Logic Notes

Aaron Anderson

February 26, 2026

Contents

Chapter 1

Introduction and Definable Sets

1.1 Introduction

In this class, we will explore techniques that let us apply logic, and model theory in particular, to everyday mathematics. Let's start in a context that should already be familiar, and will only become moreso: structures in the language of ordered rings.

This is centainly an everyday mathematical context. Structures in the language $\{0, 1, +, \times, \leq\}$ include $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. But what can we actually say about these ordered rings (or in the case of $\mathbb{N}$, a semiring) just using first-order logic in this language?

At the most basic level, we can ask which sentences these different structures satisfy. It's a pretty straightforward exercise to determine that these structures differ already at that level.

Exercise 1.1.1. For each pair of structures in the list $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, find a sentence that one satisfies, while the other does not.

Meanwhile, if we change the language by dropping multiplication (or down to just $\leq$), we find that in the languages of linear orders $\{\leq\}$ and of ordered (semi)groups $\{0, +, \leq\}$, we can distinguish $\mathbb{N}$ from $\mathbb{Z}$ from $\mathbb{Q}$ and $\mathbb{R}$, but $\mathbb{Q}$ and $\mathbb{R}$ satisfy exactly the same sentences.

Definition 1.1.2. Given a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$, let $\mathsf{Th}(\mathcal{M})$ (the *complete theory of* $\mathcal{M}$) denote the set of all $\mathcal{L}$-sentences ϕ such that $\mathcal{M} \models \phi$.
Given $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, say that they are *elementarily equivalent*, denoted $\mathcal{M} \equiv \mathcal{N}$, when $\mathsf{Th}(\mathcal{M}) = \mathsf{Th}(\mathcal{N})$. That is, when for each sentence ϕ, $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$.

To show that $\mathbb{Q}$ and $\mathbb{R}$ are elementarily equivalent in either of the above languages, we use the same strategy: find an easily-axiomatized complete theory that they both model.

Lemma 1.1.3. *If T is a complete theory, and $\mathcal{M} \models T$, then $\mathsf{Th}(\mathcal{M})$ is the set of all consequences of T. Thus if $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \equiv \mathcal{N}$.*

In the case of the language of linear orders, the theory they both model and its completeness may be familiar from 5700:

Fact 1.1.4. *There is a complete theory* DLO *(dense linear orders) in the language* $\{\leq\}$ *of linear orders whose models include* $\mathbb{Q}$ *and* $\mathbb{R}$.

In the language of ordered groups, things get a little trickier. We will prove the following in this class.

Fact 1.1.5. *There is a complete theory* ODAG *(ordered divisible abelian groups) in the language* $\{0, +, \leq\}$ *of ordered groups whose models include* $\mathbb{Q}$ *and* $\mathbb{R}$.

In the rest of this class, we will not care too much about specific structures - we will care more about their complete theories. We will develop tools for proving completeness of such theories, for classifying them based on complexity, and for evaluating formulas and sentences modulo these theories.

We will find that some theories are inherently difficult to understand, because of Gödelian phenomena you have seen in 5700, while others are actually very nice!

Incompleteness gives us some contrived examples of difficult-to-resolve sentences in the structure $(\mathbb{N}; 0, 1, +, \times, \leq)$. Historically, a huge fraction of mathematical effort has been spent trying to resolve the truth of sentences in this structure. To state these, recall that there is a formula in this structure that determines whether a number is prime:

$$\mathsf{Prime}(x) := 1 < x \wedge \forall y, \forall z, (y \times z = x) \to (y = 1 \vee z = 1).$$

Given this, we can state the following sentences:

$$\text{The Twin Primes Conjecture: } \forall n, \exists p, n \leq p \wedge \mathsf{Prime}(p) \wedge \mathsf{Prime}(p+2)$$
$$\text{The Goldbach Conjecture: } \forall n, 1 < n \to \exists p, \exists q, \mathsf{Prime}(p) \wedge \mathsf{Prime}(q) \wedge p + q = n + n.$$

We can view this as a consequence of the *definable set* of primes being somewhat complicated, and the extra quantifiers $(\forall, \exists)$ applied to make it into these sentences drives the complexity higher.

Meanwhile, in $(\mathbb{R}; 0, 1, +, \times, \leq)$, the story is very different. Consider the set in $\mathbb{R}^3$ defined by the formula

$$\phi(a, b, c) \iff \exists x, ax^2 + bx + c = 0.$$

We find that $\phi(a, b, c)$ is equivalent to

$$(a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0)).$$

To evaluate this, we only need to perform a handful of algebraic operations, checks of equality, and boolean operations.

We will find finitely-axiomatized and decidable complete theories, where all of the "paradoxes" of incompleteness are irrelevant. In order to determine which *sentences* are true in a given theory, we will want to find out how to evaluate all formulas, including those with free variables. The main way we will do this is by *eliminating quantifiers*, allowing us to turn the complexity of first-order logic into something tractable, and much closer to propositional logic.

Furthermore, these nice theories come in a variety of different flavors - it is easy to work with vector spaces over a field, and it is easy to work with algebra over the real numbers, but for somewhat different reasons. These subtleties will come out when we look at the combinatorics inherent in these structures.

For all of these purposes, we need to understand formulas, not only syntactically, but semantically in terms of the sets they define.

1.2 Definability

Definition 1.2.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $A \subseteq M$. Then a set $D \subseteq M^n$ is called A-*definable* when there is a formula $\phi(\bar{x};\bar{y}) = \phi(x_1,\dots,x_n,y_1,\dots,y_m)$ and parameters $\bar{b} \in A^m$ such that for all $\bar{a} \in M^n$, $\bar{a} \in D \iff \mathcal{M} \vDash \phi(\bar{a};\bar{b})$.

For some basic examples, let's look at the language $\{0,1,+,\times\}$ of (semi)rings. We will see that in each of the structures $\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{R}$, the set $\{(x,y) : x \leq y\}$ is definable.

This is easy for $\mathbb{N}$: $\exists z, y = x + z$.

This is almost as easy for $\mathbb{R}$: $\exists z, y = x + z^2$.

For $\mathbb{Z}$ and $\mathbb{Q}$, we have to use Lagrange's Four-Square Theorem: Every $n \in \mathbb{Z}$ with $n \geq 0$ can be written as the sum of four perfect squares. Thus for $\mathbb{Z}$, we can use

$$\exists z_1, z_2, z_3, z_4, y = x + z_1^2 + z_2^2 + z_3^2 + z_4^2.$$

This will actually also work in $\mathbb{Q}$ - exercise if you want!

This means that in each of these structures, everything definable in the language $\{0,1,+,\times,\leq\}$ is already definable without the symbol $\leq$. That's because if there's a formula $\phi(x;y)$ in the ring language equivalent to $x \leq y$, we can just replace every instance of $t_1 \leq t_2$ with $\phi(t_1;t_2)$.

This is an instance of an *expansion by definitions*, which is where we add a symbol whose interpretation is already definable to the language. This does not change the definable sets.

Another example of expansion by definition is in arithmetic. In $(\mathbb{N};0,1,+,\times,\leq)$, the exponentiation operation is definable, so we frequently add it to the language for convenience.

1.2.1 Undefinability

It can be harder to show that sets are *not* definable, but this is just as informative.

In the realm of arithmetic, we will be able to show undefinability by diagonalization:

Fact 1.2.2 (Tarski's Undefinability of Truth (Simple Version))**.** *The set*

$$\{\ulcorner\phi\urcorner : \phi \textit{ is a sentence such that } (\mathbb{N};0,1,+,\times,\leq) \vDash \phi\}$$

of natural numbers is not definable in $(\mathbb{N};0,1,+,\times,\leq)$.

Other structures that model theorists prefer tend to lack the expressive power to even do this diagonalization proof. In these cases, we will be able to place limits on definability by proving structural theorems about definable sets.

This starts by proving *quantifier elimination* in an appropriate language. That is, we will show that, after possibly expanding by definitions a little bit, every formula is equivalent to one that can be written without $\forall$ or $\exists$. This is true, for instance, in the familiar structures

$$\begin{gathered}(\mathbb{Q},<)\\(\mathbb{R},<)\\(\mathbb{Z},0,+,\leq)\\(\mathbb{C},0,1,+,\times)\\(\mathbb{R},0,1,+,\times,\leq),\end{gathered}$$

as we will show.

Because formulas without quantifiers are much easier to study, quantifier elimination will allow us to characterize definable sets quite easily. For instance, over the complex numbers (or any other algebraically closed field):

Fact 1.2.3. *The quantifier-free definable sets (with parameters) in* $(\mathbb{C}; 0, 1, +, \times)$ *are exactly the* constructible *sets - that is, boolean combinations of zerosets of polynomials with coefficients in* $\mathbb{C}$.

Once we characterize definable sets in this way, we can start proving more interesting properties of definable sets, and easily show that other sets are not definable. Staying with the algebraically closed field example:

Corollary 1.2.4. *The structure* $(\mathbb{C}; 0, 1, +, \times)$ *is* strongly minimal*: any definable subset of* $\mathbb{C}$ *(in one dimension) is either finite or cofinite.*

Proof. Zerosets of polynomials are either finite or cofinite, and any boolean combination of finite and cofinite sets is also finite or cofinite. □

Corollary 1.2.5. *There is no definable linear order on* $\mathbb{C}$*. In fact,* $\mathbb{C}$ *is stable - there cannot be an infinite linear order* I*, sequences* $(a_i, b_i : i \in I)$ *with* $a_i \in \mathbb{C}^m, b_i \in \mathbb{C}^n$ *and a formula* $\phi(x, y)$*, even with parameters, such that*

$$i \leq j \iff \mathbb{C} \vDash \phi(a_i, b_j).$$

1.3 Overview

Our first objective in this course is to build up a library of easy-to-understand structures. We will start with countable structures we can explicitly construct from finite structures: *Fraïssé limits*. These include structures such as $(\mathbb{Q}, <)$, the random graph, and the countable atomless Boolean algebra. We will learn how to show completeness, $\aleph_0$-categoricity, and then quantifier elimination, for theories of Fraïssé limits, building on the back-and-forth technique mentioned in 5700.

We will then develop a more comprehensive toolkit for showing quantifier elimination in more complicated structures, such as algebraically closed and real closed fields, at which point we can really begin applying model theory to these structures.

Once we have seen how simple definable sets can be, we will contrast with how *complicated* they can be when we don't have quantifier elimination, such as in arithmetic.

We will also review compactness somewhere around here, including a semantic proof featuring the ultraproduct construction.

Then we will pick up the story of definable sets in specifically *ordered* structures such as $(\mathbb{R}; 0, 1, +, \times, \leq)$. We will see how *o-minimality*: the simplest case for *one-dimensional* definable sets in an ordered structure, implies a powerful structural theorem (the *cell decomposition theorem*) for definable sets in all dimensions, even without quantifier elimination. This will let us work with structures such as $\mathbb{R}_{\text{an,exp}}$, which at the moment is the most fruitful context for applying model theory to other branches of math.

We then have some choices for where to go next. Some of my ideas include the following:

- Dimension theory, and in particular, pregeometries/matroids, in strongly minimal and o-minimal structures

- Incidence combinatorics and distal cell decompositions (the combinatorics of definable sets over $(\mathbb{R}; 0, 1, +, \times, \leq)$)
- NIP, VC-dimension and connections with statistical learning theory.

Please let me know if you have preferences about what you'd like me to cover.

If I *don't* get to cover what you want, I have good news: you can cover it yourself! For students enrolled in graduate class course numbers (perhaps we will change the exact mechanism, but certainly for graduate students who want to), there is a presentation option for grading in this class, requiring one (or possibly two if time allows) 45-minute presentations. I have a list of suggested papers for presenting on the course website, and will add to it over time.

Chapter 2

Fraïssé Limits

2.1 Fraïssé Classes

Before we get into the weeds of model theory, we should spend some time developing a library of examples. These can include famous structures, like the algebraic ones we have seen so far, but should also include complete theories.

We will start with a method for constructing particularly nice countable structures, called *Fraïssé limits*. These are constructed as limits of families of finite substructures. To ground *this* construction in an example to start, recall dense linear orders from 5700:

Fact 2.1.1. *Let* $\mathcal{L}_< = \{<\}$.
The $\mathcal{L}_<$*-theory of* dense linear orders without endpoints*, abbreviated* DLO*, is complete,* $\aleph_0$*-categorical, and* $(\mathbb{Q}, <) \vDash \mathsf{DLO}$.

To generate a structure like $(\mathbb{Q}, <)$, we start by looking at its finite substructures.

Definition 2.1.2. If $\mathcal{L}$ is a relational language (that is, has no function symbols), and $\mathcal{M}$ is an $\mathcal{L}$-structure, let $\mathsf{Age}(\mathcal{M})$ be the class of all $\mathcal{L}$-structures isomorphic to a finite substructure of $\mathcal{M}$.

As described, the age is a proper class. If you don't like this, you can use the set of *isomorphism types* of finite substructures of $\mathcal{M}$ instead.

Example 2.1.3. $\mathsf{Age}(\mathbb{Q}, <)$ consists of all finite linear orders.

Proof. Any finite linear order is isomorphic to any other finite linear order of the same cardinality, and for every finite cardinality n, $(\mathbb{Q}, <)$ contains a finite subset, and thus a finite substructure which is a linear order, of cardinality n. Also, any structure in $\mathsf{Age}(\mathbb{Q}, <)$ must be isomorphic to a finite substructure, and thus must be a finite linear order. $\square$

We can make a few observations about the class of all finite linear orders, which we will describe as properties:

Example 2.1.4. Let $\mathcal{K}$ be the class of all finite linear orders.

- **Essential Countability (EC):** Up to isomorphism, there are only countably many structures in $\mathcal{K}$.
- **Hereditary Property (HP):** If $A \in \mathcal{K}$, and B is a finite substructure of A, then $B \in \mathcal{K}$.
- **Joint Embedding Property (JEP):** If $A, B \in \mathcal{K}$, then there is some $C \in \mathcal{K}$ into which both A and B embed.
- **Amalgamation Property (AP):** If $A, B_1, B_2 \in \mathcal{K}$, and there are embeddings $f_i : A \hookrightarrow B_i$, then there is $C \in \mathcal{K}$ with embeddings $g_i : B_i \hookrightarrow C$ such that $g_1 \circ f_1 = g_2 \circ f_2$, making the following diagram commute:

$$\begin{tikzcd} & C & \\ B_1 \arrow[ur, "g_1"] & & B_2 \arrow[ul, "g_2"'] \\ & A \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

In fact, the first three of these hold for the age of *any* countable structure.

Proof. We will prove the first three for $\mathsf{Age}(\mathcal{M})$ where $\mathcal{M}$ is an arbitrary countable structure.

- **EC:** A countable structure has only countably many finite subsets, and thus countably many finite substructures.
- **HP:** Any finite substructure of a finite substructure of $\mathcal{M}$ is also a finite substructure of $\mathcal{M}$.
- **JEP:** If A, B are finite substructures of $\mathcal{M}$, then (in a relational language) $A \cup B$ is also a finite substructure, into which both embed.
- **AP:** This one we're only proving for $\mathcal{K} = \mathsf{Age}(\mathbb{Q}, <)$. Enumerate A as $a_1 < a_2 < \cdots < a_n$. Then to define C, we will place elements in the gaps between elements of A. To extend the embedding f_1, we need to make sure that there are at least as many elements of C between a_i, a_{i+1}, as there are in B_1 between $f_1(a_i)$ and $f_1(a_{i+1})$, and similarly for f_2.

□

Definition 2.1.5. When $\mathcal{L}$ is a finite relational language, we call any class $\mathcal{K}$ of finite $\mathcal{L}$-structures a *Fraïssé class* when it satisfies EC, HP, JEP, and AP.

We now turn to another familiar example of a Fraïssë class of finite structures: all finite graphs.

Theorem 2.1.6. *The class of all finite graphs is a Fraïssë class.*

Proof.
- **EC:** For any n, there are finitely many graphs on n vertices up to isomorphism, so there are countably many when we union over all $n \in \mathbb{N}$.

- **HP:** A substructure of a graph is a graph.
- **JEP:** We can just put the two graphs next to each other, and choose arbitrarily whether to put edges between the two graphs.
- **AP:** If A embeds into B_1 and B_2, then we can add both sets of vertices $B_1 \setminus A$ and $B_2 \setminus A$ to A. We know which edges we need between elements of A and B_i, and can choose what edges to put between B_1 and B_2 arbitrarily.

□

These are the two most canonical examples - there are others, but many of those require adding function symbols into the language, which makes things a little more complicated. This complicates the definitions slightly, but the idea of everything we do can be extended to that with a few more assumptions.

Exercise 2.1.7. Show that the class of all finite *triangle-free* graphs is a Fraïssé class.

2.2 Fraïssé Limits

Now that we've noticed that $(\mathbb{Q}, <)$ is a countable structure whose age is a Fraïssé class, I can explain why this structure is special. After all, $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$ are also countable linear orders with the same age, but their theories are not $\aleph_0$-categorical.

The critical idea is *homogeneity*:

Definition 2.2.1. Call a structure $\mathcal{M}$ *ultrahomogeneous* when for any finite substructures A, B of $\mathcal{M}$ and an isomorphism $f : A \to B$, there is an isomorphism $g : M \to M$ that extends f.
If $\mathcal{L}$ is a relational language, $\mathcal{K}$ is a class of finite $\mathcal{L}$-structures, and $\mathcal{M}$ is a countably infinite ultrahomogeneous $\mathcal{L}$-structure whose age is $\mathcal{K}$, we call $\mathcal{M}$ a *Fraïssé limit* of $\mathcal{K}$.

Example 2.2.2. It is not hard to check that $(\mathbb{Q}, <)$ is ultrahomogeneous, and thus a Fraïssé limit for the class of finite linear orders.

We can start to connect Fraïssé limits to Fraïssé classes:

Theorem 2.2.3. *If $\mathcal{K}$ is a class of finite structures in a relational language with a Fraïssé limit $\mathcal{M}$, then $\mathcal{K}$ is a Fraïssé class.*

Proof. Because $\mathcal{K} = \mathsf{Age}(\mathcal{M})$, it must satisfy **EC**, **HP**, and **JEP**, so we just need to check **AP**. Suppose A, B_1, B_2 are isomorphic to finite substructures of $\mathcal{M}$, and $f_i : A \to B_i$ are embeddings. We can assume (up to an isomorphism of everything involved) that A, B_1, B_2 are actual substructures of $\mathcal{M}$, and that f_1 is the inclusion map of A into B_1, but we can't simultaneously assume that A is a substructure of B_2, or that f_2 is an inclusion.
We can view f_2 as an isomorphism from A to its image, a substructure of B_2, and by ultrahomogeneity, we can let $h : M \to M$ be an automorphism extending f_2. Then $h^{-1} \circ f_2$ is the inclusion map of A into M, and its image is contained in $h^{-1}(B_2)$. We now let C be a finite substructure containing both B_1 and $h^{-1}(B_2)$. We then let $g_1 : B_1 \hookrightarrow C$ be the inclusion map, and let $g_2 : B_2 \hookrightarrow C$ be the inclusion map ι composed with the restriction of h^{-1} to B_2. Then $g_1 \circ f_1$ is the inclusion map of A into C, while in $g_2 \circ f_2$, the restriction of h^{-1} to B_2 cancels with

f_2 to form another inclusion map, giving us the same map in the end. □

Meanwhile, to check that something is a Fraïssé limit, we can check homogeneity step-by-step, in a back-and-forth procedure that may look familiar:

Lemma 2.2.4. *A countable structure $\mathcal{M}$ is ultrahomogeneous if and only if the following holds: For any isomorphism $f : A \to B$ of finite substructures of $\mathcal{M}$, and any finite substructure $A \subseteq C$, the isomorphism f extends to an isomorphism $g : C \to D$ of finite substructures.*

Proof. If $\mathcal{M}$ is ultrahomogeneous, then given $f : A \to B$ and C, we simply let $h : M \to M$ extend f, and then restrict h to C.
For the main direction, we're actually going to prove something a smidgen more general:

Lemma 2.2.5. *Suppose $\mathcal{M}, \mathcal{N}$ are countable structures such that for every isomorphism $f : A \to B$ of finite substructures $A \subset M, B \subset N$, and any finite substructure $A \subseteq C \subset M$, the isomorphism f extends to an isomorphism $g : C \to D \leq N$ of finite substructures, and the same holds with $\mathcal{M}, \mathcal{N}$ switched.*
Then f extends to an isomorphism $h : M \to N$.

All we need for ultrahomogeneity is to apply this to $N = M$.
Now assume $\mathcal{M}, \mathcal{N}$ has this property, and let $f : A \to B$ be an isomorphism of finite substructures. We can construct an isomorphism $h : M \to N$ extending f recursively, as the union of isomorphisms $f_k : A_k \to B_k$, each of which extends the last, where $\bigcup_k A_k = M$ and $\bigcup_k B_k = N$. Enumerate $M = \{m_0, m_1, \dots\}, N = \{n_0, n_1, \dots\}$. We start with $f_0 = f$. Then we recurse a bit differently in even and odd steps. Assuming we have defined $f_{2k} : A_{2k} \to B_{2k}$, we want to make sure that A_{2k+1} contains m_k, which in the long run, will ensure that $\bigcup_k A_k = M$. If $m_k \in A_{2k}$, we don't need to do anything. If not, then let A_{2k+1} be a finite substructure with $A_{2k} \cup \{m_k\} \subseteq A'$. By our assumption, we can then extend to an isomorphism $f : A_{2k+1} \to B_{2k+1}$. Now assuming we have $f_{2k+1} : A_{2k+1} \to B_{2k+1}$ defined, let's make sure that B_{2k+2} contains n_k, ensuring that $\bigcup_k B_k = N$. We pick B_{2k+2} containing $B_{2k+1} \cup \{n_k\}$, then we extend f_{2k+1}^{-1} to an isomorphism $B_{2k+2} \to A_{2k+2}$, whose inverse extends f_{2k+1}. □

2.3 Random Graphs

Let's advance our other example. We had another Fraïssé class, the class of finite graphs. Does this have a Fraïssé limit? If we have no idea how to build one, we may as well try to build it at random. Let's just take a countably infinite set of vertices, and for each pair of vertices, we'll flip a coin to determine whether there should be an edge.

This generates a random countably infinite graph - what's its age? For any finite graph A with n vertices, the probability of a given n vertices forming an isomorphic graph is going to be positive. In fact, it's at least $2^{-\binom{n}{2}}$, because that's the probability of each precise configuration of edges and non-edges. Just call it $p > 0$.

Then we can bound the probability that *no* set of n vertices in the infinite random graph is isomorphic to A. To simplify things, let's split the vertices into an infinite sequence of disjoint sets of n vertices. Then let E_k be the event that the kth set is isomorphic to A - this is p. Then the probability that *none* of these events happen is $\lim_{k\to\infty}(1-p)^k = 0$, so with probability 1, at least one of these events happens, so A is in the age.

Taking the intersection over countably many graphs, we see that the probability of *every* finite graph being in the age is also 1.

But what's the probability of ultrahomogeneity? By (Lemma 2.2.4), to have homogeneity, we just need to check that for every isomorphism between finite subgraphs, and every extension of one to a larger finite subgraph, the isomorphism extends. There are countably many choices of $f : A \to B$ and $C \geq A$, so let's find the probability of each one extending. We just need to find $|C \setminus A|$ vertices that have the same relationship to the vertices of B, and to each other, as the vertices of $C \setminus A$ have to A. The probability of this happening, for any ordered list of $|C \setminus A|$ vertices, is positive, as it is 2^{-n}, where n is the number of edges we need. As before, the probability of this positive probability failing for each of an infinite sequence of independent ordered lists is 0, so with probability 1, we can extend this map. The probability that each of these countably many probability-1 properties holds is 1, so with probabilistic certainty, we have generated a Fraïssé limit, completely at random.

2.4 Uniqueness

We know that $(\mathbb{Q}, <)$ is a Fraïssé limit of the class of finite linear orders, but what about other countable structures with that age?

We can rule out $(\mathbb{N}, <)$, as any automorphism will leave 0 as the left endpoint This means that the substructure $\{0\}$ can't be mapped by an automorphism to any of the isomorphic substructures $\{n\}$. In general, any Fraïssé limit of this class must not have a left endpoint, or for that matter, a right endpoint.

We can also rule out $(\mathbb{Z}, <)$, as any automorphism will leave consecutive elements consecutive. This means that the substructure $\{0, 1\}$ can't be mapped by an automorphism to the isomorphic substructure $\{0, 2\}$.

We can extend that argument. Let $\mathcal{M}$ is a Fraïssé limit of this class with elements $a < b$. As there is no right endpoint, let $a < b < c$. As $\{a, b\}$ and $\{a, c\}$ are isomorphic, there must be an automorphism $h : M \to M$ with $h(a) = a$ and $h(c) = b$. It must thus send b to some $h(b)$ with $a = h(a) < h(b) < h(c) = b$, showing that $\mathcal{M}$ is dense.

We have thus shown that any Fraïssé limit of this class is a countable model of DLO - we know there's only one of these up to isomorphism. We will now provide another proof of this which works much more generally.

Theorem 2.4.1 (Uniqueness of Fraïssé limits)**.** *If $\mathcal{M}, \mathcal{N}$ are both Fraïssé limits of a class $\mathcal{K}$, then they are isomorphic.*
In fact, if $f : A \to B$ is an isomorphism between a finite substructure of $\mathcal{M}$ and a finite substructure of $\mathcal{N}$, then there is an isomorphism extending f.

Proof. By (Lemma 2.2.5) and symmetry, we only need to check that if $C \supset A$ is a finite substructure of $\mathcal{M}$, then f extends to C.
We know by amalgamation that there is some $D \in \mathcal{K}$, with embeddings $g_B : B \to D$ and $g_C : C \to D$, such that $g_B \circ f$ equals the composition of g_C with the inclusion map, which means g_C extends $g_B \circ f$. We can shrink D to be the range of g_C, in which case g_C is an isomorphism. This would be perfect, with g_C extending f, if $B \subseteq D \subset N$. All we know is that D is isomorphic to a substructure of N - but we may as well assume it actually is one, because composing with that isomorphism won't change anything so far. Then g_B, restricted to its image, is an isomorphism $B \to g_B(B) \subseteq D$, which must extend to an automorphism $h : N \to N$. Thus the substructure we're looking for is $B \subseteq h^{-1}(D) \subset N$, as $h^{-1} \circ g_B$ is just the inclusion map, and then $h^{-1} \circ g_C$, appropriately restricted, gives an isomorphism $C \to h^{-1}(D)$ extending $h^{-1} \circ g_B \circ f$, and thus f. □

To the category theory enthusiasts in the audience, note that this uniqueness is *not* uniqueness up to unique isomorphism. Fraïssé theory is otherwise very categorical, but this provides an unusual wrinkle in categorical presentations of this topic. If you're interested in giving a presentation on the categorical aspects of Fraïssé theory later in the semester, I have some recommended papers on the course website.

2.5 Examples

Example 2.5.1. The class of finite equivalence relations (in the language $\{E\}$ of one binary relation symbol for equivalence) is Fraïssé, and its Fraïssé limit is the equivalence relation with $\aleph_0$ many $\aleph_0$-sized classes.

Example 2.5.2. The class of finite triangle-free graphs has a Fraïssé limit, one of a family of *Henson graphs*.
One can construct this by ordering the random graph in a particular way, and removing the third vertex of every triangle that appears. (We will see a general Fraïssé-theoretic construction shortly.)

2.5.1 Now With Function Symbols

Most of the construction we've done so far works with function symbols also, although we'll have to reckon with the fact that not every subset is a substructure. For full generality, we'd have to work with *finitely generated* rather than finite substructures, but here are some examples that don't need that:

Example 2.5.3. The class of finite fields of characteristic p is Fraïssé, and its Fraïssé limit is the algebraic closure of F_p.

Example 2.5.4. If K is a finite field, then the class of finite-dimensional K-vector spaces is Fraïssé, and the $\aleph_0$-dimensional K-vector space is its Fraïssé limit.

Example 2.5.5. The class of finite boolean algebras (in the language $\{\top, \bot, \wedge, \vee, \cdot^c\}$) is Fraïssé, and it has a Fraïssé limit, the unique countable atomless boolean algebra.

Example 2.5.6. The classes of finite groups and finite abelian p-groups are also Fraïssé, although their limits are harder to introduce.

Example 2.5.7. If we generalize vastly to allow *continuous logic*, there are quite a few more familiar examples, such as

- the Urysohn space is the Fraïssé limit of finite metric spaces
- the separable Hilbert space is the Fraïssé limit of finite-dimensional/Euclidean Hilbert spaces
- $[0, 1]$ with Lebesgue measure is the Fraïssé limit of finite probability algebras
- various interesting Banach spaces are also Fraïssé limits

Take note that some things will be more complicated - only some of these have $\aleph_0$-categorical theories.

2.6 Existence

Now let's actually construct Fraïssé limits in general.

Theorem 2.6.1 (Existence of Fraïssé limits)**.** *If $\mathcal{K}$ is a Fraïssé class, then it has a Fraïssé limit.*

Proof. We will prove this assuming a relational language, just for simplicity.
We will construct this as a direct limit (union) of a chain of structures in $\mathcal{K}$. If we have a chain

$$D_0 \hookrightarrow D_1 \hookrightarrow D_2 \hookrightarrow \dots$$

of structures, and we think of each embedding as inclusion, then it is easy to check that $\bigcup_n D_n$ is also a structure in a sensible way.

Lemma 2.6.2. *In this construction,* $\mathsf{Age}(\bigcup_n D_n) = \bigcup_n \mathsf{Age}(D_n)$.

Proof. Any finite subset of $\bigcup_n D_n$ must already be contained in some D_n, and is thus already a finite substructure of some element of the chain. $\square$

Thus to get the correct age, we only need to make sure that each element of $\mathcal{K}$ embeds into some element of our chain.
To get ultrahomogeneity, we need to make sure that for each D_n, and all substructures $A, B, C \subseteq D_n$ with $A \subseteq C$ and $f : A \to B$ an isomorphism, f can be extended to an isomorphism from C to some other substructure of the direct limit. For this, we will need to make sure that for some $m > n$, f can be extended to an isomorphism from C to another substructure of D_m.
To do all this, use essential countability to enumerate a representative of each isomorphism type in $\mathcal{K}$ as $A_0, A_1, A_2, \dots$. Then for each structure $D \in \mathcal{K}$, enumerate the triples (f, A, C) where $A \subseteq C$ are substructures and f is an isomorphism from A to another substructure. as $T(D)$.
Now fix a bijection $\pi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that if $\pi(n) = (j, k)$, we always have $j \leq n$.
Let $D_0 \in \mathcal{K}$. For our recursive construction, assume we have defined the chain through D_n. Then to define D_{n+1} extending D_n, let $(j, k) = \pi(n+1)$. We will make sure that A_n embeds into D_{n+1}, and that if (f, A, C) is the kth triple in $T(D_j)$, then f extends to an isomorphism from C into another substructure of D_{n+1}.
To do this, we first use JEP to find E into which both D_n and A_n embed, and for notational purposes, assume $D_n \subseteq E$. Then f embeds A into D_n, and thus into E, while inclusion embeds A into C. By AP, we can find embed C and E into another structure F making this square commute, and we may assume that the map from E to F is inclusion. We then let $D_{n+1} = F$. $\square$

2.7 Axiomatizability and Categoricity

So, how do we get from all of this to the theory DLO being complete?

Lemma 2.7.1. *Any countable model of* DLO *is a Fraïssé limit for the class of finite linear orders.*

Proof. Any infinite linear order has this age, so it suffices to check ultrahomogeneity.
Suppose $\mathcal{M} \models \mathsf{DLO}$, $f : A \to B$ is an isomorphism of finite substructures, and $C \supseteq A$ is a finite substructure. Then we can extend the isomorphism one element at a time. To add $c \notin A$ to the domain A, we first see how it compares to A in the order. It must be in some interval with endpoints in A, either between consecutive elements, or above or below all elements of A. In each of these cases, by density, we can find an element d that fits in the corresponding interval of B by density, and by sending $c \mapsto d$, we can extend the isomorphism. □

Recall that if κ is an infinite cardinal, a theory is *κ-categorical* when it has precisely one model of cardinality κ up to isomorphism.

Theorem 2.7.2. DLO *is $\aleph_0$-categorical.*

Proof. Any two models of cardinality $\aleph_0$ are Fraïssé limits for the same class, and are thus isomorphic. □

This also gives us completeness.

Theorem 2.7.3 (Łoś-Vaught Test)**.** *If an $\mathcal{L}$-theory T is κ-categorical, and $\kappa \geq \aleph_0, |\mathcal{L}|$, then T is complete.*

Proof. Let $\mathcal{M}, \mathcal{N} \models T$ be two models. We can then find, by Löwenheim-Skolem, two models $\mathcal{M}', \mathcal{N}' \models T$ with $\mathcal{M}' \equiv \mathcal{M}, \mathcal{N}' \equiv \mathcal{N}$, of size κ. These must be isomorphic, and thus elementarily equivalent, so

$$\mathcal{M} \equiv \mathcal{M}' \equiv \mathcal{N}' \equiv \mathcal{N}.$$

□

What we can generalize from this is that if there is a theory whose countable models are precisely the Fraïssé limits of $\mathcal{K}$, then that theory is $\aleph_0$-categorical and thus complete.
We will now examine when this is possible.

Lemma 2.7.4. *Let $\mathcal{L}$ be a finite language, and $A = \{a_1, \dots, a_n\}$ a finite $\mathcal{L}$-structure. Then there is a formula $\phi(\bar{x})$ with $\bar{x} = (x_1, \dots, x_n)$ such that in any $\mathcal{L}$-structure $\mathcal{M}$, if $\bar{m} \in M^n$, then $\mathcal{M} \models \phi(\bar{m})$ if and only if the map $a_i \mapsto m_i$ defines an isomorphism from A to the set $\{m_1, \dots, m_n\}$, which is a substructure.*
Furthermore, ϕ doesn't use any quantifiers.

Proof. We need to construct $\phi(\bar{x})$ to check that each of the finitely many symbols of $\mathcal{L}$ is respected. We can write down a formula checking this for each symbol separately, and then take a conjunction.
For a relation symbol $R(x_1, \dots, x_k)$, we will build a large conjunction. For each of the n^k tuples $(i_1, \dots, i_k) \in [n]^k$, if $A \models R(a_{i_1}, \dots, a_{i_k})$, we include the literal $R(x_{i_1}, \dots, x_{i_k})$, and if not, we include the literal $\neg R(x_{i_1}, \dots, x_{i_k})$.
For a function symbol $f(x_1, \dots, x_k)$, we still build a large conjunction. For each of the n^k tuples $(i_1, \dots, i_k) \in [n]^k$, there is a unique $j \in [n]$ such that if $A \models f(a_{i_1}, \dots, a_{i_k}) = a_j$, so we include the literal $f(x_{i_1}, \dots, x_{i_k}) = x_j$. □

Theorem 2.7.5. *Let $\mathcal{L}$ be a finite language, and let $\mathcal{K}$ be a Fraïssé class of finite $\mathcal{L}$-structures that is* uniformly locally finite*: there is a function $F : \mathbb{N} \to \mathbb{N}$ such that for any n and any subset A of a structure $C \in \mathcal{K}$, there is a substructure B of C with $A \subseteq B \subseteq C$ and $|B| \leq F(|A|)$.*
Then there is a theory T whose models are precisely Fraïssé limits of $\mathcal{K}$.

Proof. First, we will want to limit the age of a model of T. One consequence of uniform local finiteness is that for each n, there is a finite subclass of $\mathcal{K}$ (say, those of size at most $F(n)$), which we call $\mathcal{K}_n$, such that for any n elements in a structure in $\mathcal{K}$, or with age a subsclass of $\mathcal{K}$, those n elements must belong to a substructure isomorphic to one in $\mathcal{K}_n$.
We can then add to T the sentences, for each n,

$$\forall \bar{x}, \bigvee_{A \in \mathcal{K}} \exists \bar{y} \phi_A(\bar{x}, \bar{y}),$$

where $|\bar{x}| = n$, and $\phi_A(\bar{x}, \bar{y})$ is the formula indicating that the elements of $\bar{x}, \bar{y}$ enumerate a structure isomorphic to A.
Any finite tuple of elements in a model of T will now have to be contained in a substructure isomorphic to one of $\mathcal{K}$, ensuring that the age of this model is a subclass of $\mathcal{K}$.
To make sure the age is *all* of $\mathcal{K}$, we can add the sentences

$$\exists \bar{x}, \phi_A(\bar{x})$$

for each $A \in \mathcal{K}$.
Now we wish to ensure ultrahomogeneity. We need to show that if $A \subseteq C \in \mathcal{K}$, then any substructure isomorphic to A extends to one isomorphic to C. We can do this with the sentence

$$\forall \bar{x}, \phi_A(\bar{x}) \to \exists \bar{y}, \phi_C(\bar{x}, \bar{y}).$$

□

Corollary 2.7.6. *Each of the following Fraïssé limits is the unique countable model of a complete, $\aleph_0$-categorical theory:*

- *the countable random graph*
- *the $\aleph_0$-dimensional K-vector space whenever K is a finite field*
- *the countable atomless boolean algebra.*

We should also note that the *quantifiers* $(\forall, \exists)$ appeared in every sentence of these theories in about the same way. Every single sentence can be written (with perhaps a bit of rewriting effort) as $\forall \bar{x}, \exists \bar{y}, \phi(\bar{x}, \bar{y})$, where ϕ has no quantifiers at all.

Chapter 3

Quantifier Elimination

Most of this chapter is based on the approach in [3].

3.1 Quantifier-Free Formulas

Formulas and definable sets can be complicated, but *quantifier-free* formulas and definable sets are much simpler to work with.

Definition 3.1.1. A formula $\phi(\bar{x})$ is *quantifier-free* when, as a string, it does not contain the symbols $\exists$ and $\forall$.
A set $D \subseteq M^n$ is *quantifier-free definable* (with parameters in $A \subseteq M$) when there is a quantifier-free formula $\phi(\bar{x})$ with parameters from A such that $\bar{a} \in D \iff M \vDash \phi(\bar{a})$.

Lemma 3.1.2. *Any quantifier-free formula is equivalent to a boolean combination of atomic formulas.*

Proof. Exercise. □

Quantifier-free definable sets in algebra are easy to understand:

Lemma 3.1.3. *The quantifier-free definable sets (with parameters) in a (semi)ring $(R; 0, 1, +, \times)$ are exactly the* constructible *sets - that is, boolean combinations of zerosets of polynomials with coefficients in R.*

Proof. These sets are precisely the boolean combinations of sets definable by atomic formulas. As the only relation is $=$, we find that the only atomic formulas are of the forms $t_1(\bar{x}) = t_2(\bar{x})$ Each of these terms evaluates to a polynomial - if we allow all parameters, these can be any polynomial in $R[\bar{x}]$. □

Many of the definable sets we care about are not quantifier-free definable, even with parameters. Recall that the set of prime numbers is defined in $(\mathbb{N}; 0, 1, +, \times, \leq)$ by

$$\mathsf{Prime}(x) := 1 < x \land \forall y, \forall z, (y \times z = x) \to (y = 1 \lor z = 1).$$

It certainly seems important to state this with quantifiers, but often formulas with quantifiers can be nonobviously equivalent to quantifier-free formulas. Consider the theory of $(\mathbb{R}; 0, 1, +, \times, \leq)$

and the formulas

$$x \neq 0 \iff \exists y, x \times y = 1.$$

To confirm that the set of primes is not quantifier-free definable, let's briefly explore what quantifier-free definable sets look like in this language.

Lemma 3.1.4. *In the ordered rings* $(\mathbb{R}; 0, 1, +, \times, \leq)$ *and* $(\mathbb{Z}; 0, 1, +, \times, \leq)$ *the only definable subsets of the ring (not its cartesian product) are finite unions of intervals and points.*

Proof. Let $\mathcal{M}$ be one of these structures.
First, let's figure out which subsets are definable by atomic formulas (with parameters). These formulas are all of the form either $t_1(x, \bar{a}) = t_2(x, \bar{a})$ or $t_1(x, \bar{a}) \leq t_2(x, \bar{a})$. By subtracting, we may assume $t_1 = 0$, so each of these formulas is equivalent to one of the form either $t(x, \bar{a}) = 0$ or $t(x, \bar{a}) \geq 0$.
As the term $t(x, \bar{a})$ must be interpreted as a single-variable polynomial in $M[x]$, we see that $t(x, \bar{a}) = 0$ defines a finite set, which is indeed a finite union of intervals and points. If $\mathcal{M} = (\mathbb{R}; 0, 1, +, \times, \leq)$, then $t(x, \bar{a}) \geq 0$ defines a union of intervals whose endpoints satisfy $t(x, \bar{a}) = 0$, which is thus finite. If $\mathcal{M} = (\mathbb{Z}; 0, 1, +, \times, \leq)$, then $t(x, \bar{a}) \geq 0$ defines the intersection of a finite union of intervals in $\mathbb{R}$ with $\mathbb{Z}$, which is also a finite union of intervals.
The set of finite unions of intervals and points is closed under boolean combinations, finishing the proof. $\square$

Applying this lemma to the ordered ring $(\mathbb{Z}; 0, 1, +, \times, \leq)$, in which $\mathbb{N}$ is quantifier-free definable, we see that all quantifier-free definable subsets of $(\mathbb{N}; 0, 1, +, \times, \leq)$ must also be quantifier-free definable subsets of $\mathbb{Z}$, satisfying the result of the lemma.

It is now easy to check that the set of primes is not a finite union of intervals and points, as there are infinitely many primes, but (with the exception of 2 and 3) no two can belong to the same interval contained in the set of primes.

3.2 Quantifier Elimination

Definition 3.2.1. A theory T *eliminates quantifiers*, or *has quantifier elimination (QE)*, if every formula $\phi(\bar{x})$ is equivalent modulo T to some quantifier-free formula $\psi(\bar{x})$.
Equivalently, every $\emptyset$-definable set in a model of T is $\emptyset$-quantifier-free definable.

In such a theory, we will be able to understand formulas and definable sets easily, by eliminating quantifiers first. It may be necessary to add a few extra symbols to the language first, as this definition is language-dependent.

3.3 Theories of Fraïssé Limits

Our first examples of theories with QE come from Fraïssé theory.

Theorem 3.3.1. *Let $\mathcal{L}$ be a finite language, let $\mathcal{K}$ be a Fraïssé class of finite $\mathcal{L}$-structures that is uniformly locally finite. By (Theorem 2.7.5), there is a theory T whose countable models are precisely Fraïssé limits of $\mathcal{K}$. This theory eliminates quantifiers.*

Proof. Let $\mathcal{M} \vDash T$ be countable, and thus a Fraïssé limit of $\mathcal{K}$.
Let $\phi(\bar{x})$ be a formula, where $|\bar{x}| = n$. Recall that there is a finite subclass $\mathcal{K}_n \subseteq \mathcal{K}$ such that

any tuple of n elements in a model of T (or any other structure with age contained in $\mathcal{K}$) is contained in a substructure isomorphic to a structure in $\mathcal{K}_n$. We may furthermore assume that this substructure is the minimal substructure containing these elements, without compromising finiteness. This means that any element of the substructure can be generated from those elements by applying function symbols, which we can summarize as saying that for any element in this substructure, there is some term $t(\bar{x})$ which evaluates to that element.

For each structure $A \in \mathcal{K}_n$, recall that there is a quantifier-free formula $\phi_A(\bar{x})$ such that $\mathcal{M} \vDash \phi_A(\bar{m})$ if and only if $a_i \mapsto m_i$ is an isomorphism of finite structures. If A is the minimal substructure containing a tuple $\bar{b}$, then we may write every element in $A = \{a_1, \dots, a_r\}$ as $a_i = t_i(\bar{b})$. We then see that

$$\phi_{A,\bar{a}}(\bar{x}) = \phi_A(t_1(\bar{x}), \dots, t_r(\bar{x}))$$

is true of $\bar{m} \in M^n$ if and only if $\bar{m}$ generates a substructure isomorphic to A, with the isomorphism sending a_i to m_i.

We now see that for each $\bar{m} \in M^n$, there is some one $A \in \mathcal{K}_n$ such that $\mathcal{M} \vDash \phi_{A,\bar{a}}(\bar{m})$. We may then let $\psi(\bar{x})$ be the disjunction of all of the finitely-many formulas $\phi_{A,\bar{a}}(x)$ where there is some $\bar{m}$ such that both $\mathcal{M} \vDash \phi_{A,\bar{a}}(\bar{m})$ and $\mathcal{M} \vDash \phi(\bar{m})$.

This formula $\psi(\bar{x})$ is quantifier-free, let's check that it is equivalent to $\phi(x)$. For each $\bar{m} \in M^n$, if $\mathcal{M} \vDash \phi(\bar{m})$ holds, then we know that there are some $A, \bar{a}$ for which $\mathcal{M} \vDash \phi_{A,\bar{a}}(\bar{m})$, so $\phi_{A,\bar{a}}(\bar{x})$ is in the disjunction, and $\mathcal{M} \vDash \psi(\bar{m})$. Conversely, if $\mathcal{M} \vDash \phi(\bar{m})$, then it is because there are some $A, \bar{a}$ such that $\mathcal{M} \vDash \phi_{A,\bar{a}}(\bar{m})$ and $\phi_{A,\bar{a}}(\bar{x})$ is in the disjunction. This means there is some $\bar{m}'$ with $\mathcal{M} \vDash \phi_{A,\bar{a}}(\bar{m})$ and $\mathcal{M} \vDash \phi(\bar{m}')$. This means that A is isomorphic to the substructures generated by $\bar{m}, \bar{m}'$, and the isomorphisms send $\bar{a}$ to $\bar{m}, \bar{m}'$ respectively. Thus there is an isomorphism between these substructures sending $\bar{m}$ to $\bar{m}'$, which can be extended to an automorphism of M, showing that $\mathcal{M} \vDash \phi(\bar{m}')$ implies $\mathcal{M} \vDash \phi(\bar{m})$ as desired. ☐

3.4 Tests for Quantifier Elimination

To go beyond Fraïssé theory, we're going to need to develop more tests for quantifier elimination.

Lemma 3.4.1. *Let T be a theory, and assume that for every quantifier-free formula $\theta(\bar{x}, y)$, the formula $\exists y, \theta(\bar{x}, y)$ is equivalent mod T to a quantifier-free formula.*
Then T eliminates quantifiers.

Proof. The set of formulas equivalent to quantifier-free formulas includes all quantifier-free formulas and is closed under boolean combinations, so we only need to show that it is closed under quantification. In fact, as $\forall y, \phi(\bar{x}, y)$ is equivalent to $\neg\exists y, \neg\phi(\bar{x}, y)$, we only need closure under existential quantification, which is our assumption. ☐

To prove our main semantic test for quantifier elimination, we want to see how quantifier-free formulas pass between structures and substructures, and also how they can determine if a structure has a given substructure.

Lemma 3.4.2. *Let $\phi(\bar{x})$ be quantifier-free. If $f : \mathcal{A} \hookrightarrow \mathcal{M}$ is an embedding, then*

$$\mathcal{A} \vDash \phi(\bar{a}) \iff \mathcal{M} \vDash \phi(f(\bar{a})).$$

In particular, if $\mathcal{A}$ is a substructure of $\mathcal{M}$, then

$$\mathcal{A} \vDash \phi(\bar{a}) \iff \mathcal{M} \vDash \phi(\bar{a}).$$

Proof. We can check this by induction on the construction of ϕ. This is true for any atomic formula by the definition of an embedding, and is preserved under boolean combinations. $\square$

Lemma 3.4.3. *Let $\mathcal{A}$ be an $\mathcal{L}$-structure generated by $B \subseteq A$, and let $\mathcal{L}_B$ be the language extending $\mathcal{L}$ by additional constants $\{c_b : b \in B\}$. Interpret $\mathcal{A}$ as a $\mathcal{L}_B$-structure so that $c_b^{\mathcal{A}} = b$.*
Then let $\mathsf{Diag}_{\mathcal{A}}$, the atomic diagram *of $\mathcal{A}$, be the theory consisting of all quantifier-free $\phi(\bar{c})$ such that $\mathcal{A} \vDash \phi(\bar{c})$.*
If $\mathcal{M} \vDash \mathsf{Diag}_{\mathcal{A}}$, then $\mathcal{A}$ embeds into $\mathcal{M}$.

Proof. Because each element $a \in \mathcal{A}$ can be written as $t(\bar{c})$ for some term t, we can send $t(\bar{c})^{\mathcal{A}}$ to $t(\bar{c})^{\mathcal{N}}$.

Exercise 3.4.4. Show that this is well-defined and a valid embedding. All requirements will be given by quantifier-free sentences in $\bar{c}$.

$\square$

There are syntactic tests we can try, but here's the semantic test that we will usually apply:

Theorem 3.4.5. *Let T be a theory, $\phi(\bar{x})$ a formula. TFAE:*

1. *$\phi(\bar{x})$ is equivalent to a quantifier-free formula*
2. *Whenever a finitely generated structure $\mathcal{A}$ embeds into both $\mathcal{M}, \mathcal{N} \vDash T$ with embeddings f, g, then for any $\bar{a} \in A^n$, $\mathcal{M} \vDash \phi(f(\bar{a}))$ if and only if $\mathcal{N} \vDash \phi(g(\bar{a}))$.*
3. *Whenever a $\mathcal{M}, \mathcal{N} \vDash T$ admit a common substructure $\mathcal{A}$, then for any $\bar{a} \in A^n$, $\mathcal{M} \vDash \phi(\bar{a})$ if and only if $\mathcal{N} \vDash \phi(\bar{a})$.*

Proof. It is simple to see that 1 implies 2 and 3 - suppose $\phi(\bar{x})$ is equivalent to $\psi(\bar{x})$ which is quantifier-free. Then because ψ is quantifier-free, it doesn't matter which structure we evaluate $\phi(\bar{a})$ in, so

$$\mathcal{M} \vDash \psi(\bar{a}) \iff \mathcal{A} \vDash \psi(\bar{a}) \iff \mathcal{N} \vDash \psi(\bar{a}).$$

Statements 2 and 3 are equivalent, as it doesn't really matter for our statement whether the embeddings are inclusions.
To get 1 from 2, we start by adding extra constant symbols for the variables of $\bar{x}$. Call these $c_1, \ldots, c_n$, and call the tuple of all of them $\bar{c}$. We'll replace $\bar{x}$ with $\bar{c}$, and use the compactness theorem a bunch of times to build the structures $\mathcal{M}, \mathcal{A}, \mathcal{N}$ we need.
Let T_1 be the set of all sentences $\psi(\bar{c})$ where ψ is quantifier-free and $T \vDash \phi(\bar{x}) \to \psi(\bar{x})$. We will prove that $T \cup T_1 \vDash \phi(\bar{c})$ in a bit. From this, we can conclude that a finite subtheory of $T \cup T_1$ also implies $\phi(\bar{c})$, so there is a finite subtheory $T_0 = \{\psi_1(\bar{c}), \ldots, \psi_k(\bar{c})\} \subseteq T_1$ such that $T \cup T_0 \vDash \phi(\bar{c})$ also. Then we find that $T \vDash \bigwedge_i \psi_i(\bar{x}) \to \phi(\bar{x})$, but also, by construction, for each i, $T \vDash \phi(\bar{x}) \to \psi_i(\bar{x})$, so $T \vDash \phi(\bar{x}) \leftrightarrow \bigwedge_i \psi_i(\bar{x})$.
We now finish proving the remaining claim that $T \cup T_1 \vDash \phi(\bar{c})$. To prove this, we show by contradiction that $T \cup T_1 \cup \{\neg\phi(\bar{c})\}$ is inconsistent. If not, then let $\mathcal{M} \vDash T \cup T_1 \cup \{\neg\phi(\bar{c})\}$, and let $\mathcal{A}$ be the substructure generated by $\bar{c}^{\mathcal{M}}$.
We now consider whether

$$T \cup \mathsf{Diag}_{\mathcal{A}} \cup \{\phi(\bar{c})\}$$

is consistent. If it's not consistent, we can find an inconsistent finite subtheory $T \cup \{\psi_i(\bar{a}) : 1 \leq$

$i \le k\} \cup \{\phi(\bar{c})\}$. Then we have

$$T \vDash \bigwedge_i \psi_i(\bar{c}) \to \neg\phi(\bar{c}),$$

so by contrapositive,

$$T \vDash \phi(\bar{c}) \to \bigvee_i \neg\psi_i(\bar{c}),$$

so the quantifier-free sentence $\bigvee_i \neg\psi_i(\bar{c})$ is in T_1 by construction. This means that $\mathcal{M} \vDash \bigvee_i \neg\psi_i(\bar{c})$, and because of quantifier-free transfer, $\mathcal{A} \vDash \bigvee_i \neg\psi_i(\bar{c})$. That gives us a contradiction, as by construction of each ψ_i, $\mathcal{A} \vDash \psi_i(\bar{c})$.
Now we can conclude

$$T \cup \mathsf{Diag}_{\mathcal{A}} \cup \{\phi(\bar{c})\}$$

actually is consistent, so it has a model, $\mathcal{N}$. Because $\mathcal{N} \vDash \mathsf{Diag}_{\mathcal{A}}$, there is an embedding $g : \mathcal{A} \hookrightarrow \mathcal{N}$, sending $\bar{c}^{\mathcal{A}}$ to $\bar{c}^{\mathcal{N}}$.
The models $\mathcal{M}, \mathcal{N} \vDash T$ have $\mathcal{A}$ as a common substructure, but $\mathcal{M} \vDash \neg\phi(\bar{c})$ and $\mathcal{N} \vDash \phi(\bar{c})$, contradicting our assumption. □

Corollary 3.4.6. *To check that a theory T eliminates quantifiers, we only need to check the following:*
Whenever $\phi(\bar{x}, y)$ is a formula, and $\mathcal{M}, \mathcal{N} \vDash T$ share a common finitely-generated substructure $\mathcal{A}$, then for any $\bar{a} \in A^n$, if there is $b \in M$ such that $\mathcal{M} \vDash \phi(\bar{a}, b)$, then there is $c \in N$ with $\mathcal{N} \vDash \phi(\bar{a}, c)$.

Proof. This allows us to apply (Theorem 3.4.5) to the formulas we need in (Lemma 3.4.1). □

We can even restrict further than checking this for quantifier-free ϕ, but first we need a lemma, essentially from propositional logic:

Lemma 3.4.7. *If $\phi(\bar{x})$ is a quantifier-free formula, then it is equivalent to a quantifier-free formula in* disjunctive normal form

$$\bigvee_{i=1}^{r} \bigwedge_{j=1}^{s} \theta_{i,j}(\bar{x})$$

where each $\theta_{i,j}$ is a literal*: either an atomic formula or the negation of one.*

Proof. We know that $\phi(\bar{x}, y)$ is quantifier-free, so it is some boolean combination of atomic formulas (this is basically propositional logic of atomic formulas). Up to logical equivalence, we have some flexibility as to the structure of this boolean combination.
First, a moment of notation. If $\theta(\bar{x})$ is a formula, and $b \in \{0, 1\}$, let $(\neg)^b\theta(\bar{x})$ denote $\neg\theta(\bar{x})$ if $b = 1$ and $\theta(\bar{x})$ if $b = 0$.
Let $\theta_1, \ldots, \theta_k$ be the atomic formulas that appear in ϕ. Then the truth value of ϕ only depends on the truth values of the θ_is, so for any $(b_1, \ldots, b_k) \in \{0, 1\}^k$, the conjunction $\bigwedge_{i=1}^k (\neg)^{b_i}\theta_i(\bar{x})$ implies either $\phi(\bar{x})$ or $\neg\phi(\bar{x})$. Thus ϕ is equivalent to

$$\bigvee_{(b_1,\ldots,b_k)\in S} \bigwedge_{i=1}^{k} (\neg)^{b_i}\theta_i(\bar{x})$$

for some set $S \subseteq \{0, 1\}^k$. □

Theorem 3.4.8. *To check that a theory T eliminates quantifiers, we only need to check the following:*
Whenever $\phi(\bar{x}, y)$ is a conjunction of literals, and $\mathcal{M}, \mathcal{N} \vDash T$ share a common finitely-generated substructure $\mathcal{A}$, then for any $\bar{a} \in A^n$, if there is $b \in M$ such that $\mathcal{M} \vDash \phi(\bar{a}, b)$, then there is $c \in N$ with $\mathcal{N} \vDash \phi(\bar{a}, c)$.

Proof. By (Corollary 3.4.6), it suffices to prove this also holds for all quantifier-free ϕ, and by (Lemma 3.4.7), we may assume that ϕ is of the form $\bigvee_{i=1}^r \bigwedge_{j=1}^s \theta_{i,j}(\bar{x})$.
If

$$\mathcal{M} \vDash \bigvee_{i=1}^r \bigwedge_{j=1}^s \theta_{i,j}(\bar{a}, b),$$

then there is some i such that

$$\mathcal{M} \vDash \bigwedge_{j=1}^s \theta_{i,j}(\bar{a}, b).$$

By assumption, there is some $c \in N$ with

$$\mathcal{N} \vDash \bigwedge_{j=1}^s \theta_{i,j}(\bar{a}, c),$$

implying

$$\mathcal{N} \vDash \bigvee_{i=1}^r \bigwedge_{j=1}^s \theta_{i,j}(\bar{a}, c).$$

□

3.5 Vector Spaces

We know that theories of vector spaces over a finite field eliminate quantifiers, as each of these theories is $\aleph_0$-categorical, and its unique countable model is a Fraïssé Limit. To extend this to *all* theories of vector spaces, we will need to develop some more techniques, which we will reapply to other theories. This will also give us quantifier elimination for several examples of theories of abelian groups which are basically vector spaces.

We study additive abelian groups in the language $\mathcal{L}_+ = \{0, +, -\}$. To study vector spaces, we extend this language with scalar multiplication, giving us a language $\mathcal{L}_{K,+} = \{0, +, -, c\cdot : c \in K\}$ where K is some field of scalars.

Definition 3.5.1. Given a field K, let VS_K be the $\mathcal{L}_{K,+}$-theory of nontrivial K-vector spaces.

Note: This requires separate axioms for each instance of scalar multiplication. For instance, for each $c, d \in K$, we need an axiom $\forall x, c \cdot (d \cdot x) = (cd) \cdot x$.

Lemma 3.5.2. *If K is an infinite field, the theory VS_K has only infinite models, and is $|\kappa|$-categorical for $\kappa > |\aleph_0|$. If K is countably infinite, then the theory VS_K is κ-categorical if and only if $\kappa > \aleph_0$.*

Proof. Suppose K is an infinite field. Any nontrivial K-vector space is infinite, and if it has dimension κ, then it has cardinality $\max(\kappa, |K|)$. Thus if $\kappa > |K|$, any K-vector space of

cardinality κ must also have dimension κ, and all such vector spaces are isomorphic.
If K is countably infinite, then any finite or $\aleph_0$-dimensional vector space has cardinality $\aleph_0$, so VS_K is not $\aleph_0$-categorical. □

By Łoś-Vaught Test (Theorem 2.7.3), VS_K is complete. We will now show that it eliminates quantifiers.

Theorem 3.5.3. *Each theory* VS_K *eliminates quantifiers.*

Proof. Suppose $\mathcal{M}, \mathcal{N} \vDash \mathsf{VS}_K$ have a common substructure $\mathcal{A}$. This substructure must also be a K-vector space, which will be nontrivial as long as $\mathcal{A}$ contains a nonzero element. We may assume it does, because if $A = \{0\}$, we can just extend it to a 1-dimensional subspace of both $\mathcal{M}, \mathcal{N}$.
Now suppose that $\phi(\bar{x}, y)$ is a conjunction of literals $\bigwedge_{i=1}^{r} \theta_i(\bar{x}, y)$, and $\bar{a} \in A^n, b \in M$ satisfy $\mathcal{M} \vDash \phi(\bar{a}, b)$. We will show that there is $c \in A$ such that $\mathcal{A} \vDash \phi(\bar{a}, c)$, so as $A \subseteq N$, there is $c \in N$ such that $\mathcal{N} \vDash \phi(\bar{a}, c)$, completing our test.
Now we have to think about what atomic formulas actually look like in a vector space. There are no relation symbols, so any atomic formula is an equation of terms, and any term is a linear combination of variables. By subtracting and dividing, we see that each such atomic formula is equivalent to one of one of the forms

$$t(\bar{x}) = 0, y = t(\bar{x}),$$

where t is a term/linear combination.
We may ignore any terms of the former category, as they don't depend on the value of y. For the others, we see that $\theta_i(\bar{a}, y)$ is equivalent to either $y = a$ or $y \neq a$ for some $a \in \mathcal{A}$. If any of these is of the form $y = a$, then we know that $b = a$, so $b \in \mathcal{A}$ already, and

$$\mathcal{A} \vDash \bigwedge_{i=1}^{r} \theta_i(\bar{a}, b).$$

Otherwise, we see that

$$\mathcal{M} \vDash \bigwedge_{i=1}^{r} \theta_i(\bar{a}, b),$$

just means $b \notin S$ for a particular finite set $S \subset A$, and we can find some $c \in A \setminus S$ thus satisfying

$$\mathcal{A} \vDash \bigwedge_{i=1}^{r} \theta_i(\bar{a}, c)$$

as $\mathcal{A}$ is infinite. □

This gives us a good picture of the definable sets in any vector space. They are going to be the boolean combinations of those definable by atomic formulas, which are going to be of the form $t(\bar{x}) = 0$, where t is a term representing a linear combination. These define hyperplanes.

In the one-dimensional case, hyperplanes are just single points, so the definable subsets of a model $\mathcal{M}$ itself are exactly finite or cofinite sets. (This is a hugely important property of these theories: they are *strongly minimal.*)

3.5.1 Lessons

We can summarize the key steps of this proof in a way that will let us replicate them. First, we've seen that any substructure of a model $\mathcal{VS}_K$ basically generates a model. In this case, any such substructure *is* a model unless it's the trivial substructure $\{0\}$, but we can come up with a property that describes both. To do this, let's note that we can axiomatize substructures of models of a theory:

Exercise 3.5.4. Given a theory T, let $T_\forall$ be the theory consisting of all universal sentences (of the form $\forall x, \phi(\bar{x})$ with ϕ quantifier-free) that are consequences of T.
Then $\mathcal{A}$ embeds into a model of T if and only if $\mathcal{A} \vDash T_\forall$.

Proof. See Marker 2.5.10. Hint: Use the theory $\mathsf{Diag}_\mathcal{A}$ and compactness. □

We can now cleanly define the property we've seen.

Definition 3.5.5. A theory T has *algebraically prime models* when for every $\mathcal{A} \vDash T_\forall$, there is a model $\mathcal{M} \vDash T$ and an embedding $i : \mathcal{A} \hookrightarrow \mathcal{M}$ that every other embedding $j : \mathcal{A} \hookrightarrow \mathcal{N} \vDash T$ factors through: $j = h \circ i$ for some $h : \mathcal{M} \hookrightarrow \mathcal{N}$.

Corollary 3.5.6. *Each theory VS_K has algebraically prime models.*

We can now describe the rest of the proof with the following test:

Lemma 3.5.7. *If T has algebraically prime models, then T has quantifier elimination if and only if the following holds:*
For any models $\mathcal{M}, \mathcal{N} \vDash T$ with $M \subseteq N$, if $\phi(\bar{x}, y)$ is a conjunction of literals, and $\bar{a} \in M^{|\bar{x}|}$, then $\mathcal{N} \vDash \exists y, \phi(\bar{a}, y)$ implies $\mathcal{M} \vDash \exists y, \phi(\bar{a}, y)$.

Proof. We've seen in the proof of (Theorem 3.5.3) that this suffices for precisely the reason of algebraically prime models. Meanwhile, if T has quantifier elimination, then $\exists y, \phi(\bar{x}, y)$ is equivalent to a quantifier-free formula, so this transfer between models is trivial. □

Lastly, we often won't have to prove quantifier elimination and completeness separately.

Lemma 3.5.8. *If a theory T eliminates quantifiers, it is also* model complete*: If $f : \mathcal{M} \hookrightarrow \mathcal{N}$ is an embedding with $\mathcal{M}, \mathcal{N} \vDash T$, then f is elementary.*

Proof. We know that f preserves all quantifier-free formulas, and all other formulas are equivalent to those, and thus preserved. □

The following test gives another proof of completeness of VS_K. It also justifies the term "model complete," which says that all you need to get completeness is to work in extensions of a specific model.

Lemma 3.5.9. *Suppose that T is a theory with quantifier elimination, and $\mathcal{M} \vDash T$ embeds into every other model of T. Then T is complete.*

Proof. If $\mathcal{N} \vDash T$, then $\mathcal{M}$ embeds *elementarily* into $\mathcal{N}$ by model-completeness, so $\mathsf{Th}(\mathcal{N}) = \mathsf{Th}(\mathcal{M})$, and T is complete. □

This applies to VS_K, because the 1-dimensional K-vector space K embeds into all other nontrivial vector spaces.

3.6 Divisible Abelian Groups

The next theory we'll consider is the theory of *torsion-free divisible abelian groups*, or DAG.

Definition 3.6.1. If t is a term and $n \in \mathbb{N}$, let $n \cdot t$ abbreviate the term $\underbrace{t + t + \cdots + t}_{n \text{ times}}$.

Let DAG be the $\mathcal{L}_+$-theory consisting of the axioms of abelian groups, as well as, for each $n \in \mathbb{N}$, the torsion-free axioms

$$\forall x, x \neq 0 \to n \cdot x \neq 0$$

and the divisibility axioms

$$\forall y, \exists x, n \cdot x = y.$$

Lemma 3.6.2. *A nontrivial abelian group models* DAG *if and only if it admits a $\mathbb{Q}$-vector space structure. In this case, that vector space structure is unique.*

Proof. It is easy to see that a $\mathbb{Q}$-vector space models DAG.
Suppose $\mathcal{M} \vDash \mathsf{DAG}$. Then for $\frac{p}{q} \in \mathbb{Q}$, we can define $\frac{p}{q}\cdot$ on M through divisibility, letting $\frac{p}{q} \cdot x$ be such that $q \cdot \left(\frac{p}{q} \cdot x\right) = p \cdot x$. This is well-defined, because if y, z both satisfy these assumptions, we find that

$$q \cdot (y - z) = q \cdot y - q \cdot y = p \cdot x - p \cdot x = 0,$$

so by the torsion-free assumption, $y - z = 0$. It is then easy to check the vector space axioms.
This vector space structure is also unique, because any $\mathbb{Q}$-vector space structure must satisfy the equations $q \cdot \left(\frac{p}{q} \cdot x\right) = p \cdot x$. □

This lets us transfer results between groups and $\mathbb{Q}$-vector spaces.

Lemma 3.6.3. *The theory* DAG *has algebraically prime models.*

Proof. If $\mathcal{A} \vDash \mathsf{DAG}_\forall$, then $\mathcal{A}$ is a torsion-free abelian group. We also know that $\mathcal{A}$ embeds into a divisible abelian group $\mathcal{M} \vDash T$, which has a natural $\mathbb{Q}$-vector space structure. We assume $A \subseteq M$. Let D be the $\mathbb{Q}$-span of A in M - we call the group structure $\mathcal{D}$ the *divisible hull* of $\mathcal{A}$.
If A is nontrivial, then we can see that any embedding of A into a $\mathbb{Q}$-vector space must factor through $\mathcal{D}$. If A is trivial, then instead, any embedding of A into a $\mathbb{Q}$-vector space must factor through the 1-dimensional vector space $\mathbb{Q}$. □

Theorem 3.6.4. *The theory* DAG *eliminates quantifiers and is complete.*

Proof. Suppose $\mathcal{M}, \mathcal{N} \vDash \mathsf{DAG}$ are such that $M \subseteq N$, $\bar{a} \in M^n$, $b \in N$, and $\mathcal{N} \vDash \phi(\bar{a}, b)$ where ϕ is a conjunction of literals.
Then if we expand the language, we know that ϕ is also a formula in $\mathcal{L}_{+,\mathbb{Q}}$, so it is equivalent in models of DAG, which are in this extended language, uniquely models of $\mathsf{VS}_\mathbb{Q}$, to a quantifier-free $\mathcal{L}_{+,\mathbb{Q}}$-formula ψ. Thus there is $c \in N$ with $\mathcal{M} \vDash \phi(\bar{a}, c)$.
Completeness follows because the group $\mathbb{Q}$ embeds into all other models, or by reference to $\mathsf{VS}_\mathbb{Q}$. □

3.7 Ordered Divisible Abelian Groups and Vector Spaces

We can tack $<$ onto these languages to get the languages $\mathcal{L}_{+,<}$ and $\mathcal{L}_{+,K,<}$ of *ordered* groups and vector spaces.

By swapping the torsion-free axioms in DAG for the axioms of a linearly ordered group, we construct the theory ODAG of ordered divisible abelian groups. If K is an ordered field, we add the axioms of a linearly ordered group to VS_K along with axioms (one per scalar) that multiplication by any positive scalar is order-preserving, and axioms that multiplication by any negative scalar is order-reversing, getting the theory OVS_K of ordered K-vector spaces.

Lemma 3.7.1. *The theories* OVS_K *for any ordered field* K *imply* ODAG*, which in turn implies* DLO*.*

Proof. The ordered field $\mathbb{Q}$ embeds uniquely into any ordered field K. Using this embedding, any ordered K-vector space is uniquely an ordered $\mathbb{Q}$-vector space. Then any $\mathbb{Q}$-vector space is a divisible abelian group, and an ordered one is an ordered divisible abelian group.
The axioms of ordered divisible abelian groups include those of linear orders, so it suffices to check density and lack of endpoints. Given $a, b \in M$ with $\mathcal{M} \vDash \mathsf{ODAG}$, there is some $c \in M$ with $c + c = a + b$ by divisibility. We can check that if $a < b$, $a < c < b$.
Similarly, we check that $a \in M$ is not a minimum or a maximum. If $a > 0$, $-a < a < 2a$, and $a < 0$, then $2a < a < -a$. If $a = 0$, nontriviality lets us find $b \neq 0$, and either $-b < 0 < b$ or $b < 0 < -b$. □

Theorem 3.7.2. *If K is an ordered field,* OVS_K *is complete and has quantifier elimination.*

Proof. This has algebraically prime models, just as in the non-ordered case, as all nontrivial substructures are models, and the trivial one factors through K itself.
Now let $\mathcal{M}, \mathcal{N} \vDash \mathsf{OVS}_K$ be such that $M \subseteq N$, and let $\phi(\bar{x}, y)$ be a conjunction of literals, $\bar{a} \in M^{|\bar{x}|}, b \in N$ with $\mathcal{N} \vDash \phi(\bar{a}, b)$. Each literal is equivalent (repeating the argument from before) to either a form that doesn't depend on the value of y, which we can ignore, or something of the form $t(\bar{x}) < y$, $t(\bar{x}) = y$, or the negation of one of these. If $t(\bar{x}) = y$ ever appears in this conjunction, then $b = t(\bar{a}) \in M$ already. Thus we may assume that each of these literals is of one of the forms $y < t(\bar{x}), y \neq t(\bar{x}), y > t(\bar{x})$. We may even replace any instance of $y \neq t(\bar{x})$ with one of the inequalities $<, >$, depending on which b satisfies. We thus find ourselves with a conjunction of formulas of the form $y < a_i$ where $b < a_i = t_i(\bar{a})$, or $y > a_i$ where $b > a_i = t_i(\bar{a})$. If a_+ is the smallest a_i of the former category, and a_- is the largest of the latter, we find that all we need to satisfy this formula is $a_- < y < a_+$. As $a_- < b < a_+$, this is possible, as we can simply take $\frac{1}{2} \cdot (a_- + a_+) \in M$.
Completeness follows by embedding K itself into all other models. □

Theorem 3.7.3. *The theory* ODAG *is complete and has quantifier elimination.*

Proof. Any model of ODAG is itself torsion-free (all ordered groups are) and thus models DAG, and thus extends to a unique model of $\mathsf{VS}_{\mathbb{Q}}$. This is actually a model of $\mathsf{OVS}_{\mathbb{Q}}$, as it's not hard to check that multiplication by any positive scalar is order-preserving and any negative scalar is order-reversing. As in the non-ordered case, it will thus suffice to prove algebraically prime models and refer to $\mathsf{OVS}_{\mathbb{Q}}$.
Suppose that $\mathcal{A} \vDash \mathsf{ODAG}_\forall$. Embedding this into $\mathcal{M} \vDash \mathsf{ODAG}$, which we view as a model of $\mathsf{VS}_{\mathbb{Q}}$, we see that the divisible hull or $\mathbb{Q}$-span of $\mathcal{A}$ models ODAG, and embeddings into other models factor through this as before. We deal with the trivial case as before. □

These theories are *not* strongly minimal. In any model $\mathcal{M}$ of any of these theories, the interval $(-\infty, 0)$ is infinite but not cofinite.

However, they are o-minimal:

Definition 3.7.4. A $\mathcal{L}$-theory T is o-minimal when $\mathcal{L}$ contains a binary symbol $<$ and any model $M \vDash T$ is a dense linear order (sometimes the density assumption is not included) such that all definable subsets of M are finite unions of (points and) intervals.

Corollary 3.7.5. *The theories* ODAG *and* OVS_K *for any ordered field* K *are o-minimal.*

Proof. By quantifier elimination, and the fact that finite unions of points and intervals are closed under boolean combinations, it suffices to check that any atomic formula defines a finite union of points and intervals. In OVS_K, these are all of the form (after subtraction) $x = t(\bar{a})$, $x < t(\bar{a})$, or $x > t(\bar{a})$ for some term t, each of which defines an interval. In ODAG, we note that any definable set is definable in $\mathsf{OVS}_{\mathbb{Q}}$. $\square$

3.8 Presburger Arithmetic

We know that the theory of $(\mathbb{N}; 0, 1, +, \times, \leq)$ doesn't have quantifier elimination, and a similar story applies to $(\mathbb{Z}; 0, 1, +, -, \times, \leq)$, but what happens if we remove the troublesome $\times$? This gives us an ordered abelian group (although with an extra constant 1) $(\mathbb{Z}; 0, 1, +, -, \leq)$, and we've had some success with those before.

We will not immediately succeed in this case either. As we have already seen, the quantifier-free definable subsets of $\mathbb{Z}$ itself in this language are finite unions of intervals. However, there are other definable sets. Take for instance, the even numbers, definable by $\exists y, y + y = x$, or by similar logic, the subgroup $n\mathbb{Z}$ for any $n \geq 2$.

Perhaps these are the only obstructions. If we add to the language $\mathcal{L}_{\mathsf{or}}$ of ordered rings extra predicates P_n for these definable sets, we don't change the algebra of definable sets overall, as this is an expansion by definitions.

We call this language the *Presburger language* $\mathcal{L}_{\mathsf{Pr}}$, and in this language, we can more easily describe the peculiarities of this structure with a theory T_{Pr}, known as *Presburger arithmetic.*

We start with the theory of ordered abelian groups, and add to it first the sentence

$$\forall x, x \leq 0 \vee 1 \leq x,$$

which ensures that this is a discrete linear order. Then we add sentences

$$\forall x, P_n(x) \leftrightarrow \exists y, n \cdot y = x,$$

ensuring that these symbols are just abbreviations for these already existing formulas.

Lastly, since we know that these finite-index subgroups are definable, we can add sentences ensuring that the quotients by them are indeed isomorphic to $\mathbb{Z}/n\mathbb{Z}$. That is, we can make sure that for every $a \in \mathcal{M} \vDash T_{\mathsf{Pr}}$, and every $n \geq 2$, there is a unique $0 \leq i < n$ such that $a \equiv i \pmod{n}$:

$$\forall x, \bigvee_{i=0}^{n-1} \left(P_n\left(a - i \cdot 1\right) \wedge \bigwedge_{j \neq i} \neg P_n\left(a - j \cdot 1\right) \right).$$

3.8.1 The below was not proven in class, but the results were discussed.

Lemma 3.8.1. *Presburger arithmetic T_{Pr} has algebraically prime models.*

Proof. Let $\mathcal{M} \vDash T_{\mathsf{Pr}}$, $\mathcal{A} \subseteq \mathcal{M}$. Then let D be the *relative divisible hull of $\mathcal{A}$ in $\mathcal{M}$*, that is, the set of all $m \in M$ such that for some $n \in \mathbb{N}$, $n \cdot m \in A$. We claim that $\mathcal{D}$ is a substructure with $\mathcal{D} \vDash T_{\mathsf{Pr}}$, and that all embeddings $\mathcal{A} \hookrightarrow \mathcal{N}$ factor through the inclusion map $\mathcal{A} \hookrightarrow \mathcal{D}$.
To check that $\mathcal{D}$ is a substructure, we only need it to be a subgroup, as it already contains $\mathcal{A}$ and thus both constants. This holds, because if $d_1, d_2 \in D$ are such that $n_1 \cdot d_1, n_2 \cdot d_2 \in A$, we have $(n_1 n_2) \cdot (d_1 - d_2) \in A$, so $d_1 - d_2 \in D$.
As $\mathcal{D}$ is a substructure of $\mathcal{M} \vDash T_{\mathsf{Pr}}$, it is an ordered divisible group satisfying the discreteness axiom, as well as the universal axioms of T_{Pr} governing the quotients $\mathcal{D}/P_n(\mathcal{D})$. We only need to check that

$$\mathcal{D} \vDash \forall x, P_n(x) \leftrightarrow \exists y, n \cdot y = x$$

for each $n \geq 2$. For one direction, it suffices to check that if $d \in D$, $n \geq 2$, then $\mathcal{D} \vDash P_n(n \cdot d)$. This is true because $\mathcal{M} \vDash P_n(n \cdot d)$. For the other direction, suppose $d \in D$ satisfies $\mathcal{D} \vDash P_n(d)$. Then there is some $m \in M$ with $n \cdot m = d$, and some $k \in \mathbb{N}$ with $k \cdot d \in A$, so $(kn) \cdot m \in A$, and thus $m \in D$, so $\mathcal{D} \vDash \exists y, n \cdot y = d$.
Now suppose $f : \mathcal{A} \hookrightarrow \mathcal{N} \vDash T_{\mathsf{Pr}}$ is an embedding into a model of T_{Pr}. We will try to extend f to $\mathcal{D} \hookrightarrow \mathcal{N}$ by mapping any d with $k \cdot d = a \in A$ to some $g(d) = n \in N$ with $k \cdot n = f(a)$. There is always such an n, as for any such a, $\mathcal{A} \vDash P_k(a)$, so $\mathcal{N} \vDash P_k(f(a))$. In fact there is a unique such n, (ensuring in particular that this is an extension of f,) because $\mathcal{N}$ is ordered and thus torsion-free. It is then straightforward to check that this map is injective, a group homomorphism, and order-preserving, ensuring that it is an embedding in the language of ordered abelian groups. (These checks are decent exercises.) We still need to check that each relation P_k is preserved. If $d \in D$ satisfies $\mathcal{D} \vDash P_k(d)$, then d is actually divisible by k in $\mathcal{D}$ and thus $g(d)$ must be divisible by k in N. If $\mathcal{N} \vDash P_k(g(d))$, then let j be such that $j \cdot d \in A$. We see that $\mathcal{N} \vDash P_{jk}(g(j \cdot d))$, so $\mathcal{A} \vDash P_{jk}(j \cdot d)$, so there is actually an element $d' \in D$ with $(jk) \cdot d' = j \cdot d$. The lack of torsion implies that $k \cdot d' = d$. ☐

Theorem 3.8.2. *Presburger arithmetic T_{Pr} eliminates quantifiers and is complete.*

Proof. Suppose $\mathcal{M}, \mathcal{N} \vDash T_{\mathsf{Pr}}$, $M \subseteq N$, $\phi(\bar{x}, y)$ is a conjunction of literals, and $\bar{a} \in M^{|\bar{x}|}, b \in N$ are such that $\mathcal{N} \vDash \phi(\bar{a}, b)$.
Every term $t(\bar{x}, y)$ is interpreted as a $\mathbb{Z}$-linear combination of variables, so $t(\bar{a}, y)$ can always be interpreted as $m + k \cdot y$ for some $m \in M, k \in \mathbb{Z}$. Any equation of terms is then equivalent to something of the form $k \cdot y = m$, which by divisibility is equivalent to $y = m'$ for some $m' \in M$, any atomic formula using $<$ is equivalent to $k \cdot y < m$, which by divisibility is equivalent to either $y < m'$ or $y > m'$ for some $m' \in M$. As usual, we ignore any atomic formulas that don't depend on y, and if any of the equations $y = m$ appear unnegated in the conjunction, we are done. Any inequality $y \neq m$ can be replaced with $y < m \vee y > m$, and we can get rid of this $\vee$ using the same argument by which we reduced to a conjunction of literals in the first place, so we can assume that the conjunction of all of the literals based on $=$ or $<$ is a conjunction of inequalities, which is to say, an interval (m_-, m_+) with $m_-, m_+ \in M$.
Meanwhile, any atomic formula using P_n is equivalent to $P_n(k \cdot y - m)$. As there is some $0 \leq i < n$ with $P_n(m - i \cdot 1)$, and P_n is interpreted as a subgroup, this is equivalent to $P_n(k \cdot y - i \cdot 1)$, or in a more normal phrasing, $k \cdot y \equiv i \pmod{n}$. If we multiply by a nonzero integer d, we find that this is equivalent to $P_{dn}((dk) \cdot y - di \cdot 1)$, so we may assume that all atomic formulas of this form use the same symbol P_L, where L is the least common denominator of the ns used.
Any inequality $y \neq m$ can be replaced with $y < m \vee y > m$, and each $\neg P_L(k \cdot y - i \cdot 1)$ can be

replaced with a disjunction of atomic formulas $\bigvee_{j\neq i} P_L(k \cdot y - j \cdot 1)$. We can get rid of all of these disjunctions using the same argument by which we reduced to a conjunction of literals in the first place. Thus we may assume that ϕ is of the form

$$m_- < y < m_+ \wedge \bigwedge_{i=1}^{r} P_L(k_i \cdot y - j_i \cdot 1).$$

As there is some $j \in \mathbb{Z}$ with $\mathcal{N} \models P_L(b - j \cdot 1)$, we find that anything equivalent to j mod L must satisfy all these P_L congruences. Then there is some $k \in \mathbb{N}$ such that $\mathcal{N} \models P_L(m_- + (k - j) \cdot 1)$.
If $m_- + k \cdot 1 < m_+$, we can let this be our $c \in \mathcal{M}$, and get $\mathcal{M} \models \phi(\bar{a}, c)$.
If not, then $m_- < b < m_+ \leq m_- + k \cdot 1$, so $0 \leq b - m_- < k \cdot 1$. We can prove by induction on k that any such element of a model of T_{Pr} must be itself an integer multiple of 1, so we have $b - m_- = j \cdot 1$, and thus $b = m_- + j \cdot 1 \in M$ already.
This concludes quantifier elimination, and we get completeness by seeing that $\mathbb{Z}$ embeds into any $\mathcal{M} \models T_{\mathsf{Pr}}$. ☐

Corollary 3.8.3. *Presburger arithmetic T_{Pr} is decidable, and quantifier elimination is also computable.*

Proof. The set of axioms we gave for T_{Pr} is computable and T_{Pr} is complete, so the set of consequences of T_{Pr} is computable: We simultaneously enumerate possible proofs of ϕ and $\neg\phi$ from the axioms, and eventually find that one works, resolving whether $T_{\mathsf{Pr}} \vdash \phi$ or $T_{\mathsf{Pr}} \vdash \neg\phi$.
For quantifier elimination, we can enumerate quantifier-free formulas ψ until we find one such that $T_{\mathsf{Pr}} \vdash \forall \bar{x}, \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$. ☐

For references on the actual algorithmic complexity of Presburger arithmetic, see [4] and other papers by Danny Nguyen and Igor Pak.

3.9 Algebraically Closed Fields

We now turn our attention back to rings and fields, in the language $\mathcal{L}_{+,\times} = \{0, 1, +, -, \times\}$.

Algebraically closed fields can be axiomatized in this language. We start with the theory of fields, and then add axioms, for each $d > 0$, indicating that monic polynomials of degree d have zeroes:

$$\forall x_0, \ldots, x_{d-1}, \exists y, y^d + x_{d-1} \times y^{d-1} + \cdots + x_1 \times y + x_0 = 0,$$

giving the theory ACF.

Theorem 3.9.1. *The theory* ACF *eliminates quantifiers.*

Proof. We start by checking that this has algebraically prime models. Any substructure of an algebraically closed field is a subring, and thus an integral domain. Any embedding from an integral domain to a field factors through its fraction field, and any embedding from a field to an algebraically closed field factors (not necessarily uniquely) through its algebraic closure. In summary, any embedding from an integral domain to an algebraically closed field factors through the algebraic closure of the fraction field.
Now let $\mathcal{M}, \mathcal{N} \models \mathsf{ACF}$ with $M \subseteq N$, and let $\bigwedge_{i=1}^{k} \theta_i(\bar{x}, y)$ be a conjunction of literals such that $\bar{a} \in M^{|\bar{x}|}, b \in N$ satisfy $\mathcal{N} \models \bigwedge_{i=1}^{k} \theta_i(\bar{a}, b)$.
An atomic formula here is an equation of terms, and each term without parameters is interpreted as an integer polynomial, so each atomic formula in $\bar{x}, y$ is equivalent to $P(\bar{x}, y) = 0$ for some polynomial $P \in \mathbb{Z}[\bar{x}, y]$.

Now we can write our conjunction of literals as

$$\bigwedge_{i=1}^{m} P_i(\bar{a}, y) = 0 \wedge \bigwedge_{j=1}^{n} Q_j(\bar{a}, y) \neq 0$$

where $P_i, Q_j \in \mathbb{Z}[\bar{x}, y]$, or plugging in our parameters $\bar{a}$, $P_i(\bar{a}, y), Q_j(\bar{a}, y) \in \mathcal{M}[y]$. If there are any literals in the first category, we see that either $P_i = 0$ (we can ignore these),

$$\mathcal{N} \vDash P_i(\bar{a}, b) = 0$$

so b is algebraic over $\mathcal{M}$, and must already be in $\mathcal{M}$. Thus we may consider the case of the formula

$$\bigwedge_{j=1}^{n} Q_j(\bar{a}, y) \neq 0,$$

or equivalently,

$$\neg \bigvee_{j=1}^{n} Q_j(\bar{a}, y) = 0.$$

We may also assume that none of the polynomials Q_j is constant, so that each $Q_j(\bar{a}, y)$ has only finitely many solutions in either $\mathcal{M}$ or $\mathcal{N}$. Thus there are at most finitely many solutions to $\bigvee_{j=1}^{n} Q_j(\bar{a}, y) = 0$ in $\mathcal{M}$, and as $\mathcal{M}$ is infinite, there must be infinitely many $c \in M$ with

$$\neg \bigvee_{j=1}^{n} Q_j(\bar{a}, c) = 0.$$

□

As the quantifier-free definable sets in a field are precisely the constructible sets, this can be phrased as a version of Chevalley's Theorem from algebraic geometry:

Corollary 3.9.2. *If $K \vDash \mathsf{ACF}$ and $D \subseteq K^{m+n}$ is a constructible set, then the projection $\pi(D) \subseteq K^m$ is also constructible.*

Proof. If D is defined by $\phi(\bar{x}, \bar{y})$, then $\pi(D)$ is defined by $\exists \bar{y}, \phi(\bar{x}, \bar{y})$, so the rest is quantifier elimination. □

Exercise 3.9.3. Prove a stronger version of Chevalley's Theorem: For any multivariable polynomial $p : K^m \to K^n$, if $D \subseteq K^m$ is constructible, then so is its image $p(D)$.

Corollary 3.9.4. *The theory ACF is strongly minimal.*

Proof. The only constructible subsets of $K \vDash \mathsf{ACF}$ are finite or cofinite, as the zero set of any one polynomial is finite or K. □

Unlike the previous theories we've shown to have quantifier elimination, ACF is *not* complete. Essentially this is because there is no substructure common to all algebraically closed fields. We can fix this easily by adding the sentence $p \cdot 1 = 0$ for a prime $p > 0$ or all the sentences $p \cdot 1 \neq 0$ for $p = 0$ to form the theory ACF_p of algebraically closed fields of characteristic p.

Theorem 3.9.5. *For p a prime (or 0), ACF_p is complete.*

Proof. Each of these theories eliminates quantifiers, so to show completeness, it suffices to recall that each has a substructure common to all models.
If $p > 0$, then any model $\mathcal{M} \vDash \mathsf{ACF}_p$ contains a substructure isomorphic to $\mathbb{F}_p$. If $p = 0$, then any model $\mathcal{M} \vDash \mathsf{ACF}_0$ contains a substructure isomorphic to $\mathbb{Q}$. □

3.9.1 Ax-Grothendieck

We now have the tools to prove a classic result, the model-theoretic way. (It was proven independently by Ax, this way, and Grothendieck, using more classic algebraic geometry.)

Theorem 3.9.6 (Ax-Grothendieck)**.** *Any injective polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is also surjective. (Actually, this holds for any algebraically closed field in place of $\mathbb{C}$.)*

Proof. For a fixed degree d and dimension n, there is a sentence ϕ such that

$$(\mathbb{C}; 0, 1, +, -, \times) \vDash \phi$$

is equivalent to all injective polynomial maps of degree at most d being injective. To see this, first note that there's a term $t(\bar{x}, \bar{y})$ with $|\bar{x}| = n$ such that by plugging in different coefficients $\bar{c}$, $t(\bar{x}, \bar{c})$ can be interpreted as any polynomial of degree d. Then we can write our ϕ as

$$\forall \bar{y}_1, \ldots, \bar{y}_n, \left(\forall \bar{x}_1, \bar{x}_2, \bigwedge_{i=1}^{n} t(\bar{x}_1, \bar{y}_i) = t(\bar{x}_2, \bar{y}_i) \to \bar{x}_1 = \bar{x}_2\right) \to \left(\forall \bar{z}, \exists \bar{x}, \bigwedge_{i=1}^{n} t(\bar{x}, \bar{y}_i) = z_i\right).$$

We can thus frame the theorem as proving $\mathbb{C} \vDash \phi$ for each d, n, or indeed, $\mathsf{ACF}_0 \vDash \phi$.
We start this process by observing that for any finite field k, $k \vDash \phi$, as any injective function $k^n \to k^n$ is surjective.
We now try to go from finite fields to their algebraic closures. For any prime p, we can construct $\mathbb{F}_p^{\mathrm{alg}}$, the algebraic closure of $\mathbb{F}_p$, as a union of an ascending chain of finite subfields - call one such chain $k_0 \subseteq k_1 \subseteq \ldots$. To lift our sentence up this chain, we characterize the quantifier complexity of ϕ, and use the following general result:

Exercise 3.9.7 (See [3, Exercise 2.5.15])**.** If $\mathcal{M}$ is the union of an ascending chain of substructures $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots$, $\phi = \forall \bar{y}, \exists \bar{x}, \psi(\bar{x}, \bar{y})$ with ψ quantifier-free, and for each i, $\mathcal{A}_i \vDash \phi$, then $\mathcal{M} \vDash \phi$.
(We call such a ϕ an $\forall\exists$ formula, or Π_2.)

With a little bit of effort, we can rewrite ϕ as an $\forall\exists$ formula, and for each i, $k_i \vDash \phi$, so $\mathbb{F}_p^{\mathrm{alg}} \vDash \phi$ and by completeness, $\mathsf{ACF}_p \vDash \phi$.
Now we move to characteristic 0. We know that ACF_0 is complete, so either $\mathsf{ACF}_0 \vDash \phi$ (in which case we're done) or $\mathsf{ACF}_0 \vDash \neg\phi$. In that case, by compactness a finite subtheory of ACF_0 proves $\neg\phi$. This subtheory is a subtheory of $\mathsf{ACF} \cup \{p \cdot 1 \neq 0 : p \leq M\}$ for some M, the maximum of all the finitely many primes such that $p \cdot 1 \neq 0$ appears in the subtheory. However, for any prime $P > M$, $\mathbb{F}_p^{\mathrm{alg}} \vDash \mathsf{ACF} \cup \{p \cdot 1 \neq 0 : p \leq M\}$, so this theory cannot imply $\neg\phi$. □

We can extract a general principle from this compactness argument:

Corollary 3.9.8. *If ϕ is a sentence such that for infinitely many primes p, $\mathsf{ACF}_p \vDash \phi$, then $\mathsf{ACF}_0 \vDash \phi$.*

3.10 Real Closed Fields

We were able to axiomatize $\mathbb{C}$ with ACF_0, and then eliminate quantifiers in that theory. How about $\mathbb{R}$?

First, we observe that to get quantifier elimination, we *must* add a symbol for $<$ or $\leq$. In a ring in the language $\mathcal{L}_{+,\times}$, the only quantifier-free definable sets are constructible sets, and the only one-dimensional constructible sets are finite or cofinite, but $[0,\infty)$ is defined by $\exists y, y^2 = x$. Thus we will work mostly in $\mathcal{L}_{+,\times,<} = \mathcal{L}_{+,\times} \cup \{<\}$.

We will attempt to axiomatize $\mathbb{R}$ in a similar way to how we axiomatized $\mathbb{C}$, but making use of the ordering.

3.10.1 Formally Real Fields

We will start with ordered fields, and what these imply in the plain language of rings. This will let us do as much work as possible to study $\mathbb{R}$ in the language of rings, and use the familiar tools of field theory.

A field being ordered implies, for instance, characteristic 0, and that all squares are nonnegative. We can sum up some of these purely algebraic consequences as follows:

Definition 3.10.1. A field F is *formally real* if -1 is not a finite sum of squares.

Lemma 3.10.2. *Any ordered field is formally real.*

Proof. Easy. □

In fact, we can show the converse in a bit:

Theorem 3.10.3. *Any formally real field is* orderable*, that is, there is a linear ordering that makes it an ordered field.*

We would love to define the ordering as we do in $\mathbb{R}$, by the formula $\exists z, x + z^2 = y$, but this only defines a linear ordering when every element or its negative is a square. Instead, we will incrementally build up to an extension where that's the case, order canonically, and inherit the order from the extension.

Lemma 3.10.4. *If F is formally real and $-a$ is not a sum of squares, then $F(\sqrt{a})$ is formally real.*

Proof. If not, let b_i, c_i be such that

$$-1 = \sum_i (b_i + c_i\sqrt{a})^2 = \sum_i b_i^2 + a\sum_i c_i^2 + 2\sqrt{a}\sum_i b_i c_i.$$

As 1 and $\sqrt{a}$ are F-linearly independent, we find that $\sum_i b_i c_i = 0$ and

$$-a = \frac{1 + \sum_i b_i^2}{\sum_i c_i^2} = \frac{\sum_i c_i^2 + \left(\sum_i b_i^2\right)\left(\sum_i c_i^2\right)}{\left(\sum_i c_i^2\right)^2},$$

which we can see is a sum of squares, giving a contradiction. □

Definition 3.10.5. Call a field *real closed* if it is formally real and has no proper formally real algebraic extensions.

Lemma 3.10.6. *Any real closed field F is uniquely orderable.*

Proof. Because F is real closed, for every $a \in F$, either $-a$ is a sum of squares, or a is already a square. Otherwise, we could extend properly and algebraically by $\sqrt{a}$ and still be formally real. In the former case, any ordering must have $a < 0$, and in the latter case, we must have $a \geq 0$. This tells us that if $a < 0$, then $-a$ is not just a sum of squares, but a square itself.

Exercise 3.10.7. Verify that this is a valid field ordering.

□

We can now prove the theorem:

Proof. If F is formally real, then by Zorn's lemma or transfinite recursion, find a maximal formally real algebraic extension of F. We call such a field a *real closure* of F. This must be real closed by maximality, and thus orderable, so F is. □

3.10.2 Real Closed Fields

We know that $\mathbb{R}$ itself is a real closed field - it only has one proper algebraic extension, $\mathbb{C}$, which is not formally real. We can now collect a few other properties of real closed fields, in order to axiomatize them.

Fact 3.10.8 (Artin-Schreier Theorem)**.** *If R is a formally real field, the following are equivalent:*

1. *R is real closed.*
2. *For every $a \in R$, either $-a$ or a is a square, and every odd-degree polynomial $P(x) \in R[x]$ has a zero in R.*
3. *$R(i)$ is algebraically closed.*
4. *R admits a field ordering under which it has the intermediate value property: If $P(x) \in R[x]$ is a polynomial, and $a < b$ are such that $P(a)P(b) < 0$, then there is $c \in (a, b)$ with $P(c) = 0$.*

Proof. (3) implies that the only irreducible polynomials in $R[x]$ are of degree at most 2. From this, it is fairly easy to show that (3) implies (1) and (4). For (1), it now suffices to rule out proper formally real quadratic extensions, and for (4), it suffices to show the intermediate value property holds for irreducible polynomials.
Now, each of (1) and (4) implies (2) fairly easily. For the former implication, we must show that

adjoining a root of an odd-degree irreducible maintains formal realness. That means that a real closed field must already contain all such roots. For the latter, it we can apply the intermediate value property to polynomials of the form $x^2 - a$ where $a > 0$, and to any polynomial of odd degree.
To complete the equivalence, it suffices to show that (2) implies (3). This requires some light Galois theory.
For more details, see [3, Appendix B], [7, Appendix B.1], or for stronger algebraic statements of the theorem proven with more involved (but still finite-dimensional) Galois Theory, see [2]. ☐

This now means we can axiomatize real closed fields, two different ways. Let FRCF, the theory of *formally real closed fields*, be the theory of fields together with

- axioms of formal realness: for each n, $\forall \bar{x}, 1 + \sum_{i=1}^{n} x_i^2 \neq 0$
- $\forall x, \exists y, y^2 = x \vee y^2 + x = 0$
- for all odd n, an axiom that any polynomial of degree n always has a root.

Let RCF be the theory *in the ordered language* $\mathcal{L}_{+,\times,<}$ consisting of the axioms of ordered fields together with the axioms of FRCF (we can skip the formally real axioms). Alternately, we could take the axioms of ordered fields together with an axiom, for each n, that polynomials of degree n have the intermediate value property.

We would love to proceed to showing that real closed fields have algebraically prime models, using real closures to generate the common submodels. However, real closures aren't obviously unique.

Example 3.10.9. The field $\mathbb{Q}(x)$ is formally real, but does not have a unique real closure (up to isomorphism of field extensions).

Proof. We can order $\mathbb{Q}(x)$, for instance, by making either x larger than all elements of $\mathbb{Q}$, so it is formally real. However, $\mathbb{Q}(\sqrt{x})$ and $\mathbb{Q}(\sqrt{-x})$ are distinct formally real extensions, and each has a real closure. These real closures must be different extensions, as in one, x is a square, and in the other, $-x$ is. ☐

We can fix this just be requiring that the real closure respect the ordering.

Lemma 3.10.10. *Every ordered field F has a real closure F^{re} whose canonical ordering extends that of F. Furthermore, this is unique up to unique F-preserving isomorphism.*

Proof. Constructing such a real closure is easy enough. For each $a > 0$, we can extend the ordering to $F(\sqrt{a})$. By Zorn's Lemma/transfinite recursion, we can eventually get to an ordered field extension where every element is a square. If we take the real closure of *that*, algebraically speaking, the set of squares is preserved, so we must have preserved the ordering.
Proving that this is unique takes some work - it's basically a back-and-forth argument. If R, S are both real closures of F, then we can (transfinitely) recursively construct larger and larger isomorphisms $\phi : R' \to S'$, where $F \leq R' \leq R$ and $F \leq S' \leq S$. To check that we can always extend this isomorphism, we need to check that, say, if we extend R' by one element, by adjoining the roots of an irreducible polynomial, that S has the same number of roots of that polynomial. Essentially, we need the following lemma:

Lemma 3.10.11. *If F is formally real, and $P(x) \in F[x]$ is irreducible, then $P(x)$ has the same number of roots in any real closure of F.*

Proof. We can calculate the number of roots required either by

- reducing to quadratics by Artin-Schreier and using the discriminant
- Sturm's algorithm (see [3, Appendix B]).

□

□

3.10.3 Quantifier Elimination

That was a lot of real algebra, let's eliminate some quantifiers.

Corollary 3.10.12. *Models of the theory* $\mathsf{RCF}_\forall$ *are precisely ordered integral domains, and the theory* RCF *has algebraically prime models.*

Proof. For the first claim, it is easy to see that any substructure of an ordered field is an ordered integral domain. Conversely, any ordered integral domain $\mathcal{A}$ embeds into an ordered fraction field, which then has a real closure modeling RCF.
We then see that if $\mathcal{A} \vDash \mathsf{RCF}_\forall$ embeds into both $\mathcal{M}, \mathcal{N} \vDash \mathsf{RCF}$, the relative algebraic closure of the image of A in M or N (that is, the subfield consisting of elements algebraic over A) must be real closed. These real closures must be isomorphic by unique A-preserving isomorphism, so we can extend our embedding of $\mathcal{A}$ into an embedding of its unique real closure. □

Theorem 3.10.13. *The theory* RCF *eliminates quantifiers and is complete.*

Proof. Suppose $\mathcal{M} \leq \mathcal{N}$, with $\mathcal{M}, \mathcal{N} \vDash \mathsf{RCF}$, and $\mathcal{N} \vDash \phi(\bar{a}, b)$ where $\bar{a} \in M^{|\bar{x}|}, b \in N$, and $\phi(\bar{x}, y)$ is a conjunction of literals. We may assume, by the same argument as for OVS_K, that each literal is of the form $t_1(\bar{x}, y) = t_2(\bar{x}, y)$ or $t_1(\bar{x}, y) < t_2(\bar{x}, y)$ for some terms t_1, t_2, which in turn, because of subtraction, we may write as $t(\bar{x}, y) = 0$, $t(\bar{x}, y) < 0$ or $t(\bar{x}, y) > 0$.
For each such literal, $t(\bar{a}, y)$ is interpreted as a polynomial in $M[y]$, which has finitely many zeroes in N. By real-closedness, each of these zeroes is also in M, so let $Z \subset M$ be the union of these finitely many sets of zeroes. If any of our literals is an equation, we know that $b \in Z \subset M$, so let $c = b$. Otherwise, assume they are all inequalities.
We know that for each of these inequalities, $t(\bar{a}, b) \neq 0$, so $b \notin Z$. Let $I \subseteq N$ be the interval between consecutive elements of Z such that $b \in I$. Because this is an interval between elements of M, $I \cap M$ is an open interval of a dense linear order, so let $c \in I$. If for any of the terms $t(\bar{x}, y)$ in ϕ, $t(\bar{a}, b)$ and $t(\bar{a}, c)$ had different signs, there would be a zero, that is, an element of Z, between them by IVP. This cannot happen as $b, c \in I$, so $\mathcal{M} \vDash \phi(\bar{a}, c)$.
For completeness, we show that $\mathbb{Q}^{\mathrm{re}} = \mathbb{Q}^{\mathrm{alg}} \cap \mathbb{R}$ embeds into every model of RCF. As every model has characteristic 0, it admits an embedding of $\mathbb{Q}$, and thus by algebraically prime models, $\mathbb{Q}^{\mathrm{re}}$. □

3.10.4 Hilbert's 17th Problem

Just as QE and completeness of ACF_p gave us the transfer principles needed to prove Ax-Grothendieck, QE and completeness of RCF gives us the transfer principle we need to prove Hilbert's 17th Problem (or rather, its solution, a theorem of Artin) very quickly.

Theorem 3.10.14. *If* $f \in \mathbb{R}(\bar{X}) = \mathbb{R}(X_1, \ldots, X_n)$ *is a rational function such that whenever* $f(\bar{a})$ *with* $\bar{a} \in \mathbb{R}^n$ *is defined,* $f(\bar{a}) \geq 0$*, then* f *is a sum of squares in the field* $\mathbb{R}(\bar{x})$*.*

Proof. Actually, this works if we replace $\mathbb{R}$ with any other real closed field $R \vDash \mathsf{RCF}$. First, we note that there is a formula $\phi(\bar{x})$ (with parameters in R) such that for $\bar{a} \in R^n, R \vDash \phi(\bar{a}) \iff f(\bar{a}) \geq 0$. If $f(\bar{X}) = \frac{p(\bar{X})}{q(\bar{X})}$ is an expression as a reduced fraction of polynomials, we can use $p(\bar{x})q(\bar{x}) \geq 0$ as a proxy.

We then consider the field $R(\bar{X})$ of rational functions with coefficients in R. If $f(\bar{X})$ is not a sum of squares, then we can order $R(\bar{X})$, extending the ordering on R, so that $f(\bar{X}) < 0$. We can then extend this ordering to a real closure of $R(\bar{X})$, which we call F.

By quantifier elimination, or more directly, model completeness, any embedding, including that of R into F, is elementary. Thus

$$F \vDash \forall \bar{x}, \phi(\bar{x}) \geq 0 \iff R \vDash \forall \bar{x}, \phi(\bar{x}) \geq 0.$$

However, this sentence is true by assumption in R, while in F, we find that plugging in the variables themselves makes $f(\bar{X}) < 0$, so $F \vDash \phi(\bar{X}) < 0$, giving us a contradiction. □

3.10.5 Semialgebraic Sets

Quantifier elimination and the structure of atomic formulas tells us that the definable sets in a model of RCF are precisely the *semialgebraic* sets:

Definition 3.10.15. If F is an ordered field, then a subset $D \subseteq F^n$ is *semialgebraic* when it is a boolean combination of sets given by $P(\bar{x}) > 0$ where $P(\bar{x}) \in F[\bar{x}]$.

Theorem 3.10.16. *The theory* RCF *is o-minimal.*

Proof. By quantifier elimination, it suffices to show that any semialgebraic set $D \subseteq \mathcal{M} \vDash \mathsf{RCF}$ is a finite union of intervals, and as these sets are closed under boolean combinations, it suffices to show that each set defined by $P(x) > 0$ is. This polynomial has finitely many zeroes which split the remainder of M into finitely many open intervals. On each of these intervals, by the intermediate value property, either $P(x) > 0$ or $P(x) < 0$, so our set is the union of the finitely many such intervals where $P(x) > 0$. □

There are a variety of results, often stated in terms of semialgebraic sets specifically, which turn out to be true more broadly for definable sets in an o-minimal structure.

Chapter 4

o-Minimality

For this chapter, I will mostly follow the textbook [8], but I will also make use of online notes at [5, 6].

Let's clarify the definitions again.

Definition 4.0.1. A structure $\mathcal{M}$ in a language $\mathcal{L}$ including the symbol $<$ (or $\leq$) is *o-minimal* if $\mathcal{M} \vDash \mathsf{DLO}$ and every definable set $D \subseteq M$ is a finite union of (points and) intervals.
A $\mathcal{L}$-theory T is then o-minimal if every model of T is o-minimal.

4.1 Tame Topology of Ordered Structures

We'll now take a look at how definability and topology relate in ordered and o-minimal structures. If $\mathcal{M} \vDash \mathsf{DLO}$, we always give M the order topology, and M^n the resulting product topology. This topology has as a basis *open boxes*, that is, products of open intervals. To better write formulas about these, let's introduce some notation: Given variable tuples $\bar{x}, \bar{y}$ of length n, let $\bar{x} < \bar{y}$ denote $\bigwedge_{i=1}^n x_i < y_i$ and similarly for $\bar{x} \leq \bar{y}$. Then any open box in M^n can be defined by

$$\bar{a} < \bar{x} < \bar{b},$$

where $\bar{a}, \bar{b} \in M^n$. Because of this, we say that the collection of open boxes in M^n is *uniformly definable*: they are all definable by the same formula, but with different parameters plugged in. Because product-order topology has a uniformly definable basis, we call this a *definable topology*. Let's start to see what this definability property does for us:

Lemma 4.1.1. *If $\mathcal{M} \vDash \mathsf{DLO}$ in a language including the symbol $<$, then*

- *whenever $D \subseteq M^n$ is definable, so are $\mathsf{cl}(D), \mathsf{int}(D)$*
- *whenever $A \subseteq B \subseteq M^n$ are definable and A is relatively open in B, this can be witnessed by a definable open $U \subseteq M^n$ with $A = U \cap B$.*

Proof. For the first bullet point, it's enough to show interiors of definable sets are definable, as definable sets are closed under complement. We find that $\bar{x} \in \mathsf{int}(D)$ if there is an open box containing $\bar{x}$ and contained in D:

$$\exists \bar{y}, \bar{z}, (\bar{y} < \bar{x} < \bar{z}) \wedge (\forall \bar{w}, \bar{y} < \bar{w} < \bar{z} \rightarrow \bar{w} \in D).$$

We have defined $\mathsf{int}(D)$ here as the union of all open boxes contained in D. In general, a union

of definable sets is not definable, but uniform definability makes this work.
For the second, we can let U be the union of all open boxes whose intersection with B is contained in A. Because of uniform definability of open boxes, this union is definable:

$$\exists \bar{y}, \bar{z}, (\bar{y} < \bar{x} < \bar{z}) \wedge (\forall \bar{w}, \bar{y} < \bar{w} < \bar{z} \wedge \bar{w} \in B \rightarrow \bar{w} \in A).$$

□

Corollary 4.1.2. *If $D \subseteq \mathbb{R}^n$ is semialgebraic, so are* $\mathsf{cl}(D), \mathsf{int}(D)$.

4.1.1 Definable Completeness

If we add in the assumption of o-minimality, we can phrase some facts about definability in one dimension more topologically. These follow pretty quickly from D being a finite union of intervals.

Lemma 4.1.3. *Suppose $D \subseteq M$ is a finite union of intervals, such as a definable set in an o-minimal structure. Then the boundary* $\mathsf{bd}(D)$ *is finite, and if a, b are consecutive points of* $\mathsf{bd}(D)$*, then either $(a, b) \subseteq D$ or $(a, b) \cap D = \emptyset$.*

Lemma 4.1.4. *If M is o-minimal, then it is* definably complete*:*
If $D \subseteq M$ is definable, then

- *if D is bounded above, D has a least upper bound* $\sup D$
- *if D is bounded below, D has a greatest lower bound* $\inf D$.

4.1.2 Uniform Definability

We have seen how the family of open balls being uniformly definable helps us define topological ideas. We can also view uniformly definable families as the family of fibers of a larger set. To see this, suppose that $|\bar{x}| = m$, $|\bar{y}| = n$, and $\phi(\bar{x}, \bar{y})$ defines a family $\mathcal{F}$ of subsets of M^n, with different parameters $\bar{a} \in M^m$ plugged in for $\bar{x}$ to define the definable sets in $\mathcal{F}$. Then $\phi(\bar{x}, \bar{y})$ itself defines a set $D \subseteq M^{m+n}$, whose *fibers* over different $\bar{x}$-values range over $\mathcal{F}$:

Definition 4.1.5. Given $D \subseteq M^{m+n}$, $\bar{a} \in M^m$, let $D_{\bar{a}} = \{\bar{b} \in M^n : (a, b) \in D\}$. Call this the *fiber over $\bar{a}$*.

As an exercise, show that if D defines a uniformly definable family of fibers, then the following properties are also definable:

Exercise 4.1.6. Given $D \subseteq M^{m+n}$ definable in an ordered structure $\mathcal{M}$, show that

- $\{\bar{a} \in M^m : D_{\bar{a}} \text{ is open}\}$
- $\{(\bar{a}, \bar{b}) \in M^{m+n} : \bar{b} \in \mathsf{int}(D_{\bar{a}})\}$

are both definable.

Exercise 4.1.7. Assume $\mathcal{M}$ is o-minimal.

- Show that if $D \subseteq M$ is definable and infinite, it contains an open interval.
- Show that if $D \subseteq M^{m+1}$, then $\{\bar{a} \in M^m : D_{\bar{a}} \text{ is finite}\}$ is definable.

4.1.3 Why topology of definable sets?

Fact 4.1.8. *There exists a real closed field, for instance* real Puiseux series*, which is not a complete order and is totally disconnected in the order topology.*

For the construction of this example, see [6] and [1, Section 2.6].

This shows us that we not all sets in this real closed field behave topologically as they do in $\mathbb{R}$, so we cannot use the order topology in the way we are accustomed to. However, if we restrict our attention to definable sets, their topological behavior, sometimes called *tame topology*, is much better, as we know this o-minimal structure must be definably complete with definably connected intervals.

Bibliography

[1] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in real algebraic geometry*. Springer, 2006.

[2] Keith Conrad. The Artin–Schreier theorem. Keith Conrad's Expository Papers, 2011. URL: http://www.math.uconn.edu/~kconrad/blurbs/galoistheory/artinschreier.pdf.

[3] David Marker. *Model theory: an introduction*. Springer, 2002.

[4] Danny Nguyen and Igor Pak. Complexity of short presburger arithmetic. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2017, page 812–820, New York, NY, USA, 2017. Association for Computing Machinery. doi:10.1145/3055399.3055435.

[5] Tom Scanlon. Berkeley learning seminar on o-minimality. Online course notes, 2024. URL: https://wp.math.berkeley.edu/model-theory/2024/08/29/introduction-to-o-minimality/.

[6] Patrick Speissegger. Introduction to o-minimal structures. Online course notes, 2015. URL: https://ms.mcmaster.ca/~speisseg/712/intro.pdf.

[7] Katrin Tent and Martin Ziegler. *A course in model theory*. Number 40 in Lecture Notes in Logic. Cambridge University Press, 2012.

[8] Lou Van den Dries. *Tame topology and o-minimal structures*, volume 248. Cambridge university press, 1998.