Finding Order in Metric Structures

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Metric Structures

Definition

A *metric language* is just like a regular first-order language, consisting of functions and relations.

Definition

A *metric structure* consists of:

- A complete metric space of diameter 1
- For each n-ary function symbol, a uniformly continuous function $M^n \to M$
- For each n-ary relation symbol, a uniformly continuous function $M^n \to [0, 1]$

Formulas

Definition

An *atomic formula* is defined as usual, except instead of $=$, the basic relation is $d(x, y)$.

Definition

A *formula* is

- An atomic formula
- $u(\phi_1, \ldots, \phi_n)$ where ϕ_is are formulas and $u : [0,1]^n \to [0,1]$ is continuous
- $\sup_x \phi$ or $\inf_x \phi$

Definition

A *definable predicate* is a uniform limit of formulas.

Making Linear Orders Metric

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- Itaï Ben Yaacov has described Ordered Real Closed Metric Valued Fields, but making the metric space bounded complicated things.
- Diego Bejarano and I are working to simplify this approach.

Metric Linear Orders

- Call M a *metric linear order* if
  - M has a bounded complete metric
  - M has a linear order
  - open balls are order-convex.
- M is a metric structure in the language $\{r\}$, with

$$r(x,y) = \begin{cases} 0 & x \leq y \\ d(x,y) & y \leq x \end{cases}$$

- Think of $r(x,y)$ as "the amount x is greater than y."

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- $\sup_{x,y,z} r(x,z) \dot{-} (r(x,y) + r(y,z)) = 0$

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Definition

Let UDLO be the theory of *ultrametric-dense linear orders*, consisting of MLO with the following axioms:

- $d(x, z) \leq \max(d(x, y), d(y, z))$
- For any rational $p \in \mathbb{Q} \cap [0, 1]$, $\sup_x \inf_y |r(x, y) - p| = 0$
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- *The metric and order topologies agree in a model of UDLO*
- dcl *in UDLO is metric closure*
- *Orders Van Thé put on Urysohn ultrametric spaces model UDLO*

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Lemma (A., Bejarano)

$U_S \vDash \mathrm{UDLO}$.

Theorem (Van Thé)

U_S has extremely amenable automorphism group, following from Fraïssé theory in the discrete-logic language of ordered S-valued metric spaces.

o-Minimality in Discrete Logic

Fact

If M expands a linear order, TFAE:

- *every formula $\phi(x)$ in one variable is qf-definable in $\{<\}$*
- *every formula $\phi(x)$ in one variable is a finite union of intervals.*

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- How do we describe these properties for MLOs?

Metric o-Minimality

Theorem (A., Bejarano)

If M expands a metric linear order, TFAE:

- *every predicate $\phi(x)$ in one variable is qf-definable in $\{r\}$*
- *every predicate $\phi(x)$ in one variable is regulated (a uniform limit of step functions).*

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Definition

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By QE, a model of UDLO is o-minimal.

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Proof.

The definable predicate $r(f(x), a)$ is regulated. Approximate this with an appropriate step function, and partition into the intervals on which it is constant. $\square$

Definable Sets

Definition

A set $D \subseteq M$ is *definable* when it is closed and $\inf_{y \in D} d(x, y)$ is a definable predicate.

Theorem (Definable completeness: A., Bejarano)

Let M be an o-minimal metric structure and $D \subset M$ a definable set. If D is bounded above (resp. below), then D has a least upper bound (resp. greatest lower bound).

Theorem (A., Bejarano)

Let M be an o-minimal expansion of MDLO and $D \subset M$ a definable set. The complement of D is a union of countably many intervals, with only finitely many of diameter $\geq \varepsilon$ for any $\varepsilon > 0$.

Cell Decomposition

By using the bounded alternation numbers of (weakly) regulated functions, we can build distal cell decompositions:

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Our main goal is to find more specific o-minimal cell decompositions for definable predicates and sets.

Thank you, ASL Model Theory Session!