Distality in Combinatorics and Continuous Logic

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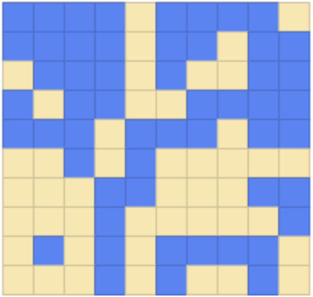
Ramsey Theory

Fact

For every n, there is a number $R(n)$ such that if $|X| = |Y| = n$ and $E \subseteq X \times Y$ is a graph relation, then there are subsets $A \subseteq X$, $B \subseteq Y$, each of size at least $R(n)$, that are E-homogeneous: either $A \cap B \subseteq E$, or $A \times B \cap E = \emptyset$. The numbers $R(n)$ grow to infinity, but very, very slowly.



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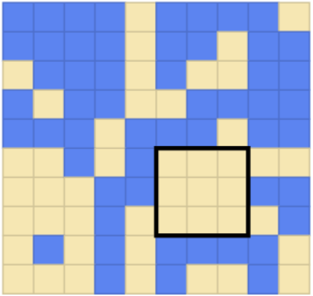


Semialgebraic Graphs

In many examples we actually care about, X and Y are both subsets of Euclidean spaces - $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$, and $E \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is given by a system of polynomial (in)equalities. We call this case *semialgebraic*.

Example

Let $\mathcal{F}$ be some family of polynomial curves of degree n. Let X be a set of points, Y a set of (coefficients of) curves in $\mathcal{F}$, and let E be the incidence graph.

Semialgebraic Strong Erdős-Hajnal

Fact (Alon, Pach, Pinchasi, Radoičić, and Sharir)

Let $E \subseteq \mathbb{R}^{m+n}$ be a semialgebraic graph. There is some $\delta > 0$ such that if $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$ are finite, then there are subsets $A \subseteq X, B \subseteq Y$ of size $|A| \geq \delta|X|, |B| \geq \delta|Y|$, that are E-homogeneous.

We call this property of E the *strong Erdős-Hajnal property*.

Definability

Semialgebraic sets are just the *definable sets* in the structure $(\mathbb{R}; 0, 1, +, \times, \leq)$. Definable sets are the solution sets of *formulas*:

- A *structure* interprets the symbols of a *language* (e.g. $\mathcal{L} = \{0, 1, +, \times, \leq\}$)
- *Terms* are built from variables and function symbols (e.g. $x^2 + y^2$)
- *Atomic formulas* are built from terms and relation symbols (e.g. $x^2 + y^2 \leq 1$)
- Atomic formulas are combined with logical symbols such as $\vee, \wedge, \neg, \exists, \forall$ to make *formulas*.

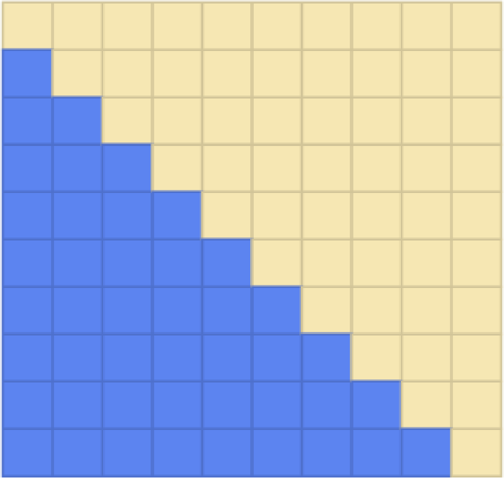
Distality

The structure $(\mathbb{R}; 0, 1, +, \times, \leq)$ is *distal*.
A structure M is distal if and only if it has the *definable strong Erdős-Hajnal property*:

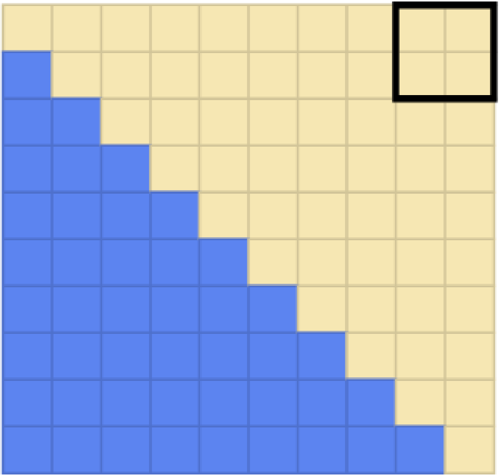
Fact (Chernikov-Starchenko)

Let $E \subseteq M^{m+n}$ be definable.
There is some $\delta > 0$ such that if $X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$ are finite, then there are subsets $A \subseteq X, B \subseteq Y$ of size $|A| \geq \delta|X|, |B| \geq \delta|Y|$, that are E-homogeneous.
Furthermore, A and B are definable (within X and Y) by formulas depending only on the definition of E.

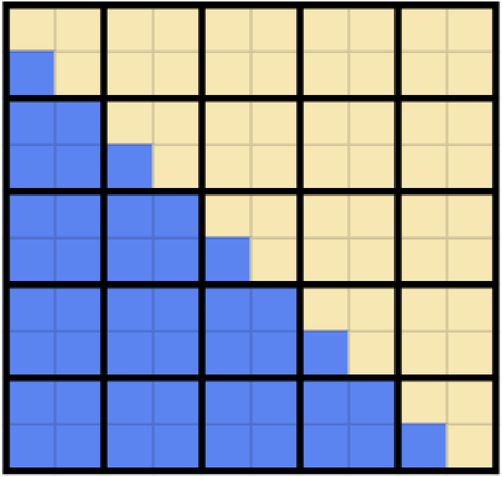
Strong Erdős-Hajnal for $x \leq y$



Strong Erdős-Hajnal for $x \leq y$



Distal Regularity for $x \leq y$



Examples of Distal Structures

- o-minimal structures like $(\mathbb{R}; 0, 1, +, \times, \leq)$ and expansions including exponential and (limited) analytic functions
- Valued field or valued vector space structures on $\mathbb{Q}_p$

To find more examples, use *continuous logic* to replace E with a real-valued function $M^{m+n} \to [0, 1]$.

- Discrete classical structures are still distal in continuous logic
- Real closed metric valued fields (think o-minimal)
- Dual linear continua

Real-Valued Strong Erdős-Hajnal

0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0
0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0
0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0
0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0
0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1
0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1
0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1
0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1
0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2
0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2
0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2
0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2
0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2
0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4
0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4
0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4
0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4
0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5

Real-Valued Strong Erdős-Hajnal

0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0
0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0
0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0
0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0
0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1
0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1
0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1
0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1
0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2
0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2
0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2
0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2
0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2
0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4
0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4
0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4
0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4
0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5

Metric Structures

Definition

A *metric language* is just like a regular first-order language, but each symbol gets a Lipschitz constant.

Definition

A *metric structure* consists of:

- A complete bounded metric space
- For each n-ary k-Lipschitz function symbol, a k-Lipschitz function $M^n \to M$
- For each n-ary k-Lipschitz relation symbol, a k-Lipschitz function $M^n \to [0, 1]$

Formulas

Definition

A *atomic formula* is defined as usual, except instead of $=$, the basic relation is $d(x, y)$.

Definition

A *formula* is

- An atomic formula
- A continuous combination $u(\phi_1, \ldots, \phi_n)$ of formulas.
- $\sup_x \phi$ or $\inf_x \phi$

Definition

A *definable predicate* is a uniform limit of formulas. This allows countably infinitely many variables.

The Lazy Caterer's Problem

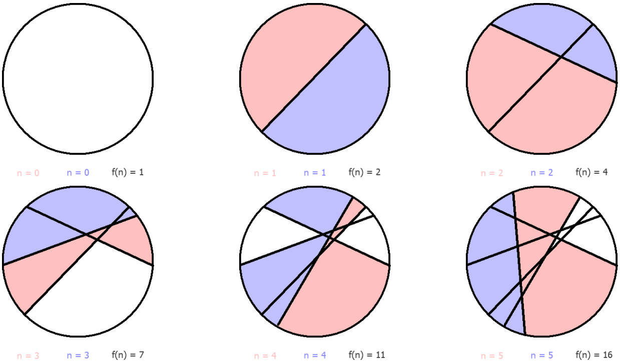


Figure: The maximum number of pieces that n slices cut a circle into

ϕ-Types and Cell Decompositions

Fix $\phi(x; y)$ such as, on the last slide, $x_1 y_1 + x_2 y_2 \leq 1$.

Definition

For $B \subseteq M^y$, let $S_\phi(B)$ be the set of consistent ways to assign a value of true or false to $\phi(x; b)$ for each $b \in B$.

The function $B \mapsto S_\phi(B)$ is an *abstract cell decomposition*:

Definition

An *abstract cell decomposition* for $\phi(x; y)$ is a function $\mathcal{T}$ that assigns to each finite $B \subset M^{|y|}$ a set $\mathcal{T}(B)$ whose elements (cells)

- cover M^x: $M^x = \bigcup \mathcal{T}(B)$
- are not crossed by $\phi(x; B)$: for each $b \in B$, $\phi(x; b)$ is true on the whole cell, or false on the whole cell.

VC Theory

Definition

Let $\pi^*_\phi(n)$ be the maximum of $|S_\phi(B)|$ over all B with $|B| = n$.

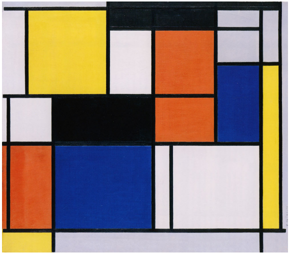
Fact (Sauer-Shelah)

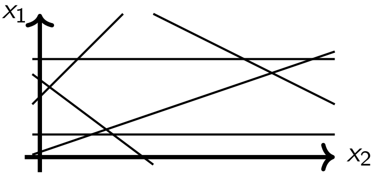
Either $\pi^*_\phi(n) = 2^n$ for all n, or there is a maximum value d where $\pi^*_\phi(d) = 2^d$. We call this the dual VC-dimension of ϕ, and get the bound $\pi^*_\phi(n) = O(n^d)$.

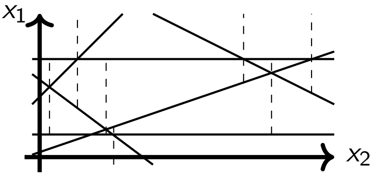
Definition

If every formula ϕ has has finite VC-dimension, we call M *NIP*.

Abstract vs. Distal Cell Decompositions







Distal Cell Decompositions

Definition

A distal cell decomposition $\mathcal{T}$ for a formula $\varphi(x; y)$ is an abstract cell decomposition where each cell of $\mathcal{T}(B)$ is defined by $\theta(x; d)$, where $\theta(x; z)$ is a fixed formula, and d is a tuple of elements of B.

Fact (Chernikov-Simon, Chernikov-Starchenko)

A structure M is distal iff every formula has a distal cell decomposition.

Theorem (A., generalizing Chernikov-Simon)

A metric structure M is distal iff every formula $\phi(x; y)$ has a strong honest definition:

A definable predicate $\theta(x; z)$ such that for any $a \in M^x$ and finite $B \subseteq M^y$, there is a tuple d in B with

- $\theta(a; d) = 0$
- *for all $a' \in M^x$, $b \in B$, $|\phi(a'; b) - \phi(a; b)| \leq \theta(a'; d)$.*

Intuitively, $\theta(x; d)$ bounds how much $\phi(x; b)$ can vary from $\phi(a; b)$, so $\{\theta(x; d) \leq \varepsilon\}$ is a good candidate for a cell containing a.

NIP in Continuous Logic

Definition

Call a predicate $\phi : M^x \times M^y \to [0,1]$ a *VC-class* when for every ε, there exists n such that for every counting measure μ on M^x, there is a tuple $(a_1, \ldots, a_n)$ such that for all $b \in M^y$,

$$\left| \frac{\sum_{i=1}^{n} \phi(a_i; b)}{n} - \mathbb{E}_\mu[\phi(x; b)] \right| \leq \varepsilon.$$

Call a structure M *NIP* when every definable predicate is a VC-class.

In classical logic, any $\phi : M^x \times M^y \to \{0,1\}$ satisfies this "uniform law of large numbers" iff it has finite VC dimension.

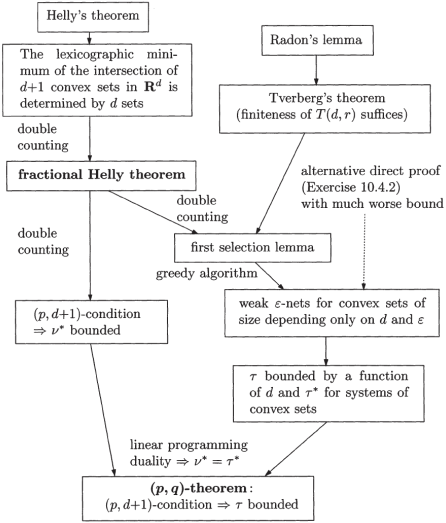
ε-nets

Theorem (A., generalizing Haussler-Welzl)

For every VC-class $\phi(x; y)$ and $\varepsilon > 0$, there is n such that for any counting measure μ on M^x, there is a tuple $(a_1, \ldots, a_n)$ which is a ε-net for ϕ with respect to μ:

For all $b \in M^y$, if $\mu(\{\phi(x; b) = 0\}) \geq \varepsilon$, then $\exists a_i, \phi(a_i; b) < 1$.

Fractional Helly Property and (p, q)-Theorem



Keisler Measures

Definition

A *Keisler measure* in x over A is a regular Borel probability measure on $S_x(A)$ - call the space of these $\mathfrak{M}_x(A)$.

We could also define these in terms of linear functionals on the space of definable predicates:

$$C(S_x(A), [0,1]) \to [0,1]$$
$$\phi(x) \mapsto \int_{S_x(A)} \phi(x)\, d\mu$$

Generically Stable Measures

Keisler measures in general behave terribly, except:

Theorem (A. generalizing Hrushovski-Pillay-Simon)

Assume NIP. The following are equivalent for $\mu \in \mathfrak{M}_x(M)$:

- $\mu(x) \otimes \mu(y) = \mu(y) \otimes \mu(x)$
- *For any $\phi(x; y)$, the "uniform law of large numbers" holds for ϕ with respect to μ.*

We call these measures *generically stable.* They are our best generalization of counting measures, because

- They include counting measures
- They are closed under ultraproducts
- Products $\mu \otimes \cdots \otimes \mu$ behave enough like i.i.d. distributions to prove the uniform LLN and existence of ε-nets.

Real-Valued Distal Cutting Lemma, Roughly

Assuming distality, find ε-nets for strong honest definitions:

Lemma (A. generalizing Chernikov-Galvin-Starchenko)

Let $\phi(x; y)$ be a definable predicate, and fix $\varepsilon > 0$.
For every generically stable $\nu \in \mathfrak{M}_y(M)$, there is a finite cover of M^x such that

- *The cover has bounded size*
- *The cells are uniformly "definable"*
- *On each cell, $\phi(x; b)$ varies by at most ε, for all but a small-measure set of $b \in M^y$.*

If we put a measure on M^x, at least one cell is reasonably large, and we can split the values of b that work for that cell into $\approx \frac{1}{\varepsilon}$ buckets.

Definable Strong Erdős-Hajnal

Theorem (A. generalizing Chernikov-Starchenko)

Assume T distal. For any $\phi(x;y)$, $\varepsilon > 0$, there are $\theta(x;w), \chi(y;z), \delta > 0$ such that for all μ, ν generically stable, there are $c \in M^w, d \in M^z$ where

- $\{\theta(x;c) > 0\}$ *and* $\{\chi(y;d) > 0\}$ *are* (ϕ, ε)*-homogeneous*
- $\int_{S_x(M)} \theta(x;c)\, d\mu \geq \delta$
- $\int_{S_y(M)} \chi(y;d)\, d\nu \geq \delta$

Real-Valued Strong Erdős-Hajnal

0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0
0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0
0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0
0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0
0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1
0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1
0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1
0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1
0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2
0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2
0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2
0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2
0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2
0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4
0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4
0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4
0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4
0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5

Real-Valued Strong Erdős-Hajnal

0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0	0.0
0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0
0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.0
0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0
0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1
0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1
0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1
0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1	0.1
0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2	0.1
0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.2
0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2
0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2	0.2
0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2	0.2
0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3	0.2
0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4	0.3
0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.4
0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4
0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4	0.4
0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.4
0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.6	0.6	0.6	0.6	0.5	0.5	0.5	0.5	0.5