}\int_0^\infty \frac{\{f(x-t)-f(x+t)\}\mathrm{d}t}{t} = g(x). \tag{3.274}$$

The conjugate series in Eq. (3.273) is often given with the opposite sign convention employed. This is obtained by considering in place of $1/2\{f(x-t)-f(x+t)\}$ the combination $1/2\{f(x+t)-f(x-t)\}$. The choice given matches the sign convention employed for the Hilbert transform. The operator that converts f to its conjugate function is sometimes called the *conjugation operator*.

Note that, in general, it is not known if the series in Eq. (3.273) is a Fourier series, but if it is the terminology "conjugate Fourier series" is employed. For example, in the interval $0 < x < 2\pi$, the series $\sum_{n=1}^{\infty}(\sin nx/n)$ is the Fourier representation of $(1/2)(\pi - x)$, while $\sum_{n=1}^{\infty}(\cos nx/n)$ represents the series for the function $-\log|2\sin(x/2)|$, which becomes unbounded as $x \to 2\pi$. See Exercise 3.10 for some additional cases to ponder. For many of the conjugate functions that arise in the physical sciences and engineering, the conjugate function and its derivative are at least piecewise continuous over the period of the function, and in these cases Eq. (3.273) will in fact define a Fourier series.

If the a_0 term is assumed to be zero, then starting with Eq. (3.273) it follows, in a similar manner to the previous equations, that

$$\frac{1}{2}\{g(x-t)-g(x+t)\} = -\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\sin nt \tag{3.275}$$

and hence

$$\frac{1}{\pi}\int_0^\infty \frac{\{g(x-t)-g(x+t)\}\mathrm{d}t}{t} = -f(x). \tag{3.276}$$

By the construction process involved, $f(x)$ is only determined to within an additive constant. Recall that the Hilbert transform of a constant is zero (see Eq. (3.105)). If the assumption that a_0 is zero is not employed, then the most general solution of the integral in Eq. (3.276) will include an arbitrary constant term. Equation (3.276) is the conjugate relationship of Eq. (3.274). Equations (3.274) and (3.276) are one form for the Hilbert transform pair.

An alternative way to view Eqs. (3.269) and (3.273) is as follows. Suppose the coefficients a_n and b_n are real, and consider the power series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n - \mathrm{i}b_n)z^n.$$

If z is restricted to the unit circle, and the substitution $z = \mathrm{e}^{\mathrm{i}x}$ is employed, then the real part of the power series can be identified with Eq. (3.269) and the imaginary part can be identified with the conjugate series Eq. (3.273).

Tauber (1891) gave the following development. Let the power series $\sum_{v=1}^{\infty} c_v z^v$ be expressed in terms of its real and imaginary components as follows:

$$\sum_{v=1}^{\infty} c_v z^v = \varphi(\theta) + \mathrm{i}\psi(\theta), \tag{3.277}$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$, and the functional dependence of φ and ψ on r has been suppressed. On writing

$$c_v r^v = a_v + \mathrm{i}b_v, \tag{3.278}$$

the real and imaginary parts in Eq. (3.277) can be written as follows:

$$\varphi(\theta) = \sum_{v=1}^{\infty} a_v \cos v\theta - b_v \sin v\theta \tag{3.279}$$

and

$$\psi(\theta) = \sum_{v=1}^{\infty} a_v \sin v\theta + b_v \cos v\theta. \tag{3.280}$$

Tauber gave the connection between the real and imaginary parts as

$$\varphi(\theta) = \frac{1}{2\pi}\int_0^{\pi} \{\psi(\theta+\phi) - \psi(\theta-\phi)\}\cot\left(\frac{\phi}{2}\right)\mathrm{d}\phi \tag{3.281}$$

and

$$\psi(\theta) = -\frac{1}{2\pi}\int_0^{\pi} \{\varphi(\theta+\phi) - \varphi(\theta-\phi)\}\cot\left(\frac{\phi}{2}\right)\mathrm{d}\phi. \tag{3.282}$$

The reader should make an effort to evaluate these last two integrals: if assistance is needed, consult Eqs. (6.51) and (6.79). Taking advantage of the periodic nature of $\varphi(\theta)$ and $\psi(\theta)$, the two preceding formulas can be expressed as follows:

$$\varphi(\theta) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(\phi)\cot\left(\frac{\theta-\phi}{2}\right)\mathrm{d}\phi \tag{3.283}$$

and

$$\psi(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\phi)\cot\left(\frac{\theta-\phi}{2}\right)\mathrm{d}\phi. \tag{3.284}$$

Apart from a sign choice, Eqs. (3.283) and (3.284) correspond to the results given in Eqs. (3.14) and (3.15), and the last two equations represent the Hilbert transform pair for the disc, Eqs. (3.185) and (3.186).

3.14 The Hilbert transform for periodic functions

The Hilbert transform for a function that is periodic with period 2π is given by

$$\mathcal{H}f(x) = \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(x-s) \cot\left(\frac{s}{2}\right) \mathrm{d}s. \tag{3.285}$$

Early references to this formula or equivalent variations include Kellogg (1904), Hilbert (1904, 1912) and Hardy and Littlewood (1929). A more general definition of the Hilbert transform is possible, when the period is taken as 2τ. In this case, the generalization of Eq. (3.285) becomes (Pandey, 1996, 1997)

$$\mathcal{H}_\tau f(x) = \frac{1}{2\tau} P \int_{-\tau}^{\tau} f(x-s) \cot\left(\frac{\pi s}{2\tau}\right) \mathrm{d}s. \tag{3.286}$$

The notational convention $\mathcal{H}_\pi \equiv \mathcal{H}$ is employed. Equation (3.286) can be rearranged as follows:

$$\mathcal{H}_\tau f(x) = \frac{1}{2\tau} P \int_{0}^{\tau} \{f(x-s) - f(x+s)\} \cot\left(\frac{\pi s}{2\tau}\right) \mathrm{d}s. \tag{3.287}$$

The case $\tau = \pi$ of Eq. (3.287) was given (apart from the choice of sign convention) by Young (1912).

In the limit $\tau \to \infty$, Eq. (3.286) reverts back to the standard definition given in Eq. (3.1):

$$\begin{aligned} \lim_{\tau\to\infty} \mathcal{H}_\tau f(x) &= \lim_{\tau\to\infty} \frac{1}{2\tau} P \int_{-\tau}^{\tau} f(x-s) \cot\left(\frac{\pi s}{2\tau}\right) \mathrm{d}s \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(x-s)\mathrm{d}s}{s} \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(s)\mathrm{d}s}{x-s}, \end{aligned} \tag{3.288}$$

where the following limit has been employed:

$$\begin{aligned} \lim_{\tau\to\infty} \frac{1}{2\tau} \cot\left(\frac{\pi s}{2\tau}\right) &= \lim_{t\to 0} t \cot(\pi s t) \\ &= \frac{1}{\pi s}. \end{aligned} \tag{3.289}$$

In a like manner, the limit $\tau \to \infty$ in Eq. (3.287) yields (apart from the sign convention employed) Eq. (3.12), which was a form also given by Young (1912).

The origin of the definition given in Eq. (3.286) is now examined, starting from the standard definition of the Hilbert transform on the real line. Suppose f is periodic with period 2τ and belongs to L^p, where $p > 1$; then

$$
\begin{aligned}
Hf(x) &= \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{x-s}\\
&= \frac{1}{\pi}\left\{\cdots + P\int_{-5\tau}^{-3\tau}\frac{f(s)\mathrm{d}s}{x-s} + P\int_{-3\tau}^{-\tau}\frac{f(s)\mathrm{d}s}{x-s} + P\int_{-\tau}^{\tau}\frac{f(s)\mathrm{d}s}{x-s}\right.\\
&\quad\left. + P\int_{\tau}^{3\tau}\frac{f(s)\mathrm{d}s}{x-s} + P\int_{3\tau}^{5\tau}\frac{f(s)\mathrm{d}s}{x-s} + P\int_{5\tau}^{7\tau}\frac{f(s)\mathrm{d}s}{x-s} + \cdots\right\}\\
&= \frac{1}{\pi}\left\{\cdots + P\int_{-\tau}^{\tau}\frac{f(s)\mathrm{d}s}{x-s-4\tau} + P\int_{-\tau}^{\tau}\frac{f(s)\mathrm{d}s}{x-s-2\tau} + P\int_{-\tau}^{\tau}\frac{f(s)\mathrm{d}s}{x-s}\right.\\
&\quad\left. + P\int_{-\tau}^{\tau}\frac{f(s)\mathrm{d}s}{x-s+2\tau} + P\int_{-\tau}^{\tau}\frac{f(s)\mathrm{d}s}{x-s+4\tau} + P\int_{-\tau}^{\tau}\frac{f(s)\mathrm{d}s}{x-s+6\tau} + \cdots\right\},
\end{aligned}
\tag{3.290}
$$

where the periodic character of $f(s)$, $f(s) = f(s + 2\tau k)$, for k a positive or negative integer, has been employed. Equation (3.290) can be expressed as follows:

$$
Hf(x) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} P\int_{-\tau}^{\tau}\frac{f(s)\mathrm{d}s}{x-s+2\tau k}. \tag{3.291}
$$

Now make use of the standard identity

$$
\cot z = z^{-1} + 2z\sum_{k=1}^{\infty}\frac{1}{z^2-\pi^2k^2}, \tag{3.292}
$$

to obtain

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty}\frac{1}{x-s+2\tau k} &= \frac{1}{x-s} + \frac{2z^2}{x-s}\sum_{k=1}^{\infty}\frac{1}{z^2-\pi^2k^2}\\
&= \frac{\pi}{2\tau}\cot z,
\end{aligned}
\tag{3.293}
$$

where $z = \pi(x-s)/2\tau$. Hence, on interchanging the order of summation and integration,

$$
Hf(x) = \frac{1}{2\tau}P\int_{-\tau}^{\tau} f(s)\cot\left(\frac{\pi(x-s)}{2\tau}\right)\mathrm{d}s. \tag{3.294}
$$

The reader is invited to provide a justification for bringing the summation under the integral sign in the last sequence of steps. Equation (3.294) can be rewritten as

$$
Hf(x) = \frac{1}{2\tau}P\int_{-\tau}^{\tau} f(x-s)\cot\left(\frac{\pi s}{2\tau}\right)\mathrm{d}s. \tag{3.295}
$$

To obtain this result, recall that, by hypothesis, f is a periodic function with period 2τ, so the integrand of Eq. (3.295) is periodic with period 2τ, and then make use of the standard result for a function of period 2τ,

$$\int_{\mathrm{d}-\tau}^{\mathrm{d}+\tau} h(x)\mathrm{d}x = \int_{-\tau}^{\tau} h(x)\mathrm{d}x, \tag{3.296}$$

where d is real and $h(x)$ has a period of 2τ.

Two examples are now evaluated, and additional cases are discussed later. Consider the Hilbert transform of $\cos ax$ for $a > 0$. The period of this function is $(2\pi/a)$, that is $\tau = \pi/a$. Therefore, from Eq. (3.286), and employing the substitutions $\tau = \pi/a$ and $y = as/2$, it follows that

$$\begin{aligned}\mathcal{H}_\tau(\cos ax) &= \frac{1}{2\tau} P\int_{-\tau}^{\tau} \cos a(x-s)\cot\left(\frac{\pi s}{2\tau}\right)\mathrm{d}s \\ &= \frac{1}{\pi} P\int_{-\pi/2}^{\pi/2} \cos(ax-2y)\cot y\,\mathrm{d}y \\ &= \frac{\cos ax}{\pi} P\int_{-\pi/2}^{\pi/2} \cos 2y \cot y\,\mathrm{d}y + \frac{\sin ax}{\pi}\int_{-\pi/2}^{\pi/2} \sin 2y\cot y\,\mathrm{d}y.\end{aligned} \tag{3.297}$$

The first integral in the final line of Eq. (3.297) vanishes (the integrand is an odd function), and the second integral can be readily evaluated to yield

$$\int_{-\pi/2}^{\pi/2} \sin 2y\cot y\,\mathrm{d}y = \pi, \tag{3.298}$$

so that

$$\mathcal{H}_{\pi/a}(\cos ax) = \sin ax, \text{ for } a > 0. \tag{3.299}$$

In a similar fashion, the Hilbert transform of $\sin ax$ for $a > 0$, also with period $2\pi/a$, is found to be

$$\begin{aligned}\mathcal{H}_{\pi/a}(\sin ax) &= \frac{1}{\pi} P\int_{-\pi/2}^{\pi/2} \sin(ax-2y)\cot y\,\mathrm{d}y \\ &= -\frac{\cos ax}{\pi}\int_{-\pi/2}^{\pi/2} \sin 2y\cot y\,\mathrm{d}y \\ &= -\cos ax, \text{ for } a > 0.\end{aligned} \tag{3.300}$$

Equations (3.299) and (3.300) are the analogs of the results given in Eqs. (3.113) and (3.311) for the Hilbert transforms of $\cos ax$ and $\sin ax$ on $\mathbb{R}$. Further discussion of the periodic case is considered in Chapter 6.

3.15 Cancellation behavior for the Hilbert transform

This chapter concludes with an examination of a fundamental cancellation property of the Hilbert transform. The form of the Hilbert transform given by Young (apart from a sign factor) is

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{f(x-t) - f(x+t)}{t} \, \mathrm{d}t. \tag{3.301}$$

A key question is: does the integral exist (for suitable f) because of the smallness of $f(x-t) - f(x+t)$ in the limit $t \to 0$, or is something more fundamental in play? Consider the following simple examples. Suppose f is a constant c, then

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{c-c}{t} \, \mathrm{d}t = 0, \tag{3.302}$$

or, if $f(x) = \sin ax$, for $a > 0$, then

$$\begin{aligned} Hf(x) &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{\sin a(x-t) - \sin a(x+t)}{t} \, \mathrm{d}t \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \left\{ \sin ax \int_{\varepsilon}^{\infty} \frac{\cos at - \cos at}{t} \, \mathrm{d}t - 2\cos ax \int_{\varepsilon}^{\infty} \frac{\sin at}{t} \, \mathrm{d}t \right\} \\ &= -\frac{2\cos ax}{\pi} \int_0^{\infty} \frac{\sin at}{t} \, \mathrm{d}t \\ &= -\cos ax. \end{aligned} \tag{3.303}$$

In the first example with f a constant, the integral does not diverge in the limit $\varepsilon \to 0$ because of the obvious cancellation that occurs. In the second example, the factor $f(x-t) - f(x+t)$ breaks up into two parts, one contribution that cancels to zero, and the second term having a limit $\varepsilon \to 0$ that is well defined. So, for both of these examples, cancellation effects in part or in whole play a key role in obtaining a non-divergent result for the Hilbert transform.

Similar considerations apply to the case of the Hilbert transform for periodic functions, which can be written assuming the function f has a period of 2π, in the form

$$\begin{aligned} \mathcal{H}f(x) &= \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(t) \cot\left\{\frac{1}{2}(x-t)\right\} \mathrm{d}t \\ &= \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi} \{f(x-t) - f(x+t)\} \cot\left(\frac{t}{2}\right) \mathrm{d}t. \end{aligned} \tag{3.304}$$

The discussion can be simplified in the following manner. First note the behavior of $\cot(t/2)$ for small values of t:

$$\cot\left(\frac{t}{2}\right) = \frac{2}{t} - \frac{t}{6} + O(t^3), \text{ as } t \to 0, \tag{3.305}$$

which can be easily obtained using the standard series expansions for $\sin x$ and $\cos x$. Hence

$$\begin{aligned}\mathcal{H}f(x) = \frac{1}{2\pi}\int_0^{\pi}\{f(x-t) - f(x+t)\}\left\{\cot\left(\frac{t}{2}\right) - \frac{2}{t}\right\}\mathrm{d}t \\ + \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi}\frac{f(x-t) - f(x+t)}{t}\,\mathrm{d}t.\end{aligned} \tag{3.306}$$

The first integral on the right-hand side of this result is bounded; there is no difficulty with the integrand as $t \to 0$, by virtue of Eq. (3.305). This means attention can be focused on the simpler looking second integral, which has an obvious similarity to Eq. (3.301). This integral is revisited momentarily.

In general, the existence of Hf or $\mathcal{H}f$ depends on an interference of positive and negative contributions, rather than the smallness of $f(x-t) - f(x+t)$ for small $|t|$. To demonstrate this, consider the second integral in Eq. (3.306), but with two changes. Set

$$I(x) = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{|f(x+t) - f(x-t)|\mathrm{d}t}{t}. \tag{3.307}$$

Changing the upper limit of integration results in no loss of generality (f now has period one). The absolute value prevents any possible interference effect between positive and negative contributions to the integral. It will now be demonstrated that there are choices for the function f for which this integral diverges for every value of x. Let $f(x)$ satisfy the following conditions:

(i) f is a continuous periodic function with period one,
(ii) $|f(x)| \le 1$,
(iii) $|f(x+t) - f(x)| \le c|t|$, for a positive constant c,
(iv) $\int_{n^{-1}}^{1}\dfrac{|f(nx+nt) - f(nx-nt)|\mathrm{d}t}{t} \sim \log n$.

As an example, consider the periodic function shown in Figure 3.10, which is

$$f(x) = \begin{cases} 4x, & 0 \le x \le 1/4 \\ \dfrac{4}{3}(1-x), & 1/4 < x \le 1. \end{cases} \tag{3.308}$$

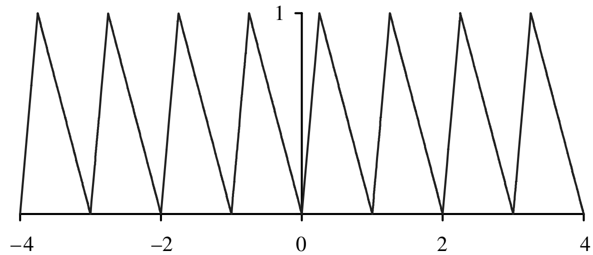

Figure 3.10. Function satisfying conditions (i)–(iv).

Let C, C_1, and C_2 denote positive constants, not necessarily the same at each occurrence. It will first be shown that

$$\int_{n^{-1}}^{1} \frac{|f(nx+nt) - f(nx-nt)|\mathrm{d}t}{t} \geq C \log n \tag{3.309}$$

and

$$\int_{0}^{1} \frac{|f(nx+nt) - f(nx-nt)|\mathrm{d}t}{t} \leq C_1 \log n, \tag{3.310}$$

for $n = 2, 3, \ldots$. The function f satisfies the following two inequalities:

$$0 < C_2 \leq \int_0^1 |f(x+t) - f(x-t)|\mathrm{d}t, \text{ for } x \in [0, 1], \tag{3.311}$$

and

$$\int_0^1 |f(x+t) - f(x-t)|\mathrm{d}t \leq 2. \tag{3.312}$$

The integral in Eq. (3.309) can be rearranged as follows:

$$\begin{aligned}
I_n &= \int_{n^{-1}}^{1} \frac{|f(nx+nt) - f(nx-nt)|\mathrm{d}t}{t} \\
&= \int_1^n \frac{|f(nx+y) - f(nx-y)|\mathrm{d}y}{y} \\
&= \int_1^2 + \int_2^3 + \int_3^4 + \cdots \left\{ \frac{|f(nx+y) - f(nx-y)|}{y} \right\} \mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{|f(nx+t+1) - f(nx-t-1)|\mathrm{d}t}{t+1} \\
&\quad + \int_0^1 \frac{|f(nx+t+2) - f(nx-t-2)|\mathrm{d}t}{t+2} \\
&\quad + \int_0^1 \frac{|f(nx+t+3) - f(nx-t-3)|\mathrm{d}t}{t+3} + \cdots \\
&= \int_0^1 |f(nx+t) - f(nx-t)| \left\{ \sum_{k=1}^{n-1} \frac{1}{t+k} \right\} \mathrm{d}t,
\end{aligned} \tag{3.313}
$$

where the periodic nature of the function has been employed in the last step. Since $k+1 \geq k+t$ for $t \in [0,1]$,

$$
\sum_{k=1}^{n-1} \frac{1}{t+k} \geq \sum_{k=2}^{n} \frac{1}{k}. \tag{3.314}
$$

Now, $\log(1+x) \leq x$ for $x > -1$, and hence

$$
\log\left(1 + \frac{1}{k}\right) \leq \frac{1}{k}, \tag{3.315}
$$

and therefore

$$
\sum_{k=2}^{n} \log\left(1 + \frac{1}{k}\right) \leq \sum_{k=2}^{n} \frac{1}{k}, \tag{3.316}
$$

and so

$$
C \log n \leq \log\left(\frac{n+1}{2}\right) \leq \sum_{k=2}^{n} \frac{1}{k}, \tag{3.317}
$$

where C is a positive constant. Inserting this result into Eq. (3.313) and using Eq. (3.311) yields

$$
I_n \geq C \log n. \tag{3.318}
$$

This completes the demonstration of Eq. (3.309) and establishes property (iv). To prove Eq. (3.310), split the integration interval as follows:

$$
\begin{aligned}
\int_0^1 \frac{|f(nx+nt) - f(nx-nt)|\mathrm{d}t}{t} &= \int_0^{n^{-1}} \frac{|f(nx+nt) - f(nx-nt)|\mathrm{d}t}{t} \\
&\quad + \int_{n^{-1}}^1 \frac{|f(nx+nt) - f(nx-nt)|\mathrm{d}t}{t}.
\end{aligned} \tag{3.319}
$$

Using Eq. (3.312) yields

$$\int_{n^{-1}}^{1} \frac{|f(nx+nt)-f(nx-nt)|\mathrm{d}t}{t} \leq C \log n\,, \tag{3.320}$$

and

$$\begin{aligned}
\int_{0}^{n^{-1}} \frac{|f(nx+nt)-f(nx-nt)|\mathrm{d}t}{t} &= \int_{0}^{1} \frac{|f(nx+y)-f(nx-y)|\mathrm{d}y}{y} \\
&= \int_{0}^{1} \frac{|f(nx+y)-f(nx)-\{f(nx-y)-f(nx)\}|\mathrm{d}y}{y} \\
&\leq \int_{0}^{1} \frac{|f(nx+y)-f(nx)|\mathrm{d}y}{y} + \int_{0}^{1} \frac{|f(nx-y)-f(nx)|\mathrm{d}y}{y} \\
&\leq 2c,
\end{aligned} \tag{3.321}$$

where condition (iii) has been used. Combining this result with Eq. (3.320) leads to Eq. (3.310).

Define the function

$$g(x) = \sum_{k=1}^{\infty} a_k f(n_k x), \tag{3.322}$$

where $a_k > 0, n_k \in \mathbb{N}$, and $n_1 < n_2 < n_3 \cdots$. Now,

$$\begin{aligned}
\int_{n_k^{-1}}^{1} \frac{|g(x+t)-g(x-t)|\mathrm{d}t}{t} &= \sum_{j=1}^{\infty} a_j \int_{n_k^{-1}}^{1} \frac{\big|f\{n_j(x+t)\}-f\{n_j(x-t)\}\big|\mathrm{d}t}{t} \\
&= a_k \int_{n_k^{-1}}^{1} \frac{|f\{n_k(x+t)\}-f\{n_k(x-t)\}|\mathrm{d}t}{t} \\
&\quad + \sum_{\substack{j=1 \\ (j\neq k)}}^{\infty} a_j \int_{n_k^{-1}}^{1} \frac{\big|f\{n_j(x+t)\}-f\{n_j(x-t)\}\big|\mathrm{d}t}{t} \\
&\geq a_k \int_{n_k^{-1}}^{1} \frac{|f\{n_k(x+t)\}-f\{n_k(x-t)\}|\mathrm{d}t}{t}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{j=1\\(j\neq k)}}^{\infty} a_j \int_{n_k^{-1}}^{1} \frac{|f\{n_j(x+t)\} - f\{n_j(x-t)\}|\mathrm{d}t}{t} \\
&\geq a_k C \log n_k - C_1 \sum_{j=1}^{k-1} a_j \log n_j \\
&- \sum_{j=k+1}^{\infty} a_j \int_{n_k^{-1}}^{1} \frac{\left|f\{n_j(x+t)\} - f\{n_j(x-t)\}\right|\mathrm{d}t}{t},
\end{aligned}
\tag{3.323}
$$

where Eqs. (3.309) and (3.310) have been used to write the last result. Making use of condition (ii) yields

$$
\int_{n_k^{-1}}^{1} \frac{|g(x+t) - g(x-t)|\mathrm{d}t}{t} \geq a_k C \log n_k - C_1 \sum_{j=1}^{k-1} a_j \log n_j - 2\log n_k \sum_{j=k+1}^{\infty} a_j.
\tag{3.324}
$$

Now consider the choice

$$
a_k = \frac{1}{k!}, \quad n_k = 2^{(k!)^2};
\tag{3.325}
$$

then,

$$
\int_{n_k^{-1}}^{1} \frac{|g(x+t) - g(x-t)|\mathrm{d}t}{t} \geq \log 2 \left\{ Ck! - C_1 \sum_{j=1}^{k-1} j! - 2(k!)^2 \sum_{j=k+1}^{\infty} \frac{1}{j!} \right\},
\tag{3.326}
$$

and hence

$$
\int_{n_k^{-1}}^{1} \frac{|g(x+t) - g(x-t)|\mathrm{d}t}{t} \geq k! \log 2 \left\{ C - C_1 \sum_{j=1}^{k-1} \frac{j!}{k!} - 2 \sum_{j=k+1}^{\infty} \frac{k!}{j!} \right\}.
\tag{3.327}
$$

In the limit $k \to \infty$, it follows that

$$
\int_{0}^{1} \frac{|g(x+t) - g(x-t)|\mathrm{d}t}{t} > \infty.
\tag{3.328}
$$

It has therefore been proved that there are cases where cancellation effects of the positive and negative contributions are an essential ingredient in obtaining a convergent result for the Hilbert transform.

Notes

§3.2 Levinson and Redheffer (1970) gave a concise readable account on the Poisson formulas. Kober (1971) discussed the Poisson operator P_a,

$$P_a f(x) = \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{(t-x)^2+a^2} = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{t-x-\mathrm{i}a} - \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{t-x+\mathrm{i}a},$$

by taking advantage of properties of the Hilbert transform.

§3.3 For further reading on issues related to the topics covered in this section, see, Weiss (1965), Zygmund (1968), and Koosis (1998). A further refinement on the developments of Section 3.3.1 is a consideration of the behavior of $f(z)$ in the limit $z \to \mathrm{e}^{\mathrm{i}\theta}$ on the boundary of the unit disc. Koosis (1998, chap. 1) gives a detailed discussion on this limit.

§3.4 A proof of the existence of the Hilbert transform, actually a finite Hilbert transform, without recourse to the theory of complex variables, is due to Besicovitch (1926). An extension of Besicovitch's approach to deal with the Stieltjes integral analog is due to Pollard (1926). Wood (1929) established the Hilbert transform pair for functions that satisfy a Lipschitz condition of order α with $0 < \alpha < 1$. The Cauchy paper cited in this section can also be found in his collected works: Cauchy (1958).

§3.6 For further reading on the Cauchy integral, see Roos (1969), Henrici (1986), and Muskhelishvili (1992). More advanced presentations can be found in Bell (1992) and Cima, Matheson, and Ross (2006).

§3.9 A review of Hilbert's contributions to integral equations, including Hilbert transforms on the circle, is given by Hellinger (1935). For some further reading on conjugate functions, see Tauber (1891), Tamarkin (1931), Tsereteli (1977), and Lukashenko (2006).

§3.12 Titchmarsh (1925a, 1948) discusses the conjugate trigonometric integral approach to evaluating Hilbert transform pairs. For a discussion of the convergence of a Fourier series and its allied series, see Young (1911) and Hobson (1926, p. 692).

§3.13 For further reading on conjugate series, see Tauber (1891), Pringsheim (1900), Priwaloff (alternative and more common spelling is Privalov) (1916a, 1916b, 1916c), Hardy (1926, 1937), Hardy and Littlewood (1929), Smirnoff (1929), Titchmarsh (1929), Paley and Wiener (1933), and Zygmund (1934, 1968). A review of some of the early work is given by Hilb and Riesz (1924).

§3.14 For additional reading on the Hilbert transform of periodic functions, see Butzer and Nessel (1971, Chap. 9) and Pandey (1996). For a discussion of the Hilbert transform of periodic functions with arbitrary period (as in Eq. (3.286)), see Pandey (1996) and also Papoulis (1973).

§3.15 The behavior of the integral given in Eq. (3.307) and the second integral in Eq. (3.306) was studied by Lusin (1913) and later by Titchmarsh (1925b), Hardy and Littlewood (1926), Kaczmarz (1931, 1932), and Marcinkiewicz (1936). Good textbook accounts can be found in particular in Zygmund (1968, Vol. I, p. 133), upon which the discussion of the second half of this section in based, and also in Torchinsky (1986, p. 62) and Koosis (1998, p. 25). See also Davis and Chang (1987, p. 34).

Exercises

3.1 Determine the Hilbert transforms $Hf(x)$ for the following choices:

(i) $f(x) = a(x^2 + a^2)^{-1}$;

(ii) $f(x) = \dfrac{x+a}{(x+a)^2 + b^2}$;

(iii) $f(x) = \dfrac{x}{(x^2 + a^2)(x^2 + b^2)}$,

making use of contour integration techniques, and specify the necessary constraints on the constants a and b that appear.

3.2 Calculate the Hilbert transforms of $\sin^{2n} ax$ and $\sin^{2n+1} ax$ for integer $n \geq 0$, where a is a constant. Hence, or otherwise, show that

(i) $H(\sin^5 ax) = -\dfrac{\operatorname{sgn} a}{16}(\cos 5ax - 5\cos 3ax + 10\cos ax)$;

(ii) $H(\sin^6 ax) = -\dfrac{\operatorname{sgn} a}{32}(\sin 6ax - 6\sin 4ax + 15\sin 2ax)$.

3.3 Determine the Hilbert transform of $\cos^n ax$ for integer $n \geq 0$, where a is a constant. Hence, or otherwise, show that

(i) $H(\cos^5 ax) = \dfrac{\operatorname{sgn} a}{16}(\sin 5ax + 5\sin 3ax + 10\sin ax)$;

(ii) $H(\cos^6 ax) = \dfrac{\operatorname{sgn} a}{32}(\sin 6ax + 6\sin 4ax + 15\sin 2ax)$.

3.4 Evaluate the Hilbert transform of the characteristic function $\chi_{[a,b]}$.

3.5 Determine the Hilbert transforms of the following functions:

(i) $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational}; \end{cases}$

(ii) $f(x) = \begin{cases} 0, & x \text{ rational} \\ 1 & x \text{ irrational}; \end{cases}$

(iii) $f(x) = \begin{cases} 0, & & |x| > 1 \\ 1, & x \text{ rational}, & |x| \leq 1 \\ 0, & x \text{ irrational}, & |x| \leq 1, \end{cases}$

(iv) $f(x) = \begin{cases} 0, & & |x| > 1 \\ 1, & x \text{ irrational}, & |x| \leq 1 \\ 0, & x \text{ rational}, & |x| \leq 1. \end{cases}$

3.6 If $P(x,y)$ and $Q(x,y)$ denote the Poisson and conjugate Poisson kernels on $\mathbb{R}$, respectively, show that $HP(t,y)(x) = Q(x,y)$ and $HQ(t,y)(x) = -P(x,y)$.

3.7 Starting with the definition

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\, \mathrm{d}t,$$

show that

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f'(t) \log\left|1 - \frac{x}{t}\right| \mathrm{d}t.$$

3.8 Using the definition of g given in Exercise 3.7, show that

$$g(x) = \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} f(t) \log\left|1 - \frac{x}{t}\right| \mathrm{d}t.$$

3.9 If $f(x)$ is periodic and

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and the conjugate series takes the form

$$g(x) \sim \sum_{n=1}^{\infty} (-a_n \sin ax + b_n \cos nx),$$

show that the following hold for almost all x:

$$g(x) = -\frac{1}{2\pi} P \int_{-\pi}^{\pi} f(t) \cot\left(\frac{x-t}{2}\right) \mathrm{d}t;$$
$$f(x) = \frac{1}{2\pi} P \int_{-\pi}^{\pi} g(t) \cot\left(\frac{x-t}{2}\right) \mathrm{d}t.$$

What additional assumptions are needed on the function f?

3.10 For the following conjugate pairs, decide if both are Fourier series. Take x to be in the interval $0 < x < 2\pi$. Comment on whether the series represent continuous or discontinuous functions.

(i) $\sum_{n=2}^{\infty} \frac{\sin nx}{n \log n}, \sum_{n=2}^{\infty} \frac{\cos nx}{n \log n};$

(ii) $\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}, \sum_{n=2}^{\infty} \frac{\cos nx}{\log n}.$

3.11 Starting with the integral

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi} \{f(s+t) - f(s-t)\} \cot\left(\frac{t}{2}\right) \mathrm{d}t,$$

cast the result in terms of the function $\psi(s,t)$, defined by

$$\psi(s,t) = \int_0^t \{f(s+y) - f(s-y)\} \mathrm{d}y,$$

and hence determine a condition on the function $\psi(s,t)$ in order that the first integral is convergent.

3.12 If $f(z)$ is a rational function that is analytic for $y \geq 0$ or for $y \leq 0$, and $f(z) = O(|z|^{-2})$ as $z \to \infty$, show that $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 0$.

3.13 Let $f(z)$ be a rational function which vanishes at infinity and is analytic for $y \geq 0$. Show that $Hf(x) = -\mathrm{i}f(x)$.

3.14 Evaluate $\mathcal{H}f(x)$ for $f(s) = \sin^2 ns$ with n a positive integer.

3.15 Evaluate $\mathcal{H}f(x)$ for $f(s) = s$.

3.16 For $f(x) = \cos\alpha x$ with α a real constant, evaluate $g_\varepsilon(x) = H_\varepsilon f(x)$ and show that $g_\varepsilon(x) \to \operatorname{sgn}\alpha \sin\alpha x$ as $\varepsilon \to 0+$.

3.17 If $f \in L^p(\mathbb{R})$ for $1 < p < \infty$, show that $g(x) = Hf(x)$ holds for almost all x and $g \in L^p(\mathbb{R})$.

3.18 If $f \in \operatorname{Lip}\alpha$ for $0 < \alpha < 1$ on every bounded interval, and vanishes regularly at infinity, show that $g(x) = Hf(x)$ holds for almost all x and $g \in \operatorname{Lip}\alpha$.

3.19 If $f(x) = (x^2 + \alpha^2)^{-1}$ with $\alpha > 0$, evaluate

$$P\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{s-x} P\int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{s-t}.$$

3.20 If $f(x) = (x^2 + \alpha^2)^{-1}$ with $\alpha > 0$, evaluate

$$\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x} P\int_{-\infty}^{\infty} \log\left|1 - \frac{x}{t}\right| f(t)\mathrm{d}t.$$

3.21 For $\alpha > 0$, evaluate

$$P\int_{-1}^{1} \cot\left[\frac{\pi(s-x)}{2}\right]\mathrm{d}x \, P\int_{-1}^{1} \cot\left[\frac{\pi(x-y)}{2}\right] \sin\alpha y \, \mathrm{d}y.$$

3.22 By choosing a suitable contour integral, determine the Hilbert transform pair arising from the function $f(z) = (z^4 + \alpha^4)^{-1}$ for $\alpha > 0$.

4

Some basic properties of the Hilbert transform

4.1 Introduction

In this chapter, the longest of the book, some of the basic properties of the Hilbert transform that will prove to be of use in later applications, or are of intrinsic interest, are collected together. In the course of examining these properties, a number of strategies for the evaluation of the Hilbert transform of different classes of functions will emerge. For quite a few properties, it is very useful to think of the Hilbert transform as an operator acting on a suitable function. This allows a number of operator-type identities to be written, without focusing on the function. The discussion of one key property, the connection between the Hilbert and Fourier transforms, is postponed until Chapter 5, where the relationship of the Hilbert transform to some of the other common transforms is treated in detail.

4.1.1 Complex conjugation property

The simplest self-evident property is that the Hilbert transform of a real function results in a real function. The Hilbert transform operator commutes with complex conjugation, that is

$$(Hf^*) = (Hf)^*, \tag{4.1}$$

where * denotes the complex conjugate operation.

4.1.2 Linearity

An important property of the Hilbert transform operator is that it is a linear operator. A linear operator L is a mapping from a vector space X into a vector space Y, written $L : X \to Y$, such that for constants $\alpha,\ \beta \in \mathbb{C}$, and functions $f, g \in X$, then

$$L\{\alpha f + \beta g\} = \alpha Lf + \beta Lg. \tag{4.2}$$

For constants $\alpha, \beta \in \mathbb{C}$ and functions f and g, it follows that

$$H\{\alpha f(x) + \beta g(x)\} = \alpha Hf(x) + \beta Hg(x). \tag{4.3}$$

In the preceding result, the separate integrals are assumed to exist, and this is true if the functions belong to the class $L^p(\mathbb{R})$ for $1 \leq p$. The latter condition can be replaced by one where the functions satisfy a suitable asymptotic constraint as $|x| \to \pm\infty$, and are uniformly Hölder continuous on every finite interval of $\mathbb{R}$.

4.2 Hilbert transforms of even or odd functions

In practical applications it is often most convenient to express the Hilbert transforms on $\mathbb{R}^+$. From the definition of the Hilbert transform, it follows that

$$Hf(x) = \frac{1}{\pi} P \int_0^\infty \left\{ \frac{f(y)}{x-y} + \frac{f(-y)}{x+y} \right\} \mathrm{d}y. \tag{4.4}$$

If $f(x)$ is an even function, $f(-x) = f(x)$, then Eq. (4.4) simplifies to

$$Hf(x) = \frac{2x}{\pi} P \int_0^\infty \frac{f(y)}{x^2 - y^2}\, \mathrm{d}y; \tag{4.5}$$

if $f(x)$ is an odd function, $f(-x) = -f(x)$, then

$$Hf(x) = \frac{2}{\pi} P \int_0^\infty \frac{yf(y)}{x^2 - y^2}\, \mathrm{d}y. \tag{4.6}$$

Hence, the Hilbert transform of an even function yields an odd function, and the Hilbert transform of an odd function gives an even function. This is the *parity* property of the Hilbert transform operator. Equations (4.5) and (4.6) are the forms of the Hilbert transforms most commonly seen in a variety of applications, where the symmetry properties of the function, that is its even or odd character, are known. In many applications the variable of interest is a frequency, in which case the interval required is $[0, \infty)$. In this situation Eqs. (4.5) and (4.6) are the most useful results.

In general, $f(x)$ can be expressed in the following form:

$$f(x) = f_\mathrm{e}(x) + f_\mathrm{o}(x), \tag{4.7}$$

where $f_\mathrm{e}(x)$ and $f_\mathrm{o}(x)$ are even and odd functions, defined by

$$f_\mathrm{e}(x) = \frac{1}{2}\{f(x) + f(-x)\}, \tag{4.8}$$

$$f_\mathrm{o}(x) = \frac{1}{2}\{f(x) - f(-x)\}. \tag{4.9}$$

The Hilbert transform of $f(x)$ can then be written as follows:

$$Hf(x) = \frac{2}{\pi} P \int_0^\infty \frac{\{xf_\mathrm{e}(s) + sf_\mathrm{o}(s)\}\mathrm{d}s}{x^2 - s^2}. \tag{4.10}$$

When dealing with the Hilbert transform of even or odd functions, it is convenient to introduce the following notation:

$$H_{\mathrm{e}} f(x) = \frac{2x}{\pi} P \int_0^\infty \frac{f(y)}{x^2 - y^2}\, \mathrm{d}y \tag{4.11}$$

and

$$H_{\mathrm{o}} f(x) = \frac{2}{\pi} P \int_0^\infty \frac{y f(y)}{x^2 - y^2}\, \mathrm{d}y. \tag{4.12}$$

If f is an even function, then $Hf \equiv H_{\mathrm{e}} f$, and if f is an odd function then $Hf \equiv H_{\mathrm{o}} f$. The reader should not confuse these abbreviations with the notation for the truncated Hilbert transform defined in Eq. (3.3). The particular choice of subscript and the context will always make it clear which operator is being discussed. These notational abbreviations will be employed in some of the following sections of this and later chapters.

A situation that arises in practical applications is the following. Suppose f is analytic in the upper half complex plane, vanishes at infinity like $z^{-1-\delta}$ with $\delta > 0$, and on the real axis $f(x) = u(x) + \mathrm{i}v(x)$, with $u \in L^p$, and $v \in L^p$ for $1 < p < \infty$. If $u(x)$ is an even function, then it follows immediately from the even–odd properties of the Hilbert transform that

$$\int_{-\infty}^{\infty} v(x) \mathrm{d}x = \int_{-\infty}^{\infty} Hu(x) \mathrm{d}x = 0. \tag{4.13}$$

Actually, a more general result is known, and this is discussed in Section 4.12.

4.3 Skew-symmetric character of Hilbert transform pairs

The Hilbert transforms of a pair of conjugate functions f and g are as follows:

$$f(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(t)\mathrm{d}t}{x - t} \tag{4.14}$$

and

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{x - t}, \tag{4.15}$$

which are obviously *skew-symmetric*, that is, if $Hf(x) = g(x)$, then $Hg(x) = -f(x)$. This sign change could be eliminated by the substitution $h(x) = f(-x)$; then

$$h(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(t)\mathrm{d}t}{t + x} \tag{4.16}$$

and

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{h(t)\mathrm{d}t}{t+x}. \tag{4.17}$$

The transform pair now has a more symmetric appearance. This form of the transforms is less commonly employed. The formulas $Hf(x) = g(x)$ and $Hg(x) = -f(x)$ are regarded as being *reciprocal relations* of one another, that is, the two equations constitute a reciprocal pair.

4.4 Inversion property

Since $Hf(x) = g(x)$ implies $Hg(x) = -f(x)$, then

$$H^2 f(x) \equiv H(Hf)(x) = -f(x), \ a.e. \tag{4.18}$$

This is referred to as the *inversion formula* for the Hilbert transform, and it is also called the *iteration property* for the Hilbert transform. In the remainder of this book both names will be used synonymously. If the definition of the Hilbert transform had been modified so that the factor π^{-1} was replaced by $(\pi \mathrm{i})^{-1}$, then the minus sign would be removed from the preceding equation. The drawback with this modification is that the Hilbert transform of a real-valued function would become a complex-valued function. The conditions that f must satisfy in order that Eq. (4.18) hold can be stated by anticipating a result that is developed fully in Section 4.20. If $f \in L^p$ for $p > 1$, then $g \equiv Hf \in L^p$. Using this result, it is clear that functions belonging to the class L^p (for $p > 1$) satisfy Eq. (4.18). If an additional assumption is made for the case $p = 1$, that is, if $f \in L$ *and* $Hf \in L$, then functions of the class L are also included. It is not difficult to find examples that do not fit this last assumption.

One approach to Eq. (4.18) is via Fourier transform methods, and this is discussed later in Section 5.2. Another approach follows directly from Eqs. (4.14) and (4.15): this pair of equations being obtained by complex variable techniques along the lines detailed in Chapter 3. Note that a complex variable proof forces additional requirements on the function f. An essential condition is that f is analytic in an appropriate region of the complex plane.

A simplified approach to obtain Eq. (4.18) makes use of the Hardy–Poincaré–Bertrand formula, which takes the following form:

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_1(x)}{x-t} \,\mathrm{d}x \, \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_2(y)}{y-x} \,\mathrm{d}y = \frac{1}{\pi} P \int_{-\infty}^{\infty} \phi_2(y) \mathrm{d}y \, \frac{1}{\pi} \times P \int_{-\infty}^{\infty} \frac{\phi_1(x)}{(x-t)(y-x)} \,\mathrm{d}x - \phi_1(t)\phi_2(t), \tag{4.19}$$

where $\phi_1(x)$ and $\phi_2(x)$ belong to the classes L^p and L^q, respectively, with the exponents satisfying $1 < p < \infty, 1 < q < \infty$, and $p^{-1} + q^{-1} = 1$. Let $\phi_1(x) = \mathrm{e}^{-ax^2}$ with $a > 0$, and set $\phi_2(y) = f(y)$. The function $\phi_1(x)$ is going to be treated as a *convergence factor*. From Eq. (4.19) it follows that

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-ax^2}\,\mathrm{d}x}{x-t}\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(y)}{y-x}\,\mathrm{d}y=\frac{1}{\pi}P\int_{-\infty}^{\infty}f(y)\mathrm{d}y\frac{1}{\pi}$$
$$\times P\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-ax^2}\,\mathrm{d}x}{(x-t)(y-x)}-\mathrm{e}^{-at^2}f(t). \tag{4.20}$$

Now,

$$P\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-ax^2}\mathrm{d}x}{(x-t)(y-x)}=\frac{1}{y-t}P\int_{-\infty}^{\infty}\left\{\frac{\mathrm{e}^{-ax^2}}{x-t}+\frac{\mathrm{e}^{-ax^2}}{y-x}\right\}\mathrm{d}x. \tag{4.21}$$

If the limit $a \to 0+$ is examined, then the last integral evaluates to zero, and Eq. (4.20) becomes

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{t-x}\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(y)}{x-y}\,\mathrm{d}y=-f(t), \tag{4.22}$$

or, in compact notation,

$$HHf(t)=-f(t), \tag{4.23}$$

which is the desired result. The reader is invited to examine critically the validity of taking lim $a \to 0+$ inside the integral in the preceding sequence of steps.

The obvious extension of Eq. (4.18) becomes, for non-negative integer n,

$$H^n f(x)=\begin{cases}(-1)^{n/2}f(x), & \text{for } n \text{ even}\\ (-1)^{(n-1)/2}g(x), & \text{for } n \text{ odd}.\end{cases} \tag{4.24}$$

This can be proved by repeated application of Eq. (4.18).

From Eq. (4.18) the operator equivalence can be written as follows:

$$H^2=-I, \tag{4.25}$$

where I denotes the identity operator. From this result the *inverse* Hilbert transform operator can be written symbolically as

$$H^{-1}=-H, \tag{4.26}$$

and so

$$H^{-1}(Hf)(x)=f(x)=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{Hf(t)\mathrm{d}t}{x-t}. \tag{4.27}$$

4.5 Scale changes

In this section the question of how the Hilbert transform changes with various linear and nonlinear changes of scale is considered. Some of the results are particularly useful to evaluate analytically a number of recalcitrant Hilbert transforms.

4.5.1 Linear scale changes

The following three properties are straightforward to prove. If $g(x) = Hf(x)$, then

$$Hf(ax) = g(ax), \quad a > 0, \tag{4.28}$$

$$Hf(-ax) = -g(-ax), \quad a > 0, \tag{4.29}$$

and

$$Hf(ax+b) = \operatorname{sgn} a\, g(ax+b), \quad b \in \mathbb{R}. \tag{4.30}$$

The last result can be established as follows. Set $h(x) \equiv f(ax+b)$; it follows that

$$\begin{aligned}
Hf(ax+b) \equiv Hh(x) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{h(s)\mathrm{d}s}{x-s} \\
&= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(as+b)\mathrm{d}s}{x-s} \\
&= \begin{cases} \dfrac{1}{\pi} P \displaystyle\int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{ax+b-t}, & \text{for } a > 0 \\ -\dfrac{1}{\pi} P \displaystyle\int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{ax+b-t}, & \text{for } a < 0 \end{cases} \\
&= \operatorname{sgn} a\, g(ax+b).
\end{aligned} \tag{4.31}$$

Equations (4.28) and (4.29) follow directly with the obvious choice of a and b.

4.5.2 Some nonlinear scale transformations for the Hilbert transform

The preceding considerations can be extended to cover some nonlinear changes of scale. Hardy (1908) and Glasser (1984) have discussed some of the results that are presented in this subsection. In the following development, it will be assumed that the Hilbert transform of the function f exists. The function $Hf[\phi(x)]$ is now investigated by starting with the case

$$\phi(s) = s^{-1}. \tag{4.32}$$

First split the integration interval, then

$$\begin{aligned}
Hf[\phi(x)] &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f[\phi(s)]\mathrm{d}s}{x-s} \\
&= \frac{1}{\pi} P \int_{-\infty}^{0} \frac{f[\phi(s)]\mathrm{d}s}{x-s} + \frac{1}{\pi} P \int_{0}^{\infty} \frac{f[\phi(s)]\mathrm{d}s}{x-s},
\end{aligned} \tag{4.33}$$

and, with the change of variable $s^{-1} = t$,

$$Hf[\phi(x)] = \frac{1}{\pi} P \int_{-\infty}^{0} \frac{f(t)\mathrm{d}t}{(tx-1)t} + \frac{1}{\pi} P \int_{0}^{\infty} \frac{f(t)\mathrm{d}t}{(tx-1)t}$$
$$= \frac{1}{\pi} P \int_{-\infty}^{\infty} \left\{ \frac{x}{tx-1} - \frac{1}{t} \right\} f(t)\mathrm{d}t, \tag{4.34}$$

and hence

$$Hf[\phi(x)] = Hf(0) - \{Hf\}[\phi(x)], \quad \text{for } \phi(s) = s^{-1}. \tag{4.35}$$

For the case $\phi(s) = as^{-1}$ with $a > 0$, the preceding argument follows in exactly the same manner and Eq. (4.35) is obtained. For the case that a is a real constant,

$$Hf[\phi(x)] = \operatorname{sgn} a\{Hf(0) - \{Hf\}[\phi(x)]\}. \tag{4.36}$$

For the choice $\phi(s) = a/(s+b)$, with $a > 0$, it follows that

$$Hf[\phi(x)] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f[\phi(s)]\mathrm{d}s}{x-s}$$
$$= \frac{1}{\pi} P \int_{-\infty}^{-b} \frac{f[\phi(s)]\mathrm{d}s}{x-s} + \frac{1}{\pi} P \int_{-b}^{\infty} \frac{f[\phi(s)]\mathrm{d}s}{x-s}, \tag{4.37}$$

and, with the change of variable $\phi(s) = t$,

$$Hf[\phi(x)] = \frac{a}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{(t(x+b)-a)t}, \tag{4.38}$$

and hence Eq. (4.35) follows for this more general choice. If $\phi(s) = a + b/(s+c)$, with $b > 0$, then

$$Hf[\phi(x)] = Hf(a) - \{Hf\}[\phi(x)]. \tag{4.39}$$

Consider the choice

$$\phi(s) = s - s^{-1}; \tag{4.40}$$

then

$$Hf[\phi(x)] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f[\phi(s)]\mathrm{d}s}{x-s}$$
$$= \frac{1}{\pi} P \int_{-\infty}^{0} \frac{f[\phi(s)]\mathrm{d}s}{x-s} + \frac{1}{\pi} P \int_{0}^{\infty} \frac{f[\phi(s)]\mathrm{d}s}{x-s}. \tag{4.41}$$

Using the change of variables $s = t/2 - \tau$ and $s = t/2 + \tau$, with $\tau = \surd(t^2/4 + 1)$, in the first and second integrals on the right-hand side of Eq. (4.41), respectively, then

$$Hf[\phi(x)] = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(t) \left\{ \frac{\psi_1'(t)}{x - \psi_1(t)} + \frac{\psi_2'(t)}{x - \psi_2(t)} \right\} \mathrm{d}t, \tag{4.42}$$

with $\psi_1(t) = t/2 + \tau$ and $\psi_2(t) = t/2 - \tau$. Hence

$$\begin{aligned} Hf[\phi(x)] &= \frac{1}{\pi} P \int_{-\infty}^{\infty} f(t) \left\{ \frac{1/2 + t/4\tau}{x - (t/2 + \tau)} + \frac{1/2 - t/4\tau}{x - (t/2 - \tau)} \right\} \mathrm{d}t \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{x - x^{-1} - t}, \end{aligned} \tag{4.43}$$

and

$$Hf[\phi(x)] = \{Hf\}[\phi(x)]. \tag{4.44}$$

The case $\phi(s) = as - bs^{-1}$, for $a > 0$, $b > 0$, leads in exactly the same manner to Eq. (4.44).

Glasser (1984) gave a further generalization of the latter case. If

$$\phi(s) = s - \sum_{j=1}^{n-1} \frac{a_j}{s - b_j}, \quad \text{for } a_j > 0, \tag{4.45}$$

then Eq. (4.44) holds. Glasser's proof is now sketched. To avoid unnecessary complications, it will be assumed that the evaluation point x does not coincide with any b_j, and, without loss of generality, the b_j are ordered so that $b_j < b_{j+1}$. From Eq. (4.45) it follows that

$$\{\phi(s) - s\} \prod_{k=1}^{n-1} (s - b_k) = -\sum_{j=1}^{n-1} a_j \prod_{\substack{k=1 \\ (k \neq j)}}^{n-1} (s - b_k). \tag{4.46}$$

The function $G(s, u)$ is defined by

$$G(s, u) = (s - u) \prod_{k=1}^{n-1} (s - b_k) - \sum_{j=1}^{n-1} a_j \prod_{\substack{k=1 \\ (k \neq j)}}^{n-1} (s - b_k). \tag{4.47}$$

Equation (4.47) is an nth degree polynomial in s, which can be rewritten as

$$G(s, u) = \prod_{k=1}^{n} (s - s_k(u)), \tag{4.48}$$

where $s_k(u)$ are the roots of the polynomial. From Eqs. (4.46)–(4.48)

$$\prod_{k=1}^{n} (s - s_k(u)) \equiv (\phi(s) - u) \prod_{k=1}^{n-1} (s - b_k), \tag{4.49}$$

and it follows that

$$\log\left\{\prod_{k=1}^{n} (s - s_k(u))\right\} = \log\{\phi(s) - u\} + \log\left\{\prod_{k=1}^{n-1} (s - b_k)\right\}, \tag{4.50}$$

and hence

$$\sum_{k=1}^{n} \log(s - s_k(u)) = \log\{\phi(s) - u\} + \sum_{k=1}^{n-1} \log(s - b_k). \tag{4.51}$$

Differentiating with respect to u yields

$$\sum_{k=1}^{n} \frac{s_k'(u)}{s - s_k(u)} = \frac{1}{\phi(s) - u}, \tag{4.52}$$

which is a key result needed in the next step. It may be helpful for the reader to examine a concrete example. If $u = s - a_1(s - b_1)^{-1}$, with $a_1 = 1$ and $b_1 = 2$, then the roots are ≈ -0.4142 and 2.4142, and the functions $s_k(u)$ are as follows: $s_1(u) = (1/2)\left\{u + b_1 - \sqrt{[(u - b_1)^2 + 4a_1]}\right\}$ and $s_2(u) = (1/2)\left\{u + b_1 + \sqrt{[(u - b_1)^2 + 4a_1]}\right\}$. These two functions are illustrated in Figure 4.1.

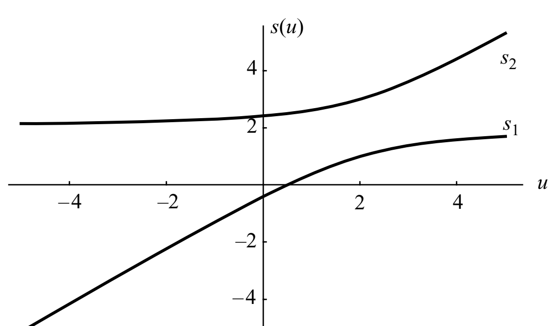

Figure 4.1. The two branches $s_1(u)$ and $s_2(u)$.

The main part of the derivation proceeds as follows. Start by writing

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f[\phi(t)]\mathrm{d}t}{x-t} = \frac{1}{\pi}\left\{P\int_{-\infty}^{b_1^-} + P\int_{b_1^+}^{b_2^-} + P\int_{b_2^+}^{b_3^-} + \cdots + P\int_{b_{n-1}^+}^{\infty}\right\}\frac{f[\phi(t)]\mathrm{d}t}{x-t}, \tag{4.53}$$

where b_j^- signifies the approach to b_j from the left and b_j^+ denotes the approach to b_j from the right. On introducing the change of variable $u = \phi(t)$,

$$\begin{aligned}\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f[\phi(t)]\mathrm{d}t}{x-t} &= \frac{1}{\pi}\left\{P\int_{-\infty}^{\infty}\frac{f(u)s_1'(u)\mathrm{d}u}{x-s_1(u)} + P\int_{-\infty}^{\infty}\frac{f(u)s_2'(u)\mathrm{d}u}{x-s_2(u)}\right.\\ &\quad\left. + P\int_{-\infty}^{\infty}\frac{f(u)s_3'(u)\mathrm{d}u}{x-s_3(u)} + \cdots + P\int_{-\infty}^{\infty}\frac{f(u)s_n'(u)\mathrm{d}u}{x-s_n(u)}\right\}\\ &= \frac{1}{\pi}P\int_{-\infty}^{\infty}\sum_{k=1}^{n}\frac{s_k'(u)f(u)\mathrm{d}u}{x-s_k(u)}\\ &= \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(u)\mathrm{d}u}{\phi(x)-u},\end{aligned} \tag{4.54}$$

which establishes Eq. (4.44).

A few applications of the key formulas just derived are now examined. From the result $H(\mathrm{e}^{\mathrm{i}ax}) = -\mathrm{i}\mathrm{e}^{\mathrm{i}ax}$ for $a > 0$, the substitution $x \to (\alpha x/a) - (\beta/ax)$ with $\alpha > 0$ and $\beta > 0$ leads to

$$H(\mathrm{e}^{\mathrm{i}(\alpha x-\beta x^{-1})}) = -\mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha x-\beta x^{-1})}. \tag{4.55}$$

On taking the real and imaginary parts,

$$H\{\cos(\alpha x-\beta x^{-1})\} = \sin(\alpha x-\beta x^{-1}), \tag{4.56}$$

and

$$H\{\sin(\alpha x-\beta x^{-1})\} = -\cos(\alpha x-\beta x^{-1}). \tag{4.57}$$

These results were given by Hardy (1901, 1908). As a second example, consider $H\{x(x^2+a^2)^{-1}\} = -a(x^2+a^2)^{-1}$ for $\mathrm{Re}\, a > 0$; then the substitution $x \to \alpha x-\beta x^{-1}$ with $\alpha > 0$, and $\beta > 0$ leads to

$$H\left[\frac{\alpha x^3-\beta x}{\alpha^2x^4+(a^2-2\alpha\beta)x^2+\beta^2}\right] = -\frac{\alpha x^2}{\alpha^2x^4+(a^2-2\alpha\beta)x^2+\beta^2}, \tag{4.58}$$

which can be evaluated in an alternative fashion using contour integration techniques or by resorting to a partial fraction expansion of the starting function. As a third example, consider the evaluation of $H(\sin\{a(x+b)^{-1}\})$ by making use of

Eq. (4.36). Start with the result $H(\mathrm{e}^{\mathrm{i}ax}) = -\mathrm{i}\mathrm{e}^{\mathrm{i}ax}$ for $a > 0$, and use the substitution $x \to (x+b)^{-1}$; then

$$\begin{aligned} H(\mathrm{e}^{\mathrm{i}a(x+b)^{-1}}) &= \{-\mathrm{i}\mathrm{e}^{\mathrm{i}ax}\}_{x=0} - \{-\mathrm{i}\mathrm{e}^{\mathrm{i}ax}\}_{x\to(x+b)^{-1}} \\ &= \mathrm{i}(\mathrm{e}^{\mathrm{i}a(x+b)^{-1}} - 1), \end{aligned} \tag{4.59}$$

which leads, on taking the real and imaginary parts, to the results for $x \neq -b$:

$$H(\cos\{a(x+b)^{-1}\}) = -\sin\{a(x+b)^{-1}\} \tag{4.60}$$

and

$$H(\sin\{a(x+b)^{-1}\}) = \cos\{a(x+b)^{-1}\} - 1. \tag{4.61}$$

This section is concluded with an observation of Glasser (1984), who gave the following result:

$$Hf(x - \cot x^{-1}) = Hf(0) - \{Hf\}(x - \cot x^{-1}) \tag{4.62}$$

for continuous functions for which the Hilbert transform is defined. This can be proved by first noting that Eq. (4.35) holds for $\phi(s) = \sum_{j=1}^{n} a_j/(s+b_j)$, with $a_j > 0$. The proof of this result runs in a very similar fashion to the approach shown in Eqs. (4.45)–(4.54). If the expansion

$$\begin{aligned} x - \cot x^{-1} &= 2x\sum_{j=1}^{\infty}\frac{1}{\pi^2 j^2 x^2 - 1} \\ &= \lim_{n\to\infty}\frac{1}{\pi^2}\sum_{j=1}^{n}\frac{1}{j^2}\left\{\frac{1}{x - j^{-1}\pi^{-1}} + \frac{1}{x + j^{-1}\pi^{-1}}\right\} \end{aligned} \tag{4.63}$$

is employed, then Eq. (4.62) follows. The details are left as an exercise for the reader.

4.6 Translation, dilation, and reflection operators

Set $g = Hf$ and let τ_a denote an operator that produces a translation by an amount a, that is

$$\tau_a f(x) = f(x-a), \quad \text{for } a \in \mathbb{R}, \tag{4.64}$$

then the Hilbert transform operator and the translation operator τ_a commute so that

$$H\tau_a f(x) = \tau_a Hf(x) = g(x-a). \tag{4.65}$$

Recall from Section 2.10 that the commutator for two operators α and β is defined by

$$[\alpha, \beta] = \alpha\beta - \beta\alpha. \tag{4.66}$$

The *anti-commutator* is defined as

$$\{\alpha, \beta\} = \alpha\beta + \beta\alpha. \tag{4.67}$$

Therefore, Eq. (4.65) can be recast in operator notation as

$$[H, \tau_a] = 0. \tag{4.68}$$

An operator T which satisfies

$$T\tau_a f(x) = \tau_a Tf(x) \tag{4.69}$$

is termed a *translation-invariant* operator. Clearly the Hilbert transform operator is an example of such an operator.

Let S_a denote the homothetic operator that produces a stretching (or shrinkage) of the coordinates, that is

$$S_a f(x) = f(ax), \quad \text{for } a > 0. \tag{4.70}$$

The operator S_a is also referred to as the *dilation* operator. The Hilbert transform operator commutes with S_a so that

$$HS_a f(x) = S_a Hf(x) = g(ax). \tag{4.71}$$

In operator notation this takes the form

$$[H, S_a] = 0. \tag{4.72}$$

Let R denote the reflection operator, defined by

$$Rg(x) = g(-x), \tag{4.73}$$

then

$$HRf(x) = -RHf(x) = -g(-x). \tag{4.74}$$

That is, the Hilbert transform operator anti-commutes with the reflection operator. In operator notation,

$$\{H, R\} = 0. \tag{4.75}$$

The translation operator is isometric; that is, for $f \in L^p(\mathbb{R})$, then

$$\|\tau_a f\|_p = \|f\|_p . \tag{4.76}$$

A slight modification of the definition of the dilation operator can be made by writing for functions in $L^p(\mathbb{R})$,

$$S'_a g(x) = a^{p^{-1}} g(ax), \quad \text{for } a > 0. \tag{4.77}$$

This modified definition of the dilation operator leads to

$$\left\|S'_a f\right\|_p = \|f\|_p . \tag{4.78}$$

An operator that commutes with the translation operator must be of the form of a difference operator. If $\mathcal{O}$ denotes the integral operator defined by

$$\mathcal{O}f(x) = P\int_{-\infty}^{\infty} k(x,y)f(y)\mathrm{d}y, \quad \text{for } x \in \mathbb{R}, \tag{4.79}$$

and $[\mathcal{O}, \tau_a] = 0$, then the kernel function $k(x,y)$ is a function of the difference coordinate $x - y$, that is,

$$k(x,y) \equiv g(x-y). \tag{4.80}$$

If $\mathcal{O}$ commutes with the dilation operator, $[\mathcal{O}, S_a] = 0$, then

$$k(ax,ay) = a^{-1}k(x,y), \quad \text{for } a > 0. \tag{4.81}$$

Clearly both these conditions hold when $\mathcal{O}$ is the Hilbert transform operator.

The translation and dilation properties are important for characterizing the Hilbert transform operator. If $1 < p < \infty$ and the bounded linear operator $\mathcal{O}$ acts on functions of the class $L^p(\mathbb{R})$ and commutes with both translations and positive dilations, then

$$\mathcal{O} = aI + bH, \tag{4.82}$$

where I is the identity operator and a and b are constants. The reader is reminded that an operator $\mathcal{O}$ in a linear space $\mathcal{L}$, whose domain is $D_{\mathcal{O}} \subset \mathcal{L}$, is bounded iff

$$\|\mathcal{O}f\| \le M\,\|f\|, \quad \text{for all } f \in D_{\mathcal{O}}, \tag{4.83}$$

where M is a non-negative constant. For functions $f \in L^p(\mathbb{R})$, the p-norm version of the last result is

$$\|\mathcal{O}f\|_p \le M\,\|f\|_p . \tag{4.84}$$

The Hilbert transform operator is a bounded operator for $p > 1$; this is proved in Section 4.20. The norm of a bounded linear operator, denoted $\|\mathcal{O}\|$, is equal to the greatest lower bound of all M (the inf) for which Eq. (4.83) holds. For functions in $L^2(\mathbb{R})$ and $\mathcal{O}$ equal to the Hilbert transform operator, Eq. (4.84) is an equality with $M = 1$ (a result discussed in Section 4.10), and so

$$\|H\|_2 = 1. \tag{4.85}$$

The proof of Eq. (4.82) runs along the followings lines (Stein (1970); see also McLean and Elliott (1988)). The following result is established in Section 5.2:

$$\mathcal{F}Hf(x) = -\mathrm{i}\,\mathrm{sgn}\,x\,\mathcal{F}f(x), \tag{4.86}$$

where $\mathcal{F}$ denotes the Fourier transform. Suppose $\mathcal{O}$ is a bounded operator on $L^2(\mathbb{R})$ that commutes with translations and positive dilations and anti-commutes with reflection and negative dilations. A key starting result is

$$\mathcal{F}\mathcal{O}f(x) = m(x)\mathcal{F}f(x). \tag{4.87}$$

If T is a continuous linear translation-invariant operator, then, for an appropriately selected function $h(x)$,the following holds:

$$Tf(x) = h(x) * f(x), \tag{4.88}$$

which is a basic result from the theory of multipliers. If the Fourier transform of this last result is taken, and making use of the standard property for the Fourier transform of a convolution leads to

$$\mathcal{F}Tf(x) = \mathcal{F}h(x)\mathcal{F}f(x). \tag{4.89}$$

On making the identifications $\mathcal{O} = T$ and $m(x) = \mathcal{F}h(x)$, Eq. (4.87) is obtained. The function $m(x)$ is called a *multiplier*. Let us set out to find its form, and along the way obtain a number of connections involving the Fourier transform operator and the dilation operator. For $a > 0$ it follows that

$$\begin{aligned}\mathcal{F}S_a f(x) &= \int_{-\infty}^{\infty} f(ay)\mathrm{e}^{-\mathrm{i}xy}\,\mathrm{d}y \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(y)\mathrm{e}^{-\mathrm{i}xya^{-1}}\,\mathrm{d}y \\ &= a^{-1}S_{a^{-1}}\mathcal{F}f(x).\end{aligned} \tag{4.90}$$

For $a < 0$, put $a = -\alpha$ with $\alpha > 0$; then

$$\begin{aligned}\mathcal{F}S_a f(x) &= \int_{-\infty}^{\infty} f(-\alpha y)\mathrm{e}^{-\mathrm{i}xy}\,\mathrm{d}y \\ &= -a^{-1}S_{a^{-1}}\mathcal{F}f(x),\end{aligned} \tag{4.91}$$

and so

$$\mathcal{F}S_a f(x) = |a|^{-1} S_{a^{-1}} \mathcal{F}f(x). \tag{4.92}$$

That is,

$$\mathcal{F}S_a = |a|^{-1} S_{a^{-1}} \mathcal{F}. \tag{4.93}$$

From Eq. (4.87), it follows that

$$m = \mathcal{F}\mathcal{O}\mathcal{F}^{-1} \tag{4.94}$$

and hence that

$$S_a m = S_a \mathcal{F}\mathcal{O}\mathcal{F}^{-1}. \tag{4.95}$$

From Eq. (4.93),

$$|a|^{-1}\mathcal{F}S_{a^{-1}} = S_a \mathcal{F}, \tag{4.96}$$

and so

$$S_a m = |a|^{-1}\mathcal{F}S_{a^{-1}}\mathcal{O}\mathcal{F}^{-1}. \tag{4.97}$$

Now, from the stated assumptions on $\mathcal{O}$,

$$\mathcal{O}S_a = \operatorname{sgn} a\, S_a \mathcal{O}, \tag{4.98}$$

and hence

$$\frac{a^{-1}}{\left|a^{-1}\right|}\mathcal{O}S_{a^{-1}} = S_{a^{-1}}\mathcal{O}. \tag{4.99}$$

Therefore,

$$S_a m = a^{-1}\mathcal{F}\mathcal{O}S_{a^{-1}}\mathcal{F}^{-1}. \tag{4.100}$$

Now, from Eq. (4.93) it follows that

$$|a|\mathcal{F}^{-1}S_a = S_{a^{-1}}\mathcal{F}^{-1}, \tag{4.101}$$

which, on insertion into Eq. (4.100), gives

$$S_a m = \operatorname{sgn} a\, \mathcal{F}\mathcal{O}\mathcal{F}^{-1}S_a. \tag{4.102}$$

So, from Eq. (4.94) it follows that

$$S_a m(x) = \operatorname{sgn} a\; m(x) S_a, \tag{4.103}$$

that is,

$$m(ax) = \operatorname{sgn} a\, m(x) S_a. \tag{4.104}$$

Applying this operator identity to the unit function leads to the following equation:

$$m(ax) = \operatorname{sgn} a\, m(x). \tag{4.105}$$

If only positive dilations are considered, the solution of this equation is given by

$$m(x) = \alpha + \beta \operatorname{sgn} x, \tag{4.106}$$

for α and β constants. From Eqs. (4.94) and (4.106),

$$\mathcal{O} f(x) = \alpha f(x) + \beta \mathcal{F}^{-1}\{\operatorname{sgn} w\, \mathcal{F} f(w)\}(x), \tag{4.107}$$

which, on comparison with Eq. (4.86), yields Eq. (4.82). If the condition that $\mathcal{O}$ also anti-commutes with the reflection operator is included, then the case $a = -1$ is allowed, hence

$$m(x) = \beta \operatorname{sgn} x \tag{4.108}$$

is the solution of Eq. (4.105), and the operator $\mathcal{O}$ is given by

$$\mathcal{O} = bH. \tag{4.109}$$

If the assumption that $\mathcal{O}$ commutes with positive dilations is modified to include both positive and negative dilations, then the only solution of Eq. (4.105) is

$$m(x) = \alpha. \tag{4.110}$$

To summarize: it has been shown that a constant multiple of the identity operator plus a constant multiple of the Hilbert transform operator is the only bounded linear operator on $L^p(\mathbb{R})$ that commutes with translations and positive dilations, and a constant multiple of the Hilbert transform operator is the only operator that commutes with translations and positive dilations and reflection. The only bounded linear operator on $L^p(\mathbb{R})$ that commutes with both translations and positive and negative dilations is a constant multiple of the identity operator.

4.7 The Hilbert transform of the product $x^n f(x)$

If $g(x) = Hf(x)$, then

$$H\{xf(x)\} = xg(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\mathrm{d}t. \tag{4.111}$$

This result is obtained in a straightforward manner:

$$\begin{aligned} H\{xf(x)\} &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{tf(t)}{x-t}\,\mathrm{d}t \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\{x-(x-t)\}f(t)}{x-t}\,\mathrm{d}t \\ &= \frac{x}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,\mathrm{d}t - \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\mathrm{d}t, \end{aligned} \tag{4.112}$$

and hence Eq. (4.111) follows. The result generalizes for integer $n \geq 0$ as follows:

$$H\{x^n f(x)\} = x^n g(x) - \frac{1}{\pi}\sum_{k=0}^{n-1} x^k \int_{-\infty}^{\infty} t^{n-1-k} f(t)\mathrm{d}t, \tag{4.113}$$

and it is assumed that all the moments $\int_{-\infty}^{\infty} t^m f(t)\mathrm{d}t$ required for the evaluation of the sum converge. Equation (4.113) will be referred to as the *moment formula* for the Hilbert transform. If $x^n f(x) \in L^p(\mathbb{R}),\ 1 < p < \infty$, then the moment formula holds. The preceding result is obtained in the following manner:

$$\begin{aligned} H\{x^n f(x)\} &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{t^n f(t)}{x-t}\,\mathrm{d}t \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\{x^n[(t/x)^n - 1] + x^n\}f(t)}{x-t}\,\mathrm{d}t \\ &= x^n g(x) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^n(t x^{-1} - 1)\sum\limits_{j=0}^{n-1}(t/x)^j f(t)}{x-t}\,\mathrm{d}t \\ &= x^n g(x) - \frac{1}{\pi}\sum_{j=0}^{n-1} x^{n-j-1}\int_{-\infty}^{\infty} t^j f(t)\mathrm{d}t, \end{aligned} \tag{4.114}$$

and, with the change of summation index $k = n - j - 1$, the result in Eq. (4.113) is obtained. In Eq. (4.113) the standard summation convention,

$$\sum_{k=m}^{p} h_k = 0, \text{ for } m > p, \tag{4.115}$$

is assumed. As an example, consider the evaluation of $H\{x^{2n}\mathrm{e}^{-ax^2}\}$ for n a non-negative integer and $a > 0$. Setting $G(a,x) = H(\mathrm{e}^{-ax^2})$ and applying Eq. (4.113)

leads to

$$H\{x^{2n}\mathrm{e}^{-ax^2}\} = x^{2n}G(a,x) - \frac{1}{\pi}\sum_{k=0}^{2n-1} x^k \int_{-\infty}^{\infty} t^{2n-1-k}\mathrm{e}^{-at^2}\,\mathrm{d}t$$
$$= x^{2n}G(a,x) - \frac{1}{\pi}\sum_{j=0}^{n-1} x^{2j+1} \int_{-\infty}^{\infty} t^{2(n-1-j)}\mathrm{e}^{-at^2}\,\mathrm{d}t, \tag{4.116}$$

where the summation index substitution $k = 2j + 1$ has been employed and the fact that the integral is zero for even values of k has been utilized. The result can be written as follows:

$$H\{x^{2n}\mathrm{e}^{-ax^2}\} = x^{2n}G(a,x) - \frac{2}{\pi}\sum_{k=0}^{n-1} x^{2n-2k-1} \int_{0}^{\infty} t^{2k}\mathrm{e}^{-at^2}\,\mathrm{d}t$$
$$= x^{2n}G(a,x) - \frac{1}{\pi}\sum_{k=0}^{n-1} x^{2n-2k-1} \frac{\Gamma(k+1/2)}{a^{k+1/2}}. \tag{4.117}$$

In this derivation $\Gamma(m)$ denotes the gamma function, defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t, \quad \text{for } \mathrm{Re}\, z > 0. \tag{4.118}$$

The gamma function appearing in Eq. (4.117) can be expressed in terms of the double factorial function as follows:

$$\frac{2^m\Gamma(m+1/2)}{\sqrt{\pi}} = (2m-1)!! = 1\cdot 3\cdot 5\cdot\cdots\cdot(2m-1), \quad \text{for } m \geq 1, \tag{4.119}$$

and, for completeness, $m!!$ is given by

$$(2m)!! = 2\cdot 4\cdot 6\cdot\cdots\cdot(2m) = 2^m m!. \tag{4.120}$$

The determination of $G(a,x)$ is considered shortly; the result will be given in Eqs. (4.148), (5.28), (5.29), and (5.33).

The extension of Eq. (4.113) to the case of negative powers of x is now considered. A function belongs to the class $L^1_{\mathrm{loc}}(\mathbb{R})$, written $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, if, $\forall a < b$,

$$\int_a^b |f(x)|\mathrm{d}x < \infty. \tag{4.121}$$

Suppose $x^{-1}f(x) \in L^1_{\mathrm{loc}}(\mathbb{R})$; from Eq. (4.111) it follows that

$$xHh(x) = H\{xh(x)\} + \frac{1}{\pi}\int_{-\infty}^{\infty} h(t)\mathrm{d}t, \tag{4.122}$$

and introducing $h(t) = t^{-1}f(t)$ leads to

$$xH\left\{\frac{f(x)}{x}\right\} = Hf(x) + \frac{1}{\pi}\int_{-\infty}^{\infty} t^{-1}f(t)\mathrm{d}t. \tag{4.123}$$

This result can be rewritten as follows:

$$H\left\{\frac{f(x)}{x}\right\} = \frac{Hf(x) - Hf(0)}{x}. \tag{4.124}$$

In a similar manner, suppose $x^{-2}f(x) \in L^1_{\mathrm{loc}}(\mathbb{R})$; from Eq. (4.113)

$$H\left\{x^2h(x)\right\} = x^2Hh(x) - \frac{x}{\pi}\int_{-\infty}^{\infty} h(t)\mathrm{d}t - \frac{1}{\pi}\int_{-\infty}^{\infty} th(t)\mathrm{d}t, \tag{4.125}$$

which, on employing the substitution $h(t) = t^{-2}f(t)$, yields

$$x^2H\left\{\frac{f(x)}{x^2}\right\} = Hf(x) + \frac{x}{\pi}\int_{-\infty}^{\infty} t^{-2}f(t)\mathrm{d}t + \frac{1}{\pi}\int_{-\infty}^{\infty} t^{-1}f(t)\mathrm{d}t. \tag{4.126}$$

This result can be recast as follows:

$$H\left\{\frac{f(x)}{x^2}\right\} = \frac{Hf(x) - xH\{t^{-1}f(t)\}(0) - Hf(0)}{x^2}. \tag{4.127}$$

A pair of examples will illustrate the application of Eqs. (4.124) and (4.127). If $f(x) = \sin ax$ for $a > 0$, then

$$\begin{aligned} xH\left(\frac{\sin ax}{x}\right) &= H(\sin ax) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ax\,\mathrm{d}x}{x} \\ &= -\cos ax + 1, \end{aligned} \tag{4.128}$$

and hence

$$H\left(\frac{\sin ax}{x}\right) = \frac{1 - \cos ax}{x}. \tag{4.129}$$

To evaluate $H(x^{-2}\sin^2 ax)$ for $a > 0$, set $f(x) = \sin^2 ax$; then

$$\begin{aligned} x^2H\left(\frac{\sin^2 ax}{x^2}\right) &= H(\sin^2 ax) + \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 ax\,\mathrm{d}x}{x^2} \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 ax\,\mathrm{d}x}{x}. \end{aligned} \tag{4.130}$$

The last integral on the right-hand side of Eq. (4.130) is zero (the integrand is an odd function) and the other integral simplifies using integration by parts to yield

$$\int_{-\infty}^{\infty} \frac{\sin^2 ax \, dx}{x^2} = a \int_{-\infty}^{\infty} \frac{\sin 2ax \, dx}{x} = \pi a. \tag{4.131}$$

Making use of $H(\sin^2 ax) = -(1/2) \sin 2ax$ leads to

$$H\left(\frac{\sin^2 ax}{x^2}\right) = \frac{2ax - \sin 2ax}{2x^2}. \tag{4.132}$$

In a similar fashion to the development of Eq. (4.127), if $x^{-3} f(x) \in L^1_{\text{loc}}(\mathbb{R})$, then

$$x^3 H\left\{\frac{f(x)}{x^3}\right\} = Hf(x) + \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \left\{\frac{1}{t} + \frac{x}{t^2} + \frac{x^2}{t^3}\right\} dt. \tag{4.133}$$

This result can be rewritten as follows:

$$H\left\{\frac{f(x)}{x^3}\right\} = \frac{Hf(x) - Hf(0) - xH\{t^{-1}f(t)\}(0) - x^2 H\{t^{-2}f(t)\}(0)}{x^3}, \tag{4.134}$$

assuming each of the separate integrals exist. For further discussion on related results, see Section 4.19. As an exercise, the reader might try the evaluation of $H\left(\sin^3 ax/x^3\right)$ for $a > 0$.

The Hilbert transform of the product $(x + a)f(x)$ can be simplified to yield

$$\begin{aligned} H\{(x + a)f(x)\} &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{(t + a)f(t)}{x - t} dt \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\{x + a - (x - t)\}f(t)}{x - t} dt, \end{aligned} \tag{4.135}$$

and hence

$$H\{(x + a)f(x)\} = (x + a)g(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) dt. \tag{4.136}$$

The generalization $H\{(x + a)^n f(x)\}$ for integer $n \geq 0$ is straightforward and is left as an exercise for the reader.

4.8 The Hilbert transform of derivatives

If $g(x) = Hf(x)$, then

$$\frac{dg(x)}{dx} = H\left\{\frac{df(x)}{dx}\right\}. \tag{4.137}$$

This result can be verified by performing integration by parts on the right-hand side of Eq. (4.137) and comparing the result with the Leibnitz derivative of $Hf(x)$. Equation (4.137) is a statement that the Hilbert transform operator commutes with the differential operator. For the latter calculation, the interchange of derivative and integral is justified if the integrals $\int_{-\infty}^{x-\varepsilon}(\partial h(x,y)/\partial x)\mathrm{d}y$ and $\int_{x+\varepsilon}^{\infty}(\partial h(x,y)/\partial x)\mathrm{d}y$ converge uniformly, where $h(x,y) = (x-y)^{-1}f(y)$. The generalization of Eq. (4.137) is as follows:

$$\frac{\mathrm{d}^n g(x)}{\mathrm{d}x} = H\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\}. \tag{4.138}$$

Some examples are now considered. The Hilbert transform of $a(a^2+x^2)^{-1}$ for $a > 0$ is $x(a^2+x^2)^{-1}$. Differentiating both functions leads to the following result:

$$H\left\{\frac{2ax}{(x^2+a^2)^2}\right\} = \frac{x^2-a^2}{(x^2+a^2)^2}. \tag{4.139}$$

Suppose the Hilbert transform of sech ax tanh ax, where a is a real constant, is to be evaluated. Noting that this function is the derivative of $-a^{-1}$ sech ax, and using the result

$$H(\operatorname{sech} ax) = 8ax\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2\pi^2+4a^2x^2}, \tag{4.140}$$

which is derived in Section 4.13, it follows that, on taking the derivative of Eq. (4.140),

$$H(\operatorname{sech} ax \tanh ax) = -8\sum_{k=0}^{\infty}\frac{(-1)^k[(2k+1)^2\pi^2-4a^2x^2]}{[(2k+1)^2\pi^2+4a^2x^2]^2}. \tag{4.141}$$

Convergence accelerator techniques would prove to be a valuable technique for the numerical calculation of the alternating series in Eq. (4.141). Recall from Section 2.16 that convergence accelerator techniques avoid the direct numerical evaluation of the sum; instead, these methods take into consideration the asymptotic behavior of the series or of sums of terms in the series (or try to approximate that behavior). This leads to algorithms that provide a much faster method of numerical evaluation for slowly converging series. In summing alternating series by convergence accelerator techniques, a very good choice is to select a convergence accelerator incorporating an embedded alternating sign; for example, the Levin (1973) transform. A cancellation of signs is thereby obtained, and this minimizes problems associated with numerical round-off. The end-notes to Chapter 2 provide some useful references to pursue on this topic.

The Hilbert transform of $\sin^{2m} ax \cos ax$, where a is a real constant, can be evaluated most directly by noting that this function is the derivative of $[(2m+1)a]^{-1}\sin^{2m+1} ax$. The Hilbert transform of $\sin^{2m+1} ax$ can be readily evaluated by using the series expansion for this function, the result is as follows:

$$H(\sin^{2m+1} ax) = \frac{\operatorname{sgn} a}{4^m}\sum_{k=0}^{m}(-1)^{m+k+1}\binom{2m+1}{k}\cos[(2m-2k+1)ax], \tag{4.142}$$

where $\begin{pmatrix} m \\ n \end{pmatrix}$ denotes a binomial coefficient. Hence, differentiating this expression leads to

$$H(\sin^{2m} ax \cos ax) = \frac{\operatorname{sgn} a}{4^m(2m+1)} \sum_{k=0}^{m} (-1)^{m+k} \begin{pmatrix} 2m+1 \\ k \end{pmatrix} (2m-2k+1) \times \sin[(2m-2k+1)ax]. \tag{4.143}$$

The following example utilizes both the derivative property of the Hilbert transform and a result from the preceding section. The Hilbert transform of the Gaussian function $f(x) = \mathrm{e}^{-ax^2}$ for $a > 0$ is the next order of business. Let $g(x) = Hf(x)$; then

$$\begin{aligned} g'(x) &= \frac{\mathrm{d}H(\mathrm{e}^{-ax^2})}{\mathrm{d}x} \\ &= -2aH(x\mathrm{e}^{-ax^2}) \\ &= -2a\left\{xH(\mathrm{e}^{-ax^2}) - \pi^{-1}\int_{-\infty}^{\infty} \mathrm{e}^{-at^2}\,\mathrm{d}t\right\}, \end{aligned} \tag{4.144}$$

where Eq. (4.111) has been used to obtain the last result. Hence

$$g'(x) + 2axg(x) = 2\sqrt{\left(\frac{a}{\pi}\right)}. \tag{4.145}$$

Multiplying through by e^{ax^2} transforms this equation into the following:

$$\frac{\mathrm{d}\{\mathrm{e}^{ax^2} g(x)\}}{\mathrm{d}x} = 2\sqrt{\left(\frac{a}{\pi}\right)}\mathrm{e}^{ax^2}. \tag{4.146}$$

This allows the solution to be written as follows:

$$g(x) = \mathrm{e}^{-ax^2}\left\{c + 2\sqrt{\left(\frac{a}{\pi}\right)}\int_0^x \mathrm{e}^{at^2}\,\mathrm{d}t\right\}, \tag{4.147}$$

where c is an arbitrary constant. Since $f(x)$ is an even function in this example, $g(x)$ must be an odd function (recall Section 4.2), and hence $c = 0$; therefore

$$H(\mathrm{e}^{-ax^2}) = 2\sqrt{\left(\frac{a}{\pi}\right)}\mathrm{e}^{-ax^2}\int_0^x \mathrm{e}^{at^2}\,\mathrm{d}t. \tag{4.148}$$

The integral appearing in this result can be expressed in terms of the error function with a complex argument; this is discussed in Section 5.2 (see Eqs. (5.27)–(5.28)), where this example is revisited using a different computational technique.

The preceding approach can also be applied to deal with a generalization of the Gaussian function. Let

$$f(x) = \mathrm{e}^{-ax^{2n}}, \quad \text{for } n = 1, 2, \ldots, \text{ and } a > 0. \tag{4.149}$$

Employing the definitions

$$M_{nk}(a) = 2n \int_0^\infty t^{2k} \mathrm{e}^{-at^{2n}} \, \mathrm{d}t = \frac{\Gamma((2k+1)/2n)}{a^{(2k+1)/2n}} \tag{4.150}$$

and

$$E_{nk}(a,x) = \int_0^x t^{2k} \mathrm{e}^{at^{2n}} \, \mathrm{d}t \tag{4.151}$$

leads to

$$Hf(x) = \frac{2a}{\pi} \mathrm{e}^{-ax^{2n}} \sum_{k=0}^{n-1} M_{nk}(a) E_{n(n-k-1)}(a,x). \tag{4.152}$$

This result is left as an exercise for the reader to confirm.

The Hilbert transform of a Gaussian is closely tied to the plasma dispersion function, which is defined by

$$Z(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-y^2} \, \mathrm{d}y}{y - a}, \quad \text{with } \operatorname{Im} a > 0. \tag{4.153}$$

The reader interested in this function can find further discussion in Fried and Conte (1961).

Examination of the table of Hilbert transforms that appears in Appendix 1 will reveal a number of examples that have been obtained by taking the derivative of a simpler transform pair. Either direct differentiation of a known transform or an indirect approach, such as the one just outlined for the Gaussian function, can be a very effective strategy for the evaluation of Hilbert transforms.

4.9 Convolution property

If $g(x)$ denotes the Hilbert transform of $f(x)$, then a convolution formula can be written as

$$g(x) = Hf(x) = \{f * k\}(x), \tag{4.154}$$

where the kernel function $k(x)$ is given by

$$k(x) = \frac{1}{\pi x}. \tag{4.155}$$

This convolution must be interpreted as a Cauchy principal value integral, since the kernel function is not integrable in E^1. In the same manner as the convolution for

Fourier transforms was defined (see Eqs. (2.53) and (2.54)), Eq. (4.154) can be written as follows:

$$g(x) = \{f * k\}(x) = P\int_{-\infty}^{\infty} f(s)k(x-s)\mathrm{d}s$$
$$= \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{f(s)\mathrm{d}s}{x-s}. \tag{4.156}$$

The Hilbert transform of a convolution of two functions can be written in terms of a convolution of one of the functions with the Hilbert transform of the other function, that is

$$H\{f * h\}(x) = \{Hf * h\}(x) = \{f * Hh\}(x). \tag{4.157}$$

The left-hand side of Eq. (4.157) can be expressed as follows:

$$H\{f * h\}(x) = \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{x-s}\int_{-\infty}^{\infty} f(u)h(s-u)\mathrm{d}u. \tag{4.158}$$

Now,

$$\begin{aligned}\{Hf * h\}(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} P\int_{-\infty}^{\infty} \frac{f(s)\mathrm{d}s}{u-s}h(x-u)\mathrm{d}u\\ &= \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{x-s}\int_{-\infty}^{\infty} f(u-x+s)h(x-u)\mathrm{d}u\\ &= \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{x-s}\int_{-\infty}^{\infty} f(u)h(s-u)\mathrm{d}u\\ &= H\{f * h\}(x).\end{aligned} \tag{4.159}$$

The change of order of integration in Eq. (4.159) is justified if f and h belong to the classes L^p and L^q, respectively, and if $p^{-1} + q^{-1} = 1$ (see Eq. (2.220)). In a similar manner,

$$\begin{aligned}\{f * Hh\}(x) &= \int_{-\infty}^{\infty} f(u)\frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{h(s)\mathrm{d}s}{x-u-s}\,\mathrm{d}u\\ &= \int_{-\infty}^{\infty} f(u)\frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{h(t-u)\mathrm{d}t}{x-t}\,\mathrm{d}u\\ &= \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{x-t}\int_{-\infty}^{\infty} f(u)h(t-u)\,\mathrm{d}u\\ &= H\{f * h\}(x),\end{aligned} \tag{4.160}$$

with the change of order on integration being justified as in Eq. (4.159).

An alternative approach to the convolution formula is to apply a Fourier transform technique. For $f \in L^2(\mathbb{R})$ and $h \in L(\mathbb{R})$,

$$\mathcal{F}\{f * h\}(x) = \mathcal{F}f(x)\,\mathcal{F}h(x), \ a.e. \tag{4.161}$$

The following identity is employed:

$$\mathcal{F}Hf(x) = -\mathrm{i}\,\mathrm{sgn}\,x\;\mathcal{F}f(x), \ a.e., \tag{4.162}$$

which is discussed in detail in Section 5.2. From the preceding two results it follows that, on setting $m(x) = -\mathrm{i}\,\mathrm{sgn}\,x$,

$$m(x)\mathcal{F}f(x)\mathcal{F}h(x) = \mathcal{F}h(x)\mathcal{F}Hf(x) = \mathcal{F}\{h * Hf\}(x) \tag{4.163}$$

and

$$m(x)\mathcal{F}f(x)\mathcal{F}h(x) = m(x)\mathcal{F}\{h * f\}(x) = \mathcal{F}H\{h * f\}(x); \tag{4.164}$$

then

$$\mathcal{F}H\{h * f\}(x) = \mathcal{F}\{h * Hf\}(x). \tag{4.165}$$

Taking the inverse Fourier transform of this equation leads to

$$H\{h * f\}(x) = \{h * Hf\}(x). \tag{4.166}$$

Setting $g = Hf$ in Eq. (4.157) and applying the Hilbert transform operator leads to

$$H^2\{f * h\}(x) = H\{g * h\}(x) = \{g * Hh\}(x), \tag{4.167}$$

and hence

$$\{Hf * Hh\}(x) = -\{f * h\}(x). \tag{4.168}$$

The Hilbert transform of the convolution of three functions can be written as

$$Hf * Hh * Hk = -H\{f * h * k\}. \tag{4.169}$$

This result follows in a straightforward fashion, since

$$-H\{f * h * k\} = -Hf * \{h * k\} = Hf * Hh * Hk. \tag{4.170}$$

Extensions of this result can be obtained in a similar manner.

4.10 Titchmarsh formulas of the Parseval type

It is possible to establish relationships for the Hilbert transform which have an analogous structure to the Parseval and Plancherel formulas of Fourier transform theory. Plancherel's (Parseval's) identity for Fourier integrals takes the following form. If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively,

$$\int_{-\infty}^{\infty} f(x)g(x)\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\overline{G(\omega)}\mathrm{d}\omega. \tag{4.171}$$

The reader is reminded that the bar signifies complex conjugate. Titchmarsh (1925a; 1926; 1948, p. 123) – see also Hardy (1932) – proved the following result. If f belongs to $L^2(\mathbb{R})$, and g is the Hilbert transform of f, then

$$\int_{-\infty}^{\infty} \{f(x)\}^2\,\mathrm{d}x = \int_{-\infty}^{\infty} \{g(x)\}^2\,\mathrm{d}x = \int_{-\infty}^{\infty} \{Hf(x)\}^2\,\mathrm{d}x. \tag{4.172}$$

There is a generalization of Eq. (4.172) (see Titchmarsh, 1926; 1948, p. 138) that takes the following form. If the functions f_1 and f_2 belong to the classes L^{p_1} and L^{p_2}, respectively, with $p_1 > 1$ and $p_2 > 1$, and

$$\frac{1}{p_1} + \frac{1}{p_2} = 1, \tag{4.173}$$

then

$$\int_{-\infty}^{\infty} f_1(x)\,f_2(x)\mathrm{d}x = \int_{-\infty}^{\infty} Hf_1(x)Hf_2(x)\mathrm{d}x. \tag{4.174}$$

This result has a form analogous to Eq. (4.171), with the Fourier transform replaced by the Hilbert transform (except for the need to take the complex conjugate). This result has been employed in practical applications, for example to find sum rules for optical constants and other properties, topics that are addressed in detail in later chapters. Using inner product notation this last result reads

$$(f_1, f_2) = (Hf_1, Hf_2). \tag{4.175}$$

Clearly, H preserves the inner product.

A related result is (Kober, 1943b)

$$\int_{-\infty}^{\infty} Hf_1(x)\,f_2(x)\mathrm{d}x = -\int_{-\infty}^{\infty} f_1(x)\,Hf_2(x)\mathrm{d}x, \tag{4.176}$$

with the same conditions for the functions f_1 and f_2 as stated in the preceding paragraph. From the formula for the interchange of integration order with a principal value integral incorporated (see Eq. (2.220)), and on setting $f(x) = f_1(x)$ and $g(x) = f_2(x)$,

yields

$$\int_{-\infty}^{\infty} f_1(x)\mathrm{d}x\, P\int_{-\infty}^{\infty} \frac{f_2(y)\mathrm{d}y}{y-x} = \int_{-\infty}^{\infty} f_2(y)\mathrm{d}y\, P\int_{-\infty}^{\infty} \frac{f_1(x)\mathrm{d}x}{y-x}. \tag{4.177}$$

Inserting the definitions for $Hf_2(x)$ and $Hf_1(y)$ into this result leads to Eq. (4.176). Nevai (1990) has given a proof, which he attributes to Harold Widom, that Eq. (4.176) can be established for the functions $f_1 \in L^\infty$ and $f_2 \in L\log^+ L(\mathbb{R})$. The latter class is defined by the requirement that

$$\int_{-\infty}^{\infty} |f_2(x)|\log^+|f_2(x)|\mathrm{d}x < \infty, \tag{4.178}$$

with $\log^+|f_2(x)| = \log|f_2(x)|$ for $|f_2(x)| \geq 1$ and zero otherwise.

Equation (4.176) can be written as follows:

$$\int_{-\infty}^{\infty} f_1(x)Hg(x)\mathrm{d}x = -\int_{-\infty}^{\infty} Hf_1(x)g(x)\mathrm{d}x, \tag{4.179}$$

On setting $f_2(x) = Hg(x)$,

$$\begin{aligned}\int_{-\infty}^{\infty} f_1(x)f_2(x)\mathrm{d}x &= -\int_{-\infty}^{\infty} Hf_1(x)g(x)\mathrm{d}x\\ &= \int_{-\infty}^{\infty} Hf_1(x)Hf_2(x)\mathrm{d}x,\end{aligned} \tag{4.180}$$

using $Hf_2(x) = H^2g(x) = -g(x)$. This is Eq. (4.174). The argument can be reversed, thereby obtaining Eq. (4.176) from Eq. (4.174).

Let $f_1 = f$ and take $f_2(t,x) = \chi_{(0,x)}(t)$, then Eq. (4.176) becomes

$$\begin{aligned}\int_0^x Hf(t)\mathrm{d}t &= -\int_{-\infty}^{\infty} f(t)H\chi_{(0,x)}(t)\mathrm{d}t\\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left\{P\int_0^x \frac{\mathrm{d}y}{t-y}\right\}\mathrm{d}t\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\log\left|1-\frac{x}{t}\right|\mathrm{d}t.\end{aligned} \tag{4.181}$$

Differentiating this with respect to x leads to

$$Hf(x) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} f(t)\log\left|1-\frac{x}{t}\right|\mathrm{d}t. \tag{4.182}$$

This constitutes a proof of Eq. (3.16). Let $f_1 = f$ and take $f_2(t,x) = H\chi_{(0,x)}(t)$, then Eq. (4.176) becomes

$$\begin{aligned}\int_0^x f(t)\mathrm{d}t &= \int_{-\infty}^{\infty} Hf(t) H\chi_{(0,x)}(t)\mathrm{d}t\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} Hf(t)\left\{P\int_0^x \frac{\mathrm{d}y}{t-y}\right\}\mathrm{d}t\\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} Hf(t)\log\left|1-\frac{x}{t}\right|\mathrm{d}t. \qquad (4.183)\end{aligned}$$

Differentiating this with respect to x leads to

$$f(x) = -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} Hf(t)\log\left|1-\frac{x}{t}\right|\mathrm{d}t. \qquad (4.184)$$

The analog of Eq. (4.176) for the even and odd Hilbert transform operators is now explored. The key result is the following: if f is an even function and g is an odd function, then it follows directly from Eq. (4.176) that

$$\int_0^{\infty} H_{\mathrm{e}}f(t)g(t)\mathrm{d}t = -\int_0^{\infty} f(t)H_{\mathrm{o}}g(t)\mathrm{d}t. \qquad (4.185)$$

To prove this result without recourse to Eq. (4.176), start with Eq. (4.11) and use Eq. (4.12), to obtain

$$\begin{aligned}\int_0^{\infty} H_{\mathrm{e}}f(t)g(t)\mathrm{d}t &= \frac{1}{\pi}\int_0^{\infty} g(t)\mathrm{d}t\left\{P\int_0^{\infty}\frac{f(s)\mathrm{d}s}{t-s} + \int_0^{\infty}\frac{f(s)\mathrm{d}s}{t+s}\right\}\\ &= -\frac{1}{\pi}\int_0^{\infty} f(s)\mathrm{d}s\left\{P\int_0^{\infty}\frac{g(t)\mathrm{d}t}{s-t} - \int_0^{\infty}\frac{g(t)\mathrm{d}t}{s+t}\right\}\\ &= -\int_0^{\infty} f(s)H_{\mathrm{o}}g(s)\mathrm{d}s, \qquad (4.186)\end{aligned}$$

which is the required result. The interchange of integration order can be justified by the use of Eq. (2.220) for the interval $(0,\infty)$, or by performing the following calculation:

$$\begin{aligned}&\frac{1}{\pi}\int_0^{\infty} g(t)\mathrm{d}t\left\{P\int_0^{\infty}\frac{f(s)\mathrm{d}s}{t-s} + \int_0^{\infty}\frac{f(s)\mathrm{d}s}{t+s}\right\}\\ &\quad= \frac{1}{\pi}\int_0^{\infty} g(t)\mathrm{d}t\int_0^{\infty}\{f(s)-f(t)\}\left\{\frac{1}{t-s}+\frac{1}{t+s}\right\}\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^\infty \mathrm{d}s \int_0^\infty \{f(s)g(t) - f(t)g(t)\}\left\{\frac{1}{t-s} + \frac{1}{t+s}\right\}\mathrm{d}t \\
&= -\frac{1}{\pi}\int_0^\infty f(s)\mathrm{d}s\, P\int_0^\infty g(t)\left\{\frac{1}{s-t} - \frac{1}{t+s}\right\}\mathrm{d}t \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}s\left\{P\int_0^\infty \frac{f(t)g(t)}{s-t}\,\mathrm{d}t + P\int_0^{-\infty} \frac{f(-t)g(-t)}{s-t}\,\mathrm{d}t\right\} \\
&= -\int_0^\infty f(s)H_\mathrm{o}g(s)\mathrm{d}s + \frac{1}{2}\int_{-\infty}^\infty H\{fg\}(s)\mathrm{d}s \\
&= -\int_0^\infty f(s)H_\mathrm{o}g(s)\mathrm{d}s, \qquad (4.187)
\end{aligned}
$$

which is the required result. The interchange of integration order does not involve a principal value integral. The last line of the sequence follows on employing the orthogonality property of the Hilbert transform, a topic that is discussed shortly (see Eq. (4.204)). Equation (4.185) applies for $f \in L^p(\mathbb{R}^+)$ and $g \in L^q(\mathbb{R}^+)$, for $p > 1$, $q > 1$, and $p^{-1} + q^{-1} = 1$.

Immediate consequences of Eq. (4.185) are the following relations:

$$
H_\mathrm{e}f(x) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty f(t)\log\left|1 - \frac{x^2}{t^2}\right|\mathrm{d}t \qquad (4.188)
$$

and

$$
H_\mathrm{o}g(x) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty g(t)\log\left|\frac{t-x}{t+x}\right|\mathrm{d}t. \qquad (4.189)
$$

To establish Eq. (4.189), set f equal to the characteristic function on the interval $(-x, x)$ and zero elsewhere; then

$$
\begin{aligned}
\int_0^\infty H_\mathrm{o}g(t)\chi_{(-x,x)}(t)\mathrm{d}t &= \int_0^x H_\mathrm{o}g(t)\mathrm{d}t \\
&= -\int_0^\infty g(t)H_\mathrm{e}\chi_{(-x,x)}(t)\mathrm{d}t \\
&= -\frac{1}{\pi}\int_0^\infty g(t)\left\{P\int_0^x \frac{\mathrm{d}s}{t-s} + \int_0^x \frac{\mathrm{d}s}{t+s}\right\}\mathrm{d}t \\
&= \frac{1}{\pi}\int_0^\infty g(t)\log\left|\frac{t-x}{t+x}\right|\mathrm{d}t. \qquad (4.190)
\end{aligned}
$$

Differentiating both sides with respect to x leads to Eq. (4.189). Equation (4.188) can be derived in a similar fashion, starting with $g(t,x) = \operatorname{sgn} t \; \chi_{(-x,x)}(t)$, so that

$$\begin{aligned}\int_0^\infty H_{\mathrm{e}} f(t)\chi(t)\mathrm{d}t &= \int_0^x H_{\mathrm{e}} f(t)\mathrm{d}t \\ &= -\int_0^\infty f(t) H_{\mathrm{o}} \chi_{(-x,x)}(t)\mathrm{d}t \\ &= -\frac{1}{\pi}\int_0^\infty f(t)\left\{P\int_0^x \frac{\mathrm{d}s}{t-s} - \int_0^x \frac{\mathrm{d}s}{t+s}\right\}\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^\infty f(t)\log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}t. \end{aligned} \tag{4.191}$$

Differentiating both sides with respect to x leads to the desired result.

4.11 Unitary property of H

Suppose f and $g \in L^2(\mathbb{R})$. Recall that the operator $\mathcal{O}^+$ is called the adjoint of a linear operator $\mathcal{O}$ if

$$(\mathcal{O}^+ g, f) = (g, \mathcal{O} f). \tag{4.192}$$

From Eq. (4.176), setting $f_1 = g^*$ and $f_2 = f$ leads to

$$(H^+ g, f) = (g, -Hf), \tag{4.193}$$

which makes it clear that the Hilbert transform operator is *skew-adjoint*, that is

$$H^+ = -H. \tag{4.194}$$

Sometimes the terminology anti-Hermitian is applied to an operator satisfying this last equation. In Eq. (4.26) it was demonstrated that $H^{-1} = -H$, and so

$$H^+ = H^{-1}. \tag{4.195}$$

Since H satisfies the isometric condition (see Eq. (4.172)),

$$\|Hf\|_2 = \|f\|_2, \tag{4.196}$$

and, together with the condition given in Eq. (4.195), this establishes that H is a unitary operator in L^2.

4.12 Orthogonality property

The Hilbert transform satisfies an orthogonality condition. Suppose $f(z)$ is analytic in the upper half complex plane and let $f_1(x)$ and $f_2(x)$ denote the real and imaginary parts of $f(x)$, respectively. If $f(z)$ vanishes like z^{-m} for $m > 1$ as $z \to \pm\infty$, and is Hölder continuous on every finite interval of the real axis, then

$$\int_{-\infty}^{\infty} f_1(x) f_2(x) \mathrm{d}x = \int_{-\infty}^{\infty} f_1(x) H f_1(x) \mathrm{d}x = 0, \tag{4.197}$$

or, in inner product notation,

$$(f_1, Hf_1) = 0. \tag{4.198}$$

This is the orthogonality condition for the Hilbert transform. This identity finds application in the development of sum rules for optical constants and other properties. The proof of Eq. (4.198) is as follows. Consider the integral $\oint_C f(z)^2 \,\mathrm{d}z$, where C is a semicircular contour in the upper half plane including the real axis and center the origin; then from the Cauchy integral theorem,

$$\oint_C f(z)^2 \,\mathrm{d}z = \int_{-\infty}^{\infty} f(x)^2 \,\mathrm{d}x = 0, \tag{4.199}$$

that is

$$\int_{-\infty}^{\infty} \left\{ f_1(x)^2 - f_2(x)^2 + 2\mathrm{i} f_1(x) f_2(x) \right\} \mathrm{d}x = 0. \tag{4.200}$$

Since $f_2(x) = Hf_1(x)$, it follows, on separating Eq. (4.200) into real and imaginary parts, that

$$\int_{-\infty}^{\infty} f_1(x)^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} \{Hf_1(x)\}^2 \,\mathrm{d}x, \tag{4.201}$$

which is Eq. (4.172), and

$$\int_{-\infty}^{\infty} f_1(x) H f_1(x) \mathrm{d}x = 0, \tag{4.202}$$

which is the orthogonality condition. A symmetry argument yields a more straightforward proof when $f_1(x)$ is an even or odd function. If $f_1(x)$ is an even or odd function, then, by the results of Section 4.2, the integrand of Eq. (4.202) is odd, and hence the integral evaluates to zero.

Let $f \in L^p$ for $1 < p < \infty$ and $g \in L^q$ with q the conjugate exponent; then

$$\int_{-\infty}^{\infty} g(x) H f(x) \mathrm{d}x = -\int_{-\infty}^{\infty} f(x) H g(x) \mathrm{d}x. \tag{4.203}$$

The choice $g = f$ yields Eq. (4.202), and the choice $g = Hf$ yields Eq. (4.201). To show that both integrals in Eq. (4.203) are bounded takes advantage of the Riesz inequality, a topic discussed in Section 4.20. The proof of Eq. (4.203) is given at the start of Section 4.23.

This section concludes with two additional formulas. For $f \in L^p(\mathbb{R})$ with $1 \le p \le 2$,

$$\int_{-\infty}^{\infty} Hf(x)\mathrm{d}x = 0. \tag{4.204}$$

A related result is as follows:

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{x} f(y)Hf(y)\mathrm{d}y = 0. \tag{4.205}$$

If f is an even function, the result in Eq. (4.204) follows immediately from the even–odd property of the Hilbert transform discussed in Section 4.2. For a general f, Eq. (4.204) can be derived in a few different ways, of which the most direct involves a Fourier transform approach. Making use of Eq. (4.162),

$$\lim_{\lambda\to 0} \mathcal{F}Hf(\lambda) = \lim_{\lambda\to 0} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\lambda x} Hf(x)\mathrm{d}x = \int_{-\infty}^{\infty} Hf(x)\mathrm{d}x. \tag{4.206}$$

Only the odd component of f need be considered, so that

$$\lim_{\lambda\to 0} \mathcal{F}Hf(\lambda) = \lim_{\lambda\to 0} \{-\mathrm{i}\,\mathrm{sgn}\,\lambda\; \mathcal{F}f(\lambda)\} = 0, \tag{4.207}$$

and Eq. (4.204) is proved.

To establish Eq. (4.205), let $F(z)$ denote a function analytic in the upper half of the complex plane and on the real axis write

$$F(x) = f(x) + \mathrm{i}\,Hf(x); \tag{4.208}$$

it follows that

$$F(x)^2 = g(x) + \mathrm{i}\,Hg(x), \tag{4.209}$$

with

$$g(x) = f(x)^2 - \{Hf(x)\}^2 \tag{4.210}$$

and

$$Hg(x) = 2f(x)Hf(x). \tag{4.211}$$

Making use of Eq. (4.201) leads to

$$\int_{-\infty}^{\infty} g(x)\mathrm{d}x = 0, \tag{4.212}$$

that is

$$\mathcal{F}g(0) = 0. \tag{4.213}$$

It follows from Eq. (4.211) that

$$\begin{aligned}
2\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{x} f(y)Hf(y)\mathrm{d}y &= \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{x} Hg(y)\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{x} -\mathrm{i}\mathcal{F}^{-1}\{\operatorname{sgn} t\, \mathcal{F}g(t)\}(y)\mathrm{d}y \\
&= \frac{-\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{x} \mathrm{d}y \int_{-\infty}^{\infty} \operatorname{sgn} t\, \mathrm{e}^{\mathrm{i}yt} \mathcal{F}g(t)\mathrm{d}t \\
&= \frac{-\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \operatorname{sgn} t\, \mathcal{F}g(t)\mathrm{d}t \int_{-\lambda}^{x} \mathrm{e}^{\mathrm{i}ty}\, \mathrm{d}y \\
&= \frac{-1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} t^{-1} \operatorname{sgn} t\, \mathcal{F}g(t)\{\mathrm{e}^{\mathrm{i}xt} - \mathrm{e}^{-\mathrm{i}\lambda t}\}\mathrm{d}t \\
&= \frac{-1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} t^{-1} \operatorname{sgn} t\, \mathrm{e}^{\mathrm{i}xt}\, \mathcal{F}g(t)\mathrm{d}t \\
&= \frac{-1}{2\pi} \int_{-\infty}^{\infty} t^{-1} \operatorname{sgn} t\, \mathcal{F}g(t)\mathrm{d}t \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}x \\
&= -\int_{-\infty}^{\infty} t^{-1} \operatorname{sgn} t\, \mathcal{F}g(t)\, \delta(t)\mathrm{d}t,
\end{aligned} \tag{4.214}$$

where the Riemann–Lebesgue lemma has been employed to handle the limit involving λ. To deal with the last integral, first note Eq. (4.212), and, for the Fourier transform component of the integrand, use $\mathrm{e}^{\mathrm{i}yt} = (\cos yt - 1) + 1 + \mathrm{i}\sin yt$, then employ the limit

$$\lim_{t\to 0} t^{-1} \operatorname{sgn} t\{\cos(yt) - 1\} = 0. \tag{4.215}$$

The integral in Eq. (4.214) then evaluates to zero, and hence Eq. (4.205) is established.

4.13 Hilbert transforms via series expansion

In a number of cases it is possible to arrive at the evaluation of a particular Hilbert transform by taking advantage of the linear nature of the Hilbert transform operator. Suppose the function f has a series expansion

$$f(x) = \sum_k f_k(x), \tag{4.216}$$

then the Hilbert transform of f can be written as follows:

$$Hf(x) = \sum_k Hf_k(x). \tag{4.217}$$

The series in Eq. (4.217) may or may not converge, and only those cases where this series does converge are of interest. The advantage of this approach is that the Hilbert transform of f_k may be much easier to evaluate than the Hilbert transform of f directly. Some examples to illustrate the approach are now examined. Suppose

$$f(x) = \operatorname{sech} ax = \frac{2}{e^{ax} + e^{-ax}}, \tag{4.218}$$

where a is a real constant. The hyperbolic secant function has the series expansion (Hansen, 1975, p. 106)

$$\operatorname{sech} ax = \frac{2}{a} \sum_{k=0}^{\infty} \frac{(-1)^k b_k}{b_k^2 + x^2}, \quad \text{with } b_k = \frac{\pi(2k+1)}{2a}. \tag{4.219}$$

Make use of the result

$$H\left(\frac{b}{b^2 + x^2}\right) = \frac{x}{b^2 + x^2}, \tag{4.220}$$

where b is a real constant, then from Eq. (4.219) it follows that

$$\begin{aligned} H\,(\operatorname{sech} ax) &= \frac{2}{a} \sum_{k=0}^{\infty} (-1)^k H\left(\frac{b_k}{b_k^2 + x^2}\right) \\ &= \frac{2x}{a} \sum_{k=0}^{\infty} \frac{(-1)^k}{b_k^2 + x^2}. \end{aligned} \tag{4.221}$$

This result can be cast in terms of special functions as follows. The psi (or digamma) function $\psi(z)$ is defined by (Abramowitz and Stegun, 1965, p. 258)

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{4.222}$$

where $\Gamma(z)$ is the gamma function. A series representation for $\psi(z)$ is given by

$$\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}, \quad z \neq -1, z \neq -2, z \neq -3, \ldots \tag{4.223}$$

Using this result and the series expansion for $\tanh x$,

$$\tanh x = 8x \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\pi^2 + 4x^2}, \tag{4.224}$$

allows $H(\operatorname{sech} ax)$ to be written as follows:

$$H(\operatorname{sech} ax) = \frac{\mathrm{i}}{\pi}\left\{\psi\left(\frac{1}{4} - \frac{\mathrm{i}ax}{2\pi}\right) - \psi\left(\frac{1}{4} + \frac{\mathrm{i}ax}{2\pi}\right)\right\} - \tanh ax. \tag{4.225}$$

Evaluation of Eq. (4.221) by a direct numerical assault is likely to be a particularly ineffective computational strategy. Instead, application of convergence accelerator techniques to sum this alternating series is a very effective evaluation approach. The end-notes to Chapter 2 include some suggested references on this computational technique.

As a second example, consider

$$f(x) = \operatorname{csch} ax - \frac{1}{ax} = \frac{2}{\mathrm{e}^{ax} - \mathrm{e}^{-ax}} - \frac{1}{ax}, \tag{4.226}$$

where a is a real constant. The hyperbolic cosecant function has the series expansion (Hansen, 1975, p. 104)

$$\operatorname{csch} ax = \frac{1}{ax} + \frac{2x}{a}\sum_{k=1}^{\infty}\frac{(-1)^k}{b_k^2 + x^2}, \quad \text{with } b_k = \pi a^{-1} k. \tag{4.227}$$

From the preceding two equations, it follows that

$$Hf(x) = H\left(\operatorname{csch} ax - \frac{1}{ax}\right) = \frac{2}{a}\sum_{k=1}^{\infty}(-1)^k H\left(\frac{x}{b_k^2 + x^2}\right). \tag{4.228}$$

Employing the result

$$H\left(\frac{x}{b^2 + x^2}\right) = -\frac{b}{b^2 + x^2}, \tag{4.229}$$

for b a real constant, leads to

$$H\left(\operatorname{csch} ax - \frac{1}{ax}\right) = -\frac{2}{a}\sum_{k=1}^{\infty}\frac{(-1)^k b_k}{b_k^2 + x^2}. \tag{4.230}$$

Generally, the approach just outlined will work for series that take one of the following simple forms:

$$\sum_{k=0}^{\infty} \frac{1}{k^2+x^2}; \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+x^2}; \quad \sum_{k=1}^{\infty} \frac{1}{k(k^2+x^2)};$$
$$\sum_{k=0}^{\infty} \frac{1}{(k^2+x^2)^2}; \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(k^2+x^2)^2}; \quad \sum_{k=1}^{\infty} \frac{1}{k^2(k^2+x^2)};$$
$$\sum_{k=1}^{\infty} \frac{1}{k^2(k^2+x^2)^2}; \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k^2+x^2)}; \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k^2+x^2)^2},$$

plus a number of related series. All of the indicated series can be evaluated in closed form in terms of hyperbolic functions or the digamma function. A number of series closely related to the preceding sums, for example,

$$\sum_{k=1}^{\infty} \frac{k}{(k^2+x^2)^2},$$

cannot be expressed in a convenient simple closed form, although it is often possible to express such series in terms of the polygamma function with a complex argument.

Hilbert transforms of more complicated functions can be found if the function of interest can be expressed in a suitable series representation. For example, consider the determination of the Hilbert transform of the function

$$f(x) = \frac{\sin x}{\cosh a - \cos x}, \tag{4.231}$$

for $a > 0$. It will first be shown that this function can be cast in the following form:

$$\frac{\sin x}{\cosh a - \cos x} = 2\sum_{n=1}^{\infty} \sin nx\, \mathrm{e}^{-an}. \tag{4.232}$$

The proof of Eq. (4.232) is straightforward:

$$\begin{aligned} 2\sum_{n=1}^{\infty} \sin nx\, \mathrm{e}^{-an} &= -\mathrm{i}\sum_{n=0}^{\infty} \{\mathrm{e}^{-(a-\mathrm{i}x)n} - \mathrm{e}^{-(a+\mathrm{i}x)n}\} \\ &= -\mathrm{i}\sum_{n=0}^{\infty} \{[\mathrm{e}^{-(a-\mathrm{i}x)}]^n - [\mathrm{e}^{-(a+\mathrm{i}x)}]^n\} \\ &= -\mathrm{i}\left\{\frac{1}{1-\mathrm{e}^{-(a-\mathrm{i}x)}} - \frac{1}{1-\mathrm{e}^{-(a+\mathrm{i}x)}}\right\} \\ &= \frac{\sin x}{\cosh a - \cos x}. \end{aligned} \tag{4.233}$$

The series expansion

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n, \quad \text{for } |z| < 1, \tag{4.234}$$

has been employed to obtain Eq. (4.233). From Eq. (4.232), it follows that

$$\begin{aligned} H\left(\frac{\sin x}{\cosh a - \cos x}\right) &= 2\sum_{n=1}^{\infty} H(\sin nx)\mathrm{e}^{-an} \\ &= 2\sum_{n=1}^{\infty} \{-\operatorname{sgn} n \cos nx\}\mathrm{e}^{-an} \\ &= -2\sum_{n=1}^{\infty} \cos nx\, \mathrm{e}^{-an}. \end{aligned} \tag{4.235}$$

It can be shown by the same approach just sketched in Eq. (4.233) that

$$\frac{\mathrm{e}^{a} - \cos x}{\cosh a - \cos x} = 2\sum_{n=0}^{\infty} \cos nx\, \mathrm{e}^{-an}. \tag{4.236}$$

Hence, from the preceding two results it follows that

$$H\left(\frac{\sin x}{\cosh a - \cos x}\right) = \frac{\mathrm{e}^{-a} - \cos x}{\cosh a - \cos x} = 1 - \frac{\sinh a}{\cosh a - \cos x}. \tag{4.237}$$

The Hilbert transform of $(\mathrm{e}^{a} - \cos x)(\cosh a - \cos x)^{-1}$ can be found in a similar manner, starting with the result given in Eq. (4.236).

4.14 The Hilbert transform of a product of functions

In this and the following two sections the topic of determining the Hilbert transform of a product of functions is considered. Unfortunately, there is no simple formula available to evaluate the Hilbert transform of a general product of functions, but in some special cases considerable simplifications can be obtained. Some of these special cases occur in applications in engineering problems.

The first class of functions considered are those that are analytic in the upper half of the complex plane, have suitable asymptotic behavior as $|z| \to \infty$, and on the real axis can be written in the following form:

$$f(x) = g(x) + \mathrm{i}h(x), \tag{4.238}$$

where g and h are real-valued functions and h is the Hilbert transform of g. The asymptotic behavior assumed for $f(z)$ is such that the Hilbert transforms of g and h both exist. In the time domain in signal processing, signals that satisfy the preceding equation are called *analytic signals*. In this section the designation analytic signal is used to describe functions that satisfy Eq. (4.238), even though time may not be the dependent variable of interest.

The Hilbert transform of f gives

$$\begin{aligned} Hf(x) &= Hg(x) + \mathrm{i}Hh(x) \\ &= h(x) + \mathrm{i}H^2 g(x), \end{aligned} \tag{4.239}$$

and, on using the iteration property of the Hilbert transform, it follows that

$$Hf(x) = -\mathrm{i}f(x). \tag{4.240}$$

Consider the product of two analytic signals f_1 and f_2, defined by

$$f_1(x) = g_1(x) + \mathrm{i}h_1(x) \tag{4.241}$$

and

$$f_2(x) = g_2(x) + \mathrm{i}h_2(x), \tag{4.242}$$

with the product $f_1(x)f_2(x)$ denoted by $f(x)$. From complex variable theory, the product of two analytic functions is also an analytic function (in the common domain for which both functions are analytic). The function f is given by

$$f(x) = f_1(x)f_2(x) = g_1(x)g_2(x) - h_1(x)h_2(x) + \mathrm{i}\{g_1(x)h_2(x) + g_2(x)h_1(x)\}. \tag{4.243}$$

If $f(x)$ is an analytic signal, then

$$H\{g_1(x)g_2(x) - h_1(x)h_2(x)\} = g_1(x)h_2(x) + g_2(x)h_1(x). \tag{4.244}$$

This result follows from Eq. (4.240) in a straightforward fashion. If $f(x)$ is an analytic signal, then

$$H\{f_1(x)f_2(x)\} = -\mathrm{i}f_1(x)f_2(x); \tag{4.245}$$

that is,

$$\begin{aligned} &H(g_1(x)g_2(x) - h_1(x)h_2(x) + \mathrm{i}\{g_1(x)h_2(x) + g_2(x)h_1(x)\}) \\ &\quad = -\mathrm{i}\{g_1(x)g_2(x) - h_1(x)h_2(x) + \mathrm{i}[g_1(x)h_2(x) + g_2(x)h_1(x)]\}. \end{aligned} \tag{4.246}$$

Equating the real and imaginary parts of Eq. (4.246) leads to Eq. (4.244) and

$$H\{g_1(x)h_2(x) + g_2(x)h_1(x)\} = -\{g_1(x)g_2(x) - h_1(x)h_2(x)\}. \tag{4.247}$$

Equation (4.247) is the second of the Hilbert transform pair.

It is possible to write the Hilbert transform of a product of analytic signals as follows:

$$H\{f_1(x)f_2(x)\} = f_1(x)Hf_2(x) = Hf_1(x)f_2(x) = -\mathrm{i}f_1(x)f_2(x). \tag{4.248}$$

The proof is straightforward:

$$\begin{aligned} f_1(x)\,Hf_2(x) &= f_1(x)H\{g_2(x) + \mathrm{i}h_2(x)\} \\ &= f_1(x)\{h_2(x) - \mathrm{i}g_2(x)\} \\ &= -\mathrm{i}f_1(x)f_2(x), \end{aligned} \tag{4.249}$$

and, similarly,

$$\begin{aligned} Hf_1(x)f_2(x) &= \{Hg_1(x) + \mathrm{i}Hh_1(x)\}f_2(x) \\ &= \{h_1(x) - \mathrm{i}g_1(x)\}f_2(x) \\ &= -\mathrm{i}f_1(x)f_2(x), \end{aligned} \tag{4.250}$$

which establishes Eq. (4.248). A useful special case of Eq. (4.248) occurs when the two functions are the same; in this case, it follows for an analytic signal f that

$$Hf^2(x) = f(x)Hf(x) = -\mathrm{i}f^2(x), \tag{4.251}$$

a result that also follows more directly from Eq. (4.240). The generalization to arbitrary powers is given by

$$Hf^n(x) = f^{n-1}(x)\,Hf(x) = -\mathrm{i}f^n(x), \tag{4.252}$$

for integer $n \geq 1$. An immediate application of this expression is the evaluation of Hilbert transforms for selected functions. As an example, consider the calculation of $H\{(x^2 - a^2)(x^2 + a^2)^{-2}\}$ for $a > 0$. This transform can be evaluated directly using contour integration techniques. However, it can be determined more quickly if it is noticed that the function of interest is the real part of the analytic signal $(x + \mathrm{i}a)^{-2}$. Hence, from Eq. (4.240),

$$H\{(x + \mathrm{i}a)^{-2}\} = -\mathrm{i}(x + \mathrm{i}\,a)^{-2}; \tag{4.253}$$

that is,

$$H\left(\frac{x^2 - a^2 - 2\mathrm{i}ax}{(x^2 + a^2)^2}\right) = \frac{-2ax - \mathrm{i}(x^2 - a^2)}{(x^2 + a^2)^2}. \tag{4.254}$$

Taking the real part of this result leads to

$$H\left(\frac{x^2 - a^2}{(x^2 + a^2)^2}\right) = \frac{-2ax}{(x^2 + a^2)^2}. \tag{4.255}$$

Equation (4.248) has an obvious generalization for more than two distinct analytic signals. Suppose $f_1(x), f_2(x)$, and $f_3(x)$ are analytic signals then

$$\begin{aligned} H\{f_1(x)f_2(x)f_2(x)\} &= \{Hf_1(x)\}f_2(x)f_3(x) = f_1(x)\{Hf_2(x)\}f_3(x) \\ &= f_1(x)f_2(x)Hf_3(x), \end{aligned} \tag{4.256}$$

and additional results for higher order products follow in a similar fashion.

4.15 The Hilbert transform product theorem (Bedrosian's theorem)

The Hilbert transform of a product of functions of a different category is now considered. The following result is due to Bedrosian (1963), and is referred to as the *Hilbert transform product theorem*, or Bedrosian's theorem. Let f and $g \in L^2(\mathbb{R})$. Suppose that the Fourier transform of $f(x)$, denoted by $F(s)$, vanishes for $|s| > a$, with $a > 0$, and the Fourier transform of $g(x)$, denoted by $G(s)$, vanishes for $|s| < a$; then

$$H\{f(x)g(x)\} = f(x)Hg(x). \tag{4.257}$$

The Hilbert transform of the product is given in terms of the Fourier transforms F and G by

$$\begin{aligned} H\{f(x)g(x)\} &= \frac{1}{(2\pi)^2} H\left[\int_{-\infty}^{\infty} F(s)e^{isx}\,ds \int_{-\infty}^{\infty} G(t)e^{itx}\,dt\right] \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} F(s)ds \int_{-\infty}^{\infty} G(t)H\{e^{i(t+s)x}\}\,dt \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} F(s)ds \int_{-\infty}^{\infty} G(t)\{-i\,\mathrm{sgn}\,(s+t)e^{i(s+t)x}\}dt. \end{aligned} \tag{4.258}$$

The reader should provide a justification for the interchange of integration order in the preceding sequence of steps. The last integral simplifies on noting that the support of $F(s)$ is $[-a, a]$ and the support of $G(t)$ is $(-\infty, -a] \bigcup [a, \infty)$; hence

$$\begin{aligned} H\{f(x)g(x)\} = \frac{-i}{(2\pi)^2} \int_{-a}^{a} F(s)e^{ixs}\,ds \left\{\int_{-\infty}^{-a} G(t)e^{ixt}\,\mathrm{sgn}(s+t)dt \right. \\ \left. + \int_{a}^{\infty} G(t)e^{ixt}\,\mathrm{sgn}\,(s+t)dt\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{-i}{(2\pi)^2}\int_{-a}^{a} F(s)\mathrm{d}s\left\{\int_{-\infty}^{-a+s} G(v-s)\mathrm{e}^{\mathrm{i}xv}\,\mathrm{sgn}(v)\mathrm{d}v\right.\\
&\quad\left. + \int_{a+s}^{\infty} G(v-s)\mathrm{e}^{\mathrm{i}xv}\,\mathrm{sgn}(v)\mathrm{d}v\right\}\\
&= \frac{-\mathrm{i}}{(2\pi)^2}\int_{-a}^{a} F(s)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s\left\{-\int_{-\infty}^{-a} G(y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y + \int_{a}^{\infty} G(y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y\right\}\\
&= \frac{1}{2\pi}f(x)\left\{\int_{-\infty}^{-a} G(y)(-\mathrm{i}\,\mathrm{sgn}\,y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y + \int_{a}^{\infty} G(y)(-\mathrm{i}\,\mathrm{sgn}\,y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y\right\}\\
&= \frac{1}{2\pi}f(x)\left\{\int_{-\infty}^{-a} G(y)H\{\mathrm{e}^{\mathrm{i}xy}\}\mathrm{d}y + \int_{a}^{\infty} G(y)H\{\mathrm{e}^{\mathrm{i}xy}\}\mathrm{d}y\right\}\\
&= \frac{1}{2\pi}f(x)H\left[\int_{-\infty}^{-a} G(y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y + \int_{a}^{\infty} G(y)\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y\right]\\
&= f(x)Hg(x),
\end{aligned}
\tag{4.259}
$$

which establishes Eq. (4.257). The restriction on the class of functions for which Eq. (4.257) holds is rather severe, but there is an important practical application. In signal processing in the frequency domain, the function F would correspond to a low-pass filter, and the function G corresponds to a high-pass filter. An illustration of the behavior of F and G is shown in Figure 4.2.

Equation (4.257) can be employed as an effective strategy to evaluate the Hilbert transform of certain functions. The following examples will illustrate the approach. The sinc function is defined by

$$\mathrm{sinc}\,x = \frac{\sin\pi x}{\pi x}, \tag{4.260}$$

and the Fourier transform of $\mathrm{sinc}\,ax$, where a is a real constant, is given by

$$\mathcal{F}\{\mathrm{sinc}\,ax\} = \frac{1}{2a}\{\mathrm{sgn}(\pi a + x) + \mathrm{sgn}(\pi a - x)\}; \tag{4.261}$$

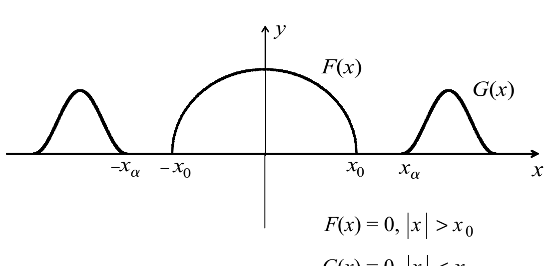

Figure 4.2. $F(x)$ is a low-pass filter and $G(x)$ is a high-pass filter.

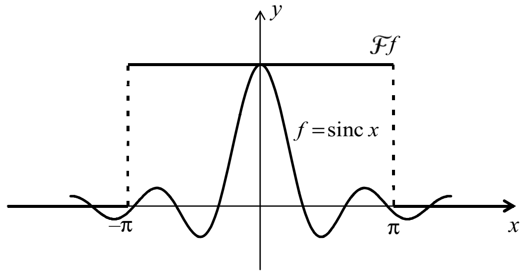

Figure 4.3. Plot of sinc x and $\mathcal{F}\{\operatorname{sinc} x\}$.

a plot of this is shown in Figure 4.3 for the choice $a = 1$. The sinc function therefore satisfies the condition that its Fourier transform has a support of a finite interval around the origin, specifically $(-\pi a, \pi a)$, from which it follows that

$$\begin{aligned} H(\sin ax \operatorname{sinc} bx) &= H(\sin ax) \operatorname{sinc} bx \\ &= -\cos ax \operatorname{sinc} bx, \quad \text{for } 0 < b\pi < a. \end{aligned} \tag{4.262}$$

In a similar fashion,

$$\begin{aligned} H(\cos ax \operatorname{sinc} bx) &= H(\cos ax) \operatorname{sinc} bx \\ &= \sin ax \operatorname{sinc} bx, \quad \text{for } 0 < b\pi < a. \end{aligned} \tag{4.263}$$

A further example is furnished by the function $\operatorname{sinc}^2 ax, a > 0$, for which the Fourier transform is given by

$$\mathcal{F}\{\operatorname{sinc}^2 ax\} = \begin{cases} \dfrac{1}{a}\left(1 - \dfrac{|x|}{2a\pi}\right), & |x| < 2a\pi \\ 0, & |x| \geq 2a\pi. \end{cases} \tag{4.264}$$

The Fourier transform of $\operatorname{sinc}^2 ax$ is often written in terms of the unit triangular function $\Lambda(x)$, which is defined by

$$\Lambda(x) = \begin{cases} 1 - |x|, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1, \end{cases} \tag{4.265}$$

and so

$$\mathcal{F}\{\operatorname{sinc}^2 ax\} = \frac{1}{a}\Lambda\left(\frac{x}{2\pi a}\right). \tag{4.266}$$

Hence,

$$\begin{aligned} H(\sin ax \operatorname{sinc}^2 bx) &= H(\sin ax) \operatorname{sinc}^2 bx \\ &= -\cos ax \operatorname{sinc}^2 bx, \quad \text{for } 0 < 2b\pi < a, \end{aligned} \tag{4.267}$$

and

$$\begin{aligned} H(\cos ax \operatorname{sinc}^2 bx) &= H(\cos ax) \operatorname{sinc}^2 bx \\ &= \sin ax \operatorname{sinc}^2 bx, \quad \text{for } 0 < 2b\pi < a. \end{aligned} \tag{4.268}$$

Further examples can be constructed using the Bessel function of the first kind, the cosine integral function Ci(x), and others. Some additional cases are discussed for special functions in Section 9.6. A number of examples have been collected in Appendix 1. The interested reader can try to find other cases where Bedrosian's approach is applicable.

Equation (4.257) arises under a different set of conditions for polynomials with vanishing moments. The details are presented in Section 7.14.

4.16 A theorem due to Tricomi

The following result was derived by Tricomi (1951a). Suppose the functions $f(x)$ and $g(x)$ belong to $L^{p_1}(\mathbb{R})$ and $L^{p_2}(\mathbb{R})$, respectively, with $p_1 > 1$ and $p_2 > 1$. If

$$\frac{1}{p_1} + \frac{1}{p_2} < 1, \tag{4.269}$$

then, almost everywhere,

$$H\{f(x)Hg(x) + g(x)Hf(x)\} = Hf(x)\, Hg(x) - f(x)g(x). \tag{4.270}$$

Equation (4.270) can be derived in the following manner. Let

$$f_1(x) = u_1(x) + \mathrm{i}v_1(x) \tag{4.271}$$

and

$$f_2(x) = u_2(x) + \mathrm{i}v_2(x), \tag{4.272}$$

with $v_1(x) = Hu_1(x)$ and $v_2(x) = Hu_2(x)$, and make use of Eq. (4.245); then

$$\begin{aligned} &H[u_1(x)u_2(x) - v_1(x)v_2(x) + \mathrm{i}\{u_1(x)v_2(x) + u_2(x)v_1(x)\}] \\ &\quad = u_1(x)v_2(x) + u_2(x)v_1(x) - \mathrm{i}\{u_1(x)u_2(x) - v_1(x)v_2(x)\}. \end{aligned} \tag{4.273}$$

Taking the imaginary part of this result leads to

$$H\{u_1(x)v_2(x) + u_2(x)v_1(x)\} = v_1(x)v_2(x) - u_1(x)u_2(x); \tag{4.274}$$

that is,

$$H\{u_1(x)Hu_2(x) + u_2(x)Hu_1(x)\} = Hu_1(x)\, Hu_2(x) - u_1(x)u_2(x). \tag{4.275}$$

With the identifications $f(x) = u_1(x)$ and $g(x) = u_2(x)$, Eq. (4.275) converts to Eq. (4.270). The real part of Eq. (4.273) leads to

$$u_1(x)Hu_2(x) + u_2(x)Hu_1(x) = H\{u_1(x)u_2(x) - Hu_1(x)\, Hu_2(x)\}. \tag{4.276}$$

Tricomi (1951a, 1985) proved Eq. (4.270) in the following manner. Define the following two functions,

$$\Phi_1(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{u_1(t) + \mathrm{i}v_1(t)}{z - t}\, \mathrm{d}t \tag{4.277}$$

and

$$\Phi_2(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{u_2(t) + \mathrm{i}v_2(t)}{z - t}\, \mathrm{d}t, \tag{4.278}$$

which are analytic for $\mathrm{Im}\, z > 0$. For positive constants K_1 and K_2, these functions satisfy

$$\int_{-\infty}^{\infty} |\Phi_1(x + \mathrm{i}y)|^{p_1}\, \mathrm{d}x < K_1 \tag{4.279}$$

and

$$\int_{-\infty}^{\infty} |\Phi_2(x + \mathrm{i}y)|^{p_2}\, \mathrm{d}x < K_2, \tag{4.280}$$

with $p_1 > 1$ and $p_2 > 1$. Setting $r = p_1 p_2 (p_1 + p_2)^{-1}$, which satisfies $r > 1$ (by Eq. (4.269)), and $\Phi(z) = \Phi_1(z)\Phi_2(z)$, application of Hölder's inequality for conjugate exponents p and q leads, on setting $p = r^{-1}p_1$ and $q = r^{-1}p_2$, to the following:

$$\begin{aligned}\int_{-\infty}^{\infty} |\Phi(x + \mathrm{i}y)|^{r}\, \mathrm{d}x &\le \left\{\int_{-\infty}^{\infty} |\Phi_1(x + \mathrm{i}y)|^{rp}\, \mathrm{d}x\right\}^{1/p} \left\{\int_{-\infty}^{\infty} |\Phi_2(x + \mathrm{i}y)|^{rq}\, \mathrm{d}x\right\}^{1/q} \\ &< K_1^{r/p_1} K_2^{r/p_2}.\end{aligned} \tag{4.281}$$

Hence, $\Phi \in L^r$ for $r > 1$. In the limit $y \to 0+$

$$\mathrm{Re}\, \Phi(x + \mathrm{i}0) = -H\{\mathrm{Im}\, \Phi(x + \mathrm{i}0)\}. \tag{4.282}$$

Equation (4.282) can be arrived at by a contour integration approach. The case $r = 2$ is discussed in some detail in Section 4.22. From Eqs. (4.277) and (4.278), it follows that

$$\lim_{y\to 0} \operatorname{Re} \Phi_1(x + \mathrm{i}y) = u_1(x) + \mathrm{i}v_1(x) \tag{4.283}$$

and

$$\lim_{y\to 0} \operatorname{Re} \Phi_2(x + \mathrm{i}y) = u_2(x) + \mathrm{i}v_2(x). \tag{4.284}$$

Hence, from Eq. (4.282) it follows that

$$u_1(x)u_2(x) - v_1(x)v_2(x) = -H\{u_1(x)v_2(x) + u_2(x)v_1(x)\}, \tag{4.285}$$

which can be rewritten as follows:

$$H\{u_1(x)Hu_2(x) + u_2(x)Hu_1(x)\} = Hu_1(x)\, Hu_2(x) - u_1(x)u_2(x). \tag{4.286}$$

With the substitutions $f(x) = u_1(x)$ and $g(x) = u_2(x)$, the required result is obtained.

An alternative approach to the Tricomi formula due to Rooney (1975) is now sketched. To proceed, suppose that $\phi_1 \in L^{p_1}(\mathbb{R})$ and $\phi_2 \in L^{p_2}(\mathbb{R})$, where

$$\frac{1}{p_1} + \frac{1}{p_2} \leq 1. \tag{4.287}$$

Let

$$p^{-1} = p_1^{-1} + p_2^{-1}, \tag{4.288}$$

then

$$\frac{1}{p_1 p^{-1}} + \frac{1}{p_2 p^{-1}} = 1, \tag{4.289}$$

and Hölder's inequality (see Eqs. (3.78) and (3.79)) applied to the two functions $f \in L^{p_1 p^{-1}}$ and $g \in L^{p_2 p^{-1}}$, with $1 < p_1 p^{-1} < \infty$, yields

$$\int_{-\infty}^{\infty} |f(x)g(x)|\mathrm{d}x \leq \left(\int_{-\infty}^{\infty} |f(x)|^{p_1 p^{-1}}\, \mathrm{d}x\right)^{pp_1^{-1}} \left(\int_{-\infty}^{\infty} |g(x)|^{p_2 p^{-1}}\, \mathrm{d}x\right)^{pp_2^{-1}}. \tag{4.290}$$

Employing the substitutions $f(x) = |\phi_1(x)|^p$ and $g(x) = |\phi_2(x)|^p$ leads to the following:

$$\left(\int_{-\infty}^{\infty} |\phi_1(x)\phi_2(x)|^p \,\mathrm{d}x\right)^{p^{-1}} \leq \left(\int_{-\infty}^{\infty} |\phi_1(x)|^{p_1} \,\mathrm{d}x\right)^{p_1^{-1}} \left(\int_{-\infty}^{\infty} |\phi_2(x)|^{p_2} \,\mathrm{d}x\right)^{p_2^{-1}}. \tag{4.291}$$

This result can be expressed in the following more compact form:

$$\|\phi_1\phi_2\|_{L^p(\mathbb{R})} \leq \|\phi_1\|_{L^{p_1}(\mathbb{R})}\|\phi_2\|_{L^{p_2}(\mathbb{R})}, \tag{4.292}$$

which can be written more concisely as

$$\|\phi_1\phi_2\|_p \leq \|\phi_1\|_{p_1}\|\phi_2\|_{p_2}, \tag{4.293}$$

with the understanding that the integrals appearing in Eq. (4.293) are taken to be Lebesgue integrals.

A second key idea in the proof that follows is the lemma: if $f_1 \in L^1(\mathbb{R})$, $f_2 \in L^2(\mathbb{R})$, and the Fourier transforms of the functions satisfy

$$\mathcal{F}f_1(x) = \mathcal{F}f_2(x), \tag{4.294}$$

almost everywhere, then $f_1(x) = f_2(x)$ *a.e.* This result can be established by taking the inverse Fourier transform of Eq. (4.294). Rooney proved the Tricomi identity for functions that satisfy $\phi_1 \in L^{p_1}(\mathbb{R})$ and $\phi_2 \in L^{p_2}(\mathbb{R})$, where

$$\frac{1}{p_1} + \frac{1}{p_2} < 1. \tag{4.295}$$

The proof for the particular case where the functions ϕ_1 and ϕ_2 are continuous with compact support, which represents a straightforward situation to deal with, is now examined. Let

$$f_1(x) = H\phi_1(x)\, H\phi_2(x) - \phi_1(x)\phi_2(x) \tag{4.296}$$

and

$$f_2(x) = H\{\phi_1(x)H\phi_2(x) + \phi_2(x)H\phi_1(x)\}; \tag{4.297}$$

then the essential feature of the strategy employed by Rooney is to show that Eq. (4.294) holds, and hence

$$H\{\phi_1(x)H\phi_2(x) + \phi_2(x)H\phi_1(x)\} = H\phi_1(x)\, H\phi_2(x) - \phi_1(x)\phi_2(x), \ a.e. \tag{4.298}$$

Let $\phi_i \in L^2$ for $i = 1,2$; because H is a bounded operator from L^p to L^p for $1 < p < \infty$ (discussed in detail in Section 4.20), $H\phi_i \in L^2$. It follows from Eq. (4.293) with $p = 1$ that

$$\phi_1(x)\phi_2(x) \in L^1 \tag{4.299}$$

and

$$H\phi_1(x)\, H\phi_2(x) \in L^1, \tag{4.300}$$

and from Eq. (4.296) it follows that $f_1 \in L^1$. The terms $\phi_1(x)H\phi_2(x)$ and $\phi_2(x)H\phi_1(x)$ are both in L^2 and hence so is the function $H\{\phi_1(x)H\phi_2(x) + \phi_2(x)H\phi_1(x)\}$; that is $f_2 \in L^2$. Recall the convolution formula for two functions f and g is:

$$\{f * g\}(x) = \int_{-\infty}^{\infty} f(s)g(x-s)\mathrm{d}s, \tag{4.301}$$

provided the integral exists. The Fourier transform of the product of f and g can be written as the convolution of the Fourier transforms of f and g,

$$\mathcal{F}\{fg\}(x) = \frac{1}{2\pi}\{\mathcal{F}f \; * \; \mathcal{F}g\}(x). \tag{4.302}$$

Using the substitution

$$g(x,t) = \mathcal{F}\phi_1(x-t)\mathcal{F}\phi_2(t), \tag{4.303}$$

the Fourier transform of f_1 is as follows:

$$\begin{aligned}\mathcal{F}f_1(x) &= \frac{1}{2\pi}\{\mathcal{F}H\phi_1 \; * \; \mathcal{F}H\phi_2 - \mathcal{F}\phi_1 \; * \; \mathcal{F}\phi_2\}(x)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [\mathcal{F}H\phi_1(x-t)\mathcal{F}H\phi_2(t) - g(x,t)]\mathrm{d}t\\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} [\operatorname{sgn}(x-t)\operatorname{sgn} t + 1]g(x,t)\mathrm{d}t,\end{aligned} \tag{4.304}$$

and Eq. (4.162) has been employed to obtain the last line. Hence,

$$\mathcal{F}f_1(x) = -\frac{1}{\pi}\begin{cases}\int_0^x g(x,t)\mathrm{d}t, & \text{for } x > 0\\ \int_x^0 g(x,t)\mathrm{d}t, & \text{for } x < 0,\end{cases} \tag{4.305}$$

and thus

$$\mathcal{F}f_1(x) = -\frac{1}{\pi}\operatorname{sgn} x \int_0^x g(x,t)\mathrm{d}t. \tag{4.306}$$

In a similar manner,

$$
\begin{aligned}
\mathcal{F}f_2(x) &= \mathcal{F}[H\{\phi_1 H\phi_2 + \phi_2 H\phi_1\}](x) \\
&= -\mathrm{i}\,\mathrm{sgn}\,x\, \mathcal{F}[\{\phi_1 H\phi_2 + \phi_2 H\phi_1\}](x) \\
&= -\frac{\mathrm{i}\,\mathrm{sgn}\,x}{2\pi}[\mathcal{F}\phi_1 \ * \ \mathcal{F}H\phi_2 + \mathcal{F}\phi_2 \ * \ \mathcal{F}H\phi_1](x) \\
&= -\frac{\mathrm{i}\,\mathrm{sgn}\,x}{2\pi}\int_{-\infty}^{\infty}[\{\mathcal{F}\phi_1\}(x-t)\{\mathcal{F}H\phi_2\}(t) \\
&\quad + \{\mathcal{F}\phi_2\}(t)\{\mathcal{F}H\phi_1\}(x-t)]\mathrm{d}t \\
&= -\frac{\mathrm{sgn}\,x}{2\pi}\int_{-\infty}^{\infty}[\mathrm{sgn}\,t + \mathrm{sgn}(x-t)]g(x,t)\mathrm{d}t \\
&= -\frac{\mathrm{sgn}\,x}{\pi}\begin{cases} \int_0^x g(x,t)\mathrm{d}t, & \text{for } x>0 \\ -\int_x^0 g(x,t)\mathrm{d}t, & \text{for } x<0, \end{cases}
\end{aligned} \tag{4.307}
$$

and hence

$$
\mathcal{F}f_2(x) = -\frac{1}{\pi}\mathrm{sgn}\,x\int_0^x g(x,t)\mathrm{d}t. \tag{4.308}
$$

From Eq. (4.294) it therefore follows that $f_1(x) = f_2(x)$ *a.e.*, which proves the Tricomi identity for continuous functions with compact support. The proof can be extended to cover the case where $\phi_i \in L^{p_i}(\mathbb{R}), i = 1, 2$, with $p_1 > 1$ and $p_1^{-1} + p_2^{-1} < 1$.

Love (1977) proved the following result: if f and g are complex-valued functions and $f \in L^p, g \in L^q$, with $p > 1$ and $p^{-1} + q^{-1} = 1$, then

$$
\begin{aligned}
P\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{x-s}P\int_{-\infty}^{\infty}\frac{g(t)\mathrm{d}t}{s-t} &= P\int_{-\infty}^{\infty}g(t)\mathrm{d}t\; P\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{(s-x)(t-s)} \\
&\quad - \pi^2 f(x)g(x),
\end{aligned} \tag{4.309}
$$

which the reader will recognize as the Hardy–Poincaré–Bertrand formula. On using a partial fraction decomposition, it follows that

$$
\begin{aligned}
P\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{x-s}P\int_{-\infty}^{\infty}\frac{g(t)\mathrm{d}t}{s-t} &= P\int_{-\infty}^{\infty}\frac{g(t)\mathrm{d}t}{t-x} \\
&\quad \times P\int_{-\infty}^{\infty}f(s)\left\{\frac{1}{s-x}+\frac{1}{t-s}\right\}\mathrm{d}s - \pi^2 f(x)g(x) \\
&= P\int_{-\infty}^{\infty}\frac{g(t)\mathrm{d}t}{x-t}P\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{x-s} \\
&\quad - P\int_{-\infty}^{\infty}\frac{g(t)\mathrm{d}t}{x-t}P\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{t-s} - \pi^2 f(x)g(x).
\end{aligned} \tag{4.310}
$$

The Tricomi identity for the Hilbert transform follows from this formula. Love's approach extends Tricomi's and Rooney's results to include the case $p^{-1} + q^{-1} = 1$.

Equation (4.270) can be recast, using the iteration property of the Hilbert transform, as follows:

$$H\{f(x)g(x)\} = H\{Hf(x)Hg(x)\} + f(x)Hg(x) + g(x)Hf(x), \tag{4.311}$$

which gives the Hilbert transform of a product of functions satisfying the conditions indicated in Tricomi's theorem. In many applications, evaluation of the right-hand side of Eq. (4.311) will not represent a computational simplification, relative to the direct evaluation of the Hilbert transform of the product. Three examples are examined; the first leads to a more involved Hilbert transform to be evaluated, then two cases are considered in which a straightforward calculation arises. As a first example, consider

$$f(x) = \frac{1}{1+x^2} \tag{4.312}$$

and

$$g(x) = \frac{1}{1+x^4}. \tag{4.313}$$

On using the results

$$H\left(\frac{1}{1+x^2}\right) = \frac{x}{1+x^2} \tag{4.314}$$

and

$$H\left(\frac{1}{1+x^4}\right) = \frac{x^3+x}{\surd 2(x^4+1)}, \tag{4.315}$$

Eq. (4.311) leads to

$$\begin{aligned} H\{f(x)g(x)\} = H\left\{\frac{x^4+x^2}{\surd(2)(x^4+1)(x^2+1)}\right\} + \frac{x^3+x}{\surd(2)(x^4+1)(x^2+1)} \\ + \frac{x}{(x^4+1)(x^2+1)}, \end{aligned} \tag{4.316}$$

which simplifies on using

$$H\left\{\frac{x^4+x^2}{\surd(2)(x^4+1)(x^2+1)}\right\} = \frac{x^3-x}{2(x^4+1)} \tag{4.317}$$

to give

$$H\{f(x)g(x)\} = \frac{x(1+\surd(2)) + x^3\surd 2 + x^5}{2(x^4+1)(x^2+1)}, \tag{4.318}$$

which can be checked to be the correct result by direct evaluation. Clearly in this example, direct evaluation of the Hilbert transform of the product of the two functions would have been quicker and simpler.

As a second example, consider the Hilbert transform of $x(1+x^2)^{-2}$. From the product formula Eq. (4.311) and the identifications $f(x) = x(1+x^2)^{-1}$ and $g(x) = (1+x^2)^{-1}$:

$$\begin{aligned} H\left(\frac{x}{(1+x^2)^2}\right) &= H\left(\frac{x}{(1+x^2)}\frac{1}{(1+x^2)}\right) \\ &= H\left\{\left(\frac{-1}{1+x^2}\right)\left(\frac{x}{1+x^2}\right)\right\} + \left(\frac{x}{1+x^2}\right)\left(\frac{x}{1+x^2}\right) \\ &\quad + \left(\frac{1}{1+x^2}\right)\left(\frac{-1}{1+x^2}\right), \end{aligned} \tag{4.319}$$

and hence

$$H\left(\frac{x}{(1+x^2)^2}\right) = \frac{x^2-1}{2(x^2+1)^2}. \tag{4.320}$$

In this example, the product formula Eq. (4.311) results in a straightforward calculation. The essential reason for the simplification in this example is the fact that

$$H\{Hf(x)Hg(x)\} = -H\{f(x)g(x)\}. \tag{4.321}$$

The obvious question is what conditions apply for this expression to hold? The answer is easy to see by taking the Hilbert transform of this equation; the result is

$$f(x)g(x) = -Hf(x)Hg(x), \tag{4.322}$$

which clearly follows if f and g form a Hilbert pair, so that $f(x) = -Hg(x)$ and $g(x) = Hf(x)$. When this situation applies, Eq. (4.311) becomes

$$H\{f(x)g(x)\} = -H\{g(x)f(x)\} - f^2(x) + g^2(x); \tag{4.323}$$

that is,

$$H\{f(x)g(x)\} = -\frac{1}{2}\{f^2(x) - g^2(x)\}. \tag{4.324}$$

The other member of the Hilbert transform pair is

$$H\{f^2(x) - g^2(x)\} = 2f(x)g(x). \tag{4.325}$$

Equations (4.324) and (4.325) could obviously have been more directly determined by considering the analytic function $h(z)$, with $h(x) = \{f(x) + \mathrm{i}g(x)\}^2$, and integrating $h(z)(z-x_0)^{-1}$ around the appropriate contour. This bypasses the Tricomi formula. Equations (4.324) and (4.325) find application in the determination of integral constraints (sum rules) for optical constants. Parenthetically it is noted that the Hilbert transform of $x(1+x^2)^{-2}$ can be most expeditiously evaluated by applying the derivative formula for the Hilbert transform:

$$\begin{aligned} H\left(\frac{x}{(1+x^2)^2}\right) &= H\left\{-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(1+x^2)^{-1}\right\} \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}H\left\{(1+x^2)^{-1}\right\} \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}x(1+x^2)^{-1} \\ &= \frac{x^2-1}{2(x^2+1)^2}. \end{aligned} \tag{4.326}$$

As a final example of the application of Eq. (4.311), consider $f(x) = \sin ax$ and $g(x) = \cos ax$ for $a > 0$. Since these two functions form a Hilbert transform pair, it follows from Eq. (4.324) that

$$H\{\sin ax \cos ax\} = \frac{1}{2}\{\sin^2 ax - \cos^2 ax\}. \tag{4.327}$$

Using an elementary trigonometric identity, Eq. (4.327) can be written in the more familiar form $H(\sin 2ax) = -\cos 2ax$.

The Tricomi formula finds applications in other areas. It can be used effectively for the solution of certain types of singular integral equations. Problems of this type are discussed in Chapter 12. An important application lies in the study of the finite Hilbert transform, and that topic is addressed in Chapter 11.

4.17 Eigenvalues and eigenfunctions of the Hilbert transform operator

The solution of the following eigenvalue equation is now considered:

$$Hf = \eta f, \tag{4.328}$$

where η is a constant. Applying the operator H to Eq. (4.328), then

$$H^2 f = \eta Hf, \tag{4.329}$$

and application of the inversion property yields

$$\eta^2 f = -f. \tag{4.330}$$

The eigenvalues of the Hilbert transform operator are therefore

$$\eta = \pm \mathrm{i}. \tag{4.331}$$

This result can be arrived at directly by contour integration techniques, without recourse to the inversion property of the Hilbert transform.

An alternative approach to the determination of the eigenvalues of H is now examined. For a function ϕ in $L^2(-\infty,\infty)$, define

$$\phi_+ = \frac{1}{2}\{\phi - \mathrm{i}H\phi\} \tag{4.332}$$

and

$$\phi_- = \frac{1}{2}\{\phi + \mathrm{i}H\phi\}. \tag{4.333}$$

Then ϕ_+ and ϕ_- are eigenfunctions of the Hilbert transform operator:

$$H\phi_+ = \mathrm{i}\phi_+ \tag{4.334}$$

and

$$H\phi_- = -\mathrm{i}\phi_-. \tag{4.335}$$

The preceding can be generalized to functions $\phi \in L^p(-\infty,\infty)$, with $1 < p < \infty$. The problem of determining the class of all functions that are eigenfunctions of the Hilbert transform operator is still an open question (Pandey, 1996, p. 215).

The set of functions defined by

$$\phi_n(x) = \frac{(1+\mathrm{i}x)^n}{(1-\mathrm{i}x)^{n+1}}, \text{ for } n \in \mathbb{Z}^+, \tag{4.336}$$

form a complete and orthogonal basis set for L^2on the real line (Higgins, 1977). These functions are also eigenfunctions of the Hilbert transform operator with eigenvalues $-\mathrm{i}\,\mathrm{sgn}\,n$, that is

$$H\phi_n = -\mathrm{i}\,\mathrm{sgn}\,n\,\phi_n, \quad \text{for } |n| \geq 1. \tag{4.337}$$

This result is now demonstrated. The case $n \geq 0$ is studied first. Consider the contour integral $\oint_C \phi_n(z)\mathrm{d}z/(x_0 - z)$, where the contour C is shown in Figure 4.4. Using the Cauchy integral formula,

$$\oint_C \frac{\phi_n(z)\mathrm{d}z}{x_0 - z} = \int_{-R}^{x_0-\varepsilon} \frac{\phi_n(x)\mathrm{d}x}{x_0 - x} + \int_{x_0+\varepsilon}^{R} \frac{\phi_n(x)\mathrm{d}x}{x_0 - x} + \int_{\Gamma_\varepsilon} \frac{\phi_n(z)\mathrm{d}z}{x_0 - z} + \int_{\Gamma_R} \frac{\phi_n(z)\mathrm{d}z}{x_0 - z} = 0. \tag{4.338}$$

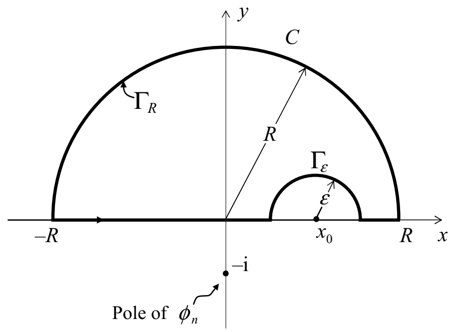

Figure 4.4. Semicircular contour in the upper half complex plane centered at the origin, with a suitable indentation to avoid the singularity at $z = x_0$.

In the limits $R \to \infty$ and $\varepsilon \to 0$, it follows that

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \left\{ \int_{-R}^{x_0-\varepsilon} \frac{\phi_n(x)\mathrm{d}x}{x_0 - x} + \int_{x_0+\varepsilon}^{R} \frac{\phi_n(x)\mathrm{d}x}{x_0 - x} \right\} = \pi H\phi_n(x_0), \tag{4.339}$$

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} \frac{\phi_n(z)\mathrm{d}z}{x_0 - z} = \pi\,\mathrm{i}\phi_n(x_0), \tag{4.340}$$

and

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{\phi_n(z)\mathrm{d}z}{x_0 - z} = 0. \tag{4.341}$$

Equation (4.338) simplifies to

$$H\phi_n(x_0) = -\mathrm{i}\phi_n(x_0), \quad \text{for } n \geq 0. \tag{4.342}$$

For the case $n < 0$, set $n = -m$ with $m > 0$, and consider the contour integral $\oint_C \phi_{-m}(z)\mathrm{d}z/(x_0 - z)$, where C is the contour shown in Figure 4.5. Evaluating this integral in a similar fashion to the steps shown in Eqs. (4.338) to (4.341) leads to the following result:

$$H\phi_n(x_0) = \mathrm{i}\,\phi_n(x_0), \quad \text{for } n < 0. \tag{4.343}$$

Combining Eqs. (4.345) and (4.346) yields

$$H\phi_n(x_0) = -\mathrm{i}\,\mathrm{sgn}\, n\; \phi_n(x_0), \quad \text{with } |n| \geq 1, \tag{4.344}$$

and $H\phi_0 = -\mathrm{i}\phi_0$, which establishes that the ϕ_n are eigenfunctions of the Hilbert transform operator with eigenvalues $-\mathrm{i}\,\mathrm{sgn}\, n$.

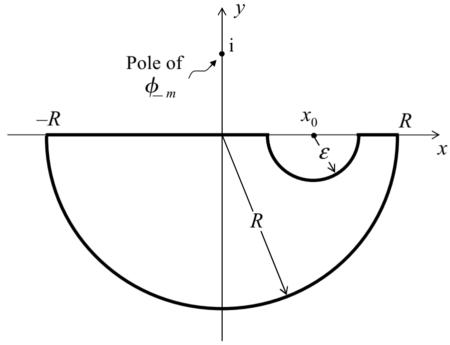

Figure 4.5. Semicircular contour in the lower half complex plane centered at the origin, with a suitable indentation to avoid the singularity at $z = x_0$.

The eigenvalue formula just given can be used to generate formulas for a number of Hilbert transforms of certain rational functions. For example, if $n = 2$ it follows that

$$H \operatorname{Re} \phi_2(x) = \operatorname{Im} \phi_2(x) \tag{4.345}$$

and

$$H \operatorname{Im} \phi_2(x) = -\operatorname{Re} \phi_2(x), \tag{4.346}$$

where

$$\phi_2(x) = \frac{1 - 10x^2 + 5x^4 + \mathrm{i}(5x - 10x^3 + x^5)}{(x^2+1)^3}. \tag{4.347}$$

Hence

$$H\left\{\frac{1 - 10x^2 + 5x^4}{(x^2+1)^3}\right\} = \frac{(5x - 10x^3 + x^5)}{(x^2+1)^3} \tag{4.348}$$

and

$$H\left\{\frac{5x - 10x^3 + x^5}{(x^2+1)^3}\right\} = -\frac{(1 - 10x^2 + 5x^4)}{(x^2+1)^3}. \tag{4.349}$$

Equation (4.348) can be quickly checked by referring to Appendix 1 using the Hilbert transform table entries (2.28), (2.35), and (2.40), and Eq. (4.349) can be verified using the entries (2.33), (2.38), and (2.42).

If f and g are both eigenfunctions of H with eigenvalues α and β, respectively, then, from the Tricomi formula for H, Eq. (4.270), and assuming $\alpha + \beta \neq 0$,

it follows that

$$H\{f(x)g(x)\} = \frac{\alpha\beta - 1}{\alpha + \beta} f(x)g(x), \tag{4.350}$$

and for the case $f(x) = g(x)$ and $\alpha = \beta, \quad$ for $\alpha \neq 0$,

$$H\{f(x)^2\} = \frac{1}{2}(\alpha - \alpha^{-1})f(x)^2. \tag{4.351}$$

The two cases are: $\alpha = \beta = \mathrm{i}$, which gives $H\{f(x)^2\} = \mathrm{i}f(x)^2$; and $\alpha = \beta = -\mathrm{i}$, leading to $H\{f(x)^2\} = -\mathrm{i}f(x)^2$. From the Tricomi formula, the case $\alpha = \pm\,\mathrm{i}, \beta = \mp\,\mathrm{i}$, leads to the identity $0 = 0$, and Eq. (4.350) does not of course apply.

4.18 Projection operators

Projection operators involving the Hilbert transform can be set up in the following way. Define the operators P_+ and P_- by

$$P_+ = \frac{1}{2}(I + \mathrm{i}H) \tag{4.352}$$

and

$$P_- = \frac{1}{2}(I - \mathrm{i}H), \tag{4.353}$$

where I denotes the unit operator. These are sometimes termed the Riesz projectors. From these definitions it follows that

$$P_+ + P_- = I. \tag{4.354}$$

These operators satisfy the idempotent conditions

$$P_+^2 = P_+ \tag{4.355}$$

and

$$P_-^2 = P_-, \tag{4.356}$$

since

$$\begin{aligned} P_+^2 &= \frac{1}{4}(I + 2\mathrm{i}H - H^2) \\ &= \frac{1}{2}(I + \mathrm{i}H) \\ &= P_+, \end{aligned} \tag{4.357}$$

and the iteration property of the Hilbert transform has been employed. The idempotent condition for P_- is demonstrated in the same straightforward manner.

The operators P_+ and P_- are an orthogonal pair of operators:

$$P_+P_- = P_-P_+ = 0. \tag{4.358}$$

Making use of the iteration property of the Hilbert transform, this result is demonstrated simply as follows:

$$\begin{aligned} P_+P_- &= \frac{1}{4}(I + \mathrm{i}H)(I - \mathrm{i}H) \\ &= \frac{1}{4}(I + H^2) = 0. \end{aligned} \tag{4.359}$$

Suppose the functions f and g satisfy $f \in L^p, g \in L^q$, for $1 < p < \infty$, and q is the conjugate exponent; then the following results hold:

$$\int_{-\infty}^{\infty} P_+f(x)P_+g(x)\mathrm{d}x = 0, \tag{4.360}$$

$$\int_{-\infty}^{\infty} P_-f(x)P_-g(x)\mathrm{d}x = 0, \tag{4.361}$$

and

$$\int_{-\infty}^{\infty} f(x)P_+P_-g(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)P_-P_+g(x)\mathrm{d}x = 0. \tag{4.362}$$

The last result follows directly from Eq. (4.358). Equation (4.360) can be established as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} P_+f(x)P_+g(x)\mathrm{d}x = \frac{1}{4}\Bigg\{&\int_{-\infty}^{\infty} f(x)g(x)\mathrm{d}x - \int_{-\infty}^{\infty} Hf(x)Hg(x)\mathrm{d}x \\ &+\mathrm{i}\int_{-\infty}^{\infty} \{f(x)Hg(x) + g(x)Hf(x)\}\mathrm{d}x\Bigg\} = 0, \end{aligned} \tag{4.363}$$

and Eqs. (4.174) and (4.176) have been employed. Equation (4.361) follows in a similar manner.

4.19 A theorem due to Akhiezer

Akhiezer (alternative spellings Achieser and Aheizer) (1956, p.129) established the following result. For $f \in L^2(\mathbb{R})$ and $\gamma = a + \mathrm{i}b$, with $b \neq 0$,

$$(x - \gamma)H\left[\frac{f(x)}{x - \gamma}\right] = Hf(x) - C(\gamma, f), \tag{4.364}$$

where $C(\gamma,f)$ is a constant depending on γ and f. Akhiezer also obtained the result that if γ is real and both f and $(x-\gamma)^{-1}(f(x)-\alpha) \in L^2(\mathbb{R})$, for α a constant,

$$(x-\gamma)H\left[\frac{f(x)-\alpha}{x-\gamma}\right] = Hf(x) - C(\alpha,\gamma,f). \tag{4.365}$$

Akhiezer actually evaluated the case $H\{(x-\mathrm{i})^{-1}f(x)\}$ utilizing Fourier transform techniques, and indicated Eq. (4.365) as a concluding comment to this example. The transform $H\{(x-\mathrm{i})^{-1}f(x)\}$ can be simplified using the following result:

$$(x-\mathrm{i})H\left\{\frac{f(x)}{x-\mathrm{i}}\right\} = Hf(x) + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{t-\mathrm{i}}. \tag{4.366}$$

The following result is needed:

$$H\{(x+a)f_1(x)\} = (x+a)Hf_1(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} f_1(t)\mathrm{d}t, \tag{4.367}$$

which was derived in Section 4.7 (see Eq. (4.136)). If the substitutions $a=-\mathrm{i}$ and $f_1(x) = f(x)/(x-\mathrm{i})$ are introduced into this equation, then Eq. (4.366) follows immediately. Assume that $f_1 \in L^1_{\mathrm{loc}}$. Equation (4.366) can also be established by considering the contour integrals of the functions $(z-\mathrm{i})^{-1}f(z)$, $(z-t)^{-1}f(z)$, and $\{(z-\mathrm{i})(z-t)\}^{-1}f(z)$, where the contour is the standard semicircle in the upper half complex plane, with center at the origin and a suitable indentation of the contour around the point t on the real axis, for the latter two functions. It is implicitly assumed that the behavior of $f(x)$ is such that by the Phragmén–Lindelöf theorem, the integral on the semicircular section of the contour can be deduced to vanish in the limit of infinite radius.

Equation (4.364) can be derived in a similar fashion:

$$H\{(x-\gamma)f_1(x)\} = (x-\gamma)Hf_1(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} f_1(t)\mathrm{d}t, \tag{4.368}$$

and hence

$$Hf(x) = (x-\gamma)H\left\{\frac{f(x)}{x-\gamma}\right\} - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{t-\gamma}, \tag{4.369}$$

which is Eq. (4.364) with the identification that $C(\gamma,f) = -\pi^{-1}\int_{-\infty}^{\infty}(t-\gamma)^{-1}f(t)\mathrm{d}t$. Equation (4.365) can be obtained in a similar fashion starting from Eq. (4.367) and using the substitutions $a=-\gamma$ and $f_1(x) = (x-\gamma)^{-1}(f(x)-\alpha)$, leading to

$$H\{f(x)-\alpha\} = (x-\gamma)H\left\{\frac{f(x)-\alpha}{x-\gamma}\right\} - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(f(t)-\alpha)\mathrm{d}t}{t-\gamma}. \tag{4.370}$$

Recalling that the Hilbert transform of a constant is zero, and identifying that $C(\alpha,\gamma,f) = -\pi^{-1}\int_{-\infty}^{\infty}(t-\gamma)^{-1}(f(t)-\alpha)\mathrm{d}t$, gives Eq. (4.365). For $\gamma \in \mathbb{R}$, the preceding integral is taken in the principal value sense, and hence

$$C(\alpha, \gamma, f) \equiv Hf(\gamma). \tag{4.371}$$

Another pair of formulas of a similar type to Eqs. (4.364) and (4.365) is as follows:

$$(x+a)(x+b)H\left[\frac{f(x)}{(x+a)(x+b)}\right] = Hf(x) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(t+x+a+b)f(t)\mathrm{d}t}{(t+a)(t+b)} \tag{4.372}$$

and

$$(x^2+a^2)H\left[\frac{f(x)}{(x^2+a^2)}\right] = Hf(x) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(t+x)f(t)\mathrm{d}t}{(t^2+a^2)}. \tag{4.373}$$

The derivation of these results makes use of Eq. (4.367), and is left as an exercise for the reader.

Equation (4.373) finds application in the determination of selected Hilbert transforms. As an example, the Hilbert transform of $f(x) = (x^2+1)^{-\alpha}$ for $\alpha > 0$ is evaluated. This case was studied by Kochneff, Sagher, and Tan (1993), and their approach is now considered. Making use of Eqs. (4.111) and (4.137), it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}H\left[\frac{1}{(x^2+1)^{\alpha}}\right] &= -2\alpha H\left[\frac{x}{(x^2+1)^{\alpha+1}}\right] \\ &= -2\alpha\left\{xH\left[\frac{1}{(x^2+1)^{\alpha+1}}\right] - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(t^2+1)^{\alpha+1}}\right\}.\end{aligned} \tag{4.374}$$

Let the constant $C(\alpha)$ denote the following integral:

$$C(\alpha) = \frac{2\alpha}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(t^2+1)^{\alpha+1}}, \tag{4.375}$$

and set $g(x) = Hf(x)$. Making use of Eq. (4.373), with $a = 1$ and $f(x) = (x^2+1)^{-\alpha}$, leads to

$$\begin{aligned}g'(x) &= -\frac{2\alpha x}{x^2+1}\left[g(x) + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x+t)\mathrm{d}t}{(t^2+1)^{\alpha+1}}\right] + C(\alpha) \\ &= -\frac{2\alpha x}{x^2+1}g(x) + \frac{C(\alpha)}{x^2+1},\end{aligned} \tag{4.376}$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^2+1)^{\alpha}g(x)\right] = (x^2+1)^{\alpha-1}C(\alpha). \tag{4.377}$$

Integration of this equation leads to

$$g(x) = \frac{C(\alpha)}{(x^2+1)^{\alpha}} \left\{ c + \int_0^x (t^2+1)^{\alpha-1} \mathrm{d}t \right\}, \tag{4.378}$$

where c is an arbitrary constant. The starting function is even, hence $g(x)$ is odd (recall Section 4.2), and therefore the constant c is zero. The constant $C(\alpha)$ can be evaluated in terms of gamma functions (see Eq. (4.118)) as follows:

$$C(\alpha) = \frac{2\alpha}{\sqrt{\pi}} \frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha+1)}, \tag{4.379}$$

and hence

$$H\{(x^2+1)^{-\alpha}\} = \frac{2\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}\Gamma(\alpha+1)(x^2+1)^{\alpha}} \int_0^x (t^2+1)^{\alpha-1}\, \mathrm{d}t. \tag{4.380}$$

The integral $\int_0^x (t^2+1)^{\alpha-1}\,\mathrm{d}t$ can be expressed in terms of the hypergeometric function ${}_2F_1(1/2, 1-\alpha; 3/2; -x^2)$. A particular simple case of Eq. (4.380) occurs for $\alpha = 1/2$, which leads to the following result:

$$H\left(\frac{1}{\sqrt{(x^2+1)}}\right) = \frac{2\sinh^{-1} x}{\pi\sqrt{(x^2+1)}} = \frac{2\log(x+\sqrt{(x^2+1)})}{\pi\sqrt{(x^2+1)}}. \tag{4.381}$$

4.20 The Riesz inequality

Marcel Riesz (1924, 1927) established the following result. If f belongs to the class $L^p(\mathbb{R})$ for $1 < p < \infty$, then

$$\int_{-\infty}^{\infty} |Hf(x)|^p\, \mathrm{d}x \le \{\Re_p\}^p \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x, \tag{4.382}$$

where $\Re_p$ is the Riesz constant, which depends only on p. The notation C_p in place of $\Re_p$ is also common. The C_p notation will be employed in this book when the Riesz inequality occurs in the middle of a derivation, and where no particular interest attaches to the value of the constants occurring. For $p = 2, \Re_p = 1$, and equality holds (see Eq. (4.172)). Inequalities of the form

$$\|Tf\|_p \le C\,\|f\|_p\,, \tag{4.383}$$

where T is an operator, such as the Hilbert transform operator, are referred to as *strong-type* estimates. The best value of the constant $\Re_p$ for $1 < p < \infty$ was found by Pichorides (1972) to be

$$\Re_p = \begin{cases} \tan \pi/2p, & 1 < p \le 2 \\ \cot \pi/2p, & 2 \le p < \infty. \end{cases} \tag{4.384}$$

The terminology *best constant* means that the result in Eq. (4.382) for some choice of f would be false if the constant in Eq. (4.384) were replaced by a smaller constant. The terminology *sharp* is applied to an inequality when the best constant is provided. The inequality constant is also called optimal when the best value is given. Equation (4.382) is one of the most important results in the theory of Hilbert transforms. Riesz's inequality establishes that the Hilbert transform is a bounded operator on $L^p(\mathbb{R})$ for $1 < p < \infty$.

For vector spaces X and Y, a bounded linear operator L from $X \to Y$ is termed continuous if, for a sequence $\{f_n\}_{n=1}^{\infty}$ in X, and as $f_n \to f$ in X, it follows that $Lf_n \to Lf$ in Y. The converse statement, a continuous linear operator is bounded, is also true. The Hilbert transform operator is a continuous operator, because as $f_n \to f$ in $L^p(\mathbb{R})$, for $1 < p < \infty, Hf_n \to Hf$ in $L^p(\mathbb{R})$. The statements that the Hilbert transform operator is continuous and that the Hilbert transform operator is bounded are used synonymously, though the latter is more prevalent in customary usage, and this practice is adhered to in the present work.

There has been a considerable amount of work associated with the Riesz result, and there exist a number of different proofs of this formula. In this section attention is focused on the result stated in Eq. (4.382), and in Section 6.17 the corresponding result for periodic functions is discussed. In both these cases complex analysis plays a central role. Riesz's inequality can also be proved using real-variable methods, and that approach is discussed in Section 7.1.

Before embarking on a proof of the general result, the particular cases $p = 2$ and $p = 4$ are examined. The condition $p = 2$ is an important special case, for, as remarked previously, this leads to an equality with $\Re_2 = 1$ (Hardy, 1924b, 1932; Kober, 1943b; Titchmarsh, 1925a), and Eq. (4.382) reduces to Eq. (4.172). This special case also shows up in a number of practical applications. Suppose $F \in L^p(\mathbb{R})$ and that $F(z)$ is analytic in the upper half complex plane. Consider the contour integral $\int_C F(z)^p \,\mathrm{d}z$, where the contour C is a semicircle in the upper half plane, radius R, centered at the origin. It is assumed that $F(z)^p$ vanishes sufficiently quickly as $|z| \to \infty$, so that the contribution from the semicircular section of the contour vanishes as $R \to \infty$. Then, by the Cauchy integral theorem,

$$\int_{-\infty}^{\infty} F(x)^p \,\mathrm{d}x = 0. \tag{4.385}$$

If

$$F(x) = u(x) + \mathrm{i}v(x), \tag{4.386}$$

where $u(x)$ and $v(x)$ are real-valued and $v(x) = Hu(x)$ (using Eq. (3.76)), then, for the case $p = 2$, it follows immediately from Eq. (4.385) that

$$\int_{-\infty}^{\infty} u(x)^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} v(x)^2 \,\mathrm{d}x, \tag{4.387}$$

which establishes the equality sign for Eq. (4.382) for this particular choice of p. Consider the case $p = 4$, then Eq. (4.385) yields

$$\int_{-\infty}^{\infty} \{u(x)^4 + v(x)^4 - 6u(x)^2 v(x)^2\} \mathrm{d}x = 0. \tag{4.388}$$

Let C denote a positive constant, not necessarily the same at each occurrence, then the Cauchy–Schwarz–Buniakowski inequality yields

$$\begin{aligned}
\int_{-\infty}^{\infty} v(x)^4 \, \mathrm{d}x &= \int_{-\infty}^{\infty} u(x)^2 \{6v(x)^2 - u(x)^2\} \, \mathrm{d}x \\
&\leq \left\{\int_{-\infty}^{\infty} u(x)^4 \, \mathrm{d}x\right\}^{1/2} \left\{\int_{-\infty}^{\infty} \{6v(x)^2 - u(x)^2\}^2 \, \mathrm{d}x\right\}^{1/2} \\
&= \left\{\int_{-\infty}^{\infty} u(x)^4 \, \mathrm{d}x\right\}^{1/2} \left\{\int_{-\infty}^{\infty} \{34v(x)^4 - u(x)^4\} \mathrm{d}x\right\}^{1/2} \\
&\leq C \left\{\int_{-\infty}^{\infty} u(x)^4 \, \mathrm{d}x\right\}^{1/2} \left\{\int_{-\infty}^{\infty} v(x)^4 \, \mathrm{d}x\right\}^{1/2},
\end{aligned} \tag{4.389}$$

and hence

$$\int_{-\infty}^{\infty} v(x)^4 \, \mathrm{d}x \leq C \int_{-\infty}^{\infty} u(x)^4 \, \mathrm{d}x, \tag{4.390}$$

which proves the Riesz inequality for the case $p = 4$. Some further discussion of this approach is deferred to the exercises.

An extension of the preceding pair of results can be given. A digression is made for a moment to discuss the Minkowski inequality. For $y \geq 0$ and $0 < \alpha < 1$,

$$(1 + y)^\alpha \leq 1 + y^\alpha, \tag{4.391}$$

and substituting $y = |f(x)||g(x)|^{-1}$ yields

$$|f(x) + g(x)|^\alpha \leq |f(x)|^\alpha + |g(x)|^\alpha . \tag{4.392}$$

Integrating this result gives (assuming the integrals exist) Minkowski's inequality:

$$\int_a^b |f(x) + g(x)|^\alpha \, \mathrm{d}x \leq \int_a^b |f(x)|^\alpha \, \mathrm{d}x + \int_a^b |g(x)|^\alpha \, \mathrm{d}x, \quad 0 < \alpha < 1. \tag{4.393}$$

For the case $\alpha \geq 1$,

$$\left\{\int_a^b |f(x) + g(x)|^\alpha \, \mathrm{d}x\right\}^{\alpha^{-1}} \leq \left\{\int_a^b |f(x)|^\alpha \, \mathrm{d}x\right\}^{\alpha^{-1}} + \left\{\int_a^b |g(x)|^\alpha \, \mathrm{d}x\right\}^{\alpha^{-1}} . \tag{4.394}$$

This result can be obtained in the following manner. Start with the identity

$$(|f| + |g|)^\alpha = |f|(|f| + |g|)^{\alpha-1} + |g|(|f| + |g|)^{\alpha-1}, \tag{4.395}$$

integrate both sides and apply the Hölder inequality to both terms on the right-hand side of the equation, to obtain

$$\begin{aligned}\int_a^b (|f(x)| + |g(x)|)^\alpha \,\mathrm{d}x &= \int_a^b |f(x)|(|f(x)| + |g(x)|)^{\alpha-1} \,\mathrm{d}x \\ &\quad + \int_a^b |g(x)|(|f(x)| + |g(x)|)^{\alpha-1} \,\mathrm{d}x \\ &\leq \left\{\int_a^b |f(x)|^p \,\mathrm{d}x\right\}^{p^{-1}} \left\{\int_a^b (|f(x)| + |g(x)|)^{(\alpha-1)q} \,\mathrm{d}x\right\}^{q^{-1}} \\ &\quad + \left\{\int_a^b |g(x)|^p \,\mathrm{d}x\right\}^{p^{-1}} \left\{\int_a^b (|f(x)| + |g(x)|)^{(\alpha-1)q} \,\mathrm{d}x\right\}^{q^{-1}},\end{aligned} \tag{4.396}$$

where p and q are conjugate exponents satisfying $p^{-1} + q^{-1} = 1$. Setting $p = \alpha$ yields

$$\begin{aligned}&\int_a^b (|f(x)| + |g(x)|)^\alpha \,\mathrm{d}x \leq \\ &\left[\left\{\int_a^b |f(x)|^\alpha \,\mathrm{d}x\right\}^{\alpha^{-1}} + \left\{\int_a^b |g(x)|^\alpha \,\mathrm{d}x\right\}^{\alpha^{-1}}\right]\left\{\int_a^b (|f(x)| + |g(x)|)^\alpha \,\mathrm{d}x\right\}^{(\alpha-1)\alpha^{-1}},\end{aligned} \tag{4.397}$$

and hence, for $\alpha \geq 1$,

$$\left\{\int_a^b (|f(x)| + |g(x)|)^\alpha \,\mathrm{d}x\right\}^{\alpha^{-1}} \leq \left\{\int_a^b |f(x)|^\alpha \,\mathrm{d}x\right\}^{\alpha^{-1}} + \left\{\int_a^b |g(x)|^\alpha \,\mathrm{d}x\right\}^{\alpha^{-1}}, \tag{4.398}$$

from which Eq. (4.394) follows.

Using the fact that $F(z)^2$ is analytic in the upper half complex plane, and noting $F(x)^2 = u(x)^2 - v(x)^2 + 2\mathrm{i}u(x)v(x)$, gives, on setting $f = u$,

$$[Hf(x)]^2 = f(x)^2 + 2H\{f(x)Hf(x)\}. \tag{4.399}$$

Suppose the following result holds:

$$\|Hf\|_p \leq C_p \|f\|_p\,. \tag{4.400}$$

Making use of Eq. (4.399) it follows that

$$\begin{aligned}
\|Hf\|_{2p}^2 &= \left\{\int_{-\infty}^{\infty} [f^2(x) + 2\{H(fHf)\}(x)]^p \, dx\right\}^{p^{-1}} \\
&\le \left\{\int_{-\infty}^{\infty} |f(x)|^{2p} \, dx\right\}^{p^{-1}} + 2\left\{\int_{-\infty}^{\infty} |\{H(fHf)\}(x)|^p \, dx\right\}^{p^{-1}} \\
&\le \left\|f^2\right\|_p + 2C_p \left\{\int_{-\infty}^{\infty} |(fHf)(x)|^p \, dx\right\}^{p^{-1}} \\
&\le \|f\|_{2p}^2 + 2C_p \|f\|_{2p} \|Hf\|_{2p}\,, \qquad (4.401)
\end{aligned}$$

and the Minkowski and Cauchy–Schwarz–Buniakowski inequalities have been employed. The final result is in the form of a quadratic inequality for the variable $\|Hf\|_{2p}$. Solving this inequality gives

$$\|Hf\|_{2p} \le C_{2p} \|f\|_{2p}\,, \qquad (4.402)$$

with

$$C_{2p} = C_p + \sqrt{(C_p^2 + 1)}. \qquad (4.403)$$

Hence, if Eq. (4.400) can be established for the special case $p = 2$, then Eq. (4.402) indicates the inequality holds for the cases $p = 2^q$ with $q \in \mathbb{N}$.

A simplified proof of Eq. (4.382), without regard to establishing the optimal value of $\Re_p$, is now examined. The following is based on a proof due to Calderón (1950, 1966). The proof is broken into two parts, the first having $1 < p \le 2$ and the second having $2 < p < \infty$. Suppose the complex function $w = u + iv$ has $u > 0$, then, for $1 < p \le 2$, the following inequality holds:

$$|v|^p \le A_p u^p - B_p \operatorname{Re}(w^p), \qquad (4.404)$$

where A_p and B_p are positive constants that depend only on p. The proof of this result is as follows. Let $w = Re^{i\theta}$, then Eq. (4.404) can be expressed as follows:

$$|\sin\theta|^p \le A_p \cos^p\theta - B_p \cos p\theta. \qquad (4.405)$$

This result can be established as follows for $-\pi/2 \le \theta \le \pi/2$ and $1 < p \le 2$. For $\theta = \pm\pi/2, \cos p\theta < 0$, and in a small neighborhood of these two values of θ the constant B_p can be selected sufficiently large such that $-B_p \cos p\theta > 1$. In any closed subinterval of $(-\pi/2, \pi/2), \cos^p\theta > 0$, and on selecting A_p sufficiently large, we have $A_p \cos^p\theta - B_p \cos p\theta > 1$, and hence Eq. (4.405) follows. This proves Eq. (4.404). A generalization of Eq. (4.405) is established in Section 6.17. Let

$$w(z) = U(x, y) + iV(x, y), \qquad (4.406)$$

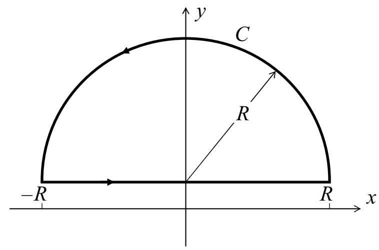

Figure 4.6. Semicircular contour with center at $(0, \mathrm{i}y)$ and diameter parallel to the x-axis.

where $w(z)$ is analytic for $y > 0$ and satisfies $w(z) \sim z^{-1}$ as $z \to \pm\infty$, and obviously $w(z)^p \sim z^{-p}$ as $z \to \pm\infty$. Apply the inequality Eq. (4.404) followed by integration over the real axis to obtain

$$\int_{-\infty}^{\infty} |V(x,y)|^p \,\mathrm{d}x \le A_p \int_{-\infty}^{\infty} |U(x,y)|^p \,\mathrm{d}x - B_p \int_{-\infty}^{\infty} \mathrm{Re}\{w(x+\mathrm{i}y)^p\}\mathrm{d}x. \quad (4.407)$$

Consider the integral $\oint_C w(z)\mathrm{d}z$, where C is the contour shown in Figure 4.6. Using the Cauchy integral theorem leads in the limit $R \to \infty$ to

$$\int_{-\infty}^{\infty} w(x+\mathrm{i}y)^p \,\mathrm{d}x = 0, \quad (4.408)$$

and hence

$$\int_{-\infty}^{\infty} \mathrm{Re}\{w(x+\mathrm{i}y)^p\}\mathrm{d}x = 0. \quad (4.409)$$

Thus Eq. (4.407) simplifies to

$$\int_{-\infty}^{\infty} |V(x,y)|^p \,\mathrm{d}x \le A_p \int_{-\infty}^{\infty} |U(x,y)|^p \,\mathrm{d}x. \quad (4.410)$$

In the limit $y \to 0+$, $U(x,y)$ and $V(x,y)$ can be expressed in terms of a function $f \in L^p(\mathbb{R})$ as follows:

$$\lim_{y\to 0+} U(x,y) = f(x), \quad a.e. \quad (4.411)$$

and

$$\lim_{y\to 0+} V(x,y) = Hf(x), \quad a.e., \quad (4.412)$$

and so Eq. (4.410) becomes Eq. (4.382) if $A_p^{p^{-1}}$ is identified with $\Re_p$.

To complete the proof, values of p in the range $2 < p < \infty$ need to be considered. If Eq. (4.382) holds for some $1 < p < \infty$, then it also holds for the conjugate exponent q. Recall that the conjugate exponent is given by $q = p(p-1)^{-1}$. The following is termed a *duality* argument. First a digression. If $f \in L^p(a,b)$ and $1 \leq p \leq \infty$, then

$$\left\{ \int_a^b |f(x)|^p \, \mathrm{d}x \right\}^{p^{-1}} = \sup_g \left| \int_a^b f(x)g(x)\mathrm{d}x \right|, \tag{4.413}$$

where the sup is taken over all g with

$$\left\{ \int_a^b |g(x)|^q \, \mathrm{d}x \right\}^{q^{-1}} \leq 1, \tag{4.414}$$

and q is the conjugate exponent of p. To see how Eq. (4.413) is obtained, proceed as follows. Using Hölder's inequality leads to

$$\left| \int_a^b f(x)g(x)\mathrm{d}x \right| \leq \left\{ \int_a^b |f(x)|^p \, \mathrm{d}x \right\}^{p^{-1}} \left\{ \int_a^b |g(x)|^q \, \mathrm{d}x \right\}^{q^{-1}} \leq \left\{ \int_a^b |f(x)|^p \, \mathrm{d}x \right\}^{p^{-1}}. \tag{4.415}$$

As a particular choice for $g(x)$ take

$$g_0(x) = \frac{|f(x)|^{p-1} \operatorname{sgn} f(x)}{\left\{ \int_a^b |f(x)|^p \, \mathrm{d}x \right\}^{(p-1)/p}}, \quad \text{for } p \geq 1; \tag{4.416}$$

it is straightforward to show that

$$\left\{ \int_a^b |g_0(x)|^q \, \mathrm{d}x \right\}^{q^{-1}} = 1. \tag{4.417}$$

Using Eq. (4.415), pick a $g(x) \to g_0(x)$ such that equality holds for

$$\begin{aligned} \sup_g \left| \int_a^b f(x)g(x)\mathrm{d}x \right| &= \left| \int_a^b f(x)|f(x)|^{p-1} \operatorname{sgn}(f(x))\mathrm{d}x \right| \left\{ \int_a^b |f(x)|^p \, \mathrm{d}x \right\}^{(1-p)/p} \\ &= \int_a^b |f(x)|^p \, \mathrm{d}x \left\{ \int_a^b |f(x)|^p \, \mathrm{d}x \right\}^{(1-p)/p} \\ &= \left\{ \int_a^b |f(x)|^p \, \mathrm{d}x \right\}^{p^{-1}}, \end{aligned} \tag{4.418}$$

which establishes Eq. (4.413). From Eq. (4.176) it follows that

$$\left| \int_{-\infty}^{\infty} Hf(x)g(x)\mathrm{d}x \right| = \left| \int_{-\infty}^{\infty} f(x)Hg(x)\mathrm{d}x \right|, \tag{4.419}$$

and, by Hölder's inequality,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f(x)Hg(x)\mathrm{d}x\right| &\le \left\{\int_{-\infty}^{\infty} |f(x)|^q \,\mathrm{d}x\right\}^{q^{-1}} \left\{\int_{-\infty}^{\infty} |Hg(x)|^p \,\mathrm{d}x\right\}^{p^{-1}} \\ &\le A_p \left\{\int_{-\infty}^{\infty} |f(x)|^q \,\mathrm{d}x\right\}^{q^{-1}} \left\{\int_{-\infty}^{\infty} |g(x)|^p \,\mathrm{d}x\right\}^{p^{-1}} \\ &\le A_p \left\{\int_{-\infty}^{\infty} |f(x)|^q \,\mathrm{d}x\right\}^{q^{-1}}, \end{aligned} \tag{4.420}$$

where Eq. (4.414) has been employed. Using Eqs. (4.413), (4.419), and the preceding result yields

$$\|Hf\|_q \le A_p \|f\|_q , \tag{4.421}$$

where q is the conjugate exponent to p. The following results have been established:

$$\|Hf\|_p \le A_p \|f\|_p , \quad 1 < p \le 2, \tag{4.422}$$

and

$$\|Hf\|_p \le C_p \,\|f\|_p , \quad 2 \le p < \infty, \tag{4.423}$$

where for convenience the constant in Eq. (4.421) has been written using a different designation, that is $C_q = A_{q(q-1)^{-1}}$ for $2 \le q < \infty$. Equations (4.422) and (4.423) represent Eq. (4.382), with

$$\Re_p = \begin{cases} A_p, & 1 < p \le 2 \\ C_p, & 2 \le p < \infty. \end{cases} \tag{4.424}$$

To determine the optimal constant requires only one of the two ranges, $1 < p \le 2$ or $2 < p < \infty$, to be investigated. Evaluation of the optimal value of $\Re_p$ is treated in Section 6.17.

Using the iteration property a simple lower bound for $\|Hf\|_p$ for $p > 1$ can be found. Starting from

$$\|H\phi\|_p \le \Re_p \|\phi\|_p , \tag{4.425}$$

and inserting $\phi = Hf$, leads to

$$\|f\|_p \le \Re_p \|Hf\|_p , \tag{4.426}$$

and hence

$$\{\Re_p\}^{-1} \|f\|_p \le \|Hf\|_p \le \Re_p \|f\|_p . \tag{4.427}$$

The isomorphic structure of the mapping of $L^p(\mathbb{R})$ onto itself under the action of the Hilbert transform operator becomes an isometry for the case $p = 2$, which is immediately apparent from the last equation, since $\Re_2 = 1$.

Because of the importance of the inequality of Marcel Riesz, there has been some interest in the chronological development of the ideas that led to Riesz's key theorems and some of the related results. For readers with a curiosity about these historical developments, there are several sources available. Gårding (1970) gives some of the details of Riesz's work; in particular, an excerpt from a letter of G. H. Hardy demanding the proof of Riesz's theorem. A concise summary of the evolution of the key ideas is given by Asmar and Hewitt (1988), including a personal recollection of the latter author on some of Riesz's remarks. A detailed discussion, including extensive quotations from personal letters between Riesz and Hardy, is given by Cartwright (1982). It took a considerable time (about three years) for Riesz to publish detailed proofs of his principal results. In the intervening years, Riesz's work was available to Hardy and others. Some questions have arisen as to whether Riesz was given appropriate acknowledgment in an early work of Titchmarsh (1926) on reciprocal formulas involving integrals and series. Titchmarsh was a student of Hardy and had read the Riesz theorems and proofs given in one of Riesz's letters to Hardy. Cartwright covers these issues in detail. As a parenthetical aside, Hardy must have been worried that Riesz would move in on one of his (and his student Titchmarsh's) areas of research, for he writes in an undated letter to Titchmarsh (Cartwright, 1982, p. 506) "You will observe that R shows symptoms of getting on to 'Hilbert transforms' himself – all the more reason for pushing on with your paper." In this quote R is Riesz. This letter of Hardy was probably written in late 1923 or early 1924.

4.21 The Hilbert transform of functions in L^1 and in L^∞

An obvious omission from the discussion of the previous section and earlier parts of the book is the case of Hilbert transforms for functions that belong to $L^1(\mathbb{R})$. The reader is reminded, following the standard custom, that L^1 is abbreviated to L. Some specialized results are now considered for this class of functions.

If f and $g = Hf$ both $\in L(\mathbb{R})$, then the following Hilbert transform pair is obtained:

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(s)\mathrm{d}s}{x - s} \tag{4.428}$$

and

$$f(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(s)\mathrm{d}s}{x - s} \tag{4.429}$$

(Hille and Tamarkin, 1935). For the case of functions in $L^p(\mathbb{R})$, $p > 1$, it is only necessary to assume that one of the functions f or g belongs to L^p, in contrast to the requirement just stated that *both* functions $\in L(\mathbb{R})$. The reason, as the reader will recall, is that, for $p > 1, f \in L^p(\mathbb{R})$ implies $Hf(x) \in L^p$ (see Section 4.20); however, for $f \in L(\mathbb{R})$, $Hf(x)$ in general does not belong to $L(\mathbb{R})$. Consider the case $f(x) = a(a^2 + x^2)^{-1}$ for $a > 0$. Now $f \in L(\mathbb{R})$ and

$$g(x) = Hf(x) = \frac{x}{a^2 + x^2}, \tag{4.430}$$

which does not belong to $L(\mathbb{R})$.

Kober (1942) gave the following result. If $f \in L(\mathbb{R})$, a necessary condition that $Hf \in L(\mathbb{R})$ is

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 0. \tag{4.431}$$

That this condition is not sufficient is attributed by Kober to H. R. Pitt. The latter result can be established as follows. Let

$$f(x) = \begin{cases} 0, & -\infty < x \le 0 \\ x^{-1}\log^{-2} x - 2/\log 2, & 0 < x < 1/2 \\ 0, & 1/2 \le x < \infty. \end{cases} \tag{4.432}$$

Using the change of variable $x = \mathrm{e}^{-y}$ (or otherwise noting that the integrand is an exact differential) leads to

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\mathrm{d}x &= \int_0^{1/2} x^{-1}\log^{-2} x\,\mathrm{d}x - \frac{1}{\log 2} \\ &= -\frac{1}{\log 2} + \int_{\log 2}^{\infty} \frac{\mathrm{d}y}{y^2} \\ &= 0, \end{aligned} \tag{4.433}$$

so Eq. (4.431) is satisfied. For $p > 1$, the integral $\int_{-\infty}^{\infty} |f(x)|^p\,\mathrm{d}x$ diverges. Now $f \in L(\mathbb{R})$ since

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|\mathrm{d}x &= \int_0^{1/2} \left| x^{-1}\log^{-2} x - \frac{2}{\log 2} \right| \mathrm{d}x \\ &= \frac{2}{\log x_2} - \frac{2}{\log x_1} + \frac{4(x_2 - x_1)}{\log 2}, \end{aligned} \tag{4.434}$$

where $x_1 \approx 0.026\,042$ and $x_2 \approx 0.389\,208$ are solutions of $x^{-1}\log^{-2} x - 2/\log 2 = 0$. Let $g(x) = Hf(x)$; for $x > 0$ it follows that

$$\begin{aligned} -g(-x) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{x+t} \\ &= \frac{1}{\pi} P \int_0^{1/2} \frac{\mathrm{d}t}{(x+t)t\log^2 t} - \frac{2}{\pi \log 2} \int_0^{1/2} \frac{\mathrm{d}t}{(x+t)}, \end{aligned} \tag{4.435}$$

and, since $(x+t)^{-1} > (2x)^{-1}$ for $t \in (0,x)$,

$$-g(-x) > \frac{1}{2\pi x}\int_0^x \frac{\mathrm{d}t}{t\log^2 t} - \frac{2}{\pi\log 2}\log\left(\frac{2x+1}{2x}\right)$$
$$= -\frac{1}{2\pi x}\int_0^x \mathrm{d}\{[\log t]^{-1}\} - \frac{2}{\pi\log 2}\log\left(\frac{2x+1}{2x}\right). \tag{4.436}$$

The second contribution in the final result is not important for the argument that follows, so this term is dropped. Hence, for $x \in (0, 1/2)$,

$$-g(-x) > -\frac{1}{2\pi x\log x}. \tag{4.437}$$

Now,

$$\int_{-1/2}^0 |g(x)|\mathrm{d}x = \int_0^{1/2} |-g(-x)|\mathrm{d}x > \frac{1}{2\pi}\int_0^{1/2} \frac{1}{x\log x}\,\mathrm{d}x = \infty; \tag{4.438}$$

that is, $Hf(x) \notin L(-1/2, 0)$, and since

$$\int_{-\infty}^{\infty} |Hf(x)|\mathrm{d}x = \int_{-\infty}^{-1/2} |g(x)|\mathrm{d}x + \int_{-1/2}^{0} |g(x)|\mathrm{d}x + \int_0^{\infty} |g(x)|\mathrm{d}x, \tag{4.439}$$

then $Hf \notin L(\mathbb{R})$, which proves that Eq. (4.431) is not a sufficient condition.

To establish Eq. (4.431), suppose $f \in L(\mathbb{R}), Hf \in L(\mathbb{R})$, and

$$F(x) = f(x) + \mathrm{i}\,Hf(x). \tag{4.440}$$

Taking the Fourier transform of this result yields

$$\mathscr{F}F(x) = \mathscr{F}f(x) + \mathscr{F}\mathrm{i}\,Hf(x)$$
$$= (1 + \operatorname{sgn} x)\mathscr{F}f(x), \tag{4.441}$$

and Eq. (4.162) has been employed. The function $F(z)$ is analytic in the upper half complex plane, and, by the Cauchy integral theorem,

$$\oint_C F(z)\mathrm{d}z = 0, \tag{4.442}$$

where the contour C is a semicircle in the upper half plane centered at the origin and including the real axis. From Eq. (4.442) it follows that

$$\int_{-\infty}^{\infty} F(x)\mathrm{d}x = 0, \tag{4.443}$$

and hence

$$\mathcal{F}F(0) = 0. \tag{4.444}$$

From Eq. (4.441) it follows that

$$\mathcal{F}f(0) = 0, \tag{4.445}$$

and this establishes Eq. (4.431).

4.21.1 The L^∞ case

The argument leading to Eq. (4.413) can be modified to include the case $p = \infty$, and the reader is left to ponder this assertion. A duality type argument is now employed. Starting with Eq. (4.413) applied to $\mathbb{R}$ with the choice $p = \infty$, and with f replaced by Hf yields

$$\|Hf\|_{L^\infty} = \sup_{\substack{g \in L \\ \|g\|_L = 1}} \left| \int_{-\infty}^{\infty} Hf(x)g(x)\mathrm{d}x \right|. \tag{4.446}$$

If $\int_{-\infty}^{\infty} Hf(x)g(x)\mathrm{d}x$ is bounded, then a Parseval-type formula (see Section 4.10) can be written:

$$\int_{-\infty}^{\infty} Hf(x)g(x)\mathrm{d}x = -\int_{-\infty}^{\infty} f(x)Hg(x)\mathrm{d}x. \tag{4.447}$$

The reader will recall that, for $g \in L^1, Hg$ does not in general belong to L^1, but the additional assumption that $Hg \in L^1$ is imposed, in which case the right-hand integral of Eq. (4.447) is bounded, and an interchange of integration order can be made (refer to Section 2.13). Equation (4.446) can be written for a positive constant C as follows:

$$\begin{aligned} \|Hf\|_{L^\infty} &= \sup_{\substack{g \in L \\ \|g\|_L = 1}} \left| \int_{-\infty}^{\infty} f(x)Hg(x)\mathrm{d}x \right| \\ &\le \sup_{\substack{g \in L \\ \|g\|_L = 1}} \|f\|_{L^\infty}\|Hg\|_{L^1}, \end{aligned} \tag{4.448}$$

and hence

$$\|Hf\|_{L^\infty} \le C\|f\|_{L^\infty}. \tag{4.449}$$

To summarize: if $f \in L^{\infty}(\mathbb{R})$, then $Hf \in L^{\infty}(\mathbb{R})$, provided that the Hilbert transform operator is bounded on L^1. In general, if $f \in L^{\infty}(\mathbb{R})$ then H is an unbounded operator. Kober (1943a) gave an extended definition of the Hilbert transform operator for the case $f \in L^{\infty}(\mathbb{R})$, and this is discussed in Section 16.3.

4.22 Connection between Hilbert transforms and causal functions

In this section a result that has found many applications is discussed. Its significance stems from the fact that it links Fourier transforms of a certain class of functions with considerable physical importance directly with the Hilbert transform. The following result is most commonly called Titchmarsh's theorem, but a number of authors have made contributions, including Hardy (1908, 1932), Titchmarsh (1925a), Riesz (1927), Paley and Wiener (1934), and particularly Hille and Tamarkin (1933, 1934, 1935), and the results are collected together in Titchmarsh's seminal work on Fourier transforms (Titchmarsh, 1948, pp.125–129).

Let $F \in L^2(\mathbb{R})$. If $F(x)$ satisfies any one of the following four conditions, then it satisfies all four conditions. The real and imaginary parts of $F(x)$, $\operatorname{Re} F(x)$, and $\operatorname{Im} F(x)$, respectively, satisfy the following:

(i)

$$\operatorname{Im} F(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Re} F(s) \mathrm{d}s}{x - s}; \tag{4.450}$$

(ii)

$$\operatorname{Re} F(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im} F(s) \mathrm{d}s}{x - s}; \tag{4.451}$$

(iii) if $f(t)$ denotes the inverse Fourier transform of $F(x)$, then

$$f(t) = 0, \quad \text{for } t < 0; \tag{4.452}$$

(iv) $F(x + \mathrm{i}y)$ is an analytic function in the upper half plane and, for almost all x,

$$F(x) = \lim_{y \to 0+} F(x + \mathrm{i}y) \tag{4.453}$$

and

$$\int_{-\infty}^{\infty} |F(x + \mathrm{i}y)|^2 \, \mathrm{d}x < \infty, \quad \text{for} \quad y > 0. \tag{4.454}$$

A simple example of a function satisfying these conditions is given by

$$F(z) = \frac{1}{z + a + \mathrm{i}b}, \tag{4.455}$$

where a and b are constants and $b > 0$. Many additional examples can be quickly obtained by examining the table of Hilbert transforms in Appendix 1. A square integrable function satisfying one of the conditions (i)–(iv), and hence satisfying all four conditions, is sometimes called a *causal transform*. That is, if the square integrable function $f(t)$ is zero for $t < 0$, then $F(x)$ is a causal transform; and conversely, if $F(x)$ is a causal transform, then $f(t)$ is a causal function.

Before embarking on a proof of these results, it is useful to have a preliminary notion of the context in which these ideas find application. Many physical processes "turn on" at some particular time, and there is no realizable information transmission prior to the turn on time. Such processes are discussed in detail in Chapter 17. If the variable t in Eq. (4.452) is interpreted as time, then the theorem makes a connection between the causal nature of a physical process and the underlying analytic structure of the function describing this process. Functions satisfying Eq. (4.452) are referred to as *causal functions*, with time usually being the implied variable when this name is attached. With t selected as a time variable, dimensional considerations imply that x is a frequency, and the interpretation of the physical meaning of negative and complex frequencies is postponed to Sections 17.7 and 19.2.

That Eq. (4.450) implies Eq. (4.451) and vice versa, the reader will instantly recognize as the Hilbert transform pair relationship; this has been discussed in Section 3.4. In Section 12.2 the same link between these two equations arises in the discussion of singular integral equations. The connections (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) have already been dealt with, and therefore no additional remarks on this part of the theorem are considered.

The connection (iii) $\Rightarrow$ (iv) is now examined. From the definition of $F(x)$,

$$\begin{aligned} F(x) &= \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t \\ &= \int_{0}^{\infty} f(t)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t, \end{aligned} \tag{4.456}$$

and hence

$$F(x+\mathrm{i}y) = \int_0^{\infty} f(t)\mathrm{e}^{\mathrm{i}xt-yt}\,\mathrm{d}t. \tag{4.457}$$

The additional factor e^{-yt} that appears as part of the integrand can only improve the convergence of the preceding integral relative to the integral in Eq. (4.456). Note that this last statement would not be true in general if the function $f(t)$ did not vanish on the interval $(-\infty, 0)$, and in fact, for most functions, the integral representation for $F(x+\mathrm{i}y)$ would diverge. So the causal nature of the function is a key ingredient in the argument. Equation (4.457) represents the analytic continuation of $F(x)$ from the real line into the upper half of the complex plane. Now,

$$\begin{aligned}\lim_{y\to+0} F(x+\mathrm{i}y) &= \lim_{y\to+0}\int_0^\infty f(t)\mathrm{e}^{\mathrm{i}xt-yt}\,\mathrm{d}t\\ &= \int_0^\infty \lim_{y\to+0} f(t)\mathrm{e}^{\mathrm{i}xt-yt}\,\mathrm{d}t\\ &= F(x). \end{aligned}\tag{4.458}$$

To justify the interchange of the limit and integral, a sequence of functions g_n is constructed, and the limit $n \to \infty$ is examined. Let $g_n(t) = f(t)\mathrm{e}^{\mathrm{i}xt-n^{-1}t}$ and suppose there exists a measurable function $h(t)$ such that $|g_n(t)| \le h(t)$ *a.e.* and that $\lim_{n\to\infty} g_n(t) = g(t)$, then Lebesgue's dominated convergence theorem allows the interchange of limit and integral. Now

$$\begin{aligned}\int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2\,\mathrm{d}x &= \int_{-\infty}^{\infty}\mathrm{d}x\int_0^\infty f(t)\mathrm{e}^{\mathrm{i}xt-yt}\,\mathrm{d}t\int_0^\infty f*(s)\mathrm{e}^{-\mathrm{i}xs-ys}\,\mathrm{d}s\\ &= 2\pi\int_0^\infty f(t)\mathrm{e}^{-yt}\,\mathrm{d}t\int_0^\infty f*(s)\mathrm{e}^{-ys}\delta(s-t)\mathrm{d}s\\ &= 2\pi\int_0^\infty |f(t)|^2\mathrm{e}^{-2yt}\,\mathrm{d}t\\ &< 2\pi\int_0^\infty |f(t)|^2\,\mathrm{d}t. \end{aligned}\tag{4.459}$$

Recalling Parseval's formula, and noting that $F \in L^2(\mathbb{R})$, yields $f \in L^2(\mathbb{R})$, and hence, for a positive constant C,

$$\int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2\,\mathrm{d}x < C. \tag{4.460}$$

The relationship (iv) ⇒ (i) and (ii) is now examined by considering the contour integral $\int_\Gamma F(z)\mathrm{d}z/(z-x_0)$, where Γ denotes the rectangular contour shown in Figure 4.7.

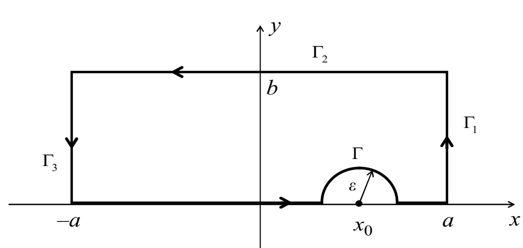

Figure 4.7. Rectangular contour with a suitable indentation to avoid the singularity at $(z = x_0)$.

Application of the Cauchy integral theorem yields

$$\int_{\Gamma} \frac{F(z)\mathrm{d}z}{z-x_0} = \int_{-a}^{x_0-\varepsilon} \frac{F(x)\mathrm{d}x}{x-x_0} + \int_{x_0+\varepsilon}^{a} \frac{F(x)\mathrm{d}x}{x-x_0} + \int_{\pi}^{0} \frac{F(z)\mathrm{d}z}{z-x_0} + \int_{\Gamma_1} \frac{F(z)\mathrm{d}z}{z-x_0} + \int_{\Gamma_2} \frac{F(z)\mathrm{d}z}{z-x_0} + \int_{\Gamma_3} \frac{F(z)\mathrm{d}z}{z-x_0} = 0. \tag{4.461}$$

The integral along Γ_2 simplifies as follows:

$$\int_{\Gamma_2} \frac{F(z)\mathrm{d}z}{z-x_0} = -\int_{-a}^{a} \frac{F(x+\mathrm{i}b)\mathrm{d}x}{x-x_0+\mathrm{i}b} = -\int_{-a}^{a} \frac{F(x+\mathrm{i}b)(x-x_0-\mathrm{i}b)\mathrm{d}x}{(x-x_0)^2+b^2}, \tag{4.462}$$

and, on using the Cauchy–Schwarz–Buniakowski inequality for a finite interval,

$$\left(\int_{\alpha}^{\beta} |f(x)g(x)|\mathrm{d}x\right)^2 \le \left[\int_{\alpha}^{\beta} |f(x)|^2\,\mathrm{d}x\right]\left[\int_{\alpha}^{\beta} |g(x)|^2\,\mathrm{d}x\right]; \tag{4.463}$$

then,

$$\left|\int_{-a}^{a} \frac{F(x+\mathrm{i}b)\mathrm{d}x}{x-x_0+\mathrm{i}b}\right|^2 \le \left[\int_{-a}^{a} |F(x+\mathrm{i}b)|^2\,\mathrm{d}x\right]\left[\int_{-a}^{a} \frac{\mathrm{d}x}{(x-x_0)^2+b^2}\right]. \tag{4.464}$$

If the limit $a \to \infty$ is examined in this last result, the first integral on the right-hand side is bounded by a constant (Eq. (4.460)), and the second integral yields πb^{-1}. Therefore the integral along Γ_2 vanishes in the lim $a, b \to \infty$. Similarly, the integral on Γ_1 simplifies as follows:

$$\left|\int_{0}^{b} \frac{F(a+\mathrm{i}y)\mathrm{i}\,\mathrm{d}y}{a-x_0+\mathrm{i}y}\right|^2 \le \left[\int_{0}^{b} |F(a+\mathrm{i}y)|^2\,\mathrm{d}y\right]\left[\int_{0}^{b} \frac{\mathrm{d}y}{(a-x_0)^2+y^2}\right] = \frac{\tan^{-1}(b/(a-x_0))}{a-x_0}\int_{0}^{b} |F(a+\mathrm{i}y)|^2\,\mathrm{d}y. \tag{4.465}$$

In the lim $a \to \infty$, the last integral must be bounded, and hence the right-hand side of Eq. (4.465) approaches zero in this limit. A similar argument applies for the integral on the contour Γ_3. Taking the lim $\varepsilon \to 0$ in Eq. (4.461) yields

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{F(s)\mathrm{d}s}{s-x} = \mathrm{i}F(x), \tag{4.466}$$

which yields Eqs. (4.450) and (4.451) on taking the real and imaginary parts. This completes the connection (iv) ⇒ (i) and (ii).

The connection between (i) and (ii) and (iii) is now investigated. Multiply Eq. (4.466) by $(2\pi)^{-1}\mathrm{e}^{-\mathrm{i}xt}$ and integrate with respect to x to obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x &= \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{F(s)\mathrm{d}s}{s-x} \\
&= \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} F(s)\mathrm{d}s\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x}{s-x} \\
&= \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} F(s)\{\mathrm{i}\ \mathrm{sgn}\, t\, \mathrm{e}^{-\mathrm{i}st}\}\mathrm{d}s \\
&= \frac{\mathrm{sgn}\, t}{2\pi}\int_{-\infty}^{\infty} F(s)\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}s,
\end{aligned}\tag{4.467}$$

and the Hilbert transform of $\mathrm{e}^{-\mathrm{i}xt}$can be evaluated most quickly by expressing the exponential in terms of $\cos xt$ and $\sin xt$ and using the results for $H(\sin xt)$ and $H(\cos xt)$. The reader is invited to justify the change of integration order in the last sequence of steps. Equation (4.467) simplifies to

$$f(t) = 0, \quad \text{for } t < 0, \tag{4.468}$$

which verifies that $f(t)$ is a casual function, and hence the connection (i) and (ii) $\Rightarrow$ (iii) is established.

In order to see what is involved in the implication (iv) $\Rightarrow$ (iii), start with the Parseval identity to obtain

$$\int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2\,\mathrm{d}x = 2\pi\int_{-\infty}^{\infty} |f(t)|^2\mathrm{e}^{-2yt}\,\mathrm{d}t. \tag{4.469}$$

For $y > 0$ the integral involving $|F(x+\mathrm{i}y)|^2$ is bounded; however, the second integral in general diverges at the lower limit. Intuitively, it is clear that one way to avoid this divergence is to have $f(t)$ vanish faster than e^{-2yt} as $t \to -\infty$. Consider the Gaussian $f(t) = \mathrm{e}^{-at^2}$, with $a > 0$, as an example of a function for which the right-hand side of Eq. (4.469) converges. The Fourier transform of a Gaussian function is a Gaussian, and hence the integral over $|F(x+\mathrm{i}y)|^2$ is unbounded as $y \to \infty$. By requiring $f(t)$ to be causal, it is clear that the right-hand side of Eq. (4.469) will converge. Suppose a function exists that is square integrable on $\mathbb{R}$ and is the boundary value of a function analytic in the upper half plane. Is this function the Fourier transform of a causal function? The aforementioned example – the Gaussian function – makes it clear that the answer is no for a general function. In addition to the two stated conditions, the key requirement that must also be satisfied in order that the function represent the Fourier transform of a causal function is that Eq. (4.454) should also hold.

A more rigorous demonstration of (iv) $\Rightarrow$ (i) is now considered following an argument of Titchmarsh (1948, pp. 125–127). If $\int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2\,\mathrm{d}x$ exists and is bounded for y in the interval $y_1 \le y \le y_2$, then, for positive δ, $\lim_{x\to\pm\infty} F(x+\mathrm{i}y) \to 0$, for $y_1 + \delta \le y \le y_2 - \delta$. This can be proved by considering the Cauchy integral

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_{|w-z|=\rho} \frac{F(w)\mathrm{d}w}{w-z}, \quad \text{with } 0 < \rho \le \delta. \tag{4.470}$$

On employing the substitution $w - z = \rho e^{\mathrm{i}\theta}$ and applying the Cauchy–Schwarz–Buniakowski inequality yields

$$|F(z)|^2 \le \frac{1}{2\pi} \int_0^{2\pi} \left| F(z + \rho e^{\mathrm{i}\theta}) \right|^2 \mathrm{d}\theta. \tag{4.471}$$

Multiplying Eq. (4.471) by ρ and integrating over this variable gives

$$\frac{1}{2}\delta^2 |F(z)|^2 \le \frac{1}{2\pi} \int_0^{\delta} \rho \mathrm{d}\rho \int_0^{2\pi} \left| F(z + \rho e^{\mathrm{i}\theta}) \right|^2 \mathrm{d}\theta. \tag{4.472}$$

Converting from polar to Cartesian coordinates with the change of variable $z + \rho e^{\mathrm{i}\theta} = u + \mathrm{i}v$, and noting that $u \in (x - \delta, x + \delta)$ and $v \in (y_1, y_2)$, leads to

$$\frac{1}{2}\delta^2 |F(z)|^2 \le \frac{1}{2\pi} \int_{y_1}^{y_2} \mathrm{d}v \int_{x-\delta}^{x+\delta} |F(u + \mathrm{i}v)|^2 \mathrm{d}u. \tag{4.473}$$

The integral $\int_{x-\delta}^{x+\delta} |F(u + \mathrm{i}v)|^2 \mathrm{d}u$ is bounded in the interval $y_1 \le y \le y_2$ and vanishes as $x \to \pm\infty$; hence,

$$\lim_{x \to \pm\infty} F(x + \mathrm{i}y) \to 0. \tag{4.474}$$

This result is now used to establish the connection (iv) $\Rightarrow$ (i). The contour integral $\int_\Gamma F(z) e^{-\mathrm{i}tz} \mathrm{d}z$ is evaluated around the rectangular contour in Figure 4.8, with

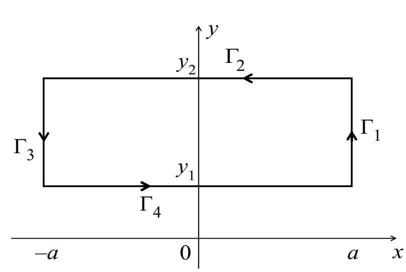

Figure 4.8. Rectangular contour with vertices located at $(-a, y_2)$, (a, y_2), (a, y_1), and $(-a, y_1)$.

the result that

$$\int_{-a}^{a} F(x+\mathrm{i}y_1)\mathrm{e}^{-\mathrm{i}tx+ty_1}\,\mathrm{d}x + \mathrm{i}\int_{y_1}^{y_2} F(a+\mathrm{i}y)e^{-\mathrm{i}ta+ty}\,\mathrm{d}y$$
$$-\int_{-a}^{a} F(x+\mathrm{i}y_2)\mathrm{e}^{-\mathrm{i}tx+ty_2}\,\mathrm{d}x - \mathrm{i}\int_{y_1}^{y_2} F(-a+\mathrm{i}y)\mathrm{e}^{\mathrm{i}ta+ty}\,\mathrm{d}y = 0. \tag{4.475}$$

Taking advantage of Eq. (4.474), then, in the lim $a \to \infty$, the integrals along Γ_1 and Γ_3 vanish, and hence

$$\lim_{a\to\infty}\left\{\int_{-a}^{a} F(x+\mathrm{i}y_1)\mathrm{e}^{-\mathrm{i}tx+ty_1}\,\mathrm{d}x - \int_{-a}^{a} F(x+\mathrm{i}y_2)\mathrm{e}^{-\mathrm{i}tx+ty_2}\,\mathrm{d}x\right\} = 0. \tag{4.476}$$

Let

$$g_a(t,y) = \frac{1}{\sqrt{(2\pi)}}\int_{-a}^{a} F(x+\mathrm{i}y)\mathrm{e}^{-\mathrm{i}tx}\,\mathrm{d}x \tag{4.477}$$

and

$$\lim_{a\to\infty} g_a(t,y) \equiv g(t,y), \tag{4.478}$$

then, from Eq. (4.476), it follows that

$$\mathrm{e}^{ty_1}g(t,y_1) = \mathrm{e}^{ty_2}g(t,y_2). \tag{4.479}$$

The right-hand side of this equation is independent of y_1 and the left-hand side is independent of y_2; this is only possible if both sides of the equation equal some function of time, hence

$$\mathrm{e}^{ty}g(t,y) = f(t), \tag{4.480}$$

and therefore

$$g(t,y) = \mathrm{e}^{-ty}f(t). \tag{4.481}$$

Parseval's theorem yields

$$\int_{-\infty}^{\infty} |f(t)|^2\mathrm{e}^{-2ty}\,\mathrm{d}t = \int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2\,\mathrm{d}x. \tag{4.482}$$

Both integrals are bounded as $y \to \infty$, and since $e^{-2(t+\delta)y} \geq 1$ for $t \in (-\infty, -\delta)$, it follows that

$$\begin{aligned}\int_{-\infty}^{-\delta} |f(t)|^2\,dt &\leq e^{-2\delta y} \int_{-\infty}^{-\delta} |f(t)|^2 e^{-2ty}\,dt \\ &\leq Ce^{-2\delta y} \\ &= 0, \quad \text{as } y \to \infty. \end{aligned} \tag{4.483}$$

Since δ is arbitrary, take $\delta \to 0+$; then

$$f(t) = 0, \quad \text{for } t < 0. \tag{4.484}$$

Equation (4.453) is now investigated. If $F \in L^2(\mathbb{R})$, and $F(z)$ is analytic in the upper half complex plane, and Eq. (4.454) holds, then

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yF(t)dt}{(t-x)^2+y^2}. \tag{4.485}$$

The reader will recognize this result as the Poisson integral formula for the half plane (see Section 3.2). Taking advantage of the Dirac delta distribution leads to

$$\begin{aligned}\lim_{y\to 0+} F(x+iy) &= \int_{-\infty}^{\infty} \lim_{y\to 0+} \frac{1}{\pi}\frac{yF(t)dt}{(t-x)^2+y^2} \\ &= \int_{-\infty}^{\infty} F(t)\delta(t-x)dt \\ &= F(x), \end{aligned} \tag{4.486}$$

and Eq. (2.252) has been employed. Here $F(t)$ is treated as a suitably well behaved test function, and the interchange of the order of the limit and the integral is done in the sense of the definition of the Dirac delta distribution. Alternatively, the following approach, which avoids the Dirac delta distribution, can be taken. Starting with Eq. (4.485) leads to

$$\begin{aligned}F(z) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,ds}{(s-x)^2+y^2}\int_{-\infty}^{\infty} e^{isu}f(u)du \\ &= \lim_{N\to\infty}\frac{1}{\pi}\int_{-N}^{N}\frac{yds}{(s-x)^2+y^2}\int_{-\infty}^{\infty} e^{isu}f(u)du \\ &= \lim_{N\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)du\int_{-N}^{N}\frac{ye^{ius}\,ds}{(s-x)^2+y^2} \\ &= \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)du\int_{-N}^{N} e^{ius}\,ds\int_{-\infty}^{\infty} e^{i(x-s)t-y|t|}\,dt \end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\mathrm{d}u \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt-y|t|}\,\mathrm{d}t \int_{-N}^{N} \mathrm{e}^{\mathrm{i}(u-t)s}\,\mathrm{d}s \\
&= \lim_{N\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(u)\mathrm{d}u \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt-y|t|} \frac{\sin(u-t)N}{(u-t)N}\,\mathrm{d}t \\
&= \lim_{N\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt-y|t|}\,\mathrm{d}t \int_{-\infty}^{\infty} \frac{f(u)\sin(u-t)N}{(u-t)N}\,\mathrm{d}u. \qquad (4.487)
\end{aligned}
$$

Let

$$
f_N(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u)\sin(u-t)N}{(u-t)N}\,\mathrm{d}u, \qquad (4.488)
$$

then

$$
\lim_{N\to\infty} \|f_N(t) - f(t)\| \to 0, \qquad (4.489)
$$

so that

$$
F(z) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt-y|t|} f(t)\mathrm{d}t. \qquad (4.490)
$$

Taking the limit $y \to 0+$ gives

$$
\begin{aligned}
\lim_{y\to 0+} F(z) &= \lim_{y\to 0+} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt-y|t|} f(t)\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt} f(t)\mathrm{d}t \\
&= F(x). \qquad (4.491)
\end{aligned}
$$

The theorem stated at the start of this section can be extended to cover the case of functions $\in L^p$, with $p > 1$. See the end-notes for references on this point. When Eq. (4.454) is replaced by

$$
\int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2\,\mathrm{d}x = O(\mathrm{e}^{2ky}), \qquad (4.492)
$$

where k is a real constant, a statement similar to Eq. (4.452) follows, but the bound on t in this equation depends on k. This is discussed in Section 17.8 (see Eq. (17.99)).

4.23 The Hardy–Poincaré–Bertrand formula revisited

In this section the Hardy–Poincaré–Bertrand formula is revisited and advantage is taken of some of the results obtained in Section 4.22. Let $f \in L^p$ for $p > 1$, $g \in L^q$, where q is the conjugate exponent, and set $\phi(s,z) = (s-z)^{-1}f(s)$, where as usual $z = x + \mathrm{i}y$. The corresponding Hilbert transforms of these three functions are denoted by $F(s), G(s)$, and $\Phi(s,z)$, respectively. The function $\Phi(s,z)$ can be written as follows:

$$\begin{aligned}\Phi(s,z) \equiv H\phi(s,z) &= \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{(t-z)(s-t)}\\ &= \frac{1}{\pi(s-z)}P\int_{-\infty}^{\infty}\left\{\frac{1}{t-z}+\frac{1}{s-t}\right\}f(t)\mathrm{d}t\\ &= \frac{1}{(s-z)}\left\{F(s)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{t-z}\right\}. \end{aligned} \tag{4.493}$$

Let

$$\Psi(z) = \{f(z)+\mathrm{i}F(z)\}\{g(z)+\mathrm{i}G(z)\}, \tag{4.494}$$

then, from the initial theorem at the start of Section 4.22 (see part (iv)), it follows that

$$\int_C \Psi(z)\mathrm{d}z = 0, \tag{4.495}$$

where C denotes a semicircular contour in the upper half complex plane, with center at the origin and diameter along the real axis. It follows directly from Eq. (4.495) that

$$\int_{-\infty}^{\infty} f(x)g(x)\mathrm{d}x = \int_{-\infty}^{\infty} F(x)G(x)\mathrm{d}x \tag{4.496}$$

and

$$\int_{-\infty}^{\infty} f(x)G(x)\mathrm{d}x = -\int_{-\infty}^{\infty} F(x)g(x)\mathrm{d}x \tag{4.497}$$

which are results that have already been encountered (see Section 4.10). Using Hölder's inequality in the form

$$\left|\int_{-\infty}^{\infty} f(x)g(x)\mathrm{d}x\right| \le \left\{\int_{-\infty}^{\infty}|f(x)|^p\,\mathrm{d}x\right\}^{p^{-1}}\left\{\int_{-\infty}^{\infty}|g(x)|^q\,\mathrm{d}x\right\}^{q^{-1}}, \tag{4.498}$$

it follows that the left-hand side of Eq. (4.496) is bounded. Similarly,

$$\left|\int_{-\infty}^{\infty} F(x)G(x)\mathrm{d}x\right| \le \left\{\int_{-\infty}^{\infty}|F(x)|^p\,\mathrm{d}x\right\}^{p^{-1}}\left\{\int_{-\infty}^{\infty}|G(x)|^q\,\mathrm{d}x\right\}^{q^{-1}}. \tag{4.499}$$

Employing the Riesz inequality in this result proves that the right-hand side of Eq. (4.496) is also bounded. A similar argument can be applied to establish that both integrals in Eq. (4.497) are bounded. Using Eqs. (4.497) and (4.493) leads to

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(s,z)G(s)\mathrm{d}s &= -\int_{-\infty}^{\infty} \Phi(s,z)g(s)\mathrm{d}s\\ &= -\int_{-\infty}^{\infty} \frac{F(s)g(s)\mathrm{d}s}{s-z} - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(s)\mathrm{d}s}{s-z}\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{t-z},\end{aligned} \tag{4.500}$$

and hence

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{(s-x+\mathrm{i}y)f(s)G(s)\mathrm{d}s}{(s-x)^2+y^2} + \int_{-\infty}^{\infty} \frac{(s-x+\mathrm{i}y)F(s)g(s)\mathrm{d}s}{(s-x)^2+y^2}\\ &\quad = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(s-x+\mathrm{i}y)g(s)\mathrm{d}s}{(s-x)^2+y^2}\int_{-\infty}^{\infty} \frac{(t-x+\mathrm{i}y)f(t)\mathrm{d}t}{(t-x)^2+y^2}.\end{aligned} \tag{4.501}$$

Consider the limit $y \to 0+$, and employ the result for this given in Section 4.22 (see Eq. (4.486) or Eqs. (4.487)–(4.491)); then

$$\begin{aligned}\frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{f(s)G(s)\mathrm{d}s}{s-x} + \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{F(s)g(s)\mathrm{d}s}{s-x} &= -\mathrm{i}\{f(x)G(x)+F(x)g(x)\}\\ &\quad - \{\mathrm{i}g(x)-G(x)\}\{\mathrm{i}f(x)-F(x)\}.\end{aligned} \tag{4.502}$$

In terms of distributions, the second integral on the right-hand side of Eq. (4.500) can be simplified by noting that

$$\frac{1}{s-z} = P\left(\frac{1}{s-x}\right) + \mathrm{i}\pi\delta(s-x), \quad \text{as } y \to 0+. \tag{4.503}$$

From Eq. (4.502), it follows that

$$\begin{aligned}\frac{1}{\pi^2}P\int_{-\infty}^{\infty} \frac{f(s)\mathrm{d}s}{s-x}P\int_{-\infty}^{\infty} \frac{g(t)\mathrm{d}t}{t-s} &= -\frac{1}{\pi^2}P\int_{-\infty}^{\infty} \frac{g(s)\mathrm{d}s}{s-x}P\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{t-s}\\ &\quad + F(x)G(x) - f(x)g(x)\\ &= -\frac{1}{\pi^2}P\int_{-\infty}^{\infty} g(s)\mathrm{d}s\, P\int_{-\infty}^{\infty} f(t)\\ &\quad \times \left\{\frac{1}{s-x}+\frac{1}{t-s}\right\}\frac{1}{t-x}\,\mathrm{d}t + F(x)G(x)\\ &\quad - f(x)g(x)\\ &= \frac{1}{\pi^2}P\int_{-\infty}^{\infty} g(s)\mathrm{d}s\, P\int_{-\infty}^{\infty}\frac{f(t)\mathrm{d}t}{(t-x)(s-t)}\\ &\quad - f(x)g(x),\end{aligned} \tag{4.504}$$

which is the Hardy–Poincaré–Bertrand formula.

4.24 A theorem due to McLean and Elliott

The norm of a linear operator $\mathcal{O} : L^p \to L^p$ is defined for $f \in L^p(\mathbb{R})$ by the relationship

$$\|\mathcal{O}\|_p = \sup_{\|f\|_p=1} \|\mathcal{O}f\|_p\,, \tag{4.505}$$

and $1 < p < \infty$ is assumed, anticipating what is to come. In this section the following result is established:

$$\|H_E\|_p = \|H\|_p\,, \tag{4.506}$$

where H_E is used to designate the truncated Hilbert operator defined by

$$H_E f(x) = \frac{1}{\pi} P \int_E \frac{f(t)\mathrm{d}t}{x-t}, \quad x \in E. \tag{4.507}$$

In Eq. (4.507) E denotes a measurable subset of $\mathbb{R}$, and it is explicitly assumed that $|E| \neq 0$. Two choices for E that show up in applications are the finite Hilbert transform defined on $(-1, 1)$, which is discussed in detail in Chapter 11, and the one-sided Hilbert transform defined on the interval $(0, \infty)$, which is treated in Section 12.7. The approach of McLean and Elliott (1988) is employed to prove Eq. (4.506). These authors use a different sign convention for their definition of $H_E f$; they also employ a factor of $\pi\mathrm{i}$ in place of π in Eq. (4.507), and they use a slightly different definition of the dilation operator. None of these changes alters the outcome in an important manner. Some preliminary results are required before proving Eq. (4.506).

Let χ_E denote the operator corresponding to multiplication by the characteristic function of the measurable set E, so that

$$\chi_E f(x) = \begin{cases} f(x), & x \in E \\ 0, & x \notin E. \end{cases} \tag{4.508}$$

McLean and Elliott gave the following result. Suppose E is a measurable subset of $\mathbb{R}$ then, (i) if the operator $\mathcal{O}$ commutes with translations,

$$\|\mathcal{O}_{a+E}\|_p = \|\mathcal{O}_E\|_p\,, \quad \text{for all } a \in \mathbb{R}, \tag{4.509}$$

and (ii) if $\mathcal{O}$ commutes with dilations, then

$$\|\mathcal{O}_{mE}\|_p = \|\mathcal{O}_E\|_p\,, \quad \text{for all } m > 0, \tag{4.510}$$

where the truncated operator $\mathcal{O}_E$ is defined by

$$\mathcal{O}_E = \chi_E \mathcal{O} \chi_E. \tag{4.511}$$

The reader is reminded that the translation operator was defined in Eq. (4.64), and the definition of the dilation operator given in Eq. (4.77) is employed (the prime superscript is dropped to simplify the notation). The term $a + E$ is interpreted as

$$a + E = \{a + x : x \in E\}; \tag{4.512}$$

and mE is interpreted as

$$mE = \{mx : x \in E\}. \tag{4.513}$$

To establish Eq. (4.509), the following preliminary result is required. Now

$$\chi_E \tau_a f(x) = \chi_E f(x-a) = \begin{cases} f(x-a), & x-a \in E \\ 0, & x-a \notin E \end{cases} \tag{4.514}$$

and

$$\tau_a \chi_{a+E} f(x) = \tau_a \begin{cases} f(x), & x \in a+E \\ 0, & x \notin a+E \end{cases} = \begin{cases} f(x-a), & x-a \in E \\ 0, & x-a \notin E, \end{cases} \tag{4.515}$$

from which it follows that

$$\tau_a \chi_{a+E} f(x) = \chi_E \tau_a f(x). \tag{4.516}$$

In a similar fashion it follows for $S_m : L^p \to L^p$ that

$$\chi_E S_m f(x) = m^{p^{-1}} \chi_E f(mx) = m^{p^{-1}} \begin{cases} f(mx), & mx \in E \\ 0, & mx \notin E, \end{cases} \tag{4.517}$$

$$S_m \chi_{mE} f(x) = S_m \begin{cases} f(x), & x \in E \\ 0, & x \notin E, \end{cases} = \begin{cases} m^{p^{-1}} f(mx), & mx \in E \\ 0, & mx \notin E, \end{cases} \tag{4.518}$$

and hence

$$\chi_E S_m f(x) = S_m \chi_{mE} f(x). \tag{4.519}$$

Carrying on with the proof of Eq. (4.509), it follows from Eqs. (4.511) and (4.516) that

$$\begin{aligned} \tau_a \mathcal{O}_{a+E} &= \tau_a \chi_{a+E} \mathcal{O} \chi_{a+E} \\ &= \chi_E \tau_a \mathcal{O} \chi_{a+E} \\ &= \chi_E \mathcal{O} \tau_a \chi_{a+E} \\ &= \chi_E \mathcal{O} \chi_E \tau_a \\ &= \mathcal{O}_E \tau_a, \end{aligned} \tag{4.520}$$

and so

$$\mathcal{O}_{a+E} = \tau_a^{-1}\mathcal{O}_E\tau_a. \tag{4.521}$$

Equation (4.521) leads to

$$\|\mathcal{O}_{a+E}f\|_p = \left\|\tau_a^{-1}\mathcal{O}_E\,\tau_a f\right\|_p. \tag{4.522}$$

Equation (4.522) simplifies on noting that $\tau_a^{-1} = \tau_{-a}$, using the isometric nature of τ_a, that is $\|\tau_a f\|_p = \|f\|_p$, and on taking the sup over all $f \in L^p(\mathbb{R})$ with $\|f\|_p = 1$. This yields

$$\|\mathcal{O}_{a+E}\|_p = \|\mathcal{O}_E\|_p, \tag{4.523}$$

which completes the proof of part (i). To deal with part (ii), start with Eq. (4.511) and use Eq. (4.519) to obtain

$$\begin{aligned} S_m\mathcal{O}_{mE} &= S_m\chi_{mE}\mathcal{O}\chi_{mE} \\ &= \chi_E S_m\mathcal{O}\chi_{mE} \\ &= \chi_E\mathcal{O}S_m\chi_{mE} \\ &= \chi_E\mathcal{O}\chi_E S_m \\ &= \mathcal{O}_E S_m, \end{aligned} \tag{4.524}$$

and hence

$$\mathcal{O}_{mE} = S_m^{-1}\mathcal{O}_E S_m. \tag{4.525}$$

Equation (4.525) leads to

$$\|\mathcal{O}_{mE}f\|_p = \left\|S_m^{-1}\mathcal{O}_E S_m f\right\|_p. \tag{4.526}$$

Equation (4.526) simplifies, on noting $S_m^{-1} = S_{m^{-1}}$, using the isometric nature of S_m, $\|S_m f\|_p = \|f\|_p$, and on taking the sup over all $f \in L^p(\mathbb{R})$ with $\|f\|_p = 1$, so that

$$\|\mathcal{O}_{mE}\|_p = \|\mathcal{O}_E\|_p, \tag{4.527}$$

and hence part (ii) is established.

The concept of the *density* of E is now introduced. The Lebesgue measure of E is denoted by $|E|$. Let J_δ denote an open interval centered at x and having length 2δ:

$$J_\delta(x) = (x-\delta, x+\delta), \quad \text{for } \delta > 0,\ x \in \mathbb{R}. \tag{4.528}$$

The density is defined by

$$d_E(x) = \lim_{\delta \to 0+} \frac{|E \cap J_\delta(x)|}{|J_\delta(x)|}, \tag{4.529}$$

assuming the limit exists. If the interval J_δ does not overlap E, then $d_E(x) = 0$, and if E includes the interval J_δ then $|E \cap J_\delta(x)| = |J_\delta(x)|$ and so $d_E(x) = 1$, and hence $0 \le d_E(x) \le 1$. If E is the interval (α, β), then $d_E(\alpha) = d_E(\beta) = 1/2$. If $|E| \ne 0$, then intuitively it is expected that the set E has a large number of points where the density is one. Just consider the evaluation of $d_E(\lambda)$, where $\alpha < \lambda < \beta$. The preceding idea is summarized in the Lebesgue density theorem, which states that

$$d_E(x) = 1, \quad \text{for almost every } x \in E. \tag{4.530}$$

Let J denote a bounded interval centered at zero, then

$$\lim_{m \to \infty} |J \cap mE| = d_E(0)\,|J|\,. \tag{4.531}$$

Take $m > 0$, then $|mE| = m\,|E|$, and for measurable sets E_1 and E_2 it follows that $m(E_1 \cap E_2) = (mE_1) \cap (mE_2)$. If J is taken as the interval $(-\alpha, \alpha)$, with $\alpha = m\delta$, then using Eq. (4.528), $J = mJ_\delta(0)$. From Eq. (4.529),

$$\begin{aligned} d_E(0) &= \lim_{\delta \to 0+} \frac{|E \cap J_\delta(0)|}{|J_\delta(0)|} \\ &= \lim_{m \to \infty} \frac{\left|E \cap m^{-1}J\right|}{\left|m^{-1}J\right|} \\ &= \lim_{m \to \infty} \frac{|mE \cap J|}{|J|}, \end{aligned} \tag{4.532}$$

and hence Eq. (4.531) follows.

The next result required is the following. If $f \in L^p(\mathbb{R})$ for $1 \le p < \infty$, then the following statements are equivalent:

$$\text{(i)} \quad d_E(0) = 1, \tag{4.533}$$

$$\text{(ii)} \quad \lim_{m \to \infty} \|\chi_{mE} f\|_p = \|f\|_p\,, \tag{4.534}$$

$$\text{(iii)} \quad \lim_{m \to \infty} \|(1 - \chi_{mE}) f\|_p = 0. \tag{4.535}$$

First note that

$$\|f\|_p^p = \|\chi_{mE} f + (1 - \chi_{mE}) f\|_p^p = \|\chi_{mE} f\|_p^p + \|(1 - \chi_{mE}) f\|_p^p\,. \tag{4.536}$$

Hence, if (iii) holds part (ii) follows, and if (ii) holds part (iii) follows. To establish (i) from (ii), let J be as defined previously and set $f = \chi_J 1$; then

$$\|f\|_p^p = \int_{-\infty}^{\infty} |\chi_J 1|^p \, dx = |J|, \tag{4.537}$$

and hence

$$\lim_{m\to\infty} |J \cap mE| = \lim_{m\to\infty} \|\chi_{mE} f\|_p^p = \|f\|_p^p = |J|, \tag{4.538}$$

and (i) now follows on using Eq. (4.531). It remains to show that (i) $\Rightarrow$ (iii) or (ii); the former of this pair is selected. Let g denote a continuous function $g : \mathbb{R} \to \mathbb{R}$ having compact support and satisfying $\|f - g\|_p < \varepsilon$, for $\varepsilon > 0$. Let J be as defined previously, and let it contain the support of g; then

$$(1 - \chi_{mE})f = (1 - \chi_{mE})(f - g) + (1 - \chi_{mE})\chi_J g. \tag{4.539}$$

Employing Minkowski's inequality followed by Hölder's inequality leads to

$$\begin{aligned} \|(1 - \chi_{mE})f\|_p &\leq \|(1 - \chi_{mE})(f - g)\|_p + \|(1 - \chi_{mE})\chi_J g\|_p \\ &< \|(f - g)\|_p + \|(1 - \chi_{mE})\chi_J\|_{pp_2} \|g\|_{pp_1} \\ &< \varepsilon + |J \backslash mE|^{1/pp_2} \|g\|_{pp_1}, \end{aligned} \tag{4.540}$$

where p_1 and p_2 are a pair of conjugate exponents. Making use of Eq. (4.531) and keeping in mind the starting assumption that $d_E(0) = 1$, yields

$$\lim_{m\to\infty} |J \backslash mE| = |J| - \lim_{m\to\infty} |J \cap mE| = |J| - d_E(0)\,|J| = 0. \tag{4.541}$$

Taking ε as an arbitrary constant, it follows that

$$\|(1 - \chi_{mE})f\|_p = 0, \tag{4.542}$$

which is the required result.

A principal result of McLean and Elliott is the following. If $d_E(0) = 1$ and $\mathcal{O}$ commutes with dilations, then

$$\|\mathcal{O}_E\|_p = \|\mathcal{O}\|_p. \tag{4.543}$$

To establish this result, let $f \in L^p(\mathbb{R})$ with $1 < p < \infty$ and satisfying $\|f\|_p = 1$, and let $\varepsilon > 0$ such that

$$\|\mathcal{O}\|_p < \|\mathcal{O}f\|_p + \varepsilon. \tag{4.544}$$

Employing Eq. (4.511),

$$\mathcal{O}f = \mathcal{O}_{mE}f + (1 - \chi_{mE})\mathcal{O}f + \chi_{mE}\mathcal{O}(1 - \chi_{mE})f. \tag{4.545}$$

Using Minkowski's inequality and applying Hölder's inequality with conjugate exponents p_1 and p_2 leads to the following:

$$\begin{aligned} \|\mathcal{O}f\|_p &\le \|\mathcal{O}_{mE}f\|_p + \|(1 - \chi_{mE})\mathcal{O}f\|_p + \|\chi_{mE}\mathcal{O}(1 - \chi_{mE})f\|_p \\ &\le \|\mathcal{O}_{mE}\|_p + \|(1 - \chi_{mE})\mathcal{O}f\|_p + \|\chi_{mE}\mathcal{O}\|_{pp_1}\|(1 - \chi_{mE})f\|_{pp_2}. \end{aligned} \tag{4.546}$$

Using Eq. (4.527), taking the limit $m \to \infty$ and employing Eq. (4.535), leads to

$$\|\mathcal{O}f\|_p \le \|\mathcal{O}_E\|_p. \tag{4.547}$$

Utilizing Eq. (4.544) gives

$$\|\mathcal{O}\|_p < \|\mathcal{O}_E\|_p + \varepsilon, \tag{4.548}$$

and since ε is an arbitrary constant, it follows that

$$\|\mathcal{O}\|_p \le \|\mathcal{O}_E\|_p. \tag{4.549}$$

But $\|\mathcal{O}_E\|_p \le \|\mathcal{O}\|_p$ is an obvious inequality, and combining this with the preceding equation leads to the desired result, Eq. (4.543).

The central result of this section can now be established. If $|E| \ne 0$ then Eq. (4.506) holds. From Eq. (4.530) for almost every $x \in E$ it follows that $d_E(x) = 1$ holds and hence $d_{-x+E}(0) = 1$. Making use of Eqs. (4.523) and (4.543), and using the fact that the Hilbert transform operator commutes with the translation and dilation operators (Section 4.6), leads to

$$\|H_E\|_p = \|H_{-x+E}\|_p = \|H\|_p, \tag{4.550}$$

which is the desired result.

4.25 The Hilbert–Stieltjes transform

The Hilbert–Stieltjes transform of the function F is defined by

$$H_S F(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}F(t)}{x - t}. \tag{4.551}$$

At the risk of making the notation overly cumbersome, an appropriate subscript has been added to designate this transform. In the literature the notation for the standard Hilbert transform is also employed to denote the Hilbert–Stieltjes transform. This transform is sometimes defined with the opposite sign convention, and occasionally the factor of pi is omitted.

Two key results are established in this section. For the appropriate class of functions F, the Hilbert–Stieltjes transform exists *a.e.* From this result the existence of Hf *a.e.* for $f \in L(\mathbb{R})$ is deduced. Further, a bound is established for the measure of the set of points where the Hilbert–Stieltjes transform exceeds a positive constant. A few background details are required.

Let $f \in L^1[a, b]$ and suppose that $F(x)$ is defined by

$$F(x) = \int_a^x f(t)\mathrm{d}t + C, \tag{4.552}$$

with C a constant. The function F is absolutely continuous on $[a, b]$, and

$$F(x) - F(a) = \int_a^x f(t)\mathrm{d}t, \quad \text{for } x \in [a, b]. \tag{4.553}$$

If for any subdivision of $[a, b]$ the sum $\sum_{k=1}^{n} \left|F(x_k) - F(x_{k-1})\right|$ is bounded, then F is said to be of *bounded variation* on the interval $[a, b]$. The *total variation* of the function on the interval $[a, b]$ is the sup of these sums. The total variation of F on $[a, b]$ is denoted by V. It can be shown that

$$V = \int_a^b |f(x)|\mathrm{d}x, \tag{4.554}$$

which is left as an exercise for the reader to consider. The class of functions that have bounded variation on $[a, b]$ is denoted by $BV([a, b])$.

To establish the results indicated for the Hilbert–Stieltjes transform, some preliminary lemmas and some background on distribution functions are needed. The notation $E_\lambda(f)$ is used to designate the set of points

$$E_\lambda(f) = \{x \in \mathbb{R} : |f(x)| > \lambda\}. \tag{4.555}$$

The distribution function of f is defined by

$$m_\lambda(f) = m(\{x \in \mathbb{R} : |f(x)| > \lambda\}), \tag{4.556}$$

where $m(\,)$ denotes the measure of the set $E_\lambda(f)$. The alternative notation $|\{\, x \in \mathbb{R} : |f(x)| > \lambda\}|$ is also commonly employed to denote the distribution function, and it is used in various sections of this book. The designation of the real line is dropped unless some misunderstanding is likely. This distribution function is not to be confused with the use of the same term employed with the connotation of a generalized function. A few of the basic properties of the distribution function that are employed in this section or in later chapters are now presented. Since $E_{\lambda_1}(f) \subset E_{\lambda_2}(f)$ for $\lambda_1 < \lambda_2$, then $m_\lambda(f)$ is a decreasing function as λ increases. For two functions f_1 and f_2, if

$$|f_1(x)| \leq |f_2(x)|, \tag{4.557}$$

then $E_\lambda(f_1) \subset E_\lambda(f_2)$, and so

$$m_\lambda(f_1) \le m_\lambda(f_2). \tag{4.558}$$

If the function f is written as a sum,

$$f(x) = f_1(x) + f_2(x), \tag{4.559}$$

then $E_{2\lambda}(f) \subset E_\lambda(f_1) \cup E_\lambda(f_2)$, and hence

$$m_{2\lambda}(f) \le m_\lambda(f_1) + m_\lambda(f_2). \tag{4.560}$$

If $f \in L^p$, then

$$\|f\|_p^p = p \int_0^\infty |\{x : |f(x)| > \lambda\}| \lambda^{p-1}\, \mathrm{d}\lambda. \tag{4.561}$$

To see what is involved in this last equation, treat f as a simple function. For simplicity take $f = \chi_I$, where I denotes an interval, and

$$|\{x : |f(x)| > \lambda\}| = \begin{cases} 0, & \text{if } \lambda > 1 \\ |I|, & \text{if } 0 < \lambda \le 1, \end{cases} \tag{4.562}$$

then $\|f\|_p^p = |I|$. The right-hand side of Eq. (4.561) simplifies as follows:

$$\begin{aligned} p \int_0^\infty |\{x : |f(x)| > \lambda\}| \lambda^{p-1}\, \mathrm{d}\lambda &= p \int_0^1 |\{x : |f(x)| > \lambda\}| \lambda^{p-1}\, \mathrm{d}\lambda \\ &\quad + p \int_1^\infty |\{x : |f(x)| > \lambda\}| \lambda^{p-1}\, \mathrm{d}\lambda \\ &= p\, |I| \int_0^1 \lambda^{p-1}\, \mathrm{d}\lambda \\ &= |I|. \end{aligned} \tag{4.563}$$

Now take a simple function to examine the general result.

The first lemma that is required to establish the key result of this section is as follows. If $b_i > 0, \lambda > 0$, and the function f is defined by

$$f(x) = \sum_{i=1}^n \frac{b_i}{x - a_i}, \tag{4.564}$$

then the set of points where $f(x) > \lambda$ consists of n intervals whose total length is $\lambda^{-1}\sum_{i=1}^{n} b_i$; that is,

$$m\{x : f(x) \geq \lambda\} = \frac{1}{\lambda}\sum_{i=1}^{n} b_i \tag{4.565}$$

and

$$m\{x : f(x) \leq -\lambda\} = \frac{1}{\lambda}\sum_{i=1}^{n} b_i. \tag{4.566}$$

This result is often referred to as Loomis' lemma (Loomis, 1946), though a number of authors call it Boole's lemma (Boole, 1857). As an example, consider

$$f(x) = \frac{3}{x-1} + \frac{2}{x-3/2} + \frac{1}{x-3}, \tag{4.567}$$

which is shown in Figure 4.9. Equation (4.565) is illustrated for the choice $\lambda = 2$. The roots of $f(x) = \lambda$ are 1.267 949 192 45, 2.5, and 4.732 050 807 41, so that the left-hand side of Eq. (4.565) is the sum of the values 0.267 949 192 45, 1, and 1.732 050 807 41, which are the three horizontal sections shown in the figure, and this sum equals $\lambda^{-1}\sum_{i=1}^{n} b_i$.

To prove Eq. (4.565), proceed as follows. First note that $\lim_{x\to a_i-} f(x) = -\infty$, $\lim_{x\to a_i+} f(x) = \infty$, and $f'(x) < 0$; that is, $f(x)$ is monotone decreasing in each interval (a_i, a_{i+1}). There are exactly n points where $f(m_i) = \lambda$, which are denoted by m_i, and the points are positioned such that $a_i < m_i < a_{i+1}$ for $i = 1, 2, \ldots, n-1$; $a_n < m_n$. Reference to Figure 4.9 should help with a visualization of the aforementioned

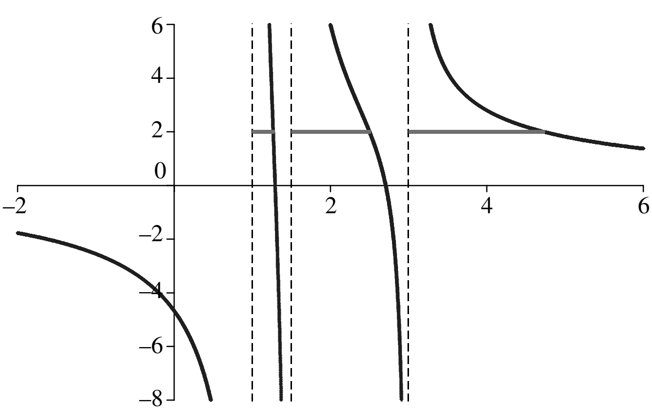

Figure 4.9. Plot of the function $f(x) = 3(x-1)^{-1} + 2(x-3/2)^{-1} + (x-3)^{-1}$ showing the points for which $f(x) \geq \lambda$.

statements. The set where $f(x) > \lambda$ is given by the sum of the intervals $(m_i - a_i)$ and has a total length of

$$m\{x : f(x) \geq \lambda\} = \sum_{i=1}^{n} (m_i - a_i). \tag{4.568}$$

The m_i values are the roots of

$$\lambda = \sum_{i=1}^{n} \frac{b_i}{x - a_i}, \tag{4.569}$$

which on multiplication by $\prod_{k=1}^{n} (x - a_k)$ gives

$$\sum_{i=1}^{n} b_i \prod_{\substack{k=1 \\ (k \neq i)}}^{n} (x - a_k) = \lambda \prod_{k=1}^{n} (x - a_k), \tag{4.570}$$

and this can be written in polynomial form as follows:

$$\lambda x^n - \left[\lambda \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right] x^{n-1} + \cdots = 0. \tag{4.571}$$

The roots m_i satisfy

$$\prod_{i=1}^{n} (x - m_i) = 0, \tag{4.572}$$

and comparing the coefficient of x^{n-1} for the last two equations leads to

$$\sum_{i=1}^{n} a_i + \frac{1}{\lambda} \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} m_i, \tag{4.573}$$

and hence

$$\sum_{i=1}^{n} (m_i - a_i) = \frac{1}{\lambda} \sum_{i=1}^{n} b_i. \tag{4.574}$$

Alternatively, Eq. (4.573) can be obtained directly from Eq. (4.571) using the Viète–Girard theorem. The required result follows immediately from Eq. (4.568).

The proof of Eq. (4.566) follows in a similar manner. Therefore

$$m\{x : |f(x)| \geq \lambda\} = \frac{2}{\lambda} \sum_{i=1}^{n} b_i. \tag{4.575}$$

The second lemma required is also due to Loomis (1946). Let $F(t)$ be a non-decreasing function with finite total variation V. Let $(x_j - \delta_j,\ x_j + \delta_j)$, $j = 1, 2, \ldots, n$, be disjoint intervals such that

$$\left| \int_{-\infty}^{x_j - \delta_j} \frac{\mathrm{d}F(t)}{t - x_j} + \int_{x_j + \delta_j}^{\infty} \frac{\mathrm{d}F(t)}{t - x_j} \right| > M > 0; \tag{4.576}$$

then

$$\sum_j \delta_j \leq \frac{8V}{M}. \tag{4.577}$$

To establish Eq. (4.577), consider

$$\int_{-\infty}^{x_j - \delta_j} \frac{\mathrm{d}F(t)}{t - x_j} + \int_{x_j + \delta_j}^{\infty} \frac{\mathrm{d}F(t)}{t - x_j} > M > 0, \tag{4.578}$$

and first note that the integrands in Eq. (4.578) are continuous over the integration intervals, and hence the integrals can be approximated by Riemann–Stieltjes sums. Let $t_i, i = 1, 2, \ldots, N$, be a subdivision of the real line that includes the points $x_j - \delta_j$, x_j, and $x_j + \delta_j$. If I_j denotes the set of indices for which $(t_i, t_{i+1}) \subseteq (x_j - \delta_j, x_j + \delta_j)$, then Eq. (4.578) can be written as follows:

$$g_j(y) \equiv \sum_{i \notin I_j} \frac{\Delta F_i}{t_i - y} > M, \tag{4.579}$$

for $y = x_j$, where $\Delta F_i = F(t_{i+1}) - F(t_i)$. Since $g_j(y)$ is an increasing function of y for $x_j - \delta_j < y < x_j + \delta_j$, Eq. (4.579) is satisfied for $x_j \leq y < x_j + \delta_j$. Let λ satisfy $0 < \lambda < 1$ and suppose that

$$h_j(y) \equiv \sum_{i \in I_j} \frac{\Delta F_i}{t_i - y} \geq -\lambda M, \tag{4.580}$$

then it follows that

$$\sum_{i=1}^{N-1} \frac{\Delta F_i}{t_i - y} = g_j(y) + h_j(y) > M - \lambda M. \tag{4.581}$$

Let S_+ denote the set where the preceding result holds, that is

$$S_+ = \left\{ y : \sum_{n=1}^{N-1} \frac{\Delta F_i}{t_i - y} > (1-\lambda)M \right\}. \tag{4.582}$$

Let S_j denote the set where

$$\sum_{i \in I_j} \frac{\Delta F_i}{t_i - y} < -\lambda M, \tag{4.583}$$

that is

$$S_j = \left\{ y : \sum_{i \in I_j} \frac{\Delta F_i}{t_i - y} < -\lambda M \right\}. \tag{4.584}$$

For $\forall y \in (x_j, x_j + \delta_j)$ implies $y \in S_+ \cup S_j$. From the first Loomis lemma it follows that the measure of the set S_+ is given by

$$\begin{aligned} m\{S_+\} &= m\left\{ y : \sum_{n=1}^{N-1} \frac{\Delta F_i}{t_i - y} > (1-\lambda)M > 0 \right\} \\ &= \frac{1}{M(1-\lambda)} \sum_{i=1}^{N-1} \Delta F_i \leq \frac{V}{M(1-\lambda)}, \end{aligned} \tag{4.585}$$

and, for the set S_j,

$$m\{S_j\} = m\left\{ y : \sum_{i \in I_j} \frac{\Delta F_i}{t_i - y} < -\lambda M < 0 \right\} = \frac{1}{\lambda M} \sum_{i \in I_j} \Delta F_i. \tag{4.586}$$

Now,

$$\bigcup_{j=1}^{n} (x_j, x_j + \delta_j) \subset S_+ \bigcup_j S_j \tag{4.587}$$

and

$$m\left\{ \bigcup_{j=1}^{n} (x_j, x_j + \delta_j) \right\} = \sum_{j=1}^{n} m\{x_j, x_j + \delta_j\} \leq m\{S_+\} + \sum_{j=1}^{n} m\{S_j\}. \tag{4.588}$$

It follows that

$$\sum_{j=1}^{n} \delta_j = \sum_{j=1}^{n} m\{(x_j, x_j + \delta_j)\} \le \frac{1}{M(1-\lambda)} \sum_{i=1}^{N-1} \Delta F_i + \frac{1}{\lambda M} \sum_{j=1}^{n} \sum_{i \in I_j} \Delta F_i. \quad (4.589)$$

This result simplifies on using the following inequality:

$$\sum_{j=1}^{n} \sum_{i \in I_j} \Delta F_i \le \sum_{i=1}^{N-1} \Delta F_i \quad (4.590)$$

to

$$\sum_{j=1}^{n} \delta_j \le \frac{1}{\lambda(1-\lambda)M} \sum_{i=1}^{N-1} \Delta F_i \le \frac{V}{\lambda(1-\lambda)M}. \quad (4.591)$$

Hence with the optimal choice of λ, that is $\lambda = 1/2$,

$$\sum_{j=1}^{n} \delta_j \le \frac{4V}{M}. \quad (4.592)$$

This covers the situation for the intervals $(x_j, x_j + \delta_j)$. Including the entire interval $(x_j - \delta_j, x_j + \delta_j)$ leads to

$$\sum_{j=1}^{n} \delta_j \le \frac{8V}{M}, \quad (4.593)$$

which corresponds to considering

$$\left| \int_{-\infty}^{x_j - \delta_j} \frac{\mathrm{d}F(t)}{t - x_j} + \int_{x_j + \delta_j}^{\infty} \frac{\mathrm{d}F(t)}{t - x_j} \right| > M > 0. \quad (4.594)$$

The preceding discussion can be repeated for the case where

$$\int_{-\infty}^{x_j - \delta_j} \frac{\mathrm{d}F(t)}{t - x_j} + \int_{x_j + \delta_j}^{\infty} \frac{\mathrm{d}F(t)}{t - x_j} < -M, \quad (4.595)$$

by making the change of variables $F(t) \to -G(-t)$ and $x_j \to -x_j$. Denote the sets that arise (the analogs of S_+ and S_j) as S_- and S'_j. The reader is asked to fill in some of the details for this case. For a given j either $(x_j, x_j + \delta_j) \subset S_+ \cup S_j$ or $(x_j - \delta_j, x_j) \subset S_- \cup S'_j$, and hence

$$\sum_{j=1}^{n} \delta_j \le \frac{2}{M} \left\{ 2 \sum_{i=1}^{N-1} \Delta F_i + \sum_{j=1}^{n} \sum_{i \in I_j} \Delta F_i \right\} \le \frac{8}{M} \sum_{i=1}^{N-1} \Delta F_i \le \frac{8V}{M}. \quad (4.596)$$

A corollary due to Loomis follows from the previous lemma. Let $F(t)$ in the preceding lemma have bounded variation on the interval $(-\infty, \infty)$; then

$$\sum_j \delta_j \leq \frac{16V}{M}. \tag{4.597}$$

A function F of bounded variation can be expressed as the difference of two increasing functions (called a Jordan decomposition) F_1 and F_2 of bounded total variation, thus, $F(t) = F_1(t) - F_2(t)$. Let $V_k, k = 1, 2$, denote the total variation of F_k, then the total variation of F is $V = V_1 + V_2$. If Eq. (4.595) is satisfied by $F(t)$, it is satisfied by $F_1(t)$ and $F_2(t)$ with M replaced by $M/2$. The preceding lemma can be applied to $F_1(t)$ and $F_2(t)$, and if the sets of j, where Eq. (4.595) holds for these two functions, are denoted by J_1 and J_2, respectively, then

$$\sum_{j \in J_k} \delta_j \leq \frac{16V_k}{M}, \quad k = 1, 2. \tag{4.598}$$

From this result it follows that

$$\sum_{j=1}^{n} \delta_j \leq \frac{16}{M}\{V_1 + V_2\} = \frac{16V}{M}, \tag{4.599}$$

which gives the desired result.

The third lemma required is classical. Let $F(t)$ be a singular function of bounded total variation that is constant almost everywhere. Given a constant $\varepsilon > 0$, there exists a function G of bounded variation on an open set S that is constant on the intervals of S, such that the total variation of $F - G$ is bounded above by ε. It is left to the reader to construct a proof of this result.

The key result of this section is the following theorem due to Loomis. The Hilbert–Stieltjes transform of the function F of bounded variation exists almost everywhere. For every $M > 0$, the set S_M, given by

$$S_M = \{x : |H_S F(x)| > M\}, \tag{4.600}$$

has measure bounded above by $32V/M$. To establish the first part of the theorem it is sufficient to show that, for a given ε and sufficiently small δ and η, with $0 < \eta < \delta$, and every x except in a set of measure less than ε,

$$\left| \int_{x-\delta}^{x-\eta} \frac{\mathrm{d}F(t)}{x - t} + \int_{x+\eta}^{x+\delta} \frac{\mathrm{d}F(t)}{x - t} \right| \leq \varepsilon. \tag{4.601}$$

The function F can be written in terms of its absolutely continuous part F_a and its singular part F_s (the Lebesgue decomposition of a function) as follows:

$$F(t) = \int_{-\infty}^{t} F'(s)\mathrm{d}s + \left\{F(t) - \int_{-\infty}^{t} F'(s)\mathrm{d}s\right\} = F_a(t) + F_s(t). \tag{4.602}$$

The function F_s is of bounded variation and has a derivative that is zero *a.e.* There exists a constant $\varepsilon' > 0$ such that F' can be approximated to within ε' by a step function h, that is

$$\int_{-\infty}^{\infty} \left|h(t) - F'(t)\right|\mathrm{d}t \le \varepsilon'. \tag{4.603}$$

The function F_1 is introduced by the definition

$$F_1(t) = \int_{-\infty}^{t} h(s)\mathrm{d}s, \tag{4.604}$$

then $F_1(t) - F_a(t)$ has a total variation bounded above by ε'. The preceding lemma implies that there exists a function F_2 whose variation is confined to a closed set of measure zero and is constant on the open intervals complementary to the set of measure zero, such that the total variation function of $F_s(t) - F_2(t)$ is also bounded above by ε'. The function F is decomposed as follows:

$$F(t) = F_1(t) + F_2(t) + F_3(t), \tag{4.605}$$

so that

$$F_3(t) = F_a(t) - F_1(t) + F_s(t) - F_2(t). \tag{4.606}$$

Equation (4.601) is satisfied when $F(t)$ is replaced by $F_1(t)$ or $F_2(t)$ with ε replaced by $\varepsilon/3$. It is therefore necessary to establish that if $F(t)$ is replaced by $F_3(t)$ in Eq. (4.601) then the set where the result fails to hold has a measure ε. Let $\varepsilon' = \varepsilon^2/384$. The reason for this choice will become apparent momentarily. The total variation of $F_3(t)$ denoted by V_3 satisfies $V_3 < \varepsilon^2/192$. Let E_ε be the set of x where the inequality

$$\left|\int_{x-\delta}^{x-\eta} \frac{\mathrm{d}F_3(t)}{x-t} + \int_{x+\eta}^{x+\delta} \frac{\mathrm{d}F_3(t)}{x-t}\right| > \frac{\varepsilon}{3} \tag{4.607}$$

holds for arbitrarily small δ and η. There exists an arbitrarily small Δ such that, for every $x \in E_\varepsilon$,

$$\left|\int_{-\infty}^{x-\Delta} \frac{\mathrm{d}F_3(t)}{x-t} + \int_{x+\Delta}^{\infty} \frac{\mathrm{d}F_3(t)}{x-t}\right| > \frac{\varepsilon}{6}. \tag{4.608}$$

A disjoint sequence of intervals $[x_i - \Delta_i, x_i + \Delta_i]$ satisfying this inequality can be found to cover the set E_ε except for a set of measure zero. This is an application of

Vitali's covering theorem, which takes the following form. If a collection of closed sets J is a Vitali covering of a set S in E^n, then there is a finite or countably infinite sequence of pairwise disjoint sets belonging to J, whose union contains all of S except a set of measure zero. A Vitali covering is defined in the following manner. For each $x \in S$ there is a positive number $a(x)$ and a sequence of sets A_n with arbitrarily small diameters each containing x and belonging to J, and for each n there is a cube C_n such that $m(A_n) \geq a(x)m(C_n)$ and C_n contains A_n.

From the corollary previously given, it follows that

$$m(E_\varepsilon) \leq 2\sum_i \Delta_i \leq 2\left\{\frac{16V_3}{\varepsilon/6}\right\} = \frac{192V_3}{\varepsilon} < \varepsilon. \tag{4.609}$$

Since the Hilbert transform of a step function exists except at a finite number of points, which can be added to the set E_ε, it follows from Eq. (4.604) that the Hilbert–Stieltjes transform of F_1 exists except at this same finite set of points. The function F_2 is constant on the intervals of an open set and its Hilbert–Stieltjes transform exists except on the complement of the open set, which may be added to E_ε. Hence, if x does not belong to the enlarged set E_ε, there exists a Δ with $0 < \delta < \eta < \Delta$ such that Eq. (4.607) holds for F_1, F_2, and F_3, and therefore Eq. (4.601) holds. This establishes that the Hilbert–Stieltjes transform exists almost everywhere.

The second part of the theorem can be proved by employing the second lemma and its corollary. The intervals $(x_j - \delta_j, x_j + \delta_j)$ are selected to cover almost all of the set of x where $|(H_S F)(x)| > M$. The measure of this set is given by

$$m(S_M) = 2\sum_j \delta_j \leq \frac{32V}{M}. \tag{4.610}$$

An immediate important consequence of the preceding theorem is the following. If $f \in L(\mathbb{R})$ and

$$F(t) = \int_{-\infty}^{t} f(s)\mathrm{d}s, \tag{4.611}$$

then $\mathrm{d}F(t) = f(t)\mathrm{d}t$ *a.e.* and hence Hf exists *a.e.*

4.26 A theorem due to Stein and Weiss

The distribution function for the Hilbert transform of the characteristic function of a Lebesgue measurable set of finite measure of the real line depends only on the measure of the set, not on how the set is spread over $\mathbb{R}$. This interesting result was established by Stein and Weiss (1959), and it is proved in this section.

The following lemma is required. Let a_k and $b_k, k = 1, \ldots, n$, be real numbers such that $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$, and let

$$f(x) = \prod_{k=1}^{n} \frac{x - a_k}{x - b_k}, \quad \text{with } x \in \mathbb{R}. \tag{4.612}$$

Suppose the real constant ξ is not equal to one, and denote the roots of the equation $f(x) = \xi$ by $r_k, k = 1, \ldots, n$, and let the roots of the equation $f(x) = -\xi$ be s_k, $k = 1, \ldots, n$. Then the following three results hold:

$$\sum_{k=1}^{n} (b_k - r_k) = \frac{1}{1 - \xi} \sum_{k=1}^{n} (b_k - a_k), \tag{4.613}$$

$$|x : f(x) > \xi| = \frac{1}{\xi - 1} \sum_{k=1}^{n} (b_k - a_k), \quad \text{for } \xi > 1, \tag{4.614}$$

and

$$|x : f(x) < -\xi| = \frac{1}{\xi + 1} \sum_{k=1}^{n} (b_k - a_k), \quad \text{for } \xi > 1. \tag{4.615}$$

Reference to Figure 4.10 shows the arrangement of the x-axis intercept points a_k, the points b_k where $f(x)$ diverges, the points r_k where $f(x) = \xi$, and the points s_k where $f(x) = -\xi$. To prove Eq. (4.613), first note that $f(x) = \xi$ can be written in the following form:

$$p(x) = \prod_{k=1}^{n} (x - a_k) - \xi \prod_{k=1}^{n} (x - b_k) = 0. \tag{4.616}$$

Alternatively, $p(x)$ can be written as follows:

$$p(x) = \prod_{k=1}^{n} (x - r_k) = 0. \tag{4.617}$$

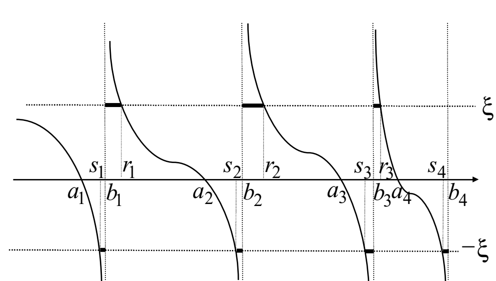

Figure 4.10. Plot of the function f defined in Eq. (4.612) showing the points for which $f(x) > \xi$ and $f(x) < -\xi$.

Expanding the two preceding equations yields

$$x^n(1-\xi) - x^{n-1}\left\{\sum_{k=1}^{n} a_k - \xi\sum_{k=1}^{n} b_k\right\} + \cdots = 0 \tag{4.618}$$

and

$$x^n - x^{n-1}\sum_{k=1}^{n} r_k + \cdots = 0. \tag{4.619}$$

Multiplying Eq. (4.618) by $(1-\xi)^{-1}$ and comparing the coefficients of the x^{n-1} term leads to

$$\sum_{k=1}^{n} r_k = \frac{1}{1-\xi}\sum_{k=1}^{n} a_k - \frac{\xi}{1-\xi}\sum_{k=1}^{n} b_k. \tag{4.620}$$

This last result can be rearranged to give

$$\sum_{k=1}^{n}(b_k - a_k) + (1-\xi)\sum_{k=1}^{n} r_k = (1-\xi)\sum_{k=1}^{n} b_k, \tag{4.621}$$

and hence Eq. (4.613) follows.

The following approach can be used to establish Eq. (4.614). From Figure 4.10 the required calculation geometrically corresponds to adding all the segments on the line $y = \xi$ that are shown in bold. Hence,

$$\begin{aligned} |x : f(x) > \xi| &= \bigcup_{k=1}^{n}(b_k, r_k) \\ &= \sum_{k=1}^{n}(r_k - b_k) \\ &= \frac{1}{\xi - 1}\sum_{k=1}^{n}(b_k - a_k), \end{aligned} \tag{4.622}$$

which is the required result. The last line follows on using Eq. (4.613). Equation (4.615) can be established in a similar manner. With reference to Figure 4.10, the calculation involves the addition of all the segments shown in bold on the line $y = -\xi$. Therefore

$$|x : f(x) < -\xi| = \bigcup_{k=1}^{n} (s_k, b_k)$$
$$= \sum_{k=1}^{n} (b_k - s_k). \tag{4.623}$$

Repeating the same type of argument used to prove Eq. (4.613) gives

$$\sum_{k=1}^{n} (b_k - s_k) = \frac{1}{1+\xi} \sum_{k=1}^{n} (b_k - a_k). \tag{4.624}$$

From this result it follows that

$$|x : f(x) < -\xi| = \frac{1}{1+\xi} \sum_{k=1}^{n} (b_k - a_k), \tag{4.625}$$

which completes the proof of the lemma.

The Stein–Weiss theorem is stated as follows. Let E denote a Lebesgue measurable set of the real line of finite measure, then the distribution function of the Hilbert transform of the characteristic function χ_E satisfies

$$|x \in \mathbb{R} : |H\chi_E| > \lambda| = \frac{2\,|E|}{\sinh \pi\lambda}, \quad \lambda > 0, \tag{4.626}$$

where $|E|$ denotes the measure of the set E, and the associated characteristic function of E is denoted by χ_E. The union of a finite number of disjoint intervals can be expressed as

$$|E| = \bigcup_{k=1}^{n} (a_k, b_k). \tag{4.627}$$

The characteristic function χ_E can be viewed as the simple function resulting from the sum of the characteristic functions for each of the intervals (a_k, b_k). The Hilbert transform of the characteristic function of E is given by

$$H\chi_E(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi_E(s)\mathrm{d}s}{x-s}$$
$$= \frac{1}{\pi} \sum_{k=1}^{n} P \int_{a_k}^{b_k} \frac{\mathrm{d}s}{x-s}$$

$$
\begin{aligned}
&= \frac{1}{\pi} \sum_{k=1}^{n} \log \left| \frac{x - a_k}{x - b_k} \right| \\
&= \frac{1}{\pi} \log \prod_{k=1}^{n} \left| \frac{x - a_k}{x - b_k} \right|. \qquad (4.628)
\end{aligned}
$$

First consider the contribution $|x \in \mathbb{R} : H\chi_E > \lambda|$. Let

$$
f(x) = \prod_{k=1}^{n} \left(\frac{x - a_k}{x - b_k} \right), \qquad (4.629)
$$

then, from Eq. (4.628),

$$
e^{2\pi H\chi_E(x)} = f(x)^2. \qquad (4.630)
$$

Taking the square root of the preceding result means that the condition $H\chi_E > \lambda$ is replaced by

$$
f(x) > e^{\pi\lambda} \qquad (4.631)
$$

and

$$
f(x) < -e^{\pi\lambda}. \qquad (4.632)
$$

From Eq. (4.627) and using $\xi = e^{\pi\lambda}$ in Eqs. (4.614) and (4.615) leads to

$$
\begin{aligned}
\left| x : f(x) > e^{\pi\lambda} \right| + \left| x : f(x) < -e^{\pi\lambda} \right| &= \frac{|E|}{(e^{\pi\lambda} - 1)} + \frac{|E|}{(e^{\pi\lambda} + 1)} \\
&= \frac{|E|}{\sinh \pi\lambda}, \qquad (4.633)
\end{aligned}
$$

and therefore

$$
|x \in \mathbb{R} : H\chi_E > \lambda| = \frac{|E|}{\sinh \pi\lambda}, \quad \lambda > 0. \qquad (4.634)
$$

Proceeding in much the same fashion it follows that

$$
\left| x : f(x) < e^{-\pi\lambda} \right| + \left| x : f(x) > -e^{-\pi\lambda} \right| = \frac{|E|}{\sinh \pi\lambda} \qquad (4.635)
$$

and hence

$$
|x \in \mathbb{R} : H\chi_E < -\lambda| = \frac{|E|}{\sinh \pi\lambda}, \quad \lambda > 0. \qquad (4.636)
$$

Combining Eqs. (4.634) and (4.636) gives the required result, Eq. (4.626).

There is a version of the Stein–Weiss theorem for the circle. Let $E \subset [0, 2\pi]$ denote a Lebesgue measurable set, then the distribution function of the Hilbert transform of the characteristic function χ_E satisfies

$$|x \in \mathbb{T} : |\mathcal{H}\chi_E| > \lambda| = 4\tan^{-1}\left(\frac{\sin(|E|/2)}{\sinh \pi\lambda}\right), \quad \lambda > 0. \tag{4.637}$$

The interested reader can pursue the proof in Stein and Weiss (1959).

Notes

§4.2 Early work on H_e was carried out by Hardy and Littlewood (1936), and there were later contributions for the even and odd Hilbert transform operators from Babenko (1948) and Rooney (1972, 1980). For the inversion of the even and odd Hilbert transformations, see Heywood and Rooney (1988).

§4.5.1 Ideas related to the work of Hardy and Glasser can be found in Boole (1857). Similar nonlinear scale changes can be made for integrals other than the Hilbert transform; see for example Glasser (1983).

§4.6 An advanced discussion of the theory of multipliers can be found in Larsen (1971). Duggal (1980) has studied operators that satisfy $Ts_a = m(a)s_a$, where $s_a f(x) = f(ax)$ for $-\infty < a < \infty$, $a \neq 0$, and $m(a) = 1$ or $\operatorname{sgn} a$. The Hilbert transform and some of its extensions satisfy this equation.

§4.8 Early work on the derivative form of the Hilbert transform pair is due to Hardy (1924a, 1932). The approach used in this section to work out the Hilbert transform of the Gaussian function is based on Kochneff *et al.* (1993). The Hilbert transform of a Gaussian function is also evaluated in Calderón and Sagher (1991). Kochneff (1992) and Kochneff, Sagher, and Zhou (1992) take advantage of some results about the Hilbert transform of the Gaussian function to establish homogeneous solutions of the heat equation, $u_t(x, t) = u_{xx}(x, t)$.

§4.9 The Hilbert transform can be studied by replacing the usual convolution with $(\pi x)^{-1}$ by convolution with $(\pi x)^{-1}|x|^{-\mathrm{i}\gamma}$, where γ is a constant. It might be expected that this would be an easier kernel to study, by virtue of the less singular nature of this choice, and with the expectation that the properties of the Hilbert transform could be recovered in the limit $\gamma \to 0$. This notion turns out not to be so simple. For some further reading on this idea, see Jones (1992).

§4.10 Equation (4.176) can be proved under different conditions than those stated in this section; see Nevai (1990).

§4.12 See Kober (1943b) and Butzer and Nessel (1971, p. 316) for different approaches to the proof of some of the results of this section. Equation (4.205) is discussed in Matsuno (1984, p. 204).

§4.13 A very good source for series expansions that would be useful for the approach discussed in this section is Hansen (1975). A well organized short collection of series, one that is particularly useful for hyperbolic functions, is Wheelon (1968). Some references that the author has found to be useful reading on using convergence accelerator techniques to sum series are given in the §2.16 end-notes.

§4.15 Bedrosian's theorem (Bedrosian, 1963) as originally stated also included a second set of conditions that would be sufficient, namely that the functions f and g are both analytic signals. This aspect of the theorem and a generalization were discussed in Section 4.14. For some additional reading, see Urkowitz (1962), Bedrosian (1966a, 1966b, 1972), Nuttall (1966), Rihacek (1966), Stark (1971, 1972), Cain (1973), Carton-Lebrun (1979), and Hahn (1996c). Brown (1986) and Xu and Yan (2006) discuss necessary and sufficient conditions for the Bedrosian theorem to apply.

§4.16 An alternative derivation of the Tricomi theorem is given by Carton-Lebrun (1977, 1987), and additional discussion can be found in Glaeske and Tuan (1995). For further reading on the convolution properties of the Fourier transform, see Titchmarsh (1948) and Champeney (1987).

§4.17 Higgins (1977) is a useful reference on basis functions for different spaces. Weideman (1995) makes use of the functions defined in Eq. (4.336) to develop an algorithm to compute the Hilbert transform on the real line, a topic explored in detail in Section 14.14. Kober (1943b) exploits the eigenfunctions of the Hilbert transform operator to prove a number of the basic properties satisfied by this operator.

§4.20 Riesz's key papers on the topic of this section can be found in his collected papers edited by Gårding and Hörmander, in Riesz (1988). An early textbook account of the Riesz inequality can be found in Hobson (1926, p. 610). A real-variable proof of the Riesz inequality for even integral values of p was given by Cossar (1939). O'Neil and Weiss (1963) and Gohberg (alternative spelling Gokhberg) and Krupnik (1968) gave upper and lower bounds, respectively, for the best value of the constant $\mathfrak{R}_p$. Pichorides (1972) proved that the Gohberg–Krupnik result is actually the best value for $\mathfrak{R}_p$ for $1 < p < \infty$. For some further reading, see Bochner (1959b), Pełczynski (1985), Krupnik (1987, chap. 2), Grafakos (1997), and Hollenbeck, Kalton, and Verbitsky (2003). The best bounds for the Riesz inequality given in Eq. (4.384) are also the best bounds for the norm of the operator defined by $Sf(x) = \int_a^b \mathcal{F}f(y)e^{2\pi ixy}\,dy$; see De Carli and Laeng (2000). For extensions beyond the Lebesgue space, see Boyd (1967).

§4.21 Additional discussion can be found in Jodeit, Kenig, and Shaw (1983).

§4.22 The key theorem of this section for functions $\in L^p$, with $p > 1$, is discussed by Hille and Tamarkin (1935) and Titchmarsh (1948, p. 139). For further reading, see Kawata (1936) and, in particular, Toll (1956), where some additional statements are established beyond the four given in the version of Titchmarsh's theorem proved in this section. The extension to cover the case where distributions are involved has been considered by Taylor (1958).

§4.23 The discussion of the Hardy–Poincaré–Bertrand formula given in this section is based on a derivation of Love (1977). For further reading, see Levinson (1965) and Okada (1992a).

§4.24 For an alternative approach to characterize the Hilbert transform, see Arcozzi and Fontana (1998). A generalization of the McLean–Elliott results to the n-dimensional Hilbert transform has been given by Pandey and Singh (1991).

§4.25 For proofs of the Loomis lemmas see, in addition to Loomis (1946), Garsia (1970, p. 113), Butzer and Nessel (1971, p. 307) Zygmund (1971, p. 7), and Bennett and Sharpley (1988, p. 128). The proof of the Loomis theorem can also be found in the latter two references. The Viète–Girard theorem says, in part, that the sum of the roots of the polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$, is equal to $-a_{n-1}/a_n$; see Girard (1629) and Viète (1646). A translation of the latter two sources can be found in Schmidt and Black (1986). Part of the discussion of this section reflects ideas given in an unpublished proof of the Loomis theorem and lemmas by Roger L. Cooke. For a proof of the Vitali covering theorem, see, for example, Cohn (1980, p. 177).

§4.26 The key paper for this section is Stein and Weiss (1959). The form of Eqs. (4.626) and (4.637) differs slightly from the final results of Stein and Weiss. Equation (4.626) includes an additional factor of π, whereas Stein and Weiss omit the factor of $1/\pi$ in their definition of the Hilbert transform on $\mathbb{R}$, and Eq. (4.637) includes a factor of 2π; Stein and Weiss omit a factor of $1/2\pi$ in their definition of the Hilbert transform on $\mathbb{T}$. Short proofs of the Stein–Weiss theorem using complex-variable methods can be found in Calderón (1966) and Garnett (1981, p. 115), and additional discussion can be found in Bennett and Sharply (1988, p. 129) and Sagher and Xiang (1996). The Stein–Weiss theorem has been discussed for the ergodic Hilbert transform by Ephremidze (2004).

Exercises

4.1 Determine the relationship between $Hf[\phi(x)]$ and $\{Hf\}[\phi(x)]$ when $\phi(s) = as - b(s+c)^{-1}$ for $a > 0, b > 0$, and c a constant.

4.2 Prove that $Hf[\phi(x)] = Hf(0) - \{Hf\}[\phi(x)]$ holds if $\phi(s) = \sum_{j=1}^{n} a_j/(s+b_j)$ with $a_j > 0$.

4.3 For constants α and $\beta, \beta > 0$, prove the following:

$$H\left[\frac{\beta}{(x+\alpha-\cot x^{-1})^2+\beta^2}\right] = \frac{(x-\cot x^{-1})\{\alpha(x+\alpha-\cot x^{-1})-\beta^2\}}{\pi(\alpha^2+\beta^2)\{(x+\alpha-\cot x^{-1})^2+\beta^2\}}. \tag{4.638}$$

4.4 Find an expression for $H[(x+a)^n f(x)]$ for integer $n \geq 0$. What assumptions need to be made on f?

4.5 Show that a linear operator that commutes with the translation operator must be of the form of a difference operator.

4.6 Taking advantage of $(Hf') = (Hf)'$ or otherwise, determine the following:

(i) $H[\cos^m ax \sin ax]$ for integer $m \geq 0$;
(ii) $H[(3x^2-a^2)(x^2+a^2)^{-3}]$ for $a > 0$,
(iii) $H[x^{-2}\{x\cos x - \sin x\}]$.

4.7 Prove that

$$H\{\operatorname{sech} ax\} = \frac{\mathrm{i}}{\pi}\left\{\psi\left(\frac{1}{4} - \frac{\mathrm{i}ax}{2\pi}\right) - \psi\left(\frac{1}{4} + \frac{\mathrm{i}ax}{2\pi}\right)\right\} - \tanh ax.$$

4.8 Prove that for $f(x) = \mathrm{e}^{-ax^{2n}}$, with $n = 1, 2, \ldots$, and $a > 0$, Eq. (4.152) follows.

4.9 Show using contour integration techniques that if $f(z)$ is analytic in the upper half plane and vanishes like z^{-p} with $p > 1$ as $|z| \to \infty$, then f is an eigenfunction of H with eigenvalue $-\mathrm{i}$. If $f(z)$ is analytic in the lower half plane and has the aforementioned asymptotic behavior, show that f is an eigenfunction of H with eigenvalue i.

4.10 Prove Eq. (4.365) using a contour integration approach.

4.11 Prove for $a \geq 0, b > 0$ that

$$H[(x^2 + b^2)^{-1}\mathrm{e}^{-ax^2}] = (x^2 + b^2)^{-1}[\mathrm{e}^{ab^2}xb^{-1}\{1 - \operatorname{erf}(b\sqrt{a})\} \\ - \mathrm{i}\mathrm{e}^{-ax^2}\operatorname{erf}(\mathrm{i}x\sqrt{a})].$$

4.12 If $p \in (0, 1]$ and

$$f(x) = \begin{cases} 0, & -\infty < x < 0 \\ \dfrac{1}{(1+x)^{2/p}}, & 0 \leq x < \infty, \end{cases}$$

determine if $Hf \in L(\mathbb{R})$.

4.13 Prove Eq. (4.302).

4.14 Let

$$f(x) = \begin{cases} 0, & -\infty < x \leq 0 \\ x^{-1}\log^{-2} x - 2/\log 2, & 0 < x < 1/2 \\ 0, & 1/2 \leq x < \infty. \end{cases}$$

Prove that the integral $\int_{-\infty}^{\infty} |f(x)|^p\,\mathrm{d}x$ diverges for $p > 1$.

4.15 Prove that $\phi_n(x) = (1 + \mathrm{i}x)^{n+\tau}/(1 - \mathrm{i}x)^{n+1+\tau}$ form a complete and orthogonal basis set for $L^2(\mathbb{R})$ for τ real and $n \in \mathbb{Z}$.

4.16 Do the functions $\phi_n(x) = (\mathrm{i} - x)^n/\sqrt{\pi}\,(\mathrm{i} + x)^{n+1}$ for $n \in \mathbb{Z}$ form an orthogonal basis on $L^p(\mathbb{R})$ for $1 \leq p < \infty$? If the basis is orthogonal, is it orthonormal?

4.17 Prove that for an $f(x) \in L^p(\mathbb{R})$ with $1 \leq p < \infty$, there are rational functions ϕ_n of the form

$$\phi_n(z) = \sum_{k=-n}^{n} \frac{(\mathrm{i} - z)^k}{(\mathrm{i} + z)^{k+a}}, \quad n = 0, 1, 2, \ldots,$$

with $a \geq 1$, for $1 < p < \infty$, and $a \geq 2$, for $p = 1$, such that $\|f(x) - \phi_n(x)\|_p \to 0$ as $n \to \infty$.

4.18 By employing the substitution $x = \tan(\theta/2)$, show that $x \in [-\infty, \infty]$ is mapped to $\theta \in [-\pi, \pi]$ and find the form of the function $(1 - \mathrm{i}x)f(x)$, where $f(x) = \sum_{n=-\infty}^{\infty} a_n\phi_n(x)$ and $\phi_n(x)$ is defined in Exercise 4.17.

4.19 If $f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})$, with $p^{-1} + q^{-1} \leq 1$, and

$$h(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} (x-y)^{-1}\{f(x)g(y) + f(y)g(x) - f(y)g(y)\}\mathrm{d}y,$$

show that $Hh(x) = Hf(x)Hg(x)$.

4.20 For the following choice of functions, determine if they are eigenfunctions of the Hilbert transform operator, and if they are, determine the corresponding eigenvalue. (i) $f(x) = (x + \mathrm{i}a)^{-1}, a > 0$; and (ii) $f(x) = x^{-1}(\mathrm{e}^{\mathrm{i}ax} - 1), a > 0$.

4.21 Let M_η denote the operator $M_\eta f(x) = x^\eta f(x)$ for $\eta \in \mathbb{R}$. Show that $M_\eta M_\rho = M_{\eta+\rho}$. Prove that the even and odd Hilbert transform operators, H_e and H_o, can be expressed as $H_\mathrm{e} = M_1 H_\mathrm{o} M_{-1}$ and $H_\mathrm{o} = M_{-1} H_\mathrm{e} M_1$.

4.22 If $f \in L^2(\mathbb{R})$ and $P_+\mathcal{F}f(x) = h(x)\mathcal{F}f(x)$, where P_+ is the Riesz projector, determine $h(x)$. If $P_-\mathcal{F}f(x) = h(x)\mathcal{F}f(x)$, find $h(x)$.

4.23 Derive Eqs. (4.372) and (4.373).

4.24 For positive constants a, b, and c, show that $2ab \leq ca^2 + c^{-1}b^2$. Starting from Eq. (4.388) and by appropriate choice of a, b, and c, show that Eq. (4.390) follows. How does the constant C in the inequality you find compare with that in Eq. (4.390) and with that given by $\Re_4$?

4.25 Let $F(z)$ be analytic in the upper half complex plane and suppose $F(z)^6$ vanishes sufficiently quickly as $z \to \infty$, so that the integral of $F(z)^6$ along a semicircular contour of radius R and center the origin goes to zero as $R \to \infty$. If f denotes the real part of $F(z)$, show that

$$\int_{-\infty}^{\infty} \{f(x)^6 - 15f(x)^4[Hf(x)]^2 + 15f(x)^2[Hf(x)]^4 - [Hf(x)]^6\}\mathrm{d}x = 0.$$

For positive a, b, and c, show that $c(a^2b - ab^2) \leq c^{-1}a^3 + c^5b^3$. For a suitable choice of a, b, and c, deduce that $\|Hf\|_6 \leq C\,\|f\|_6$.

4.26 Let $p_n(x)$ denote a polynomial and suppose the following conditions hold for the function f: $p_n(x)f(x) \in L^p$, for $1 < p < \infty$, $\int_{-\infty}^{\infty} f(x)x^k\,\mathrm{d}x = 0, k = 0, 1, \ldots, n$ and $\int_{-\infty}^{\infty} |f(x)|\,(1 + |x|)^n\,\mathrm{d}x < \infty$. Prove that

$$\int_{-\infty}^{\infty} |p_n(x)Hf(x)|^p\,\mathrm{d}x \leq C \int_{-\infty}^{\infty} |f(x)|^p\,|p_n(x)|^p\,\mathrm{d}x,$$

where C is a positive constant.

4.27 Making use of $C_2 = 1$, compare the constants obtained from Eq. (4.403) with $p = 2^{k-1}$ for $k = 1, 2, 3, \ldots$, with those obtained from $\Re_{2p} = \cot(\pi/4p)$ (Eq. (4.384)). What conclusion can you draw?

4.28 If $F(z) = \mathrm{e}^{\mathrm{i}\alpha z}/(z + a + \mathrm{i}b)$ with $\alpha < 0$, is this function the Fourier transform of a causal function?

4.29 For the function $f(x) = 3(x-1)^{-1} + 2(x-3/2)^{-1} + (x-3)^{-1}$, show that $m\{x : f(x) \leq -6\} = 1$, and hence verify

$$m\{x : f(x) \leq -\lambda\} = \frac{1}{\lambda}\sum_{i=1}^{n} b_i.$$

4.30 Show that if $f \in L^p(\mathbb{R})$ with $1 < p < \infty$, then, for $y > 0$,

$$\int_{-\infty}^{\infty} t(t^2+1)^{-1} f(x - yt)\mathrm{d}t = \int_{-\infty}^{\infty} (t^2+1)^{-1} Hf(x - yt)\mathrm{d}t.$$

4.31 Given that the one-sided Hilbert transform is defined by

$$H_1 f(x) = \frac{1}{\pi} P \int_0^{\infty} \frac{f(y)\mathrm{d}y}{x - y},$$

with $0 < x < \infty$, does there exist a Riesz-type inequality for $H_1 f$, where $f \in L^p(\mathbb{R})$ for $1 < p < \infty$?

4.32 If $f = \chi_{[0,1]}$, find Hf, and determine whether $Hf \in L^1(\mathbb{R})$. Does $Hf \in L^\infty(\mathbb{R})$?

4.33 Determine Hf if $f = \chi_{(0,1)} - \chi_{(-1,0)}$ and indicate whether $Hf \in L^1(\mathbb{R})$. Is $f \in L^1(\mathbb{R})$? Is $\mathcal{F}f(0) = 0$?

4.34 Are there Riesz-type inequalities for the operators H_{e} and H_{o}? Specify any restrictions that must be placed on the class of functions that are involved.

4.35 For any $r > 0$, define the "gap" Hilbert transform by

$$H_{(-r,r)} f(x) = -\frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \{\chi_{(r,\infty)}(y) - \chi_{(-\infty,-r)}(y)\} \mathcal{F}f(y) \mathrm{e}^{\mathrm{i}yx}\, \mathrm{d}y.$$

Determine a bound for the operator norm of the gap Hilbert transform.

4.36 Determine whether the Hilbert transform operator is a compact operator.

5

Relationship between the Hilbert transform and some common transforms

5.1 Introduction

In this chapter the relationship between the Hilbert transform and some other commonly employed transforms is considered. The most important connection is with the Fourier transform, and this relationship has already been utilized in Chapter 4. These interconnections play three important roles. First, they allow a number of Hilbert transforms to be evaluated from tables of other transforms. Second, they offer the possibility of an alternative means to the numerical evaluation of the Hilbert transform, which often avoids the issue of dealing with the singular structure of the kernel function of the Hilbert transform. In this latter case, the connection with the Fourier transform plays a particularly key role. The third area of application occurs where the Hilbert transform arises in the determination of another transform. For example, the inverse Radon transform can be written in terms of the Hilbert transform of the derivative of the projected function.

5.2 Fourier transform of the Hilbert transform

The convolution formula for the Hilbert transform was given in Eq. (4.154). On taking the Fourier transform of both sides of this equation, yields

$$\begin{aligned}\mathcal{F}\{g(x)\} &= \mathcal{F}\{f(x) * k(x)\}\\ &= \mathcal{F}\{f(x)\}\mathcal{F}\{k(x)\},\end{aligned} \tag{5.1}$$

using Eq. (2.54). The Fourier transform of $k(x)$ is given by

$$\begin{aligned}\mathcal{F}\{k(x)\} &= \int_{-\infty}^{\infty} \frac{e^{-ixs}\,ds}{\pi s}\\ &= \frac{-i}{\pi}\int_{-\infty}^{\infty} \frac{\sin xs\,ds}{s}\\ &= -i\,\mathrm{sgn}\,x.\end{aligned} \tag{5.2}$$

The cosine contribution of the Fourier transform vanishes (it is an odd function), and the remaining integral can be evaluated using a contour integration approach, or by other means. Combining Eqs. (4.154), (5.1), and (5.2) leads to

$$\mathcal{F}Hf(x) = -\mathrm{i}\ \mathrm{sgn}\, x\, \mathcal{F}f(x). \tag{5.3}$$

The reader should quickly spot the abuse of rigor that has just taken place. More care is needed with the singular structure of the kernel function that arises. To that end, a considerably more refined approach based on Neri (1971, p. 56) (see also Butzer and Nessel (1971, p. 310)) is presented. At the same time, a class of functions for which Eq. (5.3) holds is established. Momentarily, this class is generalized to cover the case that $f \in L^p(\mathbb{R})$ for $1 < p \leq 2$.

The singular structure can be dealt with by using a limiting process. Let

$$k_{\varepsilon,\eta}(x) = \begin{cases} 1/\pi x, & 0 < \varepsilon < |x| < \eta < \infty \\ 0, & \text{elsewhere,} \end{cases} \tag{5.4}$$

$$H_{\varepsilon,\eta}f(x) = \frac{1}{\pi}\int_{\varepsilon<|x-t|<\eta} \frac{f(t)\mathrm{d}t}{x-t}, \tag{5.5}$$

and $f \in L^2(\mathbb{R})$. From Eqs. (5.4) and (5.5), it follows that

$$\begin{aligned} H_{\varepsilon,\eta}f(x) &= \frac{1}{\pi}\left\{\int_{x-\eta}^{x-\varepsilon} \frac{f(t)\mathrm{d}t}{x-t} + \int_{x+\varepsilon}^{x+\eta} \frac{f(t)\mathrm{d}t}{x-t}\right\} \\ &= \int_{x-\eta}^{x-\varepsilon} f(t)k_{\varepsilon,\eta}(x-t)\mathrm{d}t + \int_{x+\varepsilon}^{x+\eta} f(t)k_{\varepsilon,\eta}(x-t)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} f(t)k_{\varepsilon,\eta}(x-t)\mathrm{d}t \\ &= (f * k_{\varepsilon,\eta})(x). \end{aligned} \tag{5.6}$$

Since $k_{\varepsilon,\eta} \in L$ and $f \in L^2$, the convolution belongs to L^2 and so $H_{\varepsilon,\eta}f \in L^2$. The assertion employed at the start of the preceding sentence is developed as Exercise 5.22. On taking the Fourier transform of Eq. (5.6), it follows that

$$\mathcal{F}H_{\varepsilon,\eta}f(x) = \mathcal{F}f(x)\, \mathcal{F}k_{\varepsilon,\eta}(x), \quad a.e. \tag{5.7}$$

The term $\mathcal{F}k_{\varepsilon,\eta}$ can be evaluated as follows:

$$\begin{aligned} \mathcal{F}k_{\varepsilon,\eta}(x) &= \int_{\varepsilon<|t|<\eta} \frac{\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t}{\pi t} \\ &= \int_{-\eta}^{-\varepsilon} \frac{\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t}{\pi t} + \int_{\varepsilon}^{\eta} \frac{\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t}{\pi t} \end{aligned}$$

$$= -\frac{2\mathrm{i}}{\pi} \int_{\varepsilon}^{\eta} \frac{\sin xt \, \mathrm{d}t}{t}$$

$$= -\frac{2\mathrm{i} \operatorname{sgn} x}{\pi} \int_{\varepsilon|x|}^{\eta|x|} \frac{\sin t \, \mathrm{d}t}{t}. \tag{5.8}$$

The integral in this last result is uniformly bounded and takes the value $\pi/2$ in the limits $\varepsilon \to 0$ and $\eta \to \infty$. Hence, there exists a constant C depending on ε and η such that

$$|\mathcal{F}k_{\varepsilon,\eta}(x)| \leq C \tag{5.9}$$

and

$$\lim_{\varepsilon \to 0} \lim_{\eta \to \infty} \mathcal{F}k_{\varepsilon,\eta}(x) = -\mathrm{i} \operatorname{sgn} x. \tag{5.10}$$

From Eqs. (5.7) and (5.9), it immediately follows that

$$\|\mathcal{F}H_{\varepsilon,\eta}f\|_2 \leq C\|\mathcal{F}f\|_2, \tag{5.11}$$

and Parseval's formula (see Eq. (2.55)) yields

$$\|H_{\varepsilon,\eta}f\|_2 \leq C\|f\|_2. \tag{5.12}$$

On taking the limit $\eta \to \infty$ in this last result, leads to

$$\|H_{\varepsilon}f\|_2 \leq C\|f\|_2, \tag{5.13}$$

which is the Riesz inequality for the truncated Hilbert transform for the choice $f \in L^2$. Equation (5.13) can be extended to the case of $f \in L^p$ with $1 < p < \infty$; this topic is considered in Section 7.10 (see Eq. (7.324)).

By the Lebesgue theorem of dominated convergence, and using Eq. (5.10), yields

$$\lim_{\varepsilon \to 0} \lim_{\eta \to \infty} \|\mathcal{F}H_{\varepsilon,\eta}f(x) - \{-\mathrm{i} \operatorname{sgn} x\}\mathcal{F}f(x)\|$$
$$= \lim_{\varepsilon \to 0} \lim_{\eta \to \infty} \|[\mathcal{F}k_{\varepsilon,\eta}(x) - \{-\mathrm{i} \operatorname{sgn} x\}]\mathcal{F}f(x)\|$$
$$= 0, \tag{5.14}$$

and hence in the sense of L^2 norm, it follows that

$$\lim_{\varepsilon \to 0} \lim_{\eta \to \infty} \mathcal{F}H_{\varepsilon,\eta}f(x) = \lim_{\varepsilon \to 0} \lim_{\eta \to \infty} \mathcal{F}k_{\varepsilon,\eta}(x)\mathcal{F}f(x). \tag{5.15}$$

Making use of Eq. (5.10) and of

$$\lim_{\varepsilon \to 0} \lim_{\eta \to \infty} \mathcal{F} H_{\varepsilon,\eta} f(x) = \mathcal{F} Hf(x), \quad a.e., \tag{5.16}$$

yields

$$\mathcal{F} Hf(x) = -\mathrm{i} \operatorname{sgn} x \; \mathcal{F} f(x), \quad a.e., \tag{5.17}$$

which holds for $f \in L^2(\mathbb{R})$. This is the desired result.

The preceding formula can be generalized. If $f \in L^p(\mathbb{R})$ for $1 < p \leq 2$, then Eq. (5.17) holds. Equation (5.17) also holds for the particular case $p = 1$ if both f and Hf belong to $L(\mathbb{R})$. Let $h \in L(\mathbb{R})$, then, from the convolution formula Eq. (4.166), it follows, on taking the Fourier transform, that

$$\mathcal{F} H\{h * f\}(x) = \mathcal{F}\{h * Hf\}(x). \tag{5.18}$$

With $h * f \in L^2$, and hence $H\{h * f\} \in L^2$ (by the Riesz inequality), and using Eq. (5.17), the left-hand side simplifies to

$$\begin{aligned} \mathcal{F}[H\{h * f\}](x) &= -\mathrm{i} \operatorname{sgn} x \; \mathcal{F}\{h * f\}(x) \\ &= -\mathrm{i} \operatorname{sgn} x \; \mathcal{F} H(x) \; \mathcal{F} f(x). \end{aligned} \tag{5.19}$$

The right-hand side of Eq. (5.18) leads to

$$\mathcal{F}[h * Hf](x) = \mathcal{F} H(x) \; \mathcal{F} Hf(x), \tag{5.20}$$

and hence

$$\mathcal{F} h(x) \; \mathcal{F} Hf(x) = -\mathrm{i} \operatorname{sgn} x \; \mathcal{F} h(x) \; \mathcal{F} f(x), \quad a.e., \tag{5.21}$$

and the required result, Eq. (5.17), follows provided $\mathcal{F} h(x) \neq 0$.

There is an issue of phase connected with the determination of Eq. (5.3). If the Fourier transform is evaluated using the opposite sign in the exponent in the integral in Eq. (5.2), then the sign in Eq. (5.3) is reversed. The version of this formula with reversed sign occurs widely in the scientific literature. In later sections, when the variables of interest are time and frequency, the convention adopted is that the Fourier transform over frequency is taken with a negative exponent, and hence the inverse Fourier transform over time has a positive exponent. This choice also has a bearing on the domain for which the function is analytic in the complex frequency plane. This issue is discussed further in Section 17.7.

Equation (5.3) is a particularly useful result. In a number of cases, it provides an effective means to evaluate Hilbert transforms. To see this, take the inverse Fourier transform of Eq. (5.3),

$$Hf(x) = -\mathrm{i} \mathcal{F}^{-1}[\operatorname{sgn} y \; \mathcal{F} f(y)](x), \tag{5.22}$$

where the integration variable for the inverse Fourier transform is y. Titchmarsh (1925a, p. 114) gave this formula, stated in a slightly different but equivalent manner, for functions $f \in L^2(\mathbb{R})$. In addition to being a valuable technique for the analytic determination of Hilbert transforms, Eq. (5.22) is also important as a computational strategy for the numerical evaluation of Hilbert transforms. When the function f is even or odd, this formula reduces the calculation of the Hilbert transform to the evaluation of Fourier sine and Fourier cosine transforms. The Fourier sine and Fourier cosine transforms are defined in Section 5.3. Extensive tabulations of these two transforms are available (see, for example, Erdélyi *et al.* (1954, Vol. I)). Two examples will illustrate the approach. As a first example, consider $f(x) = a/(a^2 + x^2)$, for $a > 0$. The Fourier transform of f is given by

$$\begin{aligned}\mathcal{F}f(x) &= a\int_{-\infty}^{\infty} \frac{e^{-ixs}\, ds}{a^2 + s^2} \\ &= 2a\int_0^{\infty} \frac{\cos xs\, ds}{a^2 + s^2} \\ &= \pi e^{-a|x|}.\end{aligned} \tag{5.23}$$

Using Eq. (5.22) leads to

$$\begin{aligned}Hf(x) &= -i\mathcal{F}^{-1}\{\operatorname{sgn} y\, \pi e^{-a|y|}\}(x) \\ &= -\frac{i}{2\pi}\int_{-\infty}^{\infty} \operatorname{sgn} y\, \pi e^{-a|y|} e^{ixy}\, dy \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{sgn} y \sin xy\, e^{-a|y|}\, dy \\ &= \int_0^{\infty} \sin xy\, e^{-ay}\, dy \\ &= \frac{x}{a^2 + x^2}.\end{aligned} \tag{5.24}$$

Equation (5.24) has been obtained by noting that the cosine contribution to the integral vanishes ($\operatorname{sgn} s \cos xs$ is an odd function in the variable s), and the final integral is the Fourier sine transform of an exponential function. Alternatively, the final integral can be evaluated simply by expressing the sine factor in terms of complex exponentials. As a second example, consider the case of a Gaussian function, $f(x) = e^{-ax^2}$, for $a > 0$. The Fourier transform of a Gaussian is a Gaussian:

$$\begin{aligned}\mathcal{F}\{e^{-ax^2}\} &= \int_{-\infty}^{\infty} e^{-as^2 - ixs}\, ds \\ &= \sqrt{\left(\frac{\pi}{a}\right)}\, e^{-x^2/4a},\end{aligned} \tag{5.25}$$

and, hence,

$$\begin{aligned} H(\mathrm{e}^{-ax^2}) &= -\mathrm{i}\mathscr{F}^{-1}\left\{\operatorname{sgn} x\sqrt{\left(\frac{\pi}{a}\right)}\,\mathrm{e}^{-x^2/4a}\right\} \\ &= \frac{1}{\sqrt{(\pi a)}}\int_0^\infty \sin xy\,\mathrm{e}^{-y^2/4a}\,\mathrm{d}y. \end{aligned} \tag{5.26}$$

Using the definition of the error function, erf(z),

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-s^2}\,\mathrm{d}s, \tag{5.27}$$

Eq. (5.26) can be written as follows:

$$H(\mathrm{e}^{-ax^2}) = -\mathrm{i}\mathrm{e}^{-ax^2}\operatorname{erf}(\mathrm{i}\sqrt{(a)}x), \tag{5.28}$$

or, in terms of Kummer's confluent hypergeometric function, ${}_1F_1(\alpha;\beta;x)$,

$$\begin{aligned} H(\mathrm{e}^{-ax^2}) &= 2\sqrt{\left(\frac{a}{\pi}\right)}\,x\mathrm{e}^{-ax^2}\;{}_1F_1\left(\frac{1}{2};\frac{3}{2};ax^2\right) \\ &= 2\sqrt{\left(\frac{a}{\pi}\right)}\,x\;{}_1F_1\left(1;\frac{3}{2};-ax^2\right). \end{aligned} \tag{5.29}$$

The hypergeometric function can be readily evaluated using the series expansion

$${}_1F_1(\alpha;\beta;z) = \sum_{k=0}^{\infty}\frac{(\alpha)_k z^k}{(\beta)_k k!}, \tag{5.30}$$

where $(\alpha)_k$ denotes a Pochhammer symbol, which is defined in terms of the gamma function by

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. \tag{5.31}$$

Dawson's integral is defined by

$$D(x) = \mathrm{e}^{-x^2}\int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t, \tag{5.32}$$

and hence the Hilbert transform of the Gaussian function e^{-ax^2} can be expressed as follows:

$$H(\mathrm{e}^{-ax^2}) = \frac{2}{\sqrt{\pi}}D(\sqrt{(a)}x). \tag{5.33}$$

From the connection between the Hilbert and Fourier transforms, the iteration property given in Section 4.4 can be obtained for functions of the class $L^p(\mathbb{R})$ for $1 < p \leq 2$. Suppose

$$Hf(x) = g(x), \tag{5.34}$$

then

$$\begin{aligned}
H^2 f(x) = Hg(x) &= -\mathrm{i}\mathcal{F}^{-1}\{\operatorname{sgn} y\, \mathcal{F}g(y)\}(x) \\
&= -\mathrm{i}\mathcal{F}^{-1}[\operatorname{sgn} y\, \{\mathcal{F}Hf\}(y)](x) \\
&= -\mathrm{i}\mathcal{F}^{-1}[\operatorname{sgn} y\, \{\mathcal{F}[-\mathrm{i}\mathcal{F}^{-1}\, \{\operatorname{sgn} w\, \mathcal{F}f(w)\}]\}(y)](x) \\
&= -\mathcal{F}^{-1}[\operatorname{sgn} y \operatorname{sgn} y\, \mathcal{F}f(y)](x) \\
&= -\mathcal{F}^{-1}[\mathcal{F}f](x) \\
&= -f(x).
\end{aligned} \tag{5.35}$$

Symbolically, the relation between the Hilbert transform operator and the Fourier transform and inverse Fourier transform can be written as

$$H = \mathcal{F}^{-1}\sigma(H)\mathcal{F}, \tag{5.36}$$

where $\sigma(H)$ is called the *symbol* of H, and

$$\sigma(H)(x) = -\mathrm{i}\operatorname{sgn} x. \tag{5.37}$$

The factor $\sigma(H)$ is the principal value Fourier transform of the kernel of H. The iteration property just proved is an immediate consequence of the result

$$\sigma(H)^2 = -1, \tag{5.38}$$

that is

$$\mathcal{F}H\{Hf\}(x) = \sigma(H)\mathcal{F}Hf(x) = \sigma(H)^2\mathcal{F}f(x) = -\mathcal{F}f(x). \tag{5.39}$$

Taking the inverse Fourier transform yields the required expression. On comparison of Eq. (5.3) and Eq. (4.87), it is clear that $\sigma(H)$ is the multiplier for the Hilbert transform operator.

5.3 Even and odd Hilbert transform operators

This section explores the connection between the even and odd Hilbert transform operators and the Fourier sine and Fourier cosine transform operators. The Fourier sine and cosine transforms are defined, respectively, as follows:

$$\mathcal{F}_{\mathrm{s}} f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \sin xt\, f(t) \mathrm{d}t \tag{5.40}$$

and

$$\mathcal{F}_{\mathrm{c}} f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \cos xt\, f(t) \mathrm{d}t, \tag{5.41}$$

and it is assumed throughout this section that $f \in L^2(\mathbb{R})$ unless an alternative statement on f is given. The Fourier sine and Fourier cosine transform operators are often defined without the prefactor $\sqrt{(2/\pi)}$ (see, for example, Erdélyi *et al.* (1954, Vol. I)), but the preceding choice simplifies the appearance of the following results. The operator identities

$$H_{\mathrm{e}} = \mathcal{F}_{\mathrm{s}} \mathcal{F}_{\mathrm{c}} \tag{5.42}$$

and

$$H_{\mathrm{o}} = -\mathcal{F}_{\mathrm{c}} \mathcal{F}_{\mathrm{s}} \tag{5.43}$$

are now established. Starting with the definitions for $\mathcal{F}_{\mathrm{s}}$ and $\mathcal{F}_{\mathrm{c}}$, it follows that

$$\begin{aligned}
\mathcal{F}_{\mathrm{s}}\, \mathcal{F}_{\mathrm{c}} f(x) &= \frac{2}{\pi} \int_0^\infty \sin xt\, \mathrm{d}t \int_0^\infty \cos ts\, f(s) \mathrm{d}s \\
&= \frac{1}{\pi} \int_0^\infty f(s) \mathrm{d}s \int_0^\infty \{\sin[(x-s)t] + \sin[(x+s)t]\} \mathrm{d}t \\
&= \frac{1}{\pi} \lim_{\lambda \to \infty} \int_0^\infty f(s) \mathrm{d}s \int_0^\lambda \{\sin[(x-s)t] + \sin[(x+s)t]\} \mathrm{d}t \\
&= \frac{1}{\pi} \lim_{\lambda \to \infty} \int_0^\infty f(s) \left\{ \frac{1-\cos[(x-s)\lambda]}{x-s} + \frac{1-\cos[(x+s)\lambda]}{x+s} \right\} \mathrm{d}s \\
&= \frac{1}{\pi} P \int_0^\infty f(s) \left\{ \frac{1}{x-s} + \frac{1}{x+s} \right\} \mathrm{d}s \\
&= H_{\mathrm{e}} f(x),
\end{aligned} \tag{5.44}$$

where the Riemann–Lebesgue lemma has been used to evaluate the $\lim_{\lambda\to\infty}$, and Eq. (4.11) has been employed. The reader should justify the interchange of integration order. Equation (5.43) is established in a similar fashion:

$$
\begin{aligned}
\mathcal{F}_{\mathrm{c}}\,\mathcal{F}_{\mathrm{s}} f(x) &= \frac{2}{\pi}\int_0^\infty \cos xt\,\mathrm{d}t \int_0^\infty \sin ts\, f(s)\mathrm{d}s \\
&= \frac{1}{\pi}\lim_{\lambda\to\infty}\int_0^\infty f(s)\mathrm{d}s\int_0^\lambda \{\sin[(s-x)t]+\sin[(s+x)t]\}\mathrm{d}t \\
&= \frac{1}{\pi}\lim_{\lambda\to\infty}\int_0^\infty f(s)\left\{\frac{1-\cos[(s-x)\lambda]}{s-x}+\frac{1-\cos[(s+x)\lambda]}{s+x}\right\}\mathrm{d}s \\
&= -\frac{1}{\pi}P\int_0^\infty f(s)\left\{\frac{1}{x-s}-\frac{1}{x+s}\right\}\mathrm{d}s \\
&= -H_{\mathrm{o}} f(x),
\end{aligned}
\tag{5.45}
$$

and Eq. (4.12) has been employed.

The operator identities

$$
H_{\mathrm{e}}H_{\mathrm{o}} = H_{\mathrm{o}}H_{\mathrm{e}} = -I, \tag{5.46}
$$

where I is the identity operator, are now established. The operator relationships

$$
\mathcal{F}_{\mathrm{c}}^2 = I \tag{5.47}
$$

and

$$
\mathcal{F}_{\mathrm{s}}^2 = I \tag{5.48}
$$

will be needed. These are proved as follows:

$$
\begin{aligned}
\mathcal{F}_{\mathrm{s}}^2 f(x) &= \frac{2}{\pi}\int_0^\infty \sin xt\,\mathrm{d}t\int_0^\infty \sin ts\, f(s)\mathrm{d}s \\
&= \lim_{\lambda\to\infty}\frac{2}{\pi}\int_0^\infty f(s)\mathrm{d}s\int_0^\lambda \sin ts\sin xt\,\mathrm{d}t \\
&= \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\infty f(s)\left\{\frac{\sin[(x-s)\lambda]}{x-s}-\frac{\sin[(x+s)\lambda]}{x+s}\right\}\mathrm{d}s \\
&= \int_0^\infty f(s)\{\delta(x-s)-\delta(x+s)\}\mathrm{d}s \\
&= f(x),
\end{aligned}
\tag{5.49}
$$

and Eq. (2.256) has been employed. In Eq. (5.49) the delta function $\delta(x-s)$ contributes for $x > 0$ and $\delta(x+s)$ contributes for $x < 0$. Equation (5.48) follows immediately. Equation (5.47) is obtained by a similar calculation. Using Eqs. (5.42), (5.43), (5.47), and (5.48),

$$H_{\mathrm{e}}H_{\mathrm{o}} = -\mathcal{F}_{\mathrm{s}}\mathcal{F}_{\mathrm{c}}\mathcal{F}_{\mathrm{c}}\mathcal{F}_{\mathrm{s}} = -I, \tag{5.50}$$

and a similar calculation proves $H_{\mathrm{o}}H_{\mathrm{e}} = -I$. Hence,

$$H_{\mathrm{e}}H_{\mathrm{o}} = H_{\mathrm{o}}H_{\mathrm{e}}, \tag{5.51}$$

and the operators H_{e} and H_{o} commute.

If f is an even function and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then it follows directly from Eq. (5.17) that

$$\mathcal{F}_{\mathrm{s}}\, Hf(x) = \operatorname{sgn} x\, \mathcal{F}_{\mathrm{c}}\, f(x), \tag{5.52}$$

and for $x > 0$ it follows that

$$\mathcal{F}_{\mathrm{s}}\, Hf(x) = \mathcal{F}_{\mathrm{c}}\, f(x). \tag{5.53}$$

Similarly, if f is an odd function and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$\mathcal{F}_{\mathrm{c}}\, Hf(x) = -\operatorname{sgn} x\, \mathcal{F}_{\mathrm{s}}\, f(x), \tag{5.54}$$

and for $x > 0$

$$\mathcal{F}_{\mathrm{c}}\, Hf(x) = -\mathcal{F}_{\mathrm{s}}\, f(x). \tag{5.55}$$

5.4 The commutator $[\mathcal{F}, H]$

A result for $H\mathcal{F}f$ will prove to be useful when the generalization of Eq. (5.3) to cover the case of distributions is considered in Section 10.11. This short section gives the required formula, and also allows the commutator $[\mathcal{F}, H]$ of the Fourier and Hilbert transform operators to be evaluated. To evaluate the commutator, it suffices to evaluate $H\mathcal{F}f$ and employ Eq. (5.3). Start with

$$\begin{aligned} H\mathcal{F}f(x) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{x-t} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ts} f(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty} f(s)\,\mathrm{d}s\, \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}t}{x-t} \\ &= \mathrm{i} \int_{-\infty}^{\infty} \operatorname{sgn} s\, f(s) \mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}s, \end{aligned} \tag{5.56}$$

and hence

$$H\mathcal{F}f(x) = \mathcal{F}[\mathrm{i}\operatorname{sgn} y\, f(y)](x). \tag{5.57}$$

The reader is requested to justify the interchange of integration order in Eq. (5.56).

5.5 Hartley transform of the Hilbert transform

In this section interest is focused on the connection between the Hilbert transform and the Hartley transform. Not surprisingly, there exists a relationship similar in form to Eq. (5.3), with the Fourier transform replaced by the Hartley transform. It is assumed throughout this section that $f \in L^2(\mathbb{R})$, although a pair of examples will be considered which fall outside this class.

The Hartley transform of a function f is defined by

$$\mathrm{H_A} f(x) = \int_{-\infty}^{\infty} \operatorname{cas} xs\, f(s)\mathrm{d}s, \tag{5.58}$$

where cas denotes the Hartley cas function, defined by

$$\operatorname{cas} x = \cos x + \sin x. \tag{5.59}$$

It is common to use the notation H to symbolize the Hartley transform, but in this work H has been reserved for the Hilbert transform, and so the notation $\mathrm{H_A}$ for the Hartley transform is adopted. The inverse Hartley transform is given by

$$\mathrm{H_A^{-1}} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{cas} xs\, f(s)\mathrm{d}s. \tag{5.60}$$

As with the definition of the Fourier transform (see Eqs. (2.48)–(2.50)), there are alternative definitions employed where the 2π factor is distributed in different ways. An obvious feature of the definition is that if f is a real-valued function, then so is its Hartley transform.

First recall that the Hilbert transform can be written as the convolution

$$g(x) = Hf(x) = f(x) * k(x), \tag{5.61}$$

with $k(x)$ defined as $(\pi x)^{-1}$. The Hartley transform of a convolution of two functions f and h can be expressed as follows:

$$\begin{aligned}\mathrm{H_A}\{f(x) * h(x)\} = \frac{1}{2}\{&\mathrm{H_A} f(x)\mathrm{H_A} h(x) + \mathrm{H_A} f(-x)\mathrm{H_A} h(x) \\ &+ \mathrm{H_A} f(x)\mathrm{H_A} h(-x) - \mathrm{H_A} f(-x)\mathrm{H_A} h(-x)\}.\end{aligned} \tag{5.62}$$

This result can be obtained as follows:

$$\begin{aligned}\mathrm{H_A}\{f(x) * h(x)\} &= \int_{-\infty}^{\infty} \operatorname{cas} xs\, \mathrm{d}s \int_{-\infty}^{\infty} f(t)h(s-t)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} f(t)\mathrm{d}t \int_{-\infty}^{\infty} h(s-t) \operatorname{cas} xs\, \mathrm{d}s \\ &= \int_{-\infty}^{\infty} f(t)\mathrm{d}t \int_{-\infty}^{\infty} h(y) \operatorname{cas}[x(y+t)]\mathrm{d}y,\end{aligned} \tag{5.63}$$

where it has been assumed that the order of integration can be interchanged. Using some elementary trigonometry,

$$\begin{aligned}\operatorname{cas} x(y+t) &= \cos(xy+xt)+\sin(xy+xt)\\ &= \cos xt(\cos xy+\sin xy)+\sin xt(\cos xy-\sin xy)\\ &= \frac{1}{2}[\{\operatorname{cas}(xt)+\operatorname{cas}(-xt)\}\operatorname{cas}(xy)+\{\operatorname{cas}(xt)-\operatorname{cas}(-xt)\}\operatorname{cas}(-xy)].\end{aligned} \tag{5.64}$$

Inserting this last result into Eq. (5.63) leads to the desired expression. Equation (5.62) simplifies as follows:

$$\mathrm{H_A}\{f(x)*h(x)\} = \mathrm{H_A}f(x)\mathrm{H_A}h(x), \qquad \text{if } f \text{ or } h \text{ is even;} \tag{5.65}$$

$$\mathrm{H_A}\{f(x)*h(x)\} = -\mathrm{H_A}f(x)\mathrm{H_A}h(x), \qquad \text{if } f \text{ and } h \text{ are both odd;} \tag{5.66}$$

$$\mathrm{H_A}\{f(x)*h(x)\} = \mathrm{H_A}f(x)\mathrm{H_A}h(-x), \qquad \text{if } f \text{ is odd and } h \text{ is neither odd nor even;} \tag{5.67}$$

$$\mathrm{H_A}\{f(x)*h(x)\} = \mathrm{H_A}\ f(-x)\mathrm{H_A}h(x), \qquad \text{if } h \text{ is odd and } f \text{ is neither odd nor even.} \tag{5.68}$$

For the particular case of interest here, that is, identifying $h(x)$ with $k(x)$, and noting that the kernel function k is odd, then the case in Eq. (5.67) does not arise. The Hartley transform of the kernel function k is given by

$$\begin{aligned}\mathrm{H_A}k(x) &= \int_{-\infty}^{\infty} \operatorname{cas} xs\, k(s)\mathrm{d}s\\ &= \int_{-\infty}^{\infty} \frac{\operatorname{cas} xs\, \mathrm{d}s}{\pi s}\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin sx\, \mathrm{d}s}{s}\\ &= \operatorname{sgn} x.\end{aligned} \tag{5.69}$$

Taking the Hartley transform of Hf, using the convolution result Eq. (5.62), and then taking the inverse Hartley transform, leads to the following result:

$$Hf(x) = \mathrm{H_A^{-1}}\{\operatorname{sgn} y\, \mathrm{H_A} f(-y)\}(x). \tag{5.70}$$

This result simplifies in an obvious fashion if f is an even or odd function. By employing the limiting argument of Section 5.2, a more satisfying and rigorous approach to Eq. (5.70) can be given, which accounts for the singular behavior of the kernel function at the origin. The reader is requested to construct the necessary details.

Equation (5.70) can be used effectively to evaluate the Hilbert transform of some functions. As a first example, consider $f(x) = \cos ax$, where a is a real constant. First note the following result:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos xs \, \mathrm{d}s; \tag{5.71}$$

then the Hartley transform of $\cos ax$ is given by

$$\begin{aligned} \mathrm{H_A}(\cos ax) &= \int_{-\infty}^{\infty} \cos as \,\mathrm{cas}\, xs \, \mathrm{d}s \\ &= \pi\{\delta(x-a) + \delta(x+a)\}. \end{aligned} \tag{5.72}$$

Since f is even,

$$\begin{aligned} Hf(x) &= \mathrm{H_A^{-1}}\{\mathrm{sgn}\, y\, \mathrm{H_A} f(y)\}(x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{cas}\, xy \,\mathrm{sgn}\, y\, \pi\{\delta(y-a) + \delta(y+a)\}\mathrm{d}y \\ &= \frac{1}{2}\{\mathrm{cas}\, ax \,\mathrm{sgn}\, a + \mathrm{cas}(-ax)\,\mathrm{sgn}(-a)\} \\ &= \mathrm{sgn}\, a \sin ax. \end{aligned} \tag{5.73}$$

In a similar fashion, it can be shown that $H\{\cos(ax+b)\} = \mathrm{sgn}\, a \sin(ax+b)$, for a and b real constants.

As a second example, consider the following function:

$$f(x) = \begin{cases} 0, & \text{for} \quad -\infty < x < a \\ h, & \text{for} \quad a < x < b \\ 0, & \text{for} \quad b < x < \infty, \end{cases} \tag{5.74}$$

where h is a constant. This function represents a rectangular pulse. The Hartley transform of $f(-s)$ is given by

$$\begin{aligned} \mathrm{H_A} f(-s) &= \int_{-\infty}^{\infty} \mathrm{cas}\,(-st) f(t)\mathrm{d}t \\ &= h \int_a^b \{\cos st - \sin st\}\mathrm{d}t \\ &= \frac{h}{s}(\mathrm{cas}\, bs - \mathrm{cas}\, as). \end{aligned} \tag{5.75}$$

From Eq. (5.70), it follows that

$$\begin{aligned} Hf(x) &= \mathrm{H_A^{-1}}\{\mathrm{sgn}\, y\, \mathrm{H_A} f(-y)\}(x) \\ &= \frac{h}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{sgn}\, y(\mathrm{cas}\, by - \mathrm{cas}\, ay)\,\mathrm{cas}\, xy \,\mathrm{d}y}{y} \end{aligned}$$

$$= \frac{2h}{\pi}\int_0^{\infty} \frac{\sin[(2x-b-a)y]\sin[(b-a)y]\mathrm{d}y}{y}$$

$$= \frac{h}{\pi}\log\left|\frac{x-a}{x-b}\right|. \tag{5.76}$$

This result is obtained by considering the different regions in Eq. (5.74) and evaluating the last integral in Eq. (5.76) with sine functions of positive argument.

5.6 Relationship between the Hilbert transform and the Stieltjes transform

The Stieltjes transform is defined as follows:

$$\mathsf{S}f(z) = \int_0^{\infty} \frac{f(x)}{x+z}\,\mathrm{d}x, \tag{5.77}$$

where z is, in general, a complex variable. The complex z-plane is assumed to be cut along the negative real axis, for a reason that will become apparent when a more general definition of this transform is given. An extended definition of the Stieltjes transform is discussed in Section 7.1. Consideration is first restricted to the class of functions that are *causal*. The support for causal functions is $[0,\infty)$, and hence

$$f(x) = 0, \quad \text{for } x < 0. \tag{5.78}$$

The connotation "causal" is employed when a temporal variable is of interest, so the reader can think of f as a function of time in the sequel. In place of the term causal function, the reader can substitute the following descriptor: function with support restricted to the interval $[0,\infty)$. For a causal function f, it follows directly from the definition of the Hilbert transform that

$$\mathsf{S}f(x) = -\pi(Hf)(-x), \quad x \geq 0, \tag{5.79}$$

provided the integral converges.

When the function f is not causal, the following connections between the Hilbert and Stieltjes transforms can be made:

$$Hf(x) = \frac{1}{\pi}(\mathsf{S}f(-t))(x) - \frac{1}{2\pi}(\mathsf{S}f)(x\mathrm{e}^{\mathrm{i}\pi}) - \frac{1}{2\pi}(\mathsf{S}f)(x\mathrm{e}^{-\mathrm{i}\pi}), \quad x > 0, \tag{5.80}$$

and

$$Hf(x) = -\frac{1}{\pi}(\mathsf{S}f(-t))(x) + \frac{1}{2\pi}(\mathsf{S}f(-t))(|x|\mathrm{e}^{\mathrm{i}\pi}) + \frac{1}{2\pi}(\mathsf{S}f(-t))(|x|\mathrm{e}^{-\mathrm{i}\pi}), \quad x < 0. \tag{5.81}$$

To obtain Eq. (5.80), proceed as follows:

$$
\begin{aligned}
Hf(x) &= \frac{1}{\pi} P \int_0^\infty \left\{ \frac{f(y)}{x-y} + \frac{f(-y)}{x+y} \right\} \mathrm{d}y \\
&= -\frac{1}{2\pi} \int_0^\infty f(y) \left\{ \frac{1}{y + x\mathrm{e}^{\mathrm{i}\pi}} + \frac{1}{y + x\mathrm{e}^{-\mathrm{i}\pi}} \right\} \mathrm{d}y + \frac{1}{\pi} (\mathsf{S} f(-t))(x) \\
&= -\frac{1}{2\pi} (\mathsf{S} f)(x\mathrm{e}^{\mathrm{i}\pi}) - \frac{1}{2\pi} (\mathsf{S} f)(x\mathrm{e}^{-\mathrm{i}\pi}) + \frac{1}{\pi} (\mathsf{S} f(-t))(x). \qquad (5.82)
\end{aligned}
$$

In the preceding sequence, the average denominator across the branch cut on the negative real axis has been taken. That is, $x\mathrm{e}^{\mathrm{i}\pi}$ is employed on the upper side of the branch cut and $x\mathrm{e}^{-\mathrm{i}\pi}$ is used on the lower side of the branch cut. Equation (5.81) can be derived in a similar manner as follows:

$$
\begin{aligned}
Hf(x) &= -\frac{1}{\pi} P \int_0^\infty \left\{ \frac{f(y)}{-x+y} - \frac{f(-y)}{x+y} \right\} \mathrm{d}y \\
&= -\frac{1}{\pi} (\mathsf{S} f)(-x) + \frac{1}{2\pi} \int_0^\infty f(-y) \left\{ \frac{1}{y + |x|\mathrm{e}^{\mathrm{i}\pi}} + \frac{1}{y + |x|\mathrm{e}^{-\mathrm{i}\pi}} \right\} \mathrm{d}y \\
&= -\frac{1}{\pi} (\mathsf{S} f)(-x) + \frac{1}{2\pi} (\mathsf{S} f(-t))(|x|\mathrm{e}^{\mathrm{i}\pi}) + \frac{1}{2\pi} (\mathsf{S} f(-t))(|x|\mathrm{e}^{-\mathrm{i}\pi}). \qquad (5.83)
\end{aligned}
$$

Equations (5.80) and (5.81) can serve as an alternative means to evaluate the Hilbert transform of a function. To illustrate the use of these formulas, two examples are considered. Let

$$
f(x) = \frac{1}{x^2 + a^2}, \quad \text{for } a > 0, \qquad (5.84)
$$

then

$$
\mathsf{S} f(x) = \frac{1}{x^2 + a^2} \left\{ \frac{\pi x}{2a} - \log(x/a) \right\}; \qquad (5.85)
$$

$(\mathsf{S} f(-t))(x)$ is also given by Eq. (5.85). Hence, from Eq. (5.80), it follows that

$$
\begin{aligned}
Hf(x) &= \frac{1}{\pi(x^2 + a^2)} \left\{ \frac{\pi x}{2a} - \log(x/a) \right\} \\
&\quad - \frac{1}{2\pi(x^2 + a^2)} \left\{ -\frac{\pi x}{2a} - \log(x\mathrm{e}^{\mathrm{i}\pi}/a) - \frac{\pi x}{2a} - \log(x\mathrm{e}^{-\mathrm{i}\pi}/a) \right\} \\
&= \frac{x}{a(x^2 + a^2)}, \quad \text{for } x > 0. \qquad (5.86)
\end{aligned}
$$

In a similar fashion, it follows from Eq. (5.81) that

$$
\begin{aligned}
Hf(x) = & -\frac{1}{\pi(x^2+a^2)}\left\{-\frac{\pi x}{2a} - \log(-x/a)\right\} \\
& + \frac{1}{2\pi(x^2+a^2)}\left\{-\frac{\pi\,|x|}{2a} - \log(|x|\mathrm{e}^{\mathrm{i}\pi}/a) - \frac{\pi\,|x|}{2a} - \log(|x|\mathrm{e}^{-\mathrm{i}\pi}/a)\right\} \\
= & \frac{x}{a(x^2+a^2)}, \quad \text{for } x < 0.
\end{aligned}
\tag{5.87}
$$

As a second example, suppose

$$
f(x) = \frac{x}{x^2+a^2}, \quad \text{for } a > 0, \tag{5.88}
$$

then

$$
\mathsf{S}f(x) = \frac{1}{x^2+a^2}\left\{\frac{\pi a}{2} + x\log(x/a)\right\}; \tag{5.89}
$$

$(\mathsf{S}f(-t))(x)$ is minus the preceding result (since f is an odd function). Hence, from Eq. (5.80) it follows that

$$
\begin{aligned}
Hf(x) = & -\frac{1}{\pi(x^2+a^2)}\left\{\frac{\pi a}{2} + x\log(x/a)\right\} \\
& - \frac{1}{2\pi(x^2+a^2)}\left\{\frac{\pi a}{2} - x\log(x\mathrm{e}^{\mathrm{i}\pi}/a) + \frac{\pi a}{2} - x\log(x\mathrm{e}^{-\mathrm{i}\pi}/a)\right\} \\
= & -\frac{a}{(x^2+a^2)},
\end{aligned}
\tag{5.90}
$$

and the same result follows from Eq. (5.81). A large number of additional Hilbert transforms can be generated using the table of Stieltjes transforms given in Erdélyi *et al*. (1954, Vol. II, p. 213).

5.7 Relationship between the Laplace transform and the Hilbert transform

The Laplace transform of the function f is defined by

$$
\mathscr{L}f(x) = \int_0^\infty f(s)\mathrm{e}^{-xs}\,\mathrm{d}s. \tag{5.91}
$$

Equation (5.91) is sometimes referred to as the unilateral Laplace transform. For comparison, the bilateral (or two-sided) Laplace transform is defined by

$$
\mathscr{L}_2 f(x) = \int_{-\infty}^\infty f(s)\mathrm{e}^{-xs}\,\mathrm{d}s. \tag{5.92}
$$

A subscript has been added to distinguish the bilateral from the conventional (unilateral) Laplace transform. Attention in the sequel is focused on the normal Laplace transform because of its much wider occurrence in practical applications. The connection between the Hilbert and Laplace transforms is now examined. Consider first the case of causal functions, as defined in Eq. (5.78). From Eq. (5.79) it follows that

$$\begin{aligned}-\pi(Hf)(-x) &= \int_0^\infty \frac{f(s)\mathrm{d}s}{x+s} \\ &= \int_0^\infty f(s)\mathrm{d}s \int_0^\infty \mathrm{e}^{-(x+s)t}\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}t \int_0^\infty f(s)\mathrm{e}^{-ts}\,\mathrm{d}s, \end{aligned} \tag{5.93}$$

where it has been assumed that f is sufficiently well behaved so that Fubini's theorem applies, and hence the order of integration can be switched to obtain the final equation. The final result can be written in condensed notation as follows:

$$\pi(Hf)(-x) = -(\mathscr{L}\mathscr{L}f)(x), \quad \text{for } x > 0. \tag{5.94}$$

The iterated Laplace transform result for the Hilbert transform given in Eq. (5.94) does not have the same practical significance as the connection between the Hilbert transform and the Fourier transform (see Eq. (5.22)). For $Hf(x)$ with $x > 0$, there is no simple result expressing this case in terms of Laplace transforms. A simple illustrative example for Eq. (5.94) is the following. Suppose

$$f(x) = \begin{cases} 0, & -\infty < x < 0 \\ \mathrm{e}^{-ax}, & 0 \le x < \infty, \end{cases} \tag{5.95}$$

where $a > 0$. The Laplace transform of f is given by

$$\begin{aligned}\mathscr{L}f(t) &= \int_0^\infty f(s)\mathrm{e}^{-ts}\,\mathrm{d}s \\ &= (a+t)^{-1}, \end{aligned} \tag{5.96}$$

and, from Eq. (5.94),

$$\begin{aligned}-\pi(Hf)(-x) &= \mathscr{L}\{(a+x)^{-1}\} \\ &= \mathrm{e}^{ax}E_1(ax), \quad x > 0, \end{aligned} \tag{5.97}$$

where $E_1(y)$ denotes the exponential integral (Abramowitz and Stegun, 1965, p. 228):

$$E_1(y) = \int_y^\infty \frac{\mathrm{e}^{-t}\,\mathrm{d}t}{t}. \tag{5.98}$$

So, from Eq. (5.97) it follows that

$$Hf(x) = -\pi^{-1}e^{-ax}E_1(-ax), \quad x < 0. \tag{5.99}$$

For completeness, the result for $(Hf)(x)$ for $x > 0$ is given by

$$Hf(x) = \pi^{-1}e^{-ax}\mathrm{Ei}(ax), \tag{5.100}$$

but this is not obtained by the use of a Laplace transform. In Eq. (5.100), $\mathrm{Ei}(y)$ denotes another exponential integral function (Abramowitz and Stegun, 1965), defined by

$$\mathrm{Ei}(y) = -P\int_{-y}^{\infty} \frac{e^{-t}\,dt}{t}, \quad y > 0. \tag{5.101}$$

An obvious question that might occur to the reader is the following: can the class of functions be broadened beyond those of the causal type, while maintaining the relationship given in Eq. (5.94), or leading to a modified connection between the Hilbert and Laplace transforms? There appears to be no simple connection between the Hilbert transform and the Laplace transform for a general function. Clearly, it is not possible to express the kernel function $(x - s)^{-1}$ as an integral of an exponential function, as was done in Eq. (5.93).

5.8 Mellin transform of the Hilbert transform

The Mellin transform of the function f is defined by

$$g(z) = Mf(z) = \int_0^{\infty} t^{z-1}f(t)\,dt, \tag{5.102}$$

for Re z satisfying $a < \mathrm{Re}\, z < b$, where the constants a and b depend on f. The Mellin transform of Hf can be written as

$$MHf(\rho) = \int_0^{\infty} x^{\rho-1}\left\{\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(t)\,dt}{x-t}\right\}dx. \tag{5.103}$$

For the class of functions such that

$$f(t) = 0, \quad \text{for } t < 0, \tag{5.104}$$

Eq. (5.103) can be simplified in the following manner. Assuming the interchange of integration order is justified (refer to Section 2.13), then, for $0 < \mathrm{Re}\,\rho < 1$,

$$
\begin{aligned}
MHf(\rho) &= \int_0^\infty x^{\rho-1}\left\{\frac{1}{\pi}P\int_0^\infty \frac{f(t)\mathrm{d}t}{x-t}\right\}\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty f(t)\left\{P\int_0^\infty \frac{x^{\rho-1}\,\mathrm{d}x}{x-t}\right\}\mathrm{d}t \\
&= \frac{1}{\pi}\int_0^\infty t^{\rho-1}f(t)\mathrm{d}t\,P\int_0^\infty \frac{u^{\rho-1}\,\mathrm{d}u}{u-1}.
\end{aligned}
\tag{5.105}
$$

The reader is requested to attempt to evaluate the integral

$$
P\int_0^\infty \frac{u^{\rho-1}\,\mathrm{d}u}{u-1}
$$

that arises. From Eq. (5.105) it follows that

$$
MHf(\rho) = -\cot\pi\rho\, Mf(\rho). \tag{5.106}
$$

This result holds for $0 < \rho < 1$.

The variable ρ can be written as $\rho = \alpha + \mathrm{i}\beta$, for α and β real, with $\alpha_1 < \alpha < \alpha_2$. The constants α_1 and α_2, which depend on the function f, define the strip of definition for the Mellin transform. The inverse Mellin transform, denoted M^{-1}, is given by the result

$$
f(x) = M^{-1}g(x) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} g(s)x^{-s}\,\mathrm{d}s, \tag{5.107}
$$

where c is a real constant and it is assumed that $g(s)$ is absolutely integrable on the line $\mathrm{Re}\,s = c$. Taking the inverse Mellin transform of Eq. (5.106) leads to

$$
Hf(t) = M^{-1}\{-\cot\pi\rho\, Mf(\rho)\}(t), \tag{5.108}
$$

which can be written as follows:

$$
Hf(t) = -\frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} t^{-\rho}\cot\pi\rho\, Mf(\rho)\mathrm{d}\rho, \tag{5.109}
$$

subject of course to the condition given for f in Eq. (5.104). The term $\cot\pi\rho\, Mf(\rho)$ of the inverse Mellin transform will be holomorphic in some strip of the complex plane, say $\mathrm{Re}\,\rho \in (a_1, a_2)$, and the value of a is selected so that $a_1 < a < a_2$.

An example will illustrate the key formula just established. Let $f(t)$ be given by

$$
f(t) = \begin{cases} 0, & t < 0 \\ 1/(1+t), & 0 \le t < \infty, \end{cases} \tag{5.110}
$$

then a straightforward direct calculation of the Hilbert transform yields

$$Hf(t) = \frac{\log t}{\pi(1+t)}, \quad \text{for } t > 0. \tag{5.111}$$

The evaluation of the integral in Eq. (5.109) requires a somewhat higher energy investment. The transform Mf is needed, which can be evaluated by contour integration techniques or, more directly, by a change of integration variable. If the definition of the beta function is introduced as

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \tag{5.112}$$

and the substitution $(1+x)^{-1} = (1-t)$ employed, then, for $0 < \operatorname{Re}\rho < 1$,

$$\begin{aligned} Mf(\rho) &= \int_0^\infty x^{\rho-1}(1+x)^{-1}\,\mathrm{d}x \\ &= \int_0^1 t^{\rho-1}(1-t)^{-\rho}\,\mathrm{d}t \\ &= B(\rho, 1-\rho). \end{aligned} \tag{5.113}$$

The beta function in Eq. (5.113) can be related to the gamma function via

$$B(\rho, 1-\rho) = \Gamma(\rho)\Gamma(1-\rho), \tag{5.114}$$

and using a well known relationship for the gamma function,

$$\Gamma(\rho)\Gamma(1-\rho) = \frac{\pi}{\sin \pi\rho}, \tag{5.115}$$

this leads to

$$Mf(\rho) = \frac{\pi}{\sin \pi\rho}. \tag{5.116}$$

To evaluate the integral in Eq. (5.109) the case $t > 1$ is examined first. Consider the contour shown in Figure 5.1. The integrand has poles of order two at $\rho = 0, \pm 1, \pm 2, \pm 3, \ldots$ arising from the term $\sin^{-2}\pi\rho$. The constant a is selected such that $0 < a < 1$, and with this choice the required inverse Mellin transform is independent of the value of a. Set

$$g(z) = -\pi t^{-z} \cot \pi z \, \csc \pi z, \tag{5.117}$$

then, from the Cauchy integral formula,

$$\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} g(z)\mathrm{d}z + \int_{\Gamma_R} g(z)\mathrm{d}z + 2\pi\mathrm{i}\sum_{n=1}^{N} \operatorname{Res}\{g(z)\}_{z=n} = 0, \tag{5.118}$$

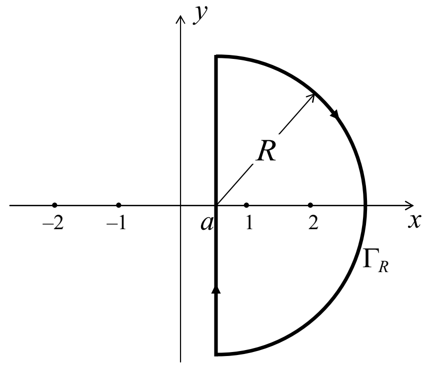

Figure 5.1. Semicircular contour with the diameter taken along $x = a$ and enclosing the poles at $\rho = 1, 2, \ldots$

where Γ_R denotes the semicircular section of the contour and N denotes the number of poles enclosed within the semicircle of radius R. For $t > 1$ and $0 \le \theta < \pi/2$, or $3\pi/2 < \theta \le 2\pi$, it can be shown that the contribution from the semicircular arc vanishes in the limit $R \to \infty$. The residue at the pole $z = n$ of order two is given by

$$\begin{aligned}\operatorname{Res}\{g(z)\}_{z=n} &= \lim_{z\to n} \frac{\mathrm{d}}{\mathrm{d}z}\{(z-n)^2 g(z)\} \\ &= \frac{(-1)^n t^{-n} \log t}{\pi},\end{aligned} \tag{5.119}$$

and so in the limit $R \to \infty$ it follows that

$$\begin{aligned}\sum_{n=1}^{\infty} \operatorname{Res}\{g(z)\}_{z=n} &= \sum_{n=1}^{\infty} \frac{(-1)^n t^{-n} \log t}{\pi} \\ &= -\frac{\log t}{\pi(1+t)},\end{aligned} \tag{5.120}$$

where the following result has been employed:

$$\frac{1}{1+t^{-1}} = \sum_{n=0}^{\infty} (-t)^{-n}, \quad \text{for } t > 1. \tag{5.121}$$

From Eq. (5.118) it follows that

$$-\frac{1}{2\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} t^{-s} \cot \pi s \csc \pi s \, \mathrm{d}s = \frac{\log t}{\pi(1+t)}, \tag{5.122}$$

which is in agreement with Eq. (5.111). For the case $0 < t < 1$, the integration contour is modified as shown in Figure 5.2.

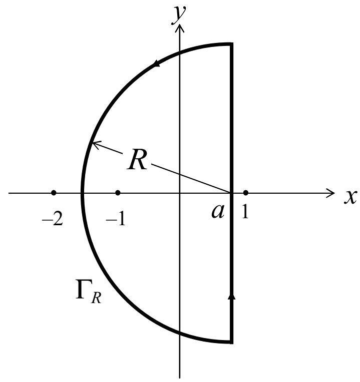

Figure 5.2. Semicircular contour with the diameter taken along $x = a$ and enclosing the poles at $\rho = 0, -1, -2, \ldots$

Applying the Cauchy integral formula yields

$$\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} g(z)\mathrm{d}z + \int_{\Gamma_R} g(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{n=0}^{N} \mathrm{Res}\{g(z)\}_{z=-n}. \tag{5.123}$$

For $0 < t < 1$ and $\pi/2 < \theta < 3\pi/2$, the integral around the semicircular contour vanishes in the limit $R \to \infty$. Making use of the expansion

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-t)^n, \quad \text{for} \quad |t| < 1, \tag{5.124}$$

and taking the limit $R \to \infty$, allows Eq. (5.123) to be simplified to Eq. (5.122) for $0 < t < 1$.

As the preceding example illustrates, working out Hilbert transforms via Eq. (5.109) is not likely to be the most useful evaluation strategy. There may be circumstances when this relationship can be effectively utilized to evaluate inverse Mellin transforms. For integrand factors of the inverse Mellin transform that take the form $\cot \pi x\, g(x)$, where g can be recognized as the Mellin transform of some function f, then the inverse Mellin transform can be obtained directly from Hf. The latter transform may be somewhat easier to evaluate than the inverse Mellin transform, depending on the complexity of the function f.

The Mellin transforms of the functions $H_{\mathrm{e}} f$ and $H_{\mathrm{o}} f$, where the operators H_{e} and H_{o} are defined in Eqs. (4.11) and (4.12), can be worked out in a similar manner to the approach taken for Hf. The transform $MH_{\mathrm{e}}f$ is evaluated as follows:

$$\begin{aligned}
MH_{\mathrm{e}}f(\rho) &= \int_0^\infty x^{\rho-1}\left\{\frac{2x}{\pi}P\int_0^\infty \frac{f(t)\mathrm{d}t}{x^2-t^2}\right\}\mathrm{d}x\\
&= \frac{2}{\pi}\int_0^\infty f(t)\left\{P\int_0^\infty \frac{x^\rho\,\mathrm{d}x}{x^2-t^2}\right\}\mathrm{d}t\\
&= \frac{1}{\pi}\int_0^\infty t^{\rho-1}f(t)\mathrm{d}t\left\{P\int_0^\infty \frac{u^\rho\,\mathrm{d}u}{u-1}-\int_0^\infty \frac{u^\rho\,\mathrm{d}u}{u+1}\right\}\\
&= \{\csc\rho\pi-\cot\rho\pi\}\int_0^\infty t^{\rho-1}f(t)\mathrm{d}t, \qquad (5.125)
\end{aligned}$$

and hence

$$MH_{\mathrm{e}}f(\rho) = \tan(\pi\rho/2)Mf(\rho). \qquad (5.126)$$

This result holds for $-1 < \rho < 0$. If f is an even function, then

$$MHf(\rho) = \tan(\pi\rho/2)Mf(\rho). \qquad (5.127)$$

Similarly,

$$\begin{aligned}
MH_{\mathrm{o}}f(\rho) &= \int_0^\infty x^{\rho-1}\left\{\frac{2}{\pi}P\int_0^\infty \frac{tf(t)\mathrm{d}t}{x^2-t^2}\right\}\mathrm{d}x\\
&= \frac{2}{\pi}\int_0^\infty tf(t)\left\{P\int_0^\infty \frac{x^{\rho-1}\,\mathrm{d}x}{x^2-t^2}\right\}\mathrm{d}t\\
&= \frac{1}{\pi}\int_0^\infty t^{\rho-1}f(t)\mathrm{d}t\left\{P\int_0^\infty \frac{u^{\rho-1}\,\mathrm{d}u}{u-1}-\int_0^\infty \frac{u^{\rho-1}\,\mathrm{d}u}{u+1}\right\}\\
&= -\{\cot\rho\pi+\csc\rho\pi\}\int_0^\infty t^{\rho-1}f(t)\mathrm{d}t, \qquad (5.128)
\end{aligned}$$

and hence

$$MH_{\mathrm{o}}f(\rho) = -\cot(\pi\rho/2)Mf(\rho). \qquad (5.129)$$

This result holds for $0 < \rho < 1$. If f is an odd function, then

$$MHf(\rho) = -\cot(\pi\rho/2)Mf(\rho). \qquad (5.130)$$

Consider the example $f(t) = (1+t^2)^{-1}$. Then Eq. (5.126) yields

$$\begin{aligned}
MH_{\mathrm{e}}f(\rho) &= \tan(\pi\rho/2)Mf(\rho)\\
&= \frac{\pi}{2}\sec(\rho\pi/2). \qquad (5.131)
\end{aligned}$$

For an example of an odd function, let $f(t) = t(1+t^2)^{-1}$; then

$$\begin{aligned} MH_{\text{o}} f(\rho) &= -\cot(\pi\rho/2) Mf(\rho) \\ &= -\frac{\pi}{2}\csc(\rho\pi/2). \end{aligned} \tag{5.132}$$

If the functions f and g form a transform pair, with $g(x) = H_{\text{e}} f(x)$ and $f(x) = -H_{\text{o}} g(x)$, then it follows from Eqs. (5.126) or (5.129) that the ratio of the Mellin transforms of f and g satisfies the following:

$$\frac{Mg(\rho)}{Mf(\rho)} = \tan(\pi\rho/2). \tag{5.133}$$

For the example discussed in the preceding paragraph, $f(t) = (1+t^2)^{-1}$,

$$Mf(\rho) = \frac{\pi}{2}\csc(\rho\pi/2), \tag{5.134}$$

and, with $g(t) = t(1+t^2)^{-1}$,

$$Mg(\rho) = \frac{\pi}{2}\sec(\rho\pi/2), \tag{5.135}$$

and Eq. (5.133) follows.

5.9 The Fourier allied integral

Closely associated to the Fourier integral approach discussed in Section 5.2 are the Fourier allied integral formulas. These were developed in Section 3.12 and take the following form:

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos xt \,\mathrm{d}t \int_0^\infty g(s)\sin st \,\mathrm{d}s, \quad \text{for } g(s) \text{ odd}, \tag{5.136}$$

and

$$g(x) = \frac{2}{\pi}\int_0^\infty \sin xt \,\mathrm{d}t \int_0^\infty f(s)\cos st \,\mathrm{d}s, \quad \text{for } f(s) \text{ even}. \tag{5.137}$$

These expressions are equivalent to the formulas for the Hilbert transform pair of conjugate functions given in Eqs. (4.14) and (4.15), with the additional simplifications resulting from the even–odd character of the functions just stated. That is, for g an odd function, Eq. (5.136) is equivalent to $f = Hg$, and, for f an even function, Eq. (5.137) is equivalent to $g = Hf$. These results also connect with the formulas given in Eqs. (4.11) and (4.12). In many cases the allied integral formulas are often very useful in the evaluation of the Hilbert transform of more involved functions. Typically, Fourier cosine, Fourier sine, or Fourier transforms are required, and there exist some extensive tabulations of these transforms for many of the common functions (see, for example, Erdélyi *et al.* (1954, Vol. I)). An advantage of these formulas is that the singular structure is absent. The obvious disadvantage is the need to solve a double integral.

Consider the evaluation of $H(\cos bx\, \mathrm{e}^{-ax^2})$, with $a > 0$. This represents a somewhat more complicated case than most of the examples previously considered in this book. Since the function is even, it follows from Eq. (5.137) that

$$\begin{aligned} H(\cos bx\, \mathrm{e}^{-ax^2}) &= \frac{2}{\pi} \int_0^{\infty} \sin xt\, \mathrm{d}t \int_0^{\infty} \cos st\, \mathrm{e}^{-as^2} \cos bs\, \mathrm{d}s \\ &= \frac{1}{2\pi} \int_0^{\infty} \sin xt\, \mathrm{d}t \int_{-\infty}^{\infty} \{\mathrm{e}^{\mathrm{i}(t+b)s} + \mathrm{e}^{\mathrm{i}(t-b)s}\}\mathrm{e}^{-as^2}\, \mathrm{d}s \\ &= \frac{1}{2\sqrt{(\pi a)}} \int_0^{\infty} \sin xt \{\mathrm{e}^{-(t+b)^2/4a} + \mathrm{e}^{-(t-b)^2/4a}\}\mathrm{d}t, \end{aligned} \tag{5.138}$$

where the last result has been obtained using

$$\mathcal{F}\{\mathrm{e}^{-ax^2}\} = \int_{-\infty}^{\infty} \mathrm{e}^{-as^2-\mathrm{i}xs}\, \mathrm{d}s = \sqrt{\left(\frac{\pi}{a}\right)}\mathrm{e}^{-x^2/4a}. \tag{5.139}$$

Further simplification of Eq. (5.138) can be made using the following Fourier sine transform (Erdélyi *et al.*, 1954, Vol. I, p. 74):

$$\begin{aligned} \int_0^{\infty} \mathrm{e}^{-\alpha s^2-\beta s} \sin xs\, \mathrm{d}s = -\frac{\mathrm{i}}{4}\sqrt{\left(\frac{\pi}{\alpha}\right)}\{&\mathrm{e}^{(\beta-\mathrm{i}x)^2/(4\alpha)}\, \mathrm{erfc}[2^{-1}\alpha^{-1/2}(\beta-\mathrm{i}x)] \\ &- \mathrm{e}^{(\beta+\mathrm{i}x)^2/(4\alpha)}\, \mathrm{erfc}[2^{-1}\alpha^{-1/2}(\beta+\mathrm{i}x)]\}, \end{aligned} \tag{5.140}$$

where $\mathrm{erfc}(z)$ denotes the complementary error function, which is defined by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \mathrm{e}^{-t^2}\, \mathrm{d}t. \tag{5.141}$$

This function is related to the error function, $\mathrm{erf}(z)$, which was defined in Eq. (5.27), by the following formula:

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z). \tag{5.142}$$

The alternative notations $\mathrm{Erfc}(z)$ and $\mathrm{Erf}(z)$ are in wide usage. Two properties of the error function that will be useful are:

$$\mathrm{erf}(-z) = -\mathrm{erf}(z) \tag{5.143}$$

and

$$\mathrm{erf}(z^*) = \{\mathrm{erf}(z)\}^*. \tag{5.144}$$

Equation (5.138) can be simplified using Eq. (5.140) to yield

$$
\begin{aligned}
H(\mathrm{e}^{-ax^2}\cos bx) = &-\frac{\mathrm{i}}{4}\mathrm{e}^{-ax^2}\left\{\mathrm{e}^{-\mathrm{i}bx}\operatorname{erfc}\left[\sqrt{(a)}\left(\frac{b}{2a}-\mathrm{i}x\right)\right]\right.\\
&-\mathrm{e}^{\mathrm{i}bx}\operatorname{erfc}\left[\sqrt{(a)}\left(\frac{b}{2a}+\mathrm{i}x\right)\right]+\mathrm{e}^{\mathrm{i}bx}\operatorname{erfc}\left[-\sqrt{(a)}\left(\frac{b}{2a}+\mathrm{i}x\right)\right]\\
&\left.-\mathrm{e}^{-\mathrm{i}bx}\operatorname{erfc}\left[-\sqrt{(a)}\left(\frac{b}{2a}-\mathrm{i}x\right)\right]\right\}\\
= &-\frac{\mathrm{i}}{2}\mathrm{e}^{-ax^2}\left\{\mathrm{e}^{\mathrm{i}bx}\operatorname{erf}\left[\sqrt{(a)}\left(\mathrm{i}x+\frac{b}{2a}\right)\right]\right.\\
&\left.+\mathrm{e}^{-\mathrm{i}bx}\operatorname{erf}\left[\sqrt{(a)}\left(\mathrm{i}x-\frac{b}{2a}\right)\right]\right\},
\end{aligned}
\tag{5.145}
$$

which is the required result. Using the connection given in Eq. (5.144), it can be shown that the term in braces in the preceding formula is purely imaginary, so Eq. (5.145) can be written as follows:

$$
H(\mathrm{e}^{-ax^2}\cos bx) = \mathrm{e}^{-ax^2}\,\mathrm{Im}\left\{\mathrm{e}^{\mathrm{i}bx}\operatorname{erf}\left[\sqrt{(a)}\left(\frac{b}{2a}+\mathrm{i}x\right)\right]\right\}. \tag{5.146}
$$

5.10 The Radon transform

The fundamental problem of determining a function $f(x,y)$ in the plane from line integrals over the function arises in important applications. The solution to the problem was published by Johannes Radon in 1917. The related problem of determining a function on a sphere from line integrals of the function over the great circles was published by Funk (1916). About forty years after the appearance of Radon's work, Allan Cormack, initially working alone, and then in collaboration with Godfrey Hounsfield, was interested in refining X-ray radiography. Cormack wanted to determine a two-dimensional cross-sectional map of the absorption of X-rays for a slice of the human body, from the absorption of X-rays along lines through each thin slice. It was some years before the importance of Radon's early work on this problem was recognized. The significance of the work of Cormack and Hounsfield was acknowledged by the award of the Nobel Prize in Physiology or Medicine in 1979.

From these beginnings, and in parallel with them, computer aided tomography emerged, where sequences of projections are used to reconstruct an object. This technique plays a pivotal role in medical imaging. The reader may be familiar with this by the well known acronym "CAT scan."

Important applications of the Radon transform occur in image reconstruction and image analysis. The Radon transform can be used to locate lines in an image, which has application in computer vision, seismic work, and image processing. The Radon transform also finds application in astronomy. Here the focus of attention is centered on the inverse Radon transform, which allows the reconstruction of the function $f(x, y)$ to be carried out. This reconstruction can be cast in terms of the Hilbert transform.

The motivating problem is as follows. Consider an X-ray passing through a slice of inhomogeneous material along a line of length L. Suppose over the line segment $\Delta\ell_j$ the absorption coefficient is α_j, then the emerging beam intensity is given by

$$I = I_0 \exp\left[-(\alpha_1 \Delta\ell_1 + \alpha_2 \Delta\ell_2 + \alpha_3 \Delta\ell_3 + \cdots + \alpha_N \Delta\ell_N)\right], \tag{5.147}$$

where I_0 is the incident beam intensity and

$$L = \sum_{j=1}^{N} \Delta\ell_j. \tag{5.148}$$

Proceeding to the limit $\Delta\ell_j \to 0$ allows Eq. (5.147) to be written in terms of a line integral:

$$I = I_0 e^{-\int_L \alpha \, d\ell}, \tag{5.149}$$

where the absorption coefficient α depends on ℓ.

A line in the xy-plane can be represented by

$$p = x \cos\theta + y \sin\theta, \tag{5.150}$$

which is illustrated in Figure 5.3.

Taking the logarithm of Eq. (5.149) and introducing the definition $P(p, \theta) = -\log(I/I_0)$ yields

$$P(p, \theta) = \int_L \alpha(x, y) d\ell. \tag{5.151}$$

The quantity $P(p, \theta)$ can be thought of as a projection of the absorption coefficient in the xy-plane, along the line L, to the point (p, θ). Keeping θ fixed, p can be varied by moving the radiation source and the detector, which are located at the terminal points of the line L, and constructing a profile of the projections, that is, a plot of $P(p, \theta)$ versus p for the material under investigation.

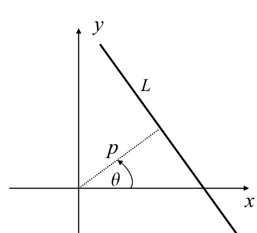

Figure 5.3. Variables p and θ characterizing the line L in the xy-plane.

Generalizing the preceding problem leads to

$$\mathrm{A}f(p,\theta) = \int_L f(x,y)\mathrm{d}s, \tag{5.152}$$

where the integration is taken along the line L. This equation defines the Radon transform operator A. It is common practice to work in the rotated coordinate system characterized by (p,s). In matrix representation, this is

$$\begin{pmatrix} p \\ s \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.153}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p \\ s \end{pmatrix}. \tag{5.154}$$

These variables are illustrated in Figure 5.4.

The definition of the Radon transform can then be written as follows:

$$\mathrm{A}f(p,\theta) = \int_{-\infty}^{\infty} f(p\cos\theta - s\sin\theta,\, p\sin\theta + s\cos\theta)\mathrm{d}s. \tag{5.155}$$

As an example, the Radon transform of the function $f(x,y) = (1/\sqrt{\pi})\mathrm{e}^{-(x^2+y^2)}$ is evaluated. From Eq. (5.155) it follows that

$$\begin{aligned}
\mathrm{A}f(p,\theta) &= \int_{-\infty}^{\infty} f(p\cos\theta - s\sin\theta, p\sin\theta + s\cos\theta)\mathrm{d}s \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(p^2+s^2)}\,\mathrm{d}s \\
&= \frac{\mathrm{e}^{-p^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s \\
&= \mathrm{e}^{-p^2}.
\end{aligned} \tag{5.156}$$

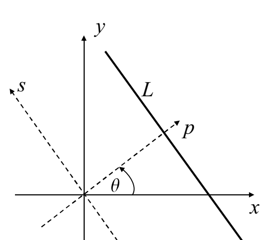

Figure 5.4. The line in the rotated coordinate system.

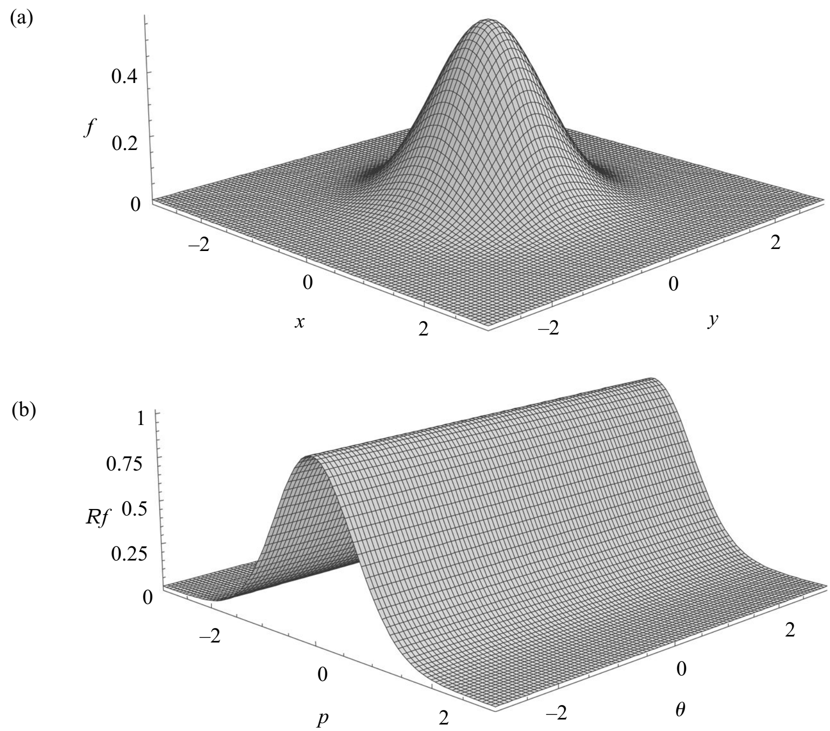

Figure 5.5. (a) The function $f(x,y)$ and (b) its Radon transform $\mathrm{A}f(p,\theta)$.

Figure 5.5 illustrates the behavior for $f(x,y)$ and the corresponding Radon transform.

Equivalently, the Radon transform is also defined in terms of the Dirac delta function as follows:

$$\mathrm{A}f(p,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\delta(p-x\cos\theta-y\sin\theta)\mathrm{d}x\,\mathrm{d}y. \tag{5.157}$$

This particular form is useful when the inversion of the Radon transform is considered. The Radon transform can be generalized further to an n-dimensional operator, but this topic is not pursued here.

In order to guarantee that the Radon transform exists, it is assumed that $f\in L^1(\mathbb{R}^2)$. In practical applications it is common to require that f is infinitely differentiable with an appropriately rapid decay at infinity. Alternatively, f is supposed to be locally integrable with compact support. An important question concerns the information required in order to recover f given the function $g=\mathrm{A}f$.

The connection between the Fourier transform of the Radon transform and the two-dimensional Fourier transform of the function is considered first. The notation $\mathcal{F}_2$ is used to denote a two-dimensional Fourier transform. Employing the definition of the two-dimensional Fourier transform of f, introducing $u=q\cos\phi$ and $v=q\sin\phi$, where $q > 0$, and making use of Eq. (2.258), leads to

$$\begin{aligned}
\mathcal{F}_2 f(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(ux+vy)} f(x,y)\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x\,\mathrm{d}y \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t}\delta(t - qx\cos\phi - qy\sin\phi)\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x\,\mathrm{d}y \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}qp}\delta(qp - qx\cos\phi - qy\sin\phi)q\,\mathrm{d}p \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x\,\mathrm{d}y \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}qp}\delta(p - x\cos\phi - y\sin\phi)\mathrm{d}p \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}qp}\,\mathrm{d}p \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(p - x\cos\phi - y\sin\phi)\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}qp}\mathrm{A}f(p,\phi)\mathrm{d}p, \qquad (5.158)
\end{aligned}$$

and so

$$\mathcal{F}_2 f(u,v) = \mathcal{F}\mathrm{A}f(q,\phi). \qquad (5.159)$$

This result is called the *central slice theorem*.

Inverting the relation given in Eq. (5.155) is now considered; that is, the inverse Radon transform is determined. Formally, writing $g = \mathrm{A}f$, yields

$$f = \mathrm{A}^{-1}g. \qquad (5.160)$$

Starting from Eq. (5.159) and writing $g(u,v) = (\mathcal{F}_2 f)(u,v)$, introducing $u = q\cos\phi$ and $v = q\sin\phi$, employing the simplifications $\rho = s\cos\phi + t\sin\phi$ and $w = x\cos\phi + y\sin\phi$, leads to

$$\begin{aligned}
f(x,y) &= \{\mathcal{F}_2^{-1} g(u,v)\}(x,y) \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xu}\,\mathrm{d}u \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}yv} g(u,v)\mathrm{d}v \\
&= \frac{1}{4\pi^2}\int_{0}^{\infty} q\,\mathrm{d}q \int_{0}^{2\pi} \mathrm{e}^{\mathrm{i}q(x\cos\phi + y\sin\phi)} g(q\cos\phi, q\sin\phi)\mathrm{d}\phi \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} |q|\mathrm{d}q \int_{0}^{\pi} \mathrm{e}^{\mathrm{i}qw} g(q\cos\phi, q\sin\phi)\mathrm{d}\phi \\
&= \frac{1}{4\pi^2}\int_{0}^{\pi} \mathrm{d}\phi \int_{-\infty}^{\infty} |q|\mathrm{e}^{\mathrm{i}qw} g(q\cos\phi, q\sin\phi)\mathrm{d}q
\end{aligned}$$

$$= \frac{1}{4\pi^2}\int_0^{\pi} \mathrm{d}\phi \int_{-\infty}^{\infty} |q|\mathrm{e}^{\mathrm{i}qw}\,\mathrm{d}q \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sq\cos\phi}\,\mathrm{d}s \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tq\sin\phi} f(s,t)\mathrm{d}t$$

$$= \frac{1}{4\pi^2}\int_0^{\pi} \mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} f(s,t)\mathrm{d}t \int_{-\infty}^{\infty} |q|\mathrm{e}^{\mathrm{i}q(w-\rho)}\,\mathrm{d}q. \tag{5.161}$$

Making use of $H(\mathrm{e}^{\mathrm{i}qx}) = -\mathrm{i}\,\mathrm{sgn}\,q\,\mathrm{e}^{\mathrm{i}qx}$, yields

$$\begin{aligned}
f(x,y) &= \frac{\mathrm{i}}{4\pi^2}\int_0^{\pi} \mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} f(s,t)\mathrm{d}t \int_{-\infty}^{\infty} q\mathrm{e}^{-\mathrm{i}q\rho} H(\mathrm{e}^{\mathrm{i}qw})\mathrm{d}q \\
&= \frac{\mathrm{i}}{4\pi^2}\int_0^{\pi} \mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} f(s,t)\mathrm{d}t \int_{-\infty}^{\infty} q\mathrm{e}^{-\mathrm{i}q\rho}\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}qp}\,\mathrm{d}p}{w-p}\,\mathrm{d}q \\
&= \frac{\mathrm{i}}{4\pi^2}\int_0^{\pi} \mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} f(s,t)\mathrm{d}t\,\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{w-p}\int_{-\infty}^{\infty} q\mathrm{e}^{\mathrm{i}q(p-\rho)}\,\mathrm{d}q \\
&= \frac{\mathrm{i}}{4\pi^2}\int_0^{\pi} \mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} f(s,t)\mathrm{d}t\,\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{w-p}\frac{(-2\pi\mathrm{i})\partial\delta(p-\rho)}{\partial p} \\
&= \frac{1}{2\pi^2}\int_0^{\pi} \mathrm{d}\phi\, P\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{w-p}\frac{\partial}{\partial p}\int_{-\infty}^{\infty}\mathrm{d}s\int_{-\infty}^{\infty} f(s,t)\delta(p-s\cos\phi-t\sin\phi)\mathrm{d}t \\
&= \frac{1}{2\pi^2}\int_0^{\pi} \mathrm{d}\phi\, P\int_{-\infty}^{\infty}\frac{\mathrm{A}f_p(p,\phi)\mathrm{d}p}{w-p},
\end{aligned} \tag{5.162}$$

where the derivative $\partial(\mathrm{A}f)/\partial p$ is indicated by the appropriate subscript, and hence

$$\begin{aligned}
f(x,y) &= \frac{1}{2\pi}\int_0^{\pi} \{H\mathrm{A}f_p(p,\phi)\}(w)\mathrm{d}\phi \\
&= \frac{1}{2\pi}\int_0^{\pi} \frac{\partial\{H\mathrm{A}f(p,\phi)\}(w)}{\partial w}\,\mathrm{d}\phi.
\end{aligned} \tag{5.163}$$

The last line follows using the derivative property of the Hilbert transform (see Section 4.8). Several changes of integration order have been made in Eqs. (5.161) and (5.162), and the reader is requested to make some justification for these switches.

The reconstruction of the function $f(x,y)$ from its Radon transform $\mathrm{A}f(p,\theta)$ is now investigated using a pair of examples. Suppose $\mathrm{A}f(p,\theta) = \mathrm{e}^{-p^2}$, then it follows from Eq. (5.162) that

$$\begin{aligned}
f(x,y) &= \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\phi\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{A}f_p(p,\theta)\mathrm{d}p}{r\cos(\phi-\theta)-p} \\
&= \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\{r\cos(\phi-\theta)-p-r\cos(\phi-\theta)\}\mathrm{e}^{-p^2}\,\mathrm{d}p}{r\cos(\phi-\theta)-p}
\end{aligned}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-p^2}\,\mathrm{d}p - \frac{1}{\pi}\int_0^{\pi} r\cos(\phi-\theta)\mathrm{d}\phi\,\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-p^2}\,\mathrm{d}p}{r\cos(\phi-\theta)-p}$$
$$= \frac{1}{\sqrt{\pi}} - \frac{1}{\pi}\int_0^{\pi} r\cos(\phi-\theta)\{H\mathrm{e}^{-p^2}\}(r\cos(\phi-\theta))\mathrm{d}\phi. \tag{5.164}$$

The integrand of the last integral can be simplified, on using Eqs. (5.29) – (5.31), to yield

$$\{H\mathrm{e}^{-p^2}\}(r\cos(\phi-\theta)) = \sum_{k=0}^{\infty}\frac{(-1)^k r^{2k+1}\cos^{2k+1}(\phi-\theta)}{\Gamma(k+3/2)}. \tag{5.165}$$

Hence,

$$-\frac{1}{\pi}\int_0^{\pi} r\cos(\phi-\theta)\{H\mathrm{e}^{-p^2}\}(r\cos(\phi-\theta))\mathrm{d}\phi$$
$$= \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k r^{2k}}{\Gamma(k+1/2)}\int_0^{\pi}\cos^{2k}(\phi-\theta)\mathrm{d}\phi. \tag{5.166}$$

A straightforward calculation yields

$$\int_0^{\pi}\cos^{2k}(\phi-\theta)\mathrm{d}\phi = \int_{-\theta}^{\pi-\theta}\cos^{2k} t\,\mathrm{d}t$$
$$= \int_{-\theta}^{\pi-\theta}\left\{\frac{1}{4^k}\binom{2k}{k} + \frac{1}{2^{2k-1}}\sum_{j=0}^{k-1}\binom{2k}{k}\cos(2k-2j)t\right\}\mathrm{d}t$$
$$= \frac{\pi}{4^k}\binom{2k}{k}, \tag{5.167}$$

and hence

$$f(x,y) = \frac{1}{\sqrt{\pi}} + \sum_{k=1}^{\infty}\frac{(-1)^k r^{2k}}{4^k\Gamma(k+1/2)}\binom{2k}{k}$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}}{4^k\Gamma(k+1/2)}\binom{2k}{k}$$
$$= \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}}{k!}. \tag{5.168}$$

The duplication formula for the gamma function,

$$\Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma(z+1/2)}{\sqrt{\pi}}, \tag{5.169}$$

has been used to simplify the second last sum in Eq. (5.168). Hence,

$$f(x,y) = \frac{\mathrm{e}^{-r^2}}{\sqrt{\pi}} = \frac{\mathrm{e}^{-(x^2+y^2)}}{\sqrt{\pi}}, \tag{5.170}$$

which is the desired result. As a second example, consider the inversion of the Radon transform given by

$$(\mathrm{A}f)(p,\phi) = \begin{cases} 2\sqrt{(1-p^2)}, & |p| \le 1 \\ 0, & |p| > 1. \end{cases} \tag{5.171}$$

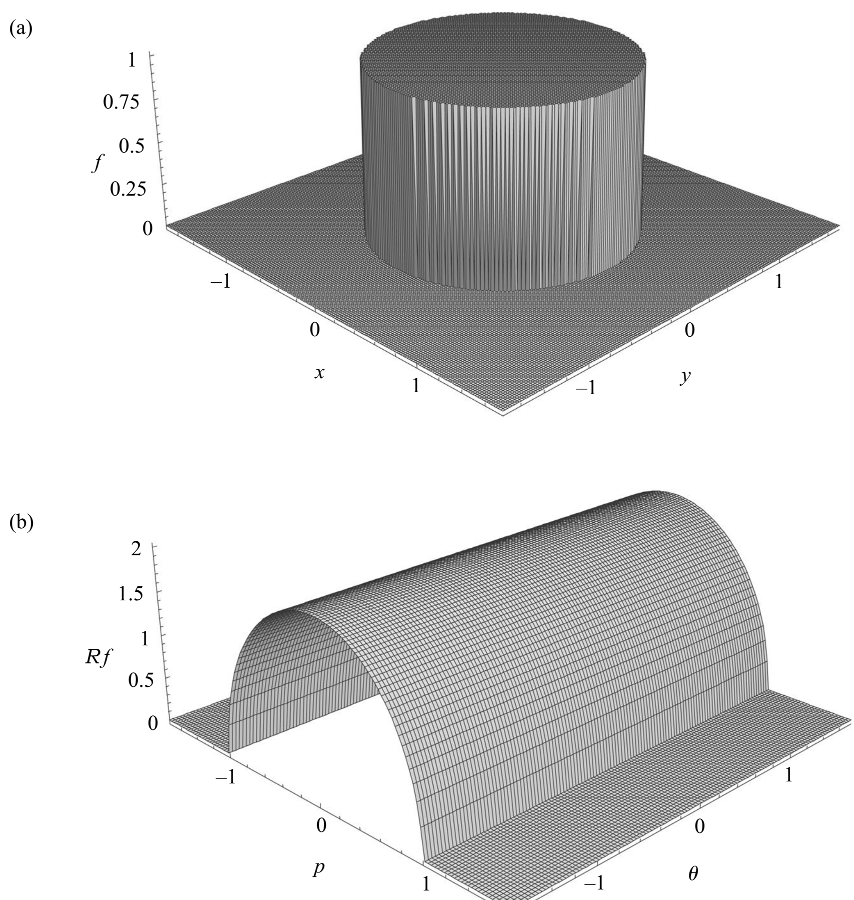

Figure 5.6. (a) The function $f(x,y)$ in Eq. (5.173) and (b) its Radon transform $\mathrm{A}f(p,\theta)$.

From Eq. (5.162), and making use of entry (12A.6) from the Table of Hilbert transforms in Appendix 1, yields

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\phi \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{(\mathrm{A}f)_p(p,\theta)\mathrm{d}p}{r\cos(\phi-\theta)-p} \\ &= -\frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi \frac{1}{\pi} P \int_{-1}^{1} \frac{p\,\mathrm{d}p}{\surd(1-p^2)\,(r\cos(\phi-\theta)-p)} \\ &= 1. \end{aligned} \tag{5.172}$$

In the preceding calculation, the inequality $|r\cos(\phi-\theta)| \leq 1$ has been employed. The inversion formula can be written as follows:

$$f(x,y) = \begin{cases} 1, & x^2+y^2 \leq 1 \\ 0, & x^2+y^2 > 1. \end{cases} \tag{5.173}$$

Figure 5.6 illustrates the behavior for $f(x,y)$ and the corresponding Radon transform.

Notes

§5.2 A good source for further reading on the subjects of this section is Butzer and Nessel (1971, chap. 8). There is a generalization of the key result in this section to an n-dimensional Euclidean space, and that is considered later in Chapter 15. For a numerical evaluation of Dawson's integral, the interested reader can pursue further reading in Cody, Paciorek, and Thacher (1970), McCabe (1974), Press *et al.* (1992, p. 252), or Weideman (1994).

§5.3 A number of results for the even and odd Hilbert transform operators can be found in Rooney (1980). The inversion of some complicated integral transforms involving $\mathcal{F}_\mathrm{s}$ and $\mathcal{F}_\mathrm{c}$ in terms of H_e and H_o can be found in Kilbas and Saigo (2004, p. 275).

§5.4 Under certain conditions on the operator A, the general commutator $[A, H]$ can be shown to be a bounded operator; see Segovia and Wheeden (1971) and Segovia and Torrea (1990).

§5.5 The Hartley transform was originally given in Hartley (1942). For further reading on this transform, see Bracewell (1986) and Olejniczak (1996).

§5.7 Wolf (1967) attempted to write the Hilbert transform of a general function as an iterated Laplace transform; however, his analysis was flawed (see Crystal (1968)). For further discussion, see Kak (1968).

§5.8 The relationship between Mellin transforms of a transform pair, Eq. (5.133), was derived in Brachman and MacDonald (1954). Ursell (1983) applied the Mellin transform to the one-sided Hilbert transform, a strategy that proved to be useful for studying the asymptotic expansion of certain integrals. An extensive table of Mellin transforms can be found in Marichev (1983), and Erdélyi *et al.* (1954, Vol. I) gives a useful collection of formulas.

§5.10 Radon's original paper (Radon, 1917) is published in a journal that is not readily accessible; however, the paper is reprinted in a book, Helgason (1980, p. 177), in a conference proceedings, Gindikin and Michor (1994, p. 324), and as an English translation in Deans (1983, p. 204) and Radon (1986). For an alternative derivation of the two-dimensional inversion formula for the Radon transform, see Deans (1996, p. 664). A key early paper for medical applications is Cormack (1963).

Exercises

5.1 Taking advantage of the relationship $Hf(x) = -\mathrm{i}\mathcal{F}^{-1}\{\operatorname{sgn} y\ \mathcal{F}f(y)\}(x)$, or otherwise, evaluate the Hilbert transform of $f(x) = \sin ax\, \mathrm{e}^{-bx^2}$ for constants $a > 0$ and $b > 0$.

5.2 Evaluate, using Fourier transform methods or otherwise, the Hilbert transform of the following functions: (i) $f(x) = x^{-2}\sin^2 ax$, for $a > 0$; (ii) $f(x) = \mathrm{e}^{-a|x|}$, for $a > 0$.

5.3 Determine Hf given $f(x) = \sin ax\, \mathrm{e}^{-|x|}$ for $a > 0$.

5.4 For a function $f \in L^2(\mathbb{R})$, express $(1/\pi)\int_0^\infty \mathrm{d}u \int_{-\infty}^\infty \sin u(t-x) f(t)\mathrm{d}t$ in terms of the Fourier transform operator.

5.5 How is the integral of Exercise 5.4 related to the integral

$$\frac{1}{\pi}\int_0^\infty \frac{f(x+t) - f(x-t)}{t}\,\mathrm{d}t?$$

Indicate clearly any assumptions you need to make. Hence, establish the central result of Section 5.2.

5.6 If the operator M_α is defined by $M_\alpha f(x) = x^\alpha f(x), \alpha \in \mathbb{R}$, evaluate $M_{-1}H_\mathrm{e}M_1$ and $M_1H_\mathrm{o}M_{-1}$.

5.7 What conditions must be satisfied by the two functions f and h in order to derive Eq. (5.62)?

5.8 Srivastav (1997) studied the following transform operators:

$$\mathrm{B}f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{\{c\cos xt - t\sin xt\}f(t)\mathrm{d}t}{t^2+c^2},$$

and

$$\mathrm{C}f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{\{c\sin xt + t\cos xt\}f(t)\mathrm{d}t}{t^2+c^2},$$

for c a constant and $f \in L^2(0,\infty)$. Do any relationships exist between the operators B and C and the operators H_e and H_o?

5.9 Prove that the operators H_e and H_o satisfy $H_\mathrm{e}^{-1} = -H_\mathrm{o}$ and $H_\mathrm{o}^{-1} = -H_\mathrm{e}$.

5.10 Consider the transform operator $Rf(x) = \int_0^\infty \{t\cos xt + c\sin xt\}f(t)\,\mathrm{d}t$, for c a constant and $f \in L^2(0,\infty)$. Taking advantage of the operators introduced in Exercise 5.8, find relationships between H_e and R and between H_o and R.

5.11 Let P_+ and P_- denote the projection operators defined in Section 4.18 and let χ_+ and χ_- denote the operators defined by multiplication by the characteristic functions $\chi_{(0,\infty)}$ and $\chi_{(-\infty,0)}$ respectively; that is,

$$\chi_+ f(x) = \begin{cases} f(x), & x \in (0,\infty) \\ 0, & x \notin (0,\infty) \end{cases}, \quad \chi_- f(x) = \begin{cases} f(x), & x \in (-\infty,0) \\ 0, & x \notin (-\infty,0). \end{cases}$$

Evaluate $P_+\mathcal{F}$ and $P_-\mathcal{F}$ in terms of the operators $\mathcal{F}$, χ_+, and χ_- for $f \in L^2(\mathbb{R})$.

5.12 For the operators defined by

$$\mathcal{P}_+ = \frac{1}{2}(I + \mathrm{i}\mathcal{F}H\mathcal{F}^{-1}) \quad \text{and} \quad \mathcal{P}_- = \frac{1}{2}(I - \mathrm{i}\mathcal{F}H\mathcal{F}^{-1}),$$

show that they are idempotent.

5.13 For the pair of operators in Exercise 5.12, evaluate $\mathcal{P}_+\mathcal{F}$ and $\mathcal{P}_-\mathcal{F}$ for functions of the class $L^2(\mathbb{R})$ in terms of $\chi_{(-\infty,0)}$, $\chi_{(0,\infty)}$, and $\mathcal{F}$.

5.14 Evaluate $[\mathcal{F}, H]f$ for $f(x) = (x^2 + a^2)^{-1}$, for $a > 0$.

5.15 Evaluate $\mathcal{F}^{-1}\{\operatorname{sgn} y\, \mathcal{F}f(y)\}(x)$. [Hint: The Riemann–Lebesgue lemma may be useful to you.]

5.16 Evaluate the Stieltjes transform of the function $f(x) = x^{-1}\sin ax, a > 0$, and hence evaluate the Hilbert transform of this function.

5.17 For $a > 0, b > 0$, determine the Radon transform of $f(x,y) = \mathrm{e}^{-ax^2-by^2}$.

5.18 Determine the Radon transform of the following function:

$$f(x,y) = \begin{cases} 1, & x^2 + y^2 \le 1 \\ 0, & x^2 + y^2 > 1. \end{cases}$$

5.19 Does the Hilbert transform arise in the inversion formula for the Radon transform in $\mathbb{R}^3$?

5.20 If $\mathrm{A}f(p,\phi) = \operatorname{cas}\phi\, p\mathrm{e}^{-p^2}$, determine $f(x,y)$.

5.21 Determine the function $f(x,y)$, given that the Radon transform $\mathrm{A}f$ takes the form

$$\mathrm{A}f(p,\phi) = \begin{cases} 2\,|ab|s^{-2}\sqrt{(s^2 - p^2)}, & |p| \le s \\ 0, & |p| > s, \end{cases}$$

where a and b are constants and $s = \sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)}$.

5.22 If $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$, show that $\{f * g\} \in L^2(\mathbb{R})$. [Hint: The identity $|f(t)g(x-t)| = (|f(t)|\,|g(x-t)|^2)^{1/2}|f(t)|^{1/2}$ and the Cauchy–Schwarz–Buniakowski inequality may prove useful to you.] A more general result can be proved. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, for $1 \le p \le \infty$ and $1 \le q \le \infty$, what can be established for $\{f * g\}$?

5.23 Determine whether the operator $\operatorname{sgn} x\, \mathcal{F}$ has any eigenfunctions. If it does, find the eigenvalues.

6

The Hilbert transform of periodic functions

6.1 Introduction

The necessity of finding the Hilbert transform of periodic functions arises in a number of applications. By a conformal mapping of the upper half plane into the interior of the unit disc, it is possible to express the frequency dependence of a number of functions such as generalized susceptibilities, scattering amplitudes, and other properties, as Fourier series. From the series representation of the dispersive component, that is the component representing the real part of the function of interest, the dissipative component, or the component representing the imaginary part of the function, is determined from the conjugate series, which is the Hilbert transform of the original series. The converse of the latter statement also applies. Some of the background on the Hilbert transform of periodic functions has already been introduced in Section 3.14. The reader is reminded that the symbol $\mathcal{H}$ in place of H is used to denote the Hilbert transform on the circle. A widely used notation to designate the Hilbert transform operator on the circle is $\tilde{H}$. It is also quite common to denote $\mathcal{H}f$ by $\tilde{f}$, which is read as f *wiggle* or f *twiddle*. Much of the focus of this chapter is Hilbert transforms on the circle, although some results for periodic functions on $\mathbb{R}$ are also considered.

There are several ways to approach the calculation of the Hilbert transform of a periodic function. In the following two sections the techniques involved are illustrated for the following two related functions:

$$f(x) = \operatorname{sgn}(\sin ax) \tag{6.1}$$

and

$$f(x) = \operatorname{sgn}(\cos ax), \tag{6.2}$$

for $a > 0$. These two functions, which represent important pulse sequences, are the square waves shown in Figure 6.1.

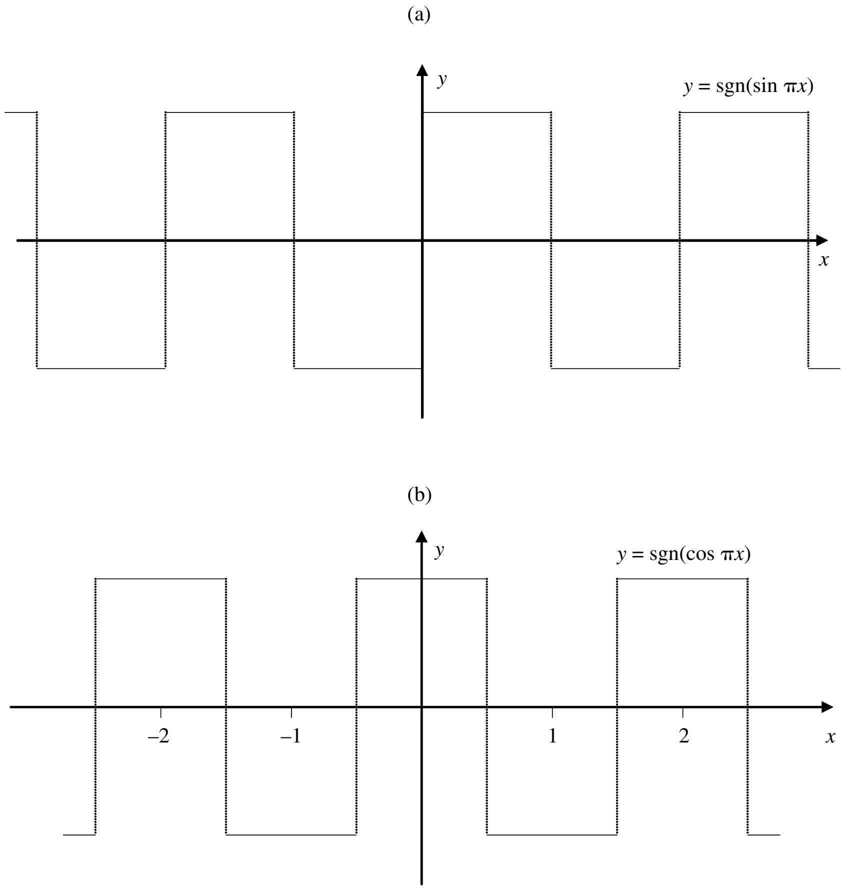

Figure 6.1. Square wave functions (a) sgn(sin ax) and (b) sgn(cos ax), for $a = \pi$.

6.2 Approach using infinite product expansions

The Hilbert transform $H\{\mathrm{sgn}(\cos ax)\}$, for $a > 0$, is considered first. By reference to Figure 6.1 and setting $b = \pi/2a$, it follows that

$$\begin{aligned}\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(\cos as)\mathrm{d}s}{x-s} = \frac{1}{\pi}\Bigg\{\cdots + P\int_{-5b}^{-3b}\frac{\mathrm{sgn}(\cos as)\mathrm{d}s}{x-s}\\ + P\int_{-3b}^{-b}\frac{\mathrm{sgn}(\cos as)\mathrm{d}s}{x-s} + P\int_{-b}^{b}\frac{\mathrm{sgn}(\cos as)\mathrm{d}s}{x-s}\\ + P\int_{b}^{3b}\frac{\mathrm{sgn}(\cos as)\mathrm{d}s}{x-s} + \cdots\Bigg\},\end{aligned} \tag{6.3}$$

which can be written compactly as follows:

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\operatorname{sgn}(\cos as)\mathrm{d}s}{x-s} = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\left\{P\int_{(4k-1)b}^{(4k+1)b}\frac{\operatorname{sgn}(\cos as)\mathrm{d}s}{x-s}\right.$$
$$\left.+P\int_{(4k+1)b}^{(4k+3)b}\frac{\operatorname{sgn}(\cos as)\mathrm{d}s}{x-s}\right\}$$
$$= \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\left\{P\int_{(4k-1)b}^{(4k+1)b}\frac{\mathrm{d}s}{x-s} - P\int_{(4k+1)b}^{(4k+3)b}\frac{\mathrm{d}s}{x-s}\right\}$$
$$= \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\log\left|\frac{\{(4k-1)b-x\}\{(4k+3)b-x\}}{\{(4k+1)b-x\}\{(4k+1)b-x\}}\right|. \tag{6.4}$$

The final summation can be simplified by isolating the log term involving $\{(4k+3)b - x\}$, making the summation index change $j = k + 1$, to give

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\operatorname{sgn}(\cos as)\mathrm{d}s}{x-s} = \frac{2}{\pi}\sum_{k=-\infty}^{\infty}\log\left|\frac{\{(4k-1)b-x\}}{\{(4k+1)b-x\}}\right|$$
$$= \frac{2}{\pi}\left\{\log\left|\frac{x+b}{x-b}\right| + \sum_{k=1}^{\infty}\log\left|\frac{16k^2b^2-(b+x)^2}{16k^2b^2-(b-x)^2}\right|\right\}. \tag{6.5}$$

The final summation can be simplified in the following way. The sine function has the infinite product expansion given by

$$\sin x = x\prod_{k=1}^{\infty}\left\{1-\frac{x^2}{k^2\pi^2}\right\} \tag{6.6}$$

(recall Eq.(2.106)), and hence

$$\log|\sin x| = \log|x| + \sum_{k=1}^{\infty}\log\left|1-\frac{x^2}{k^2\pi^2}\right|. \tag{6.7}$$

Using this result with $x \to (x+b)\pi/4b$, and then with $x \to (x-b)\pi/4b$, allows Eq. (6.5) to be simplified as follows:

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\operatorname{sgn}(\cos as)\mathrm{d}s}{x-s} = \frac{2}{\pi}\left\{\log\left|\frac{x+b}{x-b}\right| + \log\left|\sin\left(\frac{(x+b)\pi}{4b}\right)\right| - \log\left|\frac{(x+b)\pi}{4b}\right|\right.$$
$$\left.-\log\left|\sin\left(\frac{(x-b)\pi}{4b}\right)\right| + \log\left|\frac{(x-b)\pi}{4b}\right|\right\}$$

$$= \frac{2}{\pi} \log \left| \frac{\sin(x\pi/4b + \pi/4)}{\sin(x\pi/4b - \pi/4)} \right|$$
$$= \frac{2}{\pi} \log |\tan ax + \sec ax|$$
$$= \frac{2}{\pi} \log \left| \tan\left(\frac{ax}{2} + \frac{\pi}{4}\right) \right|. \tag{6.8}$$

The Hilbert transform $H\{\mathrm{sgn}(\sin ax)\}$, for $a > 0$, can be evaluated in a similar fashion, but can be more quickly established using Eq. 6.8 and elementary trigonometry. On making use of Eq. (4.30),

$$H\{\mathrm{sgn}(\sin ax)\} = H\left[\mathrm{sgn}\cos\left(ax - \frac{\pi}{2}\right)\right]$$
$$= \frac{2}{\pi} \log \left| \tan\left(\frac{(ax - \pi/2)}{2} + \frac{\pi}{4}\right) \right|$$
$$= \frac{2}{\pi} \log \left| \tan\left(\frac{ax}{2}\right) \right|. \tag{6.9}$$

6.3 Fourier series approach

An alternative approach to evaluating the Hilbert transform of periodic functions takes advantage of the Fourier series expansion of the function. This is a relatively straightforward approach, subject to the ease with which the Fourier series resulting from the application of the Hilbert transform can be summed, either analytically or numerically. The actual Hilbert transform is simple, since only $H(\sin ax)$ and $H(\cos ax)$ are required. The square wave example, $H\{\mathrm{sgn}(\sin ax)\}$, is now reconsidered. The Fourier series expansion of $\mathrm{sgn}(\sin ax)$ with $y = ax$ is given by

$$\mathrm{sgn}(\sin y) = \begin{cases} 1, & 0 < y < \pi \\ -1, & -\pi < y < 0 \end{cases}$$
$$= \frac{4}{\pi}\left\{ \sin y + \frac{\sin 3y}{3} + \frac{\sin 5y}{5} + \cdots \right\}. \tag{6.10}$$

From Eq. (6.10) it follows that

$$H\{\mathrm{sgn}(\sin y)\} = -\frac{4}{\pi}\left\{ \cos y + \frac{\cos 3y}{3} + \frac{\cos 5y}{5} + \cdots \right\}. \tag{6.11}$$

Using the standard result (see, for example, Jolly (1961, p. 97))

$$\log\left|\cot\left(\frac{y}{2}\right)\right| = 2\left\{ \cos y + \frac{\cos 3y}{3} + \frac{\cos 5y}{5} + \cdots \right\}, \quad \text{for } 0 < |y| < \pi, \tag{6.12}$$

then Eq. (6.11) can be written as follows:

$$H\{\operatorname{sgn}(\sin y)\} = -\frac{2}{\pi}\log\left|\cot\left(\frac{y}{2}\right)\right|, \tag{6.13}$$

and hence

$$H\{\operatorname{sgn}(\sin ax)\} = \frac{2}{\pi}\log\left|\tan\left(\frac{ax}{2}\right)\right|. \tag{6.14}$$

The symmetric square wave can be treated in a similar fashion. The Fourier series expansion for this function is

$$\operatorname{sgn}(\cos y) = \frac{4}{\pi}\left\{\cos y - \frac{\cos 3y}{3} + \frac{\cos 5y}{5} - \cdots\right\}, \tag{6.15}$$

from which it follows that

$$H\{\operatorname{sgn}(\cos y)\} = \frac{4}{\pi}\left\{\sin y - \frac{\sin 3y}{3} + \frac{\sin 5y}{5} - \cdots\right\}. \tag{6.16}$$

Using the standard series expansion (Jolly, 1961, p. 97),

$$\log\left\{\frac{1+\sin y}{\cos y}\right\} = 2\left\{\sin y - \frac{\sin 3y}{3} + \frac{\sin 5y}{5} - \cdots\right\}, \quad \text{for } -\frac{\pi}{2} < y < \frac{\pi}{2}, \tag{6.17}$$

allows Eq. (6.16) to be written as follows:

$$H\{\operatorname{sgn}(\cos y)\} = \frac{2}{\pi}\log\left|\frac{1+\sin y}{\cos y}\right|, \tag{6.18}$$

and hence

$$H\{\operatorname{sgn}(\cos ax)\} = \frac{2}{\pi}\log\left|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right|. \tag{6.19}$$

6.4 An operator approach to the Hilbert transform on the circle

This section discusses how the definition of the Hilbert transform on the circle emerges from an appropriately chosen operator acting on a periodic function, and considers how this can be rewritten in the conventional integral form for the Hilbert transform on the circle given in Section 3.14. Let f be a periodic function with period 2π that is absolutely continuous on $[-\pi, \pi]$ and has a square integrable derivative on this interval. The complex Fourier series expansion of $f(\theta)$ is given by

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \tag{6.20}$$

with the coefficients determined from the following formula:

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta. \tag{6.21}$$

The series expansion for $f(\theta)$ converges absolutely, which can be demonstrated in the following manner. Let

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta, \tag{6.22}$$

where the prime denotes the derivative with respect to θ; integrating Eq. (6.21) by parts yields

$$a_n = \frac{b_n}{\mathrm{i}n}. \tag{6.23}$$

Starting from the inequality

$$\left(|b_n| - \frac{1}{|n|}\right)^2 \geq 0, \tag{6.24}$$

it follows that

$$\left|\frac{b_n}{n}\right| \leq \frac{1}{2}\left(\frac{1}{n^2} + |b_n|^2\right), \tag{6.25}$$

and hence

$$|a_n| \leq \frac{1}{2}\left(\frac{1}{n^2} + |b_n|^2\right). \tag{6.26}$$

Using Bessel's inequality to establish the convergence of $\sum_{n=-\infty}^{\infty}|b_n|^2$, it follows from Eq. (6.26) that $\sum_{n=-\infty}^{\infty}|a_n|$ is convergent, and hence the absolute convergence of Eq. (6.20) follows.

If $f(\theta)$ is a real function, then from the condition

$$f(\theta) = \{f(\theta)\}^* \tag{6.27}$$

it follows that the coefficients a_n satisfy

$$a_{-n} = a_n^*. \tag{6.28}$$

The operator $\mathcal{H}$, sometimes called the circular Hilbert transform, is introduced by the following definition:

$$\mathcal{H}f(\theta) = \mathrm{i}\sum_{n=1}^{\infty}\{a_{-n}\mathrm{e}^{-\mathrm{i}n\theta} - a_n\mathrm{e}^{\mathrm{i}n\theta}\}, \tag{6.29}$$

where Eq. (6.20) has been employed for $f(\theta)$. It is demonstrated in the sequel that this series representation defines the Hilbert transform operator for periodic functions. Let $g(\theta)$ denote the series

$$g(\theta) = \mathrm{i}\sum_{k=1}^{\infty}\{b_{-k}\mathrm{e}^{-\mathrm{i}k\theta} - b_k\mathrm{e}^{\mathrm{i}k\theta}\}, \tag{6.30}$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\theta)|^2\mathrm{d}\theta = \sum_{k=-\infty}^{\infty}|b_k|^2 - |b_0|^2. \tag{6.31}$$

Noting the convergence of $\sum_{n=-\infty}^{\infty}|b_n|^2$, the series in Eq. (6.30) is convergent in the mean square sense. It is straightforward to show that

$$\int_0^{\theta} g(x)\mathrm{d}x = \mathcal{H}f(\theta) + c, \tag{6.32}$$

where c is a constant. The function $\mathcal{H}f$ is absolutely continuous on $[-\pi, \pi]$.

Application of the operator $\mathcal{H}$ on $\mathcal{H}f$ leads to a continuous function with a square integrable derivative. This follows from Eqs. (6.20) and (6.29) in the following manner:

$$\begin{aligned}\mathcal{H}^2f(\theta) &= \mathcal{H}^2\left[a_0 + \sum_{k=1}^{\infty}\{a_{-k}\mathrm{e}^{-\mathrm{i}k\theta} + a_k\mathrm{e}^{\mathrm{i}k\theta}\}\right]\\ &= \mathcal{H}\mathrm{i}\sum_{k=1}^{\infty}\{a_{-k}\mathrm{e}^{-\mathrm{i}k\theta} - a_k\mathrm{e}^{\mathrm{i}k\theta}\}\\ &= -\sum_{k=1}^{\infty}\{a_{-k}\mathrm{e}^{-\mathrm{i}k\theta} + a_k\mathrm{e}^{\mathrm{i}k\theta}\},\end{aligned} \tag{6.33}$$

and hence

$$\mathcal{H}^2f(\theta) = -f(\theta) + \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\mathrm{d}x. \tag{6.34}$$

This is the analog for periodic functions (of period 2π) of the inversion property discussed in Section 4.4. Equation (6.34) holds for the general case that f is periodic with period 2π and belongs to $L^p_{2\pi}, 1 < p < \infty$. If the period of f is 2τ, then Eq. (6.34) takes the following form:

$$\mathcal{H}_\tau^2f(\theta) = -f(\theta) + \frac{1}{2\tau}\int_{-\tau}^{\tau}f(x)\mathrm{d}x, \tag{6.35}$$

where the notation introduced in Eq. (3.286) is employed. This holds for f periodic with period 2τ and $f \in L^p_{2\tau}$, $1 < p < \infty$.

It will now be demonstrated how the series expansion for $\mathcal{H}f$ given in Eq. (6.29) can be represented as a principal value integral. Starting from Eqs. (6.21) and (6.29),

$$\begin{aligned}\mathcal{H}f(\theta) &= \frac{\mathrm{i}}{2\pi}\sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}nx-\mathrm{i}n\theta}\,\mathrm{d}x - \int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}n\theta-\mathrm{i}nx}\,\mathrm{d}x\right\}\\ &= -\frac{\mathrm{i}}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(x)\{\mathrm{e}^{\mathrm{i}n(\theta-x)} - \mathrm{e}^{-\mathrm{i}n(\theta-x)}\}\mathrm{d}x\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sum_{n=1}^{\infty} 2\sin n(\theta-x)\mathrm{d}x. \end{aligned} \tag{6.36}$$

The sum in the preceding result converges in the distributional sense. The function f can be considered a suitably well behaved test function, and so this result can be treated as the scalar product between f and the distributional value of the sum. On using the identification

$$\cot\left(\frac{\theta}{2}\right) = 2\sum_{m=1}^{\infty}\sin m\theta, \tag{6.37}$$

in the distributional sense, Eq. (6.36) becomes

$$\mathcal{H}f(\theta) = \frac{1}{2\pi}P\int_{-\pi}^{\pi} f(x)\cot\left(\frac{\theta-x}{2}\right)\mathrm{d}x, \tag{6.38}$$

which is the form given in Section 3.14 for the Hilbert transform on the circle for a periodic function with period 2π.

The link from the series given in Eq. (6.29) to functions analytic on the unit disc in the complex plane is now made, and this leads to the Plemelj formulas. Suppose that the function f is real-valued. Then it is straightforward to show, using Eqs. (6.28) and (6.29), or directly from Eq. (6.38), that $\mathcal{H}f$ is also real-valued. First set

$$f(\theta) + \mathrm{i}\mathcal{H}f(\theta) = a_0 + 2\sum_{n=1}^{\infty} a_n\mathrm{e}^{\mathrm{i}n\theta}. \tag{6.39}$$

Let $z = \mathrm{e}^{\mathrm{i}\theta}$, then a function f^+ can be defined as follows:

$$f^+(z) = a_0 + 2\sum_{n=1}^{\infty} a_n z^n, \tag{6.40}$$

which is analytic in the interior of the unit disc. In a similar manner,

$$f(\theta) - \mathrm{i}\mathcal{H}f(\theta) = a_0 + 2\sum_{n=1}^{\infty} a_{-n}\mathrm{e}^{-\mathrm{i}n\theta}, \tag{6.41}$$

and with the same substitution $z = \mathrm{e}^{\mathrm{i}\theta}$ a function f^- can be defined by

$$f^-(z) = a_0 + 2\sum_{n=1}^{\infty} a_{-n}z^{-n}, \tag{6.42}$$

which is analytic in the exterior to the unit disc. From Eqs. (6.39) and (6.41), it follows that

$$f(\theta) = \frac{1}{2}\{f^+(z) + f^-(z)\} \tag{6.43}$$

and

$$\mathcal{H}f(\theta) = \frac{1}{2\mathrm{i}}\{f^+(z) - f^-(z)\}. \tag{6.44}$$

These results are the analog for a periodic function of the Plemelj formulas discussed in Section 3.7.

6.5 Hilbert transforms of some standard kernels

This section deals with the evaluation of the Hilbert transform of some common kernel functions that occur frequently in Fourier analysis. The Poisson, Dirichlet, and Fejér kernels are examined. The partial sum of a conjugate Fourier series can be related to the Hilbert transform of the Dirichlet kernel, and the Cesàro average – the average of the first n partial sums of the Fourier conjugate series – can be connected with the Hilbert transform of the Fejér kernel. The first kernel investigated is given by

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad \text{for } 0 \le r < 1, \tag{6.45}$$

which is called the Poisson kernel (or sometimes the Abel–Poisson kernel) for the disc. It will prove useful in what follows to express $P(r,\theta)$ as an infinite series. Recalling the series expansion

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n, \quad \text{for } |z| < 1 \tag{6.46}$$

which simplifies on making use of the substitution $z = re^{i\theta}$ to yield

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n e^{in\theta} = \frac{(1+z)}{2(1-z)}$$
$$= \frac{1 - r^2 + 2ir\sin\theta}{2(1 - 2r\cos\theta + r^2)}. \tag{6.47}$$

Taking the real and imaginary parts leads to

$$1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \tag{6.48}$$

and

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}. \tag{6.49}$$

The Hilbert transform of $P(r, \theta)$ can be found on making use of the following results:

$$\frac{1}{2\pi} P \int_{-\pi}^{\pi} \cot\left(\frac{\theta - s}{2}\right) ds = 0 \tag{6.50}$$

and the more general formula

$$\frac{1}{2\pi} P \int_{-\pi}^{\pi} \cos ks \cot\left(\frac{\theta - s}{2}\right) ds = \sin k\theta, \quad \text{for integer } k \geq 0. \tag{6.51}$$

These results can be derived in the following manner. Making use of the following expansion (in the distributional sense):

$$\cot\frac{\theta}{2} = 2\sum_{m=1}^{\infty} \sin m\theta, \tag{6.52}$$

for integer $n \geq 0$, yields

$$\frac{1}{2\pi} P \int_{-\pi}^{\pi} \cos ns \cot\left(\frac{x - s}{2}\right) ds = \frac{1}{\pi} P \int_{-\pi}^{\pi} \cos ns \sum_{m=1}^{\infty} \sin(mx - ms) ds$$
$$= \frac{1}{\pi} \sum_{m=1}^{\infty} \sin mx \int_{-\pi}^{\pi} \cos ns \, \cos ms \, ds$$
$$= \sum_{m=1}^{\infty} \sin mx \, \delta_{mn}$$
$$= \sin nx, \tag{6.53}$$

where δ_{nm} denotes the Kronecker delta (recall Eq. (2.38)). The particular case $n = 0$ establishes Eq. (6.50). So $\mathcal{H}P(r,\theta)$ can be evaluated as follows:

$$\begin{aligned}
\mathcal{H}P(r,\theta) &= \frac{1}{2\pi} P\int_{-\pi}^{\pi} P(r,s)\cot\left(\frac{\theta - s}{2}\right)\mathrm{d}s \\
&= \frac{1}{2\pi} P\int_{-\pi}^{\pi} \left\{1 + 2\sum_{k=1}^{\infty} r^k \cos k\theta\right\}\cot\left(\frac{\theta - s}{2}\right)\mathrm{d}s \\
&= \frac{1}{2\pi} P\int_{-\pi}^{\pi} \cot\left(\frac{\theta - s}{2}\right)\mathrm{d}s + \frac{1}{\pi}\sum_{k=1}^{\infty} r^k P\int_{-\pi}^{\pi} \cos k\theta \cot\left(\frac{\theta - s}{2}\right)\mathrm{d}s \\
&= 2\sum_{k=1}^{\infty} r^k \sin k\theta, \qquad (6.54)
\end{aligned}$$

and using Eq. (6.49) leads to

$$\mathcal{H}P(r,\theta) = \frac{2r\sin\theta}{1 - 2r\cos\theta + r^2}. \qquad (6.55)$$

The Dirichlet kernel is defined by

$$D_n(\theta) = 1 + 2\sum_{k=1}^{n} \cos k\theta = \frac{\sin\{(2n+1)\theta/2\}}{\sin(\theta/2)}. \qquad (6.56)$$

When $\theta = 2\pi m$ with $m \in \mathbb{Z}$, $D_n(\theta) = 2n + 1$. In order to evaluate the Hilbert transform of $D_n(\theta)$, it is first useful to evaluate a finite sum of sine terms of the form $\sum_{k=1}^{n} \sin k\theta$, and in the process prove Eq. (6.56). Use the standard identity

$$z^n - 1 = (z - 1)\sum_{k=0}^{n-1} z^k, \qquad (6.57)$$

which simplifies, with the substitution $z = \mathrm{e}^{\mathrm{i}\theta}$, to

$$\begin{aligned}
\sum_{k=0}^{n} \mathrm{e}^{\mathrm{i}k\theta} &= \frac{1 - \mathrm{e}^{\mathrm{i}(n+1)\theta}}{1 - \mathrm{e}^{\mathrm{i}\theta}} \\
&= \frac{1 - \mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}n\theta} - \mathrm{e}^{\mathrm{i}(n+1)\theta}}{2 - (\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})} \\
&= \frac{1}{2}\left[1 + \frac{\sin\{(2n+1)\theta/2\}}{\sin(\theta/2)}\right] + \frac{\mathrm{i}[\cos(\theta/2) - \cos\{(2n+1)\theta/2\}]}{2\sin(\theta/2)}. \qquad (6.58)
\end{aligned}$$

Taking the real and imaginary parts yields

$$1+2\sum_{k=1}^{n}\cos k\theta = \frac{\sin\{(2n+1)\theta/2\}}{\sin(\theta/2)}, \tag{6.59}$$

which proves the right-hand side of Eq. (6.56), and

$$\begin{aligned} 2\sum_{k=1}^{n}\sin k\theta &= \cot\left(\frac{\theta}{2}\right) - \frac{\cos\{(2n+1)\theta/2\}}{\sin(\theta/2)} \\ &= \sin n\theta + (1-\cos n\theta)\cot(\theta/2). \end{aligned} \tag{6.60}$$

From Eq. (6.56), and using Eqs. (6.50) and (6.51),

$$\begin{aligned} \mathcal{H}D_n(\theta) &= \frac{1}{2\pi}P\int_{-\pi}^{\pi}\left\{1+2\sum_{k=1}^{n}\cos ks\right\}\cot\left(\frac{\theta-s}{2}\right)\mathrm{d}s \\ &= 2\sum_{k=1}^{n}\frac{1}{2\pi}P\int_{-\pi}^{\pi}\cos ks\cot\left(\frac{\theta-s}{2}\right)\mathrm{d}s \\ &= 2\sum_{k=1}^{n}\sin k\theta, \end{aligned} \tag{6.61}$$

and hence

$$\mathcal{H}D_n(\theta) = \sin n\theta + (1-\cos n\theta)\cot(\theta/2). \tag{6.62}$$

If a periodic function f is expressed as a Fourier series, then the partial sum of the first n terms of the series can be written in terms of an integral involving f and the Dirichlet kernel. A closely related result can be given for the sum of the first n terms of the conjugate Fourier series. In this case, the Dirichlet kernel is replaced by Dirichlet's conjugate kernel given in Eq. (6.62). The reader can try to construct the appropriate formulas, or pursue further discussion in Zygmund (1968, Vol. I, p. 49).

The Fejér kernel is considered next, and this is defined by

$$F_n(\theta) = \frac{1}{n+1}\sum_{k=0}^{n}D_k(\theta). \tag{6.63}$$

The sum involved in this definition can be simplified as follows:

$$F_n(\theta) = \frac{1}{n+1}\left[\frac{\sin\{(n+1)\theta/2\}}{\sin(\theta/2)}\right]^2. \tag{6.64}$$

To prove this, start with

$$\begin{aligned}\sum_{k=0}^{n} \sin\left\{\frac{(2k+1)\theta}{2}\right\} \sin\left(\frac{\theta}{2}\right) &= \frac{1}{2}\sum_{k=0}^{n} \cos k\theta - \frac{1}{2}\sum_{k=1}^{n+1} \cos k\theta \\ &= \frac{1}{2}\{1 - \cos(n+1)\theta\} \\ &= \sin^2\left(\frac{(n+1)\theta}{2}\right). \end{aligned} \tag{6.65}$$

Dividing by $\sin^2(\theta/2)$ gives the required result. The Hilbert transform of $F_n(\theta)$ can be evaluated by taking advantage of the result derived for $\mathcal{H}D_n(\theta)$, that is

$$\begin{aligned}\mathcal{H}F_n(\theta) &= \frac{1}{n+1}\sum_{k=0}^{n} \mathcal{H}D_k(\theta) \\ &= \frac{1}{n+1}\sum_{k=0}^{n} \left\{\sin k\theta + (1 - \cos k\theta)\cot\left(\frac{\theta}{2}\right)\right\}, \end{aligned} \tag{6.66}$$

which simplifies, on taking note of Eqs. (6.59) and (6.60), to

$$\mathcal{H}F_n(\theta) = \cot\left(\frac{\theta}{2}\right) - \frac{\sin(n+1)\theta}{2(n+1)\sin^2(\theta/2)}. \tag{6.67}$$

This result can also be written as follows:

$$\mathcal{H}F_n(\theta) = \cot\left(\frac{\theta}{2}\right) - \frac{\cot\{(n+1)\theta/2\}}{n+1}\left[\frac{\sin\{(n+1)\theta/2\}}{\sin(\theta/2)}\right]^2, \tag{6.68}$$

from which it follows that

$$\mathcal{H}F_n(\theta) = \cot\left(\frac{\theta}{2}\right) - \cot\left(\frac{(n+1)\theta}{2}\right)F_n(\theta). \tag{6.69}$$

Motivation for the determination of $\mathcal{H}F_n(\theta)$ is given by the following development. Consider the series

$$S = \sum_{k=0}^{\infty} a_k, \tag{6.70}$$

and let the partial sums be given by

$$s_n = \sum_{k=0}^{n} a_k. \tag{6.71}$$

Let

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}, \tag{6.72}$$

then if

$$\lim_{n\to\infty} \sigma_n = S, \tag{6.73}$$

the series $\sum_{k=0}^{\infty} a_k$ is Cesàro summable or $(C,1)$ summable.

Let f represent a periodic function with period 2π that is integrable on the interval $[0, 2\pi]$, and denote the partial sums of the Fourier series expansion of f by

$$s_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}. \tag{6.74}$$

Employing Eqs. (6.56) and (6.63), the σ_n are evaluated to be

$$\begin{aligned}\sigma_n(x) &= \frac{1}{2\pi(n+1)} \int_0^{2\pi} f(t) \sum_{k=0}^{n} \frac{\sin\{(2n+1)(x-t)/2\}}{\sin\{(x-t)/2\}}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t) F_n(x-t) dt. \end{aligned} \tag{6.75}$$

An important result is that the Cesàro means of the Fourier series of f converge to $(1/2)\{f(x_0+) + f(x_0-)\}$ for every x_0 where the right and left limits at the point x_0 exist. This result has implications for the approximation of continuous functions on the interval $[0, 2\pi]$ by trigonometric polynomials. An analog of the preceding result applies to the conjugate Fourier series of f, with f replaced by $\mathcal{H}f$, $\sigma_n(x)$ by $\tilde{\sigma}_n(x)$, and $F_n(x-t)$ by $\mathcal{H}F_n(x-t)$.

6.6 The inversion formula

In this and the next few sections, some of the basic properties for the Hilbert transform of periodic functions are enunciated. Not surprisingly, a number of the formulas derived have a close similarity or are identical in form, allowing for the obvious difference in integration range, to results given in Chapter 4. A number of the more important relationships are covered, and the reader is left to construct some of the other results, or to consult a few of the references given in the chapter end-notes to obtain additional results. In what follows, f is a periodic function with period 2π and it is assumed that $f \in L^p(\mathbb{T})$, with $1 < p < \infty$. Because of the periodic nature of f, integrals can be evaluated over the interval $[\alpha, \alpha + 2\pi)$, with α a constant, which in a number of places will be taken as $-\pi$. The Fourier series representation of f is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{6.76}$$

and the Hilbert transform of f is given by Eq. (6.38). The *allied series* of Eq. (6.76) is given by

$$g(x) \sim \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx), \tag{6.77}$$

and the *conjugate* function g is given by

$$g(x) = \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(s) \cot\left(\frac{x-s}{2}\right) \mathrm{d}s. \tag{6.78}$$

The terms "allied series" and "conjugate series" are used synonymously. The allied series is also given in the literature with the opposite sign, and that reflects the opposite sign choice to that employed in Eq. (6.77). The allied series Eq. (6.77) follows from Eq. (6.78) on noting the result given in Eq. (6.51) and

$$\frac{1}{2\pi} P \int_{-\pi}^{\pi} \sin ns \cot\left(\frac{x-s}{2}\right) \mathrm{d}s = -\cos nx, \quad \text{for } n \in \mathbb{Z}^+. \tag{6.79}$$

Equation (6.79) can be established in the following manner. For integer $n \geq 1$,

$$\begin{aligned}
\frac{1}{2\pi} P \int_{-\pi}^{\pi} \sin ns \cot\left(\frac{x-s}{2}\right) \mathrm{d}s &= \frac{1}{\pi} P \int_{-\pi}^{\pi} \sin ns \sum_{m=1}^{\infty} \sin(mx - ms) \mathrm{d}s \\
&= -\frac{1}{\pi} \sum_{m=1}^{\infty} \cos mx \int_{-\pi}^{\pi} \sin ns \, \sin ms \, \mathrm{d}s \\
&= -\sum_{m=1}^{\infty} \cos mx \, \delta_{mn} \\
&= -\cos nx.
\end{aligned} \tag{6.80}$$

The reciprocal formula to Eq. (6.78) holds for $g \in L^p(-\pi, \pi)$ with $p > 1$, and is given by

$$f(x) - \frac{1}{2}a_0 = -\frac{1}{2\pi} P \int_{-\pi}^{\pi} g(s) \cot\left(\frac{x-s}{2}\right) \mathrm{d}s, \tag{6.81}$$

that is,

$$f(x) = -\frac{1}{2\pi} P \int_{-\pi}^{\pi} g(s) \cot\left(\frac{x-s}{2}\right) \mathrm{d}s + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \mathrm{d}s. \tag{6.82}$$

This represents the inversion formula for Eq. (6.78). When the constant a_0 equals zero, the skew-symmetric character of the Hilbert transform reciprocal relations (see Section 4.3) is obtained for the case of periodic functions. Alternatively, a new function $h = f - (1/2)a_0$ can be defined, thereby obtaining a skew-symmetric pair of reciprocal relations involving g and h. The analog of Eq. (6.82) for the case $g \in L^p_{2\tau}$ and $p > 1$ (period 2τ) is given by

$$f(x) = -\frac{1}{2\tau} P \int_{-\tau}^{\tau} g(s) \cot\left(\frac{x-s}{2}\right) \mathrm{d}s + \frac{1}{2\tau} \int_{-\tau}^{\tau} f(s) \mathrm{d}s. \tag{6.83}$$

6.7 Even and odd periodic functions

In this section the standard Hilbert transform on the circle is put into a slightly different form, taking account of the even or odd character of the periodic function f. Let $g(x) = \mathcal{H}f(x)$, then

$$\begin{aligned} g(x) &= \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(s) \cot\left(\frac{x-s}{2}\right) \mathrm{d}s \\ &= -\frac{1}{2\pi} P \int_{-\pi}^{\pi} f(-s) \cot\left(\frac{-x-s}{2}\right) \mathrm{d}s. \end{aligned} \tag{6.84}$$

If f is an even function, $g(x) = -g(-x)$, that is, $\mathcal{H}f$ is an odd function; if f is an odd function, $g(x) = g(-x)$, and $\mathcal{H}f$ is an even function.

If f is an even function, then, from Eq. (6.38) and the trigonometric identity

$$\cot\left(\frac{\theta+\phi}{2}\right) + \cot\left(\frac{\theta-\phi}{2}\right) = \frac{2\sin\theta}{\cos\phi - \cos\theta}, \tag{6.85}$$

it follows that

$$\mathcal{H}f(\theta) = \frac{\sin\theta}{\pi} P \int_0^{\pi} \frac{f(\phi)\mathrm{d}\phi}{\cos\phi - \cos\theta}. \tag{6.86}$$

If f is an odd function, then, on employing the identity

$$\cot\left(\frac{\theta-\phi}{2}\right) - \cot\left(\frac{\theta+\phi}{2}\right) = \frac{2\sin\phi}{\cos\phi - \cos\theta}, \tag{6.87}$$

it follows from Eq. (6.38) that

$$\mathcal{H}f(\theta) = \frac{1}{\pi} P \int_0^{\pi} \frac{f(\phi)\sin\phi\,\mathrm{d}\phi}{\cos\phi - \cos\theta}. \tag{6.88}$$

Equations (6.86) and (6.88) represent the Hilbert transforms $\mathcal{H}_{\mathrm{e}}f$ and $\mathcal{H}_{\mathrm{o}}f$ for even and odd periodic functions, respectively. The context should make it clear to the reader whether the subscript on $\mathcal{H}$ designates the even operator (subscript e), the odd operator (subscript o), or a general period other than 2π; recall Eq. (3.286). Typically, the subscripts p and τ are used to specify the period. These formulas are the analogs of $H_{\mathrm{e}}f$ and $H_{\mathrm{o}}f$ for functions in $L^p(\mathbb{R})$ defined in Section 4.2.

6.8 Scale changes

Let $g(x) = \mathcal{H}f(x)$ and $h(x) = f(x+a)$, where a is a real constant; then

$$\begin{aligned}\mathcal{H}f(x+a) \equiv \mathcal{H}h(x) &= \frac{1}{2\pi}P\int_{-\pi}^{\pi} h(s)\cot\left(\frac{x-s}{2}\right)\mathrm{d}s\\ &= \frac{1}{2\pi}P\int_{-\pi}^{\pi} f(s+a)\cot\left(\frac{x-s}{2}\right)\mathrm{d}s\\ &= \frac{1}{2\pi}P\int_{-\pi+a}^{\pi+a} f(s)\cot\left(\frac{x+a-s}{2}\right)\mathrm{d}s\\ &= \frac{1}{2\pi}P\int_{-\pi}^{\pi} f(s)\cot\left(\frac{x+a-s}{2}\right)\mathrm{d}s, \end{aligned} \tag{6.89}$$

where the periodic property of f has been employed, and so

$$\mathcal{H}f(x+a) = g(x+a). \tag{6.90}$$

If τ_a denotes the translation operator,

$$\mathcal{H}\tau_a f(x) = \tau_a \mathcal{H}f(x) = g(x-a), \tag{6.91}$$

that is, the following commutator condition holds:

$$[\mathcal{H}, \tau_a] = 0. \tag{6.92}$$

If R denotes the reflection operator, then $Rf(x) = f(-x)$, and, on setting $h(x) = Rf(x)$,

$$\begin{aligned}\mathcal{H}Rf(x) \equiv \mathcal{H}h(x) &= \frac{1}{2\pi}P\int_{-\pi}^{\pi} h(s)\cot\left(\frac{x-s}{2}\right)\mathrm{d}s\\ &= \frac{1}{2\pi}P\int_{-\pi}^{\pi} f(-s)\cot\left(\frac{x-s}{2}\right)\mathrm{d}s\end{aligned}$$

$$= -\frac{1}{2\pi} P \int_{-\pi}^{\pi} f(s) \cot\left(\frac{-x-s}{2}\right) ds$$

$$= -g(-x), \tag{6.93}$$

and so

$$\mathcal{H}Rf(x) = -R\mathcal{H}f(x). \tag{6.94}$$

Hence the following anticommutator result holds:

$$\{\mathcal{H}, R\} = 0. \tag{6.95}$$

The reader is invited to explore whether or not the dilation operator commutes with the Hilbert transform operator on the circle.

6.9 Parseval-type formulas

It is straightforward to show, using Eqs. (6.76) and (6.78), that

$$\int_{-\pi}^{\pi} |f(s)|^2 \, ds - \frac{\pi a_0^2}{2} = \int_{-\pi}^{\pi} |\mathcal{H}f(s)|^2 \, ds, \tag{6.96}$$

which is the analog of the Parseval formula (see Eq. (4.172)) for the case where f is a periodic function. This result can also be written in the following form:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(s)|^2 \, ds = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) ds \right|^2 + \frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathcal{H}f(s)|^2 \, ds, \tag{6.97}$$

which is sometimes referred to as a special case of *Hilbert's formula*.

Let $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$, for $1 < p < \infty$, and let $p^{-1} + q^{-1} = 1$; then

$$\int_{-\pi}^{\pi} g(s)\mathcal{H}f(s) ds = -\int_{-\pi}^{\pi} f(s)\mathcal{H}g(s) ds. \tag{6.98}$$

This is the analog of the Parseval-type formula Eq. (4.176). Also,

$$\int_{-\pi}^{\pi} f(s)\overline{g(s)} ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) ds \int_{-\pi}^{\pi} \overline{g(s)} ds + \int_{-\pi}^{\pi} \mathcal{H}f(s)\overline{\mathcal{H}g(s)} ds. \tag{6.99}$$

Equation (6.97) follows directly from this result. Equations (6.98) and (6.99) are referred to as Hilbert's formulas. Equation (6.98) can be established by starting with

$$\mathcal{F}\mathcal{H}f(n) = -\mathrm{i}\,\mathrm{sgn}\, n\hat{f}(n), \quad \text{for } n \in \mathbb{Z}, \tag{6.100}$$

where $\hat{f}$ is used as a shorthand to denote the Fourier transform:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\mathrm{e}^{-\mathrm{i}ns}\,\mathrm{d}s, \quad \text{for } n \in \mathbb{Z}. \tag{6.101}$$

Equation (6.100) is discussed in detail in Section 6.11. It follows from Eq. (6.100) and the result

$$\mathcal{H}(\mathrm{e}^{-\mathrm{i}nx})(s) = \mathrm{i}\ \mathrm{sgn}\ n\,\mathrm{e}^{-\mathrm{i}ns}, \tag{6.102}$$

that

$$\int_{-\pi}^{\pi} \mathcal{H}f(s)\mathrm{e}^{-\mathrm{i}ns}\mathrm{d}s = -\int_{-\pi}^{\pi} f(s)\mathcal{H}(\mathrm{e}^{-\mathrm{i}ns})\mathrm{d}s. \tag{6.103}$$

Equation (6.103) can be generalized as follows:

$$\int_{-\pi}^{\pi} \mathcal{H}f(s)\sum_{n=-\infty}^{\infty} a_n\mathrm{e}^{-\mathrm{i}ns}\,\mathrm{d}s = -\int_{-\pi}^{\pi} f(s)\mathcal{H}\left(\sum_{n=-\infty}^{\infty} a_n\mathrm{e}^{-\mathrm{i}ns}\right)\mathrm{d}s. \tag{6.104}$$

Let

$$g(s) = \sum_{n=-\infty}^{\infty} a_n\mathrm{e}^{-\mathrm{i}ns}, \tag{6.105}$$

then

$$\int_{-\pi}^{\pi} \mathcal{H}f(s)g(s)\mathrm{d}s = -\int_{-\pi}^{\pi} f(s)\mathcal{H}g(s)\mathrm{d}s. \tag{6.106}$$

To prove Eq. (6.99), consider the following two trigonometric polynomials:

$$p_n(x) = \sum_{k=-n}^{n} a_n\mathrm{e}^{\mathrm{i}kx} \quad \text{and} \quad q_m(x) = \sum_{j=-m}^{m} b_j\mathrm{e}^{\mathrm{i}jx}, \tag{6.107}$$

with $m > n$. First note that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} p_n(x)\mathrm{d}x \tag{6.108}$$

and

$$\bar{b}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{q_m(x)}\mathrm{d}x, \tag{6.109}$$

and further that

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{H}p_n(x)\,\overline{\mathcal{H}q_m(x)}\mathrm{d}x &= \frac{1}{2\pi}\sum_{k=-n}^{n}a_k\sum_{j=-m}^{m}\bar{b}_j\int_{-\pi}^{\pi}\mathcal{H}(\mathrm{e}^{\mathrm{i}\,kx})\mathcal{H}(\mathrm{e}^{-\mathrm{i}jx})\mathrm{d}x \\
&= \frac{1}{2\pi}\sum_{k=-n}^{n}a_k\ \operatorname{sgn}\,k\sum_{j=-m}^{m}\bar{b}_j\operatorname{sgn}j\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(k-j)x}\,\mathrm{d}x \\
&= \sum_{k=-n}^{n}a_k\ \operatorname{sgn}\,k\sum_{j=-m}^{m}\bar{b}_j\operatorname{sgn}j\,\delta_{kj} \\
&= \sum_{k=-n}^{n}\{a_k\ \operatorname{sgn}\,k\}\{\bar{b}_k\ \operatorname{sgn}k\} \\
&= \sum_{\substack{k=-n\\(k\neq 0)}}^{n}a_k\bar{b}_k,
\end{aligned}
\tag{6.110}
$$

using $\operatorname{sgn}0 = 0$. It follows from Eq. (6.107) that

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}p_n(x)\overline{q_m(x)}\mathrm{d}x &= \sum_{k=-n}^{n}a_k\sum_{j=-m}^{m}\bar{b}_j\delta_{kj} \\
&= \sum_{k=-n}^{n}a_k\bar{b}_k \\
&= a_0\bar{b}_0 + \sum_{\substack{k=-n\\(k\neq 0)}}^{n}a_k\bar{b}_k \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}p_n(x)\mathrm{d}x\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{q_n(x)}\mathrm{d}x \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{H}p_n(x)\,\overline{\mathcal{H}q_n(x)}\mathrm{d}x
\end{aligned}
\tag{6.111}
$$

The final result can be generalized by the replacements $p_n \to f$ and $q_n \to g$ leading to the required result, Eq. (6.99).

6.10 Convolution property

If the functions f and h are periodic with period 2π and belong to $L^p_{2\pi}$, $1 < p < \infty$, and $L^q_{2\pi}$, respectively, with q the conjugate exponent, then

$$
\mathcal{H}\{f * h\}(x) = \mathcal{H}f(x) * h(x) = f(x) * \mathcal{H}h(x). \tag{6.112}
$$

If $f \in L^1_{2\pi}$ and $h \in L^p_{2\pi}$, $1 < p < \infty$, then

$$\mathcal{H}\{f * h\}(x) = f(x) * \mathcal{H}h(x). \tag{6.113}$$

If $f \in L^1_{2\pi}$, then Eq. (6.112) also holds provided that $\mathcal{H}f \in L^1_{2\pi}$. From the definition of $\mathcal{H}f$, Eq. (6.38),

$$\begin{aligned}
\mathcal{H}\{f * h\}(x) &= \frac{1}{2\pi} P \int_{-\pi}^{\pi} \left\{ \int_{-\pi}^{\pi} f(s) h(t-s) \mathrm{d}s \right\} \cot\left(\frac{x-t}{2}\right) \mathrm{d}t \\
&= \int_{-\pi}^{\pi} f(s) \left\{ \frac{1}{2\pi} P \int_{-\pi}^{\pi} h(t-s) \cot\left(\frac{x-t}{2}\right) \mathrm{d}t \right\} \mathrm{d}s \\
&= \int_{-\pi}^{\pi} f(s) \left\{ \frac{1}{2\pi} P \int_{-\pi-s}^{\pi-s} h(w) \cot\left(\frac{x-s-w}{2}\right) \mathrm{d}w \right\} \mathrm{d}s \\
&= \int_{-\pi}^{\pi} f(s) \left\{ \frac{1}{2\pi} P \int_{-\pi}^{\pi} h(w) \cot\left(\frac{x-s-w}{2}\right) \mathrm{d}w \right\} \mathrm{d}s \\
&= \int_{-\pi}^{\pi} f(s) \mathcal{H}h(x-s) \mathrm{d}s \\
&= \{f * \mathcal{H}h\}(x).
\end{aligned} \tag{6.114}$$

The interchange of integration order is justified since f, h, and $\mathcal{H}h$ are measurable functions and only a single principal value integral is involved (see Section 2.13); also the periodic property of the functions has been used to simplify the limits. The other half of Eq. (6.112) can be established as follows:

$$\begin{aligned}
\{\mathcal{H}f * h\}(x) &= \int_{-\pi}^{\pi} \mathcal{H}f(s) h(x-s) \mathrm{d}s \\
&= \int_{-\pi}^{\pi} \left\{ \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(w) \cot\left(\frac{s-w}{2}\right) \mathrm{d}w \right\} h(x-s) \mathrm{d}s \\
&= \int_{-\pi}^{\pi} \left\{ \frac{1}{2\pi} P \int_{-\pi}^{\pi} h(x-s) \cot\left(\frac{s-w}{2}\right) \mathrm{d}s \right\} f(w) \mathrm{d}w \\
&= \int_{-\pi}^{\pi} \left\{ \frac{1}{2\pi} P \int_{-\pi+x}^{\pi+x} h(t) \cot\left(\frac{x-t-w}{2}\right) \mathrm{d}t \right\} f(w) \mathrm{d}w \\
&= \int_{-\pi}^{\pi} f(w) \mathcal{H}h(x-t) \mathrm{d}w,
\end{aligned} \tag{6.115}$$

and so

$$\{\mathcal{H}f * h\}(x) = \{f * \mathcal{H}h\}(x). \tag{6.116}$$

6.11 Connection with Fourier transforms

Let f denote a complex-valued function in $L(-\pi,\pi)$ and let the Fourier transform of f be defined by Eq. (6.101). Formally, the Fourier series of f can be written as follows:

$$f(s) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\mathrm{e}^{\mathrm{i}ns}. \tag{6.117}$$

The conjugate series of $f(s)$, denoted by $\tilde{f}(s)$, which is a commonly employed shorthand for $\mathcal{H}f(s)$, is formally

$$\tilde{f}(s) \sim -\mathrm{i} \sum_{n=-\infty}^{\infty} \hat{f}(n)\,\mathrm{sgn}\,n\,\mathrm{e}^{\mathrm{i}ns}. \tag{6.118}$$

From the discussion of Section 3.13, it is clear that Eq. (6.118) may not in general be a Fourier series.

If f is real-valued, then

$$\begin{aligned}\bar{\hat{f}}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(s)\mathrm{e}^{-\mathrm{i}ns}}\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\mathrm{e}^{\mathrm{i}ns}\,\mathrm{d}s \\ &= \hat{f}(-n).\end{aligned} \tag{6.119}$$

Hence,

$$\hat{f}(n) + \hat{f}(-n) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos ns\,\mathrm{d}s \tag{6.120}$$

and

$$\mathrm{i}\{\hat{f}(n) - \hat{f}(-n)\} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin ns\,\mathrm{d}s. \tag{6.121}$$

If the integrals appearing in Eqs. (6.120) and (6.121) are denoted by the coefficients a_n and b_n, respectively, then Eq. (6.117) can be written, on replacing the correspondence with an equality symbol, as follows:

$$f(s) = \hat{f}(0) + \sum_{n=1}^{\infty}\{a_n \cos ns + b_n \sin ns\}, \tag{6.122}$$

with $\hat{f}(0) = a_0/2$. The conjugate series is given by

$$\tilde{f}(s) = \sum_{n=1}^{\infty} \{a_n \sin ns - b_n \cos ns\}. \tag{6.123}$$

If $f \in L^p(\mathbb{T})$ with $1 < p < \infty$,

$$\hat{\tilde{f}}(n) = -\mathrm{i}\,\mathrm{sgn}\, n \hat{f}(n), \quad \text{for } n \in \mathbb{Z}, \tag{6.124}$$

which is Eq. (6.100) written in compact notation. To establish this result, start with Eq. (6.118); then

$$\begin{aligned}
\hat{\tilde{f}}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(s) \mathrm{e}^{-\mathrm{i}ns} \mathrm{d}s \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ -\mathrm{i} \sum_{m=-\infty}^{\infty} \mathrm{sgn}\, m\, \hat{f}(m) \mathrm{e}^{\mathrm{i}ms} \right\} \mathrm{e}^{-\mathrm{i}ns}\, \mathrm{d}s \\
&= -\frac{\mathrm{i}}{2\pi} \sum_{m=-\infty}^{\infty} \mathrm{sgn}\, m \hat{f}(m) \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(m-n)s}\, \mathrm{d}s \\
&= -\mathrm{i} \sum_{m=-\infty}^{\infty} \mathrm{sgn}\ m \hat{f}\ (m) \delta_{m,n} \\
&= -\mathrm{i}\,\mathrm{sgn}\, n \hat{f}(n),
\end{aligned} \tag{6.125}$$

which is the desired result. From Eq. (6.124), with $n = 0$ and the assignment $\mathrm{sgn}\, 0 = 0$, it follows that

$$\int_{-\pi}^{\pi} \tilde{f}(s) \mathrm{d}s = 0. \tag{6.126}$$

6.12 Orthogonality property

If $f \in L^p(-\pi, \pi)$ with $1 < p < \infty$, then

$$\int_{-\pi}^{\pi} f(s) \tilde{f}(s) \mathrm{d}s = 0. \tag{6.127}$$

This result is called the orthogonality property. Equation (6.127) is derived in the following manner:

$$\begin{aligned}
\int_{-\pi}^{\pi} f(s)\tilde{f}(s)\mathrm{d}s &= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \hat{f}(n)\mathrm{e}^{\mathrm{i}ns} \sum_{m=-\infty}^{\infty} \hat{\tilde{f}}(m)\mathrm{e}^{\mathrm{i}ms}\,\mathrm{d}s \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n) \sum_{m=-\infty}^{\infty} \hat{\tilde{f}}(m) \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(n+m)s}\,\mathrm{d}s \\
&= 2\pi \sum_{n=-\infty}^{\infty} \hat{f}(n) \sum_{m=-\infty}^{\infty} \hat{\tilde{f}}(m)\delta_{n,-m} \\
&= 2\pi \sum_{m=-\infty}^{\infty} \hat{f}(-m)\,\hat{\tilde{f}}(m) \\
&= 2\pi \sum_{m=-\infty}^{\infty} \hat{f}(-m)\{-\mathrm{i}\,\mathrm{sgn}\,m\hat{f}(m)\} \\
&= -2\pi\mathrm{i}\left\{ \sum_{m=-\infty}^{-1} \mathrm{sgn}\,m\hat{f}(-m)\hat{f}(m) + \sum_{m=1}^{\infty}\hat{f}(-m)\hat{f}(m)\right\} \\
&= 0,
\end{aligned} \tag{6.128}$$

and Eq. (6.124) has been employed.

6.13 Eigenvalues and eigenfunctions of the Hilbert transform operator

On the unit circle the functions

$$\phi_n(\theta) = \frac{1}{\surd(2\pi)}\mathrm{e}^{\mathrm{i}n\theta} \tag{6.129}$$

form an orthonormal basis set, that is

$$\int_{-\pi}^{\pi} \phi_n^*(\theta)\phi_m(\theta)\mathrm{d}\theta = \delta_{nm}. \tag{6.130}$$

Applying the Hilbert transform operator to $\phi_n(\theta)$ yields

$$\begin{aligned}
\mathcal{H}\phi_n(\theta) &= \mathcal{H}\frac{1}{\surd(2\pi)}\mathrm{e}^{\mathrm{i}n\theta} \\
&= \frac{1}{\surd(2\pi)}\{-\mathrm{i}\,\mathrm{sgn}\,n\,\mathrm{e}^{\mathrm{i}n\theta}\} \\
&= -\mathrm{i}\,\mathrm{sgn}\,n\,\phi_n(\theta).
\end{aligned} \tag{6.131}$$

For $n > 0, \mathcal{H}\phi_n(\theta) = -\mathrm{i}\phi_n(\theta)$, that is the eigenvalue of $\mathcal{H}$ is $-\mathrm{i}$, and, for $n < 0$, $\mathcal{H}\phi_n(\theta) = \mathrm{i}\phi_n(\theta)$, and the eigenvalue of $\mathcal{H}$ is i. For c a constant, it follows from Eq. (6.50) that $\mathcal{H}c = 0$, and hence $\mathcal{H}\phi_0(\theta) = 0$, which can be rewritten as $\mathcal{H}\phi_0(\theta) = 0\phi_0(\theta)$, and so ϕ_0 is an eigenfunction of $\mathcal{H}$ with eigenvalue zero. This latter situation is the most restrictive case.

6.14 Projection operators

On the circle $\mathbb{T}$ the projector P_+ is defined by

$$P_+f = \frac{1}{2}(I + \mathrm{i}\mathcal{H})f + \frac{1}{2}\mathcal{F}f(0) \tag{6.132}$$

and P_- is given by

$$P_-f = \frac{1}{2}(I - \mathrm{i}\mathcal{H})f - \frac{1}{2}\mathcal{F}f(0). \tag{6.133}$$

The action of P_+ and P_- on $\sum_{n=-\infty}^{\infty}\hat{f}(n)\mathrm{e}^{\mathrm{i}nt}$ gives

$$P_+\sum_{n=-\infty}^{\infty}\hat{f}(n)\mathrm{e}^{\mathrm{i}nt} = \sum_{n=0}^{\infty}\hat{f}(n)\mathrm{e}^{\mathrm{i}nt} \tag{6.134}$$

and

$$P_-\sum_{n=-\infty}^{\infty}\hat{f}(n)\mathrm{e}^{\mathrm{i}nt} = \sum_{n=-\infty}^{-1}\hat{f}(n)\mathrm{e}^{\mathrm{i}nt}. \tag{6.135}$$

The operators P_+ and P_- satisfy the idempotent conditions

$$P_+^2 = P_+ \tag{6.136}$$

and

$$P_-^2 = P_-. \tag{6.137}$$

To establish Eq. (6.136), let $\mathcal{F}_{n=0}$ signify the evaluation of the Fourier transform at $n = 0$ (recall Eq. (6.101)), then

$$\begin{aligned}
P_+^2 &= \frac{1}{4}\{(I + \mathrm{i}\mathcal{H}) + \mathcal{F}_{n=0}\}\{(I + \mathrm{i}\mathcal{H}) + \mathcal{F}_{n=0}\} \\
&= \frac{1}{4}\{I + 2\mathrm{i}\mathcal{H} - \mathcal{H}^2 + 2\mathcal{F}_{n=0} + \mathcal{F}_{n=0}\mathrm{i}\mathcal{H} + \mathrm{i}\mathcal{H}\mathcal{F}_{n=0} + \mathcal{F}_{n=0}^2\}
\end{aligned}$$

$$= P_+ + \frac{1}{4}\left\{\mathcal{F}_{n=0}\mathrm{i}\mathcal{H} + \mathrm{i}\mathcal{H}\mathcal{F}_{n=0} - \mathcal{F}_{n=0} + \mathcal{F}_{n=0}^2\right\}$$

$$= P_+ + \frac{1}{4}\{\mathcal{F}_{n=0}\mathrm{i}\mathcal{H} + \mathrm{i}\mathcal{H}\mathcal{F}_{n=0}\}$$

$$= P_+, \tag{6.138}$$

where Eqs. (6.34), (6.126), and (6.50) have been used to simplify $\mathcal{H}^2$, $\mathcal{F}_{n=0}\mathcal{H}$, and $\mathcal{H}\mathcal{F}_{n=0}$, respectively. The corresponding result for P_-^2 can be proved in a similar fashion.

6.15 The Hardy–Poincaré–Bertrand formula

In this section, the Tricomi relation for the Hilbert transform on the circle is derived. Also, the Hardy–Poincaré–Bertrand formula for the circle is discussed, and the inversion property of the Hilbert transform on the circle obtained. The Hardy–Poincaré–Bertrand formula for interchanging the order of integration when Cauchy principal value integrals are involved was introduced in Section 2.13. For the real line this takes the following form:

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\phi_1(x)}{x-t}\,\mathrm{d}x\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\phi_2(y)}{y-x}\,\mathrm{d}y$$
$$= \frac{1}{\pi}P\int_{-\infty}^{\infty}\phi_2(y)\mathrm{d}y\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\phi_1(x)}{(x-t)(y-x)}\,\mathrm{d}x - \phi_1(t)\phi_2(t), \tag{6.139}$$

where $\phi_1(x)$ and $\phi_2(x)$ belong to the classes L^p and L^q, respectively, and $p^{-1} + q^{-1} \le 1$. The analog of this formula for the circle is now sought. The key result is:

$$\int_C \frac{\phi_1(x)}{x-t}\,\mathrm{d}x\int_C\frac{\phi_2(y)}{y-x}\,\mathrm{d}y = \int_C \phi_2(y)\mathrm{d}y\int_C\frac{\phi_1(x)}{(x-t)(y-x)}\,\mathrm{d}x - \pi^2\phi_1(t)\phi_2(t), \tag{6.140}$$

where the contour C is the unit circle and the appropriate principal values are understood. A more general statement of the Hardy–Poincaré–Bertrand formula allows C to be a smooth arc or contour. Muskhelishvili (1992, chap. 3) gives a detailed discussion of this form. Starting from Eq. (6.140) and introducing the substitutions

$$x = \mathrm{e}^{\mathrm{i}\theta}, \quad y = \mathrm{e}^{\mathrm{i}\theta_1}, \quad t = \mathrm{e}^{\mathrm{i}\theta_0}, \tag{6.141}$$

$$f(\theta) = \phi_2(\mathrm{e}^{\mathrm{i}\theta}), \quad g(\theta) = \phi_1(\mathrm{e}^{\mathrm{i}\theta}), \tag{6.142}$$

and noting that

$$\frac{\mathrm{d}x}{x-t} = \frac{1}{2}\left\{\cot\left(\frac{\theta-\theta_0}{2}\right) + \mathrm{i}\right\}\mathrm{d}\theta, \tag{6.143}$$

leads to

$$P\int_0^{2\pi} g(\theta)\frac{1}{2}\left\{\cot\left(\frac{\theta-\theta_0}{2}\right)+\mathrm{i}\right\}\mathrm{d}\,\theta P\int_0^{2\pi} f(\theta_1)\frac{1}{2}\left\{\cot\left(\frac{\theta_1-\theta}{2}\right)+\mathrm{i}\right\}\mathrm{d}\theta_1$$
$$=\frac{1}{4}P\int_0^{2\pi} f(\theta_1)\mathrm{d}\theta_1\,P\int_0^{2\pi} g(\theta)\left[\cot\left(\frac{\theta_1-\theta}{2}\right)\cot\left(\frac{\theta-\theta_0}{2}\right)-1\right.$$
$$\left.+\mathrm{i}\left\{\cot\left(\frac{\theta_1-\theta}{2}\right)+\cot\left(\frac{\theta-\theta_0}{2}\right)\right\}\right]\mathrm{d}\theta-\pi^2 f(\theta_0)g(\theta_0). \tag{6.144}$$

If the following relations hold:

$$\int_0^{2\pi} f(\theta)\mathrm{d}\theta=0 \tag{6.145}$$

and

$$\int_0^{2\pi} g(\theta)\mathrm{d}\theta=0, \tag{6.146}$$

then Eq. (6.144) simplifies as follows:

$$\frac{1}{2\pi}P\int_0^{2\pi} g(\theta)\cot\left(\frac{\theta-\theta_0}{2}\right)\mathrm{d}\theta\,\frac{1}{2\pi}P\int_0^{2\pi} f(\theta_1)\cot\left(\frac{\theta_1-\theta}{2}\right)\mathrm{d}\theta_1$$
$$=-\frac{1}{2\pi}P\int_0^{2\pi} f(\theta_1)\cot\left(\frac{\theta_0-\theta_1}{2}\right)\mathrm{d}\theta_1\,\frac{1}{2\pi}P\int_0^{2\pi} g(\theta)\cot\left(\frac{\theta_1-\theta}{2}\right)\mathrm{d}\theta$$
$$-\frac{1}{2\pi}P\int_0^{2\pi} f(\theta_1)\cot\left(\frac{\theta_0-\theta_1}{2}\right)\mathrm{d}\theta_1\,\frac{1}{2\pi}P\int_0^{2\pi} g(\theta)\cot\left(\frac{\theta-\theta_0}{2}\right)\mathrm{d}\theta$$
$$-f(\theta_0)g(\theta_0). \tag{6.147}$$

The trigonometric identity

$$\cot\left(\frac{\theta-\theta_1}{2}\right)\cot\left(\frac{\theta-\theta_0}{2}\right)=\cot\left(\frac{\theta_1-\theta_0}{2}\right)\left\{\cot\left(\frac{\theta-\theta_1}{2}\right)-\cot\left(\frac{\theta-\theta_0}{2}\right)\right\}-1 \tag{6.148}$$

and the result

$$\int_0^{2\pi} g(\theta)\mathrm{d}\,\theta P\int_0^{2\pi} f(\theta_1)\cot\left(\frac{\theta_1-\theta}{2}\right)\mathrm{d}\theta_1=\int_0^{2\pi} f(\theta_1)\mathrm{d}\theta_1\,P\int_0^{2\pi}$$
$$g(\theta)\cot\left(\frac{\theta_1-\theta}{2}\right)\mathrm{d}\theta \tag{6.149}$$

have been employed to obtain Eq. (6.147). This result can be recast as follows:

$$\mathcal{H}\{g\mathcal{H}f + f\mathcal{H}g\}(\theta) = \mathcal{H}f(\theta)\,\mathcal{H}g(\theta) - f(\theta)g(\theta), \tag{6.150}$$

which can be recognized as the Tricomi relation (see Section 4.16) for the Hilbert transform on the circle.

Start from the relationship

$$\int_C \frac{\mathrm{d}x}{x-t}\int_C \frac{\varphi(x,y)\mathrm{d}y}{y-x} = \int_C \mathrm{d}y\int_C \frac{\varphi(x,y)\mathrm{d}x}{(x-t)(y-x)} - \pi^2\varphi(t,t), \tag{6.151}$$

where the contour C is the unit circle, and the appropriate principal values are understood. This is just an obvious rewriting of Eq. (6.140). Suppose $\varphi(x,y) = \varphi(y)$, then, using the result

$$P\int_C \frac{\mathrm{d}x}{(x-t)(y-x)} = \frac{1}{(y-t)}P\int_C \left\{\frac{1}{x-t} - \frac{1}{x-y}\right\}\mathrm{d}x = 0, \tag{6.152}$$

which follows on employing Eqs. (6.141), (6.143), and (6.50), leads to

$$\int_C \frac{\mathrm{d}x}{x-t}\int_C \frac{\varphi(y)\mathrm{d}y}{y-x} = -\pi^2\varphi(t). \tag{6.153}$$

The substitutions given in Eq. (6.141) are introduced together with $f(\theta) = \varphi(\mathrm{e}^{\mathrm{i}\theta})$. Taking note of the result

$$\int_0^{2\pi} \mathrm{d}\theta P\int_0^{2\pi} \cot\left(\frac{\theta_1-\theta}{2}\right) f(\theta_1)\mathrm{d}\theta_1 = \int_0^{2\pi} f(\theta_1)\mathrm{d}\theta_1 P\int_0^{2\pi} \cot\left(\frac{\theta_1-\theta}{2}\right)\mathrm{d}\theta = 0 \tag{6.154}$$

allows Eq. (6.151) to be simplified as follows:

$$\frac{1}{4}P\int_0^{2\pi} \cot\left(\frac{\theta-\theta_0}{2}\right)\mathrm{d}\theta P\int_0^{2\pi} \cot\left(\frac{\theta_1-\theta}{2}\right) f(\theta_1)\mathrm{d}\theta_1 - \frac{\pi}{2}\int_0^{2\pi} f(\theta)\mathrm{d}\theta = -\pi^2 f(\theta_0), \tag{6.155}$$

which leads to the inversion property of the Hilbert transform on the circle:

$$\mathcal{H}^2 f(\theta) = -f(\theta) + \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{d}\theta. \tag{6.156}$$

For comparison purposes, the inversion property of the Hilbert transform on $\mathbb{R}$ is $H^2 f = -f$ (see Eq. (4.18)). If Eq. (6.145) holds, then Eq. (6.156) simplifies to the same form as the inversion property for the Hilbert transform on $\mathbb{R}$, and repeated application of $\mathcal{H}$ leads to

$$(\mathcal{H}^n f)(x) = \begin{cases} (-1)^{n/2} f(x), & \text{for } n \text{ even} \\ (-1)^{(n-1)/2} (\mathcal{H}f)(x), & \text{for } n \text{ odd,} \end{cases} \tag{6.157}$$

with n an integer ≥ 0.

6.16 A theorem due to Privalov

In this section a theorem due to Privalov is considered. Let Λ_α denote the Lipschitz class of functions that satisfy

$$|f(\theta) - f(\theta_0)| \leq C_f |\theta - \theta_0|^\alpha, \quad \forall \theta, \theta_0 \in \mathbb{T}, \text{ with } 0 < \alpha < 1, \tag{6.158}$$

where C_f is a positive constant depending on f. Privalov (1916c) proved the following. If $f \in \Lambda_\alpha$, then $\mathcal{H}f \in \Lambda_\alpha$. The analogous result for the Hilbert transform on the line was indicated in Section 3.4.1. Let $g(\theta) = \mathcal{H}f(\theta)$. To prove

$$|g(\theta + \Delta\theta) - g(\theta| = O(|\Delta\theta|^\alpha), \tag{6.159}$$

consider first the behavior of $\mathcal{H}f$ near the singularity. Let $h > 0$, and in the sequel C will denote a positive constant, not necessarily the same at each occurrence; then

$$\begin{aligned} &\left| \lim_{\varepsilon \to 0} \int_\varepsilon^h \{[f(\theta + \Delta\theta - t) - f(\theta + \Delta\theta + t)] - [f(\theta - t) - f(\theta + t)]\} \cot\left(\frac{t}{2}\right) dt \right| \\ &\quad \leq \lim_{\varepsilon \to 0} \int_\varepsilon^h \left| [f(\theta + \Delta\theta - t) - f(\theta + \Delta\theta + t)] \cot\left(\frac{t}{2}\right) \right| dt \\ &\quad\quad + \lim_{\varepsilon \to 0} \int_\varepsilon^h \left| [f(\theta - t) - f(\theta + t)] \cot\left(\frac{t}{2}\right) \right| dt. \end{aligned} \tag{6.160}$$

Making use of the condition

$$|f(\theta) - f(\theta_0)| \leq C\, |\theta - \theta_0|^\alpha, \quad \text{for } 0 < \alpha < 1, \tag{6.161}$$

and the inequality $\cot(t/2) < 2/t$, for $0 < t < 2\pi$, yields

$$\begin{aligned} &\left| \lim_{\varepsilon \to 0} \int_\varepsilon^h \{[f(\theta + \Delta\theta - t) - f(\theta + \Delta\theta + t)] - [f(\theta - t) - f(\theta + t)]\} \cot\left(\frac{t}{2}\right) dt \right| \\ &\quad \leq \lim_{\varepsilon \to 0} 2^{2+\alpha} C \int_\varepsilon^h t^{\alpha - 1}\, dt \\ &\quad = O(h^\alpha). \end{aligned} \tag{6.162}$$

Since h is arbitrary, select $h = |\Delta\theta|$, so the behavior of $\mathcal{H}f$ near the singularity is $O(|\Delta\theta|^\alpha)$.

Now,

$$\begin{aligned}
g(\theta + \Delta\theta) - g(\theta) &= \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(t) \left\{ \cot\left(\frac{\theta + \Delta\theta - t}{2}\right) - \cot\left(\frac{\theta - t}{2}\right) \right\} \mathrm{d}t \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \{f(t) - f(\theta + \Delta\theta)\} \cot\left(\frac{\theta + \Delta\theta - t}{2}\right) \mathrm{d}t \\
&\quad - \frac{1}{2\pi} \int_{-\pi}^{\pi} \{f(t) - f(\theta)\} \cot\left(\frac{\theta - t}{2}\right) \mathrm{d}t. \qquad (6.163)
\end{aligned}$$

A consequence of Eq. (6.162) is that the limits for the latter pair of integrals can be restricted to the region away from the singularity, that is, to the regions $(-\pi, \theta - h)$ and $(\theta + h, \pi)$, where $h > 0$. Employing the trigonometric identity

$$\cot\left(\frac{\theta + \Delta\theta - t}{2}\right) = -\frac{\sin(\Delta\theta/2)}{\sin((t - \theta)/2)\sin((t - \theta - \Delta\theta)/2)} + \cot\left(\frac{\theta - t}{2}\right) \qquad (6.164)$$

leads to

$$\begin{aligned}
g(\theta + \Delta\theta) - g(\theta) &= -\frac{1}{2\pi} \left\{ \int_{-\pi}^{\theta - h} + \int_{\theta + h}^{\pi} \left\{ \frac{\{f(t) - f(\theta + \Delta\theta)\}\sin(\Delta\theta/2)}{\sin((t - \theta)/2)\sin((t - \theta - \Delta\theta)/2)} \right\} \mathrm{d}t \right\} \\
&\quad + \frac{\{f(\theta) - f(\theta + \Delta\theta)\}}{2\pi} \left\{ \int_{-\pi}^{\theta - h} + \int_{\theta + h}^{\pi} \left\{ \cot\left(\frac{\theta - t}{2}\right) \right\} \mathrm{d}t \right\} \\
&= -\frac{1}{2\pi} \left\{ \int_{-\pi}^{\theta - h} + \int_{\theta + h}^{\pi} \left\{ \frac{\{f(t) - f(\theta + \Delta\theta)\}\sin(\Delta\theta/2)}{\sin((t - \theta)/2)\sin((t - \theta - \Delta\theta)/2)} \right\} \mathrm{d}t \right\}. \qquad (6.165)
\end{aligned}$$

Making use of the inequalities $\sin x \le x$, for $0 \le x$, $x/\pi \le \sin(x/2)$, for $0 \le x \le \pi$, and the Lipschitz condition for f, yields

$$\begin{aligned}
\left|g(\theta + \Delta\theta) - g(\theta)\right| &\le \frac{1}{2\pi} \int_{-\pi}^{\theta - h} \left| \frac{\{f(t) - f(\theta + \Delta\theta)\}\sin(\Delta\theta/2)}{\sin((t - \theta)/2)\sin((t - \theta - \Delta\theta)/2)} \right| \mathrm{d}t \\
&\quad + \frac{1}{2\pi} \int_{\theta + h}^{\pi} \left| \frac{\{f(t) - f(\theta + \Delta\theta)\}\sin(\Delta\theta/2)}{\sin((t - \theta)/2)\sin((t - \theta - \Delta\theta)/2)} \right| \mathrm{d}t \\
&\le C\,|\Delta\theta| \int_{-\pi}^{\theta - h} \frac{|t - \theta - \Delta\theta|^\alpha \,\mathrm{d}t}{|t - \theta|\,|t - \theta - \Delta\theta|} + C\,|\Delta\theta| \int_{\theta + h}^{\pi} \frac{|t - \theta - \Delta\theta|^\alpha \,\mathrm{d}t}{|t - \theta|\,|t - \theta - \Delta\theta|}
\end{aligned}$$

$$
\begin{aligned}
&= C|\Delta\theta| \left[\int_{-\pi}^{\theta-h} + \int_{\theta+h}^{\pi} \left\{ \frac{1}{|t-\theta|^{2-\alpha}\,|1-\Delta\theta/(t-\theta)|^{1-\alpha}} \right\} \mathrm{d}t \right] \\
&\leq C|\Delta\theta| \left[\int_{-\pi}^{\theta-h} \frac{\mathrm{d}t}{(\theta-t)^{2-\alpha}} + \int_{\theta+h}^{\pi} \frac{\mathrm{d}t}{(t-\theta)^{2-\alpha}} \right] \\
&= \frac{C|\Delta\theta|}{1-\alpha} [2h^{\alpha-1} - (\pi+\theta)^{\alpha-1} - (\pi-\theta)^{\alpha-1}].
\end{aligned}
\tag{6.166}
$$

Since h is arbitrary, select $h = a|\Delta\theta|$, where $a > 1$; Eq. (6.159) follows and hence $\mathcal{H}f \in \Lambda_\alpha$, which is the required result.

6.17 The Marcel Riesz inequality

If $f \in L^p(-\pi,\pi)$, for $1 < p < \infty$, then the Marcel Riesz formula for the case where $f(x)$ is a periodic function (period 2π) is given by the following result:

$$\int_{-\pi}^{\pi} |\mathcal{H}f(s)|^p \, \mathrm{d}s \leq \{\mathfrak{R}_p\}^p \int_{-\pi}^{\pi} |f(s)|^p \, \mathrm{d}s, \tag{6.167}$$

where $\mathfrak{R}_p$ is the Riesz constant, which depends only on p. The proof of this result is examined in four stages. A generalization of the trigonometric inequality given in Eq. (4.405) is first derived, and from this the desired formula for the case $1 < p \leq 2$ is proved. The result is then established for the case $2 < p < \infty$. Finally, the constant $\mathfrak{R}_p$ that is employed is proved to be the best possible, that is (Pichorides, 1972),

$$\mathfrak{R}_p = \begin{cases} \tan(\pi/2p), & 1 < p \leq 2 \\ \cot(\pi/2p), & 2 \leq p < \infty. \end{cases} \tag{6.168}$$

The best value of the constant $\mathfrak{R}_p$ for the case where f is a periodic function coincides with the case for functions defined on $\mathbb{R}$.

If $-\pi/2 \leq \theta \leq \pi/2$, $0 < \gamma < \pi/2$, and $0 < p \leq 2$, $p \neq 1$, then

$$|\sin\theta|^p \leq A_p(\gamma)\cos^p\theta - B_p(\gamma)\cos p\theta, \tag{6.169}$$

with

$$A_p(\gamma) = \frac{\tan^{p-1}\gamma}{\tan(p-1)\gamma} \tag{6.170}$$

and

$$B_p(\gamma) = \frac{\sin^{p-1}\gamma}{\sin(p-1)\gamma}. \tag{6.171}$$

Equation (6.169) is a refined statement of Eq. (4.405). The proof of Eq. (6.169) that follows is based on Pichorides' work. First note that, for $p = 2$, Eq. (6.169) reduces to the obvious trigonometric equality

$$\sin^2\theta = \cos^2\theta - \cos 2\theta, \tag{6.172}$$

and it suffices to restrict consideration to the case $0 < p < 2$ ($p \neq 1$). The functions appearing in Eq. (6.169) are even, and hence the discussion may be restricted to the interval $0 < \theta \leq \pi/2$. Consider the function

$$h(\theta) = \frac{\sin^p\theta - a\cos p\theta}{\cos^p\theta}, \tag{6.173}$$

where a is a real constant. Now,

$$h'(\theta) = \frac{p\sin^{p-1}\theta g(\theta)}{\cos^{p+1}\theta}, \tag{6.174}$$

where

$$g(\theta) = 1 + \frac{a\sin(p-1)\theta}{\sin^{p-1}\theta}. \tag{6.175}$$

If $a \neq 0$, then $g(\theta)$ is strictly monotonic on the interval $0 < \theta \leq \pi/2$, since $g'(\theta)$, given by

$$g'(\theta) = \frac{a(p-1)\sin(2-p)\theta}{\sin^p\theta}, \tag{6.176}$$

is of constant sign in the specified interval. Select $a = -B_p(\gamma)$, then the only solution of $g(\theta) = 0$ is $\theta = \gamma$, and hence $h'(\theta)$ vanishes only at $\theta = \gamma$. Now,

$$h''(\theta)\big|_{\theta=\gamma} = -\frac{p(p-1)\sin^{p-2}\gamma\,\sin(2-p)\gamma}{\sin(p-1)\gamma\,\cos^{p+1}\gamma}, \tag{6.177}$$

which is negative for $0 < \gamma < \pi/2$ and $0 < p < 2$, $p \neq 1$, and hence the maximum of $h(\theta)$ occurs for

$$h(\theta)|_{\theta=\gamma} = \frac{\tan^{p-1}\gamma}{\tan(p-1)\gamma} = A_p(\gamma). \tag{6.178}$$

It follows that

$$h(\theta) \leq A_p(\gamma), \tag{6.179}$$

and hence

$$\frac{\sin^p\theta + B_p(\gamma)\cos p\theta}{\cos^p\theta} \leq A_p(\gamma), \tag{6.180}$$

and therefore Eq. (6.169) is established.

The trigonometric identity just proved can be used to derive Eq. (6.167) for $0 < p \leq 2$. The approach employed is based on a proof due to Calderón (1950). Let $F(z)$ be analytic inside the unit disc $|z| < 1$, and suppose $F(z) = u(z) + \mathrm{i}v(z)$, with $u(z) > 0$ and $v(0) = 0$. On the boundary of the unit disc, the real part of F is denoted by the function f and the imaginary part is denoted by $\mathcal{H}f$. Let $F(z) = R\mathrm{e}^{\mathrm{i}\theta}$, then $u(z) = R\cos\theta$ and $v(z) = R\sin\theta$. Because of the restriction $u(z) > 0$, $-\pi/2 < \theta < \pi/2$. Consider the contour integral $\int_C F^p(z)\mathrm{d}z/z$, where C is the circular contour $|z| = r, r < 1$, centered at the origin. The Cauchy integral formula gives

$$\int_C \frac{F^p(z)\mathrm{d}z}{z} = 2\pi\mathrm{i}F^p(0), \tag{6.181}$$

which, on using $z = r\mathrm{e}^{\mathrm{i}\phi}$, gives

$$\int_0^{2\pi} F^p(r\mathrm{e}^{\mathrm{i}\phi})\mathrm{i}\,\mathrm{d}\phi = 2\pi\mathrm{i}F^p(0), \tag{6.182}$$

and hence

$$\int_0^{2\pi} R^p\mathrm{e}^{\mathrm{i}p\theta}\,\mathrm{d}\phi = 2\pi F^p(0). \tag{6.183}$$

The reader should take note that two distinct polar substitutions are in use: one for the function F, the other for the point z, so that the integrand of the preceding integral has a dependence on ϕ. Taking the real part gives

$$\int_0^{2\pi} R^p\cos p\theta\,\mathrm{d}\phi = 2\pi u^p(0) > 0. \tag{6.184}$$

Now multiply Eq. (6.169) by R^p and integrate over ϕ, then

$$\int_0^{2\pi} |R\sin\theta|^p\,\mathrm{d}\phi \leq A_p(\gamma)\int_0^{2\pi} R^p\cos^p\theta\,\mathrm{d}\phi - B_p(\gamma)\int_0^{2\pi} R^p\cos p\theta\,\mathrm{d}\phi, \tag{6.185}$$

that is,

$$\int_0^{2\pi}\left|v(r\mathrm{e}^{\mathrm{i}\phi})\right|^p\mathrm{d}\phi \leq A_p(\gamma)\int_0^{2\pi}\left|u(r\mathrm{e}^{\mathrm{i}\phi})\right|^p\mathrm{d}\phi - B_p(\gamma)\int_0^{2\pi} R^p\cos p\theta\,\mathrm{d}\phi. \tag{6.186}$$

Since $B_p(\gamma) > 0$ and the final integral is greater than zero (via Eq. (6.184)), the final term may be dropped; hence,

$$\int_0^{2\pi}\left|v(r\mathrm{e}^{\mathrm{i}\phi})\right|^p\mathrm{d}\phi < A_p(\gamma)\int_0^{2\pi}\left|u(r\mathrm{e}^{\mathrm{i}\phi})\right|^p\mathrm{d}\phi. \tag{6.187}$$

If the case $p = 2$ is included, the strict inequality sign in the preceding result is replaced by $\leq$. On taking the lim $r \to 1$ and using Fatou's theorem (see Section 3.3.1), the desired result then follows.

The minimum value for $A_p(\gamma)$ with $1 < p < 2$ is determined in the following manner. Starting from Eq. (6.170) leads to

$$\frac{\partial A_p(\gamma)}{\partial \gamma} = \frac{(p-1)\tan^p \gamma \, \sin(p-2)\gamma \, \cos p\gamma}{\sin^2 \gamma \, \sin^2(p-1)\gamma}, \tag{6.188}$$

which equals zero for $\gamma = \pi/2p$, and

$$\begin{aligned} \frac{\partial^2 A_p(\gamma)}{\partial^2 \gamma} = \; & \frac{\partial A_p(\gamma)}{\partial \gamma}\{2p \csc 2\gamma - 2\cot\gamma + (p-2)\cot(p-2)\gamma \\ & - 2(p-1)\cot(p-1)\gamma\} \\ & + \frac{p(p-1)\tan^p \gamma \sin(2-p)\gamma \sin p\gamma}{\sin^2 \gamma \, \sin^2(p-1)\gamma}, \end{aligned} \tag{6.189}$$

so that

$$\left.\frac{\partial^2 A_p(\gamma)}{\partial^2 \gamma}\right|_{\gamma=\pi/2p} = -\frac{p(p-1)\tan^p(\pi/2p)\cos(\pi/p)}{\sin^2(\pi/2p)\cos^2(\pi/2p)} > 0. \tag{6.190}$$

Hence, the optimal value of the constant $A_p(\gamma)$ is given by

$$A_p\Big(\frac{\pi}{2p}\Big) = \tan^p\left(\frac{\pi}{2p}\right). \tag{6.191}$$

In the preceding discussion the focus was on a function of positive sign. This restriction can be removed in a straightforward manner, so that the proof can be employed to cover functions of variable sign. This is dealt with in Section 7.1.

The absence of the case $p = 1$ in the Reisz inequality is due to the existence of examples where $f \in L(-\pi,\pi)$ but $\mathcal{H}f \notin L(-\pi,\pi)$. The following example is given by Wheeden and Zygmund (1977, p. 250). Suppose f is periodic with period 2π and is defined by

$$f(t) = \begin{cases} 0, & -\pi < t \le 0 \\ h(t), & 0 < t < \pi/2 \\ 0, & \pi/2 \le t < \pi, \end{cases} \tag{6.192}$$

where h is a non-negative function, and assume $f \in L(-\pi,\pi)$. It follows that

$$\mathcal{H}f(\theta) = \frac{1}{2\pi}P\int_{-\pi}^{\pi} f(t)\cot\Big(\frac{\theta-t}{2}\Big)\mathrm{d}t = \frac{1}{2\pi}P\int_0^{\pi/2} f(t)\cot\Big(\frac{\theta-t}{2}\Big)\mathrm{d}t. \tag{6.193}$$

Select $\theta \in (-\pi/2, 0)$; splitting the preceding integral:

$$P\int_0^{\pi/2} f(t)\cot\Big(\frac{\theta-t}{2}\Big)\mathrm{d}t = P\int_0^{|\theta|} f(t)\cot\Big(\frac{\theta-t}{2}\Big)\mathrm{d}t + P\int_{|\theta|}^{\pi/2} f(t)\cot\Big(\frac{\theta-t}{2}\Big)\mathrm{d}t, \tag{6.194}$$

and noting that the second integral on the right-hand side must be negative, yields

$$P\int_0^{\pi/2} f(t)\cot\Big(\frac{\theta-t}{2}\Big)\mathrm{d}t \le P\int_0^{|\theta|} f(t)\cot\Big(\frac{\theta-t}{2}\Big)\mathrm{d}t, \tag{6.195}$$

and thus

$$\mathcal{H}f(\theta) \le -\frac{1}{2\pi}P\int_0^{|\theta|} f(t)\cot\Big(\frac{t-\theta}{2}\Big)\mathrm{d}t. \tag{6.196}$$

For $t \in (0, |\theta|)$, it follows that $\tan[(t-\theta)/2] = \tan[(t+|\theta|)/2] \le \tan(|\theta|)$, and hence

$$\cot(|\theta|) \le \cot\Big(\frac{t-\theta}{2}\Big). \tag{6.197}$$

Using the preceding inequality allows Eq. (6.196) to be written as follows:

$$\frac{\cot(|\theta|)}{2\pi}\int_0^{|\theta|} f(t)\mathrm{d}t \le -\mathcal{H}f(\theta), \tag{6.198}$$

that is

$$\frac{\cot(|\theta|)}{2\pi}\int_0^{|\theta|} f(t)\mathrm{d}t \le |\mathcal{H}f(\theta)|\,. \tag{6.199}$$

As a particular example, let $f(t) = (t\log^2 t)^{-1}$, for $t \in (0, 1/2)$, and $f(t) = 0$, for $t \in [1/2, \pi/2)$; then

$$\int_{-\pi}^{\pi} |f(t)|\mathrm{d}t = \int_0^{1/2} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\log t^{-1}}\Big)\mathrm{d}t = \frac{1}{\log 2}, \tag{6.200}$$

so that $f \in L(-\pi, \pi)$. From Eq. (6.199), it follows that

$$|\mathcal{H}f(\theta)| \ge -\frac{\cot(|\theta|)}{2\pi\log|\theta|}, \tag{6.201}$$

which in the vicinity of $\theta = 0$ can be expressed as follows:

$$|\mathcal{H}f(\theta)| \ge -\frac{1}{2\pi\,|\theta|\log|\theta|}. \tag{6.202}$$

Now, for small $\alpha > 0$, write

$$\int_{-\pi}^{\pi} |\mathcal{H}f(\theta)|\mathrm{d}\theta = \int_{-\pi}^{-\alpha} |\mathcal{H}f(\theta)|\mathrm{d}\theta + \int_{-\alpha}^{\alpha} |\mathcal{H}f(\theta)|\mathrm{d}\theta + \int_{\alpha}^{\pi} |\mathcal{H}f(\theta)|\mathrm{d}\theta. \quad (6.203)$$

The second integral on the right-hand side can be written, using Eq. (6.202), as follows:

$$\begin{aligned} \int_{-\alpha}^{\alpha} |\mathcal{H}f(\theta)|\mathrm{d}\theta &\geq \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \frac{1}{|\theta| \log |\theta|^{-1}} \,\mathrm{d}\theta \\ &= -\frac{1}{\pi} \int_{0}^{\alpha} \frac{\mathrm{d}}{\mathrm{d}\theta} \big(\log\log \theta^{-1} \big) \mathrm{d}\theta \\ &= \infty, \end{aligned} \quad (6.204)$$

and hence $\mathcal{H}f \notin L(-\pi, \pi)$.

A lower bound for $\|\mathcal{H}f\|_p^p$ can be written in the following manner. If the function f satisfies

$$\int_{0}^{2\pi} f(\theta)\mathrm{d}\theta = 0, \quad (6.205)$$

then, using the inversion property $\mathcal{H}^2 f = -f$ allows $\|\mathcal{H}f\|_p^p$ to be bounded from below by $\|f\|_p^p$:

$$\{\Re_p\}^{-p} \int_{-\pi}^{\pi} |f(s)|^p \,\mathrm{d}s \leq \int_{-\pi}^{\pi} |\mathcal{H}f(s)|^p \,\mathrm{d}s. \quad (6.206)$$

6.18 The partial sum of a Fourier series

The objective in this section is to relate the partial sum of a Fourier series to the Hilbert transform of the conjugate function together with some additional integrals. Let $f(x)$ be periodic with period 2π and $f \in L^2(-\pi, \pi)$. Suppose f has the Fourier series representation

$$f(x) \sim \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx. \quad (6.207)$$

Let g, which belongs in $L^2(-\pi, \pi)$, denote the conjugate series of f, that is

$$g(x) \sim \sum_{k=1}^{\infty} a_k \sin kx - b_k \cos kx. \quad (6.208)$$

The coefficients a_k and b_k are given by

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos ks\,\mathrm{d}s = \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\sin ks\,\mathrm{d}s \tag{6.209}$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin ks\,\mathrm{d}s = -\frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\cos ks\,\mathrm{d}s. \tag{6.210}$$

Let $S_n f$ denote the sum of the first n terms of the Fourier series of f, that is

$$S_n f(x) = \sum_{k=1}^{n} a_k\cos kx + b_k\sin kx. \tag{6.211}$$

If the results for the coefficients a_k and b_k in terms of $g(x)$ are inserted in Eq. (6.211), then.

$$\begin{aligned}
S_n f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\sum_{k=1}^{n}\big(\sin ks\cos kx - \cos ks\sin kx\big)\mathrm{d}s\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\sum_{k=1}^{n}\sin(s-x)k\,\mathrm{d}s\\
&= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} g(\theta+x)\sum_{k=1}^{n}\sin k\theta\,\mathrm{d}\theta\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta+x)\left\{\cot\left(\frac{\theta}{2}\right) - \frac{\cos(n+1/2)\theta}{\sin(\theta/2)}\right\}\mathrm{d}\theta,
\end{aligned} \tag{6.212}$$

where the periodic property of the function has been employed to change the integration limits and Eq. (6.58) has been used to evaluate the sum. The preceding result can be rearranged to read

$$\begin{aligned}
S_n f(x) =& -\frac{1}{2\pi}P\int_{-\pi}^{\pi} g(s)\cot\left(\frac{x-s}{2}\right)\mathrm{d}s\\
&- \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta+x)\left\{\cos n\theta\cot\left(\frac{\theta}{2}\right) - \sin n\theta\right\}\mathrm{d}\theta\\
=& -\mathcal{H}g(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin n\theta\, g(\theta+x)\mathrm{d}\theta\\
&+ \frac{1}{\pi}P\int_{-\pi}^{\pi}\frac{\cos n\theta\, g(\theta+x)}{\theta}\mathrm{d}\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos n\theta\, g(\theta+x)\left\{\cot\left(\frac{\theta}{2}\right) - \frac{2}{\theta}\right\}\mathrm{d}\theta,
\end{aligned} \tag{6.213}$$

which simplifies further on noting that $f(x) = -\mathcal{H}g(x)$. When the integral in Eq. (6.212) is split up, it may be necessary, depending on the nature of the function g, to interpret one of the resulting integrals as a Cauchy principal value. Since

$$\lim_{\theta\to 0}\left\{\cot\left(\frac{\theta}{2}\right)-\frac{2}{\theta}\right\}=O(\theta), \tag{6.214}$$

the final integral on the right-hand side of Eq. (6.213) does not involve a Cauchy principal value. This calculation is continued in Section 6.19, and the role of the conjugate function in determining $\lim_{n\to\infty} S_n f(x)$ is established.

6.19 Lusin's conjecture

Lusin (1915) (see also Lusin (1913)) gave the result that if $f \in L^2(-\pi,\pi)$, then the conjugate function

$$\begin{aligned}\tilde{f}(x) = \lim_{\varepsilon\to 0+}\tilde{f}_\varepsilon(x) &= \lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{\varepsilon\le|\theta|\le\pi} f(x-\theta)\cot\left(\frac{\theta}{2}\right)\mathrm{d}\theta\\ &= \lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{\varepsilon}^{\pi}\{f(x-\theta)-f(x+\theta)\}\cot\left(\frac{\theta}{2}\right)\mathrm{d}\theta,\end{aligned} \tag{6.215}$$

exists *a.e.* Starting with Eq. (6.208),

$$\begin{aligned}\sum_{k=1}^{\infty}\{a_k\sin kx-b_k\cos kx\}r^k &= \frac{1}{\pi}\sum_{k=1}^{\infty}\left\{\sin kx\int_{-\pi}^{\pi}f(\theta)\cos k\theta\,\mathrm{d}\theta\right.\\ &\qquad\left.-\cos kx\int_{-\pi}^{\pi}f(\theta)\sin k\theta\,\mathrm{d}\theta\right\}r^k\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\mathrm{d}\theta\sum_{k=1}^{\infty}\left\{\sin kx\cos k\theta-\sin k\theta\cos kx\right\}r^k\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\mathrm{d}\theta\sum_{k=1}^{\infty}\sin k(x-\theta)r^k\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{r\sin(x-\theta)}{1-2r\cos(x-\theta)+r^2}f(\theta)\mathrm{d}\theta,\end{aligned} \tag{6.216}$$

where the last result follows from Eq. (6.49). Now,

$$\begin{aligned}\lim_{r\to 1}\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{r\sin(x-\theta)}{1-2r\cos(x-\theta)+r^2}f(\theta)\mathrm{d}\theta &= \frac{1}{2\pi}P\int_{-\pi}^{\pi}f(\theta)\cot\left(\frac{x-\theta}{2}\right)\mathrm{d}\theta\\ &= \tilde{f}(x),\quad a.e.\end{aligned} \tag{6.217}$$

This calculation can be justified by the use of Fatou's theorem. The function $\tilde{f}$ is, of course, identified with the conjugate series given in Eq. (6.208). Privalov (1919) extended Lusin's result to cover the case that if $f \in L(\mathbb{T})$ then $\tilde{f}$ exists *a.e.*, and the combined result is often called the Lusin–Privalov theorem.

In a similar fashion,

$$\begin{aligned}\sum_{k=1}^{\infty}\{a_k \cos kx + b_k \sin kx\}r^k &= \frac{1}{\pi}\sum_{k=1}^{\infty}\left\{\cos kx \int_{-\pi}^{\pi} g(\theta)\sin k\theta \, \mathrm{d}\theta \right.\\ &\quad \left. - \sin kx \int_{-\pi}^{\pi} g(\theta)\cos k\theta \, \mathrm{d}\theta\right\} r^k\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\mathrm{d}\theta \sum_{k=1}^{\infty}\{\sin k\theta \cos kx - \cos k\theta \sin kx\}r^k\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\mathrm{d}\theta \sum_{k=1}^{\infty} \sin k(\theta - x)r^k\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{r\sin(\theta - x)}{1 - 2r\cos(\theta - x) + r^2} g(\theta)\mathrm{d}\theta. \qquad (6.218)\end{aligned}$$

From this result, it follows that

$$\begin{aligned}\lim_{r\to 1}\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{r\sin(\theta - x)}{1 - 2r\cos(\theta - x) + r^2} g(\theta)\mathrm{d}\theta &= -\frac{1}{2\pi}P\int_{-\pi}^{\pi} g(\theta)\cot\left(\frac{x-\theta}{2}\right)\mathrm{d}\theta\\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\varepsilon\le|\theta|\le\pi} g(x+\theta)\cot\left(\frac{\theta}{2}\right)\mathrm{d}\theta\\ &= f(x), \quad a.e., \qquad (6.219)\end{aligned}$$

and the calculation is justified by employing Fatou's theorem.

Proceeding in a similar manner,

$$\begin{aligned}\sum_{k=1}^{\infty}\{a_k \cos kx + b_k \sin kx\}r^k &= \frac{1}{\pi}\sum_{k=1}^{\infty}\left\{\cos kx \int_{-\pi}^{\pi} f(\theta)\cos k\theta \, \mathrm{d}\theta \right.\\ &\quad \left. + \sin kx \int_{-\pi}^{\pi} f(\theta)\sin k\theta \, \mathrm{d}\theta\right\} r^k\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{d}\theta \sum_{k=1}^{\infty}\{\cos k\theta \cos kx + \sin k\theta \sin kx\}r^k\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{d}\theta \sum_{k=1}^{\infty} \cos k(\theta - x)r^k\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-x)+r^2} f(\theta)\mathrm{d}\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{d}\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-x)+r^2} f(\theta)\mathrm{d}\theta, \quad (6.220)$$

where Eq. (6.48) has been employed, and the integral of f over one complete period is assumed to be zero. From this last result it follows, on recalling Eq. (3.57), that

$$\lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-x)+r^2} f(\theta)\mathrm{d}\theta = f(x), \ a.e. \quad (6.221)$$

The question concerning the value of the partial sum of a Fourier series is now revisited. It was plausible to Lusin that the nth partial sum of terms of f, $S_n f$, satisfied

$$\lim_{n\to\infty} S_n f = f, \ a.e., \quad (6.222)$$

for all $f \in L^2$. This result came to be known as the *Lusin conjecture*. Following directly from Eq. (6.213),

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin n\theta \, g(\theta+x)\mathrm{d}\theta = 0 \quad (6.223)$$

and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \cos n\theta \, g(\theta+x)\left\{\cot\left(\frac{\theta}{2}\right) - \frac{2}{\theta}\right\}\mathrm{d}\theta = 0, \quad (6.224)$$

which follow directly from the Riemann–Lebesgue lemma. Lusin's conjecture then depends on establishing that

$$\lim_{n\to\infty} P\int_{-\pi}^{\pi} \frac{\cos n\theta \, g(\theta+x)}{\theta}\mathrm{d}\theta = 0, \ a.e., \quad (6.225)$$

which would appear to be the case for functions $g \in L^2$. For $g \in L^2(\mathbb{T})$, it was known that integrals such as $P\int_0^{\pi}, [g(x+\theta) - g(x-\theta)]/\theta \mathrm{d}\theta$ existed *a.e.*, and, further, that there were continuous g for which $\int_0^{\pi} |g(x+\theta) - g(x-\theta)/\theta| \,\mathrm{d}\theta$ diverged. Therefore, in general, cancellation effects play an important role in obtaining a non-divergent value for the principal value integral in Eq. (6.225). This is the same idea discussed in detail in Section 3.15 under the title "Cancellation behavior for the Hilbert transform." Lusin understood that cancellation effects must play a pivotal role in determining the convergence of trigonometric series. Over fifty years passed from the time of Lusin's conjecture to the seminal work of Carleson (1966), who proved that the Fourier series of every function $f \in L^2$ converges *a.e.* A discussion of Carleson's work lies beyond the scope of the present discussion, but it is noted that conjugate functions played an important role in his proof.

Notes

§6.1 There is a long history associated with the study of the properties of the Hilbert transform of periodic functions; see, for example, Tauber (1891), Hilbert (1904, 1905, 1912), Titchmarsh (1929), Cossar (1960), Eastham (1962), Zygmund (1968), Butzer and Nessel (1971), Pandey (1996), and Zhizhiashvili (1996).

§6.5 For some additional examples, see Hauss (1997, 1998). For further details on the partial sums of conjugate Fourier series, see Zygmund (1968, Vol. I, p. 88).

§6.7 The forms $\mathcal{H}_e f$ and $\mathcal{H}_o f$ for periodic functions have been discussed by many authors, including Hardy and Littlewood (1936), K.-K. Chen (1944), Y.-M. Chen (1963), Flett (1958), and Andersen (1976a). For some results on transformations of conjugate functions, see Kinukawa and Igari (1961), Igari (1962), and Wang (1965).

§6.9 See Butzer and Nessel (1971, chap. 9) and Lasser (1996, chap. 8) for further reading.

§6.10 For additional discussion, see Butzer and Nessel (1971, chap. 9).

§6.11 For a well written exposition, see Krantz (2006).

§6.13 Dunkl (1985) has studied properties of the operators $\mathcal{HM}$ and $\mathcal{MH}$ with $\mathcal{M}f(\theta) = \operatorname{sgn}(\theta)f(\theta)$, for even $f \in L^2(\mathbb{T})$. The eigenfunctions of these two operators are also considered.

§6.16 See Fatou (1906, p. 361) for some early discussion. Hardy and Littlewood (1928) prove a generalization of the result in this section. Additional reading can be found in Zygmund (1968, Vol. 1, p. 121), Koosis (1998, p. 100), and Cima *et al.* (2006, p. 62). The situation for double conjugate series is treated by Cesari (1938) and Žak (1950, 1952), and the multi-dimensional case is discussed by Lekishvili (1978).

§6.17 Hobson (1926, p. 610) discusses the Riesz inequality and some related developments. Generalizations have been derived by Bochner (1939), Helson (1958), and by Hewitt and Ritter (1983). For some additional discussion, including historical remarks, see Asmar and Hewitt (1988). Besides the work of Pichorides (1972) on the best constants in the Riesz inequality, B. Cole independently obtained the sharpest constants, and his work is discussed in Gamelin (1978, chap. 7). For an alternative proof of Pichorides' result (Eq. (6.168)) for the case $1 < p < 2$, see Essén, Shea, and Stanton (**?**). For further work, see Verbitsky and Krupnik (1994), Papadopoulos (1999), and Hollenbeck and Verbitsky (2000). Lamperti (1959) considered a number of different classes of functions for which the Riesz inequality holds.

§6.19 The reader should note that a common alternative spelling for Lusin employed in the mathematics literature is Luzin. For a concise discussion of Lusin's conjecture, including the historical flow of results, and an outline of the essential elements involved in Carleson's proof, Hunt (1974) is recommended.

Exercises

6.1 By taking advantage of the trigonometric identity

$$\cot\left(\frac{x-y}{2}\right)\cot\left(\frac{x-w}{2}\right)=\cot\left(\frac{y-w}{2}\right)\left\{\cot\left(\frac{x-y}{2}\right)-\cot\left(\frac{x-w}{2}\right)\right\}-1,$$

or otherwise, show that for $f(\theta) = \cot((\theta-\theta_0))/2$, with θ_0 a constant, then $\mathcal{H}f(\theta) = 1$.

6.2 Show that the Fourier series expansion of $\operatorname{sgn}(\sin x)$ with $-\pi < x < \pi$ is given by

$$\operatorname{sgn}(\sin x) = \frac{4}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right\}.$$

6.3 Can a simple form for $\mathcal{H}\{\theta f\}(\theta)$ in terms of $\theta\mathcal{H}f(\theta)$ be found?

6.4 Evaluate $\mathcal{H}\{\operatorname{sgn}(\operatorname{cas}\theta)\}$.

6.5 Evaluate (i) $\mathcal{H}\{\operatorname{sgn}(\sin\theta + \sin 2\theta)\}$ and (ii) $\mathcal{H}\{\operatorname{sgn}(\cos\theta + \cos 2\theta)\}$.

6.6 Determine if Eq. (6.213) holds for the series

$$\sum_{k=1}^{\infty}\frac{2(-1)^{k+1}\cos kx}{k}, \quad \text{for } -\pi < x < \pi.$$

[Hint: For the integrals involving $g(x+\theta)$, remember to take account of the 2π periodicity by splitting the integrals over $(-\pi,\pi)$ appropriately.]

6.7 If $f(x) = \sum_{n=3}^{\infty}\cos nx/(\log n \log\log n)$, determine if the conjugate series is integrable and is a Fourier series.

6.8 For a periodic function (period 2π) $f \in L^p(-\pi,\pi)$ with $p > 1$, show that $\int_{-\pi}^{\pi}\mathcal{H}f(\theta)\mathrm{d}\theta = 0$.

6.9 Evaluate $(1/2\pi)P\int_{-\pi}^{\pi}\cos\alpha s\cot\{(1/2)(x-s)\}\mathrm{d}s$ for α a real constant that is not necessarily an integer.

6.10 Evaluate $(1/2\pi)P\int_{-\pi}^{\pi}\sin\alpha s\cot\{(1/2)(x-s)\}\mathrm{d}s$ for α a real constant.

6.11 What results can be obtained for $\mathcal{H}f$, when non-linear scale transformations of the type discussed in Section 4.5.2 for the Hilbert transform on $\mathbb{R}$ are considered?

6.12 For $f = \sin ks$ and $k \in \mathbb{Z}$, evaluate $\mathcal{H}f$ from the Fourier transform formula, Eq. (6.100).

6.13 Using $f = \sin ks$, with $k \in \mathbb{Z}$, verify the Parseval-type formula Eq. (6.97).

6.14 Using $f = \sin ks$, with $k \in \mathbb{Z}$, verify the Parseval-type formula Eq. (6.98).

6.15 For $\tau > 0$, evaluate $(1/2\tau)P\int_{-\tau}^{\tau}\sin ks\cot\{(1/2)(x-s)\}\mathrm{d}s$ for $k \in \mathbb{Z}$. What is the result for $(1/2\tau)P\int_{-\tau}^{\tau}\sin\alpha s\cot\{(1/2)(x-s)\}\mathrm{d}s$ with α a real constant?

6.16 For the following choices, where $k \in \mathbb{Z}$, determine if the function is an eigenfunction of $\mathcal{H}$: (i) $f(\theta) = \sin k\theta$; (ii) $f(\theta) = \cos k\theta$; (iii) $f(\theta) = \operatorname{cas} k\theta$; and (iv) $f(\theta) = (\cos k\theta)\mathrm{e}^{\mathrm{i}\theta}$. For the cases where the function is an eigenfunction, find the corresponding eigenvalue. What happens if k is a general real constant, and not necessarily an integer?

6.17 By taking advantage of the Tricomi formula, or otherwise, evaluate $\mathcal{H}f$, where $f = \sin mx \sin nx - \cos mx \cos nx$ and $m, n \in \mathbb{Z}^+$.

6.18 Which of the following series (if any)

(i) $\displaystyle\sum_{k=2}^{\infty} \frac{\sin kx}{\log k}$;

(ii) $\displaystyle\sum_{k=2}^{\infty} \frac{\cos kx}{\log k}$;

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k}$;

(iv) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos kx}{k}$; and

(v) $\displaystyle\sum_{k=2}^{\infty} \frac{\sin kx}{k \log k}$,

corresponds to the Fourier series of a function of the class $L^1(\mathbb{T})$?

6.19 For the cases in Exercise 6.18 that correspond to functions $f \in L^1(\mathbb{T})$, determine which of these (if any) also have $\mathcal{H}f \in L^1(\mathbb{T})$.

6.20 If the nth partial sum of a Fourier series is denoted by $S_n f$, is there a constant C_p depending on p, but independent of f, such that $\|S_n f\|_p \le C_p \|f\|_p$, for $f \in L^p(\mathbb{T})$ with $1 < p < \infty$?

6.21 Evaluate $\mathcal{H}f$, where $f(x) = \sin bcx/\bigl(1 - 2a\cos bx + a^2\bigr)$, for $a^2 < 1, b > 0$, and allowing for the cases $0 < c < 1$ and $1 \le c$.

6.22 Calculate $\mathcal{H}f$, where

(i) $f(x) = (5 - 4\cos x)^{-1}$,

(ii) $f(x) = (5 - 4\cos x)^{-1} \sin x$, and

(iii) $f(x) = (5 - 4\cos x)^{-1} \cos x$.

7

Inequalities for the Hilbert transform

7.1 The Marcel Riesz inequality revisited

The purpose of this chapter is to explore a number of inequalities satisfied by the Hilbert transform. A detailed treatment is presented for weighted inequalities.

So far, the principal inequality for the Hilbert transform that has been discussed is the Riesz inequality, given by

$$\|Hf\|_p \leq \Re_p \|f\|_p, \quad \text{for } 1 < p < \infty. \tag{7.1}$$

Recall that this is an equality for $p = 2$. In this section the Riesz formula is revisited, with the focus on an alternative method to establish the result. The question of what can be established when p falls outside the stated interval is also examined.

Let $F(z) = u(x, y) + \mathrm{i}v(x, y)$ be analytic in the unit disc and at the origin $v(0) = 0$. The case $1 < p \leq 2$ and $u > 0$ is considered first. The approach presented employs Green's theorem as a key ingredient in the derivation. This line of reasoning was suggested by P. Stein (1933). Some useful preliminary results are required. The Cauchy–Riemann equations are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{7.2}$$

from which it follows that

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{7.3}$$

and

$$\nabla^2 v \equiv \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{7.4}$$

Now,

$$\frac{\partial^2 |u|^p}{\partial x^2} = p(p-1)|u|^{p-2} \left[\frac{\partial |u|}{\partial x}\right]^2 + p|u|^{p-1} \frac{\partial^2 |u|}{\partial x^2}, \tag{7.5}$$

and on using Eq. (7.3) it follows that

$$\nabla^2|u|^p = p(p-1)|u|^{p-2}\left\{\left[\frac{\partial|u|}{\partial x}\right]^2 + \left[\frac{\partial|u|}{\partial y}\right]^2\right\}. \tag{7.6}$$

Another useful result is the following:

$$|F'| = \sqrt{(\{u'\}^2 + \{v'\}^2)}, \tag{7.7}$$

where the prime denotes a derivative with respect to x or y, and

$$\nabla^2|u|^p = p(p-1)|u|^{p-2}|F'|^2. \tag{7.8}$$

A similar calculation gives

$$\frac{\partial^2|F|^p}{\partial x^2} = p(p-1)|F|^{p-2}\left[\frac{\partial|F|}{\partial x}\right]^2 + p|F|^{p-1}\frac{\partial^2|F|}{\partial x^2}, \tag{7.9}$$

and hence

$$\nabla^2|F|^p = p(p-1)|F|^{p-2}\left\{\left[\frac{\partial|F|}{\partial x}\right]^2 + \left[\frac{\partial|F|}{\partial y}\right]^2\right\} + p|F|^{p-1}\nabla^2|F|. \tag{7.10}$$

Making use of the Cauchy–Riemann equations yields

$$\nabla^2|F| = |F|^{-1}|F'|^2. \tag{7.11}$$

Note that

$$|F'|^2 = \left[\frac{\partial u}{\partial x}\right]^2 + \left[\frac{\partial v}{\partial x}\right]^2 = \left[\frac{\partial u}{\partial x}\right]^2 + \left[\frac{\partial u}{\partial y}\right]^2; \tag{7.12}$$

then

$$\left[\frac{\partial|F|}{\partial x}\right]^2 + \left[\frac{\partial|F|}{\partial y}\right]^2 = |F'|^2, \tag{7.13}$$

and hence

$$\nabla^2|F|^p = p^2|F|^{p-2}|F'|^2. \tag{7.14}$$

Since $|F| \geq |u|$, and it will be assumed that $u > 0$, it follows that

$$|F|^{p-2} \leq |u|^{p-2}, \quad \text{for } 1 < p \leq 2, \tag{7.15}$$

and therefore

$$p^2|F|^{p-2}\left|F'\right|^2 \le \frac{p}{p-1}\{p(p-1)\}|u|^{p-2}\left|F'\right|^2. \tag{7.16}$$

This inequality simplifies, on using Eqs. (7.8) and (7.14), to yield the following:

$$\nabla^2|F|^p \le \frac{p}{p-1}\nabla^2|u|^p. \tag{7.17}$$

Integrating both sides of this inequality leads to

$$\iint\limits_{|z|\le r} \nabla^2|F|^p \, \mathrm{d}x \, \mathrm{d}y \le \frac{p}{p-1}\iint\limits_{|z|\le r} \nabla^2|u|^p \, \mathrm{d}x \, \mathrm{d}y. \tag{7.18}$$

Making use of Green's theorem in the form

$$r\int_0^{2\pi} \frac{\partial\varphi}{\partial r}\, \mathrm{d}\theta = \iint\limits_{|z|\le r} \nabla^2\varphi \, \mathrm{d}x \, \mathrm{d}y, \tag{7.19}$$

where the derivative $\partial/\partial r$ is taken along the radius vector, and identifying φ first with $\left|F(r\mathrm{e}^{\mathrm{i}\theta})\right|^p$ and then with $|u|^p$, allows Eq. (7.18) to be written as follows:

$$\int_0^{2\pi} \frac{\partial\left|F(r\mathrm{e}^{\mathrm{i}\theta})\right|^p}{\partial r}\, \mathrm{d}\theta \le \frac{p}{p-1}\int_0^{2\pi} \frac{\partial\left|u(r\mathrm{e}^{\mathrm{i}\theta})\right|^p}{\partial r}\, \mathrm{d}\theta. \tag{7.20}$$

Integrating both sides of this inequality, from 0 to r, leads to

$$\int_0^{2\pi} \left|F(r\mathrm{e}^{\mathrm{i}\theta})\right|^p \mathrm{d}\theta \le \frac{p}{p-1}\int_0^{2\pi} \left|u(r\mathrm{e}^{\mathrm{i}\theta})\right|^p \mathrm{d}\theta - \frac{2\pi u^p(0)}{p-1}, \tag{7.21}$$

where the condition $v(0) = 0$ has been employed. Since $u(0)$ is positive, setting $C_p = \{p(p-1)^{-1}\}^{p^{-1}}$ yields

$$\|F\|_p \le C_p\|u\|_p\,, \quad \text{for } 1 < p \le 2. \tag{7.22}$$

This result is frequently called Riesz's inequality. The sharpest value of the constant C_p is (Essén, 1984; Verbitskiĭ, 1984)

$$C_p = \sec\left(\frac{\pi}{2p}\right), \quad \text{for } 1 < p \le 2, \tag{7.23}$$

and

$$C_p = \csc\left(\frac{\pi}{2p}\right), \quad \text{for } 2 < p < \infty. \tag{7.24}$$

To link up with the Hilbert transform, first note that $|v|^p \leq |F|^p$, and hence

$$\int_0^{2\pi} \left|v(re^{i\theta})\right|^p d\theta \leq C_p^p \int_0^{2\pi} \left|u(re^{i\theta})\right|^p d\theta. \tag{7.25}$$

Let

$$f(\theta) = \lim_{r\to 1} u(re^{i\theta}) \tag{7.26}$$

and

$$\widetilde{f}(\theta) = \lim_{r\to 1} v(re^{i\theta}). \tag{7.27}$$

For further discussion on these limits, see Fatou's theorem in Section 3.3.1. From Eq. (7.25), it follows that

$$\left\|\widetilde{f}\right\|_p \leq C_p \|f\|_p, \quad \text{for } 1 < p \leq 2, \tag{7.28}$$

which is the required result. The duality argument of Section 4.20 can be used to extend this result to cover the case $2 < p < \infty$.

The restriction that f is a positive function can be easily removed. Denote the positive part of f by f_1 and the absolute value of the negative part of f by f_2 so that

$$f(\theta) = f_1(\theta) - f_2(\theta), \quad \widetilde{f}(\theta) = \widetilde{f}_1(\theta) - \widetilde{f}_2(\theta), \tag{7.29}$$

and

$$|f(\theta)| = f_1(\theta) + f_2(\theta). \tag{7.30}$$

For $p > 1$, use Minkowski's inequality, Eq. (4.394), to write

$$\begin{aligned} \left\|\widetilde{f}\right\|_p &= \left\|\widetilde{f}_1 + \widetilde{f}_2\right\|_p \\ &\leq \left\|\widetilde{f}_1\right\|_p + \left\|\widetilde{f}_2\right\|_p; \end{aligned} \tag{7.31}$$

then make use of Eqs. (7.28) and (7.30),

$$\begin{aligned} \left\|\widetilde{f}\right\|_p &\leq C_p\{\|f_1\|_p + \|f_2\|_p\} \\ &\leq 2C_p \|f\|_p, \end{aligned} \tag{7.32}$$

which establishes the desired result.

7.1.1 Hilbert's integral

The Hilbert integral operator maps functions on $\mathbb{R}^+$ to functions on $\mathbb{R}^+$ according to

$$H_{\mathrm{I}} f(x) = \int_0^\infty \frac{f(t)\mathrm{d}t}{x+t}, \quad \text{for } x > 0. \tag{7.33}$$

In the literature H_{I} is denoted by H, but the latter symbol has been reserved for the Hilbert transform operator in the present work. The reader can compare this result with the definition given in Section 5.6 for the Stieltjes transform. The latter transform can be defined more generally than was given previously (see Eq. (5.77)). For a function f of bounded variation, the Stieltjes transform is defined as follows:

$$\mathsf{S}f(x) = \int_0^\infty \frac{\mathrm{d}f(t)}{(x+t)^\rho}, \quad \text{for } \rho > 0. \tag{7.34}$$

Consider the special case $\rho = 1$ with $\mathrm{d}f(t) = f(t)\mathrm{d}t$; then Eq. (7.34) reduces to Eq. (7.33). Because of the restriction to positive values of x, the integral in Eq. (7.33) is not a principal value integral, provided f has no non-integrable singularities on the interval $[0, \infty)$.

The norm $\|H_{\mathrm{I}} f\|_p$ can be bounded by $\|f\|_p$. First note the following result:

$$\int_0^\infty \frac{t^{-\alpha}\mathrm{d}t}{x+t} = \pi \csc \alpha\pi\, x^{-\alpha}, \quad \text{for } 0 < \alpha < 1, \tag{7.35}$$

which can be evaluated by contour integration (try it!). Applying Hölder's inequality (with conjugate exponents p and q) to Eq. (7.33) gives

$$|H_{\mathrm{I}} f(x)| \le \left(\int_0^\infty \frac{t^{\alpha p q^{-1}} |f(t)|^p \, \mathrm{d}t}{x+t} \right)^{1/p} \left(\int_0^\infty \frac{t^{-\alpha}\, \mathrm{d}t}{x+t} \right)^{1/q}. \tag{7.36}$$

Raising Eq. (7.36) to the pth power, integrating over x, and choosing α appropriately, leads to the inequality

$$\|H_{\mathrm{I}} f\|_p \le C_{p,\alpha} \|f\|_p \,, \tag{7.37}$$

where $C_{p,\alpha}$ depends on α and p, but is independent of f. Equation (7.37) is Hilbert's inequality.

7.2 A Kolmogorov inequality

The Riesz inequality conveys no information about the case $0 < p < 1$. An inequality due to Kolmogorov fills this gap. The Kolmogorov inequality takes the form

$$\left\| \widetilde{f} \right\|_p \le B_p \|f\|_1 \,, \quad \text{for } 0 < p < 1, \tag{7.38}$$

where B_p is a constant depending only on p. There are several different proofs of this result. The original demonstration of this inequality by Kolmogorov (see under the alternative spelling Kolmogoroff (1925)) used real-variable methods. Littlewood (1926), followed shortly by Hardy (1928b), gave proofs using complex-variable methods. The following development is based on Hardy's approach. Define

$$F(z) = u(z) + \mathrm{i}v(z), \tag{7.39}$$

and suppose $F(z)$ is analytic inside the unit disc, with $v(0) = 0$, and assume $u > 0$. Let the function G be analytic inside the unit disc; then, using the Cauchy integral formula to evaluate $\int_C G(z)\mathrm{d}z/z$, where the contour C is a closed circular arc of radius r, leads to

$$\int_{|z|=r} \frac{G(z)\mathrm{d}z}{z} = 2\pi \mathrm{i} G(0), \quad \text{for } 0 < r < 1, \tag{7.40}$$

and hence

$$\int_0^{2\pi} G(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta = 2\pi G(0). \tag{7.41}$$

Identify $G(z)$ with $F^p(z)$ and express $F(z)$ in the following form:

$$F(z) = R\mathrm{e}^{\mathrm{i}\varphi}, \quad \text{with } R > 0 \text{ and } |\varphi| \leq \pi/2. \tag{7.42}$$

On taking the real part, Eq. (7.41) leads to

$$\frac{1}{2\pi}\int_0^{2\pi} R^p \cos p\varphi \,\mathrm{d}\theta = F^p(0). \tag{7.43}$$

Now, $u(z)$ can be written as follows:

$$u(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta), \tag{7.44}$$

and the $\lim_{r\to 1} u(z)$ is identified with $f(\theta)$; hence,

$$F^p(0) = u^p(0) = \left\{\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{d}\theta\right\}^p. \tag{7.45}$$

From Eq. (7.42) it follows that $|v| \leq R$, and because of the restriction on φ the obvious inequality $0 < \cos\,(p\pi/2) \leq \cos p\varphi$ holds, and so Eq. (7.43) becomes

$$\frac{1}{2\pi}\cos\left(\frac{p\pi}{2}\right)\int_0^{2\pi} \left|v(r\mathrm{e}^{\mathrm{i}\theta})\right|^p \mathrm{d}\theta \leq \left\{\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{d}\theta\right\}^p. \tag{7.46}$$

Let

$$B_p = (2\pi)^{(1-p)/p} \sec^{p^{-1}}\left(\frac{p\pi}{2}\right), \tag{7.47}$$

and employ

$$\widetilde{f}(\theta) = \lim_{r\to 1} v(re^{i\theta}), \tag{7.48}$$

then Eq. (7.38) is obtained.

If the restriction that $f > 0$ is removed, and use is made of the definitions of f_1 and f_2, Minkowski's inequality, Eq. (4.393), and Eqs. (7.29), (7.30), and (7.38), then it follows that:

$$\begin{aligned}
\left\|\widetilde{f}\right\|_p^p &= \int_0^{2\pi} \{\widetilde{f}_1(\theta) + \widetilde{f}_2(\theta)\}^p d\theta \\
&\le \int_0^{2\pi} \left|\widetilde{f}_1(\theta)\right|^p d\theta + \int_0^{2\pi} \left|\widetilde{f}_2(\theta)\right|^p d\theta \\
&\le B_p^p \left(\int_0^{2\pi} f_1(\theta) d\theta\right)^p + B_p^p \left(\int_0^{2\pi} f_2(\theta) d\theta\right)^p \\
&\le 2B_p^p \left(\int_0^{2\pi} f(\theta) d\theta\right)^p.
\end{aligned} \tag{7.49}$$

Hence,

$$\left\|\widetilde{f}\right\|_p \le 2^{p^{-1}} B_p \|f\|_1, \quad \text{for } 0 < p < 1, \tag{7.50}$$

which proves the Kolmogorov inequality for a general function. The best constant in Kolmogorov's inequality, Eq. (7.38), for non-negative functions was given by Pichorides (1972) and by Duren in 1970 (see Duren (2000, p. 67)) as

$$B_p = \left(\sec \frac{p\pi}{2}\right)^{1/p}. \tag{7.51}$$

For a general function $f \in L^1(\mathbb{T})$, Davis (1976) studied the problem by probabilistic arguments and Baernstein (1978, 1979) used non-probability techniques. The best constant obtained is

$$B_p^p = \frac{1}{2\pi} \int_0^{2\pi} |\sin\theta|^{-p} d\theta, \tag{7.52}$$

which can be written in terms of the gamma function as follows:

$$B_p^p = \frac{\Gamma[(1-p)/2]}{\sqrt{(\pi)}\,\Gamma[(2-p)/2]}. \tag{7.53}$$

Previously it was shown that there is a version of the Riesz inequality for both the line and the circle. No direct analog of Kolmogorov's inequality (Eq. (7.38)) for the Hilbert transform on the line is known. There is another result, known as Kolmogorov's theorem (also referred to as Kolmogorov's inequality), which takes the following form. Let $f \in L^1(\mathbb{T})$ and $\lambda > 0$; then

$$m\{\widetilde{f}(\lambda)\} \leq \frac{K}{\lambda}\|f\|_1, \tag{7.54}$$

where $\widetilde{f}$ denotes as usual the conjugate function to f, K is a constant independent of f and λ, and $m\{g(\lambda)\}$ is the distribution function of g, given by

$$m\{g(\lambda)\} = |\theta \in [-\pi,\pi]\colon\ |g(\theta)| \geq \lambda|. \tag{7.55}$$

The designation for $m\{g(\lambda)\}$ is not to be confused with the terminology employed earlier for distributions functions, as used in Section 2.15 to refer to generalized functions. Equation (7.54) is sometimes termed the *weak-type* (1,1) norm of the conjugate function. Weak-type notation is discussed in Section 7.8. There is a direct analog of Eq. (7.54) for the Hilbert transform on $\mathbb{R}$. For this case, Kolmogorov's theorem is as follows: for $f \in L^1(\mathbb{R})$ and $\lambda > 0$,

$$m\{Hf(\lambda)\} \leq \frac{K}{\lambda}\|f\|_1. \tag{7.56}$$

The best constant K for both Eqs. (7.54) and (7.56) has been found by Davis (1974) to be

$$K = \frac{\pi^2}{8\beta(2)}, \tag{7.57}$$

where $\beta(2)$ is Catalan's constant:

$$\beta(2) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} \approx 0.915\,966\ldots \tag{7.58}$$

For non-negative functions the sharpest constant K in Eqs. (7.54) and (7.56) is known to be $K = 1$ (Davis, 1974). A proof of Eq. (7.54) is sketched following Koosis (1998), considering only the case of a non-negative $f \in L^1(\mathbb{T})$, and without regard to obtaining the sharpest possible constant K. The case of a general function $f \in L^1(\mathbb{T})$ is left as an exercise for the interested reader. For the choice $f \in L^1(\mathbb{R})$, Eq. (7.56) is discussed further in Section 7.8.

Let

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{e^{it}+z}{e^{it}-z}\right\}f(t)dt, \quad \text{for } |z| < 1, \tag{7.59}$$

with

$$\lim_{r \to 1} F(r\mathrm{e}^{\mathrm{i}\theta}) = f(\theta) + \mathrm{i}\widetilde{f}(\theta). \tag{7.60}$$

The factor in {} in Eq. (7.59) can be recognized as a combination of the Poisson and conjugate Poisson kernel for the disc:

$$\begin{aligned} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z} &= \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} + \frac{2\mathrm{i}r\sin(\theta - t)}{1 - 2r\cos(\theta - t) + r^2} \\ &= P(r, \theta - t) + \mathrm{i}Q(r, \theta - t). \end{aligned} \tag{7.61}$$

The function $w(z)$ is introduced using the following definition:

$$w(z) = 1 + \frac{F(z) - \lambda}{F(z) + \lambda}, \quad \text{for } \lambda > 0. \tag{7.62}$$

In the unit disc, $w(z) - 1$ is analytic, and, from the Cauchy integral formula with $|z_0| < 1$,

$$\begin{aligned} w(z_0) - 1 &= \frac{1}{2\pi\mathrm{i}} \oint_C \frac{\{w(z) - 1\}\mathrm{d}z}{z - z_0} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\{w(\mathrm{e}^{\mathrm{i}\theta}) - 1\}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{\mathrm{e}^{\mathrm{i}\theta} - z_0}. \end{aligned} \tag{7.63}$$

Taking the limit $z_0 \to 0$ gives

$$2\pi w(0) = \int_{-\pi}^{\pi} w(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta. \tag{7.64}$$

From Eq. (7.64) it follows that

$$2\pi \operatorname{Re} w(0) = \int_{-\pi}^{\pi} \operatorname{Re}\{w(\mathrm{e}^{\mathrm{i}\theta})\}\mathrm{d}\theta. \tag{7.65}$$

Now,

$$F(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\mathrm{d}t = \frac{1}{2\pi} \|f\|_1, \tag{7.66}$$

and using the non-negativity of f leads to

$$2\pi \operatorname{Re} w(0) = \frac{4\pi \|f\|_1}{\|f\|_1 + 2\pi\lambda}, \tag{7.67}$$

and therefore

$$\int_{-\pi}^{\pi} \operatorname{Re}\{w(\mathrm{e}^{\mathrm{i}\theta})\}\,\mathrm{d}\theta \leq \frac{2\|f\|_1}{\lambda}. \tag{7.68}$$

If $\left|\tilde{f}\right| \geq \lambda$, then using Eq. (7.60) implies $\left|F(\mathrm{e}^{\mathrm{i}\theta})\right| \geq \lambda$, and employing Eq. (7.62) gives $\mathrm{Re}\{w(\mathrm{e}^{\mathrm{i}\theta})\} \geq 1$. Using Eq. (7.68) yields

$$\left|\theta \in [-\pi, \pi] : \mathrm{Re}\{w(\mathrm{e}^{\mathrm{i}\theta})\} \geq 1\right| \leq \frac{2\|f\|_1}{\lambda}. \tag{7.69}$$

Since $\mathrm{Re}\{w(\mathrm{e}^{\mathrm{i}\theta})\} \geq 1$ implies $\left|F(\mathrm{e}^{\mathrm{i}\theta})\right| \geq \lambda$,

$$\mathrm{Re}\left\{1 + \frac{F(\mathrm{e}^{\mathrm{i}\theta}) - \lambda}{F(\mathrm{e}^{\mathrm{i}\theta}) + \lambda}\right\} = \mathrm{Re}\{w(\mathrm{e}^{\mathrm{i}\theta})\} \geq 1; \tag{7.70}$$

that is,

$$\mathrm{Re}\left\{\frac{[F(\mathrm{e}^{\mathrm{i}\theta}) - \lambda][F(\mathrm{e}^{\mathrm{i}\theta})^* + \lambda]}{[F(\mathrm{e}^{\mathrm{i}\theta}) + \lambda][F(\mathrm{e}^{\mathrm{i}\theta})^* + \lambda}\right\} \geq 0, \tag{7.71}$$

and hence $\left|F(\mathrm{e}^{\mathrm{i}\theta})\right|^2 - \lambda^2 \geq 0$. It follows that

$$\begin{aligned} \left|\theta \in [-\pi, \pi] : \left|\tilde{f}(\theta)\right| \geq \lambda\right| &\subset \left|\theta \in [-\pi, \pi] : \left|F(\mathrm{e}^{\mathrm{i}\theta})\right| \geq \lambda\right| \\ &\subseteq \left|\theta \in [-\pi, \pi] : \mathrm{Re}\{w(\mathrm{e}^{\mathrm{i}\theta})\} \geq 1\right|. \end{aligned} \tag{7.72}$$

Equation (7.72) yields

$$\left|\theta \in [-\pi, \pi] : \left|\tilde{f}(\theta)\right| \geq \lambda\right| \leq \frac{2\|f\|_1}{\lambda}, \tag{7.73}$$

which is the required result.

7.3 A Zygmund inequality

The next result discussed is an inequality due to Zygmund, which has a close connection with the two principal inequalities considered so far in this chapter. It is known from the results in Section 6.17 on the Riesz inequality that $f \in L^1(\mathbb{T})$ does not guarantee that $\mathcal{H}f \in L^1(\mathbb{T})$, which raises the obvious question as to what must be the least rate of growth for the function f in order for $\mathcal{H}f \in L^1$. An inequality due to Zygmund (1929, 1932, 1968) addresses this issue. The following notational device will be useful:

$$\log^+ f = \begin{cases} \log f, & f \geq 1 \\ 0, & 0 \leq f < 1. \end{cases} \tag{7.74}$$

A function f belongs to the class $L\log L(\mathbb{T})$ (sometimes referred to as the Zygmund $L\log L$ class) if

$$\int_0^{2\pi} |f(s)|\log^+|f(s)|\mathrm{d}s < \infty. \tag{7.75}$$

The Zygmund inequality reads as follows: if $f \in L\log L(\mathbb{T})$, then $\widetilde{f} \in L^1$ and

$$\int_0^{2\pi} \left|\widetilde{f}(\theta)\right|\mathrm{d}\theta \le A\int_0^{2\pi} |f(\theta)|\log^+|f(\theta)|\mathrm{d}\theta + B, \tag{7.76}$$

where A and B are constants. This is sometimes referred to as Zygmund's $L\log L$ theorem. There are different methods to establish this result; the following discussion employs a Green's theorem approach. The same notation as in the preceding section is employed, but in place of $u(z) > 0$, the constraint $u(z) \ge \mathrm{e} = 2.718\ldots$ is employed. First, note that

$$\nabla^2 u\log u = u^{-1}\left|F'\right|^2, \tag{7.77}$$

where $\left|F'\right|$ is defined in Eq. (7.7), and use the result $|F| \ge u$. Then

$$|F|^{-1}\left|F'\right|^2 \le u^{-1}\left|F'\right|^2. \tag{7.78}$$

This result can be rewritten using Eqs. (7.11) and (7.77), and noting that $u(z) \ge \mathrm{e}$, to give

$$\nabla^2|F| \le \nabla^2 u\log u. \tag{7.79}$$

Integrating this inequality yields

$$\iint\limits_{|z|\le r} \nabla^2|F|\,\mathrm{d}x\,\mathrm{d}y \le \iint\limits_{|z|\le r} \nabla^2 u\log u\,\mathrm{d}x\,\mathrm{d}y. \tag{7.80}$$

On using Green's theorem in the form of Eq. (7.19), with φ identified successively with $\left|F(r\mathrm{e}^{\mathrm{i}\theta})\right|$ and then with $u\log u$, Eq. (7.80) simplifies as follows:

$$\int_0^{2\pi} \frac{\partial\left|F(r\mathrm{e}^{\mathrm{i}\theta})\right|}{\partial r}\mathrm{d}\theta \le \int_0^{2\pi} \frac{\partial\{u(r\mathrm{e}^{\mathrm{i}\theta})\log u(r\mathrm{e}^{\mathrm{i}\theta})\}}{\partial r}\mathrm{d}\theta, \tag{7.81}$$

which, on integrating from 0 to r, yields

$$\int_0^{2\pi} \left|F(r\mathrm{e}^{\mathrm{i}\theta})\right|\mathrm{d}\theta = \int_0^{2\pi} u(r\mathrm{e}^{\mathrm{i}\theta})\log u(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta - 2\pi u(0)\{\log u(0) - 1\}. \tag{7.82}$$

With the restriction that $u(0) > \mathrm{e}$, and noting that $\left|F(r\mathrm{e}^{\mathrm{i}\theta})\right| > \left|v(r\mathrm{e}^{\mathrm{i}\theta})\right|$, leads to

$$\int_0^{2\pi} \left|v(r\mathrm{e}^{\mathrm{i}\theta})\right| \mathrm{d}\theta \leq \int_0^{2\pi} u(r\mathrm{e}^{\mathrm{i}\theta}) \log u(r\mathrm{e}^{\mathrm{i}\theta}) \mathrm{d}\theta, \tag{7.83}$$

which can be expressed as follows:

$$\int_0^{2\pi} \left|v(r\mathrm{e}^{\mathrm{i}\theta})\right| \mathrm{d}\theta \leq \int_0^{2\pi} \left|u(r\mathrm{e}^{\mathrm{i}\theta})\right| \log^+ \left|u(r\mathrm{e}^{\mathrm{i}\theta})\right| \mathrm{d}\theta. \tag{7.84}$$

Using Eqs. (7.26) and (7.27),

$$\int_0^{2\pi} \left|\widetilde{f}(\theta)\right| \mathrm{d}\theta \leq \int_0^{2\pi} |f(\theta)| \log^+ |f(\theta)| \mathrm{d}\theta. \tag{7.85}$$

This is the desired Eq. (7.76), with $A = 1$ and $B = 0$.

To deal with a general function of variable sign, proceed as follows. Let

$$f(\theta) = f_1(\theta) + f_2(\theta) + f_3(\theta), \tag{7.86}$$

where

$$f_1(\theta) = \max\{f(\theta), \mathrm{e}\}, \tag{7.87}$$

$$f_2(\theta) = \min\{f(\theta), -\mathrm{e}\}, \tag{7.88}$$

and

$$-\mathrm{e} \leq f_3(\theta) \leq \mathrm{e}. \tag{7.89}$$

Now,

$$\widetilde{f}(\theta) = \widetilde{f}_1(\theta) + \widetilde{f}_2(\theta) + \widetilde{f}_3(\theta), \tag{7.90}$$

which yields

$$\left|\widetilde{f}(\theta)\right| \leq \left|\widetilde{f}_1(\theta)\right| + \left|\widetilde{f}_2(\theta)\right| + \left|\widetilde{f}_3(\theta)\right|, \tag{7.91}$$

and hence

$$\int_0^{2\pi} \left|\widetilde{f}(\theta)\right| \mathrm{d}\theta \leq \int_0^{2\pi} \left|\widetilde{f}_1(\theta)\right| \mathrm{d}\theta + \int_0^{2\pi} \left|\widetilde{f}_2(\theta)\right| \mathrm{d}\theta + \int_0^{2\pi} \left|\widetilde{f}_3(\theta)\right| \mathrm{d}\theta. \tag{7.92}$$

Using the Cauchy–Schwarz–Buniakowski inequality and employing Eq. (7.89) gives

$$\begin{aligned}\int_0^{2\pi} \left|\widetilde{f}_3(\theta)\right| \mathrm{d}\theta &\leq \left\{\int_0^{2\pi} \mathrm{d}\theta\right\}^{1/2} \left\{\int_0^{2\pi} \left|\widetilde{f}_3(\theta)\right|^2 \mathrm{d}\theta\right\}^{1/2} \\ &= \sqrt{(2\pi)} \left\{\int_0^{2\pi} |f_3(\theta)|^2 \,\mathrm{d}\theta\right\}^{1/2} \\ &\leq \sqrt{(2\pi)} \left(2\pi \mathrm{e}^2\right)^{1/2}, \end{aligned} \tag{7.93}$$

that is

$$\int_0^{2\pi} \left|\widetilde{f}_3(\theta)\right| \mathrm{d}\theta \leq 2\pi \mathrm{e}. \tag{7.94}$$

Denote the set of points where $f \geq \mathrm{e}$ by E_1, and the complement set by $E_{1,\mathrm{c}}$, and denote the set of points where $f \leq -\mathrm{e}$ by E_2 and the complement by $E_{2,\mathrm{c}}$. Then

$$\begin{aligned}\sum_{k=1}^{2} \int_0^{2\pi} \left|\widetilde{f}_k(\theta)\right| \mathrm{d}\theta &\leq \sum_{k=1}^{2} \int_0^{2\pi} |f_k(\theta)| \log|f_k(\theta)| \mathrm{d}\theta \\ &= \sum_{k=1}^{2} \left\{\int_{E_k} |f_k(\theta)| \log|f_k(\theta)| \mathrm{d}\theta + \int_{E_{k,\mathrm{c}}} |f_k(\theta)| \log|f_k(\theta)| \mathrm{d}\theta\right\} \\ &\leq \sum_{k=1}^{2} \left\{\int_{E_k} |f(\theta)| \log|f(\theta)| \mathrm{d}\theta + \int_{E_{k,\mathrm{c}}} |f_k(\theta)| \log|f_k(\theta)| \mathrm{d}\theta\right\} \\ &\leq \int_0^{2\pi} |f(\theta)| \log^+|f(\theta)| \mathrm{d}\theta + \sum_{k=1}^{2} \int_{E_{k,\mathrm{c}}} \mathrm{e} \log \mathrm{e} \,\mathrm{d}\theta \\ &\leq \int_0^{2\pi} |f(\theta)| \log^+|f(\theta)| \mathrm{d}\theta + 4\pi \mathrm{e}. \end{aligned} \tag{7.95}$$

Substituting the inequalities Eqs. (7.94) and (7.95) into Eq. (7.92) yields the required result with $A = 1$ and $B = 6\pi \mathrm{e}$. Pichorides (1972) gave the bound $A > 2\pi^{-1}$ for the constant appearing in Zygmund's inequality.

No direct analog of the Zygmund inequality for the Hilbert transform on the line is known. The reader is invited to suggest an explanation for this fact.

7.4 A Bernstein inequality

One form of the classical Bernstein inequality reads as follows. if $f(z)$ is an entire function of exponential type σ (see Section 2.8.7), and is bounded on the real axis, then

$$\left|f'(x)\right| \le \sigma \sup|f(x)|\,. \tag{7.96}$$

The class of entire functions of exponential type $\le \sigma$ is denoted by E^σ. The equality sign in Eq. (7.96) holds if

$$f(z) = \alpha e^{i\sigma z} + \beta e^{-i\sigma z}, \tag{7.97}$$

where α and β are arbitrary constants. Alternatively, it follows for $f \in E^\sigma \cap L^\infty(\mathbb{R})$ that

$$\left\|f'\right\|_\infty \le \sigma \|f\|_\infty\,. \tag{7.98}$$

The objective of this section is to establish the following result:

$$\sin\pi\alpha\, f'(x) + \cos\pi\alpha\,(Hf)'(x) = \sum_{k=-\infty}^{\infty} \frac{\{(-1)^k - \cos\pi\alpha\} f(k+\alpha+x)}{\pi(k+\alpha)^2}, \tag{7.99}$$

where $\alpha \in \mathbb{R}$ and $f \in E^\pi \cap L^2(\mathbb{R})$. Equation (7.99) is a rather useful result since it allows both $f'(x)$ and $(Hf)'(x)$ to be determined from a sequence of discrete values of f. Furthermore, it leads to a generalization of the classical Bernstein inequality. From Eq. (7.99), the following Bernstein-type inequality can be obtained:

$$\left\|\sin\pi\alpha f'(x) + \cos\pi\alpha\,(Hf)'(x)\right\|_\infty \le \pi\|f\|_\infty\,. \tag{7.100}$$

The particular case $\alpha = 1/2$ leads to the classical Bernstein inequality for the case $\sigma = \pi$.

The starting point in the derivation is the Whittaker–Shannon–Kotel'nikov theorem: if the function $f \in L^2(\mathbb{R})$ has a Fourier transform $\hat{f}(x)$ having compact support in the interval $[-\sigma, \sigma]$, then

$$f(x) = \sum_{k=-\infty}^{\infty} f(x_k)\,\mathrm{sinc}\{\sigma\pi^{-1}(x - x_k)\}, \quad \text{for } x \in \mathbb{R}, \tag{7.101}$$

where $x_k = k\pi/\sigma$, recalling from Section 4.15 that the sinc function is defined by

$$\mathrm{sinc}\,x = \frac{\sin\pi x}{\pi x}. \tag{7.102}$$

The preceding theorem is known by engineers as the Shannon *sampling theorem*, though it was discovered (actually rediscovered) earlier by the Russian engineer Kotel'nikov. Mathematicians frequently call it Whittaker's cardinal series, or simply the cardinal series. In engineering applications, functions that have their support restricted to some finite interval are referred to as *band-limited* functions. The importance of Eq. (7.101) stems from the fact that the continuous function f can be determined uniquely from a sequence of discrete points spaced π/σ apart. This has significant applications in signal processing. Equation (7.101) can be established in the following manner. Since $\hat{f}(x)$ vanishes for $|x| > \sigma$, the Fourier inversion formula can be written as follows:

$$f(x) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} \hat{f}(s)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s. \tag{7.103}$$

It might be useful to think of the function $\hat{f}(s)$ as having a periodic extension beyond $[-\sigma,\sigma]$ and over $\mathbb{R}$. Assume $\hat{f}$ has a Fourier series representation,

$$\hat{f}(s) \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}k\pi s/\sigma}, \tag{7.104}$$

with

$$\begin{aligned} c_k &= \frac{1}{2\sigma}\int_{-\sigma}^{\sigma} \hat{f}(s)\mathrm{e}^{-\mathrm{i}k\pi s/\sigma}\,\mathrm{d}s \\ &= \frac{\pi}{\sigma} f\left(-\frac{k\pi}{\sigma}\right). \end{aligned} \tag{7.105}$$

Now,

$$\begin{aligned} \int_{-\sigma}^{\sigma} \hat{f}(s)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s &= \sum_{k=-\infty}^{\infty} c_k \int_{-\sigma}^{\sigma} \mathrm{e}^{\mathrm{i}(x+k\pi\sigma^{-1})s}\,\mathrm{d}s \\ &= 2\sigma \sum_{k=-\infty}^{\infty} c_k \operatorname{sinc}\Big[(x+k\pi\sigma^{-1})\frac{\sigma}{\pi}\Big] \\ &= 2\pi \sum_{k=-\infty}^{\infty} f\left(-\frac{k\pi}{\sigma}\right) \operatorname{sinc}\Big[(x+k\pi\sigma^{-1})\frac{\sigma}{\pi}\Big], \end{aligned} \tag{7.106}$$

and hence

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} \hat{f}(s)\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s \\ &= \sum_{k=-\infty}^{\infty} f\left(-\frac{k\pi}{\sigma}\right) \operatorname{sinc}\Big[(x+k\pi\sigma^{-1})\frac{\sigma}{\pi}\Big]. \end{aligned} \tag{7.107}$$

Making the replacement $k \to -k$ leads to

$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc}\Big[(x-k\pi\sigma^{-1})\frac{\sigma}{\pi}\Big], \tag{7.108}$$

which is the required result.

The proofs of Eqs. (7.99) and (7.100) are now considered, and the approach employed follows the work of Kochneff, Sagher, and Tan (1992). Suppose that $f \in E^{\pi} \cap L^2(\mathbb{R})$, since $(f(z) - f(0))/z \in E^{\pi} \cap L^2$, then Eq. (7.108) can be used for the case $\sigma = \pi$ to obtain

$$\begin{aligned} \frac{f(x) - f(0)}{x} &= \sum_{k=-\infty}^{\infty} \frac{f(k) - f(0)}{k} \operatorname{sinc}(x - k) \\ &= \operatorname{sinc} x f'(0) + \sum_{k=-\infty}^{\infty}{}' \frac{f(k) - f(0)}{k} \operatorname{sinc}(x - k), \end{aligned} \tag{7.109}$$

where the prime on the summation signifies that the term $k = 0$ is omitted. From this result it follows that

$$f(x) = x \operatorname{sinc} x\, f'(0) + \operatorname{sinc} x f(0) + x \sum_{k=-\infty}^{\infty}{}' \frac{f(k)\operatorname{sinc}(x - k)}{k}. \tag{7.110}$$

Now, $H(\operatorname{sinc} x)$ can be evaluated as follows:

$$\begin{aligned} H(\operatorname{sinc} x) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{sinc} s\, \mathrm{d}s}{x - s} \\ &= \frac{1}{x\pi^2} P \int_{-\infty}^{\infty} \sin y \left\{ \frac{1}{y} + \frac{1}{\pi x - y} \right\} \mathrm{d}y \\ &= \frac{1}{x\pi} \{1 - \cos \pi x\}, \end{aligned} \tag{7.111}$$

and hence $H\{\operatorname{sinc}(x - a)\} = \{\pi(x - a)\}^{-1}\{1 - \cos \pi(x - a)\}$. Using this result and Eq. (7.109) gives

$$\begin{aligned} H\left\{\frac{f(x) - f(0)}{x}\right\} &= H\{\operatorname{sinc} x\} f'(0) + \sum_{k=-\infty}^{\infty}{}' \frac{f(k) - f(0)}{k} H\{\operatorname{sinc}(x - k)\} \\ &= \frac{1 - \cos \pi x}{\pi x} f'(0) + \sum_{k=-\infty}^{\infty}{}' \frac{\{f(k) - f(0)\}(1 - \cos[\pi(x - k)])}{k\pi(x - k)}. \end{aligned} \tag{7.112}$$

Hence,

$$\begin{aligned} xH\left\{\frac{f(x) - f(0)}{x}\right\} &= \left\{\frac{1 - \cos \pi x}{\pi x}\right\} \{x f'(0) + f(0)\} \\ &\quad + x \sum_{k=-\infty}^{\infty}{}' \frac{f(k)(1 - \cos[\pi(x - k)])}{k\pi(x - k)}, \end{aligned} \tag{7.113}$$

where the following result has been employed:

$$\sum_{k=-\infty}^{\infty}{}' \frac{(1-\cos[\pi(x-k)])}{k\pi(x-k)} = \frac{\cos\pi x - 1}{\pi x^2}. \tag{7.114}$$

This last equation can be derived in the following manner. Expanding the cosine function gives

$$\sum_{k=-\infty}^{\infty}{}' \frac{(1-\cos[\pi(x-k)])}{k\pi(x-k)} = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}{}' \frac{1}{k(x-k)} - \frac{\cos\pi x}{\pi}\sum_{k=-\infty}^{\infty}{}' \frac{(-1)^k}{k(x-k)}, \tag{7.115}$$

and employing the standard series results (Hansen, 1975, p. 102),

$$\sum_{k=-\infty}^{\infty}{}' \frac{1}{k(x-k)} = \frac{\pi x\cot\pi x - 1}{x^2} \tag{7.116}$$

and

$$\sum_{k=-\infty}^{\infty}{}' \frac{(-1)^k}{k(x-k)} = \frac{\pi x\csc\pi x - 1}{x^2}, \tag{7.117}$$

yields

$$\sum_{k=-\infty}^{\infty}{}' \frac{(1-\cos[\pi(x-k)])}{k\pi(x-k)} = \frac{\pi x\cot\pi x - 1}{\pi x^2} - \frac{(\pi x\csc\pi x - 1)\cos\pi x}{\pi x^2}, \tag{7.118}$$

which leads to Eq. (7.114). Equations (7.116) and (7.117) can be established by contour integration techniques.

Using Eqs. (7.110) and (7.113) gives

$$\begin{aligned}
&\sin\pi\alpha\, f(x) + \cos\pi\alpha\, xH\left\{\frac{f(x)-f(0)}{x}\right\}\\
&\quad= \frac{f(0)}{\pi x}(\sin\pi\alpha\sin\pi x + \cos\pi\alpha\{1-\cos\pi x\})\\
&\quad\quad+ \frac{f'(0)}{\pi}(\sin\pi\alpha\sin\pi x + \cos\pi\alpha\{1-\cos\pi x\})\\
&\quad\quad+ x\sum_{k=-\infty}^{\infty}{}' \frac{f(k)}{k\pi(x-k)}\{\sin\pi\alpha\sin\pi(x-k)\\
&\quad\quad+ \cos\pi\alpha - \cos\pi\alpha\cos\pi(x-k)\}
\end{aligned}$$

$$= x \sum_{k=-\infty}^{\infty}{}' \frac{f(k)}{k\pi(x-k)}\{\cos\pi\alpha - \cos\pi(\alpha+x-k)\}$$
$$+ \frac{\{\cos\pi\alpha - \cos\pi(\alpha+x)\}}{\pi x}\{f(0)+xf'(0)\}. \tag{7.119}$$

Now make use of Eq. (4.124), with $f(x)$ replaced by $f(x) - f(0)$; then

$$xH\left\{\frac{f(x)-f(0)}{x}\right\} = H\{f(x)-f(0)\} - H\{f(x)-f(0)\}(0)$$
$$= Hf(x) - Hf(0), \tag{7.120}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xH\left\{\frac{f(x)-f(0)}{x}\right\}\right] = Hf'(x). \tag{7.121}$$

Taking the derivative of Eq. (7.119) and making use of Eq. (7.121) yields

$$\sin\pi\alpha\, f'(x) + \cos\pi\alpha\, Hf'(x) = \sum_{k=-\infty}^{\infty}{}' \frac{f(k)}{k\pi(x-k)^2}\{\pi x(x-k)\sin\pi(\alpha+x-k)$$
$$+ k\cos\pi(\alpha+x-k) - k\cos\pi\alpha\}$$
$$+ \frac{f(0)}{\pi x^2}\{\pi x\sin\pi(\alpha+x) - \cos\pi\alpha + \cos\pi(\alpha+x)\}$$
$$+ f'(0)\sin\pi(\alpha+x). \tag{7.122}$$

Now insert $x = -\alpha$; then

$$\sin\pi\alpha\, f'(-\alpha) + \cos\pi\alpha\, Hf'(-\alpha) = \frac{f(0)}{\pi\alpha^2}(1-\cos\pi\alpha)$$
$$+ \sum_{k=-\infty}^{\infty}{}' \frac{f(k)\{(-1)^k - \cos\pi\alpha\}}{\pi(\alpha+k)^2}$$
$$= \sum_{k=-\infty}^{\infty} \frac{f(k)\{(-1)^k - \cos\pi\alpha\}}{\pi(\alpha+k)^2}. \tag{7.123}$$

For a function $g(z) \in E^{\pi} \cap L^2$, Eq. (7.123) can be rewritten with f replaced by the function g. Making the identification $g(z) = f(z+\alpha+x)$, with $x \in \mathbb{R}$, then $g(k) = f(k+\alpha+x)$ and $g(-\alpha) \equiv f(x)$; hence it follows that

$$\sin\pi\alpha\, f'(x) + \cos\pi\alpha\, Hf'(x) = \sum_{k=-\infty}^{\infty} \frac{f(k+\alpha+x)\{(-1)^k - \cos\pi\alpha\}}{\pi(\alpha+k)^2}, \tag{7.124}$$

which is the first key required result. From this formula it follows by inspection that, for $\alpha = 1/2$,

$$f'(x) = \frac{4}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k f(k + (1/2) + x)}{(2k+1)^2}. \tag{7.125}$$

For $\alpha = 0$,

$$\begin{aligned} Hf'(x) &= \sum_{k=-\infty}^{\infty}{}' \frac{f(k+x)\{(-1)^k - 1\}}{\pi k^2} + \lim_{\alpha \to 0} \frac{f(\alpha + x)\{1 - \cos \pi\alpha\}}{\pi \alpha^2} \\ &= \frac{\pi f(x)}{2} - \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{f(2k+1+x)}{(2k+1)^2}. \end{aligned} \tag{7.126}$$

To establish Eq. (7.100) requires a proof of the following equation:

$$\sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k \cos \pi\alpha}{\pi(k+\alpha)^2} = \pi. \tag{7.127}$$

Making use of Eq. (7.123) with $f(z) = \cos z\pi$, hence $f'(-\alpha) = \pi \sin \pi\alpha$ and $Hf'(-\alpha) = \pi \cos \pi\alpha$, leads to

$$\pi \sin^2 \pi\alpha + \pi \cos^2 \pi\alpha = \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k \cos \pi\alpha}{\pi(\alpha+k)^2}, \tag{7.128}$$

and Eq. (7.127) follows. From Eq. (7.124),

$$\begin{aligned} \left|\sin \pi\alpha f'(x) + \cos \pi\alpha\, Hf'(x)\right| &= \left| \sum_{k=-\infty}^{\infty} \frac{f(k+\alpha+x)\{(-1)^k - \cos \pi\alpha\}}{\pi(\alpha+k)^2} \right| \\ &\le \sum_{k=-\infty}^{\infty} |f(k+\alpha+x)| \left| \frac{\{(-1)^k - \cos \pi\alpha\}}{\pi(\alpha+k)^2} \right|, \end{aligned} \tag{7.129}$$

and hence

$$\left|\sin \pi\alpha f'(x) + \cos \pi\alpha\, (Hf')(x)\right| \le \sup f \sum_{k=-\infty}^{\infty} \frac{\{1 - (-1)^k \cos \pi\alpha\}}{\pi(\alpha+k)^2}. \tag{7.130}$$

Making use of Eq. (7.127) yields

$$\left\|\sin \pi\alpha f'(x) + \cos \pi\alpha\, Hf'(x)\right\|_\infty \le \pi \|f\|_\infty, \tag{7.131}$$

which is the required result.

7.5 The Hilbert transform of a function having a bounded integral and derivative

It has been shown previously that the p-norm of the Hilbert transform is bounded above and below for $1 < p < \infty$. It is of interest to know if any bounds of a general nature can be found for the Hilbert transform. The answer is affirmative with appropriate restrictions imposed on the function. In this section an inequality derived by Logan (1983a) is discussed. Let f be defined on the real line and suppose that, for some constant M and any interval (a, b),

$$\left| \int_a^b f(x)\mathrm{d}x \right| \le M, \tag{7.132}$$

that $f'(x)$ is a bounded function satisfying, for some constant m,

$$\left| f'(x) \right| \le m, \tag{7.133}$$

and the integral of $f'(x)$ is f. For many different types of experimental data, both of the last two conditions are often satisfied. Given a function f that satisfies these conditions, then Hf is bounded above, and the following result holds:

$$|Hf(0)| \le \frac{4\log 2}{\pi}\sqrt{(mM)}. \tag{7.134}$$

To establish this inequality, set

$$F(t) = \int_0^t f(x)\mathrm{d}x. \tag{7.135}$$

It is then straightforward to show for a positive τ that

$$F(\tau) + F(-\tau) + \int_{-\tau}^{\tau} \{|t| - \tau\} f'(t)\mathrm{d}t = 0. \tag{7.136}$$

Note that

$$\begin{aligned}\int_{|t|>\tau} \frac{F(t)}{t^2}\,\mathrm{d}t &\equiv \int_{-\infty}^{-\tau} \frac{F(t)}{t^2}\,\mathrm{d}t + \int_{\tau}^{\infty} \frac{F(t)}{t^2}\,\mathrm{d}t \\ &= \int_{-\infty}^{-\tau} \frac{f(t)}{t}\,\mathrm{d}t + \int_{\tau}^{\infty} \frac{f(t)}{t}\,\mathrm{d}t + \tau^{-1}\{F(\tau) + F(-\tau)\}.\end{aligned} \tag{7.137}$$

Now,

$$
\begin{aligned}
P\int_{-\infty}^{\infty}\frac{f(t)}{t}\,\mathrm{d}t &= \lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{-\tau}\frac{f(t)}{t}\,\mathrm{d}t+\int_{-\tau}^{-\varepsilon}\frac{f(t)}{t}\,\mathrm{d}t+\int_{\varepsilon}^{\tau}\frac{f(t)}{t}\,\mathrm{d}t+\int_{\tau}^{\infty}\frac{f(t)}{t}\,\mathrm{d}t\right\}\\
&= \int_{|t|>\tau}\frac{F(t)}{t^2}\,\mathrm{d}t-\tau^{-1}\{F(\tau)+F(-\tau)\}+\int_{-\tau}^{\tau}f'(t)\log\left|\frac{\tau}{t}\right|\mathrm{d}t\\
&= \int_{-\tau}^{\tau}f'(t)\left\{\log\left|\frac{\tau}{t}\right|-c\{\tau-|t|\}\right\}\mathrm{d}t\\
&\quad+\int_{|t|>\tau}\frac{F(t)}{t^2}\,\mathrm{d}t+\{c-\tau^{-1}\}\{F(\tau)+F(-\tau)\},
\end{aligned}
\tag{7.138}
$$

where Eq. (7.136) has been employed and c is an arbitrary constant. From the preceding equation it follows that

$$
\left|P\int_{-\infty}^{\infty}\frac{f(t)}{t}\,\mathrm{d}t\right|\le m\int_{-\tau}^{\tau}\left|\log\left|\frac{\tau}{t}\right|-c\{\tau-|t|\}\right|\mathrm{d}t+2M\left\{\tau^{-1}+\left|c-\tau^{-1}\right|\right\}.
\tag{7.139}
$$

Put $\alpha=c\tau$, $\beta=2M\tau^{-1}$, and $\gamma=4mM$; then the minimum of

$$
I(\alpha,\beta)=\{1+|\alpha-1|\}\beta+\gamma\beta^{-1}\int_0^1|\log x+\alpha(1-x)|\mathrm{d}x
\tag{7.140}
$$

is required with respect to α and β. Setting the derivatives of $I(\alpha,\beta)$ with respect to α and β equal to zero, it follows for $\alpha\ge 1$ that

$$
\beta+\gamma\beta^{-1}\{x_0^2-2x_0+1/2\}=0
\tag{7.141}
$$

and

$$
\alpha-\gamma\beta^{-2}\{2x_0-\alpha x_0^2-1+\alpha/2\}=0,
\tag{7.142}
$$

where x_0 denotes the point where $\log x+\alpha(1-x)$ changes sign on the interval $(0,1)$. The solution of these two equations is given by $\alpha=2\log 2$ and $\beta=(1/2)\surd\gamma$, with $x_0=1/2$. Expressing c and τ in terms of α and β and inserting the results into Eq. (7.139) leads to Eq. (7.134). The result in Eq. (7.134) is trivial if f is an even function.

Consider an odd function f for which $\mathcal{F}f$ is of fixed sign on the interval $(0,\infty)$; from Eq. (5.3) it follows that

$$
\begin{aligned}
Hf(x) &= \mathcal{F}^{-1}\{-\mathrm{i}\,\mathrm{sgn}\,y\,(\mathcal{F}f)(y)\}(x)\\
&= -\frac{\mathrm{i}}{\pi}\int_0^{\infty}\cos xy\,(\mathcal{F}f)(y)\mathrm{d}y,
\end{aligned}
\tag{7.143}
$$

and hence

$$\begin{aligned}
|Hf(x)| &= \frac{1}{\pi}\left|\int_0^\infty \cos xy\,(\mathcal{F}f)(y)\mathrm{d}y\right| \\
&\le \frac{1}{\pi}\int_0^\infty |\cos xy\,(\mathcal{F}f)(y)|\mathrm{d}y \\
&\le \frac{1}{\pi}\int_0^\infty |(\mathcal{F}f)(y)|\mathrm{d}y \\
&= \frac{1}{\pi}\left|\int_0^\infty (\mathcal{F}f)(y)\mathrm{d}y\right|,
\end{aligned} \tag{7.144}$$

and therefore

$$|Hf(x)| \le |Hf(0)|\,. \tag{7.145}$$

Combining the preceding result with Eq. (7.134) gives

$$|Hf(x)| \le \frac{4\log 2}{\pi}\sqrt{(mM)}, \tag{7.146}$$

assuming the specified conditions on f hold.

Toland (1997b) indicates that if $f \in \mathcal{D}$, with m designating the $L^\infty(\mathbb{R})$ norm of f' and M denoting $\sup_{a,b}\left\{\left|\int_a^b f(x)\mathrm{d}x\right|\right\}$, then

$$\|Hf\|_{L^\infty(\mathbb{R})} \le 4\pi^{-1}\log 2\sqrt{(mM)}. \tag{7.147}$$

7.6 Connections between the Hilbert transform on $\mathbb{R}$ and $\mathbb{T}$

This section considers the connections between the Hilbert transforms on $\mathbb{R}$ and $\mathbb{T}$. Using an idea of Zygmund (1955, p. 317) define

$$g_n(x) = \frac{1}{2n\pi}P\int_{-n\pi}^{n\pi} f(t)\cot\left(\frac{x-t}{2n}\right)\mathrm{d}t, \tag{7.148}$$

where $f \in L^p(\mathbb{R})$, for $1 < p < \infty$, and let $g(x) = (Hf)(x)$. Recalling that

$$\lim_{y\to 0}\cot y = y^{-1} + O(y), \tag{7.149}$$

then taking the lim $n \to \infty$ in Eq. (7.148) leads to

$$\lim_{n\to\infty} g_n(x) = g(x). \tag{7.150}$$

From the definition of g_n it follows that

$$\begin{aligned} g_n(nx) &= \frac{1}{2n\pi} P \int_{-n\pi}^{n\pi} f(t) \cot\left(\frac{nx - t}{2n}\right) \mathrm{d}t \\ &= \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(nt) \cot\left(\frac{x - t}{2}\right) \mathrm{d}t \\ &= \mathcal{H}\phi_n(x), \end{aligned} \tag{7.151}$$

where $\phi_n(x) = f(nx)$ for $|x| < \pi$. Hence

$$n \int_{-\pi}^{\pi} |g_n(nx)|^p \, \mathrm{d}x = n \int_{-\pi}^{\pi} |\mathcal{H}\phi_n(x)|^p \, \mathrm{d}x, \tag{7.152}$$

which simplifies, on making the change of variable $y = nx$, to give

$$\int_{-n\pi}^{n\pi} |g_n(y)|^p \, \mathrm{d}y = \int_{-n\pi}^{n\pi} |\mathcal{H}f(y)|^p \, \mathrm{d}y. \tag{7.153}$$

Fatou's lemma takes the following form: if the non-negative functions f_n are measurable, and if $\lim_{n\to\infty} f_n = f$ *a.e.*, then

$$\int f(x)\mathrm{d}x = \int \lim_{n\to\infty} \inf f_n(x)\mathrm{d}x \le \lim_{n\to\infty} \inf \int f_n(x)\mathrm{d}x. \tag{7.154}$$

Now take the limit $n \to \infty$ in Eq. (7.153) and apply Fatou's lemma, yielding

$$\int_{-\infty}^{\infty} |g(y)|^p \, \mathrm{d}y \le \int_{-\infty}^{\infty} |\mathcal{H}f(y)|^p \, \mathrm{d}y; \tag{7.155}$$

assuming both integrals exist,

$$\|Hf\|_p \le \|\mathcal{H}f\|_p, \tag{7.156}$$

which is the desired result.

Set $h_n = |g_n|^p$. Suppose the following conditions apply: h_n are measurable functions on $\mathbb{R}$, for which $\lim_{n\to\infty} h_n = h$ *a.e.*, and there exists a non-negative function $k(x)$ such that $\int_{-\infty}^{\infty} k(x)\mathrm{d}x < \infty$ and $|h_n| \le k$ for all n. Then $\lim_{n\to\infty} \int h_n(x)\mathrm{d}x = \int h(x)\mathrm{d}x$. This is the Lebesgue dominated convergence theorem. Then from Eq. (7.153), on taking the limit $n \to \infty$ and applying the Lebesgue dominated convergence theorem, leads to

$$\int_{-\infty}^{\infty} |g(y)|^p \, \mathrm{d}y = \int_{-\infty}^{\infty} |\mathcal{H}f(y)|^p \, \mathrm{d}y; \tag{7.157}$$

that is,

$$\|Hf\|_p = \|\mathcal{H}f\|_p. \tag{7.158}$$

It is left as an exercise for the reader to determine if weaker conditions can be found such that this final result holds.

7.7 Weighted norm inequalities for the Hilbert transform

In this section some inequality results for the Hilbert transform are discussed that include an additional multiplicative function and take the following general form:

$$\int_{-\infty}^{\infty} |Hf(x)|^p\, w(x)\mathrm{d}x \le C(p,w)\int_{-\infty}^{\infty} |f(x)|^p\, w(x)\mathrm{d}x, \tag{7.159}$$

where $C(p,w)$ is a constant depending on p and w but not on f, and $1 < p < \infty$. Formulas of this type are generally referred to as *weighted norm inequalities*. The simplest case is $w(x) = 1$, for $1 < p < \infty$, which corresponds to the Riesz inequality discussed in Section 4.20. There are two principal issues associated with these inequalities. What conditions are needed to specify the function $w(x)$, and what is the sharpest value for the constant C? Inequalities of this form have attracted a considerable amount of interest. Hunt, Muckenhoupt, and Wheeden (1973) found the general solution to this problem. Before discussing the work of these authors, a few particular cases are examined, thus exhibiting some of the types of inequalities that can be found. A later section will take up a number of the more general issues associated with the work of the aforementioned authors. Weighted inequalities for the Hilbert transform arise in the study of weighted mean convergence of Fourier series, and lead to results on the convergence of certain orthogonal expansions. They also arise for a particular problem in prediction theory.

Historically, the first result of the type given in Eq. (7.159) was obtained by Hardy and Littlewood (1936). They gave the following result: if f is an even function and $x^\alpha f(x) \in L^p(0,\infty)$ with

$$-1 - \frac{1}{p} < \alpha < 1 - \frac{1}{p}, \tag{7.160}$$

then $Hf \in L^p(0,\infty)$ and

$$\int_0^{\infty} \left|x^\alpha Hf(x)\right|^p \mathrm{d}x \le C\int_0^{\infty} \left|x^\alpha f(x)\right|^p \mathrm{d}x, \tag{7.161}$$

where $C \equiv C_{p,\alpha}$. In the following development, the standard convention is employed that C denotes a positive constant, which is not necessarily the same at each occurrence. Often the parameters on which C depends are not displayed until the completion of the derivation. However, if the parameter dependence of the constant is obvious, then it is often not explicitly stated. To establish Eq. (7.161), first note that if $g(x)$ denotes the Hilbert transform of $|x|^\alpha f(x)$, then, on making use of Minkowski's inequality and the Riesz inequality, it follows for $p > 1$ that

$$\begin{aligned}\left\{\int_0^\infty \left|x^\alpha Hf(x)\right|^p \mathrm{d}x\right\}^{p^{-1}} &= \left\{\int_0^\infty \left|g(x) - \{g(x) - x^\alpha Hf(x)\}\right|^p \mathrm{d}x\right\}^{p^{-1}} \\ &\leq \left\{\int_0^\infty |g(x)|^p \,\mathrm{d}x\right\}^{p^{-1}} + \left\{\int_0^\infty \left|g(x) - x^\alpha Hf(x)\right|^p \mathrm{d}x\right\}^{p^{-1}} \\ &\leq C\left\{\int_0^\infty \left|x^\alpha f(x)\right|^p \mathrm{d}x\right\}^{p^{-1}} \\ &\quad + \left\{\int_0^\infty \left|g(x) - x^\alpha Hf(x)\right|^p \mathrm{d}x\right\}^{p^{-1}}. \end{aligned} \tag{7.162}$$

To prove Eq. (7.161) it is therefore sufficient to prove

$$\int_0^\infty \left|g(x) - x^\alpha Hf(x)\right|^p \mathrm{d}x \leq C\int_0^\infty \left|x^\alpha f(x)\right|^p \mathrm{d}x. \tag{7.163}$$

Now,

$$\begin{aligned} g(x) - x^\alpha Hf(x) &= \frac{2x}{\pi}P\int_0^\infty \frac{|y|^\alpha f(y)\mathrm{d}y}{x^2 - y^2} - \frac{2x^{1+\alpha}}{\pi}P\int_0^\infty \frac{f(y)\mathrm{d}y}{x^2 - y^2} \\ &= \frac{2}{\pi}P\int_0^\infty y^\alpha f(y)M(y,x)\mathrm{d}y, \end{aligned} \tag{7.164}$$

with

$$M(y,x) = -\frac{x}{y^\alpha}\left(\frac{x^\alpha - y^\alpha}{x^2 - y^2}\right). \tag{7.165}$$

If

$$f(kx) = k^n f(x), \tag{7.166}$$

then f is said to be *homogeneous* of degree n. With the substitutions $x \to kx$ and $y \to ky$, then clearly $M(y,x)$ is homogeneous of degree –1. Also,

$$\int_0^\infty |M(y,1)|y^{-p^{-1}}\mathrm{d}y = \int_0^\infty \left|\frac{y^\alpha - 1}{y^2 - 1}\right| y^{-\alpha-p^{-1}}\mathrm{d}y < \infty, \tag{7.167}$$

provided α satisfies Eq. (7.160). To proceed, a result from Hardy *et al.* (1952, p. 229) is required: let $K(x,y)$ be non-negative and homogeneous of degree -1, $p > 1$, with

$$\int_0^\infty K(x,1)x^{-p^{-1}}\,\mathrm{d}x = \int_0^\infty K(1,y)y^{-q^{-1}}\,\mathrm{d}y = C, \tag{7.168}$$

where q is the conjugate exponent of p; then, for a non-negative function $g \in L^q(0,\infty)$, it follows that

$$\int_0^\infty \mathrm{d}x \left(\int_0^\infty K(x,y) g(y) \mathrm{d}y \right)^q \le C^q \int_0^\infty g(y)^q \, \mathrm{d}y. \tag{7.169}$$

To establish this result, start with

$$\begin{aligned} \int_0^\infty \mathrm{d}x \left(\int_0^\infty K(x,y) g(y) \mathrm{d}y \right)^q &= \int_0^\infty \mathrm{d}x \left(\int_0^\infty K(x,y)^{p^{-1}} \left(\frac{x}{y}\right)^{1/pq} \right. \\ &\qquad \left. K(x,y)^{q^{-1}} \left(\frac{y}{x}\right)^{1/pq} g(y) \mathrm{d}y \right)^q \\ &\le \int_0^\infty \mathrm{d}x \left\{ \int_0^\infty K(x,y) \left(\frac{x}{y}\right)^{1/q} \mathrm{d}y \right\}^{q/p} \\ &\qquad \int_0^\infty K(x,y) \left(\frac{y}{x}\right)^{1/p} g(y)^q \, \mathrm{d}y, \end{aligned} \tag{7.170}$$

where the Hölder inequality has been employed. With the change of integration variable $y = xw$ in the first integral on the right-hand side of the preceding equation, and making use of the fact that

$$xK(x, xw) = K(1, w), \tag{7.171}$$

this leads to

$$\begin{aligned} \int_0^\infty \mathrm{d}x \left(\int_0^\infty K(x,y) g(y) \mathrm{d}y \right)^q &\le \int_0^\infty \mathrm{d}x \left\{ \int_0^\infty K(1,w) w^{-q^{-1}} \, \mathrm{d}w \right\}^{q/p} \\ &\qquad \int_0^\infty K(x,y) \left(\frac{y}{x}\right)^{1/p} g(y)^q \, \mathrm{d}y \\ &= C^{q/p} \int_0^\infty \mathrm{d}x \int_0^\infty K(x,y) \left(\frac{y}{x}\right)^{1/p} g(y)^q \, \mathrm{d}y \\ &= C^{q/p} \int_0^\infty g(y)^q \, \mathrm{d}y \int_0^\infty yK(yw, y) w^{-p^{-1}} \mathrm{d}w \\ &= C^{q/p} \int_0^\infty g(y)^q \, \mathrm{d}y \int_0^\infty K(w, 1) w^{-p^{-1}} \mathrm{d}w \\ &= C^{1+q/p} \int_0^\infty g(y)^q \, \mathrm{d}y \\ &= C^q \int_0^\infty g(y)^q \, \mathrm{d}y. \end{aligned} \tag{7.172}$$

Equation (7.168) and the following result:

$$yK(yw, y) = K(w, 1), \tag{7.173}$$

have been employed in the derivation of Eq. (7.172). The interchange of integration order can be justified since both g and K are non-negative and measurable and $K(x, y)g(y)$ is measurable in the xy-plane, and so Fubini's theorem can be applied. From Eq. (7.164), and on utilizing Eq. (7.169), we have

$$\begin{aligned}\int_0^\infty \left|g(x) - x^\alpha Hf(x)\right|^p \mathrm{d}x &= \int_0^\infty \mathrm{d}x \left|\frac{2}{\pi}\int_0^\infty y^\alpha f(y)M(y, x)\mathrm{d}y\right|^p \\ &\le \int_0^\infty \mathrm{d}x \left\{\frac{2}{\pi}\int_0^\infty |M(y, x)| \left|y^\alpha f(y)\right|\mathrm{d}y\right\}^p \\ &\le C\int_0^\infty \left|y^\alpha f(y)\right|^p \mathrm{d}y, \end{aligned}\tag{7.174}$$

which proves Eq. (7.163), and hence Eq. (7.161) is established. A simple case of this inequality occurs when $\alpha = -p^{-1}$, leading to the following result:

$$\int_0^\infty \frac{|Hf(x)|^p \,\mathrm{d}x}{x} \le C_p \int_0^\infty \frac{|f(x)|^p \,\mathrm{d}x}{x}. \tag{7.175}$$

There is a related inequality for the periodic case, due also to Hardy and Littlewood (1936). Let $f(\theta)$ be an even periodic function (period 2π) and suppose $\theta^{-1}|f(\theta)|$ is integrable over $[0, \pi)$; then

$$\int_0^\pi \frac{\left|\widetilde{f}(\theta)\right|^p}{\theta}\mathrm{d}\theta \le C_p \int_0^\pi \frac{|f(\theta)|^p}{\theta}\mathrm{d}\theta, \tag{7.176}$$

where $\widetilde{f}$ is the conjugate function of f. To prove this result, let g be an even function; then

$$Hg(x) = \frac{2x}{\pi} P\int_0^\infty \frac{g(y)\mathrm{d}y}{x^2 - y^2}. \tag{7.177}$$

Employ the substitutions

$$x = \tan\left(\frac{\theta}{2}\right), \quad y = \tan\left(\frac{\phi}{2}\right), \tag{7.178}$$

and

$$f(\phi) = g(y), \quad \widetilde{f}(\theta) = Hg(x), \tag{7.179}$$

then

$$\widetilde{f}(\theta) = \frac{\sin\theta}{\pi} P \int_0^{\pi} \frac{f(\phi)\mathrm{d}\phi}{\cos\phi - \cos\theta}. \tag{7.180}$$

Hence,

$$\int_0^{\pi} \frac{|f(\theta)|^p}{\sin\theta}\,\mathrm{d}\theta = \int_0^{\infty} \frac{|g(x)|^p}{x}\,\mathrm{d}x \tag{7.181}$$

and

$$\int_0^{\pi} \frac{\left|\widetilde{f}(\theta)\right|^p}{\sin\theta}\,\mathrm{d}\theta = \int_0^{\infty} \frac{|Hg(x)|^p}{x}\,\mathrm{d}x. \tag{7.182}$$

Employing Eq. (7.176), with f replaced by g, it follows immediately from the preceding two results that

$$\int_0^{\pi} \frac{\left|\widetilde{f}(\theta)\right|^p}{\sin\theta}\,\mathrm{d}\theta \le C_p \int_0^{\pi} \frac{|f(\theta)|^p}{\sin\theta}\,\mathrm{d}\theta. \tag{7.183}$$

Making use of the elementary trigonometric inequality

$$\frac{\sin\theta}{\theta} \le 1, \quad \text{for } 0 \le \theta \le \pi, \tag{7.184}$$

leads to

$$\int_0^{\pi} \frac{C_p\,|f(\theta)|^p - \left|\widetilde{f}(\theta)\right|^p}{\sin\theta}\,\mathrm{d}\theta \ge \int_0^{\pi} \frac{C_p\,|f(\theta)|^p - \left|\widetilde{f}(\theta)\right|^p}{\theta}\,\mathrm{d}\theta, \tag{7.185}$$

and by a suitable choice of the constant C_p, the right-hand side of this inequality is ≥ 0, and hence Eq. (7.176) follows.

There has been considerable interest in finding extensions of the preceding results. One particular generalization of the Hardy–Littlewood work is now examined. If the function $f(\theta)$ has a finite weighted norm defined by

$$\|f(\theta)\|_{\alpha,p} = \left\{\int_{-\pi}^{\pi} |f(\theta)|^p |\theta|^\alpha\,\mathrm{d}\theta\right\}^{1/p}, \quad \text{with } 1 \le p < \infty, \tag{7.186}$$

and

$$\|f(\theta)\|_{\alpha,\infty} = \operatorname*{ess\,sup}_{|\theta|\le\pi}\{|f(\theta)|\,|\theta|^\alpha\}, \tag{7.187}$$

then the notation $f(\theta) \in L^{\alpha,p}$ is employed. For a periodic function f with period 2π, it is common practice to define the p-norm with an additional 2π factor; that is,

$$\|f(\theta)\|_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta\right\}^{1/p}, \quad \text{with } 1 \leq p < \infty. \tag{7.188}$$

A result first announced by Hardy and Littlewood (1936) and reconsidered by Babenko (1948) is as follows: if $\widetilde{f}(\theta)$ is periodic with period 2π, $f(\theta) \in L^{\alpha,p}$ with $1 < p < \infty$, and $-1/p < \alpha < 1 - 1/p$, then $\widetilde{f}(\theta)$ is defined and

$$\left\|\widetilde{f}(\theta)\right\|_{\alpha,p} \leq C_{p,\alpha} \|f(\theta)\|_{\alpha,p}, \tag{7.189}$$

and the constant that occurs depends only on the indicated variables. There is no restriction on the even–odd character of the function $\widetilde{f}(\theta)$. The choice $\alpha = 0$ reduces directly to the Riesz inequality. The following approach to establish Eq. (7.189) is based on Hirschman (1955). From the definition of $\widetilde{f}(\theta)$, it follows that

$$\begin{aligned}
\widetilde{f}(\theta)|\theta|^\alpha &= \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(t)|t|^\alpha \cot\left\{\frac{1}{2}(\theta - t)\right\} |\theta|^\alpha |t|^{-\alpha} \, dt \\
&= \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(t)|t|^\alpha \cot\left\{\frac{1}{2}(\theta - t)\right\} dt \\
&\quad + \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(t)|t|^\alpha \cot\left\{\frac{1}{2}(\theta - t)\right\} \{|\theta|^\alpha |t|^{-\alpha} - 1\} dt \\
&= \mathcal{H}\{f(\theta)|\theta|^\alpha\} + F(\theta).
\end{aligned} \tag{7.190}$$

From this result it follows, on applying the Minkowski and Riesz inequalities, that

$$\begin{aligned}
\left\{\int_{-\pi}^{\pi} \left\{\left|\widetilde{f}(\theta)\right| |\theta|^\alpha\right\}^p d\theta\right\}^{1/p} &= \left\{\int_{-\pi}^{\pi} \left|\mathcal{H}\{f(\theta)\,|\theta|^\alpha\} + F(\theta)\right|^p d\theta\right\}^{1/p} \\
&\leq \left\{\int_{-\pi}^{\pi} \left|\mathcal{H}\{f(\theta)\,|\theta|^\alpha\}\right|^p d\theta\right\}^{1/p} + \left\{\int_{-\pi}^{\pi} |F(\theta)|^p \, d\theta\right\}^{1/p} \\
&\leq \left\{C_p \int_{-\pi}^{\pi} \left|f(\theta)\,|\theta|^\alpha\right|^p d\theta\right\}^{1/p} + \left\{\int_{-\pi}^{\pi} |F(\theta)|^p \, d\theta\right\}^{1/p},
\end{aligned} \tag{7.191}$$

and it therefore remains only to prove that $\int_{-\pi}^{\pi} |F(\theta)|^p \, d\theta$ is bounded above by $\int_{-\pi}^{\pi} |f(\theta)|\theta|^\alpha|^p \, d\theta$, and Eq. (7.181) is established. To deal with the integral over $F(\theta)$, let

$$k(\theta, t) = \frac{1}{2\pi} \left|\cot\left\{\frac{1}{2}(\theta - t)\right\}\right| \left||\theta|^\alpha \, |t|^{-\alpha} - 1\right|; \tag{7.192}$$

then

$$|F(\theta)| \leq \int_{-\pi}^{\pi} |f(t)|\,|t|^{\alpha} k(\theta,t)\mathrm{d}t, \tag{7.193}$$

where the last integral is a Cauchy principal value integral. On applying the Hölder inequality with conjugate exponents p and q,

$$\begin{aligned} |F(\theta)| &\leq \int_{-\pi}^{\pi} |f(t)|\,|t|^{\alpha} \left\{k(\theta,t)\left|\frac{t}{\theta}\right|^{1/q}\right\}^{1/p} \left\{k(\theta,t)\left|\frac{\theta}{t}\right|^{1/p}\right\}^{1/q} \mathrm{d}t \\ &\leq \left\{\int_{-\pi}^{\pi} |f(t)|^{p}|t|^{\alpha p}k(\theta,t)\left|\frac{t}{\theta}\right|^{1/q}\mathrm{d}t\right\}^{1/p} \left\{\int_{-\pi}^{\pi} k(\theta,t)\left|\frac{\theta}{t}\right|^{1/p}\mathrm{d}t\right\}^{1/q}. \end{aligned} \tag{7.194}$$

For $\theta,\ t \in (-\pi,\pi)$, employ the following inequality:

$$\big||\theta| - |t|\big| \left|\cot\left\{\frac{1}{2}(\theta - t)\right\}\right| \leq C, \tag{7.195}$$

with C a positive constant; then,

$$P\int_{-\pi}^{\pi} k(\theta,t)\left|\frac{\theta}{t}\right|^{1/p}\mathrm{d}t \leq CP\int_{-\pi}^{\pi} \big||\theta| - |t|\big|^{-1}\big||\theta|^{\alpha}\,|t|^{-\alpha} - 1\big|\,|\theta|^{p^{-1}}\,|t|^{-p^{-1}}\,\mathrm{d}t. \tag{7.196}$$

Using the change of variable $t = |\theta|s$ leads to

$$\begin{aligned} P\int_{-\pi}^{\pi} k(\theta,t)\left|\frac{\theta}{t}\right|^{1/p}\mathrm{d}t &\leq CP\int_{-\pi/|\theta|}^{\pi/|\theta|} \frac{\big|1 - |s|^{\alpha}\big|\,|s|^{-\alpha-p^{-1}}\,\mathrm{d}s}{\big|1 - |s|\big|} \\ &\leq CP\int_{-\infty}^{\infty} \frac{\big|1 - |s|^{\alpha}\big|\,|s|^{-\alpha-p^{-1}}\,\mathrm{d}s}{\big|1 - |s|\big|} \\ &= 2CP\int_{0}^{\infty} \frac{|1 - s^{\alpha}|\mathrm{d}s}{|1 - s|\,s^{\alpha+p^{-1}}}. \end{aligned} \tag{7.197}$$

Since $0 < \alpha + p^{-1} < 1$, no problem arises for the integral at the lower limit, and as $s \to \infty$ the integrand behaves like $s^{-(1+p^{-1})}$; hence,

$$P\int_{-\pi}^{\pi} k(\theta,t)\left|\frac{\theta}{t}\right|^{1/p}\mathrm{d}t \leq C_{p,\alpha}. \tag{7.198}$$

Inserting this result into Eq. (7.194) gives

$$\left\{\int_{-\pi}^{\pi} |F(\theta)|^p \mathrm{d}\theta\right\}^{1/p} \le C_{p,\alpha}\left\{\int_{-\pi}^{\pi} \mathrm{d}\theta \int_{-\pi}^{\pi} |f(t)|^p |t|^{\alpha p}\, k(\theta,t)\left|\frac{t}{\theta}\right|^{1/q} \mathrm{d}t\right\}^{1/p}$$
$$= C_{p,\alpha}\left\{\int_{-\pi}^{\pi} |f(t)|^p |t|^{\alpha p}\, \mathrm{d}t \int_{-\pi}^{\pi} k(\theta,t)\left|\frac{t}{\theta}\right|^{1/q} \mathrm{d}\theta\right\}^{1/p}. \tag{7.199}$$

It follows, on using Eq. (7.195) and employing the change of variable $\theta = |t|s$, that

$$P\int_{-\pi}^{\pi} k(\theta,t)\left|\frac{t}{\theta}\right|^{1/q} \mathrm{d}\theta \le CP\int_{-\pi}^{\pi} \big||\theta| - |t|\big|^{-1}\big||\theta|^{\alpha}|t|^{-\alpha} - 1\big|\, |t|^{q^{-1}}|\theta|^{-q^{-1}}\, \mathrm{d}\theta$$
$$= CP\int_{-\pi/|t|}^{\pi/|t|} \frac{\big||s|^{\alpha} - 1\big|\mathrm{d}s}{\big||s| - 1\big|\,|s|^{1-p^{-1}}}$$
$$\le CP\int_{0}^{\infty} \frac{|s^{\alpha} - 1|\mathrm{d}s}{|s - 1|\, s^{1-p^{-1}}}, \tag{7.200}$$

which for the stated conditions on α gives a convergent integral. Hence,

$$P\int_{-\pi}^{\pi} k(\theta,t)\left|\frac{t}{\theta}\right|^{1/q} \mathrm{d}\theta \le C_{p,\alpha}. \tag{7.201}$$

From Eq. (7.199), the following result is obtained:

$$\int_{-\pi}^{\pi} |F(\theta)|^p\, \mathrm{d}\theta \le C_{p,\alpha}\int_{-\pi}^{\pi} \big|f(t)|t|^{\alpha}\big|^p\, \mathrm{d}t, \tag{7.202}$$

and employing this connection in Eq. (7.191) gives the desired result, Eq. (7.189).

Two results due to Pichorides (1975a) are now considered. These are not weighted inequalities in the strict sense of the form given in Eq. (7.159), but are nevertheless of intrinsic interest. if $f \in L^p(\mathbb{R})$ for $p > 1$ and $0 \le \alpha \le p$, then

$$\int_{-\infty}^{\infty} |Hf(x)|^{\alpha}|f(x)|^{p-\alpha}\, \mathrm{d}x \le C_p\int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x. \tag{7.203}$$

If $p > 2,\ 0 \le \alpha \le p$, and $0 \le \beta \le p$, then, for $f \in L^p(\mathbb{R})$,

$$\int_{-\infty}^{\infty} |Hf(x)|^{\alpha}|f(x)|^{p-\alpha}\, \mathrm{d}x \le C_{p,\alpha,\beta}\int_{-\infty}^{\infty} |Hf(x)|^{\beta}|f(x)|^{p-\beta}\, \mathrm{d}x. \tag{7.204}$$

To establish Eq. (7.203), apply the Hölder inequality with exponents $p/2\alpha$ and $p/(p-2\alpha)$ and then apply the Cauchy–Schwarz–Buniakowski inequality, so that

$$\begin{aligned}
\int_{-\infty}^{\infty} |Hf(x)|^{\alpha} |f(x)|^{p-\alpha}\,\mathrm{d}x &= \int_{-\infty}^{\infty} |f(x)Hf(x)|^{\alpha} |f(x)|^{p-2\alpha}\,\mathrm{d}x \\
&\leq \left\{\int_{-\infty}^{\infty} |f(x)Hf(x)|^{p/2}\,\mathrm{d}x\right\}^{2\alpha/p} \left\{\int_{-\infty}^{\infty} |f(x)|^{p}\,\mathrm{d}x\right\}^{1-2\alpha/p} \\
&\leq \left\{\left\{\int_{-\infty}^{\infty} |Hf(x)|^{p}\,\mathrm{d}x\right\}^{1/2} \left\{\int_{-\infty}^{\infty} |f(x)|^{p}\,\mathrm{d}x\right\}^{1/2}\right\}^{2\alpha/p} \\
&\quad \left\{\int_{-\infty}^{\infty} |f(x)|^{p}\mathrm{d}x\right\}^{1-2\alpha/p} \\
&\leq \mathfrak{R}_p^{\alpha} \left\{\int_{-\infty}^{\infty} |f(x)|^{p}\,\mathrm{d}x\right\}^{\alpha/p} \left\{\int_{-\infty}^{\infty} |f(x)|^{p}\,\mathrm{d}x\right\}^{\alpha/p} \\
&\quad \left\{\int_{-\infty}^{\infty} |f(x)|^{p}\,\mathrm{d}x\right\}^{1-2\alpha/p}, \qquad (7.205)
\end{aligned}$$

and hence Eq. (7.203) follows. The Riesz inequality has been used in the final step. The particular case of Eq. (7.204) for $\beta = p/2$ is now considered. if $f(z)$ is analytic in the upper half complex plane, and on the real axis

$$F(x)^2 = \{f(x) + \mathrm{i}Hf(x)\}^2 = f(x)^2 - \{Hf(x)\}^2 + 2\mathrm{i}f(x)Hf(x), \qquad (7.206)$$

then

$$H[f(x)^2 - \{Hf(x)\}^2] = 2f(x)Hf(x) \qquad (7.207)$$

and

$$2H\{f(x)Hf(x)\} = \{Hf(x)\}^2 - f(x)^2. \qquad (7.208)$$

Now,

$$f(x)^2 \leq \left|\{f(x) + \mathrm{i}Hf(x)\}^2\right|; \qquad (7.209)$$

that is,

$$f(x)^2 \leq 2\left|H\{f(x)Hf(x)\} - \mathrm{i}f(x)Hf(x)\right|. \qquad (7.210)$$

From this result it follows, on applying Minkowski's inequality, that

$$\begin{aligned}\left\{\int_{-\infty}^{\infty} |f(x)|^{2p}\,\mathrm{d}x\right\}^{1/p} &\le \left\{2\int_{-\infty}^{\infty} |H\{f(x)Hf(x)\} - \mathrm{i}f(x)Hf(x)|^p\,\mathrm{d}x\right\}^{1/p}\\ &\le \left\{2\int_{-\infty}^{\infty} |H\{f(x)Hf(x)\}|^p\,\mathrm{d}x\right\}^{1/p}\\ &\quad+\left\{2\int_{-\infty}^{\infty} |f(x)Hf(x)|^p\,\mathrm{d}x\right\}^{1/p}\\ &\le \left\{2C_p\int_{-\infty}^{\infty} |f(x)Hf(x)|^p\,\mathrm{d}x\right\}^{1/p}\\ &\quad+\left\{2\int_{-\infty}^{\infty} |f(x)Hf(x)|^p\,\mathrm{d}x\right\}^{1/p}\end{aligned} \tag{7.211}$$

and hence,

$$\int_{-\infty}^{\infty} |f(x)|^{2p}\,\mathrm{d}x \le C_p\int_{-\infty}^{\infty} |f(x)Hf(x)|^p\,\mathrm{d}x. \tag{7.212}$$

From Eq. (7.203) and on using Eq. (7.212), it follows that

$$\int_{-\infty}^{\infty} |Hf(x)|^{\alpha}|f(x)|^{p-\alpha}\mathrm{d}x \le C_{p,\alpha}\int_{-\infty}^{\infty} |f(x)Hf(x)|^{p/2}\,\mathrm{d}x, \tag{7.213}$$

which establishes Eq. (7.204) for the case $\beta = p/2$. The interested reader might like to establish the case of general β satisfying $0 \le \beta \le p$.

There is a weighted version of the Zygmund inequality. If $-1 < \alpha \le 0$, then

$$\int_{-\pi}^{\pi} |x|^{\alpha}|\mathcal{H}f(x)|\mathrm{d}x \le A_\alpha\int_{-\pi}^{\pi} |x|^{\alpha}|f(x)|\log^+(|x|^{\alpha}|f(x)|)\mathrm{d}x + A_\alpha, \tag{7.214}$$

assuming the integral on the right-hand side is finite, and the constant A_α depends only on α. There is also a weighted version of the Kolmogorov inequality. If $0 < p < 1$, then

$$\left\{\int_{-\pi}^{\pi} \left||x|^{\alpha}\,\mathcal{H}f(x)\right|^p\mathrm{d}x\right\}^{1/p} \le C_{p,\alpha}\int_{-\pi}^{\pi} |x|^{\alpha}|f(x)|\mathrm{d}x, \tag{7.215}$$

once again assuming the integral on the right-hand side is finite. These two inequalities have been discussed by Flett (1958) and can be proved by methods very similar to those discussed previously for the Hardy–Littlewood weighted inequality. If the function f is known to be either even or odd, then the inequalities can be established for a wider range of values of α. The reader interested in these inequalities can try to construct the necessary proofs.

7.8 Weak-type inequalities

It has been noted previously that for $f \in L^1$ then, in general, $Hf \notin L^1$. It is however true that if $f \in L^1$, then Hf exists *a.e.* It is possible to state a weaker result than the Riesz inequality for the case $f \in L^1$. First, some definitions.

If T is an operator such that $T : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ for $1 \le p \le \infty, 1 \le q \le \infty$, and if

$$\|Tf\|_q \le A\|f\|_p, \tag{7.216}$$

where the constant A is independent of f, then T is said to be of *type* (p, q). Sometimes the terminology *strong-type* estimate is employed to describe an inequality of the form just stated. If for $\lambda > 0$

$$m\{x : |Tf(x)| > \lambda\} \le \left\{\frac{A\|f\|_p}{\lambda}\right\}^q, \tag{7.217}$$

where the constant A is independent of f and λ, then T is said to be of *weak-type* (p, q). The term on the left of the preceding result is the distribution function of $|Tf|$. A function f, for which the distribution function satisfies

$$m\{x : |f(x)| > \lambda\} \le A\lambda^{-p}, \tag{7.218}$$

is said to be a weak L^p function.

For the case where T is the Hilbert transform operator, and for $f \in L^p(\mathbb{R})$, then clearly for $1 < p < \infty$ the Hilbert transform is of type (p, p), an observation following directly from the Riesz inequality. Also, it can be stated that the Hilbert transform is not of type $(1, 1)$. This follows from the fact that there exist examples of functions $f \in L^1$ for which $Hf \notin L^1$ (see Section 4.21).

if $f \in L^1(\mathbb{T})$, then for $\lambda > 0$ and K a positive constant,

$$\left|\theta \in [-\pi, \pi] : \left|\widetilde{f}(\theta)\right| \ge \lambda\right| \le \frac{K\|f\|_1}{\lambda}, \tag{7.219}$$

which is Kolmogorov's theorem, and establishes that $\mathcal{H}f$ on $\mathbb{T}$ is of weak-type (1,1). The proof of this is given at the end of Section 7.2. For the case $f \in L^1(\mathbb{R})$ and $\lambda > 0$,

$$|x \in \mathbb{R} : |Hf(x)| \ge \lambda| \le \frac{K\|f\|_1}{\lambda}, \tag{7.220}$$

which is Kolmogorov's theorem for the line. This result is sometimes referred to as Kolmogorov's weak-type inequality. Hence, Hf is of weak-type $(1, 1)$. A sketch of the proof of this formula follows. Starting from the Hilbert–Stieltjes transform result in Section 4.25: if f is of bounded variation on $\mathbb{R}$ and its total variation is denoted by V, then, for $\lambda > 0$,

$$|x \in \mathbb{R} : |H_S F(x)| > \lambda| \leq \frac{KV}{\lambda}. \tag{7.221}$$

if $f \in L^1(\mathbb{R})$ and $F(t)$ is absolutely continuous on $\mathbb{R}$, then

$$F(t) = \int_{-\infty}^{t} f(s)\mathrm{d}s, \tag{7.222}$$

and $\mathrm{d}F(t) = f(t)\mathrm{d}t$, *a.e.* The function $F(t)$ is of bounded variation on $\mathbb{R}$, and the total variation of $F(t)$ is given by

$$V = \int_{-\infty}^{\infty} \mathrm{d}F(t) = \int_{-\infty}^{\infty} |f(t)|\mathrm{d}t = \|f\|_1. \tag{7.223}$$

Inserting this result and $\mathrm{d}F(t) = f(t)\mathrm{d}t$ into Eq. (7.221) leads to

$$|x \in \mathbb{R} : |Hf(x)| > \lambda| \leq \frac{K\|f\|_1}{\lambda}, \tag{7.224}$$

which establishes that Hf is of weak-type $(1, 1)$.

The Riesz inequality is now revisited one more time. First, some preliminaries are required. Chebyshev's inequality takes the following form: let $f \in L^p$, with $0 < p < \infty$, then, for any $\lambda > 0$,

$$m\{x \in \mathbb{R} : |f(x)| > \lambda\} \leq \frac{\|f\|_p^p}{\lambda^p}. \tag{7.225}$$

To prove Eq. (7.225), proceed as follows:

$$\begin{aligned} \lambda^p m\{x \in \mathbb{R} : |f(x)| > \lambda\} &= \lambda^p \int_{|f(x)|>\lambda} \mathrm{d}x \\ &\leq \int_{|f(x)|>\lambda} |f(x)|^p \,\mathrm{d}x \\ &\leq \int_{-\infty}^{\infty} |f(x)|^p \,\mathrm{d}x, \end{aligned} \tag{7.226}$$

which gives the required result.

If T is a linear operator of weak-type (p_1, p_1) and is also of weak-type (p_2, p_2), then T is bounded operator on L^p for $p_1 < p < p_2$. This is an *interpolation* idea, and the statement just presented is a particular version of the Marcinkiewicz interpolation theorem. Let the bounded function f be written as follows:

$$f(x) = g(x) + h(x), \tag{7.227}$$

with

$$g(x) = \begin{cases} f(x), & |f(x)| \le \lambda \\ 0, & |f(x)| > \lambda \end{cases} \tag{7.228}$$

and

$$h(x) = \begin{cases} 0, & |f(x)| \le \lambda \\ f(x), & |f(x)| > \lambda. \end{cases} \tag{7.229}$$

If T is of weak-type (p_1, p_1) and weak-type (p_2, p_2), then

$$m\{x : |Tf(x)| > \lambda\} \le \frac{C_{p_1} \|f\|_{p_1}^{p_1}}{\lambda^{p_1}} \tag{7.230}$$

and

$$m\{x : |Tf(x)| > \lambda\} \le \frac{C_{p_2} \|f\|_{p_2}^{p_2}}{\lambda^{p_2}}, \tag{7.231}$$

where C_{p_1} and C_{p_2} are constants depending only on p_1 and p_2, respectively. Applying Eq. (7.230) to $h(x)$ and Eq. (7.231) to $g(x)$ leads to

$$m\{x : |Tf(x)| > 2\lambda\} \le \frac{C_{p_1} \|h\|_{p_1}^{p_1}}{\lambda^{p_1}} + \frac{C_{p_2} \|g\|_{p_2}^{p_2}}{\lambda^{p_2}}, \tag{7.232}$$

and Eq. (4.560) has been employed. Multiplying both sides by λ^{p-1} and integrating over λ leads to

$$\begin{aligned} \int_0^\infty \lambda^{p-1} m\{x : |Tf(x)| > 2\lambda\} \mathrm{d}\lambda \le C_{p_1} \int_0^\infty \lambda^{p-p_1-1}\, \mathrm{d}\lambda \int_{-\infty}^\infty |h(x)|^{p_1}\, \mathrm{d}x \\ + C_{p_2} \int_0^\infty \lambda^{p-p_2-1}\, \mathrm{d}\lambda \int_{-\infty}^\infty |g(x)|^{p_2}\, \mathrm{d}x; \end{aligned} \tag{7.233}$$

hence,

$$\begin{aligned} \int_0^\infty \lambda^{p-1}\, \mathrm{d}\lambda \int_{|Tf(x)|>2\lambda} \mathrm{d}x \le C_{p_1} \int_{-\infty}^\infty \mathrm{d}x \int_0^\infty |h(x)|^{p_1} \lambda^{p-p_1-1}\, \mathrm{d}\lambda \\ + C_{p_2} \int_{-\infty}^\infty \mathrm{d}x \int_0^\infty |g(x)|^{p_2} \lambda^{p-p_2-1}\, \mathrm{d}\lambda. \end{aligned} \tag{7.234}$$

The preceding result can be rewritten as follows:

$$\int_{-\infty}^{\infty} \mathrm{d}x \left\{ \int_0^{|Tf(x)|/2} + \int_{|Tf(x)|/2}^{\infty} \right\} \lambda^{p-1}\, \mathrm{d}\lambda$$
$$\leq C_{p_1} \int_{-\infty}^{\infty} \mathrm{d}x \left\{ \int_0^{|f(x)|} + \int_{|f(x)|}^{\infty} \right\} |h(x)|^{p_1}\, \lambda^{p-p_1-1}\, \mathrm{d}\lambda$$
$$+ C_{p_2} \int_{-\infty}^{\infty} \mathrm{d}x \left\{ \int_0^{|f(x)|} + \int_{|f(x)|}^{\infty} \right\} |g(x)|^{p_2}\, \lambda^{p-p_2-1}\, \mathrm{d}\lambda, \tag{7.235}$$

and hence

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{|Tf(x)|/2} \lambda^{p-1}\, \mathrm{d}\lambda \leq C_{p_1} \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{|f(x)|} |f(x)|^{p_1} \lambda^{p-p_1-1}\, \mathrm{d}\lambda$$
$$+ C_{p_2} \int_{-\infty}^{\infty} \mathrm{d}x \int_{|f(x)|}^{\infty} |f(x)|^{p_2} \lambda^{p-p_2-1}\, \mathrm{d}\lambda; \tag{7.236}$$

that is

$$\frac{1}{2^p p} \int_{-\infty}^{\infty} |Tf(x)|^p\, \mathrm{d}x \leq C_{p_1} \int_{-\infty}^{\infty} |f(x)|^{p_1}\, \mathrm{d}x \int_0^{|f(x)|} \lambda^{p-p_1-1}\, \mathrm{d}\lambda$$
$$+ C_{p_2} \int_{-\infty}^{\infty} |f(x)|^{p_2}\, \mathrm{d}x \int_{|f(x)|}^{\infty} \lambda^{p-p_2-1}\, \mathrm{d}\lambda$$
$$= \frac{C_{p_1}}{p-p_1} \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x + \frac{C_{p_2}}{p_2-p} \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x$$
$$= \left\{ \frac{C_{p_1}}{p-p_1} + \frac{C_{p_2}}{p_2-p} \right\} \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x, \tag{7.237}$$

and so

$$\int_{-\infty}^{\infty} |Tf(x)|^p\, \mathrm{d}x \leq 2^p \left\{ \frac{pC_{p_1}}{p-p_1} + \frac{pC_{p_2}}{p_2-p} \right\} \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x. \tag{7.238}$$

Fubini's theorem is employed to make the necessary change in integration order in the previous sequence of steps. The preceding result can be used to establish the Riesz inequality. First, identify T with the Hilbert transform operator. Equation (7.230) holds for $H = T$ and $p_1 = 1$ (by Kolmogorov's theorem, Section 7.2). In Section 4.12 the equality $\|Hf\|_2 = \|f\|_2$ was proved, that is $Hf \in L^2$, and, by Chebyshev's inequality, Eq. (7.231) follows with $H = T$ and $p_2 = 2$. Insert $H = T, p_1 = 1$, and $p_2 = 2$ into Eq. (7.238), then

$$\int_{-\infty}^{\infty} |Hf(x)|^p\, \mathrm{d}x \leq 2^p C \left\{ \frac{p}{p-1} + \frac{p}{2-p} \right\} \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x; \tag{7.239}$$

that is,

$$\|Hf\|_p \leq C_p \|f\|_p, \tag{7.240}$$

which is valid for $1 < p < 2$. A duality argument (see Section 4.20) completes the proof for $2 < p < \infty$. The one deficiency that should be immediately apparent in this approach is that the quality of the bound deteriorates significantly as $p \to 2$.

7.9 The Hardy–Littlewood maximal function

In this section the Hardy–Littlewood maximal function is introduced. This is done primarily as a lead-in to the maximal Hilbert transform. if $f(x)$ is locally integrable on $\mathbb{R}$, then for $x \in \mathbb{R}$, the fundamental theorem of Lebesgue is given by

$$\lim_{\substack{a\to 0\\ b\to 0}} \frac{1}{a+b} \int_{x-a}^{x+b} f(t)\mathrm{d}t = f(x), \quad a.e., \tag{7.241}$$

which can be put in a slightly more compact form as follows:

$$\lim_{h\to 0} \frac{1}{h} \int_{x}^{x+h} f(t)\mathrm{d}t = f(x), \quad a.e. \tag{7.242}$$

A related function can be formed by making two changes: replacing the limit by a sup, and, to avoid the possibility of cancellation of positive and negative contributions, substituting $|f|$ for f. The Hardy–Littlewood maximal function is defined by

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|} \int_I |f(t)|\mathrm{d}t. \tag{7.243}$$

In this definition $|I|$ denotes the length of an interval I. Other notations are employed for Mf, of which the most widely used is f^*.

For a bounded function it follows that

$$Mf(x) \leq \|f\|_\infty. \tag{7.244}$$

A central result is the Hardy–Littlewood maximal theorem: if $f \in L^p(\mathbb{R})$ for $1 \leq p \leq \infty$, then $Mf(x)$ is finite *a.e.* Further, if $f \in L^1(\mathbb{R})$, then, for $\lambda > 0$,

$$|x \in \mathbb{R} : Mf(x) > \lambda| \leq \frac{A\|f\|_1}{\lambda}, \tag{7.245}$$

where A is a constant independent of f. if $f \in L^p(\mathbb{R})$ for $1 < p \leq \infty$, then $Mf \in L^p(\mathbb{R})$ and

$$\|Mf\|_p \leq C_p \|f\|_p, \tag{7.246}$$

where the constant C_p depends only on p. That $Mf(x)$ is finite *a.e.* follows from Eq. (7.245) if $f \in L^1(\mathbb{R})$ and from Eq. (7.246) if $f \in L^p(\mathbb{R})$ for $1 < p \leq \infty$. There are different ways to establish Eq. (7.245); the following approach is based on a proof by Koosis (1998). Without loss of generality, assume $f(x) \geq 0$. For a general function the decomposition $f(x) = f_+(x) - f_-(x)$ can be made, where $f_+(x)$ denotes the positive part of f and $f_-(x)$ designates the absolute value of the negative part of f, so that

$$m\{x : Mf(x) > \lambda\} \leq m\{x : Mf_+(x) > \lambda/2\} + m\{x : Mf_-(x) > \lambda/2\}, \tag{7.247}$$

which allows the result for a general f to be deduced. Let

$$f_1(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} f(s)\mathrm{d}s \tag{7.248}$$

and

$$f_2(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} f(s)\mathrm{d}s, \tag{7.249}$$

and, for $\lambda > 0$, let $E_1 = \{x : f_1(x) > \lambda\}$, $E_2 = \{x : f_2(x) > \lambda\}$, and $E = \{x : Mf(x) > \lambda\}$. Since $E \subseteq E_1 \cup E_2$, then

$$m\{x : Mf(x) > \lambda\} \leq m\{x : f_1(x) > \lambda\} + m\{x : f_2(x) > \lambda\}, \tag{7.250}$$

and it suffices to prove

$$m\{x : f_1(x) > \lambda\} = \frac{1}{\lambda} \int_{E_1} f(x)\mathrm{d}x \tag{7.251}$$

and

$$m\{x : f_2(x) > \lambda\} = \frac{1}{\lambda} \int_{E_2} f(x)\mathrm{d}x. \tag{7.252}$$

Let

$$F(x) = \int_0^x f(s)\mathrm{d}s, \tag{7.253}$$

and take $F(x)$ to be finite everywhere; since $f(x) \geq 0$, $F(x)$ is an increasing function of x. A graph of a typical function $F(x)$ is shown in Figure 7.1.

The intervals I_n in Figure 7.1 are disjoint, and are formed by projecting onto the x-axis the segments formed by drawing tangent lines, all of slope λ, with the curve $y = F(x)$. Suppose the interval I_n starts at the position x_n, then by inspection of

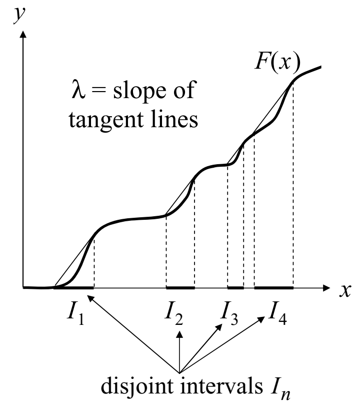

Figure 7.1. Projection of the tangent line sections to form the intervals I_n.

Figure 7.1 it follows that

$$\lambda = \frac{F(x_n + I_n) - F(x_n)}{|I_n|}$$
$$= \frac{\int_0^{x_n+I_n} f(t)\mathrm{d}t - \int_0^{x_n} f(t)\mathrm{d}t}{|I_n|}, \tag{7.254}$$

and so

$$\lambda\,|I_n| = \int_{I_n} f(t)\mathrm{d}t. \tag{7.255}$$

From the definition of $f_1(x)$,

$$f_1(x) = \sup_{h>0} \left\{ \frac{F(x+h) - F(x)}{h} \right\}. \tag{7.256}$$

Let h be any subinterval of I_n, then the least upper bound for the slope in Eq. (7.256) must exceed λ, by the way in which the intervals are constructed. On summing over all the intervals I_n, Eq. (7.255) becomes

$$\lambda|E_1| = \int_{E_1} f(t)\mathrm{d}t, \tag{7.257}$$

and so

$$|E_1| = m\{x : f_1(x) > \lambda\} = \frac{1}{\lambda} \int_{E_1} f(t)\mathrm{d}t. \tag{7.258}$$

A similar argument gives

$$|E_2| = m\{x : f_2(x) > \lambda\} = \frac{1}{\lambda}\int_{E_2} f(t)\mathrm{d}t. \tag{7.259}$$

Using Eq. (7.250) leads to

$$\begin{aligned} m\{x : Mf(x) > \lambda\} &\leq \frac{1}{\lambda}\int_{E_1} f(t)\mathrm{d}t + \frac{1}{\lambda}\int_{E_2} f(t)\mathrm{d}t \\ &\leq \frac{1}{\lambda}\int_{E} f(t)\mathrm{d}t + \frac{1}{\lambda}\int_{E} f(t)\mathrm{d}t; \end{aligned} \tag{7.260}$$

hence,

$$m\{x : Mf(x) > \lambda\} \leq \frac{2}{\lambda}\int_{Mf>\lambda} f(t)\mathrm{d}t, \tag{7.261}$$

and Eq. (7.245) follows immediately.

Equation (7.246) can be proved in several ways. The two approaches illustrated are among the easiest based on the material already covered. Write f in the following form:

$$f(x) = g(x) + h(x), \tag{7.262}$$

with

$$g(x) = \begin{cases} f(x), & |f(x)| \geq \lambda \\ 0, & |f(x)| < \lambda \end{cases} \tag{7.263}$$

and

$$h(x) = \begin{cases} 0, & |f(x)| > \lambda \\ f(x), & |f(x)| \leq \lambda. \end{cases} \tag{7.264}$$

From the preceding definition,

$$Mh(x) \leq \lambda, \tag{7.265}$$

and so

$$m\{x : Mh(x) > \lambda\} = 0. \tag{7.266}$$

Making use of Eqs. (4.560) and (7.262) leads to

$$m\{x : Mf(x) > 2\lambda\} \leq m\{x : Mg(x) > \lambda\}, \tag{7.267}$$

which simplifies, on using Eq. (7.261), to give

$$m\{x : Mf(x) > 2\lambda\} \le \frac{2}{\lambda}\int_{-\infty}^{\infty} |g(x)|\mathrm{d}x. \tag{7.268}$$

Multiplying Eq. (7.268) by λ^{p-1} and integrating from 0 to ∞ leads to

$$\int_0^{\infty} \lambda^{p-1} m\{x : Mf(x) > 2\lambda\}\mathrm{d}\lambda \le 2\int_0^{\infty} \lambda^{p-2}\,\mathrm{d}\lambda \int_{-\infty}^{\infty} |g(x)|\mathrm{d}x, \tag{7.269}$$

and so

$$\int_0^{\infty} \lambda^{p-1}\,\mathrm{d}\lambda \int_{Mf(x)>2\lambda} \mathrm{d}x \le 2\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \lambda^{p-2}\,|g(x)|\mathrm{d}\lambda. \tag{7.270}$$

The preceding inequality can be written as follows:

$$\int_{-\infty}^{\infty} \mathrm{d}x\left\{\int_0^{Mf(x)/2} + \int_{Mf(x)/2}^{\infty}\right\}\lambda^{p-1}\,\mathrm{d}\lambda \le 2\int_{-\infty}^{\infty} \mathrm{d}x\left\{\int_0^{|f(x)|} + \int_{|f(x)|}^{\infty}\right\} |g(x)|\lambda^{p-2}\,\mathrm{d}\lambda, \tag{7.271}$$

and hence

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{Mf(x)/2} \lambda^{p-1}\,\mathrm{d}\lambda \le 2\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x \int_0^{|f(x)|} \lambda^{p-2}\,\mathrm{d}\lambda. \tag{7.272}$$

It follows that

$$\frac{1}{2^p p}\int_{-\infty}^{\infty} |Mf(x)|^p\,\mathrm{d}x \le \frac{2}{p-1}\int_{-\infty}^{\infty} |f(x)|^p\,\mathrm{d}x; \tag{7.273}$$

therefore,

$$\|Mf\|_p^p \le \frac{2^{p+1}}{p-1}\|f\|_p^p, \tag{7.274}$$

and Eq. (7.246) is proved.

An alternative way to prove Eq. (7.246) makes use of the Marcinkiewicz interpolation theorem treated in Section 7.8. There is a complication that the Hardy–Littlewood maximal operator M is not linear. Suppose for an operator T that $|Tf_1|$ and $|Tf_2|$ are defined, then T is called *sublinear* if

$$|T(f_1 + f_2)| \le |Tf_1| + |Tf_2| \tag{7.275}$$

and, for a constant $c > 0$,

$$|T(cf_1)| = c\,|Tf_1|. \tag{7.276}$$

The Hardy–Littlewood maximal operator satisfies this condition. The Marcinkiewicz interpolation theorem can also be derived for sublinear operators. The reader should explore this as an exercise.

The quantity $\|Mf\|_\infty$ is bounded as follows:

$$\|Mf\|_\infty \leq \|f\|_\infty, \tag{7.277}$$

for all $f \in L^\infty$. The latter result is used as a substitute condition for Eq. (7.231) for the case $p_2 = \infty$ in the derivation of the Marcinkiewicz interpolation formula Eq. (7.238). On setting $p_1 = 1, p_2 \to \infty$ (with $C_{p_2} = 1$), identifying T with the Hardy–Littlewood maximal operator and setting the constant $A = 2$ (from Eq. (7.261)), then Eq. (7.238) yields

$$\int_{-\infty}^{\infty} |Mf(x)|^p \, dx \leq \frac{2^{p+1}p}{p-1} \int_{-\infty}^{\infty} |f(x)|^p \, dx, \tag{7.278}$$

which is the required result. The Hardy–Littlewood maximal function finds a number of applications in analysis, and these can be explored by checking the references given in the chapter end-notes. The following section shows one application to a related maximal function connected with the Hilbert transform.

7.10 The maximal Hilbert transform function

Recalling the definition of the truncated Hilbert transform, $H_\varepsilon f$,

$$H_\varepsilon f(x) = \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t} \, dt, \tag{7.279}$$

the maximal Hilbert transform function, $H_{\mathrm{M}} f$, is defined as follows:

$$H_{\mathrm{M}} f(x) = \sup_{\varepsilon>0} |H_\varepsilon f(x)|. \tag{7.280}$$

The maximal Hilbert transform operator is written as H_{M}. The notation H^*f is also frequently employed to denote the maximal Hilbert transform function. For a function $f \in L(\mathbb{T})$ the truncated Hilbert transform is given by

$$\mathcal{H}_\varepsilon f(x) = \frac{1}{2\pi} \int_\varepsilon^\pi \{f(x-t) - f(x+t)\} \cot\left(\frac{t}{2}\right) dt. \tag{7.281}$$

The maximal Hilbert transform function is then defined in a similar fashion to Eq. (7.280):

$$\mathcal{H}_{\mathrm{M}} f(x) = \sup_{0<\varepsilon<\pi} |\mathcal{H}_\varepsilon f(x)|. \tag{7.282}$$

If the function g is greater than or equal to the function f for all x, then f is *majorized* by g. From the definitions just given for the maximal Hilbert transform, then clearly the absolute value of the Hilbert transform of f is majorized by the maximal Hilbert transform, that is,

$$|Hf(x)| \leq H_{\mathrm{M}} f(x). \tag{7.283}$$

The maximal Hilbert transform can be bounded in terms of the Hardy–Littlewood maximal function. Let $f \in L^p(\mathbb{T})$ for $1 < p < \infty$; then

$$\mathcal{H}_{\mathrm{M}} f(x) \leq M\mathcal{H}f(x) + CMf(x), \tag{7.284}$$

where the constant C is independent of f. An analogous result holds for H_{M} acting on functions $f \in L^p(\mathbb{R})$:

$$H_{\mathrm{M}} f(x) \leq MHf(x) + CMf(x), \tag{7.285}$$

and this is referred to as Cotlar's inequality. Another important result is that the norm $\|H_M f\|_p$ is bounded by $\|f\|_p$, thus

$$\|H_{\mathrm{M}} f\|_p \leq C_p \|f\|_p \,. \tag{7.286}$$

To prove Eq. (7.286) for the case $f \in L^p(\mathbb{R})$ with $p > 1$, start with Cotlar's inequality to obtain

$$|H_{\mathrm{M}} f(x)|^p \leq |MHf(x) + CMf(x)|^p \,. \tag{7.287}$$

Integrating over $(-\infty, \infty)$ and making use of Minkowski's inequality leads to

$$\begin{aligned}\left\{\int_{-\infty}^{\infty} |H_{\mathrm{M}} f(x)|^p \,\mathrm{d}x\right\}^{p^{-1}} &\leq \left\{\int_{-\infty}^{\infty} |MHf(x) + CMf(x)|^p \,\mathrm{d}x\right\}^{p^{-1}} \\ &\leq \left\{\int_{-\infty}^{\infty} |MHf(x)|^p \,\mathrm{d}x\right\}^{p^{-1}} + C\left\{\int_{-\infty}^{\infty} |Mf(x)|^p \,\mathrm{d}x\right\}^{p^{-1}}.\end{aligned} \tag{7.288}$$

Employing Eq. (7.246) and the Riesz inequality yields

$$\begin{aligned}\|H_{\mathrm{M}} f\|_p &\leq C'_p \|Hf\|_p + C''_p \|f\|_p \\ &\leq C_p \|f\|_p + C''_p \|f\|_p \,,\end{aligned} \tag{7.289}$$

where C'_p and $C'{}'_p$ are constants depending only on p; hence, Eq. (7.286) follows. To complete the proof it is necessary to establish Eq. (7.285). To handle this a digression is made to treat some properties of the Poisson and conjugate Poisson operators.

The Poisson and conjugate Poisson operators for the upper half plane are defined by

$$P_\varepsilon f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon}{(x-t)^2+\varepsilon^2} f(t)\mathrm{d}t \tag{7.290}$$

and

$$Q_\varepsilon f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2+\varepsilon^2} f(t)\mathrm{d}t, \tag{7.291}$$

for $\varepsilon > 0$. The conventional variable that would appear in these definitions is y in place of ε, but the latter choice has been selected because connections will be made between these operators and the truncated Hilbert transform operator, for which the notation H_ε has been employed. Using the definitions of the Poisson and conjugate Poisson kernels for the half plane given in Eqs. (3.31) and (3.32), respectively, the Poisson and conjugate Poisson operators can be written as the following convolutions:

$$P_\varepsilon f(x) = (P(\cdot,\varepsilon) * f)(x) \tag{7.292}$$

and

$$Q_\varepsilon f(x) = (Q(\cdot,\varepsilon) * f)(x). \tag{7.293}$$

It will first be demonstrated that

$$|H_\varepsilon f(x) - Q_\varepsilon f(x)| \le P_\varepsilon |f|(x). \tag{7.294}$$

Starting with

$$\begin{aligned} H_\varepsilon f(x) - Q_\varepsilon f(x) &= \frac{1}{\pi}\int_{|x-t|\ge\varepsilon} \frac{f(t)\mathrm{d}t}{x-t} - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-t)f(t)\mathrm{d}t}{(x-t)^2+\varepsilon^2} \\ &= \frac{1}{\pi}\int_{|x-t|\ge\varepsilon} f(t)\left[\frac{1}{x-t} - \frac{(x-t)}{(x-t)^2+\varepsilon^2}\right]\mathrm{d}t \\ &\quad - \frac{1}{\pi}\int_{|x-t|<\varepsilon} \frac{(x-t)f(t)\mathrm{d}t}{(x-t)^2+\varepsilon^2}, \end{aligned} \tag{7.295}$$

it follows that

$$\begin{aligned} |H_\varepsilon f(x) - Q_\varepsilon f(x)| &\le \frac{1}{\pi}\int_{|x-t|\ge\varepsilon} \frac{\varepsilon^2|f(t)|\mathrm{d}t}{|x-t|\,\{(x-t)^2+\varepsilon^2\}} + \frac{1}{\pi}\int_{|x-t|<\varepsilon} \frac{|x-t|\,|f(t)|\mathrm{d}t}{(x-t)^2+\varepsilon^2} \\ &\le \frac{1}{\pi}\int_{|x-t|\ge\varepsilon} \frac{\varepsilon|f(t)|\mathrm{d}t}{\{(x-t)^2+\varepsilon^2\}} + \frac{1}{\pi}\int_{|x-t|<\varepsilon} \frac{\varepsilon\,|f(t)|\mathrm{d}t}{(x-t)^2+\varepsilon^2}, \end{aligned} \tag{7.296}$$

where the inequalities $\varepsilon \le |x-t|$ and $|x-t| < \varepsilon$ have been used to simplify the first and second integrals, respectively. From the preceding inequality it follows that

$$|H_\varepsilon f(x) - Q_\varepsilon f(x)\,| \le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon |f(t)|\mathrm{d}t}{(x-t)^2+\varepsilon^2} = P_\varepsilon |f|(x), \tag{7.297}$$

which is the required result. A related formula is

$$|H_\varepsilon f(x) - Q_\varepsilon f(x)\,| \le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon |f(t) - f(x)|\mathrm{d}t}{(x-t)^2+\varepsilon^2}. \tag{7.298}$$

This is established by starting with Eq. (7.295):

$$\begin{aligned} H_\varepsilon f(x) - Q_\varepsilon f(x) &= \frac{1}{\pi}\int_{|x-t|\ge\varepsilon} \{f(t) - f(x)\}\left[\frac{1}{x-t} - \frac{(x-t)}{(x-t)^2+\varepsilon^2}\right]\mathrm{d}t \\ &\quad - \frac{1}{\pi}\int_{|x-t|<\varepsilon} \frac{(x-t)\{f(t)-f(x)\}\mathrm{d}t}{(x-t)^2+\varepsilon^2} \\ &\quad + \frac{f(x)}{\pi}\int_{|x-t|\ge\varepsilon}\left[\frac{1}{x-t} - \frac{(x-t)}{(x-t)^2+\varepsilon^2}\right]\mathrm{d}t \\ &\quad - \frac{f(x)}{\pi}\int_{|x-t|<\varepsilon}\frac{(x-t)\mathrm{d}t}{(x-t)^2+\varepsilon^2}. \end{aligned} \tag{7.299}$$

The last pair of integrals in Eq. (7.299) simplify as follows:

$$\begin{aligned} &\int_{|x-t|\ge\varepsilon}\left\{\frac{1}{x-t} - \frac{(x-t)}{(x-t)^2+\varepsilon^2}\right\}\mathrm{d}t - \int_{|x-t|<\varepsilon}\frac{(x-t)\mathrm{d}t}{(x-t)^2+\varepsilon^2} \\ &\qquad = \int_{-\infty}^{x-\varepsilon}\frac{1}{x-t}\,\mathrm{d}t + \int_{x+\varepsilon}^{\infty}\frac{1}{x-t}\,\mathrm{d}t - \int_{-\infty}^{\infty}\frac{w\,\mathrm{d}w}{w^2+\varepsilon^2} \\ &\qquad = 0, \end{aligned} \tag{7.300}$$

and repeating the approach leading to Eq. (7.297) for the first two integrals in Eq. (7.299) gives Eq. (7.298).

The next result of interest is

$$\lim_{\varepsilon\to 0}\{H_\varepsilon f(x) - Q_\varepsilon f(x)\} = 0, \quad a.e. \tag{7.301}$$

Using Eq. (7.298) and Eq. (4.486) or the approach in Eqs. (4.487) – (4.491) leads to this result.

if $f \in L^p$ with $1 < p < \infty$, then

$$\lim_{\varepsilon\to 0} P_\varepsilon f(x) = f(x), \quad a.e., \tag{7.302}$$

and

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} |P_\varepsilon f(x) - f(x)|^p \,\mathrm{d}x = 0. \tag{7.303}$$

Equation (7.302) is proved using the approach discussed in Section 4.22. Equation (7.303) can be established as follows:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} |P_\varepsilon f(x) - f(x)|^p \,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \left|\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon\{f(t)-f(x)\}\mathrm{d}t}{(x-t)^2+\varepsilon^2}\right|^p \mathrm{d}x \\
&\le \int_{-\infty}^{\infty} \left|\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{\varepsilon|f(t)-f(x)|\mathrm{d}t}{(x-t)^2+\varepsilon^2}\right|^p \mathrm{d}x \\
&= 0,
\end{aligned} \tag{7.304}$$

where either Eq. (4.486) or the approach in Eqs. (4.487) – (4.491) is employed. The reader is asked to justify the switch of integral and limit that have been made.

Another result of interest is

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} |Q_\varepsilon f(x) - Hf(x)|^2 \,\mathrm{d}x = 0. \tag{7.305}$$

Recalling that $Hf(x) = \lim_{\varepsilon\to 0} H_\varepsilon f(x)$ *a.e.*, then, from Eq. (7.301), it follows that

$$Hf(x) = \lim_{\varepsilon\to 0} Q_\varepsilon f(x), \quad a.e. \tag{7.306}$$

If the following identity is employed:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon_1}{(x-s)^2+\varepsilon_1^2}\frac{s-t}{(s-t)^2+\varepsilon_2^2}\,\mathrm{d}s = \frac{x-t}{(x-t)^2+(\varepsilon_1+\varepsilon_2)^2}, \tag{7.307}$$

for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, then

$$\begin{aligned}
Q_{\varepsilon_1+\varepsilon_2} f(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2+(\varepsilon_1+\varepsilon_2)^2} f(t)\mathrm{d}t \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty} f(t)\mathrm{d}t \int_{-\infty}^{\infty} \frac{\varepsilon_1}{(x-s)^2+\varepsilon_1^2}\frac{s-t}{(s-t)^2+\varepsilon_2^2}\,\mathrm{d}s \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\varepsilon_1}{(x-s)^2+\varepsilon_1^2}\,\mathrm{d}s \int_{-\infty}^{\infty} \frac{s-t}{(s-t)^2+\varepsilon_2^2} f(t)\mathrm{d}t \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\varepsilon_1}{(x-s)^2+\varepsilon_1^2} Q_{\varepsilon_2} f(s)\mathrm{d}s \\
&= P_{\varepsilon_1} Q_{\varepsilon_2} f(x),
\end{aligned} \tag{7.308}$$

and the interchange of integration order can be justified by Fubini's theorem. The reader is requested to specify the conditions on the function f that supports making this step. if f is square-integrable, then taking the limit $\varepsilon_2 \to 0$ in the preceding result and using Eq. (7.306) yields

$$Q_\varepsilon f(x) = P_\varepsilon Hf(x); \tag{7.309}$$

that is,

$$\int_{-\infty}^{\infty} \frac{tf(x-\varepsilon t)\mathrm{d}t}{t^2+1} = \int_{-\infty}^{\infty} \frac{Hf(x-\varepsilon t)\mathrm{d}t}{t^2+1}, \quad \text{for } \varepsilon > 0, \tag{7.310}$$

a result holding for $f \in L^p(\mathbb{R})$ with $1 < p < \infty$. Note that Eq. (7.310) can be verified directly by reversing the order of integration for the integral on the right-hand side of the equation. This is left as an exercise for the reader to check. From Eq. (7.303), with f replaced by Hf and $p = 2$,

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} |P_\varepsilon Hf(x) - Hf(x)|^2 \,\mathrm{d}x = 0. \tag{7.311}$$

Inserting Eq. (7.309) into this result leads to Eq. (7.305).

Using the triangle inequality leads to the following useful result:

$$\begin{aligned} |H_\varepsilon f(x)| &= |H_\varepsilon f(x) - Q_\varepsilon f(x) + Q_\varepsilon f(x)| \\ &\le |H_\varepsilon f(x) - Q_\varepsilon f(x)| + |Q_\varepsilon f(x)| \,. \end{aligned} \tag{7.312}$$

Making use of Eqs. (7.294) and (7.309), the preceding result can be written as follows:

$$|H_\varepsilon f(x)| \le P_\varepsilon\, |f(x)| + |P_\varepsilon Hf(x)| \,. \tag{7.313}$$

In Section 5.2 (Eq. (5.13)) a bound for the norm of the truncated Hilbert transform for functions satisfying $f \in L^2$ was established, a result that is now generalized to cover the case $f \in L^p$ for $1 < p < \infty$. Start with Eq. (7.292); to deal with the convolution, make use of Young's inequality (also called Young's theorem). Young's inequality takes the following form: if $f \in L^p(\mathbb{R})$ and $h \in L^q(\mathbb{R})$, then $f * h \in L^r$, where $r^{-1} = p^{-1} + q^{-1} - 1$, and

$$\|f * h\|_r \le \|f\|_p \|h\|_q \,. \tag{7.314}$$

For $\varepsilon > 0$,

$$\|P_\varepsilon f\|_p \le \|P(x,\varepsilon)\|_1 \|f\|_p \,; \tag{7.315}$$

that is

$$\|P_\varepsilon f\|_p \le C\|f\|_p \,, \tag{7.316}$$

where

$$C = \|P(x,\varepsilon)\|_1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon}{t^2+\varepsilon^2}\,\mathrm{d}t = 1. \tag{7.317}$$

Equation (7.295) can be written in the following alternative form:

$$H_\varepsilon f(x) - Q_\varepsilon f(x) = \int_{-\infty}^{\infty} k_\varepsilon(x-t) f(t)\mathrm{d}t, \tag{7.318}$$

where

$$k_\varepsilon(x) = \begin{cases} 1/x - x/(x^2+\varepsilon^2), & \text{for } |x| > \varepsilon \\ -x/(x^2+\varepsilon^2), & \text{for } |x| < \varepsilon. \end{cases} \tag{7.319}$$

Applying Young's theorem to Eq. (7.318) gives

$$\|H_\varepsilon f - Q_\varepsilon f\|_p \le \|f\|_p \|k_\varepsilon\|_1 = C\|f\|_p\,. \tag{7.320}$$

From Eq. (7.309), and using Eq. (7.316) and the Riesz inequality, we have

$$\|Q_\varepsilon f\|_p \le \|P_\varepsilon Hf\|_p \le C_p\|f\|_p\,. \tag{7.321}$$

Writing

$$H_\varepsilon f(x) = H_\varepsilon f(x) - Q_\varepsilon f(x) + Q_\varepsilon f(x), \tag{7.322}$$

then applying Minkowski's inequality, yields

$$\|H_\varepsilon f\|_p \le \|H_\varepsilon f - Q_\varepsilon f\|_p + \|Q_\varepsilon f\|_p\,. \tag{7.323}$$

Inserting the bounds given in Eqs. (7.320) and (7.321) into the preceding inequality leads to

$$\|H_\varepsilon f\|_p \le C_p\|f\|_p\,, \tag{7.324}$$

which is the required result.

A bound for the convolution of the Poisson kernel with the function f in terms of the Hardy–Littlewood maximal function is now determined. Let $\phi_n(s)$ be a sequence of step functions selected such that in the limit $n \to \infty$, $\phi_n(s) \to$ Poisson kernel. The situation is depicted in Figure 7.2. The simple functions $\phi_n(s)$ can be written in terms of characteristic functions as follows:

$$\phi_n(s,x_0) = \sum_{k=1}^{N} a_k \chi_{(-h_k+x_0,\ h_k+x_0)}(s), \tag{7.325}$$

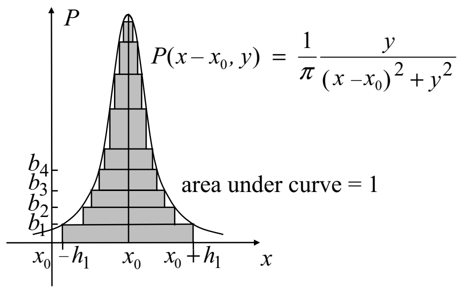

Figure 7.2. Sequence of step functions representing the Poisson kernel.

where the coefficients a_k satisfy $a_1 = b_1,\ a_k = b_k - b_{k-1}$, for $k \geq 2$, and $a_k \geq 0$. Since the Poisson kernel is normalized to unit area,

$$\int_{-\infty}^{\infty} \phi_n(s, x_0)\mathrm{d}s = \sum_{k=1}^{N} 2h_k\, a_k \leq 1. \tag{7.326}$$

It follows that

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} \phi_n(s, x_0) f(s)\mathrm{d}s\right| &\leq \int_{-\infty}^{\infty} \phi_n(s, x_0)|f(s)|\mathrm{d}s \\
&= \sum_{k=1}^{N} a_k \int_{-\infty}^{\infty} \chi_{(-h_k+x_0, h_k+x_0)}(s)|f(s)|\mathrm{d}s \\
&= \sum_{k=1}^{N} 2h_k a_k \frac{1}{2h_k}\int_{-h_k+x_0}^{h_k+x_0} |f(s)|\mathrm{d}s \\
&\leq \sum_{k=1}^{N} 2h_k a_k \sup_{h_k>0} \frac{1}{2h_k}\int_{-h_k+x_0}^{h_k+x_0} |f(s)|\mathrm{d}s \\
&= \sum_{k=1}^{N} 2h_k\, a_k Mf(x_0);
\end{aligned} \tag{7.327}$$

hence,

$$\left|\int_{-\infty}^{\infty} \phi_n(s, x) f(s)\mathrm{d}s\right| \leq Mf(x), \tag{7.328}$$

and so

$$|P_\varepsilon f(x)| \leq Mf(x). \tag{7.329}$$

Employing this result together with Eq. (7.313) leads, on taking $\sup_{\varepsilon>0}$, to

$$H_{\mathrm{M}} f(x) \leq Mf(x) + MHf(x), \tag{7.330}$$

which is the required result.

The maximal Hilbert transform is bounded in the following manner:

$$\|H_{\mathrm{M}}f\|_p \leq C_p\|Mf\|_p\,. \tag{7.331}$$

To prove this result, make use of the following inequality:

$$|x : H_{\mathrm{M}}f(x) > 2\lambda \text{ and } Mf(x) \leq \gamma\lambda| \leq C\gamma|x : H_{\mathrm{M}}f(x) > \lambda|\,, \tag{7.332}$$

for $\lambda > 0$, where C and γ are positive constants. This is called a *relative distributional inequality*, or sometimes a *good λ inequality*. The chapter end-notes provide references to locate the proof of this result. Different sources employ a comma or semi-colon in place of the "and" in Eq. (7.332). Now

$$\begin{aligned}
|x : H_{\mathrm{M}}f(x) > 2\lambda| &= |x : H_{\mathrm{M}}f(x) > 2\lambda \text{ and } Mf(x) \leq \gamma\lambda| \\
&\quad + |x : H_{\mathrm{M}}f(x) > 2\lambda \text{ and } Mf(x) > \gamma\lambda| \\
&\leq |x : H_{\mathrm{M}}f(x) > 2\lambda \text{ and } Mf(x) \leq \gamma\lambda| \\
&\quad + |x : Mf(x) > \gamma\lambda|\,.
\end{aligned} \tag{7.333}$$

Making use of Eq. (4.560) leads to

$$\|H_{\mathrm{M}}f\|_p^p = p\int_0^\infty |x : H_{\mathrm{M}}f(x) > \lambda|\lambda^{p-1}\,\mathrm{d}\lambda, \tag{7.334}$$

and, with the appropriate change of variable, it follows that

$$\begin{aligned}
\|H_{\mathrm{M}}f\|_p^p &= 2^p p\int_0^\infty |x : H_{\mathrm{M}}f(x) > 2\lambda|\lambda^{p-1}\,\mathrm{d}\lambda \\
&\leq 2^p p\left\{\int_0^\infty |x : H_{\mathrm{M}}f(x) > 2\lambda \text{ and } Mf(x) \leq \gamma\lambda|\lambda^{p-1}\,\mathrm{d}\lambda\right. \\
&\quad \left.+\int_0^\infty |x : Mf(x) > \gamma\lambda|\,\lambda^{p-1}\,\mathrm{d}\lambda\right\} \\
&\leq 2^p p\left\{C\gamma\int_0^\infty |x : H_{\mathrm{M}}f(x) > \lambda|\lambda^{p-1}\,\mathrm{d}\lambda\right. \\
&\quad \left.+\frac{1}{\gamma^p}\int_0^\infty |x : Mf(x) > \lambda|\,\lambda^{p-1}\,\mathrm{d}\lambda\right\},
\end{aligned} \tag{7.335}$$

where Eqs. (7.332) and (7.333) have been employed. Hence

$$\|H_{\mathrm{M}}f\|_p^p \leq 2^p\left\{C\gamma\,\|H_{\mathrm{M}}f\|_p^p + \frac{1}{\gamma^p}\|Mf\|_p^p\right\}. \tag{7.336}$$

From Eq. (7.336) it follows that

$$\|H_{\mathrm{M}} f\|_p^p \leq \frac{2^p}{\gamma^p(1-2^p C\gamma)} \|Mf\|_p^p . \tag{7.337}$$

Equation (7.331) follows by selecting the constant γ so that $(1-2^p C\gamma) > 0$.

Taking advantage of Eqs. (7.283) and (7.331) leads to the following result:

$$\|Hf\|_p \leq C_p \|Mf\|_p . \tag{7.338}$$

If Eq. (7.246) is employed, then Eq. (7.338) reduces to the Riesz inequality, and this approach provides another derivation of the key inequality of this chapter.

This section is concluded by considering a weak-type inequality for the maximal Hilbert transform. if $f \in L(\mathbb{R})$, then, for $\lambda > 0$,

$$m\{x : H_{\mathrm{M}} f(x) > \lambda\} \leq \frac{C}{\lambda} \int_{-\infty}^{\infty} |f(x)| \mathrm{d}x, \tag{7.339}$$

where C is a constant independent of f. To prove this result, the approach of Garsia (1970, p. 113) is closely followed. Since the proof depends on ideas developed by Loomis, the reader would find a review of parts of Section 4.25 to be valuable. First, a lemma is presented that is attributed by Garsia to W. H. Young, which is a simple form of a more general result of Vitali. A covering lemma: let $\{J_1, J_2, \ldots, J_p\}$ denote a finite family of open intervals in $\mathbb{R}$, then a finite subfamily of disjoint intervals $\{I_1, I_2, \ldots, I_n\}$ can be found such that

$$m\left(\bigcup_{k=1}^{p} J_k\right) \leq 2 \sum_{k=1}^{n} m(I_k). \tag{7.340}$$

It is assumed that $\{J_1, J_2, \ldots, J_p\}$ contains no intervals that can be written as the union of the other intervals; if this is not so, then these particular intervals are deleted. In this new family, set

$$J_k = (\alpha_k, \beta_k), \tag{7.341}$$

and index the intervals so that $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_p$, then

$$\beta_{2k-1} < \alpha_{2k+1} \tag{7.342}$$

and

$$\beta_{2k} < \alpha_{2k+2}. \tag{7.343}$$

The situation $\alpha_k = \alpha_{k+1}$ for some k is excluded, since one of the intervals J_k or J_{k+1} contains the other, which, by construction, is excluded in the new family of intervals. The case $\beta_{2k-1} \geq \alpha_{2k+1}$ is also excluded, otherwise the union of two of the intervals

J_{2k-1}, J_{2k}, or J_{2k+1} would contain the third, and a similar situation applies to the counter statement of Eq. (7.343). Therefore the even and odd numbered intervals form pairwise disjoint subfamilies, and so it follows that

$$m\left(\bigcup_{k=1}^{p} J_k\right) \leq \sum_{k \text{ even}}^{n} m(J_k) + \sum_{k \text{ odd}}^{n} m(J_k). \tag{7.344}$$

Identifying the larger of the even numbered or odd numbered intervals with the family $\{I_1, I_2, \ldots, I_n\}$, Eq. (7.340) follows.

For $\lambda > 0$, define the sets

$$E_+ = \{x : \sup_{\eta \geq \varepsilon}[H_\eta f(x)] > \lambda\} \tag{7.345}$$

and

$$E_- = \{x : \sup_{\eta \geq \varepsilon}[H_\eta f(x)] < -\lambda\}. \tag{7.346}$$

Let J_j denote the set of intervals $(x_j - \tau_j, x_j + \tau_j)$. The intervals J_j for which

$$\frac{1}{\pi}\left\{\int_{-\infty}^{x_j - \tau_j} + \int_{x_j + \tau_j}^{\infty}\right\} \frac{f(t)\mathrm{d}t}{x_j - t} > \lambda \tag{7.347}$$

cover E_+. A subfamily of disjoint intervals I_j of the form $(x_j - \delta_j, x_j + \delta_j)$, $j = 1, 2, \ldots, n$, can be selected so that

$$\frac{1}{\pi}\left\{\int_{-\infty}^{x_j - \delta_j} + \int_{x_j + \delta_j}^{\infty}\right\} \frac{f(t)\mathrm{d}t}{x_j - t} > \lambda, \tag{7.348}$$

and by the covering lemma, Eq. (7.340),

$$m(E_+) \leq 2\sum_{j=1}^{n} m(I_j). \tag{7.349}$$

The real axis can be partitioned into small intervals so that, for a given $\delta > 0$,

$$\left|\frac{1}{\pi}\left\{\int_{-\infty}^{x_j - \delta_j} + \int_{x_j + \delta_j}^{\infty}\right\} \frac{f(t)\mathrm{d}t}{x_j - t} - \frac{1}{\pi}\sum_{\sigma \notin I_j} \frac{\int_\sigma f(t)\mathrm{d}t}{x_j - x_\sigma}\right| < \delta\lambda, \tag{7.350}$$

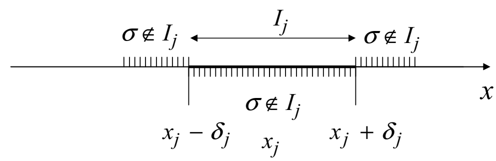

Figure 7.3. Partition of the real axis into the intervals σ and I_j.

where x_σ is the center of the interval σ. Two functions are introduced by the following definitions:

$$g(x) = \frac{1}{\pi} \sum_{\sigma} \frac{\int_\sigma f(t)\mathrm{d}t}{x - x_\sigma} \tag{7.351}$$

and

$$g_j(x) = \frac{1}{\pi} \sum_{\sigma \in I_j} \frac{\int_\sigma f(t)\mathrm{d}t}{x - x_\sigma}. \tag{7.352}$$

The sum $\sum_{\sigma \notin I_j}$ is evaluated over the intervals shown in Figure 7.3.

The difference of the two functions is given by

$$h(x) = g(x) - g_j(x) = \frac{1}{\pi} \sum_{\sigma \notin I_j} \frac{\int_\sigma f(t)\mathrm{d}t}{x - x_\sigma}, \tag{7.353}$$

and this is clearly decreasing for $x \in I_j$. Since

$$\delta\lambda > \frac{1}{\pi} \left\{ \int_{-\infty}^{x_j - \delta_j} + \int_{x_j + \delta_j}^{\infty} \right\} \frac{f(t)\mathrm{d}t}{x_j - t} - \frac{1}{\pi} \sum_{\sigma \notin I_j} \frac{\int_\sigma f(t)\mathrm{d}t}{x_j - x_\sigma} > -\delta\lambda, \tag{7.354}$$

then

$$\frac{1}{\pi} \sum_{\sigma \notin I_j} \frac{\int_\sigma f(t)\mathrm{d}t}{x_j - x_\sigma} > (1 - \delta)\lambda, \tag{7.355}$$

and hence, for $x = x_j$

$$h(x) > (1 - \delta)\lambda, \tag{7.356}$$

which remains true for x in the left half of I_j. Making use of Eq. (4.560), it follows that

$$\sum_{j=1}^{n} \frac{1}{2} m(I_j) = m\{x : h(x) > (1-\delta)\lambda\} \leq m\left\{x : g(x) > \frac{1}{2}(1-\delta)\lambda\right\} + m\left\{x : \sum_{j=1} -g_j(x) > \frac{1}{2}(1-\delta)\lambda\right\}. \tag{7.357}$$

Applying Loomis' lemma (see Section 4.25), leads to

$$\sum_{j=1}^{n} \frac{1}{2} m(I_j) \leq \frac{2}{(1-\delta)\lambda} \sum_{\sigma} \int_{\sigma} f(t)\mathrm{d}t + \sum_{j=1}^{n} \frac{2}{(1-\delta)\lambda} \sum_{\sigma\in I_j} \int_{\sigma} f(t)\mathrm{d}t. \tag{7.358}$$

Taking the limit $\delta \to 0$ and making use of Eq. (7.349) leads to

$$m(E_+) \leq \frac{16}{\lambda} \int_{-\infty}^{\infty} f(t)\mathrm{d}t, \tag{7.359}$$

and a similar approach gives

$$m(E_-) \leq \frac{16}{\lambda} \int_{-\infty}^{\infty} f(t)\mathrm{d}t. \tag{7.360}$$

Taking the limit $\varepsilon \to 0+$, it follows that

$$m\left\{x : \sup_{\varepsilon>0} |H_\varepsilon f(x)| > \lambda\right\} \leq \frac{C}{\lambda} \int_{-\infty}^{\infty} |f(x)|\mathrm{d}x. \tag{7.361}$$

Recalling the definition in Eq. (7.280) establishes Eq. (7.339).

Let $f(z)$ be defined in the upper half plane, then the non-tangential maximal function of f is introduced as

$$N_\alpha(f)(x) = \sup_{z\in\Gamma_\alpha(x)} |f(z)|, \tag{7.362}$$

where $\Gamma_\alpha(x)$ is the cone $z = \{s + \mathrm{i}y : |s - x| < \alpha y\}$. An interesting result is the following. Let $0 < p < \infty$ and suppose $u(z)$ is a real-valued harmonic function in the upper half plane with $v(z)$ its harmonic conjugate; then $F(z) = u(z) + \mathrm{i}v(z)$ is analytic in the upper half plane, and for $\alpha > 0$ and $N_\alpha(u) \in L^p$,

$$\sup_{y>0} \int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^p \,\mathrm{d}x \leq C_{\alpha,p} \int_{-\infty}^{\infty} |N_\alpha(u)(x)|^p \,\mathrm{d}x \tag{7.363}$$

and

$$\sup_{y>0} \int_{-\infty}^{\infty} |v(x+\mathrm{i}y)|^p \, \mathrm{d}x \leq c_{\alpha,p} \int_{-\infty}^{\infty} |N_\alpha(u)(x)|^p \, \mathrm{d}x. \tag{7.364}$$

This theorem is due to Burkholder, Gundy, and Silverstein (1971). The special part of the result is the case $0 < p \leq 1$. The reader interested in the proof can consult Burkholder *et al.* (1971), or for a simplier proof see Garnett (1981, p. 116) or Koosis (1998, p. 178). There is an analog of the preceding result for the unit disc.

7.11 A theorem due to Helson and Szegö

Recall from the discussion in Sections 4.10 and 4.20 that $\|Hf\|_{L^2}$ is bounded. Let μ denote a positive measure on $\mathbb{R}$, then a question of interest is to decide whether $\|Hf\|_{L^2(\mu)}$ is bounded for a suitably chosen f. This problem was answered in the affirmative by Helson and Szegö (1960). Their key result is as follows. The Hilbert transform operator is bounded in $L^2(\mu)$ iff the measure μ is absolutely continuous, so that $\mathrm{d}\mu = w(x)\mathrm{d}x$, where the weight function $w(x)$ is given by

$$w(x) = \mathrm{e}^{\varphi(x)+H\psi(x)}, \tag{7.365}$$

and the functions φ and ψ satisfy $\varphi(x) \in L^\infty(\mathbb{R})$, $\psi(x) \in L^\infty(\mathbb{R})$, and $\|\psi\|_{L^\infty} < \pi/2$. To establish this result, let $\Psi(z)$ denote the analytic function such that, on the real axis,

$$\Psi(x) = \mathrm{e}^{-\mathrm{i}\{\psi(x)+\mathrm{i}(H\psi)(x)\}}. \tag{7.366}$$

Let $F(z)$ denote a function analytic in the upper half plane and vanishing sufficiently quickly as $|z| \to \infty$ so that $\int_{-\infty}^{\infty} F(x)^2 \, \mathrm{d}x$ is bounded; hence,

$$\int_{-\infty}^{\infty} F(x)^2 \Psi(x) \mathrm{d}x = 0. \tag{7.367}$$

Let $F(x) = f(x) + \mathrm{i}(Hf)(x)$, then, on taking the real part and using Eq. (7.365), with $\varphi(x)$ assumed to be zero, it follows that

$$\int_{-\infty}^{\infty} \{f(x)^2 - [Hf(x)]^2\}\{\cos\psi(x) + 2f(x)Hf(x)\sin\psi(x)\}w(x)\mathrm{d}x = 0, \tag{7.368}$$

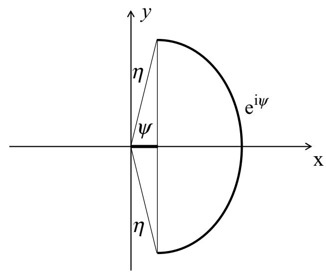

Figure 7.4. Evaluation of $\cos\psi(x)$.

which can be rearranged to yield

$$\begin{aligned}\int_{-\infty}^{\infty}[Hf(x)]^2\cos\psi(x)w(x)\mathrm{d}x &= \int_{-\infty}^{\infty}\{f(x)^2\cos\psi(x)+2f(x)Hf(x)\sin\psi(x)\}w(x)\mathrm{d}x\\ &\le \int_{-\infty}^{\infty}f(x)^2w(x)\mathrm{d}x+2\left(\int_{-\infty}^{\infty}f(x)^2w(x)\mathrm{d}x\right)^{1/2}\\ &\quad\left(\int_{-\infty}^{\infty}\{Hf(x)\}^2w(x)\mathrm{d}x\right)^{1/2}, \end{aligned}\tag{7.369}$$

where the Cauchy–Schwarz–Buniakowski inequality has been employed and the inequalities $\sin^2\psi(x)\le 1$ and $\cos\psi(x)\le 1$ are also used. Let $\|\psi\|_{L^\infty}=\pi/2-\eta$, with $\eta>0$, then it follows that there is a constant κ such that

$$0<\kappa=\cos\|\psi\|_{L^\infty}\le\cos\psi(x). \tag{7.370}$$

This can be readily seen from the geometric construction shown in Figure 7.4, where $\cos\psi(x)$ is the heavy line segment on the x-axis, and clearly this is always greater than zero if $\eta>0$. Using Eqs. (7.369) and (7.370), setting $\lambda=\int_{-\infty}^{\infty}\{Hf(x)\}^2w(x)\mathrm{d}x$ and $\alpha=\int_{-\infty}^{\infty}f(x)^2w(x)\mathrm{d}x$, leads to

$$\kappa\lambda\le\alpha+2\surd(\alpha\lambda). \tag{7.371}$$

Set $x=\surd(\alpha\lambda)$, then

$$\left\{x-\frac{\alpha}{\kappa}(1+\surd(1+\kappa))\right\}\left\{x+\frac{\alpha}{\kappa}(\surd(1+\kappa)-1)\right\}\le 0. \tag{7.372}$$

Since $\{x+(\alpha/\kappa)(\surd(1+\kappa)-1)\}$ is positive, x is bounded above in the following way:

$$x\le\frac{\alpha}{\kappa}(1+\surd(1+\kappa)); \tag{7.373}$$

that is,

$$\lambda \leq C\alpha, \tag{7.374}$$

which can be rewritten immediately as

$$\|Hf\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mu)}, \tag{7.375}$$

where C denotes a positive constant independent of f. This is the required result.

It turned out to be difficult to take the Helson–Szegö result in different directions. An alternative approach has been developed by Hunt, Muckenhoupt, and Wheeden to handle weighted inequalities for the Hilbert transform, and this is discussed in the following two sections.

7.12 The A_p condition

A fundamental issue connected with Eq. (7.159) is the following: for what class of weight functions is this inequality satisfied? In different language, if $L^p(\mu)$ denotes the class of μ-measurable functions f, for which the norm $\|f\|_{p,\mu}$ satisfies

$$\left(\int |f|^p \,\mathrm{d}\mu\right)^{1/p} < \infty,\ 1 < p < \infty, \tag{7.376}$$

what characterization of the weight functions can be found to ensure that $\|Hf\|_{p,\mu}$ is bounded on $\mathbb{R}$? For an absolutely continuous positive Borel measure μ, $\mathrm{d}\mu = w(x)\mathrm{d}x$, and this form is employed in subsequent developments. It turns out that a relatively simple condition – called the A_p condition – can be found that determines what weights allow Eq. (7.159) to be satisfied. The A_p condition is discussed in this section and some applications are illustrated in Section 7.13.

The principal interest here is the determination of a number of properties that are satisfied by functions of the class A_p. In this section, a convention commonly employed in this area of mathematics is adopted, that $0 \cdot \infty$ when it arises has the value 0. Let $w(x)$ denote a non-negative function that is locally integrable and defined on $(-\infty, \infty)$, which satisfies the inequality

$$\int_I w(x)\mathrm{d}x \left(\int_I w(x)^{-(p-1)^{-1}} \mathrm{d}x\right)^{p-1} \leq C\,|I|^p, \tag{7.377}$$

for $1 < p < \infty$. The constant C satisfies $0 < C < \infty$, and is independent of I, with I any interval such that $I \subset (-\infty, \infty)$, and $|I|$ is the length of the interval. Equation (7.377) is called the A_p condition. Functions satisfying this condition are referred to as belonging to the A_p class. From the definition it is clear that both $w(x)$ and $w(x)^{-(p-1)^{-1}}$ are required to be locally summable and finite *a.e.* For the case $p = 1$, the A_p condition is given by

$$\frac{1}{|I|}\int_I w(x)\mathrm{d}x \le C \operatorname*{ess\,inf}_{x\in I} w(x) \tag{7.378}$$

or, as sometimes written in terms of the Hardy–Littlewood maximal function,

$$Mw(x) \le Cw(x), \tag{7.379}$$

with C independent of I. For the case $p = \infty$, proceed as follows. Let $w(x)$ be a non-negative locally summable function on $\mathbb{R}$, let the measure $\mathrm{d}\mu = w(x)\mathrm{d}x$, and denote

$$\int_E w(x)\mathrm{d}x = \mu(E) = |E|_w. \tag{7.380}$$

Suppose there are real constants α and β, with $0 < \alpha < 1$ and $0 < \beta < 1$, such that

$$\frac{|E|}{|I|} < \alpha \Rightarrow \frac{\mu(E)}{\mu(I)} < \beta, \quad \text{for } E \subset I; \tag{7.381}$$

then the weight $w(x)$ satisfies the A_∞ condition.

If the Hölder inequality with conjugate exponents p and q is applied to the weight function $w(x)$, then

$$\int_I w(x)\,\mathrm{d}x \le \left(\int_I w(x)^q\,\mathrm{d}x\right)^{q^{-1}} \left(\int_I \mathrm{d}x\right)^{p^{-1}}, \tag{7.382}$$

and so

$$\frac{1}{|I|}\int_I w(x)\mathrm{d}x \le \left(\frac{1}{|I|}\int_I w(x)^q\,\mathrm{d}x\right)^{q^{-1}}. \tag{7.383}$$

By analogy with this last result, the weight function $w(x)$ is said to satisfy the *reverse Hölder inequality* if, for $q > 1$ and non-negative C independent of I,

$$\left(\frac{1}{|I|}\int_I w(x)^q\,\mathrm{d}x\right)^{q^{-1}} \le C\frac{1}{|I|}\int_I w(x)\mathrm{d}x. \tag{7.384}$$

A number of the properties associated with the A_p condition are now summarized. Several of these results find application in deriving weighted inequalities involving singular integral operators.

(i) If $w_1(x) \in A_1$ and $w_2(x) \in A_1$, and let $w(x) = w_1(x)\{w_2(x)\}^{1-p}$ for $1 < p < \infty$, then $w(x) \in A_p$.

(ii) If $w(x) \in A_p$ for $1 < p < \infty$, then there exist weights $w_1(x) \in A_1, w_2(x) \in A_1$, such that $w(x) = w_1(x)\{w_2(x)\}^{1-p}$.

(iii) If $w_1(x) \in A_p$ and $w_2(x) \in A_p$, for $1 \le p < \infty$, and let $w(x) = w_1(x)^\alpha\{w_2(x)\}^{1-\alpha}$, then $w(x) \in A_p$ for $0 \le \alpha \le 1$.
(iv) If $w(x) \in A_p$ for $1 \le p < \infty$, then $w(x) \in A_{p-\varepsilon}$, for some $\varepsilon > 0$.
(v) If $w(x) \in A_p$ for $1 \le p < \infty$, then $w(x)^\alpha \in A_p$, for α satisfying $0 < \alpha < 1$.
(vi) If the weight $w(x) \in A_p$ with constant C for $1 < p < \infty$, then $w(x)^{-(p-1)^{-1}} \in A_q$ with constant $C^{(p-1)^{-1}}$, where q is the conjugate exponent to p.
(vii) If the weight $w(x) \in A_1$, then $w(x) \in A_p$.
(viii) If the weight $w(x) \in A_p$, then $w(x) \in A_\infty$.
(ix) If the weight $w(x) \in A_p$ for some constant C and $p \ge 1$, then $w(x) \in A_r$ with constant C and $r > p$.
(x) If the weight $w(x) \in A_\infty$ and is locally integrable, then $w(x) \in A_p$ for $1 < p < \infty$.
(xi) If the weight $w(x) \in A_p$ for $1 < p < \infty$, then $w(x)$ satisfies the reverse Hölder inequality for constants q and C independent of I, with $q > 1$ and $C > 0$.
(xii) If $w(x) \in A_p$ for $1 < p < \infty$, and $w_I = (1/|I|)\int_I w(x)\mathrm{d}x$, then, for $\beta > 0, |\{x \in I : w(x) < \beta w_I\}| \le C\beta^{(p-1)^{-1}}|I|$.
(xiii) If $w(x) \in A_1$, then $w(x)$ satisfies the reverse Hölder inequality.

Some proofs for several of these statements are sketched in the remainder of this section, and the interested reader is left to pursue details for the others in the references cited in the chapter end-notes. For property (i), start with the following inequality:

$$\frac{1}{|I|}\int_I w_1(x)w_2(x)^{1-p}\,\mathrm{d}x \le \frac{1}{\inf\limits_{x\in I}\{w_2(x)^{p-1}\}}\frac{1}{|I|}\int_I w_1(x)\mathrm{d}x, \tag{7.385}$$

and

$$\begin{aligned}\left(\frac{1}{|I|}\int_I \{w_1(x)w_2(x)^{1-p}\}^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p-1} &= \left(\frac{1}{|I|}\int_I w_1(x)^{-(p-1)^{-1}}w_2(x)\mathrm{d}x\right)^{p-1}\\ &\le \frac{1}{\inf\limits_{x\in I}\{w_1(x)\}}\left(\frac{1}{|I|}\int_I w_2(x)\mathrm{d}x\right)^{p-1}.\end{aligned} \tag{7.386}$$

Multiplying the last two results, and using the fact that $w_1 \in A_1$ and $w_2 \in A_1$, gives

$$\begin{aligned}&\frac{1}{|I|}\int_I w_1(x)w_2(x)^{1-p}\,\mathrm{d}x\left(\frac{1}{|I|}\int_I \{w_1(x)w_2(x)^{1-p}\}^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p-1}\\ &\quad\le \frac{1}{\inf\limits_{x\in I}\{w_1(x)\}}\frac{1}{|I|}\int_I w_1(x)\mathrm{d}x\,\frac{1}{\inf\limits_{x\in I}\{w_2(x)^{p-1}\}}\left(\frac{1}{|I|}\int_I w_2(x)\mathrm{d}x\right)^{p-1}\\ &\quad\le C_1C_2^{p-1} = C,\end{aligned} \tag{7.387}$$

and it therefore follows that $w_1(x)w_2(x)^{1-p} \in A_p$, which is the required result.

To prove property (iii), apply Hölder's inequality to $\{w_1(x)^\alpha w_2(x)^{1-\alpha}\}^{-(p-1)^{-1}}$ with the exponent choice $p_2 = (1-\alpha)^{-1}$; then

$$\int_I \{w_1(x)^\alpha w_2(x)^{1-\alpha}\}^{-(p-1)^{-1}} \,\mathrm{d}x \le \left(\int_I w_1(x)^{-(p-1)^{-1}} \,\mathrm{d}x\right)^\alpha \left(\int_I w_2(x)^{-(p-1)^{-1}} \,\mathrm{d}x\right)^{1-\alpha}. \tag{7.388}$$

Applying Hölder's inequality to $w_1(x)^\alpha w_2(x)^{1-\alpha}$ gives

$$\int_I w_1(x)^\alpha w_2(x)^{1-\alpha} \,\mathrm{d}x \le \left(\int_I w_1(x)\mathrm{d}x\right)^\alpha \left(\int_I w_2(x)\mathrm{d}x\right)^{1-\alpha}. \tag{7.389}$$

From these preceding two inequalities, it follows that

$$\begin{aligned}
&\frac{1}{|I|}\int_I w_1(x)^\alpha w_2(x)^{1-\alpha}\mathrm{d}x \left(\frac{1}{|I|}\int_I \{w_1(x)^\alpha w_2(x)^{1-\alpha}\}^{-(p-1)^{-1}} \,\mathrm{d}x\right)^{p-1} \\
&\le \left(\frac{1}{|I|}\int_I w_1(x)\mathrm{d}x\right)^\alpha \left(\frac{1}{|I|}\int_I w_2(x)\mathrm{d}x\right)^{1-\alpha} \left(\frac{1}{|I|}\int_I w_1(x)^{-(p-1)^{-1}}\mathrm{d}x\right)^{\alpha(p-1)} \\
&\quad \left(\frac{1}{|I|}\int_I w_2(x)^{-(p-1)^{-1}} \,\mathrm{d}x\right)^{(1-\alpha)(p-1)} \\
&= \left[\frac{1}{|I|}\int_I w_1(x)\mathrm{d}x \left(\frac{1}{|I|}\int_I w_1(x)^{-(p-1)^{-1}} \,\mathrm{d}x\right)^{p-1}\right]^\alpha \\
&\quad \left[\frac{1}{|I|}\int_I w_2(x)\mathrm{d}x \left(\frac{1}{|I|}\int_I w_2(x)^{-(p-1)^{-1}} \,\mathrm{d}x\right)^{p-1}\right]^{1-\alpha} \\
&\le C_1^\alpha C_2^{1-\alpha} = C,
\end{aligned} \tag{7.390}$$

and hence $w_1(x)^\alpha w_2(x)^{1-\alpha} \in A_p$, which is the required result.

To see how property (iv) arises, start with property (xi) and apply the reverse Hölder inequality to the function $w(x)^{-(p-1)^{-1}}$ to obtain, for $\delta > 0$,

$$\left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}(1+\delta)} \,\mathrm{d}x\right)^{(1+\delta)^{-1}} \le C_1 \frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}} \,\mathrm{d}x, \tag{7.391}$$

which becomes, on substituting $1+\delta = (p-1)(p-1-\varepsilon)^{-1}$, with $\varepsilon > 0$, the following:

$$\left(\frac{1}{|I|}\int_I w(x)^{-(p-\varepsilon-1)^{-1}} \,\mathrm{d}x\right)^{p-\varepsilon-1} \le C_1^{p-1} \left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}} \,\mathrm{d}x\right)^{p-1}. \tag{7.392}$$

Multiply the preceding result by $(1/|I|)\int_I w(x)\mathrm{d}x$ to obtain

$$\begin{aligned}\frac{1}{|I|}\int_I w(x)\mathrm{d}x\left(\frac{1}{|I|}\int_I w(x)^{-(p-\varepsilon-1)^{-1}}\,\mathrm{d}x\right)^{p-\varepsilon-1}\\ \leq C_1^{p-1}\frac{1}{|I|}\int_I w(x)\mathrm{d}x\left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p-1}\\ \leq C_1^{p-1}C_2 = C,\end{aligned} \tag{7.393}$$

and hence $w(x) \in A_{p-\varepsilon}$.

Property (v) is established by starting with Hölder's inequality with exponents p_1 and p_2,

$$\int_I w(x)^{-\alpha(p-1)^{-1}}\mathrm{d}x \leq \left(\int_I \mathrm{d}x\right)^{p_1^{-1}}\left(\int_I w(x)^{-\alpha p_2(p-1)^{-1}}\,\mathrm{d}x\right)^{p_2^{-1}}, \tag{7.394}$$

and letting $p_2 = \alpha^{-1}$ with $0 < \alpha < 1$. Take the $\alpha^{-1}(p-1)$ power to obtain

$$\left(\frac{1}{|I|}\int_I w(x)^{-\alpha(p-1)^{-1}}\,\mathrm{d}x\right)^{\alpha^{-1}(p-1)} \leq \left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p-1}. \tag{7.395}$$

Since $(p-1) < \alpha^{-1}(p-1)$ and $\int_I w(x)^{\alpha}\,\mathrm{d}x < \int_I w(x)\mathrm{d}x$,

$$\begin{aligned}\frac{1}{|I|}\int_I w(x)^{\alpha}\,\mathrm{d}x\left(\frac{1}{|I|}\int_I \{w(x)^{\alpha}\}^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{(p-1)}\\ \leq \frac{1}{|I|}\int_I w(x)\mathrm{d}x\left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p-1},\end{aligned} \tag{7.396}$$

and the proof is complete.

To prove (vi), take the $(p-1)^{-1}$ power of Eq. (7.377). Using $(p-1)(q-1) = 1$ leads to

$$\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\left(\int_I w(x)^{[-(p-1)^{-1}][-(q-1)^{-1}]}\,\mathrm{d}x\right)^{(p-1)^{-1}} \leq C^{(p-1)^{-1}}|I|^{p(p-1)^{-1}}, \tag{7.397}$$

and this simplifies, on noting that $(p-1)^{-1} = (q-1)$ and $p(p-1)^{-1} = q$, to give

$$\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\left(\int_I \{w(x)^{-(p-1)^{-1}}\}^{-(q-1)^{-1}}\,\mathrm{d}x\right)^{q-1} \leq C^{(p-1)^{-1}}|I|^{q}, \tag{7.398}$$

which is the required result.

To prove property (vii), start with Hölder's inequality to obtain, for $p > 1$,

$$\left(\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^p \le |I|^{(p-1)} \int_I w(x)^{-(p-1)^{-1}} w(x)^{-1}\,\mathrm{d}x, \tag{7.399}$$

and so

$$\left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^p \le \frac{1}{|I|}\sup_{x\in I}\{w(x)^{-1}\}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x, \tag{7.400}$$

from which it follows that

$$\left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p-1} \le \frac{1}{\inf\limits_{x\in I}\{w(x)\}}. \tag{7.401}$$

Multiplying by $(1/\,|I|)\int_I w(x)\mathrm{d}x$ leads to

$$\frac{1}{|I|}\int_I w(x)\mathrm{d}x\left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p-1} \le \frac{(1/\,|I|)\int_I w(x)\mathrm{d}x}{\inf\limits_{x\in I}\{w(x)\}}. \tag{7.402}$$

If $w \in A_1$, the right-hand side of the last inequality is bounded by a constant C, and hence $w \in A_p$.

Property (viii) is established by starting with Hölder's inequality applied to $\chi_E(x)w(x)$ and using the definitions

$$w_E = \int \chi_E(x)w(x)\mathrm{d}x; \quad w_I = \int_I w(x)\mathrm{d}x. \tag{7.403}$$

So, for $\eta > 0$,

$$w_E \le |I|^{(\eta+1)^{-1}}\left(\int \chi_E(x)\mathrm{d}x\right)^{\eta(\eta+1)^{-1}}\left(\frac{1}{|I|}\int_I w(x)^{\eta+1}\,\mathrm{d}x\right)^{(\eta+1)^{-1}}. \tag{7.404}$$

This result simplifies as follows:

$$w_E \le |E|^{\eta(\eta+1)^{-1}}|I|^{(\eta+1)^{-1}}\, C\frac{1}{|I|}\int_I w(x)\mathrm{d}x, \tag{7.405}$$

where the reverse Hölder inequality has been employed. From the preceding result it follows, on setting $\delta = \eta(\eta+1)^{-1}$, that

$$\frac{w_E}{w_I} \le C\left(\frac{|E|}{|I|}\right)^{\delta}. \tag{7.406}$$

Therefore, if $w(x) \in A_p$, property (xi) states that the reverse Hölder inequality is satisfied and it therefore follows from Eq. (7.406) that $w(x) \in A_\infty$.

Property (ix) is established by starting with Hölder's inequality,

$$\int_I w(x)^{-(r-1)^{-1}}\,\mathrm{d}x \le \left(\int_I \mathrm{d}x\right)^{p^{-1}} \left(\int_I w(x)^{-q(r-1)^{-1}}\,\mathrm{d}x\right)^{q^{-1}}, \tag{7.407}$$

and setting $q = (r-1)(p-1)^{-1}$ with $r > p$; hence,

$$\int_I w(x)^{-(r-1)^{-1}}\,\mathrm{d}x \le |I|^{(r-p)(r-1)^{-1}} \left(\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{(p-1)(r-1)^{-1}}. \tag{7.408}$$

Now take the $(r-1)$ power and multiply by $\int_I w(x)\mathrm{d}x$ to obtain

$$\begin{aligned}&|I|^{-r}\left(\int_I w(x)\mathrm{d}x\right)\left(\int_I w(x)^{-(r-1)^{-1}}\,\mathrm{d}x\right)^{(r-1)}\\ &\quad\le |I|^{-p}\left(\int_I w(x)\mathrm{d}x\right)\left(\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{(p-1)}.\end{aligned} \tag{7.409}$$

Since the right-hand side of this inequality is bounded by C, property (ix) is proved.

The last property considered is (xii), which can be established using the A_p condition for the function $w(x)$ and employing the abbreviation

$$w_I = \frac{1}{|I|}\int_I w(x)\mathrm{d}x, \tag{7.410}$$

hence,

$$w_I^{(p-1)^{-1}} \int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x \le C\,|I|\,. \tag{7.411}$$

Select a $\beta > 0$, then

$$\int_{\{w<\beta w_I\}} w(x)^{-(p-1)^{-1}}\,\mathrm{d}x \le \int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x, \tag{7.412}$$

and employing the inequality $(\beta w_I)^{-(p-1)^{-1}} < w(x)^{-(p-1)^{-1}}$ leads to the following result:

$$\beta^{-(p-1)^{-1}} \int_{\{w<\beta w_I\}} \mathrm{d}x \le C\,|I|\,; \tag{7.413}$$

that is,

$$|\{x \in I : w(x) < \beta w_I\}| \le C\beta^{(p-1)^{-1}}|I|\,, \tag{7.414}$$

which is the required result.

7.13 A theorem due to Hunt, Muckenhoupt, and Wheeden

This section considers weighted inequalities of the following form:

$$\int_{-\infty}^{\infty} |Hf(x)|^p w(x)\mathrm{d}x \leq C(p,w) \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x. \tag{7.415}$$

The key issue is the determination of the general conditions that the weight function $w(x)$ must satisfy in order for Eq. (7.415) to hold. The weight function is assumed to be non-negative throughout. An obvious condition, albeit trivial, occurs when $w(x)$ is a constant for $x \in \mathbb{R}$, in which case the preceding equation reduces to the Riesz inequality. Results corresponding to Eq. (7.415) with Hf replaced by the maximal functions Mf and $H_{\mathrm{M}}f$ are also considered.

The first result to be established is the following. If $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $p > 1$, then

$$\int_{-\infty}^{\infty} |Mf(x)|^p w(x)\mathrm{d}x \leq C(p,w) \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x \tag{7.416}$$

implies the A_p condition. The approach of Coifman and Fefferman (1974) is followed. Let

$$m_I(f) = \frac{1}{|I|}\int_I |f(t)|\mathrm{d}t, \tag{7.417}$$

fix an interval I and take $f \geq 0$; then the following inequality holds:

$$\chi_I(x)\frac{1}{|I|}\int_I |f(t)|\mathrm{d}t \leq Mf(x). \tag{7.418}$$

Hence, from Eq. (7.416) it follows that

$$m_I(f)\left(\int_I w(x)\mathrm{d}x\right)^{p^{-1}} \leq C\left(\int_I |f(x)|^p w(x)\mathrm{d}x\right)^{p^{-1}}. \tag{7.419}$$

Take an interval $J \subseteq I$ and set

$$f(x) = \begin{cases} Cw(x)^{-(p-1)^{-1}}, & \text{for } x \in J \\ 0, & \text{for } x \notin J, \end{cases} \tag{7.420}$$

then Eq. (7.419) becomes

$$\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\left(\int_I w(x)\mathrm{d}x\right)^{p^{-1}} \leq C\left(\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p^{-1}}, \tag{7.421}$$

and hence

$$\frac{1}{|I|}\int_I w(x)\mathrm{d}x\left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p-1} \leq C, \tag{7.422}$$

which is the A_p condition.

To prove that the A_p condition implies Eq. (7.416), it is first necessary to show that the A_p condition implies Eq. (7.419). Using Hölder's inequality leads to

$$\begin{aligned}\int_I f(x)\mathrm{d}x &= \int_I f(x)w(x)^{p^{-1}}w(x)^{-p^{-1}}\,\mathrm{d}x\\ &\leq \left(\int_I |f(x)|^p w(x)\mathrm{d}x\right)^{p^{-1}}\left(\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{(p-1)/p},\end{aligned} \tag{7.423}$$

and hence

$$\begin{aligned}\int_I w(x)\mathrm{d}x\left(\frac{1}{|I|}\int_I f(x)\mathrm{d}x\right)^p \leq \left(\int_I |f(x)|^p w(x)\mathrm{d}x\right)\frac{1}{|I|}\int_I w(x)\mathrm{d}x\\ \left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x\right)^{p-1}.\end{aligned} \tag{7.424}$$

If the A_p condition holds, the last result simplifies to

$$(m_I(f))^p \leq C\left(\int_I w(x)\mathrm{d}x\right)^{-1}\left(\int_I |f(x)|^p w(x)\mathrm{d}x\right), \tag{7.425}$$

which gives Eq. (7.419). Taking the p^{-1} power of Eq. (7.425) and then the suppremum over all intervals yields

$$Mf(x) \leq \sup_{x\in I} C\left(\int_I w(x)\mathrm{d}x\right)^{-p^{-1}}\left(\int_I |f(x)|^p w(x)\mathrm{d}x\right)^{p^{-1}}. \tag{7.426}$$

Introducing the definition

$$M_\mu f(x) = \sup_{x\in I}\frac{1}{\mu(I)}\int_I |f|\mathrm{d}\mu, \tag{7.427}$$

and writing $\mathrm{d}\mu = w(x)\mathrm{d}x$, allows Eq. (7.426) to be expressed as follows:

$$Mf(x) \leq C\left(M_\mu|f(x)|^p\right)^{p^{-1}}. \tag{7.428}$$

The proof of the maximal theorem (see Section 7.9) can be extended to cover the case for $M_\mu f(x)$, with the result being

$$\int \left|M_\mu f\right|^r \mathrm{d}\mu \leq C\int |f|^r\,\mathrm{d}\mu, \quad \text{for } 1 < r < \infty. \tag{7.429}$$

Recall that if $w(x) \in A_p$ then $w(x) \in A_{p-\varepsilon}$ for $\varepsilon > 0$ (property (iv) in Section 7.12), and hence Eq. (7.428) can be recast as follows:

$$Mf(x) \leq C \left(M_\mu |f(x)|^{p-\varepsilon}\right)^{(p-\varepsilon)^{-1}}. \tag{7.430}$$

From this inequality and applying Eq. (7.429), it follows that

$$\begin{aligned}\int |Mf|^p \, \mathrm{d}\mu &\leq C \int \left(M_\mu |f|^{p-\varepsilon}\right)^{p(p-\varepsilon)^{-1}} \mathrm{d}\mu \\ &= C \int \left(M_\mu |f|^{p/r}\right)^r \mathrm{d}\mu \\ &\leq C \int |f|^p \, \mathrm{d}\mu,\end{aligned} \tag{7.431}$$

which is the desired result.

A theorem due to Hunt, Muckenhoupt, and Wheeden (1973) states the following. If $1 < p < \infty$, and $w(x)$ is non-negative, then the following statements are equivalent:

(a)
$$\int_I w(x)\mathrm{d}x \left(\int_I w(x)^{-(p-1)^{-1}} \mathrm{d}x\right)^{p-1} \leq C\,|I|^p, \tag{7.432}$$

(b)
$$\int_{-\infty}^{\infty} |Hf(x)|^p w(x)\mathrm{d}x \leq C \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \tag{7.433}$$

(c)
$$\int_{-\infty}^{\infty} |H_{\mathrm{M}} f(x)|^p w(x)\mathrm{d}x \leq C \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \tag{7.434}$$

and, for $\lambda > 0$,

(d)
$$\int_{|Hf(x)|>\lambda} w(x)\mathrm{d}x \leq C\lambda^{-p} \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \tag{7.435}$$

and

(e)
$$\int_{H_{\mathrm{M}} f(x)>\lambda} w(x)\mathrm{d}x \leq C\lambda^{-p} \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x. \tag{7.436}$$

A second theorem due to the same authors reads: if f is a periodic function with period 2π and for a non-negative periodic $w(\theta)$ having period 2π, then for $1 < p < \infty$ the following statements are equivalent:

(a)
$$\int_I w(\theta)\mathrm{d}\theta \left(\int_I w(\theta)^{-(p-1)^{-1}} \mathrm{d}\theta\right)^{p-1} \leq C\,|I|^p, \tag{7.437}$$

(b)
$$\int_{-\pi}^{\pi} |\mathcal{H}f(\theta)|^p w(\theta)\mathrm{d}\theta \leq C \int_{-\pi}^{\pi} |f(\theta)|^p w(\theta)\mathrm{d}\theta, \tag{7.438}$$

(c)
$$\int_{-\pi}^{\pi} |\mathcal{H}_{\mathrm{M}} f(\theta)|^p w(\theta)\mathrm{d}\theta \leq C \int_{-\pi}^{\pi} |f(\theta)|^p w(\theta)\mathrm{d}\theta, \tag{7.439}$$

and, for $\lambda > 0$,

(d)
$$\int_{|\mathcal{H}f(\theta)|>\lambda} w(\theta)\mathrm{d}\theta \leq C\lambda^{-p} \int_{-\pi}^{\pi} |f(\theta)|^p w(\theta)\mathrm{d}\theta, \tag{7.440}$$

and

(e)
$$\int_{\mathcal{H}_{\mathrm{M}} f(\theta)>\lambda} w(\theta)\mathrm{d}\theta \leq C\lambda^{-p} \int_{-\pi}^{\pi} |f(\theta)|^p w(\theta)\mathrm{d}\theta. \tag{7.441}$$

In this second theorem, the reader is reminded that $\mathcal{H}_{\mathrm{M}}$ denotes the maximal Hilbert transform operator on the circle. Just a couple of these interrelationships are proved. Note that some of the connections have already been established in earlier sections of this chapter. In particular, using Eq. (7.283) then Eq. (7.434) immediately implies Eq. (7.433) and Eq. (7.439) implies Eq. (7.438). Also the weak-type inequality Eq. (7.436) implies Eq. (7.435), and Eq. (7.441) implies Eq. (7.440).

To establish that Eq. (7.433) implies Eq. (7.432), a proof based on the work of Coifman and Fefferman (1974) is presented. Let I_1 and I_2 denote the two halves of an interval I and consider a function satisfying $f(x) \geq 0$ and having support in I_1. Since $|Hf(x)|$ is positive, there is a positive constant C_1 such that

$$C_1 \left(\frac{1}{|I_1|} \int_{I_1} f(y)\mathrm{d}y \right) \chi_{I_2}(x) \leq |Hf(x)|\,. \tag{7.442}$$

Inserting this result in Eq. (7.433) gives

$$C_1^p m_{I_1}(f)^p \int_{-\infty}^{\infty} \chi_{I_2}(x) w(x)\mathrm{d}x \leq C \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \tag{7.443}$$

where $m_I(f)$ is defined in Eq. (7.417). The preceding inequality reduces to

$$m_{I_1}(f)^p \int_{I_2} w(x)\mathrm{d}x \leq C \int_{I_1} |f(x)|^p w(x)\mathrm{d}x. \tag{7.444}$$

Since this last result is true for any f satisfying the conditions stated previously, take $f = 1$, and therefore Eq. (7.444) leads to

$$\int_{I_2} w(x)\mathrm{d}x \leq C \int_{I_1} w(x)\mathrm{d}x. \tag{7.445}$$

Switching the intervals I_1 and I_2 gives, by a repeat of the same argument,

$$\int_{I_1} w(x)\mathrm{d}x \leq C \int_{I_2} w(x)\mathrm{d}x. \tag{7.446}$$

Let $f(x) = w(x)^{-(p-1)^{-1}}$, then Eq. (7.444) becomes

$$\int_{I_2} w(x)\mathrm{d}x \left(\frac{1}{|I_1|} \int_{I_1} w(x)^{-(p-1)^{-1}} \, \mathrm{d}x \right)^p \leq C \int_{I_1} w(x)^{-p(p-1)^{-1}} w(x)\mathrm{d}x. \quad (7.447)$$

Employing Eq. (7.446) gives

$$C' \int_{I_1} w(x)\mathrm{d}x \left(\frac{1}{|I_1|} \int_{I_1} w(x)^{-(p-1)^{-1}} \, \mathrm{d}x \right)^p \leq C \int_{I_1} w(x)^{-(p-1)^{-1}} \, \mathrm{d}x, \quad (7.448)$$

and hence

$$\frac{1}{|I_1|} \int_{I_1} w(x)\mathrm{d}x \left(\frac{1}{|I_1|} \int_{I_1} w(x)^{-(p-1)^{-1}} \, \mathrm{d}x \right)^{p-1} \leq C, \quad (7.449)$$

which completes the proof that Eq. (7.433) implies the A_p condition, Eq. (7.432).

To prove that Eq. (7.432) implies Eq. (7.433), note that if $w(x)$ satisfies the A_p condition for $1 < p < \infty$, then the measure $\mathrm{d}\mu$ ($\mathrm{d}\mu = w(x)\mathrm{d}x$) satisfies the A_∞ condition. If the measure $\mathrm{d}\mu$ satisfies the A_∞ condition, then for $1 < p < \infty$ it can be shown that

$$\int |H_{\mathrm{M}} f|^p \, \mathrm{d}\mu \leq C_p \int |Mf|^p \, \mathrm{d}\mu. \quad (7.450)$$

The proof is similar to the derivation of Eq. (7.331) given previously, and the reader is requested to fill in the necessary steps. Equation (7.450) reduces to Eq. (7.331) when $w(x) = 1$ on setting $\mathrm{d}\mu = \mathrm{d}x$. Making use of Eq. (7.283) yields

$$\int |Hf|^p \, \mathrm{d}\mu \leq C_p \int |Mf|^p \, \mathrm{d}\mu. \quad (7.451)$$

Since

$$\int |Mf|^p \, \mathrm{d}\mu \leq C_p \int |f|^p \, \mathrm{d}\mu, \quad (7.452)$$

it follows that

$$\int |Hf|^p \, \mathrm{d}\mu \leq C_p \int |f|^p \, \mathrm{d}\mu. \quad (7.453)$$

To summarize, if $w(x)$ belongs to A_p then Eq. (7.447) holds, and this result, together with Eq. (7.450), indicates that the A_p condition implies Eq. (7.453).

7.13.1 Weighted norm inequalities for H_{e} and H_{o}

There is an extension of the Hunt–Muckenhoupt–Wheeden theorem to deal with the operators H_{e} and H_{o}. In the sequel it is assumed that the constant C depends on p, but is independent of the constant λ that appears in a few formulas, and is also independent of the function f. The constant C is not necessarily the same at each occurrence. The following results apply for the operators H_o and H_e (Andersen, 1976a). For a non-negative and measurable weight $w(x)$ on $\mathbb{R}^+$ with $1 < p < \infty$, then the subsequent statements are equivalent:

(a) for every interval $[a, b] \subset (0, \infty)$,

$$\int_a^b w(x)\mathrm{d}x \left(\int_a^b x^{p(p-1)^{-1}} w(x)^{-(p-1)^{-1}} \,\mathrm{d}x \right)^{p-1} \leq C \left(\frac{a^2 - b^2}{2} \right)^p; \quad (7.454)$$

(b) for a $\lambda > 0$,

$$\int_{|H_o f(x)|>\lambda} w(x)\mathrm{d}x \leq C\lambda^{-p} \int_0^\infty |f(x)|^p w(x)\mathrm{d}x; \quad (7.455)$$

and

(c)

$$\int_0^\infty |H_o f(x)|^p w(x)\mathrm{d}x \leq C \int_0^\infty |f(x)|^p w(x)\mathrm{d}x. \quad (7.456)$$

For the case $p = 1$, statements (a) and (b) are equivalent. A related result holds for H_e (Andersen, 1976a). For a non-negative and measurable weight $w(x)$ on $\mathbb{R}^+$, with $1 < p < \infty$, then the following statements are equivalent:

(a) for every interval $[a, b] \subset (0, \infty)$,

$$\int_a^b x^p w(x)\mathrm{d}x \left(\int_a^b w(x)^{-(p-1)^{-1}} \,\mathrm{d}x \right)^{p-1} \leq C \left(\frac{a^2 - b^2}{2} \right)^p; \quad (7.457)$$

(b) for a $\lambda > 0$,

$$\int_{|H_e f(x)|>\lambda} w(x)\mathrm{d}x \leq C\lambda^{-p} \int_0^\infty |f(x)|^p w(x)\mathrm{d}x; \quad (7.458)$$

and

(c)

$$\int_0^\infty |H_e f(x)|^p w(x)\mathrm{d}x \leq C \int_0^\infty |f(x)|^p w(x)\mathrm{d}x. \quad (7.459)$$

For the case $p = 1$, statement (b) implies statement (a). Equations (7.454) and (7.457) are the analogs of the A_p condition for the odd and even operators H_o and H_e on $\mathbb{R}^+$, and these are denoted by A_p^o and A_p^e, respectively.

There are related results for the Hilbert transform operators $\mathcal{H}_o$ and $\mathcal{H}_e$ on the unit disc. Recall on the unit disc for $\theta \in [0, \pi)$ that

$$\mathcal{H}_e f(\theta) = \frac{\sin\theta}{\pi} P \int_0^{\pi} \frac{f(\phi)\mathrm{d}\phi}{\cos\phi - \cos\theta} \tag{7.460}$$

and

$$\mathcal{H}_o f(\theta) = \frac{1}{\pi} P \int_0^{\pi} \frac{f(\phi)\sin\phi\,\mathrm{d}\phi}{\cos\phi - \cos\theta}. \tag{7.461}$$

In the preceding two formulas and for what follows, the function f is periodic with period 2π. The following theorems apply for the operators $\mathcal{H}_e$ and $\mathcal{H}_o$ (Andersen, 1976a). If $w(\theta)$ is a non-negative periodic (period 2π) measurable weight on $(0, \pi)$ and $1 < p < \infty$, then the following statements are equivalent:

(a) for each interval $(a, b) \subset (0, \pi)$,

$$\int_a^b w(\theta)\mathrm{d}\theta \left(\int_a^b \{\sin\theta\}^{p(p-1)^{-1}} w(\theta)^{-(p-1)^{-1}}\,\mathrm{d}\theta \right)^{p-1} \leq C \sin^p\left(\frac{b+a}{2}\right) \sin^p\left(\frac{b-a}{2}\right); \tag{7.462}$$

(b) for a $\lambda > 0$,

$$\int_{|\mathcal{H}_o f(\theta)| > \lambda} w(\theta)\mathrm{d}\theta \leq C\lambda^{-p} \int_0^{\pi} |f(\theta)|^p w(\theta)\mathrm{d}\theta; \tag{7.463}$$

and

(c)

$$\int_0^{\pi} |\mathcal{H}_o f(\theta)|^p w(\theta)\mathrm{d}\theta \leq C \int_0^{\pi} |f(\theta)|^p w(\theta)\mathrm{d}\theta. \tag{7.464}$$

For the case $p = 1$, statements (a) and (b) are equivalent. For a non-negative and measurable periodic weight $w(\theta)$ with period 2π on the interval $(0, \pi)$, and for $1 < p < \infty$, then the following statements are equivalent:

(a) for every interval $[a, b] \subset (0, \infty)$,

$$\int_a^b \sin^p\theta\, w(\theta)\mathrm{d}\theta \left(\int_a^b w(\theta)^{-(p-1)^{-1}}\,\mathrm{d}\theta \right)^{p-1} \leq C \sin^p\left(\frac{b+a}{2}\right) \sin^p\left(\frac{b-a}{2}\right); \tag{7.465}$$

(b) for a $\lambda > 0$,

$$\int_{|\mathcal{H}_e f(\theta)|>\lambda} w(\theta)\mathrm{d}\theta \le C_1\lambda^{-p} \int_0^{\pi} |f(\theta)|^p w(\theta)\mathrm{d}\theta; \tag{7.466}$$

and

(c)

$$\int_0^{\pi} |\mathcal{H}_e f(\theta)|^p w(\theta)\mathrm{d}\theta \le C \int_0^{\pi} |f(\theta)|^p\, w(\theta)\mathrm{d}\theta. \tag{7.467}$$

For the case $p = 1$, statement (b) implies statement (a). The reader interested in the preceding formulas for the operators $H_e, H_o, \mathcal{H}_e$, and $\mathcal{H}_o$, can pursue the proofs of the key results in Andersen (1976a).

7.14 Weighted norm inequalities for the Hilbert transform of functions with vanishing moments

If additional restrictions are placed on the functions that satisfy Eq. (7.159), then one might expect to learn more about the types of weight functions that apply. In this section, one such example is considered, where the extra information specified concerns the moments of the function f. The moments of f are defined for $m \in \mathbb{Z}$ by

$$I_m = \int_{-\infty}^{\infty} x^m f(x)\mathrm{d}x. \tag{7.468}$$

Consideration is now given to the types of weight functions that satisfy the weighted norm inequality, if the first N such moments are zero; that is

$$I_m = 0, \text{ for all } 0 \le m \le N. \tag{7.469}$$

Let $q(x)$ denote a polynomial of order less than or equal to $N + 1$. Then

$$H\{q(x)f(x)\} = q(x)Hf(x). \tag{7.470}$$

This result follows directly from Eq. (4.113) on utilizing Eq. (7.469). Alternatively, by first recalling the expansion

$$y^n - 1 = (y-1)\sum_{k=0}^{n-1} y^k, \tag{7.471}$$

it is straightforward to demonstrate that $(q(x) - q(t))/(x - t)$ is a polynomial with a maximum degree N. Then, using Eq. (7.469),

$$\int_{-\infty}^{\infty} \frac{q(x) - q(t)}{x - t} f(t)\mathrm{d}t = 0, \tag{7.472}$$

and hence

$$\frac{q(x)}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{x - t} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{q(t)f(t)\mathrm{d}t}{x - t}, \tag{7.473}$$

which is Eq. (7.470). Equation (7.470) is a particular example of the Bedrosian formula given in Eq. (4.257), but obviously requiring rather different conditions than those given in the statement of Bedrosian's theorem in Section 4.15.

If the function u satisfies $u(x) \in A_p$, it follows that

$$\begin{aligned}\int_{-\infty}^{\infty} |Hf(x)|^p |q(x)|^p u(x)\mathrm{d}x &= \int_{-\infty}^{\infty} |H\{q(x)f(x)\}|^p u(x)\mathrm{d}x \\ &\leq C \int_{-\infty}^{\infty} |f(x)|^p |q(x)|^p u(x)\mathrm{d}x,\end{aligned} \tag{7.474}$$

where the Hunt–Muckenhoupt–Wheeden theorem has been employed. Thus, for the class of functions under consideration, Eq. (7.474) indicates that the weight function in the weighted norm inequality for the Hilbert transform can be expressed as follows:

$$w(x) = |q(x)|^p u(x). \tag{7.475}$$

The argument just presented establishes the sufficiency of the form just given for the weight function, for the restricted class of f under discussion. The interested reader might like to try to establish a necessity argument.

7.15 Weighted norm inequalities for the Hilbert transform with two weights

A generalization of Eq. (7.415) is to seek conditions for which the following inequality is true:

$$\int_{-\infty}^{\infty} |Hf(x)|^p v(x)\mathrm{d}x \leq C(p, w, v) \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x. \tag{7.476}$$

This may be regarded as the two-weights version of the weighted norm inequalities discussed in Section 7.13, and reduces to that situation with the obvious choice

$v(x) = w(x)$. Apart from their intrinsic interest, inequalities of this type are important because they represent a generalization of the one-weight version discussed in Section 7.13.

Equation (7.476) can of course be regarded as a special case of the more general inequality

$$\int_{-\infty}^{\infty} |Sf(x)|^p v(x)\mathrm{d}x \leq C \int_{-\infty}^{\infty} |Tf(x)|^p w(x)\mathrm{d}x, \tag{7.477}$$

for $1 \leq p < \infty$, where S and T are two specified operators, and the constant C depends on v, w, and p, but is independent of f. This in turn can be regarded as a special case of the even more general result

$$\left\{\int_{-\infty}^{\infty} |Sf(x)|^p v(x)\mathrm{d}x\right\}^{p^{-1}} \leq C \left\{\int_{-\infty}^{\infty} |Tf(x)|^q w(x)\mathrm{d}x\right\}^{q^{-1}}, \tag{7.478}$$

where in general p and q may take different values.

A further type of inequality is of weak-type, and takes the following form:

$$\int_{|Sf(x)|>\lambda} v(x)\mathrm{d}x \leq \frac{C}{\lambda^p} \int_{-\infty}^{\infty} |Tf(x)|^p w(x)\mathrm{d}x, \tag{7.479}$$

for $\lambda > 0$. Results of this type may be useful when the strong-type inequality (Eq. (7.477)) does not hold. The weak-type result can also serve as a vehicle for establishing strong-type results using the Marcinkiewicz interpolation theorem. An example of this approach was illustrated in Section 7.8.

Equations (7.477)–(7.479) are of fundamental significance for the information they reveal about the connection between the operators S and T. Characterizing the classes of functions $v(x)$ and $w(x)$ for which these inequalities hold is a significant problem.

Attention will be focused on Eq. (7.476). It has already been remarked in Section 7.7 that the simplest case of this equation, $v(x) = w(x) = 1$, is of importance in connection with the study of the mean convergence of Fourier series. The more general case, $v(x) \neq 1$ and $w(x) \neq 1$, can be applied in the study of weighted mean convergence for Fourier series.

A two-weights version of the A_p condition is as follows:

$$\int_I v(x)\mathrm{d}x \left(\int_I w(x)^{-(p-1)^{-1}} \mathrm{d}x\right)^{p-1} \leq C|I|^p, \tag{7.480}$$

where C is independent of I and $1 < p < \infty$. Equation (7.480) is a necessary condition for the following equations:

$$\int_{-\infty}^{\infty} |Mf(x)|^p v(x)\mathrm{d}x \leq C \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \tag{7.481}$$

$$\int_{-\infty}^{\infty} |Hf(x)|^p v(x)\mathrm{d}x \le C \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \tag{7.482}$$

$$\int_{Mf(x)>\lambda} v(x)\mathrm{d}x \le C\lambda^{-p} \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \quad \text{for } \lambda > 0, \tag{7.483}$$

and

$$\int_{|Hf(x)|>\lambda} v(x)\mathrm{d}x \le C\lambda^{-p} \int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \quad \text{for } \lambda > 0. \tag{7.484}$$

Equation (7.480) is a sufficient condition for Eq. (7.483) (Muckenhoupt, 1972). Consider the following examples due to Muckenhoupt and Wheeden (1976b). In the first example, for $p = 2$, let

$$v(x) = \begin{cases} -x\log x, & x \in (0, 1/2] \\ 0, & \text{elsewhere,} \end{cases} \tag{7.485}$$

$$w(x) = \begin{cases} x\log^2 x, & x \in (0, 1/2] \\ \infty, & \text{elsewhere,} \end{cases} \tag{7.486}$$

and

$$f(x) = \begin{cases} \{x\log^2 x\}^{-1}, & x \in (0, 1/2] \\ 0, & \text{elsewhere.} \end{cases} \tag{7.487}$$

For I any interval on $\mathbb{R}$, Eq. (7.480) holds. In particular, for any interval that includes $(0, 1/2]$ as a subinterval, and using

$$-\int_0^{2^{-1}} x\log x\,\mathrm{d}x = \frac{2\log 2 + 1}{16} \tag{7.488}$$

and

$$\int_0^{2^{-1}} \{x\log^2 x\}^{-1}\,\mathrm{d}x = \frac{1}{\log 2}, \tag{7.489}$$

it follows that Eq. (7.480) simplifies as follows:

$$\frac{2\log 2 + 1}{16\log 2} \le C|I|^2\,. \tag{7.490}$$

The right-hand side of Eq. (7.481) yields

$$C\int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x = C\int_0^{1/2} \frac{1}{x\log^2 x}\,\mathrm{d}x = \frac{C}{\log 2}. \tag{7.491}$$

The following result also holds:

$$\begin{aligned} Mf(x) &= \sup_{a<x<b} \frac{1}{b-a} \int_a^b |f(t)|\mathrm{d}t \\ &\Rightarrow \frac{-1}{x\log x}, \quad \text{for } x \in (0, 1/2]. \end{aligned} \tag{7.492}$$

Hence,

$$\int_{-\infty}^{\infty} |Mf(x)|^2 v(x)\mathrm{d}x = \int_0^{1/2} \frac{-1}{x\log x}\,\mathrm{d}x = \infty, \tag{7.493}$$

which makes it clear that Eq. (7.480) is not a sufficient condition for Eq. (7.481).

As a second example, also for $p = 2$, consider

$$v(x) = \begin{cases} x^{-1}|\log x|^{-5/2}, & x \in (0, 1/2] \\ 0, & \text{elsewhere}, \end{cases} \tag{7.494}$$

$$w(x) = \begin{cases} x^{-1}|\log x|^{-3/2}, & x \in (0, 1/2] \\ \infty, & \text{elsewhere}, \end{cases} \tag{7.495}$$

and

$$f(x) = \begin{cases} 1, & x \in (0, 1/2] \\ 0, & \text{elsewhere}. \end{cases} \tag{7.496}$$

Making use of

$$\int_0^{2^{-1}} x^{-1}|\log x|^{-5/2}\,\mathrm{d}x = \frac{2}{3\{\log(2)\}^{3/2}} \tag{7.497}$$

and

$$\int_0^{2^{-1}} x\,|\log x|^{3/2}\,\mathrm{d}x = \frac{3}{16}\sqrt{\left(\frac{\pi}{2}\right)}\{1 - \operatorname{erf}(\sqrt{\log 4})\} + \frac{\sqrt{\log 2}}{8}\left\{\log 2 + \frac{3}{4}\right\}, \tag{7.498}$$

then clearly Eq. (7.480) is satisfied. The right-hand side of Eq. (7.482) is given by

$$C\int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x = C\int_0^{1/2} \frac{\mathrm{d}x}{x\,|\log x|^{3/2}} = \frac{2C}{\sqrt{\log 2}}. \tag{7.499}$$

On making use of

$$Hf(x) = \frac{1}{\pi}\log\left(\frac{2x}{1-2x}\right), \quad \text{for } x \in (0, 1/2), \tag{7.500}$$

the right-hand side of Eq. (7.482) simplifies to

$$\int_{-\infty}^{\infty} |Hf(x)|^p v(x)\mathrm{d}x = \frac{1}{\pi^2}\int_0^{1/2} \frac{\{\log x - \log(1/2 - x)\}^2}{x\,|\log x|^{5/2}}\,\mathrm{d}x. \tag{7.501}$$

The first contribution to the integral on the right-hand side of this last result becomes, on using the change of variable $x = \mathrm{e}^{-y}$,

$$\frac{1}{\pi^2}\int_0^{1/2} \frac{\{\log x\}^2}{x\,|\log x|^{5/2}}\,\mathrm{d}x = \frac{1}{\pi^2}\int_{\log 2}^{\infty} \frac{1}{\sqrt{y}}\,\mathrm{d}y = \infty. \tag{7.502}$$

Since the other factors in Eq. (7.501) cannot cancel the result in Eq. (7.502), it is clear that $\int_{-\infty}^{\infty} |Hf(x)|^p v(x)\mathrm{d}x$ diverges, and hence this example shows that Eq. (7.480) cannot be a sufficient condition for Eq. (7.482). The reader is invited to test the weak-type inequality, Eq. (7.484), to see if it is satisfied by this second example.

This section is concluded by reporting a few key results for two-weight inequalities, and the interested reader is left to pursue the detailed proofs in the work of Muckenhoupt and Wheeden (1976b), or to follow the hints provided in a few of the exercises at the end of the chapter. For non-negative weights $v(x)$ and $w(x)$, with $1 < p < \infty$, and

$$\int_{-\infty}^{\infty} |Mf(x)|^p v(x)\mathrm{d}x \le C\int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \tag{7.503}$$

for each interval $I \subset \mathbb{R}$,

$$|I|^{p-1}\int_{-\infty}^{\infty} \frac{v(x)\mathrm{d}x}{(|I| + |x - x_I|)^p}\left(\frac{1}{|I|}\int_I w(x)^{-(p-1)^{-1}}\right)^{p-1} \le C_1, \tag{7.504}$$

where x_I is the center of I. If

$$\int_{-\infty}^{\infty} |Hf(x)|^p v(x)\mathrm{d}x \le C\int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \tag{7.505}$$

then for each interval $I \subset \mathbb{R}$, Eq. (7.504) holds. If

$$\int_{|Hf(x)|>\lambda} v(x)\mathrm{d}x \le C\lambda^{-p}\int_{-\infty}^{\infty} |f(x)|^p w(x)\mathrm{d}x, \quad \text{for } \lambda > 0, \tag{7.506}$$

then for each interval $I \subset \mathbb{R}$, Eq. (7.504) holds.

If, for non-negative weights $v(x)$ and $w(x)$ and for almost all x,

$$\int_{-\infty}^{\infty} \frac{v(t)\mathrm{d}t}{|x - t|} \le Cw(x), \tag{7.507}$$

then

$$\int_{-\infty}^{\infty} Mf(x)v(x)\mathrm{d}x \le C\int_{-\infty}^{\infty} |f(x)|w(x)\mathrm{d}x, \tag{7.508}$$

and

$$\int_{-\infty}^{\infty} |Hf(x)|v(x)\mathrm{d}x \le C\int_{-\infty}^{\infty} |f(x)|w(x)\mathrm{d}x. \tag{7.509}$$

7.16 Some miscellaneous inequalities for the Hilbert transform

This chapter concludes with a few miscellaneous inequalities that are satisfied by the Hilbert transform. On the unit circle the following result applies. If $0 < \lambda < 1$ and $\nu < 1 - \lambda$, then (Kober, 1943a)

$$\left(\int_{-\pi}^{\pi} |\mathcal{H}f(\theta)|^{\lambda} \cos^{-\nu}(\theta/2)\mathrm{d}\theta\right)^{\lambda^{-1}} \le C_{\lambda,\nu}\int_{-\pi}^{\pi} |f(\theta)|\mathrm{d}\theta. \tag{7.510}$$

For $\nu = 0$, the preceding formula is just the Kolmogorov inequality discussed in Section 7.2. Hence, the inequality is obvious for $\nu < 0$. Apply Hölder's inequality to the left-hand side of Eq. (7.510) and assume $\lambda < p^{-1} < 1 - \nu$, then

$$\begin{aligned}\int_{-\pi}^{\pi} |\mathcal{H}f(\theta)|^{\lambda} \cos^{-\nu}(\theta/2)\mathrm{d}\theta &\le \left(\int_{-\pi}^{\pi} |\mathcal{H}f(\theta)|^{\lambda p}\,\mathrm{d}\theta\right)^{p^{-1}} \\ &\quad \left(\int_{-\pi}^{\pi} \cos^{-\nu p/(p-1)}(\theta/2)\mathrm{d}\theta\right)^{(p-1)/p} \\ &\le C\left(\int_{-\pi}^{\pi} |f(\theta)|\mathrm{d}\theta\right)^{\lambda}\left(\int_{-\pi}^{\pi} \cos^{-\nu p/(p-1)}(\theta/2)\mathrm{d}\theta\right)^{(p-1)/p},\end{aligned} \tag{7.511}$$

where the Kolmogorov inequality, Eq. (7.38), has been employed in the preceding step. Since $-1 < -\nu p/(p-1)$, the integral over $\cos^{-\nu p/(p-1)}(\theta/2)$ is bounded, and, on taking the λ^{-1} power of both sides, Eq. (7.510) follows. Apart from intrinsic interest, the inequality in Eq. (7.510) is useful to prove a particular weighted inequality for the Hilbert transform on $\mathbb{R}$, a topic that is now addressed.

if $f \in L(\mathbb{R})$, $0 < \lambda < 1$, and $1 - \lambda < 2\mu$, then (Kober, 1943a)

$$\left(\int_{-\infty}^{\infty} \frac{|Hf(x)|^{\lambda}\,\mathrm{d}x}{(1+x^2)^{\mu}}\right)^{\lambda^{-1}} \le C_{\lambda,\mu}\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x. \tag{7.512}$$

To prove this result requires the use of both the Hilbert transform on $\mathbb{R}$ and on the unit disc. The following proof is due to Kober. Starting with Eq. (3.133) and noting the change of variables employed, Eqs. (3.128) and (3.129), and using the substitutions $\theta = 2\beta$ and $\phi = 2\alpha$, with $-\pi < \phi < \pi$, leads to

$$\begin{aligned} Hf(x) &= \frac{1}{2\pi} P \int_{-\pi}^{\pi} f\left(\tan\left(\frac{\theta}{2}\right)\right) \cot\left(\frac{\phi-\theta}{2}\right) \mathrm{d}\theta \\ &\quad - \frac{1}{2\pi} P \int_{-\pi}^{\pi} f\left(\tan\left(\frac{\theta}{2}\right)\right) \tan\left(\frac{\theta}{2}\right) \mathrm{d}\theta. \end{aligned} \tag{7.513}$$

If the following substitution is employed:

$$g(\theta) = (1+\mathrm{e}^{\mathrm{i}\theta})^{-2} f\left(\tan\left(\frac{\theta}{2}\right)\right), \tag{7.514}$$

the first integral in Eq. (7.513) can be written as follows:

$$\begin{aligned} &\frac{1}{2\pi} P \int_{-\pi}^{\pi} f\left(\tan\left(\frac{\theta}{2}\right)\right) \cot\left(\frac{\phi-\theta}{2}\right) \mathrm{d}\theta = \frac{1}{2\pi} P \int_{-\pi}^{\pi} (1+\mathrm{e}^{\mathrm{i}\theta})^2 \cot\left(\frac{\phi-\theta}{2}\right) g(\theta) \mathrm{d}\theta \\ &\quad = (1+\mathrm{e}^{\mathrm{i}\phi})^2 \left\{ \mathcal{H}g(\phi) + \frac{1}{2\pi} P \int_{-\pi}^{\pi} \left\{ \left(\frac{1+\mathrm{e}^{\mathrm{i}\theta}}{1+\mathrm{e}^{\mathrm{i}\phi}}\right)^2 - 1 \right\} \cot\left(\frac{\phi-\theta}{2}\right) g(\theta) \mathrm{d}\theta \right\} \\ &\quad = (1+\mathrm{e}^{\mathrm{i}\phi})^2 \left\{ \mathcal{H}g(\phi) - \frac{\mathrm{i}}{2\pi} \int_{-\pi}^{\pi} g(\theta) \mathrm{d}\theta \right. \\ &\qquad \left. + \frac{\mathrm{i}}{2\pi} P \int_{-\pi}^{\pi} \frac{[2-(1+\mathrm{e}^{\mathrm{i}\theta})^2 - 2\mathrm{e}^{\mathrm{i}(\phi+\theta)}]}{(1+\mathrm{e}^{\mathrm{i}\phi})^2} g(\theta) \mathrm{d}\theta \right\}. \end{aligned} \tag{7.515}$$

The second integral in Eq. (7.513) can be written as follows:

$$\begin{aligned} \frac{1}{2\pi} P \int_{-\pi}^{\pi} f\left(\tan\left(\frac{\theta}{2}\right)\right) \tan\left(\frac{\theta}{2}\right) \mathrm{d}\theta &= \frac{1}{2\pi} P \int_{-\pi}^{\pi} (1+\mathrm{e}^{\mathrm{i}\theta})^2 \tan\left(\frac{\theta}{2}\right) g(\theta) \mathrm{d}\theta \\ &= \frac{\mathrm{i}}{2\pi} P \int_{-\pi}^{\pi} (1-\mathrm{e}^{2\mathrm{i}\theta}) g(\theta) \mathrm{d}\theta. \end{aligned} \tag{7.516}$$

Introducing the definition

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n\theta} g(\theta) \mathrm{d}\theta, \tag{7.517}$$

and on collecting the results in Eqs. (7.515) and (7.516), then Eq. (7.513) becomes

$$\begin{aligned} Hf(x) &= (1+\mathrm{e}^{\mathrm{i}\phi})^2 \{\mathcal{H}g(\phi) - \mathrm{i}c_0\} \\ &\quad + \frac{\mathrm{i}}{2\pi} P \int_{-\pi}^{\pi} [2-(1+\mathrm{e}^{\mathrm{i}\theta})^2 - 2\mathrm{e}^{\mathrm{i}(\phi+\theta)} - (1-\mathrm{e}^{2\mathrm{i}\theta})] g(\theta) \mathrm{d}\theta \\ &= (1+\mathrm{e}^{\mathrm{i}\phi})^2 \{\mathcal{H}g(\phi) - \mathrm{i}c_0\} - 2\mathrm{i}(1+\mathrm{e}^{\mathrm{i}\phi}) c_1. \end{aligned} \tag{7.518}$$

Now,

$$\begin{aligned}|Hf(x)|^{\lambda} &\le \left|(1+e^{i\phi})^2\mathcal{H}g(\phi)\right|^{\lambda} + \left|(1+e^{i\phi})^2 ic_0\right|^{\lambda} + \left|2i(1+e^{i\phi})c_1\right|^{\lambda} \\ &= 2^{2\lambda}\cos^{2\lambda}\left(\frac{\phi}{2}\right)\left\{|\mathcal{H}g(\phi)|^{\lambda} + |c_0|^{\lambda} + \cos^{-\lambda}\left(\frac{\phi}{2}\right)|c_1|^{\lambda}\right\}. \end{aligned} \tag{7.519}$$

Using the change of variable $t = \tan(\theta/2), c_1$ can be written as follows:

$$\begin{aligned}|c_1| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta}g(\theta)d\theta\right| \\ &= \left|\frac{1}{4\pi}\int_{-\infty}^{\infty} f(t)dt\right| \\ &\le \frac{1}{4\pi}\int_{-\infty}^{\infty} |f(t)|\, dt. \end{aligned} \tag{7.520}$$

Noting

$$e^{-i\theta} = \frac{1-it}{1+it}, \tag{7.521}$$

then

$$\begin{aligned}|c_0| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta\right| \\ &= \left|\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{1-it}{1+it} f(t)dt\right| \\ &\le \frac{1}{4\pi}\int_{-\infty}^{\infty} |f(t)|\, dt. \end{aligned} \tag{7.522}$$

Making use of Eq. (7.519), it follows that

$$\begin{aligned}2^{1-2\lambda}\int_{-\infty}^{\infty} \frac{|Hf(x)|^{\lambda}\, dx}{(1+x^2)^{\mu}} &= 2^{-2\lambda}\int_{-\pi}^{\pi} \frac{|Hf(\tan(\phi/2))|^{\lambda}\, d\phi}{\cos^{2-2\mu}(\phi/2)} \\ &\le \int_{-\pi}^{\pi} \frac{|\mathcal{H}g(\phi)|^{\lambda}\, d\phi}{\cos^{2-2\mu-2\lambda}(\phi/2)} + |c_0|^{\lambda}\int_{-\pi}^{\pi} \frac{d\phi}{\cos^{2-2\mu-2\lambda}(\phi/2)} \\ &\quad + |c_1|^{\lambda}\int_{-\pi}^{\pi} \frac{d\phi}{\cos^{2-2\mu-\lambda}(\phi/2)}. \end{aligned} \tag{7.523}$$

Noting that $2 - 2\mu - \lambda < 1$, the final two integrals in Eq. (7.523) are bounded. Identifying $\nu = 2 - 2\mu - 2\lambda$ and making use of Eq. (7.510) allows Eq. (7.523) to be written as follows:

$$2^{1-2\lambda}\int_{-\infty}^{\infty} \frac{|Hf(x)|^{\lambda}\, dx}{(1+x^2)^{\mu}} \le C\left(\int_{-\pi}^{\pi} |g(\phi)|d\phi\right)^{\lambda} + C_0|c_0|^{\lambda} + C_1|c_1|^{\lambda}, \tag{7.524}$$

where C_0 and C_1 denote the second and third integrals involving the cosine function in Eq. (7.523). Employing Eqs. (7.520) and (7.522) and noting that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\phi)|\mathrm{d}\phi = \frac{1}{4\pi}\int_{-\infty}^{\infty}|f(x)|\mathrm{d}x, \tag{7.525}$$

allows Eq. (7.524) to be written as follows:

$$\int_{-\infty}^{\infty}\frac{|Hf(x)|^{\lambda}\,\mathrm{d}x}{(1+x^2)^{\mu}} \le C\left(\int_{-\infty}^{\infty}|f(x)|\mathrm{d}x\right)^{\lambda}, \tag{7.526}$$

which is the required result.

Some inequalities of the Zygmund-type for the Hilbert transform on the circle are now considered. The first result is given by (Essén, Shea, and Stanton, 1999)

$$\begin{aligned}\left\|\widetilde{f}\right\|_1 \le \frac{2}{\pi}&\int_0^{2\pi}|f(\theta)|\log(\mathrm{e}+|f(\theta)|)\mathrm{d}\theta\\ &+B_0\int_0^{2\pi}|f(\theta)|\log\log(\mathrm{e}+|f(\theta)|)\mathrm{d}\theta + B_1\|f\|_1,\end{aligned} \tag{7.527}$$

where B_0 and B_1 are constants. The constant $2/\pi$ is optimal, and $B_0 \le 4\pi^{-1}$. If $\alpha > 1$, then

$$\begin{aligned}\int_0^{2\pi}\left|\widetilde{f}(\theta)\right|\{\log(\mathrm{e}+\left|\widetilde{f}(\theta)\right|)\}^{\alpha-1}\,\mathrm{d}\theta \le{}& \frac{2}{\pi\alpha}\int_0^{2\pi}|f(\theta)|\{\log(\mathrm{e}+|f(\theta)|)\}^{\alpha}\,\mathrm{d}\theta\\ &+\frac{2}{\pi}\int_0^{2\pi}|f(\theta)|\{\log(\mathrm{e}+|f(\theta)|)\}^{\alpha-1}\\ &\times\log\log(\mathrm{e}+|f(\theta)|)\mathrm{d}\theta\\ &+A\int_0^{2\pi}|f(\theta)|\{\log(\mathrm{e}+f(\theta))\}^{\alpha-1}\,\mathrm{d}\theta,\end{aligned} \tag{7.528}$$

with A a positive constant. The constants $2/\pi\alpha$ and $2/\pi$ are optimal. A related result for $0 < \alpha < 1$ is as follows:

$$\begin{aligned}\int_0^{2\pi}\left|\widetilde{f}(\theta)\right|\{\log(\mathrm{e}+\left|\widetilde{f}(\theta)\right|)\}^{\alpha-1}\,\mathrm{d}\theta \le{}& \frac{2}{\pi\alpha}\int_0^{2\pi}|f(\theta)|\{\log(\mathrm{e}+|f(\theta)|)\}^{\alpha}\,\mathrm{d}\theta\\ &+A\int_0^{2\pi}|f(\theta)|\log\log(\mathrm{e}+|f(\theta)|)\mathrm{d}\theta.\end{aligned} \tag{7.529}$$

Also,

$$\int_0^{2\pi} \left|\widetilde{f}(\theta)\right| \{\log(\mathrm{e} + \left|\widetilde{f}(\theta)\right|)\}^{-1}\, \mathrm{d}\theta \leq \frac{2}{\pi} \int_0^{2\pi} |f(\theta)| \log\log(\mathrm{e} + |f(\theta)|)\mathrm{d}\theta + A\|f\|_1\,, \tag{7.530}$$

and, if f is non-negative,

$$\int_0^{2\pi} f(\theta)\log(1 + f(\theta))\mathrm{d}\theta \leq \frac{\pi}{2} \int_0^{2\pi} \left|\widetilde{f}(\theta)\right| \mathrm{d}\theta + 2\pi f(0)\log(1 + f(0)). \tag{7.531}$$

The reader interested in further details on these, as well as additional, related inequalities, should consult the work of Essén *et al.* (1999) and the more recent effort by the same authors, Essén *et al.* (2002), where further study is carried out on obtaining improved inequalities and the determination of sharp bounds.

A rather different type of inequality involving the Hilbert transform on the unit disc takes the following form. Let $\varphi(x)$ denote a non-negative, continuous, and increasing function for $0 \leq x < \infty$. Suppose $\varphi(x)$ satisfies the following for $1 \leq a < b < \infty$:

$$\varphi(0) = 0; \tag{7.532}$$

and, for $x \to \infty$,

$$\varphi(2x) = 0(\varphi(x)), \tag{7.533}$$

$$\int_x^\infty \frac{\varphi(t)\mathrm{d}t}{t^{b+1}} = 0\left(\frac{\varphi(x)}{x^b}\right), \tag{7.534}$$

and

$$\int_1^x \frac{\varphi(t)\mathrm{d}t}{t^{a+1}} = 0\left(\frac{\varphi(x)}{x^a}\right). \tag{7.535}$$

Functions having the preceding set of properties will be designated as belonging to the class $M(a, b)$. If $\varphi(x) \in M(a, b)$ and satisfies the following additional conditions as $x \to 0$:

$$\varphi(2x) = 0(\varphi(x)), \tag{7.536}$$

$$\int_x^1 \frac{\varphi(t)\mathrm{d}t}{t^{b+1}} = 0\left(\frac{\varphi(x)}{x^b}\right), \tag{7.537}$$

and

$$\int_0^x \frac{\varphi(t)\mathrm{d}t}{t^{a+1}} = 0\left(\frac{\varphi(x)}{x^a}\right), \tag{7.538}$$

then the following is employed: $\varphi(x) \in Z(a,b)$. If $\varphi(\theta) \in M(a,b)$ for $1 < a < b < \infty$, then

$$\int_0^{2\pi} \varphi\left[\left|\widetilde{f}(\theta)\right|\right]\mathrm{d}\theta \le C \int_0^{2\pi} \varphi[|f(\theta)|]\mathrm{d}\theta + C_1, \tag{7.539}$$

where C and C_1 are constants independent of f. If $\varphi(x) \in Z(a,b)$, the constant C_1 in the last inequality is zero. This formula is associated with Marcinkiewicz (1939). Zygmund (1956a) and Koizumi (1958a, 1959b) have given proofs, and the reader is directed to these two authors for further details. For $a < r < b$, both the functions

$$\varphi(u) = u^r \tag{7.540}$$

and

$$\varphi(u) = u^r \log(1+u) \tag{7.541}$$

belong to the class $Z(a,b)$. The choice in Eq. (7.540) leads directly to the Riesz inequality. Equation (7.541) yields

$$\int_0^{2\pi} \left|\widetilde{f}(\theta)\right|^r \log\left(1+\left|\widetilde{f}(\theta)\right|\right)\mathrm{d}\theta \le C \int_0^{2\pi} |f(\theta)|^r \log(1+|f(\theta)|)\mathrm{d}\theta. \tag{7.542}$$

An interesting inequality of Pichorides (1975b) is as follows. Let f be a real periodic function with period 2π, and bounded such that $|f| \le k < \frac{\pi}{2}$, then

$$\|\sinh(\mathcal{H}f/2)\|_2 \le \frac{1}{\sqrt{\cos k}} \, \|f/2\|_2 \, . \tag{7.543}$$

A lower bound can be given for the conjugate function (Calderón, Weiss, and Zygmund, 1967) that takes the following form. Let $f(x)$ be periodic with period 2π and integrable over a period, and restrict f so that $f(x) \ge 1$ for $x \in \mathbb{R}$, then, with $I = (1/2\pi) \int_0^{2\pi} f(x)\mathrm{d}x$,

$$\frac{1}{2\pi} \int_0^{2\pi} f(x) \log f(x)\mathrm{d}x \le \frac{1}{2\pi} \int_0^{2\pi} |(\mathcal{H}f)(x)|\mathrm{d}x + I \log I. \tag{7.544}$$

This can be proved with a short argument starting from the Poisson integral of f. The details are left for the reader.

Notes

§7.1 For an early extension of the Riesz inequality, see Forelli (1963). Some additional reading on the Hilbert integral and Hilbert's inequality can be found in Phong and Stein (1986). Essén (1992) discusses inequalities of the form of Eq. (7.22) and some generalizations.

§7.2 For further discussion on Kolomogorov's theorem involving the distribution function of $\widetilde{f}$, see Koosis (1998, p. 92), and for a real-variable proof, see Loomis (1946). For a generalization to several variables, see Zhizhiashvili (1983). For work on the best constants in some one-sided weak-type inequalities involving the Hilbert transform, see Kuijlaars (1998). Further reading on the approach of Davis can be found in Davis (1973, 1974, 1976, 1979). For a review, see Tsereteli (1977), and for some additional reading consult Grushevskii (1986). The situation where $f \in L^p$, with $0 < p < 1$, is discussed by Hardy and Littlewood (1932).

§7.3 For further reading on Zygmund-type inequalities, see Littlewood (1929), Pichorides (1972), Bennett (1976), and Essén *et al*. (1999, 2002).

§7.4 There is a concise account of some of the history associated with the Shannon sampling theorem in Zayed (1993), and the essay of Higgins (1985) provides some interesting historical notes on the cardinal series. Two of the key original references are Whittaker (1915) and Shannon (1949). For further reading on the sampling theorem, see Kohlenberg (1953), Stens (1983, 1984), Butzer, Splettstösser, and Stens (1988), and Butzer, Higgins, and Stens (2000). Reading on the situation where distributions are considered can be found in Campbell (1968) and Pfaffelhuber (1971b). The cardinal series can serve as a useful method for approximating the Hilbert transform; see Stenger (1976, 2000). Boas (1954, sect. 11.4) provides a source for further reading on Bernstein's inequalities involving conjugate functions.

§7.5 A related bound is given in Buffoni (2004), and some further discussion can be found in Logan (1978).

§7.6 For an extension of the inequality in Eq. (7.156) for operator norms, see Berkson and Gillespie (1985).

§7.7 Helson and Szegö (1960) discuss weighted norm results for the Hilbert transform in the context of a problem in prediction theory. Muckenhoupt (1969, 1970) exploits weighted norm inequalities in studies of the convergence of some classical polynomials. Chen (1944) and Babenko (1948) also gave proofs of the Hardy–Littlewood inequality. Gapoškin (see alternative spelling Gaposhkin (1958)) gave an extension of the Hardy–Littlewood inequality to include more general weight functions. Flett (1958), Chen (1963), and Andersen (1976a) gave a number of extensions involving even and odd functions as well as some additional related results, and Stein (1957) gave a generalization to $\mathbb{R}^n$. Dyn'kin and Osilenker (1985) have reviewed contributions to the theory of weighted singular integrals, with particular emphasis given to work from the 1970s to the early 1980s. For some further reading, see the following: Cotlar and Sadosky (1975), Muckenhoupt and Wheeden (1976a), Kaneko and Yano (1975), Arocena, Cotlar, and Sadosky (1981), Treil (1983), and Gurielashvili (1987).

§7.8 Among the first papers on weak-type inequalities for the Hilbert transform are the works of Besicovitch (see alternative spelling Besikovitch (1923)), Kolmogorov (see Kolmogoroff (1925)), Littlewood (1926), Hardy (1928b), and Titchmarsh (1929). For some weighted weak-type inequalities, see Muckenhoupt and Wheeden (1977). A generalization of Eq. (7.220) has recently been obtained by Pinsky (2001). His principal result is as follows: if $f \in B_1$, with $B_1 = \left\{f : \int_{-\infty}^{\infty} |f(x)| \mathrm{d}x/(1+|x|) < \infty\right\}$, then Hf exists *a.e.* if $f \geq 0$, and $\mu(A)$ is the weighted measure $\mu(A) = \pi^{-1}\int_A (1+x^2)^{-1}\,\mathrm{d}x$, then he finds an upper bound for $\mu\{x : |Hf| \geq \alpha\}$ which contains the Kolmogorov weak-type inequality as a special case.

§7.9 The key paper for the Hardy–Littlewood maximal function is Hardy and Littlewood (1930). See Phillips (1967) for another exposition on the Hardy–Littlewood maximal function. For additional reading on the Marcinkiewicz interpolation theorem, see Zygmund (1968, Vol. II, p. 111), Sadosky (1979), Bennett and Sharpley (1988, p. 216), and Folland (1999, p. 203). For a connection with the Hilbert transform, see Cordoba and Fefferman (1977). The representation of the function $F(x)$ in Figure 7.1 is based on Koosis (1998, p. 173), and the construction, originally due to F. Riesz, forms part of the rising sun lemma, or, in a slightly modified form, the flowing water lemma.

§7.10 Additional discussion including proofs and generalizations of Cotlar's inequality can be found in Kaneko (1970), Garcia-Cuerva and Rubio de Francia (1985, p. 204), Davis and Chang (1987, p. 57), Meyer and Coifman (1997), and Duoandikoetxea (2001). The derivation of the relative distributional inequality, Eq. (7.332), is discussed in Davis and Chang (1987, p. 61), and for generalizations see Garnett (1981, p. 265), and Stein (1993, pp. 151, 206). Hunt (1972) gave an exponential-type bound of the form $m\{x \in (-\pi, \pi) : Mf(x) \leq y, |\mathcal{H}f(x)| > \lambda y\} \leq C\mathrm{e}^{-C\lambda}$, for $\lambda > 0$ and $f \in L^1(-\pi, \pi)$. The following related exponential-type bounds are discussed in Garsia (1970, pp. 119, 123). if f is an essentially bounded function with support restricted to the interval A, then $m\{x : |Hf(x)| > \lambda\} \leq C\lambda^{-1}A\mathrm{e}^{-C'\lambda}$ for constants C, $C' > 0$, $\|f\|_\infty \leq 1$, and λ selected sufficiently large. For the maximal Hilbert transform, $m\{x : H_\mathrm{M}f(x) > C\lambda\} \leq \lambda^{-1}A\mathrm{e}^{-\lambda}$, and $m\{x : H_\mathrm{M}f(x) \geq C\lambda\|f\|_\infty\} \leq \lambda^{-1}A\mathrm{e}^{-\lambda}$. Samotij (1991) discussed an example where a non-negative integrable function on the real line has a Hilbert transform that cannot be *a.e.* dominated by the Hardy–Littlewood maximal function. The approximation of the Poisson kernel shown in Figure 7.2 is based on Garnett (1981, p. 23).

§7.11 For an extension, see Forelli (1963).

§7.12 Some general references for discussion of A_p weights are: Garnett (1981), Dyn'kin and Osilenker (1985), Garcia-Cuerva and Rubio de Francia (1985), Torchinsky (1986, chap. 9), and Stein (1993). Some specific references for the properties (i)–(xiii) are: (i) Torchinsky (1986, p. 236), Dyn'kin and Osilenker (1985, p. 2096); (ii) Torchinsky (1986, p. 239), Jones (1980), Rubio de Francia (1982); (iii) Dyn'kin and Osilenker (1985, p. 2096); (iv) Coifman and Fefferman (1974); (v) Dyn'kin and Osilenker (1985, p. 2096); (vi) Hunt *et al*. (1973); (vii) Davis and Chang (1987, p. 72); (viii) Muckenhoupt (1972), Coifman and Fefferman (1974); (ix) Hunt *et al*. (1973); (x) Muckenhoupt (1974a); (xi) Coifman and Fefferman (1974); (xii) Davis and Chang (1987, p. 67); (xiii) Torchinsky (1986, p. 230).

§7.13 For further reading on weighted norm inequalities, see Kahanpää and Mejlbro (1984), Garcia-Cuerva and Rubio de Francia (1985), Kokilashvili (1980), Kokilashvili and Krbec (1991), Pick (1994), Treil and Volberg (1995, 1997), Nazarov and Treil (1996), Kokilashvili and Meskhi (1997), Katz and Pereyra (1997), and Petermichl and Wittwer (2002). For the case of the finite Hilbert transform, see Wegert and Wolfersdorf (1988), and Astala, Päivärinta, and Saksman. (1996). A proof of Eq. (7.450) can be found in Garnett (1981, p. 265). Löfström (1983) has shown that there are no non-trivial translation invariant operators on weighted L^p spaces for which the weight functions are rapidly varying.

§7.14 The case discussed in this section has been treated in detail by Adams (1982), where both the sufficiency and necessity of Eq. (7.475) are proved.

§7.15 The material of this section is authoritatively reviewed by Muckenhoupt (1974b, 1979). For some further work on weighted inequalities with two weights, see Andersen (1977a), Andersen and Muckenhoupt (1982), Domínguez (1990a, 1990b), Fernández-Cabrera and Torrea (1993), Edmunds and Kokilashvili (1995), and Treil, Volberg, and Zheng. (1997).

§7.16 For extensions and further discussion of Eq. (7.539), see Chen (1960). Some bounds for the integral $(1/|I|)\int |Hf - (Hf)_I|\mathrm{d}x$, where $f_I = (1/|I|)\int_I f\,\mathrm{d}x$ and I denotes an interval, can be found in Jiang (1991). See Córdoba, Córdoba, and Fontelos (2006) for some further inequalities.

Exercises

7.1 Determine a value for the constant $C_{p,\alpha}$ in Eq. (7.37).

7.2 if f is a complex-valued function, does Eq. (7.51) provide the optimal constant in the Kolmogorov inequality in Eq. (7.38)? If it does not, can you find such a constant?

7.3 Determine if the Fourier transform of the function $f(x) = (\sin \pi x - \pi x)/\pi x^2$ is band-limited.

7.4 if $f(x) = (1 - \cos \pi x)/\pi x^2$, find the support of $\hat{f}$.

7.5 Does the function in the preceding question $\in E^\pi \cap L^2$?

7.6 Evaluate the contour integral $\int_C f(z)\mathrm{d}z$, where $f(z) = \cot z/(z(z-\beta))$ for β a constant, and C is the square with corners located at $z = \pi(\pm 1, \pm \mathrm{i})(N + 1/2)$. By making a suitable choice for β and taking the limit $N \to \infty$, show that Eq. (7.116) is obtained.

7.7 Using a similar approach to Exercise 7.6, but with the choice $f(z) = \csc z - z^{-1}$, derive Eq. (7.117).

7.8 For $f \in E^\pi \cap L^2(\mathbb{R})$, and making use of Eq. (7.108), show that

$$Hf(x) = \sum_{k=-\infty}^{\infty} \frac{f(k)\{1 - \cos \pi(x-k)\}}{\pi(x-k)}.$$

7.9 Find the analog of Eq. (7.101) when the function f has a Fourier transform having compact support in the asymmetric interval $[-\sigma, \eta]$.

7.10 Taking advantage of Eq. (4.127) or otherwise, show that $H\{(\sin \pi x - \pi x)/x^2\} = (1 - \cos \pi x)/x^2$.

7.11 Attempt to derive Eq. (7.158) using less stringent conditions than those stated in Section 7.6.

7.12 Verify that the condition expressed by Eq. (7.160) is required in order that Eq. (7.167) holds.

7.13 If $0 < a < b, \quad p > 1$, and $-1 < \alpha < p - 1$, show that there exists a constant C_p depending only on p such that

$$\int_{-\infty}^{\infty} \left| \log \left| \frac{b-x}{a-x} \right| \right|^p |x|^\alpha \,\mathrm{d}x \leq \frac{C_p}{p-1-\alpha} \int_a^b |x|^\alpha \,\mathrm{d}x.$$

7.14 By taking advantage of the result in Exercise 7.13, or otherwise, prove Eq. (7.161). [Hint: Approximate the function by a sequence of step functions.]

7.15 Prove that the integrals occurring in Eqs. (7.196) and (7.199) do not become unbounded due to the singularity in the integrand, and that the integrals are bounded by a constant $C_{p,\alpha} < \infty$.

7.16 Prove Eq. (7.204).

7.17 If $f \in L^1(\mathbb{R})$, and assuming that f is not equal to zero almost everywhere, is $Mf \in L^1(\mathbb{R})$?

7.18 If $f \in L^p$, for $p \geq 1$, show that $\|P_\varepsilon f\|_p \leq \|f\|_p$, where

$$P_\varepsilon f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon f(t)\mathrm{d}t}{(x-t)^2 + \varepsilon^2}.$$

7.19 If $f(x) = (1 + x^2)^{-1}$, show that

$$\int_{-\infty}^{\infty} \frac{\{tf(a-bt) - Hf(a-bt)\}\mathrm{d}t}{(t^2+1)} = 0, \text{ for } a, b > 0.$$

7.20 Verify Eq. (7.310) by evaluating the integral on the right-hand side of the formula.

7.21 Check that Eq. (7.310) is satisfied when $f(x) = x^{-1} \sin x$.

7.22 If f is a real periodic function with period 2π and bounded above as $|f| \leq k$ with $k < \pi/2$, show that

$$\left\| \sinh(\widetilde{f}/2) \right\|_2^2 \leq \frac{1}{\cos k} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \sin^2\{f(\theta)/2\}\mathrm{d}\theta - \sin^2\left\{ \frac{1}{4\pi} \int_0^{2\pi} f(\theta)\mathrm{d}\theta \right\} \right\},$$

and hence

$$\left\| \sinh(\widetilde{f}/2) \right\|_2 \leq \frac{1}{\sqrt{\cos k}} \|f/2\|_2,$$

where $\widetilde{f}$ is the conjugate of f. [Hint: See Pichorides (1975b).]

7.23 If f is even, $f \geq 0$, and $f \in B_1$ for $B_1 = \{f : \int_{-\infty}^{\infty} |f(x)|\mathrm{d}x/(1+|x|) < \infty\}$, then, with $\mu(A) = \pi^{-1} \int_A (1+x^2)^{-1}\,\mathrm{d}x$, show that, for $\alpha > 0$,

$$\mu\{x : |Hf| \geq \alpha\} \leq \frac{4}{\pi^2\alpha} \int_{-\infty}^{\infty} \frac{|f(x)|\mathrm{d}x}{1+x^2}.$$

7.24 Determine if $|x|^\alpha$ is an A_1 weight; if it is, specify the range of values for the constant α.

7.25 For the finite Hilbert transform $Tf(x) = \pi^{-1}P\int_a^b f(t)\mathrm{d}t/(x-t)$, with $x \in (a,b)$, is $\int_a^b w(x)|(Tf)(x)|^p\,\mathrm{d}x$ bounded above by $\int_a^b w(x)|f(x)|^p\,\mathrm{d}x$? What condition(s) must be satisfied by the weight function $w(x)$?

7.26 Prove Eq. (7.504), assuming Eq. (7.503) holds. [Hint: Treat the cases where $\int_I w(x)^{-(p-1)^{-1}}\,\mathrm{d}x$ is zero, infinite, and lies between these limits, separately, and set $f(x) = w(x)^{-(p-1)^{-1}}$ on I and zero elsewhere.]

7.27 Prove Eq. (7.504), assuming Eq. (7.505) holds. Use the same hint as in Exercise 7.26.

7.28 Using the fact that Mf and Hf are bounded above by $\int_{-\infty}^{\infty} |f(t)|\mathrm{d}t/|x-t|$, prove that Eqs. (7.508) and (7.509) follow if Eq. (7.507) holds.

8

Asymptotic behavior of the Hilbert transform

8.1 Asymptotic expansions

In this chapter the behavior of the Hilbert transform $Hf(x)$ as $x \to \infty$ is examined for some different choices for the asymptotic behavior of the function f. Problems of this type are of particular importance when the asymptotic behavior of a function is of interest, and the function is only known via its relationship to a Hilbert transform of a different function for which the asymptotic behavior is known.

The tilde sign $\sim$ (also termed the twiddle symbol) is employed in the following manner: if

$$f(x) \sim g(x), \text{ as } x \to \infty, \tag{8.1}$$

then

$$\frac{f(x)}{g(x)} \to 1, \text{ as } x \to \infty. \tag{8.2}$$

Based on Eq. (8.1), $f(x)$ is termed asymptotically equal to $g(x)$, as $x \to \infty$.

If $f(x)$ can be written in the following form:

$$f(x) = \sum_{k=0}^{n} a_k x^{-k} + O(x^{-n-1}), \text{ for integer } n \geq 0, \text{ as } x \to \infty, \tag{8.3}$$

then $f(x)$ can be written as the power series

$$f(x) \sim \sum_{k=0}^{\infty} a_k x^{-k}, \tag{8.4}$$

and this is called the asymptotic expansion of $f(x)$. The power series may be either convergent or divergent. The key is that the error involved in truncating the expansion with the term $a_n x^{-n}$ in Eq. (8.3) goes to zero as $x \to \infty$ faster than the last term included in the series. The equations just presented can be readily extended to functions of a complex variable.

For the functions f and g, the asymptotic series can be added and multiplied to give the corresponding asymptotic series for $f(x) + g(x)$ and $f(x)g(x)$. The asymptotic series for the integral of $f(x)$ can be obtained by integration of the asymptotic series. In general, the asymptotic series for $f'(x)$ cannot be found by the termwise differentiation of the asymptotic series for $f(x)$. The asymptotic expansion for a given function is unique, but an asymptotic series does not uniquely determine the function.

The approach presented in this section to study the asymptotic behavior of the Hilbert transform is due to Wong (1980b, 1989). The Hilbert transform Hf is first split as follows:

$$Hf(x) = \frac{1}{\pi}\int_0^{\infty}\frac{f(-s)\mathrm{d}s}{x+s} + \frac{1}{\pi}P\int_0^{\infty}\frac{f(s)\mathrm{d}s}{x-s}. \tag{8.5}$$

In what follows it will be assumed, without any loss of generality, that $x > 0$. The first integral in Eq. (8.5) is the Stieltjes transform of $f(-s)$, and the second is the one-sided Hilbert transform of f. The asymptotic behavior of Hf can be obtained by separately examining the asymptotic behavior of each of the integrals in Eq. (8.5).

8.2 Asymptotic expansion of the Stieltjes transform

Recall that the Stieltjes transform can be written in the following form:

$$\mathsf{S}f(z) = \int_0^{\infty}\frac{f(s)\mathrm{d}s}{z+s}, \text{ with } |\arg z| < \pi. \tag{8.6}$$

In the following developments, consideration is restricted to the case where $\mathrm{Im}\, z = 0$ and $\mathrm{Re}\, z > 0$. The asymptotic behavior of $\mathsf{S}f(x)$ is examined for three separate cases: when f has an exponential decay; when f has an algebraic asymptotic decay; and when f has an asymptotic decay which is the product of an oscillatory component multiplied by an algebraic term.

If the asymptotic behavior of f is an exponential decay, and f is integrable on $[0, \infty)$, then the moments of this function:

$$m_n = \int_0^{\infty} s^n f(s)\mathrm{d}s, \quad n = 0, 1, 2, \ldots, \tag{8.7}$$

will be finite. Employing the expression

$$\frac{1}{x+s} = \sum_{k=0}^{n-1}(-s)^k x^{-k-1} + \frac{(-s)^n x^{-n-1}}{(1+sx^{-1})}, \tag{8.8}$$

Eq. (8.6) can be written, for $\operatorname{Im} z = 0$, as follows:

$$\begin{aligned}
\mathsf{S}f(x) &= \int_0^\infty \frac{f(s)\mathrm{d}s}{x+s} \\
&= \sum_{k=0}^{n-1} (-1)^k x^{-k-1} \int_0^\infty s^k f(s)\mathrm{d}s + (-1)^n x^{-n-1} \int_0^\infty \frac{s^n f(s)\mathrm{d}s}{(1+sx^{-1})} \\
&= \sum_{k=0}^{n-1} (-1)^k \, x^{-k-1} m_k + \varepsilon_n(x).
\end{aligned} \tag{8.9}$$

The error term $\varepsilon_n(x)$ can be expressed as follows:

$$|\varepsilon_n(x)| \le x^{-n-1} \sup_{t\in(0,\,\infty)} \left| \int_0^t s^n f(s)\mathrm{d}s \right|. \tag{8.10}$$

This result can be obtained as follows. Let $g(s)$ be defined by

$$g(s) = \int_0^s t^n f(t)\mathrm{d}t. \tag{8.11}$$

Using integration by parts leads to

$$\begin{aligned}
\int_0^\infty \frac{s^n f(s)\mathrm{d}s}{(1+sx^{-1})} &= x \int_0^\infty \frac{1}{x+s} \frac{\mathrm{d}g(s)}{} \mathrm{d}s \\
&= x \int_0^\infty \frac{g(s)\mathrm{d}s}{(x+s)^2} \\
&\le x \left\{ \int_0^\infty \frac{\mathrm{d}s}{(x+s)^2} \right\} \sup_{s\,\in\,(0,\,\infty)} |g(s)| \\
&= \sup_{s\,\in\,(0,\,\infty)} |g(s)|,
\end{aligned} \tag{8.12}$$

and Eq. (8.10) follows.

Consider the case where the asymptotic behavior of f is algebraic; that is, suppose

$$f(s) \sim \sum_{k=0}^{\infty} a_k s^{-k-\alpha}, \quad \text{for} \quad 0 < \alpha \le 1. \tag{8.13}$$

With this type of asymptotic expansion for f, the same approach employed in Eqs. (8.8)–(8.10) cannot be used, since the required moments m_n would in general be divergent. For the choice given in Eq. (8.13), the asymptotic expression for the Stieltjes transform is given, for the case $0 < \alpha < 1$, by

$$\mathsf{S}f(x) = \frac{\pi}{\sin \pi\alpha} \sum_{k=0}^{n-1} \frac{(-1)^k a_k}{x^{k+\alpha}} - \sum_{k=1}^{n} \frac{(-1)^k b_k}{x^k} + R_n(x), \tag{8.14}$$

where

$$b_k = \int_0^\infty s^{k-1}\psi_k(s)\mathrm{d}s \tag{8.15}$$

and

$$\psi_n(s) = f(s) - \sum_{k=0}^{n-1} a_k s^{-k-\alpha}. \tag{8.16}$$

The term $R_n(x)$ represents a remainder contribution. Equation (8.14) can be derived by the methods set forth in Section 8.3. This result was obtained by McClure and Wong (1978) using techniques based on the theory of distributions. These authors have also studied the case $\alpha = 1$ in Eq. (8.13), as well as the case where $f(x)$ is oscillatory near infinity, that is

$$f(x) \sim \mathrm{e}^{\mathrm{i}cx}\sum_{k=0}^{\infty} a_k x^{-k-1}, \tag{8.17}$$

where c is real and not equal to zero.

8.3 Asymptotic expansion of the one-sided Hilbert transform

In this section the asymptotic behavior of the one-sided Hilbert transform is examined. This transform is written as follows:

$$H_1 f(x) = \frac{1}{\pi} P \int_0^\infty \frac{f(s)\mathrm{d}s}{x-s}, \tag{8.18}$$

with $x \in (0,\infty)$. The approach employed is based on Wong's (1980b) analysis of the asymptotic expansion of the one-sided Hilbert transform, but the present work adopts the opposite sign convention to Wong, and a factor of π^{-1} is introduced to conform with the definition of the Hilbert transform used previously. For a real constant c, the asymptotic form of f is taken to be

$$f(s) \sim \mathrm{e}^{\mathrm{i}cs}\sum_{k=0}^{\infty} a_k s^{-k-\alpha}, \text{ as } s \to \infty, \text{ with } 0 < \alpha \le 1. \tag{8.19}$$

After considering the general result for $0<\alpha<1$, two special cases where $\alpha = 1, c = 0$, and $\alpha = 1, c \neq 0$, are examined. The functions $\psi_n(s)$ and $\delta_n(x)$ are introduced by the following equations:

$$\psi_n(s) = f(s) - \mathrm{e}^{\mathrm{i}cs}\sum_{k=0}^{n-1} a_k s^{-k-\alpha}, \text{ for } 0 < \alpha < 1, \tag{8.20}$$

and

$$\delta_n(x) = \frac{1}{\pi} P \int_0^\infty \frac{s^n\, \psi_n(s) \mathrm{d}s}{x - s}, \tag{8.21}$$

and the coefficients b_n are defined by

$$b_n = -\frac{1}{\pi} \int_0^\infty s^{n-1} \psi_n(s) \mathrm{d}s. \tag{8.22}$$

In Eq. (8.20) the standard summation convention is employed, that empty sums $\sum_{k=i}^{j} \{\}$, for $j < i$, are zero. If n is replaced by $n - 1$ in Eq. (8.20) and the resulting equation subtracted from Eq. (8.20), the following recurrence formula is obtained:

$$\psi_n(s) = \psi_{n-1}(s) - a_{n-1} \mathrm{e}^{\mathrm{i}cs} s^{-n+1-\alpha}. \tag{8.23}$$

Making use of the identity

$$\frac{s^n}{x - s} = s^{n-1} \left\{ \frac{x}{x - s} - 1 \right\}, \tag{8.24}$$

it follows from Eq. (8.21) that

$$\delta_n(x) = \frac{x}{\pi} P \int_0^\infty \frac{s^{n-1}\, \psi_n(s) \mathrm{d}s}{x - s} - \frac{1}{\pi} \int_0^\infty s^{n-1}\, \psi_n(s) \mathrm{d}s. \tag{8.25}$$

This result can be simplified by inserting the definition of b_n and employing Eq. (8.23); hence

$$\begin{aligned} \delta_n(x) &= \frac{x}{\pi} P \int_0^\infty \frac{s^{n-1}\, \psi_{n-1}(s) \mathrm{d}s}{x - s} - \frac{x a_{n-1}}{\pi} P \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}cs} s^{-\alpha}\, \mathrm{d}s}{x - s} - \frac{1}{\pi} \int_0^\infty s^{n-1}\, \psi_n(s) \mathrm{d}s \\ &= x\delta_{n-1}(x) + b_n - x a_{n-1} E_{\alpha,c}(x), \end{aligned} \tag{8.26}$$

where

$$E_{\alpha,c}(x) = \frac{1}{\pi} P \int_0^\infty \frac{\mathrm{e}^{\mathrm{ics}}\, s^{-\alpha}\, \mathrm{d}s}{x - s}. \tag{8.27}$$

From Eq. (8.26) it follows, for $n = 1, 2$, and 3, that

$$\delta_1(x) = b_1 - a_0\, x E_{\alpha,c}(x) + x\delta_0(x), \tag{8.28}$$

$$\delta_2(x) = x b_1 + b_2 - \{x a_1 + x^2 a_0\} E_{\alpha,c}(x) + x^2 \delta_0(x), \tag{8.29}$$

and

$$\delta_3(x) = x^2 b_1 + x b_2 + b_3 - \{x a_2 + x^2 a_1 + x^3 a_0\} E_{\alpha,c}(x) + x^3 \delta_0(x). \tag{8.30}$$

Clearly the nth term is given by

$$\delta_n(x) = \sum_{k=1}^{n} b_k\, x^{n-k} - E_{\alpha,c}(x) \sum_{k=0}^{n-1} a_k\, x^{n-k} + x^n \delta_0(x); \tag{8.31}$$

hence,

$$x^{-n}\, \delta_n(x) = \sum_{k=1}^{n} b_k\, x^{-k} - E_{\alpha,c}(x) \sum_{k=0}^{n-1} a_k\, x^{-k} + \delta_0(x). \tag{8.32}$$

From Eq. (8.20) and the definition given in Eq. (8.21),

$$\delta_0(x) \equiv H_1 f(x), \tag{8.33}$$

and using Eq. (8.32) it follows that

$$H_1 f(x) = E_{\alpha,c}(x) \sum_{k=0}^{n-1} \frac{a_k}{x^k} - \sum_{k=1}^{n} \frac{b_k}{x^k} + \frac{\delta_n(x)}{x^n}, \tag{8.34}$$

which is the required result. To complete the analysis, it is necessary to determine the asymptotic expansion for $E_{\alpha,c}(x)$ and to find a bound for the error term $x^{-n}\delta_n(x)$. The first order of business is to evaluate the integral in Eq. (8.27). This is done by considering the integral

$$\frac{1}{\pi} \int_C \frac{\mathrm{e}^{\mathrm{i}cz} z^{-\alpha}\, \mathrm{d}z}{t - z},$$

where c and t are both real and greater than zero, and C is the contour shown in Figure 8.1.

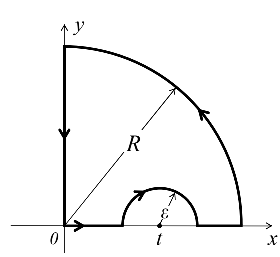

Figure 8.1. Quarter-circle contour centered at the origin with an indentation at the point t.

From the Cauchy integral theorem it follows that

$$\int_0^{t-\varepsilon} \frac{e^{icx}x^{-\alpha}\,dx}{t-x} + i\int_0^{\pi} \frac{e^{ic(t+\varepsilon e^{i\theta})}\,d\theta}{(t+\varepsilon e^{i\theta})^{\alpha}} + \int_{t+\varepsilon}^{R} \frac{e^{icx}x^{-\alpha}\,dx}{t-x}$$
$$+\int_0^{\pi/2} \frac{e^{icRe^{i\theta}}(Re^{i\theta})^{-\alpha}\,i\,Re^{i\theta}\,d\theta}{t-Re^{i\theta}} + \int_R^0 \frac{e^{-cy}(iy)^{-\alpha}\,i\,dy}{t-iy} = 0. \tag{8.35}$$

In the limit $R \to \infty$, Jordan's lemma can be applied to the fourth integral to show that it vanishes, and hence in the limits $R \to \infty$ and $\varepsilon \to 0$ it follows that

$$E_{\alpha,c}(t) + it^{-\alpha}e^{ict} + \pi^{-1}i^{-\alpha}\int_0^{\infty} \frac{e^{-cy}\,dy}{y^{\alpha}(y+it)} = 0. \tag{8.36}$$

Recalling the definition of the gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt, \quad \operatorname{Re} z > 0, \tag{8.37}$$

and employing the incomplete gamma function,

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1}e^{-t}\,dt, \tag{8.38}$$

yields

$$\begin{aligned}\int_0^{\infty} \frac{e^{-cy}\,dy}{y^{\alpha}(y+it)} &= c^{\alpha}\int_0^{\infty} u^{-\alpha}e^{-u}\,du\int_0^{\infty} e^{-(u+itc)s}\,ds\\ &= c^{\alpha}\int_0^{\infty} e^{-itcs}\,ds\int_0^{\infty} u^{-\alpha}e^{-(1+s)u}\,du\\ &= c^{\alpha}\int_0^{\infty} e^{-icts}(1+s)^{\alpha-1}\,ds\int_0^{\infty} w^{-\alpha}e^{-w}\,dw\\ &= \Gamma(1-\alpha)e^{itc}(it)^{-\alpha}\int_{itc}^{\infty} s^{\alpha-1}e^{-s}\,ds\\ &= \Gamma(1-\alpha)\Gamma(\alpha,itc)e^{itc}(it)^{-\alpha}.\end{aligned} \tag{8.39}$$

Therefore, Eq. (8.36) simplifies to

$$E_{\alpha,c}(t) = -\frac{e^{ict}}{\pi t^{\alpha}}\{\pi i + e^{-i\alpha\pi}\Gamma(1-\alpha)\Gamma(\alpha,ict)\}. \tag{8.40}$$

The asymptotic behavior of the gamma and incomplete gamma functions have been extensively studied (see Abramowitz and Stegun (1965, pp. 257–263) and

Olver (1997, pp. 66, 88, 293)), and are given by

$$\Gamma(z) \sim \mathrm{e}^{-z} z^{z-1} \surd(2\pi z) \left\{ 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51\,840z^3} + \cdots \right\},$$
$$z \to \infty, |\arg\ z| < \pi, \tag{8.41}$$

and

$$\Gamma(a,z) \sim \mathrm{e}^{-z} z^{a-1} \left\{ 1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \cdots \right\}, \quad z \to \infty, |\arg\ z| < \frac{3\pi}{2}. \tag{8.42}$$

These results allow the asymptotic behavior of $E_{\alpha,c}(t)$ to be determined.

To establish how the error term is bounded, the integral for $\delta_n(x)$ is split into the three integration ranges: $(0, x-1)$, $(x-1, x+1)$, and $(x+1, \infty)$, and the contributions denoted as $\delta_{n,1}(x)$, $\delta_{n,2}(x)$, and $\delta_{n,3}(x)$, respectively, so that

$$\begin{aligned} \delta_n(x) &= \delta_{n,1}(x) + \delta_{n,2}(x) + \delta_{n,3}(x) \\ &= \frac{1}{\pi} \int_0^{x-1} \frac{s^n \psi_n(s)\mathrm{d}s}{x-s} + \frac{1}{\pi} P \int_{x-1}^{x+1} \frac{s^n \psi_n(s)\mathrm{d}s}{x-s} \\ &\quad + \frac{1}{\pi} \int_{x+1}^{\infty} \frac{s^n \psi_n(s)\mathrm{d}s}{x-s}. \end{aligned} \tag{8.43}$$

The following definitions are introduced:

$$M_{n,1} = \int_0^1 s^n\, |\psi_n(s)| \mathrm{d}s, \tag{8.44}$$

$$M_{n,2} = \sup\{s^{n+\alpha} |\psi_n(s)| : s \geq 1\}, \tag{8.45}$$

and

$$M_{n,3} = \sup\{s^{n+\alpha} \left|\psi_n'(s)\right| : s \geq 1\}. \tag{8.46}$$

For $x > 1$,

$$\begin{aligned} \int_0^1 \frac{s^n \psi_n(s)\mathrm{d}s}{x-s} &\leq \int_0^1 \left| \frac{s^n \psi_n(s)}{x-s} \right| \mathrm{d}s \\ &\leq \frac{1}{x-1} \int_0^1 \left| s^n \psi_n(s) \right| \mathrm{d}s = \frac{M_{n,1}}{x-1} \end{aligned} \tag{8.47}$$

and

$$\begin{aligned}
\int_1^{x-1} \frac{s^n \psi_n(s)\mathrm{d}s}{x-s} &\leq \int_1^{x-1} \frac{\left|s^{n+\alpha}\psi_n(s)\right|}{s^\alpha(x-s)}\mathrm{d}s \\
&\leq M_{n,2} \int_1^{x-1} \frac{\mathrm{d}s}{s^\alpha(x-s)} \\
&= \frac{M_{n,2}}{x^\alpha} \int_{1/x}^{(x-1)/x} \frac{\mathrm{d}t}{t^\alpha(1-t)} \\
&\leq \frac{M_{n,2}}{x^\alpha} \int_{1/x}^{(x-1)/x} \frac{\mathrm{d}t}{t(1-t)} \\
&\leq \frac{2M_{n,2}\log x}{x^\alpha}. \qquad (8.48)
\end{aligned}$$

Now,

$$\delta_{n,1}(x) = \pi^{-1} \int_0^1 \frac{s^n\psi_n(s)\mathrm{d}s}{x-s} + \pi^{-1} \int_1^{x-1} \frac{s^n\psi_n(s)\mathrm{d}s}{x-s}; \qquad (8.49)$$

hence, it follows, from Eqs. (8.47) and (8.49) and for $x > 2.38$, that

$$\left|\delta_{n,1}(x)\right| \leq 2\pi^{-1}(M_{n,1} + M_{n,2})x^{-\alpha} \log x, \qquad (8.50)$$

where the inequality $(x-1)^{-1} \leq 2x^{-\alpha} \log x$, which holds for $0 < \alpha \leq 1$ and $x > 2.38$, has been employed.

The integral over the interval $(x+1, \infty)$ can be expressed as follows:

$$\begin{aligned}
\left|\delta_{n,3}(x)\right| &\leq \pi^{-1} \int_{x+1}^\infty \left|\frac{s^{n+\alpha}\psi_n(s)}{s^\alpha(x-s)}\right|\mathrm{d}s \\
&\leq \pi^{-1}M_{n,2} \int_{x+1}^\infty \frac{\mathrm{d}s}{s^\alpha(s-x)} \\
&= \pi^{-1}x^{-\alpha}M_{n,2}\left\{\int_{1/x}^1 \frac{\mathrm{d}s}{s(1+s)^\alpha} + \int_1^\infty \frac{\mathrm{d}s}{s(1+s)^\alpha}\right\} \\
&\leq \pi^{-1}x^{-\alpha}M_{n,2}\left\{\int_{1/x}^1 \frac{\mathrm{d}s}{s} + \int_1^\infty \frac{\mathrm{d}s}{s^{\alpha+1}}\right\} \\
&\leq \pi^{-1}(1+\alpha^{-1})M_{n,2}\frac{\log x}{x^\alpha}. \qquad (8.51)
\end{aligned}$$

To obtain a bound for $\delta_{n,3}(x)$, the inequalities $1 \leq (1+s)^\alpha$ for the integration interval $(x^{-1}, 1)$ and $s^\alpha \leq (s+1)^\alpha$ for the interval $(1, \infty)$ have been employed, and $x > \mathrm{e}$ is assumed. To complete the derivation, a bound for $\left|\delta_{n,2}(x)\right|$ is required. On writing

$$\varphi_n(s) = s^n\psi_n(s), \qquad (8.52)$$

then

$$\begin{aligned}\delta_{n,2}(x) &= \pi^{-1}P\int_{x-1}^{x+1}\frac{s^n\psi_n(s)\mathrm{d}s}{x-s}\\ &= \pi^{-1}P\int_{x-1}^{x+1}\frac{\varphi_n(s)\mathrm{d}s}{x-s}\\ &= -\pi^{-1}\int_{x-1}^{x+1}\frac{\{\varphi_n(s)-\varphi_n(x)\}\mathrm{d}s}{s-x}\\ &= -\pi^{-1}\int_{x-1}^{x+1}\varphi_n'(\xi)\mathrm{d}s, \qquad (8.53)\end{aligned}$$

where the mean value theorem has been employed and ξ lies between s and x. From the definition of $\varphi_n(s)$ it follows that

$$\begin{aligned}\left|\varphi_{n,2}'(s)\right| &\le \left\{n\left|s^{n-1+\alpha}\psi_n(s)\right| + \left|s^{n+\alpha}\psi_n'(s)\right|\right\}s^{-\alpha}\\ &\le \{nM_{n,2}+M_{n,3}\}s^{-\alpha}, \quad \text{for } s\ge 1. \qquad (8.54)\end{aligned}$$

From Eq. (8.53) it follows that

$$\left|\delta_{n,2}(x)\right| \le \pi^{-1}\int_{x-1}^{x}\left|\varphi_n'(\xi)\right|\mathrm{d}s + \pi^{-1}\int_{x}^{x+1}\left|\varphi_n'(\xi)\right|\mathrm{d}s. \qquad (8.55)$$

Note that ξ lies between s and x, $\xi^{-\alpha}\le s^{-\alpha}$ in the first integral in Eq. (8.55); for the second integral use $\xi^{-\alpha}\le x^{-\alpha}$. Hence,

$$\begin{aligned}\left|\delta_{n,2}(x)\right| &\le \pi^{-1}\{nM_{n,2}+M_{n,3}\}x^{-\alpha}\{1+(1-\alpha)^{-1}x[1-(1-x^{-1})^{1-\alpha}]\}\\ &= \pi^{-1}x^{-\alpha}\{nM_{n,2}+M_{n,3}\}\left\{1+x\sum_{k=1}^{\infty}\frac{\alpha_{k-1}}{k!\,x^k}\right\}, \qquad (8.56)\end{aligned}$$

where a_m denotes a Pochhammer symbol. On making use of

$$\sum_{k=1}^{\infty}\frac{\alpha_{k-1}}{k!x^k}\le\sum_{k=1}^{\infty}\frac{1}{k\,x^k}=\log\left\{\frac{x}{x-1}\right\}, \text{ for } x>1, \qquad (8.57)$$

and the inequality for $x > 3.3$,

$$\log\{x(x-1)^{-1}\} < x^{-1}\log x, \qquad (8.58)$$

Eq. (8.56) can be written as follows:

$$\left|\delta_{n,2}(x)\right| \le 2\pi^{-1}\{nM_{n,2}+M_{n,3}\}x^{-\alpha}\log x. \qquad (8.59)$$

The constant in Eq. (8.59) could be improved by sharpening the inequality used for the sum in Eq. (8.56); however, this is not pursued further, since the overall approach employed only produces a rough bound for $|\delta_n(x)|$.

Combining Eqs. (8.50), (8.51), and (8.59) leads to the following result:

$$|\delta_n(x)| \leq M\frac{\log x}{x^\alpha}, \tag{8.60}$$

where M is the obvious collection of constants from these three inequalities. This bound establishes the condition

$$\delta_n(x) = o(1), \text{ as } x \to \infty. \tag{8.61}$$

Two special cases of Eq. (8.19) are now considered: $\alpha = 1, c = 0$ and $\alpha = 1, c \neq 0$. These particular cases cannot be evaluated from Eq. (8.34) because $E_{\alpha,c}(x)$ diverges for $\alpha = 1$. The case $\alpha = 1, c = 0$ is considered first. From Eq. (8.23)

$$\psi_n(s) = \psi_{n-1}(s) - a_{n-1}s^{-n}, \tag{8.62}$$

and the constants c_n are introduced by

$$c_n = -\frac{1}{\pi}\int_0^1 s^{n-1}\psi_{n-1}(s)\mathrm{d}s - \frac{1}{\pi}\int_1^\infty s^{n-1}\psi_n(s)\mathrm{d}s. \tag{8.63}$$

Starting with the definition for $\delta_n(x)$, Eq. (8.21), and making use of Eqs. (8.62) and (8.24), yields

$$\begin{aligned}\delta_n(x) &= \frac{1}{\pi}P\int_0^1 \frac{s^n\psi_n(s)\mathrm{d}s}{x-s} + \frac{1}{\pi}P\int_1^\infty \frac{s^n\psi_n(s)\mathrm{d}s}{x-s} \\ &= \frac{x}{\pi}P\int_0^1 \frac{s^{n-1}\psi_{n-1}(s)\mathrm{d}s}{x-s} - \frac{1}{\pi}\int_0^1 s^{n-1}\psi_{n-1}(s)\mathrm{d}s + a_{n-1}\pi^{-1}\log\left\{\frac{x-1}{x}\right\} \\ &\quad + \frac{x}{\pi}P\int_1^\infty \frac{s^{n-1}\psi_{n-1}(s)\mathrm{d}s}{x-s} - \frac{1}{\pi}\int_1^\infty s^{n-1}\psi_n(s)\mathrm{d}s - a_{n-1}\pi^{-1}\log(x-1),\end{aligned} \tag{8.64}$$

and hence

$$\delta_n(x) = x\delta_{n-1}(x) + c_n - a_{n-1}\pi^{-1}\log x. \tag{8.65}$$

From this result it follows, for $n = 1, 2$, and 3, that

$$\delta_1(x) = c_1 + x\delta_0(x) - a_0\pi^{-1}\log x, \tag{8.66}$$

$$\delta_2(x) = c_2 + xc_1 + x^2\delta_0(x) - \{a_0x + a_1\}\pi^{-1}\log x, \tag{8.67}$$

and

$$\delta_3(x) = c_3 + xc_2 + x^2c_1 + x^3\delta_0(x) - \{a_0x^2 + a_1x + a_2\}\pi^{-1}\log x, \tag{8.68}$$

and hence for the nth term it follows that

$$\delta_n(x) = \sum_{k=1}^{n} c_k x^{n-k} + x^n\delta_0(x) - \pi^{-1}\log x \sum_{k=0}^{n-1} a_k x^{n-1-k}, \tag{8.69}$$

which can be rearranged to read

$$H_1 f(x) = -\sum_{k=1}^{n} c_k x^{-k} + \pi^{-1}\log x \sum_{k=0}^{n-1} a_k x^{-k-1} + \frac{\delta_n(x)}{x^n}. \tag{8.70}$$

For the case $\alpha = 1, c \neq 0$ a similar approach can be employed, so that

$$\psi_n(s) = \psi_{n-1}(s) - a_{n-1}\mathrm{e}^{\mathrm{i}cs}s^{-n}, \tag{8.71}$$

$$\begin{aligned}\frac{1}{\pi}P\int_0^1 \frac{s^n\psi_n(s)\mathrm{d}s}{x-s} &= \frac{x}{\pi}P\int_0^1 \frac{s^{n-1}\psi_{n-1}(s)\mathrm{d}s}{x-s} - \frac{1}{\pi}\int_0^1 s^{n-1}\psi_{n-1}(s)\mathrm{d}s \\ &\quad - a_{n-1}\pi^{-1}P\int_0^1 \frac{\mathrm{e}^{\mathrm{i}cs}\,\mathrm{d}s}{x-s},\end{aligned} \tag{8.72}$$

and

$$\begin{aligned}\frac{1}{\pi}P\int_1^\infty \frac{s^n\psi_n(s)\mathrm{d}s}{x-s} &= \frac{x}{\pi}P\int_1^\infty \frac{s^{n-1}\psi_{n-1}(s)\mathrm{d}s}{x-s} - \frac{1}{\pi}\int_1^\infty s^{n-1}\psi_n(s)\mathrm{d}s \\ &\quad - a_{n-1}\pi^{-1}xP\int_1^\infty \frac{\mathrm{e}^{\mathrm{i}cs}\,\mathrm{d}s}{s(x-s)},\end{aligned} \tag{8.73}$$

and hence

$$\delta_n(x) = c_n + x\delta_{n-1}(x) - a_{n-1}E_c(x), \tag{8.74}$$

where

$$E_c(x) = \pi^{-1}P\int_0^1 \frac{\mathrm{e}^{\mathrm{i}cs}\,\mathrm{d}s}{x-s} + x\pi^{-1}P\int_1^\infty \frac{\mathrm{e}^{\mathrm{i}cs}\,\mathrm{d}s}{s(x-s)}. \tag{8.75}$$

Following the same procedure discussed for the preceding case,

$$H_1 f(x) = E_c(x)\sum_{k=0}^{n-1} a_k x^{-k-1} - \sum_{k=1}^{n} c_k x^{-k} + \frac{\delta_n(x)}{x^n}. \tag{8.76}$$

To complete this case the asymptotic expansion for $E_c(x)$ is required. From Eq. (8.75) it follows that

$$\begin{aligned} E_c(x) &= \pi^{-1}P\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}cs}\,\mathrm{d}s}{x-s} + \pi^{-1}\int_1^\infty \frac{\mathrm{e}^{\mathrm{i}cs}\,\mathrm{d}s}{s} \\ &= \pi^{-1}\mathrm{e}^{\mathrm{i}cx}\{\mathrm{Ci}(|c|\,x) - \mathrm{i}\ \mathrm{Si}(cx)\} - \pi^{-1}\{\mathrm{Ci}(|c|) + \mathrm{i}\ \mathrm{Si}(c)\} \\ &\quad + \frac{\mathrm{i}}{2}(1-\mathrm{e}^{\mathrm{i}cx})\ \mathrm{sgn}\ c, \end{aligned} \tag{8.77}$$

where $\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$ denote, respectively, the cosine and sine integrals:

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos y\,\mathrm{d}y}{y}, \tag{8.78}$$

$$\mathrm{Si}(x) = \int_0^x \frac{\sin y\,\mathrm{d}y}{y}. \tag{8.79}$$

The asymptotic behavior of $\mathrm{Ci}(z)$ and $\mathrm{Si}(z)$ for $|\arg z| < \pi$ are given by (Abramowitz and Stegun, 1965, p. 232)

$$\mathrm{Ci}(z) = f(z)\sin z - g(z)\cos z \tag{8.80}$$

and

$$\mathrm{Si}(z) = \frac{\pi}{2} - f(z)\cos z - g(z)\sin z, \tag{8.81}$$

where

$$f(z) \sim \frac{1}{z}\sum_{k=0}^\infty \frac{(-1)^k(2k)!}{z^{2k}} \tag{8.82}$$

and

$$g(z) \sim \frac{1}{z^2}\sum_{k=0}^\infty \frac{(-1)^k(2k+1)!}{z^{2k}}. \tag{8.83}$$

From these results and Eq. (8.77), the asymptotic behavior of $E_c(x)$ can be constructed.

Ursell (1983) considered the case where the asymptotic form for f is given by

$$f(x) \sim \sum_{k=1}^\infty \frac{a_k}{x^k} + \cos\omega x\sum_{k=1}^\infty \frac{b_k}{x^k} + \sin\omega x\sum_{k=1}^\infty \frac{c_k}{x^k},\ \text{as } x\to\infty, \tag{8.84}$$

where $\omega > 0$. Defining

$$f(x) = \psi_n(x) + \sum_{k=1}^{n} \frac{a_k}{x^k} + \cos\omega x \sum_{k=1}^{n} \frac{b_k}{x^k} + \sin\omega x \sum_{k=1}^{n} \frac{c_k}{x^k}, \text{ for } n \geq 1, \tag{8.85}$$

$$d_n = \int_0^1 s^{n-1}\psi_{n-1}(s)\mathrm{d}s + \int_1^\infty s^{n-1}\{\psi_{n-1}(s) - a_n s^{-n}\}\mathrm{d}s, \text{ for } n \geq 1, \tag{8.86}$$

and

$$\delta_n(x) = \frac{1}{\pi} P \int_0^\infty \frac{s^n \psi_n(s)\mathrm{d}s}{x-s}, \tag{8.87}$$

it follows from Eq. (8.85) that

$$\psi_n(s) = \psi_{n-1}(s) - a_n s^{-n} - b_n \cos\omega s\, s^{-n} - c_n \sin\,\omega s\, s^{-n}. \tag{8.88}$$

Equations (8.87) and (8.88) lead to

$$\delta_n(x) = \frac{1}{\pi} P \int_0^\infty \frac{s^n\{\psi_{n-1}(s) - a_n s^{-n}\}\mathrm{d}s}{x-s} - \frac{1}{\pi} P \int_0^\infty \frac{\{b_n \cos\omega s + c_n \sin\,\omega s\}\mathrm{d}s}{x-s}. \tag{8.89}$$

The first of these integrals can be simplified as follows:

$$\begin{aligned}
\frac{1}{\pi} P \int_0^\infty \frac{s^n\{\psi_{n-1}(s) - a_n s^{-n}\}\mathrm{d}s}{x-s} &= \frac{1}{\pi} P \int_0^1 s^{n-1}\left(\frac{x}{x-s} - 1\right)(\psi_{n-1}(s) - a_n s^{-n})\mathrm{d}s \\
&\quad + \frac{1}{\pi} P \int_1^\infty s^{n-1}\left(\frac{x}{x-s} - 1\right)(\psi_{n-1}(s) - a_n s^{-n})\mathrm{d}s \\
&= x\delta_{n-1}(x) - \pi^{-1} d_n \\
&\quad - a_n \pi^{-1} \lim_{T\to\infty} \left\{ P \int_0^T \frac{\mathrm{d}s}{x-s} + \int_1^T \frac{\mathrm{d}s}{s} \right\} \\
&= x\delta_{n-1}(x) - \pi^{-1} d_n - \pi^{-1} a_n \log x.
\end{aligned} \tag{8.90}$$

Inserting this result into Eq. (8.89), and following the procedure previously discussed, yields

$$\begin{aligned}
H_1 f(x) = \frac{1}{\pi} \sum_{k=1}^{n} x^{-k} &\left\{ \mathrm{d}_k + a_k \log x + b_k P \int_0^\infty \frac{\cos\omega s\,\mathrm{d}s}{x-s} + c_k P \int_0^\infty \frac{\sin\,\omega s\,\mathrm{d}s}{x-s} \right\} \\
&+ \frac{1}{x^n \pi} P \int_0^\infty \frac{s^n \psi_n(s)\mathrm{d}s}{x-s},
\end{aligned} \tag{8.91}$$

where the final integral represents a remainder term. To complete this result, the following formulas are required:

$$P\int_0^\infty \frac{\cos \omega s\,\mathrm{d}s}{x-s} \sim \pi \sin \omega x + \sum_{k=0}^\infty k!(\omega x)^{-k-1}\cos[(k+1)\pi/2] \tag{8.92}$$

and

$$P\int_0^\infty \frac{\sin \omega s\,\mathrm{d}s}{x-s} \sim -\pi \cos \omega x + \sum_{k=0}^\infty k!(\omega x)^{-k-1}\sin[(k+1)\pi/2]. \tag{8.93}$$

These results are obtained in the following manner. Consider the evaluation of the contour integral $\oint_C \mathrm{e}^{\mathrm{i}\omega z}\mathrm{d}z/(z-\xi)$, for $\omega > 0$, where C is a quarter-circle contour centered at the origin as shown in Figure 8.1 (with the circular section of the arc denoted by Γ and ξ replacing t). Hence, by the Cauchy integral theorem and taking the limit $\varepsilon \to 0$, it follows that

$$P\int_0^R \frac{\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x}{x-\xi} - \mathrm{i}\pi\mathrm{e}^{\mathrm{i}\omega\xi} + \int_\Gamma \frac{\mathrm{e}^{\mathrm{i}\omega z}\,\mathrm{d}z}{z-\xi} - \int_0^R \frac{\mathrm{e}^{-\omega y}\,\mathrm{d}y}{y+\mathrm{i}\xi} = 0. \tag{8.94}$$

In the limit $R \to \infty$, the integral over Γ vanishes by Jordan's lemma, and hence

$$P\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x}{x-\xi} = \mathrm{i}\pi\mathrm{e}^{\mathrm{i}\omega\xi} - \mathrm{i}\int_0^\infty \frac{\mathrm{e}^{-\omega\xi u}\,\mathrm{d}u}{1-\mathrm{i}u}. \tag{8.95}$$

The last integral in Eq. (8.95) can be written as follows:

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{-\omega\xi u}\,\mathrm{d}u}{1-\mathrm{i}u} &= \int_0^\infty \mathrm{e}^{-\omega\xi u}\left\{\sum_{k=0}^{n-1}(\mathrm{i}u)^k + \frac{(\mathrm{i}u)^n}{1-\mathrm{i}u}\right\}\mathrm{d}u \\ &= \sum_{k=0}^{n-1}\frac{\mathrm{i}^k k!}{(\omega\xi)^{k+1}} + R_n(\omega\xi),\end{aligned} \tag{8.96}$$

where the remainder term is given by

$$R_n(\omega\xi) = \mathrm{i}^n\int_0^\infty \frac{(1+\mathrm{i}u)u^n\mathrm{e}^{-\omega\xi u}\,\mathrm{d}u}{(1+u^2)}. \tag{8.97}$$

This term can be bounded as follows:

$$\begin{aligned}|R_n(\omega\xi)| &< \int_0^\infty \left|\frac{(1+\mathrm{i}u)u^n\mathrm{e}^{-\omega\xi u}}{(1+u^2)}\right|\mathrm{d}u \\ &< \frac{n!}{(\omega\xi)^{n+1}}.\end{aligned} \tag{8.98}$$

In the asymptotic limit $\xi \to \infty$, and taking the limit $n \to \infty$ in Eq. (8.96),

$$\int_0^\infty \frac{e^{-\omega\xi u}\,du}{1 - iu} \sim \sum_{k=0}^{\infty} \frac{i^k k!}{(\omega\xi)^{k+1}}. \tag{8.99}$$

An alternative approach can be made by employing Watson's lemma (Watson, 1944, p. 236). Hence, it follows that

$$P\int_0^\infty \frac{e^{i\omega x}\,dx}{x - \xi} \sim i\pi\, e^{i\omega\xi} - \sum_{k=0}^{\infty} \frac{e^{(k+1)\pi i/2} k!}{(\omega\xi)^{k+1}}. \tag{8.100}$$

Taking the real and imaginary parts of this result leads to Eqs. (8.92) and (8.93).

Notes

§8.1 A good source for general information on the asymptotic approximation of integrals is Wong (1989).

§8.2 For further reading on the asymptotic expansion of the Stieltjes transform, see the books by Bleistein and Handelsman (1986) and Wong (1989), and the journal articles by Woolcock (1967, 1968), Zimering (1969), McClure and Wong (1978), and Wong (1980a). A useful reference for the asymptotic expansion of various special functions is Olver (1997).

§8.3 For additional reading on the asymptotic expansion of the one-sided Hilbert transform, see Wong (1980b, 1989). A detailed study of the coefficients d_n appearing in Eq. (8.86) can be found in Ursell (1983).

Exercises

8.1 Assuming $a > 0$ and $x > 0$, determine the asymptotic behavior for large x for the following integrals:

(i) $\int_0^\infty \dfrac{e^{-as}\,ds}{x+s}$;

(ii) $\int_0^\infty \dfrac{\sin s\, e^{-as}\,ds}{x+s}$;

(iii) $\int_0^\infty \dfrac{e^{-as^2}\,ds}{x+s}$;

(iv) $\int_0^\infty \dfrac{e^{-as}\,ds}{(x+s)^2}$;

(v) $\int_0^\infty \dfrac{e^{-ias}\,ds}{x+s}$;

(vi) $\int_0^\infty \dfrac{ds}{(s^2+1)(x+s)}$.

8.2 Determine the asymptotic behavior for large x for the following integrals assuming $a > 0$ and $x > 0$:

(i) $P\int_0^\infty \dfrac{e^{-as}\,ds}{x-s}$;

(ii) $P\int_0^\infty \dfrac{\sin s\, e^{-as}\,ds}{x-s}$;

(iii) $P\int_0^\infty \dfrac{e^{-as^2}\,ds}{x-s}$;

(iv) $P\int_0^\infty \dfrac{ds}{(s^2+1)(x-s)}$;

(v) $P\int_0^\infty \dfrac{e^{-ias}\,ds}{x-s}$.

8.3 Determine the asymptotic behavior for large x for the following integrals assuming $a > 0$ and $x > 0$:

(i) $P\int_{-\infty}^\infty \dfrac{e^{-a|s|}\,ds}{x-s}$;

(ii) $P\int_{-\infty}^\infty \dfrac{\sin s\, e^{-a|s|}\,ds}{x-s}$;

(iii) $P\int_{-\infty}^\infty \dfrac{e^{-as^2}\,ds}{x-s}$;

(iv) $P\int_{-\infty}^\infty \dfrac{ds}{(s^2+1)(x-s)}$;

(v) $P\int_{-\infty}^\infty \dfrac{e^{-ias}\,ds}{x-s}$.

8.4 Evaluate the asymptotic behavior of $(Hf)(x)$ for large x with $f(x) = \operatorname{sinc} x\, e^{-ax^2}$, for $a > 0$.

8.5 Evaluate the asymptotic behavior for large x for the integral

$$\frac{2x}{\pi} P \int_0^\infty \frac{f(s)\,ds}{x^2 - s^2}$$

for the even functions (i) $f(x) = 1/(x^2 + a^2)$, $a > 0$, and (ii) $f(x) = e^{-ax^2}$, $a > 0$. These integrals correspond to $H_e f$ and arise in the Kramers–Kronig transforms of even functions.

8.6 Evaluate the asymptotic behavior for large x for the integral

$$\frac{2}{\pi} P \int_0^\infty \frac{s f(s)\,ds}{x^2 - s^2}$$

for the odd functions (i) $f(x) = x/(x^2 + a^2)$, $a > 0$, and (ii) $f(x) = xe^{-ax^2}$, $a > 0$. These integrals correspond to $H_o f$ and arise in the Kramers–Kronig transforms of odd functions.

8.7 For $f(x) = x\mathrm{e}^{-a|x|}$, with $a > 0$, determine the asymptotic behavior for large x of the allied integral $(2/\pi)\int_0^\infty \cos xt\,\mathrm{d}t \int_0^\infty f(s)\sin st\,\mathrm{d}s$.

8.8 For $f(x) = \mathrm{e}^{-a|x|}$, with $a > 0$, determine the asymptotic behavior for large x of the allied integral $(2/\pi)\int_0^\infty \sin xt\,\mathrm{d}t \int_0^\infty f(s)\cos st\,\mathrm{d}s$.

8.9 Evaluate the asymptotic behavior for large x for the integral

$$\frac{1}{\pi}\int_0^\infty \frac{\{f(x-s) - f(x+s)\}\mathrm{d}s}{s}$$

for the functions

(i) $f(x) = \sin ax/(x^2+b^2), a > 0,\ b > 0$, and
(ii) $f(x) = x\mathrm{e}^{-a|x|},\ a > 0$.

8.10 Determine the asymptotic behavior for large x for the integral

$$\frac{1}{\pi}P\int_{-\infty}^\infty \frac{f(x+s)\sin s}{s^2}\,\mathrm{d}s,$$

which is the Boas transform (discussed later in Section 16.4), for the functions

(i) $f(x) = \sum_{k=1}^\infty \{\alpha_k \sin kx + \beta_k \cos kx\}$, where α_k and β_k are constants, and
(ii) $f(x) = \mathrm{e}^{-a|x|},\ a > 0$.

8.11 The Bessel function of the first kind $J_0(x)$ can be written as

$$J_0(x) = \frac{1}{\pi}\int_{-1}^1 \frac{\cos xt\,\mathrm{d}t}{\surd(1-t^2)}.$$

Find an expression for the asymptotic behavior of $J_0(x)$ for large positive values of x.

8.12 Show that, for large positive x,

$$P\int_0^\infty \frac{J_0^2(s)\mathrm{d}s}{x-s} \sim \frac{\log x}{\pi x} + \frac{(3\log 2+\gamma)}{\pi x} - \frac{\cos 2x}{x} - \frac{\sin 2x}{4x^2} - \frac{\log x}{8\pi x^3} - \frac{(3\log 2 - 5/2 + \gamma)}{\pi x^3} + \frac{5\cos 2x}{32x^3},$$

where γ is Euler's constant. [Hint: See Ursell (1983).]

8.13 Show that, for small positive x,

$$P\int_0^\infty \frac{J_0^2(s)\mathrm{d}s}{x-s} \sim \frac{\pi J_0(x)Y_0(x)}{2} + \surd\pi\sum_{k=0}^\infty \cos k\pi\,\frac{\Gamma(k+1)\cos k\pi\, x^{2k+1}}{\{\Gamma(k+3/2)\}^3},$$

where $Y_0(x)$ denotes a Bessel function of the second kind and $\Gamma(k)$ designates a gamma function. Make a numerical comparison between the result just given and the formula from Exercise 8.12. [Hint: See Ursell (1983).]

8.14 Evaluate the following one-sided Hilbert transforms:

$$\pi^{-1} P \int_0^\infty \frac{s^{-\alpha} \cos as \, \mathrm{d}s}{x - s}$$

and

$$\pi^{-1} P \int_0^\infty \frac{s^{-\alpha} \sin as \, \mathrm{d}s}{x - s}, \text{ for } a > 0 \text{ and } 0 < \alpha < 1.$$

Determine the asymptotic behavior for each integral for large values of x.

9

Hilbert transforms of some special functions

9.1 Hilbert transforms of special functions

The Hilbert transforms of some of the common special functions that occur widely in various applications are investigated in this chapter. The Hilbert transforms of the classical orthogonal polynomials are considered in the first few sections. These polynomials are orthogonal on the interval (α, β) with a weight function $w(x)$, that is

$$\int_{\alpha}^{\beta} w(x)\mathrm{P}_n(x)\mathrm{P}_m(x)\mathrm{d}x = \begin{cases} c_n, & n = m \\ 0, & n \neq m, \end{cases} \tag{9.1}$$

where c_n is a constant and $\mathrm{P}_n(x)$ denotes one of the orthogonal polynomials. In discussing the Hilbert transforms of the classical polynomials, it is necessary to multiply the polynomial by a suitable weight function, to ensure that the Hilbert transform converges. The weight function selected is usually the $w(x)$ given in Eq. (9.1), multiplied where appropriate by a rectangular pulse function or step function, to restrict the integration interval to a required range. The polynomials that are considered are shown in Table 9.1. The notation $C_n^{(\lambda)}(x)$ is also commonly employed for the Gegenbauer polynomials.

It is a standard technique in solving many problems to expand a function under investigation on a particular interval, in terms of polynomials orthogonal on the same interval. When it is desired to determine the Hilbert transform, it may be considerably more convenient to work with the series representation of the function, rather than directly with the function. This leads to a number of strategies to carry out the numerical evaluation of the Hilbert transform.

9.2 Hilbert transforms involving Legendre polynomials

Legendre's differential equation of order n is given by

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + n(n+1)y = 0. \tag{9.2}$$

Table 9.1. *The important classical orthogonal polynomials*

Polynomial	Symbol	Interval (α, β)	Weight $w(x)$
Legendre	$P_n(x)$	$(-1, 1)$	1
Hermite	$H_n(x)$	$(-\infty, \infty)$	e^{-x^2}
Laguerre	$L_n(x)$	$(0, \infty)$	e^{-x}
Chebyshev (of the first kind)	$T_n(x)$	$(-1, 1)$	$\sqrt{[(1-x^2)^{-1}]}$
Chebyshev (of the second kind)	$U_n(x)$	$(-1, 1)$	$\sqrt{(1-x^2)}$
Gegenbauer (ultraspherical)	$C_n^\lambda(x)$	$(-1, 1)$	$(1-x^2)^{\lambda-1/2}$
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(-1, 1)$	$(1-x)^\alpha(1+x)^\beta$

For integer $n \geq 0$, the solutions of this equation are the Legendre polynomials, denoted by $P_n(x)$. A second set of solutions of this equation are the Legendre functions of the second kind, $Q_n(x)$, and these are discussed in a later section. This differential equation arises in a number of applications; for example, the solution of the one-particle Schrödinger equation in three dimensions is reducible to Eq. (9.2) for certain quantum numbers. The related problem of finding the eigenfunctions of the square of the angular momentum operator for the same problem also reduces to the Legendre differential equation. The generating function for the Legendre polynomials is given by

$$\frac{1}{\sqrt{(1-2tx+t^2)}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{9.3}$$

The evaluation of the Hilbert transforms of the functions $P_n(\cos\theta)$ and $P_n(\sin\theta)$ are examined. A waveform that can be expanded in terms of a series of sine or cosine functions can be rearranged to a series of Legendre polynomials. When this rearrangement is carried out, it is of interest to have results for the Hilbert transform. The original series could, for example, represent a dissipative process; the Hilbert transform would then correspond to the dispersive component of the process, or vice versa. A continuous function with support in the interval $[-1, 1]$ has a unique series expansion in terms of Legendre polynomials, and the details are discussed momentarily. Functions with support extending outside the interval $[-1, 1]$ can be multiplied by an appropriate step function to achieve restriction to this interval. It is also of interest to be able to evaluate the Hilbert transform of these step function–Legendre polynomial combinations.

From the generating function, a compact formula for $P_n(\cos\theta)$ can be determined in the following manner. Starting with

$$\begin{aligned}\frac{1}{\sqrt{(1-2tx+t^2)}} &= \frac{1}{\sqrt{(1-2t\cos\theta+t^2)}}\\ &= \frac{1}{\sqrt{(1-te^{i\theta}-te^{-i\theta}+t^2)}}\end{aligned}$$

$$= \frac{1}{\sqrt{(1 - t\mathrm{e}^{\mathrm{i}\theta})}} \frac{1}{\sqrt{(1 - t\mathrm{e}^{-\mathrm{i}\theta})}}$$

$$= \sum_{k=0}^{\infty} \eta_k \, t^k \mathrm{e}^{\mathrm{i}k\theta} \sum_{j=0}^{\infty} \eta_j \, t^j \mathrm{e}^{-\mathrm{i}j\theta}, \tag{9.4}$$

with

$$\eta_k = \begin{cases} 1, & k = 0 \\ \dfrac{(2k-1)!!}{(2k)!!}, & k \geq 1, \end{cases} \tag{9.5}$$

and the reader is reminded of the definitions for the double factorials:

$$(2k-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1) \tag{9.6}$$

and

$$(2k)!! = 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k). \tag{9.7}$$

Let $\alpha_k = t^k \mathrm{e}^{\mathrm{i}k\theta} \eta_k$ and $\beta_j = t^j \mathrm{e}^{-\mathrm{i}j\theta} \eta_j$, then the double infinite summation in Eq. (9.4) is of the following form:

$$\sum_{k=0}^{\infty} \alpha_k \sum_{j=0}^{\infty} \beta_j = \begin{array}{l} \alpha_0\beta_0 + \alpha_0\beta_1 + \alpha_0\beta_2 + \alpha_0\beta_3 + \cdots \\ + \alpha_1\beta_0 + \alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_1\beta_3 + \cdots \\ + \alpha_2\beta_0 + \alpha_2\beta_1 + \alpha_2\beta_2 + \cdots \\ + \alpha_3\beta_0 + \alpha_3\beta_1 + \alpha_3\beta_2 + \cdots, \end{array} \tag{9.8}$$

which can be rearranged by summing along the diagonals (left to right) to yield

$$\sum_{k=0}^{\infty} \alpha_k \sum_{j=0}^{\infty} \beta_j = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha_{n-m} \, \beta_m$$

$$= \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} \mathrm{e}^{\mathrm{i}(n-2m)\theta} \eta_{n-m} \, \eta_m. \tag{9.9}$$

Comparing this expression with the result from the generating function yields, on taking the real part,

$$P_n(\cos\theta) = \sum_{m=0}^{n} \cos[(n-2m)\theta] \eta_{n-m} \, \eta_m. \tag{9.10}$$

The product $\eta_{n-m} \, \eta_m$ can be written in terms of binomial coefficients as follows:

$$\eta_{n-m} \, \eta_m = \frac{1}{4^n} \binom{2m}{m} \binom{2n-2m}{n-m}. \tag{9.11}$$

Equation (9.10) can be written in a form that avoids the repetition of cosine terms with the same argument, as follows:

$$P_n(\cos\theta) = \begin{cases} 4^{-n}\binom{n}{n/2}^2 + \dfrac{1}{2^{2n-1}}\displaystyle\sum_{m=1}^{n/2}\binom{n+2m}{n/2+m}\binom{n-2m}{n/2-m}\cos 2m\theta, & \text{for } n \text{ even} \\ \dfrac{1}{2^{2n-1}}\displaystyle\sum_{m=0}^{(n-1)/2}\binom{n+2m+1}{(n+1)/2+m}\binom{n-2m-1}{(n-1)/2-m}\cos[(2m+1)\theta], & \text{for } n \text{ odd.} \end{cases} \tag{9.12}$$

This result is most suitable for determining the Hilbert transforms of the Legendre polynomials. To evaluate $H\{P_n(\cos\theta)\}$, it follows, from Eq. (9.12) and the linear property of the Hilbert transform, that

$$HP_n(\cos\theta) = \begin{cases} \dfrac{1}{2^{2n-1}}\displaystyle\sum_{m=1}^{n/2}\binom{n+2m}{n/2+m}\binom{n-2m}{n/2-m}\sin 2m\theta, & \text{for } n \text{ even} \\ \dfrac{1}{2^{2n-1}}\displaystyle\sum_{m=0}^{(n-1)/2}\binom{n+2m+1}{(n+1)/2+m}\binom{n-2m-1}{(n-1)/2-m}\sin[(2m+1)\theta], & \text{for } n \text{ odd.} \end{cases} \tag{9.13}$$

From Eq. (9.13), a table of Hilbert transforms for the Legendre polynomials can be constructed (Table 9.2). Some additional entries are given in Appendix 1.

The determination of $H\{P_n(\sin\theta)\}$ is now considered. From Eq. (9.12) it follows that

$$P_n(\sin\theta) = \begin{cases} \dfrac{1}{4^n}\binom{n}{n/2}^2 + \dfrac{1}{2^{2n-1}}\displaystyle\sum_{m=1}^{n/2}\binom{n+2m}{n/2+m}\binom{n-2m}{n/2-m}(-1)^m\cos 2m\theta, & \text{for } n \text{ even} \\ \dfrac{1}{2^{2n-1}}\displaystyle\sum_{m=0}^{(n-1)/2}\binom{n+2m+1}{(n+1)/2+m}\binom{n-2m-1}{(n-1)/2-m}(-1)^m\sin[(2m+1)\theta], & \text{for } n \text{ odd,} \end{cases} \tag{9.14}$$

and hence

$$HP_n(\sin\theta) = \begin{cases} \dfrac{1}{2^{2n-1}}\displaystyle\sum_{m=1}^{n/2}\binom{n+2m}{n/2+m}\binom{n-2m}{n/2-m}(-1)^m\sin 2m\theta, & \text{for } n \text{ even} \\ \dfrac{1}{2^{2n-1}}\displaystyle\sum_{m=0}^{(n-1)/2}\binom{n+2m+1}{(n+1)/2+m}\binom{n-2m-1}{(n-1)/2-m}(-1)^{m+1}\cos[(2m+1)\theta], & \text{for } n \text{ odd.} \end{cases} \tag{9.15}$$

Table 9.2. *Hilbert transforms of the Legendre polynomials* $P_n(\cos\theta)$

$P_n(\cos\theta)$	$HP_n(\cos\theta)$
$P_0(\cos\theta)$	0
$P_1(\cos\theta)$	$\sin\theta$
$P_2(\cos\theta)$	$(3/4)\sin 2\theta$
$P_3(\cos\theta)$	$(1/8)[3\sin\theta + 5\sin 3\theta]$
$P_4(\cos\theta)$	$(5/64)[4\sin 2\theta + 7\sin 4\theta]$
$P_5(\cos\theta)$	$(1/128)[30\sin\theta + 35\sin 3\theta + 63\sin 5\theta]$

Table 9.3. *Hilbert transforms of the Legendre polynomials* $P_n(\sin\theta)$

$P_n(\sin\theta)$	$HP_n(\sin\theta)$
$P_0(\sin\theta)$	0
$P_1(\sin\theta)$	$-\cos\theta$
$P_2(\sin\theta)$	$-(3/4)\sin 2\theta$
$P_3(\sin\theta)$	$(1/8)[5\cos 3\theta - 3\cos\theta]$
$P_4(\sin\theta)$	$(5/64)[7\sin 4\theta - 4\sin 2\theta]$
$P_5(\sin\theta)$	$-(1/128)[63\cos 5\theta - 35\cos 3\theta + 30\cos\theta]$

Some values for $H\{P_n(\sin\theta)\}$ are presented in Table 9.3, and additional values are given in Appendix 1.

The Legendre polynomials form an orthogonal set of functions on the interval $[-1, 1]$, and satisfy the following condition:

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = \frac{2}{2n+1}\delta_{nm}, \tag{9.16}$$

where δ_{nm} is the Kronecker delta (see Eq. (2.38)). On the interval $[-1, 1]$, a continuous function can be expanded in terms of a series of Legendre polynomials as follows:

$$f(x) = \sum_{k=0}^{\infty} a_k P_k(x), \tag{9.17}$$

and the coefficients a_k are given by

$$a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)\mathrm{d}x, \tag{9.18}$$

which follows from Eq. (9.16).

The most direct manner in which to restrict the support of a function to the interval $[-1, 1]$ is to multiply it by a step function. The unit rectangular step function is

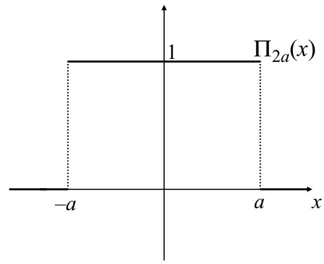

Figure 9.1. Rectangular step function $\Pi_{2a}(x)$.

defined, for $a > 0$, by

$$\Pi_{2a}(x) = \begin{cases} 0, & \text{for } |x| > a \\ 1, & \text{for } |x| < a, \end{cases} \tag{9.19}$$

and it has the appearance shown in Figure 9.1. In some books the vertical sections at $x = -a$ and $x = a$ are drawn, to give the appearance of a rectangular form. The definition given in Eq. (9.19) is sometimes augmented by the assignment $\Pi_{2a}(x) = 1/2$ for $|x| = a$. For $a = 1/2$, the abbreviation $\Pi(x) \equiv \Pi_1(x)$ is widely employed.

The Hilbert transform of the function $P_n(x)\Pi_2(x)$ is now considered. The presence of the factor $\Pi_2(x)$ has the effect of reducing the Hilbert transform on $\mathbb{R}$ to a finite Hilbert transform on $[-1, 1]$. Finite Hilbert transforms are explored further in Chapter 11. Two approaches are shown to work out the desired Hilbert transform. The first method involves setting up a recursive scheme, based on the well known recurrence formula for the Legendre polynomials:

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x), \quad \text{for } n \geq 1. \tag{9.20}$$

The Hilbert transform of $xP_n(x)\Pi_2(x)$ is considered first. From the Hilbert transform of the product $x^n f(x)$ (see Section 4.7, Eq. (4.113)),

$$H[xP_n(x)\Pi_2(x)] = xH[P_n(x)\Pi_2(x)] - \frac{1}{\pi}\int_{-\infty}^{\infty} P_n(x)\Pi_2(x)\mathrm{d}x. \tag{9.21}$$

The last integral simplifies, using Eq. (9.16) and recalling $P_0(x) = 1$, as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} P_n(x)\Pi_2(x)\mathrm{d}x &= \int_{-1}^{1} P_n(x)\mathrm{d}x \\ &= \int_{-1}^{1} P_0(x)P_n(x)\mathrm{d}x = 2\delta_{n0}. \end{aligned} \tag{9.22}$$

Hence,

$$H[xP_n(x)\Pi_2(x)] = xH[P_n(x)\Pi_2(x)] - \frac{2}{\pi}\delta_{n0}. \tag{9.23}$$

So, from the recurrence relation for the Legendre polynomials, it follows that

$$\begin{aligned} H[P_{n+1}(x)\Pi_2(x)] = {} & \frac{2n+1}{n+1} x\, H[P_n(x)\Pi_2(x)] \\ & - \frac{n}{n+1} H[P_{n-1}(x)\Pi_2(x)], \quad \text{for } n \geq 1. \end{aligned} \tag{9.24}$$

This relationship can be employed with the following starting values:

$$\begin{aligned} H[P_0(x)\Pi_2(x)] &= H[\Pi_2(x)] \\ &= \frac{1}{\pi}\log\left|\frac{1+x}{1-x}\right|, \quad |x| \neq 1, \end{aligned} \tag{9.25}$$

and

$$\begin{aligned} H[P_1(x)\Pi_2(x)] &= x\, H[\Pi_2(x)] - \frac{2}{\pi} \\ &= \frac{1}{\pi}\left\{x \log\left|\frac{1+x}{1-x}\right| - 2\right\}, \quad |x| \neq 1. \end{aligned} \tag{9.26}$$

The algebra involved in this type of recurrence scheme can be handled effectively with modern symbolic packages such as *Mathematica*.

An alternative approach to evaluating $H[\mathrm{P}_n(x)\Pi_2(x)]$ starts directly with the series expansion for the Legendre polynomials:

$$P_n(x) = 2^{-n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{2n-2k}{n} \binom{n}{k} x^{n-2k}, \tag{9.27}$$

where $\lfloor x \rfloor$ denotes the *floor* function, the greatest integer not larger than x; that is, for integer m,

$$\left\lfloor \frac{m}{2} \right\rfloor = \begin{cases} m/2, & \text{for } m \text{ even} \\ (m-1)/2, & \text{for } m \text{ odd.} \end{cases} \tag{9.28}$$

It is slightly more convenient to consider the cases n even and n odd separately. For n even, set $n = 2j$, then Eq. (9.27) can be written as

$$P_{2j}(x) = \sum_{k=0}^{j} a_{jk}\, x^{2k}, \tag{9.29}$$

with

$$a_{jk} = 4^{-j}(-1)^{k+j}\binom{2j+2k}{2j}\binom{2j}{j-k}. \tag{9.30}$$

From Eq. (9.29) it follows that

$$H[P_{2j}(x)\Pi_2(x)] = \frac{1}{\pi}\sum_{k=0}^{j} a_{jk}\, P\int_{-\infty}^{\infty} \frac{t^{2k}\Pi_2(t)\mathrm{d}t}{x-t}, \tag{9.31}$$

which simplifies, on using a summation rearrangement of the type

$$\sum_{m=1}^{j} \alpha_{jm}\, x^{2m} \sum_{n=1}^{m} \beta_n\, x^{-2n} = \sum_{m=1}^{j} x^{2m-2} \sum_{n=m}^{j} \alpha_{jn}\, \beta_{n-m+1}, \tag{9.32}$$

to yield

$$\begin{aligned} H[P_{2j}(x)\Pi_2(x)] &= \frac{1}{\pi}\log\left|\frac{1+x}{1-x}\right| \sum_{k=0}^{j} a_{jk}\, x^{2k} \\ &\quad - \frac{2}{\pi}\sum_{k=1}^{j} x^{2k-1} \sum_{m=k}^{j} \frac{a_{jm}}{2m-2k+1}, \text{ with } |x| \neq 1. \end{aligned} \tag{9.33}$$

In a similar manner for odd n, on setting $n = 2j+1$ in Eq. (9.29),

$$P_{2j+1}(x) = \sum_{k=0}^{j} b_{jk}\, x^{2k+1}, \tag{9.34}$$

with

$$b_{jk} = 2^{-2j-1}(-1)^{k+j}\binom{2j+2k+2}{2j+1}\binom{2j+1}{j-k}. \tag{9.35}$$

From Eq. (9.34) it follows by a straightforward calculation that

$$\begin{aligned} H[P_{2j+1}(x)\Pi_2(x)] &= \frac{1}{\pi}\log\left|\frac{1+x}{1-x}\right| \sum_{k=0}^{j} b_{jk}\, x^{2k+1} \\ &\quad - \frac{2}{\pi}\sum_{k=0}^{j} x^{2k} \sum_{m=k}^{j} \frac{b_{jm}}{2m-2k+1}, \text{ with } |x| \neq 1. \end{aligned} \tag{9.36}$$

From Eqs. (9.33) and (9.36), the following table of Hilbert transforms has been constructed (Table 9.4). Additional cases are given in Appendix 1.

Table 9.4. *Hilbert transforms of the functions $P_n(x)\Pi_2(x)$*

$P_n(x)$	$H[P_n(x)\Pi_2(x)]$, $(\lvert x\rvert \neq 1)$
$P_0(x)$	$\pi^{-1}\log\left\lvert\dfrac{1+x}{1-x}\right\rvert$
$P_1(x)$	$\pi^{-1}\left\{x\log\left\lvert\dfrac{1+x}{1-x}\right\rvert - 2\right\}$
$P_2(x)$	$\pi^{-1}\left\{\left[\dfrac{3}{2}x^2-\dfrac{1}{2}\right]\log\left\lvert\dfrac{1+x}{1-x}\right\rvert - 3x\right\}$
$P_3(x)$	$\pi^{-1}\left\{\left[\dfrac{5}{2}x^3-\dfrac{3}{2}x\right]\log\left\lvert\dfrac{1+x}{1-x}\right\rvert - 5x^2+\dfrac{4}{3}\right\}$
$P_4(x)$	$\pi^{-1}\left\{\left[\dfrac{35}{8}x^4-\dfrac{15}{4}x^2+\dfrac{3}{8}\right]\log\left\lvert\dfrac{1+x}{1-x}\right\rvert - \dfrac{35}{4}x^3+\dfrac{55}{12}x\right\}$
$P_5(x)$	$\pi^{-1}\left\{\left[\dfrac{63}{8}x^5-\dfrac{35}{4}x^3+\dfrac{15}{8}x\right]\log\left\lvert\dfrac{1+x}{1-x}\right\rvert - \dfrac{63}{4}x^4+\dfrac{49}{4}x^2-\dfrac{16}{15}\right\}$

9.3 Hilbert transforms of the Hermite polynomials with a Gaussian weight

The calculation of the Hilbert transforms of the Hermite polynomials using a weight function of e^{-x^2} are considered in this section. Waveforms involving a product of Hermite polynomials with a Gaussian have found application in wavelet analysis. For example, the whimsically named Mexican-hat wavelet, introduced by Gabor, can be written as follows:

$$f(x) = \left\{\frac{1}{2}H_0(x) - \frac{1}{4}H_2(x)\right\} e^{-(1/2)x^2}. \tag{9.37}$$

This function is illustrated in Figure 9.2.

The Gaussian function is useful in waveform analysis because it is localized in both the time and frequency domains. The range where the function is significant can be modified by multiplication by appropriate functions, of which the Hermite polynomials are one choice. The functions generated from the product of a Hermite polynomial and a Gaussian function arise in a number of applications, because they are eigenfunctions of the Fourier transform operator.

Hermite's differential equation is given by

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0. \tag{9.38}$$

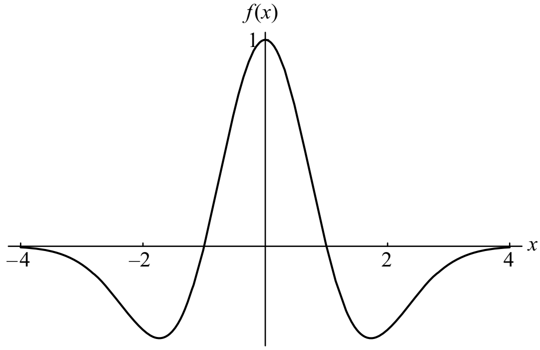

Figure 9.2. The Mexican-hat wavelet.

For integer $n \geq 0$, the solutions of this equation are the Hermite polynomials, $H_n(x)$, which have the following explicit representation:

$$H_n(x) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (2x)^{n-2m}}{m!(n-2m)!}. \tag{9.39}$$

The Hermite differential equation occurs in several important applications. The first encounter with this equation often arises in a quantum mechanical study of the one-dimensional harmonic oscillator, which serves, among other uses, as a model for vibrational motion. The Hermite polynomials satisfy a recurrence relation of the following form:

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x), \quad \text{for } n \geq 1. \tag{9.40}$$

A continuous function $f(x)$ with appropriate asymptotic behavior as $x \to \pm\infty$ can be cast in terms of a series of Hermite polynomials multiplied by the Gaussian weight function e^{-x^2}.

The calculation of the Hilbert transforms of the Hermite polynomials using a weight function of e^{-x^2} can be effectively carried out in at least two different ways. The first involves the use of the recurrence formula just given, and the second method involves finding an explicit expression based on the moment formula for the Hilbert transform of $x^n f(x)$ developed in Section 4.7. From the recurrence formula for the Hermite polynomials it follows that

$$H[H_{n+1}(x)\mathrm{e}^{-x^2}] = 2H[x H_n(x)\mathrm{e}^{-x^2}] - 2n H[H_{n-1}(x)\mathrm{e}^{-x^2}], \quad \text{for } n \geq 1. \tag{9.41}$$

Using the moment formula for the Hilbert transform of $x^n f(x)$, yields

$$H[x H_n(x)\mathrm{e}^{-x^2}] = x H[H_n(x)\mathrm{e}^{-x^2}] - \frac{1}{\pi} \int_{-\infty}^{\infty} H_n(t)\mathrm{e}^{-t^2}\, \mathrm{d}t. \tag{9.42}$$

The final term in Eq. (9.42) can be evaluated using the orthogonality condition for the Hermite polynomials:

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = \begin{cases} 2^n n!\sqrt{\pi}, & n = m \\ 0, & n \neq m. \end{cases} \tag{9.43}$$

Since $H_0(x) = 1$,

$$\int_{-\infty}^{\infty} H_n(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} H_0(x)H_n(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = \delta_{n0}\sqrt{\pi}. \tag{9.44}$$

The recurrence result in Eq. (9.41) thus simplifies to

$$H[H_{n+1}(x)\mathrm{e}^{-x^2}] = 2xH[H_n(x)\mathrm{e}^{-x^2}] - 2n\,H[H_{n-1}(x)\mathrm{e}^{-x^2}], \quad \text{for } n \geq 1. \tag{9.45}$$

To use this relationship the first two values of $H[H_n(x)\mathrm{e}^{-x^2}]$ for $n = 0$ and $n = 1$ are required. In Section 5.2 the Hilbert transform of the Gaussian function was evaluated to be

$$H(\mathrm{e}^{-ax^2}) = -\mathrm{i}\mathrm{e}^{-ax^2}\,\mathrm{erf}(\mathrm{i}\sqrt{(a)}\ x). \tag{9.46}$$

This Hilbert transform is denoted by $G(a, x)$, and the abbreviation $G(1, x) \equiv G(x)$ is employed. Thus,

$$H[H_0(x)\mathrm{e}^{-x^2}] = G(x) \tag{9.47}$$

and

$$H[H_1(x)\mathrm{e}^{-x^2}] = 2x\,G(x) - \frac{2}{\sqrt{\pi}}. \tag{9.48}$$

Equation (9.48) is obtained using Eq. (4.111). With these two results, $H[H_n(x)\mathrm{e}^{-x^2}]$ can be evaluated using Eq. (9.45).

Direct evaluation of the Hilbert transform $H[H_n(x)\mathrm{e}^{-x^2}]$ starts with the expansion Eq. (9.39). Using this result and the moment formula for the Hilbert transform of the product $x^n f(x)$ and with $g(x) = Hf(x)$, yields

$$H[x^n f(x)] = x^n g(x) - \frac{1}{\pi}\sum_{k=0}^{n-1} x^k \int_{-\infty}^{\infty} t^{n-1-k} f(t)\mathrm{d}t. \tag{9.49}$$

From this it follows that

$$H[H_{2j}(x)\mathrm{e}^{-x^2}] = G(x)\sum_{m=0}^{j} a_{jm}\,x^{2m} - \sum_{m=1}^{j} b_{jm}\,x^{2m-1} \tag{9.50}$$

and

$$H[H_{2j+1}(x)\mathrm{e}^{-x^2}] = \sum_{m=0}^{j} \left\{ \frac{(4j+2)a_{jm}G(x)}{2m+1} x^{2m+1} - c_{jm}x^{2m} \right\}, \tag{9.51}$$

where

$$a_{jm} = \frac{4^m(-1)^{j+m}(2j)!}{(2m)!(j-m)!}, \tag{9.52}$$

$$\begin{aligned} b_{jm} &= \pi^{-1} \sum_{n=m}^{j} a_{jn}\Gamma(n-m+1/2) \\ &= 4^j\pi^{-1}(2j)! \sum_{k=0}^{j-m} \frac{\Gamma(j-m-k+1/2)}{(-4)^k\, k!(2j-2k)!}, \end{aligned} \tag{9.53}$$

and

$$\begin{aligned} c_{jm} &= \pi^{-1}(4j+2) \sum_{n=m}^{j} \frac{a_{jn}\Gamma(n-m+1/2)}{2n+1} \\ &= 2^{2j+1}\pi^{-1}(2j+1)! \sum_{k=0}^{j-m} \frac{\Gamma(j-m-k+1/2)}{(-4)^k\, k!(2j-2k+1)!}. \end{aligned} \tag{9.54}$$

In Eqs. (9.53) and (9.54), $\Gamma(n)$ denotes the gamma function. To obtain Eqs. (9.50) and (9.51), summation rearrangements of the type

$$\sum_{m=1}^{j} \alpha_m \sum_{n=1}^{m} \beta_{mn} = \sum_{m=1}^{j} \sum_{n=m}^{j} \alpha_n \beta_{nm} \tag{9.55}$$

have been employed. Evaluation of Eqs. (9.50) and (9.51) leads to the results presented in Table 9.5, and additional values are given in Appendix 1.

9.4 Hilbert transforms of the Laguerre polynomials with a weight function $\mathrm{H}(x)\mathrm{e}^{-x}$

Laguerre's differential equation is given by

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1-x)\frac{\mathrm{d}y}{\mathrm{d}x} + ny = 0. \tag{9.56}$$

The solutions of this equation for integer $n \geq 0$ are the Laguerre polynomials $L_n(x)$. These polynomials can be evaluated using Rodrigues' formula:

$$L_n(x) = \frac{\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n\mathrm{e}^{-x}), \tag{9.57}$$

Table 9.5. *Hilbert transforms of the Hermite polynomials with a Gaussian weight function*

The function $G(x)$ is defined in Eq. (9.46) with $a = 1$

$H_n(x)$	$H[H_n(x)e^{-x^2}]$
$H_0(x)$	$G(x)$
$H_1(x)$	$2xG(x) - \dfrac{2}{\sqrt{\pi}}$
$H_2(x)$	$(4x^2 - 2)G(x) - \dfrac{4x}{\sqrt{\pi}}$
$H_3(x)$	$(8x^3 - 12x)G(x) + \dfrac{8(1 - x^2)}{\sqrt{\pi}}$
$H_4(x)$	$(16x^4 - 48x^2 + 12)G(x) + \dfrac{8(5x - 2x^3)}{\sqrt{\pi}}$
$H_5(x)$	$(32x^5 - 160x^3 + 120x)G(x) - \dfrac{16(4 - 9x^2 + 2x^4)}{\sqrt{\pi}}$

and they satisfy the recurrence formula,

$$L_{n+1}(x) - (2n + 1 - x)L_n(x) + n^2 L_{n-1}(x) = 0. \tag{9.58}$$

A continuous function $f(x)$ with appropriate asymptotic behavior as $x \to \infty$ can be cast in terms of a series of Laguerre polynomials multiplied by the weight function e^{-x}.

Clearly, both $H[L_n(x)]$ (for $n \geq 1$) and $H[L_n(x)e^{-x}]$ diverge. The Hilbert transform of $H[L_n(x)\mathrm{H}(x)e^{-x}]$, where $\mathrm{H}(x)$ denotes the Heaviside step function, is well behaved. The Heaviside step function is defined by

$$\mathrm{H}(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases} \tag{9.59}$$

This definition is sometimes supplemented with the assignment $\mathrm{H}(x) = 1/2$ for $x = 0$. On making use of the expansion

$$L_n(x) = \sum_{m=0}^{n} a_{nm}\, x^m, \tag{9.60}$$

with

$$a_{nm} = \frac{(-1)^m}{m!} \binom{n}{n-m}, \tag{9.61}$$

it follows that

$$
\begin{aligned}
H[L_n(x)\mathrm{H}(x)\mathrm{e}^{-x}] &= \frac{1}{\pi}\sum_{m=0}^{n} a_{nm}\, P\int_0^\infty \frac{s^m \mathrm{e}^{-s}\,\mathrm{d}s}{x-s} \\
&= \frac{1}{\pi}\sum_{m=0}^{n} a_{nm}\, x^m \left\{ \int_0^\infty \frac{((s/x)^m - 1)\,\mathrm{e}^{-s}\,\mathrm{d}s}{x-s} + P\int_0^\infty \frac{\mathrm{e}^{-s}\,\mathrm{d}s}{x-s} \right\} \\
&= E(x)L_n(x) - \frac{1}{\pi}\sum_{m=1}^{n} a_{nm}\, x^{m-1} \sum_{k=0}^{m-1} x^{-k} \int_0^\infty s^k \mathrm{e}^{-s}\,\mathrm{d}s \\
&= E(x)L_n(x) - \frac{1}{\pi}\sum_{m=1}^{n} a_{nm}\, x^{m-1} \sum_{k=0}^{m-1} x^{-k} k!,
\end{aligned} \tag{9.62}
$$

and the definition

$$
E(x) = \frac{1}{\pi} P\int_0^\infty \frac{\mathrm{e}^{-s}\,\mathrm{d}s}{x-s} \tag{9.63}
$$

has been introduced. The integral in Eq. (9.63) can be easily recast in terms of the exponential integral function:

$$
E(x) = \pi^{-1}\mathrm{e}^{-x}\,\mathrm{Ei}(x). \tag{9.64}
$$

If a summation rearrangement of the following form is employed:

$$
\sum_{m=1}^{n} \alpha_{nm}\, x^m \sum_{k=0}^{m-1} \beta_k x^{-k} = \sum_{j=1}^{n} x^j \sum_{m=j}^{n} \alpha_{nm}\beta_{m-j}, \tag{9.65}
$$

then

$$
H[L_n(x)\mathrm{H}(x)\mathrm{e}^{-x}] = E(x)L_n(x) - \sum_{j=1}^{n} c_{nj}\, x^{j-1}, \tag{9.66}
$$

with

$$
c_{nj} = \frac{1}{\pi}\sum_{m=j}^{n} \frac{(-1)^m (m-j)!}{m!} \binom{n}{n-m}. \tag{9.67}
$$

The first few results generated from Eq. (9.66) are shown in Table 9.6, and some additional values are given in Appendix 1.

Table 9.6. *Hilbert transforms of* $L_n(x)\mathrm{H}(x)\mathrm{e}^{-x}$

$L_n(x)$	$H[L_n(x)\mathrm{H}(x)\mathrm{e}^{-x}]$
$L_0(x)$	$E(x)$
$L_1(x)$	$(1-x)E(x)+\pi^{-1}$
$L_2(x)$	$[(2-4x+x^2)E(x)+\pi^{-1}(3-x)]/2$
$L_3(x)$	$[(6-18x+9x^2-x^3)E(x)+\pi^{-1}(11-8x+x^2)]/6$
$L_4(x)$	$[(24-96x+72x^2-16x^3+x^4)E(x)+\pi^{-1}(50-58x+15x^2-x^3)]/24$

9.5 Other orthogonal polynomials

There are a number of polynomials that occur widely in applications that are orthogonal on the interval $(-1, 1)$ with the appropriately chosen weight function. The Hilbert transform of the product of each of these polynomials with its corresponding weight function, multiplied by a suitable step function, can be worked out in a manner related to the discussion given earlier for the Legendre polynomials. For the Chebyshev, Gegenbauer, and Jacobi polynomials,

$$H\left[\frac{T_n(x)\Pi_2(x)}{\sqrt{(1-x^2)}}\right] = \frac{1}{\pi}P\int_{-1}^{1}\frac{T_n(t)\mathrm{d}t}{(x-t)\sqrt{(1-t^2)}}, \tag{9.68}$$

$$H[U_n(x)\Pi_2(x)\sqrt{(1-x^2)}] = \frac{1}{\pi}P\int_{-1}^{1}\frac{U_n(t)\sqrt{(1-t^2)}\mathrm{d}t}{x-t}, \tag{9.69}$$

$$H[C_n^{\lambda}(x)\Pi_2(x)(1-x^2)^{\lambda-1/2}] = \frac{1}{\pi}P\int_{-1}^{1}\frac{C_n^{\lambda}(t)\,(1-t^2)^{\lambda-1/2}\,\mathrm{d}t}{x-t}, \tag{9.70}$$

and

$$H[P_n^{(\alpha,\beta)}(x)\Pi_2(x)(1-x)^{\alpha}(1+x)^{\beta}] = \frac{1}{\pi}P\int_{-1}^{1}\frac{P_n^{(\alpha,\beta)}(t)(1-t)^{\alpha}(1+t)^{\beta}\,\mathrm{d}t}{x-t}. \tag{9.71}$$

In Eqs. (9.68)–(9.71) $x \in (-1, 1)$, so that the principal value of the integral is needed in each case. These four integrals can be most conveniently regarded as particular cases of the finite Hilbert transform, a topic discussed in Chapter 11. The integrals given in Eqs. (9.68) and (9.69), as well as some related cases, are considered in Section 11.12, and particular cases of the other two integrals are also examined in Chapter 11.

9.6 Bessel functions of the first kind

Bessel functions arise frequently in problems in physics and applied mathematics. The solutions of the differential equation

$$z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+z\frac{\mathrm{d}w}{\mathrm{d}z}+(z^2-v^2)w=0 \tag{9.72}$$

are Bessel functions of the first kind $J_{\pm v}(z)$, Bessel functions of the second kind $Y_v(z)$, and Bessel functions of the third kind $H^{(1)}_{\pm v}(z)$, $H^{(2)}_{\pm v}(z)$. Other names and notations are employed for these functions. The $Y_v(z)$ are also called Weber's functions, and $H^{(1)}_{\pm v}(z)$, $H^{(2)}_{\pm v}(z)$ are called Hankel functions. A summary of some of the more common alternative notations can be found in Abramowitz and Stegun (1965, p. 358) or in Erdélyi *et al.* (1953, Vol. II, p. 3).

Attention is first directed to Bessel's function of the first kind. This function has the following series expansion:

$$J_v(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k+v}}{k!\,\Gamma(v+k+1)}, \tag{9.73}$$

where $\Gamma(n)$ denotes the gamma function. The determination of the Hilbert transform of Bessel functions is now considered. The simple example $J_0(ax)$ is examined first. This calculation can be approached in at least two ways, the first involving the Fourier transform method discussed in Section 5.2. The Fourier transform of $J_0(ax)$ for $a>0$ is given by

$$\mathscr{F}[J_0(ax)]=\begin{cases}\dfrac{2}{\sqrt{(a^2-x^2)}}, & 0<|x|<a\\ 0, & a<|x|<\infty,\end{cases} \tag{9.74}$$

for $a>0$. The Hilbert transform of $J_0(ax)$ for $a>0$ is thus determined as follows:

$$\begin{aligned}H[J_0(ax)]&=-\mathrm{i}\mathscr{F}^{-1}\left[\operatorname{sgn}x\begin{cases}2/\sqrt{(a^2-x^2)}, & 0<|x|<a\\ 0, & a<|x|<\infty\end{cases}\right]\\
&=-\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\operatorname{sgn}t\,\mathrm{e}^{\mathrm{i}xt}\begin{Bmatrix}2/\sqrt{(a^2-t^2)}, & 0<|t|<a\\ 0, & a<|t|<\infty\end{Bmatrix}\mathrm{d}t\\
&=-\frac{\mathrm{i}}{\pi}\int_{-a}^{a}\frac{\operatorname{sgn}t\,\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t}{\sqrt{(a^2-t^2)}}\\
&=\frac{2}{\pi}\int_0^a\frac{\sin xt\,\mathrm{d}t}{\sqrt{(a^2-t^2)}}.\end{aligned} \tag{9.75}$$

The latter integral can be expressed as follows:

$$\int_0^a\frac{\sin xy\,\mathrm{d}y}{\sqrt{(a^2-y^2)}}=\int_0^{\pi/2}\sin[ax\sin\theta]\mathrm{d}\theta=\int_0^{\pi/2}\sin[ax\cos\theta]\mathrm{d}\theta. \tag{9.76}$$

A more general form of this integral arises in applications. Struve's function $\mathbf{H}_v(z)$ has the following integral representation (Abramowitz and Stegun, 1965, p. 496):

$$\mathbf{H}_v(z) = \frac{2(z/2)^v}{\sqrt{(\pi)}\,\Gamma(v+1/2)} \int_0^{\pi/2} \sin[z\cos\theta]\sin^{2v}\theta \mathrm{d}\theta. \tag{9.77}$$

Using this result, and noting that $\Gamma(1/2) = \sqrt{\pi}$, yields

$$H[J_0(ax)] = \mathbf{H}_0(ax). \tag{9.78}$$

The Struve function can be evaluated from the power series representation:

$$\mathbf{H}_v(z) = (z/2)^{v+1} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(k+3/2)\Gamma(k+v+3/2)}. \tag{9.79}$$

The determination of $H[J_0(ax)]$ can be approached in a more direct manner using an integral representation for $J_0(ax)$. The Bessel function $J_0(ax)$ can be expressed as follows (Abramowitz and Stegun, 1965, p. 360):

$$J_0(ax) = \frac{1}{\pi} \int_0^{\pi} \cos(ax\sin\theta)\mathrm{d}\theta. \tag{9.80}$$

From this result, and making use of $H(\cos ax) = \operatorname{sgn} a \sin ax$, it follows for $a > 0$ that

$$\begin{aligned} H[J_0(ax)] &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{x-t} \frac{1}{\pi} \int_0^{\pi} \cos(at\sin\theta)\mathrm{d}\theta \\ &= \frac{1}{\pi} \int_0^{\pi} \mathrm{d}\theta \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\cos(at\sin\theta)}{x-t}\,\mathrm{d}t \\ &= \frac{1}{\pi} \int_0^{\pi} \operatorname{sgn}(a\sin\theta)\sin(ax\sin\theta)\mathrm{d}\theta \\ &= \frac{1}{\pi} \int_0^{\pi} \sin(ax\sin\theta)\mathrm{d}\theta. \end{aligned} \tag{9.81}$$

The fact that $a\sin\theta$ is positive for $0 < \theta < \pi$ has been employed in the preceding sequence. The reader is requested to justify the interchange of integration order in the steps given. The last integral in Eq. (9.81) can be written as follows:

$$\begin{aligned} \int_0^{\pi} \sin(ax\sin\theta)\mathrm{d}\theta &= \int_0^{\pi/2} \sin(ax\sin\theta)\mathrm{d}\theta + \int_{\pi/2}^{\pi} \sin(ax\sin\theta)\mathrm{d}\theta \\ &= \int_0^{\pi/2} \sin(ax\sin\theta)\mathrm{d}\theta + \int_0^{\pi/2} \sin(ax\cos\theta)\mathrm{d}\theta \\ &= 2\int_0^{\pi/2} \sin(ax\sin\theta)\mathrm{d}\theta, \end{aligned} \tag{9.82}$$

and Eq. (9.76) has been employed in the last step. Equation (9.78) is obtained from Eqs. (9.77), (9.81), and (9.82).

To approach the calculation of the Hilbert transform of functions like $H[\sin(ax)J_n(bx)]$ and $H[\cos(ax)J_n(bx)]$ for $0 < b < a$ and n taking integer values 0, 1, 2, . . . , it is worthwhile to check if Bedrosian's theorem (see Section 4.15) will work. The underlying reason for examining this approach is that a number of Fourier transforms involving Bessel functions vanish in some interval, which corresponds to the support of the Fourier transform of $\sin ax$ or $\cos ax$. It will prove useful to consider Bedrosian's theorem for functions that are even or odd. The Fourier, Fourier cosine, and Fourier sine transforms of $f(x)$ and $g(x)$ are denoted by $F(x), F_c(x), F_s(x), G(x), G_c(x)$, and $G_s(x)$, respectively. If $f(x)$ is an even function with $F_c(x) = 0$ for $x > a$ $(a > 0)$, and

(i) $g(x)$ is an even function with $G_c(x) = 0$ for $0 < x < a$,

or

(ii) $g(x)$ is an odd function with $G_s(x) = 0$ for $0 < x < a$,

or

(iii) $G(x) = 0$ for $|x| < a$;

or $f(x)$ is an odd function with $F_s(x) = 0$ for $x > a$, and either (i), (ii), or (iii) applies, or $F(x) = 0$ for $|x| > a$, and either (i), (ii), or (iii) applies, then

$$H[f(x)g(x)] = f(x)Hg(x). \tag{9.83}$$

The reader is requested to supply a proof of the preceding result. Some examples are now examined. The first choice is

$$f(x) = \frac{\sin^3 bx}{x^3}, \quad \text{for } b > 0, \tag{9.84}$$

which does not involve special functions. This function is even and satisfies

$$F_c(x) = 0, \quad \text{for } x > 3b. \tag{9.85}$$

Now select $g(x) = \sin ax$ $(a > 0)$, which is an odd function and satisfies (in the distributional sense)

$$G_s(x) = 0, \quad \text{for } 0 < x < a, \tag{9.86}$$

then if $a > 3b$, it follows using Eq. (9.83) that

$$\begin{aligned} H\left[\frac{\sin ax \sin^3 bx}{x^3}\right] &= H[\sin ax]\frac{\sin^3 bx}{x^3} \\ &= -\frac{\cos ax\ \sin^3 bx}{x^3}. \end{aligned} \tag{9.87}$$

Note the importance of the condition $a > 3b$, without which Eq. (9.83) does not hold. In a similar manner, if $g(x) = \cos ax$ $(a > 0)$, then

$$G_c(x) = 0, \quad \text{for } 0 < x < a, \tag{9.88}$$

then, for $a > 3b$,

$$\begin{aligned} H\left[\frac{\cos ax \sin^3 bx}{x^3}\right] &= H[\cos ax]\frac{\sin^3 bx}{x^3} \\ &= \frac{\sin ax\ \sin^3 bx}{x^3}. \end{aligned} \tag{9.89}$$

A number of applications of this approach can be found for Bessel functions, and the cases $H[\sin ax\, J_n(bx)]$ and $H[\cos ax\, J_n(bx)]$ for $n = 0, 1, 2, \ldots$ and $a > b > 0$ are now examined. The case $H[\sin ax\, J_{2n}(bx)]$ is considered first. From Eq. (9.73) it is clear that $J_{2n}(bx)$ is an even function, and the Fourier cosine transform of this function can be expressed in terms of Chebyshev (alternative spelling Tchebichef) polynomials $T_m(x)$ as follows (see Erdélyi *et al.* (1953, Vol. 1, p. 43); note that there is a typographical error in the range given in this standard reference, and this is corrected in the next equation):

$$\mathcal{F}_c\, J_{2n}(bx) = \begin{cases} \dfrac{(-1)^n T_{2n}(b^{-1}x)}{\sqrt{(b^2 - x^2)}}, & 0 < x < b \\ 0, & b < x < \infty. \end{cases} \tag{9.90}$$

With the condition $a > b > 0$ and the results given in Eqs. (9.90) and (9.86),

$$\begin{aligned} H[\sin ax\, J_{2n}(bx)] &= H[\sin ax]\, J_{2n}(bx) \\ &= -\cos ax\, J_{2n}(bx). \end{aligned} \tag{9.91}$$

To evaluate $H[\sin ax\, J_{2n+1}(bx)]$, first note from Eq. (9.73) that $J_{2n+1}(bx)$ is an odd function, and the following result holds (Erdélyi *et al.*, 1953, Vol. 1, p. 99):

$$\mathcal{F}_s\, J_{2n+1}(bx) = \begin{cases} \dfrac{(-1)^n T_{2n+1}(b^{-1}x)}{\sqrt{(b^2 - x^2)}}, & 0 < x < b \\ 0, & b < x < \infty. \end{cases} \tag{9.92}$$

Using this result and Eq. (9.86), it follows that

$$\begin{aligned} H[\sin ax\, J_{2n+1}(bx)] &= H[\sin ax]\, J_{2n+1}(bx) \\ &= -\cos ax\, J_{2n+1}(bx), \quad \text{for } 0 < b < a. \end{aligned} \tag{9.93}$$

This result can be combined with Eq. (9.91) to give

$$H[\sin ax\, J_n(bx)] = -\cos ax\, J_n(bx), \quad \text{for } 0 < b < a, \tag{9.94}$$

which holds for $n = 0, 1, 2, \ldots$ In a similar manner,

$$H[\cos ax\, J_n(bx)] = \sin ax\, J_n(bx), \quad \text{for } 0 < b < a. \tag{9.95}$$

Obviously, this approach to evaluating the Hilbert transforms of these functions requires significantly less labor than a direct evaluation. A number of Hilbert transforms of Bessel functions that have been worked out in this manner are presented in Appendix 1, Table 1.8.

An alternative evaluation approach for $H[\sin ax\, J_n(bx)]$ is now considered, and in the process some complications that arise when the condition $b < a$ is not satisfied are examined. A similar approach to that discussed previously for $H[J_0(ax)]$ is adopted. The function $J_n(ax)$ for integer n can be expressed as follows (Abramowitz and Stegun, 1965, p. 360):

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta - n\theta)\mathrm{d}\theta. \tag{9.96}$$

From this result it follows that

$$\begin{aligned}
H[\sin ax\, J_n(bx)] &= \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\sin at\ \mathrm{d}t}{x-t}\frac{1}{\pi}\int_0^{\pi} \cos(bt\sin\theta - n\theta)\mathrm{d}\theta \\
&= \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\theta \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{\sin at\cos(bt\sin\theta - n\theta)\mathrm{d}t}{x-t} \\
&= \frac{1}{\pi}\int_0^{\pi} \frac{1}{\pi}\left\{\cos n\theta\, P\int_{-\infty}^{\infty} \frac{\sin at\cos(bt\sin\theta)\mathrm{d}t}{x-t}\right. \\
&\quad \left. + \sin n\theta\, P\int_{-\infty}^{\infty} \frac{\sin at\sin(bt\sin\theta)\mathrm{d}t}{x-t}\right\}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{\pi} \frac{1}{\pi}\left\{\cos n\theta\, P\int_{-\infty}^{\infty} \frac{\sin(a - b\sin\theta)t\,\mathrm{d}t}{x-t}\right. \\
&\quad + \cos n\theta\, P\int_{-\infty}^{\infty} \frac{\sin(a + b\sin\theta)t\,\mathrm{d}t}{x-t} \\
&\quad + \sin n\theta\, P\int_{-\infty}^{\infty} \frac{\cos(a - b\sin\theta)t\,\mathrm{d}t}{x-t} \\
&\quad \left. - \sin n\theta\, P\int_{-\infty}^{\infty} \frac{\cos(a + b\sin\theta)t\,\mathrm{d}t}{x-t}\right\}\mathrm{d}\theta
\end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{\pi}\{\operatorname{sgn}(a-b\sin\theta)[-\cos n\theta\cos(a-b\sin\theta)x$$
$$+\sin n\theta\sin(a-b\sin\theta)x]$$
$$-\operatorname{sgn}(a+b\sin\theta)[\cos n\theta\cos(a+b\sin\theta)x$$
$$+\sin n\theta\sin(a+b\sin\theta)x]\}\mathrm{d}\theta. \tag{9.97}$$

The different parameter ranges can be examined by consideration of this result. If $a > b > 0$, then sgn $(a - b\sin\theta) = 1$ for $0 < \theta < \pi$, and so Eq. (9.97) simplifies to give

$$H[\sin ax\, J_n(bx)] = \frac{1}{2\pi}\int_0^{\pi}\{-\cos n\theta\cos(a-b\sin\theta)x+\sin n\theta\sin(a-b\sin\theta)x$$
$$-\cos n\theta\cos(a+b\sin\theta)x-\sin n\theta\sin(a+b\sin\theta)x\}\mathrm{d}\theta$$
$$= -\frac{\cos ax}{\pi}\int_0^{\pi}\{\cos n\theta\cos(bx\sin\theta)+\sin n\theta\sin(bx\sin\theta)\}\mathrm{d}\theta$$
$$= -\frac{\cos ax}{\pi}\int_0^{\pi}\cos(n\theta-bx\sin\theta)\mathrm{d}\theta$$
$$= -\cos ax\, J_n(bx), \tag{9.98}$$

which is in agreement with the result obtained previously using Bedrosian's theorem. For the case $a = b$, a similar argument applies, and it follows from Eq. (9.97) that

$$H[\sin ax\, J_n(ax)] = -\frac{1}{2\pi}\int_0^{\pi}\{\cos n\theta\cos(a-a\sin\theta)x-\sin n\theta\sin(a-a\sin\theta)x$$
$$+\cos n\theta\cos(a+a\sin\theta)x+\sin n\theta\sin(a+a\sin\theta)x\}\mathrm{d}\theta$$
$$= -\frac{\cos ax}{\pi}\int_0^{\pi}\cos(n\theta-ax\sin\theta)\mathrm{d}\theta$$
$$= -\cos ax\, J_n(ax). \tag{9.99}$$

To illustrate the problem that arises when $b > a > 0$, consider the case $H[\sin ax\, J_0(bx)]$. Employing Eq. (9.97) leads to

$$H[\sin ax\, J_0(bx)] = -\frac{1}{2\pi}\int_0^{\pi}\operatorname{sgn}(a-b\sin\theta)\cos(a-b\sin\theta)x\,\mathrm{d}\theta$$
$$-\frac{1}{2\pi}\int_0^{\pi}\operatorname{sgn}(a+b\sin\theta)\cos(a+b\sin\theta)x\,\mathrm{d}\theta. \tag{9.100}$$

Since $\operatorname{sgn}(a + b\sin\theta) = 1$ for $0 < \theta < \pi$, the second integral on the right-hand side of Eq. (9.100) is given by

$$\begin{aligned}
-\frac{1}{2\pi}\int_0^{\pi} \operatorname{sgn}(a+b\sin\theta)\cos(a+b\sin\theta)x\,\mathrm{d}\theta &= -\frac{\cos ax}{2\pi}\int_0^{\pi}\cos(bx\sin\theta)\mathrm{d}\theta \\
&\quad + \frac{\sin ax}{2\pi}\int_0^{\pi}\sin(bx\sin\theta)\mathrm{d}\theta \\
&= -\frac{1}{2}\cos ax\, J_0(bx) \\
&\quad + \frac{1}{2}\sin ax\ \mathbf{H}_0(bx).
\end{aligned} \tag{9.101}$$

The first integral on the right-hand side of Eq. (9.100) is somewhat more tedious to evaluate. On writing $c = a/b$, it follows that

$$\begin{aligned}
&-\frac{1}{2\pi}\int_0^{\pi}\operatorname{sgn}(a-b\sin\theta)\cos(a-b\sin\theta)x\,\mathrm{d}\theta \\
&= -\frac{1}{\pi}\int_0^{\pi/2}\operatorname{sgn}(a-b\sin\theta)\cos(a-b\sin\theta)x\,\mathrm{d}\theta \\
&= -\frac{\cos ax}{\pi}\int_0^{\pi/2}\operatorname{sgn}(a-b\sin\theta)\cos(bx\sin\theta)\mathrm{d}\theta \\
&\quad - \frac{\sin ax}{\pi}\int_0^{\pi/2}\operatorname{sgn}(a-b\sin\theta)\sin(bx\sin\theta)\mathrm{d}\theta \\
&= -\frac{\cos ax}{\pi}\left\{\int_0^{\sin^{-1}c}\cos(bx\sin\theta)\mathrm{d}\theta - \int_{\sin^{-1}c}^{\pi/2}\cos(bx\sin\theta)\mathrm{d}\theta\right\} \\
&\quad - \frac{\sin ax}{\pi}\left\{\int_0^{\sin^{-1}c}\sin(bx\sin\theta)\mathrm{d}\theta - \int_{\sin^{-1}c}^{\pi/2}\sin(bx\sin\theta)\mathrm{d}\theta\right\}.
\end{aligned} \tag{9.102}$$

The integrals in the preceding expression can be evaluated by an appropriate series development, or dealt with by numerical quadrature techniques.

9.7 Bessel functions of the first and second kind for non-integer index

In this section the evaluation of the Hilbert transforms of Bessel function of the first and second kind is illustrated using integral identities for $J_v(x)$ and $Y_v(x)$. For $x > 0$ and $-1/2 < \operatorname{Re} v < 1/2$ (Watson, 1944, p. 170),

$$J_v(x) = \frac{2}{\sqrt{(\pi)}\,\Gamma(1/2 - v)(x/2)^v}\int_1^{\infty}\frac{\sin xt\,\mathrm{d}t}{(t^2-1)^{v+1/2}} \tag{9.103}$$

and

$$Y_v(x) = -\frac{2}{\surd(\pi)\,\Gamma(1/2 - v)(x/2)^v}\int_1^\infty \frac{\cos xt\,\mathrm{d}t}{(t^2-1)^{v+1/2}}. \tag{9.104}$$

The preceding two results are called the Mehler–Sonine integrals. To evaluate $H[|x|^v Y_v(a|x|)]$ for $a > 0$, it follows from the preceding equation that

$$\begin{aligned}
H[|x|^v Y_v(a|x|)] &= -\frac{2^{v+1}}{\surd(\pi)\,\Gamma(1/2 - v)} H\left[\int_1^\infty \frac{\cos(at|x|)\,\mathrm{d}t}{(t^2-1)^{v+1/2}}\right] \\
&= -\frac{2^{v+1}}{\surd(\pi)\,\Gamma(1/2 - v)}\int_1^\infty \frac{H[\cos(atx)]\mathrm{d}t}{(t^2-1)^{v+1/2}} \\
&= -\frac{2^{v+1}}{\surd(\pi)\,\Gamma(1/2 - v)}\int_1^\infty \frac{\operatorname{sgn}(at)\sin(axt)\mathrm{d}t}{(t^2-1)^{v+1/2}} \\
&= -\frac{\operatorname{sgn} x\, 2^{v+1}}{\surd(\pi)\,\Gamma(1/2 - v)}\int_1^\infty \frac{\sin(a|x|t)\mathrm{d}t}{(t^2-1)^{v+1/2}},
\end{aligned} \tag{9.105}$$

and hence

$$H[|x|^v Y_v(a|x|)] = -\operatorname{sgn} x\,|x|^v J_v(a|x|), \quad \text{for } a > 0, \text{ and } -1/2 < \operatorname{Re} v < 1/2. \tag{9.106}$$

Starting from Eq. (9.103) yields, in a similar fashion, the following result:

$$H[\operatorname{sgn} x\,|x|^v J_v(a|x|)] = |x|^v Y_v(a|x|), \quad \text{for } a > 0, \text{ and } -1/2 < \operatorname{Re} v < 1/2. \tag{9.107}$$

The reader is left to determine if Hilbert transforms of functions like $|x|^v Y_v(a|x|)\sin bx$ and $|x|^v Y_v(a|x|)\cos bx$, where b is a constant, can be worked out by employing the integral relationships for $J_v(x)$ and $Y_v(x)$ given at the start of this section. The case $H[|x|^v J_v(a|x|)]$ is worked out in Section 9.8.

9.8 The Struve function

The Struve function $\mathbf{H}_n(x)$ is closely related to the Bessel function $J_n(x)$. The Hilbert transform of the Struve function can be worked out in much the same manner as described for the Bessel function $J_n(x)$. This is illustrated by considering $H\mathbf{H}_0(ax)$ for $a > 0$:

$$\begin{aligned}
H\mathbf{H}_0(ax) &= \frac{1}{\pi}P\int_{-\infty}^\infty \frac{\mathrm{d}s}{x-s}\frac{2}{\pi}\int_0^{\pi/2}\sin(as\cos\theta)\mathrm{d}\theta \\
&= \frac{2}{\pi}\int_0^{\pi/2}\mathrm{d}\theta\frac{1}{\pi}P\int_{-\infty}^\infty \frac{\sin(as\cos\theta)\mathrm{d}s}{x-s}
\end{aligned}$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} \operatorname{sgn}(a\cos\theta)\cos(ax\cos\theta)\mathrm{d}\theta$$

$$= -\frac{1}{\pi}\int_0^{\pi} \cos(ax\sin\theta)\mathrm{d}\theta$$

$$= -J_0(ax), \tag{9.108}$$

which is the skew-symmetric partner of Eq. (9.78). An example where the Hilbert transform of the Struve function shows up is in the determination of the radiation impedance for a rigid circular piston (Mangulis, 1964).

The result in the next example, involves the Struve function. Consider the Hilbert transform $H[|x|^{-\nu}J_\nu(a|x|)]$, for the constant $a > 0$. This can be evaluated by reference to the following integral identity (Abramowitz and Stegun, 1965, p. 360):

$$J_\nu(z) = \frac{2(z/2)^\nu}{\surd(\pi)\,\Gamma(1/2+\nu)}\int_0^1 (1-t^2)^{\nu-1/2}\cos(zt)\mathrm{d}t, \quad \text{for } \operatorname{Re}\nu > -1/2. \tag{9.109}$$

Hence,

$$\begin{aligned} H[|x|^{-\nu}J_\nu(a|x|)] &= \frac{2(a/2)^\nu}{\surd(\pi)\,\Gamma(1/2+\nu)} H\int_0^1 (1-t^2)^{\nu-1/2}\cos(axt)\mathrm{d}t \\ &= \frac{2(a/2)^\nu}{\surd(\pi)\,\Gamma(1/2+\nu)}\int_0^1 (1-t^2)^{\nu-1/2}\sin(axt)\mathrm{d}t \\ &= \frac{2(a/2)^\nu \operatorname{sgn} x}{\surd(\pi)\,\Gamma(1/2+\nu)}\int_0^1 (1-t^2)^{\nu-1/2}\sin(a|x|t)\mathrm{d}t. \end{aligned} \tag{9.110}$$

The Struve function can be represented by the following integral identity (Abramowitz and Stegun, 1965, p. 496):

$$\mathbf{H}_\nu(z) = \frac{2(z/2)^\nu}{\surd(\pi)\,\Gamma(1/2+\nu)}\int_0^1 (1-t^2)^{\nu-1/2}\sin(zt)\mathrm{d}t, \quad \text{for } \operatorname{Re}\nu > -1/2, \tag{9.111}$$

so that Eq. (9.110) simplifies to give

$$H[|x|^{-\nu}J_\nu(a|x|)] = \operatorname{sgn} x\, |x|^{-\nu}\, \mathbf{H}_\nu(a|x|), \quad \text{for } \operatorname{Re}\nu > -1/2. \tag{9.112}$$

The preceding result can be used to evaluate $H[|x|^{-\nu}J_\nu(a|x|)]$, for $a > 0$. Starting from the identity (Abramowitz and Stegun, 1965, p. 358) given by

$$Y_\nu(z) = \frac{\cos\nu\pi\, J_\nu(z) - J_\nu(z)}{\sin\nu\pi} \tag{9.113}$$

and employing Eqs. (9.112) and (9.106) yields

$$
\begin{aligned}
H[|x|^{\nu} J_{\nu}(a|x|)] &= \sec \nu\pi\, H[|x|^{\nu} J_{-\nu}(a|x|)] + \tan \nu\pi\, H[|x|^{\nu} Y_{\nu}(a|x|)] \\
&= \operatorname{sgn} x\, |x|^{\nu} \{\sec \nu\pi\, \mathbf{H}_{-\nu}(a|x|) - \tan \nu\pi\, J_{\nu}(a|x|)\}, \qquad (9.114)
\end{aligned}
$$

which holds for $a > 0$, and $-1/2 < \operatorname{Re} \nu < 3/2$.

9.9 Spherical Bessel functions

The spherical Bessel functions of the first kind are defined in terms of the Bessel functions of the first kind by

$$
j_n(z) = \sqrt{\left(\frac{\pi}{2z}\right)} J_{n+1/2}(z), \tag{9.115}
$$

and the spherical Bessel functions of the second kind are given in terms of the Bessel functions of the second kind by

$$
y_n(z) = \sqrt{\left(\frac{\pi}{2z}\right)} Y_{n+1/2}(z). \tag{9.116}
$$

These functions arise in the solution of the following differential equation:

$$
x^2 \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 2x \frac{\mathrm{d}w}{\mathrm{d}x} + (x^2 - n(n+1))w = 0, \quad \text{for } n \in \mathbb{Z}; \tag{9.117}
$$

that is,

$$
w = \alpha j_n(x) + \beta y_n(x), \tag{9.118}
$$

where α and β are constants.

The Hilbert transform of the spherical Bessel functions can be most quickly determined by employing the Rayleigh formulas:

$$
j_n(x) = (-x)^n \left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sin x}{x}\right), \quad \text{for } n \in \mathbb{Z}^+, \tag{9.119}
$$

and

$$
y_n(x) = x^n (-1)^{n+1} \left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\cos x}{x}\right), \quad \text{for } n \in \mathbb{Z}^+. \tag{9.120}
$$

Making use of the table of Hilbert transforms (Appendix 1, Table 1.5) yields

$$
H j_0(x) = \frac{1 - \cos x}{x} \tag{9.121}
$$

and

$$\begin{aligned} Hj_1(x) &= -\frac{\mathrm{d}}{\mathrm{d}x} H\left[\frac{\sin x}{x}\right] \\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-\cos x}{x}\right) \\ &= -\frac{\sin x}{x} + \frac{1-\cos x}{x^2}. \end{aligned} \tag{9.122}$$

To evaluate $H j_2(x)$, make use of Eqs. (4.111) and (4.137), so that

$$\begin{aligned} Hj_2(x) &= H\left[x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x}\right)\right] \\ &= xH\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x}\right)\right] - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sin t}{t}\right)\mathrm{d}t \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}H\left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x}\right] \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}H\left[\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x}\right] + \frac{1}{\pi x}\int_{-\infty}^{\infty}\frac{1}{t}\left(\frac{\cos t}{t} - \frac{\sin t}{t^2}\right)\mathrm{d}t\right\} \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}H\left[\frac{\sin x}{x}\right] - \frac{1}{2x}\right\} \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-\cos x}{x}\right) - \frac{1}{2x}\right\} \\ &= \frac{1}{2x} + \frac{3}{x^3} + \left(\frac{1}{x} - \frac{3}{x^3}\right)\cos x - \frac{3\sin x}{x^2}. \end{aligned} \tag{9.123}$$

To evaluate $Hj_n(x)$ for higher values of n, the most direct approach is to make use of the recurrence formula for $j_n(x)$, which takes the following form (Abramowitz and Stegun, 1965, p. 439):

$$xj_{n+1}(x) = (2n+1)j_n(x) - xj_{n-1}(x), \quad \text{for } n \in \mathbb{N}. \tag{9.124}$$

Taking the Hilbert transform of this result and employing Eq. (4.111) yields

$$Hj_{n+1}(x) = \frac{(2n+1)}{x}Hj_n(x) - Hj_{n-1}(x) + \frac{1}{\pi x}\int_{-\infty}^{\infty}[j_{n+1}(t) + j_{n-1}(t)]\mathrm{d}t. \tag{9.125}$$

As an example, consider the case $n = 1$. Employing Eqs. (9.121) and (9.122),

$$\begin{aligned} Hj_2(x) &= \frac{3}{x}Hj_1(x) - Hj_0(x) + \frac{1}{\pi x}\int_{-\infty}^{\infty}[j_2(t) + j_0(t)]\mathrm{d}t \\ &= \frac{3}{x}\left(-\frac{\sin x}{x} + \frac{1-\cos x}{x^2}\right) - \frac{(1-\cos x)}{x} + \frac{3}{\pi x}\int_{-\infty}^{\infty}\frac{[\sin t - t\cos t]}{t^3}\,\mathrm{d}t. \end{aligned} \tag{9.126}$$

Making use of the result

$$\int_{-\infty}^{\infty} \frac{[\sin t - t\cos t]}{t^3}\,\mathrm{d}t = \frac{\pi}{2} \tag{9.127}$$

allows Eq. (9.126) to be simplified to give Eq. (9.123).

The Hilbert transforms of the spherical Bessel functions of the second kind involve the Dirac delta distribution. Making use of results from the table of Hilbert transforms (Appendix 1, Table 1.5) yields

$$Hy_0(x) = -H\left[\frac{\cos x}{x}\right] = \pi\delta(x) - \frac{\sin x}{x}, \tag{9.128}$$

$$\begin{aligned}
Hy_1(x) &= H\left[\frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos x}{x}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}H\left[\frac{\cos x}{x}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin x}{x} - \pi\delta(x)\right\} \\
&= -\pi\delta'(x) + \frac{\cos x}{x} - \frac{\sin x}{x^2}
\end{aligned} \tag{9.129}$$

and

$$\begin{aligned}
Hy_2(x) &= -H\left[x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos x}{x}\right)\right] \\
&= H\left[\frac{\cos x}{x} - \frac{3\cos x}{x^3} - \frac{3\sin x}{x^3}\right] \\
&= \frac{\pi}{2}(\delta(x) + 3\delta''(x)) + \frac{3\cos x}{x^2} + \left(\frac{1}{x} - \frac{3}{x^3}\right)\sin x.
\end{aligned} \tag{9.130}$$

Additional examples can be worked out in a similar fashion. The appearance of the Dirac delta distribution is discussed in detail in Chapter 10. The Hilbert transform of the spherical Bessel functions of the third kind can be evaluated directly in terms of $H[j_n(x)]$ and $H[y_n(x)]$. The spherical Bessel functions of the third kind are defined by

$$h_n^{(1)}(z) = j_n(z) + \mathrm{i}y_n(z) = \sqrt{\left(\frac{\pi}{2z}\right)}H^{(1)}_{(2n+1)/2}(z) \tag{9.131}$$

and

$$h_n^{(2)}(z) = j_n(z) - \mathrm{i}y_n(z) = \sqrt{\left(\frac{\pi}{2z}\right)}H^{(2)}_{(2n+1)/2}(z), \tag{9.132}$$

where $H_m^{(n)}(z)$ are the Bessel functions of the third kind. The functions $h_n^{(1)}(z)$ and $h_n^{(2)}(z)$ are also referred to as the spherical Hankel functions of the first kind and the spherical Hankel functions of the second kind, respectively, and the functions

$H_m^{(1)}(z)$ and $H_m^{(2)}(z)$ are also called the Hankel function of the first kind and the Hankel function of the second kind, respectively.

9.10 Modified Bessel functions of the first and second kind

The solutions of the differential equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2 + \nu^2)w = 0 \tag{9.133}$$

are the modified Bessel functions of the first kind, $I_{\pm\nu}(z)$, and the modified Bessel functions of the second kind, $K_\nu(z)$. Series expansion formulas for these functions are given in Appendix 1, p. 455–6. The two functions are connected by the following result:

$$K_\nu(z) = \frac{\pi\{I_{-\nu}(z) - I_\nu(z)\}}{2\sin\pi\nu}. \tag{9.134}$$

Hilbert transforms of the modified Bessel functions are most conveniently worked out by taking advantage of various integral identities. For cases where the function of interest belongs to $L^p(\mathbb{R})$ for $1 < p < \infty$, the inversion property of the Hilbert transform can be employed to generate a second Hilbert transform. In the sequel, the inversion property approach is not exploited; the Hilbert transforms are directly calculated using several integral identities satisfied by the modified Bessel functions. The evaluation process is illustrated using several examples. For the modified Bessel function $K_0(ax)$, the following integral identity due to Mehler holds (Watson, 1944, p. 425; Gray, Mathews, and MacRobert, 1966, p. 74):

$$K_0(ax) = \int_0^\infty \frac{yJ_0(ay)dy}{y^2 + x^2}, \quad \text{for } a > 0,\ x > 0. \tag{9.135}$$

From this identity, for $a > 0$,

$$\begin{aligned} H[K_0(a|x|)] &= H\left\{\int_0^\infty \frac{yJ_0(ay)dy}{y^2 + x^2}\right\} \\ &= \int_0^\infty yJ_0(ay)dy\, H\left[\frac{1}{y^2 + x^2}\right] \\ &= x\int_0^\infty \frac{J_0(ay)dy}{y^2 + x^2}. \end{aligned} \tag{9.136}$$

The Nicholson integral identity (Nicholson, 1911; Watson, 1944, p. 425) is given by

$$\int_0^\infty \frac{J_0(ay)dy}{y^2 + x^2} = \frac{\pi}{2x}\{I_0(ax) - \mathbf{L}_0(ax)\}, \quad \text{for } a > 0, \text{ and } \operatorname{Re} x > 0, \tag{9.137}$$

where $\mathbf{L}_v(x)$ denotes the modified Struve function, and this is defined in the table of Hilbert transforms (Appendix 1, p. 456). Only the situation of real x is of interest for the application at hand, and to include the case $x < 0$, the preceding formula can be rewritten as follows:

$$\int_0^\infty \frac{J_0(ay)\mathrm{d}y}{y^2 + x^2} = \frac{\pi}{2x}\{\operatorname{sgn} x\, I_0(ax) - \mathbf{L}_0(ax)\}, \quad \text{for } a > 0. \tag{9.138}$$

Employing this result allows Eq. (9.136) to be simplified to

$$H[K_0(a|x|)] = \frac{\pi}{2}\{\operatorname{sgn} x\, I_0(ax) - \mathbf{L}_0(ax)\}, \quad \text{for } a > 0. \tag{9.139}$$

The Hilbert transform of the functions $\cosh(ax)K_0(a|x|)$ and $\sinh(ax)K_0(a|x|)$ can be evaluated by taking advantage of the following integral identities (Gray *et al.*, 1966, p. 78):

$$\cosh(ax)K_0(bx) = \int_0^\infty \frac{y\cos(ay)\, J_0(by)\mathrm{d}y}{x^2 + y^2}, \quad b > a, \tag{9.140}$$

and

$$\frac{\sinh(ax)K_0(bx)}{x} = \int_0^\infty \frac{\sin(ay)\, J_0(by)\mathrm{d}y}{x^2 + y^2}, \quad b > a. \tag{9.141}$$

The limit $b \to a$ can be taken in both of the preceding integrals. From Eq. (9.140) it follows that

$$\begin{aligned} H[\cosh(ax)K_0(a|x|)] &= H\left\{\int_0^\infty \frac{y\cos(ay)J_0(ay)\mathrm{d}y}{x^2 + y^2}\right\} \\ &= \int_0^\infty y\cos(ay)\, J_0(ay)\mathrm{d}y\, H\left[\frac{1}{x^2 + y^2}\right] \\ &= x\int_0^\infty \frac{\cos(ay)\, J_0(ay)\mathrm{d}y}{x^2 + y^2}. \end{aligned} \tag{9.142}$$

The following integral identity holds (Gray *et al.*, 1966, p. 78):

$$\int_0^\infty \frac{\cos(ay)\, J_0(by)\mathrm{d}y}{x^2 + y^2} = \frac{\pi}{2x}\mathrm{e}^{-ax} I_0(bx), \quad \text{for } a > 0,\ -a < b < a, \tag{9.143}$$

which can be re-expressed for general x as follows:

$$\int_0^\infty \frac{\cos(ay)\, J_0(by)\mathrm{d}y}{x^2 + y^2} = \frac{\pi}{2x}\operatorname{sgn} x\, \mathrm{e}^{-a|x|} I_0(bx). \tag{9.144}$$

The limit $b \to a$ holds for the preceding integral. Equation (9.142) simplifies to yield

$$H[\cosh(ax)K_0(a|x|)] = \frac{\pi}{2}\operatorname{sgn} x\, \mathrm{e}^{-a|x|} I_0(ax), \quad \text{for } a > 0. \tag{9.145}$$

In a similar fashion,

$$H[\sinh(ax)K_0(a|x|)] = H\left\{x\int_0^\infty \frac{\sin(ay)\,J_0(ay)\mathrm{d}y}{x^2+y^2}\right\}$$
$$= \int_0^\infty \sin(ay)\,J_0(ay)\mathrm{d}y\, H\left[\frac{x}{x^2+y^2}\right]$$
$$= -\int_0^\infty \frac{y\sin(ay)\,J_0(ay)\mathrm{d}y}{x^2+y^2}. \tag{9.146}$$

The following integral identity holds (Gray *et al.*, 1966, p. 78):

$$\int_0^\infty \frac{y\sin(ay)\,J_0(by)\mathrm{d}y}{x^2+y^2} = \frac{\pi}{2}\mathrm{e}^{-ax}I_0(bx), \quad \text{for } a>0,\ -a<b<a, \tag{9.147}$$

which can be rewritten for general x as follows:

$$\int_0^\infty \frac{y\sin(ay)\,J_0(by)\mathrm{d}y}{x^2+y^2} = \frac{\pi}{2}\mathrm{e}^{-a|x|}I_0(bx). \tag{9.148}$$

The limit $b \to a$ holds in the preceding result. Equation (9.146) simplifies to

$$H[\sinh(ax)K_0(a|x|)] = -\frac{\pi}{2}\mathrm{e}^{-a|x|}I_0(ax), \quad \text{for } a>0. \tag{9.149}$$

Taking the sum of Eqs. (9.145) and (9.149) yields

$$H[\mathrm{e}^{ax}K_0(a|x|)] = \begin{cases} -\pi\,\mathrm{e}^{ax}I_0(ax), & x<0 \\ 0, & x>0 \end{cases}, \quad \text{for } a>0. \tag{9.150}$$

Taking the difference of Eq. (9.149) and (9.145) leads to

$$H[\mathrm{e}^{-ax}K_0(a|x|)] = \begin{cases} 0, & x<0 \\ \pi\,\mathrm{e}^{-ax}I_0(ax), & x>0, \end{cases} \tag{9.151}$$

for $a > 0$. To deal with the Hilbert transform of functions like $\mathrm{e}^{-a|x|}I_0(ax)$ and $\operatorname{sgn} x\,\mathrm{e}^{-a|x|}I_0(ax)$, the integral identities given in Eqs. (9.144) and (9.148) can be employed. Hence,

$$H[\mathrm{e}^{-a|x|}I_0(ax)] = \frac{2}{\pi}H\left\{\int_0^\infty \frac{y\sin(ay)\,J_0(ay)\mathrm{d}y}{x^2+y^2}\right\}$$
$$= \frac{2}{\pi}\int_0^\infty y\sin(ay)\,J_0(ay)\mathrm{d}y\, H\left[\frac{1}{x^2+y^2}\right]$$
$$= \frac{2x}{\pi}\int_0^\infty \frac{\sin(ay)\,J_0(ay)\mathrm{d}y}{x^2+y^2}, \tag{9.152}$$

which simplifies on using Eq. (9.141) to yield

$$H[\mathrm{e}^{-a|x|}I_0(ax)] = \frac{2}{\pi}\sinh(ax)K_0(a|x|), \quad \text{for } a > 0. \tag{9.153}$$

In a similar fashion, starting with Eq. (9.144),

$$\begin{aligned} H[\operatorname{sgn} x\, \mathrm{e}^{-a|x|}I_0(ax)] &= \frac{2}{\pi}H\left\{x\int_0^\infty \frac{\cos(ay)\, J_0(ay)\mathrm{d}y}{x^2+y^2}\right\} \\ &= \frac{2}{\pi}\int_0^\infty \cos(ay)\, J_0(ay)\mathrm{d}y\, H\left[\frac{x}{x^2+y^2}\right] \\ &= -\frac{2}{\pi}\int_0^\infty \frac{y\cos(ay)\, J_0(ay)\mathrm{d}y}{x^2+y^2}, \end{aligned} \tag{9.154}$$

which simplifies on employing Eq. (9.140) to give

$$H[\operatorname{sgn} x\, \mathrm{e}^{-a|x|}I_0(ax)] = -\frac{2}{\pi}\cosh(ax)K_0(a|x|), \quad \text{for } a > 0. \tag{9.155}$$

The sum of Eqs. (9.153) and (9.155) leads to

$$H[(1+\operatorname{sgn} x)\mathrm{e}^{-a|x|}I_0(ax)] = -\frac{2}{\pi}\mathrm{e}^{-ax}K_0(a|x|), \quad \text{for } a > 0, \tag{9.156}$$

and the difference of Eqs. (9.153) and (9.155) yields the following result:

$$H[(1-\operatorname{sgn} x)\mathrm{e}^{-a|x|}I_0(ax)] = \frac{2}{\pi}\mathrm{e}^{ax}K_0(a|x|), \quad \text{for } a > 0. \tag{9.157}$$

The Hilbert transform of $|x|^\nu K_\nu(a|x|)$, with $a > 0$, can be evaluated by taking advantage of two integral identities. From Watson (1944, pp. 172, 185)

$$K_\nu(xz) = \frac{\Gamma(\nu+1/2)(2z)^\nu}{x^\nu\sqrt{\pi}}\int_0^\infty \frac{\cos(xt)\mathrm{d}t}{(t^2+z^2)^{\nu+1/2}}, \quad \text{for } x > 0,\ -1/2 < \operatorname{Re}\nu,\ |\arg z| < \pi/2, \tag{9.158}$$

which is due to Poisson, with later contributions from Basset and Malmstén. The second integral identity required is given by (Erdélyi *et al.*, 1953, Vol. II, p. 38)

$$\mathbf{L}_\nu(x) = I_{-\nu}(x) - \frac{2(x/2)^\nu}{\sqrt{(\pi)}\,\Gamma(\nu+1/2)}\int_0^\infty (t^2+1)^{\nu-1/2}\sin(xt)\mathrm{d}t, \quad \text{for } x > 0,\ -1/2 < \operatorname{Re}\nu < 1/2. \tag{9.159}$$

Setting $z = 1$ and $x \to a|x|$ with $a > 0$ in Eq. (9.158) leads to

$$|x|^\nu K_\nu(a|x|) = \frac{\Gamma(\nu+1/2)(2/a)^\nu}{\sqrt{\pi}}\int_0^\infty \frac{\cos(a|x|t)\mathrm{d}t}{(t^2+1)^{\nu+1/2}}, \tag{9.160}$$

and setting $x \to a|x|$, with $a > 0$ and $\nu \to -\nu$, in Eq. (9.159) gives, for $-1/2 < \operatorname{Re}\nu < 1/2$,

$$\int_0^\infty \frac{\sin(a|x|t)\mathrm{d}t}{(t^2+1)^{\nu+1/2}} = \frac{(a|x|)^\nu}{2^{\nu+1}}\sqrt{(\pi)}\,\Gamma(1/2-\nu)\{I_\nu(a|x|) - \mathbf{L}_{-\nu}(a|x|)\}. \quad (9.161)$$

From Eq. (9.160), the Hilbert transform of $|x|^\nu K_\nu(a|x|)$, with $a > 0$, can be calculated as follows:

$$\begin{aligned} H[|x|^\nu K_\nu(a|x|)] &= \frac{\Gamma(\nu+1/2)(2/a)^\nu}{\sqrt{\pi}} H\left\{\int_0^\infty \frac{\cos(a|x|t)\mathrm{d}t}{(t^2+1)^{\nu+1/2}}\right\} \\ &= \frac{\Gamma(\nu+1/2)(2/a)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\mathrm{d}t}{(t^2+1)^{\nu+1/2}} H[\cos(axt)] \\ &= \frac{\Gamma(\nu+1/2)(2/a)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\sin(axt)\mathrm{d}t}{(t^2+1)^{\nu+1/2}} \\ &= \frac{\Gamma(\nu+1/2)(2/a)^\nu \operatorname{sgn} x}{\sqrt{\pi}} \int_0^\infty \frac{\sin(a|x|t)\mathrm{d}t}{(t^2+1)^{\nu+1/2}}. \end{aligned} \quad (9.162)$$

Employing Eq. (9.161) leads to

$$H[|x|^\nu K_\nu(a|x|)] = \frac{1}{2}\Gamma(\nu+1/2)\Gamma(1/2-\nu)\operatorname{sgn} x\,|x|^\nu\{I_\nu(a|x|) - \mathbf{L}_{-\nu}(a|x|)\}. \quad (9.163)$$

The preceding result can be simplified by making use of the reflection formula for the gamma function:

$$\Gamma(z)\,\Gamma(1-z) = \pi\,\csc(\pi z), \quad (9.164)$$

so that, for $a > 0$ and $-1/2 < \operatorname{Re}\nu < 1/2$,

$$H[|x|^\nu K_\nu(a|x|)] = \frac{\pi\,\sec(\pi\nu)}{2}\operatorname{sgn} x\,|x|^\nu\{I_\nu(a|x|) - \mathbf{L}_{-\nu}(a|x|)\}. \quad (9.165)$$

The limit $\nu \to 0$ in the preceding result reduces to Eq. (9.139).

One approach to the evaluation of the Hilbert transform of products of modified Bessel functions is the following scheme based on the Tricomi identity. From the Tricomi relation, Eq. (4.270), with $g = f$,

$$H\{f\,Hf\} = \frac{1}{2}(Hf)^2 - \frac{1}{2}f^2. \quad (9.166)$$

Consider the case $f(x) = \mathrm{e}^{-a|x|}I_0(ax)$, for $a > 0$; employing Eq. (9.153) leads to

$$H[\mathrm{e}^{-a|x|}\sinh(ax)I_0(ax)K_0(a|x|)] = \frac{1}{\pi}\sinh^2(ax)\{K_0(a|x|)\}^2 - \frac{\pi}{4}\mathrm{e}^{-2a|x|}\{I_0(ax)\}^2. \quad (9.167)$$

9.11 The cosine and sine integral functions

Recall from Section 8.3 that the cosine integral function $\mathrm{Ci}(x)$ is defined as follows:

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos y \, \mathrm{d}y}{y}. \tag{9.168}$$

To evaluate the Hilbert transform of $\mathrm{Ci}(a|x|)$ for $a > 0$, the following approach can be employed:

$$\begin{aligned} H[\mathrm{Ci}(a|x|)] &= -H \int_{a|x|}^\infty \frac{\cos y \, \mathrm{d}y}{y} \\ &= -\int_1^\infty H[\cos(aw|x|)] \frac{\mathrm{d}w}{w} \\ &= -\int_1^\infty H[\cos(awx)] \frac{\mathrm{d}w}{w} \\ &= -\int_1^\infty \frac{\mathrm{sgn}(aw) \sin(axw) \mathrm{d}w}{w} \\ &= -\mathrm{sgn}\, x \int_1^\infty \frac{\sin(a|x|w) \mathrm{d}w}{w} \\ &= -\mathrm{sgn}\, x \int_{a|x|}^\infty \frac{\sin(y) \mathrm{d}y}{y}. \end{aligned} \tag{9.169}$$

The reader is asked to justify the switch of integration order that has been employed. The sine integral function $\mathrm{si}(x)$ is introduced by the definition

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin y \, \mathrm{d}y}{y}, \tag{9.170}$$

and this is connected to the function $\mathrm{Si}(x)$ by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin y \, \mathrm{d}y}{y} = \frac{\pi}{2} + \mathrm{si}(x). \tag{9.171}$$

Hence, $H[\mathrm{Ci}(a|x|)]$ can be expressed as follows:

$$H\,\mathrm{Ci}(ax) = -\mathrm{sgn}\, x\, \mathrm{si}(a|x|). \tag{9.172}$$

The evaluation of $H[\mathrm{sgn}\, x\, \mathrm{si}(a|x|)]$ for $a > 0$ is left as an exercise for the reader.

9.12 The Weber and Anger functions

The Anger function can be defined by the integral

$$\mathbf{J}_\upsilon(z) = \frac{1}{\pi} \int_0^\pi \cos(\upsilon\theta - z \sin\theta) \mathrm{d}\theta, \tag{9.173}$$

and the related Weber function is defined by the integral

$$\mathbf{E}_v(z) = \frac{1}{\pi}\int_0^{\pi} \sin(v\theta - z\sin\theta)\mathrm{d}\theta. \tag{9.174}$$

These integral representations are particularly suitable for evaluating the Hilbert transforms of $\mathbf{J}_v(x)$ and $\mathbf{E}_v(x)$. To illustrate the procedure, suppose the constant α satisfies $\alpha > 0$; then

$$\begin{aligned}
H\mathbf{J}_v(\alpha x) &= \frac{1}{\pi}H\left[\int_0^{\pi} \cos(v\theta - \alpha x\sin\theta)\mathrm{d}\theta\right] \\
&= \frac{1}{\pi}\int_0^{\pi} \{\cos(v\theta)H[\cos(\alpha x\sin\theta)] + \sin(v\theta)H[\sin(\alpha x\sin\theta)]\}\mathrm{d}\theta \\
&= \frac{1}{\pi}\int_0^{\pi} \operatorname{sgn}(\alpha\sin\theta)\{\cos(v\theta)\sin(\alpha x\sin\theta) - \sin(v\theta)\cos(\alpha x\sin\theta)\}\mathrm{d}\theta \\
&= \frac{1}{\pi}\int_0^{\pi} \{\cos(v\theta)\sin(\alpha x\sin\theta) - \sin(v\theta)\cos(\alpha x\sin\theta)\}\mathrm{d}\theta \\
&= -\frac{1}{\pi}\int_0^{\pi} \sin(v\theta - \alpha x\sin\theta)\mathrm{d}\theta,
\end{aligned} \tag{9.175}$$

and hence

$$H\mathbf{J}_v(\alpha x) = -\mathbf{E}_v(\alpha x), \quad \text{for } \alpha > 0. \tag{9.176}$$

Starting from Eq. (9.174), it follows in a similar fashion that

$$H\mathbf{E}_v(\alpha x) = \mathbf{J}_v(\alpha x), \quad \text{for } \alpha > 0. \tag{9.177}$$

The reader is left to decide if a similar approach can be employed to evaluate the Hilbert transform of examples such as $\mathbf{E}_v(\alpha x)\sin\beta x$, $\mathbf{E}_v(\alpha x)\cos\beta x$, $\mathbf{J}_v(\alpha x)\sin\beta x$, and $\mathbf{J}_v(\alpha x)\cos\beta x$, where β is a constant.

Notes

Two major sources on information for special functions are Abramowitz and Stegun (1965) and Erdélyi *et al.* (1953). For those having access to the web, Eric Weisstein's World of Mathematics (supported by Wolfram Research) is a very useful resource: www.mathworld.wolfram.com. The Hilbert transform of some relatively simple functions can be evaluated in terms of special functions. For example, the Hilbert transform of a number of logarithmic functions can be evaluated by employing results for the dilogarithm function; see Lee (1996). Tables of selected Hilbert transforms of special functions can be found in Erdélyi *et al.* (1954, Vol. II, p. 239), Hahn (1996a, p. 397, 1996b), and Poularikas (1999). The opposite sign convention to that employed in the present book is used in Erdélyi *et al.* (1954).

Exercises

9.1 Determine the Hilbert transform for the following functions:

(i) $f(x) = \begin{cases} T_n(x), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$

(ii) $f(x) = \begin{cases} \dfrac{T_n(x)}{\sqrt{(1-x^2)}}, & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$

for $n \in \mathbb{Z}^+$.

9.2 Is there a simple recurrence scheme for $Hf(x)$ as a function of n using the choices for f given in Exercise 9.1?

9.3 Determine the Hilbert transform for the following functions:

(i) $f(x) = \begin{cases} U_n(x), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$

(ii) $f(x) = \begin{cases} \sqrt{(1-x^2)}U_n(x), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$

for $n \in \mathbb{Z}^+$.

9.4 Using the functions in Exercise 9.3, can a simple recurrence relationship in the index n be established for $Hf(x)$?

9.5 Determine the Hilbert transform for the following functions involving Gegenbauer polynomials:

(i) $f(x) = \begin{cases} C_n^\alpha(x), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$

(ii) $f(x) = \begin{cases} (1-x^2)^{\alpha-1/2}C_n^\alpha(x), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$

for $n \in \mathbb{Z}^+$.

9.6 For the following functions involving Jacobi polynomials, determine the Hilbert transform:

(i) $f(x) = \begin{cases} P_n^{(\alpha,\beta)}(x), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$

(ii) $f(x) = \begin{cases} (1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$

for $n \in \mathbb{Z}^+$.

9.7 Evaluate $HT_n(\cos x)$ for $n \in \mathbb{Z}^+$.

9.8 Determine $HU_n(\cos x)$ for $n \in \mathbb{Z}^+$.

9.9 For the Gegenbauer polynomials, calculate $HC_n^\alpha(\cos x)$, for $n \in \mathbb{N}$, allowing for both the cases $\alpha = 0$ and $\alpha \neq 0$.

9.10 Evaluate $H[{}_2F_1(-n, n+1; 1; \cos x)]$, for $n \in \mathbb{Z}^+$, where ${}_2F_1(a, b; c; z)$ denotes a Gauss hypergeometric function.

9.11 Determine $H[{}_2F_1(-n, n; 1/2; \cos x)]$, for $n \in \mathbb{Z}^+$.

9.12 Calculate $H\,P_n^{(\alpha,\beta)}(\cos x)$, for $n \in \mathbb{Z}^+$, and hence determine $HP_2^{(1,1)}(\cos x)$.

9.13 Evaluate $HH_n(\cos x)$, for $n \in \mathbb{Z}^+$.

9.14 Determine $H[P_n(x)\mathrm{e}^{-x^2}\Pi_2(x)]$, for $n \in \mathbb{Z}^+$.

9.15 Evaluate $H[|x|^v J_v(a|x|)]$, for $-1/2 < \operatorname{Re} v < 1/2$ and $a > 0$.

9.16 Determine $H[\sin ax\, J_0(b\surd(x^2 + c^2))]$, where $0 < b < a$ and $c \geq 0$.

9.17 Calculate $H[\cos ax\, J_\nu(x) J_{-\nu}(x)]$, for $a > 2$.

9.18 Evaluate $H[\sin ax\, J_{n+1/2}(bx) J_{n+1/2}(cx)]$, for $b > 0$, $c > 0$, $a > b + c$, and $n = 0, 1, 2, \ldots$.

9.19 Determine $H[\cos ax\, J_{\mu+x}(b) J_{\nu-x}(b)]$, for $\operatorname{Re}(\mu + \nu) > 1$ and $a > \pi$.

9.20 Calculate $H[\sin ax\{\mathbf{J}_\nu(bx) - \mathbf{J}_{-\nu}(bx)\}]$, for $0 < b < a$, where $\mathbf{J}_\nu(z)$ denotes Anger's function, defined in Eq. (9.173).

9.21 Evaluate $H\, \mathrm{Si}(a|x|)$, for $a > 0$.

9.22 Determine $H[\surd(|x|) J_{1/2}(ax)]$, for $a > 0$.

9.23 Calculate $H[\surd(|x|) J_{-1/2}(ax)]$, for $a > 0$.

9.24 Evaluate $H[\surd(|x|) Y_{1/2}(ax)]$, for $a > 0$.

9.25 Determine $H[\surd(|x|) Y_{-1/2}(ax)]$, for $a > 0$.

9.26 Calculate $H[\surd(|x|) J_{3/2}(ax)]$, for $a > 0$.

9.27 Evaluate $H[\surd(|x|) J_{-3/2}(ax)]$, for $a > 0$.

9.28 Determine $H[\surd(|x|) K_{1/2}(a|x|)]$, for $a > 0$.

9.29 Calculate $H[\surd(|x|) K_{-1/2}(a|x|)]$, for $a > 0$.

10

Hilbert transforms involving distributions

10.1 Some basic distributions

The Hilbert transform of a distribution can be formally defined in more than one way, and this chapter is concerned with setting forth some of the most commonly employed definitions. Some more formal properties are also considered, and the foundations for establishing the key properties of the Hilbert transform of distributions are discussed. Some distributions occur very widely in applications in physics, applied mathematics, and branches of engineering. It is of interest, both practical and intrinsic, to have methods available for the evaluation of the Hilbert transform of distributions of different classes. This chapter will treat some of the more important cases that arise.

An introductory non-rigorous evaluation of the Hilbert transform of a pair of distributions, one of which occurs frequently in applications, is given first. The emphasis is on doing some informal operations that at least have some intuitive appeal, but are definitely not to be regarded as representing a rigorous derivation. Later sections will deal with a more formal and rigorous approach. A definition of the Dirac delta distribution that will be particularly useful, is as follows:

$$\pi\delta(x) = \lim_{a\to 0} \frac{a}{a^2 + x^2}. \tag{10.1}$$

This formula was discussed in Section 2.15. The reader is reminded (and detailed elaboration will be given in the following section) that $\delta(x)$ is not an ordinary function on $\mathbb{R}$. Consider the evaluation of the Hilbert transform of x^{-1}. The Hilbert transform of this function can be evaluated by considering the following result:

$$H\left(\frac{x}{a^2 + x^2}\right) = -\frac{a}{a^2 + x^2}. \tag{10.2}$$

Now examine the limit $a \to 0$ in this expression. Then it follows that

$$\lim_{a\to 0} H\left(\frac{x}{a^2 + x^2}\right) = H\left(\lim_{a\to 0} \frac{x}{a^2 + x^2}\right) = H\left(\frac{1}{x}\right) = -\lim_{a\to 0} \frac{a}{a^2 + x^2}, \tag{10.3}$$

and therefore

$$H\left(\frac{1}{x}\right) = -\pi\delta(x). \tag{10.4}$$

To make sense of this calculation, it is necessary to justify the interchange of limit and integral. For the moment it will be assumed that this interchange can be performed. Before proceeding further it is necessary to give some interpretation of the equation. Equation (10.4) is to be understood in the distributional sense. That is, for a suitable test function $\phi(x)$, then

$$\left\langle H\left(\frac{1}{x}\right), \phi(x)\right\rangle = \langle -\pi\delta(x), \phi(x)\rangle. \tag{10.5}$$

A moment's reflection will also indicate that the Hilbert transform of x^{-1} is divergent at $x = 0$, and further that the limiting operation will not be defined in the sense of ordinary functions. To sharpen, at least a little, the intended meaning of $H(x^{-1})$, it is necessary to discuss some points on distributions. The generalized function $\mathcal{P}(x^{-n})$ for $n \geq 1$, termed the principal part of x^{-n}, is introduced by the following definition:

$$\begin{aligned}\langle \mathcal{P}(x^{-n}), \phi(x)\rangle &= \int_{-\infty}^{\infty} \mathcal{P}(x^{-n})\phi(x)\mathrm{d}x \\ &= \lim_{\varepsilon\to 0+}\int_{-\infty}^{\infty} \phi(x)\frac{1}{2}\left\{\frac{1}{(x+\mathrm{i}\varepsilon)^n} + \frac{1}{(x-\mathrm{i}\varepsilon)^n}\right\}\mathrm{d}x,\end{aligned} \tag{10.6}$$

where ϕ is a suitable test function. Note formally that the limit can be removed outside the integral, and this is part of the definition. More on this issue will be discussed in the following section. The case of immediate interest is $n = 1$, and, on using Eq. (10.1), the following two formulas are obtained:

$$\lim_{\varepsilon\to 0+}\frac{1}{(x+\mathrm{i}\varepsilon)} = \mathcal{P}\left(\frac{1}{x}\right) - \pi\mathrm{i}\delta(x) \tag{10.7}$$

and

$$\lim_{\varepsilon\to 0+}\frac{1}{(x-\mathrm{i}\varepsilon)} = \mathcal{P}\left(\frac{1}{x}\right) + \pi\mathrm{i}\delta(x). \tag{10.8}$$

These occur widely in applications in physics. A derivation of these two results is given in Section 10.3. The Heisenberg delta functions, defined by

$$\delta^{+}(x) = \frac{1}{2}\delta(x) - \frac{1}{2\pi\mathrm{i}}\mathcal{P}\left(\frac{1}{x}\right) = -\frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\to 0+}\frac{1}{(x+\mathrm{i}\varepsilon)} \tag{10.9}$$

and

$$\delta^{-}(x) = \frac{1}{2}\delta(x) + \frac{1}{2\pi \mathrm{i}}\mathcal{P}\left(\frac{1}{x}\right) = \frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon\to 0+} \frac{1}{(x - \mathrm{i}\varepsilon)}, \tag{10.10}$$

also show up in practical applications. These distributions are sometimes defined with the meaning of δ^+ and δ^- interchanged; see, for example, Schwartz (1962). The distribution $\mathcal{P}(x^{-1})$ is occasionally termed a *pseudofunction* and is at times denoted by $Pf(x^{-1})$. The distribution $\mathcal{P}(x^{-1})$ is also written as $p.v.(x^{-1})$, where $p.v.$ is read as principal value, so that $p.v.(x^{-1})$ is called the principal value distribution. This choice comes from the connection of Eq. (10.6) for the case $n = 1$, with the Cauchy principal value integral. For a suitable function $\phi(x)$ and $a > 0$,

$$P\int_{-\infty}^{\infty} \frac{\phi(x)\mathrm{d}x}{x} = \int_{|x|\geq a} \frac{\phi(x)\mathrm{d}x}{x} + \int_{-a}^{a} \frac{\{\phi(x) - \phi(0)\}\mathrm{d}x}{x} + \phi(0)P\int_{-a}^{a} \frac{\mathrm{d}x}{x}. \tag{10.11}$$

The first two integrals on the right-hand side are well behaved, and the third integral satisfies

$$P\int_{-a}^{a} \frac{\mathrm{d}x}{x} = 0. \tag{10.12}$$

Now, from Eq. (10.6)

$$\begin{aligned}
\langle \mathcal{P}(x^{-1}), \phi(x)\rangle &= \int_{-\infty}^{\infty} \mathcal{P}(x^{-1})\phi(x)\mathrm{d}x \\
&= \lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} \frac{\phi(x)}{2}\left\{\frac{1}{(x + \mathrm{i}\varepsilon)} + \frac{1}{(x - \mathrm{i}\varepsilon)}\right\}\mathrm{d}x \\
&= \int_{|x|\geq a} \frac{\phi(x)\mathrm{d}x}{x} + \int_{-a}^{a} \frac{\{\phi(x) - \phi(0)\}\mathrm{d}x}{x} \\
&\quad + \lim_{\varepsilon\to 0+} \frac{\phi(0)}{2} \int_{-a}^{a} \left\{\frac{1}{x + \mathrm{i}\varepsilon} + \frac{1}{x - \mathrm{i}\varepsilon}\right\}\mathrm{d}x.
\end{aligned} \tag{10.13}$$

The last integral simplifies to

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \int_{-a}^{a} \left\{\frac{1}{x + \mathrm{i}\varepsilon} + \frac{1}{x - \mathrm{i}\varepsilon}\right\}\mathrm{d}x &= \lim_{\varepsilon\to 0+} \left\{ \log(x + \mathrm{i}\varepsilon)\big|_{-a}^{a} + \log(x - \mathrm{i}\varepsilon))\big|_{-a}^{a} \right\} \\
&= 0,
\end{aligned} \tag{10.14}$$

which completes the demonstration of the association of $\mathcal{P}(x^{-1})$ with $p.v.(x^{-1})$. Henceforth, the notation $p.v.(x^{-1})$ will be employed to describe the particular distribution under discussion, although the notation $\mathcal{P}(x^{-1})$ or just $P(1/x)$ is very common. With the latter choice, the symbol P is then used to signify a distribution and, as part of the symbol $P\int$, is employed to indicate a Cauchy principal value integral. The choice $P(1/x)$ has been avoided to minimize any possible confusion for the reader. Equation (10.4) will be written in the following form:

$$H\left(p.v.\frac{1}{x}\right) = -\pi\delta(x), \tag{10.15}$$

and this will be regarded as a symbolic result. The discussion later in this chapter will amplify this point.

A second example is now examined which is a slightly more complicated case, $H(\cos ax/x)$, assuming $\operatorname{Im} a = 0$. If the limit $a \to 0$ is investigated, it is clear that this Hilbert transform must involve a delta function. Based on the preceding comments this is written as $H\{p.v.(\cos ax/x)\}$. Three approaches are examined for the evaluation of this transform, the more lengthy method being considered first. In the same spirit as Eq. (10.3) , start with the result

$$H\left(\frac{x\cos ax}{b^2+x^2}\right) = \frac{x\operatorname{sgn} a\sin ax}{b^2+x^2} - \frac{|b|\mathrm{e}^{-|ab|}}{b^2+x^2}. \tag{10.16}$$

This formula can be established by considering the contour integral of the function

$$f(z) = \frac{z\mathrm{e}^{\mathrm{i}az}}{(z^2+b^2)(z-x_0)} \tag{10.17}$$

and integrating around a semicircular contour, center the origin, with a semicircular indentation around x_0, choosing the upper or lower half planes depending on the sign of a. If the limit $b \to 0+$ in Eq. (10.16) is considered, then

$$\begin{aligned} H\left\{p.v.\left(\frac{\cos ax}{x}\right)\right\} &= H\left(\lim_{b\to 0+}\frac{x\cos ax}{b^2+x^2}\right) \\ &= \lim_{b\to 0+} H\left(\frac{x\cos ax}{b^2+x^2}\right) \\ &= \frac{\operatorname{sgn} a\sin ax}{x} + \lim_{b\to 0+}\frac{b}{b^2+x^2}\{\operatorname{sgn} a\sinh ab - \cosh ab\} \\ &= \frac{\operatorname{sgn} a\sin ax}{x} - \pi\delta(x). \end{aligned} \tag{10.18}$$

The last line follows using Eq. (10.1) .

The second approach to the evaluation of $H\{p.v.(x^{-1}\cos ax)\}$ starts with the result

$$H\left(\frac{\sin ax}{x}\right) = \operatorname{sgn} a\left\{\frac{1-\cos ax}{x}\right\}, \tag{10.19}$$

which can be derived in a straightforward manner. The right-hand side of Eq. (10.19) is well behaved in the vicinity of $x = 0$, but if it is separated into two terms, then each is understood in the sense of $p.v.()$. Taking the Hilbert transform of both sides of Eq. (10.19), and making use of the inversion formula for the Hilbert transform, leads to

$$\operatorname{sgn} a\, H\left\{ p.v.\left(\frac{\cos ax}{x}\right)\right\} = \operatorname{sgn} a\, H\left(p.v.\frac{1}{x}\right) + \frac{\sin ax}{x}, \tag{10.20}$$

which simplifies on employing Eq. (10.15) to give

$$H\left\{p.v.\left(\frac{\cos ax}{x}\right)\right\} = -\pi\delta(x) + \operatorname{sgn} a \frac{\sin ax}{x}. \tag{10.21}$$

The preceding scheme involves somewhat less labor than the first approach described.

A third procedure involves splitting this transform into two terms, that is

$$H\left\{p.v.\left(\frac{\cos ax}{x}\right)\right\} = H\left\{p.v.\frac{1}{x}\right\} + H\left\{\frac{\cos ax - 1}{x}\right\}, \tag{10.22}$$

and evaluating $H\{(\cos ax - 1)/x\}$ (see Appendix 1, entry (5.113) of Table 1.5), leads to Eq. (10.21).

10.2 Some important spaces for distributions

This section introduces some of the important spaces that are employed to discuss the basic properties of distributions. Further elaboration is provided in the following sections as the need arises. The focus of the presentation in this chapter will be the Hilbert transform of distributions on $\mathbb{R}$. Accordingly, most of the following discussion is restricted to one-dimensional spaces. Much of what is given can be generalized to $\mathbb{R}^n$ in a straightforward fashion. In Section 2.15.2 the space $\mathcal{D}(\mathbb{R})$ of all C^∞ functions with compact support was introduced. This space is often abbreviated as $\mathcal{D}$. The alternative notation C_0^∞, or sometimes C_c^∞, is occasionally used in place of the Schwartz notation $\mathcal{D}$, but the latter is more common when the focus is the discussion of distributions, and will be employed in this book. A real-valued function $\phi(x)$ belongs to $\mathcal{D}$ if $\phi(x)$ and all its derivatives $\partial^k\phi(x)/\partial x^k$, $k = 1, 2, 3, \ldots$, exist, and there is a constant x_c such that for $|x| > x_c, \phi(x)$ vanishes. To help the reader in the remainder of this chapter, general distributions are represented usually by T and S, which is a commonly employed notation, and ordinary functions are typically designated in the familiar way as f and g. Test functions are commonly denoted by ϕ, φ, or θ.

Recall that a linear functional T on the vector space $\mathcal{D}$ satisfies

$$\langle T, \alpha\phi + \beta\varphi\rangle = \alpha\langle T, \phi\rangle + \beta\langle T, \varphi\rangle, \tag{10.23}$$

for $\alpha, \beta \in \mathbb{C}$ and $\phi, \varphi \in \mathcal{D}$, and that a continuous linear functional on the vector space $\mathcal{D}$ is called a distribution. The set of all distributions on $\mathcal{D}$ form a vector space which is denoted by $\mathcal{D}'$. The reader will need to be a little alert with the appearance of a prime, since it is used as shorthand for a derivative as well as a designation for a space of distributions. Context should make the intended meaning fairly obvious.

To characterize a particular distribution space it is necessary to specify a topology for the vector space of test functions. This is done by defining the concept of convergence on the vector space. For the Schwartz space $\mathcal{D}$ the following requirements define convergence. Let $\{\varphi_j\}$ be a sequence of functions such that $\varphi_j \in \mathcal{D}$ for all j. Then $\varphi_j \to \varphi$ with $\varphi \in \mathcal{D}$ as $j \to \infty$ if: (i) the supports of φ_j belong to a common bounded set, independent of j; (ii) the derivatives $\partial^m \varphi_j$ for any m converge uniformly to $\partial^m \varphi$ as $j \to \infty$. For a distribution space a topology may also be specified. If $\{T_j\}$ denotes a sequence of distributions, then T_j converges to T, $T_j \to T$ as $j \to \infty$, if for a $\phi \in \mathcal{D}$ the $\lim_{j\to\infty}\langle T_j, \phi\rangle = \langle T, \phi\rangle$. That is, the limit of the sequence T_j is a $T \in \mathcal{D}'$. For the following spaces that are introduced briefly, the topology is not specified. The reader is urged to consult one of the several books mentioned in the chapter end-notes for the appropriate details on this aspect of characterizing test-function and distributional spaces.

A space related to $\mathcal{D}$ is $\mathcal{D}_{L^p}$, which is defined for functions satisfying the two conditions:

(i) $\phi(x) \in C^\infty$,

(ii) $\dfrac{\partial^k \phi(x)}{\partial x^k} \in L^p(\mathbb{R}),\ k \in \mathbb{Z}^+, \quad \text{for } 1 < p < \infty.$

The dual space of $\mathcal{D}_{L^p}$ is denoted by $\mathcal{D}'_{L^q}$, where q is the conjugate exponent of p: $q = p(p-1)^{-1}$. A caution on notation is appropriate. It is wiser to use $\mathcal{D}'_{L^1}$ rather than writing $\mathcal{D}'_L$, since the latter notation is also used with another meaning (the space of left-sided distributions, that is those with supports bounded on the right) . The space $\mathcal{D}'_{L^q}$ satisfies the inclusion relation $\mathcal{D}'_{L^q} \subset \mathcal{D}'$. All summable functions belong to $\mathcal{D}'_{L^1}$, and any distribution with compact support also belongs to this space. The reader might anticipate, from the central role played by functions of the class $L^p(\mathbb{R})$ in the earlier development of the properties of the Hilbert transform, that the space $\mathcal{D}'_{L^q}$ will occupy an important place in the following discussions for the Hilbert transform of distributions. The space $\mathcal{D}'_{L^p}$ satisfies the following inclusion relation

$$\mathcal{D}'_{L^p} \subset \mathcal{D}'_{L^q}, \quad \forall p \le q. \tag{10.24}$$

The following result will prove to be useful for defining the Hilbert transform for a class of distributions. If $T \in \mathcal{D}'_{L^2}$ then

$$T(x) = \sum_j^m \frac{\partial^j h_j(x)}{\partial x^j}, \quad \text{for } h_j \in L^2. \tag{10.25}$$

A proof of this result can be found in Beltrami and Wohlers (1966a, p. 33). A more general result is the following. The distribution $T \in \mathcal{D}'_{L^p}$ with $1 \le p \le \infty$, iff there exists an $m > 0$ (depending on T) such that

$$T(x) = \sum_j^m \frac{\partial^j h_j(x)}{\partial x^j}, \quad \text{for } h_j \in L^p. \tag{10.26}$$

A proof for the case $1 < p \leq \infty$ is given by Barros-Neto (1973, p. 173). An example for the case $p = 1$ is as follows:

$$\delta(x) = \frac{1}{2}\left\{ e^{-|x|} - \frac{d^2 e^{-|x|}}{dx^2} \right\}. \tag{10.27}$$

First note that $e^{-|x|} \in L^1$. Making use of the two results

$$\frac{d}{dx}|x|^\alpha = \alpha|x|^{\alpha-1} \operatorname{sgn} x, \quad \text{for } \alpha > 0, \tag{10.28}$$

and

$$\frac{d}{dx} \operatorname{sgn} x = 2\delta(x), \tag{10.29}$$

then

$$\begin{aligned}
\frac{1}{2}\left\{ e^{-|x|} - \frac{d^2 e^{-|x|}}{dx^2} \right\} &= \frac{1}{2}\{e^{-|x|} - e^{-|x|}(\operatorname{sgn} x)^2 + 2e^{-|x|}\delta(x)\} \\
&= e^{-|x|}\delta(x) \\
&= \delta(x).
\end{aligned} \tag{10.30}$$

The space $\mathcal{D}'_+$ denotes those distributions on the space $\mathcal{D}'$ whose supports are to the right of some specified point. If $T \in \mathcal{D}'_+$, then $\operatorname{supp}(T) \in [a, \infty)$ for $a \in \mathbb{R}$. The situation of most interest is $a = 0$, and these distributions vanish on the negative real axis. In this case the distributions are sometimes referred to as *causal distributions* when the variable dependence is time. The space $\mathcal{D}'_+$ will be useful for discussing the connection of causal behavior and Hilbert transforms of distributions, a topic treated in Section 17.13.

The space $\mathscr{S}$ of test functions that have rapid decay is now introduced. A test function belonging to $\mathscr{S}$ satisfies the following:

(i) $\phi(x) \in C^\infty$,
(ii) $\phi(x)$ and all the derivatives $\phi^{(k)}(x), k = 1, 2, \ldots,$ vanish faster than any inverse power of $|x|$ as $|x| \to \infty$.

A comment on the notation for derivatives is appropriate. The customary prime notation becomes awkward for higher derivatives, and so the notation $\phi^{(k)}(x)$ is preferred. This also has the advantage of minimizing any possible conflicts with the use of a prime to denote a dual space. Where there is no risk of confusion, and lower-order derivatives are involved, prime notation is often employed. Condition (ii) can be cast as: for $k \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^+$, then

$$\lim_{|x|\to\infty} x^m \frac{\partial^k \phi(x)}{\partial x^k} = 0. \tag{10.31}$$

A good example of a function belonging to $\mathscr{S}$ is the Gaussian $\phi(x) = \mathrm{e}^{-x^2}$. From the definitions of the spaces $\mathcal{D}$ and $\mathscr{S}$, it is clear that

$$\mathcal{D} \subset \mathscr{S}. \tag{10.32}$$

A linear continuous functional T on the space $\mathscr{S}$ is called a *tempered distribution*, sometimes a distribution of *slow growth*, or a *temperate distribution*. The set of all such tempered distributions is denoted by $\mathscr{S}'$. The inclusion relation for the spaces $\mathcal{D}'$ and $\mathscr{S}'$ is

$$\mathscr{S}' \subset \mathcal{D}'. \tag{10.33}$$

An example of a distribution that belongs to $\mathcal{D}'$ but not to $\mathscr{S}'$ is $\sum_{k=-\infty}^{\infty} \delta(x-k)\mathrm{e}^{k^2}$, which can be seen by evaluating $\left\langle \sum_{k=-\infty}^{\infty} \delta(x-k), \phi(x) \right\rangle$ using a $\phi \in \mathcal{D}$ and a $\phi \in \mathscr{S}$ (try $\phi(x) = \mathrm{e}^{-x^2}$). Every slowly increasing function belongs to the space $\mathscr{S}'$. Also, the functions belonging to L^p, for $1 \leq p$, belong to $\mathscr{S}'$. One important property of the space $\mathscr{S}'$ is that Fourier transforms of distributions in $\mathscr{S}'$ also belong to $\mathscr{S}'$ (Zemanian, 1965, p. 185), which was the principal motivation for the introduction of this space. The Fourier transform of a distribution belonging to $\mathcal{D}'$ does not result in a distribution in the space $\mathcal{D}'$.

Some further spaces of interest will be employed in the discussion of distributions. The space of all C^∞ functions with arbitrary support on $\mathbb{R}$ is denoted by $\mathcal{E}$. The inclusion relation for the spaces $\mathcal{D}$, $\mathscr{S}$, and $\mathcal{E}$ is

$$\mathcal{D} \subset \mathscr{S} \subset \mathcal{E}. \tag{10.34}$$

The space of distributions having compact support is denoted by $\mathcal{E}'$. That is, $\mathcal{E}'$ is a subspace of $\mathcal{D}'$, and it is also a subspace of $\mathcal{D}'_{L^p}$. The inclusion connection is

$$\mathcal{E}' \subset \mathcal{D}'_{L^p} \subset \mathscr{S}' \subset \mathcal{D}'. \tag{10.35}$$

To facilitate discussion of Fourier transforms of distributions belonging to the space $\mathcal{D}'$, the space $\mathcal{Z}'$ is defined in the following way: if the distribution $T \in \mathcal{D}'$ then $\mathcal{F}T \in \mathcal{Z}'$. Because $\mathcal{F}T$ is not in general a distribution for an arbitrary $T \in \mathcal{D}', \mathcal{Z}'$ is called the space of *ultradistributions*. The space of functions whose Fourier transforms belong to $\mathcal{D}$ is denoted by $\mathcal{Z}$. These are regarded as testing functions. For the spaces $\mathcal{Z}$ and $\mathscr{S}$, the inclusion connection is

$$\mathcal{Z} \subset \mathscr{S}. \tag{10.36}$$

and for $\mathscr{S}'$ and $\mathcal{Z}'$ it is

$$\mathscr{S}' \subset \mathcal{Z}', \tag{10.37}$$

The reader is left to decide if $\mathcal{Z} \subset \mathcal{D}, \mathcal{D} \subset \mathcal{Z}$, or neither result holds.

To discuss the case of periodic distributions, the space $\mathcal{P}_\tau$ of periodic test functions of period τ is employed. The functions in $\mathcal{P}_\tau$ are taken to be infinitely smooth. The space of periodic distributions of period τ is denoted by $\mathcal{P}'_\tau$. The periodic property for a periodic distribution T is understood in the sense that, for every $\phi \in \mathcal{D}$,

$$\langle T(t-\tau), \phi(t)\rangle = \langle T(t), \phi(t)\rangle. \tag{10.38}$$

The space $\mathcal{D}_{L^p}$ and its dual $\mathcal{D}'_{L^q}$ have a close connection with Sobolev spaces. The Sobolev space is denoted by $W^{p,m}$, and the notations L^p_m and $H_{m,p}$ are also commonly employed. It is defined as the space of functions f for which $f \in L^p$ and $\partial^k f/\partial x^k \in L^p$, for $k \leq m$ with $m \geq 0$ and $1 \leq p$. The derivatives are taken in the distributional sense, so $g = \partial^k f/\partial x^k$ is taken to be shorthand for

$$\int_{-\infty}^{\infty} g(x)\varphi(x)\mathrm{d}x = (-1)^k \int_{-\infty}^{\infty} f(x)\frac{\partial^k \varphi(x)}{\partial x^k}\,\mathrm{d}x, \tag{10.39}$$

with $\varphi \in \mathcal{D}$. An integration by parts k times connects the two integrals. Derivatives taken in this manner are termed *in the weak sense* or simply *weak derivatives*. Equation (10.39) can be used to define a generalized derivative. The Sobolev norm is defined by

$$\|f\|_{W^{p,m}} \equiv \left(\sum_{k=0}^{m}\int_{-\infty}^{\infty}\left|\frac{\partial^k f}{\partial x^k}\right|^p \mathrm{d}x\right)^{1/p} = \left(\sum_{k=0}^{m}\left\|\frac{\partial^k f}{\partial x^k}\right\|_{L^p}^p\right)^{1/p}. \tag{10.40}$$

The Sobolev spaces serve as a valuable means by which to study the smoothness of functions.

The particular case $W^{p,0}$ is just L^p. A useful result is the fact that a distribution T belongs to the space $\mathcal{D}'_{L^2}$ if and only if $T \in W'^{2,m}$ for some m, where $W'^{2,m}$ is the dual space of $W^{2,m}$. The space $\mathcal{D}'_{L^2}$ can be equated to the union of the spaces $W'^{2,m}$ for $m \geq 0$.

Another space that arises is $\mathcal{O}'_C$, the space of distributions that decrease rapidly at infinity. A distribution that decreases faster at infinity than any inverse power is said to be rapidly decreasing at infinity. For example, if $(1+x^2)^k T$ is bounded for any k, then T is rapidly decreasing. This space is useful for mapping the Fourier transform of a convolution of two distributions into the product of the Fourier transforms of the individual distributions. The space $\mathcal{O}'_C$ is defined as follows. If the distribution $T \in \mathcal{D}'$ satisfies, for real numbers k, the following,

$$(1 + x^2)^k T \in \mathcal{D}'_{L^\infty}, \tag{10.41}$$

then $T \in \mathcal{O}'_C$. Every distribution with compact support belongs to $\mathcal{O}'_C$. The inclusion connection for $\mathcal{O}'_C$ is

$$\mathcal{E}' \subset \mathcal{O}'_C \subset \mathcal{D}'_{L^p}. \tag{10.42}$$

This section concludes with a comment on terminology. It was indicated in Section 2.15 that the terms distribution and generalized function are often used synonymously by many writers. Some authors make a distinction by using the terminology distribution, tempered distribution, and ultradistribution to refer, respectively, to functionals belonging to the spaces $\mathcal{D}'$, $\mathcal{S}'$, and $\mathcal{Z}'$, and the terminology generalized function is used to refer collectively to these three cases, the other functionals that have been introduced, as well as some that have not been discussed.

10.3 Some key distributions

A few of the properties of some of the key distributions that arise in practical applications are discussed in this section. An important distribution employed in discussions of the Hilbert transform is $p.v.(1/x)$, and two other distributions that enter into many problems in physics and engineering are the Dirac delta distribution $\delta(x)$ and the Heaviside step distribution. In the physical sciences literature the term "distribution" in both of the preceding names is most commonly replaced by "function." The principal objectives of this section are to establish that these are indeed distributions and to ascertain the appropriate spaces to which they belong. The reader is reminded that, while it is often useful to write distributions in the form just given, in formal calculations the forms $\langle p.v.(1/x), \phi\rangle$ and $\langle \delta, \phi\rangle$ are employed, where ϕ is a suitable test function. A word on notation is in order. Writing $\delta(x)$ gives the appearance that the Dirac delta distribution is a function, which of course it is not. It is clearly better to write $\langle \delta, \phi\rangle$ rather than $\langle \delta(x), \phi(x)\rangle$. In the equation

$$\langle \delta, \phi\rangle = \langle \delta(x), \phi(x)\rangle = \phi(0), \tag{10.43}$$

the middle factor is an informal mode of writing, and, although it occurs widely, it is better simply to write

$$\langle \delta, \phi\rangle = \phi(0). \tag{10.44}$$

It is rather convenient in particular cases to write a variable dependence, as in the case $\langle p.v.(1/x), \phi(x)\rangle$. The reader needs to keep in mind that this notational expediency does not of course imply that $p.v.(1/x)$ is an ordinary function. Suppose T is a distribution that has a dependence on the variable x, then one commonly employed notational mode signifies this dependence by writing T_x. The use of the subscript avoids writing $T(x)$, and giving T the appearance of being an ordinary function, as the notation might suggest. Having indicated the ideal formal symbolism, it is to be noted that the notation $T(x)$ is in very widespread use, and will also be employed in the following discussion. This puts an added burden on the reader to pay careful attention to distinguish distributions from ordinary functions. The context should always make this distinction readily apparent. It will be convenient in some situations to carry out symbolic manipulations with distributions. In these calculations the action of the distribution on some test function is not explicitly displayed. However, it is assumed that the reader can readily formalize the calculation by inserting the appropriate $\langle T, \phi\rangle$ factors, where ϕ is a suitable test function and T is the distribution under consideration.

The function x^{-1} does not define a distribution because it is not integrable in the neighborhood of $x = 0$. The distribution $p.v.(1/x)$ is defined by

$$\left\langle p.v.\frac{1}{x}, \phi\right\rangle = P\int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,\mathrm{d}x = \lim_{\varepsilon\to 0+}\int_{|x|\geq\varepsilon} \frac{\phi(x)}{x}\,\mathrm{d}x. \tag{10.45}$$

This definition of $p.v.(1/x)$ clearly satisfies a linear condition on the ϕ. Continuity is examined in the following manner. Recall the convergence requirement for distributions. A sequence of distributions T_n converges to a distribution $T \in \mathcal{D}'$ if

$$\lim_{n\to\infty}\langle T_n, \phi\rangle = \langle T, \phi\rangle, \quad \text{for every } \phi \in \mathcal{D}. \tag{10.46}$$

Analogously for the case of a continuous variable, the following can be stated: if T_ε for $0 < \varepsilon < a$ denotes a family of distributions, then, for each $\phi \in \mathcal{D}$, T_ε converges to a distribution $T \in \mathcal{D}'$ if

$$\lim_{\varepsilon\to 0}\langle T_\varepsilon, \phi\rangle = \langle T, \phi\rangle, \tag{10.47}$$

assuming $\langle T, \phi\rangle$ exists.

Suppose ϕ_k converges to ϕ as $k \to \infty$ in $\mathcal{D}$. Assume the interval $(-a, a)$ contains the supports of the ϕ_k, and that $\psi_k = \phi_k - \phi$; then

$$P\int_{-\infty}^{\infty} \frac{\psi_k(x)}{x}\,\mathrm{d}x = P\int_{-a}^{a} \frac{\psi_k(0)}{x}\,\mathrm{d}x + \int_{-a}^{a} \frac{\psi_k(x) - \psi_k(0)}{x}\,\mathrm{d}x. \tag{10.48}$$

Employing the *mean value theorem* leads to

$$\left|\frac{\psi_k(x) - \psi_k(0)}{x}\right| \leq \max\left|\psi_k'(\xi)\right|, \tag{10.49}$$

with $\xi \in (0, x)$, so that Eq. (10.48) can be rewritten as follows:

$$\left| P \int_{-\infty}^{\infty} \frac{\psi_k(x)}{x} \, \mathrm{d}x \right| \le 2a \max \left| \psi_k'(\xi) \right|, \tag{10.50}$$

and $\xi \in (-a, a)$. The right-hand side of this result $\to 0$ as $k \to \infty$. It follows that $p.v.(1/x)$ is a distribution belonging to $\mathcal{D}'$.

The definition of the derivative of a distribution T takes the following form:

$$\left\langle \frac{\partial T}{\partial x}, \phi \right\rangle = -\left\langle T, \frac{\partial \phi}{\partial x} \right\rangle. \tag{10.51}$$

This definition is consistent with the standard formula from calculus for integration by parts. The derivative of the distribution $\log|x|$ leads to

$$\left\langle \frac{\partial \log|x|}{\partial x}, \phi \right\rangle = \left\langle p.v.\frac{1}{x}, \phi \right\rangle, \quad \forall \phi \in \mathcal{D}, \tag{10.52}$$

which is often written symbolically as follows:

$$\frac{\partial \log|x|}{\partial x} = p.v.\frac{1}{x}. \tag{10.53}$$

The Heaviside distribution, frequently referred to as the unit step function, is defined by

$$\mathrm{H}(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases} \tag{10.54}$$

A second definition is also in widespread use where a value is assigned for $x = 0$; this is discussed further in Section 18.7.1. The derivative of the Heaviside distribution can be written as

$$\langle \mathrm{H}', \phi \rangle = -\langle \mathrm{H}, \phi' \rangle = -\int_0^{\infty} \phi'(x) \mathrm{d}x = \phi(0) = \langle \delta, \phi \rangle; \tag{10.55}$$

that is, symbolically,

$$\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{H}(x). \tag{10.56}$$

The Dirac delta distribution belongs to $\mathcal{D}'$. Since $\delta(x) = 0$ for $x \neq 0$, it follows that $\operatorname{supp} \delta = \{0\}$. In Section 2.15.2 it was indicated that δ is a singular distribution. To see this, suppose there is a function f that belongs to L^1_{loc} and a test function $\phi \in \mathcal{D}$, such that

$$\langle \delta, \phi \rangle = \int_{-\infty}^{\infty} f(x) \phi(x) \mathrm{d}x = \phi(0). \tag{10.57}$$

Now, $|x|^2\,\phi \in \mathcal{D}$, and so it follows that

$$\int_{-\infty}^{\infty} f(x)|x|^2\phi(x)\mathrm{d}x = 0 = \langle |x|^2 f(x), \phi(x)\rangle. \tag{10.58}$$

This requires $|x|^2 f(x) = 0\,a.e.$, and therefore $f(x) = 0\,a.e.$ This leads to a contradiction: there is no function that belongs to L^1_{loc} for which Eq. (10.57) holds, and hence, δ is not a regular distribution.

The Heisenberg delta distributions given in Eqs. (10.9) and (10.10) can be obtained in the following manner:

$$\begin{aligned} \lim_{y\to\pm 0} \log(x+\mathrm{i}y) &= \lim_{y\to\pm 0}\{\log|x+\mathrm{i}y| + \mathrm{i}\arg(x+\mathrm{i}y)\} \\ &= \log|x| + \mathrm{i}\lim_{y\to\pm 0}\tan^{-1}\left(\frac{y}{x}\right) \\ &= \log|x| \pm \mathrm{i}\pi\mathrm{H}(-x). \end{aligned} \tag{10.59}$$

To see the preceding limit, it may be helpful to the reader to examine Figure 10.1, which illustrates the behavior of $\pi^{-1}\tan^{-1}(\varepsilon/x)$ for different small values of ε.

Taking the derivative of Eq. (10.59) leads to

$$\lim_{y\to\pm 0}\frac{1}{x+\mathrm{i}y} = p.v.\frac{1}{x} \mp \mathrm{i}\pi\delta(x), \tag{10.60}$$

and hence Eqs. (10.9) and (10.10) follow.

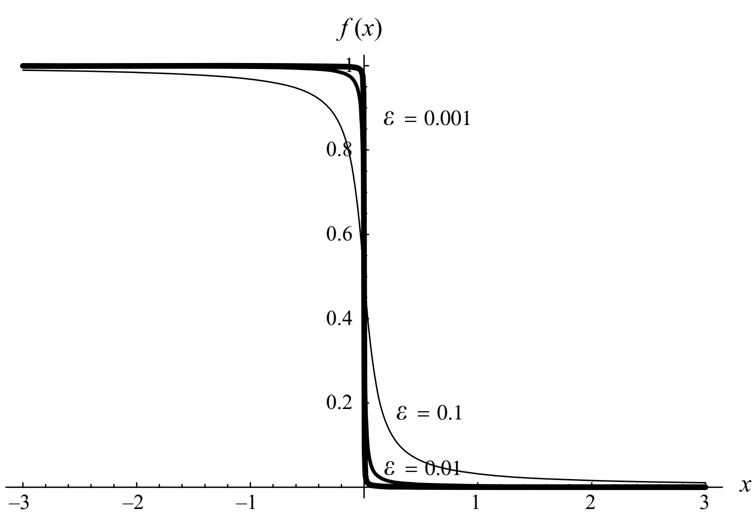

Figure 10.1. Plot of $f(x) = \pi^{-1}\tan^{-1}(\varepsilon/x)$ for ε = 0.1, 0.01, and 0.001.

10.4 The Fourier transform of some key distributions

This section examines the Fourier transform of several distributions that arise widely in problems in the physical sciences and engineering. The focus will be on a couple of examples that are important for discussing the Hilbert transform. The key spaces for treating Fourier transforms are the Schwartz spaces $\mathscr{S}$ and $\mathscr{S}'$. There is a simple reason for introducing these spaces. If $\phi \in \mathcal{D}$, it would be desirable if $\mathcal{F}\phi \in \mathcal{D}$. However, this result is in general false, which can be seen from the following argument. Let the supp of ϕ be the interval $[a, b]$; then

$$\mathcal{F}\phi(s) = \int_a^b \mathrm{e}^{-\mathrm{i}xs}\phi(x)\mathrm{d}x, \tag{10.61}$$

and $(\mathcal{F}\phi)(s)$ is an entire function of s. It follows that the supp of $(\mathcal{F}\phi)(s)$ is $(-\infty, \infty)$. An entire function cannot vanish over any finite interval unless it vanishes identically, which leads to the restriction $(\mathcal{F}\phi)(s) = 0$, that is $\phi(s) = 0$. Hence, in general, $(\mathcal{F}\phi)(s) \notin \mathcal{D}$. To surmount this difficulty, Schwartz introduced the spaces $\mathscr{S}$ and $\mathscr{S}'$. Some examples of distributions in the space $\mathscr{S}'$ are $\log|x|$, δ, and $p.v.(1/x)$.

If $T \in \mathscr{S}'$ and $\phi \in \mathscr{S}$, then $\mathcal{F}T$ is defined by

$$\langle \mathcal{F}T, \phi\rangle = \langle T, \mathcal{F}\phi\rangle, \quad \forall\phi \in \mathscr{S}, \tag{10.62}$$

which has a formal correspondence to the Parseval relation for ordinary functions. In analogy once again with Parseval's relation for ordinary functions, it follows that

$$\langle T, \phi\rangle = \langle \mathcal{F}T, \mathcal{F}\phi\rangle. \tag{10.63}$$

A distribution T with bounded support is a tempered distribution. If T has bounded support, then $T \in \mathcal{E}'$, and, from the inclusion connection $\mathcal{E}' \subset \mathscr{S}'$, the preceding statement follows. An important property is that the Fourier transform of every tempered distribution is also a tempered distribution (Bremermann, 1965a, p. 86; Zemanian, 1965, p. 185). A function $f(x)$ is said to be of *slow growth* if f and its derivatives are of polynomial growth as $x \to \infty$, that is

$$\left|f^{(k)}(x)\right| \le C|x|^m, \text{ as } x \to \infty, \quad \text{for } k \ge 0, \tag{10.64}$$

where C and m are constants and $m > 0$. A tempered distribution can be constructed from every function of slow growth via the formula

$$\langle T, \phi\rangle = \int_{-\infty}^{\infty} T(x)\phi(x)\mathrm{d}x, \quad \forall\phi \in \mathscr{S}. \tag{10.65}$$

Every tempered distribution T can be expressed as a finite order derivative:

$$T = f^{(k)}(x), \tag{10.66}$$

where f is a continuous function of polynomial growth. Consider the example

$$f(x) = x \log |x| - x; \tag{10.67}$$

then

$$p.v.\frac{1}{x} = f''(x). \tag{10.68}$$

As a second example, consider the function

$$f(x) = \begin{cases} x, & x > 0 \\ 0, & x < 0; \end{cases} \tag{10.69}$$

then

$$\delta = f'', \tag{10.70}$$

and Eqs. (10.54) and (10.56) have been employed.

Attention is now directed to the evaluation of the Fourier transform of some distributions that have wide occurrence in problems. For the Dirac delta distribution, it follows, on using Eq. (10.62), that

$$\begin{aligned} \langle \mathcal{F}\delta, \phi \rangle &= \left\langle \delta(x), \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xt} \phi(t) \mathrm{d}t \right\rangle \\ &= \int_{-\infty}^{\infty} \phi(t) \mathrm{d}t \\ &= \langle 1, \phi \rangle, \end{aligned} \tag{10.71}$$

and in symbolic form it follows that

$$\mathcal{F}\delta = 1, \tag{10.72}$$

where 1 is the unit distribution. In a similar fashion,

$$\begin{aligned} \langle \mathcal{F}\mathrm{e}^{\mathrm{i}ax}, \phi \rangle &= \left\langle \mathrm{e}^{\mathrm{i}ax}, \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xt} \phi(t) \mathrm{d}t \right\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ax} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xt} \phi(t) \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \phi(t) \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xt+\mathrm{i}ax} \, \mathrm{d}x \\ &= \langle 2\pi \delta(t-a), \phi(t) \rangle; \end{aligned} \tag{10.73}$$

that is, symbolically,

$$\mathcal{F}(e^{iat}) = 2\pi\,\delta(x-a), \tag{10.74}$$

and, for $a = 0$,

$$\mathcal{F}(1) = 2\pi\,\delta. \tag{10.75}$$

When comparing with other sources, the reader might note some differences for the Fourier transform of a particular distribution. Most common is either the absence or appearance of a factor of 2π, or a difference in sign. These changes reflect the use of a different definition of the Fourier transform; see Eqs. (2.46) and (2.48)–(2.50).

An important distribution employed in discussing Hilbert transforms is $p.v.(1/x)$, and the Fourier transform of this singular distribution is evaluated as follows:

$$\begin{aligned}\left\langle \mathcal{F}p.v.\frac{1}{x}, \phi \right\rangle &= \left\langle p.v.\frac{1}{x}, \int_{-\infty}^{\infty} e^{-ixt}\phi(t)\mathrm{d}t \right\rangle \\ &= \int_{-\infty}^{\infty} \phi(t)\mathrm{d}t\, P\int_{-\infty}^{\infty} \frac{e^{-ixt}}{x}\,\mathrm{d}x \\ &= -i\int_{-\infty}^{\infty} \phi(t)\mathrm{d}t \int_{-\infty}^{\infty} \frac{\sin tx}{x}\,\mathrm{d}x \\ &= -i\pi \int_{-\infty}^{\infty} \operatorname{sgn} t\,\phi(t)\mathrm{d}t \\ &= \langle -i\pi \operatorname{sgn} t, \phi(t)\rangle, \end{aligned} \tag{10.76}$$

and, in symbolic form,

$$\mathcal{F}p.v.\frac{1}{x} = -i\pi \operatorname{sgn} x. \tag{10.77}$$

The Fourier transform of the distribution $\operatorname{sgn} x$ is given by

$$\begin{aligned}\langle \mathcal{F}\operatorname{sgn} x, \phi\rangle &= \left\langle \operatorname{sgn} x, \int_{-\infty}^{\infty} e^{-ixt}\phi(t)\mathrm{d}t \right\rangle \\ &= \int_{-\infty}^{\infty} \operatorname{sgn} x\,\mathrm{d}x \int_{-\infty}^{\infty} e^{-ixt}\phi(t)\mathrm{d}t \\ &= \frac{1}{i\pi}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \frac{e^{ixy}}{y}\,\mathrm{d}y \int_{-\infty}^{\infty} e^{-ixt}\phi(t)\mathrm{d}t \\ &= \frac{1}{i\pi}\int_{-\infty}^{\infty} \phi(t)\mathrm{d}t \int_{-\infty}^{\infty} p.v.\frac{1}{y}\,\mathrm{d}y \int_{-\infty}^{\infty} e^{-ix(t-y)}\,\mathrm{d}x \end{aligned}$$

$$= -2\mathrm{i}\int_{-\infty}^{\infty}\phi(t)\mathrm{d}t\int_{-\infty}^{\infty}p.v.\frac{1}{y}\delta(t-y)\mathrm{d}y$$

$$= -2\mathrm{i}\int_{-\infty}^{\infty}p.v.\frac{1}{t}\phi(t)\mathrm{d}t$$

$$= \left\langle -2\mathrm{i}\,p.v.\frac{1}{t},\phi(t)\right\rangle, \tag{10.78}$$

and hence

$$\mathcal{F}\operatorname{sgn}x = -2\mathrm{i}\;p.v.\frac{1}{x}. \tag{10.79}$$

The Fourier transform of the Heaviside distribution follows in a straightforward fashion:

$$\mathcal{F}H(x) = \mathcal{F}\frac{1}{2}\{1+\operatorname{sgn}y\}(x)$$
$$= \pi\left\{\delta(x)+\frac{1}{\pi\mathrm{i}}p.v.\frac{1}{x}\right\}. \tag{10.80}$$

10.5 A Parseval-type formula approach to *HT*

The Hilbert transform of a distribution T that satisfies $T \in \mathcal{D}'_{L^p}$ is now examined. First recall one of the classical results satisfied by the Hilbert transform operator. If $f_1 \in L^p(\mathbb{R})$ and $f_2 \in L^q(\mathbb{R})$, with $1 < p < \infty$, and q is the conjugate exponent of p, then the Parseval-type formula for the Hilbert transform is given by

$$\int_{-\infty}^{\infty}Hf_1(x)f_2(x)\mathrm{d}x = -\int_{-\infty}^{\infty}f_1(x)\,Hf_2(x)\mathrm{d}x, \tag{10.81}$$

or, in compact notation, anticipating the jump to distributions,

$$\langle Hf_1,f_2\rangle = \langle f_1,-Hf_2\rangle. \tag{10.82}$$

In the remainder of this chapter the symbols p and q will be employed as defined in the preceding sentence. For distributions $T \in \mathcal{D}'_{L^p}$, the Hilbert transform HT can be defined (see Pandey (1996), p. 96), by the following relationship;

$$\langle HT,\phi\rangle = \langle T,-H\phi\rangle, \tag{10.83}$$

for all test functions ϕ that satisfy $\phi \in \mathcal{D}_{L^q}$. Hence, the distribution that results when the Hilbert transform operator is applied to the distribution T is calculated formally by forming the inner product between T and the new test function $-H\phi$.

Consider the evaluation of $H\delta$. It follows $\forall\phi \in \mathcal{D}_{L^q}$ that

$$\begin{aligned}\langle H\delta, \phi\rangle &= \langle \delta, -H\phi\rangle \\ &= -H\phi(0) \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,\mathrm{d}x \\ &= \left\langle \frac{1}{\pi} p.v.\frac{1}{x}, \phi(x) \right\rangle,\end{aligned} \tag{10.84}$$

and, in symbolic form,

$$H\delta = \frac{1}{\pi} p.v.\frac{1}{x}. \tag{10.85}$$

For the evaluation of $H(p.v.(1/x))$, it follows, on writing $\varphi(x) = H\phi(x)$, that

$$\begin{aligned}\left\langle H\left(p.v.\frac{1}{x}\right), \phi \right\rangle &= -\left\langle p.v.\frac{1}{x}, \varphi(x) \right\rangle \\ &= -P \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t\, P \int_{-\infty}^{\infty} \frac{\varphi(x)}{t-x}\,\mathrm{d}x \\ &= \pi \int_{-\infty}^{\infty} \delta(t)\, H\varphi(t)\mathrm{d}t \\ &= \pi \int_{-\infty}^{\infty} \delta(t)\, H^2\phi(t)\mathrm{d}t \\ &= -\pi\,\langle \delta, \phi\rangle,\end{aligned} \tag{10.86}$$

and, in symbolic form,

$$H\, p.v.\frac{1}{x} = -\pi\,\delta. \tag{10.87}$$

The inversion property of the Hilbert transform has been employed to obtain the final step in Eq. (10.86).

10.6 Convolution operation for distributions

In this section the definition of the Hilbert transform of a distribution is established by way of a convolution formula. Some preliminary results are needed. Given a test function $\phi(x,y) \in \mathcal{D}(\mathbb{R}^2)$ and distributions $T(x) \in \mathcal{D}'$ and $S(y) \in \mathcal{D}'$, the *tensor product* of these two distributions is denoted $T \otimes S$, and represents a distribution belonging to $\mathcal{D}'(\mathbb{R}^2)$ that is defined by

$$\langle T(x) \otimes S(y), \phi(x,y)\rangle = \langle T(x), \langle S(y), \phi(x,y)\rangle\rangle. \tag{10.88}$$

The tensor product is also called the *direct product*, denoted by $T \times S$. This definition is intuitive provided the function $\psi(x)$ defined by

$$\psi(x) = \langle S(y), \phi(x,y)\rangle \tag{10.89}$$

satisfies $\psi(x) \in \mathcal{D}$. For a demonstration of this, see Kanwal (1998, p. 173). As an example, consider the tensor product of the Dirac delta distribution with itself; then,

$$\begin{aligned}\langle \delta(x) \otimes \delta(y), \phi(x,y)\rangle &= \langle \delta(x), \langle \delta(y), \phi(x,y)\rangle\rangle \\ &= \langle \delta(x), \phi(x,0)\rangle \\ &= \phi(0,0). \end{aligned} \tag{10.90}$$

The tensor product $\delta(x) \otimes \delta(y)$ is sometimes written as $\delta(x,y)$, which is the delta distribution on $\mathbb{R}^2$. The reader should note that nothing can be inferred from this example about the meaning of the product of two Dirac delta distributions, that is $\delta^2(x)$. No useful way has been found to attach a meaning to this product.

The tensor product satisfies the following commutative condition:

$$\langle T(x) \otimes S(y), \phi(x,y)\rangle = \langle S(y) \otimes T(x), \phi(x,y)\rangle. \tag{10.91}$$

The associative condition for the tensor product is given by

$$\{T(x) \otimes S(y)\} \otimes R(z) = T(x) \otimes \{S(y) \otimes R(z)\}. \tag{10.92}$$

The support of $T(x) \otimes S(y)$ is given by

$$\operatorname{supp}\{T(x) \otimes S(y)\} = \operatorname{supp}\{T(x)\} \times \operatorname{supp}\{S(y)\}. \tag{10.93}$$

The reader should note that the ordinary product of distributions is not in general defined, but when such a product exists it is not necessarily associative. That is, if R, S, and T denote distributions, then in general

$$R(ST) \neq (RS)T. \tag{10.94}$$

For example,

$$\delta\left\{x\, p.v.\frac{1}{x}\right\} = \delta \cdot 1 = \delta, \tag{10.95}$$

whereas

$$\{\delta x\} p.v.\frac{1}{x} = 0 \cdot p.v.\frac{1}{x} = 0. \tag{10.96}$$

The following results have been employed to obtain the preceding two formulas:

$$x\, p.v.\frac{1}{x} = 1 \tag{10.97}$$

and

$$x\delta(x) = 0. \tag{10.98}$$

An example where the product of distributions leads to an obvious contradiction is the following situation:

$$\delta(x) = \delta(x) \cdot 1 = \delta(x) \left\{ x\, p.v.\frac{1}{x} \right\} = \{\delta(x) x\}\; p.v.\frac{1}{x} = \{x\delta(x)\}\; p.v.\frac{1}{x} = 0 \cdot p.v.\frac{1}{x} = 0. \tag{10.99}$$

The fallacy in Eq. (10.99) is assuming the associative property holds, but Eqs. (10.95) and (10.96) demonstrate that it does not.

The concept of the tensor product can be employed to attach a meaning to the convolution of distributions. The tensor product concept will be applied in Section 15.13 to discuss the Hilbert transform of distributions in n dimensions.

In Section 4.9 the convolution property for ordinary functions was considered, and that discussion is now extended to include distributions. Let $\phi \in \mathcal{D}$, then the convolution of two distributions T and S is defined by

$$\langle T * S, \phi \rangle = \langle T(x) \otimes S(y), \phi(x+y) \rangle. \tag{10.100}$$

The convolution of two distributions can also be defined in the following way:

$$\begin{aligned}
\langle T * S, \phi \rangle &= \int_{-\infty}^{\infty} (T * S)(t)\phi(t)\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} T(\tau)S(t-\tau)\mathrm{d}\tau \right\} \phi(t)\mathrm{d}t \\
&= \int_{-\infty}^{\infty} T(\tau) \left\{ \int_{-\infty}^{\infty} S(t-\tau)\phi(t)\mathrm{d}t \right\} \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} T(\tau) \left\{ \int_{-\infty}^{\infty} S(x)\phi(x+\tau)\mathrm{d}x \right\} \mathrm{d}\tau.
\end{aligned} \tag{10.101}$$

The last line of Eq. (10.101) can be written in more compact notation as follows:

$$\langle T * S, \phi \rangle = \langle T(x), \langle S(y), \phi(x+y) \rangle \rangle. \tag{10.102}$$

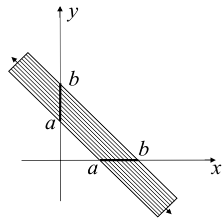

Figure 10.2. Support for the function $\phi(x+y)$.

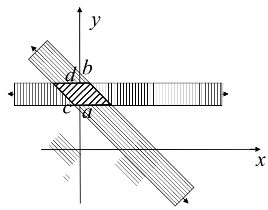

Figure 10.3. Region of bounded support for supp $\{T(x) \otimes S(y)\} \cap$ supp $\phi(x+y)$.

This formula makes the connection with the tensor product definition given in Eq. (10.100). One issue remains: the function $\phi(x+y)$ is not a test function, since it does not have compact support. The support of $\phi(x+y)$ is not bounded in the xy-plane. If supp $\phi(x) = [a, b]$, then supp $\phi(x+y)$ is the strip shown in Figure 10.2, where $a \leq x+y \leq b$.

Suppose supp $T(x)$ is unbounded but that supp $S(y) = [c, d]$; then, from Eq. (10.93), it follows that supp$\{T(x) \otimes S(y)\} \cap$ supp $\phi(x+y)$ will correspond to the cross-hatched region in Figure 10.3.

The factor supp$\{T(x) \otimes S(y)\} \cap$ supp $\phi(x+y)$ is bounded for any $\phi \in \mathcal{D}$, and hence $\phi(x+y)$ in Eq. (10.100) can be replaced by $\eta(x,y)\phi(x+y)$, where the function $\eta(x,y)$ is defined to be one in the region where supp $\{T(x) \otimes S(y)\} \cap$ supp $\phi(x+y)$ is bounded, and zero otherwise. The function $\eta(x,y)$ is a test function in $\mathcal{D}(\mathbb{R}^2)$.

Let R, S, and T denote three distributions; then, two different convolution products, $R * (S * T)$ and $(R * S) * T$, can be formed. Assuming for the moment that both of these exist, then in general

$$R * (S * T) \neq (R * S) * T, \tag{10.103}$$

and the associative condition for the convolution of distributions does not hold. A well known example involving commonly occurring distributions is as follows:

$$(1 * \delta') * \mathrm{H} = 0 * \mathrm{H} = 0 \tag{10.104}$$

and

$$1 * (\delta' * \mathrm{H}) = 1 * \delta = 1. \tag{10.105}$$

The terminology bounded from the left and bounded from the right are defined as follows. A distribution is said to have its support bounded from the left if it is contained in the interval (a, ∞), and it is bounded from the right if its support is contained in $(-\infty, b)$. If the supports of all but possibly one of several distributions are bounded, then the convolution product of the distributions is both commutative and associative. The same results holds if all the distributions have their supports bounded from the left or, similarly, bounded from the right. As an exercise for the reader, consider the situations where supp $T(x)$ and supp $S(y)$ are both bounded, or where supp $T(x)$ and supp $S(y)$ are both bounded on the left, or both bounded on the right, determine for any $\phi \in \mathcal{D}$ the region where supp $\{T(x) \otimes S(y)\} \cap \text{supp}\, \phi(x+y)$ is bounded.

If S and T are two distributions and $S \in \mathcal{D}'$ and $T \in \mathcal{E}'$, then $S * T \in \mathcal{D}'$. If $S \in \mathcal{S}'$ and $T \in \mathcal{E}'$, then $S * T \in \mathcal{S}'$. If $S \in \mathcal{E}'$ and $T \in \mathcal{E}'$, then $S * T \in \mathcal{E}'$. If S and T belong to $\mathcal{S}'$, then $S * T$ is not necessarily a distribution. For further discussion and proofs of some of these assertions, see Zemanian (1965, p. 122) and Gasquet and Witomski (1999, p. 297). The convolution of distributions is now exploited to set up a definition of the Hilbert transform of a distribution.

10.7 Convolution and the Hilbert transform

In this section a common definition of the Hilbert transform of a distribution is introduced. The distribution $p.v.(1/x)$ enters in an obvious way, so some results about this distribution are considered first. The distribution T belongs to $\mathcal{D}'_{L^p}$ for $1 \leq p \leq \infty$, iff there exists an expansion of the following form:

$$T = \sum_{k}^{m} h_k^{(k)}, \quad \text{for } h_k \in L^p. \tag{10.106}$$

For the distribution $p.v.(1/x)$,

$$p.v.\frac{1}{x} = h(x) - h''(x), \tag{10.107}$$

where

$$h(x) = \int_0^\infty \frac{\sin xt \, \mathrm{d}t}{1+t^2}. \tag{10.108}$$

To establish Eq. (10.107), start with Eq. (10.77) and take the inverse Fourier transform to obtain the symbolic correspondence

$$p.v.\frac{1}{x} = -\frac{\mathrm{i}}{2} \int_{-\infty}^{\infty} \operatorname{sgn} t \, \mathrm{e}^{\mathrm{i}xt} \, \mathrm{d}t, \tag{10.109}$$

which can be recast as follows:

$$
\begin{aligned}
p.v.\frac{1}{x} &= -\frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} \frac{\operatorname{sgn} t \left(1 - \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\right) \mathrm{e}^{\mathrm{i}xt}}{1+t^2}\,\mathrm{d}t \\
&= -\frac{\mathrm{i}}{2}\left(1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\int_{-\infty}^{\infty} \frac{\operatorname{sgn} t\, \mathrm{e}^{\mathrm{i}xt}}{1+t^2}\,\mathrm{d}t \\
&= \left(1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\int_{0}^{\infty} \frac{\sin xt}{1+t^2}\,\mathrm{d}t.
\end{aligned}
\tag{10.110}
$$

Using the identification in Eq. (10.83), it remains to show that $h \in L^p$ for $1 < p$. Now,

$$
|h(x)| = \left|\int_0^{\infty} \frac{\sin xt\,\mathrm{d}t}{1+t^2}\right| \le \left|\int_0^{\infty} \frac{\mathrm{d}t}{1+t^2}\right| = \frac{\pi}{2},
\tag{10.111}
$$

and, using an integration by parts, it follows for $x \neq 0$ that

$$
\begin{aligned}
|h(x)| &= \frac{1}{|x|}\left|1 - \int_0^{\infty} \frac{2t\cos xt\,\mathrm{d}t}{(1+t^2)^2}\right| \\
&\le \frac{C}{|x|},
\end{aligned}
\tag{10.112}
$$

where C is a constant. Hence, for some $\alpha > 0$ and making use of Eqs. (10.111) and (10.112), leads to

$$
\begin{aligned}
\int_{-\infty}^{\infty} |h(x)|^p\,\mathrm{d}x &= \int_{-\infty}^{-\alpha} |h(x)|^p\,\mathrm{d}x + \int_{\alpha}^{\infty} |h(x)|^p\,\mathrm{d}x + \int_{-\alpha}^{\alpha} |h(x)|^p\,\mathrm{d}x \\
&\le C\int_{-\infty}^{-\alpha} |x|^{-p}\,\mathrm{d}x + C\int_{\alpha}^{\infty} |x|^{-p}\,\mathrm{d}x + \frac{\pi^p}{2^p}\int_{-\alpha}^{\alpha}\mathrm{d}x \\
&< \infty,
\end{aligned}
\tag{10.113}
$$

and hence $h \in L^p$ for $1 < p$. It therefore follows that $p.v.(1/x) \in \mathcal{D}'_{L^p}$.

An important result is the following. If $T \in \mathcal{D}'_{L^2}$ and $S \in \mathcal{D}'_{L^2}$, then the convolution $T * S$ exists and

$$
\mathcal{F}\{T * S\} = \mathcal{F}T\mathcal{F}S.
\tag{10.114}
$$

This can be established by writing

$$
T = \sum_{k}^{m} \frac{\partial^k}{\partial x^k} f_k(x)
\tag{10.115}
$$

and

$$S = \sum_{l}^{m} \frac{\partial^l}{\partial x^l} g_l(x), \tag{10.116}$$

where f_k and g_l belong to L^2. Since the individual terms $(\partial^{k+l}/\partial x^{k+l})\{f_k * g_l\}$ are defined, the sum also exists as a distribution. The Fourier transform connection can be established similarly by appeal to the classical result for the Fourier transform of a convolution of L^2 functions. Equation (10.114) will be used in the sequel, but immediate use of the existence of $T * S$ can be made if $T, S \in \mathcal{D}'_{L^2}$. In particular, for $T \in \mathcal{D}'_{L^2}$ and using the fact that $p.v.(1/x) \in \mathcal{D}'_{L^2}$, then the convolution $T * p.v.(1/x)$ exists.

The Hilbert transform of a distribution $T \in \mathcal{D}'_{L^2}$ can be defined by the convolution

$$S = HT = \frac{1}{\pi} T * p.v.\frac{1}{x}. \tag{10.117}$$

The second of the Hilbert transform pair can be written as

$$T = -HS = -\frac{1}{\pi} S * p.v.\frac{1}{x}. \tag{10.118}$$

This definition can be formalized with the following result. If $T \in \mathcal{D}'_{L^2}$, then HT exists and $HT \in \mathcal{D}'_{L^2}$. To establish this result, first note that if $T \in \mathcal{D}'_{L^2}$, then

$$T = \sum_{k=0}^{m} \frac{\partial^k h_k(x)}{\partial x^k}, \tag{10.119}$$

with $h_k \in L^2$. It follows that

$$\begin{aligned} HT &= \frac{1}{\pi} \left(\sum_{k=0}^{m} \frac{\partial^k h_k(x)}{\partial x^k} \right) * p.v.\frac{1}{x} \\ &= \frac{1}{\pi} \left(\sum_{k=0}^{m} \frac{\partial^k}{\partial x^k} \right) h_k(x) * p.v.\frac{1}{x} \\ &= \left(\sum_{k=0}^{m} \frac{\partial^k}{\partial x^k} \right) Hh_k(x). \end{aligned} \tag{10.120}$$

Using the Riesz inequality, $Hh_k \in L^2$ since $h_k \in L^2$, and hence $HT \in \mathcal{D}'_{L^2}$.

The preceding result can be generalized. If $T \in \mathcal{D}'_{L^p}$ and $S \in \mathcal{D}'_{L^q}$ with $1/p + 1/q \geq 1$, then the convolution $S * T$ exists and $S * T \in \mathcal{D}'_{L^r}$ with $1/r = 1/p + 1/q - 1$. This result is due to Schwartz (1966a, p. 203). The Hilbert transform of the distribution

$T \in \mathcal{D}'_{L^p}$ can therefore be defined in terms of the convolution with the distribution $p.v.(1/x)$ for $1 < p < \infty$, so that

$$HT = \frac{1}{\pi} T * p.v.\frac{1}{x}. \tag{10.121}$$

10.8 Analytic representation of distributions

In this section an alternative definition of the Hilbert transform of a distribution in $\mathcal{D}'$ is given by making use of the analytic representation of distributions. The discussion follows the approach of Orton (1973), and also Lauwerier (1963). Unless there is a statement to the contrary, distributions and test functions are assumed to be real-valued in this section.

Suppose $g^+(z)$ and $g^-(z)$ are analytic functions for $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$, respectively. On the real axis let

$$g^+(x) = \lim_{y \to 0+} g^+(x + iy) \tag{10.122}$$

and

$$g^-(x) = \lim_{y \to 0-} g^-(x + iy). \tag{10.123}$$

The Hilbert problem is the determination of analytic functions $g^+(z)$ and $g^-(z)$ such that

$$g(x) = g^+(x) + g^-(x), \tag{10.124}$$

where $g(x)$ is some given function defined on $\mathbb{R}$. This problem can be generalized to more complicated situations, and these are considered in Chapter 11.

Attention is now directed to the analog of the preceding problem for distributions. Let $g \in \mathcal{D}'_{L^p}$ for $1 < p < \infty$, and define

$$g^+(x) = \lim_{y \to 0+} G^+(x + iy) \tag{10.125}$$

and

$$g^-(x) = \lim_{y \to 0-} G^-(x + iy), \tag{10.126}$$

where the analytic function $G(z)$ is such that

$$g(x) = g^+(x) + g^-(x). \tag{10.127}$$

The function $G(z)$ for $\operatorname{Im} z > 0$ is given by

$$G(z) = \frac{1}{2\pi \mathrm{i}} \left\langle g(t), \frac{1}{t - z} \right\rangle. \tag{10.128}$$

This is a generalized Cauchy integral representation of the distribution g. Mostly upper case letters have been employed to designate distributions in this chapter; the preceding choice is a departure from this convention. A commonly employed approach to denote the generalized Cauchy integral representation of a distribution T is given by

$$\hat{T}(z) = \frac{1}{2\pi \mathrm{i}} \left\langle T(t), \frac{1}{t - z} \right\rangle. \tag{10.129}$$

The *analytic representation* of a distribution T will be denoted by $\hat{T}(z)$. The reader will recall that the symbol ˆ is also commonly used to denote the Fourier transform; however, that usage is not employed in this chapter, so there should be no confusion about the intended meaning. Equation (10.128) can be rewritten as follows:

$$G(z) = \frac{1}{2\pi \mathrm{i}} \left\langle g(t), \frac{t - x}{(t - x)^2 + y^2} \right\rangle + \frac{1}{2\pi} \left\langle g(t), \frac{y}{(t - x)^2 + y^2} \right\rangle. \tag{10.130}$$

Two problems are posed for the reader. For $g \in \mathcal{D}'_{L^p}$ with $1 < p < \infty$, show that

$$\lim_{y \to 0+} \left\langle g(t), \frac{1}{\pi} \frac{y}{(t - x)^2 + y^2} \right\rangle = g(x) \tag{10.131}$$

and

$$\lim_{y \to 0+} \left\langle g(t), \frac{1}{\pi} \frac{t - x}{(t - x)^2 + y^2} \right\rangle = -Hg(x). \tag{10.132}$$

With these two results it follows from Eq. (10.125) that

$$g^+(x) = \frac{1}{2} g(x) + \frac{\mathrm{i}}{2} Hg(x). \tag{10.133}$$

In a similar fashion, define $G(z)$ for $\operatorname{Im} z < 0$ by

$$G(z) = -\frac{1}{2\pi \mathrm{i}} \left\langle g(t), \frac{1}{t - z} \right\rangle, \tag{10.134}$$

which can be rewritten as follows:

$$G(z) = \frac{1}{2\pi \mathrm{i}} \left\langle g(t), \frac{x - t}{(x - t)^2 + y^2} \right\rangle - \frac{1}{2\pi} \left\langle g(t), \frac{y}{(t - x)^2 + y^2} \right\rangle. \tag{10.135}$$

Hence, on noting $y < 0$,

$$g^{-}(x) = \frac{1}{2}g(x) - \frac{\mathrm{i}}{2}Hg(x). \tag{10.136}$$

Equation (10.127) is obtained from Eqs. (10.133) and (10.136), and also

$$Hg(x) = -\mathrm{i}\{g^{+}(x) - g^{-}(x)\}. \tag{10.137}$$

The important skew-reciprocal structure of the Hilbert transform of distributions is now considered. Let $u(x, y)$ be harmonic in the upper half plane and let $v(x, y)$ be its harmonic conjugate. Let u and v be distributions that belong to $\mathcal{D}'_{L^p}$ with $1 < p < \infty$, and let

$$g(x) = \lim_{y \to 0+} u(x, y), \tag{10.138}$$

$$h(x) = \lim_{y \to 0+} v(x, y), \tag{10.139}$$

and

$$G(z) = u + \mathrm{i}v. \tag{10.140}$$

Since $G(z)$ is an analytic function in the upper half plane,

$$\begin{aligned} G(z) &= \frac{1}{2\pi\mathrm{i}}\left\langle u + \mathrm{i}v, \frac{1}{t - z}\right\rangle \\ &= \frac{1}{2\pi\mathrm{i}}\left\langle u + \mathrm{i}v, \frac{t - x + \mathrm{i}y}{(t - x)^2 + y^2}\right\rangle. \end{aligned} \tag{10.141}$$

Taking the limit $y \to 0+$ in this result leads to

$$\begin{aligned} g(x) + \mathrm{i}h(x) = &\frac{1}{2\pi\mathrm{i}} \lim_{y \to 0+}\left\langle u + \mathrm{i}v, \frac{t - x}{(t - x)^2 + y^2}\right\rangle \\ &+ \frac{1}{2\pi} \lim_{y \to 0+}\left\langle u + \mathrm{i}v, \frac{y}{(t - x)^2 + y^2}\right\rangle \\ = &-\frac{1}{2\mathrm{i}}\{Hg(x) + \mathrm{i}\,Hh(x)\} + \frac{1}{2}\{g(x) + \mathrm{i}h(x)\}, \end{aligned} \tag{10.142}$$

and hence

$$g(x) + Hh(x) + \mathrm{i}\{h(x) - Hg(x)\} = 0. \tag{10.143}$$

From the real and imaginary parts it follows that

$$h(x) = Hg(x) \tag{10.144}$$

and

$$g(x) = -Hh(x), \tag{10.145}$$

which are the skew-reciprocal formulas for the Hilbert transform of distributions belonging to $\mathcal{D}'_{L^p}$ with $1 < p < \infty$.

If $T \in \mathcal{D}'$, then the analytic representation of T is defined by

$$\lim_{y \to 0+} \int_{-\infty}^{\infty} \{\hat{T}(x + \mathrm{i}y) - \hat{T}(x - \mathrm{i}y)\}\phi(x)\mathrm{d}x = \langle T(x), \phi(x)\rangle, \quad \forall \phi \in \mathcal{D}. \tag{10.146}$$

The generalized Cauchy integral representation of T defined in Eq. (10.129) will now be utilized. The functions $u(x, y)$ and $v(x, y)$ for $y > 0$ are introduced by the following definitions:

$$u(x, y) = \hat{T}(x + \mathrm{i}y) - \hat{T}(x - \mathrm{i}y) \tag{10.147}$$

and

$$v(x, y) = -\mathrm{i}\{\hat{T}(x + \mathrm{i}y) + \hat{T}(x - \mathrm{i}y)\}. \tag{10.148}$$

The functions u and v are conjugate harmonic functions in the upper half plane.

The distributional limit of a function $u(x, y)$ is defined, for a distribution $T \in \mathcal{D}'$, as follows:

$$\lim_{y \to 0+} \int_{-\infty}^{\infty} u(x, y)\phi(x)\mathrm{d}x = \langle T(x), \phi(x)\rangle, \quad \forall \phi \in \mathcal{D}. \tag{10.149}$$

The Hilbert transform of a distribution $T \in \mathcal{D}'$ can be written as follows:

$$\lim_{y \to 0+} \int_{-\infty}^{\infty} v(x, y)\phi(x)\mathrm{d}x = -\langle H\hat{T}(x), \phi(x)\rangle, \quad \forall \phi \in \mathcal{D}. \tag{10.150}$$

The Hilbert transform can be defined in the following manner. Let $\hat{T}(z)$ denote the analytic representation of a distribution $T \in \mathcal{D}'$. The Hilbert transform of T relative to $\hat{T}(z)$ is defined by

$$H\hat{T}(x) = \lim_{y \to 0+} \mathrm{i}\{\hat{T}(x + \mathrm{i}y) + \hat{T}(x - \mathrm{i}y)\}. \tag{10.151}$$

To see how this definition is suggested, consider the following sequence:

$$\begin{aligned}
H\hat{T}(x) &= \lim_{y\to 0+} \mathrm{i}\{\hat{T}(x+\mathrm{i}y) + \hat{T}(x-\mathrm{i}y)\} \\
&= \lim_{y\to 0+} \frac{1}{\pi}\left\langle T(t), \frac{x-t}{(x-t)^2+y^2}\right\rangle \\
&= \lim_{y\to 0+} \frac{1}{\pi} T(x) * \frac{x}{x^2+y^2} \\
&= \frac{1}{\pi} T(x) * p.v.\frac{1}{x}. \qquad (10.152)
\end{aligned}$$

The final result agrees with the convolution definition developed in Section 10.7.

10.9 The inversion formula

A fundamental property of the classical Hilbert transform is the inversion formula $H^2 f = -f$. The details on this were discussed in Section 4.4 for ordinary functions, including the restrictions on the functions for which this identity applies. The analog of this result for distributions is now considered. Suppose that the distribution T satisfies $T \in \mathcal{D}'_{L^p}$, with $1 < p < \infty$; then it follows that

$$\begin{aligned}
\langle H^2 T, \phi\rangle &= \langle HT, -H\phi\rangle \\
&= \langle T, H^2\phi\rangle, \qquad (10.153)
\end{aligned}$$

where all test functions satisfy $\phi \in \mathcal{D}_{L^q}$. Because of the latter condition for the test functions, the classical inversion formula $H^2\phi = -\phi$ can be applied; hence,

$$\langle H^2 T, \phi\rangle = \langle -T, \phi\rangle, \qquad (10.154)$$

which can be written as the symbolic result

$$H^2 T = -T, \qquad (10.155)$$

for distributions satisfying $T \in \mathcal{D}'_{L^p}$.

A short digression is made to derive the following result:

$$p.v.\frac{1}{x} * p.v.\frac{1}{x} = -\pi^2\delta(x), \qquad (10.156)$$

which will be employed in the derivation of the inversion formula for the Hilbert transform. For two distributions $T \in \mathcal{E}'$ and $S \in \mathcal{E}'$, or $S \in \mathcal{S}'$, or $T \in \mathcal{O}'_C$ and $S \in \mathcal{S}'$, it follows that

$$\mathcal{F}\{T * S\} = \mathcal{F}T\mathcal{F}S. \qquad (10.157)$$

This result also holds for the case that $T \in \mathcal{D}'_{L^p}$ and $S \in \mathcal{D}'_{L^q}$, with $p^{-1} + q^{-1} - 1 \geq 0$. In Section 10.7 it was shown that $p.v.(1/x) \in \mathcal{D}'_{L^p}$, so the left-hand side of Eq. (10.156) exists. Now,

$$\mathcal{F}\left\{p.v.\frac{1}{x} * p.v.\frac{1}{x}\right\} = \mathcal{F}\left\{p.v.\frac{1}{x}\right\} \mathcal{F}\left\{p.v.\frac{1}{x}\right\}, \tag{10.158}$$

and employing

$$\mathcal{F}\left\{p.v.\frac{1}{x}\right\} = -\mathrm{i}\pi \operatorname{sgn} x, \tag{10.159}$$

leads to

$$\mathcal{F}\left\{p.v.\frac{1}{x} * p.v.\frac{1}{x}\right\} = -\pi^2. \tag{10.160}$$

Taking the inverse Fourier transform of this result and making use of

$$\mathcal{F}^{-1}\{1\}(x) = \delta(x) \tag{10.161}$$

leads to Eq. (10.156).

A convolution argument is now employed to obtain the inversion formula for the Hilbert transform of distributions. Let $S = HT$, where $T \in \mathcal{E}'$; then

$$\begin{aligned} H^2 T &= HS \\ &= \frac{1}{\pi} S * p.v.\frac{1}{x} \\ &= \frac{1}{\pi^2}\left\{T * p.v.\frac{1}{x}\right\} * p.v.\frac{1}{x} \\ &= \frac{1}{\pi^2} T * \left\{p.v.\frac{1}{x} * p.v.\frac{1}{x}\right\}. \end{aligned} \tag{10.162}$$

The associative property of three distributions T, S, and R, namely

$$T * (S * R) = (T * S) * R, \tag{10.163}$$

holds if the supports of at least two of the distributions are bounded. This statement generalizes to a convolution of n distributions, with the associative property holding if $n - 1$ of the distributions have bounded supports. The associative condition also holds under the conditions that the supports of all distributions are bounded on the left, or all the supports are bounded on the right. Since the associative property does not hold for distributions in general, it is necessary to justify the application of the associative condition in the preceding step. The distribution $p.v.(1/x)$ can be written in the following manner:

$$p.v.\frac{1}{x} = S + f, \tag{10.164}$$

where S is a distribution that belongs to $\mathcal{E}'$ and f is a function satisfying $f \in L^2$. Hence, it follows that

$$p.v.\frac{1}{x} * p.v.\frac{1}{x} = S * S + S * f + f * S + f * f. \tag{10.165}$$

Since $T \in \mathcal{E}'$,

$$T * (S * S) = (T * S) * S, \tag{10.166}$$

$$T * (S * f) = (T * S) * f, \tag{10.167}$$

$$T * (f * S) = (T * f) * S, \tag{10.168}$$

and, if the support of T is taken to be $(-\alpha, \alpha)$, the result is as follows:

$$\begin{aligned} T * (f * f) &= \int_{-\infty}^{\infty} T(t)\mathrm{d}t \int_{-\infty}^{\infty} f(u)f(x-t-u)\mathrm{d}u \\ &= \int_{-\alpha}^{\alpha} T(t)\mathrm{d}t \int_{-\infty}^{\infty} f(u)f(x-t-u)\mathrm{d}u \\ &= \int_{-\infty}^{\infty} f(u)\mathrm{d}u \int_{-\alpha}^{\alpha} T(t)f(x-t-u)\mathrm{d}t \\ &= (T * f) * f, \end{aligned} \tag{10.169}$$

where Fubini's theorem has been employed to reverse the order of integration. Hence, the application of the associative property in Eq. (10.162) is justified. Employing Eq. (10.156) and using

$$T * \delta = T, \tag{10.170}$$

means Eq. (10.162) simplifies to

$$H^2 T = -T, \tag{10.171}$$

which is the required result.

10.10 The derivative property

Let ∂ denote the derivative operator and suppose the distribution T satisfies $T \in \mathcal{D}'_{L^p}$ for $1 < p < \infty$; then the distribution formed by applying the derivative operator, ∂T, is defined for all test functions $\phi \in \mathcal{D}_{L^q}$ by

$$\langle \partial T, \phi \rangle = \langle T, -\partial \phi \rangle. \tag{10.172}$$

The reader should note that this definition of the distributional derivative is consistent with what would be expected if T were replaced by an ordinary function. The result for a normal function can be seen by doing an integration by parts.

The commutation property of the derivative and Hilbert transform operators is now established for distributions belonging to $\mathcal{D}'_{L^p}$; that is, symbolically,

$$\partial H \equiv H\partial. \tag{10.173}$$

To derive this result, employ Eq. (10.83) with T replaced by ∂T, so that

$$\begin{aligned} \langle H\partial T, \phi \rangle &= \langle \partial T, -H\phi \rangle \\ &= \langle T, \partial H\phi \rangle. \end{aligned} \tag{10.174}$$

For the test function ϕ, using Eq. (4.137) leads to

$$\partial H\phi = H\partial\phi, \tag{10.175}$$

and hence, on reversing the steps in Eq. (10.174), that

$$\begin{aligned} \langle H\partial T, \phi \rangle &= \langle T, H\partial\phi \rangle \\ &= \langle -HT, \partial\phi \rangle \\ &= \langle \partial HT, \phi \rangle, \end{aligned} \tag{10.176}$$

which establishes the symbolic correspondence

$$\partial HT = H\partial T. \tag{10.177}$$

By an induction argument, the preceding result can be generalized to deal with the kth distributional derivative:

$$\partial^k HT = H\partial^k T, \ \text{ for } k \in \mathbb{N}. \tag{10.178}$$

10.11 The Fourier transform connection

One of the key properties satisfied by the Hilbert transform is

$$\mathcal{F}Hf(x) = -\mathrm{i}\,\mathrm{sgn}\,x\,\mathcal{F}f(x), \tag{10.179}$$

where $\mathcal{F}$ denotes the Fourier transform operator and f is a function satisfying $f \in L^p$ with $1 < p \leq 2$. In this section the extension of Eq. (10.179) to the case of distributions is examined. Recalling the definition of the Fourier transform of a tempered distribution, and making use of Parseval's formula (Section 10.4), it follows for $T \in \mathscr{S}'$, that

$$\langle \mathcal{F}T, \phi \rangle = \langle T, \mathcal{F}\phi \rangle, \quad \text{for } \phi \in \mathscr{S}. \tag{10.180}$$

In this result $\mathcal{F}T$ denotes a new distribution that is also a tempered distribution. From Eq. (10.180) it follows, using Fubini's theorem, that

$$\begin{aligned}\langle T, \mathcal{F}\phi \rangle &= \int_{-\infty}^{\infty} T(x)\mathrm{d}x \int_{-\infty}^{\infty} \phi(s)\mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}s \quad = \int_{-\infty}^{\infty} \phi(s)\mathrm{d}s \int_{-\infty}^{\infty} T(x)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x \\ &= \langle \mathcal{F}T, \phi \rangle. \end{aligned} \tag{10.181}$$

A key point to note is the obvious shift from the space $\mathcal{D}$ to the space $\mathscr{S}$ for the test functions under consideration. The Parseval formula corresponding to Eq. (10.180) for $\phi \in \mathcal{D}$ is not useful as a definition of the Fourier transform of a distribution, since the function $\mathcal{F}\phi$ may not be a test function satisfying $\mathcal{F}\phi \in \mathcal{D}$. See Eq. (10.61) and the sequel explanation for a discussion of this point. It is therefore clear that it is necessary to expand the set of functions beyond those in $\mathcal{D}$ to include test functions having suitable decay characteristics as $|x| \to \pm\infty$. Equation (10.181) makes the connection with the definition of the Fourier transform of a classical function.

By analogy with Eq. (10.180), the inverse Fourier transform operation can be established for distributions T satisfying $T \in \mathscr{S}'$, with the result that

$$\langle \mathcal{F}^{-1}T, \phi \rangle = \langle T, \mathcal{F}^{-1}\phi \rangle, \quad \text{for } \phi \in \mathscr{S}. \tag{10.182}$$

It is not difficult to establish, for $T \in \mathscr{S}'$, the operator equivalence:

$$\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = I. \tag{10.183}$$

To arrive at an analog of Eq. (10.179) for distributions, start with test functions satisfying $\phi \in \mathscr{S}$ and let $T \in \mathcal{D}'_{L^p}$; then,

$$\begin{aligned}\langle \mathcal{F}HT, \phi \rangle &= \langle HT, \mathcal{F}\phi \rangle \\ &= \langle T, -H\mathcal{F}\phi \rangle, \end{aligned} \tag{10.184}$$

where both Eqs. (10.180) and (10.83) have been employed. On making use of Eq. (5.57), it follows that

$$\begin{aligned}\langle \mathcal{F}HT,\, \phi \rangle &= \langle T(x), -\mathcal{F}[\mathrm{i}\,\mathrm{sgn}\, y\, \phi(y)](x) \rangle \\ &= \langle \mathcal{F}T(x), -\mathrm{i}\,\mathrm{sgn}\, x\, \phi(x) \rangle \\ &= \langle -\mathrm{i}\,\mathrm{sgn}\, x\, \mathcal{F}T(x), \phi(x) \rangle, \end{aligned} \tag{10.185}$$

which yields the following symbolic relationship:

$$\mathcal{F}HT(x) = -\mathrm{i}\,\mathrm{sgn}\,x\,\mathcal{F}T(x), \quad \text{for } T \in \mathcal{D}'_{L^p}. \tag{10.186}$$

This section concludes by revisiting the evaluation of the Hilbert transform of a couple of key distributions. To calculate $H\delta$, start with Eq. (10.185):

$$\begin{aligned}\langle \mathcal{F}H\delta, \phi\rangle &= \langle -\mathrm{i}\,\mathrm{sgn}\,x\,\mathcal{F}\delta, \phi(x)\rangle \\ &= \langle -\mathrm{i}\,\mathrm{sgn}\,x, \phi(x)\rangle,\end{aligned} \tag{10.187}$$

and Eq. (10.72) has been employed. Using Eq. (10.77), the inverse Fourier transform of sgn x is given by

$$\mathcal{F}^{-1}\{-\mathrm{i}\pi\,\mathrm{sgn}\,x\} = p.v.\frac{1}{x}. \tag{10.188}$$

If $T = \mathcal{F}H\delta$ is employed in Eq. (10.182), and Eqs. (10.187) and (10.188) are utilized, it follows that

$$\begin{aligned}\langle H\delta, \phi\rangle &= \langle \mathcal{F}H\delta,\ \mathcal{F}^{-1}\phi\rangle \\ &= \langle -\mathrm{i}\,\mathrm{sgn}\,x,\ \mathcal{F}^{-1}\phi\rangle \\ &= \langle \mathcal{F}^{-1}\{-\mathrm{i}\,\mathrm{sgn}\,x\}, \phi\rangle \\ &= \langle p.v.\frac{1}{\pi x}, \phi\rangle,\end{aligned} \tag{10.189}$$

and, in symbolic form,

$$H\delta = p.v.\frac{1}{\pi x}. \tag{10.190}$$

To evaluate $H(p.v.(1/x))$, use Eq. (10.77), so that

$$\begin{aligned}\left\langle \mathcal{F}H\left(p.v.\frac{1}{x}\right), \phi\right\rangle &= \left\langle -\mathrm{i}\,\mathrm{sgn}\,x\,\mathcal{F}\left(p.v.\frac{1}{x}\right), \phi(x)\right\rangle \\ &= \langle -\mathrm{i}\,\mathrm{sgn}\,x\,(-\mathrm{i}\pi\,\mathrm{sgn}\,x), \phi(x)\rangle \\ &= \langle -\pi, \phi(x)\rangle,\end{aligned} \tag{10.191}$$

and it follows, on using Eq. (10.182) with $T = \mathcal{F}H(p.v.(1/x))$, that

$$\begin{aligned}\left\langle H\left(p.v.\frac{1}{x}\right), \phi\right\rangle &= \left\langle \mathcal{F}H\left(p.v.\frac{1}{x}\right), \mathcal{F}^{-1}\phi\right\rangle \\ &= \langle -\pi, \mathcal{F}^{-1}\phi\rangle \\ &= \pi\langle \mathcal{F}^{-1}\{-1\}, \phi(x)\rangle \\ &= \langle -\pi\delta, \phi\rangle. \end{aligned} \tag{10.192}$$

Symbolically, this gives the now familiar result

$$H\left(p.v.\frac{1}{x}\right) = -\pi\delta(x). \tag{10.193}$$

10.12 Periodic distributions: some preliminary notions

The goal of this section is to introduce the idea of a periodic distribution. A particular focus is the connection with the Fourier series expansion of an ordinary function. This section considers the set of conditions that allow a periodic distribution to be written as a general trigonometric series of the form

$$T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{2in\pi t/\tau}, \tag{10.194}$$

where τ is employed to denote the period. The meaning of the coefficients c_n and how they are determined will be given in this section. The following section will deal with the determination of the Hilbert transform of a periodic distribution.

First, some preliminary ideas. Given a $\phi \in \mathcal{D}$, then a unique function $\theta(t)$ belonging to the space $\mathcal{P}_\tau$ can be generated from the following expansion:

$$\theta(t) = \sum_{k=-\infty}^{\infty} \phi(t - k\tau). \tag{10.195}$$

Since ϕ has compact support there are only a finite number of non-vanishing terms in this expansion. It will be useful for later purposes to have a test function that vanishes for $|t| \geq 1$ and satisfies the condition

$$\sum_{k=-\infty}^{\infty} U(t - k\tau) = 1, \tag{10.196}$$

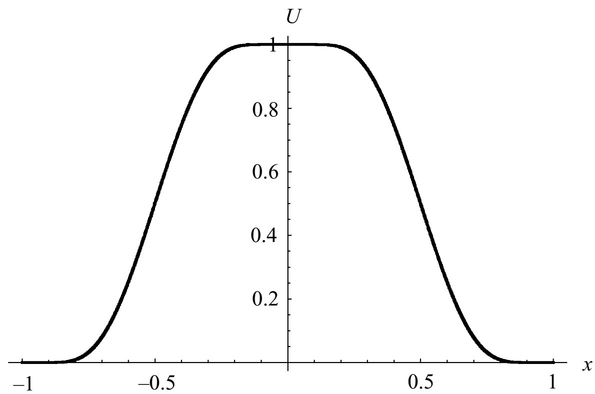

Figure 10.4. Behavior of the unitary function $U(x)$ on the interval $|x| < 1$.

for all values of t. Lighthill (1958, p. 61) called this a *unitary function*. An example for the case $\tau = 1$ is:

$$U(x) = \begin{cases} \dfrac{\int_{|x|}^{1} e^{-y^{-1}(1-y)^{-1}}\, dy}{\int_0^1 e^{-y^{-1}(1-y)^{-1}}\, dy}, & -1 < x < 1 \\ 0, & |x| \geq 1, \end{cases} \tag{10.197}$$

which is displayed in Figure 10.4. As for the case of Eq. (10.195), there are only a finite number of non-vanishing terms in the sum in Eq. (10.196). The unitary function just given is constructed so that the derivatives of $U(x)$ vanish at $x = \pm 1$. The function $U(x)$ satisfies

$$U(x) + U(x-1) = 1, \quad \text{for } 0 \leq x < 1. \tag{10.198}$$

An immediate consequence of the definition of a unitary function is that, for a suitable regular periodic distribution T with period τ,

$$\int_{-\infty}^{\infty} T(t)U(t)dt = \int_0^{\tau} T(t)U(t)dt. \tag{10.199}$$

The Fourier transform of U is given by

$$V(t) \equiv \mathcal{F}U(t) = \int_{-\infty}^{\infty} U(x)e^{-ixt}\, dx$$

$$= \cdots + \int_{-3}^{-2} U(x)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x + \int_{-2}^{-1} U(x)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x + \int_{-1}^{0} U(x)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x$$
$$+ \int_{0}^{1} U(x)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x + \int_{1}^{2} U(x)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x + \int_{2}^{3} U(x)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x + \cdots$$
$$= \sum_{k=-\infty}^{\infty} \int_0^1 \mathrm{e}^{-\mathrm{i}(x+k)t} U(x+k)\mathrm{d}x, \tag{10.200}$$

where the appropriate change of integration variable in each integral has been made to obtain the last summation. If $t = 2\pi n$ with $n \in \mathbb{Z}$, using Eq. (10.196) leads to

$$V(2\pi n) = \int_0^1 \mathrm{e}^{-2\pi i n x}\,\mathrm{d}x = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases} \tag{10.201}$$

The notation $\mathcal{U}_\tau$ will be used to denote the space of unitary functions that satisfy Eq. (10.196). A principal utility of the unitary function is that the integration of a periodic distribution $T(t)$ over a single period can be replaced by the integration of $T(t)U(t/\tau)$ over the interval $(-\infty, \infty)$. It is left as an exercise for the reader to decide why this approach might be useful, as opposed to multiplying $T(t)$ by a suitable characteristic function for the appropriate interval. Start by thinking about the issue of multiplying two distributions and deciding if the distributions are separably locally integrable. Is the product of the two distributions also locally integrable?

For a given $\theta(t) \in \mathcal{P}_\tau$ and a $\mathrm{U}(t) \in \mathcal{U}_\tau$, $U\theta \in \mathcal{D}$. The product $U(t-k\tau)\theta(t-k\tau)$ summed over k can be simplified in the following way:

$$\sum_{k=-\infty}^{\infty} U(t-k\tau)\theta(t-k\tau) = \theta(t) \sum_{k=-\infty}^{\infty} U(t-k\tau) = \theta(t), \tag{10.202}$$

which follows from the periodic nature of $\theta(t)$.

If $T \in \mathcal{D}'$ and T is a periodic distribution with period τ for $\tau > 0$, so that

$$T(t) = T(t-\tau), \tag{10.203}$$

then, for every $\phi \in \mathcal{D}$, the following definition is employed:

$$\langle T(t), \phi(t)\rangle = \langle T(t-\tau), \phi(t)\rangle. \tag{10.204}$$

For example, $T(t) = \sin(2\pi t/\tau)$ is a regular distribution with period τ.

The standard definition for a regular distribution associates with the distribution T the value $\langle T, \phi\rangle$ for each $\phi \in \mathcal{D}$. This is replaced for a periodic distribution by the following: for a $\theta \in \mathcal{P}_\tau$, the dot product $T \cdot \theta$ associates a value for a given periodic distribution T, and this product is defined by

$$T \cdot \theta = \langle T, U\theta\rangle, \tag{10.205}$$

for any $U \in \mathcal{U}_\tau$. The terminology dot product used here is not intended to convey the usual usage that it has in vector analysis. An important feature is that this definition is not tied to a particular choice of unitary function. The distribution T can be written as

$$T(t) = \sum_{k=-\infty}^{\infty} T(t)U(t - k\tau), \tag{10.206}$$

which follows directly from Eq. (10.196). The summation in this equation will have only a finite number of terms. For any $\phi \in \mathcal{D}$, it follows that

$$\left\langle \sum_k T(t)U(t - k\tau), \phi(t) \right\rangle = \left\langle T(t), \phi(t) \sum_k U(t - k\tau) \right\rangle = \langle T, \phi \rangle. \tag{10.207}$$

From Eq. (10.205), linearity on $\mathcal{P}_\tau$ can be established. Further, if $\{\theta_n\}_{n=1}^{\infty}$ converges to θ in P_τ, then $\{U\theta_n\}_{n=1}^{\infty}$ converges to $U\theta$ in $\mathcal{D}$, from which it can be deduced that T is a continuous linear functional on $\mathcal{P}_\tau$.

The definition given in Eq. (10.205) can be replaced by

$$T \cdot \theta = \langle T, \phi \rangle, \qquad \forall \phi \in \mathcal{D}. \tag{10.208}$$

This follows by employing the periodic property of T, and using Eqs. (10.195) and (10.196), to give

$$\begin{aligned} T \cdot \theta &= \langle T, U\theta \rangle \\ &= \left\langle T(t), U(t) \sum_{k=-\infty}^{\infty} \phi(t - k\tau) \right\rangle \\ &= \sum_{k=-\infty}^{\infty} \langle T(t), U(t)\phi(t - k\tau) \rangle \\ &= \sum_{k=-\infty}^{\infty} \langle T(t + k\tau), U(t + k\tau)\phi(t) \rangle \\ &= \sum_{k=-\infty}^{\infty} \langle T(t), U(t + k\tau)\phi(t) \rangle \\ &= \langle T(t), \phi(t) \sum_{k=-\infty}^{\infty} U(t + k\tau) \rangle \\ &= \langle T(t), \phi(t) \rangle, \end{aligned} \tag{10.209}$$

which is the desired result.

If T is a periodic distribution with period τ, then it has the following series expansion:

$$T(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt/\tau}, \tag{10.210}$$

where the coefficients c_k are given by

$$c_k = \frac{1}{\tau}\int_{-\infty}^{\infty} T(t)U\left(\frac{t}{\tau}\right) e^{-2\pi ikt/\tau}\, dt. \tag{10.211}$$

Inserting Eq. (10.210) into the preceding integral and employing the properties of the unitary function shows that the right-hand side of Eq. (10.211) is indeed c_k. The sequence $\{c_k\}_{k=1}^{\infty}$ is of slow growth and the summation in Eq. (10.210) converges in $\mathscr{S}'$ to T. The proof of this can be found in Zemanian (1965, p. 332).

The Dirac *comb* distribution is defined by

$$\Delta_\tau = \sum_{k=-\infty}^{\infty} \delta(t - k\tau), \tag{10.212}$$

where the period τ satisfies $\tau > 0$. The action of Δ_τ on a test function can be written symbolically in the following manner:

$$\Delta_\tau(\phi) = \sum_{k=-\infty}^{\infty} \phi(k\tau). \tag{10.213}$$

The Dirac comb is an example of a distribution belonging to $\mathcal{P}'_\tau$. If T is a distribution and $g \in C^\infty$, then gT is a distribution defined by

$$\langle gT, \phi\rangle = \langle T, g\phi\rangle, \qquad \forall\phi \in \mathcal{D}. \tag{10.214}$$

Analogously, $g\Delta_\tau$ is a distribution defined by

$$\langle g\Delta_\tau, \phi\rangle = \langle \Delta_\tau, g\phi\rangle = \sum_{k=-\infty}^{\infty} g(k\tau)\phi(k\tau). \tag{10.215}$$

Symbolically, this can be written as

$$g\Delta_\tau = \sum_{k=-\infty}^{\infty} g(t)\delta(t - k\tau). \tag{10.216}$$

The Dirac comb represents a sequence of impulses spaced at equal intervals τ apart. If the variable t is regarded as a time, and g a signal, then from Eq. (10.216) an approximate determination of the function g can be made from the sampled values at the positions $k\tau$. The Dirac comb has a trigonometric series representation:

$$\Delta_\tau = \frac{1}{\tau} \sum_{k=-\infty}^{\infty} e^{2\pi i k t/\tau}. \tag{10.217}$$

This series does not converge in the sense of normal functions, but does in the sense of generalized functions. Consider the periodic function with period 2π defined by

$$f(x) = \frac{1}{2} - \frac{x}{2\pi}, \qquad \text{for } x \in (0, 2\pi). \tag{10.218}$$

The derivative of this function is given by (where $f \to T'$ is employed to emphasize the distributional character of the derivative)

$$T' = -\frac{1}{2\pi} + \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k). \tag{10.219}$$

The Fourier series expansion of f is given by

$$f(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{1}{2\pi i} \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty} \frac{e^{ikx}}{k}, \qquad x \in (0, 2\pi). \tag{10.220}$$

The series in Eq. (10.220) converges in $\mathscr{D}'$. Differentiation of this Fourier series in the distributional sense yields

$$T' = \frac{1}{2\pi} \sum_{\substack{k=-\infty \\ (k\neq 0)}}^{\infty} e^{ikx} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} - \frac{1}{2\pi}. \tag{10.221}$$

Comparing Eqs. (10.219) and (10.221) leads to

$$\sum_{k=-\infty}^{\infty} \delta(x - 2\pi k) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx}. \tag{10.222}$$

This is Eq. (10.217) for $\tau = 2\pi$.

If S is a periodic distribution with period $\tau > 0$, then $S \in \mathscr{S}'$. A proof of this is given by Zemanian (1965, p. 332). There is a distribution $T \in \mathscr{E}'$ such that

$$S = T * \Delta_\tau. \tag{10.223}$$

A distribution T can be written as $T = US$, where U is a unitary function. The convolution of the Dirac delta distribution with a distribution in $\mathcal{D}'$ can be written as

$$T * \delta(x-a) = \delta(x-a) * T = \tau_a T, \tag{10.224}$$

where τ_a is the translation operator (recall Eq. (4.64)). This can be seen in a straightforward fashion:

$$\begin{aligned} \langle T * \delta(x-a), \phi \rangle &= \langle T(y), \langle \delta(x-a), \phi(x+y) \rangle \rangle \\ &= \langle T(y), \phi(y+a) \rangle \\ &= \langle T, \tau_{-a}\phi \rangle \\ &= \langle \tau_a T(y), \phi(y) \rangle. \end{aligned} \tag{10.225}$$

For $\phi \in \mathcal{D}$, noting the periodic character of S and U and employing Eq. (10.207), yields

$$\begin{aligned} \langle T * \Delta_\tau, \phi \rangle &= \left\langle T(y), \left\langle \sum_{k=-\infty}^{\infty} \delta(t-k\tau), \phi(t+y) \right\rangle \right\rangle \\ &= \sum_{k=-\infty}^{\infty} \langle T(y), \phi(y+k\tau) \rangle \\ &= \sum_{k=-\infty}^{\infty} \langle SU, \tau_{-k\tau}\phi \rangle \\ &= \sum_{k=-\infty}^{\infty} \langle \tau_{k\tau} SU, \phi \rangle \\ &= \sum_{k=-\infty}^{\infty} \langle S(y-k\tau)U(y-k\tau), \phi(y) \rangle \\ &= \sum_{k=-\infty}^{\infty} \langle S(y), U(y-k\tau)\phi(y) \rangle \\ &= \langle S, \phi \rangle, \end{aligned} \tag{10.226}$$

and Eq. (10.223) is established.

If T is a periodic distribution with period τ, it has a unique Fourier series representation:

$$T = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{2\pi \mathrm{i}kt/\tau}. \tag{10.227}$$

The Fourier coefficients c_k are given by

$$c_k = \frac{1}{\tau} T \mathrm{e}^{-2\pi \mathrm{i}kt/\tau}. \tag{10.228}$$

The coefficients are of slow growth. Two important facts about this Fourier series representation are the following. (i) If the coefficients c_k are defined by Eq. (10.228), then the sequence $\{c_k\}_{k=1}^{\infty}$ is of slow growth, and the sum in Eq. (10.227) converges to a distribution $T \in \mathscr{S}'$. (ii) A series of the form of Eq. (10.227), where the coefficients c_k are of slow growth, converges to a periodic distribution (with period τ) in $\mathscr{S}'$. For the proof of these two results, the reader is referred to Zemanian (1965, pp. 331–332).

10.13 The Hilbert transform of periodic distributions

Let $T \in \mathcal{P}'_\tau$; then the Hilbert transform of T is defined by

$$(\mathcal{H}T) \cdot \theta = -T \cdot (\mathcal{H}\theta), \tag{10.229}$$

where θ is given in Eq. (10.195). Employing Eq. (10.205), it follows for all $U \in \mathcal{U}_\tau$ and $\theta \in \mathcal{P}_\tau$ that

$$\mathcal{H}T \cdot \theta = \langle T, -U\mathcal{H}\theta \rangle. \tag{10.230}$$

The inversion formula for the Hilbert transform for a distribution $T \in \mathcal{P}'_\tau$ is obtained in the following manner:

$$\begin{aligned}(\mathcal{H}^2 T) \cdot \theta &= \langle \mathcal{H}T, -U\mathcal{H}\theta \rangle \\ &= \langle T, U\mathcal{H}^2\theta \rangle.\end{aligned} \tag{10.231}$$

Since $\theta \in \mathcal{P}_\tau$, and employing Eq. (6.35) leads to

$$\mathcal{H}^2\theta(t) = -\theta(t) + \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \theta(x)\mathrm{d}x, \tag{10.232}$$

therefore Eq. (10.231) can be rewritten as follows:

$$(\mathcal{H}^2 T) \cdot \theta = \left\langle T, U\left\{-\theta(t) + \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \theta(x)\mathrm{d}x\right\}\right\rangle. \tag{10.233}$$

If the symbol $T_{\mathrm{I},\tau}$ is introduced by the formula

$$T_{\mathrm{I},\tau} \cdot \theta \equiv \left\langle T, U \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \theta(x)\mathrm{d}x \right\rangle, \tag{10.234}$$

where the subscript I signifies taking an integral over a period of θ, then Eq. (10.233) can be expressed symbolically as follows:

$$\mathcal{H}^2 T = -T + T_{\mathrm{I},\tau}. \tag{10.235}$$

Pandey (1996, 1997) introduced a subspace of $\mathcal{P}_\tau$ for functions that are infinitely differentiable, have period τ, and satisfy the requirement

$$\int_{-\tau/2}^{\tau/2} \theta(x)\mathrm{d}x = 0. \tag{10.236}$$

This subspace is denoted by $\mathcal{Q}_\tau$, and the dual space is designated $\mathcal{Q}'_\tau$. If $T \in \mathcal{Q}'_\tau$, then the Hilbert transform of T is defined by

$$\langle \mathcal{H}T, \theta \rangle = \langle T, -U\mathcal{H}\theta \rangle, \qquad \forall \theta \in \mathcal{Q}_\tau. \tag{10.237}$$

If $T \in \mathcal{Q}'_\tau$, it follows that

$$\mathcal{H}^2 T = -T, \tag{10.238}$$

which has the analogous form to the corresponding result for periodic functions when the integral of the function over a period vanishes (see Eq. (6.35)).

10.14 The Hilbert transform of ultradistributions and related ideas

In this section two extensions of some of the previous approaches given for the Hilbert transform of distributions are briefly discussed. The definition given in Eq. (10.117) applies for distributions belonging to the space $\mathcal{D}'_{L^p}$, but it does not cover the case of general distributions belonging to $\mathcal{D}'$. To rectify this situation, an approach of Gel'fand and Shilov that has been expanded upon by Pandey is briefly considered. Let $\mathscr{H}$ denote the space of test functions ϕ that satisfy the following: (i) $\phi \in C^\infty$; (ii) there exists a $\varphi \in \mathcal{D}$ with $\phi(t) = \mathcal{H}\varphi(t)$. The inclusion relation for this space is $\mathscr{H} \subset \mathcal{D}_{L^p}$. Pandey denotes this space by $H(\mathcal{D})$, but this notation is not employed to minimize any possible confusion with the symbolism employed for the Hilbert transform. The space of continuous linear functionals on $\mathscr{H}$ will be denoted by $\mathscr{H}'$, and this space satisfies the inclusion condition $\mathcal{D}'_{L^p} \subset \mathscr{H}'$, with $1 < p < \infty$. Pandey (1983, 1996, p. 115) discusses the space $\mathscr{H}$ in detail, and the reader is referred to these sources for further elaboration. The Hilbert transform of a distribution $T \in \mathcal{D}'$

is defined (Pandey, 1983) to be an ultradistribution $HT \in \mathscr{H}'$ such that

$$\langle HT, \varphi\rangle = \langle T, -H\varphi\rangle, \qquad \forall \varphi \in \mathscr{H}. \tag{10.239}$$

In this result, $H\varphi$ denotes an ordinary Hilbert transform. If $T \in \mathscr{H}'$, then the Hilbert transform of T is defined to be a Schwartz distribution given by the following formula:

$$\langle HT, \varphi\rangle = \langle T, -H\varphi\rangle, \qquad \forall \varphi \in \mathcal{D}. \tag{10.240}$$

If $T \in \mathcal{D}'$, the inversion formula $H^2T = -T$ can be established from the results just given. The reader is requested to construct the necessary argument.

If $T \in \mathcal{D}'$, the derivative operator and the Hilbert transform operator are commutative, that is

$$(HT)' = HT'. \tag{10.241}$$

This result can be established in a similar manner to the case for $T \in \mathcal{D}'_{L^p}$ given in Section 10.10. The required sequence of steps is as follows:

$$\begin{aligned}\langle HT', \varphi\rangle &= \langle T', -H\varphi\rangle \\ &= \left\langle T, \frac{\mathrm{d}}{\mathrm{d}x} H\varphi(x)\right\rangle \\ &= \left\langle T, H\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)\right\rangle \\ &= -\left\langle HT, \frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)\right\rangle \\ &= \langle (HT)', \varphi\rangle,\end{aligned} \tag{10.242}$$

where the result $(H\varphi)' = H\varphi'$ has been used (recall Eq. (4.137)). The preceding result can be generalized to

$$(HT)^{(k)} = HT^{(k)}, \qquad \text{for } k \in \mathbb{N}. \tag{10.243}$$

Pandey (1982) has extended an idea of Gel'fand and Shilov (1968) to deal with the Hilbert transform of tempered distributions. If $\phi \in \mathscr{S}$ and $\varphi \in \mathscr{S}$, then Eq. (5.3) can be employed to write

$$\langle \mathcal{F}H\phi(x), \varphi(x)\rangle = -\mathrm{i}\langle \operatorname{sgn} x\, \mathcal{F}\phi(x), \varphi(x)\rangle, \tag{10.244}$$

and this can be written as

$$\langle H\phi(x), \mathcal{F}\varphi(x)\rangle = -\mathrm{i}\langle \phi, \mathcal{F}\{\operatorname{sgn} x\, \varphi(x)\}\rangle, \tag{10.245}$$

using the Parseval-type formula for the Fourier transform.

Pandey introduced two subspaces in the following way. A function $\varphi \in C^\infty$ belongs to the space $\mathscr{S}_1$ if $\varphi \in \mathscr{S}$ and $\varphi(x) = 0$ for $x \in (-a, a)$ for some $a > 0$. The space of Fourier transforms of test functions belonging to $\mathscr{S}_1$ is denoted by $\mathcal{Z}_1$. Pandey then defined the Hilbert transform of a tempered distribution T:

$$\langle HT, \mathcal{F}\varphi(x)\rangle = -\mathrm{i}\langle T, \mathcal{F}\{\operatorname{sgn} x\, \varphi(x)\}\rangle, \qquad \forall\varphi \in \mathscr{S}_1. \tag{10.246}$$

The Hilbert transform so defined is a distribution belonging to the space $\mathcal{Z}_1'$. The inversion formula can be derived as follows:

$$\begin{aligned}\langle H^2 T, \mathcal{F}\varphi(x)\rangle &= -\mathrm{i}\langle HT, \mathcal{F}\{\operatorname{sgn} x\, \varphi(x)\}\rangle \\ &= \mathrm{i}^2\langle T, \mathcal{F}\{(\operatorname{sgn} x)^2 \varphi(x)\}\rangle \\ &= -\langle T, \mathcal{F}\varphi(x)\rangle,\end{aligned} \tag{10.247}$$

which holds for $\forall\varphi \in \mathscr{S}_1$. As an example in $\mathcal{Z}_1'$, $H\delta = p.v.(1/\pi x)$.

If $T \in \mathscr{S}'$, the derivative property for the Hilbert transform holds; that is,

$$HT' = (HT)'. \tag{10.248}$$

This can be established as follows. Let $\varphi \in \mathscr{S}$, then

$$\begin{aligned}\langle HT', \mathcal{F}\varphi(x)\rangle &= -\mathrm{i}\langle T', \mathcal{F}\{\operatorname{sgn} x\, \varphi(x)\}\rangle \\ &= \mathrm{i}\left\langle T, \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}[\operatorname{sgn} t\, \varphi(t)](x)\right\rangle \\ &= \mathrm{i}\langle T, \mathcal{F}\{-\mathrm{i}\, t \operatorname{sgn} t\, \varphi(t)\}(x)\rangle \\ &= \langle HT, \mathcal{F}\{\mathrm{i}\, t\, \varphi(t)\}(x)\rangle \\ &= -\left\langle HT, \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}\varphi(x)\right\rangle \\ &= \langle (HT)', \mathcal{F}\varphi(x)\rangle,\end{aligned} \tag{10.249}$$

which is the required result.

A number of the results of this chapter find utility in the solution of singular integral equations involving distributions of different types. Several simple cases are discussed in Section 12.14, and some examples for the reader to try are given in the Exercises section of Chapter 12. Various ideas from this chapter allow an extension of Titchmarsh's theorem to be made. This is discussed in Section 17.13.

Notes

§10.1 The topic of generalized functions is exhaustively covered in a great many sources. Good introductory accounts can be found in Lighthill (1958), Jones (1982), Kanwal (1998), and Howell (2001) (with a focus directed to Fourier analysis). Schwartz (1966a, 1966b) and Zemanian (1965) give detailed presentations.

§10.2 The authoritative source for further discussion on distribution spaces is Schwartz (1966a).

§10.4 Fourier transforms of distributions are commonly discussed in most books on distributions. In addition to the aforementioned references, further reading can be found in Bremermann (1965a), Beltrami and Wohlers (1966a), and Barros-Neto (1973).

§10.5 See Schwartz (1966b) and Gasquet and Witomski (1999) for further reading on convolutions.

§10.6 The Hilbert transform of distributions is discussed in the books by Beltrami and Wohlers (1966a), Bremermann (1965a), Schwartz (1966a), Gel'fand and Shilov (1968), Roos (1969), Nussenzveig (1972), Brychkov and Prudnikov (1989), Pathak (1997), and particularly Pandey (1996). The latter source provides details on aspects of this topic that were glossed over or not treated at all in the present chapter. Further information can be found in papers by Horváth (1953a, 1956), Lauwerier (1963), Beltrami and Wohlers (1965), Güttinger (1966, 1967), Orton (1973), Pandey and Hughes (1976), Pandey (1983, 1997), Pandey and Chaudhry (1983), Chaudhry and Pandey (1985, 1987), Ishikawa (1987), Singh and Pandey (1990a), and Carton-Lebrun (1991, 2005).

§10.7 Beltrami and Wohlers (1966a), and Horváth (1953a, 1956) and Jones (1965) would be useful starting points for extra reading. A concise discussion is given in Kierat and Sztaba (2003).

§10.8 Additional reading can be found in the books by Beltrami and Wohlers (1966a) and Bremermann (1965a), and the papers by Beltrami and Wohlers (1965, 1966b, 1967), Bremermann (1967), and Orton (1973, 1977).

§10.9 For additional reading, see Pandey (1996) and Gasquet and Witomski (1999).

§10.11 Gel'fand and Shilov (1962, 1968) and Pandey (1996) are good sources for further reading.

§10.12 Zemanian (1965) is an excellent source for additional discussion on periodic distributions.

§10.13 For additional reading, see Pandey (1996, 1997, 2001, 2004).

§10.14 Further discussion can be found in a series of papers by Pandey and coworkers: Pandey (1983), Chaudhry and Pandey (1987), Singh and Pandey (1990a, 1990b), and the books by Pandey (1996, chap. 4) and Pathak (1997, p. 202). See also the work of Carton-Lebrun (1988), Pilipović (1987), Pilipović and Sad (1990), Toland (1997b), and Chung (2001). For the definition of the Stieltjes-Hilbert transform of distributions, see Stanković (1988).

Exercises

10.1 Is $\mathcal{D}'(\mathbb{R})$ the dual of $\mathcal{D}(\mathbb{R})$? Explain.

10.2 Do any analytic functions belong to $\mathcal{D}$? Explain.

10.3 Is it possible to assign a value to $\delta(x)$ at $x = 0$? Explain.
10.4 Is δ a tempered distribution?
10.5 Does $\delta \in \mathcal{E}'$?
10.6 To what Sobolev space does $p.v.(1/x)$ belong?
10.7 What functions belong to $\mathcal{Z} \cap \mathcal{D}$?
10.8 Do the functions e^{-x} and $e^{-|x|}$ belong to $\mathscr{S}$?
10.9 Determine $\mathcal{F}p.v.(1/|x|)$.
10.10 Calculate $\mathcal{F} \log |x|$.
10.11 Evaluate $H\delta(ax + b)$, where a and b are real constants and $a \neq 0$.
10.12 Determine $H\delta'$.
10.13 Calculate $\mathcal{F}\delta^+$ and $\mathcal{F}\delta^-$.
10.14 If $T \in \mathcal{D}'$, evaluate $\delta * T$ and $\delta^{(k)} * T$.
10.15 If T_1 and T_2 belong to $\mathscr{S}'$, is $T_1 * T_2$ in general a distribution?
10.16 Do any analytic functions belong to the space $C_c^\infty(\mathbb{R})$? Explain.
10.17 If $\phi \in \mathcal{D}$, does it follow that $\mathcal{F}\phi \in \mathcal{D}$? Explain.
10.18 Determine if the following relation holds: $\delta^{(1)} * (1 * \mathrm{H}) = (\delta^{(1)} * 1) * \mathrm{H}$?
10.19 Does $(\operatorname{sgn} x * \delta') * 1 = \operatorname{sgn} x * (\delta' * 1) = (\operatorname{sgn} x * 1) * \delta'$ hold?
10.20 Evaluate $H[\operatorname{csch} x]$.
10.21 Evaluate $H[x^{-2} \sin ax]$ and $H[x^{-2} \cos ax]$, where a is a real constant.
10.22 Does the unitary function given in Eq. (10.197) belong to $\mathcal{Z}$?
10.23 The periodic distribution $\sum_{k=-\infty}^{\infty} \delta(t - ka)$, where a is a constant, belongs to $\mathscr{S}'$. Determine its Hilbert transform.
10.24 If

$$U(t) = \begin{cases} 1 - |t|, & -1 < t < 1 \\ 0, & |t| \geq 1, \end{cases}$$

is it a unitary function with period one? Explain.
10.25 If $T \in \mathscr{S}'$, simplify $H[x^m T]$ for $m \in \mathbb{N}$.

11

The finite Hilbert transform

11.1 Introduction

This chapter treats the finite Hilbert transform. This important transform arises in a number of applications, of which the best known is probably the airfoil problem. Most of the basic properties of the finite Hilbert transform are treated in detail, and some strategies are developed for the evaluation of particular cases of this transform.

For the function f, one definition of the finite Hilbert transform is given by

$$g(x) = \frac{1}{\pi} P \int_{-a}^{a} \frac{f(y)\mathrm{d}y}{x-y}, \tag{11.1}$$

where the point x lies in the interval $(-a, a)$, so the Cauchy principal value of the integral is required. The principal objective of this chapter is the exploration of the properties of the finite Hilbert transform. One important application is discussed in detail.

The definition in some sources (for example Erdélyi *et al*. (1954, Vol. II, p. 239)) uses an integration interval of (a, b), for arbitrary a and b. When (a, b) is not $(-\infty, \infty)$, the resulting transforms are often referred to as *truncated* Hilbert transforms. Frequently, the finite Hilbert transform is defined with the integration domain in Eq. (11.1) taken as $(-1, 1)$. The notation $Tf(x)$ is used to designate the finite Hilbert transform, and the following definition is employed:

$$g(x) = Tf(x) = \frac{1}{\pi} P \int_{-1}^{1} \frac{f(y)\mathrm{d}y}{x-y}, \quad \text{for } -1 < x < 1. \tag{11.2}$$

The reader is alerted to the fact that the notation T is often used in the literature to denote a more general singular integral operator than the one just defined, so some caution is required when comparing the notation in different sources. In a fashion similar to the definition of the Hilbert transform on $(-\infty, \infty)$, the finite Hilbert transform is often defined with the opposite sign convention to that employed in Eq. (11.2). The sign choice given is consistent with the earlier sign convention that was adopted for Hf. A straightforward change of integration variable converts the integration interval from $(-a, a)$ to $(-1, 1)$. This particular equation is sometimes called the *airfoil equation*, and the name gives an indication of the sub-branch of aerodynamics where this formula arises. Equation (11.2) is occasionally referred to as the *Tricomi transform*. Tricomi was one of the early investigators of the finite Hilbert transform, and some of his work is covered in the sections that follow. The function f belongs to the class $L^p(-1, 1)$, for $1 \leq p < \infty$. The existence of Tf *a.e.* can be established along lines similar to that employed earlier for Hf (see Section 4.25). A reminder to the reader on notation is in order. In a manner identical to the standard Hilbert transform, $T[f(t)](x)$, where t is the dummy integration variable, is abbreviated to $Tf(x)$ when there is no interest in specifying the integration variable for the finite Hilbert transform.

Two examples of the finite Hilbert transform are first considered, and some other cases are dealt with in due course. The first example examined is $f(x) = \sin ax$, for $a > 0$ and $-1 < x < 1$. With a change of integration variable and expansion of the sine term, it follows that

$$\begin{aligned} g(x) &= \frac{1}{\pi} P \int_{-1}^{1} \frac{\sin ay \, dy}{x - y} \\ &= \frac{\sin ax}{\pi} P \int_{a(x-1)}^{a(x+1)} \frac{\cos y \, dy}{y} - \frac{\cos ax}{\pi} \int_{a(x-1)}^{a(x+1)} \frac{\sin y \, dy}{y}. \end{aligned} \tag{11.3}$$

The integrals in Eq. (11.3) can be recast in terms of the cosine integral, defined by (see Erdélyi *et al.* (1953, Vol. II, p. 145), Abramowitz and Stegun (1965, p. 231), and Gradshteyn and Ryzhik (1965, p. xxxiii); and given in Eq. (8.78) in a slightly different form)

$$\mathrm{Ci}(z) = \gamma + \log z + \int_0^z \frac{\cos y - 1}{y} \, dy, \tag{11.4}$$

where γ is Euler's constant ($\gamma = 0.577\,215\,664\,9\ldots$), and the sine integral, defined by

$$\mathrm{Si}(z) = \int_0^z \frac{\sin y \, dy}{y}. \tag{11.5}$$

Using these two definitions, it follows that

$$\begin{aligned} \frac{1}{\pi} P \int_{-1}^{1} \frac{\sin ay \, dy}{x - y} = {} & \frac{\sin ax}{\pi} \{\mathrm{Ci}[a(1+x)] - \mathrm{Ci}[a(1-x)]\} \\ & - \frac{\cos ax}{\pi} \{\mathrm{Si}[a(1+x)] + \mathrm{Si}[a(1-x)]\}. \end{aligned} \tag{11.6}$$

In a similar manner it can be deduced, for $a > 0$ and $-1 < x < 1$, that

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{\cos ay \, \mathrm{d}y}{x-y} = \frac{\cos ax}{\pi}\{\mathrm{Ci}[a(1+x)] - \mathrm{Ci}[a(1-x)]\} + \frac{\sin ax}{\pi}\{\mathrm{Si}[a(1+x)] + \mathrm{Si}[a(1-x)]\}. \tag{11.7}$$

11.2 Alternative formulas: the cosine form

The finite Hilbert transform can be written in some slightly different ways. Using the following change of variables:

$$u = \cos x, \tag{11.8}$$

$$v = \cos y, \tag{11.9}$$

$$G(x) = g(\cos x), \tag{11.10}$$

and

$$F(y) = \sin y \, f(\cos y), \tag{11.11}$$

then

$$g(u) = \frac{1}{\pi} P \int_{-1}^{1} \frac{f(v)\mathrm{d}v}{u-v} \tag{11.12}$$

becomes

$$G(x) = \frac{1}{\pi} P \int_{0}^{\pi} \frac{F(y)\mathrm{d}y}{\cos x - \cos y}. \tag{11.13}$$

Equation (11.13) will be called the cosine form of the finite Hilbert transform. Some examples of this formula are now examined, a couple of which find application in the developments presented later. The simplest case is $F(y)$ equal to a constant. In integrals of this type, a substitution that is often successful is the following:

$$\cos y = \frac{1-t^2}{1+t^2}, \quad t > 0, \tag{11.14}$$

$$\sin y = \frac{2t}{1+t^2}, \tag{11.15}$$

with

$$\mathrm{d}y = 2(1+t^2)^{-1}\,\mathrm{d}t. \tag{11.16}$$

Employing Eqs. (11.14) and (11.16), it follows that, for a constant c,

$$\frac{1}{\pi}P\int_0^{\pi}\frac{c\,\mathrm{d}y}{\cos x-\cos y}=\frac{2c}{\pi(1+\cos x)}P\int_0^{\infty}\frac{\mathrm{d}t}{t^2-((1-\cos x)/(1+\cos x))}$$
$$=0. \tag{11.17}$$

The case $F(y)=\sin y$ is most readily evaluated using the substitution $u=\cos y$, leading to

$$\begin{aligned}\frac{1}{\pi}P\int_0^{\pi}\frac{\sin y\,\mathrm{d}y}{\cos x-\cos y}&=\frac{1}{\pi}P\int_{-1}^{1}\frac{\mathrm{d}u}{\cos x-u}\\&=\frac{1}{\pi}\log\left(\frac{1+\cos x}{1-\cos x}\right)\\&=\frac{2}{\pi}\log\cot\left(\frac{x}{2}\right),\quad\text{for } 0<x<\pi.\end{aligned}\tag{11.18}$$

As a third example, suppose $F(y)=\cos y$. Define

$$\beta=\frac{1}{\pi(1+\cos x)}\tag{11.19}$$

and

$$a^2=\frac{1-\cos x}{1+\cos x};\tag{11.20}$$

then the substitutions given in Eqs. (11.14) – (11.16) work effectively, leading to

$$\begin{aligned}\frac{1}{\pi}P\int_0^{\pi}\frac{\cos y\,\mathrm{d}y}{\cos x-\cos y}&=\beta P\int_{-\infty}^{\infty}\frac{(1-t^2)\mathrm{d}t}{(1+t^2)(t^2-a^2)}\\&=\beta P\int_{-\infty}^{\infty}\frac{\{2-(1+t^2)\}\mathrm{d}t}{(1+t^2)(t^2-a^2)}\\&=2\beta P\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(1+t^2)(t^2-a^2)}.\end{aligned}\tag{11.21}$$

The even character of the integrand has been utilized to expand the integration range to $(-\infty,\ \infty)$, and the last simplification employs Eq. (11.17). The integral in Eq. (11.21) can be most conveniently evaluated using contour integral techniques. Working with the contour in Figure 11.1 leads to the following result:

$$\begin{aligned}P\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(1+t^2)(t^2-a^2)}&=2\pi\mathrm{i}\lim_{z\to\mathrm{i}}\left\{\frac{1}{(z+\mathrm{i})(z^2-a^2)}\right\}\\&=-\frac{\pi}{a^2+1}.\end{aligned}\tag{11.22}$$

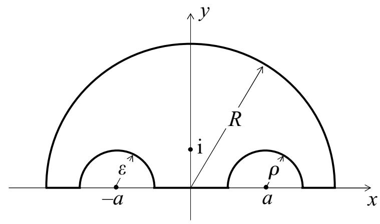

Figure 11.1. Half circle contour in the upper half plane with indentations at the points $\pm a$ on the real axis.

The contribution from the semicircular section vanishes in the limit $R \to \infty$, and the two contributions from the small semicircular arcs cancel each other in the limits $\varepsilon \to 0$ and $\rho \to 0$. Hence, it follows that

$$\frac{1}{\pi} P \int_0^{\pi} \frac{\cos y \, \mathrm{d}y}{\cos x - \cos y} = -1. \tag{11.23}$$

The preceding example is now re-examined using a different approach, one that will prove to be useful in later developments. Start with the following standard series expansion (see, for example, Jolly (1961, p. 96), Hansen (1975, p. 239), and Benedetto (1997, p. 267)):

$$\log(2 \, \sin \theta) = -\sum_{n=1}^{\infty} \frac{\cos 2n\theta}{n}, \quad 0 < \theta < \pi. \tag{11.24}$$

Now, in turn substitute $\theta = (1/2)(y - x)$ and $\theta = (1/2)(y + x)$, then add the results, to obtain

$$\log |2 \, [\cos x - \cos y]| = -2 \sum_{n=1}^{\infty} \frac{\cos ny \cos nx}{n}. \tag{11.25}$$

Differentiating both sides of this result with respect to x leads to

$$\frac{\sin x}{\cos y - \cos x} = 2 \sum_{n=1}^{\infty} \sin nx \cos ny. \tag{11.26}$$

This result can be derived in an alternative manner. Start with the trigonometric identity

$$\frac{2 \sin x}{\cos y - \cos x} = \cot\left(\frac{x - y}{2}\right) + \cot\left(\frac{x + y}{2}\right). \tag{11.27}$$

Employing entry number 686 from Jolly (1961, p. 128), setting $\beta = 0$, and taking the limit $a \to 1-$ in this entry, yields the following expansion (understood in the distributional sense):

$$\frac{1}{2}\cot\left(\frac{\theta}{2}\right) = \sum_{n=1}^{\infty} \sin n\theta; \tag{11.28}$$

then, on using $\theta = x - y$ and $\theta = x + y$ in turn, Eq. (11.26) is obtained.

From Eq. (11.26), it follows that

$$\begin{aligned}\frac{1}{\pi}P\int_0^{\pi}\frac{\cos y\,\mathrm{d}y}{\cos x - \cos y} &= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{\sin x}\int_0^{\pi}\cos y\cos ny\,\mathrm{d}y\\ &= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{\sin x}\frac{\pi}{2}\delta_{n1}\\ &= -1.\end{aligned} \tag{11.29}$$

The reader should justify the switch in order of the summation and the integration in this sequence. As a final example, consider $F(y) = \sin y \sin ny$. Using Eq. (11.26) with x and y interchanged,

$$\begin{aligned}\frac{1}{\pi}P\int_0^{\pi}\frac{\sin ny\sin y\,\mathrm{d}y}{\cos x - \cos y} &= \frac{2}{\pi}\sum_{m=1}^{\infty}\cos mx\int_0^{\pi}\sin ny\sin my\,\mathrm{d}y\\ &= \frac{2}{\pi}\sum_{m=1}^{\infty}\cos mx\frac{\pi}{2}\delta_{nm}\\ &= \cos nx.\end{aligned} \tag{11.30}$$

One form of the truncated Hilbert transform is defined by

$$g(x) = \frac{1}{\pi}P\int_0^{\infty}\frac{f(y)\mathrm{d}y}{x - y}. \tag{11.31}$$

The terminology "truncated Hilbert transform" is also employed with other limits, common among them being $(0, \infty)$, which is referred to as the one-sided Hilbert transform, and $(|x - y| > \varepsilon, \infty)$, for $\varepsilon > 0$; this has been labeled as H_ε in previous chapters. Equation (11.31) can be reduced to the cosine form discussed earlier, by introducing the following new variables:

$$y = \frac{1 - \cos v}{1 + \cos v}, \quad x = \frac{1 - \cos u}{1 + \cos u}, \tag{11.32}$$

$$G(u) = -g\left(\tan^2\left(\frac{u}{2}\right)\right)\tan\left(\frac{u}{2}\right)\csc u, \tag{11.33}$$

and

$$F(v) = f\left(\tan^2\left(\frac{v}{2}\right)\right)\tan\left(\frac{v}{2}\right), \tag{11.34}$$

so that

$$G(u) = \frac{1}{\pi} P \int_0^{\pi} \frac{F(v)\mathrm{d}v}{\cos u - \cos v}. \tag{11.35}$$

11.2.1 A result due to Hardy

By considering the function

$$h(s, t, x) = \frac{\phi(s)}{(\cos \pi s - \cos \pi t)(\cos \pi t - \cos \pi x)}, \tag{11.36}$$

Hardy (1908) established, for suitable $\phi(s)$, that

$$\begin{aligned} P \int_0^1 \frac{\mathrm{d}t}{\cos \pi x - \cos \pi t} P \int_0^1 \frac{\phi(s)\mathrm{d}s}{\cos \pi t - \cos \pi s} \\ = \csc^2 \pi x \, \{\cos^2(\pi x/2)\, \phi(0) + \sin^2(\pi x/2)\, \phi(1) - \phi(x)\}. \end{aligned} \tag{11.37}$$

The choice $\phi(s) = \sin \pi s f(\pi s)$ and the definition

$$g(t) = \frac{1}{\pi} P \int_0^{\pi} \frac{\sin s f(s)\mathrm{d}s}{\cos t - \cos s}, \tag{11.38}$$

leads to

$$f(x) = -\frac{1}{\pi} P \int_0^{\pi} \frac{\sin x\, g(s)\mathrm{d}s}{\cos x - \cos s}. \tag{11.39}$$

These two results represent a conjugate pair of Hilbert transforms on the circle.

11.3 The cotangent form

Equation (11.12) can be recast into another form that is very commonly seen in the literature. Introducing the substitutions given in Eqs. (11.8) and (11.9), and recalling the trigonometric identity

$$\frac{2 \sin y}{\cos y - \cos x} = \cot\left(\frac{x - y}{2}\right) - \cot\left(\frac{x + y}{2}\right), \tag{11.40}$$

leads to

$$g(\cos x) = \frac{1}{\pi} P \int_0^{\pi} \frac{f(\cos y) \sin y \, \mathrm{d}y}{\cos x - \cos y}$$
$$= \frac{1}{2\pi} P \int_0^{\pi} f(\cos y) \left\{ \cot\left(\frac{x+y}{2}\right) - \cot\left(\frac{x-y}{2}\right) \right\} \mathrm{d}y. \tag{11.41}$$

On introducing Eq. (11.10), the substitution

$$F(y) = f(\cos y), \tag{11.42}$$

and supposing that $F(y)$ is an odd function, then Eq. (11.41) can be written as follows:

$$G(x) = -\frac{1}{2\pi} P \int_{-\pi}^{\pi} F(y) \cot\left(\frac{x-y}{2}\right) \mathrm{d}y. \tag{11.43}$$

If $F(y)$ is a periodic function with period 2π, then Eq. (11.43) can be written as

$$G(x) = -\frac{1}{2\pi} P \int_0^{2\pi} F(y) \cot\left(\frac{x-y}{2}\right) \mathrm{d}y, \tag{11.44}$$

or, with the obvious notational simplification for the arguments of G and F,

$$G(x) = -P \int_0^1 F(y) \cot[\pi(x-y)] \mathrm{d}y, \tag{11.45}$$

which corresponds to a form given in Section 3.1. As an alternative, Eq. (11.40) can be employed with x and y interchanged; then, on setting

$$G(x) = \sin x \, g(x), \tag{11.46}$$

and using Eq. (11.42), it follows that

$$G(x) = -\frac{1}{2\pi} P \int_0^{\pi} F(y) \left\{ \cot\left(\frac{x+y}{2}\right) + \cot\left(\frac{x-y}{2}\right) \right\} \mathrm{d}y, \tag{11.47}$$

which simplifies, if $F(y)$ is an even function, to yield

$$G(x) = -\frac{1}{2\pi} P \int_{-\pi}^{\pi} F(y) \cot\left(\frac{x-y}{2}\right) \mathrm{d}y. \tag{11.48}$$

If $F(y)$ is a periodic function with period 2π, this result can be written as

$$G(x) = -\frac{1}{2\pi} P \int_0^{2\pi} F(y) \cot\left(\frac{x-y}{2}\right) \mathrm{d}y, \tag{11.49}$$

which can be put in the following form:

$$G(x) = -P\int_0^1 F(y)\cot[\pi(x-y)]\mathrm{d}y. \tag{11.50}$$

This result and Eq. (11.45) represent the connection between the finite Hilbert transform for a periodic function and the Hilbert transform on the circle.

11.4 The inversion formula: Tricomi's approach

The next issue addressed is the inversion formula for Eq. (11.2). The simple inversion symmetry $g(x) = Hf(x)$, then $f(x) = -Hg(x)$, that exists for the Hilbert transform on the line, no longer applies to the finite Hilbert transform. Finding the inverse formula can be regarded as a problem in integral equations, and that is most commonly the area of mathematics where discussion of this topic is found. There are different techniques available to carry out the inversion process. The approaches can be broadly defined as belonging to one of the following areas:

(1) methods based on the Hardy–Poincaré–Bertrand formula;
(2) techniques based on Fourier theory;
(3) approaches derived from complex variable theory;
(4) schemes that depend on transform techniques.

This section and the following sections examine some different approaches.

The following result was given in Section 4.16:

$$H\{\phi_1(x)H\phi_2(x) + \phi_2(x)H\phi_1(x)\} = H\phi_1(x)H\phi_2(x) - \phi_1(x)\phi_2(x), \tag{11.51}$$

which was proved by Tricomi (1951a). Equation (11.51) can also be generalized to cover the finite Hilbert transform; that is,

$$T\{\phi_1(x)T\phi_2(x) + \phi_2(x)T\phi_1(x)\} = T\phi_1(x)T\phi_2(x) - \phi_1(x)\phi_2(x). \tag{11.52}$$

If the support of the functions ϕ_1 and ϕ_2 is restricted to the interval $(-1, 1)$, and the Hardy–Poincaré–Bertrand formula given in Eq. (4.504) is employed with a partial fraction decomposition of the term $\{(t-x)(s-t)\}^{-1}$, then the Tricomi identity for the finite Hilbert transform follows immediately. Alternatively, the Hardy–Poincaré–Bertrand formula on a finite interval can be developed, from which the Tricomi identity follows.

Tricomi's equation, Eq. (11.52), can be employed to determine the inversion formula for the finite Hilbert transform. Tricomi started with the following choice:

$$\phi_1(x) = f(x), \quad \phi_2(x) = \surd(1-x^2). \tag{11.53}$$

To proceed, the finite Hilbert transform of ϕ_2 is required. This can be obtained in the following manner. Consider first the finite Hilbert transform of $f(x) = 1/\surd(1-x^2)$, which can be evaluated as follows:

$$
\begin{aligned}
T\left[\frac{1}{\surd(1-x^2)}\right] &= \frac{1}{\pi}P\int_{-1}^{1}\frac{\mathrm{d}y}{(x-y)\surd(1-y^2)} \\
&= \frac{2}{\pi(x+1)}P\int_0^\infty \frac{\mathrm{d}s}{s^2-(1-x)(1+x)^{-1}},
\end{aligned}
\tag{11.54}
$$

where the preceding line is obtained on using the substitution $y = (1-s^2)(1+s^2)^{-1}$. The integral can be readily evaluated using partial fractions, and hence

$$
T\left[\frac{1}{\surd(1-x^2)}\right] = 0. \tag{11.55}
$$

The finite Hilbert transform of $\surd(1-x^2)$ can be easily evaluated taking advantage of the preceding result. Employing Eq. (11.55) leads to

$$
\begin{aligned}
T[\surd(1-x^2)] &= \frac{1}{\pi}P\int_{-1}^{1}\frac{(1-y^2)\mathrm{d}y}{(x-y)\surd(1-y^2)} \\
&= \frac{1}{\pi}P\int_{-1}^{1}\frac{(x^2-y^2)\mathrm{d}y}{(x-y)\surd(1-y^2)} \\
&= \frac{1}{\pi}\int_{-1}^{1}\frac{(x+y)\mathrm{d}y}{\surd(1-y^2)},
\end{aligned}
\tag{11.56}
$$

and hence, on noting the odd behavior of part of the integrand,

$$
T[\surd(1-x^2)] = x. \tag{11.57}
$$

On inserting Eq. (11.53) and employing Eq. (11.57), Eq. (11.52) leads to

$$
T\{f(x)\ T[\surd(1-x^2)] + \surd(1-x^2)\ Tf(x)\} = Tf(x)\ T[\surd(1-x^2)] - f(x)\surd(1-x^2), \tag{11.58}
$$

and using $g(x) = (Tf)(x)$ yields

$$
T\{xf(x) + \surd(1-x^2)\ g(x)\} = x\,g(x) - f(x)\,\surd(1-x^2). \tag{11.59}
$$

The first part of Eq. (11.59) can be simplified as follows:

$$
\begin{aligned}
T[xf(x)] &= \frac{1}{\pi}P\int_{-1}^{1}\frac{sf(s)\mathrm{d}s}{x-s} \\
&= xTf(x) - \frac{1}{\pi}\int_{-1}^{1}f(s)\mathrm{d}s.
\end{aligned}
\tag{11.60}
$$

Equation (11.60) represents, for the finite Hilbert transform, the analog of the moment formula property developed for $H[xf(x)]$ (see Eq. (4.111)). The constant c is introduced by the definition

$$c = \frac{1}{\pi}\int_{-1}^{1} f(s)\mathrm{d}s; \tag{11.61}$$

then Eq. (11.59) becomes

$$x\,g(x) - c + T[\surd(1-x^2)g(x)] = x\,g(x) - f(x)\,\surd(1-x^2), \tag{11.62}$$

which rearranges to give

$$\begin{aligned} f(x) &= \frac{c}{\surd(1-x^2)} - \frac{T[\surd(1-x^2)g(x)]}{\surd(1-x^2)} \\ &= \frac{c}{\surd(1-x^2)} - \frac{1}{\pi\,\surd(1-x^2)}P\int_{-1}^{1}\frac{\surd(1-s^2)g(s)\mathrm{d}s}{x-s}. \end{aligned} \tag{11.63}$$

Equation (11.63) is the inversion formula for the finite Hilbert transform. As a consequence of Eq. (11.55), the constant c can be treated as arbitrary, rather than being defined by Eq. (11.61). Because of the arbitrariness of this constant, there is no unique inverse transformation for the finite Hilbert transform.

Before proceeding to check Eq. (11.63), the class of functions for which this equation holds needs to be considered. Let $f \in L^p(-1, 1)$, $g \in L^{p'}(-1, 1)$, where $p > 1$ and $p' > 1$. If the support of f is $(-1, 1)$ and if $f \in L^p(\mathbb{R})$, then $Tf \in L^p$ for $1 < p < \infty$, a fact that is employed momentarily. The preceding result follows directly from the Riesz inequality for Hf. The assumption on p' helps establish the correctness of the inversion formula given in Eq. (11.63). The upper limit of p is now fixed and the lower limit of p' is determined as follows. The appearance of the term $c/\surd(1-x^2)$ in the solution for $f(x)$ implies that $p < 2$. This value can be fine-tuned by examining the integral occurring in Eq. (11.63). Following an argument given by Tricomi (1985, p. 176), let

$$h(x) = -\frac{1}{\pi\,\surd(1-x^2)}P\int_{-1}^{1}\frac{\surd(1-s^2)g(s)\mathrm{d}s}{x-s}; \tag{11.64}$$

then

$$\begin{aligned} h(x) &= \frac{1}{\pi\,\surd(1-x^2)}P\int_{-1}^{1}\frac{\{\surd(1-x^2)-\surd(1-s^2)\}g(s)\mathrm{d}s}{x-s} - \frac{1}{\pi}P\int_{-1}^{1}\frac{g(s)\mathrm{d}s}{x-s} \\ &= -\frac{1}{\pi\,\surd(1-x^2)}\int_{-1}^{1}\frac{(x+s)g(s)\mathrm{d}s}{\surd(1-x^2)+\surd(1-s^2)} - (Tg)(x). \end{aligned} \tag{11.65}$$

The factor Tg belongs to the same class as g, so attention can be focused on the first integral in Eq. (11.65), which is denoted by the function $k(x)$. Thus, by Hölder's inequality,

$$|k(x)| \leq \left(\int_{-1}^{1} \left[\frac{|x+s|}{\surd(1-x^2)+\surd(1-s^2)} \right]^{p_1} \mathrm{d}s \right)^{p_1^{-1}} \left(\int_{-1}^{1} |g(s)|^{p_2}\, \mathrm{d}s \right)^{p_2^{-1}}, \tag{11.66}$$

where p_1 and p_2 are a pair of conjugate exponents; hence,

$$\int_{-1}^{1} |k(x)|^{p_1}\, \mathrm{d}x \leq \left(\int_{-1}^{1} |g(s)|^{p_2}\, \mathrm{d}s \right)^{p_1 p_2^{-1}} \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \left[\frac{|x+s|}{\surd(1-x^2)+\surd(1-s^2)} \right]^{p_1} \mathrm{d}s. \tag{11.67}$$

From the double integral it follows that $p_1 < 4$, and so $p_2 > 4/3$. To see that $p_1 < 4$, convert the double integral to the interval $(0,\ 1) \times (0,\ 1)$ and employ the change of variables $x = (1+u)^{-1}$ and $s = (1+v)^{-1}$; then

$$\begin{aligned}\int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} &\left[\frac{|x+s|}{\surd(1-x^2)+\surd(1-s^2)} \right]^{p_1} \mathrm{d}s \\ &= 4\int_{0}^{\infty} \frac{\mathrm{d}u}{(1+u)^2} \int_{0}^{\infty} \frac{1}{(1+v)^2} \left[\frac{|1-uv\,|}{(1+v)\surd u + \ (1+u)\surd v} \right]^{p_1} \mathrm{d}v. \end{aligned} \tag{11.68}$$

Now focus attention on the critical part of the integral as $u \to 0$ and $v \to 0$, ignoring the unessential factors. Using the variable change $v = ux$ leads to

$$\begin{aligned}\lim_{\varepsilon_1\to 0} \lim_{\varepsilon_2\to 0} \int_{\varepsilon_1} \mathrm{d}u \int_{\varepsilon_2} \frac{\mathrm{d}v}{[\surd u + \surd v]^{p_1}} &\to \lim_{\varepsilon_1\to 0} \lim_{\varepsilon_2\to 0} \int_{\varepsilon_1} u^{1-p_1/2}\, \mathrm{d}u \int_{\varepsilon_2} \frac{\mathrm{d}x}{(1+x)^{p_1}} \\ &\to \lim_{\varepsilon_1\to 0} \int_{\varepsilon_1} \frac{\mathrm{d}u}{u^{p_1/2-1}} \end{aligned} \tag{11.69}$$

from which the required condition on p_1 follows. The following result is needed: if $f_1 \in L^{q_1}(-1,\ 1)$ and $f_2 \in L^{q_2}(-1,\ 1)$, then $f_1 f_2 \in L^r(-1,\ 1)$, with $r^{-1} = q_1^{-1} + q_2^{-1}$. Noting that $1/\surd(1-x^2)$ is in $L^q(-1,\ 1)$ for $q < 2$, and using Eq. (11.67), leads to the result that $f \in L^p(-1,\ 1)$ for $1 < p < 4/3$ provided $g \in L^{p'}(-1,\ 1)$ for $p' > 4/3$. If $g \in L^{p'}(-1,\ 1)$ for $1 < p' \leq 4/3$, then it cannot be asserted that $f \in L^p(-1,\ 1)$ for any $1 < p$. However, it might be conjectured that, if a solution of this class exists, it will take the form given by Eq. (11.63).

It will now be verified that Eq. (11.63) is indeed a solution of Eq. (11.2). From this pair of equations it follows that

$$\begin{aligned} g(x) &= \frac{1}{\pi} P \int_{-1}^{1} \frac{f(y)\mathrm{d}y}{x-y} \\ &= \frac{1}{\pi} P \int_{-1}^{1} \frac{\mathrm{d}y}{x-y} \left\{ \frac{c}{\sqrt{(1-y^2)}} - \frac{1}{\pi\sqrt{(1-y^2)}} P \int_{-1}^{1} \frac{\sqrt{(1-s^2)}g(s)\mathrm{d}s}{y-s} \right\} \\ &= -\frac{1}{\pi^2} P \int_{-1}^{1} \frac{\mathrm{d}y}{x-y} \left\{ \frac{1}{\sqrt{(1-y^2)}} P \int_{-1}^{1} \frac{\sqrt{(1-s^2)}g(s)\mathrm{d}s}{y-s} \right\}, \end{aligned} \tag{11.70}$$

and the last line follows using Eq. (11.55). If the following identifications are made:

$$f_1(x) = \frac{1}{\sqrt{(1-x^2)}} \tag{11.71}$$

and

$$f_2(x) = -g(x)\sqrt{(1-x^2)}, \tag{11.72}$$

then

$$g(x) = \frac{1}{\pi^2} P \int_{-1}^{1} \frac{f_1(y)\mathrm{d}y}{x-y} \left\{ P \int_{-1}^{1} \frac{f_2(s)\mathrm{d}s}{y-s} \right\}. \tag{11.73}$$

The Hardy–Poincaré–Bertrand formula takes the form

$$\begin{aligned} \frac{1}{\pi^2} P \int_a^b \frac{h_1(s)\mathrm{d}s}{s-x} \left\{ P \int_a^b \frac{h_2(y)\mathrm{d}y}{y-s} \right\} &= \frac{1}{\pi^2} P \int_a^b h_2(y)\mathrm{d}y \left\{ P \int_a^b \frac{h_1(s)\mathrm{d}s}{(y-s)(s-x)} \right\} \\ &\quad - h_1(x)h_2(x) \end{aligned} \tag{11.74}$$

for functions $h_1(x)$ and $h_2(x)$, which are assumed to belong to $L^p(a, b)$, for $1 < p < \infty$, and the condition is imposed that the support of these functions is restricted to the interval (a, b). Let $a = -1$, $b = 1$, and employ

$$\frac{1}{(s-x)(y-s)} = \frac{1}{y-x} \left(\frac{1}{s-x} - \frac{1}{s-y} \right); \tag{11.75}$$

then the integral in Eq. (11.73) can be written as follows:

$$\begin{aligned} \frac{1}{\pi^2} P \int_{-1}^{1} \frac{f_1(y)\mathrm{d}y}{x-y} \left\{ P \int_{-1}^{1} \frac{f_2(s)\mathrm{d}s}{y-s} \right\} &= \frac{1}{\pi^2} P \int_{-1}^{1} \frac{f_2(s)\mathrm{d}s}{x-s} \left\{ P \int_{-1}^{1} \frac{f_1(y)\mathrm{d}y}{x-y} \right\} \\ &\quad - \frac{1}{\pi^2} P \int_{-1}^{1} \frac{f_2(s)\mathrm{d}s}{x-s} \left\{ P \int_{-1}^{1} \frac{f_1(y)\mathrm{d}y}{s-y} \right\} \\ &\quad - f_1(x)f_2(x). \end{aligned} \tag{11.76}$$

This equation simplifies on noting that

$$P\int_{-1}^{1}\frac{f_1(y)\mathrm{d}y}{x-y}=P\int_{-1}^{1}\frac{f_1(y)\mathrm{d}y}{s-y}=0, \tag{11.77}$$

using Eqs. (11.71) and (11.55); hence, from Eqs. (11.74) and (11.75) it follows that

$$g(x)=-f_1(x)f_2(x), \tag{11.78}$$

which, on noting Eqs. (11.71) and (11.72), verifies that Eq. (11.63) is a solution of Eq. (11.2).

The inversion formula for the finite Hilbert transform can be recast in several different ways. Starting with the identity

$$\frac{1-y^2}{x-y}=\frac{1-x^2}{x-y}+x+y, \tag{11.79}$$

it follows that

$$\begin{aligned}\frac{\sqrt{(1-y^2)}}{(x-y)\sqrt{(1-x^2)}}&=\frac{\sqrt{(1-x^2)}}{(x-y)\sqrt{(1-y^2)}}+\frac{x}{\sqrt{(1-x^2)}\sqrt{(1-y^2)}}\\&\quad+\frac{y}{\sqrt{(1-x^2)}\sqrt{(1-y^2)}}.\end{aligned} \tag{11.80}$$

Hence, Eq. (11.63) can be written as follows:

$$\begin{aligned}f(x)=&\frac{c}{\sqrt{(1-x^2)}}-\frac{\sqrt{(1-x^2)}}{\pi}P\int_{-1}^{1}\frac{g(y)\mathrm{d}y}{(x-y)\sqrt{(1-y^2)}}\\&-\frac{x}{\pi\sqrt{(1-x^2)}}P\int_{-1}^{1}\frac{g(y)\mathrm{d}y}{\sqrt{(1-y^2)}}-\frac{1}{\pi\sqrt{(1-x^2)}}P\int_{-1}^{1}\frac{yg(y)\mathrm{d}y}{\sqrt{(1-y^2)}}.\end{aligned} \tag{11.81}$$

If the constants c_1, c_2, and c_3 are assigned as

$$c_1=-\frac{1}{\pi}P\int_{-1}^{1}\frac{g(y)\mathrm{d}y}{\sqrt{(1-y^2)}}, \tag{11.82}$$

$$c_2=-\frac{1}{\pi}P\int_{-1}^{1}\frac{yg(y)\mathrm{d}y}{\sqrt{(1-y^2)}}, \tag{11.83}$$

and $c_3=c+c_2$, then the following result is obtained:

$$f(x)=\frac{c_3}{\sqrt{(1-x^2)}}+\frac{c_1x}{\sqrt{(1-x^2)}}-\frac{\sqrt{(1-x^2)}}{\pi}P\int_{-1}^{1}\frac{g(y)\mathrm{d}y}{(x-y)\sqrt{(1-y^2)}}. \tag{11.84}$$

A further simplification of this result is possible. If the functions $h_1(x)$ and $h_2(x)$ belong to $L^{p_1}(a,b)$ and $L^{p_2}(a,b)$, with p_1 and p_2 satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} = 1, \tag{11.85}$$

then

$$\int_a^b h_1(x)\mathrm{d}x \left\{ P\int_a^b \frac{h_2(y)\mathrm{d}y}{x-y} \right\} = \int_a^b h_2(y)\mathrm{d}y \left\{ P\int_a^b \frac{h_1(x)\mathrm{d}x}{x-y} \right\}. \tag{11.86}$$

Using the substitutions $a = -1$, $b = 1$, and setting $h_1(x) = 1/\surd(1-x^2)$, $h_2(x) = f(y)$, then Eq. (11.86) simplifies, on using Eq. (11.55), to yield

$$\int_{-1}^{1} g(x)/\surd(1-x^2)\mathrm{d}x = 0. \tag{11.87}$$

Under the conditions necessary to write Eq. (11.86) and hence Eq. (11.87), Eq. (11.84) simplifies to

$$f(x) = \frac{c_3}{\surd(1-x^2)} - \frac{\surd(1-x^2)}{\pi} P\int_{-1}^{1} \frac{g(y)\mathrm{d}y}{(x-y)\surd(1-y^2)}. \tag{11.88}$$

This result can be obtained directly from Eq. (11.52) by employing

$$\phi_1(x) = f(x), \quad \phi_2(x) = \frac{1}{\surd(1-x^2)}, \tag{11.89}$$

and, on noting Eq. (11.55),

$$T\left[\frac{g(x)}{\surd(1-x^2)}\right] = -\frac{f(x)}{\surd(1-x^2)}; \tag{11.90}$$

hence, Eq. (11.88) follows when the contribution $c_3/\surd(1-x^2)$ is added.

If instead of Eq. (11.80) the following result is employed:

$$\frac{\surd(1-y^2)}{\surd(1-x^2)} = \frac{\surd(1+x)}{\surd(1-x)}\frac{\surd(1-y)}{\surd(1+y)}\left[1 - \frac{x-y}{x+1}\right], \tag{11.91}$$

then

$$\begin{aligned} f(x) = {} & \frac{c}{\surd(1-x^2)} - \frac{\surd(1+x)}{\pi\surd(1-x)} P\int_{-1}^{1} \frac{\surd(1-y)g(y)\mathrm{d}y}{(x-y)\surd(1+y)} \\ & + \frac{1}{\pi\surd(1-x^2)} P\int_{-1}^{1} \frac{\surd(1-y)\,g(y)\mathrm{d}y}{\surd(1+y)}. \end{aligned} \tag{11.92}$$

Let

$$c_4 = \frac{1}{\pi} P \int_{-1}^{1} \frac{\sqrt{(1-y)}\, g(y) \mathrm{d}y}{\sqrt{(1+y)}} \tag{11.93}$$

and $c_5 = c + c_4$, then

$$f(x) = \frac{c_5}{\sqrt{(1-x^2)}} - \frac{\sqrt{(1+x)}}{\pi \sqrt{(1-x)}} P \int_{-1}^{1} \frac{\sqrt{(1-y)}\, g(y) \mathrm{d}y}{(x-y)\sqrt{(1+y)}}. \tag{11.94}$$

The alternative identity, given by

$$\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}} = \frac{\sqrt{(1-x)}}{\sqrt{(1+x)}} \frac{\sqrt{(1+y)}}{\sqrt{(1-y)}} \left[1 + \frac{x-y}{1-x} \right], \tag{11.95}$$

leads to

$$\begin{aligned} f(x) = &\frac{c}{\sqrt{(1-x^2)}} - \frac{\sqrt{(1-x)}}{\pi \sqrt{(1+x)}} P \int_{-1}^{1} \frac{\sqrt{(1+y)}\, g(y) \mathrm{d}y}{(x-y)\sqrt{(1-y)}} \\ &- \frac{1}{\pi \sqrt{(1-x^2)}} P \int_{-1}^{1} \frac{\sqrt{(1+y)}\, g(y) \mathrm{d}y}{\sqrt{(1-y)}}. \end{aligned} \tag{11.96}$$

Let

$$c_6 = -\frac{1}{\pi} P \int_{-1}^{1} \frac{\sqrt{(1+y)}\, g(y) \mathrm{d}y}{\sqrt{(1-y)}}, \tag{11.97}$$

and $c_7 = c + c_6$, then

$$f(x) = \frac{c_7}{\sqrt{(1-x^2)}} - \frac{\sqrt{(1-x)}}{\pi \sqrt{(1+x)}} P \int_{-1}^{1} \frac{\sqrt{(1+y)}\, g(y) \mathrm{d}y}{(x-y)\sqrt{(1-y)}}. \tag{11.98}$$

Equations (11.88), (11.94), and (11.98) represent alternative forms for the inversion of the finite Hilbert transform.

11.4.1 Inversion of the finite Hilbert transform for the interval **(0, 1)**

This subsection deals with the inversion formula for the finite Hilbert transform on the interval (0, 1), using a clever transformation approach exploited by Peters (1963). The general finite Hilbert transform on the interval (a, b), that is

$$\varphi(s) = \frac{1}{\pi} P \int_{a}^{b} \frac{\phi(t) \mathrm{d}t}{s-t}, \tag{11.99}$$

can be converted with the following changes of variables:

$$t = (b-a)\, y + a, \quad s = (b-a)\, x + a, \tag{11.100}$$

and the identifications $g(x) = \varphi(a + (b - a)x)$, and $f(y) = \phi(a + (b - a)y)$, to

$$g(x) = \frac{1}{\pi} P \int_0^1 \frac{f(y)\mathrm{d}y}{x - y}, \tag{11.101}$$

which is the form of interest for the remainder of this subsection. The functions f and g are assumed to be uniformly Hölder continuous on the interval [0, 1], and the restriction $0 < x < 1$ is adopted throughout. The problem of interest is the inversion of Eq. (11.101) to find the unknown function f.

Two preliminary items are required. In the sequel, the value of the following integral is needed:

$$I(x) = \frac{1}{\pi} P \int_0^1 \frac{\mathrm{d}y}{(x - y)\sqrt{[y(1 - y)]}}. \tag{11.102}$$

Using the substitution $y = s^2(1 + s^2)^{-1}$ leads to

$$I(x) = \frac{2}{\pi(x - 1)} P \int_0^\infty \frac{\mathrm{d}s}{s^2 - x/(1 - x)} = 0, \quad \text{for } 0 < x < 1. \tag{11.103}$$

The second preliminary item required is the inversion formula for Abel's integral equation. This equation takes the following form:

$$\psi(x) = \int_0^x \frac{h(y)\mathrm{d}y}{(x - y)^\alpha}, \quad 0 \le \alpha < 1, \tag{11.104}$$

and the solution is given by

$$h(x) = \frac{\sin \pi\alpha}{\pi} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{\psi(y)\mathrm{d}y}{(x - y)^{1-\alpha}}. \tag{11.105}$$

The reader can find this integral equation discussed in standard sources, for example Hochstadt (1973, p. 41) or Pipkin (1991, p. 99). The particular case of interest for the present discussion is $\alpha = 1/2$, and the solution of Eq. (11.104) for this choice can be inferred from the discussion given later in this subsection, so that

$$h(x) = \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{\psi(y)\mathrm{d}y}{\sqrt{(x - y)}}. \tag{11.106}$$

The following approach to the solution of Eq. (11.101) involves transforming it to an Abel integral equation, and then employing the inversion formula just given for this equation. The following development follows Peters' (1963) derivation. Equation (11.101) can be rewritten, using

$$\frac{1}{x - y} = \frac{y}{x} \frac{1}{x - y} - \frac{1}{x}, \tag{11.107}$$

as follows:

$$xg(x) = \frac{1}{\pi} P \int_0^1 \frac{y f(y)\mathrm{d}y}{x-y} - \frac{1}{\pi} \int_0^1 f(y)\mathrm{d}y. \tag{11.108}$$

Setting

$$c = \frac{1}{\pi} \int_0^1 f(y)\mathrm{d}y, \tag{11.109}$$

yields

$$\surd(x) g(x) + \frac{c}{\surd x} = \frac{1}{\pi \surd x} P \int_0^1 \frac{y f(y)\mathrm{d}y}{(\surd x - \surd y)(\surd x + \surd y)}, \tag{11.110}$$

and integration of this equation leads to

$$\begin{aligned}
\int_0^x \surd(t) g(t)\mathrm{d}t + 2c\surd x &= \int_0^x \frac{\mathrm{d}t}{\surd t} \frac{1}{\pi} P \int_0^1 \frac{y f(y)\mathrm{d}y}{(\surd t - \surd y)(\surd t + \surd y)} \\
&= \frac{1}{2\pi} \int_0^x \frac{\mathrm{d}t}{\surd t} P \int_0^1 \surd(y) f(y)\mathrm{d}y \left\{ \frac{1}{(\surd t - \surd y)} - \frac{1}{(\surd t + \surd y)} \right\} \\
&= \frac{1}{2\pi} P \int_0^1 \surd(y) f(y)\mathrm{d}y \int_0^x \left\{ \frac{1}{(\surd t - \surd y)} - \frac{1}{(\surd t + \surd y)} \right\} \frac{\mathrm{d}t}{\surd t} \\
&= \frac{1}{\pi} \int_0^1 \surd(y) f(y) \log \left| \frac{\surd x - \surd y}{\surd x + \surd y} \right| \mathrm{d}y.
\end{aligned} \tag{11.111}$$

Taking note of the result

$$2 \tanh^{-1} \surd\left(\frac{y}{x}\right) = \begin{cases} \displaystyle\int_0^y \frac{\mathrm{d}t}{\surd(y-t)\surd(x-t)}, & x > y \\ \displaystyle\int_0^x \frac{\mathrm{d}t}{\surd(y-t)\surd(x-t)}, & y > x, \end{cases} \tag{11.112}$$

and recalling that $\tanh^{-1}(x) = (1/2) \log[(1+x)/(1-x)]$, for $-1 < x < 1$, allows Eq. (11.111) to be written as follows:

$$\begin{aligned}
\int_0^x \surd(t)\, g(t)\mathrm{d}t + 2c\surd x = \frac{1}{\pi} \int_0^x \surd(y)\, f(y) \log \left| \frac{\surd x - \surd y}{\surd x + \surd y} \right| \mathrm{d}y \\
+ \frac{1}{\pi} \int_x^1 \surd(y)\, f(y) \log \left| \frac{\surd x - \surd y}{\surd x + \surd y} \right| \mathrm{d}y
\end{aligned}$$

$$= -\frac{1}{\pi}\int_0^x \surd(y)\, f(y)\mathrm{d}y \int_0^y \frac{\mathrm{d}t}{\surd(y-t)\surd(x-t)}$$
$$- \frac{1}{\pi}\int_x^1 \surd(y)\, f(y)\mathrm{d}y \int_0^x \frac{\mathrm{d}t}{\surd(y-t)\surd(x-t)}. \tag{11.113}$$

Making use of the formula for the switch of integration order,

$$\int_0^\alpha \mathrm{d}x \int_0^x F(x,y)\mathrm{d}y = \int_0^\alpha \mathrm{d}y \int_y^\alpha F(x,y)\mathrm{d}x, \tag{11.114}$$

Eq. (11.113) can be rewritten as follows:

$$\int_0^x \surd(t)\, g(t)\mathrm{d}t + 2c\surd x = -\frac{1}{\pi}\int_0^x \frac{\mathrm{d}t}{\surd(x-t)} \int_t^x \frac{\surd(y)\, f(y)\mathrm{d}y}{\surd(y-t)}$$
$$- \frac{1}{\pi}\int_0^x \frac{\mathrm{d}t}{\surd(x-t)} \int_x^1 \frac{\surd(y)\, f(y)\mathrm{d}y}{\surd(y-t)}$$
$$= -\frac{1}{\pi}\int_0^x \frac{\mathrm{d}t}{\surd(x-t)} \int_t^1 \frac{\surd(y)\, f(y)\mathrm{d}y}{\surd(y-t)}. \tag{11.115}$$

Setting $h(t) = -\frac{1}{\pi}\int_t^1 \frac{\surd(y)\, f(y)\mathrm{d}y}{\surd(y-t)}$ yields

$$\int_0^x \surd(t)\, g(t)\mathrm{d}t + 2c\surd x = \int_0^x \frac{h(t)\mathrm{d}t}{\surd(x-t)}, \tag{11.116}$$

and application of Eq. (11.106) leads to

$$h(x) = \frac{1}{\pi}\int_0^x \frac{\{\surd(y)g(y) + c/\surd y\}}{\surd(x-y)}\,\mathrm{d}y, \tag{11.117}$$

and hence

$$\int_x^1 \frac{\surd(y)\, f(y)\mathrm{d}y}{\surd(y-x)} = -\int_0^x \frac{\surd(y)\, g(y)}{\surd(x-y)}\,\mathrm{d}y - c\int_0^x \frac{\mathrm{d}y}{\surd[y(x-y)]}$$
$$= -\int_0^x \frac{\surd(y)\, g(y)}{\surd(x-y)}\,\mathrm{d}y - c\,\pi. \tag{11.118}$$

The solution of the following integral equation:

$$\psi(x) = \int_x^1 \frac{\phi(y)\mathrm{d}y}{\surd(y-x)} \tag{11.119}$$

is given by

$$\phi(x) = -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \frac{\psi(y)\mathrm{d}y}{\surd(y-x)}. \tag{11.120}$$

This can be obtained in the following manner. Multiplying Eq. (11.119) by the factor $(1/\pi)(1/\surd(x-y))$ and integrating over the interval $[y, 1]$ yields

$$\frac{1}{\pi}\int_y^1 \frac{\psi(x)\mathrm{d}x}{\surd(x-y)} = \frac{1}{\pi}\int_y^1 \frac{\mathrm{d}x}{\surd(x-y)}\int_x^1 \frac{\phi(t)\mathrm{d}t}{\surd(t-x)}. \tag{11.121}$$

Differentiating both sides with respect to y, making a change of integration order, and employing

$$\frac{1}{\pi}\int_y^t \frac{\mathrm{d}x}{\surd(x-y)\surd(t-x)} = 1, \tag{11.122}$$

leads to

$$\begin{aligned}\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\int_y^1 \frac{\psi(x)\mathrm{d}x}{\surd(x-y)} &= \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\int_y^1 \phi(t)\mathrm{d}t\int_y^t \frac{\mathrm{d}x}{\surd(t-x)\surd(x-y)}\\ &= \frac{\mathrm{d}}{\mathrm{d}y}\int_y^1 \phi(t)\mathrm{d}t\\ &= -\phi(y),\end{aligned} \tag{11.123}$$

which is the required result. A similar approach can be employed to obtain the solution given in Eq. (11.106).

Applying Eqs. (11.119) and (11.120) to Eq. (11.118) leads to

$$\begin{aligned}\surd(x)f(x) &= \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \frac{\mathrm{d}y}{\surd(y-x)}\int_0^y \frac{\surd(t)\,g(t)}{\surd(y-t)}\,\mathrm{d}t + c\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \frac{\mathrm{d}y}{\surd(y-x)}\\ &= -\frac{c}{\surd(1-x)} + \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \frac{\mathrm{d}y}{\surd(y-x)}\int_0^y \frac{\surd(t)\,g(t)}{\surd(y-t)}\,\mathrm{d}t.\end{aligned} \tag{11.124}$$

The double integral in the preceding result can be simplified in the following fashion. Noting the change of integration order formula,

$$\begin{aligned}\int_\beta^1 \mathrm{d}u\int_0^u F(u,v)\mathrm{d}v &= \int_\beta^1 \mathrm{d}u\left\{\int_0^\beta F(u,v)\mathrm{d}v + \int_\beta^u F(u,v)\mathrm{d}v\right\}\\ &= \int_0^\beta \mathrm{d}v\int_\beta^1 F(u,v)\mathrm{d}u + \int_\beta^1 \mathrm{d}v\int_v^1 F(u,v)\mathrm{d}u,\end{aligned} \tag{11.125}$$

which follows from Eq. (11.114), Eq. (11.124) can be cast as follows:

$$f(x) = -\frac{c}{\sqrt{[x(1-x)]}} + \frac{1}{\pi\sqrt{x}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sqrt{(t)}\, g(t)\mathrm{d}t \int_x^1 \frac{\mathrm{d}y}{\sqrt{(y-x)}\sqrt{(y-t)}}$$
$$+ \frac{1}{\pi\sqrt{x}}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \sqrt{(t)}\, g(t)\mathrm{d}t \int_t^1 \frac{\mathrm{d}y}{\sqrt{(y-x)}\sqrt{(y-t)}}. \tag{11.126}$$

Employing Eq. (11.112) in the form

$$\log\left|\frac{\sqrt{(1-t)}+\sqrt{(1-x)}}{\sqrt{(1-t)}-\sqrt{(1-x)}}\right|$$
$$= \begin{cases} \displaystyle\int_0^{1-x} \frac{\mathrm{d}s}{\sqrt{(1-t-s)}\sqrt{(1-x-s)}} = \int_x^1 \frac{\mathrm{d}y}{\sqrt{(y-x)}\sqrt{(y-t)}}, & x > t \\ \displaystyle\int_0^{1-t} \frac{\mathrm{d}s}{\sqrt{(1-t-s)}\sqrt{(1-x-s)}} = \int_t^1 \frac{\mathrm{d}y}{\sqrt{(y-x)}\sqrt{(y-t)}}, & t > x, \end{cases} \tag{11.127}$$

leads to

$$f(x) = -\frac{c}{\pi\sqrt{[x(1-x)]}} + \frac{1}{\pi\sqrt{x}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 \sqrt{(t)}\, g(t)\log\left|\frac{\sqrt{(1-t)}+\sqrt{(1-x)}}{\sqrt{(1-t)}-\sqrt{(1-x)}}\right|\mathrm{d}t. \tag{11.128}$$

The constant appearing in Eq. (11.128) was originally defined in Eq. (11.109); however, because of the result given in Eq. (11.103), it can be regarded as an arbitrary constant, which is denoted by C. Hence,

$$f(x) = \frac{C}{\sqrt{[x(1-x)]}} - \frac{1}{\pi\sqrt{[x(1-x)]}} P\int_0^1 \frac{\sqrt{[t(1-t)]}\, g(t)\mathrm{d}t}{x-t}. \tag{11.129}$$

This represents the solution of Eq. (11.101).

11.5 Inversion by a Fourier series approach

The inversion of the finite Hilbert transform is now considered using a Fourier series approach. The following substitutions are introduced into Eq. (11.2):

$$x = \cos u, \tag{11.130}$$

$$y = \cos v, \tag{11.131}$$

$$G(u) = g(\cos u)\ \sin u, \tag{11.132}$$

and

$$F(v) = f(\cos v)\ \sin v, \tag{11.133}$$

so that

$$G(u) = \frac{\sin u}{\pi} P \int_0^{\pi} \frac{F(v)\mathrm{d}v}{\cos u - \cos v}. \tag{11.134}$$

The introduction of the extra $\sin u$ anticipates a factor of $(\sin u)^{-1}$ that will emerge at the next step. The function $F(v)$ is expanded in terms of a complete orthonormal set as follows:

$$F(u) = \frac{1}{\sqrt{\pi}} a_0 + \sqrt{\left(\frac{2}{\pi}\right)} \sum_{n=1}^{\infty} a_n \cos nu. \tag{11.135}$$

The following integral needs to be calculated:

$$I(u) = \frac{1}{\pi} P \int_0^{\pi} \frac{\cos nv \, \mathrm{d}v}{\cos u - \cos v}. \tag{11.136}$$

This can be evaluated by contour integration (try it), or by more elementary means. Using the expansion given in Eq. (11.26),

$$\begin{aligned} \frac{1}{\pi} P \int_0^{\pi} \frac{\cos ny \, \mathrm{d}y}{\cos x - \cos y} &= -\frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\sin mx}{\sin x} \int_0^{\pi} \cos ny \cos my \, \mathrm{d}y \\ &= -\frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\sin mx}{\sin x} \frac{\pi}{2} \delta_{mn} \\ &= -\frac{\sin nx}{\sin x}, \quad n = 0,\ 1,\ 2, \ldots \end{aligned} \tag{11.137}$$

The preceding result generalizes Eqs. (11.17) and (11.23). Hence, it follows that

$$G(u) = -\sqrt{\left(\frac{2}{\pi}\right)} \sum_{n=1}^{\infty} a_n \sin nu, \tag{11.138}$$

and the a_n coefficients are determined from

$$a_n = -\sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\pi} G(v) \sin nv \, \mathrm{d}v. \tag{11.139}$$

If this result is inserted into Eq. (11.135), it follows that

$$F(u) = \frac{1}{\sqrt{\pi}} a_0 - \frac{2}{\pi} \sum_{n=1}^{\infty} \cos nu \int_0^{\pi} G(v) \sin nv \, \mathrm{d}v. \tag{11.140}$$

The summation in this expression can be evaluated by making use of Eq. (11.26); then Eq. (11.140) can be written in the following form:

$$F(u) = \frac{1}{\sqrt{\pi}} a_0 - \frac{1}{\pi} P \int_0^{\pi} \frac{\sin v \, G(v) \mathrm{d}v}{\cos u - \cos v}. \tag{11.141}$$

Reverting back to the original variable set, using Eqs. (11.130) – (11.133), leads to

$$f(x) = \frac{c}{\sqrt{(1-x^2)}} - \frac{1}{\pi \sqrt{(1-x^2)}} P \int_{-1}^{1} \frac{\sqrt{(1-y^2)}\, g(y) \mathrm{d}y}{x-y}, \tag{11.142}$$

where $a_0/\sqrt{\pi}$ has been denoted by c, which will play the role of an arbitrary constant. This result represents the inversion of the finite Hilbert transform by the Fourier series approach. The derivation avoids the use of the Tricomi identity, but does involve some dexterity with series manipulation.

11.6 The Riemann problem

This section and the following two sections will lay the groundwork for an alternative approach to the inversion of the finite Hilbert transform. The techniques also find application in the solution of more complicated singular integral equations, an example of which is given in Chapter 12.

The following is termed the *Riemann problem*: determine a function f which is analytic inside a closed contour C and for which there is a given linear relationship between its real and imaginary parts on the contour. The Riemann mapping theorem (Section 2.9) allows C to be chosen as a circle. Let $F_+(z)$ denote the function that is sought, where the subscript + denotes the interior of C, and a subscript – is used to indicate the exterior of the contour. Let

$$F_+(z) = u(x, y) + \mathrm{i}\, v(x, y), \tag{11.143}$$

and write the linear constraint that must be satisfied as follows:

$$\alpha(t)u(t) + \beta(t)v(t) = \gamma(t), \quad \forall\, t \text{ on } C, \tag{11.144}$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are given, real functions. It is assumed that $\alpha(t)$ and $\beta(t)$ do not vanish simultaneously. The special case for which $\beta = 0$, $\alpha = 1$, leads to the problem of constructing a function from the values that its real part takes on the boundary. Outside the circle, an analytic function $F_-(z)$ can be defined in terms of the complex conjugate of $F_+(z)$ as follows:

$$F_-(z) = F_+^*\left(\frac{1}{z^*}\right). \tag{11.145}$$

If C is taken as the unit circle, then, for z on C,

$$zz^* = |z|^2 = 1, \tag{11.146}$$

and hence, as $z \to t$ on C,

$$F_-(t) = F_+^*(t), \tag{11.147}$$

from which it follows that

$$F_-(t) = u(t) - \mathrm{i}v(t). \tag{11.148}$$

Using Eqs. (11.143) and (11.148), Eq. (11.144) becomes

$$\{\alpha(t) - \mathrm{i}\beta(t)\}\, F_+(t) + \{\alpha(t) + \mathrm{i}\beta(t)\}F_-(t) = 2\gamma(t). \tag{11.149}$$

Riemann's problem is therefore reduced to the determination of functions $F_+(z)$ and $F_-(z)$, analytic in the interior and exterior of C, respectively, such that Eq. (11.149) is satisfied on the boundary of C.

11.7 The Hilbert problem

A generalization of Riemann's problem was made by Hilbert. The following is generally referred to as *Hilbert's problem*: determine a function $F(z)$ that is analytic for all points z not lying on a contour C, such that, for t on C,

$$F_+(t) = g(t)F_-(t) + f(t), \tag{11.150}$$

where $F_+(t)$ and $F_-(t)$ have the same meaning as in Section 11.6, and where $g(t)$ and $f(t)$ are given complex-valued functions. Equation (11.150) is also referred to as the *inhomogeneous Hilbert problem*. The *homogeneous Hilbert problem* is obtained by setting $f(t) = 0$ in Eq. (11.150). Hilbert's focus was on situations where C is a closed contour.

11.8 The Riemann–Hilbert problem

The general problem given in Section 11.7 where C is allowed to be a closed contour, an arc, or a more generally a collection of arcs, is most frequently termed the *Riemann–Hilbert problem*. In some sources it is simply called the Hilbert problem; in others, the Hilbert–Riemann problem (Gakhov, 1966, p. 284) and, less commonly, it is called the *problem of Privalov*. The problem is often stated with the behavior of the function at infinity being given. When C is not a closed contour, it is necessary to specify the behavior of $F(z)$ near the endpoints of the arc.

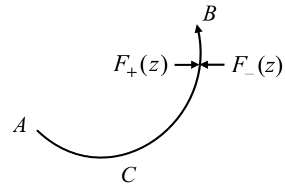

Figure 11.2. $F_+(z)$ and $F_-(z)$ are, respectively, the value of the function to the left and to the right of the contour C.

The homogeneous Riemann–Hilbert problem is considered first. The specifications necessary to define $F_+(z)$ and $F_-(z)$, when C is an arc, are indicated in Figure 11.2. Take the logarithm of the homogeneous equation, so that

$$\log F_+(t) - \log F_-(t) = \log g(t). \tag{11.151}$$

This result is of the form of the Plemelj boundary value problem (see Section 3.7), so the solution can be written as follows:

$$\log F(z) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\log g(t)\mathrm{d}t}{t - z}, \tag{11.152}$$

and hence

$$F(z) = \exp\left\{\frac{1}{2\pi \mathrm{i}} \int_C \frac{\log g(t)\mathrm{d}t}{t - z}\right\}. \tag{11.153}$$

It has been tacitly assumed that $\log g(t)$ is single-valued on C. When this is not the case, a modified function can be introduced, $g_0(t) = \log\{t^{-p} g(t)\}$, which is single-valued on C, where p, called the *index* of the homogeneous Hilbert problem, can be determined from

$$p = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\mathrm{d}\log g(t)}{\mathrm{d}t}\,\mathrm{d}t. \tag{11.154}$$

A Plemelj boundary value problem of the form of Eq. (11.151) can then be set up, with $g(t)$ replaced by $g_0(t)$. The interested reader can pursue this further by consulting the references listed in the chapter end-notes.

The solution of the inhomogeneous Riemann–Hilbert problem, Eq. (11.150), is now considered. Choose $L(z)$ such that it solves the following homogeneous problem:

$$L_+(t) = g(t)\, L_-(t), \tag{11.155}$$

with $L(z)$, $L_+(z)$, and $L_-(z)$ non-zero. The inhomogeneous problem can then be written as follows:

$$\frac{F_+(t)}{L_+(t)} - \frac{F_-(t)}{L_-(t)} = \frac{f(t)}{L_+(t)}. \tag{11.156}$$

The function $F(z)/L(z)$ is analytic, except for z on C. The function $K(z)$ is defined by

$$K(z) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(t)\mathrm{d}t}{L_+(t)(t-z)}; \tag{11.157}$$

then

$$K_+(t) - K_-(t) = \frac{f(t)}{L_+(t)}, \tag{11.158}$$

and hence

$$\frac{F_+(t)}{L_+(t)} - K_+(t) = \frac{F_-(t)}{L_-(t)} - K_-(t). \tag{11.159}$$

The analytic function $L^{-1}(z)F(z) - K(z)$ has the same boundary values from the left and right on the contour C. If $L(z)$ has been determined, then $F(z)$ can be found, but only to within a polynomial of appropriate degree. On this latter point, consult the references given in the end-notes.

11.8.1 The index of a function

The notion of index of a function is introduced in this subsection. This has considerable utility in characterizing various classes of Riemann–Hilbert problems. The properties of the index can be most readily obtained by examining a more general result called the argument principle. Specialization of the argument principle for different situations leads to some important properties of the index of a function. Application to the Riemann–Hilbert problem is then considered. The starting point for the discussion is a consideration of the *multiplicity* or *order* of a zero.

Suppose the complex-valued function f satisfies at some point z_0 the condition $\alpha = f(z_0)$; then the function $f - \alpha$ has a zero of order n or multiplicity n at the point z_0. If $n = 1$, the zero is called a simple zero. The order n can be determined in the following manner. Suppose the function f is analytic in a disc that encloses the point z_0, which is a zero f. Assume the disc encloses no other zeros and that f is non-vanishing on the boundary, then

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = n, \tag{11.160}$$

where C is the closed boundary of the disc. This result can be established as follows. If f has a zero of order n at z_0, it can be represented as

$$f(z) = (z - z_0)^n g(z), \tag{11.161}$$

where $g(z)$ is analytic in the disc and $g(z_0) \neq 0$. Hence,

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}. \tag{11.162}$$

The function $f'(z)/f(z)$ has a simple pole at z_0, and employing the Cauchy residue theorem leads to

$$\oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = 2\pi\mathrm{i}\,\mathrm{Res}\left.\frac{f'(z)}{f(z)}\right|_{z=z_0} = 2\pi\mathrm{i}n, \tag{11.163}$$

which is Eq. (11.160).

Equation (11.160) can be generalized in two ways. Consider a function f which is analytic in a disc that encloses zeros of f at the points $z_0, z_1, \ldots, z_m$, with orders $n_0, n_1, \ldots, n_m$, respectively, and assume that f is non-vanishing on the boundary; then

$$\frac{1}{2\pi\mathrm{i}}\oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = \sum_{k=0}^{m} n_k. \tag{11.164}$$

This result can be established by writing f in the following form:

$$f(z) = \prod_{k=0}^{m}(z - z_k)^{n_k} g_k(z_k), \tag{11.165}$$

where the functions g_k are analytic in the disc and $g_k(z_k) \neq 0$. It is common practice to rewrite Eq. (11.164) with the right-hand side of the equation replaced by a single letter:

$$\frac{1}{2\pi\mathrm{i}}\oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = Z, \tag{11.166}$$

with the understanding that Z denotes the number of enclosed zeros in the disc and that each zero is counted according to its particular multiplicity.

The situation for poles is considered next. Suppose f is analytic inside a disc except for a pole of order q at the point z_0. The function in this case can be represented as

$$f(z) = \frac{g(z)}{(z - z_0)^q}, \tag{11.167}$$

where $g(z)$ is analytic inside the disc, and suppose that $g(z_0) \neq 0$. It follows that

$$\frac{f'(z)}{f(z)} = -\frac{q}{z - z_0} + \frac{g'(z)}{g(z)}. \tag{11.168}$$

Applying the Cauchy residue theorem leads to

$$\oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = 2\pi \mathrm{i} \operatorname{Res} \frac{f'(z)}{f(z)}\bigg|_{z=z_0} = -2\pi \mathrm{i}\, q, \tag{11.169}$$

so that

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = -q. \tag{11.170}$$

The generalization to the case of a meromorphic function having poles at $z_0, z_1, \ldots, z_p$, with orders $q_0, q_1, \ldots, q_p$, respectively, assuming that f is non-vanishing in the disc and on the boundary and that it has no poles on the boundary, leads to

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = -\sum_{k=0}^{p} q_k. \tag{11.171}$$

It is common to see this result written in the following format:

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = -P, \tag{11.172}$$

where P signifies the number of poles of the meromorphic function enclosed by the disc, each counted according to its particular multiplicity.

The results in Eqs. (11.166) and (11.172) can be combined, to yield for a meromorphic function having both zeros and poles of differing multiplicities, the following result:

$$\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = Z - P. \tag{11.173}$$

This formula is referred to as the argument principle, or, by some authors, as the principle of the argument. The result can be expressed in the following form:

$$\frac{1}{2\pi} [\arg f(z)]_C = Z - P, \tag{11.174}$$

where the notation $[\arg f(z)]_C$ is employed to denote the change in $\arg f(z)$ as the contour is traversed in the positive sense, that is in a counter-clockwise manner. For the function f,

$$\log f(z) = \log |f(z)| + \mathrm{i} \arg f(z). \tag{11.175}$$

After traversing the closed contour C, $|f(z)|$ returns to its original value, so that

$$[\arg f(z)]_C = \frac{1}{\mathrm{i}}[\log f(z)]_C, \tag{11.176}$$

and Eq. (11.174) can be written as

$$\frac{1}{2\pi\mathrm{i}}[\log f(z)]_C = Z - P. \tag{11.177}$$

Note that

$$Z - P = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f'(z)\mathrm{d}z}{f(z)} = \frac{1}{2\pi\mathrm{i}}\oint_C \{\log f(z)\}'\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}[\log f(z)]_C. \tag{11.178}$$

The index of the function f with respect to the contour C is defined by

$$\operatorname{Ind} f(z) = \frac{1}{2\pi}[\arg f(z)]_C. \tag{11.179}$$

Employing Eq. (11.176) allows the preceding result to be written as follows:

$$\operatorname{Ind} f(z) = \frac{1}{2\pi\mathrm{i}}[\log f(z)]_C. \tag{11.180}$$

The index of a function can also be written in integral form:

$$\operatorname{Ind} f(z) = \frac{1}{2\pi}\oint_C \mathrm{d}\{\arg f(z)\} = \frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}\{\log f(z)\}. \tag{11.181}$$

If f is differentiable on C, then

$$\operatorname{Ind} f(z) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f'(z)\mathrm{d}z}{f(z)}. \tag{11.182}$$

It should be clear to the reader from the developments at the start of this subsection that, for a continuous function f that is non-vanishing on a closed contour C, $\operatorname{Ind} f$ is an integer, which may be negative, zero, or positive. From Eq. (11.180) it follows, for continuous functions f_1 and f_2 non-vanishing on the boundary of the disc, that

$$\operatorname{Ind} f_1 f_2 = \operatorname{Ind} f_1 + \operatorname{Ind} f_2 \tag{11.183}$$

and

$$\operatorname{Ind} \frac{f_1}{f_2} = \operatorname{Ind} f_1 - \operatorname{Ind} f_2. \tag{11.184}$$

The index of a function has a useful application to the classification of Riemann–Hilbert problems. Let κ denote $\operatorname{Ind} g$, where g occurs in the Riemann–Hilbert problem specified by Eq. (11.150), then the following situations apply for the Riemann–Hilbert problem for the circle (Muskhelishvili, 1992, p. 104). For $\kappa \geq 0$, the homogeneous

Riemann–Hilbert problem (set $f(t) = 0$ in Eq. (11.150)) has $\kappa + 1$ independent solutions, and they can be written as follows:

$$F(z) = L(z)\{c_0 z^\kappa + c_1 z^{\kappa-1} + \cdots + c_\kappa\}, \tag{11.185}$$

where $L(z)$ is the fundamental solution of the homogeneous Riemann–Hilbert problem $F_+(z) = g(z)F_-(z)$ and the c_k are constants. For $\kappa \le -2$, the homogeneous Riemann–Hilbert problem has no non-zero solutions. The solutions of the inhomogeneous Riemann–Hilbert problem can also be classified in terms of the index κ, as can Riemann–Hilbert problems for other domains. The interested reader can pursue the details in the references given in the end-notes.

11.9 Carleman's approach

The inversion of the finite Hilbert transform is now considered, taking advantage of the ideas developed in Sections 11.6–11.8. The objective is the determination of $f(x)$ from the equation

$$g(x) = \frac{1}{\pi} P \int_{-1}^{1} \frac{f(s)\,\mathrm{d}s}{x - s}. \tag{11.186}$$

Let

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{f(s)\,\mathrm{d}s}{s - z}, \quad \text{with } \operatorname{Im} z \neq 0. \tag{11.187}$$

From Eq. (11.187) it follows that

$$F_+(x) - F_-(x) = f(x) \tag{11.188}$$

and

$$F_+(x) + F_-(x) = \frac{1}{\pi \mathrm{i}} P \int_{-1}^{1} \frac{f(s)\,\mathrm{d}s}{s - x}. \tag{11.189}$$

From the preceding result,

$$F_+(x) = -F_-(x) + \mathrm{i}g(x). \tag{11.190}$$

The task is to determine a non-zero function $L(z)$ such that

$$\frac{L_+(x)}{L_-(x)} = -1. \tag{11.191}$$

Take

$$\log L(z) = \frac{1}{2\pi \mathrm{i}} P \int_{-1}^{1} \frac{\log(-1)\mathrm{d}s}{s-z}$$
$$= \frac{1}{2} \log\left(\frac{z-1}{z+1}\right), \tag{11.192}$$

hence,

$$L(z) = \sqrt{\left(\frac{z-1}{z+1}\right)}. \tag{11.193}$$

From Eqs. (11.190) and (11.191), it follows that

$$\frac{F_+(x)}{L_+(x)} - \frac{F_-(x)}{L_-(x)} = \frac{\mathrm{i}g(x)}{L_+(x)} \tag{11.194}$$

and

$$\frac{F(z)}{L(z)} = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\mathrm{i}g(s)\mathrm{d}s}{L_+(s)(s-z)}. \tag{11.195}$$

The general form of Eq. (11.195) would include the addition of an arbitrary constant, and Eq. (11.194) still holds. Using Eqs. (11.188) and (11.195) allows $f(x)$ to be obtained as

$$f(x) = F_+(x) - F_-(x)$$
$$= \frac{1}{2}\left[\frac{F_+(x)}{L_+(x)} + \frac{F_-(x)}{L_-(x)}\right][L_+(x) - L_-(x)]$$
$$= [L_+(x) - L_-(x)]\frac{1}{2\pi \mathrm{i}} P \int_{-1}^{1} \frac{\mathrm{i}g(s)\mathrm{d}s}{L_+(s)(s-x)}. \tag{11.196}$$

From Eq. (11.193),

$$L_+(x) = -\mathrm{i}\sqrt{\left(\frac{1-x}{1+x}\right)} \tag{11.197}$$

and

$$L_-(x) = \mathrm{i}\sqrt{\left(\frac{1-x}{1+x}\right)}. \tag{11.198}$$

Hence, Eq. (11.196) becomes

$$f(x) = -\sqrt{\left(\frac{1-x}{1+x}\right)} \frac{1}{\pi} P \int_{-1}^{1} \frac{\sqrt{(1+s)}\, g(s)\mathrm{d}s}{(x-s)\sqrt{(1-s)}}. \tag{11.199}$$

To this result may be added the solution of the following homogeneous equation:

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{f(s)\mathrm{d}s}{x-s} = 0, \tag{11.200}$$

which is

$$f(x) = \frac{c}{\surd(1-x^2)}, \tag{11.201}$$

where c is an arbitrary constant. The solution of Eq. (11.186) thus becomes

$$f(x) = \frac{c}{\surd(1-x^2)} - \surd\left(\frac{1-x}{1+x}\right) \frac{1}{\pi} P \int_{-1}^{1} \frac{\surd(1+s)\, g(s)\mathrm{d}s}{(x-s)\surd(1-s)}. \tag{11.202}$$

This result is in agreement with Eq. (11.98), which represents one of the alternative formulas that can be found for the inversion of the finite Hilbert transform.

11.10 Some basic properties of the finite Hilbert transform

In this section a number of basic properties of the finite Hilbert transform are collected together. Some resemble quite closely the corresponding properties for H on $\mathbb{R}$, but a number are also rather different, as the reader will quickly discover.

11.10.1 Even–odd character

For $f(x)$ even on the interval $(-1, 1)$, it follows that

$$g(x) = Tf(x) = \frac{2x}{\pi} P \int_{0}^{1} \frac{f(s)\mathrm{d}s}{x^2 - s^2}, \tag{11.203}$$

and for $f(x)$ an odd function on the interval $(-1, 1)$,

$$g(x) = Tf(x) = \frac{2}{\pi} P \int_{0}^{1} \frac{s f(s)\mathrm{d}s}{x^2 - s^2}. \tag{11.204}$$

That is, the finite Hilbert transform of an even function results in an odd function, and the finite Hilbert transform of an odd function results in an even function.

11.10.2 Inversion property

It follows directly from the results for the inversion formula, Eq. (11.63), that

$$f(x) = c/\surd(1-x^2) - [1/\surd(1-x^2)]\, T[\surd(1-x^2)\, Tf(x)], \tag{11.205}$$

where c is an arbitrary constant. In terms of complexity, this result contrasts sharply with the iteration property for the Hilbert transform on the real line, $f(x) = -H^2 f(x)$.

Equation (11.205) has considerably less practical significance than the iteration formula involving H.

Suppose that the support of f is restricted to the interval $(-1, 1)$, then, on making use of the Hardy–Poincaré–Bertrand formula on the interval $(-1, 1)$, it follows that

$$\begin{aligned}\frac{1}{\pi^2}P\int_{-1}^{1}\frac{\mathrm{d}y}{x-y}\left\{P\int_{-1}^{1}\frac{f(t)\mathrm{d}t}{y-t}\right\} &= \frac{1}{\pi^2}P\int_{-1}^{1}f(t)\mathrm{d}t\left\{P\int_{-1}^{1}\frac{\mathrm{d}y}{(x-y)(y-t)}\right\}-f(x)\\ &= \frac{1}{\pi^2}P\int_{-1}^{1}\frac{f(t)\mathrm{d}t}{x-t}P\int_{-1}^{1}\left\{\frac{1}{x-y}+\frac{1}{y-t}\right\}\mathrm{d}y-f(x)\\ &= \frac{1}{\pi^2}\log\left(\frac{1+x}{1-x}\right)P\int_{-1}^{1}\frac{f(t)\mathrm{d}t}{x-t}\\ &\quad -\frac{1}{\pi^2}P\int_{-1}^{1}\frac{\log[(1+t)/(1-t)]f(t)\mathrm{d}t}{x-t}-f(x),\end{aligned} \tag{11.206}$$

and hence

$$T^2f(x) = -f(x)+\pi^{-1}\log\left(\frac{1+x}{1-x}\right)Tf(x)-\pi^{-1}T\left[\log\left(\frac{1+x}{1-x}\right)f(x)\right]. \tag{11.207}$$

This formula is more complicated than the corresponding result for H: $H^2f(x) = -f(x)$.

11.10.3 Scale changes

Let $g(x) = Tf(x)$, then it is straightforward to show, for $a > 0$ and $ax \in (-1, 1)$, that

$$Tf(ax) = g(ax)+\frac{1}{\pi}\int_{1}^{a}\left\{\frac{f(y)}{ax-y}+\frac{f(-y)}{ax+y}\right\}\mathrm{d}y. \tag{11.208}$$

Because of the constraint on ax, a Cauchy principal value is not required for the integral on the right-hand side of Eq. (11.208). For $a > 0$ and $ax \in (-1, 1)$, it is also a simple calculation to prove that

$$Tf(-ax) = -g(-ax)+\frac{1}{\pi}\int_{1}^{a}\left\{\frac{f(y)}{ax+y}+\frac{f(-y)}{ax-y}\right\}\mathrm{d}y. \tag{11.209}$$

For $a > 0, b \in \mathbb{R}$, and $(ax+b) \in (-1, 1)$, it is a short calculation to show that

$$Tf(ax+b) = g(ax+b)+\frac{1}{\pi}\int_{1}^{b+a}\frac{f(y)\mathrm{d}y}{ax+b-y}-\frac{1}{\pi}\int_{-1}^{b-a}\frac{f(y)\mathrm{d}y}{ax+b-y}. \tag{11.210}$$

Because of the constraint on $ax+b$, a Cauchy principal value is not required for the integral on the right-hand side of Eq. (11.210).

To decide if the finite Hilbert transform operator and the translation operator τ_a commute, proceed as follows. From Eq. (11.210), it follows that

$$\begin{aligned} T\tau_a f(x) &= Tf(x-a) \\ &= g(x-a) + \frac{1}{\pi} P \int_1^{1-a} \frac{f(y)\mathrm{d}y}{x-a-y} - \frac{1}{\pi} P \int_{-1}^{-1-a} \frac{f(y)\mathrm{d}y}{x-a-y}; \end{aligned} \tag{11.211}$$

however,

$$\tau_a Tf(x) = \tau_a g(x) = g(x-a), \tag{11.212}$$

and clearly, in general, the commutator $[T, \tau_a] \neq 0$ for $a \neq 0$.

To see if the dilation operator S_a commutes with the finite Hilbert transform operator, first note from Eq. (11.208) that

$$TS_a f(x) = g(ax) + \frac{1}{\pi} P \int_1^a \left\{ \frac{f(y)}{ax-y} + \frac{f(-y)}{ax+y} \right\} \mathrm{d}y, \tag{11.213}$$

but,

$$S_a Tf(x) = S_a g(x) = g(ax), \tag{11.214}$$

and hence in general it follows that $[T, S_a] \neq 0$ for $a > 0$.

11.10.4 Finite Hilbert transform of the product $x^n f(x)$

Let $g(x) = Tf(x)$, then

$$\begin{aligned} T[xf(x)] &= \frac{1}{\pi} P \int_{-1}^{1} \frac{\{x-(x-s)\}f(s)\mathrm{d}s}{x-s} \\ &= xTf(x) - \frac{1}{\pi} \int_{-1}^{1} f(s)\mathrm{d}s. \end{aligned} \tag{11.215}$$

The generalization to higher powers of x is straightforward. For integer $n \geq 0$ it follows that

$$\begin{aligned} T[x^n f(x)] &= \frac{x^n}{\pi} P \int_{-1}^{1} \frac{\{1 + s^n x^{-n} - 1\} f(s)\mathrm{d}s}{x-s} \\ &= x^n\, Tf(x) + \frac{x^n}{\pi} \int_{-1}^{1} \frac{x^{-1}(s-x)\sum_{m=0}^{n-1} s^m x^{-m} f(s)\mathrm{d}s}{x-s} \\ &= x^n\, Tf(x) - \frac{x^{n-1}}{\pi} \sum_{m=0}^{n-1} x^{-m} \int_{-1}^{1} s^m f(s)\mathrm{d}s, \end{aligned} \tag{11.216}$$

and hence

$$T[x^n f(x)] = x^n \, Tf(x) - \frac{1}{\pi} \sum_{m=0}^{n-1} x^m \int_{-1}^{1} s^{n-1-m} f(s) \mathrm{d}s. \tag{11.217}$$

This last result will be called the moment formula for the finite Hilbert transform. This has a similar form to the moment formula for the Hilbert transform on $\mathbb{R}$.

11.10.5 Derivative of the finite Hilbert transform

In this subsection the derivative of the finite Hilbert transform is evaluated. Using the Leibnitz formula for the derivative of an integral leads to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} Tf(x) &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \left\{ \frac{f(x-\varepsilon)}{\varepsilon} + \frac{f(x+\varepsilon)}{\varepsilon} - \int_{-1}^{x-\varepsilon} \frac{f(s)\mathrm{d}s}{(x-s)^2} - \int_{x+\varepsilon}^{1} \frac{f(s)\mathrm{d}s}{(x-s)^2} \right\} \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \left\{ \frac{f(-1)}{x+1} - \frac{f(1)}{x-1} + \int_{-1}^{x-\varepsilon} \frac{f'(s)\mathrm{d}s}{x-s} + \int_{x+\varepsilon}^{1} \frac{f'(s)\mathrm{d}s}{x-s} \right\}, \end{aligned} \tag{11.218}$$

where an integration by parts has been employed in the last step, and hence

$$\frac{\mathrm{d}}{\mathrm{d}x} Tf(x) = T\left[\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right] + \frac{1}{\pi} \left\{ \frac{f(-1)}{x+1} - \frac{f(1)}{x-1} \right\}. \tag{11.219}$$

This result can be used as a tool for the quick evaluation of some finite Hilbert transforms. Consider the example $T[-x/\surd(1-x^2)]$, then it an easy calculation to show using $f(x) = \surd(1-x^2)$ that

$$\begin{aligned} T[-x/\surd(1-x^2)] &= T\left[\frac{\mathrm{d}}{\mathrm{d}x}\surd(1-x^2)\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x} T[\surd(1-x^2)] - \frac{1}{\pi} \left\{ \frac{f(-1)}{x+1} - \frac{f(1)}{x-1} \right\}, \end{aligned} \tag{11.220}$$

which simplifies, on noting $f(1) = f(-1) = 0$ and $T[\surd(1-x^2)] = x,$ to give

$$T[-x/\surd(1-x^2)] = 1. \tag{11.221}$$

An alternative approach for this example can be carried out using the moment formula of the preceding subsection and the result $T[1/\surd(1-x^2)] = 0$:

$$\begin{aligned} T[-x/\surd(1-x^2)] &= -x\, T[1/\surd(1-x^2)] + \frac{1}{\pi} \int_{-1}^{1} \frac{\mathrm{d}s}{\surd(1-s^2)} \\ &= 1. \end{aligned} \tag{11.222}$$

The generalization of Eq. (11.219) is straightforward to derive. For integer $n \geq 0$,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} Tf(x) = T\left[\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right] - \frac{1}{\pi}\sum_{k=0}^{n-1}(-1)^{n-k}(n-k-1)!$$
$$\times \left\{ \left.\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\right|_{x=-1} \frac{1}{(x+1)^{n-k}} - \left.\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\right|_{x=1} \frac{1}{(x-1)^{n-k}} \right\}, \quad (11.223)$$

assuming that the integral $T[\mathrm{d}^n f(x)/\mathrm{d}x^n]$ exists.

11.10.6 Convolution property

In this subsection, the convolution property for the finite Hilbert transform is considered. If supp $f \in (-1, 1)$, it can be shown that

$$\{Tf * h\}(x) = \{f * Hh\}(x). \quad (11.224)$$

The proof is straightforward:

$$\begin{aligned}\{Tf * h\}(x) &= \int_{-\infty}^{\infty} h(x-u)\left(\frac{1}{\pi}P\int_{-1}^{1}\frac{f(s)\mathrm{d}s}{u-s}\right)\mathrm{d}u \\ &= \int_{-\infty}^{\infty} h(x-u)\left(\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(s)\mathrm{d}s}{u-s}\right)\mathrm{d}u \\ &= \int_{-\infty}^{\infty} f(s)\,\mathrm{d}s\,\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{h(y)\mathrm{d}y}{x-s-y} \\ &= \{f * Hh\}(x),\end{aligned} \quad (11.225)$$

as required. The change of order of integration in the preceding sequence of steps is justified if f and h belong to the classes L^p and L^q, respectively, with $p^{-1} + q^{-1} = 1$. If supp h also belongs to $(-1, 1)$, it follows from the preceding result that

$$\{Tf * h\}(x) = \{f * Th\}(x). \quad (11.226)$$

For the case where supp $f \in (-1, 1)$ and supp $h \in (-1, 1)$,

$$T\{f * h\}(x) = \{f * Th\}(x) + \frac{1}{\pi}\int_0^1 \mathrm{d}t\, P\int_1^{1-t}\left\{\frac{f(t)h(y)}{x-t-y} + \frac{f(-t)h(-y)}{x+t+y}\right\}\mathrm{d}y, \quad (11.227)$$

which does not have the same simple form as the corresponding result for the standard Hilbert transform. The proof of Eq. (11.227) goes as follows:

$$
\begin{aligned}
T\{f * h\}(x) &= \frac{1}{\pi} P \int_{-1}^{1} \frac{\mathrm{d}s}{x-s} \int_{-\infty}^{\infty} f(t) h(s-t) \mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t) \mathrm{d}t \, \frac{1}{\pi} P \int_{-1}^{1} \frac{h(s-t) \mathrm{d}s}{x-s} \\
&= \int_{-1}^{1} f(t) \mathrm{d}t \, \frac{1}{\pi} P \int_{-1-t}^{1-t} \frac{h(y) \mathrm{d}y}{x-t-y} \\
&= \frac{1}{\pi} \int_{-1}^{0} f(t) \mathrm{d}t \;\; P \int_{0}^{1} \frac{h(y) \mathrm{d}y}{x-t-y} + \frac{1}{\pi} \int_{0}^{1} f(t) \mathrm{d}t \;\; P \int_{0}^{1-t} \frac{h(y) \mathrm{d}y}{x-t-y} \\
&\quad - \frac{1}{\pi} \int_{0}^{1} f(t) \mathrm{d}t \;\; P \int_{0}^{-1} \frac{h(y) \mathrm{d}y}{x-t-y} - \frac{1}{\pi} \int_{-1}^{0} f(t) \mathrm{d}t \;\; P \int_{0}^{-1-t} \frac{h(y) \mathrm{d}y}{x-t-y} \\
&= \{f * Th\}(x) - \frac{1}{\pi} \int_{0}^{1} f(-t) \mathrm{d}t \;\; P \int_{0}^{1} \frac{h(-y) \mathrm{d}y}{x+t+y} \\
&\quad - \frac{1}{\pi} \int_{0}^{1} f(t) \mathrm{d}t \;\; P \int_{0}^{1} \frac{h(y) \mathrm{d}y}{x-t-y} + \frac{1}{\pi} \int_{0}^{1} f(t) \mathrm{d}t \;\; P \int_{0}^{1-t} \frac{h(y) \mathrm{d}y}{x-t-y} \\
&\quad + \frac{1}{\pi} \int_{0}^{1} f(-t) \mathrm{d}t \;\; P \int_{0}^{1-t} \frac{h(-y) \mathrm{d}y}{x+t+y} \\
&= \{f * Th\}(x) + \frac{1}{\pi} \int_{0}^{1} \mathrm{d}t \;\; P \int_{1}^{1-t} \left\{ \frac{f(t)\, h(y)}{x-t-y} + \frac{f(-t)\, h(-y)}{x+t+y} \right\} \mathrm{d}y.
\end{aligned}
\tag{11.228}
$$

11.10.7 Fourier transform of the finite Hilbert transform

The Fourier transform of the finite Hilbert transform is now examined. Suppose the support of f is $(-1,\ 1)$; then

$$
\begin{aligned}
\mathcal{F} Tf(x) &= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xt} \, \mathrm{d}t \frac{1}{\pi} P \int_{-1}^{1} \frac{f(s) \mathrm{d}s}{t-s} \\
&= \int_{-1}^{1} f(s) \mathrm{d}s \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}xt} \, \mathrm{d}t}{t-s} \\
&= \int_{-1}^{1} f(s) \, \{-\mathrm{i} \operatorname{sgn} x \, \mathrm{e}^{-\mathrm{i}xs}\} \mathrm{d}s \\
&= -\mathrm{i} \operatorname{sgn} x \int_{-\infty}^{\infty} f(s) \mathrm{e}^{-\mathrm{i}xs} \, \mathrm{d}s,
\end{aligned}
\tag{11.229}
$$

and hence

$$\mathcal{F}Tf(x) = -\mathrm{i}\,\mathrm{sgn}\,x\,\mathcal{F}f(x). \tag{11.230}$$

If the support of f is not restricted to $(-1,\ 1)$, then the final step in Eq. (11.229) can be recast in terms of the rectangular function $\Pi_2(x)$ (see Eq. (9.19)) to yield

$$\mathcal{F}Tf(x) = -\mathrm{i}\,\mathrm{sgn}\,x\,\mathcal{F}\{\Pi_2 f\}(x). \tag{11.231}$$

It follows from Eq. (11.230), for functions with support on the interval $(-1,\ 1)$, that

$$Tf(x) = -\mathrm{i}\mathcal{F}^{-1}\{\mathrm{sgn}\,y\,\mathcal{F}f(y)\}(x), \tag{11.232}$$

or, for functions whose support is not restricted to the interval $(-1,\ 1)$, from Eq. (11.231),

$$Tf(x) = -\mathrm{i}\mathcal{F}^{-1}\{\mathrm{sgn}\,y\,(\mathcal{F}\Pi_2 f)(y)\}(x). \tag{11.233}$$

As an example, consider

$$f(x) = \begin{cases} x, & \text{for } -1 < x < 1 \\ 0, & \text{for } |x| \geq 1; \end{cases} \tag{11.234}$$

then from Eq. (11.232) it follows that

$$\begin{aligned} Tf(x) &= -\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt}\,\mathrm{sgn}\,t\,\mathrm{d}t \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}ts} s\,\mathrm{d}s \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt}\,\mathrm{sgn}\,t\left\{\frac{\cos t}{t} - \frac{\sin t}{t^2}\right\}\mathrm{d}t \\ &= \frac{2}{\pi}\int_{0}^{\infty}\left\{\frac{\cos t}{t} - \frac{\sin t}{t^2}\right\}\cos xt\,\mathrm{d}t \\ &= \frac{x}{\pi}\log\left(\frac{1+x}{1-x}\right) - \frac{2}{\pi}. \end{aligned} \tag{11.235}$$

This result could be obtained more readily by direct evaluation of the finite Hilbert transform. Equations (11.232) or (11.233) offer an alternative strategy to evaluate finite Hilbert transforms, and this may be an easier route if the direct evaluation of the Cauchy principal integral is particularly difficult.

11.10.8 Parseval-type identities

For functions $f \in L^p(-1,1)$ with $p > 1$ and $g \in L^q(-1,1)$ with $q > 1$ and $p^{-1} + q^{-1} \leq 1$, it follows that

$$\int_{-1}^{1} f(x)\mathrm{d}x \frac{1}{\pi} P \int_{-1}^{1} \frac{g(y)\mathrm{d}y}{y-x} = \int_{-1}^{1} g(y)\mathrm{d}y \frac{1}{\pi} P \int_{-1}^{1} \frac{f(x)\mathrm{d}x}{y-x}, \tag{11.236}$$

and hence

$$\int_{-1}^{1} f(x)Tg(x)\mathrm{d}x = -\int_{-1}^{1} g(x)Tf(x)\mathrm{d}x. \tag{11.237}$$

This is the analog for the finite Hilbert transform of the Parseval-type identity given for the Hilbert transform in Eq. (4.176). Equation (11.237) also follows directly from Eq. (4.176) if the support of both f and g is the interval $(-1,\ 1)$.

If, in addition to the aforementioned conditions given for f and g, the support of f and g is restricted to the interval $(-1,\ 1)$, then a modified Parseval-type identity for the finite Hilbert transform can be obtained. Starting from Eq. (4.174), then

$$\int_{-\infty}^{\infty} f(x)g(x)\mathrm{d}x = \int_{-\infty}^{\infty} Hf(x)\, Hg(x)\mathrm{d}x, \tag{11.238}$$

which can be rewritten as follows:

$$\begin{aligned}\int_{-1}^{1} f(x)g(x)\mathrm{d}x &= \int_{-1}^{1} Tf(x)\, Tg(x)\mathrm{d}x \\ &\quad + \int_{-\infty}^{-1} Hf(x)\, Hg(x)\mathrm{d}x + \int_{1}^{\infty} Hf(x)\, Hg(x)\mathrm{d}x\,. \end{aligned} \tag{11.239}$$

The split of the integration range has been made since the finite Hilbert transform $Tf(x)$ has been defined for $-1 < x < 1$. Note that because f has support on $(-1,\ 1)$, it does not follow that the support of $Tf(x)$ is restricted to the same interval. The final pair of integrals in Eq. (11.239) can be rearranged in the following manner:

$$\begin{aligned}&\int_{-\infty}^{-1} Hf(x)\, Hg(x)\mathrm{d}x + \int_{1}^{\infty} Hf(x)\, Hg(x)\mathrm{d}x \\ &= \int_{1}^{\infty} \{Hf(x)\, Hg(x) + Hf(-x)\, Hg(-x)\}\mathrm{d}x \\ &= \frac{1}{\pi^2}\int_{1}^{\infty} \left\{\int_{-1}^{1} \frac{f(s)\mathrm{d}s}{x-s} \int_{-1}^{1} \frac{g(t)\mathrm{d}t}{x-t} + \int_{-1}^{1} \frac{f(s)\mathrm{d}s}{x+s} \int_{-1}^{1} \frac{g(t)\mathrm{d}t}{x+t}\right\}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi^2}\int_{-1}^{1} f(s)\mathrm{d}s \int_{-1}^{1} g(t)\mathrm{d}t \int_{1}^{\infty}\left\{\frac{1}{(x-s)(x-t)}+\frac{1}{(x+s)(x+t)}\right\}\mathrm{d}x \\
&= \frac{1}{\pi^2}\int_{-1}^{1} f(s)\mathrm{d}s\, P\int_{-1}^{1} g(t)\mathrm{d}t \int_{1}^{\infty}\left[\left\{\frac{1}{(x-s)}-\frac{1}{(x-t)}\right\}\frac{1}{s-t}\right. \\
&\quad \left. +\left\{\frac{1}{(x+s)}-\frac{1}{(x+t)}\right\}\frac{1}{t-s}\right]\mathrm{d}x \\
&= \frac{1}{\pi^2}\int_{-1}^{1} f(s)\mathrm{d}s\, P\int_{-1}^{1} g(t)\left\{\frac{1}{s-t}\log\left(\frac{1-t}{1-s}\right)+\frac{1}{t-s}\log\left(\frac{1+t}{1+s}\right)\right\}\mathrm{d}t \\
&= \frac{1}{\pi^2}\int_{-1}^{1} f(s)\,\log\left(\frac{1+s}{1-s}\right)\mathrm{d}s\, P\int_{-1}^{1}\frac{g(t)\mathrm{d}t}{s-t} \\
&\quad +\frac{1}{\pi^2}\int_{-1}^{1} g(t)\log\left(\frac{1+t}{1-t}\right)\mathrm{d}t\, P\int_{-1}^{1}\frac{f(s)\mathrm{d}s}{t-s} \\
&= \frac{1}{\pi}\int_{-1}^{1}\log\left(\frac{1+x}{1-x}\right)\{f(x)(Tg)(x)+g(x)(Tf)(x)\}\mathrm{d}x. \qquad (11.240)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_{-1}^{1} f(x)g(x)\mathrm{d}x &= \int_{-1}^{1} Tf(x)\; Tg(x)\mathrm{d}x \\
&\quad +\frac{1}{\pi}\int_{-1}^{1}\log\left(\frac{1+x}{1-x}\right)\{f(x)Tg(x)+g(x)Tf(x)\}\mathrm{d}x. \qquad (11.241)
\end{aligned}
$$

This result can be obtained in an alternative manner starting from Tricomi's identity, Eq. (11.52). Integrating this equation over the interval $(-1,\ 1)$ and replacing ϕ_1 by f and ϕ_2 by g leads to

$$
\int_{-1}^{1} f(x)g(x)\mathrm{d}x = \int_{-1}^{1} Tf(x)\; Tg(x)\mathrm{d}x - \int_{-1}^{1} T\{f(x)Tg(x)+g(x)Tf(x)\}\mathrm{d}x. \qquad (11.242)
$$

Setting $\phi(x) = f(x)Tg(x) + g(x)Tf(x)$, the last integral in Eq. (11.242) can be simplified as follows:

$$
\begin{aligned}
\int_{-1}^{1} T\phi(x)\mathrm{d}x &= \int_{-1}^{1}\mathrm{d}x\frac{1}{\pi}\,P\int_{-1}^{1}\frac{\phi(s)\mathrm{d}s}{x-s} \\
&= \int_{-1}^{1}\phi(s)\mathrm{d}s\,\frac{1}{\pi}\,P\int_{-1}^{1}\frac{\mathrm{d}x}{x-s} \\
&= -\frac{1}{\pi}\int_{-1}^{1}\phi(x)\log\left(\frac{1+x}{1-x}\right)\mathrm{d}x. \qquad (11.243)
\end{aligned}
$$

Inserting this result into Eq. (11.242) yields Eq. (11.241).

A couple of examples are now considered. Suppose that f and g are given by

$$f(x) = \begin{cases} \surd(1-x^2), & \text{for } |x| < 1 \\ 0, & \text{for } |x| \geq 1 \end{cases} \tag{11.244}$$

and

$$g(x) = \begin{cases} x, & \text{for } |x| < 1 \\ 0, & \text{for } |x| \geq 1; \end{cases} \tag{11.245}$$

then

$$\begin{aligned} \int_{-1}^{1} f(x)Tg(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-1}^{1} \surd(1-x^2)\left\{x\log\left(\frac{1+x}{1-x}\right) - 2\right\}\mathrm{d}x \\ &= -\frac{2}{3}, \end{aligned} \tag{11.246}$$

$$-\int_{-1}^{1} g(x)Tf(x)\mathrm{d}x = -\int_{-1}^{1} x^2\,\mathrm{d}x = -\frac{2}{3}, \tag{11.247}$$

and hence Eq. (11.244) is satisfied. Equation (11.241) is seen to be trivially satisfied because the integrand for each of the three integrals is an odd function, and therefore the value of each integral is zero. As a second example, suppose $g(x)$ is modified to

$$g(x) = \begin{cases} \dfrac{1}{\surd(1-x^2)}, & \text{for } |x| < 1 \\ 0, & \text{for } |x| \geq 1, \end{cases} \tag{11.248}$$

f is the same as in Eq. (11.244); then Eq. (11.237) is trivially satisfied since $g(x)Tf(x)$ is odd and $Tg(x) = 0$. From Eq. (11.241), the right-hand side evaluates on noting $Tf(x) = x$ to give

$$\begin{aligned} &\int_{-1}^{1} Tf(x)Tg(x)\mathrm{d}x + \frac{1}{\pi}\int_{-1}^{1} \log\left(\frac{1+x}{1-x}\right)\{f(x)Tg(x) + g(x)Tf(x)\}\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-1}^{1} \log\left(\frac{1+x}{1-x}\right)\frac{x\,\mathrm{d}x}{\surd(1-x^2)} = 2, \end{aligned} \tag{11.249}$$

and the left-hand side evaluates by inspection to the same result; hence, Eq. (11.241) is satisfied.

11.10.9 Orthogonality property

If f is an even function, then, by Section 11.10.1 $Tf(x)$ is odd, and hence it follows that

$$\int_{-1}^{1} f(x)Tf(x)\mathrm{d}x = 0, \tag{11.250}$$

since the integrand is an odd function. This last result also holds when f is an odd function. Equation (11.250) can be regarded as the orthogonality condition for the finite Hilbert transform, and is the analog of the orthogonality condition for the Hilbert transform on $\mathbb{R}$, Eq. (4.198). For functions $f \in L^p(\mathbb{R})$ with $1 < p < \infty$, and with $\operatorname{supp} f \in (-1, 1)$, Eq. (11.250) follows directly from Eq. (4.202), without regard to the even–odd character of the function.

If f is written in terms of its even and odd components,

$$f(x) = f_{\mathrm{e}}(x) + f_{\mathrm{o}}(x), \tag{11.251}$$

it follows that

$$\int_{-1}^{1} f(x)Tf(x)\mathrm{d}x = \int_{-1}^{1} f_{\mathrm{e}}(x)Tf_{\mathrm{o}}(x)\mathrm{d}x + \int_{-1}^{1} f_{\mathrm{o}}(x)Tf_{\mathrm{e}}(x)\mathrm{d}x, \tag{11.252}$$

where the integrals with terms $f_{\mathrm{e}}(x)Tf_{\mathrm{e}}(x)$ and $f_{\mathrm{o}}(x)Tf_{\mathrm{o}}(x)$ are both zero because the integrands are odd functions. Suppose that f_{e} and f_{o} satisfy the conditions necessary to write the Parseval-type identity, Eq. (11.237); then

$$\int_{-1}^{1} f_{\mathrm{e}}(x)Tf_{\mathrm{o}}(x)\mathrm{d}x + \int_{-1}^{1} f_{\mathrm{o}}(x)Tf_{\mathrm{e}}(x)\mathrm{d}x = 0, \tag{11.253}$$

and from Eq. (11.252) it therefore follows that

$$\int_{-1}^{1} f(x)Tf(x)\mathrm{d}x = 0. \tag{11.254}$$

As an example, consider the choice

$$f(x) = \begin{cases} a\sqrt{(1-x^2)} + bx, & \text{for } |x| < 1 \\ 0, & \text{for } |x| \geq 1, \end{cases} \tag{11.255}$$

where a and b are constants. On the interval $(-1, 1)$, $f_{\mathrm{e}}(x) = a\sqrt{(1-x^2)}$ and $f_{\mathrm{o}}(x) = bx$, and Eq. (11.253) is readily checked to be satisfied. Also,

$$\int_{-1}^{1} \{a\sqrt{(1-x^2)} + bx\}T[a\sqrt{(1-x^2)} + bx]\mathrm{d}x = 0, \tag{11.256}$$

which verifies that Eq. (11.254) is satisfied.

11.10.10 Eigenfunctions and eigenvalues of the finite Hilbert transform operator

Thinking in terms of the finite Hilbert transform as an operator, the equation

$$Tf(x) = \lambda f(x), \tag{11.257}$$

where λ is a constant, is an eigenvalue equation, with f the eigenfunction and λ the eigenvalue. This equation can be solved by the method given in Section 11.9, and the interested reader might like to try that as an exercise. The equation is an example of a singular integral equation, a topic that is taken up in Chapter 12.

The solutions of Eq. (11.257) are as follows:

$$f(x) = \frac{(1-x)^{a-1}}{(1+x)^a}, \quad 0 < a < 1, \text{ and } -1 < x < 1, \tag{11.258}$$

which are the eigenfunctions of the operator T. Using this expression for f, it can be shown (try it) that

$$T\left[\frac{(1-x)^{a-1}}{(1+x)^a}\right] = -\cot a\pi \frac{(1-x)^{a-1}}{(1+x)^a}. \tag{11.259}$$

From this last result the eigenvalues can be identified as $-\cot a\pi$.

If f and g are eigenfunctions of T with eigenvalues α and β, respectively, then, from the Tricomi formula for T, Eq. (11.52), it follows immediately (assuming $\alpha + \beta \neq 0$), that

$$T[f(x)g(x)] = \frac{\alpha\beta - 1}{\alpha + \beta} f(x)g(x). \tag{11.260}$$

For the case $f(x) = g(x)$ and $\alpha = \beta$ ($\alpha \neq 0$),

$$T[f(x)^2] = \frac{1}{2}(\alpha - \alpha^{-1}) f(x)^2. \tag{11.261}$$

11.11 Finite Hilbert transform of the Legendre polynomials

The Legendre function of the second kind, $Q_n(x)$, can be related to the Legendre function of the first kind via a finite Hilbert transform relation. If z is not a real number lying between –1 and 1, and n is an integer ≥ 0, then

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(s)\mathrm{d}s}{z-s}, \tag{11.262}$$

which is F. Neumann's formula for $Q_n(z)$. The Legendre function $Q_n(z)$ satisfies the following relationship:

$$Q_n(z) = \frac{1}{2}P_n(z)\log\left(\frac{z+1}{z-1}\right) - W_{n-1}(z), \quad \text{for } n \geq 1, \tag{11.263}$$

where $W_{n-1}(z)$ is a polynomial in z. If x lies in the interval $(-1,1)$, it is customary to define

$$Q_n(x) = \frac{1}{2}\{Q_n(x+\mathrm{i}0) + Q_n(x-\mathrm{i}0)\}. \tag{11.264}$$

The function $Q_n(x)$ can be written in terms of $P_n(x)$ using the following result:

$$Q_n(x) = \frac{1}{2}P_n(x)\log\left(\frac{1+x}{1-x}\right) - W_{n-1}(x), \quad \text{for } n \geq 1, \tag{11.265}$$

where $W_{n-1}(x)$ is a polynomial in x given by

$$W_{n-1}(x) = \sum_{j=1}^{n} \frac{P_{j-1}(x)P_{n-j}(x)}{j}. \tag{11.266}$$

Equation (11.265) can be obtained by substituting $z = x + \mathrm{i}y$, and $z = x - \mathrm{i}y$ in turn into Eq. (11.263) and then evaluating the limit $y \to 0+$. Equation (11.264) then leads to Eq. (11.265). The function $W_{n-1}(x)$ can also be expressed in the form (King, 1991)

$$\begin{aligned} W_{n-1}(x) &= (2n-1)!! \sum_{j=0}^{[(n-1)/2]} x^{n-1-2j} \sum_{v=0}^{j} \frac{(-1)^v(2n-1-2v)!!}{(2j-2v+1)(2v)!!(n-2v)!(2n-1)!!} \\ &= \sum_{j=0}^{[(n-1)/2]} x^{n-1-2j} \sum_{v=0}^{j} \frac{a_{nv}}{2j-2v+1}, \end{aligned} \tag{11.267}$$

where $[m/2]$ denotes $m/2$ if m is an even integer or $(m-1)/2$ if m is an odd integer. The constants a_{nv} can be expressed in terms of binomial coefficients as follows:

$$a_{nv} = \frac{(-1)^v}{2^n}\binom{2n-2v}{n}\binom{n}{v}. \tag{11.268}$$

The function $Q_n(x)$ can be written as a finite Hilbert transform:

$$\begin{aligned} Q_n(x) &= \frac{\pi}{2}TP_n(x) \\ &= \frac{1}{2}P\int_{-1}^{1} \frac{P_n(s)\mathrm{d}s}{x-s}. \end{aligned} \tag{11.269}$$

This relationship can be proved by showing that the right-hand side of Eq. (11.269) is equal to the right-hand side of Eq. (11.265). Proceed as follows:

$$\frac{1}{2}P\int_{-1}^{1}\frac{P_n(s)\mathrm{d}s}{x-s}=\frac{1}{2}\int_{-1}^{1}\frac{\{P_n(s)-P_n(x)\}\mathrm{d}s}{x-s}+\frac{P_n(x)}{2}P\int_{-1}^{1}\frac{\mathrm{d}s}{x-s}. \tag{11.270}$$

The second term on the right-hand side of Eq. (11.270) satisfies

$$\frac{P_n(x)}{2}P\int_{-1}^{1}\frac{\mathrm{d}s}{x-s}=\frac{1}{2}P_n(x)\log\left(\frac{1+x}{1-x}\right). \tag{11.271}$$

Comparing with Eq. (11.265), it therefore remains to establish that

$$\frac{1}{2}\int_{-1}^{1}\frac{\{P_n(s)-P_n(x)\}\mathrm{d}s}{s-x}=W_{n-1}(x). \tag{11.272}$$

The standard expansion formula for the Legendre polynomials takes the following form:

$$P_n(x)=\sum_{r=0}^{[n/2]}a_{nr}x^{n-2r}, \tag{11.273}$$

where a_{nr} is defined in Eq. (11.268). Inserting this result into Eq. (11.272) leads to

$$\begin{aligned}\frac{1}{2}\int_{-1}^{1}\frac{\{P_n(s)-P_n(x)\}\mathrm{d}s}{s-x}&=\frac{1}{2}\sum_{r=0}^{[n/2]}a_{nr}x^{n-2r}\int_{-x^{-1}}^{x^{-1}}\frac{(w^{n-2r}-1)\mathrm{d}w}{w-1}\\&=\frac{1}{2}\sum_{r=0}^{[n/2]}a_{nr}x^{n-2r}\sum_{k=0}^{n-2r-1}\frac{[1+(-1)^k]}{(k+1)\,x^{k+1}}\\&=\frac{1}{2}\sum_{r=0}^{[n/2]}a_{nr}\sum_{j=0}^{n-2r-1}\frac{[1+(-1)^{n-1-j}]\,x^j}{n-2r-j}.\end{aligned} \tag{11.274}$$

The preceding expression is simplified by making a summation rearrangement:

$$\sum_{r=0}^{[n/2]}\alpha_r\sum_{j=0}^{n-2r-1}\beta_{rj}=\sum_{\mu=0}^{n-1}\sum_{v=0}^{[(n-\mu-1)/2]}\alpha_v\beta_{v\mu}. \tag{11.275}$$

If this result is not transparent, it is extremely useful to write out a section of each double sum and see how changing the direction of summation alters the summation limits. Equation (11.274) can now be written as follows:

$$\begin{aligned}
\frac{1}{2}\sum_{r=0}^{[n/2]} a_{nr} \sum_{j=0}^{n-2r-1} \frac{[1+(-1)^{n-1-j}]x^j}{n-2r-j} &= \frac{1}{2}\sum_{\mu=0}^{n-1} x^{\mu}[1-(-1)^{n-\mu}] \sum_{v=0}^{[(n-\mu-1)/2]} \frac{a_{nv}}{n-2v-\mu} \\
&= \frac{1}{2}\sum_{k=1}^{n} x^{n-k}[1-(-1)^k] \sum_{v=0}^{[(k-1)/2]} \frac{a_{nv}}{k-2v} \\
&= \sum_{j=0}^{[(n-1)/2]} x^{n-2j-1} \sum_{v=0}^{j} \frac{a_{nv}}{2j+1-2v} \\
&= W_{n-1}(x). \qquad (11.276)
\end{aligned}$$

The last line follows from Eq. (11.267). Hence, Eq. (11.269) is proved.

If $p(x)$ denotes polynomial of order less than or equal to k, then

$$2\,Q_k(x)p(x) = \pi(TP_k p)(x) = P\int_{-1}^{1} \frac{p(s)P_k(s)\mathrm{d}s}{x-s}. \qquad (11.277)$$

To prove this result start with

$$p(s) = \sum_{j=0}^{k} c_j s^j, \qquad (11.278)$$

where the c_j are appropriate constants characterizing the polynomial. The factor s^j can be written as

$$s^j = x^j - x^{j-1}(x-s)\sum_{m=0}^{j-1} x^{-m}s^m, \qquad (11.279)$$

and hence

$$\begin{aligned}
P\int_{-1}^{1} \frac{p(s)P_k(s)\mathrm{d}s}{x-s} &= \sum_{j=0}^{k} c_j P\int_{-1}^{1} \frac{s^j P_k(s)\mathrm{d}s}{x-s} \\
&= \sum_{j=0}^{k} c_j x^j \left\{ P\int_{-1}^{1} \frac{P_k(s)\mathrm{d}s}{x-s} - \sum_{m=0}^{j-1} x^{-m-1}\int_{-1}^{1} s^m P_k(s)\mathrm{d}s \right\} \\
&= 2\,Q_k(x)p(x) - \sum_{j=0}^{k} c_j \sum_{m=0}^{j-1} x^{j-m-1}\int_{-1}^{1} s^m P_k(s)\mathrm{d}s.
\end{aligned}$$

$$(11.280)$$

Making use of the formula

$$\int_{-1}^{1} s^m P_k(s)\mathrm{d}s = 0, \qquad \text{for } m < k, \tag{11.281}$$

it follows that the double sum over these integrals is zero, and the required result is proved. The preceding formula can be established by noting that s^m can be expanded as a series of Legendre polynomials, and application of the orthogonality condition for the Legendre polynomials leads directly to Eq. (11.281). Equation (11.277) can be useful when dealing with the finite Hilbert transform of polynomial functions.

11.12 Finite Hilbert transform of the Chebyshev polynomials

The finite Hilbert transform of the Chebyshev polynomials in combination with certain functions that lead to straightforward results are considered in this section. For the Chebyshev polynomial of the first kind, $T_n(x)$, for $n \in \mathbb{Z}^+$, the change of variables $u = \cos x$ and $v = \cos y$ leads to

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{T_n(v)\mathrm{d}v}{(u-v)\sqrt{(1-v^2)}} = \frac{1}{\pi} P \int_{0}^{\pi} \frac{T_n(\cos y)\mathrm{d}y}{\cos x - \cos y}. \tag{11.282}$$

Inserting the standard relation for $T_n(\cos y)$,

$$T_n(\cos y) = \cos ny, \tag{11.283}$$

and employing Eq. (11.26), yields

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{T_n(v)\mathrm{d}v}{(u-v)\sqrt{(1-v^2)}} = -\frac{\sin nx}{\sin x}. \tag{11.284}$$

The Chebyshev polynomial of the second kind, $U_n(x)$, for $n \in \mathbb{Z}^+$, satisfies

$$U_n(\cos x) = \frac{\sin(n+1)x}{\sin x}, \tag{11.285}$$

and so Eq. (11.284) can be expressed as

$$T\left[\frac{T_n(x)}{\sqrt{(1-x^2)}}\right] = -U_{n-1}(x). \tag{11.286}$$

In a similar manner, it follows that

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{\sqrt{(1-v^2)}\, U_{n-1}(v)\mathrm{d}v}{u-v} = \frac{1}{\pi} P \int_{0}^{\pi} \frac{\sin ny \sin y\, \mathrm{d}y}{\cos x - \cos y}. \tag{11.287}$$

Utilizing Eq. (11.30) leads to

$$T[\surd(1-x^2)\,U_{n-1}(x)] = T_n(x). \tag{11.288}$$

The finite Hilbert transform of $U_n(x)$ is now considered. Perhaps the easiest approach is to employ the series expansion for the Chebyshev polynomial (Erdélyi *et al.*, 1953, Vol. II, p. 185),

$$U_n(x) = \sum_{m=0}^{[n/2]} b_{nm}\, x^{n-2m}, \tag{11.289}$$

where

$$b_{nm} = \frac{2^{n-2m}(-1)^m (n-m)!}{m!(n-2m)!}. \tag{11.290}$$

Hence,

$$\begin{aligned}
TU_n(x) &= \frac{1}{\pi}\sum_{m=0}^{[n/2]} b_{nm} P\int_{-1}^{1} \frac{y^{n-2m}\,\mathrm{d}y}{x-y} \\
&= \frac{1}{\pi}\sum_{m=0}^{[n/2]} b_{nm} x^{n-2m}\left\{P\int_{-x^{-1}}^{x^{-1}} \frac{\mathrm{d}y}{1-y} - \int_{-x^{-1}}^{x^{-1}} \frac{\{y^{n-2m}-1\}\mathrm{d}y}{y-1}\right\} \\
&= \frac{1}{\pi}\sum_{m=0}^{[n/2]} b_{nm} x^{n-2m}\left\{\log\left(\frac{1+x}{1-x}\right) - \sum_{k=0}^{n-2m-1} \frac{[1+(-1)^k]}{(k+1)\,x^{k+1}}\right\} \\
&= \frac{1}{\pi} U_n(x)\log\left(\frac{1+x}{1-x}\right) - \frac{1}{\pi}\sum_{m=0}^{[n/2]} b_{nm} \sum_{k=0}^{n-2m-1} \frac{[1+(-1)^k]\,x^{n-2m-k-1}}{k+1}.
\end{aligned} \tag{11.291}$$

The double summation in this expression can be simplified using the result in Eq. (11.275). Hence, Eq. (11.291) yields

$$TU_n(x) = \frac{1}{\pi} U_n(x)\,\log\left(\frac{1+x}{1-x}\right) - \frac{1}{\pi}\sum_{j=0}^{[(n-1)/2]} x^{n-2j-1}\sum_{v=0}^{j} \frac{b_{nv}}{2j+1-2v}. \tag{11.292}$$

In a similar manner, $T[T_n(x)]$ can be determined. Starting with the expansion formula for the Chebyshev polynomial (Erdélyi *et al.*, 1953, Vol. II, p. 185),

$$T_n(x) = \sum_{m=0}^{[n/2]} c_{nm} x^{n-2m}, \tag{11.293}$$

where

$$c_{nm} = \begin{cases} \dfrac{2^{n-2m-1}(-1)^m n(n-m-1)!}{m!(n-2m)!}, & n \geq 1 \\ 1, & n = 0, \end{cases} \tag{11.294}$$

and following the same steps as previously indicated, leads to

$$TT_n(x) = \frac{1}{\pi} T_n(x) \log\left(\frac{1+x}{1-x}\right) - \frac{1}{\pi} \sum_{j=0}^{[(n-1)/2]} x^{n-2j-1} \sum_{v=0}^{j} \frac{c_{nv}}{2j+1-2v}. \tag{11.295}$$

From the results for $TT_n(x), TU_n(x)$, and the identities given in Eqs. (11.283) and (11.285), the following formulas for the alternative cosine form of the finite Hilbert transform introduced in Section 11.2 can be deduced:

$$\begin{aligned} \frac{1}{\pi} P \int_0^{\pi} \frac{\sin y \cos ny \, dy}{\cos x - \cos y} &= \frac{1}{\pi} P \int_{-1}^{1} \frac{T_n(y) dy}{\cos x - y} \\ &= \frac{2}{\pi} \cos nx \log \cot\left(\frac{x}{2}\right) \\ &\quad - \frac{1}{\pi} \sum_{j=0}^{[(n-1)/2]} (\cos x)^{n-2j-1} \sum_{v=0}^{j} \frac{c_{nv}}{2j+1-2v}, \end{aligned} \tag{11.296}$$

and

$$\begin{aligned} \frac{1}{\pi} P \int_0^{\pi} \frac{\sin ny dy}{\cos x - \cos y} &= \frac{1}{\pi} P \int_{-1}^{1} \frac{U_{n-1}(y) dy}{\cos x - y} \\ &= \frac{2 \sin nx}{\pi \sin x} \log \cot\left(\frac{x}{2}\right) \\ &\quad - \frac{1}{\pi} \sum_{j=0}^{[(n-2)/2]} (\cos x)^{n-2j-2} \sum_{v=0}^{j} \frac{b_{(n-1)v}}{2j+1-2v}. \end{aligned} \tag{11.297}$$

There is a formula with a similar structure to Eq. (11.286) for the Gegenbauer polynomials, $C_n^\lambda(x)$, also termed the ultraspherical polynomials. These polynomials have some importance because of the utility of the generating function relationship:

$$(1-2xr+r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) r^n. \tag{11.298}$$

The finite Hilbert transform of the weighted Gegenbauer polynomials $C_n^\lambda(x)$ defines the ultraspherical function of the second kind $D_n^\lambda(x)$ (Andrews, Askey, and Roy, 1999, p. 322),

$$(1-x^2)^{\lambda-1/2} D_n^\lambda(x) = \frac{1}{\pi} P \int_{-1}^{1} \frac{(1-s^2)^{\lambda-1/2} C_n^\lambda(s)\mathrm{d}s}{x-s}. \tag{11.299}$$

11.13 Contour integration approach to the derivation of some finite Hilbert transforms

A number of finite Hilbert transforms can be conveniently evaluated by contour integration techniques. In this section an illustration of this technique is provided for a class of transforms. Suppose the following conditions hold for the function $f(z)$.

(1) The function $f(z)$ is analytic in the entire complex plane, except at the singular points $z_1, z_2, \ldots, z_n$, which are assumed to satisfy the condition $z_k \notin [a, b]$, and at the singular points $\alpha_1, \alpha_2, \ldots, \alpha_m$, which satisfy $\alpha_k \in [a, b]$.
(2) The following condition holds:

$$\lim_{z\to\infty} z^{p+q+1} f(z) = c, \tag{11.300}$$

where p and q satisfy $p \in (-1,\ 1), q \in (-1,\ 1), p+q = -1, 0$, or 1, and c is a constant.

Then

$$\frac{1}{\pi} P \int_a^b (s-a)^p (b-s)^q f(s)\mathrm{d}s = \csc \pi q \left\{ \sum_{k=1}^{n} \mathrm{Res}_{z_k} (z-a)^p\,(z-b)^q f(s) - c \right\}$$
$$+ \cot \pi q \left\{ \sum_{k=1}^{m} \mathrm{Res}_{\alpha_k} (z-a)^p (b-z)^q f(z) \right\}. \tag{11.301}$$

In this equation, the notation $\mathrm{Res}_{z_k}()$ means "evaluate the residue of () at the point z_k." Equation (11.301) can be obtained in the following manner. Consider the contour shown in Figure 11.3, where only a single singular point $\alpha_k \Rightarrow \alpha$ on the segment

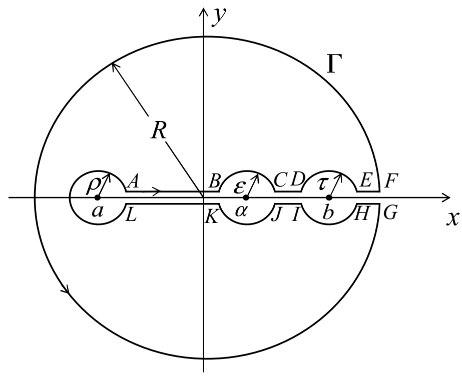

Figure 11.3. Circular contour, center the origin, with a cut along the x-axis from $[a, \infty)$.

(a, b) is shown. The generalization to a sum of such points is immediate. Using the Cauchy residue theorem, it follows that

$$\begin{aligned}\oint_{\Gamma} (z-a)^p(z-b)^q f(z)\mathrm{d}z &= \int_{\Gamma_{AB}} \{\}\mathrm{d}z + \int_{\Gamma_{BC}} \{\}\mathrm{d}z + \int_{\Gamma_{CD}} \{\}\mathrm{d}z + \int_{\Gamma_{DE}} \{\}\mathrm{d}z \\ &\quad + \int_{\Gamma_{EF}} \{\}\mathrm{d}z + \int_{\Gamma_{FG}} \{\}\mathrm{d}z + \int_{\Gamma_{GH}} \{\}\mathrm{d}z + \int_{\Gamma_{HI}} \{\}\mathrm{d}z \\ &\quad + \int_{\Gamma_{IJ}} \{\}\mathrm{d}z + \int_{\Gamma_{JK}} \{\}\mathrm{d}z + \int_{\Gamma_{KL}} \{\}\mathrm{d}z + \int_{\Gamma_{LA}} \{\}\mathrm{d}z \\ &= 2\pi\mathrm{i} \sum_{k=1}^{n} \mathrm{Res}_{z_k}(z-a)^p(z-b)^q f(z), \end{aligned} \tag{11.302}$$

where $\{\} \equiv (z-a)^p(z-b)^q f(z)$ and the $\Gamma_{\alpha\beta}$ refer to the various segments along the contour Γ. The individual contributions in Eq. (11.302) are now evaluated. The first pair of integrals gives

$$\begin{aligned}&\int_{\Gamma_{AB}} \{\}\mathrm{d}z + \int_{\Gamma_{CD}} \{\}\mathrm{d}z \\ &\quad = \mathrm{e}^{\mathrm{i}\pi q} \left\{ \int_{a+\rho}^{\alpha-\varepsilon} (x-a)^p(b-x)^q f(x)\mathrm{d}x + \int_{\alpha+\varepsilon}^{b-\tau} (x-a)^p(b-x)^q f(x)\mathrm{d}x \right\}, \end{aligned} \tag{11.303}$$

which simplifies, on taking $\lim \rho \to 0$, $\lim \varepsilon \to 0$, and $\lim \tau \to 0$, to yield

$$\int_{\Gamma_{AB}} \{\}\mathrm{d}z + \int_{\Gamma_{CD}} \{\}\mathrm{d}z = \mathrm{e}^{\mathrm{i}\pi q} P \int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x. \tag{11.304}$$

The Cauchy principal value covers the singularity at $x = \alpha$. In a similar manner

$$\int_{\Gamma_{IJ}} \{\}\mathrm{d}z + \int_{\Gamma_{KL}} \{\}\,\mathrm{d}z = -\mathrm{e}^{-\mathrm{i}\pi q} P \int_a^b (x-a)^p (b-x)^q f(x)\mathrm{d}x. \tag{11.305}$$

In the preceding two equations the result $(z-b)^q = (b-z)^q(-1)^q \Rightarrow (b-x)^q \mathrm{e}^{\mathrm{i}\pi q}$ (recalling $\alpha^\beta = \mathrm{e}^{\beta \ln \alpha}$) has been employed on the x-axis on the upper side of the branch cut, and $(z-b)^q = (b-z)^q(-1)^q \Rightarrow (b-x)^q \mathrm{e}^{\mathrm{i}\pi q - 2\mathrm{i}\pi q} = (b-x)^q \mathrm{e}^{-\mathrm{i}\pi q}$ on the x-axis on the lower side of the cut. Employing the substitutions

$$z - a = \rho \mathrm{e}^{\mathrm{i}\theta}, \quad \text{on } \Gamma_{LA}, \tag{11.306}$$

and

$$z - b = \tau \mathrm{e}^{\mathrm{i}\theta}, \quad \text{on } \Gamma_{DE} \text{ and } \Gamma_{HI}, \tag{11.307}$$

then

$$\begin{aligned}\int_{\Gamma_{LA}} \{\}\mathrm{d}z + \int_{\Gamma_{DE}} \{\}\mathrm{d}z + \int_{\Gamma_{HI}} \{\}\mathrm{d}z &= \int_{2\pi}^{0} \rho^p \mathrm{e}^{\mathrm{i}p\theta} (a - b + \rho \mathrm{e}^{\mathrm{i}\theta})^q f(a + \rho \mathrm{e}^{\mathrm{i}\theta}) \mathrm{i}\rho \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\ &+ \int_{\pi}^{0} (b - a + \tau \mathrm{e}^{\mathrm{i}\theta})^p \tau^q \mathrm{e}^{\mathrm{i}q\theta} f(b + \tau \mathrm{e}^{\mathrm{i}\theta}) \mathrm{i}\tau \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\ &+ \int_{2\pi}^{\pi} (b - a + \tau \mathrm{e}^{\mathrm{i}\theta})^p \tau^q \mathrm{e}^{\mathrm{i}q\theta} f(b + \tau \mathrm{e}^{\mathrm{i}\theta}) \mathrm{i}\tau \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta,\end{aligned} \tag{11.308}$$

which, in the lim $\rho \to 0$ and lim $\tau \to 0$, leads to

$$\int_{\Gamma_{LA}} \{\}\mathrm{d}z + \int_{\Gamma_{DE}} \{\}\mathrm{d}z + \int_{\Gamma_{HI}} \{\}\mathrm{d}z = 0, \tag{11.309}$$

since $p + 1 > 0$ and $q + 1 > 0$. To treat the singular point on the interval (a, b), the substitution $z - \alpha = \varepsilon \mathrm{e}^{\mathrm{i}\theta}$ on Γ_{BC} and Γ_{JK} leads to

$$\begin{aligned}\int_{\Gamma_{BC}} \{\}\mathrm{d}z + \int_{\Gamma_{JK}} \{\}\mathrm{d}z &= \mathrm{e}^{\mathrm{i}\pi q} \int_{\pi}^{0} (\alpha - a + \varepsilon \mathrm{e}^{\mathrm{i}\theta})^p (b - \alpha - \varepsilon \mathrm{e}^{\mathrm{i}\theta})^q f(\alpha + \varepsilon\, \mathrm{e}^{\mathrm{i}\theta}) \mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\ &+ \mathrm{e}^{-\mathrm{i}\pi q} \int_{2\pi}^{\pi} (\alpha - a + \varepsilon \mathrm{e}^{\mathrm{i}\theta})^p (b - \alpha - \varepsilon \mathrm{e}^{\mathrm{i}\theta})^q f(\alpha + \varepsilon \mathrm{e}^{\mathrm{i}\theta}) \mathrm{i}\varepsilon\, \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta,\end{aligned} \tag{11.310}$$

which, on taking $\lim \varepsilon \to 0$, yields

$$\begin{aligned}\int_{\Gamma_{BC}} \{\}\mathrm{d}z + \int_{\Gamma_{JK}} \{\}\mathrm{d}z &= -\mathrm{i}\mathrm{e}^{\mathrm{i}\pi q} \int_0^{\pi} (\alpha - a)^p (b - \alpha)^q g(\alpha)\mathrm{d}\theta \\ &\quad - \mathrm{i}\mathrm{e}^{-\mathrm{i}\pi q} \int_{\pi}^{2\pi} (\alpha - a)^p (b - \alpha)^q g(\alpha)\mathrm{d}\theta \\ &= -2\pi \mathrm{i} \cos \pi q \, (\alpha - a)^p (b - \alpha)^q g(\alpha), \end{aligned} \tag{11.311}$$

where the assumption that the singularity is a simple pole has been employed, and the function f is written as

$$f(z) = \frac{g(z)}{z - \alpha}, \tag{11.312}$$

so that

$$\begin{aligned}\lim_{\varepsilon \to 0} f(\alpha + \varepsilon \mathrm{e}^{\mathrm{i}\theta})\mathrm{d}z &= \lim_{\varepsilon \to 0} \frac{g(\alpha + \varepsilon\, \mathrm{e}^{\mathrm{i}\theta})\mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}\theta}\, \mathrm{d}\theta}{\varepsilon\, \mathrm{e}^{\mathrm{i}\theta}} \\ &= \mathrm{i}g(\alpha)\mathrm{d}\theta. \end{aligned} \tag{11.313}$$

Equation (11.311) can be written as follows:

$$\int_{\Gamma_{BC}} \{\,\}\mathrm{d}z + \int_{\Gamma_{JK}} \{\,\}\mathrm{d}z = -2\pi \mathrm{i} \cos \pi q \, \mathrm{Res}_{z\alpha} (z - a)^p (b - z)^q f(z). \tag{11.314}$$

The contributions from Γ_{EF} and Γ_{GH} cancel:

$$\begin{aligned}\int_{\Gamma_{EF}} \{\}\mathrm{d}z + \int_{\Gamma_{GH}} \{\}\mathrm{d}z &= \int_{b+\tau}^{R} (x - a)^p (x - b)^q f(x)\mathrm{d}x \\ &\quad + \int_{R}^{b+\tau} (x - a)^p (x - b)^q f(x)\mathrm{d}x \\ &= 0\,. \end{aligned} \tag{11.315}$$

The remaining contribution comes from the large circular section of the contour. Set

$$z = R\mathrm{e}^{\mathrm{i}\theta}, \tag{11.316}$$

so

$$\int_{\Gamma_{FG}} \{\}\mathrm{d}z = \int_0^{2\pi} (R\mathrm{e}^{\mathrm{i}\theta} - a)^p (R\mathrm{e}^{\mathrm{i}\theta} - b)^q f(R\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\, \mathrm{d}\theta. \tag{11.317}$$

In the $\lim R \to \infty$, Eq. (11.317) can be written as follows:

$$\int_{\Gamma_{FG}} \{\}\mathrm{d}z = \lim_{R\to\infty} \int_0^{2\pi} R^{p+q+1}\, \mathrm{e}^{\mathrm{i}(p+q+1)\theta} f(R\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}\,\mathrm{d}\theta \tag{11.318}$$

and hence

$$\begin{aligned}\int_{\Gamma_{FG}} \{\}\mathrm{d}z &= \lim_{z\to\infty} \int_0^{2\pi} z^{p+q+1} f(z)\mathrm{i}\,\mathrm{d}\theta \\ &= 2\pi\mathrm{i}c.\end{aligned} \tag{11.319}$$

This result follows from the initial hypothesis given for the behavior of $f(z)$ as $z \to \infty$. Putting it all together yields the following:

$$\begin{aligned}\oint_{\Gamma} (z-a)^p (z-b)^q f(z)\mathrm{d}z &= \int_{\Gamma_{AB}} \{\}\mathrm{d}z + \int_{\Gamma_{CD}} \{\}\mathrm{d}z + \int_{\Gamma_{IJ}} \{\}\mathrm{d}z + \int_{\Gamma_{KL}} \{\}\mathrm{d}z \\ &\quad + \int_{\Gamma_{BC}} \{\}\mathrm{d}z + \int_{\Gamma_{JK}} \{\}\mathrm{d}z + \int_{\Gamma_{FG}} \{\}\mathrm{d}z \\ &= 2\mathrm{i}\,\sin\pi q\, P\int_a^b (x-a)^p\,(b-x)^q\, f(x)\mathrm{d}x + 2\pi\mathrm{i}c \\ &\quad - 2\pi\mathrm{i}\cos\pi q\,\mathrm{Res}_{z=\alpha}(z-a)^p(b-z)^q f(z).\end{aligned} \tag{11.320}$$

Hence, on allowing for the singularities of $f(z)$ at the points z_k, $k = 1, 2, \ldots, n$,

$$\begin{aligned}\frac{1}{\pi}P\int_a^b (x-a)^p(b-x)^q f(x)\mathrm{d}x &= \cot\pi q\,\mathrm{Res}_{z=\alpha}(z-a)^p(b-z)^q f(z) \\ &\quad - \csc\pi q\left\{c - \sum_{k=1}^{n}\mathrm{Res}_{z_k}(z-a)^p(z-b)^q f(z)\right\}.\end{aligned} \tag{11.321}$$

The generalization to include a multiple number of singularities in the interval (a, b) is a straightforward extension of Eq. (11.321), so Eq. (11.301) is proved.

Some applications of the preceding formula are now considered. Equation (11.321) can be written in the form of the finite Hilbert transform by setting $a = -1$, $b = 1$ and

$$f(s) = \frac{g(s)}{x-s}, \tag{11.322}$$

so that

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{(1+s)^{\mathrm{p}}(1-s)^{q} g(s) \mathrm{d}s}{x-s} = \cot \pi q \, \mathrm{Res}_{z=x}\{(z+1)^{p}(1-z)^{q} g(z)(x-z)^{-1}\} - \csc \pi q \left\{ c - \sum_{k=1}^{n} \mathrm{Res}_{z_k}(z+1)^{p}(z-1)^{q} f(z) \right\}. \tag{11.323}$$

As a first example, let

$$p = q = -\frac{1}{2} \tag{11.324}$$

and

$$g(s) = 1. \tag{11.325}$$

The asymptotic limit is given by

$$\lim_{z\to\infty} z^{p+q+1} f(z) = \lim_{z\to\infty} (x-z)^{-1} = 0, \tag{11.326}$$

and so $c = 0$. The function $g(s)$ has no singularities in the complex plane, and hence

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{\mathrm{d}s}{(x-s)\sqrt{(1-s^2)}} = 0, \tag{11.327}$$

which is a result discussed previously in this chapter. For the second example let

$$p = -a, \quad q = a, \tag{11.328}$$

and

$$g(z) = 1. \tag{11.329}$$

The asymptotic limit is given by

$$\lim_{z\to\infty} z^{p+q+1} f(z) = \lim_{z\to\infty} z(x-z)^{-1} = -1 \tag{11.330}$$

and so $c = -1$. The function $g(z)$ has no singularities in the complex plane, and hence

$$\frac{1}{\pi} P \int_{-1}^{1} \left(\frac{1-s}{1+s}\right)^{a} \frac{\mathrm{d}s}{x-s} = \csc \pi a - \cot \pi a \left(\frac{1-x}{1+x}\right)^{a}, \quad \text{for } 0 < a < 1. \tag{11.331}$$

As a third example let

$$p = -a, \quad q = a - 1, \tag{11.332}$$

and

$$g(z) = 1. \tag{11.333}$$

The asymptotic limit is given by

$$\lim_{z\to\infty} z^{p+q+1} f(z) = \lim_{z\to\infty} (x - z)^{-1} = 0, \tag{11.334}$$

that is, $c = 0$. Again, there are no singularities arising from $g(z)$, hence

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{(1-s)^{a-1}}{(1+s)^a} \frac{\mathrm{d}s}{x-s} = -\frac{(1-x)^{a-1}}{(1+x)^a} \cot \pi a, \quad \text{for } 0 < a < 1. \tag{11.335}$$

With p and q interchanged, using the same $g(z)$ yields

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{(1+s)^{a-1}}{(1-s)^a} \frac{\mathrm{d}s}{x-s} = \frac{(1+x)^{a-1}}{(1-x)^a} \cot \pi a, \quad \text{for } 0 < a < 1. \tag{11.336}$$

As a final example, consider the evaluation of the finite Hilbert transform of $\surd(1 - x^2)$, which means selecting

$$p = q = \frac{1}{2} \tag{11.337}$$

and

$$g(z) = 1. \tag{11.338}$$

Examination of the asymptotic limit yields

$$\lim_{z\to\infty} z^{p+q+1} f(z) = \lim_{z\to\infty} z^2 (x - z)^{-1} = \infty, \tag{11.339}$$

which means that the contour integration formula developed does not apply. However, a rather simple device can be employed. In place of Eq. (11.338), suppose

$$g(z) = \frac{1}{1 - \lambda z}, \tag{11.340}$$

where λ is a complex constant. The term $(1 - \lambda z)^{-1}$ functions as a convergence factor. To obtain the desired finite Hilbert transform using this choice for $g(z)$, the $\lim\ \lambda \to 0$ is examined. With the choice of $g(z)$ given in Eq. (11.340), the asymptotic limit is given by

$$\lim_{z\to\infty} z^{p+q+1} f(z) = \lim_{z\to\infty} z^2[(x-z)(1-\lambda z)]^{-1} = \lambda^{-1}, \tag{11.341}$$

and so $c = \lambda^{-1}$. The function $g(z)$ has a simple pole at $z = \lambda^{-1}$, and, with the restriction that λ is a complex, this pole is not located on the cut or the real axis. This is essential, otherwise there is no λ contribution to the integral for the choice $q = 1/2$. Hence, it follows that

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{\sqrt{(1-s^2)}\mathrm{d}s}{(x-s)\,(1-\lambda\, s)} = \frac{\sqrt{(1-\lambda^2)}}{\lambda\,(1-x\,\lambda)} - \lambda^{-1}. \tag{11.342}$$

If the lim $\lambda \to 0$ is taken for Eq. (11.342),

$$\frac{1}{\pi} P \int_{-1}^{1} \frac{\sqrt{(1-s^2)}\mathrm{d}s}{x-s} = x. \tag{11.343}$$

The approach of choosing an appropriate convergence factor can be a very effective tool for evaluating a variety of finite Hilbert transforms.

11.14 The thin airfoil problem

In this section a concise account of the thin airfoil problem is considered. An airfoil is illustrated in Figure 11.4. The key problem of interest is to understand how the airfoil responds to a surrounding air stream. The prototypical application of the airfoil occurs in the design of plane wings. The variables that characterize the airfoil are its length c, measured along the chord between the leading edge (LE) and the trailing edge (TE), the maximum thickness $t_{\max}$ and its location $x_{\max}$, the maximum displacement of the camber line from the chord line, z_c, and its location, x_c. The camber line is shown as a dashed curve in Figure 11.4, and is placed halfway between the upper and lower edges of the airfoil. A thin airfoil has a small value for the ratio $t_{\max}/c$, and also a small value for the maximum displacement of the camber line from the chord line. The principal problem considered here is the determination of the circulation density, and from this the lift per unit span can be evaluated.

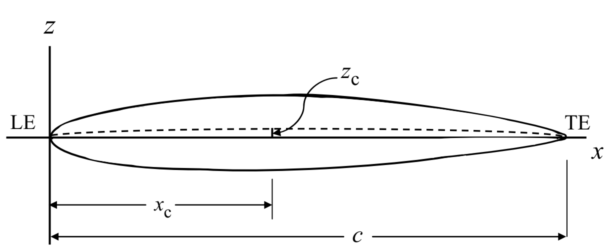

Figure 11.4. Thin airfoil showing the camber line as a dashed curve.

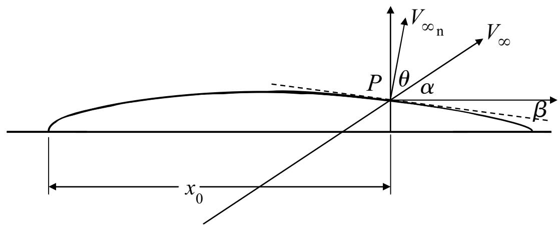

Figure 11.5. Geometric arrangement to determine V_{∞_n}.

If the airfoil is thin the problem can be simplified by replacing the airfoil by a single streamline having the same geometric shape as the camber line. The problem therefore reduces to a study of the response of this streamline to an air stream. The calculation requires the evaluation of the velocity components perpendicular to the camber line. Suppose the oncoming air stream makes an angle α with the chord line. This is called the angle of attack. The streamlines are parallel to the line labeled V_∞ in Figure 11.5. The dashed line in Figure 11.4 has been replaced by a solid curve in Figure 11.5. The velocity component of a streamline perpendicular to the camber line at some point P is denoted by V_{∞_n}, and it can be calculated in the following manner. The slope of the camber line at the point P is denoted by $(\mathrm{d}z/\mathrm{d}x)_0$, and the subscript zero signifies that the slope is evaluated at the location x_0. From Figure 11.5 it follows that

$$\begin{aligned} V_{\infty_n} &= V_\infty \cos\theta \\ &= V_\infty \cos\left\{\frac{\pi}{2} - (\alpha+\beta)\right\} \\ &= V_\infty \sin\left(\alpha - \tan^{-1}\left\{\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_0\right\}\right), \end{aligned} \tag{11.344}$$

and the last line follows using $\tan\beta = -(\mathrm{d}z/\mathrm{d}x)_0$.

The induced velocity component $\mathrm{d}V_\mathrm{i}$ at the point P located a distance r from a segment $\mathrm{d}s$, which is situated on a vortex sheet coincident with the camber curve, is given by

$$\mathrm{d}V_\mathrm{i} = -\frac{\gamma}{2\pi r}\,\mathrm{d}s, \tag{11.345}$$

and γ is called the circulation density. For further discussion on this result, consult Kuethe and Chow (1986, p. 108). The negative sign in Eq. (11.345) arises from the convention that a clockwise circulation has a positive outward normal vector. The geometric arrangement is shown in Figure 11.6. The component $\mathrm{d}V_\mathrm{i}$ normal to the camber line is denoted $\mathrm{d}V_{\mathrm{i}_\mathrm{n}}$ and is given by

$$\mathrm{d}V_{\mathrm{i}_\mathrm{n}} = -\frac{\gamma\cos\phi}{2\pi r}\,\mathrm{d}s. \tag{11.346}$$

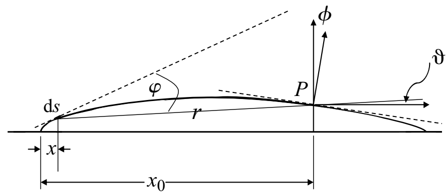

Figure 11.6. Variables to the describe the calculation of $V_{\mathrm{i_n}}$.

The quantity $V_{\mathrm{i_n}}$ can be determined by making use of the following results:

$$\cos\vartheta = \frac{x_0 - x}{r} \tag{11.347}$$

and

$$\cos\varphi = \frac{\mathrm{d}x}{\mathrm{d}s}. \tag{11.348}$$

Combining the preceding three results, and integrating from the leading edge to the trailing edge, leads to

$$V_{\mathrm{i_n}} = -\frac{1}{2\pi} P \int_{x_{\mathrm{LE}}}^{x_{\mathrm{TE}}} \frac{\gamma(x)}{x_0 - x} \frac{\cos\phi\cos\vartheta}{\cos\varphi}\,\mathrm{d}x. \tag{11.349}$$

The quantities $\gamma, \cos\phi, \cos\vartheta$, and $\cos\varphi$ are all functions of x.

In order that the camber line represents a streamline, the sum of the normal velocity components must sum to zero; that is,

$$V_{\mathrm{i_n}} + V_{\infty_{\mathrm{n}}} = 0. \tag{11.350}$$

Making using of this result leads to

$$\frac{1}{2\pi} P \int_0^c \frac{\gamma(x)}{x_0 - x} \frac{\cos\phi\cos\vartheta}{\cos\varphi}\,\mathrm{d}x = V_\infty \sin\left(\alpha - \tan^{-1}\left\{\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_0\right\}\right), \tag{11.351}$$

which represents a singular integral equation for the function of interest, γ. The reader will immediately recognize the finite Hilbert transform that appears in this formula.

A particularly simple case of Eq. (11.351) occurs when the airfoil is close to symmetric about the chord line, so that the camber line is displaced only slightly above the chord line. In this case, each of the three angles, ϕ, ϑ, and φ, are close to zero, so the angular terms in the integrand in Eq. (11.351) can be set to one. This amounts to replacing the camber line by a segment of the x-axis, the so-called *planar wing approximation*. Also, the slope $(\mathrm{d}z/\mathrm{d}x)_0$ is close to zero: it is zero in the planar wing approximation. Hence Eq. (11.351) can be approximated by

$$\frac{1}{2\pi} P \int_0^c \frac{\gamma(x)}{x_0 - x}\, dx = V_\infty \sin\left(\alpha - \left(\frac{dz}{dx}\right)_0\right), \tag{11.352}$$

which simplifies further when the angle of attack is small, to yield

$$\frac{1}{2\pi} P \int_0^c \frac{\gamma(x)}{x_0 - x}\, dx = V_\infty \left(\alpha - \left(\frac{dz}{dx}\right)_0\right). \tag{11.353}$$

The simplest case of Eq. (11.353) arises when the airfoil is symmetric, or when the planar wing approximation is employed, so that the slope term $(dz/dx)_0$ is zero, therefore leading to

$$\frac{1}{2\pi} P \int_0^c \frac{\gamma(x)}{x_0 - x}\, dx = \alpha V_\infty. \tag{11.354}$$

Introducing the change of variables

$$x = \frac{c}{2}(1 - \cos\vartheta), \quad x_0 = \frac{c}{2}(1 - \cos\vartheta_0), \tag{11.355}$$

leads to

$$\frac{1}{\pi} P \int_0^\pi \frac{\gamma(\vartheta) \sin\vartheta\, d\vartheta}{\cos\vartheta - \cos\vartheta_0} = 2\alpha V_\infty. \tag{11.356}$$

The boundary condition that needs to be satisfied is as follows:

$$\gamma(x_{\text{TE}}) = \gamma(\pi) = 0, \tag{11.357}$$

which is called the *Kutta condition*. There is no velocity discontinuity at the trailing edge of the airfoil if the Kutta condition is imposed. The numerator of the integrand in Eq. (11.356) can be expanded as follows:

$$\gamma(\vartheta) \sin\vartheta = a_0 + \sum_{n=1}^{\infty} a_n \cos n\vartheta. \tag{11.358}$$

Inserting this result in Eq. (11.356) and making use of Eq. (11.137) leads to

$$\sum_{n=1}^{\infty} a_n \sin n\vartheta_0 = 2\alpha V_\infty \sin\vartheta_0. \tag{11.359}$$

Multiplying this result by $\sin m\vartheta_0$ and integrating over $[0, \pi]$ yields

$$a_m = 2\alpha V_\infty \delta_{m1}, \quad m \geq 1, \tag{11.360}$$

and hence

$$\gamma(\vartheta_0)\sin\vartheta_0 = a_0 + 2\alpha V_\infty \cos\vartheta_0. \tag{11.361}$$

The constant a_0 in the expansion in Eq. (11.358) can be determined from the Kutta condition:

$$a_0 = 2\alpha V_\infty, \tag{11.362}$$

and therefore

$$\gamma(\vartheta_0) = \frac{2\alpha V_\infty(1+\cos\vartheta_0)}{\sin\vartheta_0} = 2\alpha V_\infty \cot\left(\frac{\vartheta_0}{2}\right). \tag{11.363}$$

This can be readily checked to be the solution of Eq. (11.356) by back insertion and employing Eq. (11.137). Converting to the original variable using Eq. (11.355) leads to

$$\gamma(x_0) = 2\alpha V_\infty \sqrt{\left(\frac{c-x_0}{x_0}\right)}. \tag{11.364}$$

The lift per unit span, L, is obtained in terms of the air density, ρ, by integrating the quantity $\rho\gamma(x)V_\infty$ over the length of the airfoil:

$$L = \int_0^c \rho\gamma(x)V_\infty\,\mathrm{d}x = \pi\alpha\rho c V_\infty^2. \tag{11.365}$$

To deal with the more complicated situation of Eq. (11.353), use the same transformation to angular variables, Eq. (11.355), and let

$$\gamma(\vartheta) = 2V_\infty\left[\frac{(1+\cos\vartheta)}{\sin\vartheta}b_0 + \sum_{n=1}^{\infty} b_n \sin n\vartheta\right]. \tag{11.366}$$

Inserting this result into the transformed version of Eq. (11.353) leads to the following:

$$\frac{1}{\pi}P\int_0^\pi \frac{[(1+\cos\vartheta)b_0 + \sum_{n=1}^\infty b_n \sin n\vartheta \sin\vartheta]\mathrm{d}\vartheta}{\cos\vartheta - \cos\vartheta_0} = \alpha - \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_0; \tag{11.367}$$

that is,

$$b_0 + \sum_{n=1}^{\infty} b_n \frac{1}{\pi}P\int_0^\pi \frac{\sin n\vartheta\ \sin\vartheta\,\mathrm{d}\vartheta}{\cos\vartheta - \cos\vartheta_0} = \alpha - \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_0, \tag{11.368}$$

which simplifies on using Eq. (11.30) to give

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_0 = (\alpha - b_0) + \sum_{n=1}^{\infty} b_n \cos n\vartheta_0. \tag{11.369}$$

From this result the coefficients b_n can be determined in terms of the geometry of the slope of the camber line, which is a known quantity. The circulation density $\gamma(\vartheta)$ can therefore be evaluated from Eq. (11.366).

11.15 The generalized airfoil problem

A two-dimensional oscillating airfoil in a wind tunnel can be described by the following equation:

$$g(x) = \frac{1}{\pi} P \int_{-1}^{1} \sqrt{\left(\frac{1-y}{1+y}\right)} \frac{f(y)\mathrm{d}y}{x-y} + \frac{v}{\pi} \int_{-1}^{1} \sqrt{\left(\frac{1-y}{1+y}\right)} \log|x-y|\, f(y)\mathrm{d}y, \tag{11.370}$$

where $x \in (-1, 1)$, v is a complex constant, $g(x)$ is a known function (the downwash velocity), and $f(x)$ is the sought function (the pressure jump across the airfoil). This particular singular integral equation is called a generalized airfoil problem. Note that for the case $v = 0$ it reverts to a standard finite Hilbert transform result, and can be solved in the manner indicated in Section 11.4.

The terminology generalized airfoil problem is also associated with an equation of the following form:

$$g(x) = \frac{1}{\pi} P \int_{-1}^{1} \frac{f(y)\mathrm{d}y}{x-y} + \frac{1}{\pi} \int_{-1}^{1} K(x,y) f(y)\mathrm{d}y. \tag{11.371}$$

The kernel function $K(x,y)$ is given by

$$K(x,y) = k_1(x,y)\ \log|x-y| + k_2(x,y), \tag{11.372}$$

where $k_1(x,y)$ and $k_2(x,y)$ are particular entire functions. Singular integral equations of the level of complexity of Eqs. (11.370) and (11.372) are usually not amendable to exact analytic solution, though for some particularly simple choices of $g(x)$, exact solutions for $f(x)$ can be constructed. Instead, numerical approaches must be employed to solve these equations. The end-notes provide some starting sources for reading on this topic.

11.16 The cofinite Hilbert transform

The cofinite Hilbert transform was introduced by Polyakov (2007). In the present work the cofinite Hilbert transform is represented by the notation H_c, and is defined by

$$h(x) = H_c f(x) = \frac{1}{\pi} \left\{ P \int_{-\infty}^{-1} \frac{f(s)\mathrm{d}s}{x-s} + P \int_{1}^{\infty} \frac{f(s)\mathrm{d}s}{x-s} \right\}, \tag{11.373}$$

with $|x| > 1$. This transform arises in aeroelasticity theory. The preceding definition can be generalized in an obvious fashion to cover the case

$$H_{\mathrm{c}_{a,b}} f(x) = \frac{1}{\pi}\left\{P\int_{-\infty}^{a}\frac{f(s)\mathrm{d}s}{x-s} + P\int_{b}^{\infty}\frac{f(s)\mathrm{d}s}{x-s}\right\}, \tag{11.374}$$

with $a < b$ and $x \in (-\infty, a) \cup (b, \infty)$. The notational simplification $H_{c_{-1,1}} \equiv H_{\mathrm{c}}$ is adopted.

The inversion formula for H_{c} can be found by elementary methods, if advantage is taken of the results for the finite Hilbert transform on the interval $(-1, 1)$. A preliminary result is needed. If $f(s) = 1/\surd(s^2 - 1)$, then the change of variable $y = \surd[(s-1)/(s+1)]$ in Eq. (11.373) yields, for $|x| > 1$,

$$\begin{aligned}
H_{\mathrm{c}}\left[\frac{1}{\surd(x^2-1)}\right] &= \frac{1}{\pi}\left\{P\int_{-\infty}^{-1}\frac{\mathrm{d}s}{(x-s)\surd(s^2-1)} + P\int_{1}^{\infty}\frac{\mathrm{d}s}{(x-s)\surd(s^2-1)}\right\}\\
&= -\frac{2}{\pi(1+x)}P\int_{0}^{\infty}\frac{\mathrm{d}y}{y^2-(x-1)/(x+1)}\\
&= 0.
\end{aligned} \tag{11.375}$$

Use the change of variables $s = y^{-1}$ and $x = t^{-1}$ in Eq. (11.373); setting

$$F(y) = \frac{1}{y}f\left(\frac{1}{y}\right), \quad g(t) = -\frac{1}{t}\,h\left(\frac{1}{t}\right), \tag{11.376}$$

leads to

$$g(t) = \frac{1}{\pi}P\int_{-1}^{1}\frac{F(y)\mathrm{d}y}{t-y}. \tag{11.377}$$

The inversion formula for this finite Hilbert transform was investigated in Section 11.4. Several inversion formulas can be given, and each is applied to obtain the inversion formula of Eq. (11.373). From Eq. (11.63)

$$F(t) = \frac{c}{\surd(1-t^2)} - \frac{1}{\pi\,\surd(1-t^2)}\,P\int_{-1}^{1}\frac{\surd(1-y^2)\,g(y)\mathrm{d}y}{t-y}. \tag{11.378}$$

Employing the reverse of the previous change of variables, starting with $t = x^{-1}$, leads to

$$f(x) = \frac{c}{\surd(x^2-1)} + \frac{|x|}{\pi\,\surd(x^2-1)}\,P\int_{-1}^{1}\frac{\surd(1-y^2)\,h(y^{-1})\mathrm{d}y}{y(1-xy)}. \tag{11.379}$$

The preceding formula represents one form of the inversion of Eq. (11.373). The factor c plays the role of an arbitrary constant, a fact that follows directly from Eq. (11.375).

With the change of variable $s^{-1} = y$, Eq. (11.379) can be cast in the following form

$$f(x) = \frac{c}{\sqrt{(x^2-1)}} - \frac{|x|}{\sqrt{(x^2-1)}} H_c[\sqrt{(1-y^{-2})}h(y)](x). \tag{11.380}$$

It follows from Eqs. (11.88), (11.94), and (11.98) that the alternative forms of the solution of Eq. (11.377) lead to the following results:

$$f(x) = \frac{c}{\sqrt{(x^2-1)}} + \frac{\sqrt{(x^2-1)}}{\pi|x|} P\int_{-1}^{1} \frac{h(s^{-1})\mathrm{d}s}{s(1-xs)\sqrt{(1-s^2)}}, \tag{11.381}$$

$$f(x) = \frac{c}{\sqrt{(x^2-1)}} + \frac{1}{\pi}\sqrt{\left(\frac{x+1}{x-1}\right)} P\int_{-1}^{1} \sqrt{\left(\frac{1-s}{1+s}\right)} \frac{h(s^{-1})\mathrm{d}s}{s\,(1-xs)}, \tag{11.382}$$

and

$$f(x) = \frac{c}{\sqrt{(x^2-1)}} + \frac{1}{\pi}\sqrt{\left(\frac{x-1}{x+1}\right)} P\int_{-1}^{1} \sqrt{\left(\frac{1+s}{1-s}\right)} \frac{h(s^{-1})\mathrm{d}s}{s\,(1-xs)}. \tag{11.383}$$

These results can in turn be written as follows:

$$f(x) = \frac{c}{\sqrt{(x^2-1)}} - \frac{\sqrt{(x^2-1)}}{|x|} H_c\left[\frac{h(y)}{\sqrt{(1-y^{-2})}}\right](x), \tag{11.384}$$

$$f(x) = \frac{c}{\sqrt{(x^2-1)}} - \sqrt{\left(\frac{x+1}{x-1}\right)} H_c\left[\sqrt{\left(\frac{y-1}{y+1}\right)}\, h(y)\right](x), \tag{11.385}$$

and

$$f(x) = \frac{c}{\sqrt{(x^2-1)}} - x\sqrt{\left(\frac{x-1}{x+1}\right)} H_c\left[\sqrt{\left(\frac{y+1}{y-1}\right)}\, h(y)\right](x). \tag{11.386}$$

On the basis of the discussion given in Section 11.4, the reader is left to determine the general class of functions that applies for the inversion process just given.

Notes

§11.1 For an application of the finite Hilbert transform to determine the stress drop from the finite displacement for a seismic event, see Kikuchi and Fukao (1976).

§11.4 The Hardy–Poincaré–Bertrand formula for the finite Hilbert transform is discussed further in Tricomi (1955) and Okada (1992a).

§11.4.1 A small refinement to the approach discussed in this subsection can be given based on knowledge of the value of the integral $\int_0^t \mathrm{d}y/\{(x-y)\sqrt{[y(t-x)]}\}$. The details can be found in Peters (1968).

§11.5 The trigonometric series approach to the solution of the airfoil equation is discussed in Hamel (1937, p. 145).

§11.6 For a brief history of the Riemann, Hilbert, and Riemann–Hilbert problems, see Gakhov (1966, p. 284).

§11.7 Additional discussion can be found in Levinson (1965).
§11.8 For further work on issues connected with multivalued functions, see Roos (1969, p. 245) and Carrier *et al*. (1983, p. 420). Additional work can be found in Khvedelidze (1977). For an application, see Onishchuk, Popov, and Farshait (1986).
§11.8.1 For further reading on the index of a function, see Gakhov (1966), Roos (1969), Muskhelishvili (1992), and Kress (1999).
§11.9 For a review on the Carleman approach, see Estrada and Kanwal (1987).
§11.10 For some further properties of the finite Hilbert transform, see Rosenblum and Rovnyak (1974), Suzuki (1976), Logan (1983b, 1984), Okada (1992b), Okada and Elliott (1991, 1994), Mastroianni and Occorsio (1996), and Elliott and Okada (2004). Discussion on the spectral representations for the finite Hilbert transform can be found in Koppelman and Pinus (1959), Widom (1960), and Rooney (1986). For further reading, see Clancey (1975). The spectrum of the one-sided Hilbert transform is discussed by Koppelman and Pinus (1959), Del Pace and Venturi (1981), and Rooney (1986).
§11.11 More complicated finite Hilbert transform examples involving Legendre, associated Legendre, and Jacobi polynomials can be found in Kuipers and Robin (1961).
§11.13 For additional reading on the principal result of this section, see Mitrinović and Kečkić (1984, chap. 5).
§11.14 For a detailed account on airfoils, see McCormick (1979), Houghton and Carruthers (1982), Kuethe and Chow (1986), and Bertin (2002), and for a discussion of the double interval problem for the airfoil equation, see Tricomi (1951b).
§11.15 For some further reading on the generalized airfoil equation, see Bland (1970), Moss (1983), Monegato and Sloan (1997), Monegato (1998), and Mastroianni and Themistoclakis (2005). For a discussion on numerical methods to solve singular integral equations of the type mentioned in this section, see Fromme and Golberg (1979) and Golberg (1990).

Exercises

11.1 Determine whether $\sum_{n=1}^{\infty} \sin n\theta$ diverges or converges for $0 < \theta < \pi$.

11.2 Evaluate the following finite Hilbert transforms for $-1 < x < 1$:

(i) $T[x^4]$,
(ii) $T[\surd(1-x^2)]$,
(iii) $T[1/\surd(1-x^2)]$,
(iv) $T[x\surd(1-x^2)]$, and
(v) $T[x/\surd(1-x^2)]$.

11.3 Calculate $Tf(x)$ given

$$f(x) = \begin{cases} 0, & |x| > 1 \\ 1, & |x| \leq 1, \text{ and } x \text{ rational} \\ 0, & |x| \leq 1, \text{ and } x \text{ irrational.} \end{cases}$$

11.4 Evaluate $Tf(x)$ using

$$f(x) = \begin{cases} 0, & |x| > 1 \\ 1, & |x| \le 1, \text{ and } x \text{ irrational.} \\ 0, & |x| \le 1, \text{ and } x \text{ rational.} \end{cases}$$

11.5 Prove Eq. (11.30) without recourse to a series expansion. [Hint: Convert the required integral to a contour integral with the contour taken as the unit circle.]

11.6 Show that, for $-1 < x < 1$, $T[\sqrt{(1-x^2)}\log(1+x)] = 1 - x\log 2 - (\pi/2 - \arcsin x)\sqrt{(1-x^2)}$.

11.7 Prove that, for $-1 < x < 1$,

$$T\left[\frac{\log(1-x)}{\sqrt{(1-x^2)}}\right] = \frac{\pi - \cos^{-1} x}{\sqrt{(1-x^2)}}.$$

11.8 Evaluate

$$P\int_0^\pi \frac{f(y)\sin y\,\mathrm{d}y}{\cos x - \cos y}$$

for $f(y) = \cos y$.

11.9 Calculate

$$P\int_0^\pi \frac{f(y)\sin y\,\mathrm{d}y}{\cos x - \cos y}$$

using $f(y) = \cos y \cos ny$, with $n \in \mathbb{Z}^+$.

11.10 Evaluate

$$P\int_0^\pi \frac{f(y)\sin y\,\mathrm{d}y}{\cos x - \cos y}$$

using $f(y) = \cos(n + 1/2)y$, with $n \in \mathbb{N}$.

11.11 Evaluate the commutator $[T, R]$, where R is the reflection operator.

11.12 Show that Eq. (11.219) holds for the case $f(x) = \sqrt{(1-x^2)}\log(1+x)$.

11.13 Show that

$$T[x^n\sqrt{(1-x^2)}] = x^{n+1} - \frac{1}{2}x^{(n-1)} - \sum_{j=1}^{[(n-1)/2]} \frac{(2j-1)!!\,x^{n-1-2j}}{2^{j+1}(j+1)!}$$

for integer $n \ge 1$.

11.14 Show that Eq. (11.223) is satisfied for $n = 4$ and $f(x) = x^5$.

11.15 Find an eigenfunction of the operator T^2. What is the corresponding eigenvalue?

11.16 Calculate $Tf(x)$ given $f(x) = (x^2 - 1)P_n'(x)$, with $n \in \mathbb{Z}^+$.

11.17 Evaluate $Tf(x)$ given $f(x) = (1 - x^2)^a$, with the parameter a real but not necessarily an integer. For what range of values of a does Tf converge?

11.18 Determine $Tf(x)$ for $f(x) = (1-x)^{a-1}(1+x)^{-a}\sin nx$, with $n \in \mathbb{N}$ and $0 < a < 1$.

11.19 If $Tf(x) = x(x+1)$, determine $f(x)$.

11.20 Is there a Riesz-type inequality for the finite Hilbert transform operator? Specify any restrictions that must be placed on the class of functions that are involved.

11.21 Given that $(Tf)(x) = x\,T_n(x)$, find $f(x)$.

12

Some singular integral equations

12.1 Introduction

Some elementary singular integral equations are examined in this chapter. The types that are considered involve the Hilbert transform, or one of the standard variants. Consider the following equation:

$$f(x) = g(x) + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(s)\mathrm{d}s}{x - s}. \tag{12.1}$$

The objective is to solve this equation for the function f. Clearly this equation is not part of a Hilbert transform pair, so a simple inversion operation cannot be applied. This chapter outlines the approaches necessary to attack the solution of equations like Eq. (12.1). Examples of singular integral equations involving the standard Hilbert transform, or one of the common modifications, arise in aerodynamics, electromagnetic theory, crack propagation, the study of water waves, as well as in other applications.

An integral equation is recognized by having the unknown function that is to be determined as part of the integrand of an integral. The most common types of integral equations take one of the following forms:

$$g(x) = \int_a^b K(x, y) f(y)\mathrm{d}y, \tag{12.2}$$

$$g(x) = f(x) - \lambda \int_a^b K(x, y) f(y)\mathrm{d}y, \tag{12.3}$$

and

$$g(x) = h(x) f(x) - \lambda \int_a^b K(x, y) f(y)\mathrm{d}y. \tag{12.4}$$

In these equations, f is the unknown function to be determined, $K(x, y)$ is the kernel of the integral equation, λ is a constant parameter, and a and b are constants. The functions g, h, and K are assumed to be known. These three equations are referred to as Fredholm equations of the first, second, and third kind, respectively. The limits may either be finite or infinite. If the limit b in the preceding equations is replaced by the variable x, the resulting formulas are termed Volterra integral equations of the first, second, and third kind, respectively. When the kernel function is singular, the resulting integral equation is termed a *singular integral equation*. Not unexpectedly, such equations in the most general form are difficult to solve. There are some cases, however, that can be treated in a straightforward manner. Suppose

$$K(x,y) = \frac{h(x,y)}{|x-y|^{\alpha}}, \tag{12.5}$$

where $h(x,y)$ is continuous and bounded in both variables over the range of integration, and α satisfies $0 < \alpha < 1$. This is a weak type of singularity, and it can be proved that if $f(y)$ is continuous over the integration interval, then Eq. (12.3) has a unique solution for all λ. The equation can be transformed into one with a bounded kernel. This is not pursued here; the interested reader is directed to the literature (Tricomi, 1985). The case of significance in this book is $\alpha = 1$, which requires a different approach.

For the remainder of this chapter the kernel function is restricted to the following form:

$$K(x,y) = \frac{1}{\pi(x-y)}, \tag{12.6}$$

and a principal focus will be the integration range $(-\infty, \infty)$, although consideration is also given to the interval $(0, \infty)$, and to the finite intervals $(-1, 1)$ and (a, b). These restrictions obviously limit deliberation to singular integral equations involving the Hilbert transform, the one-sided Hilbert transform, or the finite Hilbert transform.

12.2 Fredholm equations of the first kind

The Fredholm integral equation of the first kind (with the restriction to the kernel given in Eq. (12.6)) takes the following form:

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x-y}, \tag{12.7}$$

and the Cauchy principal value has been employed in order for the integral to be convergent, assuming f is a suitably chosen function. The solution of this equation is given by

$$f(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(y)\mathrm{d}y}{x-y}. \tag{12.8}$$

This result is just the second equation of a Hilbert transform pair. Since the Hilbert transform of a constant is zero, an arbitrary constant can be added to the right-hand side of Eq. (12.8), thereby obtaining a more general solution for Eq. (12.7). The conditions that must satisfied by f for the integral in Eq. (12.7) to converge, and for the solution given in Eq. (12.8) to be obtained, have been discussed in detail in Section 3.4.1. Chapter 10 discusses the extension to situations where f or g are generalized functions. It is convenient for what follows to abbreviate the compact notation for the Hilbert transform pair employed earlier; that is,

$$g(x) = Hf(x), \quad f(x) = -Hg(x), \tag{12.9}$$

to

$$g = Hf, \quad f = -Hg, \tag{12.10}$$

when the variable is of no particular interest, or there is not likely to be any confusion as to what the variable is. So the solution of Eq. (12.7) can be viewed as follows. Take the Hilbert transform of both sides and use the inversion property to obtain

$$Hg = H^2 f = -f. \tag{12.11}$$

Consider the following integral equation:

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x - y} = \alpha, \tag{12.12}$$

with α a constant. Recall from the Riesz inequality that if $f \in L^p(\mathbb{R})$, for $p > 1$, then $Hf \in L^p(\mathbb{R})$. Since the right-hand side of Eq. (12.12) does not meet this requirement for $\alpha \neq 0$, it is clear that the solution of this integral equation (assuming that one exists) will not satisfy $f \in L^p(\mathbb{R})$, for $p > 1$. This helps to narrow significantly the possible class of solutions, and minimizes the potential of finding false solutions. Using the derivative property of the Hilbert transform, then from Eq. (12.12) it follows that (assuming that the Hilbert transform of $f'(x)$ exists)

$$\frac{\mathrm{d}}{\mathrm{d}x}\{Hf(x)\} = Hf'(x) = 0, \tag{12.13}$$

and so

$$f'(x) = c_1, \tag{12.14}$$

where c_1 is a constant. Hence,

$$f(x) = c_1 x + c_2, \tag{12.15}$$

with c_2 a constant. Since $H[x]$ diverges, it is necessary to take $c_1 = 0$, and therefore

$$f(x) = c_2. \tag{12.16}$$

A solution of Eq. (12.12) for the case $\alpha = 0$ is therefore given by Eq. (12.16). For the case with $\alpha = 0$, a key question is whether or not there exist other solutions. The answer is affirmative if functions with restricted support are considered. Suppose

$$f(x) = \begin{cases} 1/\sqrt{(1-x^2)}, & \text{for } -1 < x < 1 \\ 0, & \text{for } |x| \geq 1, \end{cases} \tag{12.17}$$

and that x in Eq. (12.12) satisfies $-1 < x < 1$; then it follows, on using Eq. (11.55), that

$$\begin{aligned} \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x-y} &= \frac{1}{\pi} P \int_{-1}^{1} \frac{\mathrm{d}y}{(x-y)\sqrt{(1-y^2)}} \\ &= 0, \end{aligned} \tag{12.18}$$

which establishes that Eq. (12.17) is a solution of Eq. (12.12) for $\alpha = 0$. For the case where $\alpha \neq 0$, and allowing for functions with restricted support, the choice

$$f(x) = \begin{cases} -\alpha x/\sqrt{(1-x^2)}, & \text{for } -1 < x < 1 \\ 0, & \text{for } |x| \geq 1, \end{cases} \tag{12.19}$$

is a solution of Eq. (12.12), for the case $-1 < x < 1$. This can be readily verified using Eq. (11.55):

$$\begin{aligned} \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x-y} &= \frac{-\alpha}{\pi} P \int_{-1}^{1} \frac{\{x-(x-y)\}\mathrm{d}y}{(x-y)\sqrt{(1-y^2)}} \\ &= \frac{-\alpha x}{\pi} P \int_{-1}^{1} \frac{\mathrm{d}y}{(x-y)\sqrt{(1-y^2)}} + \frac{\alpha}{\pi} \int_{-1}^{1} \frac{\mathrm{d}y}{\sqrt{(1-y^2)}} \\ &= \alpha, \end{aligned} \tag{12.20}$$

which proves that Eq. (12.19) is a solution of Eq. (12.12).

As a second example, consider the following integral equation:

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x-y} = |x|^{\mu}, \quad \text{for } -1 < \mu < 0. \tag{12.21}$$

Employing entry (2.60) from Appendix 1, Table 1.2, the solution of this integral equation can be written as follows:

$$f(x) = c + \tan\left(\frac{\pi\mu}{2}\right) |x|^{\mu} \ \operatorname{sgn} x, \tag{12.22}$$

where c plays the role of an arbitrary constant. The same table (Table 1.2) provides results for the solution of Eq. (12.7) for a wide variety of functions.

12.3 Fredholm equations of the second kind

This section treats the more complicated Fredholm integral equation of the second kind, which can be written, assuming the restriction to the kernel of Eq. (12.6), as follows:

$$g(x) = f(x) - \lambda Hf(x). \tag{12.23}$$

The simplest case of Eq. (12.23) occurs when $g(x) = 0$, which is referred to as the *homogeneous equation*, and this is considered first. Setting $g = 0$ in Eq. (12.23) and taking the Hilbert transform yields

$$\begin{aligned} Hf &= \lambda H^2 f \\ &= -\lambda f. \end{aligned} \tag{12.24}$$

The last line follows from the application of the inversion formula of the Hilbert transform. This property will find frequent use in the following discussion. From Eqs. (12.23) and (12.24), it follows that

$$(1 + \lambda^2)f = 0. \tag{12.25}$$

Hence, if λ is a real constant, the only solution of $f(x) = \lambda Hf(x)$ is the trivial solution $f = 0$. For the Fredholm equation under discussion, the eigenvalues are given by

$$\lambda = \pm\mathrm{i}. \tag{12.26}$$

For the cases where $g(x) \neq 0$, the Hilbert transform is applied to Eq. (12.23), to obtain the following result:

$$Hf = Hg - \lambda f, \tag{12.27}$$

and hence Eq. (12.23) can be converted to

$$f = \frac{g + \lambda Hg}{1 + \lambda^2}. \tag{12.28}$$

As an example, consider the following integral equation:

$$f(x) = \operatorname{cas} ax + Hf(x), \quad \text{for } a > 0, \tag{12.29}$$

where $\operatorname{cas} ax = \sin ax + \cos ax$. Employing Eq. (12.28) and using $H[\cos ax] = \sin ax$ and $H[\sin ax] = -\cos ax$, it is straightforward to verify that $f(x) = \sin ax$ is the solution of this integral equation.

Two related forms of Eq. (12.23) are now considered. Suppose the solution of the following singular integral equation is sought:

$$f'(x) = g(x) + \lambda Hf(x), \tag{12.30}$$

where the prime indicates the derivative with respect to x. Taking the derivative with respect to x leads to

$$f''(x) + \lambda^2 f(x) = g'(x) + \lambda Hg(x). \tag{12.31}$$

In a similar manner, from the integral equation

$$f(x) = g(x) + \lambda Hf'(x), \tag{12.32}$$

it follows that

$$\lambda^2 f''(x) + f(x) = g(x) + \lambda Hg'(x). \tag{12.33}$$

Equations (12.30) and (12.32) are examples of *singular integro-differential* equations, since the function sought is involved in the equations both as a derivative and as part of the integrand of a singular integral. To proceed further with the solution of these two equations, some particular cases are examined. The simplest examples arise when

$$g'(x) + \lambda Hg(x) = 0, \tag{12.34}$$

for Eq. (12.31), and

$$g(x) + \lambda Hg'(x) = 0, \tag{12.35}$$

for Eq. (12.33). Functions that satisfy Eq. (12.34) lead to the differential equation given by

$$g'' + \lambda^2 g = 0, \tag{12.36}$$

and functions for which Eq. (12.35) holds satisfy

$$\lambda^2 g'' + g = 0. \tag{12.37}$$

For example, consider the solution of Eqs. (12.30) when

$$g(x) = \sin x \tag{12.38}$$

and $\lambda = 1$. Since Eq. (12.34) holds, Eq. (12.31) simplifies to

$$f''(x) + f(x) = 0, \tag{12.39}$$

which has the solution

$$\begin{aligned} f(x) &= \alpha \mathrm{e}^{\mathrm{i}x} + \beta \mathrm{e}^{-\mathrm{i}x} \\ &= a \sin x + b \cos x, \end{aligned} \tag{12.40}$$

where a, b, α, and β are arbitrary constants. Substituting Eq. (12.40) into Eq. (12.30) establishes the values for a and b, and so the solution of Eq. (12.30) is obtained:

$$f(x) = -\frac{1}{2}\cos x. \tag{12.41}$$

As a second example, consider the solution of the integral equation (12.32) for the choice $g(x) = \cos^2 x$ and $\lambda = 1$. Equation (12.33) reduces to the following differential equation:

$$f''(x) + f(x) = \frac{1}{2} + \frac{3\cos 2x}{2}, \tag{12.42}$$

for which the solution is given by

$$f(x) = a\sin x + b\cos x + \sin^2 x, \tag{12.43}$$

where a and b are constants. Substituting Eq. (12.43) into Eq. (12.32) confirms that the solution of the integral equation has been determined.

12.4 Fredholm equations of the third kind

Fredholm equations of the third kind with the singular kernel given in Eq. (12.6) are considerably more difficult to solve. The simplifying device that has been used in the preceding sections, namely the inversion property of the Hilbert transform, is now significantly more involved, because there are limited results available for the Hilbert transform of a product of functions.

A special case of the following singular integral equation is considered:

$$g(x) = h(x)f(x) - \lambda Hf(x), \tag{12.44}$$

where $g(x) = 0$ and $h(x) = x^n$, for n a positive integer and $\lambda = 1$, so that

$$x^n f(x) = Hf(x). \tag{12.45}$$

For this choice of $h(x)$, the symmetry properties of the Hilbert transform should be kept in mind. If n is even then $f(x)$ cannot be an even function, and if n is odd $f(x)$ cannot be an odd function. Two approaches can be taken to attack this type of equation; one involves converting Eq. (12.45) to a differential equation, and the second method relies on the moment formula property of the Hilbert transform (Section 4.7).

The cases $n=1$ and $n=2$ are considered separately. Let $n = 1$ and apply the Hilbert transform operator to Eq. (12.45); then, using the moment formula, Eq. (4.111), it follows that

$$H[xf(x)] = xHf(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\mathrm{d}y = -f(x). \tag{12.46}$$

Let

$$c_0 = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\mathrm{d}y; \tag{12.47}$$

then Eq. (12.45), for $n = 1$, simplifies, using Eq. (12.46), to

$$f(x) = \frac{c_0}{x^2 + 1}. \tag{12.48}$$

Since the Hilbert transform of $(x^2 + 1)^{-1}$ is $x(x^2 + 1)^{-1}$, it is clear that the general solution of Eq. (12.45) for $n = 1$ is given by

$$f(x) = \frac{c}{x^2 + 1}, \tag{12.49}$$

where c is an arbitrary constant.

The same problem is now considered from a different approach. From Eq. (12.45) for $n = 1$, applying the Hilbert transform operator followed by the derivative operator leads to

$$Hf(x) + H\left[x\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right] = -\frac{\mathrm{d}f(x)}{\mathrm{d}x}, \tag{12.50}$$

and hence

$$Hf(x) + xH\left[\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right] = -\frac{\mathrm{d}f(x)}{\mathrm{d}x}. \tag{12.51}$$

To obtain the preceding result, f' has been assumed to be integrable on $\mathbb{R}$. From Eq. (12.50) it follows that

$$H\left[\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right] = x\frac{\mathrm{d}f(x)}{\mathrm{d}x} + f(x). \tag{12.52}$$

Substituting Eq. (12.52) and $Hf(x) = xf(x)$ into Eq. (12.51) leads to

$$(x^2 + 1)\frac{\mathrm{d}f(x)}{\mathrm{d}x} + 2xf(x) = 0. \tag{12.53}$$

The solution of this differential equation is given by Eq. (12.49).

The case $n = 2$ of Eq. (12.45) is now considered, so the problem to be solved is as follows:

$$x^2 f(x) = Hf(x). \tag{12.54}$$

Noting the even–odd properties of Hf, it is clear that f cannot be simply an even or an odd function. Let

$$c_k = \frac{1}{\pi}\int_{-\infty}^{\infty} y^k f(y)\mathrm{d}y, \quad \text{for } k = 0 \text{ and } 1, \tag{12.55}$$

and suppose $y^k f(y)$ is integrable on $\mathbb{R}$, for $k = 0$ and 1. From Eq. (12.54) it follows, on making use of Eq. (4.113) and the inversion property of the Hilbert transform, that

$$x^2 Hf(x) - xc_0 - c_1 = -f(x), \tag{12.56}$$

which simplifies to

$$f(x) = \frac{c_1 + c_0 x}{x^4 + 1}. \tag{12.57}$$

Employing Eq. (12.54) and Table 1.2 entries (2.12) and (2.17) from Appendix 1 allows the coefficients to be determined as follows:

$$c_1 = c_0 = 0, \tag{12.58}$$

leading to $f(x) = 0$ as a solution of Eq. (12.54). If the approach using differential equations is employed, the following differential equation is obtained:

$$(x^4 + 1)f'' + 8x^3 f' + 12x^2 f = 0. \tag{12.59}$$

The solution of this equation is given by

$$f(x) = \frac{ax + b}{x^4 + 1}, \tag{12.60}$$

with a and b constants. Substituting Eq. (12.60) into Eq. (12.54) and equating coefficients of the powers of x leads to $a = 0$ and $b = 0$ as the solution, and therefore $f = 0$ is obtained as the solution of Eq. (12.54), albeit a trivial solution, which could have been obtained by inspection of the given singular integral equation.

For the following integral equation:

$$x^3 f(x) = Hf(x), \tag{12.61}$$

the reader is left to show that the solution is given by

$$f(x) = c\frac{x^2 + 1}{x^6 + 1}, \tag{12.62}$$

where c is an arbitrary constant, and then to verify this by using results from Table 1.2, Appendix 1. The interested reader might like to solve Eq. (12.45) for the case of general positive integer n.

The case of Eq. (12.44) for non-zero $g(x)$ and $h(x) = x^n$, for n a positive integer, is now examined. Suppose $\lambda = 1$, so that

$$x^n f(x) = g(x) + Hf(x). \tag{12.63}$$

For $n = 1$, the solution is obtained by taking the Hilbert transform, with the following result:

$$f(x) = \frac{c_0}{x^2+1} + \frac{xg(x) + Hg(x)}{x^2+1}, \tag{12.64}$$

where c_0 is defined in Eq. (12.47). The constant c_0 can be replaced by a general constant c. To verify by direct substitution that Eq. (12.64) is a solution of Eq. (12.63), for $n = 1$, requires some dexterity with the basic properties of the Hilbert transform. From Eq. (12.64), it follows, on using Eq. (4.373) and Eq. (4.111), that

$$\begin{aligned} Hf(x) &= H\left\{\frac{c_0}{x^2+1} + \frac{xg(x)+Hg(x)}{x^2+1}\right\} \\ &= \frac{xc_0 + H\{xg(x)+Hg(x)\}}{x^2+1} \\ &\quad + \frac{1}{\pi(x^2+1)}\int_{-\infty}^{\infty} \frac{(t+x)\{tg(t)+Hg(t)\}\mathrm{d}t}{t^2+1} \\ &= \frac{xc_0 - g(x) + xHg(x)}{x^2+1} \\ &\quad + \frac{1}{\pi(x^2+1)}\int_{-\infty}^{\infty}\left[-g(t) + \frac{(t+x)\{tg(t)+Hg(t)\}}{t^2+1}\right]\mathrm{d}t. \end{aligned} \tag{12.65}$$

The last integral can be written as follows:

$$\begin{aligned} \int_{-\infty}^{\infty}\left[-g(t) + \frac{(t+x)\{tg(t)+Hg(t)\}}{t^2+1}\right]\mathrm{d}t = x\int_{-\infty}^{\infty} \frac{tg(t)+(Hg)(t)}{t^2+1}\,\mathrm{d}t \\ + \int_{-\infty}^{\infty}\frac{tHg(t)-g(t)}{t^2+1}\,\mathrm{d}t. \end{aligned} \tag{12.66}$$

Employing Eq. (7.310) with $x = 0$, $\varepsilon = 1$, and $f = g$ leads to

$$\int_{-\infty}^{\infty}\frac{Hg(t)\mathrm{d}t}{t^2+1} = -\int_{-\infty}^{\infty}\frac{tg(t)\mathrm{d}t}{t^2+1}, \tag{12.67}$$

and using the same equations with $f = Hg$ yields

$$\int_{-\infty}^{\infty}\frac{tHg(t)\mathrm{d}t}{t^2+1} = \int_{-\infty}^{\infty}\frac{g(t)\mathrm{d}t}{t^2+1}. \tag{12.68}$$

So the last integral in Eq. (12.65) is zero, and using Eq. (12.64) leads to

$$Hf(x) = \frac{xc_0 - g(x) + xHg(x)}{x^2 + 1} = xf(x) - g(x), \tag{12.69}$$

which proves that Eq. (12.64) is a solution of Eq. (12.63), for $n = 1$. As an example, consider the following integral equation:

$$xf(x) = \cos ax + Hf(x), \quad \text{for } a > 0; \tag{12.70}$$

then, from Eq. (12.64),

$$f(x) = \frac{c + x\cos ax + \sin ax}{x^2 + 1}. \tag{12.71}$$

This can be checked to satisfy Eq. (12.63) for $n = 1$ using the following results:

$$H\left[\frac{\sin ax}{x^2 + 1}\right] = \frac{\mathrm{e}^{-a} - \cos ax}{x^2 + 1}, \quad \text{for } a > 0, \tag{12.72}$$

$$H\left[\frac{x\cos ax}{x^2 + 1}\right] = \frac{x\sin ax - \mathrm{e}^{-a}}{x^2 + 1}, \quad \text{for } a > 0, \tag{12.73}$$

and

$$H\left[\frac{1}{x^2 + 1}\right] = \frac{x}{x^2 + 1}. \tag{12.74}$$

An alternative approach to solving Eq. (12.44) is now considered in which λ can be treated as a small parameter. Inserting the following changes:

$$\frac{g(x)}{h(x)} \to g(x) \tag{12.75}$$

and

$$\frac{1}{h(x)} \to h(x); \tag{12.76}$$

then

$$f(x) = g(x) + \lambda h(x)Hf(x). \tag{12.77}$$

If λ is small, try $f(x) \approx g(x)$ as a first approximation, and insert this into the last result to obtain

$$f(x) \approx g(x) + \lambda h(x)Hg(x) \tag{12.78}$$

as an improved solution. Inserting this result into Eq. (12.77) and continuing the iteration process yields

$$f(x) = \sum_{n=0}^{\infty} \lambda^n \{h(x)H\}^n g(x). \tag{12.79}$$

This is a Neumann-type series for the integral equation. It is left as an exercise for the interested reader to ponder the conditions for convergence of the last series. As an example, consider the following integral equation:

$$f(x) = \cos^2 ax + \lambda \csc ax\, Hf(x), \tag{12.80}$$

for $a > 0$. From the right-hand side of Eq. (12.79), it follows that

$$\begin{aligned} \lambda h(x)Hg(x) &= \lambda \csc ax\, H\left[\frac{1+\cos 2ax}{2}\right] \\ &= \lambda \cos ax, \qquad (12.81) \\ \lambda^2 h(x)H[h(x)Hg(x)] &= \lambda^2 \csc ax\, H[\cos ax] \\ &= \lambda^2, \qquad (12.82) \end{aligned}$$

and

$$\{h(x)H\}^n g(x) = 0, \quad \text{for } n \geq 3. \tag{12.83}$$

Hence, from Eq. (12.79) the solution of Eq. (12.80) is given by

$$f(x) = \cos^2 ax + \lambda \cos ax + \lambda^2. \tag{12.84}$$

12.5 Fourier transform approach to solving singular integral equations

The evaluation of the Hilbert transform of a function in terms of the Fourier transform of the function was examined in Section 5.2. It turns out that the Fourier transform technique is a powerful method to attack certain types of singular integral equations. Consider the following integral equation:

$$f(x) = \lambda Hf(x) + g(x). \tag{12.85}$$

Taking the Fourier transform of Eq. (12.85), and using the following key property:

$$\mathcal{F}Hf(x) = -\mathrm{i}\ \mathrm{sgn}\ x\ \mathcal{F}f(x), \tag{12.86}$$

leads to

$$\begin{aligned}\mathcal{F}f(x) &= \mathcal{F}\{\lambda Hf\}(x) + \mathcal{F}g(x)\\ &= -\mathrm{i}\lambda\ \mathrm{sgn}\, x\ \mathcal{F}f(x) + \mathcal{F}g(x),\end{aligned} \tag{12.87}$$

and hence, for $x \neq 0$,

$$(1+\lambda^2)\mathcal{F}f(x) = (1 - \mathrm{i}\lambda\ \mathrm{sgn}\ x)\mathcal{F}g(x). \tag{12.88}$$

Taking the inverse Fourier transform yields

$$\begin{aligned}f(x) &= \frac{1}{1+\lambda^2}\mathcal{F}^{-1}\{(1-\mathrm{i}\lambda\ \mathrm{sgn}\ y)\mathcal{F}g(y)\}(x)\\ &= \frac{g(x)+\lambda Hg(x)}{1+\lambda^2}.\end{aligned} \tag{12.89}$$

The last line follows using Eq. (12.86).

The following example:

$$xf(x) = Hf(x), \tag{12.90}$$

previously treated in Section 12.4, is reconsidered using the Fourier transform technique. Let $h(x) = \mathcal{F}f(x)$, then from Eq. (12.90) it follows that

$$\mathcal{F}[xf(x)] = \mathcal{F}Hf(x) = -\mathrm{i}\ \mathrm{sgn}\, x\ \mathcal{F}f(x). \tag{12.91}$$

Using

$$\mathrm{i}\frac{\mathrm{d}h(x)}{\mathrm{d}x} = \mathcal{F}[xf(x)] \tag{12.92}$$

yields the following differential equation:

$$\frac{\mathrm{d}h(x)}{\mathrm{d}x} = -\mathrm{sgn}\ x\ h(x). \tag{12.93}$$

The solution of Eq. (12.93) is given by

$$h(x) = c\mathrm{e}^{-|x|}, \tag{12.94}$$

where c is an arbitrary constant. This result for $h(x)$ can be readily checked on recalling that

$$\frac{\mathrm{d}|x|}{\mathrm{d}x} = \mathrm{sgn}\ x. \tag{12.95}$$

The solution of Eq. (12.90) is now obtained by taking the inverse Fourier transform of $h(x)$:

$$f(x) = \mathcal{F}^{-1}h(x) = \frac{c}{x^2+1}, \tag{12.96}$$

and a factor of π^{-1} from the inverse Fourier transform has been swept into the arbitrary constant c. The effort required in using this approach to solve Cauchy-type integral equations is in part dependent on the ease with which the inverse Fourier transform can be evaluated. In many circumstances, this calculation will be easier than evaluating the Hilbert transform of some function.

12.6 A finite Hilbert transform integral equation

In Chapter 11 the inversion of the finite Hilbert transform was considered. This amounts to the solution of a singular integral equation. A more general case of this inversion formula, where the integration range is $[a, b]$, is now considered. That is, the solution of the following integral equation is sought:

$$g(x) = \frac{1}{\pi} P \int_a^b \frac{f(t)\mathrm{d}t}{x-t}, \quad \text{with } a < x < b. \tag{12.97}$$

Let the transform operation in Eq. (12.97) be denoted by T_{ab}, so that

$$T_{ab} f(x) = g(x). \tag{12.98}$$

Three results that will be useful for solving Eq. (12.97):

$$T_{ab}\left[\frac{1}{\surd[(b-x)(x-a)]}\right] = 0, \tag{12.99}$$

$$T_{ab}\left[\frac{x}{\surd[(b-x)(x-a)]}\right] = -1, \tag{12.100}$$

and

$$T_{ab}[\surd[(b-x)(x-a)]] = x - \frac{b+a}{2}. \tag{12.101}$$

Equation (12.99) can be proved as follows:

$$\begin{aligned} T_{ab}\left[\frac{1}{\surd[(b-x)(x-a)]}\right] &= \frac{1}{\pi} P \int_a^b \frac{\mathrm{d}y}{(x-y)\surd[(b-y)(y-a)]} \\ &= \frac{2}{\pi(b-a)} P \int_{-1}^1 \frac{\mathrm{d}w}{(\chi-w)\surd(1-w^2)}, \end{aligned} \tag{12.102}$$

where $-1 < \chi < 1$ and the change of variables

$$y = \frac{(b-a)}{2}w + \frac{b+a}{2} \tag{12.103}$$

and

$$x = \frac{(b-a)}{2}\chi + \frac{b+a}{2} \tag{12.104}$$

have been employed. The integral in Eq. (12.102) was evaluated to be zero in Eqs. (11.54) – (11.55). Equation (12.100) can be proved as follows:

$$\begin{aligned} T_{ab}\left[\frac{x}{\surd[(b-x)(x-a)]}\right] &= \frac{1}{\pi}P\int_a^b \frac{y\,\mathrm{d}y}{(x-y)\surd[(b-y)(y-a)]} \\ &= -\frac{1}{\pi}P\int_a^b \frac{\{x-y-x\}\mathrm{d}y}{(x-y)\surd[(b-y)(y-a)]} \\ &= -\frac{1}{\pi}\int_a^b \frac{\mathrm{d}y}{\surd[(b-y)(y-a)]} \\ &= -\frac{1}{\pi}\int_{-1}^1 \frac{\mathrm{d}y}{\surd(1-y^2)} \\ &= -1. \end{aligned} \tag{12.105}$$

Equation (12.101) is proved as follows:

$$\begin{aligned} T_{ab}[\surd[(b-x)(x-a)]] &= \frac{1}{\pi}P\int_a^b \frac{\surd[(b-y)(y-a)]\mathrm{d}y}{x-y} \\ &= \frac{1}{\pi}P\int_a^b \frac{\{y(b+a)-y^2\}\mathrm{d}y}{(x-y)\surd[(b-y)(y-a)]} \\ &= -(b+a) - \frac{1}{\pi}P\int_a^b \frac{\{(x-y)^2+2xy-x^2\}\mathrm{d}y}{(x-y)\surd[(b-y)(y-a)]} \\ &= -(b+a) - \frac{1}{\pi}\int_a^b \frac{(x-y)\mathrm{d}y}{\surd[(b-y)(y-a)]} \\ &\quad - \frac{2x}{\pi}P\int_a^b \frac{y\,\mathrm{d}y}{(x-y)\surd[(b-y)(y-a)]} \\ &= x - \frac{b+a}{2}. \end{aligned} \tag{12.106}$$

To solve Eq. (12.97), the Tricomi identity for the operator T_{ab} (from Section 11.4) is employed. For two functions ϕ_1 and ϕ_2 with supports in the interval (a, b), satisfying $\phi_1 \in L^{p_1}(a, b)$ and $\phi_2 \in L^{p_2}(a, b)$, with $p_1 > 1, p_2 > 1$, and $p_1^{-1} + p_2^{-1} \le 1$,

it follows that

$$T_{ab}[\phi_1(t)T_{ab}\phi_2(t) + \phi_2(t)T_{ab}\phi_1(t)] = T_{ab}\phi_1(x)T_{ab}\phi_2(x) - \phi_1(x)\phi_2(x). \quad (12.107)$$

A similar choice for the functions ϕ_1 and ϕ_2 that was made in Section 11.4 is employed, so that

$$\phi_1(x) = \begin{cases} f(x), & a < x < b \\ 0, & \text{otherwise}, \end{cases} \quad (12.108)$$

$$\phi_2(x) = \begin{cases} \sqrt{[(b-x)(x-a)]}, & a < x < b \\ 0, & \text{otherwise}, \end{cases} \quad (12.109)$$

and let $g(x) = T_{ab}f(x)$. With these choices, Eq. (12.107) becomes

$$f(x) = \frac{g(x)T_{ab}[\sqrt{[(b-x)(x-a)]}]}{\sqrt{[(b-x)(x-a)]}} - \frac{T_{ab}[f(x)T_{ab}[\sqrt{[(b-x)(x-a)]}] + g(x)\sqrt{[(b-x)(x-a)]}]}{\sqrt{[(b-x)(x-a)]}}, \quad (12.110)$$

which simplifies on using Eq. (12.101) to give

$$\begin{aligned} f(x) &= \frac{g(x)\{x-(b+a)/2\}}{\sqrt{[(b-x)(x-a)]}} - \frac{T_{ab}[f(x)\{x-(b+a)/2\} + g(x)\sqrt{[(b-x)(x-a)]}]}{\sqrt{[(b-x)(x-a)]}} \\ &= \frac{xg(x)}{\sqrt{[(b-x)(x-a)]}} - \frac{T_{ab}[xf(x)]}{\sqrt{[(b-x)(x-a)]}} - \frac{T_{ab}[g(x)\sqrt{[(b-x)(x-a)]}]}{\sqrt{[(b-x)(x-a)]}}. \end{aligned} \quad (12.111)$$

Using the moment formula for the finite Hilbert transform,

$$T_{ab}[xf(x)] = xT_{ab}f(x) - \frac{1}{\pi}\int_a^b f(x)\mathrm{d}x, \quad (12.112)$$

and setting

$$c = \frac{1}{\pi}\int_a^b f(x)\mathrm{d}x, \quad (12.113)$$

allows Eq. (12.111) to be simplified as follows:

$$f(x) = \frac{c}{\sqrt{[(b-x)(x-a)]}} - \frac{T_{ab}[g(x)\sqrt{[(b-x)(x-a)]}]}{\sqrt{[(b-x)(x-a)]}}. \quad (12.114)$$

In view of Eq. (12.99), c plays the role of an arbitrary constant in Eq. (12.114).

Consider the solution of the following integral equation:

$$\frac{1}{\pi}P\int_a^b \frac{f(t)\mathrm{d}t}{x-t} = 1, \quad \text{with } a < x < b. \tag{12.115}$$

Setting $g(x) = 1$ in Eq. (12.114) and using Eq. (12.101) leads to

$$\begin{aligned} f(x) &= \frac{c}{\sqrt{[(b-x)(x-a)]}} - \frac{\{x-(1/2)(b+a)\}}{\sqrt{(b-x)(x-a)}} \\ &= \frac{c'}{\sqrt{[(b-x)(x-a)]}} - \frac{x}{\sqrt{[(b-x)(x-a)]}}, \end{aligned} \tag{12.116}$$

where c' plays the role of an arbitrary constant. That this is a solution of Eq. (12.115) can be readily checked by making use of Eqs. (12.99) and (12.100). As a second example, consider the solution of the following integral equation:

$$\frac{1}{\pi}P\int_a^b \frac{f(t)\mathrm{d}t}{x-t} = x, \quad \text{with } a < x < b. \tag{12.117}$$

Making use of Eq. (12.114), with $g(x) = x$, yields

$$f(x) = \frac{c}{\sqrt{[(b-x)(x-a)]}} - \frac{T_{ab}[x\sqrt{[(b-x)(x-a)]}]}{\sqrt{[(b-x)(x-a)]}}, \tag{12.118}$$

and hence

$$f(x) = \frac{c + 2^{-1}(b+a)x - x^2}{\sqrt{[(b-x)(x-a)]}}, \tag{12.119}$$

where c plays the role of an arbitrary constant. This result is readily confirmed to be the solution of Eq. (12.117) by direct substitution.

A more complicated integral equation involving the finite Hilbert transform is as follows:

$$f(x) = g(x) + \frac{\lambda}{\pi}P\int_{-1}^{1} \frac{f(t)\mathrm{d}t}{x-t}, \tag{12.120}$$

with λ a constant. To solve this equation a series expansion is employed, assuming λ to be a small parameter. The approach is related to some of the ideas discussed in Section 12.4. If the approximations

$$f(x) \approx g(x), \tag{12.121}$$

$$f(x) \approx g(x) + \lambda T g(x), \tag{12.122}$$

$$f(x) \approx g(x) + \lambda T g(x) + \lambda^2 T^2 g(x), \tag{12.123}$$

and so forth, are tried, then continuing the iterative approach leads to

$$f(x) = \sum_{n=0}^{\infty} \lambda^n T^n g(x). \tag{12.124}$$

The reader is invited to explore the convergence properties of this series. There are situations where the series solution approach is effective, even when λ is not a small parameter. A pair of examples will illustrate the approach. If

$$g(x) = \frac{\alpha}{\sqrt{(1-x^2)}}, \tag{12.125}$$

where α is a constant, then

$$T^n g(x) = 0, \quad \text{for } n \geq 1, \tag{12.126}$$

which gives a solution of Eq. (12.120) as

$$f(x) = \frac{\alpha}{\sqrt{(1-x^2)}}. \tag{12.127}$$

As a second example, suppose

$$g(x) = \frac{(1-x)^{a-1}}{(1+x)^a}, \quad \text{for } 0 < a < 1; \tag{12.128}$$

then, from Eq. (12.124), it follows that

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \lambda^n T^n \{(1-x)^{a-1}(1+x)^{-a}\} \\ &= \sum_{n=0}^{\infty} (-\lambda)^n (\cot a\pi)^n (1-x)^{a-1}(1+x)^{-a} \\ &= (1-x)^{a-1}(1+x)^{-a} \sum_{n=0}^{\infty} (-\lambda \cot a\pi)^n, \end{aligned} \tag{12.129}$$

and hence

$$f(x) = \frac{(1-x)^{a-1}}{(1+x)^a(1+\lambda \cot a\pi)}. \tag{12.130}$$

The reader should instantly recognize why these two examples work out so simply. In the first example, $Tf(x) = 0$, and hence, from Eq. (12.120), $f(x) = g(x)$. The second example involves the eigenvalue equation $Tf(x) = cf(x)$, with the eigenvalue $c = -\cot a\pi$, in which case Eq. (12.120) simplifies to $f(x) = g(x) + c\lambda f(x)$, which can be readily solved for f.

A more complicated extension of Eq. (12.120) is now examined. Consider the following equation:

$$a(x)f(x) = g(x) - \frac{\lambda}{\pi} P \int_{-1}^{1} \frac{f(t)\mathrm{d}t}{x-t}, \tag{12.131}$$

where $a(x)$ and $g(x)$ are given functions and x is assumed to lie in the interval $(-1, 1)$. The approach depends on ideas leading to the Plemelj formulas (Section 3.7), the Riemann–Hilbert problem (Section 11.8), and Carleman's method (Section 11.9). Carleman's approach can be applied to solve this equation. Let the function $F(z)$ be given by

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{f(s)\mathrm{d}s}{s-z}; \tag{12.132}$$

then it follows that

$$(a(x) - \lambda \mathrm{i})F_{+}(x) = (a(x) + \lambda \mathrm{i})F_{-}(x) + g(x). \tag{12.133}$$

A function $L(z)$ is introduced such that

$$L(z) = \phi(z)\mathrm{e}^{\psi(z)}, \tag{12.134}$$

with $\phi(z)$ being chosen to control the asymptotic growth of $L(z)$. Let $\phi(z) = (z-1)^{-1}$, and choose $L(z)$ such that

$$\frac{L_{+}(x)}{L_{-}(x)} = \frac{a(x) + \lambda \mathrm{i}}{a(x) - \lambda \mathrm{i}}, \tag{12.135}$$

then Eq. (12.133) can be written as follows:

$$\frac{F_{+}(x)}{L_{+}(x)} - \frac{F_{-}(x)}{L_{-}(x)} = \frac{g(x)}{L_{+}(x)(a(x) - \lambda \mathrm{i})}. \tag{12.136}$$

From Eqs. (12.134) and (12.135), it follows that

$$\log L_{+}(x) = \log \phi(x) + \psi_{+}(x), \tag{12.137}$$

$$\log L_{-}(x) = \log \phi(x) + \psi_{-}(x), \tag{12.138}$$

$$\begin{aligned} \psi_{+}(x) - \psi_{-}(x) &= \log[L_{+}(x)/L_{-}(x)] \\ &= \log\left[\frac{a(x) + \lambda \mathrm{i}}{a(x) - \lambda i}\right], \end{aligned} \tag{12.139}$$

and hence

$$\psi(x) = \frac{\psi_{+}(x) + \psi_{-}(x)}{2} = \frac{1}{2\pi \mathrm{i}} P \int_{-1}^{1} \frac{\log[(a(s) + \lambda \mathrm{i})(a(s) - \lambda \mathrm{i})^{-1}]\mathrm{d}s}{s-x}. \tag{12.140}$$

The last integral will be denoted by $w(x)$. From Eqs. (12.134) and (12.135), it follows that

$$L_+(x)L_-(x) = \phi^2(x)\mathrm{e}^{[\psi_+(x)+\psi_-(x)]} = \phi^2(x)\mathrm{e}^{2w(x)}, \tag{12.141}$$

and also

$$\frac{a(x)+\lambda\mathrm{i}}{a(x)-\lambda\mathrm{i}}L_-^2(x) = \phi^2(x)\mathrm{e}^{2w(x)}; \tag{12.142}$$

hence,

$$L_-(x) = \sqrt{\left[\frac{a(x)-\lambda\mathrm{i}}{a(x)+\lambda\mathrm{i}}\right]}\phi(x)\mathrm{e}^{w(x)} \tag{12.143}$$

and

$$L_+(x) = \sqrt{\left[\frac{a(x)+\lambda\mathrm{i}}{a(x)-\lambda\mathrm{i}}\right]}\phi(x)\mathrm{e}^{w(x)}. \tag{12.144}$$

Using Eq. (12.136) and recalling the results from Section 3.7 leads to

$$\frac{1}{2}\left\{\frac{F_+(x)}{L_+(x)} + \frac{F_-(x)}{L_-(x)}\right\} = \frac{1}{2\pi\mathrm{i}}P\int_{-1}^{1}\frac{g(s)\mathrm{d}s}{L_+(s)(a(s)-\lambda\mathrm{i})(s-x)}. \tag{12.145}$$

Solving the pair of equations (12.136) and (12.145) yields

$$F_+(x) = \frac{g(x)}{2[a(x)-\lambda\mathrm{i}]} + \frac{L_+(x)}{2\pi\mathrm{i}}P\int_{-1}^{1}\frac{g(s)\mathrm{d}s}{L_+(s)(a(s)-\lambda\mathrm{i})(s-x)} \tag{12.146}$$

and

$$F_-(x) = -\frac{g(x)L_-(x)}{2L_+(x)[a(x)-\lambda\mathrm{i}]} + \frac{L_-(x)}{2\pi\mathrm{i}}P\int_{-1}^{1}\frac{g(s)\mathrm{d}s}{L_+(s)(a(s)-\lambda\mathrm{i})(s-x)}. \tag{12.147}$$

The function $f(x)$ is determined as follows:

$$\begin{aligned}
f(x) &= F_+(x) - F_-(x) \\
&= \frac{g(x)}{2[a(x)-\lambda\mathrm{i}]}\left\{1 + \frac{L_-(x)}{L_+(x)}\right\} + \frac{\{L_+(x)-L_-(x)\}}{2\pi\mathrm{i}} \\
&\quad \times P\int_{-1}^{1}\frac{g(s)\mathrm{d}s}{L_+(s)(a(s)-\lambda\mathrm{i})(s-x)} \\
&= \frac{a(x)g(x)}{a^2(x)+\lambda^2} + \frac{\lambda\phi(x)\mathrm{e}^{w(x)}}{\sqrt{(a^2(x)+\lambda^2)}}\frac{1}{\pi}P\int_{-1}^{1}\frac{g(s)\mathrm{d}s}{L_+(s)(a(s)-\lambda\mathrm{i})(s-x)}.
\end{aligned} \tag{12.148}$$

The last integral can be simplified as follows:

$$
\begin{aligned}
P\int_{-1}^{1}\frac{g(s)\mathrm{d}s}{L_+(s)(s-x)(a(s)-\lambda\mathrm{i})} &= P\int_{-1}^{1}\frac{g(s)\mathrm{e}^{-w(s)}\,\mathrm{d}s}{\phi(s)(s-x)\surd(a^2(s)+\lambda^2)}\\
&= P\int_{-1}^{1}\frac{[(s-x)+(x-1)]g(s)\mathrm{e}^{-w(s)}\,\mathrm{d}s}{(s-x)\surd(a^2(s)+\lambda^2)}\\
&= \int_{-1}^{1}\frac{g(s)\mathrm{e}^{-w(s)}\,\mathrm{d}s}{\surd(a^2(s)+\lambda^2)}\\
&\quad + (x-1)P\int_{-1}^{1}\frac{g(s)\mathrm{e}^{-w(s)}\,\mathrm{d}s}{(s-x)\surd(a^2(s)+\lambda^2)}\\
&= c+\frac{1}{\phi(x)}P\int_{-1}^{1}\frac{g(s)\mathrm{e}^{-w(s)}\,\mathrm{d}s}{(s-x)\surd(a^2(s)+\lambda^2)},
\end{aligned}
\tag{12.149}
$$

where c is a constant. Hence, from Eq. (12.148),

$$
\begin{aligned}
f(x) &= \frac{a(x)g(x)}{a^2(x)+\lambda^2}+\frac{\lambda\mathrm{e}^{w(x)}}{\surd(a^2(x)+\lambda^2)}\frac{1}{\pi}P\int_{-1}^{1}\frac{g(s)\mathrm{e}^{-w(s)}\,\mathrm{d}s}{(s-x)\surd(a^2(s)+\lambda^2)}\\
&\quad+\frac{c\mathrm{e}^{w(x)}}{(1-x)\surd(a^2(x)+\lambda^2)}.
\end{aligned}
\tag{12.150}
$$

This result was given by Carleman (1922) and is the general solution of Eq. (12.131), with c playing the role of an arbitrary constant. Let $a(x)=0$ and put $g(x)\to\lambda g(x)$, then the homogeneous equation is recovered from Eq. (12.131):

$$
g(x)=\frac{1}{\pi}P\int_{-1}^{1}\frac{f(t)\mathrm{d}t}{x-t}. \tag{12.151}
$$

From Eq. (12.140),

$$
w(x)=\frac{1}{2}P\int_{-1}^{1}\frac{\mathrm{d}s}{s-x}=\frac{1}{2}\log\left(\frac{1-x}{1+x}\right), \tag{12.152}
$$

and hence, from Eq. (12.150),

$$
\begin{aligned}
f(x) &= \frac{\mathrm{e}^{w(x)}}{\pi}P\int_{-1}^{1}\frac{g(s)\mathrm{e}^{-w(s)}\,\mathrm{d}s}{s-x}+\frac{c\mathrm{e}^{w(x)}}{(1-x)}\\
&= \frac{c}{\surd(1-x^2)}-\surd\left(\frac{1-x}{1+x}\right)\frac{1}{\pi}P\int_{-1}^{1}\surd\left(\frac{1+s}{1-s}\right)\frac{g(s)\mathrm{d}s}{(x-s)}.
\end{aligned}
\tag{12.153}
$$

This result represents one formula for the inversion of the finite Hilbert transform, and corresponds to the result given in Eq. (11.98).

A singular integral equation of the form

$$f(x) = g(x) - \frac{\lambda}{\pi} P \int_0^1 \frac{f(t)\mathrm{d}t}{x-t}, \tag{12.154}$$

which is of the type given in Eq. (12.131), can be solved by an extension of the approach discussed in Section 11.4.1. The interested reader can pursue the details in Peters (1968).

12.7 The one-sided Hilbert transform

In Section 8.3 the one-sided Hilbert transform was defined by

$$g(x) \equiv H_1 f(x) = \frac{1}{\pi} P \int_0^\infty \frac{f(y)\mathrm{d}y}{x-y}, \quad \text{with } 0 < x < \infty. \tag{12.155}$$

This transform is sometimes termed the *reduced* Hilbert transform, the *half*-Hilbert transform or the *semi-infinite* Hilbert transform. In some problems it may be convenient to split the Hilbert transform into two integrals. If $x > 0$, then

$$\begin{aligned} Hf(x) &= \frac{1}{\pi} \int_0^\infty \frac{f(-y)\mathrm{d}y}{x+y} + \frac{1}{\pi} P \int_0^\infty \frac{f(y)\mathrm{d}y}{x-y} \\ &= \pi^{-1} \mathsf{S} f(-x) + H_1 f(x), \end{aligned} \tag{12.156}$$

where $\mathsf{S}f$ denotes the Stieltjes transform of f.

The inversion of the one-sided Hilbert transform is a singular integral equation problem, and can be solved using the Tricomi identity in the following form:

$$H_1[\phi_1(x) H_1 \phi_2(x) + \phi_2(x) H_1 \phi_1(x)](x) = H_1 \phi_1(x) H_1 \phi_2(x) - \phi_1(x)\phi_2(x), \tag{12.157}$$

where the supports of ϕ_1 and ϕ_2 are $(0, \infty)$. Let

$$\phi_1(x) = \begin{cases} f(x), & 0 < x < \infty \\ 0, & \text{otherwise}, \end{cases} \tag{12.158}$$

and

$$\phi_2(x) = \begin{cases} \dfrac{1}{(x+a)\sqrt{x}}, & 0 < x < \infty \\ 0, & \text{otherwise}, \end{cases} \quad \text{for } a > 0, \tag{12.159}$$

and suppose that $f \in L^p(\mathbb{R}^+)$, for some $p > 1$, and $\phi_2 \in L^q(\mathbb{R}^+)$, for some $q > 1$, with $p^{-1} + q^{-1} \leq 1$. The following result is proved first:

$$(x+a) H_1 \left[\frac{f(x)}{x+a} \right] = H_1 f(x) + \frac{1}{\pi} \int_0^\infty \frac{f(s)\mathrm{d}s}{s+a}. \tag{12.160}$$

This formula can be established as follows:

$$\begin{aligned} H_1[(x+a)h(x)] &= H_1[xh(x)] + aH_1h(x) \\ &= \frac{1}{\pi}P\int_0^\infty \frac{\{s-x+x\}h(s)\mathrm{d}s}{x-s} + aH_1h(x) \\ &= (x+a)H_1h(x) - \frac{1}{\pi}P\int_0^\infty h(s)\mathrm{d}s, \end{aligned} \tag{12.161}$$

which on inserting $h(x) = (x+a)^{-1}f(x)$ leads to Eq. (12.160). The other result required is a special case of

$$H_1[x^\mu] = x^\mu \cot \pi\mu, \quad \text{for } -1 < \mu < 0. \tag{12.162}$$

Using Eqs. (12.160) and (12.162) leads to

$$(x+a)H_1\left[\frac{1}{(x+a)\sqrt{x}}\right] = H_1\left[\frac{1}{\sqrt{x}}\right] + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}s}{(s+a)\sqrt{s}} = C_a, \tag{12.163}$$

where C_a denotes the last integral and $H_1[1/\sqrt{x}] = 0$. Inserting Eqs. (12.158) and (12.159) into Eq. (12.157) and making use of Eqs. (12.160) and (12.163) leads to

$$\begin{aligned} \frac{f(x)}{(x+a)\sqrt{x}} &= \frac{C_a\, g(x)}{x+a} - C_a H_1\left[\frac{f(x)}{x+a}\right] - H_1\left[\frac{g(x)}{(x+a)\sqrt{x}}\right] \\ &= \frac{C_a\, g(x)}{x+a} - \frac{C_a}{x+a}H_1 f(x) - \frac{C_a\, C_{fa}}{x+a} - \frac{1}{x+a}H_1\left[\frac{g(x)}{\sqrt{x}}\right] - \frac{C_{ga}}{x+a} \\ &= -\frac{1}{x+a}H_1\left[\frac{g(x)}{\sqrt{x}}\right] - \frac{C}{x+a}, \end{aligned} \tag{12.164}$$

where

$$C_{fa} = \frac{1}{\pi}\int_0^\infty \frac{f(s)\mathrm{d}s}{s+a}, \tag{12.165}$$

$$C_{ga} = \frac{1}{\pi}\int_0^\infty \frac{g(s)\mathrm{d}s}{(s+a)\sqrt{s}}, \tag{12.166}$$

and $C = C_a\, C_{fa} + C_{ga}$. Hence,

$$f(x) = -\sqrt{x}\, H_1\left[\frac{g(x)}{\sqrt{x}}\right] - C\sqrt{x}. \tag{12.167}$$

Now the constant C can be evaluated as follows:

$$\begin{aligned}
C_{ga} &= \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}s}{(s+a)\sqrt{s}}\frac{1}{\pi}P\int_0^\infty \frac{f(y)\mathrm{d}y}{s-y} \\
&= \frac{1}{\pi}\int_0^\infty f(y)\mathrm{d}y\frac{1}{\pi}P\int_0^\infty \frac{\mathrm{d}s}{(s+a)(s-y)\sqrt{s}} \\
&= -\frac{1}{\pi}\int_0^\infty \frac{f(y)\mathrm{d}y}{y+a}\frac{1}{\pi}P\int_0^\infty \frac{1}{\sqrt{s}}\left\{\frac{1}{s+a}-\frac{1}{s-y}\right\}\mathrm{d}s \\
&= -\frac{1}{\pi}\int_0^\infty \frac{f(y)\mathrm{d}y}{y+a}\frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}s}{(s+a)\sqrt{s}} \\
&= -C_a C_{fa},
\end{aligned} \tag{12.168}$$

and hence $C = 0$. This is the expected result, since a non-zero C would imply that $f \notin L^p(\mathbb{R}^+)$. It follows that the solution of Eq. (12.155) is given by

$$f(x) = -\sqrt{(x)}\, H_1\left[\frac{g(x)}{\sqrt{x}}\right]. \tag{12.169}$$

If the restriction that $f \in L^p(\mathbb{R}^+)$, for some $p > 1$, is dropped, then the inversion formula for the one-sided Hilbert transform can be written as follows:

$$f(x) = \frac{c}{\sqrt{x}} - \sqrt{(x)}\, H_1\left[\frac{g(x)}{\sqrt{x}}\right]. \tag{12.170}$$

In Eq. (12.170) c plays the role of an arbitrary constant. The additional factor appears since $H_1[1/\sqrt{x}] = 0$ using Eq. (12.162). In the following section an alternative strategy is demonstrated to invert the one-sided Hilbert transform.

As an example, consider the solution of the following integral equation:

$$\frac{1}{\pi}P\int_0^\infty \frac{f(y)\mathrm{d}y}{x-y} = x^{-1/4}, \quad \text{for } 0 < x < \infty. \tag{12.171}$$

From Eq. (12.170) it follows that

$$f(x) = \frac{c}{\sqrt{x}} - \sqrt{(x)}\, H_1[x^{-3/4}], \tag{12.172}$$

which simplifies, on using the result given in Eq. (12.162), to yield

$$f(x) = \frac{c}{\sqrt{x}} - x^{-1/4}. \tag{12.173}$$

12.7.1 Eigenfunctions and eigenvalues of the one-sided Hilbert transform operator

The integral equation

$$H_1 f(x) = \lambda f(x), \tag{12.174}$$

where λ is a constant, is the eigenvalue equation for the one-sided Hilbert transform operator. From what has been developed previously in this section, the eigenfunctions of this equation take the following form:

$$f(x) = x^{\mu}, \quad \text{for } -1 < \mu < 0 \text{ and } x \in (0, \infty). \tag{12.175}$$

Since

$$H_1[x^{\mu}] = \cot \mu\pi \; x^{\mu}, \tag{12.176}$$

the eigenvalues of H_1 are $\cot \mu\pi$, for $-1 < \mu < 0$.

12.8 Fourier transform approach to the inversion of the one-sided Hilbert transform

An alternative technique that can be used to invert the one-sided Hilbert transform is the Fourier transform approach. If the following change of variables is introduced:

$$x = \mathrm{e}^{2u}, \quad y = \mathrm{e}^{2v}, \tag{12.177}$$

and

$$f(\mathrm{e}^{2v})\mathrm{e}^{v} = h(v), \quad g(\mathrm{e}^{2u})\mathrm{e}^{u} = k(u), \tag{12.178}$$

then the one-sided Hilbert transform can be written as follows:

$$k(u) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{h(v)\mathrm{d}v}{\sinh(u - v)}. \tag{12.179}$$

Taking the Fourier transform of this equation yields

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ut} k(u)\mathrm{d}u &= \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ut}\, \mathrm{d}u\, P \int_{-\infty}^{\infty} \frac{h(v)\mathrm{d}v}{\sinh(u - v)} \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} h(v)\mathrm{d}v\, P \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}ut}\, \mathrm{d}u}{\sinh(u - v)},\end{aligned} \tag{12.180}$$

where it has been assumed that Fubini's theorem can be applied to change the order of integration. On making use of the result

$$P\int_{-\infty}^{\infty} \frac{e^{itw}\,dw}{\sinh w} = \pi \mathrm{i} \tanh\left(\frac{\pi t}{2}\right), \tag{12.181}$$

Eq. (12.180) can be written as follows:

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}tu} k(u)\mathrm{d}u = -\mathrm{i}\tanh\left(\frac{\pi t}{2}\right)\int_{-\infty}^{\infty} h(v)e^{-\mathrm{i}tv}\,\mathrm{d}v, \tag{12.182}$$

and hence

$$\int_{-\infty}^{\infty} h(v)e^{-\mathrm{i}tv}\,\mathrm{d}v = \mathrm{i}\coth\left(\frac{\pi t}{2}\right)\int_{-\infty}^{\infty} e^{-\mathrm{i}tu} k(u)\mathrm{d}u. \tag{12.183}$$

On taking the inverse Fourier transform of this equation, it follows that

$$\begin{aligned} h(w) &= \frac{\mathrm{i}}{2\pi} P\int_{-\infty}^{\infty} e^{\mathrm{i}wt}\coth\left(\frac{\pi t}{2}\right)\mathrm{d}t \int_{-\infty}^{\infty} e^{-\mathrm{i}tu}k(u)\mathrm{d}u \\ &= \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} k(u)\mathrm{d}u\, P\int_{-\infty}^{\infty} e^{\mathrm{i}(w-u)t}\coth\left(\frac{\pi t}{2}\right)\mathrm{d}t. \end{aligned} \tag{12.184}$$

Making use of the result

$$P\int_{-\infty}^{\infty} e^{\mathrm{i}wt}\coth\left(\frac{\pi t}{2}\right)\mathrm{d}t = 2\mathrm{i}\coth w \tag{12.185}$$

leads to

$$h(w) = -\frac{1}{\pi}P\int_{-\infty}^{\infty} k(u)\coth(w-u)\mathrm{d}u. \tag{12.186}$$

Inserting the change of variables

$$x = e^{2w}, \quad y = e^{2u}, \tag{12.187}$$

and

$$h(w) = f(e^{2w})e^{w}, \quad k(u) = g(e^{2u})e^{u}, \tag{12.188}$$

which represents a change back to the original variables that were previously employed, yields

$$f(x) = -\frac{1}{2\pi}P\int_{0}^{\infty} \frac{(x+y)g(y)\mathrm{d}y}{(x-y)\sqrt{(xy)}}. \tag{12.189}$$

As a consequence of Eq. (12.162), a term $c/\sqrt{x}$, with c an arbitrary constant, can be added to the preceding solution, and hence

$$f(x) = \frac{c}{\sqrt{x}} - \frac{1}{2\pi} P \int_0^\infty \frac{(x+y)g(y)\mathrm{d}y}{(x-y)\sqrt{(xy)}}. \tag{12.190}$$

This result represents the inversion formula for the one-sided Hilbert transform. The solution just found takes a different appearance to the solution found in the preceding section. The equivalence of the two solutions can be established in the following straightforward manner. From Eq. (12.190),

$$\begin{aligned} f(x) &= \frac{c}{\sqrt{x}} + \frac{\sqrt{x}}{\pi} P \int_0^\infty \frac{g(y)}{(x-y)\sqrt{y}} \frac{(x-y-2x)\mathrm{d}y}{2x} \\ &= \frac{c}{\sqrt{x}} + \frac{1}{2\pi\sqrt{x}} P \int_0^\infty \frac{g(y)\mathrm{d}y}{\sqrt{y}} - \frac{\sqrt{x}}{\pi} P \int_0^\infty \frac{g(y)\mathrm{d}y}{(x-y)\sqrt{y}} \\ &= \frac{c}{\sqrt{x}} + \frac{c'}{\sqrt{x}} - \frac{\sqrt{x}}{\pi} P \int_0^\infty \frac{g(y)\mathrm{d}y}{(x-y)\sqrt{y}} \\ &= \frac{c''}{\sqrt{x}} - \frac{\sqrt{x}}{\pi} P \int_0^\infty \frac{g(y)\mathrm{d}y}{(x-y)\sqrt{y}}, \end{aligned} \tag{12.191}$$

where c' and c'' are constants. The preceding result is of the form given in Eq. (12.170).

12.9 An inhomogeneous singular integral equation for H_1

The focus of attention in this section is the inhomogeneous singular integral equation

$$f(x) = g(x) + \lambda H_1 f(x), \quad \text{for } 0 < x < \infty. \tag{12.192}$$

Suppose the function f satisfies

$$f(x) = 0, \quad \text{for } x < 0, \tag{12.193}$$

and further assume that $f \in L^p(\mathbb{R})$, for $p > 1$, and $g \in L^p(\mathbb{R})$, for $p > 1$. Since the solution of Eq. (12.192) is sought, a result like Eq. (12.193) must be inferred from the underlying physics of the problem. Using Eq. (12.193), it follows that

$$f(x) = g(x) + \lambda Hf(x), \tag{12.194}$$

and, on applying the Hilbert transform operator,

$$Hf(x) = Hg(x) - \lambda f(x), \tag{12.195}$$

hence

$$\lambda^{-1}\{f(x) - g(x)\} = Hg(x) - \lambda f(x), \tag{12.196}$$

leading to

$$f(x) = \frac{g(x)}{1+\lambda^2} + \frac{\lambda}{1+\lambda^2} Hg(x). \tag{12.197}$$

An alternative approach to the solution of Eq. (12.192) without employing the standard Hilbert transform proceeds as follows. Start with the Tricomi identity, Eq. (12.157), and employ the choice

$$\phi_1(x) = \begin{cases} f(x), & 0 < x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{12.198}$$

and

$$\phi_2(x) = \begin{cases} \dfrac{k(x)}{(x+a)}, & 0 < x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{12.199}$$

for $a > 0$, where it is assumed that $f \in L^p(\mathbb{R}^+)$, for some $p > 1$, and $\phi_2 \in L^q(\mathbb{R}^+)$, for some $q > 1$, with $p^{-1} + q^{-1} \leq 1$. Suppose further that $k(x)$ is an eigenfunction of the operator H_1 and write

$$H_1 k(x) = C_k k(x), \tag{12.200}$$

where C_k denotes the appropriate eigenvalue. The following definitions are introduced:

$$C_{ka} = \frac{1}{\pi} \int_0^\infty \frac{k(s)\mathrm{d}s}{s+a}, \tag{12.201}$$

$$C_{fa} = \frac{1}{\pi} \int_0^\infty \frac{f(s)\mathrm{d}s}{s+a}, \tag{12.202}$$

$$C_{kfa} = \frac{1}{\pi} \int_0^\infty \frac{k(s)f(s)\mathrm{d}s}{s+a}, \tag{12.203}$$

and

$$C_{kHfa} = \frac{1}{\pi} \int_0^\infty \frac{k(s)(H_1 f)(s)\mathrm{d}s}{s+a}. \tag{12.204}$$

Then from Eq. (12.157) it follows that

$$(x+a)^{-1} f(x)k(x) = H_1 f(x) H_1 \left[\frac{k(x)}{(x+a)}\right] - H_1 \left[f(s) H_1 \left\{\frac{k(s)}{(s+a)}\right\} + \frac{k(s)}{(s+a)} H_1 f(s) \right](x)$$

$$
\begin{aligned}
&= H_1 f(x)(x+a)^{-1} H_1 k(x) + H_1 f(x)\,(x+a)^{-1} C_{ka} \\
&\quad - H_1\left[\frac{f(x)}{(x+a)} H_1 k(x)\right] - C_{ka} H_1\left[\frac{f(x)}{(x+a)}\right] \\
&\quad - (x+a)^{-1} H_1[k(x)\, H_1 f(x)] - (x+a)^{-1} C_{kHfa},
\end{aligned} \tag{12.205}
$$

and hence

$$
\begin{aligned}
f(x)k(x) &= H_1 f(x)\, H_1 k(x) - H_1[f(x) H_1 k(x)] - C_k C_{kfa} \\
&\quad - C_{ka} C_{fa} - H_1[k(x) H_1 f(x)] - C_{kHfa} \\
&= C_k \lambda^{-1} k(x)\{f(x) - g(x)\} - C_k H_1[f(x)k(x)] - C_k C_{kfa} \\
&\quad - C_{ka} C_{fa} - \lambda^{-1} H_1[k(x)\{f(x) - g(x)\}] - C_{kHfa},
\end{aligned} \tag{12.206}
$$

which simplifies on setting

$$
C = C_{ka} C_{fa} + C_k C_{kfa} + C_{kHfa} \tag{12.207}
$$

and choosing

$$
C_k = -\lambda^{-1}, \tag{12.208}
$$

to yield

$$
f(x) = \frac{g(x)}{1+\lambda^2} + \frac{\lambda}{1+\lambda^2} k(x)^{-1} H_1[k(x)g(x)] - \frac{\lambda^2}{1+\lambda^2} k(x)^{-1} C. \tag{12.209}
$$

To evaluate the constant C, observe that

$$
\begin{aligned}
C_{kHfa} &= \frac{1}{\pi}\int_0^\infty \frac{k(s)\mathrm{d}s}{s+a}\,\frac{1}{\pi} P\int_0^\infty \frac{f(y)\mathrm{d}y}{s-y} \\
&= \frac{1}{\pi}\int_0^\infty f(y)\mathrm{d}y \frac{1}{\pi} P\int_0^\infty \frac{k(s)\mathrm{d}s}{(s+a)(s-y)} \\
&= -\frac{1}{\pi}\int_0^\infty \frac{f(y)\mathrm{d}y}{y+a}\,\frac{1}{\pi} P\int_0^\infty k(s)\left\{\frac{1}{s+a} - \frac{1}{s-y}\right\}\mathrm{d}s \\
&= -\frac{1}{\pi}\int_0^\infty \frac{f(y)\mathrm{d}y}{y+a}\,\frac{1}{\pi}\int_0^\infty \frac{k(s)\mathrm{d}s}{s+a} \quad - \frac{1}{\pi}\int_0^\infty \frac{f(y)\mathrm{d}y}{y+a}\,\frac{1}{\pi} P\int_0^\infty \frac{k(s)}{y-s}\,\mathrm{d}s \\
&= -C_{fa}\, C_{ka} \ - C_k\, C_{kfa},
\end{aligned} \tag{12.210}
$$

and hence $C = 0$. Therefore the solution of Eq. (12.192) becomes

$$
f(x) = \frac{g(x)}{1+\lambda^2} + \frac{\lambda}{1+\lambda^2} k(x)^{-1} H_1[k(x)g(x)]. \tag{12.211}
$$

The eigenfunctions of H_1 can be written as

$$k(x) = x^{\mu}, \quad \text{for } -1 < \mu < 0, \tag{12.212}$$

and the eigenvalues are

$$C_k = \cot \mu\pi; \tag{12.213}$$

hence Eq. (12.211) becomes

$$f(x) = \frac{g(x)}{1+\lambda^2} + \frac{\lambda}{1+\lambda^2} x^{-\mu} H_1[x^{\mu} g(x)], \tag{12.214}$$

with Eqs. (12.208) and (12.213) giving the connection between λ and μ:

$$\lambda = -\tan \pi\mu. \tag{12.215}$$

The reader is invited to examine the case when μ has an imaginary component.

12.10 A nonlinear singular integral equation

The equations dealt with in the previous sections were all examples of linear singular integral equations. The potential for complexity increases significantly when the integral equation is nonlinear. For nonlinear integral equations, the function sought occurs somewhere in the equation to a power other than one. The nonlinear term may also involve mixed partial derivatives. For example, the integral equation might contain terms such as $f_t f_{xx}$, where f is a function of the two variables x and t. In this and the following sections of this chapter, the standard subscript convention is employed for a derivative (or a partial derivative depending on the context) with respect to the indicated variable. One relatively simple case is examined in this section, and the following three sections discuss other examples that have been extensively investigated.

Consider the nonlinear singular integro-differential equation given by

$$f^2(x) - \alpha f(x) = Hf'(x), \tag{12.216}$$

where α is a constant. First consider the case where $\alpha = 1$. Here are three ideas to try to find a possible quick exact solution. (i) Use the background science that led to the equation to suggest some possible functional forms for the exact solution. (ii) Look for a suitable change of variable that puts the equation in a more useful form. (iii) Search for a change of variable that puts the equation in the form of one of the many nonlinear singular integral equations that have been well studied. The latter strategy may have significant advantages in some cases, since the integral equation of interest is transformed into an equation for which only numerical approaches to the solution are currently available. If these first three approaches are not productive, it will be necessary to resort to more general means of attack. A couple of examples of the latter are illustrated in the following sections. The approach that was often very useful in the case of linear integral equations, that of taking the Hilbert transform of the equation and making use of the inversion property, will generally not be useful for the nonlinear cases. The basic reason for this is that there is no simple formula for the Hilbert transform of a product of functions.

To solve Eq. (12.216) using strategy (i), note that the derivative of the Hilbert transform of the Cauchy pulse (a Lorentzian profile for a dissipative mode) is closely related to the square of the same pulse. This suggests looking for a possible solution of the form

$$f(x) = \frac{\beta}{x^2+1}, \tag{12.217}$$

where β denotes a constant. The right-hand side of Eq. (12.216) yields

$$Hf'(x) = \beta \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x^2+1}\right) = \frac{\beta(1-x^2)}{(1+x^2)^2}. \tag{12.218}$$

The left-hand side of Eq. (12.216), for $\alpha = 1$, yields

$$f(x)^2 - f(x) = \frac{\beta(\beta - 1 - x^2)}{(1+x^2)^2}. \tag{12.219}$$

Equating Eqs. (12.218) and (12.219) yields $\beta = 2$, and the solution of Eq. (12.216) for the case $\alpha = 1$ has been determined. Now that the correct functional form of the solution has been determined, it is a straightforward calculation to show that

$$f(x) = \frac{2\alpha}{\alpha^2 x^2 + 1} \tag{12.220}$$

is the solution of Eq. (12.216). A similar approach can be made for the equation

$$f''(x) - f(x)^2 + f(x) + H(f^2)'(x) = 0. \tag{12.221}$$

The reader is invited to check this out.

12.11 The Peierls–Nabarro equation

In a study of the dislocation of a rectangular lattice of atoms, Peierls (1940) arrived at the following equation:

$$P\int_{-\infty}^{\infty} \left.\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right|_{x=x'} \frac{\mathrm{d}x'}{x-x'} = \frac{1}{2(1-\sigma)} \sin\left(\frac{2\pi\phi(x)}{d}\right), \tag{12.222}$$

where d denotes an atomic spacing, $\phi(x)$ is related to the displacement at the dislocation, and σ is a constant (the Poisson ratio) characterizing the lattice. This is called the Peierls equation. Nabarro (1947) reanalyzed the problem and obtained

$$P\int_{-\infty}^{\infty} \left.\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right|_{x=x'} \frac{\mathrm{d}x'}{x-x'} = (1-\sigma)\sin\left(\frac{2\pi\phi(x)}{d}\right). \tag{12.223}$$

This is generally referred to as the Peierls–Nabarro equation. Peierls gave the solution of Eq. (12.222). The following change of variables is introduced:

$$u(x) = 2\pi d^{-1}\phi(x) \tag{12.224}$$

and

$$\rho = 2d^{-1}(1-\sigma); \tag{12.225}$$

then Eq. (12.223) becomes

$$Hu_x(x) = \rho \sin u(x). \tag{12.226}$$

Equation (12.226) is an example of a nonlinear singular integro-differential equation.

To approach the solution of Eq. (12.226), consider first the case $\rho = 1$. The trigonometric term is simplified by employing the change of variable

$$u(x) = 2\tan^{-1} f(x); \tag{12.227}$$

it then follows, for $f > 0$, that

$$\begin{aligned} H\left[\frac{2f_x(x)}{1+f(x)^2}\right] &= 2\sin[\tan^{-1} f(x)]\cos[\tan^{-1} f(x)] \\ &= \frac{2f(x)}{1+f(x)^2}, \end{aligned} \tag{12.228}$$

and hence

$$\{1+f(x)^2\}H\left[\frac{f_x(x)}{1+f(x)^2}\right] = f(x). \tag{12.229}$$

The solution of this last equation is not difficult to see; it is $f(x) = x$. Hence, a solution of Eq. (12.226) is determined to be

$$u(x) = 2\tan^{-1}(\rho x), \quad \text{for } x > 0. \tag{12.230}$$

For $f < 0$, the sign of the right-hand side of Eq. (12.229) is reversed, and hence $f(x) = -x$ is readily checked to be the solution of the resulting equation. This leads to the second solution of Eq. (12.226):

$$u(x) = -2\tan^{-1}(\rho x). \tag{12.231}$$

A further generalization is given by

$$u(x) = 2\pi n \pm 2\tan^{-1}(\rho x), \tag{12.232}$$

where $n \in \mathbb{Z}$.

12.12 The sine–Hilbert equation

The sine-Gordon equation, which arose originally in differential geometry, takes the following form:

$$u_{xt} = \sin u, \tag{12.233}$$

where $u \equiv u(x,t)$. The sine-Gordon equation is often given in the form

$$u_{xx} - u_{tt} = \sin u, \tag{12.234}$$

but, with an appropriate transformation, this can be converted to the result given in Eq. (12.233). Making a formal replacement of the partial derivative with respect to x by the Hilbert transform with respect to the same variable leads to

$$Hu_t(x,t) = \sin u(x,t), \tag{12.235}$$

and this is called the sine-Hilbert equation. The variables x and t are employed to denote a coordinate in the x-direction and the time, respectively. Equation (12.235) can be written more compactly as $Hu_t = \sin u$, with the understanding that the Hilbert transform is evaluated with respect to the spatial variable. The sine-Hilbert equation made its first appearance not from a representation of any physical phenomena, but as a model form of a nonlinear evolution equation which is solvable.

If the term $\sin u(x,t)$ in Eq. (12.235) is replaced by $cu(x,t)$, with c denoting a constant, then the equation is linearized:

$$Hu_t(x,t) = cu(x,t). \tag{12.236}$$

Taking the Hilbert transform of the preceding equation, followed by the partial derivative with respect to t, yields

$$u_{tt}(x,t) + c^2 u(x,t) = 0, \tag{12.237}$$

which has the solution

$$u(x,t) = \alpha(x)\mathrm{e}^{\mathrm{i}ct} + \beta(x)\mathrm{e}^{-\mathrm{i}ct}, \tag{12.238}$$

where α and β are functions of x alone.

For the particular case that the functional dependence of u on x and t is of the form $x+ct$, with c playing the role of a velocity, then the variable $\xi = x+ct$ is introduced, and $\theta(\xi) \equiv u(x,t)$ is employed; hence Eq. (12.235) can be transformed to

$$H\theta_\xi(\xi) = c\sin\theta(\xi), \tag{12.239}$$

which is recognized as the Peierls–Nabarro equation with solution

$$\theta(\xi) = 2\pi n \pm 2\tan^{-1}(c\xi), \tag{12.240}$$

with $n \in \mathbb{Z}$.

A general method to deal with Eq. (12.235) is now examined, following an approach developed by Matsuno (1995). Set

$$u(x,t) = \mathrm{i}\log(f^*/f), \tag{12.241}$$

where $*$ denotes the complex conjugate and the function f is given by

$$f(x,t) = \prod_{j=1}^{N}\{x - x_j(t)\}, \tag{12.242}$$

with $\mathrm{Im}\{x_j(t)\} > 0$ and $x_j(t) \neq x_k(t)$, for $j \neq k$. Using Eq. (12.241), it follows that

$$\begin{aligned}\sin u(x,t) &= \frac{\mathrm{e}^{\mathrm{i}u} - \mathrm{e}^{-\mathrm{i}u}}{2\mathrm{i}}\\ &= \frac{\mathrm{e}^{-\log(f^*/f)} - \mathrm{e}^{\log(f^*/f)}}{2\mathrm{i}}\\ &= \frac{1}{2\mathrm{i}}\left\{\frac{f}{f^*} - \frac{f^*}{f}\right\}.\end{aligned} \tag{12.243}$$

The following results are required in the sequel:

$$H\left[\frac{1}{x - x_j(t)}\right] = \frac{\mathrm{i}}{x - x_j(t)} \tag{12.244}$$

and

$$H\left[\frac{1}{x - x_j^*(t)}\right] = -\frac{\mathrm{i}}{x - x_j^*(t)}, \tag{12.245}$$

for $\mathrm{Im}\{x_j(t)\} > 0$ (see Eqs. (3.115) – (3.117) for details). Employing Eqs. (12.241) and (12.242) leads to

$$\begin{aligned}
Hu_t(x,t) &= \mathrm{i}\frac{\partial}{\partial t}H\{\log[f^*(x,t)/f(x,t)]\} \\
&= \mathrm{i}\frac{\partial}{\partial t}H\sum_{j=1}^{N}\{\log[x-x_j^*(t)]-\log[x-x_j(t)]\} \\
&= -\mathrm{i}H\sum_{j=1}^{N}\left\{\frac{\dot{x}_j^*(t)}{x-x_j^*(t)}-\frac{\dot{x}_j(t)}{x-x_j(t)}\right\} \\
&= -\sum_{j=1}^{N}\left\{\frac{\dot{x}_j(t)}{x-x_j(t)}+\frac{\dot{x}_j^*(t)}{x-x_j^*(t)}\right\},
\end{aligned} \tag{12.246}$$

where $\dot{x}_j$ denotes $\partial x_j/\partial t$. The term $Hu_t(x,t)$ can be evaluated in the following manner:

$$\begin{aligned}
\frac{\partial}{\partial t}\log(f^*f) &= \frac{\partial}{\partial t}\log\left\{\prod_{j=1}^{N}[x-x_j(t)]\prod_{k=1}^{N}[x-x_k^*(t)]\right\} \\
&= \frac{\partial}{\partial t}\sum_{j=1}^{N}\{\log[x-x_j(t)]+\log[x-x_j^*(t)]\} \\
&= -\sum_{j=1}^{N}\left\{\frac{\dot{x}_j(t)}{x-x_j(t)}+\frac{\dot{x}_j^*(t)}{x-x_j^*(t)}\right\},
\end{aligned} \tag{12.247}$$

and hence

$$Hu_t(x,t) = \frac{\partial}{\partial t}\{\log(f^*f)\}. \tag{12.248}$$

Making use of Eq. (12.243) leads to

$$\frac{\partial}{\partial t}(f^*f) = \frac{1}{2\mathrm{i}}(f^2-f^{*2}), \tag{12.249}$$

which can be recast as

$$f^*\left[f_t-\frac{1}{2\mathrm{i}}(f-f^*)\right]+\left\{f^*\left[f_t-\frac{1}{2\mathrm{i}}(f-f^*)\right]\right\}^* = 0. \tag{12.250}$$

Therefore Eq. (12.249) is satisfied if

$$f_t = \frac{1}{2\mathrm{i}}(f-f^*). \tag{12.251}$$

A solution of the preceding equation can be written in the following form:

$$f(x,t) = f(x,0) + \frac{1}{2\mathrm{i}}[f(x,0) - f^*(x,0)]t. \tag{12.252}$$

Suppose f is expanded in a power series as follows:

$$f(x,t) = \sum_{j=0}^{N} (-1)^j s_j(t) x^{N-j}, \tag{12.253}$$

where the $s_j(t)$ are symmetric combinations of $x_j(t)$. Thus,

$$s_0 = 1, \tag{12.254}$$

$$s_1(t) = \sum_{j=1}^{N} x_j(t), \tag{12.255}$$

$$s_2(t) = \sum_{j=1}^{N} \sum_{j<k}^{N} x_j(t) x_k(t), \tag{12.256}$$

and so on, until finally

$$s_N(t) = \prod_{j=1}^{N} x_j(t). \tag{12.257}$$

Then, from Eq. (12.251),

$$\dot{s}_j(t) = \frac{1}{2\mathrm{i}}(s_j(t) - s_j^*(t)), \tag{12.258}$$

and the solution can be written as

$$s_j(t) = b_j + \mathrm{i}a_j + a_j t, \tag{12.259}$$

with a_j and b_j constants. Noting the connection

$$\mathrm{i}\log(f^*/f) = 2\tan^{-1}(\mathrm{Im}f/\mathrm{Re}f), \tag{12.260}$$

then, for the particular case that $N = 1$, the solution of Eq. (12.235) can be expressed as follows:

$$u(x,t) = -2\tan^{-1}\left(\frac{a_1}{x - a_1 t - b_1}\right), \quad a_1 > 0, \tag{12.261}$$

which is called the one-soliton solution. The reader can explore the case for larger N.

A generalization of Eq. (12.235) is given by

$$Hu_t(x,t) = \sin u(x,t) + \varepsilon u_x(x,t), \tag{12.262}$$

where ε is a positive constant. This formula is referred to as the damped sine-Hilbert equation. A solution of this equation can be determined in a similar manner to the approach discussed in Eqs. (12.241)–(12.261). The solution of Eq. (12.262) has been given by Matsuno (1995):

$$u(x,t) = -2\tan^{-1}\left\{\frac{a+\varepsilon t}{x-at-b-\varepsilon t^2/2}\right\}, \tag{12.263}$$

where a and b are constants and $a > 0$. The reader is asked to verify that this is in fact a solution of Eq. (12.262).

12.13 The Benjamin–Ono equation

One of the well studied nonlinear partial differential equations is the Korteweg–de Vries (often abbreviated KdV) equation, which takes the general form

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \tag{12.264}$$

where α and β are real constants. This equation plays an important role in wave phenomena. As before, x and t play the role of a coordinate in the x-direction, the direction of wave propagation, and time, respectively. A common alternative form is given by

$$u_t + \alpha u_x + \beta u u_x + \gamma u_{xxx} = 0, \tag{12.265}$$

where α, β, and γ are constants directly related to the constants appearing in Eq. (12.264). This equation can be transformed into Eq. (12.264) by an appropriate scaling of the variables x, t, and the function u.

By a formal replacement of the term u_{xxx} by Hu_{xx} (and a sign change for the constant γ), the KdV equation becomes

$$u_t + \alpha u_x + \beta u u_x - \gamma Hu_{xx} = 0. \tag{12.266}$$

This is another example of a nonlinear singular integro-differential equation. Benjamin (1967) derived this form in an investigation of waves of finite amplitude in stable fluid systems. Ono (1975) studied this equation in connection with solitary waves in stratified fluids. Equation (12.266) is called the Benjamin–Ono equation. To simplify the notation a little, in the sequel a term such as $Hf_{xx}(x,t)$ may be shortened to Hf_{xx}, since the variable dependence of f should be obvious from the context. It is assumed that the Hilbert transform is taken with respect to the spatial variable.

The term "solitary wave" employed in the preceding paragraph refers to a waveform that is most commonly a single peak, and was first described in the scientific literature in 1844 by John Scott Russell. The mathematical form of the solitary wave can be extracted from the KdV equation, and the solution obtained in terms of hyperbolic functions. An outcome of this analysis, which is in accord with experimental observations, is that solitary waves of higher amplitude travel faster. A particularly interesting observation occurs when a faster traveling solitary wave bears down on a slower moving solitary wave. Here, the terminology is stretched, since neither is technically a "solitary wave." The swifter moving wave can overtake the more slowly moving wave, and essentially maintain its form, as though apparently no interaction between the passing waves transpired. The term soliton was introduced by Zabusky and Kruskal (1965) to reflect the form-preserving character of the wave during the collision process. "Soliton" has the particle-like connotation of familiar descriptive terms such as photon, proton, etc.

A simplification of Eq. (12.266) is examined first. Suppose the variables x and t in Eq. (12.266) are replaced by χ and τ, respectively, then the following change of variables is introduced:

$$\alpha + \beta u = \zeta f, \tag{12.267}$$

$$\chi = \kappa x, \tag{12.268}$$

$$\tau = \eta t, \tag{12.269}$$

where $\alpha,\ \beta,\ \zeta,\ \kappa$, and η are constants, so that Eq. (12.266) becomes

$$f_t + \lambda f f_x - \rho H f_{xx} = 0, \tag{12.270}$$

where

$$\lambda = \frac{\eta\zeta}{\kappa} \tag{12.271}$$

and

$$\rho = \frac{\gamma\eta}{\kappa^2}. \tag{12.272}$$

Solutions are examined where the functional dependence of f on x and t is of the form $x - ct$, with c playing the role of a velocity. The choice of the form $x \pm ct$ is based on solutions derived from the wave equation for one-dimensional motion. Set $\xi = x - ct$, then Eq. (12.270) is transformed as follows:

$$-c\frac{\mathrm{d}f}{\mathrm{d}\xi} + \lambda f\frac{\mathrm{d}f}{\mathrm{d}\xi} - \rho H\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} = 0. \tag{12.273}$$

Recalling the derivative property of the Hilbert transform (see Section 4.8), $(\mathrm{d}/\mathrm{d}\xi)(Hf) = H(\mathrm{d}f/\mathrm{d}\xi)$, then Eq. (12.273) can be integrated to yield

$$\int_{-\infty}^{x} \left\{ -c\frac{\mathrm{d}f}{\mathrm{d}\xi} + \lambda f\frac{\mathrm{d}f}{\mathrm{d}\xi} - \rho H\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} \right\} \mathrm{d}\,\xi = 0. \tag{12.274}$$

Employing the assumption that $f(-\infty) = 0$, and application of Fubini's theorem, leads to

$$\frac{\lambda}{2} f^2(x) - cf(x) - \rho Hf_x(x) = 0. \tag{12.275}$$

A constant of integration has been omitted. Formally, the integration constant can be eliminated by introducing a change of variables for f and c in terms of $f(-\infty)$. If Eq. (12.273) is integrated over the interval $(0, x)$, then the constant of integration depends on $f(0)$ and can be removed by an appropriate change of variables for f and c. The following variable changes are introduced:

$$f(x) = \frac{2\rho}{\lambda} g(x) + \frac{2c}{\lambda} \tag{12.276}$$

and

$$c = -\alpha\rho; \tag{12.277}$$

then Eq. (12.275) transforms as follows:

$$g^2(x) - \alpha g(x) = Hg_x(x). \tag{12.278}$$

This result is recognized as the nonlinear singular integro-differential equation from Section 12.10. The solution is given by

$$g(x) = \frac{2\alpha}{\alpha^2 x^2 + 1}, \tag{12.279}$$

and hence the solution of Eq. (12.275) is

$$f(x) = \frac{2c}{\lambda}\left\{1 - \frac{1}{\alpha} g(x)\right\} = \frac{2c}{\lambda} - \frac{4c\rho^2}{\lambda(c^2x^2 + \rho^2)}. \tag{12.280}$$

The result given in Eq. (12.279) is a Lorentzian, and not of the hyperbolic form commonly observed for solitary waves. Equation (12.279) is referred to as an algebraic solitary wave profile (Ono, 1975), with the wave shape vanishing algebraically as $|x| \to \infty$.

12.13.1 Conservation laws

Some conservation laws associated with Eq. (12.273) are considered in this subsection. Conservation laws are familiar to every student with a basic knowledge of the physical sciences. As a very simple example, consider the partial differential equation

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial f(x,t)}{\partial x} = 0, \tag{12.281}$$

where, as usual, x and t play the roles of spatial and temporal variables, respectively, $\rho(x,t)$ can be thought of as a density and $f(x,t)$ as a flux, and further suppose that neither quantity depends on partial derivatives with respect to time. Equation (12.281) then represents a conservation law. The simplest applications of this type are, for example, the treatment of the one-dimensional steady flow of a fluid, or the equation of continuity for electric charges.

If information is available on the asymptotic form of $f(x,t)$, it may be possible to integrate the preceding equation, to arrive at additional conservation laws. For example, suppose that $f(x,t)$ satisfies

$$f(x,t) \to 0, \quad \text{as } x \to \pm\infty, \tag{12.282}$$

and vanishes sufficiently quickly that the integral of $f_x(x,t)$ over $(-\infty,\infty)$ converges. Then integrating Eq. (12.281) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \rho(x,t)\mathrm{d}x = -\int_{-\infty}^{\infty} \frac{\partial f(x,t)}{\partial x}\,\mathrm{d}x = 0, \tag{12.283}$$

and hence

$$\int_{-\infty}^{\infty} \rho(x,t)\mathrm{d}x = k, \tag{12.284}$$

where k is some constant. The integral $\int_{-\infty}^{\infty} \rho(x,t)\mathrm{d}x$ is termed a *constant of motion*.

As an example, consider the KdV equation, and set

$$\rho(x,t) = u(x,t) \tag{12.285}$$

and

$$f(x,t) = \beta u_{xx}(x,t) + \frac{\alpha}{2}u^2(x,t). \tag{12.286}$$

Assuming $u(x,t)$ vanishes sufficiently quickly as $x \to \pm\infty$, for a constant k it follows that

$$\int_{-\infty}^{\infty} u(x,t)\mathrm{d}x = k. \tag{12.287}$$

This is not the only conserved quantity for the KdV equation.

Attention is now directed to finding some of the conserved quantities for the Benjamin–Ono equation. Ono (1975) explored a number of such properties. It is assumed that $f(x,t)$ and its derivatives with respect to x vanish sufficiently quickly as $x \to \pm\infty$. Then, from Eq. (12.270), it follows that

$$\int_{-\infty}^{\infty} \{f_t(x,t) + \lambda f(x,t)f_x(x,t) - \rho H f_{xx}(x,t)\} \mathrm{d}x = 0, \tag{12.288}$$

and, on carrying out an integration by parts, this leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} f(x,t) \mathrm{d}x = 0, \tag{12.289}$$

and hence the integral $\int_{-\infty}^{\infty} f(x,t)\mathrm{d}x$ is time-invariant.

If Eq. (12.270) is multiplied by $f(x,t)$ and integrated over $\mathbb{R}$, then

$$\int_{-\infty}^{\infty} \{f(x,t)f_t(x,t) + \lambda f^2(x,t)f_x(x,t) - \rho f(x,t) H f_{xx}(x,t)\} \mathrm{d}x = 0. \tag{12.290}$$

An integration by parts yields

$$\int_{-\infty}^{\infty} f^2(x,t)f_x(x,t) \mathrm{d}x = 0, \tag{12.291}$$

and setting $g(x,t) = f_x(x,t)$ and integrating by parts leads to

$$\int_{-\infty}^{\infty} f(x,t) \frac{\partial}{\partial x} H g(x,t) \mathrm{d}x = -\int_{-\infty}^{\infty} g(x,t) H g(x,t) \mathrm{d}x = 0. \tag{12.292}$$

The orthogonality property of the Hilbert transform has been employed in the preceding equation. It follows, from Eq. (12.290), that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} f^2(x,t) \mathrm{d}x = 0, \tag{12.293}$$

and hence the integral $\int_{-\infty}^{\infty} f^2(x,t)\mathrm{d}x$ is time-invariant.

If Eq. (12.270) is multiplied by $f^2(x,t)$ and integrated over $\mathbb{R}$, then

$$\int_{-\infty}^{\infty} \{f^2(x,t)f_t(x,t) + \lambda f^3(x,t)f_x(x,t) - \rho f^2(x,t) H f_{xx}(x,t)\} \mathrm{d}x = 0. \tag{12.294}$$

An integration by parts yields

$$\int_{-\infty}^{\infty} f^3(x,t)f_x(x,t) \mathrm{d}x = 0, \tag{12.295}$$

and also

$$\begin{aligned}\int_{-\infty}^{\infty} f^2(x,t)Hf_{xx}(x,t)\mathrm{d}x &= -2\int_{-\infty}^{\infty} f(x,t)f_x(x,t)Hf_x(x,t)\mathrm{d}x \\ &= -2\lambda^{-1}\rho\int_{-\infty}^{\infty} Hf_{xx}(x,t)Hf_x(x,t)\mathrm{d}x \\ &\quad + 2\lambda^{-1}\int_{-\infty}^{\infty} f_t(x,t)Hf_x(x,t)\mathrm{d}x \\ &= -2\lambda^{-1}\rho\int_{-\infty}^{\infty} f_{xx}(x,t)f_x(x,t)\mathrm{d}x \\ &\quad - 2\lambda^{-1}\int_{-\infty}^{\infty} f_{tx}(x,t)Hf(x,t)\mathrm{d}x \\ &= -2\lambda^{-1}\int_{-\infty}^{\infty} f_{tx}(x,t)Hf(x,t)\mathrm{d}x, \end{aligned} \tag{12.296}$$

where the Parseval-type property of the Hilbert transform (see Eq. (4.174)) has been employed. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\frac{f^3(x,t)}{3}\,\mathrm{d}x + 2\rho\lambda^{-1}\int_{-\infty}^{\infty} f_{tx}(x,t)Hf(x,t)\mathrm{d}x = 0. \tag{12.297}$$

Applying the Parseval-type property of the Hilbert transform (see Eq. (4.176)) and an integration by parts yields the following result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} f_x(x,t)Hf(x,t)\mathrm{d}x = 2\int_{-\infty}^{\infty} f_{tx}(x,t)Hf(x,t)\mathrm{d}x, \tag{12.298}$$

and so Eq. (12.297) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\left\{\frac{f^3(x,t)}{3} + \rho\lambda^{-1}f_x(x,t)Hf(x,t)\right\}\mathrm{d}x = 0, \tag{12.299}$$

and hence

$$\int_{-\infty}^{\infty}\left\{\frac{f^3(x,t)}{3} + \rho\lambda^{-1}f_x(x,t)Hf(x,t)\right\}\mathrm{d}x$$

is another time-invariant integral for the Benjamin–Ono equation.

If Eq. (12.270) is multiplied by x and integrated over $\mathbb{R}$, then

$$\int_{-\infty}^{\infty}\{xf_t(x,t) + \lambda xf(x,t)f_x(x,t) - \rho xHf_{xx}(x,t)\}\mathrm{d}x = 0, \tag{12.300}$$

and an integration by parts yields

$$\int_{-\infty}^{\infty} xf(x,t)f_x(x,t)\mathrm{d}x = -\frac{1}{2}\int_{-\infty}^{\infty} f^2(x,t)\mathrm{d}x \tag{12.301}$$

and

$$\int_{-\infty}^{\infty} xHf_{xx}(x,t)\mathrm{d}x = 0; \tag{12.302}$$

hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} xf(x,t)\mathrm{d}x = \frac{\lambda}{2}\int_{-\infty}^{\infty} f^2(x,t)\mathrm{d}x. \tag{12.303}$$

Recall from Eq. (12.293) that $\int_{-\infty}^{\infty} f^2(x,t)\mathrm{d}x$ is time-invariant, and therefore

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{-\infty}^{\infty} xf(x,t)\mathrm{d}x = 0. \tag{12.304}$$

The reader is invited to find other time-invariant quantities for the Benjamin–Ono equation. Another useful exercise is to attach a physical interpretation to the several time-invariant quantities derived in this section. Such an interpretation usually becomes much more difficult, if not impossible, as the level of complexity of the time-invariant quantity becomes increasingly involved.

12.14 Singular integral equations involving distributions

In this section attention is directed at a few simple examples of singular integral equations involving the Hilbert transform in which a distribution function occurs in the equation. The first example is the singular integral equation given by

$$f(x) = \delta(x) + Hf(x). \tag{12.305}$$

In this section, equations containing distributions are intended in the symbolic sense only. A more rigorous account of what follows can be given by using the appropriate procedures outlined in Chapter 10. Employing the same strategy indicated in Section 12.3, operate on the preceding equation with H to obtain

$$Hf(x) = p.v.\frac{1}{\pi x} - f(x), \tag{12.306}$$

and, on using Eq. (12.305), this leads to

$$f(x) = \frac{1}{2}\delta(x) + \frac{1}{2\pi}p.v.\frac{1}{x}. \tag{12.307}$$

This result can be verified by direct substitution into Eq. (12.305).

As a second example, consider the solution of the following equation:

$$f'(x) = \delta'(x) - Hf(x) + p.v.\frac{1}{\pi x}. \tag{12.308}$$

Applying the Hilbert transform operator yields

$$Hg'(x) = g(x), \tag{12.309}$$

where $g(x) = f(x) - \delta(x)$. Taking the derivative of Eq. (12.309) and applying the Hilbert transform operator leads to

$$g''(x) + g(x) = 0, \tag{12.310}$$

and hence

$$g(x) = \alpha \mathrm{e}^{\mathrm{i}x} + \beta \mathrm{e}^{-\mathrm{i}x} = a \sin x + b \cos x, \tag{12.311}$$

where α, β, a, and b are constants. Therefore

$$f(x) = \delta(x) + a \sin x + b \cos x, \tag{12.312}$$

which can be readily verified by direct substitution to be the solution of Eq. (12.308).

As a final example, consider the singular integro-differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x} Hf(x) + H\{xf(x)\} = \lambda\delta(x), \tag{12.313}$$

where λ is a constant. Employing the derivative property of the Hilbert transform, $Hf' = (Hf)'$, yields

$$H\{f'(x) + xf(x)\} = \lambda\delta(x). \tag{12.314}$$

Applying the Hilbert transform operator and using the inversion property gives

$$f'(x) + xf(x) = -\frac{\lambda}{\pi} p.v.\frac{1}{x}, \tag{12.315}$$

and, on multiplying by $\mathrm{e}^{x^2/2}$, this leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\mathrm{e}^{x^2/2} f(x)\} = -\frac{\lambda}{\pi} \mathrm{e}^{x^2/2} p.v.\frac{1}{x}, \tag{12.316}$$

and hence

$$f(x) = C\mathrm{e}^{-x^2/2} - \frac{\lambda}{\pi} \mathrm{e}^{-x^2/2} \int_{\alpha}^{x} \mathrm{e}^{t^2/2} p.v.\frac{1}{t}\, \mathrm{d}t, \tag{12.317}$$

where α and C are constants. Equation (12.315) can be readily checked to be satisfied using the f from Eq. (12.317).

Notes

§12.1 For further material on singular integral equations, see Pipkin (1991), Hochstadt (1973), and Tricomi (1985), which are highly recommended for introductory reading. For more advanced texts, consult Mikhlin (1965), Gakhov (1966), Mikhlin and Prössdorf (1986), Dzhuraev (1992), Gohberg and Krupnik (1992a, 1992b), Muskhelishvili (1992), Lifanov (1996), and Kress (1999). For further reading, see Wolfersdorf (1985) and Gera (1986).

§12.4 For a connection to the weighted norm inequalities for the Hilbert transform discussed in Chapter 7, see Widom (1960).

§12.6 The solution of Eq. (12.131) is discussed by Carleman (1922), Carrier *et al.* (1983, p. 422) and Tricomi (1985, p. 185). For further discussion and an application to water wave scattering, see Mandal (2001), and for applications to waveguide problems, see Lewin (1961). The solution of Eq. (12.97) is discussed by Cooke (1970), and the case $a = 0$ and $b = 1$ has been given by Bueckner (1966), in which a particular generalization is discussed in detail. For a special case of the problem studied by Bueckner, Williams (1978) has shown how to transform the key formula to a Carleman integral equation, and Smith (1969) has applied it to a dislocation problem in metals. See also Söhngen (1939, 1954), Nickel (1951), Tricomi (1951a, 1951b, 1985), Estrada and Kanwal (1985), and Jiang and Rokhlin (2003), for further elaboration on singular integral equations involving the finite Hilbert transform. For a related singular integral equation, see Bierman (1971). The singular integral equation

$$g(r) = T\left[\frac{\mathrm{d}}{\mathrm{d}r} f(r)\sqrt{(1-r)}\right] + \frac{1}{2}T\left[\frac{f(r)}{\sqrt{(1-r)}}\right]$$

has been considered by Olmstead and Raynor (1964). For discussion on a numerically based approach to the solution of extended versions of Eq. (12.97), see Capobianco (1993). For numerical approaches to the solution of singular integral equations involving the finite Hilbert transform, see, for example, Dow and Elliott (1979), Fromme and Golberg (1979), Elliott (1982), and Golberg (1990). Singular integral equations involving the Hilbert transform on the circle, and close variants, have been considered by Hilbert (1912), Noether (1921), and Nickel (1953).

§12.7 For a discussion of singular integral equations involving the one-sided Hilbert transform, see Shamir (1964), Hochstadt (1973, p. 159), Pipkin (1991, chap. 11) and Paveri-Fontana and Zweifel (1994). For the solution of the singular integral equation

$$g(y) = \frac{2}{\pi}\int_0^{\infty} \frac{f(x)\mathrm{d}x}{1 - y^2x^2},$$

see Goodspeed (1939).

§12.9 An alternative discussion of the singular integral equation considered in this section can be found in Hochstadt (1973, p. 187). Srivastava and Tuan (1995) have developed a different solution strategy based on a convolution theorem for the Stieltjes transform. Gakhov (1966, p. 148) gives the solution for a more general class of integral equations, of which Eq. (12.192) is a special case.

§12.10 An example of a nonlinear integral equation involving the Hilbert transform that has been investigated is $Hu_x - u + u^p = 0$; see Alfimov, Usero, and Vázquez (2000). For an example in elastodynamics, see Ladopoulos (2005). The reader should expect that many nonlinear singular integral equations involving the Hilbert transform cannot be solved in closed form. In those cases it is necessary to resort to numerical methods. For one such case study, see Chandler and Graham (1993).

§12.11 The solution of Eq. (12.222) reported by Peierls has a misplaced factor of $(1 - \sigma)$ in the numerator of the arctan term; it should appear in the denominator. For an early modification of the Peierls–Nabarro equation, see Huntington (1955), and for a review of fifty years of work on the Peierls–Nabarro stress, see Nabarro (1997). A generalization is discussed by Schoeck (1994), and for further reading see Foreman, Jaswon, and Wood (1951).

§12.12 For discussion on the sine-Hilbert equation and some related forms, see Degasperis and Santini (1983), Degasperis, Santini, and Ablowitz (1985), Matsuno (1986, 1987a, 1987b, 1990, 1991, 1995), and Santini, Ablowitz, and Fokas (1987).

§12.13 For an excellent introductory exposition on solitons, see Drazin and Johnson (1989); a more detailed mathematical presentation is given in Ablowitz and Clarkson (1991). For some additional reading, see Abdelouhab *et al.* (1989), Debnath (1990), Fuchssteiner and Schulze (1995), Matsuno (1996), and Matsuno and Kaup (1997). There is a linearized version of the Benjamin–Ono equation that can be written in the form $u_t - \gamma Hu_{xx} = 0$. The solution of this equation is known in closed form; see Matsuno (1980). Numerical investigation of the generalized Benjamin–Ono equation, $u_t + u^p u_x - Hu_{xx} = 0$, has been carried out by Bona and Kalisch (2004). The singular integral equations, $u_t - \nu u_{xx} + \{(Hu)u\}_x = 0$ and $u_t - \nu u_{xx} - \{Hu\}u_x = 0$, with the Hilbert transform taken in the spatial variable, have some formal similarities with the Benjamin–Ono equation. These have been studied by Baker, Li, and Morlet (1996), and Morlet (1997) has also considered the latter equation. The related equation, $u_t - \nu u_{xx} - \{Hu\}u = 0$, has been investigated by Schochet (1986). Moore (1983) has studied the equation $u_{tt} - u^2 - \langle u^2 \rangle - Hu_x = 0$, where $\langle\,\rangle$ denotes a spatial average. A modified Kuramoto–Sivashinsky equation of the form $u_t + u_{xx} + uu_x + u_{xxxx} + \alpha Hu_{xxx} = 0$ has been studied by Duan and Ervin (1998), and Feng and Kawahara (2000) considered the related form $u_t + u_{xx} + uu_x + u_{xxxx} - \eta\{Hu_{xxx} + Hu_x\} = 0$. Lerche (1986) discusses some complicated nonlinear singular integral equations involving the Hilbert transform that arise in electromagnetic theory. Further discussion on the Benjamin–Ono equation can be found in Case (1979), Kenig, Ponce, and Vega (1994), Toland (1997a), Thomée and Murthy (1998), and Hayashi and Naumkin (1999).

§12.14 For some additional reading, see Orton (1979), Estrada and Kanwal (1985), and Pandey (1996).

Exercises

12.1 Prove that

$$P\int_{-\infty}^{\infty} \frac{e^{itw}\,dw}{\sinh w} = \pi\,i \tanh\left(\frac{\pi t}{2}\right).$$

12.2 Show that

$$P\int_{-\infty}^{\infty} e^{iwt}\coth\left(\frac{\pi t}{2}\right)dt = 2i\coth w.$$

12.3 For a periodic function (period 2π) $f\in L^p(-\pi,\pi)$, with $p>1$, find a solution of the equation $f(\theta)+\mathcal{H}f(\theta)=0$.

12.4 For a periodic function (period 2π) $f\in L^p(-\pi,\pi)$, with $p>1$, what is the relationship between the operator K defined by

$$Kf(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\log\left\{4\sin^2\left(\frac{s-\theta}{2}\right)\right\}ds$$

and $\mathcal{H}$?

12.5 If the periodic functions f and g with period 2π belong to $L^p(-\pi,\pi)$, with $p>1$, determine the solution of the equation $g(\theta)+\mathcal{H}f(\theta)+\lambda Kf(\theta)=\rho f(\theta)$, where λ and ρ are constants and $Kf(\theta)$ is defined in Exercise 12.4. Examine the particular cases $\lambda=\rho=0, \lambda=0, \rho\neq 0$, and $\lambda\neq 0, \rho=0$.

12.6 Determine if there are any solutions of $f(x)=\sin x+Hf'(x)$.

12.7 What, if any, solutions exist for the equation $f(x)=\sin^2 x+Hf'(x)$?

12.8 Find a solution of

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f'(y)dy}{x-y}=\alpha.$$

12.9 Determine a solution of $2\sin ax f(x)=\sin 2ax+2\lambda Hf(x)$, where a and λ are constants and $a>0$.

12.10 Find a solution of $\sin ax\, f(x)=\sin 2\,ax\cos ax+\lambda Hf(x)$, where a and λ are constants and $a>0$.

12.11 Determine a solution of the equation $f(x)=\cos^3 ax-(1/2)\cos ax+\lambda\csc ax\, Hf(x)$, where a and λ are constants and $a>0$.

12.12 Constantin, Lax, and Majda (1985) proposed the model one-dimensional vorticity equation for incompressible flow: $\partial\omega(x,t)/\partial t=\omega(x,t)H[\omega(x,t)]$. If $\omega_0(x)$ denotes a smooth function with sufficiently rapid decay as $|x|\to\infty$, determine the solution of the vorticity equation in terms of $\omega_0(x)$. [Hint: Try the substitution $z(x,t)=i\,\omega(x,t)+H[\omega(x,t)]$.]

12.13 Show by direct substitution that Eq. (12.150) is a solution of Eq. (12.131).

12.14 Determine the solution of

$$P\int_{\alpha}^{\infty}\frac{f(t)dt}{x^2-t^2}=0,\quad \text{for } \alpha<x<\infty.$$

12.15 Suppose $g(x)$ is continuous on (a,b). Determine the solution of

$$P\int_a^b\frac{f(t)dt}{x-t}=g(x),\quad \text{for } a<x<b,$$

and each of the following conditions: (i) $f(t) = O(1/\surd|t-a|)$ and $f(t) = O(1/\surd|t-b|)$ as $t \to a$ and $t \to b$, respectively; (ii) $f(t) = O(1/\surd|t-a|)$ as $t \to a$ and f is bounded like $f(t) = O(\surd|t-b|)$ as $t \to b$; (iii) f is bounded like $f(t) = O(\surd|t-a|)$ and $f(t) = O(\surd|t-b|)$ as $t \to a$ and $t \to b$, respectively.

12.16 Could the choice $\phi_2(x) = 1/\surd x$ be used in place of Eq. (12.159) to find an inversion formula for the one-sided Hilbert transform? Explain.

12.17 For what values of p, if any, does $\phi \in L^p(\mathbb{R}^+)$, where $\phi(x) = (x+a)^{-1}x^{\mu}$, with $a > 0$ and $-1 < \mu < 0$?

12.18 For a simple closed contour C, find a solution of the integral equation

$$a(x)f(x) = g(x) - \frac{\lambda}{\pi} P \int_C \frac{f(t)\mathrm{d}t}{x-t},$$

where x lies on C. Specify any required conditions for the functions and the constant λ that appear in the equation.

12.19 Determine f given

$$P\int_{-a}^{a} \frac{f(s)\mathrm{d}s}{x^2 - s^2} = g(s), \quad \text{for } -a < x < a.$$

12.20 Find the solution f of the equation

$$\frac{1}{\pi} P \int_0^1 \frac{f(s)\mathrm{d}s}{x-s} = g(s), \quad \text{for } 0 < x < 1,$$

and $g \in L^p(0,1)$, with $p > 1$.

12.21 If $|f(x)| < 1$, find a solution of the integral equation $H[f'(x)/\surd\{1 - f(x)^2\}] = \lambda f(x)$, where λ is a constant.

12.22 What transformation of variables takes the equation $u_{xx}(x,t) - u_{tt}(x,t) = \sin u(x,t)$ to the form $u_{xt}(x,t) = \sin u(x,t)$?

12.23 By direct substitution, show that $u(x,t) = -2\tan^{-1}[a/(x - at - b)]$, with a and b constants and $a > 0$, is a solution of the equation $Hu_t(x,t) = \sin u(x,t)$, where the Hilbert transform is with respect to the variable x.

12.24 The Hilbert transform $H[(1 - b\cos ax)^{-1}]$, for $|b| < 1$, arises in the study of periodic solutions of the Benjamin–Ono equation (Ono, 1975). Show that

$$H[(1 - b\cos ax)^{-1}] = b \operatorname{sgn} a \sin ax\{(1 - b\cos ax)\surd(1 - b^2)\}^{-1}.$$

12.25 Show that the solution of $Hu_t(x,t) = \sin u(x,t) + \varepsilon u(x,t)$ is given by

$$u(x,t) = -2\tan^{-1}\left\{(a_1 + \varepsilon t)\left(x - a_1 t - b_1 - \frac{\varepsilon^2 t}{2}\right)^{-1}\right\},$$

where a and b are constants and $a > 0$.

12.26 For a real constant λ and a function $g \in L^p(\mathbb{R})$ with $p > 1$, determine the solution of the equation $\lambda f(x) = g(x) - 2f(x)Hf(x) + Hf^2(x)$.

12.27 Determine if the equation $Hf'(x) = \delta(x) + \cos x$ has a solution.

12.28 For a real constant λ, find the solution for f given $\mathrm{d}f/\mathrm{d}x - Hg'(x) = \delta(x)$ and $g \in L^p$, $g' \in L^p$, for $p > 1$.

12.29 For λ a real constant, $g \in \mathcal{D}'$ and $Hg \in \mathcal{D}'$, determine f given: $f - Hf = \lambda g$.

12.30 If $f \in \mathscr{S}'$ and λ is a real constant, determine the solution of $f' + f = \lambda H\delta$.

13

Discrete Hilbert transforms

13.1 Introduction

The terminology "discrete Hilbert transform" arises in two distinct contexts. The first occurrence is in the study of certain types of series with a denominator of the form $n - m$, where m is the summation index and $n \in \mathbb{Z}$. The second area is in signal processing in engineering problems. The first part of this chapter is devoted to a discussion of the discrete Hilbert transform as it is commonly employed in engineering applications, and to some related topics including the discrete Fourier transform. The focus of the second part of the chapter is a discussion of the analog of the Hilbert transform on $\mathbb{R}$ for a discrete series. Two related definitions are considered. The key inequality for the discrete Hilbert transform, called Hilbert's inequality, is established and a generalization given by M. Riesz is discussed. Since much of the data collected in both the physical sciences and engineering are discrete, the discrete Hilbert transform is a rather useful tool in these areas for the general analysis of this type of data.

13.2 The discrete Fourier transform

The discrete Fourier transform is commonly denoted in the scientific literature by the acronym DFT. The ideas that evolve for this transform provide a foundation for the development of the discrete Hilbert transform. The essential motivation behind thinking about discrete transforms is that experimental data are most frequently not taken in a continuous manner, but sampled at discrete time values. This is the fundamental feature of digital signal processing. An important concern in signal processing is the connection between the time and frequency representations of a system. This topic is used as a lead into the discrete Fourier transform.

Suppose the following sequence of samples is available: $h(0), h(1), h(2), \ldots$; to make the discussion concrete, think of the argument of h as an integer time variable. The notation can be made more compact by the association $h(n) = h_n$, which also serves to remind the reader that a discretely measured function is being sampled. An alternative notation, which is employed later in this chapter, is to write discrete sequences using the notation $h[n]$. If a sampled sinusoidal input is applied to a linear system, then the output is a sampled sinusoidal function multiplied by a system response function. Linear systems are discussed in Chapter 17, so a justification of the preceding statement is left for that discussion, together with some of the more detailed points. The response function is denoted by $\mathsf{H}(\omega)$, and for the case of interest here is given by

$$\mathsf{H}(\omega) = \sum_{n=0}^{N-1} h_n e^{-i\omega n}, \tag{13.1}$$

where ω denotes a circular frequency, which is $2\pi\nu$, and ν is the frequency. Many physics sources use the term angular frequency synonymously with circular frequency; this book will do the same. From Eq. (13.1) it is evident that $\mathsf{H}(\omega)$ is a continuous function of frequency, and that it is a periodic function of ω with period 2π. It is assumed that $\mathsf{H}(\omega)$ is sampled at M equally spaced points in the frequency variable, with

$$\omega = \frac{2\pi\nu}{M}, \quad \text{for } \nu = 0, 1, \ldots, M-1. \tag{13.2}$$

Since $\mathsf{H}(\omega)$ is periodic, the sampling can be restricted to a frequency interval of 2π, as there is no additional information to be obtained beyond this frequency range. For this sample set, Eq. (13.1) can be written as follows:

$$\mathsf{H}_\nu = \sum_{n=0}^{N-1} h_n e^{-2\pi i\nu n/M}, \quad \text{for } \nu = 0, 1, \ldots, M-1. \tag{13.3}$$

This result gives a transform of the N values h_n in the time domain to the M values H_ν in the frequency domain. It is highly desirable if this formula can be inverted to yield h_n as a function of H_ν. The inversion can be made particularly simple by restricting the sample size in the frequency domain to N points. So, Eq. (13.3) takes the following simplified form:

$$\mathsf{H}_\nu = \sum_{n=0}^{N-1} h_n e^{-2\pi i\nu n/N}, \quad \text{for } \nu = 0, 1, \ldots, N-1. \tag{13.4}$$

This result defines the discrete Fourier transform of the function h.

Equation (13.4) can be written more compactly by making the substitution

$$W = e^{-2\pi i/N}. \tag{13.5}$$

The inversion process is as follows. From Eq. (13.4), it follows that

$$\begin{aligned}
\sum_{\nu=0}^{N-1} \mathsf{H}_\nu \mathrm{e}^{2\pi\mathrm{i}\nu k/N} &= \sum_{\nu=0}^{N-1} \mathrm{e}^{2\pi\mathrm{i}\nu k/N} \sum_{n=0}^{N-1} h_n \mathrm{e}^{-2\pi\mathrm{i}\nu n/N} \\
&= \sum_{\nu=0}^{N-1} W^{-\nu k} \sum_{n=0}^{N-1} h_n W^{\nu n} \\
&= \sum_{n=0}^{N-1} h_n \sum_{\nu=0}^{N-1} W^{\nu(n-k)}. \qquad (13.6)
\end{aligned}$$

The preceding result can be simplified using the following standard expansion:

$$x^m - 1 = (x-1)\sum_{v=0}^{m-1} x^v. \qquad (13.7)$$

For the case $n \neq k$, and the assignment $m = N$ and $x = W^{n-k}$ in Eq. (13.7),

$$\begin{aligned}
\sum_{\nu=0}^{N-1} W^{\nu(n-k)} &= \frac{W^{(n-k)N} - 1}{W^{n-k} - 1} \\
&= 0, \qquad (13.8)
\end{aligned}$$

where the following result has been employed:

$$W^{jN} = 1, \quad \text{for integer } j. \qquad (13.9)$$

For the case $n = k$, it follows that

$$\sum_{\nu=0}^{N-1} W^{\nu(n-k)} = N, \qquad (13.10)$$

and so Eq. (13.6) simplifies to

$$\sum_{\nu=0}^{N-1} \mathsf{H}_\nu \mathrm{e}^{2\pi\mathrm{i}\nu k/N} = \sum_{n=0}^{N-1} h_n N \delta_{nk}, \qquad (13.11)$$

where δ_{nk} is the Kronecker delta, and hence

$$h_k = \frac{1}{N}\sum_{\nu=0}^{N-1} \mathsf{H}_\nu \mathrm{e}^{2\pi\mathrm{i}\nu k/N}. \qquad (13.12)$$

This result represents the inversion formula for the discrete Fourier transform. While the preceding discussion used the time and frequency domains as an example, the

DFT formula (13.4) and its inversion (13.12) can obviously be extended to other pairs of variables.

13.3 Some properties of the discrete Fourier transform

The DFT satisfies the periodicity condition

$$\mathsf{H}_m = \mathsf{H}_{m+N}. \tag{13.13}$$

This result can be established as follows:

$$\begin{aligned}\mathsf{H}_{m+N} &= \sum_{n=0}^{N-1} h_n W^{(m+N)n} \\ &= \sum_{n=0}^{N-1} h_n W^{mn} W^{Nn} \\ &= \sum_{n=0}^{N-1} h_n W^{mn} \\ &= \mathsf{H}_m.\end{aligned} \tag{13.14}$$

A result which follows from the periodicity property is

$$\mathsf{H}_{N-m} = \mathsf{H}_{-m}, \tag{13.15}$$

which can be established along the same lines shown in Eq. (13.14). As a consequence of the periodicity property, the DFT can be written in the following form:

$$\mathsf{H}_\nu = \sum_{n=-N/2}^{N/2-1} h_n \mathrm{e}^{-2\pi \mathrm{i}\nu n/N}, \tag{13.16}$$

and the inverse is given by

$$h_k = \frac{1}{N} \sum_{\nu=-N/2}^{N/2-1} \mathsf{H}_\nu \mathrm{e}^{2\pi \mathrm{i}\nu k/N}. \tag{13.17}$$

Equations (13.4) and (13.12) are frequently referred to as the one-sided DFT and inverse DFT, respectively, while Eqs. (13.16) and (13.17) are called the two-sided or centered DFT and inverse DFT, respectively. The one-sided format is slightly more convenient for computer calculations, while the symmetry aspects are more readily discussed using the two-sided forms.

From the definition of the DFT the following symmetry property can be established. If h_n is real, then, on taking the complex conjugate,

$$\mathsf{H}_{-m}^{*} = \mathsf{H}_m, \tag{13.18}$$

and if h_n is purely imaginary,

$$\mathsf{H}_{-m}^{*} = -\mathsf{H}_m. \tag{13.19}$$

If h_n is both real and an even function of the index about $n = 0$, then

$$\mathsf{H}_m^{*} = \mathsf{H}_m. \tag{13.20}$$

Parseval's relation for the DFT is given by

$$\sum_{n=0}^{N-1} |h_n|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\mathsf{H}_m|^2. \tag{13.21}$$

This result can be obtained in the following manner:

$$\begin{aligned}\sum_{n=0}^{N-1} |h_n|^2 &= \sum_{n=0}^{N-1} \frac{1}{N}\sum_{\nu=0}^{N-1} \mathsf{H}_\nu \mathrm{e}^{2\pi\mathrm{i}\nu n/N}\frac{1}{N}\sum_{m=0}^{N-1} \mathsf{H}_m^{*}\mathrm{e}^{-2\pi\mathrm{i}mn/N}\\ &= \frac{1}{N^2}\sum_{\nu=0}^{N-1}\sum_{m=0}^{N-1} \mathsf{H}_\nu \mathsf{H}_m^{*}\sum_{n=0}^{N-1} W^{n(m-\nu)},\end{aligned} \tag{13.22}$$

which simplifies, on using Eqs. (13.8) and (13.10), as follows:

$$\begin{aligned}\sum_{n=0}^{N-1} |h_n|^2 &= \frac{1}{N^2}\sum_{\nu=0}^{N-1}\sum_{m=0}^{N-1} \mathsf{H}_\nu \mathsf{H}_m^{*} N\delta_{m\nu}\\ &= \frac{1}{N}\sum_{m=0}^{N-1} |\mathsf{H}_m|^2.\end{aligned} \tag{13.23}$$

13.4 Evaluation of the DFT

The DFT can be written in matrix notation as follows:

$$\mathsf{H} = \mathscr{F}_N \mathsf{h}, \tag{13.24}$$

where $\mathcal{F}_N$ denotes the N-point DFT operator, and, assuming that the one-sided form is employed,

$$\mathsf{H} = \begin{pmatrix} \mathsf{H}_0 \\ \mathsf{H}_1 \\ \mathsf{H}_2 \\ \vdots \\ \mathsf{H}_{N-1} \end{pmatrix}, \quad \mathsf{h} = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{N-1} \end{pmatrix}, \tag{13.25}$$

and

$$\mathcal{F}_N = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{(N-1)} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)^2} \end{pmatrix}. \tag{13.26}$$

The individual elements of the matrix are given by $(\mathcal{F}_N)_{mn} = \mathrm{e}^{-2\pi \mathrm{i} mn/N} = W^{mn}$, for $0 \le m, n \le N-1$. It is common practice also to define the matrix elements via $(\mathcal{F}_N)_{mn} = (1/\sqrt{N})\mathrm{e}^{-2\pi \mathrm{i} mn/N}$, which leads to the eigenvalue set of $\{1, -1, \mathrm{i}, -\mathrm{i}\}$, avoiding factors of $\sqrt{N}$. In matrix form the inverse DFT operator is given by

$$\mathcal{F}_N^{-1} = N^{-1} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W^{-1} & W^{-2} & \cdots & W^{-(N-1)} \\ 1 & W^{-2} & W^{-4} & \cdots & W^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{-(N-1)} & W^{-2(N-1)} & \cdots & W^{-(N-1)^2} \end{pmatrix}. \tag{13.27}$$

As an example, DFT{1, 0, 0, 1} is evaluated from

$$\begin{pmatrix} \mathsf{H}_0 \\ \mathsf{H}_1 \\ \mathsf{H}_2 \\ \mathsf{H}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1+\mathrm{i} \\ 0 \\ 1-\mathrm{i} \end{pmatrix}, \tag{13.28}$$

and hence

$$\mathrm{DFT}\{1, 0, 0, 1\} = \{2, 1+\mathrm{i}, 0, 1-\mathrm{i}\}. \tag{13.29}$$

The inverse DFT, which is abbreviated as IDFT, of $\{2, 1+\mathrm{i}, 0, 1-\mathrm{i}\}$ is given by

$$\begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{pmatrix}\begin{pmatrix} 2 \\ 1+\mathrm{i} \\ 0 \\ 1-\mathrm{i} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \tag{13.30}$$

and hence

$$\mathrm{IDFT}\{2, 1+\mathrm{i}, 0, 1-\mathrm{i}\} = \{1, 0, 0, 1\}. \tag{13.31}$$

13.5 Relationship between the DFT and the Fourier transform

The principal focus of the following chapter is the numerical evaluation of the Hilbert transform. The reader will recall from Section 5.2 that there is a close connection between the Hilbert and Fourier transforms, so an effective numerical route to the Fourier transform is of importance for evaluating the Hilbert transform.

In order that the Fourier transform of f converges, it is required that $f(s)$ vanishes sufficiently quickly as $s \to \pm\infty$; hence, the integration range may be truncated, and thus

$$\mathcal{F}f(x) = \int_{-\infty}^{\infty} f(s)\mathrm{e}^{-2\pi\mathrm{i}xs}\,\mathrm{d}s \approx \int_{-L/2}^{L/2} f(s)\mathrm{e}^{-2\pi\mathrm{i}xs}\,\mathrm{d}s. \tag{13.32}$$

If L is selected to be sufficiently large, a reasonable numerical approximation for the integral can be obtained, subject to the caveat that the oscillatory behavior of the integrand is not too extreme. A slight rearrangement of this integral yields

$$\begin{aligned}\int_{-L/2}^{L/2} f(s)\mathrm{e}^{-2\pi\mathrm{i}xs}\,\mathrm{d}s &= \int_{-L/2}^{0} f(s)\mathrm{e}^{-2\pi\mathrm{i}xs}\,\mathrm{d}s + \int_{0}^{L/2} f(s)\mathrm{e}^{-2\pi\mathrm{i}xs}\,\mathrm{d}s \\ &= \mathrm{e}^{2\pi\mathrm{i}xL}\int_{L/2}^{L} f(s-L)\mathrm{e}^{-2\pi\mathrm{i}xs}\,\mathrm{d}s \\ &\quad + \int_{0}^{L/2} f(s)\mathrm{e}^{-2\pi\mathrm{i}xs}\,\mathrm{d}s. \end{aligned} \tag{13.33}$$

If x is selected at the discrete points k/L, with $k = -N/2, \ldots, N/2-1$ (assuming N to be even), then

$$\int_{-L/2}^{L/2} f(s)\mathrm{e}^{-2\pi\mathrm{i}xs}\,\mathrm{d}s = \int_{0}^{L} g(s)\mathrm{e}^{-2\pi\mathrm{i}ks/L}\,\mathrm{d}s, \tag{13.34}$$

where

$$g(s) = \begin{cases} f(s), & \text{for } 0 \le s < L/2 \\ \{f(L/2) + f(-L/2)\}/2, & \text{for } s = L/2 \\ f(s-L), & \text{for } L/2 < s \le L. \end{cases} \tag{13.35}$$

The integral in Eq. (13.34) can be approximated by a uniform left-endpoint sum of the form

$$\int_0^L g(s)\mathrm{e}^{-2\pi \mathrm{i}ks/L}\,\mathrm{d}s \approx \Delta \sum_{n=0}^{N-1} g(s_n)\mathrm{e}^{-2\pi \mathrm{i}ks_n/L}, \tag{13.36}$$

with the sampling interval $\Delta = L/N$ and $s_n = nL/N$. Hence,

$$\begin{aligned}(\mathcal{F}f)(k/L) &\approx \Delta \sum_{n=0}^{N-1} g(nL/N)\mathrm{e}^{-2\pi \mathrm{i}nk/N} \\ &= \Delta G_k,\end{aligned} \tag{13.37}$$

where G_k denotes the DFT of g. Because of the periodic property of the DFT, the range of the index k, from $-N/2$ to $N/2 - 1$, can be replaced by 0 to $N - 1$.

13.6 The *Z* transform

Let $\{x_n\}$ denote a sequence of numbers and let $x[n]$ designate the nth value; then, the Z transform of this sequence, denoted $Z\{x_n\}$, also frequently represented as $X(z)$, is defined by

$$Z\{x_n\} \equiv X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \tag{13.38}$$

This definition is usually referred to as the *one-sided* Z transform; one-sided referring to the fact that the summation index includes only non-negative integer indices. When there is interest in a sequence $\{x_n\}$ defined for both positive and negative values of n, then the *two-sided* Z transform is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \tag{13.39}$$

Common usage is that Eq. (13.38) is frequently referred to as the Z transform and Eq. (13.39) called the *two-sided* Z transform, although one could make a good argument for calling the more general result in Eq. (13.39) simply the Z transform. In some sources the sequence members are designated by x_n, which does have some pedagogical value, but has definite typesetting disadvantages when complicated indices are involved. A version of the Z transform that is useful in some applications is the N-point Z transform, defined by

$$X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}. \tag{13.40}$$

Some authors make use of a convention where the variable z^{-1} is replaced by z, so that Eq. (13.40) becomes a polynomial in the variable z.

One rather simple example is now examined, which illustrates the advantage of thinking in terms of the Z transform. A *shift-invariant* system is characterized in the following way. Suppose $y[n]$ is the response to $x[n]$, then $y[n-m]$ is the response to $x[n-m]$. The sum

$$y[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m], \tag{13.41}$$

where $x_1[n]$ and $x_2[n]$ are elements of two different sequences, arises frequently in signal processing and other applications, and is usually written in the compact notation

$$y[n] = x_1[n] * x_2[n]. \tag{13.42}$$

Equation (13.42) is referred to as the convolution of $x_1[n]$ with $x_2[n]$. This is the discrete analog of the convolution theorem discussed for continuous functions in Section 2.6.2. As an example, consider the convolution of the two sequences of data measured at equally spaced time intervals, starting at time zero, which corresponds to the first entry in each sequence:

$$x_1 \equiv \{1, 2, 3\} \tag{13.43}$$

and

$$x_2 \equiv \{2, 0, 3, 3\}. \tag{13.44}$$

The convolution product, using Eq. (13.41), leads to

$$\left.\begin{aligned}
y[0] &= x_1[0]x_2[0] = 2,\\
y[1] &= x_1[0]x_2[1] + x_1[1]x_2[0] = 4,\\
y[2] &= x_1[0]x_2[2] + x_1[1]x_2[1] + x_1[2]x_2[0] = 9,\\
y[3] &= x_1[0]x_2[3] + x_1[1]x_2[2] + x_1[2]x_2[1] = 9,\\
y[4] &= x_1[1]x_2[3] + x_1[2]x_2[2] = 15,\\
y[5] &= x_1[2]x_2[3] = 9.
\end{aligned}\right\} \tag{13.45}$$

The preceding calculation can be replaced by the more compact approach:

$$x_1 = \{1, 2, 3\} \overset{Z}{\Rightarrow} X_1(z) = 1 + 2z^{-1} + 3z^{-2} \tag{13.46}$$

and

$$x_2 = \{2, 0, 3, 3\} \stackrel{Z}{\Rightarrow} X_2(z) = 2 + 3z^{-2} + 3z^{-3}, \tag{13.47}$$

and hence

$$\begin{aligned} X_1(z)X_2(z) &= 2 + 4z^{-1} + 9z^{-2} + 9z^{-3} + 15z^{-4} + 9z^{-5} \Rightarrow y \equiv x_1 * x_2 \\ &= \{2, 4, 9, 9, 15, 9\}. \end{aligned} \tag{13.48}$$

Multiplication of the Z transforms allows the convolution product to be determined in a direct manner, a result arising from the collection of like powers in the polynomial multiplication operation in the variable z^{-1}. That is, advantage is being taken of the following formula:

$$x_1 * x_2 \stackrel{Z}{\rightarrow} X_1(z)X_2(z) = Y(z) \stackrel{\text{inverse Z}}{\longrightarrow} y[n]. \tag{13.49}$$

The Z transform can be viewed as a formal power series in the complex variable z, and is thus a Laurent series. A number of the properties of the Z transform can be taken directly from known results for the Laurent series. Several of those properties were discussed in Section 2.8.5. An important special case for the N-point Z transform is obtained on making the identification

$$z = \mathrm{e}^{\mathrm{i}\theta}, \quad \text{with } \theta = \frac{2\pi k}{N}. \tag{13.50}$$

At equally spaced discrete points on the unit circle, $k = 0, 1, \ldots, N-1$, the N-point Z transform equals the DFT.

The inverse of the Z transform can be determined in the following manner. Consider the Cauchy integral theorem applied to the function $f(z) = z^m$ for integer $m \geq 0$; then

$$\oint_C f(z)\mathrm{d}z = 0, \tag{13.51}$$

where C is the contour shown in Figure 13.6.

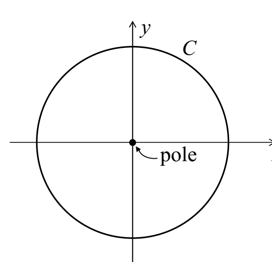

Figure 13.1. Simple circular contour enclosing the origin.

With the choice of function $f(z) = z^{-m-1}$ for integer $m \geq 0$, and the same contour, the residue theorem leads to

$$\frac{1}{2\pi \mathrm{i}} \oint_C z^{-m-1}\,\mathrm{d}z = \mathrm{Res}\{z^{-m-1}\}_{z\to 0} = \begin{cases} 1, & \text{for } m = 0 \\ 0, & \text{for } m \geq 1. \end{cases} \tag{13.52}$$

From the definition of the Z transform, it follows that

$$z^{m-1}X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{m-n-1}, \tag{13.53}$$

and on integrating this expression using the same contour shown in Figure 13.6, leads to the result

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \oint_C z^{m-1}X(z)\mathrm{d}z &= \frac{1}{2\pi \mathrm{i}} \oint_C \sum_{n=-\infty}^{\infty} x[n]\, z^{m-n-1}\,\mathrm{d}z \\ &= \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi i} \oint_C z^{m-n-1}\,\mathrm{d}z \\ &= \sum_{n=-\infty}^{\infty} x[n]\delta_{mn}, \end{aligned} \tag{13.54}$$

and hence

$$x[m] = \frac{1}{2\pi \mathrm{i}} \oint_C z^{m-1}X(z)\mathrm{d}z, \tag{13.55}$$

which represents the required inversion formula. Making the interchange of summation and integration in Eq. (13.54) assumes that the series is uniformly convergent for values of z involved in the integration.

13.7 *Z* transform of a product

Denote the Z transforms of the sequences $x_1[n], x_2[n]$, and $x_3[n]$, by $X_1(z), X_2(z)$, and $X_3(z)$, respectively, and suppose that the sequence $x_3[n]$ is formed as a product of the other two sequences:

$$x_3[n] = x_1[n]\, x_2[n]. \tag{13.56}$$

Let r_1 and r_2 denote the radius of convergence for $X_1(z)$ and $X_2(z)$, respectively, so that $X_1(z)$ converges for $|z| > r_1$ and $X_2(z)$ converges for $|z| > r_2$. The Z transform $X_3(z)$ can be obtained using the following approach. Start with

$$Z\{x_3[n]\} = Z\{x_1[n]x_2[n]\}; \tag{13.57}$$

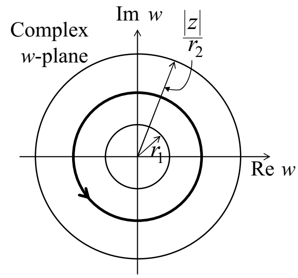

Figure 13.2. Circular contour center the origin with radius between r_1 and $|z|/r_2$.

hence,

$$X_3(z) = \sum_{n=0}^{\infty} x_1[n]x_2[n]\, z^{-n}. \tag{13.58}$$

The preceding sum can be simplified in the following way. Consider the contour integral

$$\frac{1}{2\pi \mathrm{i}} \oint_C X_1(w)X_2(z/w)w^{-1}\,\mathrm{d}w$$

for $|z| > r_1r_2$, where C is the circular contour shown in Figure 13.7, which lies in the region where the sums represented by $X_1(w)$ and $X_2(z/w)$ converge.

Then

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \oint_C X_1(w)X_2\left(\frac{z}{w}\right) w^{-1}\,\mathrm{d}w &= \frac{1}{2\pi \mathrm{i}} \oint_C \sum_{n=0}^{\infty} x_1[n]w^{-n} \sum_{m=0}^{\infty} x_2[m]\left(\frac{z}{w}\right)^{-m} w^{-1}\,\mathrm{d}w \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_1[n]\, x_2[m]\, z^{-m} \frac{1}{2\pi \mathrm{i}} \oint_C w^{m-n-1}\,\mathrm{d}w \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_1[n]\, x_2[m]\, z^{-m}\delta_{mn} \\
&= \sum_{n=0}^{\infty} x_1[n]\, x_2[n]\, z^{-n} \\
&= X_3(z).
\end{aligned} \tag{13.59}$$

The interchange of the order of integration and summation is made assuming that the series converges uniformly. Equations (13.50) and (13.51) have been used to evaluate the last integral in the preceding sequence of steps. Equation (13.59) provides the connection between the Z transform of a product of sequences and the individual Z transforms of the sequences. This result is employed in the following section.

13.8 The Hilbert transform of a discrete time signal

The relationships that exist between the real and imaginary parts of a sequence are developed in this section. Some definitions are required in the sequel, and these are elaborated upon in Chapter 17. In what follows the focus is on linear, stable, and causal digital systems. If the inputs to a system are $x_1[n]$ and $x_2[n]$, and the corresponding outputs are $y_1[n]$ and $y_2[n]$, respectively, then the system is described as *linear* if the input $\alpha x_1[n]+\beta x_2[n]$ leads to an output $\alpha y_1[n]+\beta y_2[n]$, for arbitrary constants α and β. If the sequence index describes time, then a *time-invariant* system is one where if an input $x[n]$ produces an output $y[n]$, then an input $x[n-m]$ produces an output $y[n-m]$ for all m. The system is termed *stable* if every bounded input produces a bounded output. The *causal* condition for a time-sequence is that the output $y[m]$ is determined by input values satisfying $n \leq m$, that is, the output at some time does not depend on sequence data in the future.

A time-sequence is denoted by $x[n]$, and it is written in terms of its real and imaginary parts as follows:

$$x[n] = x_{\mathrm{r}}[n] + \mathrm{i}\, x_{\mathrm{i}}[n], \tag{13.60}$$

where the subscripts r and i designate the real and imaginary parts, respectively. The Fourier transform of the time-sequence $x[n]$ is denoted by $X(\omega)$, which is a periodic function with period 2π, and $X_{\mathrm{r}}(\omega)$ and $X_{\mathrm{i}}(\omega)$ are used to represent the Fourier transforms of $x_{\mathrm{r}}[n]$ and $x_{\mathrm{i}}[n]$, respectively. If $x[n]$ is real, it follows from the form of the Fourier transform that the following symmetry condition holds:

$$X^*(-\omega) = X(\omega). \tag{13.61}$$

The Fourier transforms $X_{\mathrm{r}}(\omega)$ and $X_{\mathrm{i}}(\omega)$ for the general sequence given in Eq. (13.60) satisfy the following:

$$X_{\mathrm{r}}^*(-\omega) = X_{\mathrm{r}}(\omega) \tag{13.62}$$

and

$$X_{\mathrm{i}}^*(-\omega) = X_{\mathrm{i}}(\omega). \tag{13.63}$$

From these last two relationships, it follows that

$$X_{\mathrm{r}}(\omega) = \frac{1}{2}\{X(\omega) + X^*(-\omega)\} \tag{13.64}$$

and

$$X_{\mathrm{i}}(\omega) = \frac{1}{2\mathrm{i}}\{X(\omega) - X^*(-\omega)\}. \tag{13.65}$$

The following condition is imposed on $X(\omega)$:

$$X(\omega) = 0, \quad \text{for } -\pi \le \omega < 0. \tag{13.66}$$

The immediate implication of this requirement is that

$$X_{\mathrm{r}}(\omega) = \begin{cases} \dfrac{1}{2}X(\omega), & \text{for } 0 < \omega < \pi \\ X(0), & \text{for } \omega = 0 \end{cases} \tag{13.67}$$

and

$$X_{\mathrm{i}}(\omega) = \begin{cases} \dfrac{1}{2\mathrm{i}}X(\omega), & \text{for } 0 < \omega < \pi \\ 0, & \text{for } \omega = 0, \end{cases} \tag{13.68}$$

and hence

$$X_{\mathrm{i}}(\omega) = -\mathrm{i}X_r(\omega), \quad \text{for } 0 < \omega < \pi \tag{13.69}$$

and

$$X_{\mathrm{i}}(\omega) = \mathrm{i}X_r(\omega), \quad \text{for } -\pi \le \omega < 0. \tag{13.70}$$

The function $\mathsf{H}(\omega)$ is introduced via the following definition:

$$\mathsf{H}(\omega) = \begin{cases} \mathrm{i}, & \text{for } -\pi \le \omega < 0 \\ -\mathrm{i}, & \text{for } 0 < \omega < \pi \end{cases} \equiv -\mathrm{i}\,\mathrm{sgn}\,\omega; \tag{13.71}$$

then it follows that

$$X_{\mathrm{i}}(\omega) = \begin{cases} \mathsf{H}(\omega)X_{\mathrm{r}}(\omega), & 0 < |\omega| \le \pi \\ 0, & \omega = 0. \end{cases} \tag{13.72}$$

It also follows that

$$X_{\mathrm{r}}(\omega) = \begin{cases} X(0), & \omega = 0 \\ \mathsf{H}^{-1}(\omega)X_{\mathrm{i}}(\omega), & 0 < |\omega| \le \pi, \end{cases} \tag{13.73}$$

or

$$X_{\mathrm{r}}(\omega) = \mathsf{H}_{\mathrm{I}}(\omega)X_{\mathrm{i}}(\omega), \quad 0 < |\omega| \le \pi, \tag{13.74}$$

where $\mathsf{H}_{\mathrm{I}}(\omega)$, the inverse of $\mathsf{H}(\omega)$, is given by

$$\mathsf{H}_{\mathrm{I}}(\omega) = \begin{cases} \mathrm{i}, & \text{for } 0 < \omega < \pi \\ -\mathrm{i}, & \text{for } -\pi \le \omega < 0 \end{cases} \equiv \mathrm{i}\,\mathrm{sgn}\,\omega. \tag{13.75}$$

Consider the following sequence:

$$h[k] = \begin{cases} 0, & k = 0 \\ \dfrac{2\sin^2(\pi k/2)}{\pi k}, & k \neq 0; \end{cases} \tag{13.76}$$

then

$$\mathsf{H}(\omega) = \sum_{k=-\infty}^{\infty} h[k]\mathrm{e}^{-\mathrm{i}\omega k} = -\mathrm{i}\,\mathrm{sgn}\,\omega. \tag{13.77}$$

Using this result, together with the following discrete Fourier transforms:

$$X_{\mathrm{i}}(\omega) = \sum_{k=-\infty}^{\infty} x_{\mathrm{i}}[k]\mathrm{e}^{-\mathrm{i}\omega k} \tag{13.78}$$

and

$$X_{\mathrm{r}}(\omega) = \sum_{k=-\infty}^{\infty} x_{\mathrm{r}}[k]\mathrm{e}^{-\mathrm{i}\omega k}, \tag{13.79}$$

Eq. (13.72) can be written as

$$\sum_{k=-\infty}^{\infty} x_i[k]\mathrm{e}^{-\mathrm{i}\omega k} = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h[m]x_r[k]\mathrm{e}^{-\mathrm{i}\omega(m+k)}. \tag{13.80}$$

Multiplying the preceding formula by $\mathrm{e}^{\mathrm{i}\omega n}$ and integrating over a period leads to

$$x_{\mathrm{i}}[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x_{\mathrm{r}}[m] = \sum_{m=-\infty}^{\infty} h[m]\,x_{\mathrm{r}}[n-m]. \tag{13.81}$$

In a similar manner, let

$$h_{\mathrm{I}}[k] = \begin{cases} X(0), & k = 0 \\ -\dfrac{2\sin^2(\pi k/2)}{\pi k}, & k \neq 0, \end{cases} \tag{13.82}$$

then

$$H_{\mathrm{I}}(\omega) = \sum_{k=-\infty}^{\infty} h_{\mathrm{I}}[k]\mathrm{e}^{-\mathrm{i}\omega k} = \begin{cases} X(0), & \omega = 0 \\ \mathrm{i}\,\mathrm{sgn}\,\omega, & 0 < |\omega| < \pi. \end{cases} \tag{13.83}$$

From Eq. (13.73) it follows that

$$x_{\mathrm{r}}[n] = \sum_{m=-\infty}^{\infty} h_{\mathrm{I}}[n-m]\, x_{\mathrm{i}}[m] = \sum_{m=-\infty}^{\infty} h_{\mathrm{I}}[m]\, x_{\mathrm{i}}[n-m]. \tag{13.84}$$

This expression can be rewritten as follows:

$$x_r[n] = h_{\mathrm{I}}[0] - \sum_{\substack{m=-\infty \\ (m\neq 0)}}^{\infty} h[m]\, x_{\mathrm{i}}[n-m]. \tag{13.85}$$

In signal processing, the pair Eq. (13.81) and Eq. (13.85) are referred to as Hilbert transform relations for the discrete-time signal.

13.9 *Z* transform of a causal sequence

The connection between the real and imaginary parts of the Z transform of a causal sequence is determined in this section. Let $x[n]$ denote a causal sequence, so that

$$x[n] = 0, \quad \text{for } n < 0. \tag{13.86}$$

Let $X(z)$ be the Z transform of this sequence, and suppose this function is analytic outside the unit circle. From $x[n]$, the even and odd sequences $x_{\mathrm{e}}[n]$ and $x_{\mathrm{o}}[n]$ can be constructed using

$$x_{\mathrm{e}}[n] = \frac{1}{2}\{x[n] + x[-n]\} \tag{13.87}$$

and

$$x_{\mathrm{o}}[n] = \frac{1}{2}\{x[n] - x[-n]\}; \tag{13.88}$$

then the sequence $x[n]$ can be expressed as follows:

$$x[n] = 2x_{\mathrm{e}}[n]u[n] - x_{\mathrm{e}}[0]\delta[n], \tag{13.89}$$

or

$$x[n] = 2x_{\mathrm{o}}[n]u[n] + x[0]\delta[n], \tag{13.90}$$

where $u[n]$ is the unit step sequence defined by

$$u[n] = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0, \end{cases} \tag{13.91}$$

and this is displayed in Figure 13.9.

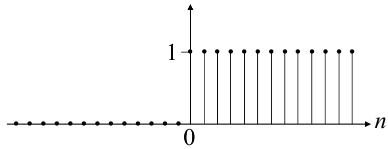

Figure 13.3. Unit step sequence.

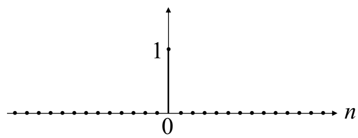

Figure 13.4. Unit sample sequence.

The unit sample sequence, also sometimes termed the unit impulse, $\delta[n]$, is given by

$$\delta[n] = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0, \end{cases} \tag{13.92}$$

and this is illustrated in Figure 13.4.

If the Z transform of Eq. (13.89) is taken, then

$$X(z) = \sum_{n=-\infty}^{\infty} \{2x_e[n]u[n] - x[0]\delta[n]\}z^{-n}. \tag{13.93}$$

This is of the form of a Z transform of a product. The Z transform of $u[n]$ is given by

$$\begin{aligned} Z\{u[n]\} \equiv X_u(z) &= \sum_{n=-\infty}^{\infty} u[n]z^{-n} \\ &= \sum_{n=0}^{\infty} z^{-n} \\ &= \frac{z}{z-1}. \end{aligned} \tag{13.94}$$

In what follows, attention is focused on the evaluation of Z transform on the unit circle, which can be implemented by writing $z = e^{i\phi}$. On the unit circle, $X(z)$ can be decomposed into its real and imaginary parts:

$$X(e^{i\phi}) = X_r(e^{i\phi}) + iX_i(e^{i\phi}), \tag{13.95}$$

and the following relationships hold:

$$X_{\mathrm{r}}(\mathrm{e}^{\mathrm{i}\phi}) = \sum_{n=-\infty}^{\infty} x_{\mathrm{e}}[n] z^{-n} \bigg|_{z=\mathrm{e}^{\mathrm{i}\phi}} \tag{13.96}$$

and

$$\mathrm{i}X_{\mathrm{i}}(\mathrm{e}^{\mathrm{i}\phi}) = \sum_{n=-\infty}^{\infty} x_{\mathrm{o}}[n] z^{-n} \bigg|_{z=\mathrm{e}^{\mathrm{i}\phi}}. \tag{13.97}$$

The preceding associations can be seen by writing

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x_{\mathrm{e}}[n] z^{-n} \bigg|_{z=\mathrm{e}^{\mathrm{i}\phi}} &= x_{\mathrm{e}}[0] + \sum_{n=-\infty}^{-1} x_{\mathrm{e}}[n] \mathrm{e}^{-\mathrm{i}n\phi} + \sum_{n=1}^{\infty} x_{\mathrm{e}}[n] \mathrm{e}^{-\mathrm{i}n\phi} \\
&= x_{\mathrm{e}}[0] + \sum_{n=1}^{\infty} \{x_{\mathrm{e}}[-n] \mathrm{e}^{\mathrm{i}n\phi} + x_{\mathrm{e}}[n] \mathrm{e}^{-\mathrm{i}n\phi}\} \\
&= x_{\mathrm{e}}[0] + 2\sum_{n=1}^{\infty} x_{\mathrm{e}}[n] \cos n\phi
\end{aligned} \tag{13.98}$$

and

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x_{\mathrm{o}}[n] z^{-n} \bigg|_{z=\mathrm{e}^{\mathrm{i}\phi}} &= \sum_{n=-\infty}^{-1} x_{\mathrm{o}}[n] \mathrm{e}^{-\mathrm{i}n\phi} + \sum_{n=1}^{\infty} x_{\mathrm{o}}[n] \mathrm{e}^{-\mathrm{i}n\phi} \\
&= \sum_{n=1}^{\infty} \{x_{\mathrm{o}}[n] \mathrm{e}^{-\mathrm{i}n\phi} + x_{\mathrm{o}}[-n] \mathrm{e}^{\mathrm{i}n\phi}\} \\
&= -2\mathrm{i}\sum_{n=1}^{\infty} x_{\mathrm{o}}[n] \sin n\phi.
\end{aligned} \tag{13.99}$$

Using the result for the Z transform of a product of sequences given in Section 13.7, it follows, from Eq. (13.93) and using $z = \mathrm{e}^{\mathrm{i}\phi}$, that

$$X(\mathrm{e}^{\mathrm{i}\phi}) = \frac{1}{2\pi\mathrm{i}} \oint_C 2X_{\mathrm{r}}(w) X_u\left(\frac{\mathrm{e}^{\mathrm{i}\phi}}{w}\right) w^{-1}\,\mathrm{d}w - x[0]. \tag{13.100}$$

In deriving the product formula given in Eq. (13.59), it was assumed that the singular points of $X_1(w)$ are enclosed by the contour and that the singular points of $X_2(zw^{-1})$ lie outside the contour. Let $w = r\mathrm{e}^{\mathrm{i}\theta}$, and take the limit $r \to 1$; then, using

Eq. (13.94), it follows that

$$X(e^{i\phi}) = \frac{1}{\pi i} P \int_0^{2\pi} X_r(e^{i\theta}) \frac{e^{i\phi}}{e^{i\phi} - e^{i\theta}} i\, d\theta + \frac{1}{\pi i} \int_{2\pi}^{\pi} X_r(w) \frac{z w^{-1} dw}{z - w} - x[0]. \tag{13.101}$$

Since there is a singularity on the unit disc for $w = z$, the circular contour is modified to include a semicircular indentation around the point $w = z$, so that the singularity is located outside the contour. The contour employed is shown in Figure 3.8. The second integral on the right-hand side of Eq. (13.101) arises from this modification of the circular contour. The Cauchy principal value appears because of the singular structure of the integrand. To simplify the second integral, set $w - z = \varepsilon e^{i\theta}$ and let $\varepsilon \to 0$; hence,

$$\frac{1}{\pi i} \int_{2\pi}^{\pi} X_r(w) \frac{z}{z - w} \frac{dw}{w} = X_r(z) = X_r(e^{i\phi}). \tag{13.102}$$

The factor $e^{i\phi}(e^{i\phi} - e^{i\theta})^{-1}$ can be expressed as follows:

$$\frac{e^{i\phi}}{e^{i\phi} - e^{i\theta}} = \frac{1}{2} - \frac{i}{2} \cot\left(\frac{\phi - \theta}{2}\right), \tag{13.103}$$

and therefore Eq. (13.101) can be written as

$$\begin{aligned} X(e^{i\phi}) &= X_r(e^{i\phi}) + \frac{1}{2\pi} \int_0^{2\pi} X_r(e^{i\theta}) d\theta - \frac{i}{2\pi} \int_0^{2\pi} X_r(e^{i\theta}) \cot\left(\frac{\phi - \theta}{2}\right) d\theta - x[0] \\ &= X_r(e^{i\phi}) - \frac{i}{2\pi} \int_0^{2\pi} X_r(e^{i\theta}) \cot\left(\frac{\phi - \theta}{2}\right) d\theta. \end{aligned} \tag{13.104}$$

Employing Eq. (13.95) leads to

$$X_i(e^{i\phi}) = -\frac{1}{2\pi} P \int_0^{2\pi} X_r(e^{i\theta}) \cot\left(\frac{\phi - \theta}{2}\right) d\theta. \tag{13.105}$$

An expression for $X_r(e^{i\phi})$ can be found in an analogous manner starting from Eq. (13.90). Taking the Z transform of Eq. (13.90) yields

$$X(z) = \sum_{n=-\infty}^{\infty} \{2x_o[n]u[n] + x[0]\delta[n]\} z^{-n}. \tag{13.106}$$

It follows that

$$\begin{aligned}
X(\mathrm{e}^{\mathrm{i}\phi}) &= \frac{1}{2\pi\mathrm{i}}P\int_0^{2\pi} 2\mathrm{i}X_\mathrm{i}(\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{e}^{\mathrm{i}\phi}}{\mathrm{e}^{\mathrm{i}\phi}-\mathrm{e}^{\mathrm{i}\theta}}\mathrm{i}\mathrm{d}\theta + \frac{1}{2\pi\mathrm{i}}\int_{2\pi}^{\pi} 2\mathrm{i}X_\mathrm{i}(w)\frac{z}{z-w}\frac{\mathrm{d}w}{w} + x[0] \\
&= \frac{\mathrm{i}}{2\pi}P\int_0^{2\pi} X_\mathrm{i}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta + \frac{1}{2\pi}P\int_0^{2\pi} X_\mathrm{i}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\cot\left(\frac{\phi-\theta}{2}\right)\mathrm{d}\theta \\
&\quad + \mathrm{i}X_\mathrm{i}(\mathrm{e}^{\mathrm{i}\phi}) + x[0] \\
&= \mathrm{i}x_\mathrm{o}[0] + \frac{1}{2\pi}P\int_0^{2\pi} X_\mathrm{i}(\mathrm{e}^{\mathrm{i}\theta})\cot\left(\frac{\phi-\theta}{2}\right)\mathrm{d}\theta + \mathrm{i}X_\mathrm{i}(\mathrm{e}^{\mathrm{i}\phi}) + x[0].
\end{aligned} \tag{13.107}$$

Noting the fact that $x_\mathrm{o}[0] = 0$ and using Eq. (13.95) leads to

$$X_\mathrm{r}(\mathrm{e}^{\mathrm{i}\phi}) = x[0] + \frac{1}{2\pi}P\int_0^{2\pi} X_\mathrm{i}(\mathrm{e}^{\mathrm{i}\theta})\cot\left(\frac{\phi-\theta}{2}\right)\mathrm{d}\theta. \tag{13.108}$$

Equations (13.105) and (13.108) are referred to as the discrete Hilbert transform relationships. It is clear from these formulas that knowledge of $X_\mathrm{r}(\mathrm{e}^{\mathrm{i}\phi})$ can be used to determine $X_\mathrm{i}(\mathrm{e}^{\mathrm{i}\phi})$, from which the original signal $x[n]$ can be constructed. However, knowledge of $X_\mathrm{i}(\mathrm{e}^{\mathrm{i}\phi})$ must be supplemented by the value of $x[0]$ in order to determine $X_\mathrm{r}(\mathrm{e}^{\mathrm{i}\phi})$.

13.10 Fourier transform of a causal sequence

In this section a Fourier transform approach is employed to obtain the connections between the real and imaginary parts of a causal stable real sequence $x[n]$. A few preliminary results are now collected together. The Fourier transform of the sequence $x[n]$ is given by

$$X(\mathrm{e}^{\mathrm{i}\omega}) = \sum_{k=-\infty}^{\infty} x[k]\mathrm{e}^{-\mathrm{i}\omega k}, \tag{13.109}$$

and $X(\mathrm{e}^{\mathrm{i}\omega})$ is sometimes referred to as the discrete time Fourier transform. The inverse Fourier transform relationship is given by

$$x[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{i}\omega})\mathrm{e}^{\mathrm{i}\omega k}\,\mathrm{d}\omega. \tag{13.110}$$

If $x[n]$ and $y[n]$ are two discrete sequences, and the Fourier transforms are given by, $X(\mathrm{e}^{\mathrm{i}\omega})$ and $Y(\mathrm{e}^{\mathrm{i}\omega})$, respectively, then the Fourier transform of the product $x[n]y[n]$ takes the form

$$\mathcal{F}\{x[n]y[n]\}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{i}w})Y(\mathrm{e}^{\mathrm{i}(\omega-w)})\mathrm{d}w. \tag{13.111}$$

This result can be established as follows. Making use of Eq. (13.109), it follows that

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{i}w})Y(\mathrm{e}^{\mathrm{i}(\omega-w)})\mathrm{d}w &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} x[k]\mathrm{e}^{-\mathrm{i}wk}\sum_{m=-\infty}^{\infty} y[m]\mathrm{e}^{-\mathrm{i}(\omega-w)m}\,\mathrm{d}w \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x[k]\sum_{m=-\infty}^{\infty} y[m]\mathrm{e}^{-\mathrm{i}\omega m}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(m-k)w}\,\mathrm{d}w \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x[k]\sum_{m=-\infty}^{\infty} y[m]\mathrm{e}^{-\mathrm{i}\omega m}2\pi\,\delta_{mk} \\
&= \sum_{k=-\infty}^{\infty} x[k]y[k]\mathrm{e}^{-\mathrm{i}\omega k} \\
&= \mathcal{F}\{x[k]y[k]\}(\omega).
\end{aligned}
\tag{13.112}
$$

The Fourier transform of the unit step sequence is given by

$$
U(\mathrm{e}^{\mathrm{i}\omega}) = \frac{1}{1-\mathrm{e}^{-\mathrm{i}\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k). \tag{13.113}
$$

The first factor on the right-hand side of this result can be expressed as follows:

$$
\frac{1}{1-\mathrm{e}^{-\mathrm{i}\omega}} = \frac{1}{2} - \frac{\mathrm{i}}{2}\cot\left(\frac{\omega}{2}\right). \tag{13.114}
$$

To see that $u[n]$ can be recovered from Eq. (13.113), the following approach can be applied:

$$
\begin{aligned}
u[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\mathrm{e}^{\mathrm{i}\omega})\mathrm{e}^{\mathrm{i}\omega n}\,\mathrm{d}\omega \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}\omega n}\mathrm{d}\omega - \frac{\mathrm{i}}{4\pi}P\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}\omega n}\cot\left(\frac{\omega}{2}\right)\mathrm{d}\omega \\
&\quad + \frac{1}{2}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)\mathrm{e}^{\mathrm{i}\omega n}\,\mathrm{d}\omega \\
&= \frac{1}{2}\delta[n] + \frac{\operatorname{sgn} n}{2} + \frac{1}{2}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}\delta(\omega-2\pi k)\mathrm{e}^{\mathrm{i}\omega n}\,\mathrm{d}\omega,
\end{aligned}
\tag{13.115}
$$

where Eq. (6.102) has been employed, and the definition of the sgn function used has $\operatorname{sgn} 0 = 0$. The final integral can be evaluated as follows:

$$\begin{aligned}
\frac{1}{2}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}\delta(\omega-2\pi k)\mathrm{e}^{\mathrm{i}\omega n}\,\mathrm{d}\omega &= \frac{1}{2}\sum_{k=-\infty}^{\infty}\int_{-\pi-2\pi k}^{\pi-2\pi k}\delta(w)\mathrm{e}^{\mathrm{i}(w+2\pi k)n}\,\mathrm{d}w \\
&= \frac{1}{2}\left\{\cdots+\int_{-3\pi}^{-\pi}\delta(w)\mathrm{e}^{\mathrm{i}(w+2\pi)n}\,\mathrm{d}w\right. \\
&\quad+\int_{-\pi}^{\pi}\delta(w)\mathrm{e}^{\mathrm{i}wn}\,\mathrm{d}w \\
&\quad\left.+\int_{\pi}^{3\pi}\delta(w)\mathrm{e}^{\mathrm{i}(w-2\pi)n}\,\mathrm{d}w+\cdots\right\} \\
&= \frac{1}{2}\int_{-\pi}^{\pi}\delta(w)\mathrm{e}^{\mathrm{i}wn}\,\mathrm{d}w \\
&= \frac{1}{2},
\end{aligned} \tag{13.116}$$

and hence

$$u[n] = \frac{1}{2}\{1+\operatorname{sgn} n+\delta[n]\} = \begin{cases}1, & n \geq 1\\ 1, & n = 0\\ 0, & n < 0,\end{cases} \tag{13.117}$$

which is the required result.

From Eqs. (13.89) and (13.111), it follows that

$$X(\mathrm{e}^{\mathrm{i}\omega}) = \frac{1}{\pi}P\int_{-\pi}^{\pi}X_{\mathrm{r}}(\mathrm{e}^{\mathrm{i}\theta})U(\mathrm{e}^{\mathrm{i}(\omega-w)})\mathrm{d}w - x[0], \tag{13.118}$$

where $X_{\mathrm{r}}(\mathrm{e}^{\mathrm{i}\omega})$ is the Fourier transform of $x_{\mathrm{e}}[n]$, $U(\mathrm{e}^{\mathrm{i}\omega})$ designates the Fourier transform of $u[n]$, and, in the sequel, $\mathrm{i}X_{\mathrm{i}}(\mathrm{e}^{\mathrm{i}\omega})$ denotes the Fourier transform of $x_{\mathrm{o}}[n]$. The Fourier transform $X(\mathrm{e}^{\mathrm{i}\omega})$ can be expressed in terms of its real and imaginary parts:

$$X(\mathrm{e}^{\mathrm{i}\omega}) = X_{\mathrm{r}}(\mathrm{e}^{\mathrm{i}\omega}) + \mathrm{i}X_{\mathrm{i}}(\mathrm{e}^{\mathrm{i}\omega}). \tag{13.119}$$

The reason for taking the Cauchy principal value in Eq. (13.118) reflects the singular nature of the kernel function $U(\mathrm{e}^{\mathrm{i}(\omega-w)})$. Inserting Eq. (13.113) into Eq. (13.118), and making use of Eqs. (13.114), (13.110), and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}X_{\mathrm{r}}(\mathrm{e}^{\mathrm{i}w})\mathrm{d}w = x[0], \tag{13.120}$$

yields

$$X(e^{i\omega}) = -\frac{i}{2\pi} P \int_{-\pi}^{\pi} X_r(e^{iw}) \cot\left(\frac{\omega - w}{2}\right) dw + \int_{-\pi}^{\pi} X_r(e^{iw}) \sum_{k=-\infty}^{\infty} \delta(\omega - w - 2\pi k) dw. \tag{13.121}$$

The preceding integral can be written as follows:

$$\begin{aligned}
\int_{-\pi}^{\pi} X_r(e^{iw}) \sum_{k=-\infty}^{\infty} \delta(\omega - w - 2\pi k) dw &= \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi}^{(2k+1)\pi} X_r(e^{i(w-2\pi k)}) \delta(\omega - w) dw \\
&= \cdots + \int_{-3\pi}^{-\pi} X_r(e^{i(w+2\pi)}) \delta(\omega - w) dw \\
&\quad + \int_{-\pi}^{\pi} X_r(e^{iw}) \delta(\omega - w) dw \\
&\quad + \int_{\pi}^{3\pi} X_r(e^{i(w-2\pi)}) \delta(\omega - w) dw + \cdots \\
&= X_r(e^{i\omega}),
\end{aligned} \tag{13.122}$$

and hence Eq. (13.121) becomes

$$X(e^{i\omega}) = X_r(e^{i\omega}) - \frac{i}{2\pi} P \int_{-\pi}^{\pi} X_r(e^{iw}) \cot\left(\frac{\omega - w}{2}\right) dw. \tag{13.123}$$

Employing Eq. (13.119) yields

$$X_i(e^{i\omega}) = -\frac{1}{2\pi} P \int_{-\pi}^{\pi} X_r(e^{iw}) \cot\left(\frac{\omega - w}{2}\right) dw. \tag{13.124}$$

In a similar fashion, starting from Eq. (13.90), it follows that

$$\begin{aligned}
X(e^{i\omega}) &= \frac{1}{\pi} P \int_{-\pi}^{\pi} iX_i(e^{iw}) U(e^{i(\omega-w)}) d\theta + x[0] \\
&= \frac{i}{2\pi} \int_{-\pi}^{\pi} X_i(e^{iw}) dw + \frac{1}{2\pi} P \int_{-\pi}^{\pi} X_i(e^{iw}) \cot\left(\frac{\omega - w}{2}\right) dw \\
&\quad + i \int_{-\pi}^{\pi} X_i(e^{iw}) \sum_{k=-\infty}^{\infty} \delta(\omega - w - 2\pi k) dw + x[0] \\
&= x[0] + \frac{1}{2\pi} P \int_{-\pi}^{\pi} X_i(e^{iw}) \cot\left(\frac{\omega - w}{2}\right) dw + iX_i(e^{i\omega}).
\end{aligned} \tag{13.125}$$

Inserting Eq. (13.119) yields

$$X_\mathrm{r}(\mathrm{e}^{\mathrm{i}\omega}) = x[0] + \frac{1}{2\pi} P \int_{-\pi}^{\pi} X_\mathrm{i}(\mathrm{e}^{\mathrm{i}w}) \cot\left(\frac{\omega - w}{2}\right) \mathrm{d}w. \tag{13.126}$$

In the engineering literature, Eqs. (13.124) and (13.126) are termed the "discrete Hilbert transform relations." From Eq. (13.123), it is clear that $X_\mathrm{r}(\mathrm{e}^{\mathrm{i}\omega})$ completely determines $X(\mathrm{e}^{\mathrm{i}\omega})$. However, to determine $X(\mathrm{e}^{\mathrm{i}\omega})$ from $X_\mathrm{i}(\mathrm{e}^{\mathrm{i}\omega})$ requires the value of $x[0]$. The reader should note the obvious connection with the key results of the preceding section.

13.11 The discrete Hilbert transform in analysis

This section and the reminder of the chapter examines the discrete Hilbert transform as it is discussed in harmonic analysis and related branches of mathematics. The notation $f = \{f[n]\}$ is employed to denote a discrete sequence. The discrete Hilbert transform is defined in the following way:

$$H_\mathrm{D} f[n] = \frac{1}{\pi} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{f[m]}{n - m}. \tag{13.127}$$

While it is advantageous to keep superscript and subscript labels on symbols to the absolute minimum, a subscript D has been introduced to distinguish the discrete Hilbert transform from the Hilbert transform for continuous functions. Other writers omit this subscript. With the preceding definition, the analogy with the standard definition of the Hilbert transform on $\mathbb{R}$ should be apparent. The qualifier $m \neq n$ on the summation symbol in a sense plays a role analogous to that of the P in the integral transform formula. The discrete Hilbert transform H_D is also written using the following alternative notation

$$H_\mathrm{D} f[n] = \frac{1}{\pi} \sum_{m=-\infty}^{\infty}{}' \frac{f[m]}{n - m}, \tag{13.128}$$

where the prime signifies the term $m = n$ is omitted. In harmonic analysis the factor of π^{-1} is often not incorporated in the definition of the discrete Hilbert transform.

Let X denote the infinite sequence $\{\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots\}$, where the members are real or complex numbers. For $p > 0$, the pth norm of X is denoted as $\|X\|_p$ and defined by

$$\|X\|_p = \left(\sum_{n=-\infty}^{\infty} |x_n|^p \right)^{1/p}. \tag{13.129}$$

The class of sequences for which each $\|X\|_p$ is finite is denoted by l^p.

If $\{f[n]\}$ is an odd sequence, then, with $f[0] = 0$,

$$H_D f[n] = \frac{2}{\pi} \sum_{m=1}^{\infty}{}' \frac{mf[m]}{n^2 - m^2}, \tag{13.130}$$

that is, $H_D f[n]$ is an even sequence. If $\{f[n]\}$ is an even sequence, then

$$H_D f[n] = \frac{2n}{\pi} \sum_{m=1}^{\infty}{}' \frac{f[m]}{n^2 - m^2}, \tag{13.131}$$

and hence $H_D f[n]$ is an odd sequence for $n \neq 0$.

There are other variants of the discrete Hilbert transform, the most important of which are discussed in Section 13.13. One extension takes the following form (Hardy *et al.*, 1952, p. 222):

$$H_D f(n, \lambda) = \frac{1}{\pi} \sum_{m=-\infty}^{\infty}{}' \frac{f[m]}{n - m + \lambda}, \tag{13.132}$$

where the prime qualifier is required only for integer λ.

A semi-discrete Hilbert transform (also termed the mixed Hilbert transform) can be defined by

$$\{H_{sD} f\}(x) = \frac{1}{\pi} \sum_{\substack{m=-\infty \\ m \neq m(x)}}^{\infty} \frac{f[m]}{x - m}, \quad \text{for } x \in \mathbb{R}, \tag{13.133}$$

where $m(x)$ is an element of $\mathbb{Z}$ such that $m(x) - 1/2 \leq x < m(x) + 1/2$. It can be proved that the semi-discrete Hilbert transform is a bounded linear transform from $l^p(\mathbb{Z})$ into $L^p(\mathbb{R})$ for $1 < p < \infty$.

13.12 Hilbert's inequality

Hilbert's inequality for discrete sequences is given by

$$\|H_D f\|_2 \leq C \|f\|_2, \tag{13.134}$$

where C is a positive constant. This result appeared first in Hilbert's lectures, and was later published in 1908 by Hermann Weyl in his doctoral thesis. Riesz (1927) established the following more general result:

$$\|H_D f\|_p \leq C_p \|f\|_p, \quad \text{for } p > 1. \tag{13.135}$$

This formula is the discrete analog of the Riesz inequality for the Hilbert transform on $\mathbb{R}$. The optimal constant in Eq. (13.134) is $C = 1$ (this would be $C = \pi$ if H_D were defined without the factor of π^{-1}), and for Eq. (13.135) the optimal constant C_p appears not to be known.

A derivation of Eq. (13.134) based on elementary methods is examined, using an approach due to Grafakos (1994). The first step is the evaluation of the sum $\sum_{\substack{j=-\infty \\ (j\neq m,n)}}^{\infty} 1/[(j-m)(j-n)]$, which can be treated by selecting a J such that $|m| < J$ and $|n| < J$; then,

$$\begin{aligned}
\sum_{j=-\infty}^{\infty}{}' \frac{1}{(j-m)(j-n)} &= \frac{1}{(m-n)} \lim_{J\to\infty} \sum_{\substack{j=-J \\ (j\neq m,n)}}^{J} \left\{ \frac{1}{j-m} - \frac{1}{j-n} \right\} \\
&= \frac{1}{(m-n)} \lim_{J\to\infty} \left\{ \sum_{\substack{j=-J \\ (j\neq m)}}^{J} \frac{1}{j-m} - \frac{1}{n-m} \right. \\
&\qquad \left. - \left(\sum_{\substack{j=-J \\ (j\neq n)}}^{J} \frac{1}{j-n} - \frac{1}{m-n} \right) \right\} \\
&= \frac{2}{(m-n)^2} + \frac{1}{(m-n)} \lim_{J\to\infty} \left\{ \sum_{j=J-n+1}^{J+n} \frac{1}{j} - \sum_{j=J-m+1}^{J+m} \frac{1}{j} \right\} \\
&= \frac{2}{(m-n)^2}.
\end{aligned} \tag{13.136}$$

The second result required is as follows:

$$\sum_{j=-\infty}^{\infty}{}' \frac{1}{(j-n)^2} = 2\sum_{j=1}^{\infty} \frac{1}{j^2} = 2\varsigma(2) = \frac{\pi^2}{3}, \tag{13.137}$$

where $\varsigma(n)$ denotes the Riemann zeta function, which (the reader will recall from Section 2.16), is defined for integer argument by

$$\varsigma(n) = \sum_{j=1}^{\infty} \frac{1}{j^n}, \quad \text{for } n = 2, 3, \ldots \tag{13.138}$$

The factor $\|H_{\mathrm{D}}f\|_2^2$ can be simplified as follows:

$$\begin{aligned}
\|H_{\mathrm{D}}f\|_2^2 &= \frac{1}{\pi^2} \sum_{j=-\infty}^{\infty} \left| \sum_{\substack{m=-\infty \\ (m\neq j)}}^{\infty} \frac{f[m]}{j-m} \right|^2 \\
&= \frac{1}{\pi^2} \sum_{j=-\infty}^{\infty} \sum_{\substack{m=-\infty \\ (m\neq j)}}^{\infty} \frac{f[m]}{j-m} \sum_{\substack{n=-\infty \\ (n\neq j)}}^{\infty} \frac{f[n]}{j-n}
\end{aligned}$$

$$= \frac{1}{\pi^2} \sum_{j=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ (n \neq j)}}^{\infty} \frac{f[n]^2}{(j-n)^2} + \frac{1}{\pi^2} \sum_{j=-\infty}^{\infty} \sum_{\substack{m=-\infty \\ (m \neq j)}}^{\infty} \frac{f[m]}{(j-m)} \sum_{\substack{n=-\infty \\ (n \neq j) \\ (n \neq m)}}^{\infty} \frac{f[n]}{(j-n)}$$

$$= \frac{1}{\pi^2} \sum_{n=-\infty}^{\infty} f[n]^2 \sum_{\substack{j=-\infty \\ (j \neq n)}}^{\infty} \frac{1}{(j-n)^2} + \frac{1}{\pi^2} \sum_{m=-\infty}^{\infty} f[m]$$

$$\times \sum_{\substack{n=-\infty \\ (n \neq m)}}^{\infty} f[n] \sum_{\substack{j=-\infty \\ (j \neq m, n)}}^{\infty} \frac{1}{(j-m)(j-n)}$$

$$= \frac{1}{3} \sum_{n=-\infty}^{\infty} f[n]^2 + \frac{2}{\pi^2} \sum_{m=-\infty}^{\infty} f[m] \sum_{\substack{n=-\infty \\ (n \neq m)}}^{\infty} \frac{f[n]}{(m-n)^2}. \tag{13.139}$$

From the elementary inequality $(a-b)^2 \geq 0$, it follows that

$$2f[m]f[n] \leq f[m]^2 + f[n]^2, \tag{13.140}$$

and hence Eq. (13.139) can be recast as follows:

$$\left\| H_{\mathrm{D}} f \right\|_2^2 \leq \frac{1}{3} \sum_{n=-\infty}^{\infty} f[n]^2 + \frac{1}{\pi^2} \sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ (n \neq m)}}^{\infty} \frac{(f[m]^2 + f[n]^2)}{(m-n)^2}$$

$$= \frac{1}{3} \sum_{n=-\infty}^{\infty} f[n]^2 + \frac{1}{\pi^2} \sum_{m=-\infty}^{\infty} f[m]^2 \sum_{\substack{n=-\infty \\ (n \neq m)}}^{\infty} \frac{1}{(m-n)^2} + \frac{1}{\pi^2} \sum_{n=-\infty}^{\infty} f[n]^2$$

$$\times \sum_{\substack{m=-\infty \\ (m \neq n)}}^{\infty} \frac{1}{(m-n)^2}$$

$$= \frac{1}{3} \sum_{n=-\infty}^{\infty} f[n]^2 + \frac{2}{3} \sum_{m=-\infty}^{\infty} f[m]^2. \tag{13.141}$$

Therefore,

$$\left\| H_{\mathrm{D}} f \right\|_2^2 \leq \sum_{n=-\infty}^{\infty} f[n]^2, \tag{13.142}$$

and Eq. (13.134) is established and a value for the constant C is determined.

To determine the best constant in Eq. (13.134), the approach of Grafakos (1994) is employed. Start with the following ratio:

$$\alpha_N = \frac{\dfrac{2}{\pi^2}\displaystyle\sum_{n=-N}^{N} f[n] \sum_{\substack{m=-N\\(m\neq n)}}^{N} \frac{f[m]}{(m-n)^2}}{\displaystyle\sum_{n=-N}^{N} f[n]^2}, \tag{13.143}$$

and employ the sequence

$$f[n] = \begin{cases} 1, & \text{for } |n| \leq N \\ 0, & \text{for } |n| > N. \end{cases} \tag{13.144}$$

Equation (13.143) can be rewritten as follows:

$$\alpha_N = \frac{4}{\pi^2(2N+1)} \sum_{n=1}^{N} \left\{ \frac{2}{n^2} + \sum_{\substack{m=1\\(m\neq n)}}^{N} \frac{1}{(m-n)^2} + \sum_{m=1}^{N} \frac{1}{(m+n)^2} \right\}. \tag{13.145}$$

Making use of the following results:

$$\sum_{n=1}^{N}\sum_{m=1}^{N} \frac{1}{(m+n)^2} = N\sum_{n=1}^{N}\frac{1}{n^2} + \sum_{n=1}^{N}\sum_{m=N+1}^{N+n}\frac{1}{m^2} - \sum_{n=1}^{N}\sum_{m=1}^{n}\frac{1}{m^2} \tag{13.146}$$

and

$$\sum_{n=1}^{N}\sum_{\substack{m=1\\(m\neq n)}}^{N} \frac{1}{(m-n)^2} = \sum_{n=1}^{N-1}\frac{1}{n^2} + \sum_{n=2}^{N}\sum_{m=1}^{N-n}\frac{1}{m^2} + \sum_{n=2}^{N}\sum_{m=1}^{n-1}\frac{1}{m^2}, \tag{13.147}$$

yields

$$\alpha_N = \frac{4}{\pi^2}\sum_{n=1}^{N}\frac{1}{n^2} + \frac{4}{\pi^2(2N+1)}\sum_{n=1}^{N}\left\{ \sum_{m=N+1}^{N+n}\frac{1}{m^2} - \sum_{m=N+1-n}^{N}\frac{1}{m^2} \right\}. \tag{13.148}$$

The preceding double sum can be written as follows:

$$\begin{aligned}
&\frac{1}{(2N+1)}\sum_{n=1}^{N}\left\{ \sum_{m=N+1}^{N+n}\frac{1}{m^2} - \sum_{m=N+1-n}^{N}\frac{1}{m^2} \right\} \\
&\quad = \frac{1}{(2N+1)}\left\{ N\left\{\frac{1}{(N+1)^2} - \frac{1}{N^2}\right\} + (N-1)\left\{\frac{1}{(N+2)^2} - \frac{1}{(N-1)^2}\right\} \right. \\
&\qquad \left. + (N-2)\left\{\frac{1}{(N+3)^2} - \frac{1}{(N-2)^2}\right\} + \cdots \right\}
\end{aligned}$$

$$= -\left\{ \frac{1}{(N+1)^2 N} + \frac{3}{(N+2)^2(N-1)} + \frac{5}{(N+3)^2(N-2)} + \cdots + \frac{(2N-3)}{(2N-1)^2 2} + \frac{(2N-1)}{(2N)^2} \right\}. \tag{13.149}$$

On taking the limit $N \to \infty$, the preceding sum vanishes and Eq. (13.148) yields

$$\lim_{N\to\infty} \alpha_N = \frac{2}{3}. \tag{13.150}$$

From Eq. (13.141) the optimal constant is determined as $C = 1$ in Eq. (13.134). A clarifying comment on this approach should be useful to some readers. Assume there is a constant C_1 satisfying $C_1 < C$, so that Eq. (13.134) is satisfied for all sequences with the "improved" constant. Selecting the series given in Eq. (13.144) leads to $C_1 = 1$, which contradicts the starting assumption that $C_1 < C$. Hence, the constant determined in the preceding development is optimal.

The result in Eq. (13.134) can be refined by using an extension of the inequality $2ab \leq a^2 + b^2$, for example (Alzer, 1997)

$$2ab \leq a^2 + b^2 - \delta(b-a)^2, \quad \text{for } \delta < 1; \tag{13.151}$$

hence,

$$\frac{1}{\pi^2}\sum_{j=-\infty}^{\infty}\left(\sum_{\substack{m=-\infty\\(m\neq j)}}^{\infty}\frac{f[m]}{j-m}\right)^2 + \frac{\delta}{\pi^2}\sum_{n=-\infty}^{\infty}\sum_{\substack{m=-\infty\\(m\neq n)}}^{\infty}\left(\frac{f[m]-f[n]}{m-n}\right)^2 \leq \sum_{n=-\infty}^{\infty} f[n]^2. \tag{13.152}$$

Let $a_1, a_2, \ldots$ be arbitrary complex numbers. An expression related to Eq. (13.134) of the following form can be obtained:

$$\left|\sum_{m=-\infty}^{\infty} \bar{a}[m] H_{\mathrm{D}} a[m]\right| \leq \sum_{n=-\infty}^{\infty} a[n]^2, \tag{13.153}$$

where the bar denotes the complex conjugate. This result is also referred to as Hilbert's inequality. To prove this result, the approach of Ivić (1985, p. 130) is followed. Let S denote the double series given by

$$S = \sum_{m=-\infty}^{\infty}\sum_{\substack{n=-\infty\\(n\neq m)}}^{\infty}\frac{a[m]\bar{a}[n]}{m-n}. \tag{13.154}$$

It follows immediately that $S = -\bar{S}$, and hence that S/i is real. Let I denote the following integral:

$$I = \int_0^1 \mathrm{d}x \int_0^x \left| \sum_{n=-\infty}^{\infty} a[n] \mathrm{e}^{2\pi \mathrm{i} ny} \right|^2 \mathrm{d}y, \tag{13.155}$$

which obviously satisfies $I \geq 0$. Then it follows that

$$\begin{aligned} I &= \int_0^1 \mathrm{d}x \int_0^x \sum_{m=-\infty}^{\infty} a[m] \mathrm{e}^{2\pi \mathrm{i} my} \sum_{n=-\infty}^{\infty} \bar{a}[n] \mathrm{e}^{-2\pi \mathrm{i} ny} \mathrm{d}y \\ &= \sum_{n=-\infty}^{\infty} |a[n]|^2 \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y + \sum_{m=-\infty}^{\infty} a[m] \sum_{\substack{n=-\infty \\ (n\neq m)}}^{\infty} \bar{a}[n] \int_0^1 \mathrm{d}x \int_0^x \mathrm{e}^{2\pi \mathrm{i}(m-n)y} \mathrm{d}y \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} |a[n]|^2 + \frac{1}{2\pi \mathrm{i}} \sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ (n\neq m)}}^{\infty} \frac{a[m]\bar{a}[n]}{m-n} \int_0^1 \{\mathrm{e}^{2\pi \mathrm{i}(m-n)x} - 1\} \mathrm{d}x \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} |a[n]|^2 - \frac{S}{2\pi \mathrm{i}}, \end{aligned} \tag{13.156}$$

and, since $I \geq 0$,

$$\left| \sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ (n\neq m)}}^{\infty} \frac{a[m]\bar{a}[n]}{m-n} \right| \leq \pi \sum_{n=-\infty}^{\infty} |a[n]|^2, \tag{13.157}$$

and Eq. (13.153) follows on employing the definition of $H_{\mathrm{D}} a[m]$ given in Eq. (13.127).

13.13 Alternative approach to the discrete Hilbert transform

An alternative definition that has been employed for the discrete Hilbert transform is considered in this section. The discrete Hilbert transform of the sequence $f[n]$ is defined by

$$\mathcal{H}_{\mathrm{D}} f[n] = \frac{1}{\pi} \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} \frac{f[m]\{1 - (-1)^{n-m}\}}{n-m}. \tag{13.158}$$

Different notation is employed to distinguish the two forms of the discrete Hilbert transform. The reader needs to be aware that both Eqs. (13.158) and (13.127) are employed by different authors to denote the discrete Hilbert transform. While the two definitions in Eq. (13.158) and Eq. (13.127) are similar, the one given in Eq. (13.158) leads to a number of results that have a very similar structure to the Hilbert transform on $\mathbb{R}$. For this reason, a good case can be made to adopt Eq. (13.158) as the definition of the discrete Hilbert transform, and to regard Eq. (13.127) as defining a related discrete Hilbert-type operation. In particular, note that the analog of the Hilbert transform pair on $\mathbb{R}$

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\, \mathrm{d}t, \tag{13.159}$$

and

$$f(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(t)}{x-t}\, \mathrm{d}t, \tag{13.160}$$

is *not* the pair

$$g[n] = \frac{1}{\pi} \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} \frac{f[m]}{n-m} \tag{13.161}$$

and

$$f[n] = -\frac{1}{\pi} \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} \frac{g[m]}{n-m} \tag{13.162}$$

for the discrete case. For example, if $f[m] = \sin \alpha m$, with the constant α satisfying $0 < \alpha < 2\pi$, and employing the result

$$\sum_{n=1}^{\infty} \frac{\sin \alpha n}{n} = \frac{\pi - \alpha}{2}, \quad \text{for } 0 < \alpha < 2\pi, \tag{13.163}$$

it follows that

$$g[n] = \frac{1}{\pi} \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} \frac{\sin \alpha m}{n-m} = -\frac{\pi - \alpha}{\pi} \cos \alpha n, \tag{13.164}$$

while Eq. (13.162) yields

$$f[n] = \frac{\pi - \alpha}{\pi^2} \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} \frac{\cos \alpha m}{n-m} = \frac{(\pi - \alpha)^2}{\pi^2} \sin \alpha n. \tag{13.165}$$

This example shows that the skew-reciprocal behavior that characterizes the Hilbert transform on $\mathbb{R}$ is not satisfied by Eqs. (13.161) and (13.162).

The motivation for the definition given in Eq. (13.158) is first examined, and then some of the properties satisfied by $\mathcal{H}_D f$ are derived. The standard Fourier transform pair for continuous functions is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{13.166}$$

and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{13.167}$$

which becomes, for the discrete case,

$$F(\omega) = \Delta t \sum_{n=-\infty}^{\infty} f(n\Delta t)\mathrm{e}^{-\mathrm{i}\omega n\Delta t} \tag{13.168}$$

and

$$f(n\Delta t) = \frac{1}{2\pi}\int_{-\pi/\Delta t}^{\pi/\Delta t} F(\omega)\mathrm{e}^{\mathrm{i}\omega n\Delta t}\,\mathrm{d}\omega. \tag{13.169}$$

If the sample interval Δt is fixed as unity, and the function $f[t]$ is sampled at integer values of t, then

$$F(\omega) = \sum_{t=-\infty}^{\infty} f[t]\mathrm{e}^{-\mathrm{i}\omega t} \tag{13.170}$$

and

$$f[t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega. \tag{13.171}$$

The term $\mathcal{H}_D f[t]$ can be evaluated from Eq. (13.171) in the following manner:

$$\begin{aligned}
\mathcal{H}_D f[t] &= \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{t-s}\frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\mathrm{d}\omega\,\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s}{t-s} \\
&= -\frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi} \operatorname{sgn}\omega\;\mathrm{e}^{\mathrm{i}\omega t}F(\omega)\mathrm{d}\omega
\end{aligned}$$

$$= -\frac{\mathrm{i}}{2\pi} \sum_{k=-\infty}^{\infty} f[k] \int_{-\pi}^{\pi} \operatorname{sgn} \omega \, \mathrm{e}^{\mathrm{i}(t-k)\omega} \, \mathrm{d}\omega$$

$$= \frac{1}{\pi} \sum_{\substack{k=-\infty \\ (k \neq t)}}^{\infty} \frac{1 - \cos \pi (t-k)}{t-k} f[k], \tag{13.172}$$

and hence

$$\mathcal{H}_{\mathrm{D}} f[t] = \frac{1}{\pi} \sum_{k=-\infty}^{\infty}{}' f[k] \frac{(1 - (-1)^{t-k})}{t-k}, \tag{13.173}$$

or

$$\mathcal{H}_{\mathrm{D}} f[t] = \frac{1}{\pi} \sum_{k=-\infty}^{\infty}{}' f[t-k] \frac{(1 - (-1)^{k})}{k}. \tag{13.174}$$

The primes on the preceding two summations signify that the term with $k = t$ is omitted for Eq. (13.173) and that the term with $k = 0$ is omitted for Eq. (13.174).

The inversion of Eq. (13.174) is given by

$$f[t] = \mathcal{H}_{\mathrm{D}} g[t] = -\frac{1}{\pi} \sum_{k=-\infty}^{\infty}{}' g[k] \frac{(1 - (-1)^{t-k})}{t-k}, \tag{13.175}$$

where

$$g[t] = \mathcal{H}_{\mathrm{D}} f[t]. \tag{13.176}$$

A restriction is made to cases where the functions f and g are square summable. Making use of the steps in Eq. (13.172) leads to

$$\sum_{k=-\infty}^{\infty}{}' g[k] \frac{(1 - (-1)^{t-k})}{t-k} = -\frac{\mathrm{i}}{2\pi} \sum_{k=-\infty}^{\infty}{}' \frac{(1 - (-1)^{t-k})}{t-k} \int_{-\pi}^{\pi} \operatorname{sgn} \omega \, F(\omega) \mathrm{e}^{\mathrm{i}\omega k} \, \mathrm{d}\omega$$

$$= -\frac{\mathrm{i}}{2\pi} \int_{-\pi}^{\pi} \operatorname{sgn} \omega \, F(\omega) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega \sum_{k=-\infty}^{\infty}{}' \frac{1 - \cos k\pi}{k} \mathrm{e}^{-\mathrm{i}\omega k}. \tag{13.177}$$

The summation in the preceding result is evaluated as follows:

$$\sum_{k=-\infty}^{\infty}{}' \frac{1 - \cos k\pi}{k} \mathrm{e}^{-\mathrm{i}\omega k} = -\mathrm{i} \sum_{k=-\infty}^{\infty}{}' \frac{(1 - \cos k\pi) \sin k\omega}{k}$$

$$= -4\mathrm{i} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\omega}{2k+1}$$

$$
\begin{aligned}
&= -\mathrm{i}\pi \ \mathrm{sgn}\,(\sin\omega) \\
&= -\mathrm{i}\pi \ \mathrm{sgn}\,\omega, \quad \text{for } -\pi < \omega < \pi,
\end{aligned} \tag{13.178}
$$

and hence

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty}{}' g[k]\frac{(1-(-1)^{t-k})}{t-k} &= -\frac{1}{2}\int_{-\pi}^{\pi} \{\mathrm{sgn}\ \omega\}^2 F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \\
&= -\pi f[t],
\end{aligned} \tag{13.179}
$$

which establishes Eq. (13.175). Equations (13.173) and (13.175) display the same skew-reciprocal behavior that characterizes the Hilbert transform pair on $\mathbb{R}$.

Consider the example

$$
f[t] = \cos(\omega t + \phi), \tag{13.180}
$$

where ω and ϕ are time-independent constants then, from Eq. (13.174), it follows for $-\pi < \omega < \pi$ that

$$
\begin{aligned}
\mathcal{H}_{\mathrm{D}} f[t] &= \frac{1}{\pi}\sum_{k=-\infty}^{\infty}{}' \cos(\omega(t-k)+\phi)\frac{(1-(-1)^k)}{k} \\
&= \frac{1}{\pi}\sin(\omega t+\phi)\sum_{k=-\infty}^{\infty}{}' \frac{(1-(-1)^k)\sin k\omega}{k} \\
&= \frac{4}{\pi}\sin(\omega t+\phi)\sum_{k=0}^{\infty}\frac{\sin(2k+1)\omega}{2k+1} \\
&= \sin(\omega t+\phi)\ \mathrm{sgn}\,(\sin\omega) \\
&= \ \mathrm{sgn}\ (\omega)\sin(\omega t+\phi).
\end{aligned} \tag{13.181}
$$

If $f[t] = \sin(\omega t + \phi)$, for $-\pi < \omega < \pi$, then, in a similar manner,

$$
\begin{aligned}
\mathcal{H}_{\mathrm{D}} f[t] &= \frac{1}{\pi}\sum_{k=-\infty}^{\infty}{}' \sin(\omega(t-k)+\phi)\frac{(1-(-1)^k)}{k} \\
&= -\frac{1}{\pi}\cos(\omega t+\phi)\sum_{k=-\infty}^{\infty}{}' \frac{(1-(-1)^k)\sin k\omega}{k} \\
&= -\mathrm{sgn}\ (\omega)\cos(\omega t+\phi).
\end{aligned} \tag{13.182}
$$

Both of these examples produce results identical to the corresponding Hilbert transforms on $\mathbb{R}$.

When f is sampled at even integer values, then from Eqs. (13.173) and (13.175) it follows that

$$\mathcal{H}_{\mathrm{D}} f[t] = \frac{2}{\pi} \sum_{\substack{k=-\infty \\ (k \text{ odd})}}^{\infty} \frac{f[k]}{t-k}, \quad \text{for } t \text{ even}, \tag{13.183}$$

and

$$f[t] = -\frac{2}{\pi} \sum_{\substack{k=-\infty \\ (k \text{ odd})}}^{\infty} \frac{\mathcal{H}_{\mathrm{D}} f[k]}{t-k}, \quad \text{for } t \text{ even}. \tag{13.184}$$

When f is sampled at odd integer values, it follows that

$$\mathcal{H}_{\mathrm{D}} f[t] = \frac{2}{\pi} \sum_{\substack{k=-\infty \\ (k \text{ even})}}^{\infty} \frac{f[k]}{t-k}, \quad \text{for } t \text{ odd}, \tag{13.185}$$

and

$$f[t] = -\frac{2}{\pi} \sum_{\substack{k=-\infty \\ (k \text{ even})}}^{\infty} \frac{\mathcal{H}_{\mathrm{D}} f[k]}{t-k}, \quad \text{for } t \text{ odd}. \tag{13.186}$$

If $\{f[n]\}$ is an odd sequence, then, with $f[0] = 0$,

$$\mathcal{H}_{\mathrm{D}} f[n] = \frac{2}{\pi} \sum_{k=1}^{\infty}{}' \frac{kf[k](1-(-1)^{n-k})}{n^2-k^2}, \tag{13.187}$$

that is, $\mathcal{H}_{\mathrm{D}} f[n]$ is an even sequence. If $\{f[n]\}$ is an even sequence, then

$$\mathcal{H}_{\mathrm{D}} f[n] = \frac{2n}{\pi} \sum_{m=1}^{\infty}{}' \frac{f[m](1-(-1)^{n-m})}{n^2-m^2}, \tag{13.188}$$

and hence $\mathcal{H}_{\mathrm{D}} f[n]$ is odd for $n \neq 0$. The designations $\mathcal{H}_{\mathrm{D_o}}$ and $\mathcal{H}_{\mathrm{D_e}}$ are used to denote the operators appearing in Eqs. (13.187) and (13.188), respectively. The subscripts imply the action of the operators on odd and even sequences, respectively.

The discrete Hilbert transform satisfies a Parseval-type identity that takes the form

$$\sum_{k=-\infty}^{\infty} |\mathcal{H}_{\mathrm{D}} f[k]|^2 = \sum_{k=-\infty}^{\infty} |f[k]|^2, \tag{13.189}$$

assuming both of these sums converge. This is the isometric property for the discrete Hilbert transform. To establish Eq. (13.189), first note that

$$\sum_{k=-\infty}^{\infty}{}' \frac{(1-(-1)^{k-j})^2}{(k-j)^2} = 8\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \pi^2 \tag{13.190}$$

and

$$\begin{aligned}\sum_{\substack{k=-\infty \\ (j\neq m)}}^{\infty}{}' \frac{(1-(-1)^{k-j})(1-(-1)^{k-m})}{(k-j)(k-m)} &= \frac{1}{(j-m)} \\ &\quad\times\left\{\sum_{n=-\infty}^{\infty}{}' \frac{(1-(-1)^{n})(1-(-1)^{n+j-m})}{n} - \frac{(1-(-1)^{n})(1-(-1)^{n+j-m})}{n}\right\} \\ &= 0.\end{aligned} \tag{13.191}$$

Using Eq. (13.175) leads to

$$\begin{aligned}\sum_{k=-\infty}^{\infty}|f[k]|^2 &= \frac{1}{\pi^2}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}{}' \mathcal{H}_{\mathrm{D}}f[j]\frac{(1-(-1)^{k-j})}{k-j}\sum_{m=-\infty}^{\infty}{}' \mathcal{H}_{\mathrm{D}}f[m]^*\frac{(1-(-1)^{k-m})}{k-m} \\ &= \frac{1}{\pi^2}\sum_{j=-\infty}^{\infty}\mathcal{H}_{\mathrm{D}}f[j]\sum_{m=-\infty}^{\infty}\mathcal{H}_{\mathrm{D}}f[m]^*\sum_{\substack{k=-\infty \\ (k\neq j,m)}}^{\infty}\frac{(1-(-1)^{k-j})(1-(-1)^{k-m})}{(k-j)(k-m)} \\ &= \frac{1}{\pi^2}\sum_{j=-\infty}^{\infty}|\mathcal{H}_{\mathrm{D}}f[j]|^2\sum_{k=-\infty}^{\infty}{}' \frac{(1-(-1)^{k-j})^2}{(k-j)^2} \\ &\quad+\frac{1}{\pi^2}\sum_{j=-\infty}^{\infty}\mathcal{H}_{\mathrm{D}}f[j]\sum_{m=-\infty}^{\infty}\mathcal{H}_{\mathrm{D}}f[m]^*\sum_{\substack{k=-\infty \\ (j\neq m)}}^{\infty}{}' \frac{(1-(-1)^{k-j})(1-(-1)^{k-m})}{(k-j)(k-m)} \\ &= \sum_{j=-\infty}^{\infty}|\mathcal{H}_{\mathrm{D}}f[j]|^2,\end{aligned} \tag{13.192}$$

which is the desired result.

A key property for the Hilbert transform on $\mathbb{R}$ is the inversion formula $H^2 f = -f$ (see Section 4.4), and the corresponding result for the discrete Hilbert transform is given by

$$\mathcal{H}_{\mathrm{D}}^2 f[k] = -f[k]. \tag{13.193}$$

This result can be established as follows:

$$\begin{aligned}
\mathcal{H}_{\mathrm{D}}^2 f[k] &= \frac{1}{\pi^2} \sum_{j=-\infty}^{\infty}{}' \frac{(1-(-1)^{k-j})}{k-j} \sum_{m=-\infty}^{\infty}{}' f[m] \frac{(1-(-1)^{j-m})}{j-m} \\
&= \frac{1}{\pi^2} \sum_{j=-\infty}^{\infty}{}' \frac{(1-(-1)^{k-j})}{k-j} \\
&\quad \times \left\{ f[k] \frac{(1-(-1)^{j-k})}{j-k} + \sum_{\substack{m=-\infty \\ (m\neq k)}}^{\infty}{}' f[m] \frac{(1-(-1)^{j-m})}{j-m} \right\} \\
&= -\frac{f[k]}{\pi^2} \sum_{j=-\infty}^{\infty}{}' \frac{(1-(-1)^{k-j})^2}{(k-j)^2} \\
&\quad + \frac{1}{\pi^2} \sum_{\substack{m=-\infty \\ (m\neq k,j)}}^{\infty} f[m] \sum_{j=-\infty}^{\infty}{}' \frac{(1-(-1)^{k-j})(1-(-1)^{j-m})}{(k-j)(j-m)} \\
&= -f[k], \qquad (13.194)
\end{aligned}$$

where the final equation follows using Eqs. (13.190) and (13.191), and the function f is assumed to be square summable. This establishes the desired result.

The discrete Hilbert transform can be written in terms of the Fourier transform operation:

$$\mathcal{H}_{\mathrm{D}} f[t] = -\mathrm{i}\mathcal{F}^{-1}\{\operatorname{sgn}\omega\ \mathcal{F}f(\omega)\}[t], \qquad (13.195)$$

where the forms discussed in Eqs. (13.170) and (13.171) are assumed for the discrete Fourier transforms, and the function f is understood to be square summable. From Eq. (13.158),

$$\begin{aligned}
\mathcal{H}_D f[t] &= \frac{1}{\pi} \sum_{\tau=-\infty}^{\infty}{}' \frac{f[\tau]\{1-(-1)^{t-\tau}\}}{t-\tau} \\
&= \frac{1}{2\pi\mathrm{i}} \sum_{\tau=-\infty}^{\infty} f[\tau] \int_0^{\pi} \{\mathrm{e}^{\mathrm{i}(t-\tau)\omega} - \mathrm{e}^{-\mathrm{i}(t-\tau)\omega}\} \mathrm{d}\omega \\
&= \frac{1}{2\pi\mathrm{i}} \sum_{\tau=-\infty}^{\infty} f[\tau] \left\{ \int_0^{\pi} \mathrm{e}^{\mathrm{i}(t-\tau)\omega}\,\mathrm{d}\omega - \int_{-\pi}^{0} \mathrm{e}^{\mathrm{i}(t-\tau)\omega}\,\mathrm{d}\omega \right\} \\
&= -\frac{\mathrm{i}}{2\pi} \sum_{\tau=-\infty}^{\infty} f[\tau] \int_{-\pi}^{\pi} \operatorname{sgn}\omega\, \mathrm{e}^{\mathrm{i}(t-\tau)\omega}\,\mathrm{d}\omega
\end{aligned}$$

$$= -\frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi} \operatorname{sgn}\omega\, \mathrm{e}^{\mathrm{i}t\omega}\, \mathrm{d}\omega \sum_{\tau=-\infty}^{\infty} f[\tau]\mathrm{e}^{-\mathrm{i}\tau\omega}$$

$$= -\frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi} \operatorname{sgn}\omega\, \mathrm{e}^{\mathrm{i}t\omega}\mathcal{F}f(\omega)\mathrm{d}\omega$$

$$= -\mathrm{i}\mathcal{F}^{-1}\{\operatorname{sgn}\omega\, \mathcal{F}f(\omega)\}[t], \tag{13.196}$$

where Eqs. (13.170) and (13.171) have been employed in the final two steps.

If $\{f[n]\}$ and $\{g[n]\}$ are two sequences, there is a Parseval-type identity of the following form:

$$\sum_{n=-\infty}^{\infty} g[n]\mathcal{H}_\mathrm{D}f[n] = -\sum_{n=-\infty}^{\infty} f[n]\mathcal{H}_\mathrm{D}g[n]. \tag{13.197}$$

This result can be established as follows:

$$\sum_{n=-\infty}^{\infty} g[n]\mathcal{H}_\mathrm{D}f[n] = \sum_{n=-\infty}^{\infty} g[n]\frac{1}{\pi}\sum_{m=-\infty}^{\infty}{}' f[m]\frac{(1-(-1)^{n-m})}{n-m}$$

$$= \sum_{m=-\infty}^{\infty} f[m]\frac{1}{\pi}\sum_{n=-\infty}^{\infty}{}' g[n]\frac{(1-(-1)^{n-m})}{n-m}$$

$$= -\sum_{m=-\infty}^{\infty} f[m]\mathcal{H}_\mathrm{D}g[m], \tag{13.198}$$

which is the required result. The summations can be interchanged if $\{f[n]\}$ and $\{g[n]\}$ are square summable. More generally, the result can be established if $f \in l^p$ and $g \in l^q$ with $p > 1$ and $p^{-1} + q^{-1} = 1$. Equation (13.189) is an immediate consequence of Eq. (13.197) if $g[n] = \mathcal{H}_\mathrm{D}f[n]$ is inserted and Eq. (13.193) employed.

The analog of Bedrosian's theorem (see Section 4.15) is now applied to the discrete Hilbert transform. Consider a signal $a[t]$ whose Fourier transform $A(\omega)$ is band-limited such that

$$A(\omega) = 0, \quad \text{for } 0 < b < |\omega| < \pi. \tag{13.199}$$

Given a signal of the form

$$f[t] = a[t]\cos\omega_0 t, \tag{13.200}$$

then the discrete Hilbert transform for $0 < b < \omega_0 < \pi$ is given by

$$\mathcal{H}_\mathrm{D}f[t] = a[t]\sin\omega_0 t. \tag{13.201}$$

To establish this result, first note from Eq. (13.170) that

$$F(\omega) = \sum_{t=-\infty}^{\infty} f[t]\mathrm{e}^{-\mathrm{i}\omega t} = \frac{1}{2}\{A(\omega - \omega_0) + A(\omega + \omega_0)\}. \tag{13.202}$$

The function $F(\omega)$ satisfies

$$F(\omega) = \begin{cases} (1/2)\, A(\omega - \omega_0), & \omega_0 - b < \omega < \omega_0 + b \\ (1/2)\, A(\omega + \omega_0), & -\omega_0 - b < \omega < -\omega_0 + b \\ 0, & \text{otherwise.} \end{cases} \tag{13.203}$$

From Eq. (13.172),

$$\begin{aligned} \mathscr{H}_{\mathrm{D}} f[t] &= -\frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi} \operatorname{sgn}\omega\, \mathrm{e}^{\mathrm{i}\omega t} F(\omega)\mathrm{d}\omega \\ &= -\frac{\mathrm{i}}{4\pi}\int_{-\pi}^{\pi} \operatorname{sgn}\omega\, \mathrm{e}^{\mathrm{i}\omega t}\{A(\omega - \omega_0) + A(\omega + \omega_0)\}\mathrm{d}\omega, \end{aligned} \tag{13.204}$$

which simplifies, for $\omega_0 + b < \pi$, to

$$\begin{aligned} \mathscr{H}_D f[t] = &-\frac{\mathrm{i}}{4\pi}\int_{\omega_0 - b}^{\omega_0 + b} \operatorname{sgn}\omega\, \mathrm{e}^{\mathrm{i}\omega t} A(\omega - \omega_0)\mathrm{d}\omega \\ &-\frac{\mathrm{i}}{4\pi}\int_{-\omega_0 - b}^{-\omega_0 + b} \operatorname{sgn}\omega\, \mathrm{e}^{\mathrm{i}\omega t} A(\omega + \omega_0)\mathrm{d}\omega \\ = &-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\omega_0 t}}{4\pi}\int_{-b}^{b} \mathrm{e}^{\mathrm{i}\omega t} A(\omega)\mathrm{d}\omega + \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega_0 t}}{4\pi}\int_{-b}^{b} \mathrm{e}^{\mathrm{i}\omega t} A(\omega)\mathrm{d}\omega \\ = &\sin\omega_0 t \frac{1}{2\pi}\int_{-b}^{b} \mathrm{e}^{\mathrm{i}\omega t} A(\omega)\mathrm{d}\omega \\ = &\, a[t]\sin\omega_0 t. \end{aligned} \tag{13.205}$$

In a similar manner, for the choice

$$f[t] = a[t]\sin\omega_0 t, \tag{13.206}$$

the discrete Hilbert transform, for $0 < b < \omega_0 < \pi$, is given by

$$\mathscr{H}_{\mathrm{D}} f[t] = -a[t]\cos\omega_0 t. \tag{13.207}$$

13.14 Discrete analytic functions

Some of the basic ideas connected with discrete analytic functions are introduced in this section. The presentation is based on Duffin (1956). The focus is on the essentials that allow the analog of the Hilbert transform pair for discrete functions to be developed.

Complex-valued functions, defined on points of the complex plane that are integers, are the focus of this section. These points form a lattice that allows the complex plane to be represented by a collection of squares. If in one of these squares a function f satisfies

$$\frac{f(z+1+\mathrm{i})-f(z)}{1+\mathrm{i}}=\frac{f(z+\mathrm{i})-f(z+1)}{\mathrm{i}-1}, \tag{13.208}$$

then the function is called *discrete analytic*. Equation (13.208) is a statement that the difference quotient across one diagonal is equal to the difference quotient across the other diagonal. Other definitions of a discrete analytic function can be given and these are briefly mentioned by Duffin (1956).

The complex plane is partitioned into squares, as shown in Figure 13.5. The complex variable z is given by $z = m + \mathrm{i}n$, where $m, n \in \mathbb{Z}$. Associated with a lattice point z_0 are the points $z_0 + 1, z_0 + 1 + \mathrm{i}$, and $z_0 + \mathrm{i}$, which are the lattice points of the vertices of the unit square. Regions are the union of unit squares. Using the compact notation

$$f_k = f(z_k), \tag{13.209}$$

a function is defined to be analytic on the square associated with z_0 if

$$f_0 + \mathrm{i}f_1 + \mathrm{i}^2 f_2 + \mathrm{i}^3 f_3 = 0, \tag{13.210}$$

where the indices 0, 1, 2, and 3 signify the points z_k in the order just indicated. This result is equivalent to the definition in Eq. (13.208), and can be written in the more compact form

$$Lf(z_0) = 0, \tag{13.211}$$

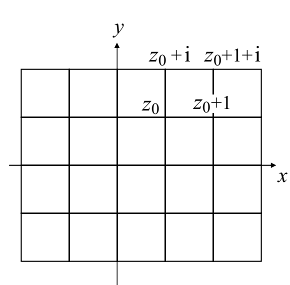

Figure 13.5. Lattice for the complex plane.

where the action of the operator L is given by

$$Lf(z_0) = f_0 + \mathrm{i}f_1 + \mathrm{i}^2 f_2 + \mathrm{i}^3 f_3. \tag{13.212}$$

If f_k is written in terms of its real and imaginary components:

$$f_k = u_k + \mathrm{i}v_k, \tag{13.213}$$

then Eq. (13.210) reduces to

$$u_2 - u_0 = v_3 - v_1 \tag{13.214}$$

and

$$u_3 - u_1 = v_0 - v_2. \tag{13.215}$$

This pair of results might be regarded as the analogs of the Cauchy–Riemann equations.

Let $q(z)$ denote a lattice function that satisfies

$$Lq(z) = \begin{cases} 1, & z = 0 \\ 0, & z \neq 0. \end{cases} \tag{13.216}$$

Suppose $f(z)$ is analytic in each unit square of the upper half complex plane and that, for a given z_0 and $\operatorname{Im} z \geq 0$,

$$f(z)q(z - z_0) = o(|z|^{-1}). \tag{13.217}$$

Duffin showed, for $\operatorname{Im} z \geq 0$, that

$$f(z_0) = \sum_{m=-\infty}^{\infty} f(m)\theta(z_0 - m), \tag{13.218}$$

where

$$\theta(z) = q(-z) + \mathrm{i}q(1 - z). \tag{13.219}$$

For $\operatorname{Im} z < 0$,

$$\sum_{m=-\infty}^{\infty} f(m)\theta(z_0 - m) = 0. \tag{13.220}$$

Noting that $\theta(z) = \theta(m, n)$, with $z = m + \mathrm{i}n$, Duffin then obtained the following result:

$$\theta(m, n) = \frac{1}{\pi} \int_{-\pi/4}^{\pi/4} \mathrm{e}^{\mathrm{i}m(\pi/2 - 2t)} \tan^n t \, \mathrm{d}t. \tag{13.221}$$

The reader interested in the details of the derivation should consult Duffin (1956).

The discrete Hilbert transform can be derived from the preceding results. Using Eqs. (13.218) and (13.220), it follows that

$$f(z) = \sum_{m=-\infty}^{\infty} f(m)\{\theta(z-m) - \theta(z^* - m)\}. \tag{13.222}$$

On setting

$$2\theta(z) = h(z) + \mathrm{i}k(z), \tag{13.223}$$

where $h(m,n) = 2\mathrm{Re}\,\theta(m,n)$ and $k(m,n) = 2\mathrm{Im}\,\theta(m,n)$, and employing $\mathrm{Im}\, z > 0$,

$$\theta(z-m) - \theta(z^* - m) = h(z-m). \tag{13.224}$$

Using Eq. (13.218) leads to

$$f(z) = \mathrm{i}\sum_{m=-\infty}^{\infty} f(m)k(z-m). \tag{13.225}$$

If z is restricted to be real, so that $n = 0$, and the real value is taken to be j, then from Eq. (13.221) it follows that

$$\begin{aligned} k(j,0) &= \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \sin\left[j\left\{\frac{\pi}{2} - 2t\right\}\right]\mathrm{d}t \\ &= \frac{1}{\pi j}[1-(-1)^j]. \end{aligned} \tag{13.226}$$

Equation (13.225) can be written as follows:

$$f(j) = \frac{\mathrm{i}}{\pi}\sum_{m=-\infty}^{\infty}{}' \frac{f(m)[1-(-1)^{j-m}]}{j-m}. \tag{13.227}$$

Let

$$f(z) = u(z) + \mathrm{i}v(z), \tag{13.228}$$

where u and v are real functions. Inserting Eq. (13.228) into Eq. (13.227), and taking the real and imaginary parts, leads to

$$v(j) = \frac{1}{\pi}\sum_{m=-\infty}^{\infty}{}' \frac{u(m)[1-(-1)^{j-m}]}{j-m} \tag{13.229}$$

and

$$u(j) = -\frac{1}{\pi} \sum_{m=-\infty}^{\infty}{}' \frac{v(m)[1-(-1)^{j-m}]}{j-m}. \tag{13.230}$$

The reader will recognize this pair of results as the discrete Hilbert transforms discussed in Section 13.13.

13.15 Weighted discrete Hilbert transform inequalities

The weighted norm inequalities for the Hilbert transform on $\mathbb{R}$ discussed in Section 7.13 have analogs for the case of the discrete Hilbert transform. The following theorem is due to Hunt *et al.* (1973). Let T denote the operator defined by

$$Tf[n] = \sup_{k>0} \left| \frac{1}{\pi} \sum_{|n-m|\geq k} \frac{f[m]}{n-m} \right|. \tag{13.231}$$

This can be viewed as the discrete analog of the maximal Hilbert transform operator defined in Section 7.10. Suppose $1 < p < \infty$, and let the weight function $w_k \geq 0$. Then there is a constant C_p, independent of f, such that the following are equivalent.

(a) For $m \leq n$,

$$\sum_{k=m}^{n} w_k \left\{ \sum_{k=m}^{n} (w_k)^{-(p-1)^{-1}} \right\}^{p-1} \leq C_p(n-m+1)^p. \tag{13.232}$$

(b) For every sequence $f[k]$,

$$\sum_{k=-\infty}^{\infty} |H_{\mathrm{D}} f[k]|^p w_k \leq C_p \sum_{k=-\infty}^{\infty} |f[k]|^p w_k, \tag{13.233}$$

(c) and

$$\sum_{k=-\infty}^{\infty} |Tf[k]|^p w_k \leq C_p \sum_{k=-\infty}^{\infty} |f[k]|^p w_k. \tag{13.234}$$

(d) For $a > 0$, and every sequence $f[k]$,

$$\sum_{k=-\infty}^{\infty}{}_{|H_{\mathrm{D}} f[k]|\,>\,a} w_k \leq C_p a^{-p} \sum_{k=-\infty}^{\infty} |f[k]|^p w_k, \tag{13.235}$$

(e) and

$$\sum_{k=-\infty}^{\infty}{}_{Tf[k] > a} w_k \leq C_p a^{-p} \sum_{k=-\infty}^{\infty} |f[k]|^p w_k. \tag{13.236}$$

In Eqs. (13.235) and (13.236) the sum only includes terms for which the inequality stated in the subscript holds. For the case $p = 1$, the statements (a), (d), and (e) are equivalent. Equation (13.232) represents the discrete analog of the A_p condition discussed in Section 7.12. Equations (13.235) and (13.236) are the discrete analogs of the weak-type inequalities given for the classical Hilbert transform operator in Eqs. (7.435) and (7.436), respectively. The last four results are, together, the discrete version of the Hunt–Muckenhoupt–Wheeden results given for the continuous case (see Eqs. (7.433)–(7.436)). Part of the proof of this theorem can be carried over from the results of the continuous case discussed in Section 7.13. The interested reader might like to construct the proof of the equivalence statements, or consult the sketch of the proof of these results in Hunt *et al.* (1973).

Let the operators appearing in Eqs. (13.130) and (13.131) be denoted by $H_{\mathrm{D_o}}$ and $H_{\mathrm{D_e}}$, respectively. The additional subscripts denote odd and even. A result similar to the Hunt–Muckenhoupt–Wheeden theorem can be proved with H_{D} interchanged for each of these operators in turn (Andersen, 1977b). The A_p condition is slightly modified. The operator H_{D} in Eqs. (13.233) and (13.235) can be replaced by the operator $\mathcal{H}_{\mathrm{D}}$. A similar situation also applies for the operators $\mathcal{H}_{\mathrm{D_o}}$ and $\mathcal{H}_{\mathrm{D_e}}$ defined in Eqs. (13.187) and (13.188).

Notes

§13.2 The discrete Fourier transform is discussed in a large number of sources. The following references should prove useful to the reader for additional information: Williams (1980), Čížek (1986), Hahn (1996a), Walker (1996), Oppenheim *et al.* (1999), and Howell (2001).

§13.4 For concise comments on the evaluation of the discrete Fourier transform, see Press *et al.* (1992).

§13.6 For further reading on the Z transform, see Poularikas (1996b) and Oppenheim *et al.* (1999, chap. 3).

§13.8 The authoritative source for information and further study on the topics in this section is Oppenheim *et al.* (1999, chap. 11), and the presentation given has relied on this work. (The first edition of that book, titled *Digital Signal Processing*, also contains an account on the discrete Hilbert transform.) Further reading on the discrete Hilbert transform, with an emphasis on signal processing, can be found in Čížek (**?**), Gold, Oppenheim, and Radar (1970), Kak (1970, 1972, 1973, 1977), Bonzanigo (1972), Read and Treitel (1973), Burris (1975), Sabri and Steenaart

(1975, 1976, 1977), Blyumin and Trakhtman (1977), Dutta Roy and Agrawal (1978), Bose and Prabhu (1979), Pei and Jaw (1989), Reddy, Sathyanarayana, and Swamy (1991a, 1991b), Witte *et al.* (1991), Zanotti, Fogale, and Capitani (1996), Padala and Prabhu (1997), Stanomir, Negrescu, and Pârvu (1997), and Damera-Venkata, Evans, and McCaslin (2000). For applications in magnetic resonance, see Ernst (1969) and Bartholdi and Ernst (1973).

§13.9 For further reading, see Gold *et al.* (1970) and Oppenheim *et al.* (1999).

§13.10 Oppenheim *et al.* (1999) is the next stop for additional discussion on the issues associated with this section.

§13.11 For further reading, see Laeng (2007). Some extensions of the standard discrete Hilbert transform can be found in Koizumi (1958b, 1959a) and Komori (2001). For the discrete Hilbert transform of imaginary exponentials, and the connection to several topics including a time-dependent Schrödinger equation, see Oskolkov (1998). For further reading on the semi-discrete Hilbert transform, see Marsden, Richards, and Riemenschneider (1975) and Bardaro *et al*. (2006).

§13.12 For a version of the Hilbert inequality for discrete sequences different to that given in Eq. (13.134), see Hardy *et al*. (1952), and an elementary proof can be found in Oleszkiewicz (1993). A refinement of Hilbert's inequality can be found in Montgomery and Vaughan (1974). For the version of Hilbert's inequality given in Exercise 13.10, there are a number of extensions, see Gao (1996, 1997), Jichang and Debnath (2000), and Yang (2000). A generalization of the Riesz theorem for the discrete Hilbert transform to higher dimension spaces is discussed in Calderón and Zygmund (1954) and Zygmund (1957).

§13.13 A useful account of the discrete Hilbert transform as defined in this section can be found in Saito (1974). Further discussion can be found in Varsavsky (1949) and Andersen (1976b).

§13.14 In addition to the items mentioned in this section, Duffin (1956) also develops the analogs of the Cauchy integral formula for discrete analytic functions. For an earlier discussion of some of the ideas of this section, see Ferrand (1944).

§13.15 For further reading, see Andersen (1977b), Lyubarskii and Seip (1997), Gabisonija and Meskhi (1998), and Rakotondratsimba (1999).

Exercises

13.1 Determine DFT $\{1, 1, 0, 0, 1, 1\}$.

13.2 Evaluate the IDFT of the answer in Exercise 13.1.

13.3 The IDFT of a sequence is given by $\{10, -2-2\mathrm{i}, -2, -2+2\mathrm{i}\}$. What was the starting sequence?

13.4 If $x[n]$ is a complex periodic sequence with period N, and X_n denotes its DFT, prove the following: (i) if $x[n]$ is even, then X_n is even, and if $x[n]$ is odd, then X_n is odd; (ii) if $x[n]$ is real, then $X_{-n} = X_n^*$.

13.5 If $x[n]$ is a real and even periodic sequence with period N, what can be said about X_n? If $x[n]$ is real and odd, what can be said about X_n?

13.6 The Dirac comb is defined by $\Delta_\tau \equiv \sum_{k=-\infty}^{\infty} \delta(t - k\tau)$, where the period $\tau > 0$. Does there exist a Z transform for the Dirac comb? If there does, find it.

13.7 Suppose $a > 0$ and $b > 0$ and

$$x[n] = \begin{cases} a^n, & n < 0 \\ b^n, & n \geq 0. \end{cases}$$

Evaluate the Z transform of the sequence $x[n]$.

13.8 Determine the Z transform of the sequence $x[n] = \mathrm{e}^{-nt}$ for $n \geq 0$, where t is a positive sampling time.

13.9 Determine the convolution $x_1(n) * x_2(n)$, where x_1 and x_2 represent data sequences of equally spaced time measurements starting at time zero: $x_1 = \{1, 2, 0, 2, 0\}$ and $x_2 = \{0, 0, 2, 2, 2, 0, 0\}$.

13.10 If $\{a_m\}$ and $\{b_n\}$ are square summable sequences of real numbers, prove that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m + n} \leq \pi \sqrt{\left[\sum_{m=1}^{\infty} a_m^2\right]} \sqrt{\left[\sum_{n=1}^{\infty} a_n^2\right]}.$$

This is called Hilbert's inequality (and is distinct from Eq. (13.134)).

13.11 For a real number $p > 1$ and a positive m, show that

$$\sum_{n=1}^{\infty} \frac{m^{\mathrm{p}^{-1}}}{n^{\mathrm{p}^{-1}}(m + n)} \leq \pi \csc\left(\frac{\pi}{p}\right).$$

13.12 Suppose $\sum_{m=1}^{\infty} a_m^p$ and $\sum_{n=1}^{\infty} b_n^q$ are convergent for sequences $\{a_m\}$ and $\{b_n\}$ of non-negative numbers, where $p > 1, q > 1$, and $p^{-1} + q^{-1} = 1$. Prove that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m + n} \leq \pi \csc\left(\frac{\pi}{p}\right) \left(\sum_{m=1}^{\infty} a_m^p\right)^{p^{-1}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{q^{-1}}.$$

13.13 Determine whether the constants appearing in Exercises 13.10 and 13.12 are the best possible.

13.14 If $x[n]$ denotes a causal sequence for which $X_{\mathrm{r}}(\mathrm{e}^{\mathrm{i}\theta}) = 2\cos^2\theta$, determine $X_{\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta})$. Is there sufficient information to determine the sequence $x[n]$? If there is, find the sequence.

13.15 Determine the discrete Hilbert transform of the following sequences whose real part is given by (i) $x_{\mathrm{r}}[n] = \sin(\omega_0 n + \phi)$, (ii) $x_{\mathrm{r}}[n] = \cos(\omega_0 n + \phi)$, where ω_0 and ϕ are constants.

13.16 Prove that H_{D} is a bounded linear operator mapping $l^p(\mathbb{Z})$ into $l^p(\mathbb{Z})$ for $1 < p < \infty$.

13.17 Evaluate $\mathcal{H}_{\mathrm{D}} f[n]$, where (i) $f[n] = \operatorname{cas}(\omega_0 n + \phi)$ and (ii) $f[n] = n^{-1} \sin \omega_0 n$.

13.18 Prove Eq. (13.163).

13.19 What form does the Hunt–Muckenhoupt–Wheeden theorem of Section 13.14 take for the operators $\mathcal{H}_{\mathrm{D}}$ and $\mathcal{T}$, where

$$\mathcal{T} f[n] = \sup_{k>0} \left| \frac{1}{\pi} \sum_{|n-m| \geq k} \frac{f[m](1 - (-1)^{n-m})}{n - m} \right| ?$$

14

Numerical evaluation of Hilbert transforms

14.1 Introduction

Principal value integrals arise in a wide variety of applications, and in many cases it is not possible to evaluate such integrals in a simple closed analytic form. Consequently, there has been a significant investment of research effort devoted to the numerical evaluation of principal value integrals. The expression "numerical quadrature", or frequently just the term "quadrature", is used synonymously with numerical integration.

Some of the numerical integration approaches that have been developed for principal value integrals are outlined in this chapter. These range from rather simple schemes, which are sometimes quite effective, to approaches that yield fairly precise results and can be implemented in a high-speed calculation. Methods that are discussed include Maclaurin's formula, the trapezoidal rule, Simpson's formula, specialized Gaussian quadrature methods, and techniques involving Fourier transforms, including the fast Fourier transform, Fourier allied integral approaches, and methods based on conjugate Fourier series. Since a number of principal value integral problems arise in the context of transforming experimental data, some attention is devoted to the discretized nature of the data and how this can be handled.

Even if an analytic solution can be found for a particular principal value integral, numerical methods can be employed as a very useful check on the closed form result.

14.2 Some elementary transformations for Cauchy principal value integrals

Two straightforward, but potentially very useful, transformations that may be employed to simplify the evaluation of the Hilbert transform are discussed in this section. The first approach is to subtract a contribution evaluated at the singularity.

This approach yields the following:

$$\begin{aligned} Hf(x) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x-y} \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\{f(y)-f(x)\}\mathrm{d}y}{x-y} + \frac{f(x)}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{x-y}. \end{aligned} \tag{14.1}$$

Recall that

$$P \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{x-y} = 0, \tag{14.2}$$

so that Eq. (14.1) simplifies to

$$Hf(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\{f(y)-f(x)\}\mathrm{d}y}{x-y}. \tag{14.3}$$

If the function $f(x)$ is Hölder continuous in the vicinity of the singularity, then the numerical evaluation of this preceding integral will in general be a simpler proposition than dealing with the standard Cauchy principal value form of the Hilbert transform. The idea just described is often referred to as a *subtracted dispersion relation* in the physics literature, and this terminology is used synonymously with subtracted Hilbert transform.

Exactly the same idea can be applied to the finite Hilbert transform. For $x \in (a, b)$,

$$\begin{aligned} Hf(x) &= \frac{1}{\pi} P \int_{a}^{b} \frac{f(y)\mathrm{d}y}{x-y} \\ &= \frac{1}{\pi} \int_{a}^{b} \frac{\{f(y)-f(x)\}\mathrm{d}y}{x-y} + \pi^{-1} f(x) \log\left|\frac{x-a}{b-x}\right|, \end{aligned} \tag{14.4}$$

where once again it is assumed that near the singular point x the function is Hölder continuous. From a numerical standpoint, one has to be on the lookout for possible numerical instabilities. For example, in the second integral in Eq. (14.4), evaluation at a grid point located near $y = x$ has the potential to magnify errors in the difference $f(y) - f(x)$, due to the size of the factor $(y - x)^{-1}$.

To see how the subtraction idea applies, two examples are considered that are sufficiently simple that the resulting subtracted integrals can be performed analytically.

Consider the case $f(x) = \sin \alpha x$, with α denoting a constant; then,

$$
\begin{aligned}
\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\sin \alpha y \, dy}{x-y} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{[\sin \alpha y - \sin \alpha x] dy}{x-y} \\
&= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{[\sin \alpha(w+x) - \sin \alpha x] dw}{w} \\
&= -\frac{\cos \alpha x}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha w \, dw}{w} - \frac{\sin \alpha x}{\pi} \int_{-\infty}^{\infty} \frac{[1 - \cos \alpha w] dw}{w} \\
&= -\frac{\cos \alpha x}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha w \, dw}{w} \\
&= -\operatorname{sgn} \alpha \cos \alpha x. \qquad (14.5)
\end{aligned}
$$

The integrand $w^{-1} \sin \alpha w$ is well behaved in the vicinity of $w \to 0$. As a second example, consider $f(x) = (1 + x^2)^{-1}$; then,

$$
\begin{aligned}
\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{(1+y^2)^{-1} \, dy}{x-y} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x-y} \left\{ \frac{1}{1+y^2} - \frac{1}{1+x^2} \right\} dy \\
&= \frac{1}{\pi(1+x^2)} \int_{-\infty}^{\infty} \frac{1}{x-y} \left\{ \frac{x^2 - y^2}{1+y^2} \right\} dy \\
&= \frac{x}{\pi(1+x^2)} \int_{-\infty}^{\infty} \frac{dy}{1+y^2} \\
&= \frac{x}{1+x^2}. \qquad (14.6)
\end{aligned}
$$

In this example, the resulting integrand is now significantly simpler than the starting Cauchy principal value integral. If the preceding example were replaced by $f(x) = (1 + x^2)^{-1} g(x)$, then

$$
\begin{aligned}
\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y) dy}{x-y} &= \frac{x}{\pi(1+x^2)} \int_{-\infty}^{\infty} \frac{g(y) dy}{1+y^2} + \frac{1}{\pi(1+x^2)} \int_{-\infty}^{\infty} \frac{y g(y) dy}{1+y^2} \\
&\quad - \frac{1}{\pi(1+x^2)} \int_{-\infty}^{\infty} \frac{\{g(y) - g(x)\} dy}{y-x}. \qquad (14.7)
\end{aligned}
$$

If $g(x)$ is an even or odd function, then one of the first two integrals on the right-hand side of Eq. (14.7) is zero, and, as before, it is to be expected that the third integral would be more stable towards numerical evaluation. It is necessary of course to be alert to the possibility of significant loss in accuracy in numerical calculations, when the difference is taken of quantities that are very close in magnitude. For example, this might be the case if the result for the third integral in Eq. (14.7) is determined principally by the values of the function $g(y)$ in the region near the point $y = x$. A function that is sharply peaked near $y = x$, and falls off very quickly away from this point, is an example where numerical round-off errors would likely arise.

The second simplification involves splitting the function of interest into its even and odd components. It follows in this case that

$$
\begin{aligned}
Hf(x) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\{f_{\mathrm{e}}(y) + f_{\mathrm{o}}(y)\}\mathrm{d}y}{x - y} \\
&= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left\{ \int_{\varepsilon}^{\infty} \frac{\{f_{\mathrm{e}}(y - x) - f_{\mathrm{e}}(y + x)\}\mathrm{d}y}{y} \right. \\
&\quad \left. - \int_{\varepsilon}^{\infty} \frac{\{f_{\mathrm{o}}(y - x) + f_{\mathrm{o}}(y + x)\}\mathrm{d}y}{y} \right\},
\end{aligned} \tag{14.8}
$$

where f_{e} and f_{o} are the even and odd parts of f, respectively. Consider for the moment the case $x = 0$; then,

$$
Hf(0) = -\frac{2}{\pi} \lim_{\varepsilon \to 0} + \int_{\varepsilon}^{\infty} \frac{f_{\mathrm{o}}(y)\mathrm{d}y}{y}. \tag{14.9}
$$

In the resulting integral, it will often be easier to deal with the behavior as $y \to 0+$, and in some cases there is no singularity at $y = 0$. As an example, consider

$$
f(y) = \begin{cases} \mathrm{e}^{y}, & \text{for } |y| \leq 1 \\ 0, & \text{for } |y| > 1, \end{cases} \tag{14.10}
$$

then

$$
f_{\mathrm{o}}(y) = \frac{1}{2}\{f(y) - f(-y)\} = \sinh y, \tag{14.11}
$$

and hence

$$
Hf(0) = -\frac{2}{\pi} \int_{0}^{1} \frac{\sinh y \, \mathrm{d}y}{y}. \tag{14.12}
$$

The integrand of this integral is well behaved as $y \to 0$, so it would be expected that this form is more amenable to numerical evaluation, rather than trying to deal with the original Cauchy principal value integral. Consider the case $x \neq 0$, and suppose that $|x| < 1$; then, for the example given in Eq. (14.10), it follows that

$$
\begin{aligned}
Hf(x) &= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1-x} \frac{\{f_{\mathrm{e}}(y - x) - f_{\mathrm{o}}(y - x) - f(y + x)\}\mathrm{d}y}{y} \\
&\quad + \frac{1}{\pi} \int_{1-x}^{1+x} \frac{\{f_{\mathrm{e}}(y - x) - f_{\mathrm{o}}(y - x)\}\mathrm{d}y}{y},
\end{aligned} \tag{14.13}
$$

which simplifies to

$$
Hf(x) = -\frac{2\mathrm{e}^{x}}{\pi} \int_{0}^{1-x} \frac{\sinh y \, \mathrm{d}y}{y} + \frac{\mathrm{e}^{x}}{\pi} \int_{1-x}^{1+x} \frac{\mathrm{e}^{-y} \, \mathrm{d}y}{y}. \tag{14.14}
$$

The integrand of the first integral on the right-hand side of Eq. (14.14) is well behaved as $y \to 0$, and there is no singularity present for the second integral. As before, this form would be expected to be more suitable for numerical evaluation than the standard Cauchy principal value integral.

14.3 Some classical formulas for numerical quadrature

A few of the most elementary approaches available for carrying out a numerical quadrature are considered in this section. The principal advantage of these techniques is that they are very simple to implement. The main drawback is that the accuracy of the integral evaluation may be limited. This can be the case particularly when the function under consideration, or the experimental data available to model the function, are not smoothly varying over the integration interval in question.

The classical formulas for the numerical evaluation of an integral take the following form:

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{i=1}^{N} w_i f(x_i), \tag{14.15}$$

where N is the number of sample points in the interval, which may be open or closed, x_i denotes the points at which the integrand is sampled, and the w_i represent weighting coefficients at the sampling points. The simplest examples of this approach are the trapezoidal rule and Simpson's rule, which are discussed in a number of introductory calculus texts. A common feature of some of the simpler numerical quadrature approaches is that the abscissa values, the x_i, are selected in an equally spaced fashion, as illustrated in Figure 14.1.

14.3.1 A Maclaurin-type formula

Consider the evaluation of the integral

$$Hf(x_i) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{x_i - x}, \tag{14.16}$$

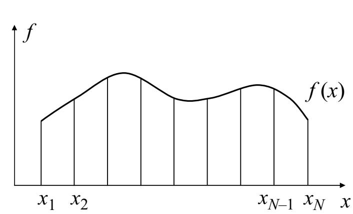

Figure 14.1. Partitioning of the integration range into equally spaced intervals.

where the function $f(x)$ is obtained experimentally as a discrete set of measurements at equally spaced intervals:

$$x_1, x_2, x_3, x_4, x_5, \ldots, x_n,$$
$$f_1, f_2, f_3, f_4, f_5, \ldots, f_n,$$

where $f_k \equiv f(x_k)$. With experimental data in this form, there are two approaches that may be taken to evaluate the integral. The first scheme is to fit the data to some functional form, the particular choice usually being dictated by some understanding of the physics of the model that underlies the collected data. Since the data have only been collected over a subset of the integration interval, ideally the underlying model will also help deal with the asymptotic behavior. That is, a presumably reliable extrapolation of the data outside the measurement domain can be made. If the particular fitting function is simple, so that the Hilbert transform can be obtained in closed form, then the problem is solved. If the selected function is rather involved, then a numerical quadrature is required, but there is the added flexibility to partition the integration interval in different ways.

The second approach covers those cases where the data are treated directly, without any recourse to curve fitting. The problem of what to do for the absence of data outside the measured range remains. The simplest assumption to make is that the true function represented by the data vanishes outside the measured interval. This could be a relatively poor choice if the function representing the data dies off rather slowly. Using this assumption, Eq. (14.16) is replaced by

$$Hf(x_i) \approx \frac{1}{\pi} P \int_{x_1}^{x_n} \frac{f(x)\mathrm{d}x}{x_i - x}, \tag{14.17}$$

where it is assumed that x_i coincides with one of the sampled values of x.

Let h denote the interval between consecutive data points, that is

$$h = x_{j+1} - x_j, \quad \text{for } j = 1, 2, \ldots, n-1. \tag{14.18}$$

On setting

$$g_j = \frac{f_j}{\pi(x_i - x_j)}, \tag{14.19}$$

it follows that

$$Hf(x_i) \approx 2h \sum_j^n g_j, \tag{14.20}$$

where the sum is taken over every second data point, starting with $j = 1$ if the index i is even, and $j = 2$ if the index i is odd. In this way the singular point $j = i$ is avoided. This elementary method would be expected to perform poorly if the underlying function is not smooth on the interval for which the measurements are obtained. The result given by Eq. (14.20) would of course be poor if the measurement interval were selected too narrowly, reflecting the fact that the asymptotic behavior of the function plays an important role in evaluating the Hilbert transform of the function.

If in place of the preceding situation, which focused on measured experimental data, the objective is the numerical evaluation of Hf for a given function, then a fairly large value for n can be employed together with a very small step size h. The accuracy of the calculation depends in an important manner on the overall smoothness of the function, and in particular, on the behavior of the function near the singularity.

14.3.2 The trapezoidal rule

Following on from Section 14.3.1, it is assumed that a discrete array of experimental values is available. The integration interval $[x_1, x_n]$ is divided into $n-1$ subintervals of length $h = (x_n - x_1)/n$. In each subinterval the function is assumed to have a constant value, so that

$$Hf(x_i) \approx h\left\{\frac{1}{2}g(x_1) + g(x_1+h) + g(x_1+2h) + \cdots + g(x_n-h) + \frac{1}{2}g(x_n)\right\}, \tag{14.21}$$

and the particular term involving the singularity is excluded. Equation (14.21) is obtained by using, on the interval $[x_i, x_{i+1}]$, the following approximation:

$$\int_{x_i}^{x_{i+1}} g(x)\mathrm{d}x \approx \frac{h}{2}\{g(x_i) + g(x_{i+1})\}, \tag{14.22}$$

which is the trapezoidal rule for $n = 1$. The outcome from the trapezoidal formula given in Eq. (14.21) can be improved by adding a contribution symmetric about the singular point, that is

$$\pi^{-1}P\int_{x_i-h/2}^{x_i+h/2} \frac{f(x)\mathrm{d}x}{x_i - x}.$$

This term can be approximated by

$$\pi^{-1}P\int_{x_i-h/2}^{x_i+h/2} \frac{f(x)\mathrm{d}x}{x_i - x} \approx -\pi^{-1}hf'(x_i), \tag{14.23}$$

where the prime denotes the derivative. This result is obtained by making a Taylor series expansion about the point x_i and dropping the higher-order derivatives beyond the first. As a first approximation, the derivative can be written as follows:

$$f'(x_i) \approx \{f(x_{i+1}) - f(x_{i-1})\}/(2h). \tag{14.24}$$

14.3.3 Simpson's rule

Consider an interval $[a, b]$ and subdivide it into an even number of intervals with a step size of $2h$. Suppose a quadratic fit is made to the points within each interval. For the first subinterval this means fitting a quadratic function to the values $f(a)$, $f(a+h)$, and $f(a+2h)$. The integral over this subinterval, denoted I_1, is given by

$$I_1 = h\{\alpha f(a) + \beta f(a+h) + \gamma f(a+2h)\}, \tag{14.25}$$

where the coefficients α, β, and γ are to be determined. A simple way to find these constants is to consider the case $f(x) = x^2$, which, on insertion into Eq. (14.25), leads to the following set of equations:

$$\begin{aligned} 2 &= \alpha + \beta + \gamma, \\ 2 &= \beta + 2\gamma, \\ \frac{8}{3} &= \beta + 4\gamma, \end{aligned} \tag{14.26}$$

and hence $\alpha = 1/3, \beta = 4/3$, and $\gamma = 1/3$. It therefore follows that

$$I_1 = \frac{h}{3}\{f(a) + 4f(a+h) + f(a+2h)\}. \tag{14.27}$$

Continuing this process for the entire interval $[a, b]$ leads to a formula of the type

$$\begin{aligned} Hf(x_i) \approx \frac{h}{3}\{&g(x_1) + 4g(x_1+h) + 2g(x_1+2h) + 4g(x_1+3h) \\ &+ \cdots + 4g(x_n - h) + g(x_n)\}. \end{aligned} \tag{14.28}$$

This is Simpson's rule applied to the integral in question. The singular point is omitted in this sum. The sum can be supplemented by adding a contribution centered on the singular point, in much the same manner as was indicated for the trapezoidal rule. Refinements to Simpson's rule could be made by assuming fits to higher-order polynomials, but that topic is not pursued here.

14.4 Gaussian quadrature: some basics

A general method for the numerical evaluation of integrals that is in widespread use is Gaussian quadrature. The purpose of this section is to discuss the Gaussian quadrature technique. The essential features of this procedure are outlined first, and the following section gives additional details and refinements of the approach. Sections 14.6 and 14.7 treat applications of the technique to the evaluation of the Hilbert transform and to one-sided Hilbert transforms, respectively.

In Gaussian quadrature schemes, the restriction to equally spaced evaluation points is dropped, which contrasts sharply with the approaches discussed in Section 14.3.

This has the immediate effect of doubling the number of variables that can be used to optimize the calculation of the integral. Since the assumption of equally spaced intervals is now abandoned, the function to be numerically integrated is assumed to be available. If experimental results have been collected as a discrete data set, this data set must first be fitted to an appropriate functional form. Any data extrapolation beyond the measured range must also be handled.

The second feature of considerable importance is that the weights and abscissa values can be determined so that the quadrature is exact (to approximately whatever machine precision is implied by the computer employed) for integrands of the form

$$f(x) = W(x)p(x), \tag{14.29}$$

where $p(x)$ denotes a polynomial and $W(x)$ is called a weight function. In what follows, the convention is adopted that the weight function satisfies

$$W(x) \geq 0, \quad \text{for } x \in [a, b]. \tag{14.30}$$

The moments of the weight function, m_j, defined by

$$m_j = \int_a^b W(x)x^j \, \mathrm{d}x, \tag{14.31}$$

are all assumed to be finite. If the weights and abscissa values are specifically tailored for the function $W(x)$, it follows that

$$\int_a^b f(x)\mathrm{d}x = \int_a^b W(x)p(x)\mathrm{d}x \approx \sum_{i=1}^{N} w_i p(x_i). \tag{14.32}$$

The $\approx$ sign is maintained, since approximate computer evaluations are employed. The important observation to note is that the function $W(x)$ no longer occurs explicitly in the summation term of the quadraure formula, Eq. (14.32). It appears implicitly in the values of $\{w_i, x_i\}$. By construction, a polynomial of order $2N - 1$ can be evaluated exactly, or more precisely to the machine precision of the computer employed, when N quadrature points are utilized in Eq. (14.32).

The procedure is illustrated with a simple example for the case $N = 2$. Suppose

$$\int_a^b f(x)\mathrm{d}x \approx w_1 f(x_1) + w_2 f(x_2), \tag{14.33}$$

and the weights w_1 and w_2 and the abscissa values x_1 and x_2 are selected so that the preceding equation is exact for polynomial functions up to third-order. If Eq. (14.33) is employed directly, then four equations in four unknowns can be obtained. For the choices $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$, the following equations are obtained:

$$b - a = w_1 + w_2, \tag{14.34}$$

$$\frac{1}{2}(b^2 - a^2) = w_1 x_1 + w_2 x_2, \tag{14.35}$$

$$\frac{1}{3}(b^3 - a^3) = w_1 x_1^2 + w_2 x_2^2, \tag{14.36}$$

and

$$\frac{1}{4}(b^4 - a^4) = w_1 x_1^3 + w_2 x_2^3. \tag{14.37}$$

Alternatively, these four equations can be obtained from the choice

$$f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3, \tag{14.38}$$

where the α_k are general constants. Equations (14.34)–(14.37) are nonlinear, so finding the solution of this set of equations is problematic. Extension beyond this simple example would lead to a highly intractable set of nonlinear equations. Instead of this direct approach, the following scheme is employed. Start with the polynomial function

$$g(x) = (x - x_1)(x - x_2), \tag{14.39}$$

which has the roots x_1 and x_2. If the function f is represented by g, the following results follow directly from Eq. (14.33):

$$\int_a^b g(x)\mathrm{d}x = 0 \tag{14.40}$$

and

$$\int_a^b x g(x)\mathrm{d}x = 0. \tag{14.41}$$

The integral $\int_a^b x^2 g(x)\mathrm{d}x$ is not necessarily zero, since the integrand is a fourth-order polynomial, and Eq. (14.33) is no longer exactly true in this case. To within an arbitrary multiplicative constant, Eqs. (14.40) and (14.41) can be used to determine $g(x)$. To simplify, suppose the integration interval is $[-1, 1]$. The elementary transformation $x = (1/2)(b - a)y + (1/2)(b + a)$ converts Eq. (14.33) to this integration interval. For this interval,

$$g(x) = P_2(x) = \frac{3}{2}\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right), \tag{14.42}$$

where $P_n(x)$ denotes a Legendre polynomial. Equation (14.42) follows from the fact that the Legendre polynomials form an orthogonal system on the interval $[-1, 1]$. For the integration interval $[-1, 1]$, Eqs. (14.34)–(14.37) simplify to

$$w_1 + w_2 = 2, \tag{14.43}$$

$$w_1x_1 + w_2x_2 = 0, \tag{14.44}$$

$$w_1x_1^2 + w_2x_2^2 = \frac{2}{3}, \tag{14.45}$$

and

$$w_1x_1^3 + w_2x_2^3 = 0. \tag{14.46}$$

Since the abscissa values are known to be $x_1 = -1/\surd 3$ and $x_2 = 1/\surd 3$, only the first two of the preceding set of four equations need be considered. This leaves a linear system of equations to solve for the weights w_1 and w_2, which contrasts sharply with the nonlinear system of equations that must be solved if the abscissa values are also treated as unknowns. From the preceding analysis, it follows that $w_1 = 1$ and $w_2 = 1$, and hence Eq. (14.33) simplifies to

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx f(-1/\surd 3) + f(1/\surd 3). \tag{14.47}$$

The example just treated can be readily generalized to include more terms in the expansion in Eq. (14.33). The following section discusses this generalization.

14.5 Gaussian quadrature: implementation procedures

If the integration range is fixed, it is possible to tabulate the values $\{w_i, x_i\}$ for different values of N for a variety of common functional forms. This has been done for the functions shown in Table 14.1 (Stroud and Secrest, 1966).

There is one fairly obvious question that the reader should ask at this point: what is the situation when the function of interest cannot be expressed in the form of Eq. (14.29)? As an example, suppose the required integral over the interval $[0, \infty)$ involves the function

$$f(x) = \frac{\mathrm{e}^{-ax}}{\surd(b + x^3)}, \quad a, b > 0. \tag{14.48}$$

A change of integration variable suggests the use of Gauss–Laguerre quadrature. The function $g(x) = \surd[a/(a^3b + x^3)]$ is not a polynomial function; hence, the formula

$$\int_0^{\infty} f(x)\mathrm{d}x = \int_0^{\infty} \frac{\mathrm{e}^{-ax}}{\surd(b + x^3)}\mathrm{d}x = \surd(a)\int_0^{\infty} \frac{\mathrm{e}^{-x}}{\surd(a^3b + x^3)}\mathrm{d}x \approx \sum_{i=1}^{N} w_i g(x_i) \tag{14.49}$$

Table 14.1. *Common functions with the associated integration ranges for which* $\{w_i, x_i\}$ *are available as a function of* N

Integration range	Weight function $W(x)$	Name
$[-1, 1]$	1	Gauss–Legendre quadrature (Gaussian quadrature)
$[-1, 1]$	$(1-x)^{\alpha}(1+x)^{\beta}$	Gauss–Jacobi quadrature
$[-1, 1]$	$\dfrac{1}{\sqrt{(1-x^2)}}$	Gauss–Chebyshev quadrature
$[0, \infty)$	e^{-x}	Gauss–Laguerre quadrature
$(-\infty, \infty)$	e^{-x^2}	Gauss–Hermite quadrature

gives an approximate result, independent of any errors associated with machine round-off. The extent to which $g(x)$ can be approximated by a polynomial of order $2N-1$ will give a direct indication of the expected effectiveness of a numerical quadrature on a function of the preceding type. For example, consider the evaluation of the following integral:

$$I = \int_0^1 \log\left(\frac{1+x}{1-x}\right)\mathrm{d}x. \tag{14.50}$$

This integral can be evaluated in closed form, with the result $I = 2\log 2 = 1.386\,294\,36\ldots$ If the weight function $W(x) = 1$ is used and a 32-point Gaussian quadrature employed, then the value obtained is 1.385 696, while a 384-point Gaussian quadrature yields the value 1.386 290 090 (King, Dykema, and Lund, 1992). Neither numerical value can be regarded as a highly accurate approximation to the exact result. In a situation like this, it is possible to resort to interval dissection techniques, where the integral is split into several integrals, and Gaussian quadrature applied to each interval. An alternative strategy is to keep expanding the size of the quadrature until convergence at some desired level is obtained. There is of course a limit to the size of N for which tabulated values of $\{w_i, x_i\}$ can be found in the literature, or can be conveniently computed.

The reason for the relatively low accuracy obtained for the preceding example is not difficult to uncover. The integrand has a slowly converging series representation, and is not well approximated by a simple polynomial function. A similar situation would apply to extensions of the form

$$I = \int_0^1 \log\left(\frac{1+x}{1-x}\right) f(x)\mathrm{d}x, \tag{14.51}$$

even if $f(x)$ is a relatively smooth function in the interval $[0, 1]$. Examples of the preceding type are candidates for specialized Gaussian quadrature. They are specialized in the sense that they are not presently part of the literature of commonly tabulated Gaussian quadrature values for $\{w_i, x_i\}$.

The approach for the evaluation of the values $\{w_i, x_i\}$ is now outlined. The scalar product for two functions $f(x)$ and $g(x)$ with weight function $W(x)$ on the interval $[a, b]$ is defined by

$$(f, g) = \int_a^b W(x)f(x)g(x)\mathrm{d}x. \tag{14.52}$$

If (f, g) is zero, the two functions are said to be orthogonal, and if (f, f) is unity, the function f is normalized. The set of polynomials that are orthogonal on the interval $[a, b]$ with the weight function $W(x)$ must be determined. Let p_j denote the jth-order polynomial

$$p(x) = x^j + a_1 x^{j-1} + \cdots + a_j; \tag{14.53}$$

then

$$(p_j, p_k) = 0, \quad \text{for } j \neq k. \tag{14.54}$$

The polynomial $p(x)$ is termed *monic*, because the leading coefficient (of the jth power of x) is unity. These polynomials can be constructed by a recursive scheme called the Gram–Schmidt orthogonalization. The first two polynomials are given by

$$p_0(x) = 1 \tag{14.55}$$

and

$$p_1(x) = x - \alpha_0, \tag{14.56}$$

with the higher-order polynomials constructed using the following recurrence relation:

$$p_{i+1}(x) = (x - \alpha_i)p_i(x) - \beta_i p_{i-1}(x), \quad \text{for } i \geq 1. \tag{14.57}$$

The constants α_i and β_i appearing in Eqs. (14.56) and (14.57) are calculated from the following expressions:

$$\alpha_i = \frac{(xp_i, p_i)}{(p_i, p_i)}, \quad \text{for } i \geq 0, \tag{14.58}$$

and

$$\beta_i = \frac{(p_i, p_i)}{(p_{i-1}, p_{i-1})}, \quad \text{for } i \geq 1. \tag{14.59}$$

The roots of the polynomial $p_N(x)$ are denoted by $x_1, x_2, \ldots, x_N$. The x_i are real, simple, and lie in the interval (a, b). A simple root has a multiplicity of one, that is, it occurs once. The solution of the system of equations

$$\sum_{i=1}^{N} p_j(x)w_i = \begin{cases} 0, & \text{for } j = 1, 2, \ldots, N-1 \\ (p_0, p_0), & \text{for } j = 0, \end{cases} \tag{14.60}$$

is denoted by $w_1, w_2, \ldots, w_N$. These weights can be evaluated using the standard methods of solving a system of linear equations, or by employing the alternative result

$$w_i = \frac{(p_{N-1}, p_{N-1})}{p_{N-1}(x_i)p'_N(x_i)}, \tag{14.61}$$

where the prime denotes a derivative. The key result is then given by

$$\int_a^b W(x)p(x)\mathrm{d}x = \sum_{i=1}^{N} w_i p(x_i), \tag{14.62}$$

where $p(x)$ is a normed polynomial of degree $2N - 1$.

One of the simplest cases is $W(x) = 1$ for the interval $[-1, 1]$. This is called Gaussian quadrature, in honor of Gauss, who formulated Eq. (14.61). It is also referred to as a Gauss–Legendre quadrature, which reflects the fact that the system of polynomials orthogonal on the interval $[-1, 1]$ with weight function one are the Legendre polynomials.

The problem therefore breaks down into three steps, which are as follows. (1) Determine the polynomial, that is find the coefficients a_i in Eq. (14.53). This is carried out using Eqs. (14.55)–(14.59). (2) Find the roots x_i of the polynomial. (3) Evaluate the weights w_i. For the classical polynomials, the recursive formulas have been studied in detail, and there are well known expressions for the coefficients a_i. On departing from the standard choices of weight functions, the Gaussian quadrature approach can become significantly non-trivial.

Consider, for example, the numerical evaluation of the integral given by

$$I = \int_0^1 \log(1/x)f(x)\mathrm{d}x, \tag{14.63}$$

and suppose that $f(x)$ is continuous in the interval $[0, 1]$. This integral will find application in the following section. Based on what has been described for the example in Eq. (14.51), a normal Gaussian quadrature is not expected to be particularly effective for the evaluation of Eq. (14.63). To deal with Eq. (14.63), the following weight function is employed:

$$W(x) = \log(1/x). \tag{14.64}$$

The set of orthogonal polynomials on the interval [0, 1] with the preceding weight function must be determined. From a numerical viewpoint, the unfavorable part of the integrand in this example is the region near $x \to 0+$. By sweeping this poor behavior up in the $\{w_i, x_i\}$ values, it is possible to obtain a very effective evaluation scheme when $f(x)$ is a smooth function on the interval [0, 1]. Unfortunately, this was an extremely difficult task to accomplish for a very long time, the reason being that the recursive system of Eqs. (14.55)–(14.59) becomes highly unstable for a numerical evaluation as the polynomial degree increases. The problem is often termed as being extremely *ill-conditioned*. Until very recently, the only values that have been published giving a reasonable number of digits, for the weights and abscissa for the choice of weight function given in Eq. (14.64), were restricted to $N \leq 16$ (Stroud and Secrest, 1966). This is not a particularly large value of N for a quadrature scheme. An example to substantiate this statement will be illustrated later in Table 14.3. The general advice that is often given is to avoid the recursive approach. This view is not entirely correct. With the development of modern symbolic algebra packages, it is possible to work with the recursive scheme given, either in analytic mode, or in numerical form using high-precision arithmetic.

To illustrate the approach just outlined, the numerical evaluation of the integral $\int_0^1 x^2 \log x^{-1}\, \mathrm{d}x$ is considered using a two-point Gaussian quadrature formula:

$$\int_0^1 x^2 \log x^{-1} \mathrm{d}x \approx \sum_{k=1}^{2} w_k f(x_k). \tag{14.65}$$

The monic polynomials $p_1(x)$ and $p_2(x)$ are given by

$$p_1(x) = x - \frac{1}{4} \tag{14.66}$$

and

$$p_2(x) = x^2 - \frac{5}{7}x + \frac{17}{252}. \tag{14.67}$$

From the latter equation, the roots x_1 and x_2 are determined to be as follows:

$$x_1 = \frac{5}{14} - \sqrt{\left(\frac{53}{882}\right)} \approx 0.112\,008\,806 \tag{14.68}$$

and

$$x_2 = \frac{5}{14} + \sqrt{\left(\frac{53}{882}\right)} \approx 0.602\,276\,908. \tag{14.69}$$

The weights are determined from Eq. (14.61) as follows:

$$w_i = \frac{(p_1, p_1)}{p_1(x_i)p_2'(x_i)} = \frac{49}{36(4x_i - 1)(14x_i - 5)}, \tag{14.70}$$

and so

$$w_1 \approx 0.718\,539\,319, \quad w_2 \approx 0.281\,460\,681. \tag{14.71}$$

Hence, from Eq. (14.65),

$$\begin{aligned}\int_0^1 x^2 \log x^{-1}\,\mathrm{d}x &\approx 0.718\,539\,319\,(0.112\,008\,806)^2 \\ &\quad + 0.281\,460\,681\,(0.602\,276\,908)^2 \\ &\approx 0.111\,111\,111,\end{aligned} \tag{14.72}$$

which compares very favorably, as expected, with the exact result of 1/9, which can be obtained from the following formula:

$$\int_0^1 s^m \log s^{-1}\,\mathrm{d}s = \frac{1}{(m+1)^2}. \tag{14.73}$$

Some significant refinements have been discovered which improve upon the recursive scheme, particularly when the integration interval is finite. One idea that has proved very useful is to replace the powers of x in Eq. (14.53) by known polynomials which form an orthogonal set. This has the potential to lead to enhanced numerical stability. The approach can be found in numerous references, but the following are particularly useful: Sack and Donovan (1972), Wheeler (1974), Gautschi (1990), and Press *et al.* (1992, p. 151). Gautschi supplies some historical comments, and Press *et al*. provide a Fortran source code to evaluate the following scheme. Let the required polynomials be denoted by $\rho_i(x)$, and assume that the moments m_i, defined by

$$m_i = \int_a^b W(x)\rho_i(x)\mathrm{d}x, \tag{14.74}$$

can be accurately determined. The new polynomials $\rho_i(x)$ satisfy a recursive scheme similar to Eqs. (14.55)–(14.57). The first two polynomials are given by:

$$\rho_0(x) = 1 \tag{14.75}$$

and

$$\rho_1(x) = x - c_0, \tag{14.76}$$

and the higher-order polynomials are found recursively using

$$\rho_{i+1}(x) = (x - c_i)\rho_i(x) - d_i\rho_{i-1}(x), \quad \text{for } i \geq 1. \tag{14.77}$$

The coefficients c_i and d_i are explicitly known because of the particular choice of $\rho_i(x)$. The coefficients α_i and β_i in the desired polynomial can be found via the following results:

$$\alpha_i = c_i - \frac{\sigma_{i-1,i}}{\sigma_{i-1,i-1}} + \frac{\sigma_{i,i+1}}{\sigma_{i,i}} \tag{14.78}$$

and

$$\beta_i = \frac{\sigma_{i,i}}{\sigma_{i-1,i-1}}, \tag{14.79}$$

where the $\sigma_{i,j}$ satisfy the recursive scheme

$$\sigma_{i,j} = \sigma_{i-1,j+1} - (\alpha_{i-1} - c_j)\sigma_{i-1,j} - \beta_{i-1}\sigma_{i-2,j} + d_j\sigma_{i-1,j-1}, \tag{14.80}$$

and are given by

$$\sigma_{i,j} = (p_i, \rho_j). \tag{14.81}$$

The initial values required to execute the recursive scheme are given by

$$\sigma_{0,i} = m_i, \tag{14.82}$$

$$\beta_0 = 0, \tag{14.83}$$

and

$$\alpha_0 = c_0 + (m_1/m_0). \tag{14.84}$$

The normalization factors for the original polynomial can be determined from

$$(p_0, p_0) = m_0 \tag{14.85}$$

and

$$(p_i, p_i) = \beta_i(p_{i-1}, p_{i-1}), \quad \text{for } i \geq 1. \tag{14.86}$$

Programming up the scheme just given is a straightforward exercise.

For the example given in Eq. (14.64), the following choice is employed:

$$\rho_i(x) = \frac{(i!)^2}{(2i)!} P_i(2x - 1), \tag{14.87}$$

where $P_i(x)$ is a Legendre polynomial and the prefactor of $P_i(2x - 1)$ ensures that $\rho_i(x)$ is monic. The function $P_i(2x - 1)$ is a called a shifted Legendre polynomial of degree i. The c_i and d_i coefficients in Eq. (14.77) and the modified moments of $\rho_i(x)$ in Eq. (14.74) are not difficult to determine. The results are as follows:

$$c_i = \frac{1}{2}, \quad i = 0, 1, 2, \ldots, \tag{14.88}$$

$$d_i = \frac{i^2}{4(4i^2 - 1)}, \quad i = 0, 1, 2, \ldots, \tag{14.89}$$

and

$$m_i = \begin{cases} 1, & \text{for } i = 0 \\ \dfrac{(-1)^i (i!)^2}{i(i+1)(2i!)^2}, & \text{for } i \geq 1 \end{cases}. \tag{14.90}$$

14.6 Specialized Gaussian quadrature: application to the Hilbert transform

Let the function $f(x)$ satisfy the following conditions: $f(x)$ is continuous in the interval $(-\infty, \infty)$ and, for a constant c,

$$\lim_{x \to 0} \{f[(1+x)c] - f[(1-x)c]\} = O(x^m), \quad \text{with } m > 0. \tag{14.91}$$

Using a change of variable, the Hilbert transform for $x \neq 0$ can be written as follows:

$$\begin{aligned} Hf(x) &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f[x(s+1)]\mathrm{d}s}{s} \\ &= -\frac{1}{\pi} \left\{ P \int_{-\infty}^{-1} \frac{f[x(s+1)]\mathrm{d}s}{s} + P \int_{-1}^{1} \frac{f[x(s+1)]\mathrm{d}s}{s} + P \int_{1}^{\infty} \frac{f[x(s+1)]\mathrm{d}s}{s} \right\} \\ &= -\frac{1}{\pi} \left\{ P \int_{-1}^{1} \frac{f[x(s+1)]\mathrm{d}s}{s} + P \int_{1}^{\infty} \frac{f[x(s+1)]\mathrm{d}s}{s} - P \int_{1}^{\infty} \frac{f[x(1-s)]\mathrm{d}s}{s} \right\}. \end{aligned} \tag{14.92}$$

In the preceding equation, the first integral simplifies to

$$\begin{aligned} P \int_{-1}^{1} \frac{f[x(s+1)]\mathrm{d}s}{s} &= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{\{f[x(s+1)] - f[x(1-s)]\}\mathrm{d}s}{s} \\ &= \int_{0}^{1} \frac{\{f[x(s+1)] - f[x(1-s)]\}\mathrm{d}s}{s}, \end{aligned} \tag{14.93}$$

where the initial assumption on the behavior of the numerator in the lim $s \to 0$ has been employed. Utilizing an integration by parts yields

$$\begin{aligned}
P\int_{-1}^{1}\frac{f[x(s+1)]\mathrm{d}s}{s} &= \int_0^1 \{f[x(1+s)] - f[x(1-s)]\}\frac{\mathrm{d}\log s}{\mathrm{d}s}\,\mathrm{d}s \\
&= \{f[x(1+s)] - f[x(1-s)]\}\log s|_0^1 \\
&\quad + \int_0^1 \log s^{-1}\{f'[x(1+s)] - f'[x(1-s)]\}\mathrm{d}s \\
&= \int_0^1 \log s^{-1}\{f'[x(1+s)] - f'[x(1-s)]\}\mathrm{d}s, \qquad (14.94)
\end{aligned}$$

where the prime denotes differentiation with respect to s, and the following result has been employed:

$$\lim_{s\to 0}\{f[(1+s)x] - f[(1-s)x]\}\log s = \lim_{s\to 0} s^m \log s = 0, \quad \text{since } m > 0. \qquad (14.95)$$

The logarithmic factor in Eq. (14.94) is treated as a weight function, and has therefore been put into a form such that it satisfies the positive requirement indicated in Eq. (14.30). Employing the change of variable $s = t^{-1}$, the second and third integrals in Eq. (14.92) can be recast as follows:

$$\begin{aligned}
&P\int_1^{\infty}\frac{f[x(s+1)]\mathrm{d}s}{s} - P\int_1^{\infty}\frac{f[x(1-s)]\mathrm{d}s}{s} \\
&= P\int_0^1 \frac{\{f[x(1+t^{-1})] - f[x(1-t^{-1})]\}\mathrm{d}t}{t} \\
&= \int_0^1 \{f[x(1+t^{-1})] - f[x(1-t^{-1})]\}\frac{\mathrm{d}\log t}{\mathrm{d}t}\,\mathrm{d}t \\
&= \{f[x(1+t^{-1})] - f[x(1-t^{-1})]\}\log t|_0^1 \\
&\quad + \int_0^1 \log t^{-1}\{f'[x(1+t^{-1})] - f'[x(1-t^{-1})]\}\mathrm{d}t. \qquad (14.96)
\end{aligned}$$

If $f(x)$ is $L^2(-\infty,\infty)$, then

$$\lim_{t\to 0}\{f[x(1+t^{-1})] - f[x(1-t^{-1})]\}\log t = \lim_{t\to 0} t^n \log t = 0, \quad \text{since } n > 0. \qquad (14.97)$$

Equation (14.92) can be expressed as follows:

$$Hf(x) = \int_0^1 \log s^{-1} K(s,x)\mathrm{d}s, \qquad (14.98)$$

where

$$K(s,x) = \pi^{-1}\{f'[x(1-s)] - f'[x(1+s)] \\ + f'[x(1-s^{-1})] - f'[x(1+s^{-1})]\}, \quad \text{for } x \neq 0. \tag{14.99}$$

In the preceding result the derivatives are with respect to the variable s. For the case where $x = 0$, it follows that

$$Hf(0) = \int_0^1 \log s^{-1} K(s,0)\mathrm{d}s, \tag{14.100}$$

with

$$K(s,0) = \pi^{-1}\{f'(-s) - f'(s) + f'(-s^{-1}) - f'(s^{-1})\}. \tag{14.101}$$

Some simplification can be obtained using the even–odd character of $f(x)$. If $f(x)$ is an even function,

$$K(s,0) = 0 \tag{14.102}$$

and

$$K(s,x) = \pi^{-1}\{f'[x(1-s)] - f'[x(1+s)] \\ + f'[x(s^{-1}-1)] - f'[x(s^{-1}+1)]\}, \quad \text{for } x \neq 0; \tag{14.103}$$

if $f(x)$ is an odd function,

$$K(s,0) = -2\pi^{-1}\{f'(s) + f'(s^{-1})\} \tag{14.104}$$

and

$$K(s,x) = \pi^{-1}\{f'[x(1-s)] - f'[x(1+s)] \\ - f'[x(s^{-1}-1)] - f'[x(s^{-1}+1)]\}, \quad \text{for } x \neq 0. \tag{14.105}$$

Equations (14.98) and (14.100) are in a form suitable for Gaussian quadrature, where the weight function is identified with $\log s^{-1}$. The result for the Hilbert transform then takes the following form:

$$Hf(x) \approx \sum_{i=1}^{N} w_i K(x_i, x), \tag{14.106}$$

where N denotes the number of evaluation points, and the weights w_i and evaluation points x_i are determined from the set of polynomials based on the weight function $\log s^{-1}$. The principal advantage of this approach is that the singularity in the original integral is now incorporated in the set $\{x_i, w_i\}$. While the determination of the weights and evaluation points is a major computational assignment, this calculation need only be carried out once. There is the possibility for loss of accuracy in the evaluation of $K(x_i, x)$, but some numerical experiments with representative functional forms indicate that this problem does not arise to any significant extent. Test functions could, however, be constructed where loss of accuracy is likely during the evaluation of $K(x_i, x)$.

The moments that are needed to determine the weights and evaluation points are very easy to evaluate; they are given by Eq. (14.73). The solution of the recursion scheme given in Eqs. (14.55) – (14.59) for the case where the weight function is $\log s^{-1}$ is numerically highly unstable. Several authors have tabulated $\{w_i, x_i\}$ for the case when $\log s^{-1}$ is the weight function (Krylov and Pal'cev, 1963; Anderson, 1965; Stroud and Secrest, 1966), but either the accuracy is limited, or N is not very large. The numerical instability of the recursion scheme explains the limitations of earlier calculations of $\{w_i, x_i\}$. The numerical difficulties can be circumvented by working in higher-precision arithmetic using packages such as *Mathematica*, or working with codes capable of performing high-precision calculations. The author has calculated the required $\{w_i, x_i\}$ up to $N = 100$ in steps of 10. These weights and abscissa points are given in Table 14.1.1 in Appendix 14.1 of this chapter. They were evaluated using *Mathematica* software. The associated error factors are given in Table 14.2.1.

Applications of the use of Eq. (14.106) fall into two main types. The first are those problems where the function is specified, but the Hilbert transform cannot be evaluated in terms of known functions. The second group of examples comprise those where the function is unknown, but is instead represented by a set of discrete experimental data points. A representative case is now considered.

The selected example can be evaluated in closed form. The availability of the analytic solution serves as a valuable comparison point for the numerical quadrature approach. Suppose a collection of experimental data, which is of the form of a set of discrete points $\{I_i, \omega_i\}$, where I_i might, for example, represent an intensity measurement, is fitted to a Lorentzian profile. The Lorentzian function takes the form

$$I(\omega) = \frac{1}{\pi} \frac{a}{a^2 + (\omega - \omega_0)^2}, \tag{14.107}$$

where a and ω_0 are constants, and ω is an angular frequency. The factor of π^{-1} in Eq. (14.107) is selected so that the Lorentzian encloses unit area on the interval $(-\infty, \infty)$. An alternative normalization for the Lorentzian can be chosen so that the curve encloses unit area on $[0, \infty)$. There are several issues associated with the fitting process. What underlying physical reasoning leads the experimentalist to believe that a Lorentzian profile will provide a satisfactory fit to the experimental data? Since the data are collected over a finite range, can the experimentalist be sure that, outside the measurement interval, the Lorentzian will be a reasonable representation of the data? While these are important issues, they are separate from the actual numerical transformation of the data that is now considered. The Hilbert transform

Table 14.2. *Application of a logarithmic Gaussian quadrature to evaluate the Hilbert transform of the Lorentzian*
The values $a = 1$ and $\omega_0 = 1$ have been employed

ω	Quadrature result for Hilbert transform	Percentage error
0.1	−0.158 275 634 0	2.1×10^{-8}
0.2	−0.155 273 115 211 605 207	-8.0×10^{-16}
0.5	−0.127 323 954 473 516 268 615 1	-1.2×10^{-30}
0.9	−0.315 158 303 152 267 991 621 5	-3.1×10^{-30}
1.0	-3×10^{-30}	–
2.0	0.159 154 943 091 845 335 768 8	0
5.0	0.074 896 443 807 950 746 244 176	-6.2×10^{-21}
10.0	0.034 936 450 922 611	-4.1×10^{-13}
20.0	0.016 706 872 47	-4.0×10^{-8}
30.0	0.010 963 167	3.7×10^{-6}
40.0	0.008 156 432	3.0×10^{-5}
50.0	0.006 493 4	4.6×10^{-4}

of the Lorentzian can be obtained in closed form:

$$HI(\omega) = \frac{1}{\pi} \frac{\omega - \omega_0}{a^2 + (\omega - \omega_0)^2}. \tag{14.108}$$

A comparison of the numerical quadrature formula versus the exact result is shown in Table 14.2 as a function of angular frequency based on a quadrature with $N = 60$. The calculations were carried out in quadruple precision using a 32 bit word (about 30–31 digits) with the weights and abscissas input with 30 digit accuracy. It is useful to keep in mind that experimental data are being dealt with, where typically no more than three to five digits of precision are available for the data, and therefore an error of around 10^{-2} percent in the Hilbert transformation would be adequate. This condition is met, except at $\omega = 1$. The percentage error at $\omega = 1$ is governed by machine round-off. The exact value for the Hilbert transform at this point is zero, and the calculated quadrature value is -3×10^{-30}, which is in excellent agreement with the true result. The accuracy of the numerical quadrature result improves with increasing N. A comparison of Eq. (14.106) with the exact result in Eq. (14.108) is shown in Table 14.3 for two selected values of ω. From these results it appears that values of around $N = 20$ to $N = 30$ are sufficient to obtain the Hilbert transformation to about the accuracy of the experimental data. Since the time for the numerical evaluation of Eq. (14.106) is relatively trivial, assuming the availability of the abscissas points and weights, the safest approach is to employ the largest size quadrature possible.

Table 14.3. *Application of a logarithmic Gaussian quadrature to evaluate the Hilbert transform of the Lorentzian*
The values $a = 1$ and $\omega_0 = 1$ have been employed

Test value: $\omega = 0.1$, exact result = −0.158 275 634 014 039 560 433 144 1

N	Quadrature result for Hilbert transform	Percentage error
10	−0.2	32
20	−0.1575	-4.7×10^{-1}
30	−0.158 279	2.4×10^{-3}
40	−0.158 275 70	4.3×10^{-5}
50	−0.158 275 632	-1.4×10^{-6}
60	−0.158 275 634 05	2.1×10^{-8}

Test value: $\omega = 10$; exact result = 0.034 936 450 922 611 171 266 340 338 301 28

N	Quadrature result for Hilbert transform	Percentage error
10	0.034 2	−2.2
20	0.034 935 8	-1.8×10^{-3}
30	0.034 936 46	3.0×10^{-5}
40	0.034 936 450 94	3.8×10^{-8}
50	0.034 936 450 922 53	2.2×10^{-10}
60	0.034 936 450 922 611 0	-4.1×10^{-13}

14.6.1 Error estimates

When quadrature points are available for large values of N, a useful approach to assess the accuracy of the calculation is to examine the rate of convergence of the numerical quadrature as an increasing number of quadrature points are employed. An alternative approach is to estimate the error in terms of information available for the function to be integrated. A standard result in Gaussian quadrature theory is as follows:

$$\int_a^b W(x)f(x)\mathrm{d}x - \sum_{n=1}^{N} w_i f(x_i) = \frac{f^{(2n)}(\xi)}{(2n)!}(p_n, p_n), \tag{14.109}$$

where $\xi \in (a, b)$ and p_n is the monic polynomial orthogonal on the interval (a, b) with weight function $W(x)$. The proof of this result is straightforward (see, for example, Stoer and Bulirsch (1980), p. 151). Let e_r designate an error factor and let M_r denote the maximum of $f^{(r)}(\xi)$; then

$$\left| \int_a^b W(x)f(x)\mathrm{d}x - \sum_{n=1}^{N} w_i f(x_i) \right| \leq e_r M_r. \tag{14.110}$$

This formula assumes that the rth derivative of the function f is continuous on the interval $[a, b]$. A sequence of error estimates involving progressively higher derivatives up to $2n$ can be given, the most useful of which is the error estimate with $r = 2n$, for which

$$e_{2n} = \frac{1}{(2n)!} \int_a^b W(x)[p_n(x)]^2 \, \mathrm{d}x. \tag{14.111}$$

Following on from the example considered in Section 14.5, the errors associated with the evaluation of the integral $\int_0^1 x^m \log x^{-1} \, \mathrm{d}x$ are considered for $m = 2$ and $m = 4$, using a two-point Gaussian quadrature formula. It is useful to keep in mind that the two-point formula is exact for polynomials up to order three; that is, that the numerical accuracy is limited only by computer round-off errors. For the case $m = 2$, the previously computed (see Eq. (14.72)) numerical quadrature value for the integral was determined as 0.111 111 111, and, using the fact that $M_{2n} \equiv M_4 = f^{(4)}(\xi) = 0$, the associated error with this result is zero, and it is concluded that the result is accurate to the number of significant digits carried in the calculation. For the case $m = 4$, the quadrature estimate is given by

$$\begin{aligned} \int_0^1 x^4 \log x^{-1} \, \mathrm{d}x &\approx 0.718\,539\,319(0.112\,008\,806)^4 \\ &\quad + 0.281\,460\,681(0.602\,276\,908)^4 \\ &\approx 0.037\,147\,266\,3. \end{aligned} \tag{14.112}$$

The exact result (from Eq. (14.73)) is 0.04, and $M_{2n} \equiv M_4 = f^{(4)}(\xi) = 24$. From Eqs. (14.111), (14.67), and (14.73), it follows that

$$4!e_4 = \int_0^1 \log x^{-1} \left[x^2 - \frac{5}{7}x + \frac{17}{252}\right]^2 \mathrm{d}x = \frac{647}{226\,800} \approx 0.002\,852\,733\,686, \tag{14.113}$$

and hence the error term is

$$e_4 M_4 \approx 0.002\,852\,733\,686. \tag{14.114}$$

The integral $\int_0^1 x^4 \log x^{-1} \, \mathrm{d}x$ is therefore bounded above by the value

$$\int_0^1 x^4 \log x^{-1} \, \mathrm{d}x \leq 0.037\,147\,266\,3 + 0.002\,852\,733\,686 \approx 0.040\,000\,000\,0, \tag{14.115}$$

and hence, from Eq. (14.109), this result gives an improved value for the numerical quadrature of the integral. This estimate is observed to be in agreement with the exact result to the stated number of digits.

14.7 Specialized Gaussian quadrature: application to H_e and H_o

For the analysis of functions that have a particular even or odd symmetry, and depend on a variable that takes on positive values, for example a frequency, it is more common to write the Hilbert transforms as follows:

$$Hf(x) = \frac{2x}{\pi} P \int_0^\infty \frac{f(s)\mathrm{d}s}{x^2 - s^2}, \quad \text{for } f(x) \text{ even,} \tag{14.116}$$

and

$$Hf(x) = \frac{2}{\pi} P \int_0^\infty \frac{sf(s)\mathrm{d}s}{x^2 - s^2}, \quad \text{for } f(x) \text{ odd.} \tag{14.117}$$

Recall $Hf \equiv H_e f$ if f is an even function and $Hf \equiv H_o f$ if f is an odd function. These are the forms (to within a sign for Eq. (14.117)) which are referred to in the literature as the *Kramers–Kronig transforms* or the *Kramers–Kronig relations*. The factor $(x^2 - s^2)^{-1}$ can be resolved into partial fractions in two different ways. The combination $\{(x-s)^{-1} + (x+s)^{-1}\}(2x)^{-1}$ leads to

$$\frac{2x}{\pi} P \int_0^\infty \frac{f(s)\mathrm{d}s}{x^2 - s^2} = Hf(x), \quad \text{for } f(x) \text{ even,} \tag{14.118}$$

or, using the combination $\{(x-s)^{-1} - (x+s)^{-1}\}(2s)^{-1}$,

$$\frac{2x}{\pi} P \int_0^\infty \frac{f(s)\mathrm{d}s}{x^2 - s^2} = xH\left\{\frac{f(x)}{x}\right\}, \quad \text{for } f(x) \text{ even.} \tag{14.119}$$

Similarly,

$$\frac{2}{\pi} P \int_0^\infty \frac{sf(s)\mathrm{d}s}{x^2 - s^2} = Hf(x), \quad \text{for } f(x) \text{ odd,} \tag{14.120}$$

or

$$\frac{2}{\pi} P \int_0^\infty \frac{sf(s)\mathrm{d}s}{x^2 - s^2} = x^{-1}H\{xf(x)\}, \quad \text{for } f(x) \text{ odd.} \tag{14.121}$$

Repeating the analysis outlined in Section 14.6 for Eq. (14.118) leads to

$$\frac{2x}{\pi} P \int_0^\infty \frac{f(s)\mathrm{d}s}{x^2 - s^2} = \int_0^1 \log s^{-1} K(s,x)\mathrm{d}s, \quad \text{for } f(x) \text{ even,} \tag{14.122}$$

with $K(s,x)$ defined by Eq. (14.103) for $x \neq 0$ and by Eq. (14.102) for $x = 0$. From Eq. (14.120), it follows that

$$\frac{2}{\pi} P \int_0^\infty \frac{sf(s)\mathrm{d}s}{x^2 - s^2} = \int_0^1 \log s^{-1} K(s,x)\mathrm{d}s, \quad \text{for } f(x) \text{ odd,} \tag{14.123}$$

with $K(s,x)$ defined by Eq. (14.105) for $x \neq 0$ and by Eq. (14.104) for $x = 0$. Equation (14.121) leads to

$$\frac{2}{\pi} P \int_0^\infty \frac{sf(s)\mathrm{d}s}{x^2 - s^2} = \int_0^1 \log s^{-1} K_1(s,x)\mathrm{d}s, \quad \text{for } f(x) \text{ odd}, \tag{14.124}$$

with $K_1(s,x)$ defined by

$$\begin{aligned} K_1(s,x) = \frac{1}{\pi x} \{ & g'[x(1-s)] - g'[x(1+s)] \\ & + g'[x(1-s^{-1})] - g'[x(s^{-1}+1)]\}, \quad \text{for } x \neq 0, \end{aligned} \tag{14.125}$$

with $g(s) = sf(s)$. For $x = 0$, set $K_1 \equiv K$ and employ Eq. (14.104). Starting with Eq. (14.119) leads to

$$\frac{2x}{\pi} P \int_0^\infty \frac{f(s)\mathrm{d}s}{x^2 - s^2} = \int_0^1 \log s^{-1} K_2(s,x)\mathrm{d}s, \quad \text{for } f(x) \text{ even}, \tag{14.126}$$

with $K_2(s,x)$ defined by

$$\begin{aligned} K_2(s,x) = x\pi^{-1} \{ & g'[x(1-s)] - g'[x(1+s)] \\ & + g'[x(1-s^{-1})] - g'[x(1+s^{-1})]\}, \quad \text{for } x \neq 0, \end{aligned} \tag{14.127}$$

and $g(s) = s^{-1}f(s)$. The advantage of Eq. (14.123) versus Eq. (14.124), or of Eq. (14.122) versus Eq. (14.126), depends in part on the behavior of the function f in the vicinity of the origin. The forms given in Eqs. (14.122) and (14.124) would be more favorable to use when avoiding numerical problems near the origin, while the forms given in Eqs. (14.123) and (14.126) would be more useful to improve the numerical accuracy as $|x| \to \infty$. For some choices of f, only one of the two forms might lead to a convergent integral.

14.8 Numerical integration of the Fourier transform

Because of the very wide occurrence of Fourier transforms in a multitude of applications, extensive work has been devoted to the numerical evaluation of the Fourier transform. Only a few of the basics are touched upon in this section; Section 14.9 will elaborate on the numerical implementation of the fast Fourier transform; and Section 14.10 exploits these ideas to evaluate the Hilbert transform numerically.

If the function f has a continuous nth derivative on the interval $[a, b]$, then repeated application of integration by parts for a truncated Fourier transform leads to

$$\int_a^b e^{i\omega t} f(t)\mathrm{d}t = \frac{e^{i\omega b}}{i\omega}\sum_{k=0}^{n-1}\left(\frac{i}{\omega}\right)^k f^{(k)}(b) - \frac{e^{i\omega a}}{i\omega}\sum_{k=0}^{n-1}\left(\frac{i}{\omega}\right)^k f^{(k)}(a)$$
$$+ (-i\omega)^{-n}\int_a^b e^{i\omega t} f^{(n)}(t)\mathrm{d}t, \tag{14.128}$$

where $f^{(k)}$ signifies the kth derivative of f. The limits can be extended to cover the range $(-\infty, \infty)$ if some additional constraints on the asymptotic behavior of f and its derivatives are imposed, which will be done momentarily. Given the structure of Eq. (14.128), a numerical evaluation is not likely to be successful when ω is large. There is bound to be significant cancellation of terms of opposite sign, which always provides an opportunity for loss of numerical accuracy. For large values of the parameter ω, other numerical approaches should be explored.

The connection between the Fourier transform and the discrete Fourier transform was made in Section 13.5, and that discussion assumed that the Fourier transform was truncated at some upper limit. The upper limit b would be chosen such that the remaining integrals,

$$I(\omega) = \int_{-\infty}^{-b} g(t)e^{i\omega t}\,\mathrm{d}t + \int_b^\infty g(t)e^{i\omega t}\,\mathrm{d}t, \tag{14.129}$$

are small relative to the other integral contributing to the Fourier transform, Eq. (14.128). If $g(t)$ falls off sufficiently quickly as $t \to \pm\infty$, then the value of b does not need to be chosen to be very large. In some cases it may be desirable to make an effort to evaluate $I(\omega)$ numerically. One approach is to employ an asymptotic expansion. Suppose $g(t)$ is infinitely differentiable on the interval $(-\infty, -b] \cup [b, \infty)$ and satisfies

$$g^{(n)}(t) = O(t^{-1-\varepsilon}), \quad \text{as } t \to \infty, \tag{14.130}$$

where $\varepsilon > 0$ and $g^{(n)}$ signifies the nth derivative of g. Application of integration by parts leads to the following result:

$$\int_{-\infty}^{-b} g(t)e^{i\omega t}\,\mathrm{d}t + \int_b^\infty g(t)e^{i\omega t}\,\mathrm{d}t \sim \sum_{k=0}\left(\frac{i}{\omega}\right)^{k+1}[e^{ib\omega}g^{(k)}(b) - e^{-ib\omega}g^{(k)}(-b)]. \tag{14.131}$$

If ω is not too small and the derivatives of the function g fall off quickly for large arguments, the sum in Eq. (14.131) would be expected to be amenable to numerical evaluation. Note that there is still the possibility for loss of accuracy due to partial cancellation of terms with opposite signs. For very large values of ω, the result for the sum of the two integrals would be expected to be fairly close to zero, and there is also the likelihood for loss of numerical accuracy due to cancellation of contributions of opposite sign.

There are many additional ideas that have been developed to evaluate the Fourier transform numerically. The interested reader is referred to the references given in the chapter end-notes. Perhaps the most popular approach is the fast Fourier transform method, which is now revisited.

14.9 The fast Fourier transform: numerical implementation

The following result was given in Section 13.5:

$$\mathcal{F}[f(k/L)] \approx \Delta \sum_{n=0}^{N-1} g(nL/N) \mathrm{e}^{-\mathrm{i}2\pi nk/N} \equiv \Delta G_k, \tag{14.132}$$

where the function g is defined in Eq. (13.35) and G_k denotes the DFT of g. This result can be modified in a straightforward manner to cover the case where the lower limit of the integration range for the truncated Fourier transform is taken to be α rather than zero. Setting $s_n = \alpha + n\Delta$ and $\omega_k = 2\pi k/L$, then

$$\mathcal{F}[f(k/L)] \approx \Delta \mathrm{e}^{-\mathrm{i}\alpha\omega_k} \sum_{n=0}^{N-1} g(s_n) \mathrm{e}^{-\mathrm{i}2\pi nk/N} \equiv \Delta \mathrm{e}^{-\mathrm{i}\alpha\omega_k} G_k, \tag{14.133}$$

which for $\alpha = 0$ reduces to Eq. (14.132). How useful this or the preceding formula will be depends in a critical way on the size of ω_k. For fairly small values of ω_k, the results given are numerically stable with satisfactory accuracy obtainable; however, for large values of this parameter, the formulas are not suitable. Large values of ω_k imply significant oscillation for the integrand of the truncated Fourier transform. The value of the integral is likely to be small in such cases (think about the Riemann–Lebesgue lemma in the limit of extremely large oscillatory behavior), and truncation errors then play an important role in determining the accuracy of the final numerical result.

When it is necessary to go beyond Eq. (14.133), which may often be the case, an interpolation scheme can be constructed that leads to Eq. (14.133) multiplied by an appropriate weighting function plus a correction term for endpoint evaluations. The exact form of the weighting function and the endpoint corrections depend on the specific type of interpolation process employed. The interested reader can pursue the details of such formulas in Press *et al*. (1992, p. 578). It is not necessary to perform any major software development to implement these procedures, as standard codes are available (see, for example, Press *et al*. (1992), p. 580).

14.10 Hilbert transform via the fast Fourier transform

Recall from Section 5.2 that

$$Hf(x) = -\mathrm{i}\mathcal{F}^{-1}\{\operatorname{sgn} t\, \mathcal{F}f(t)\}(x). \tag{14.134}$$

From this result, the numerical evaluation of the Hilbert transform is therefore replaced by a numerical evaluation of a Fourier transform and an inverse Fourier transform. Since well developed and computationally fast algorithms are available to evaluate the Fourier transform, that is the FFT, then Eq. (14.134) may allow the numerical evaluation of the Hilbert transform to be carried forward in a very effective manner.

A key ingredient is that the function f should vanish sufficient quickly for large values of the argument so that the Fourier transform can be truncated to some convenient finite interval, and further that the resulting integrand incorporating the $\operatorname{sgn} x$ contribution should also vanish sufficiently quickly for large arguments. When these conditions are not satisfied, an attempt to estimate the remaining integral after truncation should be carried out, along the lines discussed in Section 14.8.

Software is available in a number of sources to set up this type of calculation. See, for example, Bertie and Zhang (1992) or Press *et al.* (1992). An Internet search will also locate depositories of software to carry out these calculations.

14.11 The Hilbert transform via the allied Fourier integral

Let the function $f(z)$ be analytic in the upper half complex plane and vanish sufficiently quickly as $|z| \to \infty$ (say z^{-m} with $m > 1$), and suppose that on the real axis the following holds:

$$f(x) = g(x) + \mathrm{i}h(x), \tag{14.135}$$

where g and h are real functions and satisfy the even and odd symmetry conditions

$$g(-x) = g(x) \tag{14.136}$$

and

$$h(-x) = -h(x). \tag{14.137}$$

Giving one of these symmetry results suffices to establish the other, since, under the stated conditions on $f(x)$, h is the Hilbert transform of g, and the even–odd property of the Hilbert transform (Section 4.2) implies that h is an odd function if g is an even function, and vice versa. In place of the standard Hilbert transform pair, relations connecting g and h can be written in the following alternative form:

$$g(x) = \frac{2}{\pi}\int_0^\infty \cos xt\,\mathrm{d}t \int_0^\infty \sin st\, h(s)\mathrm{d}s \tag{14.138}$$

and

$$h(x) = \frac{2}{\pi}\int_0^\infty \sin xt\,\mathrm{d}t \int_0^\infty \cos st\, g(s)\mathrm{d}s, \tag{14.139}$$

which are the allied Fourier integral forms discussed in Section 3.12.

These results can be readily derived by first considering the contour integral $\oint_C \mathrm{e}^{\mathrm{i}tz} f(z)\mathrm{d}z$, where the contour C is taken as a semicircle, center the origin with diameter located along the x-axis and radius R, which is allowed to become infinite. It then follows, by applying Jordan's lemma, that

$$\int_0^\infty g(x)\cos xt\,\mathrm{d}x = \int_0^\infty h(x)\sin xt\,\mathrm{d}x, \quad \text{for } t > 0. \tag{14.140}$$

The Fourier integral formula is given by

$$\mathrm{f}(x) = \frac{1}{\pi}\int_0^\infty \mathrm{d}u \int_{-\infty}^\infty \cos[u(x-t)]\,\mathrm{f}(t)\mathrm{d}t. \tag{14.141}$$

Employing Eq. (14.141), with $\mathrm{f}(x)$ taken first as $g(x)$ and then $h(x)$, and utilizing Eq. (14.140) and the symmetry conditions Eqs. (14.136) and (14.137), leads to Eqs. (14.138) and (14.139).

The FFT technique is ideally suited for the numerical evaluation of the integrals in Eqs. (14.138) and (14.139). The one and obvious drawback is that the one-dimensional quadrature required to evaluate a standard Hilbert transform is replaced by the necessity of carrying out a two-dimensional quadrature. The obvious positive feature is that it is no longer required to deal with a singularity in the integrand. Even allowing for the limitation of using the FFT twice, numerical quadrature on the allied Fourier integral forms can be carried out computationally with very good efficiency.

Software to carry out the quadratures of the key formulas is available, and the interested reader might begin with the very popular book by Press *et al.* (1992, p. 577). For studies on the implementation of this approach (including a source listing) Bertie and Zhang (1992) is a useful starting point.

14.12 The Hilbert transform via conjugate Fourier series

Let $f(z)$ satisfy the same conditions as stated at the start of the preceding section and assume that g and h have the same symmetry properties as given in Eqs. (14.136) and (14.137). A conformal mapping of the upper half complex plane into the interior of the unit disc in the complex w plane is performed. This is accomplished by the transformation

$$w = \frac{z - \mathrm{i}}{z + \mathrm{i}}. \tag{14.142}$$

Points on the real axis in the z-plane map onto the boundary of the unit disc in the w-plane. The function $f(z)$ can be written as

$$f(z) \to f\left[-\mathrm{i}\frac{w+1}{w-1}\right] = \sum_{n=0}^{\infty} c_n w^n, \quad \text{for } |w| \leq 1. \tag{14.143}$$

Making use of the change of variable $w = \mathrm{e}^{\mathrm{i}\theta}$ yields

$$f\left[-\mathrm{i}\frac{w+1}{w-1}\right] = \sum_{n=0}^{\infty} c_n \cos n\theta + \mathrm{i}\sum_{n=1}^{\infty} c_n \sin n\theta. \tag{14.144}$$

The following definitions are introduced:

$$g_w(\theta) = g\left(\cot\left(\frac{\theta}{2}\right)\right) \tag{14.145}$$

and

$$h_w(\theta) = h\left(\cot\left(\frac{\theta}{2}\right)\right), \tag{14.146}$$

then it follows that

$$f(z) \to f\left(-\cot\frac{\theta}{2}\right) = g_w(\theta) - \mathrm{i}h_w(\theta). \tag{14.147}$$

If it is now assumed that a Fourier series expansion can be made for the functions g_w and h_w, hence

$$g_w(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \tag{14.148}$$

and

$$h_w(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta, \tag{14.149}$$

where it has been assumed that the even–odd symmetry properties given in Eqs. (14.136) and (14.137) apply. If these last two results are inserted into

Eq. (14.147), then

$$f\left(-\cot\frac{\theta}{2}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta - \mathrm{i}\sum_{n=1}^{\infty} b_n \sin n\theta. \tag{14.150}$$

Comparing Eqs. (14.150) and (14.144) leads to

$$a_0 = 2c_0, \tag{14.151}$$

$$a_n = c_n, \quad b_n = -c_n, \quad \text{for } n \neq 0, \tag{14.152}$$

and hence

$$a_n = -b_n, \quad \text{for } n \neq 0. \tag{14.153}$$

The series representations for g_w and h_w then become

$$g_w(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \tag{14.154}$$

and

$$h_w(\theta) = -\sum_{n=1}^{\infty} a_n \sin n\theta, \tag{14.155}$$

which are recognized as a conjugate Fourier series pair.

In a practical application, g_w represents a dispersive mode and h_w is the corresponding dissipative mode. If experimental data are available to construct the series expansion for g_w, then h_w is determined directly from the conjugate series. If a Fourier series representation for h_w is available, then g_w is determined to within the constant a_0 from the conjugate Fourier series. To fix the constant a_0, the following approach might be applied. The asymptotic behavior for g, and hence g_w, will invariably be known from the physics of the problem. For example, if

$$\lim_{x\to\infty} g(x) = \lim_{\theta\to 0} g\left(\cot\frac{\theta}{2}\right) = 0 \tag{14.156}$$

which is the behavior typically required to write the Hilbert transform relations in the first place, then

$$a_0 = -2\sum_{n=1}^{\infty} a_n. \tag{14.157}$$

If the coefficients a_n, for $n \neq 0$, can be evaluated from the data available for h_w, then the conjugate Fourier series together with the asymptotic constraint given in

Eq. (14.157) determines g_w. Evaluating $g(0)$ directly from the appropriate Hilbert transform relation for g yields

$$a_0 = \frac{4}{\pi} P \int_0^\infty \frac{h(x)\mathrm{d}x}{x} - 2\sum_{n=1}^\infty (-1)^n a_n, \tag{14.158}$$

or the allied Fourier integral representation yields

$$a_0 = \frac{4}{\pi} \int_0^\infty \mathrm{d}t \int_0^\infty h(x) \sin xt \, \mathrm{d}x - 2\sum_{n=1}^\infty (-1)^n a_n. \tag{14.159}$$

These formulas may be used as a check on the value of a_0 determined from Eq. (14.157).

To implement the conjugate series algorithm for a set of experimental data, two approaches are possible. The first procedure is to fit the experimental data to a functional form suggested by the physics of the problem, one for which the series expansion can be determined analytically. For example, suppose the experimental data is fit to a function g given by

$$g(x) = \frac{1}{1+x^2}. \tag{14.160}$$

For the purposes of this simple example, the fact that $h(x)$ can be readily determined by direct evaluation of the Hilbert transform is ignored. Now

$$g_w(\theta) = \sin^2\left(\frac{\theta}{2}\right), \tag{14.161}$$

and from Eq. (14.148) it follows that

$$a_0 = 1, \quad a_n = -\frac{\delta_{n1}}{2}, \quad \text{for } n \neq 0. \tag{14.162}$$

Equation (14.155) then yields

$$h_w(\theta) = \frac{\sin\theta}{2}, \tag{14.163}$$

and from Eq. (14.146) the following result is obtained:

$$h(x) = \frac{x}{1+x^2}. \tag{14.164}$$

The second approach is to fit the data directly in terms of a trigonometric sine and/or cosine series expansion, by making use of the trigonometric transformation $\theta = 2\cot^{-1} x$. The Hilbert transform is obtained immediately as the conjugate series. Since $\sin n\theta$ can be expanded as $\sin\theta$ multiplied by a finite series in powers of $\cos\theta$, and $\cos n\theta$ can be expanded in terms of a finite series in powers of $\cos\theta$, then use of the transformations

$$\cos\theta \to \frac{x^2-1}{x^2+1}, \quad \sin\theta \to \frac{2x}{x^2+1} \tag{14.165}$$

converts the data back to the real line representation. The success of this type of series approach depends in large part on how accurately the data can be fit by a compact trigonometric expansion.

14.13 The Hilbert transform of oscillatory functions

The general issues associated with the numerical evaluation of integrals with oscillatory integrands were touched upon in Section 14.8. Similar issues emerge for the evaluation of the Hilbert transform of functions with oscillatory integrands. This section examines an approach which is fairly robust and which could be extended in several different ways to deal with a variety of functions. The key to the method employed is the application of convergence accelerator techniques (see Section 2.16) to handle the summations that arise.

Hilbert transforms of functions with oscillatory behavior arise in a number of problems in the physical sciences, engineering, and applied mathematics. Some of the simplest examples are the sine, cosine, and other simple combinations of these trigonometric functions, which occur widely in applications.

Recall two important results given previously: for a function f satisfying $f \in L^p(\mathbb{R})$ for $1 < p < \infty$, the Hilbert transform of f satisfies $Hf \in L^p(\mathbb{R})$; and, for the case $p = 1$, Hf exists almost everywhere, but in general is not integrable. The focus of most of the published literature on the numerical evaluation of Hilbert transforms and other singular integrals covers the case of functions belonging to the class L^p (for $p \geq 1$). In this section, some examples that fall outside the aforementioned class $L^p(\mathbb{R})$ are also considered. For example, the elementary choice $f(x) = \sin ax$, with a denoting a real constant, yields $\int_{-\infty}^{\infty} |\sin ax|^p \, dx = \infty$ for $p > 0$.

The method presented is based on an approach taken by King *et al.* (2002). The integration domain is split as follows:

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(x)}{x_0 - x} \, dx = \frac{1}{\pi} \int_{-\infty}^{x_0-\tau} \frac{f(x)}{x_0 - x} \, dx + \frac{1}{\pi} P \int_{x_0-\tau}^{x_0+\tau} \frac{f(x)}{x_0 - x} \, dx + \frac{1}{\pi} \int_{x_0+\tau}^{\infty} \frac{f(x)}{x_0 - x} \, dx, \tag{14.166}$$

and it is assumed that the function f is oscillatory in nature. A standard numerical quadrature approach can be employed to evaluate the Cauchy principal value integral on the right-hand side of the preceding equation. In the examples to be presented shortly, the widely available software package *Mathematica* was employed. The *CauchyPrincipalValue* object in *Mathematica* can be used to evaluate the second integral on the right-hand side of Eq. (14.166) over a small interval $(x_0 - \tau, x_0 + \tau)$, where τ was selected to be 10^{-3} on the basis of some trial calculations. This particular choice yields high accuracy for a number of test functions that were examined. The integrals over $(-\infty, x_0 - \tau)$ and $(x_0 + \tau, \infty)$ are separately partitioned into two sets of integrals. The range for each integral in the set is determined from the oscillatory characteristics of the function, which requires the determination of the roots of the function. Each integral in a set is treated as a term of a slowly converging infinite series. Convergence acceleration techniques (recall Section 2.16) are then applied to obtain the desired accuracy. The Levin-u transformation is fairly robust and appears to work well in applications. This convergence accelerator takes the following form:

$$u_k = \frac{\sum_{j=0}^{k} (-1)^j \binom{k}{j} (j+1)^{k-2} (S_{j+1}/a_{j+1})}{\sum_{j=0}^{k} (-1)^j \binom{k}{j} (j+1)^{k-2} (1/a_{j+1})}, \tag{14.167}$$

where $\binom{k}{j}$ is a binomial coefficient, a_j denotes the jth term in the series to be summed, and S_j designates the jth partial sum, that is

$$S_j = \sum_{n=1}^{j} a_n. \tag{14.168}$$

An additional factor of $(k+1)^{k-2}$ can be inserted into the denominator of both sums in Eq. (14.167) to improve the stability of the calculation.

To illustrate the technique, some test functions are examined that satisfy the following basic requirements:

(a) $Hf(x)$ does not diverge for all x, and
(b) $Hf(x)$ can be evaluated in analytic form.

The latter condition has been selected for the obvious reason that it allows an important check to be made on the quality of the results from the numerical evaluations.

When Hilbert transforms of the form $H\{f(x)g(x)\}$ are encountered, where f exhibits oscillatory behavior and g is a continuous function with a suitable asymptotic behavior as $x \to \pm\infty$, then a viable computational approach is to try the technique just outlined on Hf. This assumes that Hf does not diverge, and further assumes that it is possible to evaluate Hf analytically, so the quality of the numerical result can be evaluated. Then, depending on the smoothness properties of g, a reasonable assumption is that, if Hf can be successfully evaluated numerically, the same would probably be true for $H\{f(x)g(x)\}$. Note that when f exhibits oscillatory behavior, there are a very large number of examples that can be worked out in analytic form; however, the inclusion of the function g, even for some fairly simple choices, can very quickly make the solution of $H\{f(x)g(x)\}$ in terms of standard special functions a very difficult, if not impossible, assignment. Series techniques may be a feasible evaluation strategy in such cases. Some test results are presented in Table 14.4.

In the entries in Table 14.4 the evaluation point x_0 employed in Eq. (14.166) has been simplified to x. The working precision was set at four times the requested precision required for the final answer. This particular scaling of the working precision could be decreased for most of the examples examined, as the accuracy obtained for the numerical results exceeded what has been reported in Table 14.4. The following notation has been employed: the imaginary part is denoted by Im; recall from Section 5.2 that the error function is defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-s^2}\,\mathrm{d}s; \tag{14.169}$$

sgn denotes the signum function (sign function); and recall from Section 4.15 that the sinc function is given by

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}, \tag{14.170}$$

and that the Fresnel cosine and sine integrals are defined, respectively, by

$$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right)\mathrm{d}t \tag{14.171}$$

and

$$S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right)\mathrm{d}t. \tag{14.172}$$

For the Hilbert transform of oscillatory functions whose integrands resemble those of Figures 14.2 and 14.3, the approach outlined is expected to work fairly well, which is supported by the results presented in Table 14.4. In the preceding two cases the oscillations die off fairly uniformly after some distance on both sides of the singularity at $x = 2$.

The choice $(\sin bx)^{-1} \sin 2abx$, for a and b constants, represents a more difficult case. A plot of $\{\pi(2 - x)\sin bx\}^{-1} \sin 2abx$ is shown in Figure 14.4. In this example there are major oscillations centered about the singular point, but as Figure 14.4 clearly illustrates, there is a non-monotonic decay of the oscillatory behavior away from the singular point. The approach described handles this situation, but the accuracy obtained is not as high as that determined for some of the other examples reported in Table 14.4. Further refinements in the partitioning technique employed, coupled with a different selection of convergence accelerator techniques, would likely lead to improved accuracy.

Two considerably more difficult examples are now examined. The first is $f(x) = \cos(ax^{-1})$, where a is a constant, for which $Hf(x) = -\operatorname{sgn} a \sin(ax^{-1})$. The function $\pi^{-1}(2 - x)^{-1}\cos(x^{-1})$ is displayed in the vicinity of the origin in Figure 14.5.

Table 14.4. *Numerical quadrature values for the Hilbert transform versus exact evaluation for some oscillatory functions*
The values $x = 2$, $a = 5$, and $b = 3$ have been employed

$f(x)$	$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(s)\mathrm{d}s}{x-s}$ (exact result)	Numerical quadrature result	From the exact result
$\sin ax$	$-\mathrm{sgn}\, a \cos ax$	0.839 071 529 076 452	0.839 071 529 076 452
$\cos ax$	$\mathrm{sgn}\, a \sin ax$	−0.544 021 110 889 369	−0.544 021 110 889 369
$\frac{\cos x}{x^2+a^2}$	$\frac{x\mathrm{e}^{-a} + a \sin x}{a(x^2+a^2)}$	0.031 448 020 883 631 6	0.031 448 020 883 631 6
$\frac{\sin^2(ax)}{x^2}$	$\frac{\mathrm{sgn}\, a\{2ax - \sin 2ax\}}{2x^2}$	2.385 881 843 659 046	2.385 881 843 659 046
$\frac{\sin^5 x}{x^5}$	$\frac{6 + 5x^2 + (115/12)x^4 - 10\cos x + 5\cos 3x - \cos 5x}{16x^5}$	0.369 403 759 103 775	0.369 403 759 103 775
$\cos \pi x\, \mathrm{e}^{-x^2}$	$\mathrm{e}^{-x^2} \mathrm{Im}\left\{\mathrm{e}^{\mathrm{i}\pi x} \mathrm{erf}\,(\pi/2 + \mathrm{i}x)\right\}$	0.013 915 590 535 066	0.013 915 590 535 066
$\mathrm{sinc}\, x$	$\frac{1 - \cos \pi x}{\pi x}$	$6.275\,95 \times 10^{-34}$	0
$\sin ax^2 (a > 0)$	$-\mathrm{sgn}\, x\{S(\surd(2a/\pi)\,\lvert x\rvert)[\cos ax^2 + \sin ax^2]$ $+C(\surd(2a/\pi)\,\lvert x\rvert)[\cos ax^2 - \sin ax^2]\}$	−0.316 830 1	−0.316 829 655 318 096
$\frac{\sin 2abx}{\sin bx} (a > 0, b > 0)$	$\frac{2\sin^2(abx)}{\sin(bx)}$	−6.987 17	−6.987 489 931 661 428
$\cos x^{-1}$	$-\sin x^{-1}$	−0.479 369	−0.479 425 538 604 203

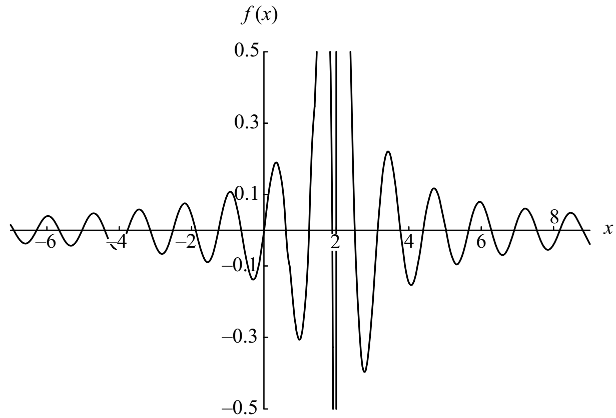

Figure 14.2. Plot of the integrand function $\pi^{-1}(2-x)^{-1}\sin 5x$.

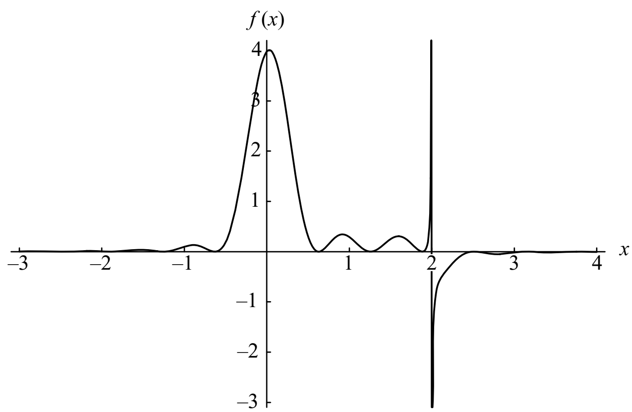

Figure 14.3. Plot of the integrand function $\pi^{-1}x^{-2}(2-x)^{-1}\sin^2 5x$.

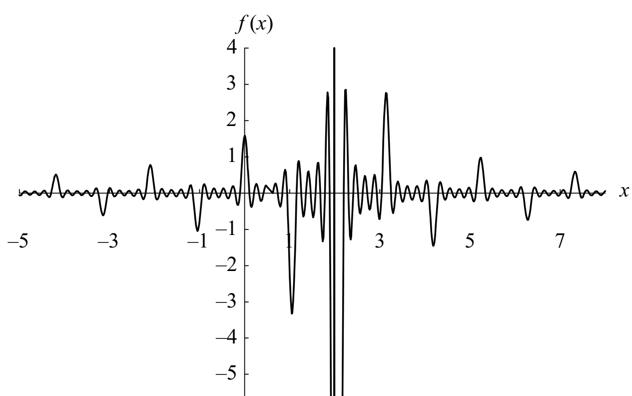

Figure 14.4. Plot of the integrand function $\{\pi(2-x)\sin bx\}^{-1}\sin 2abx$ for $a = 5$ and $b = 3$.

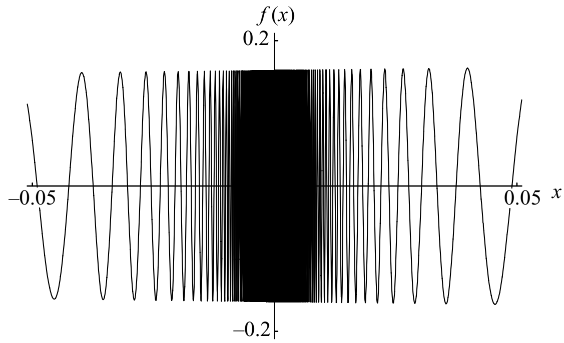

Figure 14.5. Plot of the integrand function $\pi^{-1}(2-x)^{-1}\cos(x^{-1})$.

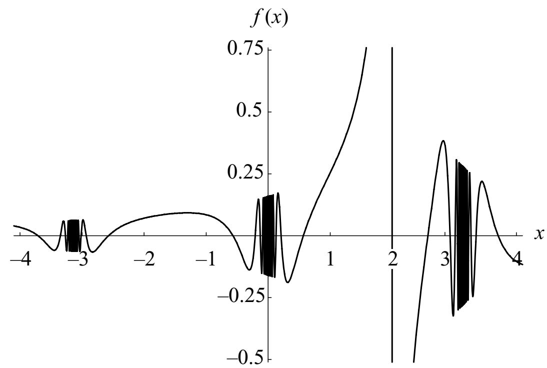

Figure 14.6. Plot of the integrand function $\{\pi(2-x)\}^{-1}\cos(\alpha\cot x)$ for $\alpha = 1$.

Inspection of Figure 14.5 immediately reveals the expected difficulties to be associated with this example. The extreme oscillation as $x \to 0$ makes this case rather recalcitrant to solve with a general robust type algorithm. A modification of the previously described procedure can be applied to cases such as this example. For convenience, assume the singularity is located in $(1, \infty)$, a choice that results in no loss of generality. The integration is split into $(-\infty, -1)$, $(-1, 1)$, $(1, x_0-\tau)$, $(x_0-\tau, x_0+\tau)$, and $(x+\tau, \infty)$, and the first, third, fourth, and fifth intervals are treated as previously described. The same procedure is also applied to the second interval, but the origin is approached from both the points –1 and 1. As the center of the pocket of extreme oscillation is approached, the number of roots spirals so significantly that a cutoff in the number of determined roots must obviously be made. Using this modification a modest accuracy level is obtained, as evidenced by the values reported for the appropriate entry in Table 14.4.

As a final example, consider $f(x) = \cos(\alpha\cot x)$, where α is a constant and $\alpha > 0$, for which $Hf(x) = -\sin(\alpha\cot x)$. Figure 14.6 illustrates a plot of $\{\pi(2 - x)\}^{-1} \cos(\alpha \cot x)$. It should be clear that it will be more difficult to handle the multiple packets of extreme oscillations that occur periodically beyond the singular point. A modification of the procedure used to deal with the case $\cos x^{-1}$ should work on functions that have periodically placed pockets of extreme oscillation, provided that the overall decay characteristics of the function are not too slow. Oscillatory functions with non-periodically placed pockets of extreme oscillation would be significantly more difficult to deal with using a general robust type of algorithm.

For all the examples presented in Table 14.4, an attempt was made to evaluate numerically the integrals directly using *Mathematica*. It is left as an exercise for the reader to find which of the examples presented in the table can be numerically evaluated directly employing the *Mathematica* software.

For wildly oscillatory integrands, or for integrands whose asymptotic nature is not well approximated as a suitably convergent infinite series, the direct application of the approach described is likely to be unsatisfactory. Such cases need to be dealt with on a case-by-case basis. An examination of the asymptotic characteristics may be suggestive of two or more slowly convergent series. If this is the situation, the principal series may be broken up into multiple infinite series and convergence accelerated individually. Here it may be possible to use the larger oscillations in a separate series from the smaller oscillations.

The technique discussed can also be utilized with very little change to cover the case of non-oscillatory continuous functions with suitable decay characteristics. Discussion of this possibility is not pursued, but is left for the interested reader to explore independently.

14.14 An eigenfunction expansion

An approach developed by Weideman (1995) to evaluate numerically the Hilbert transform in terms of an eigenfunction expansion is outlined in this section. Recall from Section 4.17 that the eigenfunctions of the Hilbert transform operator can be written as follows:

$$\phi_n(x) = \frac{(1 + \mathrm{i}x)^n}{(1 - \mathrm{i}x)^{n+1}}, \quad \text{for } n \in \mathbb{Z}^+, \tag{14.173}$$

with

$$H\phi_n = -\mathrm{i} \operatorname{sgn} n\, \phi_n, \quad \text{for } |n| \geq 1, \tag{14.174}$$

and

$$H\phi_0 = -\mathrm{i}\phi_0. \tag{14.175}$$

In this section the last two equations are combined using the following notation:

$$sign(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ -1, & \text{for } n < 0. \end{cases} \tag{14.176}$$

The preceding function is introduced since the signum function for $n = 0$ is typically not defined, but when it is the most common assignment is $\operatorname{sgn} 0 = 0$. The functions ϕ_n form an orthogonal basis for $L^2(\mathbb{R})$. This is easily verified by using the change of variable

$$x = \tan(\theta/2), \tag{14.177}$$

so that

$$\frac{1 + \mathrm{i}x}{1 - \mathrm{i}x} = \mathrm{e}^{\mathrm{i}\theta}, \tag{14.178}$$

and hence

$$\begin{aligned} \int_{-\infty}^{\infty} \phi_m^*(x)\phi_n(x)\mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta \\ &= \pi\delta_{nm}. \end{aligned} \tag{14.179}$$

Weideman expands the function of interest:

$$f(x) = \sum_{n=-\infty}^{\infty} a_n\phi_n(x), \tag{14.180}$$

where a_n are expansion coefficients. From this result it follows that

$$Hf(x) = \sum_{n=-\infty}^{\infty} a_n H\phi_n(x) = -\mathrm{i}\sum_{n=-\infty}^{\infty} a_n\, sign(n)\, \phi_n(x). \tag{14.181}$$

Weidman examined several examples, and determined the form of the coefficients a_n. The simplest example to consider is as follows:

$$f(x) = \frac{1}{1 + x^2} = \frac{1/2}{1 - \mathrm{i}x} + \frac{1/2}{1 + \mathrm{i}x}, \tag{14.182}$$

which allows the expansion coefficients to be evaluated by inspection:

$$a_{-1} = 1/2, \quad a_0 = 1/2, \quad a_n = 0, \quad \text{for } n \neq 0, -1. \tag{14.183}$$

Hence, from Eq. (14.181),

$$\begin{aligned}
Hf(x) &= -\mathrm{i} \sum_{n=-\infty}^{\infty} a_n \, sign(n) \, \phi_n(x) \\
&= \frac{\mathrm{i}}{2(1+\mathrm{i}x)} - \frac{\mathrm{i}}{2(1-\mathrm{i}x)} \\
&= \frac{x}{1+x^2}.
\end{aligned} \tag{14.184}$$

A more complicated case is the following example:

$$f(x) = \frac{1}{1+x^4}. \tag{14.185}$$

The expansion coefficients are obtained from

$$a_n = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_n^*(x) f(x) \mathrm{d}x. \tag{14.186}$$

First, note that the coefficients a_n satisfy

$$a_{-n} = a_{n-1}, \tag{14.187}$$

$$a_{2n} = -a_{2n-1}, \quad \text{for } n \geq 1, \tag{14.188}$$

and

$$a_{2n} = \frac{1}{2}(-1)^n (\sqrt{(2)} - 1)^{2n}. \tag{14.189}$$

From Eqs. (14.187) and (14.188), Hf is determined as follows:

$$\begin{aligned}
Hf(x) &= -\mathrm{i} \sum_{n=-\infty}^{\infty} a_n \, sign(n) \phi_n(x) \\
&= -\mathrm{i} \sum_{n=0}^{\infty} a_n \{\phi_n(x) - \phi_{-n-1}(x)\} \\
&= -\mathrm{i} \sum_{n=0}^{\infty} a_{2n} \{\phi_{2n}(x) - \phi_{2n-1}(x) + \phi_{-2n}(x) - \phi_{-2n-1}(x)\} \\
&\quad - \mathrm{i} a_0 \{\phi_{-1}(x) - \phi_0(x)\},
\end{aligned} \tag{14.190}$$

Table 14.5. *Evaluation of the Hilbert transform of a Gaussian function using an eigenfunction expansion*

Number of terms in expansion	$Hf(1/2)$
10	0.478 763 067
20	0.478 927 901
30	0.478 925 108
40	0.478 925 172
50	0.478 925 173

which simplifies to

$$\begin{aligned} Hf(x) &= -\frac{x}{1+x^2} + \frac{2x}{1+x^2}\sum_{n=0}^{\infty} a_{2n}\left\{\frac{(1+\mathrm{i}x)^{2n}}{(1-\mathrm{i}x)^{2n}} + \frac{(1-\mathrm{i}x)^{2n}}{(1+\mathrm{i}x)^{2n}}\right\} \\ &= -\frac{x}{1+x^2} + \frac{2x}{1+x^2}\left\{\frac{(1+x^2)^2+\alpha^2(1-6x^2+x^4)}{(1+\alpha^2)^2(1+x^4)}\right\}, \end{aligned} \tag{14.191}$$

where $\alpha = \surd(2) - 1$ and the following expansion has been employed:

$$(1+z)^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad \text{for } |z| < 1. \tag{14.192}$$

Hence,

$$Hf(x) = \frac{x(1+x^2)}{\surd(2)\,(1+x^4)}. \tag{14.193}$$

The two examples just considered demonstrate how the approach is implemented. However, in a practical application, analytic formulas are not generally available for the expansion coefficients a_n; it is then necessary to obtain reasonable estimates for these coefficients by numerical integration procedures. Weideman considered several examples, including $\mathrm{e}^{-|x|}, \mathrm{e}^{-x^2}$, and $(1+x^4)^{-1}\sin x$, and the focus of the method revolves around the construction of suitable estimates for the coefficients a_n. Asymptotic-type formulas for the coefficients a_n for several examples were given by Weideman, and the convergence properties of the coefficients for each example were investigated. Weideman (1995) may be consulted for further details.

Consider the example $f(x) = \mathrm{e}^{-x^2}$; then, for $x = 1/2$, $Hf(1/2) \approx 0.478\,925\,172\,9\ldots$ Table 14.5 shows the results for the numerical evaluation of $Hf(1/2)$ using the expansion formula Eq. (14.181) truncated at N terms. The coefficients used to determine the table entries were worked out by numerical quadrature. The convergence is observed to be reasonably quick for this example.

14.15 The finite Hilbert transform

This chapter concludes with some comments on the numerical evaluation of the finite Hilbert transform. Since the limits on the integration are now finite, the reader should expect that the approach taken to carry out the numerical evaluation would be easier than for the Hilbert transform on the real line. And indeed, this is the case. Several of the approaches that were discussed earlier in the chapter can be applied directly to the finite Hilbert transform. One technique is considered, and the interested reader is referred to the references given in the chapter end-notes for other approaches.

The finite Hilbert transform on the interval $[-1,1]$ can be expressed, for $x \in (-1,1)$, as follows:

$$\begin{aligned} Tf(x) &= \frac{1}{\pi} P \int_{-1}^{1} \frac{f(s)\mathrm{d}s}{x-s} \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \left\{ \int_{\varepsilon}^{1+x} \frac{f(x-s)\mathrm{d}s}{s} - \int_{\varepsilon}^{1-x} \frac{f(x+s)\mathrm{d}s}{s} \right\} \\ &= \frac{1}{\pi} \int_{0}^{1} \log t^{-1} \{ f'(x-t) - f'(x+t) \} \mathrm{d}t \\ &\quad + \frac{x}{\pi} \int_{-1}^{1} \left\{ \frac{f[(x/2)(1-t)-1]}{x(t+1)+2} - \frac{f[(x/2)(1-t)+1]}{x(t+1)-2} \right\} \mathrm{d}t, \end{aligned} \tag{14.194}$$

where the prime denotes derivative with respect to t. The standard change of variable

$$s = \frac{(b-a)}{2} t + \frac{b+a}{2} \tag{14.195}$$

has been employed to convert

$$\int_{a}^{b} f(s)\mathrm{d}s \to \int_{-1}^{1} g(t)\mathrm{d}t, \tag{14.196}$$

with

$$g(t) = \frac{(b-a)}{2} f \left\{ \frac{(b-a)t}{2} + \frac{(b+a)}{2} \right\}. \tag{14.197}$$

To carry out the integration by parts, the function f is assumed to be Hölder continuous on the interval $(-1,1)$ with an exponent m, so that

$$|f(x-\varepsilon) - f(x+\varepsilon)| \leq C\varepsilon^{m}, \tag{14.198}$$

with C a constant and $m > 0$. This condition allows the following limit to be employed:

$$\lim_{\varepsilon \to 0} [f(x-\varepsilon) - f(x+\varepsilon)] \log \varepsilon = 0, \tag{14.199}$$

which is required to obtain Eq. (14.194). In the form given in Eq. (14.194), the first integral can be evaluated using a Gaussian quadrature with the weight function $W(t) = \log t^{-1}$, and the second integral can be treated using a standard Gauss–Legendre quadrature. Since $x \in (-1, 1)$, the second of the two integrals in Eq. (14.194) is not a Cauchy principal value integral. If the evaluation point x in the second integral takes a value very close to one of the endpoints, the Gauss-Legendre quadrature may be rather susceptible to errors in the vicinity of $t \approx \pm 1$, depending on the behavior of $f(t)$ as $t \to \pm 1$.

Table 14.6 shows the results obtained from Eq. (14.194) for some simple test cases. The special functions appearing in the table are the cosine integral, the sine integral, recall Eqs. (8.78) and (8.79), respectively, the exponential integral, defined by

$$E_n(z) = \int_1^\infty \frac{e^{-zy}\, dy}{y^n}, \quad \text{for } n = 0, 1, 2, \ldots, \quad \text{with } \operatorname{Re} z > 0, \tag{14.200}$$

and the hyperbolic sine integral function,

$$\operatorname{Shi}(z) = \int_0^z \frac{\sinh t\, dt}{t}. \tag{14.201}$$

The specialized Gaussian quadrature with the log weight was carried out using $N = 60$ and the Gauss–Legendre quadrature was carried using $N = 384$. The number of points employed for the latter quadrature should be more than sufficient to achieve a reasonable level of accuracy for the second integral in Eq. (14.194), provided the function has an acceptable polynomial approximation on the interval $(-1, 1)$. For the results presented in Table 14.6, the accuracy of the results obtained corresponds to the accuracy of the weights and abscissa values employed.

A more difficult case is represented by the function $f(x) = \surd(1 - x^2)$. For this multivalued function the change of integration variables that yields Eq. (14.194) now leads to the separate integrals being complex-valued. This is not a particular difficulty. However, the resulting integrand in the first integral in Eq. (14.194) is not well approximated by a polynomial function, even a high-order one, and as a consequence the accuracy obtained is significantly diminished. Table 14.7 shows some representative values as a function of the number of quadrature points used for the specialized Gaussian quadrature. No interval dissection has been employed. The imaginary component should of course be zero. The errors resulting from the second integral in Eq. (14.194) are significantly smaller than those arising from the specialized quadrature of the first integral. These results could be readily improved

Table 14.6. *Comparison of numerical quadrature values for the finite Hilbert transform versus exact evaluation*
The evaluation point $x = 1/2$ has been employed

$f(x)$	$\frac{1}{\pi} P \int_{-1}^{1} \frac{f(s)\mathrm{d}s}{x-s}$ (exact result)	Numerical quadrature result	From the exact result
x	$\frac{x}{\pi} \log \left\lvert \frac{1+x}{1-x} \right\rvert - \frac{2}{\pi}$	−0.461 770 196 084 551 454 076 799 412 43	−0.461 770 196 084 551 454 076 799 412 43
x^2	$\frac{x^2}{\pi} \log \left\lvert \frac{1+x}{1-x} \right\rvert - \frac{2x}{\pi}$	−0.230 885 098 042 275 727 038 399 706 2	−0.230 885 098 042 275 727 038 399 706 2
e^x	$\frac{\mathrm{e}^x}{\pi} \{E_1(1-x) - E_1(1+x) - 2\,\mathrm{Shi}(1-x)\}$	−0.290 867 255 078 251 192 988 580 844 10	−0.290 867 255 078 251 192 988 580 844 10
$x\mathrm{e}^x$	$\frac{x\mathrm{e}^x}{\pi} \{E_1(1-x) - E_1(1+x) - 2\,\mathrm{Shi}(1-x)\} + \frac{1}{\pi}(\mathrm{e}^{-1} - \mathrm{e})$	−0.893 589 943 922 752 362 331 580 674 76	−0.893 589 943 922 752 362 331 580 674 76
$\sin x$	$\frac{\sin x}{\pi} \{\mathrm{Ci}(1+x) - \mathrm{Ci}(1-x)\} - \frac{\cos x}{\pi} \{\mathrm{Si}(1+x) + \mathrm{Si}(1-x)\}$	−0.408 877 509 399 954 681 778 827 405 25	−0.408 877 509 399 954 681 778 827 405 25
$\cos x$	$\frac{\cos x}{\pi} \{\mathrm{Ci}(1+x) - \mathrm{Ci}(1-x)\} + \frac{\sin x}{\pi} \{\mathrm{Si}(1+x) + \mathrm{Si}(1-x)\}$	0.458 459 218 905 942 313 272 080 055 092	0.458 459 218 905 942 313 272 080 055 092

Table 14.7. *Numerical quadrature of the finite Hilbert transform for the function* $f(x) = \sqrt{(1 - x^2)}$
The evaluation point $x = 0.1$ has been employed and the exact result is $(Tf)(x) = x$, and hence the exact numerical result is 0.1

Number of quadrature points	Numerical quadrature for $\frac{1}{\pi} P \int_{-1}^{1} \frac{f(s)\,ds}{x-s}$	Percentage error, real part	Imaginary component
10	0.137 536 660 598 4	37.5	0.085 6
20	0.092 158 493 445 2	−7.84	−0.029 0
30	0.094 029 198 118 1	−5.97	−0.009 9
40	0.095 143 703 429 0	−4.86	−0.005 0
50	0.095 922 078 892 8	−4.08	−0.002 7
60	0.096 518 134 571 6	−3.48	−0.001 4
70	0.097 003 524 441 9	−3.00	−0.000 55
80	0.097 417 061 264 4	−2.58	0.000 038
90	0.097 782 196 885 0	−2.22	0.000 47
100	0.098 114 445 667 3	−1.89	0.000 79

by carrying out appropriate subdivisions of the integration interval, and refining the results until some pre-selected cutoff tolerance has been achieved.

To summarize, when the function and its derivatives are likely to be well approximated by a modest sized polynomial expansion, then the quadrature approach just outlined will be reasonably effective. For functions that cannot be so represented, the accuracy of the approach is likely to be modest at best. In this case, a rather large number of specialized quadrature points are required to achieve a reasonably accurate result, or it will be necessary to resort to interval dissection techniques.

Notes

§14.1 A particularly good source of information on the fundamentals of numerical integration can be found in Davis and Rabinowitz (1975). Press *et al.* (1992, 2002) are valuable and widely employed references for advice on the practical implementation of a large number of numerical methods, including those pertinent to the topics covered in this chapter. The latter also gives source code for the algorithms they discuss. Some general references on the numerical evaluation of Cauchy principal value integrals are Delves (1967), Piessens (1970), Paget and Elliott (1972), Hunter (1972, 1973), Elliott and Paget (1979), Lubinsky and Rabinowitz (1984), and Kolm and Rokhlin (2001). For the particular case of the Hilbert transform on the circle, see Schneider (1998). The numerical treatment of singular integrals of different types has attracted a great deal of attention. Some references for the interested reader to pursue for further study are: Bareiss and Neuman (1965), Morawitz (1970), Atkinson (1972), Elliott and Paget (1975, 1976), Rabinowitz (1978, 1983, 1986), Chawla and Kumar (1979), Kumar (1980), Criscuolo and Mastroianni (1987, 1989), Rabinowitz and Lubinsky (1989), Amari (1994), Diethelm (1994a–d, 1995a–c, 1996a–c, 1997, 1999), Hasegawa and Torii (1994), Criscuolo *et al.* (1995), Mastroianni and Occorsio (1995), Criscuolo and Scuderi (1998), Bialecki and Keast (1999), Diethelm and Köhler (2000), Capobianco, Criscuolo, and Giova (2001), Damelin and Diethelm (2001), De Bonis and Mastroianni (2003), Damelin (2003), and Yamamoto (2006). For an interesting application of Hilbert transform theory to the treatment of non-singular integrals, see Smith and Lyness (1969).

§14.2 Subtracted dispersion relations are treated in many sources; see, for example, Nussenzveig (1972). Sloan (1968) gives a concise commentary on numerical evaluation of a subtracted finite Hilbert transform. Longman (1958) discusses examples of splitting Cauchy principal value integrals over the even and odd components of the function.

§14.3 The trapezoidal approach to the numerical evaluation of the Kramers–Kronig transforms is discussed by Collocott and Troup (1979). A comparison of the Maclaurin, trapezoidal, and Simpson's rule approaches with the Fourier technique of Section 14.11 is carried out in Ohta and Ishida (1988).

§14.4 For some further reading, see Stroud and Secrest (1966) and Gautschi (1981).

§14.5 For further discussion on using a polynomial basis to determine the weights and abscissas for a Gaussian quadrature, see Sack and Donovan (1972) and Gautschi (1970, 1990). Blue (1979) discusses the integral for the shifted Legendre polynomial $P_n(2x-1)$ multiplied by the function $\log x^{-1}$. This integral arises in the construction of the weights and abscissas for a Gaussian quadrature employing the weight function $\log x^{-1}$ using the Sack–Donovan approach. A generalization of this integral is given by Gautschi (1979).

§14.6 Quadrature points $\{w_i, x_i\}$ for the related functions $s^\alpha \log s^{-1}$ have been discussed by Danloy (1973). The numerical evaluation of the one-sided Hilbert transform using a Gauss–Jacobi quadrature formula is discussed by Della Vecchia (1994). For another study of the one-sided Hilbert transform, see Gautschi and Waldvogel (2001). The results based on a specialized Gaussian quadrature are discussed in King (2002).

§14.8 Discussion on the asymptotic behavior of the Fourier transform and other oscillatory integrals can be found in Wong (1989).

§14.10 This approach to the numerical transform is mentioned concisely in Henrici (1986, p. 203) and is discussed by Henery (1984). For additional reading, see Liu and Kosloff (1981) and Gazonas (1986). For an application to immittance data, see VanderNoot (1992).

§14.11 Peterson and Knight (1973) pointed out that Eqs. (14.138) and (14.139) could serve as a very effective starting point for the numerical inversion of the Hilbert transform pair. Gross (1941) derived this form for the dielectric loss function, but then converted the relations to the usual Hilbert transform, which he considered more suitable for numerical applications. This was the pre-FFT era. Other references where this form has been employed can be found in King (1978a). For some practical applications, see Bertie and Eysel (1985), Harbecke (1986), and Bertie and Zhang (1992).

§14.12 For further reading, see Johnson (1975) and King (1977, 1978a). A numerical implementation has been discussed by Collocott (1977), and for an application to color centers in KCl_xBr_{1-x}, see Ketolainen, Peiponen, and Karttuner (1991).

§14.14 For a particular series decomposition of the Hilbert transform of a suitable function, see Kress and Martensen (1970), and for an alternative approach see Gregor (1961). Afshar, Mueller, and Shaffer (1973) have considered a numerical approach where the Hilbert transform kernel is expanded as an infinite sum of Hermite functions. Sidky and Pan (2005) have developed an expansion approach for the inversion of the finite Hilbert transform on the interval $(-1, 1)$.

§14.15 For some further discussion on the numerical treatment of finite Hilbert transforms, see Pykhteev (1959), Stewart (1960), Lebedev and Baburin (1965), Piessens (1970), Hunter (1972), Elliott and Paget (1979), Taurian (1980), Monegato (1982), Gautschi and Wimp (1987), Mastroianni and Occorsio (1990), Criscuolo and Mastroianni (1991), Hasegawa and Torii (1991), Gori and Santi (1995), Natarajan and Mohankumar (1995), Gopinath and Nayak (1996), Criscuolo and Giova (1999), Dragomir, Dragomir, and Farrell (2002), Dragomir (2002a, 2002b), Nadir (2003), and Hasegawa (2004). Some of the references indicated at the start of these end-notes for the treatment of Cauchy principal value integral cover explicitly the case of the finite Hilbert transform. For a numerical treatment of the truncated Kramers–Kronig transforms, see King (2007).

Exercises

14.1 Manipulate

$$P\int_0^{\pi/2} \frac{e^t\,dt}{\sin t - \cos t}$$

into a form more suitable for numerical evaluation, and then evaluate the integral numerically.

14.2 Simplify the integral

$$P\int_{-\pi/2}^{\pi/2} \frac{dt}{\sqrt{(1+\sin t)} - 1}$$

to avoid the Cauchy principal value, then evaluate the integral numerically.

14.3 Evaluate the Hilbert transform $Hf(x)$ for $f(x) = \cos\alpha x$ with α a constant, by making an appropriate subtraction.

14.4 By making an appropriate subtraction, rearrange the Hilbert transform

$$\pi^{-1}P\int_{-\infty}^{\infty} \frac{\sin y\,dy}{(x-y)(1+y^2)}$$

into a form avoiding the Cauchy principal value integral, and then evaluate the resulting integral numerically for $x = \pi$. Evaluate the integral analytically and compare the results.

14.5 Evaluate the integral

$$P \int_{-\infty}^{\infty} \frac{e^{-(y/10)^2}\, dy}{2 - y}$$

using Simpson's rule and compare the result obtained with a direct evaluation of the hypergeometric function expansion given in Eqs. (5.29) and (5.30).

14.6 Evaluate the integral of Exercise 14.5 using the trapezoidal rule.

14.7 What error results for the numerical integration of (i) $f(x) = ax^3 + bx^2 + cx + d$ and (ii) $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, and e are constants, using the two-point Gaussian quadrature formula $\int_{-1}^{1} f(x)dx \approx f(-1/\sqrt{3}) + f(1/\sqrt{3})$?

14.8 Evaluate the integral $\int_a^b e^{i\omega t} f(t)dt$ for $f(t) = e^{-t^2}$ analytically in terms of special functions. Determine the numerical value of the integral $\int_{-10}^{10} e^{i\omega t} f(t)dt$ using Eq. (14.128) for $\omega = 1, 10$, and 100, and for $n = 1, 5$, and 10. Compare the resulting values with those obtained from a direct evaluation of the analytic formula.

14.9 Suggest a reason why a polynomial basis might do better in terms of numerical stability than a simple power basis to determine the weights and abscissas for a Gaussian quadrature.

14.10 Carry out a numerical evaluation of the Hilbert transform of the function $\alpha e^{-\beta(x-x_0)^2}$, for the values $\alpha = 2, \beta = 3$, and $x_0 = 1$, using the specialized Gaussian quadrature of Section 14.6 for $N = 10$ and $N = 20$. Compare the numerical results with the value obtained from the analytic formula for the Hilbert transform of a Gaussian function. For readers with Internet access, the required quadrature points can be downloaded from the author's home page at the URL given in the Preface.

14.11 Evaluate numerically the Hilbert transform of the function $f(x) = \sin x\, e^{-x^2}$ at $x = 3$ using the specialized Gaussian quadrature approach of Section 14.6 for $N = 10$. Give an estimate of the error involved.

14.12 Determine to an accuracy of six digits the Hilbert transform of the function $f(x) = (1 + x^2)^{-1} \cos x\, e^{-|x|}$ at the point $x = 4$.

14.13 Evaluate numerically the integrals $(H_e f)(x)$ and $(H_o g)(x)$ at the point $x = 4$, where $f(x) = (1 + x^4)^{-1}$ and $g(x) = xe^{-|x|}$, to an accuracy of five digits.

14.14 Evaluate numerically by the FFT technique, the integral $\int_{-\infty}^{\infty} e^{i\omega t} e^{-t^2}\, dt$ for $\omega = 2$. Compare your answer with the value obtained from the analytic solution of this integral.

14.15 Explore the numerical stability of Eq. (14.133) for the evaluation of the integral $\int_{-\pi}^{\pi} e^{i\omega t}(1 + t + t^2)dt$ using the values $\omega = 1, 5, 10$, and 20. Compare the results obtained with the exact result for the integral.

14.16 Evaluate numerically the Hilbert transform of $g(x) = e^{-x^2}$ for $x = 2$ using the allied Fourier integral formula, Eq. (14.139), and compare the value obtained with the exact result.

14.17 Fit the following data, given as $\{x, f(x)\}$: $(-100, 9.998\,000 \times 10^{-5})$, $(-50, 3.996\,803 \times 10^{-4})$, $(-40, 6.242\,197 \times 10^{-4})$, $(-20, 2.487\,562 \times 10^{-3})$, $(-10, 9.803\,922 \times 10^{-3})$, $(-5, 3.703\,704 \times 10^{-2})$, $(-2, 1.666\,667 \times 10^{-1})$, $(-1, 3.333\,333 \times 10^{-1})$, $(0, 5.000\,000 \times 10^{-1})$, $(1, 3.333\,333 \times 10^{-1})$, $(2, 1.666\,667 \times 10^{-1})$, $(5, 3.703\,704 \times 10^{-2})$, $(10, 9.803\,922 \times 10^{-3})$, $(20, 2.487\,562 \times 10^{-3})$, $(40, 6.242\,197 \times 10^{-4})$, $(50, 3.996\,803 \times 10^{-4})$, and $(100, 9.998\,000 \times 10^{-5})$, to any suitable functional form. Determine the Hilbert transform of the data set and evaluate this at the points $x = 1$ and $x = 5$. Use any reasonable extrapolation of the data that you feel is appropriate.

14.18 For the same data set as given in Exercise 14.7, fit the data to a sine or cosine series as appropriate, then determine the Hilbert transform of the data set and evaluate this at the points $x = 1$ and $x = 5$.

14.19 For the following data set: $(-5, 4.579\,470 \times 10^{-11})$, $(-4, 3.045\,996 \times 10^{-8})$, $(-3, 7.453\,306 \times 10^{-6})$, $(-2, 6.709\,253 \times 10^{-4})$, $(-1, 2.221\,799 \times 10^{-2})$, $(0, 2.706\,706 \times 10^{-1})$, $(1, 1.213061)$, $(2, 2)$, $(3, 1.213\,061)$, $(4, 2.706\,706 \times 10^{-1})$, $(5, 2.221\,799 \times 10^{-2})$, $(6, 6.709\,253 \times 10^{-4})$, and $(7, 7.453\,306 \times 10^{-6})$, determine the Hilbert transform at the points $x = 1, x = 5$, and $x = 10$. Use any reasonable extrapolation of the data that you feel is appropriate.

14.20 Using a suitable quadrature approach, verify the given values for $(Hf)(x)$:

(i) $f(x) = (\sin^{2n} ax)/x^p$, for $p = 5$, $n = 3$, $a = 11/10$, and $x = 7/10$, then $Hf(x) \approx -0.136\,767\,043\,017\,876\ldots$;

(ii) $f(x) = \operatorname{sgn} x \sin(a\,|x|^{1/2})$, for $a = 14/5$ and $x = 6/5$, then $Hf(x) \approx 0.950\,688\,424\,322\,45\ldots$;

(iii) $f(x) = (\sin^3 ax)/x^3$, for $a = -7/5$ and $x = 11/5$, then $Hf(x) \approx -0.762\,362\,280\,393\,94\ldots$

14.21 Determine numerically the Hilbert transform of the following functions at the points $x = 0, x = \pi$, and $x = 10$: $f(x) = \sin x$, $f(x) = \sin 12x\, e^{-x^2}$, and $f(x) = \sin(x^{-1})$.

14.22 Prove Eqs. (14.187)–(14.189).

14.23 Evaluate numerically the finite Hilbert transform $\pi^{-1}P\int_{-1}^{1} e^{-y}dy/(x-y)$ to an accuracy of six digits at the point $x = 1/2$.

14.24 Evaluate numerically the finite Hilbert transform $\pi^{-1}P\int_{-1}^{1} dy/[(x-y)\surd(1-y^2)]$ at the point $x = 3/4$ and estimate the error. Compare the numerical value determined with the exact value of the integral.

Appendix 14.1

Points and weights for quadrature with the function $\log_e x^{-1}$

Table 14.1.1. *Abscissas and weights for a Gaussian quadrature with a weight function of* $\log x^{-1}$

Abscissas (x_i) for $N = 10$	Weights (w_i)
0.9042630962 1996506369 4662745251 E-02	0.1209551319 5457051498 8582388553
0.5397126622 2500629504 2012263704 E-01	0.1863635425 6407187032 7415761808
0.1353118246 3925077487 0232111120	0.1956608732 7775998271 0442500258
0.2470524162 8715982422 2545443293	0.1735771421 8290692084 0382679110
0.3802125396 0933233397 2341265091	0.1356956729 9548420166 9911621386
0.5237923179 7184320116 1163824640	0.9364675853 8110525987 2906483510 E-01
0.6657752055 1642459722 2381296904	0.5578772735 1415874075 9075889448 E-01
0.7941904160 1196621735 8509331891	0.2715981089 9233331145 9339153986 E-01
0.8981610912 1900353816 6953162341	0.9515182602 8485149992 5482488176 E-02
0.9688479887 1863353939 1504526904	0.1638157633 5982632548 7807130901 E-02

Abscissas (x_i) for $N = 20$	Weights (w_i)
0.2588327955 9219554283 3273585516 E-02	0.4314275213 3208078578 9708432187 E-01
0.1520966234 9560231720 6559633041 E-01	0.7538370990 8589359550 4548598668 E-01
0.3853655037 2165327959 8479658091 E-01	0.9305326745 1663051372 6903186899 E-02
0.7218161381 5873906434 9676492723 E-01	0.1014567118 4982975443 6918308132
0.1154605264 8763315055 8894044220	0.1032017620 5607206905 7820954502
0.1674428562 7532968571 8283894558	0.1000225498 0527316653 2795906096
0.2269837872 6020250336 1295077813	0.9325979930 0297678083 6606811451 E-01
0.2927549609 4154583299 1980256477	0.8402895287 1941056497 0846278107 E-01
0.3632774298 5785890453 7983651857	0.7328558913 0030740962 8311569280 E-01
0.4369571400 9076831848 6637247297	0.6185033691 3730289957 2280804070 E-01
0.5121225946 7896733619 5665907884	0.5041660443 8374677637 0507997375 E-01
0.5870640449 1440991513 2493034911	0.3955137000 5298385332 9369671524 E-01
0.6600734133 1490941391 2098073061	0.2969407789 5812844804 6071929235 E-01
0.7294840839 2968749887 1046415640	0.2115631535 5427097673 0226771161 E-01
0.7937096719 8708581774 3689513314	0.1412373293 8964020436 5980864816 E-01
0.8512808927 8912572722 2167469702	0.8660974504 3354986282 3251287219 E-02
0.9008796808 5441759422 3438757398	0.4719940146 2036049543 6708198937 E-02
0.9413697491 2909167630 2651369673	0.2151397403 9652061146 7792004210 E-02
0.9718227410 7526319373 8477379400	0.7197282146 5320264635 8261281031 E-03
0.9915380814 3871197265 2461357188	0.1204276763 3021674169 2763608185 E-03

Abscissas (x_i) for $N = 30$	Weights (w_i)
0.1214771793 0219078619 7210056168 E-02	0.2264278801 5571797236 4752743610 E-01
0.7069884839 8636750737 5339397224 E-02	0.4134095517 9739982010 1542980129 E-01
0.1790653342 0994657096 6305841135 E-01	0.5355528621 2183818418 1036240807 E-01
0.3366258077 2383811116 5915284571 E-01	0.6164047019 7788675155 7191408424 E-01
0.5420249083 9775346046 3388697518 E-01	0.6667164696 4571022403 3528534635 E-01
0.7933221307 8120485064 3511402413 E-01	0.6929923022 7482324676 0522914652 E-01
0.1088050595 2687031563 1452767611	0.6998386731 5380167508 9576038787 E-01
0.1423259410 5937096499 2248091535	0.6908576924 2359783727 9540197332 E-01
0.1795553150 1845001459 1789878312	0.6690513778 7833853626 0187225547 E-01
0.2201132403 1520041071 5741652967	0.6370238277 2500878085 9398917898 E-01
0.2635836721 0583161584 8456897097	0.5970879743 7219206660 9431041174 E-01
0.3095190319 2138466537 4436440497	0.5513230699 6722809028 9562999748 E-01
0.3574450468 6630507371 8433787175	0.5016052032 8708261853 7455725842 E-01
0.4068658301 3376562043 4811358249	0.4496225847 2567535011 0335564008 E-01
0.4572691630 3373918556 8295196594	0.3968822159 2717629191 6741415521 E-01
0.5081319313 8782666446 0702814366	0.3447118897 6248186284 6590558527 E-01
0.5589256644 9383561042 4974263003	0.2942599834 0654702965 3365543738 E-01
0.6091221219 4825689613 0920428274	0.2464946367 5952071110 5755975547 E-01
0.6581988719 5442728078 0548775491	0.2022033702 2865462674 0508914264 E-01
0.7056448040 7965709045 0124431741	0.1619938460 4765072041 3967623125 E-01
0.7509655196 0639493231 2823995045	0.1262962385 2900291747 9516805106 E-01

Table 14.1.1. (*Cont.*)

0.7936885435 5423584191 2021375456	0.9536750777 4674102056 8365426104 E-02
0.8333683040 5688779798 7909328730	0.6929774356 9083090191 8227766558 E-02
0.8695908269 8330304387 1404713436	0.4801864677 6125538937 2836644511 E-02
0.9019780963 6940038873 7230019025	0.3131413572 4879546569 4996339492 E-02
0.9301920340 5884910781 9921790096	0.1883299940 5013986144 2018245662 E-02
0.9539380538 0031780657 4895575291	0.1010346449 5323391500 6968871788 E-02
0.9729681404 4405869493 6301126629	0.4549497337 6116500824 7575717690 E-03
0.9870833569 5450125802 7836227390	0.1508624177 2490582783 8193255488 E-03
0.9961351660 1930615674 5803584885	0.2510285727 0432204829 1612436750 E-04
Abscissas (x_i) for $N = 40$	Weights (w_i)
0.7046660728 9806776877 0465135290 E-03	0.1413104711 2154336885 7126192303 E-01
0.4075166664 2903088343 2682825966 E-02	0.2642081370 8677438220 4921530474 E-01
0.1030931590 3585798771 7782621191 E-01	0.3506262466 4940104524 8422798869 E-01
0.1939437075 3851413949 3078502042 E-01	0.4139440342 8997035863 1966641388 E-01
0.3128971098 8574044869 8363051969 E-01	0.4601366464 5454154379 7897298147 E-01
0.4593391289 3468428012 4733641187 E-01	0.4926865952 4000027419 4240875633 E-01
0.6324702157 0223317125 7642181378 E-01	0.5139483861 8449263799 4888996990 E-01
0.8313184048 8351332412 1186536850 E-01	0.5256759781 6075194897 2855332145 E-01
0.1054749307 6737254686 1734330126	0.5292681216 6054503431 6901860044 E-01
0.1301475307 4306188695 5440521203	0.5258964795 3543630901 8875634611 E-01
0.1570064761 6822216852 0791908966	0.5165778811 4682875557 1804079674 E-01
0.1858951551 8124845866 3369209377	0.5022172357 4294366528 7970966730 E-01
0.2166445126 6714569024 1988390590	0.4836338900 2787326287 3095731868 E-01
0.2490741092 8832117803 9734505261	0.4615781250 7400207884 7271855677 E-01
0.2829932355 0578902022 0415970592	0.4367415376 9433977838 2847366996 E-01
0.3182020780 4852502979 6090120800	0.4097635176 5601425377 9907941645 E-01
0.3544929345 1002272164 7938014556	0.3812351872 1755054635 6871712864 E-01
0.3916514705 7704607702 0424975350	0.3517016808 6147301043 6037620882 E-01
0.4294580135 8364636821 9258279784	0.3216633484 6554071431 5099561188 E-01
0.4676888755 0460728338 6076361776	0.2915762808 3415987365 0644549517 E-01
0.5061176980 8483887605 9776578429	0.2618524372 9978201590 0453124488 E-01
0.5445168125 0343845338 3664780070	0.2328595755 9188973706 9060584277 E-01
0.5826586057 6844019126 5646632628	0.2049211295 6586892238 7539141072 E-01
0.6203168859 1540276775 8243216253	0.1783161418 0784787499 8683450927 E-01
0.6572682380 3013127181 4920472436	0.1532793301 1189067712 6205932953 E-01
0.6932933631 2640704036 7502132539	0.1300013459 0892194572 4960386827 E-01
0.7281783919 7890140601 2631813122	0.1086292666 8708043190 5530070223 E-01
0.7617161661 3583834715 8838018391	0.8926735182 0890672244 0031249082 E-02
0.7937074785 1233752063 1817087173	0.7197808104 8832170849 4739239833 E-02
0.8239622661 9070991415 4080252975	0.5678348644 8660541724 4681354151 E-02
0.8523007483 2491471238 0362257534	0.4366678170 8792243840 0720870359 E-02
0.8785545023 5829601678 7747160082	0.3257428647 3540031528 0477304453 E-02
0.9025674721 0859487301 4613553450	0.2341763833 6252753544 1974327813 E-02
0.9241969016 3440653639 2323852601	0.1607628051 9029849744 4108276251 E-02
0.9433141891 2450529406 2638404727	0.1040020930 7644968532 7319205459 E-02
0.9598056551 8753125059 8316996883	0.6212961834 9170739633 5588773931 E-03
0.9735732192 3910325870 7592863317	0.3314821779 5167145464 1358486315 E-03
0.9845349730 9399938849 5806513167	0.1486217957 4663384854 6274932875 E-03
0.9926256113 9280558642 4350983364	0.4912887652 6090375650 9811346671 E-04
0.9977963821 8088712167 7954556255	0.8158592173 1289388771 5455715330 E-05
Abscissas (x_i) for $N = 50$	Weights (w_i)
0.4602237980 7543176252 3689948144 E-03	0.9737186532 1791647216 2872355565 E-02
0.2649407617 8752296664 3639190164 E-02	0.1848851141 5214896193 0934623146 E-01
0.6694874036 2647640902 6048386743 E-02	0.2490389727 1803905141 6146390818 E-01
0.1259491189 3333483102 4055772142 E-01	0.2984738144 3860458563 3409534430 E-01
0.2033432241 8643314136 1180789958 E-01	0.3370186990 3663231141 3798956249 E-01
0.2988852264 9297635201 0236983267 E-01	0.3668706965 5490437128 4913930403 E-01
0.4122474684 0972248615 2239547250 E-01	0.3894824561 9164614061 6745080878 E-01
0.5430264104 1815387853 3367567735 E-01	0.4059071985 9423565569 9214473177 E-01
0.6907465940 2075175246 0660493824 E-01	0.4169605777 8593021521 0553106398 E-01

Table 14.1.1. (*Cont.*)

0.8548638641 8784446428 7707378394 E-01	0.4233063196 8705093443 3885251726 E-01
0.1034768334 1710700566 4968643208	0.4255054436 9165843562 5165349910 E-01
0.1229787309 0639284060 0350243017	0.4240462894 4241259500 4773321915 E-01
0.1439188273 3800588270 4055448435	0.4193636524 4970701232 5753148444 E-01
0.1662181995 6640041722 9988975614	0.4118513761 2294696338 2901670929 E-01
0.1897925776 1684356424 8668205135	0.4018708297 3015679926 9286051463 E-01
0.2145526848 5918808442 2200029721	0.3897567039 0367890445 0658964677 E-01
0.2404045937 9168601931 3222070668	0.3758210054 1607069869 7100336365 E-01
0.2672500970 7379447319 2277955879	0.3603558150 6616625011 3267842020 E-01
0.2949870930 7135499167 8641554658	0.3436351811 3262175733 1936512559 E-01
0.3235099849 1619244987 7495920426	0.3259164015 2348922765 5775353953 E-01
0.3527100918 9308440355 0554841817	0.3074408710 3305657644 6076203111 E-01
0.3824760718 2585602367 3771836129	0.2884346194 3559338025 8026734350 E-01
0.4126943530 2648390749 5709460267	0.2691086318 3402776153 2833172857 E-01
0.4432495742 8756689285 7930850733	0.2496590189 2668769654 8902215294 E-01
0.4740250313 3367416445 7474797878	0.2302670880 6366263358 7912151935 E-01
0.5049031280 9724807289 0947696460	0.2110993538 5988432192 7601472004 E-01
0.5357658311 4750398601 5435488839	0.1923075182 4091027715 0592082255 E-01
0.5664951255 7450184942 1903537970	0.1740284431 4636554857 0310393540 E-01
0.5969734706 1413810847 8269852211	0.1563841340 5078778329 4566015676 E-01
0.6270842532 9242292736 0412061421	0.1394817485 3525100763 5846275150 E-01
0.6567122383 6847904805 8983514006	0.1234136410 4577666048 6969627513 E-01
0.6857440128 6477513054 7771723297	0.1082574524 8855880588 1475644706 E-01
0.7140684234 8982498352 5466989365	0.9407625128 2772520105 4869902591 E-02
0.7415770052 8263434035 6072707024	0.8091873080 8115770188 6017681971 E-02
0.7681643998 3927636621 7821370154	0.6881946676 5722545276 1129416725 E-02
0.7937287615 1985220594 8518013978	0.5779923675 8818027713 1092398729 E-02
0.8181721500 7846037654 4142531908	0.4786540335 0146044349 2299637708 E-02
0.8414009082 0934290714 9859379541	0.3901236093 4721346294 9681231210 E-02
0.8633260225 5870945011 9874983862	0.3122204595 5006123285 5783450477 E-02
0.8838634668 1332303257 8726185127	0.2446450926 3402821075 2977530393 E-02
0.9029345255 4288018057 1342501086	0.1869854879 0711259991 4343552879 E-02
0.9204660975 4182573797 3955252538	0.1387240009 8992275608 9887185274 E-02
0.9363909774 8364198399 5662012175	0.9924481875 7107016126 3909168265 E-03
0.9506481147 5679820138 2783838404	0.6784192957 6098202180 0046597237 E-03
0.9631828483 6676365582 5145012700	0.4372757053 5069486193 8942794294 E-03
0.9739471166 6220633851 6314115963	0.2604110962 4506262360 3353278154 E-03
0.9828996399 7399616372 5181401321	0.1385831756 8490619308 3799565972 E-03
0.9900060710 5000763187 2905800214	0.6200981264 4545172136 4195698085 E-04
0.9952390888 9723649172 5735768725	0.2046809227 9884450967 6460342070 E-04
0.9985782202 5654899164 0715853532	0.3395871576 4839229631 5077173895 E-05

Abscissas (x_i) for $N = 60$	Weights (w_i)
0.3242993314 4248107289 1208675318 E-03	0.7155181881 1300426417 6256977727 E-02
0.1860416905 5784504158 1595088031 E-02	0.1373694600 1689768187 3911967870 E-01
0.4696573639 0991652078 3942114106 E-02	0.1869537888 1776323261 4062949421 E-01
0.8833904836 9572925024 0525480692 E-02	0.2263474845 1526893488 0317606364 E-01
0.1426594295 0014852993 5907548326 E-01	0.2582207148 4993127288 0978814286 E-01
0.2098127619 9176948993 7591897557 E-01	0.2840945573 6376243830 2374085373 E-01
0.2896429015 1974968507 7779661262 E-01	0.3049634644 8417452363 6463561347 E-01
0.3819550862 7577167845 7652451676 E-01	0.3215378277 2867450134 8312868467 E-01
0.4865179933 4554167127 1789258721 E-01	0.3343582004 8381905227 2392755345 E-01
0.6030652447 3874537777 9958103054 E-01	0.3438560534 1658051990 3925727055 E-01
0.7312966757 2377715272 1956094888 E-01	0.3503889570 1292293530 7218884009 E-01
0.8708795071 6818465500 4786847836 E-01	0.3542622661 8137415111 6083454265 E-01
0.1021449492 7718703334 3753434520	0.3557431317 9722807461 6893994436 E-01
0.1182612079 0207493527 6736535984	0.3550698901 9998310718 1910918798 E-01
0.1353943598 7942803410 6215052743	0.3524585360 1232078426 2583289705 E-01
0.1534992510 2791649982 4410393161	0.3481072831 2964683853 7711628370 E-01
0.1725280687 4247676881 0751615165	0.3421998324 7375104565 3352025842 E-01
0.1924304764 8440543587 9406047262	0.3349077416 9485313548 9920695492 E-01
0.2131537537 9162642166 4979147456	0.3263921574 4805421225 3160877862 E-01
0.2346429417 4030179976 8882039685	0.3168050869 5948533893 6334090503 E-01
0.2568409936 8793538512 0164097312	0.3062903316 6486108988 6453688808 E-01

Table 14.1.1. (*Cont.*)

0.2796889310 5491968614 5535122197	0.2949841701 0523366161 8396303268 E-01
0.3031260038 4673172558 9059742126	0.2830158532 0746897358 5418980118 E-01
0.3270898555 7690101950 7161861217	0.2705079584 7117274794 6108097139 E-01
0.3515166922 1778264157 3738970549	0.2575766378 9968035821 5190658984 E-01
0.3763414547 7968952572 1739776737	0.2443317861 4624767855 6738996878 E-01
0.4014979950 9749646457 9556250996	0.2308771492 5795409049 5938964335 E-01
0.4269192543 8636326868 5393651628	0.2173103899 0143052525 7821613466 E-01
0.4525374441 1401036027 9319612067	0.2037231215 8329769188 9902889367 E-01
0.4782842287 2551210517 5277620657	0.1902009218 1711193980 2560973046 E-01
0.5040909097 4749546810 8991182831	0.1768233322 1730712784 2423172572 E-01
0.5298886107 9168916656 7162116955	0.1636638519 6378294852 7375439102 E-01
0.5556084629 7277580090 6100560207	0.1507899298 6756158028 5921614000 E-01
0.5811817902 5231879837 8984973374	0.1382629592 9824698944 6072243759 E-01
0.6065402942 1906302504 3114930066	0.1261382794 4941953790 7613510518 E-01
0.6316162378 1606124476 9252256358	0.1144651857 7548128854 4695718259 E-01
0.6563426275 2679276478 3050538426	0.1032869519 0076314604 9380496786 E-01
0.6806533935 3566088849 8606347738	0.9264086485 4849783331 6520249540 E-02
0.7044835673 8293455484 5820704397	0.8255827510 8878508270 6956810267 E-02
0.7277694566 4029349689 0444115937	0.7306466256 2173184923 0168673692 E-02
0.7504488161 4060444129 2657229655	0.6417971934 6374123518 2638024787 E-02
0.7724610153 0435804028 9376716565	0.5591745006 7047158592 3892455562 E-02
0.7937472011 1529142145 3534593237	0.4828628988 4274780594 3947262176 E-02
0.8142504563 0906633852 2258726872	0.4128924063 9033490732 2780640830 E-02
0.8339159523 5142074989 4567260213	0.3492402505 7411225839 4440584756 E-02
0.8526910967 9590805452 6992204307	0.2918325890 6919565532 4125813196 E-02
0.8705256746 2611647352 9997905774	0.2405464083 5959599528 5230001148 E-02
0.8873719832 0303087598 5577376353	0.1952115949 7066545695 7389780512 E-02
0.9031849604 5482295781 0309457342	0.1556131743 5442202694 3526361422 E-02
0.9179223059 6359360185 3333605043	0.1214937111 5354054258 4498065752 E-02
0.9315445946 2097135151 5016487895	0.9255586357 3497100062 2201153601 E-03
0.9440153825 4095212996 7863965113	0.6846508368 2151985039 6682993332 E-03
0.9553013049 3140747807 6299728999	0.4885245462 6620370345 9548064624 E-03
0.9653721656 2844728181 5869017569	0.3331765500 4587598419 8963697843 E-03
0.9742010179 6975726056 1635095850	0.2143203994 8545842187 3659404527 E-03
0.9817642365 5412274904 0506665789	0.1274182787 6086249031 9137668869 E-03
0.9880415789 2436265047 7686198034	0.6771381333 3216217294 8175479532 E-04
0.9930162339 2887477895 2832112688	0.3026569953 0513963209 4285127954 E-04
0.9966748400 1858749516 0243928475	0.9982035828 0485052441 3948530437 E-05
0.9990073230 8937402711 3229359469	0.1655280633 5491778653 4893629025 E-05

Abscissas (x_i) for $N = 70$	Weights (w_i)
0.2409287033 0539821298 2960344914 E-03	0.5500885988 4653895211 2478378755 E-02
0.1378298837 5262804797 2208442584 E-02	0.1065044896 4034216351 8009170044 E-01
0.3476599342 5284480581 2663139571 E-02	0.1460645691 1317445924 5995291435 E-01
0.6537630260 4274966431 2530522341 E-02	0.1781528007 3220929107 8120510153 E-01
0.1055843700 6840515156 7081950715 E-01	0.2047396562 3506836371 3500880484 E-01
0.1553316117 1125593148 9846186664 E-01	0.2269435972 4911042077 5365220623 E-01
0.2145354169 9410085070 5079925268 E-01	0.2454914605 0202411263 9310857245 E-01
0.2830913593 4705571293 0893250893 E-01	0.2608981114 1688504808 2347329166 E-01
0.3608744494 9094046568 2857717559 E-01	0.2735512002 8677712577 3982138388 E-01
0.4477399858 1598905970 7739449727 E-01	0.2837563622 0854101477 0246850691 E-01
0.5435242173 8087888873 1159275140 E-01	0.2917634857 8853299249 4882341402 E-01
0.6480449174 5886409170 1264663492 E-01	0.2977829820 1010432285 7064512447 E-01
0.7611019170 1350903298 9565382129 E-01	0.3019963633 6127836612 8831017714 E-01
0.8824776249 1136588305 4543131697 E-01	0.3045633916 7949488291 4297166370 E-01
0.1011937550 1972451227 5015615488	0.3056270575 3112220649 5309488193 E-01
0.1149230835 4905432437 1838918603	0.3053171354 2524644887 8809454747 E-01
0.1294090806 9356840152 0528120370	0.3037527731 3595416880 3898991634 E-01
0.1446235543 9364632792 0197583929	0.3010444078 9699524563 5183909562 E-01
0.1605368470 5060635315 2444592607	0.2972952025 0213287913 4117478641 E-01
0.1771178969 1424521130 0680006534	0.2926021321 2617400952 3599944004 E-01
0.1943343017 5088756245 5366371935	0.2870568126 7730416687 6549215835 E-01
0.2121523847 7544014800 0623008832	0.2807461350 8019022304 1185368587 E-01
0.2305372627 9857741422 5422141017	0.2737527520 3927736430 3507911734 E-01

Table 14.1.1. (*Cont.*)

0.2494529165 1595518200 3532466480	0.2661554515 1290503324 6974901219 E-01
0.2688622628 4773609115 2961609761	0.2580294424 6601741364 3353904788 E-01
0.2887272292 2207686023 5598942353	0.2494465722 7274667015 0936356583 E-01
0.3090088296 8448619104 8771836371	0.2404754906 3846648307 6827876641 E-01
0.3296672427 0537916806 4278304804	0.2311817715 9353005390 1128236462 E-01
0.3506618905 5022487954 6096449372	0.2216280026 3380461781 8067927424 E-01
0.3719515200 7005703828 4753418700	0.2118738482 1058975749 3240022122 E-01
0.3934942847 6454404902 8880190203	0.2019760933 4058627748 4810346628 E-01
0.4152478279 6511738844 0372405383	0.1919886719 9958412935 8017260722 E-01
0.4371693669 8170656735 1460045533	0.1819626840 9878715429 5975941483 E-01
0.4592157780 5333108457 0952467910	0.1719464041 6039317691 7180340152 E-01
0.4813436819 4008491734 7728627281	0.1619852842 6546961023 9530032923 E-01
0.5035095299 9186504497 2311406254	0.1521219534 0999482809 4830709708 E-01
0.5256696905 2750275443 6619021852	0.1423962150 5006898173 2601605932 E-01
0.5477805353 5672470059 6818325730	0.1328450443 2659118982 6120907890 E-01
0.5697985262 7657623712 8331186590	0.1235025862 1938301190 8548524774 E-01
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0.6348631529 9584961813 6958549064	0.9702650852 8273593277 6736097173 E-02
0.6560791577 6979187786 9866101158	0.8880381721 6259094212 6806210045 E-02
0.6769889717 7113409692 6683388631	0.8091823039 9659893625 1257129774 E-02
0.6975513904 8980576990 0467839277	0.7338703832 9321695742 9989803629 E-02
0.7177258900 4028739008 1987168388	0.6622478393 0312765300 9023388730 E-02
0.7374727074 2918283925 9893618632	0.5944329467 0966759686 3006973286 E-02
0.7567529193 1109754289 7166669215	0.5305172036 7214025783 3450165696 E-02
0.7755285190 7757578760 9508115275	0.4705657706 4024191222 5012034481 E-02
0.7937624921 2706039338 8619000326	0.4146179708 3823349226 8216953386 E-02
0.8114188891 6735844577 6709359812	0.3626878528 4544433838 7352138237 E-02
0.8284628974 0591837055 0632749618	0.3147648152 6872152265 6447567973 E-02
0.8448609094 8733561050 5101770413	0.2708142931 0729571774 2421505809 E-02
0.8605805900 4189430789 1561443034	0.2307785050 4696642919 5955154036 E-02
0.8755909397 1360589798 9020600493	0.1945772605 8506135108 6269637408 E-02
0.8898623565 4110306380 7600029795	0.1621088255 7749162918 0039433479 E-02
0.9033666945 6986140902 2929041330	0.1332508445 1213007314 1614189112 E-02
0.9160773195 7950739140 3317545747	0.1078613175 4683654600 1756487098 E-02
0.9279691618 1535153116 1125842473	0.8577963010 4249284427 2116467057 E-03
0.9390187656 1860878170 8851486058	0.6682763258 7746212641 6662492240 E-03
0.9492043358 5471093709 2658210330	0.5081076757 2786902651 6965659116 E-03
0.9585057810 4293833833 9384450120	0.3751924163 4402143571 6063295915 E-03
0.9669047530 9145488665 7914719066	0.2672923879 4306315892 8649705546 E-03
0.9743846835 3470345381 9490943945	0.1820417240 9550395938 9469568733 E-03
0.9809308161 3989096507 1894081383	0.1169597217 8677054517 0437385707 E-03
0.9865302356 4436114599 6184088205	0.6946402811 6415039061 3469668286 E-04
0.9911718919 9939591636 1195306011	0.3688410800 3257232261 8996081665 E-04
0.9948466178 0270196936 6613408965	0.1647495660 3373261607 6024733105 E-04
0.9975471266 3698326460 7801104511	0.5431021655 1462165577 9980095959 E-05
0.9992678814 6516393640 0211528180	0.9003261417 4235389746 6507662703 E-06

Abscissas (x_i) for $N = 80$	Weights (w_i)
0.1861004756 9983688361 6320864487 E-03	0.4373242147 9209686238 5014685789 E-02
0.1062200079 3081544509 2446686979 E-02	0.8524353363 4096843641 1477039997 E-02
0.2677378618 3154933018 8105553608 E-02	0.1176097144 5048977338 5941124247 E-01
0.5033457063 9681333654 4555068132 E-02	0.1442630086 9942707912 8689120670 E-01
0.8129053354 2066593565 5015396429 E-02	0.1667181642 3239953039 9327633561 E-01
0.1196093822 0862407326 6401458821 E-01	0.1858333552 4505677213 5324005010 E-01
0.1652439488 0218351103 0082747549 E-01	0.2021643762 7622382625 6858414074 E-01
0.2181337374 6945562834 3264777027 E-01	0.2161029760 8932765393 3301041250 E-01
0.2782057685 2256188364 5955903645 E-01	0.2279421903 8392647616 9689912998 E-01
0.3453751219 5715240517 0618032758 E-01	0.2379112342 4072942005 0303725542 E-01
0.4195453363 9191376670 7377755829 E-01	0.2461958202 7067082921 0574489184 E-01
0.5006087343 1150945462 6420477798 E-01	0.2529507747 1926219015 2514178522 E-01
0.5884467094 3723237467 6207567544 E-01	0.2583082688 2165994333 7230741672 E-01
0.6829299957 8819491265 8850606096 E-01	0.2623834041 4782150491 4120509661 E-01
0.7839189297 4792832799 6985883441 E-01	0.2652781244 4520774939 0976453234 E-01

Table 14.1.1. (*Cont.*)

0.8912637119 6419004142 5620341520 E-01	0.2670840273 3754250250 2038031249 E-01
0.1004804673 3254632910 5381515737	0.2678844289 9599460412 2987310250 E-01
0.1124372547 6962832554 9942628086	0.2677559074 1482976100 7048649731 E-01
0.1249788753 1105557678 6935145646	0.2667694730 8064124850 5950615247 E-01
0.1380865682 4803277244 4793848325	0.2649914678 6828993996 4568356621 E-01
0.1517407004 4446454192 7085005874	0.2624842621 5193428608 0112415781 E-01
0.1659207974 6823463240 3213633412	0.2593067997 4888045931 9540133227 E-01
0.1806055757 7977122785 3071146286	0.2555150265 4232285479 7814219953 E-01
0.1957729759 7303053380 4711649769	0.2511622291 2253826098 2446268128 E-01
0.2114001970 5218664859 3348523126	0.2462993030 9987848588 4009615503 E-01
0.2274637317 1820817080 4884742419	0.2409749659 6015116837 7271596120 E-01
0.2439394026 3234730205 6003470491	0.2352359258 5888156045 7057410147 E-01
0.2608023996 1783509372 2572888752	0.2291270151 9180029592 6716089370 E-01
0.2780273177 5640195808 4237774530	0.2226912958 6930608527 2528572310 E-01
0.2955881963 3237051051 0741942280	0.2159701417 8069957928 1659066416 E-01
0.3134585585 7381742652 1987929838	0.2090033028 3307295781 2685022249 E-01
0.3316114521 3754251119 5383001207	0.2018289541 0051061967 8107480224 E-01
0.3500194902 8222235743 5289420893	0.1944837329 5781470262 8579100740 E-01
0.3686548936 7209366276 9483943901	0.1870027665 5307508052 6405651778 E-01
0.3874895327 5175623643 7060678640	0.1794196915 6121811907 3671077828 E-01
0.4064949706 3116963627 5676439486	0.1717666678 3116961925 7913836961 E-01
0.4256425064 1861132030 7231776134	0.1640743872 7377213527 1277527151 E-01
0.4449032189 3824619249 6995606961	0.1563720791 2202864839 7895592116 E-01
0.4642480107 6801060121 7205941595	0.1486875125 1890714665 8823280489 E-01
0.4836476525 3272516621 2145482248	0.1410469972 4264144164 2030482196 E-01
0.5030728273 8671007958 0766309323	0.1334753832 5888065267 6644491881 E-01
0.5224941756 1967571391 9308014314	0.1259960596 8826052204 5299980889 E-01
0.5418823393 1929406332 0082586837	0.1186309536 9315126945 1423989240 E-01
0.5612080070 2361755036 3600817660	0.1114005297 1545442668 5061924851 E-01
0.5804419582 9639678522 4111675012	0.1043237894 3598048985 3805597640 E-01
0.5995551081 5835438212 1898188164	0.9741827277 3234205299 8100607690 E-02
0.6185185513 0759485679 1367012707	0.9070006019 3839790504 6759606118 E-02
0.6373036060 6256825508 2929613326	0.8418377656 7128274250 9234673305 E-02
0.6558818579 6135507484 6643134195	0.7788259676 1593231856 9435285486 E-02
0.6742252029 5150000390 7401222837	0.7180825315 0196470090 9538615536 E-02
0.6923058901 0518988914 2544377431	0.6597104516 4217910449 6250476215 E-02
0.7100965637 9524512614 7522216977	0.6037985101 0959848103 3280349923 E-02
0.7275703052 6817129259 6777702314	0.5504214164 8697015374 4636861294 E-02
0.7447006735 5139731020 1695499713	0.4996399709 2450376753 1033943736 E-02
0.7614617456 3280564275 3273174581	0.4515012510 6157443638 8586764034 E-02
0.7778281558 5173689200 2772819629	0.4060388232 0369744705 2813017784 E-02
0.7937751344 4182342374 6649503847	0.3632729779 9525683631 9588475384 E-02
0.8092785451 6727201511 3035809527	0.3232109906 8955715435 3265588275 E-02
0.8243149219 9557148570 5953821411	0.2858474059 8936424397 1165133173 E-02
0.8388615047 5104519025 5421341840	0.2511643473 1204078905 8252192499 E-02
0.8528962736 9519716891 2457062078	0.2191318502 2269461404 1988030854 E-02
0.8663979829 8141133690 1785165012	0.1897082196 7563312874 4807472089 E-02
0.8793461929 3325139304 0976388537	0.1628404106 0818163186 8390504761 E-02
0.8917213010 9737015982 0606793287	0.1384644313 4101461980 1456750519 E-02
0.9035045720 2386407940 5504611619	0.1165057691 5510000126 9327706090 E-02
0.9146781657 2879157174 9914679675	0.9687983733 6777065921 5359158715 E-03
0.9252251647 9549669707 8054106268	0.7949244290 9056994341 4572801839 E-03
0.9351296000 7331346139 9629277998	0.6424027419 8681749832 7289723585 E-03
0.9443764749 3411675056 0917883920	0.5101140732 4582013394 3334025872 E-03
0.9529517880 4892234453 9099403347	0.3968583063 3955427999 0094442530 E-03
0.9608425546 4807919779 5104622413	0.3013598605 7094687480 3365012211 E-03
0.9680368262 2897956951 8745471431	0.2222732630 1213566250 5944353200 E-03
0.9745237086 7322636858 4108028771	0.1581888675 6763188909 9432156611 E-03
0.9802933787 2699603392 0148658162	0.1076387094 7065844092 1214325772 E-03
0.9853370987 7231951309 8380837535	0.6910248313 6064954766 8268765513 E-04
0.9896472297 3356141270 8252625946	0.4101363095 5293797637 4553647102 E-04
0.9932172416 6387700183 6258009104	0.2176553035 2480126379 0855234373 E-04
0.9960417202 5337293855 5939649011	0.9717766362 4788237937 7299220009 E-05
0.9981163598 4692317127 9463642148	0.3202478165 1389842977 5644846988 E-05
0.9994378578 7955450130 4510178454	0.5307830899 5638523798 5397932173 E-06

Table 14.1.1. (*Cont.*)

Abscissas (x_i) for $N = 90$	Weights (w_i)
0.1481118966 3536867925 5117178022 E-03	0.3567892283 0944520382 3758554194 E-02
0.8437398752 3358220078 6356479084 E-03	0.6993175209 5432582045 7376659248 E-02
0.2125425037 8691138826 5659501502 E-02	0.9695391395 5459156750 5934472103 E-02
0.3994823496 8125793999 8984518260 E-02	0.1194662914 8149207698 2706309935 E-01
0.6451310560 4103886498 7885689900 E-02	0.1386699423 4685463045 0323282175 E-01
0.9493012709 5147844803 2645926098 E-02	0.1552446375 8731868125 2227884152 E-01
0.1311707714 1325271183 8866846014 E-01	0.1696297507 9782616642 4582723351 E-01
0.1731978583 2335752336 1926214880 E-01	0.1821340258 2114354538 9868548055 E-01
0.2209661626 4488157114 0029107537 E-01	0.1929874532 5097616230 5517545734 E-01
0.2744227916 1925651541 3419068376 E-01	0.2023690011 0848422543 3805837032 E-01
0.3335074498 0238996220 2848678504 E-01	0.2104227817 0189529452 9234582773 E-01
0.3981526447 4076837654 7889586949 E-01	0.2172681054 3448730326 0960813381 E-01
0.4682838603 5476522301 0567343852 E-01	0.2230060531 9212102938 8192004435 E-01
0.5438197127 6143378396 0047614539 E-01	0.2277239469 5336829207 3531719243 E-01
0.6246720970 9714808030 4917026430 E-01	0.2314984904 5611127903 0181399667 E-01
0.7107463305 4966827654 5797414675 E-01	0.2343980350 7623373917 0703041101 E-01
0.8019412948 8410831034 0259418261 E-01	0.2364842513 0302637001 9819285604 E-01
0.8981495805 8789548673 8057861494 E-01	0.2378133850 0995930908 1401640957 E-01
0.9992576340 3293729994 1240721094 E-01	0.2384372167 1381694786 3380261214 E-01
0.1105145908 5812693335 4225938778	0.2384038039 3331222337 8252744630 E-01
0.1215689020 2443730695 8445781344	0.2377580622 5798963185 8410929956 E-01
0.1330755908 2886320250 0383820304	0.2365422245 5321413089 5667593374 E-01
0.1450210001 0255201796 1943848226	0.2347962067 8212059413 7821807624 E-01
0.1573909386 9137333207 9770735137	0.2325579013 6817075049 4336901191 E-01
0.1701706991 0180162625 3464805794	0.2298634137 0541853186 5258025948 E-01
0.1833450756 8070839698 0161237101	0.2267472536 1975435382 5384897673 E-01
0.1968983833 2249214419 7200242042	0.2232424908 2090701116 3903155532 E-01
0.2108144766 9319075365 9495619386	0.2193808813 4908818986 5777855884 E-01
0.2250767699 5819083184 0011868448	0.2151929705 0100326927 7450579653 E-01
0.2396682569 9768134596 8369302876	0.2107081765 7295621778 5913167455 E-01
0.2545715320 9195873233 9097655586	0.2059548588 8330568887 7535298348 E-01
0.2697688110 5697986580 1972037703	0.2009603728 6153012011 1180365034 E-01
0.2852419528 0910691867 5887753133	0.1957511144 6581018382 8415118234 E-01
0.3009724813 3674194047 4334975268	0.1903525557 7854660393 7362134208 E-01
0.3169416080 5547014783 8112853730	0.1847892733 0266694576 6897389779 E-01
0.3331302545 2239032152 6190859863	0.1790849702 2101313057 8038861688 E-01
0.3495190754 8448635174 9005858813	0.1732624936 7166005228 5583010660 E-01
0.3660884822 3516907444 3737017433	0.1673438479 2247673255 4735296066 E-01
0.3828186662 5247921307 5818261757	0.1613502041 9007634123 6065317137 E-01
0.3996896230 9188051809 4300090679	0.1553019077 3497402216 0145169017 E-01
0.4166811765 0607932145 7498986647	0.1492184827 7123477173 3785706287 E-01
0.4337730027 6387661764 7059644748	0.1431186356 5122264201 2091362582 E-01
0.4509446551 3968665038 4280240098	0.1370202567 2118437849 7726683636 E-01
0.4681755885 4503801795 9186520266	0.1309404211 8889417976 9161791964 E-01
0.4854451842 7310646347 7348874190	0.1248953892 9853262319 0649031793 E-01
0.5027327748 2711033276 8317143141	0.1189006060 6884233383 1098249684 E-01
0.5200176688 0322816268 3017743842	0.1129707008 1716988780 3579622251 E-01
0.5372791757 9857136120 9306209919	0.1071194866 6328041919 8101687627 E-01
0.5544966313 1466208829 8951350842	0.1013599601 8201938240 9480958954 E-01
0.5716494216 2682608494 1376872500	0.9570430135 2345269167 0876345041 E-02
0.5887170085 8991133342 0264953295	0.9016387393 1437358253 7835739417 E-02
0.6056789543 5078519365 8660664187	0.8474922636 6058923130 1158642678 E-02
0.6225149459 3814427218 3338264504	0.7947009333 8770858962 7540140088 E-02
0.6392048197 0029203316 9959223815	0.7433539803 3647826023 1542629056 E-02
0.6557285855 6169839566 6383766827	0.6935325519 4603768794 8227647983 E-02
0.6720664510 6935265063 1359961802	0.6453097503 9082389183 3320902085 E-02
0.6881988452 0015536897 4586875141	0.5987506808 0540991378 5449855084 E-02
0.7041064419 0086595987 7695776443	0.5539125090 4711058643 5057219505 E-02
0.7197701833 3242958071 3179233340	0.5108445293 6936909719 3877163591 E-02
0.7351713027 9084959304 0313759428	0.4695882423 0874765975 4916595002 E-02
0.7502913472 7714908844 6974728340	0.4301774430 2394284359 6653829353 E-02
0.7651121996 8937654108 0059460770	0.3926383202 6591981511 8208560195 E-02
0.7796161006 1005572615 2499827428	0.3569895661 0340897799 8958885594 E-02

Table 14.1.1. (*Cont.*)

0.7937856696 6295799438 5891649210	0.3232424964 7712453974 3471663769 E-02
0.8076039264 1358509824 8914326301	0.2914011826 0871614290 9874861485 E-02
0.8210543107 8829227583 3545131506	0.2614625932 4628593794 9921222488 E-02
0.8341207029 8755341433 7844298615	0.2334167476 8698352858 9669855048 E-02
0.8467874428 6947197984 1791134965	0.2072468794 7846593878 6460078817 E-02
0.8590393487 8027207598 7484560583	0.1829296106 6465454774 6539700265 E-02
0.8708617358 0916243093 2507959177	0.1604351364 0704484388 2459543119 E-02
0.8822404334 4565109004 5318872632	0.1397274197 8066548769 1775261381 E-02
0.8931618026 1809862508 5698000571	0.1207643965 1350254489 5542546678 E-02
0.9036127520 9303083222 6292136557	0.1034981894 0968564265 4279546652 E-02
0.9135807542 1548549516 1326398195	0.8787533216 9875755803 3357691800 E-03
0.9230538599 7143779048 9437571281	0.7383700229 7015064608 2755322657 E-03
0.9320207133 5412874095 1300639077	0.6131926275 1826539560 5092985160 E-03
0.9404705650 1689934252 3489338063	0.5025331200 0124418069 7219706420 E-03
0.9483932851 9588841568 8797265973	0.4056574207 3065901230 7648942349 E-03
0.9557793758 8664327636 6604957095	0.3217880424 1896819003 9514240279 E-03
0.9626199822 5923269482 4709510097	0.2501068189 0484547364 2539613998 E-03
0.9689069032 9665163103 8422722967	0.1897577015 1961707453 9660010178 E-03
0.9746326016 4072303989 6041110733	0.1398496186 0102962432 1145900204 E-03
0.9797902126 2721857677 5860916003	0.9945939351 6139038859 6029173387 E-04
0.9843735524 8449340762 4685707461	0.6763471642 3378972092 0255989404 E-04
0.9883771256 4837694948 5750475858	0.4339716488 5657400524 9675917933 E-04
0.9917961310 7901635721 4852345550	0.2574526834 5096826370 2478810780 E-04
0.9946264672 2722305230 8018560279	0.1365761140 1369357693 4242681824 E-04
0.9968647342 6504963190 5893807976	0.6095970885 2454367247 1758089250 E-05
0.9985082261 3367816854 2677038740	0.2008482528 9805716498 7167469719 E-05
0.9995548450 3341230388 9452654205	0.3328425529 0978548456 7607223249 E-06
Abscissas (x_i) for $N = 100$	Weights (w_i)
0.1207018461 8758285520 7384059708 E-03	0.2971391875 9864430305 3407586352 E-02
0.6864521098 4388761653 7125665129 E-03	0.5851257177 7897799996 2706578655 E-02
0.1728282056 7831611377 1382364750 E-02	0.8145060832 0728044709 5021643288 E-02
0.3247644817 5204325910 6021802245 E-02	0.1007379404 2083922077 1196296744 E-01
0.5244298638 2303223450 9465256262 E-02	0.1173503758 8772146682 3316142063 E-01
0.7717116169 6052449845 9952020780 E-02	0.1318399040 2209401547 9028600392 E-01
0.1066429320 0028611360 2560148714 E-01	0.1445628756 4362842266 7104100370 E-01
0.1408343598 4245447506 3434414943 E-01	0.1557692101 8208393685 7792528726 E-01
0.1797160698 1293881272 3033394152 E-01	0.1656445743 9840148179 4132414993 E-01
0.2232535259 8666968711 5043730176 E-01	0.1743329423 6502620882 6332157741 E-01
0.2714072204 8404951125 8043228955 E-01	0.1819497578 6736368316 4805593817 E-01
0.3241328143 9222722314 7129626750 E-01	0.1885901277 7930470185 9129827611 E-01
0.3813812518 8409450574 7031267962 E-01	0.1943341848 1095150891 9098385830 E-01
0.4430988589 4260215406 3240122188 E-01	0.1992507408 6442563628 6869068429 E-01
0.5092274333 3107639432 0470221288 E-01	0.2033998583 7383790505 6447289100 E-01
0.5797043298 6596158038 5283197292 E-01	0.2068347096 6510825228 9714405286 E-01
0.6544625435 6608541890 4629370397 E-01	0.2096029523 2842686050 1856768895 E-01
0.7334307923 6152466255 4852212508 E-01	0.2117477663 4577980898 3971082550 E-01
0.8165336004 8787527071 5112229543 E-01	0.2133086491 1824910413 8374220737 E-01
0.9036913833 3102645362 5897184344 E-01	0.2143220335 7145228406 0829017194 E-01
0.9948205342 4691566059 1368321381 E-01	0.2148217745 9109980715 2927933475 E-01
0.1089833513 7158404110 5099377805	0.2148395358 7446134194 8654193619 E-01
0.1188638941 0750992900 3818598978	0.2144051003 7813818488 8022675749 E-01
0.1291141688 9905633397 3780335712	0.2135466213 9220776211 9456973403 E-01
0.1397242980 7667837495 3685741344	0.2122908269 4306272784 1724197472 E-01
0.1506840490 5496221481 6797357568	0.2106631871 3022448272 3922458034 E-01
0.1619828446 4406289083 3568266505	0.2086880517 5193805190 2947379375 E-01
0.1736097736 5154341230 6872852525	0.2063887639 1609666537 8172617474 E-01
0.1855536017 7171480043 9940688251	0.2037877540 9547251949 3453097337 E-01
0.1978027827 5787173624 7795279546	0.2009066181 5172780649 1952058669 E-01
0.2103454698 7143050768 1029669750	0.1977661821 3935308520 9508449611 E-01
0.2231695276 0082963268 4962667296	0.1943865561 5069876343 8140555174 E-01
0.2362625436 4209398018 5356987071	0.1907871790 3525862690 7186531491 E-01
0.2496118411 3214997922 6675261317	0.1869868554 9037615561 6038685921 E-01
0.2632044911 2528250621 9534291061	0.1830037867 5464029305 2281432919 E-01

Table 14.1.1. (*Cont.*)

0.2770273253 0252066499 8506333138	0.1788555959 2318781778 3117910820 E-01
0.2910669489 0321279121 1747104175	0.1745593487 3380851263 0533503864 E-01
0.3053097538 6758753098 0168040090	0.1701315705 3501391672 2847570813 E-01
0.3197419321 7868768642 3288212811	0.1655882600 3512750051 4779337585 E-01
0.3343494893 9169887536 4458176055	0.1609449003 3965326216 5801289127 E-01
0.3491182583 3836982353 4697802797	0.1562164677 0856360238 0604139078 E-01
0.3640339129 9393050931 0100776871	0.1514174384 0251264986 9721217094 E-01
0.3790819824 9365463473 1692275904	0.1465617939 3481907561 0965030943 E-01
0.3942478652 8598100213 6233620838	0.1416630250 0239521105 9934550664 E-01
0.4095168434 0890193847 7770942102	0.1367341343 3205866324 9581095331 E-01
0.4248740968 7614400782 6290046134	0.1317876386 4759775754 8470118169 E-01
0.4403047181 5950533652 6074403127	0.1268355699 3656996815 5311915554 E-01
0.4557937267 5357368771 6613725449	0.1218894761 7328448140 4475256211 E-01
0.4713260838 0892893635 6238716001	0.1169604216 3510124444 4817632746 E-01
0.4868867068 1983196983 3900408489	0.1120589869 3253629106 7934507217 E-01
0.5024604843 5231857543 0400414663	0.1071952688 5926112596 6389722273 E-01
0.5180322907 9855099246 0056070142	0.1023788801 5556484345 9282986156 E-01
0.5335870011 4323101361 6921039127	0.9761894926 7916688330 1184015823 E-02
0.5491095057 2784639906 7996405487	0.9292412017 7680057695 8243046421 E-02
0.5645847249 9850655751 2751029533	0.8830255236 3583985596 2782758554 E-02
0.5799976242 2312363485 2434006738	0.8376192095 5086206337 3759713813 E-02
0.5953332281 6371105210 3667954118	0.7930941712 7121892448 6376596220 E-02
0.6105766356 8960289514 9673095582	0.7495174878 0805808413 1152650419 E-02
0.6257130343 1744414518 3972933559	0.7069514154 9343409524 6583945877 E-02
0.6407277146 6386333009 4277998420	0.6654534016 3624419139 4078312786 E-02
0.6556060847 9681556341 6442436907	0.6250761020 7649659368 4890499443 E-02
0.6703336844 7167491507 2659150204	0.5858674029 0018213162 8182299863 E-02
0.6848961992 3826042642 8766930368	0.5478704465 4126220447 2144898476 E-02
0.6992794744 0509964160 6975405427	0.5111236624 6457202591 5813987613 E-02
0.7134695288 4736707653 7806871953	0.4756608025 9339596602 6831231088 E-02
0.7274525686 4508238296 6639126591	0.4415109816 1778378964 9105169714 E-02
0.7412150005 3831387807 7883093589	0.4086987222 9407043954 8986320394 E-02
0.7547434451 8630738731 6274747579	0.3772440058 2230415004 8923331693 E-02
0.7680247502 1764776721 3955739978	0.3471623273 6617741861 9343601612 E-02
0.7810460030 5876080745 7702571071	0.3184647567 5942005990 1952751768 E-02
0.7937945435 2827621892 6158358048	0.2911580044 2330451930 0033946211 E-02
0.8062579761 8499784870 6577309739	0.2652444925 0180263335 7318022001 E-02
0.8184241824 1746486433 3002355893	0.2407224312 0390947773 1585869654 E-02
0.8302813322 6333714446 1681985751	0.2175859003 2661833682 9569994663 E-02
0.8418178959 4709921271 5957782849	0.1958249359 1690866462 9936243506 E-02
0.8530226551 2484944639 7604942635	0.1754256220 1682489969 8244820159 E-02
0.8638847138 2522464710 7431128577	0.1563701874 2221654620 9064231018 E-02
0.8743935090 7580400528 4568909911	0.1386371073 7292580420 4373457984 E-02
0.8845388212 0464060290 2502761852	0.1222012100 8010223813 6702597548 E-02
0.8943107838 0688237751 2888812199	0.1070337879 8485480158 6565995499 E-02
0.9036998933 6676729243 3388916397	0.9310271363 1585719565 3906978330 E-03
0.9126970185 2560849542 4925408999	0.8037256002 9058723039 9202491139 E-03
0.9212934089 8672333113 3987128171	0.6880472536 2510489452 1693428319 E-03
0.9294807040 4860342785 2014845777	0.5835756191 0896986994 5904688598 E-03
0.9372509407 5796884424 9215462663	0.4898650901 4656964880 1183077262 E-03
0.9445965616 7469253892 7156999251	0.4064422993 1156986924 5405434763 E-03
0.9515104222 4091338169 1918759421	0.3328075240 7242432012 6180473901 E-03
0.9579857977 4695982602 3021221645	0.2684361279 1045107681 7769284446 E-03
0.9640163898 8694893610 4426277880	0.2127800349 8380715708 2489899106 E-03
0.9695963328 9703717075 1192053193	0.1652692364 2700286579 3816311244 E-03
0.9747201992 6912675332 8033450433	0.1253133263 1632644393 9989844280 E-03
0.9793830050 3201494711 2289051912	0.9230306527 6663446209 8936630081 E-04
0.9835802145 8960885115 3193356490	0.6561196965 4708090999 9470956116 E-04
0.9873077450 9944609024 2461759164	0.4459792413 7626043498 9986904567 E-04
0.9905619703 5691081809 2403337804	0.2860481565 5435364154 0413206364 E-04
0.9933397240 9576536605 4138932847	0.1696418637 2045700096 1279867673 E-04
0.9956383024 2200932959 6570947394	0.8996903554 3373696500 7097312495 E-05
0.9974554642 5965149142 9812176981	0.4014844167 3672693525 3293519505 E-05
0.9987894237 6807690208 4838050003	0.1322592599 4789845850 3202269879 E-05
0.9996387800 6340674918 6513497975	0.2191562063 9622993468 1058342349 E-06

Appendix 14.2

Table 14.2.1. *Error factors* $(2N)!e_{2N}$ *for a Gaussian quadrature with a weight function of* $\log_e x^{-1}$

N value	$(2N)!e_{2N}$
10	6.1386357946 7781378635 3294949353 E-13
20	5.5139828026 3439376076 7687891851 E-25
30	4.9935598396 1872831173 4145770058 E-37
40	4.5318450772 0545774955 2193878157 E-49
50	4.1163488127 8278187103 7664612366 E-61
60	3.7405584259 3670104809 2372275674 E-73
70	3.3999142500 5861254837 1184600935 E-85
80	3.0907695014 1845459994 8847141048 E-97
90	2.8100242420 6252456908 3615096858 E-109
100	2.5549645887 9190613773 5185708344 E-121

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Author index

Only a limited number of citations are included for Hilbert, Cauchy, and Fourier, as these authors occur very prominently throughout the book. Authors cited implicitly (as part of *et al.*) are indicated by italic font. Page numbers for volume 2 are indicated by (2).

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