Constant rate rainfall infiltration: A versatile nonlinear model, I. Analytic solution

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Constant Rate Rainfall Infiltration: A Versatile Nonlinear Model
1. Analytic Solution

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Analytic solutions are presented for a nonlinear diffusion-convection model describing constant rate rainfall infiltration in uniform soils and other porous materials. The model is based on the Darcy-Buckingham approach to unsaturated water flow and assumes simple functional forms for the soil water diffusivity $D(\theta)$ and hydraulic conductivity $K(\theta)$ which depend on a single free parameter C and readily measured soil hydraulic properties. These $D(\theta)$ and $K(\theta)$ yield physically reasonable analytic moisture characteristics. The relation between this model and other models which give analytic solutions is explored. As $C \to \infty$, the model reduces to the weakly nonlinear Burgers' equation, which has been applied in certain field situations. At the other end of the range as $C \to 1$, the model approaches a Green-Ampt-like model. A wide range of realistic soil hydraulic properties is encompassed by varying the C parameter. The general features of the analytic solutions are illustrated for selected C values. Gradual and steep wetting profiles develop during rainfall, aspects seen in the laboratory and field. In addition, the time-dependent surface water content and surface water pressure potential are presented explicitly. A simple traveling wave approximation is given which agrees closely with the exact solution at comparatively early infiltration times.

1. INTRODUCTION

Over 20 years have passed since *Rubin and Steinhardt* [1963] used numerical techniques to illuminate the general features of rainfall infiltration. The search for analytical or quasi-analytical solutions has, however, continued unabated. This sustained quest is no mere dilettantism but reflects continuing needs for such solutions. These are principally fourfold; they provide general physical insight into the infiltration process. Additionally, they form the bases for rational approximations and simplifications in terms of readily measured soil water properties. They also furnish bench marks for the validation and improvement of numerical schemes. Finally, they are useful for testing various inverse techniques used in the estimation of soil hydraulic properties. We introduce here a nonlinear model of constant rate rainfall which shows considerable versatility in meeting these and related requirements.

Adopting the Darcy-Buckingham approach, we seek solutions to the highly nonlinear, one-dimensional Fokker-Planck diffusion-convection equation subject to flux boundary conditions, which is taken to describe rainfall infiltration in uniform soils [*Philip*, 1969]. *Braester* [1973], in an attempt to solve this problem, linearized the flow equation. The linear model has several severe limitations [*Philip*, 1957*a*; *Parlange*, 1976], the most notable being that the linear convection term does not permit the development of a traveling wave solution at large infiltration times. This problem does not arise in the exactly solvable Burgers' equation with its weakly nonlinear convection term [*Philip*, 1973, 1974]. The correct form of Knight's solution of Burgers' equation for the constant flux boundary condition is given by *Clothier et al.* [1981*a*]. Burgers' equation, however, treats diffusivity as constant, and while the solution satisfactorily described rainfall infiltration in an undisturbed field soil, many materials exhibit soil water diffusivities that vary over several orders of magnitude across the water content range of interest.

An alternative approach arose through the development of quasi-analytical or integral solutions. *Parlange* [1971, 1972] introduced an approximate, integral method for solving the general one-dimensional, nonlinear flow equation for the constant flux boundary condition. This quasi-analytical technique has been extended also to time-dependent rainfall rates [*Smith and Parlange*, 1978; *Morel-Seytoux*, 1978, 1982]. The extension, however, is not for general, time-dependent rainfall rates. Rather it is restricted to rates for which a one-to-one relationship between soil water flux and water content persists; that is, nonhysteretic flows.

In principle, these quasi-analytical solutions can be refined to any desired accuracy using the iterative procedure of *Philip and Knight* [1974]. Such refinements, however, have not been carried out for rainfall infiltration. This is because physically plausible initial estimates of the flux-concentration relation reproduce, within experimental error, the observed evolution of water content profiles in a wide range of materials in both the laboratory and the field [*Smiles*, 1978; *White et al.*, 1979; *White*, 1979; *Perroux et al.*, 1981; *Clothier et al.*, 1981*b*]. Despite their practical utility, estimates are approximate and essentially untested against highly nonlinear analytic models. The present exact solution offers means of "calibrating" the Philip and Knight procedure.

In paper 1 of this work we describe a highly nonlinear model of rainfall infiltration that has an exact analytic solution. We show that the model covers a wide range of soil water hydraulic properties and that the solutions describe all the principal features of one-dimensional, unsaturated flow of soil water during constant rate rainfall. Paper 2 [*White and Broadbridge*, this issue] details simple techniques for fitting the model to soils, particularly those in the field, and compares the analytic solution with laboratory and field observations.

2. NONLINEAR MODEL

The nonlinear Fokker-Planck diffusion-convection equation commonly used to describe one-dimensional, nonhysteretic infiltration in uniform nonswelling soil is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] - K'(\theta)\frac{\partial \theta}{\partial z} \tag{1}$$

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where t is time; z is depth (positive downwards); $\theta(z, t)$ is volumetric soil water content; $D(\theta)$ is the soil water diffusivity; $K(\theta)$ is the hydraulic conductivity; and $K'(\theta) = dK/d\theta$. We impose a constant flux R at the soil surface, giving the boundary condition

$$K(\theta) - D(\theta)\partial\theta/\partial z = v(z) = R \qquad z = 0 \tag{2}$$

and assume a uniform initial soil water content

$$\theta(z, 0) = \theta_n \qquad z \geq 0 \tag{3}$$

Knight and Philip [1974] found that when the soil water diffusivity is given by

$$D(\theta) = a(b - \theta)^{-2} \tag{4}$$

where a and b are constants, the nonlinear diffusion equation can be transformed to the linear form that possesses a well-known analytical solution. *Knight and Philip* [1974] argued that the functional form (4) is plausible for soil water diffusivities. We adopt (4) here as a reasonable representation of $D(\theta)$ in (1) and seek forms for $K(\theta)$ which, through the same transformation, reduce (1) to an analytically solvable form for the constant flux boundary condition (2). Two such forms are already known; the linear diffusion-convection equation

$$\partial\theta/\partial t = D_*\partial^2\theta/\partial z^2 - k\partial\theta/\partial z$$

with D_* and k constant, and the weakly nonlinear Burgers' equation

$$\partial\theta/\partial t = D_*\partial^2\theta/\partial z^2 - k\theta\partial\theta/\partial z$$

With $D(\theta)$ given by (4), (1) can be transformed to Burgers' equation whenever K is of the form

$$K(\theta) = \beta + \gamma(b - \theta) + \lambda/[2(b - \theta)] \tag{5}$$

with β, γ, and λ constants. This form for $K(\theta)$ differs from previous proposals but, as we show, it is both adequate and physically reasonable.

Fokas and Yortsos [1982] have used (4) and (5), with $\beta = \gamma = 0$, to model the displacement of one fluid by another in a porous medium in the absence of gravity. Subsequently, *Rogers et al.* [1983] incorporated gravity into the two-phase immiscible displacement problem by taking suitable values of β and γ in (5). The explicitly solvable system (1)–(5) has been noticed independently by *Rosen* [1982] also, who applied it to the transport of a chemical substance subjected to adsorption on the microsurfaces of a porous material. It must be recognized that the transformation of (1) into Burgers' equation is no guarantee of solution. The boundary conditions must also transform to conditions for which Burgers' equation can be solved. Only constant flux boundary conditions lead to solutions under these transformations.

The relationship between the physical flow problem studied by *Rogers et al.* [1983] and that considered here is not readily deduced (C. Rogers, private communication, 1987). Here we concentrate on single-phase unsaturated flow of water in soil resulting from the application of a constant flux boundary condition (2). We specifically derive and illustrate solutions in forms directly applicable to soil water flow.

Our nonlinear model for unsaturated flow in soil is therefore (1) with D given by (4), and with (5) describing the dependence of hydraulic conductivity on water content. We now examine how the five parameters in those equations, enough to fit a running elephant, may be written in terms of readily measured hydraulic properties with direct application to field situations.

3. Application to Soil Hydraulic Properties

The functional forms (4) and (5) have five parameters between them. We show here that by the use of readily measured soil hydraulic properties, these parameters can be fixed so that (4) and (5) adequately represent a wide range of $D(\theta)$ and $K(\theta)$.

3.1. *Hydraulic Conductivity*

Our first requirement is that the hydraulic conductivity at θ_s, the water content at which the soil water pressure potential is zero, be the measured saturated conductivity K_s

$$K(\theta_s) = K_s \tag{6}$$

In most practical applications, $K'(\theta_n)$ is negligible compared to $K'(\theta_s)$, so we may reasonably take

$$K'(\theta_n) = 0 \tag{7}$$

Using (7) in (5) immediately gives $\gamma = (\lambda/2)(b - \theta_n)^{-2}$. The specification of parameters is completed by our third requirement:

$$K(\theta_n) = K_n \tag{8}$$

where K_n is the measured conductivity at $\theta = \theta_n$.

It is convenient to introduce both the parameter C, with

$$C = (b - \theta_n)/(\theta_s - \theta_n)$$
$$= (b - \theta_n)/\Delta\theta \tag{9}$$

with $\Delta\theta = \theta_s - \theta_n$ and the relative water content

$$\Theta = (\theta - \theta_n)/\Delta\theta \tag{10}$$

Using (9) and (10), together with (6) to (8) we find from (5)

$$(K - K_n)/\Delta K = \Theta^2(C - 1)/(C - \Theta) \tag{11}$$

with $\Delta K = K_s - K_n$.

The three parameters in (5) are

$$\lambda = 2C^2(C - 1)\Delta\theta\Delta K \tag{12}$$

$$\beta = K_s - [1 + 2C(C - 1)]\Delta K \tag{13}$$

$$\gamma = (C - 1)\Delta K/(\Delta\theta) \tag{14}$$

We have reduced K to a form involving a single free parameter C and two readily measured conductivities, K_s and K_n.

3.2. *Soil Water Diffusivity*

The remaining parameters a and b both appear in the expression (4) for diffusivity, which we now write as

$$D = a\Delta\theta^{-2}(C - \Theta)^{-2} \tag{15}$$

One more parameter may be fixed by matching the sorptivity $S(\theta_s, \theta_n)$ [*Philip*, 1957*b*] calculated from this diffusivity with the measured value for initial and surface water content, θ_n and θ_s. Sorptivity is one of the easiest soil water hydraulic properties to measure in the field under either ponded [*Talsma*, 1969] or unsaturated conditions [*Clothier and White*, 1981]; for convenience we write $S = S(\theta_s, \theta_n)$.

Sorptivity can be related to the diffusivity function $D(\theta)$ by [*Philip and Knight*, 1974]

$$S^2 = 2(\Delta\theta)^2 \int_0^1 \frac{\Theta D(\Theta)}{F(\Theta)}\, d\Theta \tag{16}$$

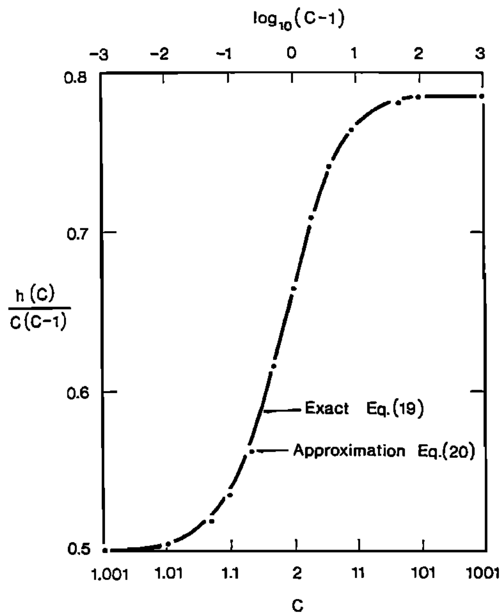

Fig. 1. Functional dependence of diffusivity parameter h on C given by (19) and approximation (20).

where F is the flux concentration relation of *Philip* [1973], the ratio of the water flux at Θ to the flux at the soil surface where $\Theta = 1$. It follows from (15) and (16) that

$$a = h(C)[S(\theta_s, \theta_n)]^2 \tag{17}$$

with

$$h(C) = \left[2 \int_0^1 \frac{\Theta(C-\Theta)^{-2}\, d\Theta}{F(C, \Theta)}\right]^{-1} \tag{18}$$

which is clearly a function of C alone. In the general case F can be evaluated by the recursive estimation scheme of *Philip and Knight* [1974]. For the $D(\Theta)$ in (15) $h(C)$ can be calculated from the exact solution of *Fujita* [1952]:

$$1/C = \{\pi/[4h(C)]\}^{1/2} \exp\{[4h(C)]^{-1}\} \operatorname{erfc}[[4h(C)]^{-1/2}] \tag{19}$$

We find to within 1%, (19) can be approximated by

$$h(C) = C(C-1)[\pi(C-1) + B]/[4(C-1) + 2B] \tag{20}$$

with $B = 1.46147$. Figure 1 shows a plot of $h(C)/C(C-1)$ together with the approximation (20). As C increases from 1 to ∞, $h(C)/C(C-1)$ increases continuously from $\frac{1}{2}$ to $\frac{\pi}{4}$. We have thus reduced D to an expression involving the single parameter C and the sorptivity. The latter is an easily measured soil water property. For simplicity, we write $h = h(C)$ so that

$$D = h\{S/[\Delta\theta(C-\Theta)]\}^2 \tag{21}$$

3.3. *Moisture Characteristic*

Up to this point the hydraulic properties have been expressed as functions of water content. Certain applications require that the soil water potential Ψ be known. In addition, an appropriately weighted mean Ψ provides a characteristic macroscopic length necessary for scaling infiltration [*Raats and Gardner*, 1971; *Philip*, 1985]. We therefore require a relationship between θ and Ψ. The moisture characteristic $\Psi(\theta)$ for the forms (21) and (11) follows from the definition of D: $D = K d\Psi/d\theta$. For $K_n/K_s < 4C(C-1)$, we obtain

$$\Psi(\Theta) = -C^{-1}\lambda_s \left\{\ln\left(\frac{C-\Theta}{C-1}\right) + \frac{1}{2}\ln\left(\frac{K_s(C-1)}{K(\Theta)(C-\Theta)}\right) + \frac{K_n - 2C(C-1)\Delta K}{\kappa}\left[\arctan\left(\frac{K_s-(2C-1)\Delta K}{\kappa}\right) - \arctan\left(\frac{K_s - [2\Theta(C-1)+1]\Delta K}{\kappa}\right)\right]\right\} \tag{22}$$

For $K_n/K_s \geq 4C(C-1)$

$$\Psi(\Theta) = -\lambda_s\left(\frac{1}{C}\ln\frac{C-\Theta}{C-1} + \frac{C(C-1)}{r_1 - C}\frac{\Delta K}{K_n}\ln\left|\frac{r_1-\Theta}{r_1-1}\right| - \frac{C(C-1)}{r_2-C}\frac{\Delta K}{K_n}\ln\left|\frac{r_2-\Theta}{r_2-1}\right|\right) \tag{23}$$

In (22) and (23),

$$\kappa = \{[(2C-1)^2\Delta K - K_s]K_n\}^{1/2}$$

$$r_1 = \{K_n/[2\Delta K(C-1)]\}[1 + (1 - 4C(C-1)\Delta K/K_n]^{1/2}$$

$$r_2 = \{K_n/[2\Delta K(C-1)]\}[1 - (1 - 4C(C-1)\Delta K/K_n]^{1/2}$$

and λ_s is a macroscopic capillary length [*Philip*, 1985] given by

$$\lambda_s = [h/C(C-1)]S^2/(\Delta\theta\Delta K) \tag{24}$$

Clearly, λ_s is a natural scaling factor for $\Psi(\Theta)$. We will also use it in the following as a length scale. Its physical significance will become apparent in later sections. When K_n is negligibly small, (22) reduces to

$$\Psi_*(\Theta) = \Psi(\Theta)/\lambda_s = -(1-\Theta)/\Theta - C^{-1}\ln\{(C-\Theta)/[(C-1)\Theta]\} \tag{25}$$

In (22), (23), and (25) as $\Theta \to 1$, $\Psi \to 0$ so that potential vanishes at saturation, as it must. The dependence of Ψ_* on Θ given by (25) is shown in Figure 2 for selected values of C.

It is encouraging to observe in Figure 2 that as $C \to 1$, the moisture characteristic effectively shows a pronounced "air entry" or "tension-saturated" zone as $\Theta \to 1$, which is typical

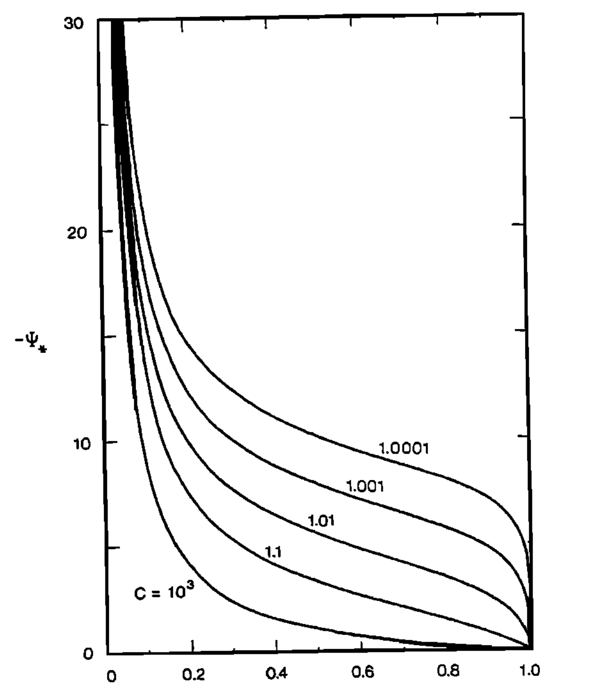

Fig. 2. Dimensionless moisture potential Ψ_* as a function of relative water content Θ and the parameter C given by (25).

of laboratory-packed soils or uniform porous materials. As $C \to \infty$ in Figure 2, the "knee" in the normalized moisture characteristic disappears and produces characteristics similar to those found in intact field soils [see, for example, *Perroux et al.*, 1982]. We are thus able to express the hydraulic properties of the model analytically in terms of θ or Ψ and have produced $\Psi(\theta)$ which simulate reality.

4. Relationship to Other Models

Equations (11), (21), and (22) represent in our model the basic hydraulic properties and are expressed in terms of the measurable soil water properties Θ, K_s, K_n, and S and the single parameter C. We now explore the relationship between our model properties and those of other models that give analytic solutions of the unsaturated flow equation.

4.1. *Burgers' Equation*

For C very large, (11) and (21) may be written as

$$D = [h/C^2][S/\Delta\theta]^2[1 + O(1/C)]$$

$$K - K_n = \Delta K\Theta^2 + O(1/C)$$

We know that as $C \to \infty$, $h/C^2 \to \pi/4$ (Figure 1), and so the D and K functions in this limit reduce to those of Burgers' equation with D constant $(=(\pi/4)(S/\Delta\theta)^2)$ and $K \propto \Theta^2$ (see section 2). In this limit the moisture characteristic (22) becomes

$$\Psi = -\frac{\pi}{4}\frac{S^2}{\Delta\theta\Delta K}\left(\frac{\Delta K}{K_n}\right)^{1/2}\left\{\arctan\left(\frac{\Delta K}{K_n}\right)^{1/2} - \arctan\left(\left[\frac{\Delta K}{K_n}\right]^{1/2}\Theta\right)\right\}$$

or for $K_n = 0$,

$$\Psi = -[\pi S^2/4\Delta\theta K_s][(1-\Theta)/\Theta]$$

4.2. *Green-Ampt or Delta Function Model*

The lowest value for C such that K is always positive is $C = 1$. As $C \to 1$, $h/C(C-1) \to \frac{1}{2}$(Figure 1). If we write $\varepsilon = C - 1 \ll 1$ in (21), then to first order in ε,

$$D(\Theta) = D(\Theta) = (S/\Delta\theta)^2\varepsilon/[2(1 + \varepsilon - \Theta)^2]$$
$$\to (S/\Delta\theta)^2\delta(\Theta - 1) \to (S^2/\Delta\theta)\delta(\theta - \theta_s)$$

where δ is the Dirac delta function. Moreover, we find that as $C \to 1$,

$$K(\Theta) \to K_n \qquad \Theta < 1$$

$$K(\Theta) \to K_s \qquad \Theta = 1$$

The similarity between this $C \to 1$ limit of the model and that of *Green and Ampt* [1911] is clear. At this limit the length scale λ_s defined in (24) is identical to the "wetting front potential" of the Green-Ampt model [*Philip*, 1958].

Therefore the extremes of the range of C in our model correspond, on one hand, to the weakly nonlinear Burgers' equation and on the other to a highly nonlinear Green-Ampt-like model.

4.3. *Quasi-Linear Model*

The quasi-linear model has been widely used to find steady solutions to the flow equation in two and three dimensions [see, for example, *Philip*, 1968; *Wooding*, 1968; *Batu*, 1978; *Philip*, 1984] and for time-dependent, multidimensional but linear problems [e.g., *Warrick*, 1974; *Philip*, 1986]. For this model the relationship between hydraulic conductivity and potential is taken to be

$$K(\Psi) = K_0 \exp[\alpha(\Psi - \Psi_0)] \tag{26}$$

with α constant. Our model has the advantage of giving analytical solutions for the transient case. It appears, however, to be limited to one dimension because of the improbability of extending the Hopf-Cole transformation to two dimensions [*Broadbridge*, 1986].

We now seek to relate α in (26) to the parameters of our model. For many soils (26) does not describe completely measured $K(\Psi)$ over the entire potential range of interest. However, *Philip* [1985] showed that any soil $K(\Psi)$ could be fitted optimally to the exponential form (26) by defining α as the reciprocal K-weighted mean potential

$$\alpha^{-1} = \int_{\Psi_n}^{\Psi_0} K(\Psi)\, d\Psi/\Delta K$$
$$= \int_{\theta_n}^{\theta_0} D(\theta)\, d\theta/\Delta K \tag{27}$$

with $\Psi_0 = \Psi(\theta_0)$ and $\theta_n < \theta_0 \le \theta_s$. Using (16) and following *White and Sully* [1987], (27) may be rewritten

$$\alpha^{-1} = \left[2\int_0^1 \frac{\Theta D\, d\Theta}{F}\right]^{-1}\int_0^1 D\, d\Theta\, S^2/(\Delta\theta\Delta K) \tag{28}$$

Substitution of the D of our model, (21), in (28) and recalling (24) leads to the identity

$$\alpha^{-1} = \lambda_s \tag{29}$$

So we find the inverse of the macroscopic capillary length of our model is simply the appropriate α of the quasi-linear model.

4.4. *Other Models*

Soil physics literature is replete with models of varying complexity and usefulness, each with their own set of adherents. A detailed comparison between our model and extant models is not possible here. Rather we comment on some general issues.

In physically based models, the soil moisture potential is scaled with some reference potential such as the displacement pressure [*Brooks and Corey*, 1964] or the critical pressure [*Bouwer*, 1964; *Raats and Gardner*, 1971] or other unnamed parameters [e.g., *van Genuchten*, 1980]. Elsewhere it is shown [*White and Sully*, 1987] that not only are these reference potentials related to λ_s, the macroscopic capillary length used here, but also λ_s is a direct measure of the soil's flow-weighted mean characteristic pore size.

We also draw attention to the fact that by scaling soil water diffusivity with sorptivity [*Brutsaert*, 1979; *Clothier and White*, 1981] as in section 3.2 above, the number of free parameters required for $D(\theta)$ may be reduced by one. In addition, the use of sorptivity provides at least partial guarantee of matching predicted and observed integral soil properties. Finally, we note that the principal advantage of our model over others, excluding those in 4.1 to 4.3 above, is that it possesses analytic solutions for constant rate rainfall.

5. Analytic Solution for Constant Rate Rainfall

We now proceed to the solution of the flow equation using dimensionless variables.

5.1. *Dimensionless Variables*

The macroscopic capillary length specified in (24) is used to define dimensionless distance z_* as

$$z_* = z/\lambda_s \tag{30}$$

The associated capillary time scale t_s is

$$t_s = hS^2/[C(C-1)\Delta K^2] \tag{31}$$

giving dimensionless time t_* as

$$t_* = t/t_s \tag{32}$$

The similarity between t_s here and the time scale t_{grav} of *Philip* [1969] is evident.

With dimensionless distance and time defined by (30) and (32), the dimensionless soil water properties of our model are

Hydraulic conductivity

$$\begin{aligned} K_* &= (K - K_n)/\Delta K \\ &= (C-1)\Theta^2/(C-\Theta) \end{aligned} \tag{33}$$

Soil water diffusivity

$$\begin{aligned} D_* &= D/D_r \\ &= C(C-1)/(C-\Theta)^2 \end{aligned} \tag{34}$$

with

$$\begin{aligned} D_r &= h[S/\Delta\theta]^2/[C(C-1)] \\ &= \lambda_s{}^2 t_s{}^{-1} \end{aligned} \tag{35}$$

Rainfall rate

$$R_* = (R - K_n)/\Delta K \tag{36}$$

Soil water potential

$$\Psi_* = \Psi/\lambda_s \tag{37}$$

which is expressible as a function of Θ through (22). Prior to ponding the dimensionless cumulative infiltration is

$$R_* t_* = (R - K_n)/(\Delta\theta\lambda_s)$$

5.2. *Analytic Solution*

With the variables (30) and (32) and the properties (33) and (34), we seek to solve the dimensionless flow equation

$$\frac{\partial\Theta}{\partial t_*} = \frac{\partial}{\partial z_*}(D_*\partial\theta/\partial z_*) - \partial K_*/\partial z_* \tag{38}$$

subject to the initial condition

$$\Theta = 0 \qquad t_* = 0 \qquad z_* \geq 0 \tag{39}$$

and the boundary condition for constant rate rainfall

$$v_*(z_*) = K_* - D_*\partial\Theta/\partial z_* = R_* \qquad t_* > 0 \qquad z_* = 0 \tag{40}$$

Here v_* is the dimensionless soil water flux $v_* = (v - K_n)/\Delta K$. Details of the solution techniques are given in the appendix, where we show that the sequential application of the *Kirchhoff* [1894], *Storm* [1951], and *Hopf-Cole* [1950] transformations reduce (30) to a linear diffusion equation. Fortuitously, the initial and boundary conditions (39), (40) are transformed also to conditions which can be handled by Laplace transforms. From the appendix the solution is

$$\Theta = C[1 - (2\rho + 1 - u^{-1}\partial u/\partial\zeta)^{-1}] \tag{41}$$

$$z_* = C^{-1}[\rho^2(1+\rho^{-1})\tau + \rho(2+\rho^{-1})\zeta - \ln u] \tag{42}$$

with

$$\begin{aligned} u = \frac{1}{2}\exp[-\zeta^2\tau^{-1}]\{&2\exp[(\zeta+\rho\tau)/\tau^{1/2}]^2 \\ &+ f([\zeta - \rho(1+\rho^{-1})^{1/2}\tau]/\tau^{1/2}) - f([\zeta - \rho\tau]/\tau^{1/2}) \\ &+ f([\zeta + \rho(1+\rho^{-1})^{1/2}\tau]/\tau^{1/2}) - f([\zeta + \rho\tau]/\tau^{1/2})\} \end{aligned} \tag{43}$$

$$\begin{aligned} \partial u/\partial\zeta = \rho\exp[-\zeta^2\tau^{-1}][&2\exp([\zeta+\rho\tau]/\tau^{1/2})^2 \\ &- (1+\rho^{-1})^{1/2}\{f([\zeta - \rho(1+\rho^{-1})^{1/2}\tau]/\tau^{1/2}) \\ &- f([\zeta + \rho(1+\rho^{-1})^{1/2}\tau]/\tau^{1/2})\} \\ &+ f([\zeta - \rho\tau]/\tau^{1/2}) - f([\zeta + \rho\tau]/\tau^{1/2})] \end{aligned} \tag{44}$$

where $f(x) = \exp(x^2)\operatorname{erfc}(x)$

$$\begin{aligned} \rho &= R_*/m \\ m &= 4C(C-1) \\ \tau &= mt_* \\ \zeta &= m^{1/2}Z/2 \\ Z &= 2m^{-1/2}\int_0^{z_*}(C-\Theta)\,dz_* \end{aligned}$$

Note in these solutions $\rho\tau = R_* t_*$, the dimensionless cumulative infiltration.

Equations (41)–(44) constitute an exact parametric solution $\{\Theta(\zeta, \tau), z_*(\zeta, \tau)\}$, with parameter ζ. Equations (43) and (44) are used in (41) and (42) to give the distribution of relative water content $\Theta(z_*, t_*)$ at any time t_* and any rainfall rate R_*.

In this solution use of the dimensionless water content Θ/C and dimensionless depth Cz_* together with ρ and τ gives solutions independent of C. Our dimensionless variables and parameters in this scheme are Θ/C, Cz_*, τ, ρ, K_*/m, C^2D_*/m and $C\Psi_*$.

5.3. *Main Features of Solutions*

To illustrate our solutions (41)–(44), we select two values of C, 1.02 and 1.5, which span the range of C found for much experimental data [*White and Broadbridge*, this issue]. We choose three dimensionless rainfall rates, $R_* = 0.2$ and 0.5 (both nonponding, i.e., Ψ at surface <0 for all time) and $R_* = 1.2$ (ponding, i.e., Ψ at surface >0 at some time during rainfall). For these three rates the development in time t_* of the soil water content profile for $C = 1.02$ and 1.5 is shown in Figures 3*a*–3*c*. For nonponding rainfall rates the surface water content approaches a limiting value at large times, as is required (Figures 3*a* and 3*b*), while for the ponding rate (Figure 3*c*), saturation at the surface, $\Theta = 1$, is reached at a finite time. The profiles for $R_* = 0.2$ (Figure 3*a*) are clearly not as steep as those for $R_* = 0.5$ and 1.2. This is reasonable, since the diffusivity varies only slowly with water content at small Θ. For $C = 1.5$ in Figures 3*a*, 3*b*, and 3*c* the water content profiles are more gradual and the approach of the water content at the soil surface to a limiting value for the nonponding rates is less dramatic.

The profiles of Figures 3*a*–3*c* for $C = 1.02$ quite clearly exhibit the same general features as those measured in the laboratory under constant flux conditions [*Haverkamp et al.*, 1977; *Perroux et al.*, 1981]. For $C = 1.5$ they mirror field observation of constant rate rainfall infiltration [*Clothier et al.*, 1981*a*, *b*]. Note for the weaker Θ dependent case, $C = 1.5$, "wetting fronts" extend deeper into the soil than for $C = 1.02$ with the application of the same quantity ($= R_* t_*$) of rainfall.

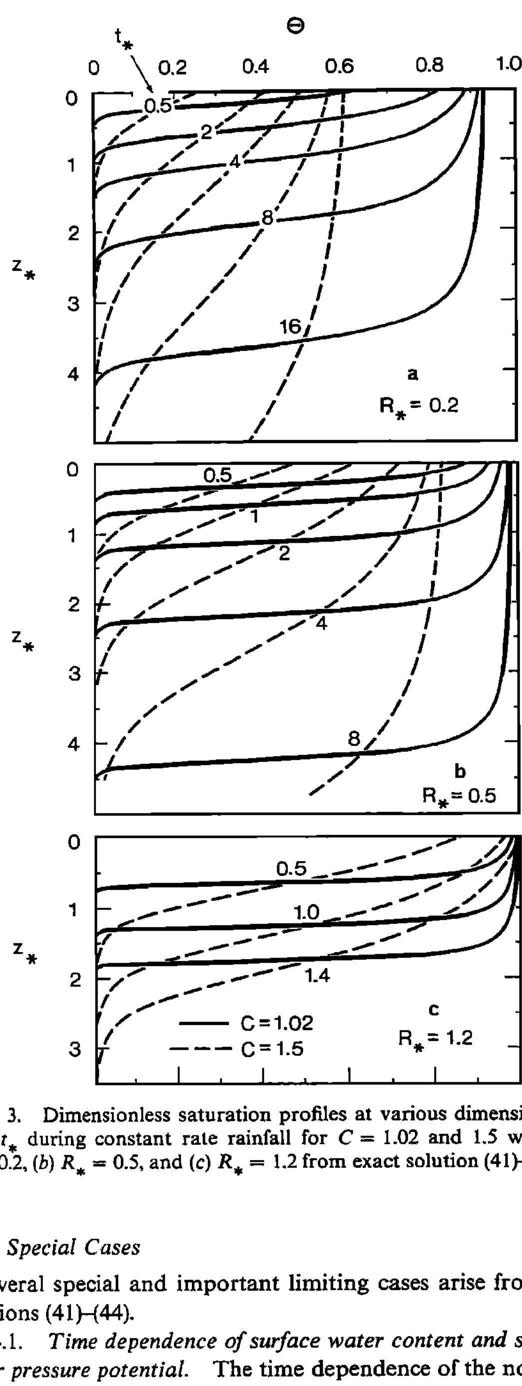

Fig. 3. Dimensionless saturation profiles at various dimensionless times t_* during constant rate rainfall for $C = 1.02$ and 1.5 with (*a*) $R_* = 0.2$, (*b*) $R_* = 0.5$, and (*c*) $R_* = 1.2$ from exact solution (41)–(44).

5.4. *Special Cases*

Several special and important limiting cases arise from the solutions (41)–(44).

5.4.1. *Time dependence of surface water content and surface water pressure potential.* The time dependence of the normalized surface water content $\Theta_0(\tau)$ during constant rate rainfall infiltration follows by putting $z_* = \zeta = 0$ in (41)–(44). We find that $\Theta_0(\tau)$ is given by

$$\Theta_0 = C\{1 - 1/[1 + 2\rho\{1 - \exp(-\rho\tau)\,\mathrm{erfc}\,(-\rho\tau^{1/2}) + (1 + \rho^{-1})^{1/2}\,\mathrm{erf}\,[\rho(\rho + 1)\tau)]^{1/2}\}]\} \qquad (45)$$

In Figures 4*a* and 4*b* we show the time dependence of the surface water content calculated from (45) for the cases shown in Figures 3*a*–3*c*. For $C = 1.02$ (Figure 4*a*), the initial rate of increase of Θ_0 is higher than that for $C = 1.5$ (Figure 4*b*), which is consistent with the stronger Θ dependence of the hydraulic properties for $C = 1.02$. Despite these differences, for the ponding rainfall rate, $R_* = 1.2$, the time at which incipient ponding occurs, that is, $\Theta = 1$, is similar for both values of C (Figures 4*a* and 4*b*). We further discuss time to ponding elsewhere [*Broadbridge and White*, 1987].

For nonponding rainfall rates, $R_* < 1$, (Figures 4*a* and 4*b*) the surface water content asymptotically approaches a limiting or equilibrium value Θ_e at large times. We find Θ_e for $R_* < 1$ by taking the limit as $t_* \to \infty$ in (45):

$$\Theta_e = 2C\rho[(1 + \rho^{-1})^{1/2} - 1] \qquad (46)$$

The appropriate values of Θ_e are indicated in Figures 4*a* and 4*b*, where it may be observed that in all cases Θ_0 is close to Θ_e at $t_* = 10$. Taking the limit of (45) for the two conditions $\rho \gg 1$ and $\rho^2\tau \ll 1$ produces the Burgers' equation solution of *Clothier et al.* [1981*a*] for θ_0; in this limit, $\Theta_e = R_*^{1/2}$.

Soil water potential close to the soil surface is relatively easy to measure during preponding infiltration in the field [*Clothier et al.*, 1981*b*; *Zegelin and White*, 1982]. This potential is of vital use in determining the time at which incipient ponding occurs [*Hamilton et al.*, 1983]. With the expression for $\Theta_0(\tau)$ and the analytic form for the moisture characteristic $\Psi_*(\Theta)$, we determine, and show in Figures 5*a* and 5*b*, the time dependence of the dimensionless surface water pressure potential, Ψ_{*0}. For simplicity it has been assumed $K_n = 0$ and the less complicated expression (25) has been used for Ψ_*. Despite the considerable difference in time dependence of surface water content for the two cases $C = 1.02$ and $C = 1.5$ (Figure 4), the differences in surface potential for the two cases are more subtle (Figure 5). Initially, $d\Psi_*/dt_*$ is greater for $C = 1.5$ than for $C = 1.02$, but eventually a crossover point is reached, beyond which values of Ψ_{*0} for $C = 1.02$ are less than those for $C = 1.5$. The calculated equilibrium values Ψ_{*e}, corresponding to (46), also are shown in Figure 5.

5.4.2. *Short time solution, $\rho\tau \ll 1$.* We obtain the gravity-free or short-time solution for constant rate rainfall infiltration from the solution by considering $\rho\tau \ll 1$, the condition for

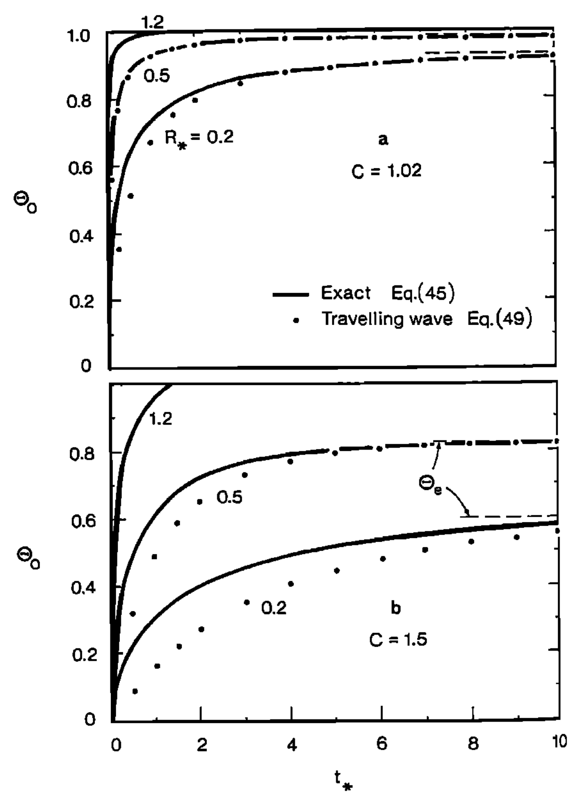

Fig. 4. Change of surface relative water content Θ_0 during constant rate rainfall for (*a*) $C = 1.02$ and (*b*) $C = 1.5$ and rainfall rates of Figure 3. Points are for the traveling wave approximation (49).

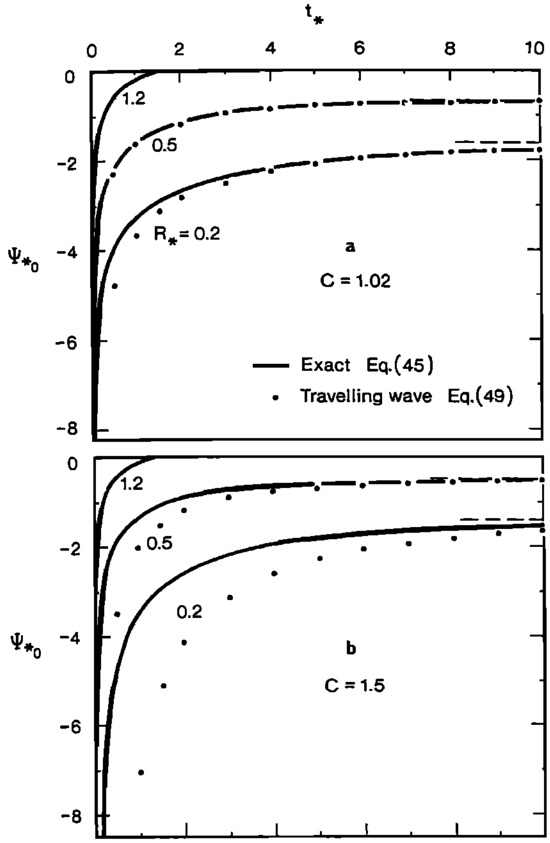

Fig. 5. Change of dimensionless water potential at the surface during constant rate rainfall for (a) $C = 1.02$ and (b) $C = 1.5$ and the rainfall rates of Figure 4. Points are from the traveling wave approximation (49) and (25) with $K_n = 0$.

small dimensionless cumulative infiltration. Reverting to dimensional variables, this condition is simply

$$t \ll t_g = hS^2/[C(C-1)(R-K_n)\Delta K] \qquad (47)$$

In (47) t_g is a "gravity time" for rainfall infiltration. For $t \ll t_g$, the gravity-free solution [*Knight*, 1973; *Knight and Philip*, 1974] for constant flux absorption emerges from our solutions (41)–(44).

5.4.3. Long-time traveling wave solution, $\rho\tau \gg 1$. As we noted earlier, a characteristic of infiltration in deep, uniform soils at large times is the development and steady convection of a soil water content profile of fixed shape. We find this traveling wave solution here for $R_* \le 1$ by considering the case $\rho\tau \gg 1$. The shape of the water content profiles as $t \to \infty$ is

$$z_* = \Theta_e^{-1}[\ln\{C(\Theta_e - \Theta)/[\Theta(C - \Theta_e)]\} - C^{-1}\ln\{\Theta_e(C - \Theta)/[\Theta(C - \Theta_e)]\} + R_* t_*]$$

with Θ_e given by (46).

A more useful form which serves as an approximation for finite t_* follows from insisting that the increase in water content in the profile be equal to the cumulative infiltration

$$z_* = \Theta_e^{-1}\ln\{\Theta_0'(\Theta_e - \Theta)/[\Theta(\Theta_e - \Theta_0')]\} - C^{-1}\ln\{\Theta_0'(C - \Theta)/[\Theta(C - \Theta_0')]\} \qquad (48)$$

with the surface water content Θ_0' given by

$$\Theta_0' = \Theta_e[\exp(R_* t_*) - 1]/[\exp(R_* t_*) - (\Theta_e/C)] \qquad (49)$$

The profile shape given by (48) remains constant and as $t_* \to \infty$ is convected with speed $dz_*/dt_* = R_*/\Theta_e$.

In Figure 6 we plot the traveling wave approximation (48) for $C = 1.02$ and $C = 1.5$ and for the reduced rainfall rate $R_* = 0.5$. For comparison we have plotted selected values of the exact solution also.

In addition we compare in Figures 4 and 5 the time dependence of Θ_0 and Ψ_{*0} given by the exact solution with those from the long-time approximation (49) together with (25). At even the lowest value of R_* for $C = 1.02$ we see excellent agreement in Figures 4, 5, and 6 between (48) and the exact solution for t_* as small as 4. On the other hand, for $C = 1.5$ and $R_* \ge 0.5$ the agreement is good for $t_* \ge 5$ but for the lowest rate considered, $R_* = 0.2$, $t_* > 10$ is required for correspondence.

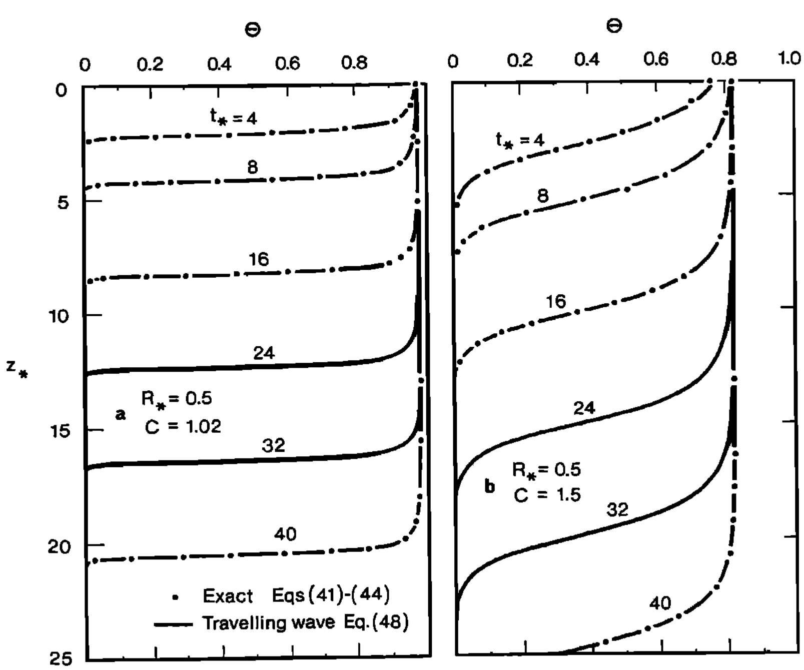

Fig. 6. Traveling wave approximation (48) for constant rate rainfall $R_* = 0.5$ for (a) $C = 1.02$ and (b) $C = 1.5$. Points are the exact values.

Clearly, the smaller the rainfall rate the longer it takes the travelling wave to develop. This arises because capillarity tends to dominate the flow process at the lower water contents generated by small rates of addition of water. Similarly, we find that in soils with identical sorptivity and hydraulic conductivities, K_s and K_n, but with differing Θ dependencies (i.e., different values of C) capillarity dominates flow longer in those soils whose hydraulic properties have weaker Θ dependencies.

For many soils the estimated value of C is close to 1.02 [*White and Broadbridge*, this issue]. For this case the evidence in Figures 4–6 suggests that at moderately small infiltration times, the simpler traveling wave solution (38) will be sufficiently accurate for many practical purposes.

6. Concluding Remarks

We have found for constant rate rainfall infiltration an analytically solvable model capable of incorporating soil properties ranging from those of the weakly nonlinear Burgers' equation to those of a highly nonlinear Green-Ampt-like model. The model's hydraulic functions have been expressed in terms of a single free parameter and soil water properties that are measurable in the field. Additionally, we have demonstrated a direct connection between the useful quasi-linear model and the capillary length scale which arises in our model.

The analytic solutions describe the temporal development of the water content profile during rainfall. They predict the time dependence of both surface moisture content and surface soil water potential and the shape of the large-time "traveling wave." A simple traveling wave approximation agreed excellently with the full solution at surprisingly short infiltration times.

For our model to have any practical utility, it is obvious that simple procedures for the estimation of the parameter need to be established. We turn to this issue in paper 2 [*White and Broadbridge*, this issue] where we also examine the utility of the solutions.

Appendix: Solving the Flow Equation

We seek to solve (38) subject to the initial condition (39) and the boundary condition (40). Application of the *Kirchhoff* [1894] transformation, which introduces the dimensionless matrix flux potential

$$\begin{aligned}\mu &= \int_{-\infty}^{\Theta} \frac{C(C-1)}{(C-w)^2}\, dw \\ &= C(C-1)/(C-\Theta) \\ &= [C(C-1)]^{1/2} D_*^{\ 1/2} \\ &= CK_*/\Theta^2 \end{aligned} \tag{A1}$$

reduces (38) to

$$D_*^{-1}\, \partial\mu/\partial t_* = \partial^2\mu/\partial z_*^{\ 2} - (dK_*/d\mu)\, \partial\mu/\partial z_* \tag{A2}$$

$$D_*^{-1}\, \partial\mu/\partial t_* = -\partial v_*/\partial z_* \tag{A3}$$

Under this transformation the initial and boundary conditions become

$$\mu = (C-1) \qquad t = 0 \qquad z_* \geq 0 \tag{A4}$$

$$v_* = R_* \quad t_* > 0 \qquad z_* = 0 \tag{A5}$$

To linearize the remaining diffusive term, we make use of the *Storm* [1951] transformation

$$\begin{aligned} Z &= \int_0^{z_*} D_*^{\ -1/2}(\mu[z_*, t_*])\, dz_* \\ &= [C(C-1)]^{-1/2} \int_0^{z_*} (C-\Theta)\, dz^* \\ &= [C(C-1)]^{1/2} \int_0^{z_*} \mu^{-1}\, dz_* \\ T &= t_* \end{aligned} \tag{A6}$$

Taking $\partial\mu/\partial T$ to mean the partial time derivative of μ with Z fixed,

$$\partial\mu/\partial t_* = \partial\mu/\partial T + (\partial\mu/\partial Z)\partial Z/\partial t_* \tag{A7}$$

Using (A6) and (A2) in (A7), together with (A5),

$$\partial\mu/\partial t_* = \partial\mu/\partial T + [C(C-1)]^{-1/2}[v_* - R_*]\partial\mu/\partial Z \tag{A8}$$

Now (A6) implies $\partial/\partial z_* = (\partial Z/\partial z_*)\, \partial/\partial Z$, so that

$$\partial^2\mu/\partial z_*^{\ 2} = C[C-1]\mu^{-2}[\partial^2\mu/\partial Z^2 - \mu^{-1}(\partial\mu/\partial Z)^2] \tag{A9}$$

It follows from (A8) and (A9) that (A2) is equivalent to

$$\partial\mu/\partial T = \partial^2\mu/\partial Z^2 - m^{1/2}[1 - 2\rho(1 + 1/\rho) + \mu/(C-1)]\partial\mu/\partial Z \tag{A10}$$

where

$$m = 4C(C-1) \tag{A11}$$

$$\rho = R_*/m \tag{A12}$$

Equation (A10) is essentially Burgers' equation. The *Storm* [1951] transformation (A6) changes the initial and boundary conditons (39) and (40) to

$$\mu = C - 1 \qquad T = 0 \qquad Z \geq 0 \tag{A13}$$

$$C\mu^2 - (m/2)[1 + 2\rho]\mu + (C-1)(m/4) - (m/4)^{1/2}\partial\mu/\partial Z = 0 \tag{A14}$$

Fortunately, the same transformation which reduces (A10) to the linear diffusion equation simplifies the boundary condition (A14) also. The transformation [*Hopf*, 1950; *Cole*, 1951]

$$1 - 2\rho(1 + 1/\rho) + \mu/(C-1) = -u^{-1}\partial u/\partial\zeta \tag{A15}$$

with $\zeta = m^{1/2}Z/2$, transforms (A10) to

$$u^{-1}[u^{-1}\partial u/\partial\zeta - \partial/\partial\zeta][\partial u/\partial\tau - \tfrac{1}{4}\partial^2 u/\partial\zeta^2] = 0 \tag{A16}$$

with $\tau = mT(= mt_*)$, and reduces the initial and boundary conditions to

$$u^{-1}\partial u/\partial\zeta = 2\rho \qquad \tau = 0 \qquad \zeta \geq 0 \tag{A17}$$

$$\partial^2 u/\partial\zeta^2 - 4\rho(\rho + 1)u = 0 \tag{A18}$$

If $u(\zeta, \tau)$ is a solution to the linear diffusion equation $\partial u/\partial\tau = \frac{1}{4}\partial^2 u/\partial\zeta^2$, then (A16) is satisfied. Using this reduces (A18) to

$$\partial u/\partial\tau = \rho(\rho + 1)u \tag{A19}$$

Without loss of generality, the boundary condition (A14) can be taken as

$$u = \exp[\rho(\rho + 1)\tau] \tag{A20}$$

and the initial condition is then

$$u = \exp(2\rho\zeta) \tag{A21}$$

The solution to the linear diffusion equation satisfying (A20) and (A21) can be found using Laplace transforms [e.g., *Carslaw and Jaeger*, 1959, chapter 12]:

$$u = \tfrac{1}{2} \exp(-\zeta^2\tau^{-1})\{2 \exp[(\zeta + \rho\tau)/\tau^{1/2}]^2 + f([\zeta - \rho(1 + \rho^{-1})^{1/2}\tau]/\tau^{1/2}) - f([\zeta - \rho\tau]/\tau^{1/2}) + f([\zeta + \rho(1 + \rho^{-1})^{1/2}\tau]/\tau^{1/2}) - f([\zeta + \rho\tau]/\tau^{1/2})\}$$

where

$$f(x) = \exp(x^2) \operatorname{erfc}(x) \tag{A22}$$

To obtain the dimensionless matrix flux potential μ from (A11), we derive

$$\partial u/\partial\zeta = \rho \exp(-\zeta^2\tau^{-1})[2 \exp([\zeta + \rho\tau]/\tau^{1/2})^2 - ([1 + \rho^{-1})^{1/2}\{f([\zeta - \rho(1 + \rho^{-1})^{1/2}\tau]/\tau^{1/2}) - f([\zeta + \rho(1 + \rho^{-1})^{1/2}\tau]/\tau^{1/2})\} + f([\zeta - \rho\tau]/\tau^{1/2}) - f([\zeta + \rho\tau]/\tau^{1/2})] \tag{A23}$$

Note that in (A22) and (A23), $\rho\tau = R_* t_*$, which is the dimensionless cumulative infiltration before ponding.

The Hopf-Cole transformation (A15) can be rewritten as

$$\Theta = C[1 - (2\rho + 1 - u^{-1}\partial u/\partial\zeta)^{-1}] \tag{A24}$$

and the dimensionless depth is recovered by inverting (A6):

$$z_* = [C(C - 1)]^{-1/2} \int_0^Z \mu \, dZ = C^{-1}[\rho^2(1 + \rho^{-1})\tau + \rho(2 + \rho^{-1})\zeta - \ln u] \tag{A25}$$

Equations (A22)–(A25) constitute an exact parameteric solution $\{\Theta(\zeta, \tau), z_*(\zeta, \tau)\}$ with parameter ζ.

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