**Lecture 11**
**05-September-2016**

# RICHARDS EQUATION:

In the last class, we were discussing about infiltration and unsaturated flow.

Using the Reynolds Transport Theorem (RTT), we derived the conservation of mass of liquid water in an unsaturated porous media having flow in one direction.

$$\frac{\partial \boldsymbol{\theta}}{\partial \boldsymbol{t}} + \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{z}} = \mathbf{0}$$

This continuity expression can be generalized as

$$\frac{\partial \boldsymbol{\theta}}{\partial \boldsymbol{t}} + \nabla . \vec{\boldsymbol{q}} = \mathbf{0}$$

where, $$\vec{\boldsymbol{q}} = \boldsymbol{q_x}\hat{\boldsymbol{\imath}} + \boldsymbol{q_y}\hat{\boldsymbol{\jmath}} + \boldsymbol{q_z}\widehat{\boldsymbol{k}}$$

In the above expression, we have now both $\boldsymbol{\theta}$ and $\boldsymbol{q}$ as the unknowns.

Adopting conservation of linear momentum principles (you will study that in the course SUBSURFACE HYDROLOGY), we can find an expression for Darcy velocity.

- This expression is also experimentally proven by Darcy

  i.e., $q = -Ki$

  where, $K$ =hydraulic conductivity $[LT^{-1}]$

  Hydraulic head, $H = \frac{p}{\rho g} + z$

$i$ = Hydraulic gradient $= \frac{\partial}{\partial z}\left(\frac{p}{\rho g} + z\right)$

The expression for Darcy $q = -Ki$ was actually derived for saturated flow through sand column.

We can extend the expression for Darcy velocity in unsaturated conditions as

$$q = -K\frac{\partial H}{\partial z}$$

Where H = Suction Head + Gravity Head

$= \psi + z$

$q = -K\frac{\partial}{\partial z}(\psi + z)$

In addition, for unsaturated flows, the hydraulic conductivity
$K \rightarrow K(\psi)$
Also, $K = K(\theta)$
i.e., $K$ can be function of $\theta$ or $\Psi$
Hence, $\theta$ and $\Psi$ are intrinsically related.

The term,

$$\frac{\partial \psi}{\partial z} = \frac{d\psi}{d\theta}\frac{\partial \theta}{\partial z}$$

where,

$\frac{\partial \theta}{\partial z}$ is the gradient of water content in vertical direction.

$\frac{d\psi}{d\theta}$ is specific water capacity

Hence,

$$q = -K(\frac{\partial \psi}{\partial z} + \frac{\partial z}{\partial z})$$

$$= -K\left(\frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial z} + 1\right)$$

$$= -K\left(\frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial z}\right) - K$$

Defining $D = K\frac{d\psi}{d\theta}$

where, $D$ is soil-water diffusivity

We get $q = -\left[D\frac{\partial \theta}{\partial z} + K\right]$

From continuity equation,

$$\frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial z}$$

$$= \frac{\partial}{\partial z}\left[D\frac{\partial \theta}{\partial z} + K\right]$$

i.e,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D\frac{\partial \theta}{\partial z} + K\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

One dimensional Richards Equation for unsaturated flow.

**EXAMPLE:**

For a soil the hydraulic conductivity K is expressed as a function of suction head $K=300(-\psi)^{-2.20}$cm/day

The observation of energy head in the experiment were

At $Z_1 = 100\ cm,\ H_1 = -170$ cm, $\psi_1 = -70$ cm

At $Z_2 = 120\ cm,\ H_2 = -200$ cm, $\psi_2 = -80$ cm

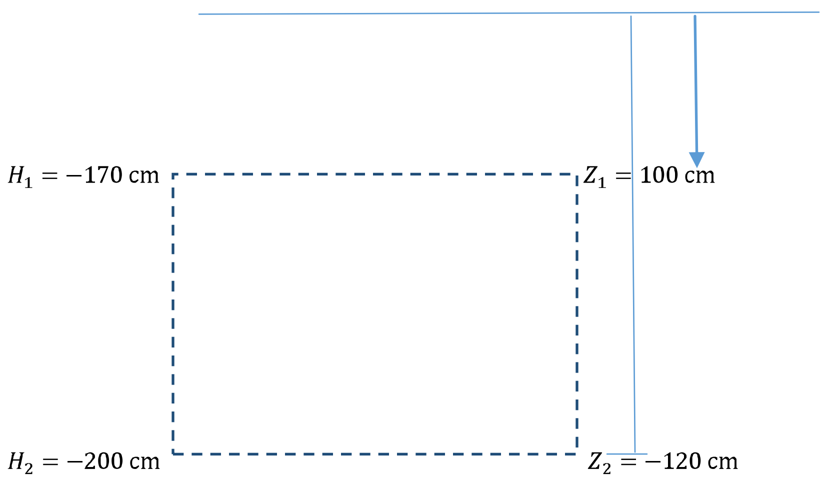
Determine the water flux

Ans:

Given energy head

$H_1 = -170$ cm

$H_2 = -200$ cm




Also
$Z_1 = -100$ cm,
$Z_2 = -120$ cm

$$q = -K\frac{\partial H}{\partial z}$$

$$= -K\left[\frac{H_2 - H_1}{Z_2 - Z_1}\right]$$

$$= -K\left[\frac{-200-(-170)}{-120-(-100)}\right]$$

$$= -K\left[\frac{-30}{-20}\right]$$

Now,

K=$300(-\psi)^{-2.20}$

Average $\psi$ between $Z_1$ $and$ $Z_2$;

$$\psi_{avg} = \frac{-70-(-80)}{2}$$

$$= -75 \text{ cm}$$

i.e. $K = 300(75)^{-2.20}$

= 0.0224 cm/day

i.e. q= 0.0337 cm/day

The flux is downwards

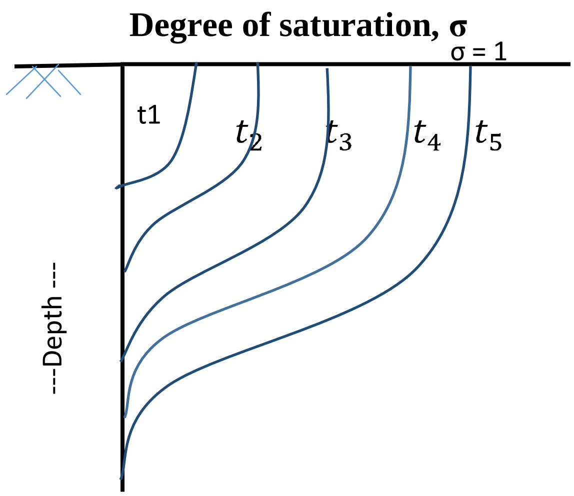
## GREEN AMPT METHOD:

When the rainfall occurs on the plain surface, the soils in the top surface may be saturated or unsaturated before rainfall.

Let us consider a dry soil.

Therefore, the top layer will be having unsaturated soils

So if we plot water content or degree of saturation with respect to time and depth, it may look like this:

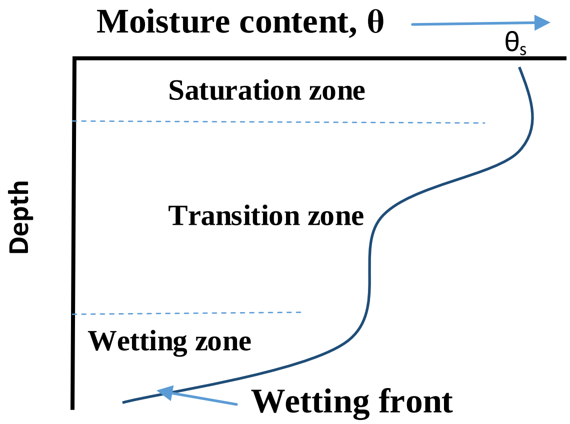



That is, the degree of saturation at the top layer will increase with respect to time and may reach maximum degree of saturation i.e. 1.0

You can also see that the variation of degree of saturation with respect to depth is that at an instant, degree of saturation decreases with depth.

The infiltration rate 'f' is the rate at which water enters the soil surface.

The movement of infiltrated water depth and moisture content can be perceived as




When the top soil becomes saturated, ponding of water occurs at the top.

Potential infiltration rate is the rate when water is ponded on the surface

Cumulative infiltration (F) is accumulated depth of water infiltrated

$$F(t) = \int_0^t f(\tau)\, d\tau$$

Where $d\tau \rightarrow$ dummy integrating parameter representing time.