# Constant rate rainfall infiltration: A versatile nonlinear model: 2. Applications of solutions

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2 authors:

Ian White
Australian National University


P. Broadbridge
La Trobe University

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# Constant Rate Rainfall Infiltration: A Versatile Nonlinear Model 2. Applications of Solutions

I. WHITE AND P. BROADBRIDGE[1]

*Division of Environmental Mechanics, CSIRO, Canberra, ACT, Australia*

In paper 1 (Broadbridge and White, this issue) an analytical nonlinear model for constant rate rainfall infiltration was proposed which promised considerable versatility. In it a wide range of soil hydraulic properties are generated through the variation of a single free parameter $C$. Here three techniques are advanced for determining this parameter. The first, a one-dimensional technique, involves simultaneous determination of sorptivity and wetting front position. The second uses measured values of surface water content at long infiltration times for rainfall rates less than the saturated conductivity. In the third, three- and one-dimensional flow rates are measured on the same soil sample. All are suitable for field applications. The practical range of the $C$ parameter, for a variety of repacked and in situ soils, is found to be restricted to between 1 and 2. The hydraulic conductivities and diffusivities of the model are in general agreement with independent measurements. Its moisture characteristics $\Psi(\theta)$, which are not matched in any way to measured characteristics, follow closely those observed. Also, the model permits predictions of the dependence of sorptivity on antecedent water content given a single measurement of sorptivity. The analytic solutions for constant flux infiltration given in paper 1 (Broadbridge and White, this issue) describe satisfactorily the evolution of water content profiles and surface water pressure potential in the laboratory and field without a posteriori adjustments. The mathematically simple, traveling wave approximation agrees well with observations at comparatively short infiltration times. Finally, field and laboratory measured times to ponding are predicted satisfactorily by the model's analytic expression.

## 1. INTRODUCTION

In paper 1 of this work [*Broadbridge and White*, this issue] we described the features of solutions to a nonlinear convection-diffusion model of constant rate rainfall infiltration. By varying a single free parameter $C$, the model corresponded at one extreme to the weakly nonlinear Burgers' equation and at the other to a Green-Ampt-like model.

In this paper we test the application of the model to a range of soils from laboratory and field studies. First, we describe methods for determining the parameter $C$ for both repacked and in situ field soils. Second, we examine the range of the parameter $C$ for a variety of soils with substantially different hydraulic properties. Third, we compare for selected soils hydraulic properties estimated from the model with measurements. Finally, we test the model's a priori analytic solutions for the evolution of soil water content profiles, surface water pressure potential, and time to ponding for constant rate rainfall infiltration against published laboratory and field observations.

## 2. HYDRAULIC PROPERTIES

The hydraulic conductivity $K(\theta)$ of the model is [*Broadbridge and White*, this issue, equation (11)]

$$[K(\theta) - K_n]/\Delta K = \Theta^2(C-1)/(C-\Theta) \tag{1}$$

and the soil water diffusivity $D(\theta)$ is [*Broadbridge and White*, this issue, equation (21)]

$$D(\theta) = h\{S/[\Delta\theta(C-\Theta)]\}^2 \tag{2}$$

We recall that $\theta$ is volumetric water content; $\Delta\theta = \theta_s - \theta_n$, with $\theta_s$ the water content at "saturation" and $\theta_n$ the initial water content; $\Theta = (\theta - \theta_n)/\Delta\theta$ is the relative water content; $\Delta K = K_s - K_n$; $K_n = K(\theta_n)$; $K_s = K(\theta_s)$; $S = S(\theta_s, \theta_n)$ is the sorptivity and $h$ is a known function of $C$ shown in Figure 1 of *Broadbridge and White* [this issue] and approximated by [*Broadbridge and White*, this issue, equation (20)]:

$$h(C) \approx C(C-1)[\pi(C-1) + B]/[4(C-1) + 2B] \tag{3}$$

with $B = 1.46147$. The moisture characteristic $\Psi(\theta)$ when $K_n = 0$ is [*Broadbridge and White*, this issue, equation (25)]

$$\Psi(\Theta) = -\lambda_s\{(1-\Theta)/\Theta + C^{-1}\ln((C-\Theta)/[\Theta(C-1)])\} \tag{4}$$

with the capillary length scale $\lambda_s$ given by

$$\lambda_s = [h/C(C-1)]S^2/[\Delta\theta\Delta K] \tag{5}$$

Since $S$ and $K_s$ are routinely measured, the problem in applying (1) to (5) reduces to that of estimating $C$.

## 3. DETERMINATION OF $C$ PARAMETER

There is no unique procedure for determining $C$ for soils. Indeed, the procedure ought to be dictated by the application required. We could simply determine $C$ by insisting that (2), with $\Theta = 1$, match the large but finite measured soil water diffusivity at saturation [*Clothier and Wooding*, 1983]. However, such measurements are difficult to make in the field. We describe here three methods for determining $C$.

### 3.1. *Constant Potential Method*

We adapt a scheme, already used in the field by *Clothier and White* [1981], in which wet front position $z_p$ and the sorptivity [*Philip*, 1957*a*] are simultaneously measured during the early stages of constant potential infiltration. The dimensionless penetration $\Phi_p$ of any plane of given relative water content $\Theta_p$ during the gravity-free early stages is

$$\Phi_p = \lambda(\Theta_p)\Delta\theta/S \tag{6}$$

Here $\lambda(\Theta_p)$ is the Boltzmann variable $z_p\, t^{-1/2}$ associated with the position of the plane $\Theta(z_p, t) = \Theta_p$ at time $t$. For the early stages $\Phi_p$ is a function of $\Theta_p$ only; $\Phi_p$ is a measure of the "shape" of the absorption profile. When $\Phi_p \to 1$ the profile approaches the step function of the Green-Ampt model. As $\Phi_p \to 3.230$ (for $\Theta_p = 0.01$), the profile approaches that given

[1] Now at Department of Mathematics, La Trobe University, Bundoora, Victoria, Australia.



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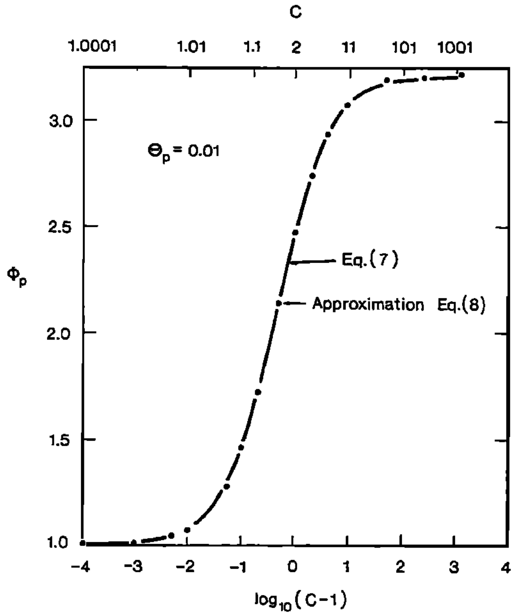



Fig. 1. Dependence of dimensionless "wetting front" position $\Phi_p$ on the parameter $C$ for $\Theta_p = 0.01$ from (7) and approximation (8).

by a constant $D$. For the diffusivity of our model, (2), $\Phi_p$ may be derived from *Fujita*'s [1952] analytic solution for constant concentration absorption:

$$\Phi_p = \{P/(C - \Theta_p) - (C - 1)^{-1} \exp [(1 - P^2)/[4h(C)]]\} \quad (7)$$

with

$$P = [4h(C)]^{1/2} \operatorname{inverfc} \{[(C-1)\Theta_p/(C-\Theta_p)] \operatorname{erfc} [4h(C)]^{-1/2}\}$$

By identifying the "wetting front" with a plane of saturation, say, $\Theta_p = 0.01$, (7) gives a one-to-one relation between $\Phi_p$ and $C$, plotted in Figure 1.

Figure 1 shows that for flows in which the wetting front shape is steep, i.e., $S \rightarrow \lambda(\Theta_p)\Delta\theta$, $\Phi_p \rightarrow 1$, so that $C \rightarrow 1$. The relationship between $\Phi_p$ and $C$ depicted in Figure 1 can be approximated to within 1% by

$$C = 1 + [0.52339(\Phi_p - 1)/(3.2303 - \Phi_p)]^{1.15} \quad (8)$$

As $C \rightarrow \infty$, small variations in $\Phi_p$ generate large variations in $C$. In this region of $C$, wetting fronts are difficult or impossible to detect. From Figure 1 we see that this method is practically restricted to $1.01 \leq C \leq 20$.

Methods for the simultaneous measurement of $z_p$ and $S$ at various times are given in the work by *Clothier and White* [1981]. Having obtained $z_p$, $S$ at selected $t$, $\lambda_p$ and $\Phi_p$ follow from (6), provided $\Delta\theta$ is known.

Values of $\Phi_p$, and hence $C$, for a range of repacked soils and in situ field soils are given in Table 1, with the surface pressure $\Psi_0$ used to supply water and the initial water content of the soil. It is apparent from Table 1 that for the soils examined the practical range of $C$ is quite small ($1 \leq C \leq 1.5$). The materials listed there include some whose properties can be considered extreme. For example, the fine sand–kaolinite mixture of *Smiles et al.* [1978] was deliberately packed to produce a gradual, near "linear" water content profile during constant potential absorption. In contrast, the Brindabella silty clay loam has the steep wetting front found in textbooks. The in situ field soils in Table 1 have the higher values of $C$, a result entirely consistent with observations that certain field soils have soil water diffusivities only weakly dependent on water content [*Clothier et al.*, 1981*a*, *b*; *Clothier and White*, 1982; *Kutílek*, 1984].

To indicate the errors in $C$ likely to be encountered in field determinations, we show in Table 1 the range arising from the observed errors in $\Phi_p$ for the field soils. We stress that the nature of $C$ is such that values between 1.5 and $\infty$ produce

TABLE 1. Values of $C$ Parameter for Selected Soils

| Soil | $\Psi_0$, m | $\theta_n$ | $\Phi_p$ | $C$ | Reference |
|---|---|---|---|---|---|
| | | *Repacked Laboratory Soils* | | | |
| Brindabella silty clay loam | −0.01 | 0.11 | 1.131 | 1.020 | *Perroux et al.* [1981] |
| Bungendore fine sand (washed) | −0.01 | 0.013 | 1.157 | 1.024 | *White et al.* [1979] |
| Bungendore fine sand (70%) kaolinite (30%) mixture | −0.01 | 0.04 | 1.639 | 1.166 | *Smiles et al.* [1978] |
| Casuarina river sand | −0.01 | 0.008 | 1.195 | 1.032 | I. White (unpublished data, 1976) |
| Cowra sandy loam | −0.01 | 0.03 | 1.122 | 1.018 | B. E. Clothier and I. White (unpublished data, 1980) |
| Molonglo loam | −0.01 | 0.023 | 1.145 | 1.022 | K. M. Perroux (private communication, 1983) |
| Pialligo sand | −0.01 | 0.01 | 1.119 | 1.017 | K. M. Perroux (private communication, 1983) |
| Yolo light clay | 0 | 0.2376 | 1.642 | 1.169 | *Philip* [1957*b*] *Moore* [1939] |
| | | *Undisturbed Field Soils* | | | |
| Bungendore fine sand | 0 to −0.03 | 0.03 | 1.82 (±0.43) | 1.26 (1.08–1.63) | *Clothier et al.* [1981*b*] |
| Cowra sandy loam | −0.03 | 0.04 | 2.04 (±0.54) | 1.41 (1.11–2.32) | *Clothier et al.* [1981*b*] |
| Herbarium loamy sand | −0.04 | 0.18 | 1.90 (±0.5) | 1.30 (1.08–1.87) | S. J. Zegelin and I. White (unpublished data, 1981) |

TABLE 2. Estimation of $C$ Parameter Using Rainfall Data

| Property | Brindabella Silty Clay Loam (Laboratory) | Bungendore Fine Sand (Field) |
|---|---|---|
| $R$, mm h$^{-1}$ | 16.5 | 12.2 |
| $K_s$, mm h$^{-1}$ | 117.7 | $43 \pm 8$ |
| $K_n$, mm h$^{-1}$ | $\sim 0$ | $\sim 0$ |
| $R_*$ | 0.140 | $0.282 \pm 0.056$ |
| $t$, hour | 4.97 | 2.0 |
| $t_g$, hour | 1.75 | 0.75 |
| $t/t_g$ | 2.84 | 2.64 |
| $\theta_s$ | 0.485 | 0.33 |
| $\theta_n$ | 0.11 | 0.03 |
| $\theta_e$ | $0.427 \pm 0.006$ | $0.21 \pm 0.01$ |
| $\Theta_e$ | $0.845 \pm 0.016$ | $0.600 \pm 0.029$ |
| $C$ | $1.038 \pm 0.006$ | 2.56<br>1.4–10 |

Data for Brindabella silty clay loam are taken from *Perroux et al.* [1981]. Data for Bungendore fine sand are taken from *Clothier et al.* [1981*b*].

only minor changes in calculated water content profiles or in hydraulic properties.

### 3.2. *Constant Flux Method*

It is equally possible to determine $C$ using constant flux infiltration measurements. For example, the rate of increase of the surface water content at early infiltration times ($t < t_g$ [*Broadbridge and White*, this issue]) may be used to determine both sorptivity and the value of $C$ provided the rainfall rate is known. But the precise and rapid determination of the evolving near-surface water content is difficult in the field.

An alternative scheme uses rainfall rates less than the saturated hydraulic conductivity. At large infiltration times for these rates the surface moisture content approaches an equilibrium value $\Theta_e < 1$. Equation (46) of *Broadbridge and White* [this issue] provides a relationship between $\Theta_e$, $C$, and the supply rate. This gives a simple method for estimating $C$.

Rearranging gives for constant rainfall rate $R$:

$$C = \Theta_e(\Theta_e - R_*)/(\Theta_e{}^2 - R_*) \tag{9}$$

where the dimensionless rainfall rate $R_*$ is

$$R_* = (R - K_n)/\Delta K \tag{10}$$

For many field situations $R$ and $K_s \gg K_n$ so $R_* = R/K_s$. To use (9) $R$, $K_s$, $K_n$, $\theta_s$, and $\theta_n$ are required and $\theta_e$, the surface volumetric water content at flow equilibrium, must be determined.

We test this procedure by using constant rate rainfall infiltration data for two of the soils of Table 1: repacked Brindabella silty loam and in situ Bungendore fine sand. Values of $\theta_e$ were estimated from water content profiles measured at large times [*Perroux et al.*, 1981; *Clothier et al.*, 1981*b*]; $\theta_e$ was taken as the water content associated with the near-surface portion of the profile where $\partial\theta/\partial z \approx 0$. The data needed for the calculation and the derived values of $C$ estimated from (9) are given in Table 2.

The value of $C$ for the repacked Brindabella silty clay loam is close to that shown in Table 1. The error in $C$ in Table 2 for this material arose solely from the determination of $\theta_e$. Since the other parameters appearing in (9) have comparable errors, we expect the range of $C$ to be larger, leading to overlap of the values in Table 1 and 2.

There is, however, a discrepancy between the mean values shown in Tables 1 and 2 for the Bungendore fine sand field data although the ranges overlap. In the method used to obtain Table 1 we found $C$ by matching integral properties over a considerable part of the soil's moisture content range. For the constant flux methods we match hydraulic conductivity at one unsaturated water content $\theta_e$. Note that for this material the range of $C$ is generated by modest experimental errors in both $\theta_e$ and $K_s$. The discrepancy then is unsurprising. Furthermore, large differences in $C$ for values above 1.5 have only minor effect on calculated water content profiles.

### 3.3. *Three-Dimensional Technique*

The recent introduction of the disc permeameter (I. White et al., unpublished data, 1985) provides an additional method for estimating $C$ from in situ soil measurements. To a good approximation the steady state, three-dimensional flow rate $q_\infty$ from a disc or shallow circular pond of radius $r$ placed on the soil surface is [*Wooding*, 1968]

$$q_\infty/\pi r^2 = \Delta K_0(1 + 4\lambda_s/\pi r) \tag{11}$$

where $\Delta K_0 = K(\theta_0) - K_n$, $\theta_0$ being the water content in equilibrium with the disc water supply potential $\Psi_0$ and $\theta_0 \le \theta_s$.

We may rearrange (11) using our definition of $\lambda_s$, (5), to obtain

$$h/[C(C-1)] = \Delta\theta_0\pi r[q_\infty/(\pi r^2) - \Delta K_0]/(2S_0)^2 \tag{12}$$

with $\Delta\theta_0 = \theta_0 - \theta_n$ and $S_0 = S(\theta_0, \theta_n)$. Provided we have independently measured values of $S(\theta_0, \theta_n)$ and $\Delta K_0$ at the location where $q_\infty$ is measured, (12) permits us to determine $h/[C(C-1)]$, and $C$ follows from (3). Details of the measurement procedure are given by I. White et al. (unpublished data, 1985). The necessary measured properties and the estimated values of $C$ at four separate sites for in situ Bungendore fine sand are shown in Table 3.

Despite the six-fold variation of $q_\infty$ across the sites evident in Table 3 and typical of the field, the calculated values of $C$ for this three-dimensional method encompass the range found for the one-dimensional technique of Table 1. Considering the 6 years and 400 m tract of dry schlerophyl forest separating the two sets of measurements and their sites, we consider the agreement to be satisfactory. These three schemes for fixing the $C$ parameter are by no means exclusive but they suffice for a variety of applications.

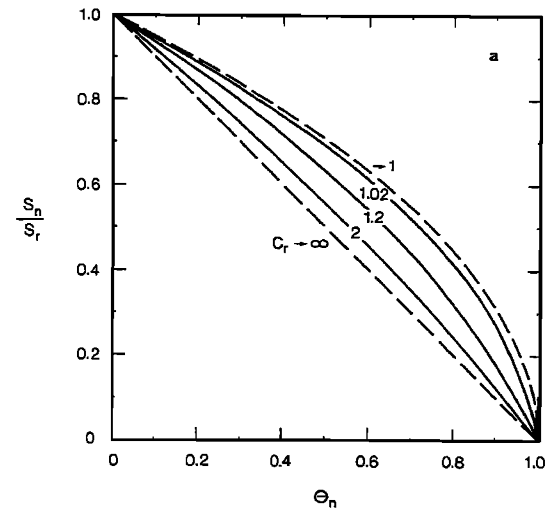
### 3.4. *Dependence of C Parameter on Soil Water Content*

It is apparent from the preceeding sections that the parameter $C$ depends on the soil water content range over which it is determined. Clearly, it is an advantage to determine $C$ over as wide a water content range as possible. Where this is not possible we need to be able to interpolate or extrapolate values of $C$ and sorptivity from a single, reference water content range.

TABLE 3. Estimation of $C$ Parameter Using Three-Dimensional Flow From a Surface Disc

| | Site 1 | Site 2 | Site 3 | Site 4 |
|---|---|---|---|---|
| $q_\infty/\pi r^2$, mm h$^{-1}$ | 28.0 | 15.6 | 70.8 | 95.3 |
| $\Delta\theta_0$ | 0.111 | 0.105 | 0.189 | 0.216 |
| $S(\theta_0, \theta_n)$, mm h$^{-1/2}$ | 11.2 | 6.94 | 32.4 | 34.2 |
| $\Delta K_0$, mm h$^{-1}$ | 17.6 | 12.1 | 25.8 | 51.1 |
| $h(C)/[C(C-1)]$ | 0.728 | 0.600 | 0.636 | 0.643 |
| $C$ | 3.90 | 1.39 | 1.67 | 1.73 |

Data is for in situ Bungendore fine sand: $r = 100$ mm and $\Psi_0 = -35$ mm $H_2O$ (from I. White et al., unpublished data, 1985).



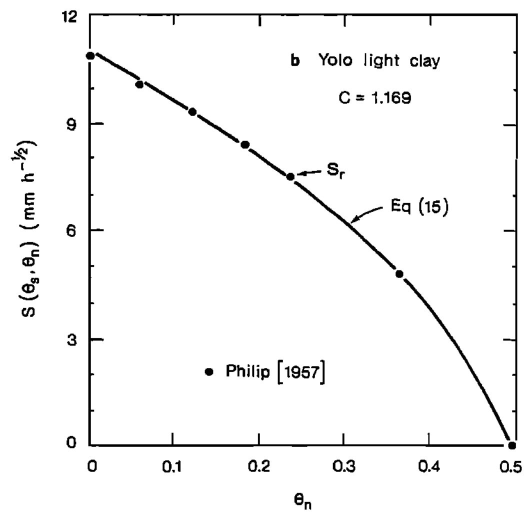




Fig. 2. Dependence of sorptivity on initial water content. (*a*) Model dependence on *C* parameter given by (15). (*b*) Comparison of predictions from (15) given a single reference value of sorptivity $S_r$ for Yolo light clay. Points are from *Philip* [1957*b*].

Let us suppose that we have measured one sorptivity $S_r$ for a particular water content range $\theta_{0r} - \theta_{nr} = \Delta\theta_r$ and we determine $C_r$ for this range. We may find the value for the parameter $C$ at any other range $\theta_0 - \theta_n = \Delta\theta_0$ from the definition of $C$ [*Broadbridge and White*, this issue, equation (9)]:

$$C = (C_r - \Theta_n)/(\Theta_0 - \Theta_n) \tag{13}$$

with

$$\Theta_0 = (\theta_0 - \theta_{nr})/\Delta\theta_r$$

$$\Theta_n = (\theta_n - \theta_{nr})/\Delta\theta_r$$

Note here that $\Theta_0$ and $\Theta_n$ are not restricted to the range 0–1.

The sorptivity at any $\theta_0$, $\theta_n$, $S(\theta_0, \theta_n)$ may be found from the reference sorptivity using

$$S(\theta_0, \theta_n) = [h(C_r)/h(C)]^{1/2} S_r \tag{14}$$

Here $h(C)$ is found from (13) and (3).

Alternatively, we find from *Fujita* [1952]

$$\frac{\Theta_0 - \Theta_n}{C_r - \Theta_n} = \left[\frac{\pi}{4h(C_r)}\right]^{1/2} \frac{S(\theta_0, \theta_n)}{S_r} \operatorname{erfc}\left[\frac{S(\theta_0, \theta_n)}{[4h(C_r)]^{1/2} S_r}\right] \cdot \exp\left[\frac{S(\theta_0, \theta_n)}{[4h(C_r)]^{1/2} S_r}\right]^2 \tag{15}$$

Using the approximation (3) we may write to an accuracy of better than 1%:

$$\left[\frac{S(\theta_0, \theta_n)}{S_r}\right]^2 = \frac{(\Theta_0 - \Theta_n)^2 C_r(C_r - 1)}{(C_r - \Theta_n)(C_r - \Theta_0)} \frac{\pi(C_r - 1) + B}{\pi(C_r - \Theta_0) + B(\Theta_0 - \Theta_n)} \cdot \frac{4(C_r - \Theta_0) + 2B(\Theta_0 - \Theta_n)}{4(C_r - 1) + 2B} \tag{16}$$

Thus through (13) and (15) or (16) we can determine $C$ and the sorptivity at any $\theta_0$, $\theta_n$, given $C_r$, $S_r$ and $\theta_{0r}$, $\theta_{nr}$. We note in (15) that as $C_r \to 1$, when $\Theta_0 = 1$, $S(\theta_0, \theta_n)/S_r \to (1 - \Theta_n)^{1/2}$, and as $C_r \to \infty$, $S(\theta_0, \theta_n)/S_r \to (\theta_0 - \theta_n)$, as they ought to be consistent with the limiting cases of a Dirac delta function soil water diffusivity and a constant diffusivity.

In many field applications $\theta_0$ is fixed by the measurement technique, usually at $\theta_s$. For these situations only $\theta_n$ varies and $\Theta_0 = 1$ in (13)–(16). We show the predictions of (15) for the range $1 \le C_r \le \infty$ with $\Theta_0 = 1$, i.e. fixed $\theta_0$ in Figure 2*a*.

To illustrate the utility of (15), we use it to calculate the sorptivity of Yolo light clay from a single measured sorptivity $S_r$. For this illustration $\theta_s = 0.495$ and we select $\theta_{nr} = 0.2376$, at which $S_r = 7.523$ mm h$^{-1/2}$ [*Philip*, 1957*b*]. At this $\theta_n$ the value of $C_r$ from Table 1 is 1.169 for Yolo light clay. The calculated dependence of sorptivity on $\theta_n$ given by (15) is compared in Figure 2*b* with the values of *Philip* [1957*b*]. The agreement is excellent with the maximum deviation between estimated and actual sorptivity being 2%. This is much less than the measurement error in determining sorptivity in the laboratory, let alone the field. We reiterate that the extrapolated values of sorptivity shown in Figure 2*b* were calculated using only a single reference value of initial water content $\theta_r$.

Finally, we note that the model's soil water diffusivity function scaled at the reference water content range

$$D(\theta) = h(C_r)\{S_r/[\Delta\theta_r(C_r - \Theta)]\}^2$$

with $\Theta = (\theta - \theta_{nr})/\Delta\theta_r$, is identical to that scaled over any other range of water content

$$D(\theta) = h\{S(\theta_s, \theta_n)/[\Delta\theta(C - \Theta^*)]\}^2$$

with $\Theta^* = (\theta - \theta_n)/\Delta\theta$.

## 4. Comparison of Model and Measured Properties

Having proposed methods for estimating the parameter $C$, we now examine the adequacy of the model's functional forms for $K(\theta)$, (1), and $D(\theta)$, (2), for representing measured soil water hydraulic properties. We select two materials in Table 1, the benchmark Yolo light clay and Brindabella silty clay loam whose properties are listed in Table 4.

TABLE 4. Data for Illustrative Application of Exact Solution

| Property | Yolo Light Clay | Brindabella Silty Clay Loam |
|---|---|---|
| $K_s - K_n$, m s$^{-1}$ | $1.226 \times 10^{-7}$ | $(3.27 \pm 0.29) \times 10^{-5}$ |
| $K_n$, m s$^{-1}$ | $1.2 \times 10^{-10}$ | $\sim 0$ |
| $S(\theta_s, \theta_n)$, m s$^{-1/2}$ | $1.254 \times 10^{-4}$ | $(1.335 \pm 0.080) \times 10^{-3}$ |
| $\theta_s - \theta_n$ | 0.2574 | 0.375 |
| $\theta_n$ | 0.2376 | 0.11 |
| $\Phi_p$ | 1.642 | 1.131 |
| $C$ | 1.169 | 1.020 |
| $h(C)/(C - 1)$ | 0.5536 | 0.5076 |
| $\lambda_s$, m | 0.276 | 0.0738 |

Data for Yolo light clay are taken from *Moore* [1939] and *Philip* [1957*b*]. Data for Brindabella silty clay loam are taken from *Perroux et al.* [1981] and K. M. Perroux (private communication, 1983).

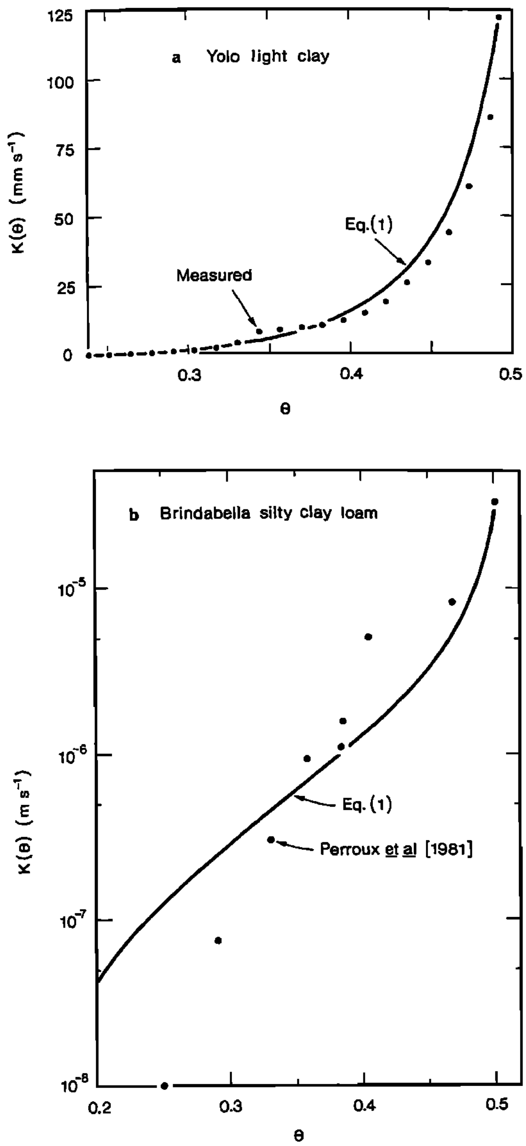



Fig. 3. Comparison between the hydraulic conductivity of the model, (1), and measurements for (*a*) Yolo light clay [*Moore*, 1939] and (*b*) Brindabella silty clay loam [*Perroux et al.*, 1981].

The resulting $K(\theta)$, (1), with the appropriate properties from Table 4, are shown in Figure 3*a* for Yolo light clay and in Figure 3*b* for Brindabella silty clay loam. The directly measured $K(\theta)$ for both soils are plotted for comparison. Agreement is reasonable for both soils although (1) overestimates $K(\theta)$ at the lower end of the water content range, particularly for the silty clay loam. Since the quantities of water transported at low $\theta$ are small, we expect this mismatch to be noticeable only at very short rainfall infiltration times or with very low water application rates.

For Yolo light clay the root mean square error between (1) and the measurements is $5.2 \times 10^{-9}$ m s$^{-1}$. Had we chosen to pursue the three parameter forms for $K(\theta)$ [*Broadbridge and White*, this issue, equation (5)], which still allows analytic solutions, we would have been required to fit the three parameters by a least squares fit of the experimental data in Figure 3*b*. Such a fit produces a lower mean square error, $3.3 \times 10^{-9}$ m s$^{-1}$, but also gives greatest deviations close to $\theta_s$, leading to unacceptably large errors for rainfall rates close to or exceeding $K_s$.

Figure 4 shows the corresponding comparison for soil water diffusivity. A pleasing feature of the comparisons for both soils is the model's satisfactory representation of the rapid increase in $D(\theta)$ as $\theta \rightarrow \theta_s$. Again the model overestimates the hydraulic properties as $\theta \rightarrow \theta_n$, particularly for the silty clay loam. We must recall, however, that the one-step method of *Bruce and Klute* [1956] used to measure $D(\theta)$ for this soil is least accurate in the region $\theta \rightarrow \theta_n$.

Finally, we compare in Figure 5 the model's moisture characteristic ($\Psi(\theta)$ [*Broadbridge and White*, this issue, equation (22)]) with experimental values. We stress here that this comparison represents a critical test of the model's capabilities since no part of the model's $\Psi(\theta)$ is matched to the experimental moisture $\Psi(\theta)$. Rather, the $\Psi(\theta)$ relies solely on independent estimates of $\lambda_s$, $K_s$, $K_n$, and $C$.

Figure 5*a* compares the model's a priori moisture characteristic and the 21 experimental values of *Moore* [1939] for Yolo light clay. The agreement, we believe, is remarkable. Figure 5*b* shows the comparison for Brindabella silty clay loam. The experimental points are those determined statically by *Perroux et al.* [1982]. Equation (22) of *Broadbridge and White* [this issue] requires a measured value of $K_n$. We have taken this to be the lowest measured value in Figure 3*b*, $K_n = 10^{-8}$ m s$^{-1}$ at $\theta_n = 0.25$. At this water content $\theta_s - \theta_n = 0.235$; (15) gives $C = 1.032$ and (3) shows $h/(C-1) = 0.5120$; (3) and (15) predict the sorptivity at this $\theta_n$ to be $1.062 \times 10^{-3}$ m s$^{-1/2}$, and $\lambda_s = 0.0751$ m. Note that despite the considerable change of $\theta_n$ from Table 4, the new $\lambda_s$ is very close to that listed in Table 4.

Apart from the region $0 > \Psi > -0.20$ in Figure 2*b*, the model predictions are close to the measurements. When the $\pm 20\%$ measurement error in $\lambda_s$ is taken into account as indicated by the shaded region in Figure 5*b* the overall agreement is satisfactory.

We conclude that the model's functional forms for the hydraulic properties, (1) and (2), not only reduce to well-known models at the extremes of the $C$ parameter range but appear to adequately represent measured hydraulic properties at intermediate values.

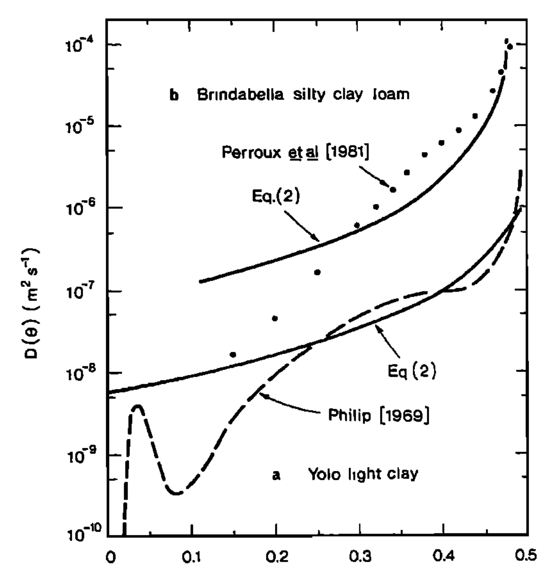



Fig. 4. Soil water diffusivity of the model, (2), compared with measurements for (*a*) Yolo light clay [*Philip*, 1969] and (*b*) Brindabella silty clay loam [*Perroux et al.*, 1982].

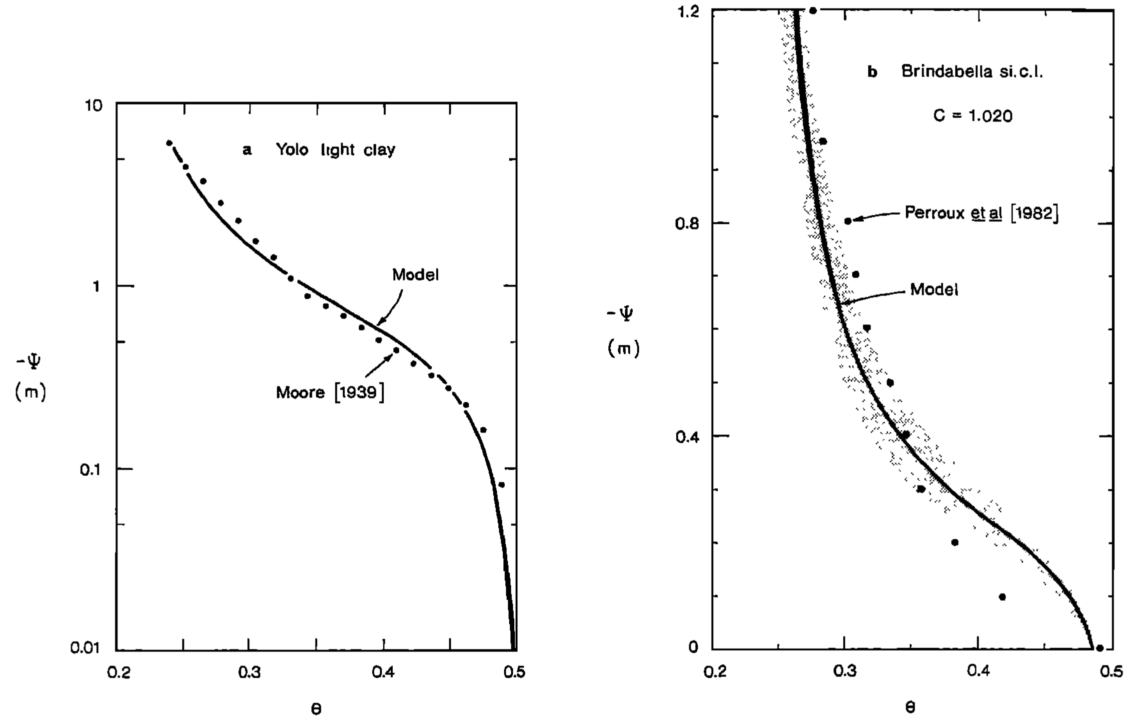



Fig. 5. Moisture characteristic of the model compared with measurements for (*a*) Yolo light clay [*Moore*, 1939] and (*b*) Brindabella silty clay loam [*Perroux et al.*, 1982].

## 5. Comparison of Model Predictions With Experiments for Constant Rate Rainfall Infiltration

### 5.1. *Evolution of Water Content Profiles*

To test our solutions for constant rate rainfall [*Broadbridge and White*, this issue, equations (41)–(44)] we compare predictions with the experimental observations of *Perroux et al.* [1981] for repacked Brindabella silty clay loam and the field data of *Clothier et al.* [1981*a*, *b*] on Bungendore fine sand. We select these studies because the properties necessary for predictions are available and because the values of their $C$ parameters (Table 1) lie at the extrema of the observed range for $C$.

We use the values of $C$ listed in Table 1 for the two soils. As is discussed in section 3.1., these are derived from independently measured $S(\theta_s, \theta_n)$ and $\Phi_p$. Our tests then involve no a posteriori fitting of the solutions to the rainfall data. The values needed for calculations are given in Tables 1 and 3 for the silty clay loam and in Figure 6*c* for the fine sand. It is reasonable to take $K_n$ as negligibly small for both cases.

Using these values in Figures 6*a* and 6*b* we contrast the predicted soil water profile development for Brindabella silty clay loam at two rainfall rates, 16.5 and 40.5 mm h$^{-1}$, with the measurements of *Perroux et al.* [1981]. In Figure 6*c* the comparison is shown for the in situ field soil with a rainfall rate of 31.6 mm h$^{-1}$.

Clearly, for the silty clay loam at the lowest rate the exact solutions overestimate the surface water content $\theta_0$. Furthermore, except for the profile at 1.15 hours, the predicted wetting front is not as steep as that measured. This discrepancy was expected for lower rainfall results since the model's hydraulic parameters showed deviations from the actual measurements at lower water contents (Figures 3*b* and 4*b*). Note that for constant flux, in any preponding situation the amount of water in the predicted and measured profiles should be identical at any time. This does not appear to be the case in Figure 6*a* for $t$ = 4.97 hours. Such mismatches are not unknown for field data where inadequate profile sampling and preferential flow present problems for quantitative recovery of water (see, for example, *Clothier et al.* [1981*b*]).

At the higher rainfall rate the model again overestimates $\theta_0$ but accurately predicts the position of the wetting front. In Figure 6*c* the agreement between predictions and field experiments is better than expected, particularly when we recall that the predictions rely on independently measured in situ values for $\Phi_p$, $S(\theta_s, \theta_n)$ and $K_s$. The simpler traveling wave approximation [*Broadbridge and White*, this issue, equation (48)] is shown in Figure 6 also. For the Brindabella soil this traveling wave approximation is close to or indistinguishable from the exact solution for the infiltration times considered. The rainfall times used for the in situ Bungendore find sand in Figure 5*c* are extremely short ($t_g \approx 0.57$ hours). Nonetheless, we see that for the profile at 0.28 hours in Figure 5*c*, the agreeement between the traveling wave approximation and observation is respectable.

The overall agreement evident in Figure 6 between measured water content profiles and the analytical solution is encouraging, particularly recalling that there has been no a posteriori adjustment of the model's parameters. These were determined by techniques independent of the rainfall measurements.

### 5.2. *Surface Soil Water Potential During Rainfall*

One of the features of the model is that the analytic form for $\Psi(\theta)$, (4), permits calculation of the change in surface water potential with time $\Psi_0(t)$ during rainfall [*Broadbridge and White*, this issue, section 5.4.1]. During the in situ field measurements of *Zegelin and White* [1982], the surface potential was recorded at two separate locations on a loamy sand pasture site for rainfall rates of 10 and 15 mm h$^{-1}$. Their observations and the predicted $\Psi_0(t)$ behavior are shown in Figure 7.

The predicted early time behavior of $\Psi_0(t)$ shown in Figure

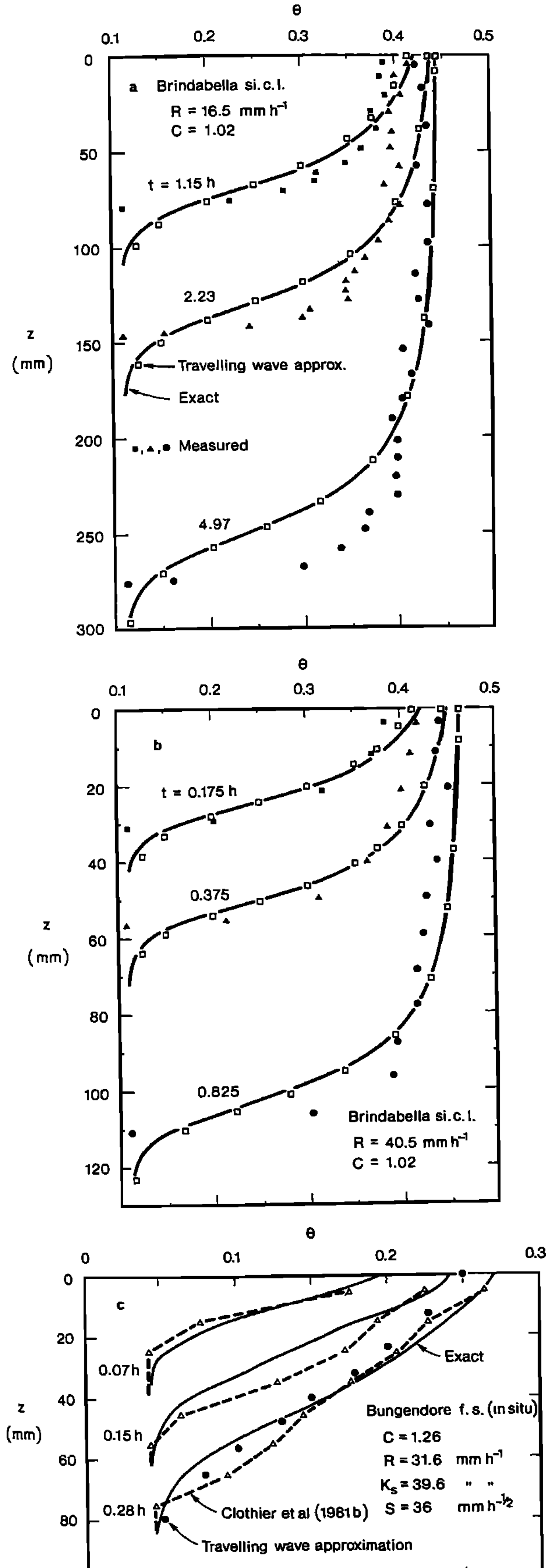



Fig. 6. Predicted evolution of soil water content profiles during constant rate rainfall infiltration compared with measurements for (*a*, *b*) Brindabella silty clay loam [*Perroux et al.*, 1981] and (*c*) in situ Bungendore fine sand [*Clothier et al.*, 1981*b*]. Also shown is the traveling wave approximation, (12).

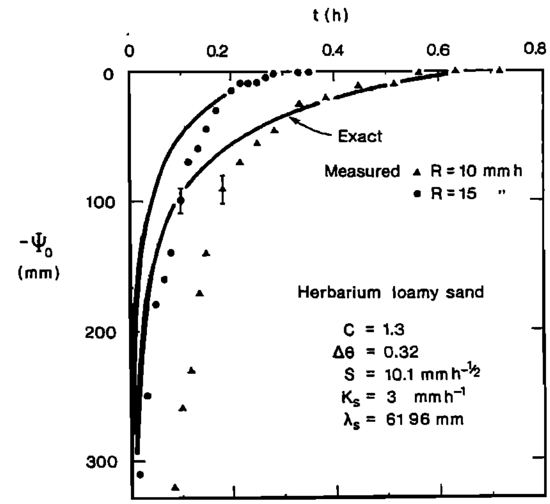



Fig. 7. Predicted time dependence of surface soil water potential during rainfall compared with in situ measurements at two sites [*Zegelin and White*, 1982].

7 gives values that are greater than the measurements. However, as the surface potential approaches zero, the predictions match the observations remarkably well. We believe the mismatch at early times reflects the intricacies of the soil's water diffusivity function at low water contents. Evidence suggests [*Clothier and White*, 1981, 1982] that the diffusivity function of light textured field soils may be segmented with the low water content region exhibiting the typical rapid change with $\theta$ while in the high water content region $D$ varies only slightly with $\theta$. A single parameter model such as ours cannot presume to mirror such details. The agreement, however, as $\Psi_0 \rightarrow 0$, for a priori independent predictions and for separate field sites is excellent.

### 5.3. *Time to Ponding for Constant Rate Rainfall*

By putting $\Theta_0 = 1$ in (45) of *Broadbridge and White* [this issue], it is possible to find an analytic form for predicting the time at which the soil surface first becomes saturated, the so called time-to-incipient ponding,

$$1 = C[1 - 1/[1 + 2\rho\{1 - \exp(-\rho\tau_p)\,\mathrm{erfc}\,(-\rho\tau_p{}^{1/2}) + (1 + \rho^{-1})^{1/2}\,\mathrm{erf}\,[\rho(\rho + 1)\tau_p]^{1/2}\}]] \qquad (17)$$

where $\rho = R_*/4C(C - 1)$ and $\tau_p$ is the dimensionless ponding time $4C(C - 1)t_p/t_s$.

Laboratory observations [*White et al.*, 1982] of time to ponding under constant rate infiltration on Molonglo loam (see Table 1) are compared in Figure 8 with the behavior expected from (17). Equation (17) predicts the time to ponding for these laboratory experiments exceedingly well.

We may use the data of *Zegelin and White* [1982] displayed in Figure 7 also to give time to ponding estimates for in situ field results. These are listed in Table 5 together with the ranges of predictions from (17) due to the experimental uncertainties in the in situ determination of sorptivity and $K_s$. Equation (17) overestimated the time to ponding at the smaller rainfall rate and underestimated it for the larger rate. Considering the errors in the in situ measured sorptivity and $K_s$, the agreement between prediction and field observation is satisfactory.

## 6. CONCLUDING REMARKS

In paper 1 [*Broadbridge and White*, this issue] we found analytic solutions for constant rate rainfall into uniform soil

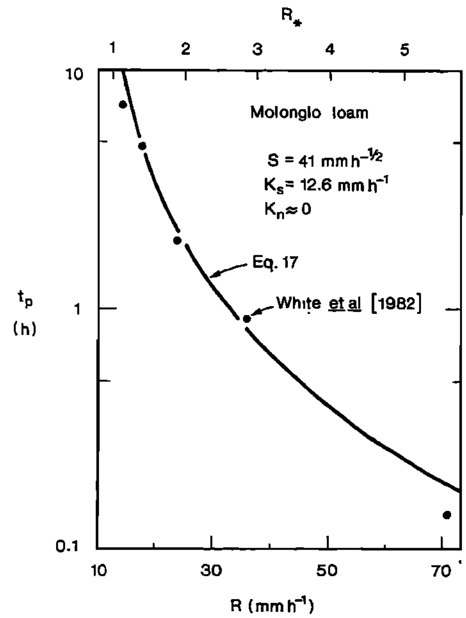



Fig. 8. Comparison of analytic expression (17) for time to ponding under constant rate infiltration with measurements for Molonglo loam [*White et al.*, 1982].

whose highly nonlinear hydraulic conductivity and soil water diffusivity were described through a single free parameter $C$. The hydraulic properties ranged from those of the weakly nonlinear Burgers' equation to a highly nonlinear Green-Ampt-like model.

In this paper, we have detailed three techniques for estimating the $C$ parameter; all may be used in the field. The choice of the particular method, and there are more than those discussed here, depends on the required application. For integral soil water properties it is probably sufficient to simply estimate $C$ as $C \approx 1.02$ for materials with narrow pore size distribution such as repacked samples, and $C = 2$ for materials with wide pore size distribution as is found in field situations.

We have shown that the model's functional forms for $D(\theta)$ and $K(\theta)$ adequately represent measured data. The comparison between the model and measured $\Psi(\theta)$ was a critical test since that of the model is calculated a priori without fitting. In this test the agreement was very satisfactory.

The analytic solutions for rainfall infiltration, without a posteriori tinkering, reasonably described the evolution of soil water content profiles and surface soil water pressure potential observed in field and laboratory experiments. In addition, the mathematically simple traveling wave approximation agreed with observations at longer infiltration times. Finally, the model satisfactorily predicted time to ponding for both laboratory and field experiments.

TABLE 5. Comparison of Observed and Predicted Times to Ponding for in Situ Herbarium Loamy Sand

| R, mm h$^{-1}$ | $t_p$, hour | |
|---|---|---|
| | Measured* | Equation (17) |
| 10 | 0.57 | $0.70 \pm 0.34$ |
| 15 | 0.32 | $0.29 \pm 0.14$ |

$S = 10.1$ mm h$^{-1/2}$, $K_s = 3$ mm h$^{-1/2}$, and $K_n \sim 0$.
*From *Zegelin and White* [1982].

These results demonstrate that the nonlinear model is indeed versatile, covering a wide range of the known behavior of soils during constant rate rainfall. In addition, we have attempted to show that the model's hydraulic properties are realistic for a range of soils. These model properties, we believe, will be useful in a wide range of studies. Currently, we are examining their use in testing rational approximations such as time to ponding: in verifying inverse techniques for measuring soil hydraulic properties, in describing soil drainage, and in developing an analytical approach to flow in heterogeneous porous materials.

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P. Broadbridge, Department of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia.

I. White, Division of Environmental Mechanics, CSIRO, GPO Box 821, Canberra, ACT 2601, Australia.

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