UNIVERSITY OF SHEFFIELD

ACS 6101: Foundation of Control Systems

Modelling & Simulation

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Section A

Task 1: Linear model of the pendulum system

Introduction

This task aimed to derive a linear approximation mathematical model of the pendulum system shown in Figure 1.

Figure 1 shows the abstraction level of the pendulum system which was considered to describe the relationship between the torque developed on the mass M and angle, θ between the rod and the vertical plane.

Following assumptions were made while evaluating the relationship between the torque and the angle,θ.

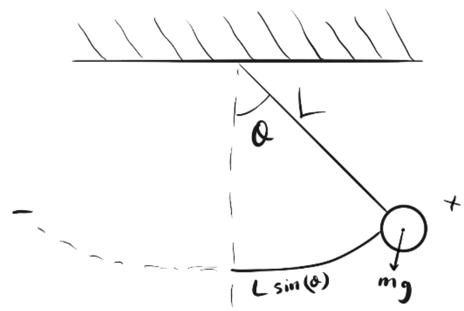

Figure 1 – Abstraction of the Pendulum System

- The pendulum system abided by the law of conservation of energy which states that 'the energy cannot be created or destroyed but can change from one form to another', in this case from kinetic to potential energy (University of CALGARY , n.d.).
- The pendulum system was ideal which meant it had no friction i.e., mechanical energy was conserved. Thus, the pendulum pivots freely around the pivot point and continue forever.
- The string with length, L, shown in Figure 1 was rigid and always in tension with no mass of itself. The overall mass of the pendulum system is represented by, m.
- Frictional force or friction coefficient resisting the motion is negligible

Methodology

Equation 1 did not pass the superposition test, therefore Taylor series expansion shown in Equation 2 was used to linearize Equation 1 into Equation 3, where θ is between $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

$$T = mg\ Lsin\theta \qquad Eq.(1)$$

$$f_x = \sum_{n=0}^{\infty} \frac{f^n a}{n!}(x-a)^n \qquad Eq.(2)$$

$$T = mgl\theta \qquad Eq.(3)$$

Newton's second law of motion described in Equation 4 was used in amalgamation with Newton's rotational law (Eq. 5) and Equation 6 to obtain Equation 7 which was transposed into Equation 9.

$$F = ma \qquad Eq.(4)$$

$$\sum T = I\ddot{\theta} \ \ \& \ \ I = ml^2 \qquad Eq.(5)$$

$$T = F \times L \qquad Eq.(6)$$

$$T - mgl\theta - ml^2\ddot{\theta} = 0 \qquad Eq.(7)$$

$$T = mgl\theta + ml^2\ddot{\theta} \qquad Eq.(8)$$

$$\ddot{\theta} = \frac{1}{ml^2}T - \frac{g}{l}\theta \qquad Eq.(9)$$

Equation 8 and Equation 9 describes the correlation between the Torque, T and angle,θ.

Task 2: Simplified Car Suspension

Introduction

In this task, a mathematical model and the state-space model of the simplified car suspension shown in (Panoutsos) was derived in terms of ODE and represented in the matrix form.

Figure 2 shows the masses and the direction of forces acting in the simplified car suspension shown in (Panoutsos). Masses; M_1, M_2 and M_3 refers to the wheel, the chassis, and the seat, respectively.

Certain assumptions were taken into consideration while deriving the mathematical model of the system and are as follows:

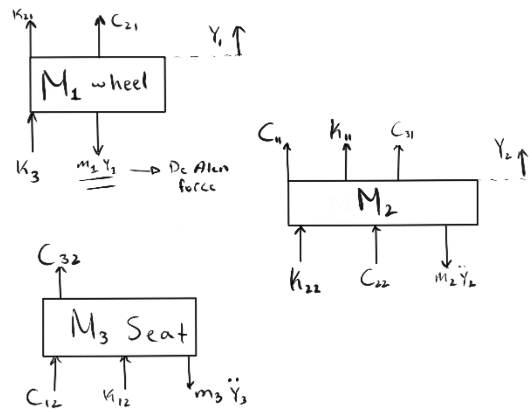

Figure 2 - Masses, M1, M2 & M3 acting in a simplified car suspension

- Springs and dampers connecting the masses follow the Hooke's law. In other words, springs used in car suspension were ideal.
- Constant relationship between coordinates set i.e., y_1, y_2 and y_3 shown in (Panoutsos).
- Inertial frame of reference was constant or moving with constant velocity. Thus, Newton's law can be applied.
- Seat-back friction, C_3 was connected to the back of the seat and the displacement of C_3 was dictated by the displacement of the seat, M_3 and the chassis, M_2. In other words, C_3 was fixed with the seat, therefore the amount of movement done by the seat would affect the displacement of C_3.
- The height of the seat was constant hence, ignored.

Methodology

$$F_s = -ky \qquad Eq.(10)$$

$$F_d = C\dot{y} \qquad Eq.(11)$$

Applying Newton's

Hooke's law shown in Equation 10 was used in conjunction with Newton's law shown in Equation 4 to derive the following equations:

$$K_2(y_1 - y_2) + C_2(\dot{y}_1 - \dot{y}_2) + K_3(y_1 - z) = -m_1\ddot{y}_1 \qquad Eq.(12)$$

$$C_1(\dot{y}_2 - \dot{y}_3) + K_1(y_2 - y_3) + K_2(y_2 - y_1) + C_2(\dot{y}_2 - \dot{y}_1) + C_3(\dot{y}_2 - \dot{y}_3) = -m_2\ddot{y}_2 \qquad Eq.(13)$$

$$K_1(y_3 - y_2) + C_1(\dot{y}_3 - \dot{y}_2) + C_3(\dot{y}_3 - \dot{y}_2) = -m_3\ddot{y}_3 \qquad Eq.(14)$$

$Where: y_1, y_2$ & y_3 $are\ the\ vertical\ displacement$

$z\ is\ displacement\ from\ the\ ground\ profile$

K_1, K_2 & K_3 $are\ the\ spring\ constant$

Rearranging the equations

$$\ddot{y_1} = \frac{K_3}{M_1}z - \frac{K_3}{M_1}y_1 + \frac{C_2}{M_1}\dot{y_2} - \frac{C_2}{M_1}\dot{y_1} + \frac{K_2}{M_1}y_2 - \frac{K_2}{M_1}y_1 \qquad Eq. (15)$$

$$\ddot{y_2} = -\frac{C_1}{M_2}\dot{y_2} + \frac{C_1}{M_2}\dot{y_3} - \frac{K_1}{M_2}y_2 + \frac{K_1}{M_2}y_3 - \frac{K_2}{M_2}y_2 + \frac{K_2}{M_2}y_1 - \frac{C_2}{M_2}\dot{y_2} + \frac{C_2}{M_2}\dot{y_1} - \frac{C_3}{M_2}\dot{y_2} + \frac{C_3}{M_2}\dot{y_3} \qquad Eq. (16)$$

$$\ddot{y_3} = -\frac{K_1}{M_3}y_3 + \frac{K_1}{M_3}y_2 - \frac{C_1}{M_3}\dot{y_3} + \frac{C_1}{M_3}\dot{y_2} - \frac{C_3}{M_3}\dot{y_3} + \frac{C_3}{M_3}\dot{y_2} \qquad Eq. (17)$$

Laplace Transform

Laplace transform was performed to convert the ODEs into s-domain. While performing Laplace transform, initial conditions were assumed to be zero.

$$s^2y_1 = \frac{K_3}{M_1}z - \frac{K_3}{M_1}y_1 + s\frac{C_2}{M_1}y_2 - s\frac{C_2}{M_1}y_1 + \frac{K_2}{M_1}y_2 - \frac{K_2}{M_1}y_1 \qquad Eq. (18)$$

$$s^2y_2 = -s\frac{C_1}{M_1}y_2 + s\frac{C_1}{M_2}y_3 - \frac{K_1}{M_2}y_2 + \frac{K_1}{M_2}y_3 - \frac{K_2}{M_2}y_2 + \frac{K_2}{M_2}y_1 - s\frac{C_2}{M_2}y_2 + s\frac{C_2}{M_2}y_1 - s\frac{C_3}{M_2}y_2 + s\frac{C_3}{M_2}y_3 \qquad Eq. (19)$$

$$s^2y_3 = -\frac{K_1}{M_3}y_3 + \frac{K_1}{M_3}y_2 - s\frac{C_1}{M_3}y_3 + s\frac{C_1}{M_3}y_2 - s\frac{C_3}{M_3}y_3 + s\frac{C_3}{M_2}y_2 \qquad Eq. (20)$$

Transfer Function

Laplace transform shown as transfer function

$$Y_{1,WHEEL}(s) = GY_1(s) \times Y_2(s) + G_z(s) \times z(s) \qquad Eq. (21)$$

$$Y_{2,CHASSIS}(s) = GY_2(s) \times Y_3(s) + GY_{21} \times Y_1(s) \qquad Eq. (22)$$

$$Y_{3,SEAT}(s) = GY_3(s) \times Y_2(s) \qquad Eq. (23)$$

$Where$:

$$GY_1(s) = \left[\frac{s\left(\frac{C_1}{M_1}\right) + \frac{K_2}{M_2}}{s^2 + s\left(\frac{C_2}{M_2}\right) + \frac{K_3}{M_1} + \frac{K_2}{M_1}}\right] \qquad Eq. (24)$$

$$G_z(s) = \left[\frac{\frac{K_3}{M_1}}{s^2 + s\left(\frac{C_2}{M_1}\right) + \frac{K_2}{M_1} + \frac{K_3}{M_1}}\right] \qquad Eq. (25)$$

$$G_{Y2}(s) = \left[\frac{s\left(\frac{C_1}{M_2} + \frac{C_3}{M_2}\right) - \frac{K_1}{M_2}}{s^2 + s\left(\frac{C_1}{M_2} + \frac{C_2}{M_2} + \frac{C_3}{M_2}\right) + \frac{K_1}{M_2} + \frac{K_2}{M_2}}\right] \qquad Eq. (26)$$

$$GY_{21}(s) = \left[\frac{s\left(\frac{C_2}{M_2}\right) + \frac{K_2}{M_2}}{s^2 + s\left(\frac{C_1}{M_2} + \frac{C_2}{M_2} + \frac{C_3}{M_2}\right) + \frac{K_1}{M_2} + \frac{K_2}{M_2}}\right] \qquad Eq. (27)$$

$$GY_3(s) = \left[\frac{s\left(\frac{C_1}{M_3} + \frac{C_3}{M_3}\right) + \frac{K_1}{M_3}}{s^2 + s\left(\frac{C_1}{3} + \frac{C_3}{M_3}\right) + \frac{K_1}{M_3}}\right] \qquad Eq.(28)$$

The system in matrix form

$$y = Ax + Bu \quad Where: u = z = Input \qquad Eq.(29)$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix} = \begin{bmatrix} 0 & GY_1(s) & 0 \\ GY_{21}(s) & 0 & G_{Y2}(s) \\ 0 & GY_3(s) & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} + \begin{bmatrix} G_z(s) \\ 0 \\ 0 \end{bmatrix} z \qquad Eq.(30)$$

The transfer function shown in Equation 24 to Equation 28 were substituted to obtain Equation 30 which shows the matrix form in the Laplace domain. This was done for better visuality.

State Space Representation of the system & its matrix

$$State\ Variables: \qquad Input = u = z$$

$$x_1 = Y_1 \qquad Output = Y_1, Y_2, Y_3$$

$$x_2 = \dot{Y_1}$$

$$x_3 = Y_2$$

$$x_4 = \dot{Y_2}$$

$$x_5 = Y_3$$

$$x_6 = \dot{Y_3}$$

$$\dot{x_2} = \frac{K_3}{M_1}z - \frac{K_3}{M_1}x_1 + \frac{C_2}{M_1}x_4 - \frac{C_2}{M_1}x_2 + \frac{K_2}{M_1}x_3 - \frac{K_2}{M_1}x_1 \qquad Eq.(31)$$

$$\dot{x_4} = -\frac{C_1}{M_2}x_4 + \frac{C_1}{M_2}x_6 - \frac{K_1}{M_2}x_3 + \frac{K_2}{M_2}x_5 - \frac{K_2}{M_2}x_3 + \frac{K_2}{M_2}x^1 - \frac{C_2}{M_2}x_4 + \frac{C_2}{M_2}x_2 - \frac{C_3}{M_2}x_4 + \frac{C_3}{M_2}x_6 \qquad Eq.(32)$$

$$\dot{x_6} = -\frac{K_1}{M_3}x_5 + \frac{K_1}{M_3}x_3 - \frac{C_1}{M_3}x_6 + \frac{C_1}{M_3}x_4 - \frac{C_3}{M_3}x_6 + \frac{C_3}{M_3}x_4 \qquad Eq.(33)$$

Converting the state space into matrix form

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{K_3}{M_1} - \frac{K_2}{M_1} & -\frac{C_2}{M_1} & \frac{K_2}{M_1} & \frac{C_2}{M_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{K_2}{M_2} & \frac{C_2}{M_2} & -\frac{K_1}{M_2} - \frac{K_2}{M_2} & -\frac{C_1}{M_2} - \frac{C_3}{M_2} - \frac{C_2}{M_2} & \frac{K_1}{M_2} & \frac{C_1}{M_2} + \frac{C_3}{M_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{K_1}{M_3} & \frac{C_1}{M_3} + \frac{C_3}{M_3} & -\frac{K_1}{M_3} & -\frac{C_1}{M_3} - \frac{C_3}{M3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_3}{M_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} z$$

Section B

Task 1: Signal flow graph of simplified car suspension

The signal flow graph is a graphical representation of algebraic equations that consist of nodes and branches. Nodes represent a signal or a variable whereas branches with arrow depict the unidirectional path of the signal. A signal flow graph is a useful tool in analyzing and modelling feedback control systems since it clearly shows the direction of signals.

Block diagram and signal flow graph are very closely related since they both provide similar function; however, a block diagram is a tedious method and requires more time and space than the simple signal flow graph as evident from Figure 3 and Figure 4. As seen from Figure 4, the signal flow graph provides the correlation between system variables without much evaluation. Furthermore, it could also be useful in picturing the output response to an input signal (Bishop, 2017)

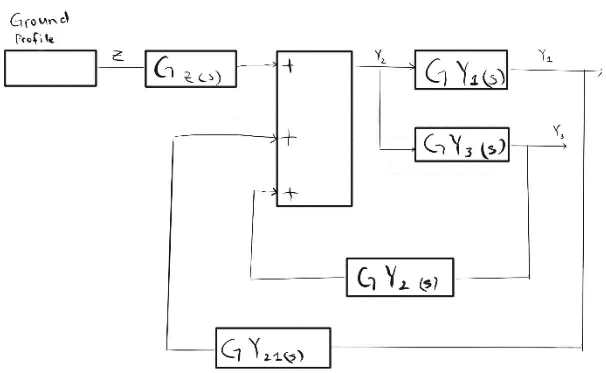

Figure 3 - Block diagram for the car suspension system

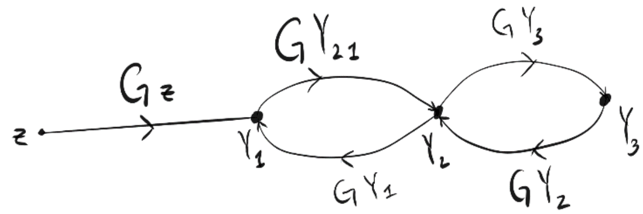

Figure 4 - Signal Flow Graph for the car suspension system

NOTE: Transfer functions shown in Equation 24 to Equation 28 were used to design the signal flow graph illustrated in Figure 4 for the sake of better visibility.

Task 2: Depth Control for a Torpedo

Step Response of the controlled torpedo

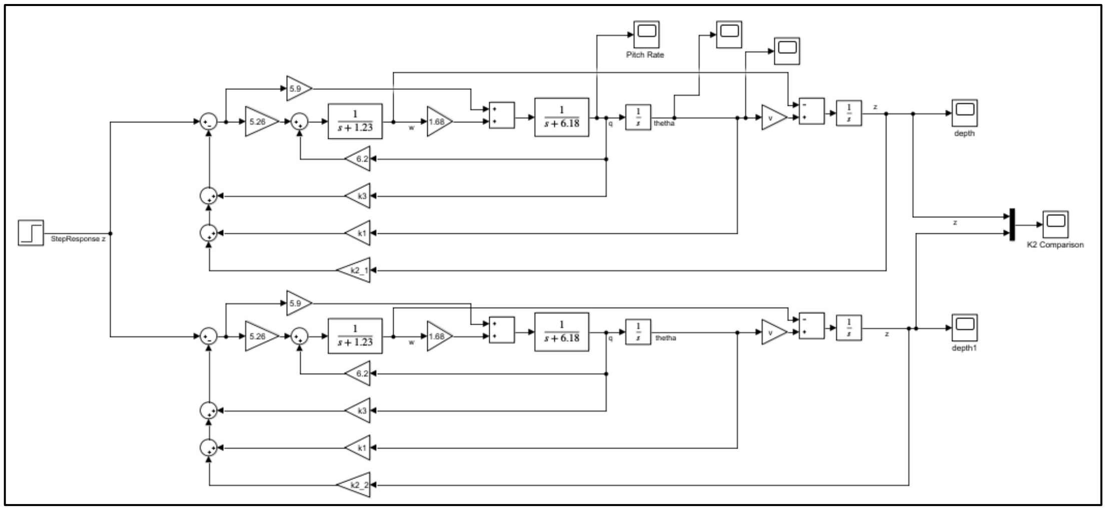

Figure 5 - Simulink Model of the Controlled Torpedo

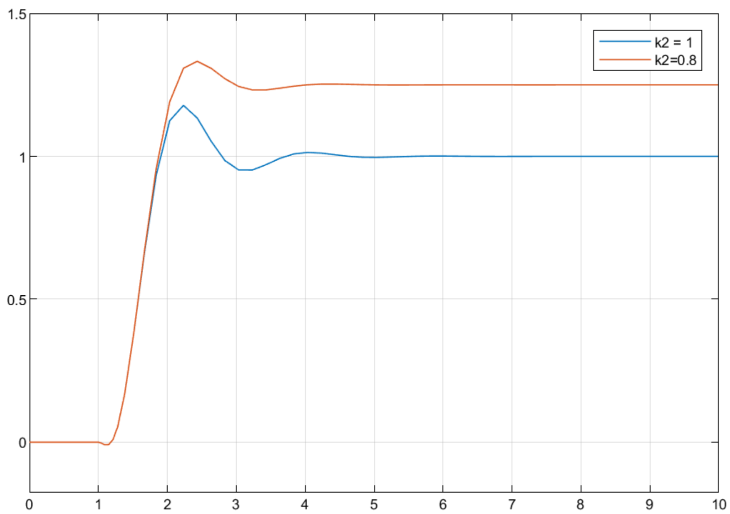

Figure 6 - Step response with K2 = 1 & k2 = 0.8 with same v of 20m/s

The step response of the controlled torpedo shown in Figure 6 was generated via Simulink displayed in Figure 5. Reducing the gain, K_2 from one to 0.8 heavily affected the overshoot of the step response while slightly impacted the settling time through oscillation. In other words, reducing the gain, K_2, increased the percentage overshoot of the step response, however, it also reduced the oscillation and settling time. It should also be noted that the rise time was not affected by the change in K_2. Knowing the correlation between the gain and the transient performance is useful in determining the necessary gain required to achieve desired performance specifications.

Effect of speed, v, of Torpedo on the step response

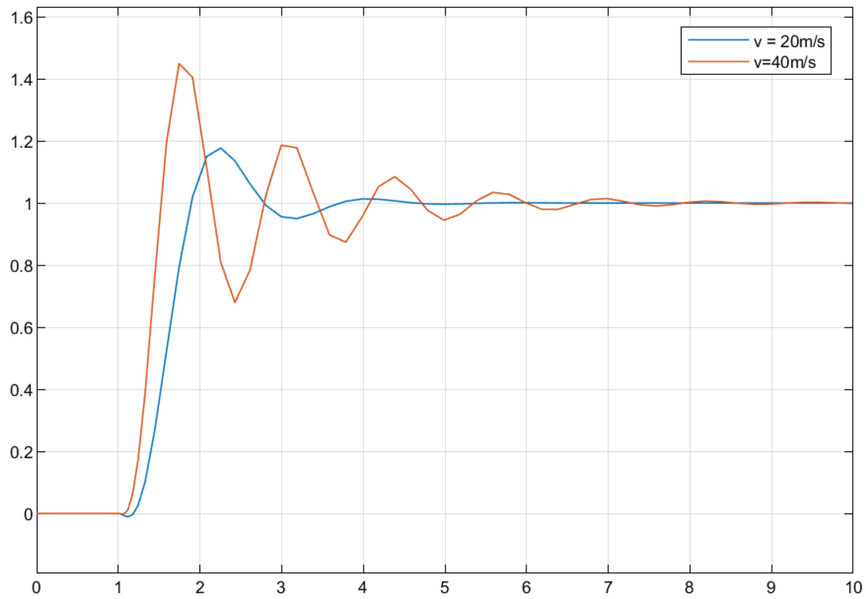

Figure 7 - Step response of v = 20m/s and v = 40m/s

Figure 7 shows the effect of change in the speed of the torpedo via the step response. It was apparent that the speed of the torpedo impacted major transient performance criteria that includes rise time, settling time, and the overshoot.

Table 1 shows the effect of speed on the transient performance of the torpedo. Increasing the speed reduced the rise time, however, increased the peak response and the settling time to a step response. The effect of the rise in the speed of the torpedo was tackled through changing the gain, K_1 and K_3.

Table 1 - Compares the Transient Performance of the step response @ v= 20m/s & v = 40m/s

Transient Performance	$v = 20\ m/s$	$v = 40m/s$
Rise Time	0.678	0.4
Peak Response	1.172	1.449
Settling Time	4.6	9.5

After several trial and error, suitable gain was determined for K_1 and K_3 that gave a desired transient performance. It was deduced that the value of K_1 to be 30 and 5 for K_3.

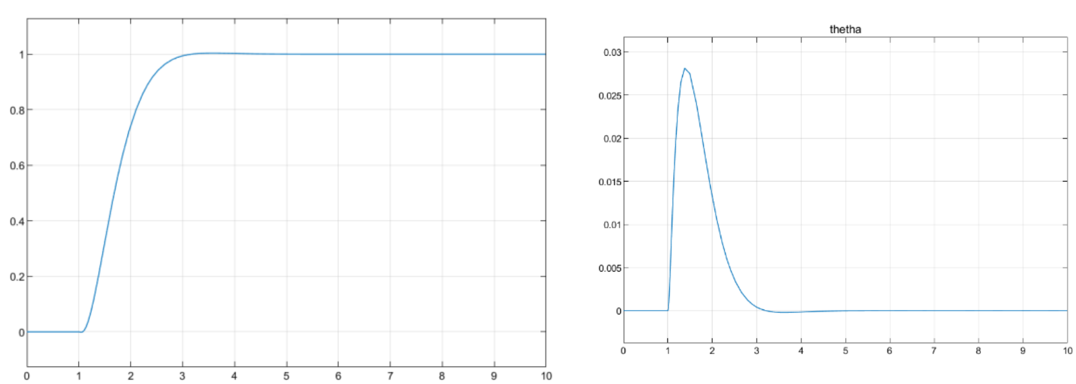

Figure 8 - shows the output response and the theta response with K_1 of 30 and K_3 of 5

Figure 8 shows the step response and the theta outcome with the adjust gain value of 30 for K_1 and 5 for K_3. The step response has no steady-state error with no overshoot and a settling time of 3 seconds. Adjusting the gain values allowed the system to cope with the effect of change in the speed.

Block diagram & step response of the non-linear system

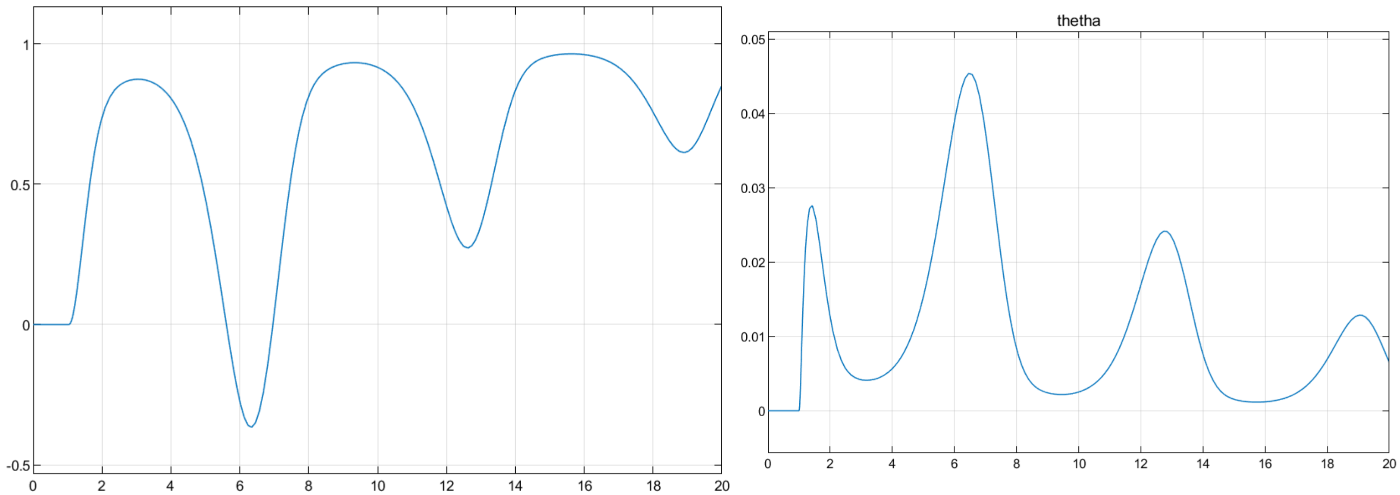

Figure 9 - Step response of z & theta of the non-linearized system

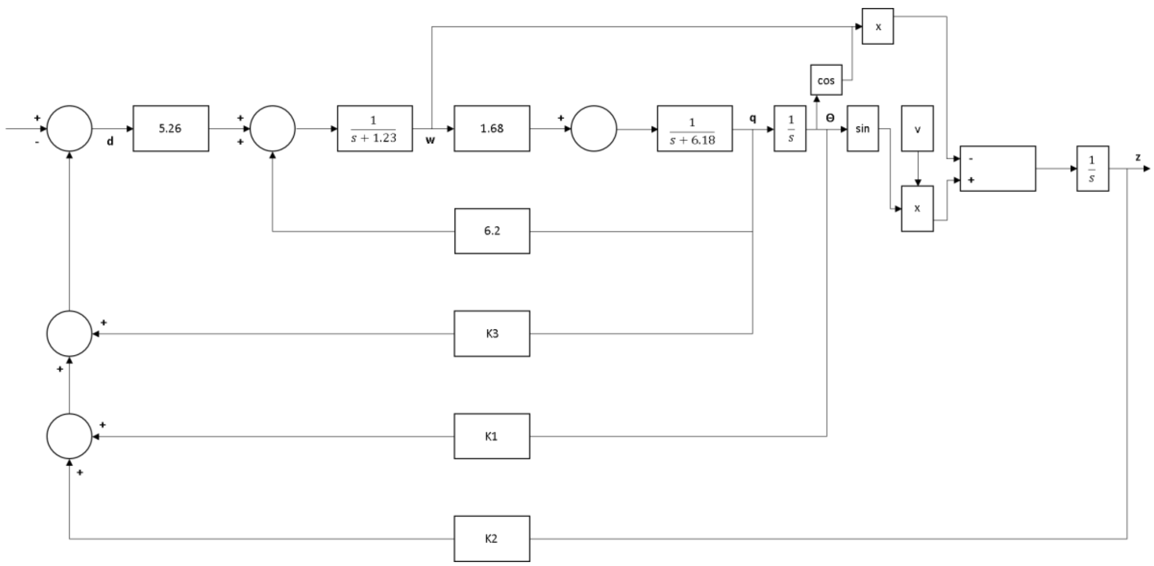

Figure 10 - Shows the block diagram of the non-linearized system

The block diagram was implemented in the MATLAB Simulink and it was deduced from Figure 9 that linearized system only considered the linearized section of the graph (the section with the positive slope) and ignored the rest since theta was capped between $\pm 5°$ in the linearized system. This is also apparent from the non-linearized theta graph shown in Figure 9 since the theta value changes with time whereas the linearized theta graph shown in Figure 8 has a constant theta value after the peak response. The linearized pendulum system depicted in Equation 3 had a similar limit on θ to keep it linear.

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