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A. W. Warrick
The University of Arizona

Alvaro Islas
University of Central Florida

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An Analytical Solution to Richards' Equation for a Draining Soil Profile

A. W. Warrick

Department of Soil and Water Science, University of Arizona, Tucson

D. O. Lomen and A. Islas

Department of Mathematics, University of Arizona, Tucson

Analytical solutions are developed for the Richards' equation following the analysis of Broadbridge and White. Included here is the solution for drainage and redistribution of a partially or deeply wetted profile. Additionally, infiltration for various initial conditions is examined as well as evaporation at the upper boundary. In all cases the surface flux is constant, whether it be zero for drainage, positive for infiltration, or negative for evaporation. The solutions assume specific forms for the soil water diffusivity and hydraulic conductivity functions: $a(b - \theta)^{-2}$ and $\beta + \gamma(b - \theta) + \lambda/[2(b - \theta)]$, respectively. Here θ is the water content and a, b, β, γ, and λ are constants.

Introduction

Recently *Broadbridge and White* [1988*a*] (B. W.) presented an analytical solution to Richards' equation for soil water infiltration. The diffusivity and hydraulic conductivity functions were by necessity

$$D(\theta) = a(b - \theta)^{-2} \tag{1}$$

$$K = \beta + \gamma(b - \theta) + \lambda/[2(b - \theta)] \tag{2}$$

where θ is the volumetric water content and a, b, β, γ, and λ are constants. They included examples demonstrating the versatility of the solution to describe the soil water hydraulic properties of a wide range of soils. *Sander et al.* [1988*a*] independently published a nearly identical solution based on the analysis by *Rogers et al.* [1983] for two-phase flow. For further discussion see also *Broadbridge and White* [1988*b*] and *Sander et al.* [1988*b*]. A closely related paper by *Broadbridge et al.* [1988] is for a similar problem but for a bounded, rather than a semiinfinite, profile.

Although quasi-analytical and analytical solutions have been studied in great detail (see, for example, the numerous references of B. W.), corresponding relationships for drainage of initially wet profiles have not received as much attention. Existing choices are largely limited to numerical solutions, those based on fixed gradient assumptions [*Nielsen et al.*, 1973; *Raats*, 1983; *Sisson*, 1987], empirical approximations [cf. *Hillel*, 1982, equation 13.11], or linearized solutions [cf. *Warrick*, 1975, equation 20]. Recently, *Novak* [1988] applied quasi-linear solutions to problems of evaporation, but gravity effects were not included in the analysis.

The primary objective of this paper is to develop an analytical solution to Richards' equation for drainage and evaporation. The development is an adaption of the solution of B. W., and the results apply for any initial condition and any constant flux density at the surface boundary.

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Theory

The nonlinear Richards' equation describing one-dimensional flow is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] - \frac{\partial K}{\partial z} \tag{3}$$

The initial and boundary conditions considered here are

$$\theta(z, 0) = g(z) \tag{4}$$

$$v(0, t) = R \tag{5}$$

with $v(z, t)$ the Darcian velocity, z the depth, and R a constant surface flux density.

The solution of (3) subject to (4) and (5) follows along the lines of the appendix of *Broadbridge and White* [1988*a*]. For their solution, $g(z)$ was the constant θ_n corresponding to dry initial conditions, and R was positive. Our main interest here is to expand the solution to include any initial distribution $g(z)$ and additional R values which are zero or negative. A list of dimensionless parameters is summarized in Appendix A.

In terms of dimensionless parameters (Appendix A) we rewrite Richards' equation as did B. W.:

$$\frac{\partial \Theta}{\partial t_*} = \frac{\partial}{\partial z_*}(D_*\partial\Theta/\partial z_*) - \partial K_*/\partial z_* \tag{6}$$

A dimensionless matrix flux potential μ is defined as the Kirchoff transformation

$$\mu = C(C - 1)\int_{-\infty}^{\Theta} (C - w)^{-2}\, dw \tag{7}$$

which becomes

$$\mu = C(C - 1)/(C - \Theta) \tag{8}$$

With this definition, (6) may be recast in terms of μ as

$$D_*^{-1}\partial\mu/\partial t_* = \partial^2\mu/\partial z_*^2 - (dK_*/d\mu)\partial\mu/\partial z_* \tag{9}$$

The initial condition becomes

$$\mu = g_*(z_*) \qquad t_* = 0 \qquad z_* \geq 0 \tag{10}$$

where

$$g_*(z_*) = C(C-1)\Delta\theta/[C\Delta\theta - g(\lambda_s z^*) + \theta_n] \tag{11}$$

and the boundary condition is

$$v_* = R_* \qquad t_* > 0 \qquad z_* = 0 \tag{12}$$

where λ_s, v_*, and R_* are given in Appendix A.

The *Storm* [1951] transformation is now used, which reduces the above to

$$Z = [C(C-1)]^{0.5} \int_0^{z_*} \mu^{-1}\, dz_* \tag{13}$$

$$T = t_* \tag{14}$$

If $\partial\mu/\partial T$ is used to refer to a partial time derivative with Z fixed, the result becomes

$$\partial\mu/\partial T = \partial^2\mu/\partial Z^2 - 2C^{0.5}(C-1)^{0.5}[1 - 2\rho(1 + 1/\rho) + \mu/(C-1)]\partial\mu/\partial Z \tag{15}$$

which is equivalent to (A10) of B. W. The transformation [*Hopf*, 1950]

$$1 - 2\rho(1 + 1/\rho) + \mu/(C-1) = -u^{-1}\partial u/\partial\zeta \tag{16}$$

with

$$\zeta = C^{0.5}(C-1)^{0.5}Z \tag{17}$$

$$\rho = R_*[4C(C-1)]^{-1} \tag{18}$$

changes (15) to

$$u^{-1}[u^{-1}(\partial u/\partial\zeta) - (\partial/\partial\zeta)][\partial u/\partial\tau - 0.25(\partial^2 u/\partial\zeta^2)] = 0 \tag{19}$$

with

$$\tau = 4C(C-1)T \tag{20}$$

The appropriate initial and boundary conditions become

$$u^{-1}(\partial u/\partial\zeta) = G(\zeta) \qquad \tau = 0 \qquad \zeta \geq 0 \tag{21}$$

$$\partial u/\partial\tau = \rho(\rho+1)u \qquad \tau > 0 \qquad \zeta = 0 \tag{22}$$

where

$$G(\zeta) = 1 + 2\rho - C\Delta\theta[C\Delta\theta - g + \theta_n]^{-1} \tag{23}$$

The function g is from (4) and can be expressed in terms of ζ with the help of (A11), (13), and (17).

The initial and boundary conditions can be replaced by equivalent expressions

$$u = \exp\left[\int_0^{\zeta} G(\zeta')\, d\zeta'\right] \qquad \tau = 0 \qquad \zeta > 0 \tag{24}$$

$$u = \exp[\rho(\rho+1)\tau] \qquad \tau > 0 \qquad \zeta = 0 \tag{25}$$

where the integral in (24) will be in terms of ζ.

The solution to the diffusion equation within the right square brackets of (19) subject to (24) and (25) is well known [cf. *Carslaw and Jaeger*, 1959, p. 357]:

$$u = U_1 + U_2 \tag{26}$$

$$U_1 = 0.5 \exp(-\zeta^2/\tau)\{f[\zeta\tau^{-0.5} - (\lambda\tau)^{0.5}] + f[\zeta\tau^{-0.5} + (\lambda\tau)^{0.5}]\} \tag{27}$$

$$U_2 = (\pi\tau)^{-0.5} \int_0^{\infty} \exp\left[\int_0^{\zeta'} G(\zeta'')\, d\zeta''\right] h(\zeta, \tau, \zeta')\, d\zeta' \tag{28}$$

with

$$\lambda = \rho(\rho+1) \tag{29a}$$

$$f(x) = \exp(x^2)\,\mathrm{erfc}\,(x) \tag{29b}$$

$$h(\zeta, \tau, \zeta') = \exp[-(\zeta - \zeta')^2/\tau)] - \exp[-(\zeta + \zeta')^2/\tau] \tag{30}$$

Additional relationships to complete the solution are from B. W. (equations (A24) and (A25) of their appendix):

$$\Theta = C[1 - (2\rho + 1 - u^{-1}\partial u/\partial\zeta)^{-1}] \tag{31}$$

$$z_* = C^{-1}[\rho(\rho+1)\tau + (2\rho+1)\zeta - \ln u] \tag{32}$$

Equations (26), (31), and (32) represent an exact parametric solution, with Θ and z_* following for a specified ζ. For numerical evaluation, first ζ and t are specified. This allows an explicit τ from (A12), (14), and (20). Next u is evaluated by (26), followed by Θ and z_* by (31) and (32). The dimensioned depth z is by (A11), resulting in corresponding z, t, and Θ values.

U_1 for $R_* > 0$, $R_* = 0$, and $R_* < 0$

The effect of $R_* = (R - K_n)/\Delta K$ is expressed in U_1. For $R_* > 1$, (27) can be used without ambiguity. This will be the normal case for infiltration. For drainage, however $R = 0$, and R_* can be less than or equal to zero. For $R_* = \rho = 0$, U_1 is evaluated with $\lambda = \rho(\rho + 1) = 0$ (see (29*a*) and (A8) for which U_1 simplifies to

$$U_1 = \mathrm{erfc}\,(\zeta\tau^{-0.5}) \qquad \rho = 0 \tag{33}$$

If $K_n > 0$ and $R \leq 0$, R_* is negative and $\lambda = \rho(\rho + 1)$ can be negative. This leads to complex arguments in (27), and for this case, μ^2 is defined by

$$\lambda = -\mu^2 \tag{34}$$

with μ positive and real. Following *Abramowitz and Stegun* [1964, equation 7.1.3], w is defined by

$$w(iZ) = f(Z) \tag{35}$$

where Z is a complex argument in (29*b*). Thus (27) is equivalent to

$$U_1 = 0.5 \exp(-\zeta^2/\tau)[w(iZ) + w(i\bar{Z})] \qquad \rho(\rho+1) < 0 \tag{36}$$

with

$$Z = \zeta\tau^{-0.5} + i\mu\tau^{0.5} \tag{37}$$

or

$$Z = r\exp(i \arg Z) \qquad 0 \leq \arg Z < 2\pi \tag{38}$$

and $\bar{Z}$ the conjugate of Z. A direct result of their equation (7.1.8) is

$$w(iZ) + w(i\bar{Z}) = 2 \sum_{n=0}^{\infty} \frac{(-r)^n}{\Gamma(0.5n+1)} \cos(n \arg Z) \tag{39}$$

with Γ the gamma function. The result from (39) will be real, giving U_1 as a real number in (36). For large arguments their equation (7.1.23) is more suitable, although (39) converges for all $r < \infty$.

U_2 for Constant Initial Condition $g(z) = \theta_0$

If the initial water content is a constant θ_0, then by (23),

$$G(\zeta) = A_0$$

with

$$A_0 = 1 + 2\rho - C\Delta\theta[C\Delta\theta - \theta_0 + \theta_n]^{-1} \tag{40}$$

or

$$\exp\left[\int_0^{\zeta'} G(\zeta'')\, d\zeta''\right] = \exp(A_0\zeta') \tag{41}$$

If (40) is used in (28), an immediate result from (B7) is

$$U_2 = 0.5 \exp(-\zeta^2/\tau)\{f(-0.5A_0\tau^{0.5} - \zeta\tau^{-0.5}) - f(-0.5A_0\tau^{0.5} + \zeta\tau^{-0.5})\} \qquad g(z) = \theta_0 \tag{42}$$

U_2 for a Initial Water Content as Step Function

If the initial condition is

$$\begin{aligned} \theta(z, 0) &= g(z) = \theta_0 \qquad 0 < z < z_0 \\ \theta(z, 0) &= g(z) = \theta_\infty \qquad z > z_0 \end{aligned} \tag{43}$$

Then by (23) we find

$$\begin{aligned} \exp\left[\int_0^{\zeta'} G(\zeta'')\, d\zeta''\right] &= \exp(A_0\zeta') \qquad 0 < \zeta' < \zeta_0 \\ \exp\left[\int_0^{\zeta'} G(\zeta'')\, d\zeta''\right] &= \exp[(A_0\zeta_0 + A_\infty(\zeta' - \zeta_0)] \qquad \zeta' > \zeta_0 \end{aligned} \tag{44}$$

where A_∞ is analogous to A_0:

$$A_\infty = 1 + 2\rho - C\Delta\theta[C\Delta\theta - \theta_\infty + \theta_n]^{-1} \tag{45}$$

By equations (28) and (B7) the result may expressed as

$$\begin{aligned} U_2 = -0.5 &\exp(-\zeta^2/\tau) \\ &\cdot\{\exp(x_1^2)[\operatorname{erfc}(\zeta_0\tau^{-0.5} + x_1) - \operatorname{erfc}(x_1)] \\ &-\exp(x_2^2)[\operatorname{erfc}(\zeta_0\tau^{-0.5} + x_2) - \operatorname{erfc}(x_2)] \\ &+0.5 \exp[(A_0 - A_\infty)\zeta_0 - \zeta^2/\tau]\{\exp(x_3^2) \\ &\cdot[\operatorname{erfc}(\zeta_0\tau^{-0.5} + x_3)] \\ &- \exp(x_4^2)[\operatorname{erfc}(\zeta_0\tau^{-0.5} + x_4)]\} \end{aligned} \tag{46}$$

$$x_1 = -0.5A_0\tau^{0.5} - \zeta\tau^{-0.5} \tag{47}$$

$$x_2 = -0.5A_0\tau^{0.5} + \zeta\tau^{-0.5} \tag{48}$$

$$x_3 = -0.5A_\infty\tau^{0.5} - \zeta\tau^{-0.5} \tag{49}$$

$$x_4 = -0.5A_\infty\tau^{0.5} + \zeta\tau^{-0.5} \tag{50}$$

Note that as $\zeta_0 \to \infty$, this reduces to the previous case (i.e., equation (42)).

TABLE 1. Hydraulic Data Used for the Examples

Property	Yolo Light Clay	Brindabella Silty Clay Loam
$K_s - K_n$, m s^{-1}	$1.226(10)^{-7}$	$3.27(10)^{-5}$
K_n, m s^{-1}	$1.2(10)^{-10}$	~ 0
$S(\theta_s, \theta_n)$, m s$^{-0.5}$	$1.254(10)^{-4}$	$1.335(10)^{-3}$
$\theta_s - \theta_n$	0.2574	0.375
θ_n	0.2376	0.11
C	1.169	1.020
$h(C)/(C - 1)$	0.5536	0.5076
λ_s, m	0.276	0.0738

See *Moore* [1939], *Perroux et al.* [1981], and *White and Broadbridge* [1988*a*].

APPLICATIONS

Examples will now be discussed for Yolo light clay and Brindabella silty clay loam using the hydraulic properties listed by *White and Broadbridge* [1988, Table 4]. The values used for these hydraulic representations are repeated here as Table 1 along with equations for calculating a, b, β, γ, and λ of equations (A1) and (A2).

The calculations were performed in the Gauss Programming Language (Gauss) (Aptech Systems Incorporated, Kent, Washington; this is not an endorsement, but only for the reader's information). For large arguments ($x > 3.5$), $f(x)$ of equation (29*b*) was evaluated with an appropriate asymptotic formula [cf. *Abramowitz and Stegun*, 1964, equation 7.1.23]. For arguments $0 < x < 3.5$, the function erfc (x) from Gauss was used. For negative arguments the identity

$$f(-x) = 2\exp(x^2) - f(x) \tag{51}$$

was used. For the related complex arguments, (39) was used along with the asymptotic formula [*Abramowitz and Stegun*, 1964, equation 7.1.23] and (51) as necessary. For convenience the derivative needed in (31) was computed numerically, although it can also be done analytically. Also, for convenience when u and $\partial u/\partial\zeta$ approached zero for large ζ, the limiting value of Θ from the initial condition was used.

For all of the examples the analytical results were compared for consistency with numerical solutions using a finite difference model developed by D. K. Kahaner, National Bureau of Standards, and C. D. Sutherland, Los Alamos National Laboratory (personal communication, 1988).

Example 1

Drainage is from a deeply wetted profile with $f(z) = \Theta_{wet}$ and $R_* = 0$. For the first example we examine drainage for the Brindabella silty clay loam, assuming an initially, deeply wetted profile $\theta(z, 0) = \theta_s$ and "no flow" or $R = 0$ at the surface. Thus $u = U_1 + U_2$ is from (33) and (42). The resulting profiles are shown in Figure 1*a* for $t = 0$, 6, 12, and 24 hours for 0–0.6 m.

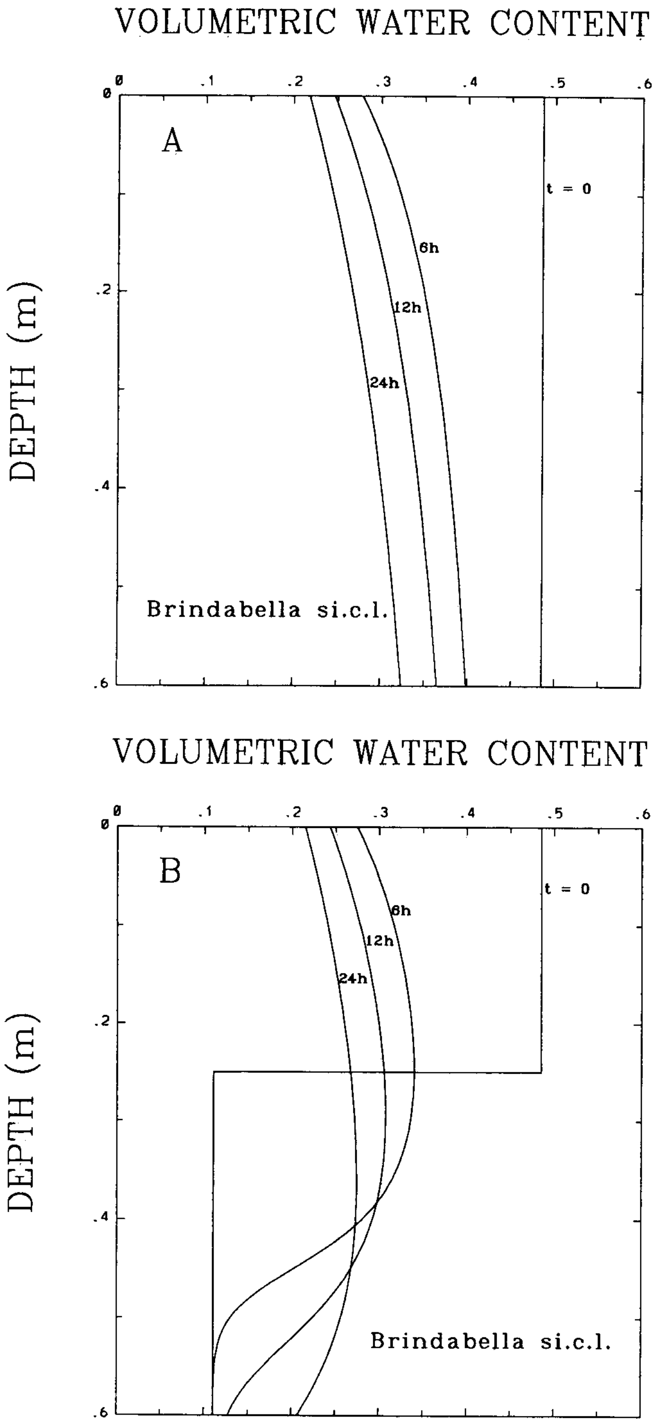

Fig. 1. (*a*) Drainage from a deeply wetted profile and (*b*) drainage and redistribution of a partially wetted profile.

Example 2

Drainage (and redistribution) is from a partially wetted profile with $R_* = 0$. Here we take

$$g(z) = \theta_s \qquad 0 < z < 0.25 \text{ m}$$

$$g(z) = \theta_n \qquad z > 0.25 \text{ m} \qquad (52)$$

and again consider the Brindabella silty clay loam. The solution is by (33) and (46) with $\theta_0 = \theta_s$ and $\theta_\infty = \theta_n$. The result is given in Figure 1*b* and also plotted for $t = 0$, 6, 12, and 24 hours. Initially the moisture profile is a "step function," and as time increases, the profile spreads and moves downward.

Example 3

Drainage is from a partially wetted profile, with and without evaporation. In this example the Yolo light clay is chosen with the initial condition of (52). Results are in Figure 2*a* for $t = 0$, 24, and 96 hours without evaporation. Additionally, calculations are performed taking an evaporative

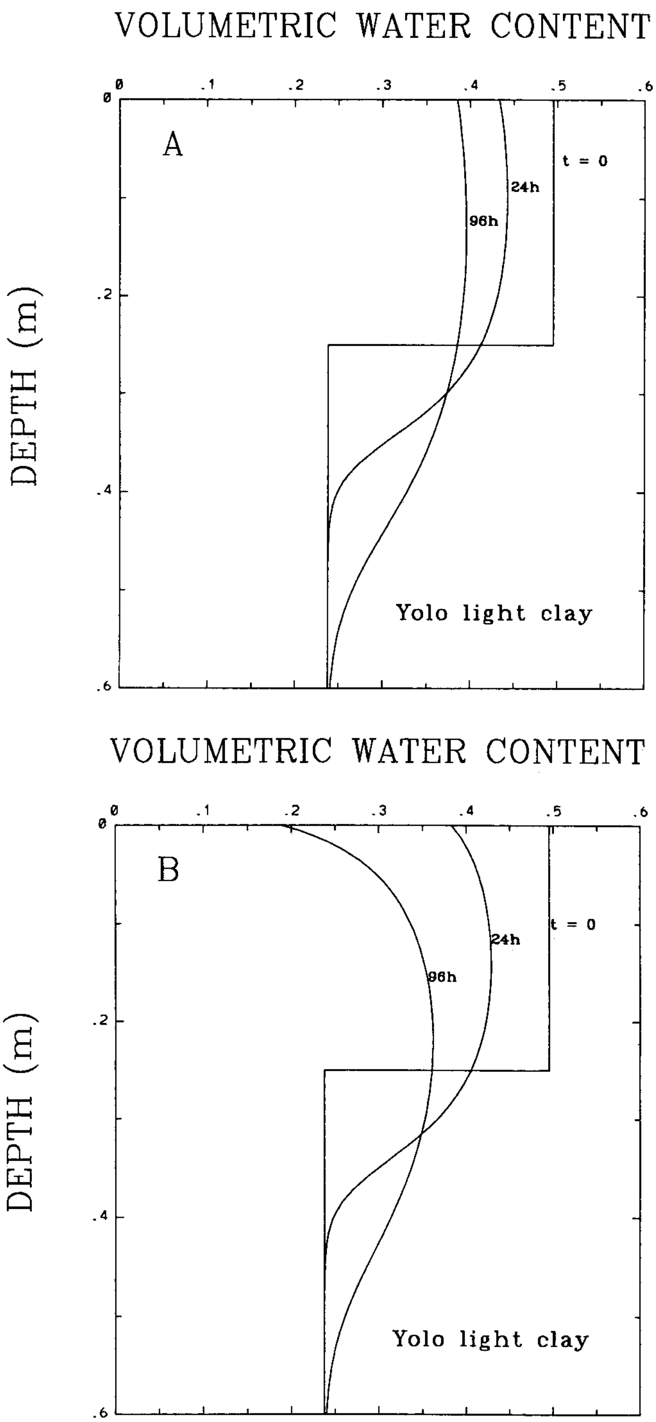

Fig. 2. (*a*) Drainage and (*b*) drainage with evaporation from a partially wetted profile.

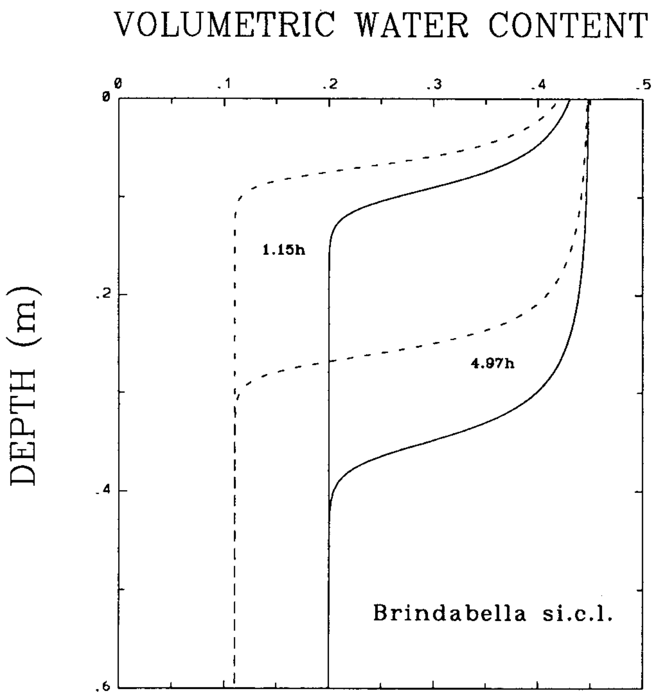

Fig. 3. Infiltration with contrasting initial water contents.

flux of $R_* = -(5.79)(10)^{-8}$ m s^{-1} (5 mm/d). Results are given in Figure 2*b* and reveal a lower surface water content than when evaporation does not occur. At larger times (about 130 hours) the surface water content approaches zero. This is to be expected, and the solution must not be used beyond such a time for which the surface water content becomes zero.

Example 4

Infiltration is shown for different initial water contents. We return to the example of B. W. (especially Figure 6). The "dashed curves" of Figure 3 show results for $\theta(z, 0) = \theta_n = 0.11$ and $R = 4.58(10)^{-6}$ m s^{-1} for $t = 1.15$ and 4.97 hours and agree with their results. Calculations for $u = U_1 + U_2$ are by equations (27) and (42).

Additionally, results are given for a wetter initial condition $\theta(z, 0) = 0.2$. These are the "solid" curves in Figure 3. The wetting profiles for each given time are deeper than the previous results for a dryer initial condition, as expected. This solution does not follow from the B. W. results or from those of *Sander et al.* [1988*a*] in an obvious fashion. Although θ_n can be changed in their solution, the hydraulic functions K and D are implicitly dependent on the same θ_n, K_n.

Summary and Conclusions

The analytical solution of *Broadbridge and White* [1988] has been extended to treat drainage from partially and deeply wetted soil profiles. Also, evaporation can be modeled by use of a negative surface flux. The time at which the evaporation case becomes invalid is approximately when the soil surface water content becomes zero, although a rigorous analysis would be complex. An analogous situation already used by B. W. is the time of ponding appropriate when a specified flux exceeds the infiltrability of the soil. A limitation is that only one form of D and K is allowed and the upper boundary condition is limited to a constant flux density. Some further generalizations may be possible, for example, an initial profile of piecewise step functions and perhaps other forms. This would allow simulation of cyclic conditions.

To our knowledge, this is the only analytical solution to Richards' equation for drainage conditions which includes the effect of gravity. Although the algebraic relationships are involved, calculations are extremely fast with minimal numerical difficulties.

Appendix A: Some Relationships Defined by Broadbridge and White [1988*a*]

Dimensionless and Other Parameters

$$\Delta\theta = \theta_s - \theta_n \qquad \text{(A1)}$$

$$\Delta K = K_s - K_n \qquad \text{(A2)}$$

$$C = (b - \theta_n)/\Delta\theta \qquad \text{(A3)}$$

$$\lambda_s = a/[C(C-1)\Delta\theta\Delta K] \qquad \text{(A4)}$$

$$t_s = a/[C(C-1)(\Delta K)^2] \qquad \text{(A5)}$$

$$D_* = (t_s/\lambda_s^2)D \qquad \text{(A6)}$$

$$K_* = (K - K_n)/\Delta K \qquad \text{(A7)}$$

$$R_* = (R - K_n)/\Delta K \qquad \text{(A8)}$$

$$v_* = (v - K_n)/\Delta K \qquad \text{(A9)}$$

$$\Theta = (\theta - \theta_n)/\Delta\theta \qquad \text{(A10)}$$

$$z_* = z/\lambda_s \qquad \text{(A11)}$$

$$t_* = t/t_s \qquad \text{(A12)}$$

$$\tau = 4C(C-1)t_* \qquad \text{(A13)}$$

Relationships for Coefficients Defining D and K

$$a = hS^2 \qquad \text{(A14)}$$

$$b = \theta_n + C\Delta\theta \qquad \text{(A15)}$$

$$\beta = K_s - [1 + 2C(C-1)]\Delta K \qquad \text{(A16)}$$

$$\gamma = (C-1)\Delta K/\Delta\theta \qquad \text{(A17)}$$

$$\lambda = 2C^2(C-1)\Delta\theta\Delta K \qquad \text{(A18)}$$

Appendix B: A Useful Integral

The integral considered is

$$I_\pm = \int \exp(-ax')\exp[-b(x \pm x')^2]\,dx' \qquad \text{(B1)}$$

where "±" can be either "+" or "−." First observe

$$\begin{aligned} ax' + b(x \pm x')^2 &= ax' + bx^2 \pm 2bxx' + bx'^2 \\ &= bx^2 + bx'^2 + 2(0.5a \pm bx)x' \\ &= bx^2 - b(c_\pm)^2 + b(x' + c_\pm)^2 \end{aligned} \qquad \text{(B2)}$$

where

$$c_{\pm} = [0.5(a/b) \pm x] \tag{B3}$$

Thus $I_{\pm}$ may be immediately expressed in terms of a complementary error function as

$$I_{\pm} = -0.5(\pi/b)^{0.5} \exp\{-b[x^2 - (c_{\pm})^2]\} \cdot \operatorname{erfc}[b(x' + c_{\pm})] + \text{const} \tag{B4}$$

An alternate form is

$$I_{\pm} = -0.5(\pi/b)^{0.5} \exp(-bx^2 + x_{\pm}^2) \cdot \operatorname{erfc}(b^{0.5}x' + x_{\pm}) + \text{const} \tag{B5}$$

where

$$x_{\pm} = 0.5ab^{-0.5} \pm b^{0.5}x \tag{B6}$$

If $b = 1/\tau$ and h is defined by (30), note that

$$(\pi\tau)^{-0.5} \int \exp(-a\zeta')h(\zeta, \tau, \zeta')\, d\zeta' = -0.5 \exp(-\zeta^2/\tau)\{\exp(x_5^2) \operatorname{erfc}(\zeta'\tau^{-0.5} + x_5) - \exp(x_6^2) \operatorname{erfc}(\zeta'\tau^{-0.5} + x_6)\} + \text{const} \tag{B7}$$

$$x_5 = 0.5a\tau^{0.5} - \zeta\tau^{-0.5} \tag{B8}$$

$$x_6 = 0.5a\tau^{0.5} + \zeta\tau^{-0.5} \tag{B9}$$

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A. Islas, and D. Lomen, Department of Mathematics, University of Arizona, Building 89, Tucson, AZ 85721.

A. W. Warrick, Department of Soil and Water Science, University of Arizona, Tucson, AZ 85721.

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