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An Analytical Solution to Richards' Equation for Time-Varying Infiltration

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An analytical solution is developed for Richards equation for time-varying infiltration. The infiltration rates must be specified as piecewise constants with time. The answer is expressed as a sum of two terms, the first is a function of the instantaneous infiltration and the second an integral which accounts for the water distribution within the profile for the previous infiltration history. The solutions assume specific forms for the soil water diffusivity and hydraulic functions used earlier by Broadbridge and White (1988).

INTRODUCTION

Broadbridge and White [1988], *Broadbridge and Rogers* [1990], and *Warrick et al.* [1990] presented analytical solutions to Richards equation for steady infiltration rates (including zero). Several related efforts are reviewed in those papers. The objective here is to extend those results for a surface input expressed as a series of constant rates stepped over arbitrary time intervals.

THEORY

The solution to Richards equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] - \frac{\partial K}{\partial z} \tag{1}$$

subject to the initial and boundary conditions

$$\theta(z, 0) = g(z) \tag{2}$$

$$v(0, t) = R \tag{3}$$

is given by *Warrick et al.* [1990] following *Broadbridge and White* [1988]. In (1)–(3), θ is the volumetric water content, t is the time, z is the depth, K is the hydraulic conductivity, D is the soil water diffusivity, $g(z)$ is the initial water content, v is the Darcian velocity, and R is a steady input rate at the soil surface. The primary assumption is that D and K be of special forms:

$$D(\theta) = a(b - \theta)^{-2} \tag{4}$$

$$K = \beta + \gamma(b - \theta) + \lambda/[2(b - \theta)] \tag{5}$$

valid for the entire profile where a, b, β, γ, and λ are constants.

The solution is of a parametric form:

$$\Theta = C[1 - (2\rho + 1 - u^{-1}\partial u/\partial \zeta)^{-1}] \tag{6}$$

$$z_* = C^{-1}[\rho(\rho + 1)\tau + (2\rho + 1)\zeta - \ln u] \tag{7}$$

In (6), Θ is a relative water content

$$\Theta = (\theta - \theta_n)/(\theta_s - \theta_n) \tag{8}$$

where θ_n is for a dry value and θ_s is the saturated value. Further definitions of constants are

$$C = (b - \theta_n)/(\theta_s - \theta_n) \tag{9}$$

$$\rho = R_*[4C(C - 1)]^{-1} \tag{10}$$

$$R_* = (R - K_n)/(K_s - K_n) \tag{11}$$

$$\lambda_s = a/[C(C - 1)(\theta_s - \theta_n)(K_s - K_n)] \tag{12}$$

$$t_s = a/[C(C - 1)(K_s - K_n)^2] \tag{13}$$

with K_s and K_n corresponding to θ_s and θ_n. The τ is directly proportional to real time

$$\tau = 4C(C - 1)t/t_s \tag{14}$$

The z_* and ζ are depth coordinates with

$$z_* = z/\lambda_s \tag{15}$$

$$\zeta = [C(C - 1)] \int_0^{z_*} \mu^{-1}\, dz_* \tag{16}$$

with μ a dimensionless matrix flux potential given by

$$\mu = C(C - 1)/(C - \Theta) \tag{17}$$

Finally, the dependent variable u is given by

$$u = U_1 + U_2 \tag{18}$$

$$U_1(\zeta, \tau, \lambda) = 0.5 \exp(-\zeta^2/\tau)\{f[\zeta\tau^{-0.5} - (\lambda\tau)^{0.5}] + f[\zeta\tau^{-0.5} + (\lambda\tau)^{0.5}]\} \tag{19}$$

$$U_2(\zeta, \tau) = (\pi\tau)^{-0.5} \cdot \int_0^{\infty} \exp\left[\int_0^{\zeta'} G(\zeta'')\, d\zeta''\right] h(\zeta, \tau, \zeta')\, d\zeta' \tag{20}$$

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with

$$\lambda = \rho(\rho + 1) \tag{21}$$

$$f(x) = \exp(x^2) \operatorname{erfc}(x) \tag{22}$$

$$h(\zeta, \tau, \zeta') = \exp[-(\zeta - \zeta')^2/\tau)] - \exp[-(\zeta + \zeta')^2/\tau] \tag{23}$$

and $G(\zeta)$ corresponding to the initial condition by

$$G(\zeta) = 1 + 2\rho - C(\theta_s - \theta_n)[C(\theta_s - \theta_n) - g + \theta_n]^{-1} \tag{24}$$

Solution for R changing with time. We now consider the solution to (1) when the infiltration rate (R) changes with time as

$$R = R_i, \qquad \begin{matrix} \tau_i < \tau < \tau_{i+1} \\ i = 1, 2, \cdots \end{matrix} \tag{25}$$

with $\tau_1 = 0$, and all of the R_i are constants. If we consider each R_i separately, the solution will be

$$u = U_1(\zeta, \tau - \tau_i, \lambda) + U_2(\zeta, \tau - \tau_i), \quad \tau_i < \tau < \tau_{i+1} \tag{26}$$

with U_1 given by (19) and U_2 following from (20) in the form

$$U_2 = [\pi(\tau - \tau_i)]^{-0.5} \int_0^\infty u(\zeta', \tau_i^+) h(\zeta, \tau - \tau_i, \zeta')\, d\zeta' \tag{27}$$

where

$$\tau_i^\pm = \lim_{\varepsilon \to 0} \tau_i \pm \varepsilon \tag{28}$$

For $i = 1$, $u(\zeta, \tau_1^+)$ follows directly from (20):

$$u(\zeta', \tau_1^+) = \exp\left[\int_0^{\zeta'} G(\zeta')\, d\zeta'\right] \tag{29}$$

Otherwise, the key relationship is

$$u(\zeta, \tau_i^+) = u(\zeta, \tau_i^-) \exp[-\rho_{i-1}(\rho_{i-1} + 1)(\tau_i - \tau_{i-1}) - 2(\rho_{i-1} - \rho_i)\zeta], \quad i = 2, 3, \cdots \tag{30}$$

where $u(\zeta, \tau_i^-)$ is the ending value from the previous time step. To verify (30) is correct, note that by (16) and (17), both μ and ζ are continuous at a given real depth for time going from τ_i^- to τ_i^+. Thus the correct "jump" in u at τ_i is defined by (7), resulting in (30). The above solution is complete, but now we derive some useful relationships for computations.

Values for u deep in the profile. Assume the limit condition

$$\lim_{\zeta \to \infty} \Theta = \Theta_i \tag{31}$$

By substitution of (31) into (6) we observe

$$\lim_{\zeta \to \infty} u^{-1}(\partial u/\partial \zeta) = A_i \tag{32}$$

$$A_i = 1 + 2\rho_i + C/(\Theta_i - C) \tag{33}$$

Thus, for ζ sufficiently large

$$\ln u = A_i\zeta + f(\tau) \tag{34}$$

where $f(\tau)$ is an arbitrary function of time. If ζ_0 is taken as a specified large value at ζ, the above equation is equivalent to

$$u(\zeta, \tau) \approx u(\zeta_0, \tau) \exp[A_i(\zeta - \zeta_0)], \quad \zeta > \zeta_0 \tag{35}$$

A useful observation in related research is that the hydraulic gradient deep in a profile often approaches a unit gradient. By (15) and (16) the derivative with respect to z satisfies

$$\frac{\partial}{\partial z} = \frac{\partial \zeta}{\partial z}\frac{\partial}{\partial \zeta} + \frac{\partial \tau}{\partial z}\frac{\partial}{\partial \tau} \tag{36a}$$

or

$$\frac{\partial}{\partial z} = (\text{const})\mu^{-1}\frac{\partial}{\partial \zeta} \tag{36b}$$

As a unit gradient corresponds to $\partial\Theta/\partial z = 0$, it follows from (6) that

$$u^{-1}(\partial u/\partial \zeta) = F(\tau) \tag{37a}$$

or

$$\ln u = F(\tau)\xi + G(\tau) \tag{37b}$$

where $F(\tau)$ and $G(\tau)$ are functions to be determined. Thus, if the unit gradient is to be satisfied for $\zeta > \zeta_0^*$ (with $\zeta_0^* > \zeta_0$), we have the following approximation

$$u(\zeta, \tau) \approx u(\zeta_0, \tau) \exp[A(\tau)(\zeta - \zeta_0)] \tag{38}$$

with

$$A(\tau) = \frac{\ln[u(\zeta_0, \tau)/u(\zeta_0^*, \tau)]}{\zeta_0 - \zeta_0^*} \tag{39}$$

NUMERICAL CALCULATIONS

Calculations follow using the hydraulic properties for the Brindabella silty clay loam as described by *White and Broadbridge* [1988, Table 4]. Values include $K_s - K_n = (3.27)(10)^{-5}$ ms^{-1}, $K_n = 0$, $S = (1.335)(10)^{-3}$ ms$^{-0.5}$, $\theta_s - \theta_n = 0.375$, $\theta_n = 0.11$, $C = 1.020$, $h(C)(/(C - 1) = 0.5076$, and $\lambda_s = 0.0738$ m.

Before discussing results, we examine particularly U_1 and U_2 of (19) and (27). Evaluation of U_1 is relatively straightforward with three forms noted by *Warrick et al.* [1990]. For $\rho > 0$, (19) is used directly with all arguments real; for $\rho = 0$ it simplifies to erfc $(\zeta\tau^{-0.5})$; and for $\rho(\rho + 1) < 0$, complex arguments are encountered. Further discussion of evaluating U_1 is in the work by *Warrick et al.* [1990]. Only the first two forms are needed here. As a caveat, the complementary error function needs to be calculated with the correct order of magnitude which generally requires asymptotic forms for large arguments [cf. *Abramowitz and Stegun*, 1964, equation (7.1.23)].

The evaluation of U_2 by (27) is made difficult in that $u(\zeta',$

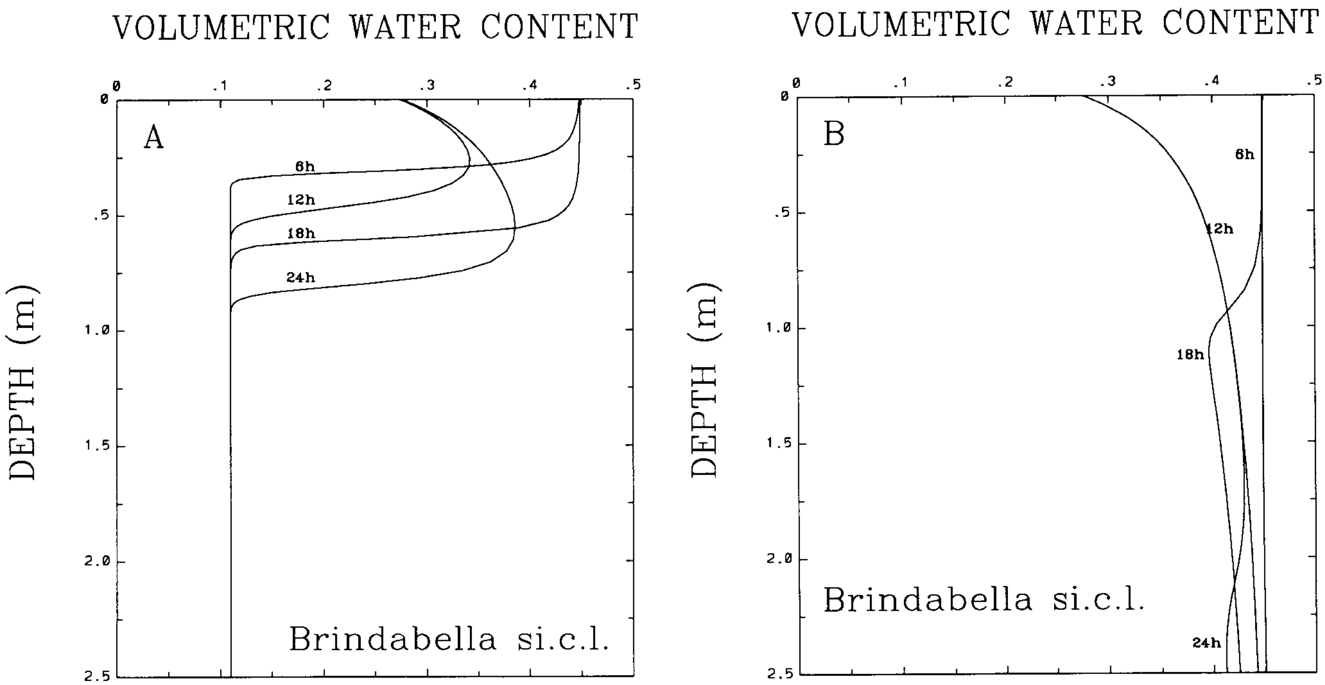

Fig. 1. Profiles for 6, 12, 18, and 24 hours with initial conditions of (*a*) 0.11 and (*b*) 0.485.

τ_i^+) in the integrand comes from numerical integration for $i > 2$. Values for ρ_i in (30) can be as large as 3 or 4 which corresponds to very large (and alternately very small) values for $u(\zeta', \tau_i^+)$. The exponents within h of (23), tend to be very large resulting in small values for the integral except near $\zeta = \zeta'$, in fact for $\tau - \tau_i$ small, the $h(\zeta, \tau - \tau_i, \zeta')/(\tau - \tau_i)^{0.5}$ approaches a delta function! For U_2 the integral was evaluated by

$$U_2(\zeta, \tau) = I_1 + I_2 \tag{40}$$

with

$$I_1 = [\pi(\tau - \tau_i)]^{-0.5} \int_0^{\zeta_0} u(\zeta', \tau_i^+) h(\zeta, \tau - \tau_i, \zeta')\, d\zeta' \tag{41}$$

$$I_2 = [\pi(\tau - \tau_i)]^{-0.5} u(\zeta_0, \tau_i^+) \int_{\zeta_0}^{\infty} \cdot \exp[A(\zeta' - \zeta_0)] h(\zeta, \tau - \tau_i, \zeta')\, d\zeta' \tag{42}$$

The first integral (I_1) was evaluated by Simpson's algorithm with a constant number of equal steps [cf. *Abramowitz and Stegun*, 1964, equation (25.4.6)]. The second integral (I_2) was evaluated in closed form [*Warrick et al.*, 1990, equation (7)]. Choices to be made include ζ_0 and number of steps in the Simpson algorithm. Also, with respect to ζ_0 a choice of either a constant water content or unit gradient condition is required.

A useful relationship for the gradient of U_2 required in (6) is

$$\frac{\partial U_2}{\partial \zeta} = \frac{2\zeta U_2}{\tau - \tau_i} + 2\pi^{-0.5}(\tau - \tau_i)^{-1.5} \int_0^{\infty} \zeta' u(\zeta', \tau_i^+) \cdot \{\exp[-(\zeta - \zeta')^2/(\tau - \tau_i)] + \exp[-(\zeta + \zeta')^2/(\tau - \tau_i)]\}\, d\zeta' \tag{43}$$

Preliminary computations showed results to be insensitive to number of terms in the Simpson approximation beyond 100 terms as well as to the form of the lower boundary condition provided ζ_0 was specified well below the depth for evaluation. For the calculations which follow, the number of terms in the Simpson approximation was taken as 200 and the "unit gradient" form of (38) assumed for $\zeta_0 = 10$. The results which follow were checked successfully against the previous analytical solution for $0 = t_1 < t < t_2$ and against a numerical solution for all values of time.

Consider cyclic input of the form

$R = (4.58)(10)^{-6}\ \mathrm{ms}^{-1}$	$0 < t < 6$ hours
$R = 0$	$6 < t < 12$ hours
$R = (4.58)(10)^{-6}\ \mathrm{ms}^{-1}$	$12 < t < 18$ hours
$R = 0$	$18 < t < 24$ hours

Results are in Figure 1*a* for an initial value $\theta_i = 0.11$ and for $t = 6$, 12, 18, and 24. For 6 hours the result may be compared approximately against *Broadbridge and White* [1988, Figure 6]. From 6 to 12 hours there is no water added to the surface and another pulse of water is added for 12 to 18 hours. Finally, drainage again occurs from 18 to 24 hours.

In Figure 1*b* the profile is initially saturated ($\theta_i = 0.485$). Profiles are also given at 6, 12, 18, and 24 hours. Water content at deep depths are wetter with near-surface values approaching the results from the initially dry soil in Figure 1*a*. Thus the history of the previous water contents is lost close to the soil surface but preserved at deeper depths.

Moisture contents at $z = 0.25$ and 0.5 m are given as a function of time in Figure 2 for a cyclic boundary condition

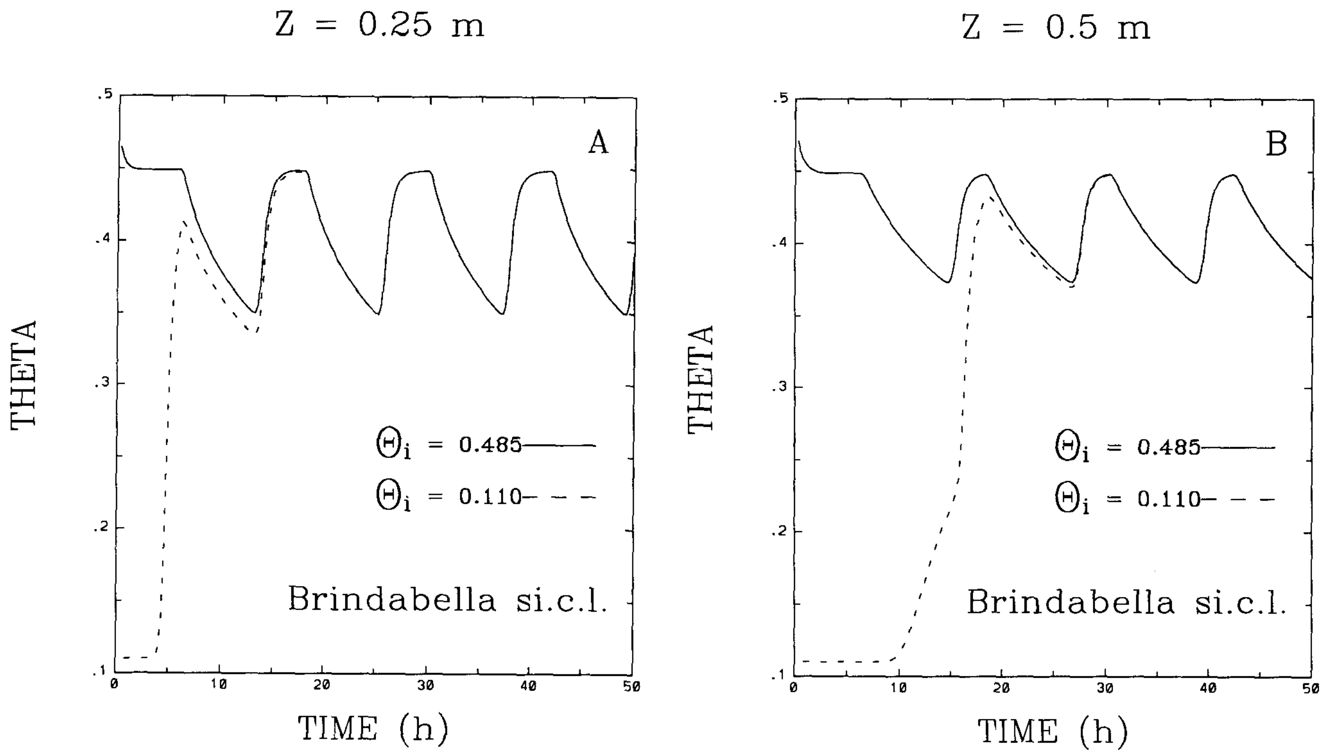

Fig. 2. Time response for (*a*) z = 0.25 and (*b*) z = 0.5 m for initial conditions of 0.11 and 0.485.

as above but continued for 50 hours. Six hours of infiltration of $4.58(10)^{-6}$ are followed by 6 hours of drainage and repeated for the entire 50 hours. For z = 0.25 m (Figure 2*a*) the results are identical for the very dry (0.11) and very wet (0.485) initial conditions after two cycles. For z = 0.5 m (Figure 2*b*) a bit longer time is required but results converge for the different initial conditions. As would be expected from a harmonic analysis, the fluctuations are slightly more for the shallower depth than for the deeper depth (about 0.1 compared to 0.06). Also, increases tend to occur rapidly within each cycle contrasted to a more gradual descent.

SUMMARY AND CONCLUSIONS

Results presented here add to the analytical solutions available for the Richards equation. Any initial condition can be included as well as arbitrary surface infiltration rates as a series of step inputs. Extensions to other hydraulic functions or type of boundary conditions are unlikely, although additional problems for a bounded profile and two-phase flow [*Broadbridge et al.*, 1988; *Rogers et al.*, 1983] may be amenable to the same general techniques.

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