Introduction to Dual Numbers

CS 3110 Final Project

April 2021

1 Introduction

I am sure we are all familiar with the complex numbers; that is, numbers in the form $a + bi$. a is the real component, and b is the imaginary component.
The dual numbers are defined similarly. They are of the form $a + b\varepsilon$, where a is the real component and b is the dual component.
Operations on the dual numbers are generally the same as complex numbers. You can think of the dual number $a + b\varepsilon$ as a tuple (a, b) and then consider operations on that tuple.
Multiplication will become clear after looking at the Formal Definition in the next section.

1.1 Formal Definition

The question then arises, what is ε? The traditional definition is $\varepsilon^2 = 0$, where $\varepsilon \neq 0$. It then follows that, if e cannot be 0, then the only way for $\varepsilon^2 = 0$ to hold is if ε is a very very small number, so small that its square is essentially 0.
Formally put, ε is an infinitesimally small number.
So to multiply two dual numbers, $a + b\varepsilon$ and $c + d\varepsilon$, it is simply given as

$$ac + (bc + ad)\varepsilon + bd\varepsilon^2,$$

but because $\varepsilon^2 = 0$, we simply discard that term.

1.2 Connection to Derivatives

Let us consider the Taylor expansion of a function $f(x)$ around a point q, which is given by

$$f(x) = f(q) + f'(q)(x - q) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots .$$

Now, let us plug in our dual number $a + b\varepsilon$ in the Taylor Expansion around a point a, which yields

$$f(a + b\varepsilon) = f(a) + f'(a)(a + b\varepsilon - a) + f''(a)\frac{(b\varepsilon)^2}{2!} + f'''(a)\frac{(b\varepsilon)^3}{3!} + \cdots .$$

But notice the third term and onward all contain ε^2 which is just 0, so they all zero out.
Hence, the evaluation of a function f at $a + b\varepsilon$ results in $f(a) + f'(a)(b)\varepsilon$ - that is, the value (in the real component) and the derivative at that point a (in the dual component).
If we let $b = 1$, then we can quite simply extract the value of the derivative from the ε portion without further operation necessary.
This gives us a formula for converting between reals and duals, at least for calculation of first derivative: r is simply $r + 1\varepsilon$.