Logic Notes

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Chapter 1

Introduction and Definable Sets

1.1 Introduction

In this class, we will explore techniques that let us apply logic, and model theory in particular, to everyday mathematics. Let's start in a context that should already be familiar, and will only become moreso: structures in the language of ordered rings.

This is centainly an everyday mathematical context. Structures in the language $\{0, 1, +, \times, \leq\}$ include $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. But what can we actually say about these ordered rings (or in the case of $\mathbb{N}$, a semiring) just using first-order logic in this language?

At the most basic level, we can ask which sentences these different structures satisfy. It's a pretty straightforward exercise to determine that these structures differ already at that level.

Exercise 1.1.1. For each pair of structures in the list $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, find a sentence that one satisfies, while the other does not.

Meanwhile, if we change the language by dropping multiplication (or down to just $\leq$), we find that in the languages of linear orders $\{\leq\}$ and of ordered (semi)groups $\{0, +, \leq\}$, we can distinguish $\mathbb{N}$ from $\mathbb{Z}$ from $\mathbb{Q}$ and $\mathbb{R}$, but $\mathbb{Q}$ and $\mathbb{R}$ satisfy exactly the same sentences.

Definition 1.1.2. Given a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$, let $\mathrm{Th}(\mathcal{M})$ (the *complete theory of* $\mathcal{M}$) denote the set of all $\mathcal{L}$-sentences ϕ such that $\mathcal{M} \vDash \phi$.
Given $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, say that they are *elementarily equivalent*, denoted $\mathcal{M} \equiv \mathcal{N}$, when $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. That is, when for each sentence ϕ, $\mathcal{M} \vDash \phi \iff \mathcal{N} \vDash \phi$.

To show that $\mathbb{Q}$ and $\mathbb{R}$ are elementarily equivalent in either of the above languages, we use the same strategy: find an easily-axiomatized complete theory that they both model.

Lemma 1.1.3. *If T is a complete theory, and $\mathcal{M} \vDash T$, then $\mathrm{Th}(\mathcal{M})$ is the set of all consequences of T. Thus if $\mathcal{M}, \mathcal{N} \vDash T$, $\mathcal{M} \equiv \mathcal{N}$.*

In the case of the language of linear orders, the theory they both model and its completeness may be familiar from 5700:

Fact 1.1.4. *There is a complete theory* DLO *(dense linear orders) in the language* $\{\leq\}$ *of linear orders whose models include* $\mathbb{Q}$ *and* $\mathbb{R}$.

In the language of ordered groups, things get a little trickier. We will prove the following in this class.

Fact 1.1.5. *There is a complete theory* ODAG *(ordered divisible abelian groups) in the language* $\{0, +, \leq\}$ *of ordered groups whose models include* $\mathbb{Q}$ *and* $\mathbb{R}$.

In the rest of this class, we will not care too much about specific structures - we will care more about their complete theories. We will develop tools for proving completeness of such theories, for classifying them based on complexity, and for evaluating formulas and sentences modulo these theories.

We will find that some theories are inherently difficult to understand, because of Gödelian phenomena you have seen in 5700, while others are actually very nice!

Incompleteness gives us some contrived examples of difficult-to-resolve sentences in the structure $(\mathbb{N}; 0, 1, +, \times, \leq)$. Historically, a huge fraction of mathematical effort has been spent trying to resolve the truth of sentences in this structure. To state these, recall that there is a formula in this structure that determines whether a number is prime:

$$\mathsf{Prime}(x) := 1 < x \wedge \forall y, \forall z, (y \times z = x) \to (y = 1 \vee z = 1).$$

Given this, we can state the following sentences:

$$\begin{aligned} \text{The Twin Primes Conjecture: } & \forall n, \exists p, n \leq p \wedge \mathsf{Prime}(p) \wedge \mathsf{Prime}(p+2) \\ \text{The Goldbach Conjecture: } & \forall n, 1 < n \to \exists p, \exists q, \mathsf{Prime}(p) \wedge \mathsf{Prime}(q) \wedge p + q = n + n. \end{aligned}$$

We can view this as a consequence of the *definable set* of primes being somewhat complicated, and the extra quantifiers $(\forall, \exists)$ applied to make it into these sentences drives the complexity higher.

Meanwhile, in $(\mathbb{R}; 0, 1, +, \times, \leq)$, the story is very different. Consider the set in $\mathbb{R}^3$ defined by the formula

$$\phi(a, b, c) \iff \exists x, ax^2 + bx + c = 0.$$

We find that $\phi(a, b, c)$ is equivalent to

$$(a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0)).$$

To evaluate this, we only need to perform a handful of algebraic operations, checks of equality, and boolean operations.

We will find finitely-axiomatized and decidable complete theories, where all of the "paradoxes" of incompleteness are irrelevant. In order to determine which *sentences* are true in a given theory, we will want to find out how to evaluate all formulas, including those with free variables. The main way we will do this is by *eliminating quantifiers*, allowing us to turn the complexity of first-order logic into something tractable, and much closer to propositional logic.

Furthermore, these nice theories come in a variety of different flavors - it is easy to work with vector spaces over a field, and it is easy to work with algebra over the real numbers, but for somewhat different reasons. These subtleties will come out when we look at the combinatorics inherent in these structures.

For all of these purposes, we need to understand formulas, not only syntactically, but semantically in terms of the sets they define.

1.2 Definability

Definition 1.2.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $A \subseteq M$. Then a set $D \subseteq M^n$ is called A-*definable* when there is a formula $\phi(\bar{x}; \bar{y}) = \phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and parameters $\bar{b} \in A^m$ such that for all $\bar{a} \in M^n$, $\bar{a} \in D \iff \mathcal{M} \vDash \phi(\bar{a}; \bar{b})$.

For some basic examples, let's look at the language $\{0, 1, +, \times\}$ of (semi)rings. We will see that in each of the structures $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, the set $\{(x, y) : x \leq y\}$ is definable.

This is easy for $\mathbb{N}$: $\exists z, y = x + z$.

This is almost as easy for $\mathbb{R}$: $\exists z, y = x + z^2$.

For $\mathbb{Z}$ and $\mathbb{Q}$, we have to use Lagrange's Four-Square Theorem: Every $n \in \mathbb{Z}$ with $n \geq 0$ can be written as the sum of four perfect squares. Thus for $\mathbb{Z}$, we can use

$$\exists z_1, z_2, z_3, z_4, y = x + z_1^2 + z_2^2 + z_3^2 + z_4^2.$$

This will actually also work in $\mathbb{Q}$ - exercise if you want!

This means that in each of these structures, everything definable in the language $\{0, 1, +, \times, \leq\}$ is already definable without the symbol $\leq$. That's because if there's a formula $\phi(x; y)$ in the ring language equivalent to $x \leq y$, we can just replace every instance of $t_1 \leq t_2$ with $\phi(t_1; t_2)$.

This is an instance of an *expansion by definitions*, which is where we add a symbol whose interpretation is already definable to the language. This does not change the definable sets.

Another example of expansion by definition is in arithmetic. In $(\mathbb{N}; 0, 1, +, \times, \leq)$, the exponentiation operation is definable, so we frequently add it to the language for convenience.

1.2.1 Undefinability

It can be harder to show that sets are *not* definable, but this is just as informative.

In the realm of arithmetic, we will be able to show undefinability by diagonalization:

Fact 1.2.2 (Tarski's Undefinability of Truth (Simple Version))**.** *The set*

$$\{\ulcorner \phi \urcorner : \phi \textit{ is a sentence such that } (\mathbb{N}; 0, 1, +, \times, \leq) \vDash \phi\}$$

of natural numbers is not definable in $(\mathbb{N}; 0, 1, +, \times, \leq)$.

Other structures that model theorists prefer tend to lack the expressive power to even do this diagonalization proof. In these cases, we will be able to place limits on definability by proving structural theorems about definable sets.

This starts by proving *quantifier elimination* in an appropriate language. That is, we will show that, after possibly expanding by definitions a little bit, every formula is equivalent to one that can be written without $\forall$ or $\exists$. This is true, for instance, in the familiar structures

$$\begin{array}{l}
(\mathbb{Q}, <) \\
(\mathbb{R}, <) \\
(\mathbb{Z}, 0, +, \leq) \\
(\mathbb{C}, 0, 1, +, \times) \\
(\mathbb{R}, 0, 1, +, \times, \leq),
\end{array}$$

as we will show.

Because formulas without quantifiers are much easier to study, quantifier elimination will allow us to characterize definable sets quite easily. For instance, over the complex numbers (or any other algebraically closed field):

Fact 1.2.3. *The quantifier-free definable sets (with parameters) in* $(\mathbb{C}; 0, 1, +, \times)$ *are exactly the* constructible *sets - that is, boolean combinations of zerosets of polynomials with coefficients in* $\mathbb{C}$.

Once we characterize definable sets in this way, we can start proving more interesting properties of definable sets, and easily show that other sets are not definable. Staying with the algebraically closed field example:

Corollary 1.2.4. *The structure* $(\mathbb{C}; 0, 1, +, \times)$ *is* strongly minimal*: any definable subset of* $\mathbb{C}$ *(in one dimension) is either finite or cofinite.*

Proof. Zerosets of polynomials are either finite or cofinite, and any boolean combination of finite and cofinite sets is also finite or cofinite. □

Corollary 1.2.5. *There is no definable linear order on* $\mathbb{C}$*. In fact,* $\mathbb{C}$ *is stable - there cannot be an infinite linear order* I*, sequences* $(a_i, b_i : i \in I)$ *with* $a_i \in \mathbb{C}^m, b_i \in \mathbb{C}^n$ *and a formula* $\phi(x, y)$*, even with parameters, such that*

$$i \leq j \iff \mathbb{C} \vDash \phi(a_i, b_j).$$

1.3 Overview

Our first objective in this course is to build up a library of easy-to-understand structures. We will start with countable structures we can explicitly construct from finite structures: *Fraïssé limits*. These include structures such as $(\mathbb{Q}, <)$, the random graph, and the countable atomless Boolean algebra. We will learn how to show completeness, $\aleph_0$-categoricity, and then quantifier elimination, for theories of Fraïssé limits, building on the back-and-forth technique mentioned in 5700.

We will then develop a more comprehensive toolkit for showing quantifier elimination in more complicated structures, such as algebraically closed and real closed fields, at which point we can really begin applying model theory to these structures.

Once we have seen how simple definable sets can be, we will contrast with how *complicated* they can be when we don't have quantifier elimination, such as in arithmetic.

We will also review compactness somewhere around here, including a semantic proof featuring the ultraproduct construction.

Then we will pick up the story of definable sets in specifically *ordered* structures such as $(\mathbb{R}; 0, 1, +, \times, \leq)$. We will see how *o-minimality*: the simplest case for *one-dimensional* definable sets in an ordered structure, implies a powerful structural theorem (the *cell decomposition theorem*) for definable sets in all dimensions, even without quantifier elimination. This will let us work with structures such as $\mathbb{R}_{\text{an,exp}}$, which at the moment is the most fruitful context for applying model theory to other branches of math.

We then have some choices for where to go next. Some of my ideas include the following:

- Dimension theory, and in particular, pregeometries/matroids, in strongly minimal and o-minimal structures

- Incidence combinatorics and distal cell decompositions (the combinatorics of definable sets over $(\mathbb{R}; 0, 1, +, \times, \leq)$)
- NIP, VC-dimension and connections with statistical learning theory.

Please let me know if you have preferences about what you'd like me to cover.

If I *don't* get to cover what you want, I have good news: you can cover it yourself! For students enrolled in graduate class course numbers (perhaps we will change the exact mechanism, but certainly for graduate students who want to), there is a presentation option for grading in this class, requiring one (or possibly two if time allows) 45-minute presentations. I have a list of suggested papers for presenting on the course website, and will add to it over time.

Chapter 2

Fraïssé Limits

2.1 Fraïssé Classes

Before we get into the weeds of model theory, we should spend some time developing a library of examples. These can include famous structures, like the algebraic ones we have seen so far, but should also include complete theories.

We will start with a method for constructing particularly nice countable structures, called *Fraïssé limits*. These are constructed as limits of families of finite substructures. To ground *this* construction in an example to start, recall dense linear orders from 5700:

Fact 2.1.1. *Let* $\mathcal{L}_< = \{<\}$.
The $\mathcal{L}_<$*-theory of* dense linear orders without endpoints*, abbreviated* DLO*, is complete,* $\aleph_0$*-categorical, and* $(\mathbb{Q}, <) \models \mathsf{DLO}$.

To generate a structure like $(\mathbb{Q}, <)$, we start by looking at its finite substructures.

Definition 2.1.2. If $\mathcal{L}$ is a relational language (that is, has no function symbols), and $\mathcal{M}$ is an $\mathcal{L}$-structure, let $\mathsf{Age}(\mathcal{M})$ be the class of all $\mathcal{L}$-structures isomorphic to a finite substructure of $\mathcal{M}$.

As described, the age is a proper class. If you don't like this, you can use the set of *isomorphism types* of finite substructures of $\mathcal{M}$ instead.

Example 2.1.3. $\mathsf{Age}(\mathbb{Q}, <)$ consists of all finite linear orders.

Proof. Any finite linear order is isomorphic to any other finite linear order of the same cardinality, and for every finite cardinality n, $(\mathbb{Q}, <)$ contains a finite subset, and thus a finite substructure which is a linear order, of cardinality n. Also, any structure in $\mathsf{Age}(\mathbb{Q}, <)$ must be isomorphic to a finite substructure, and thus must be a finite linear order. □

We can make a few observations about the class of all finite linear orders, which we will describe as properties:

Example 2.1.4. Let $\mathcal{K}$ be the class of all finite linear orders.

- **Essential Countability (EC):** Up to isomorphism, there are only countably many structures in $\mathcal{K}$.
- **Hereditary Property (HP):** If $A \in \mathcal{K}$, and B is a finite substructure of A, then $B \in \mathcal{K}$.
- **Joint Embedding Property (JEP):** If $A, B \in \mathcal{K}$, then there is some $C \in \mathcal{K}$ into which both A and B embed.
- **Amalgamation Property (AP):** If $A, B_1, B_2 \in \mathcal{K}$, and there are embeddings $f_i : A \hookrightarrow B_i$, then there is $C \in \mathcal{K}$ with embeddings $g_i : B_i \hookrightarrow C$ such that $g_1 \circ f_1 = g_2 \circ f_2$, making the following diagram commute:

$$\begin{tikzcd}
& C & \\
B_1 \arrow[ur, "g_1"] & & B_2 \arrow[ul, "g_2"'] \\
& A \arrow[ul, "f_1"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

In fact, the first three of these hold for the age of *any* countable structure.

Proof. We will prove the first three for $\mathsf{Age}(\mathcal{M})$ where $\mathcal{M}$ is an arbitrary countable structure.

- **EC:** A countable structure has only countably many finite subsets, and thus countably many finite substructures.
- **HP:** Any finite substructure of a finite substructure of $\mathcal{M}$ is also a finite substructure of $\mathcal{M}$.
- **JEP:** If A, B are finite substructures of $\mathcal{M}$, then (in a relational language) $A \cup B$ is also a finite substructure, into which both embed.
- **AP:** This one we're only proving for $\mathcal{K} = \mathsf{Age}(\mathbb{Q}, <)$. Enumerate A as $a_1 < a_2 < \cdots < a_n$. Then to define C, we will place elements in the gaps between elements of A. To extend the embedding f_1, we need to make sure that there are at least as many elements of C between a_i, a_{i+1}, as there are in B_1 between $f_1(a_i)$ and $f_1(a_{i+1})$, and similarly for f_2.

$\square$

Definition 2.1.5. When $\mathcal{L}$ is a finite relational language, we call any class $\mathcal{K}$ of finite $\mathcal{L}$-structures a *Fraïssé class* when it satisfies EC, HP, JEP, and AP.

We now turn to another familiar example of a Fraïssë class of finite structures: all finite graphs.

Theorem 2.1.6. *The class of all finite graphs is a Fraïssë class.*

Proof.

- **EC:** For any n, there are finitely many graphs on n vertices up to isomorphism, so there are countably many when we union over all $n \in \mathbb{N}$.

- **HP:** A substructure of a graph is a graph.
- **JEP:** We can just put the two graphs next to each other, and choose arbitrarily whether to put edges between the two graphs.
- **AP:** If A embeds into B_1 and B_2, then we can add both sets of vertices $B_1 \setminus A$ and $B_2 \setminus A$ to A. We know which edges we need between elements of A and B_i, and can choose what edges to put between B_1 and B_2 arbitrarily.

$\square$

These are the two most canonical examples - there are others, but many of those require adding function symbols into the language, which makes things a little more complicated. This complicates the definitions slightly, but the idea of everything we do can be extended to that with a few more assumptions.

Exercise 2.1.7. Show that the class of all finite *triangle-free* graphs is a Fraïssé class.

2.2 Fraïssé Limits

Now that we've noticed that $(\mathbb{Q}, <)$ is a countable structure whose age is a Fraïssé class, I can explain why this structure is special. After all, $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$ are also countable linear orders with the same age, but their theories are not $\aleph_0$-categorical.

The critical idea is *homogeneity*:

Definition 2.2.1. Call a structure $\mathcal{M}$ *ultrahomogeneous* when for any finite substructures A, B of $\mathcal{M}$ and an isomorphism $f : A \to B$, there is an isomorphism $g : M \to M$ that extends f.
If $\mathcal{L}$ is a relational language, $\mathcal{K}$ is a class of finite $\mathcal{L}$-structures, and $\mathcal{M}$ is a countably infinite ultrahomogeneous $\mathcal{L}$-structure whose age is $\mathcal{K}$, we call $\mathcal{M}$ a *Fraïssé limit* of $\mathcal{K}$.

Example 2.2.2. It is not hard to check that $(\mathbb{Q}, <)$ is ultrahomogeneous, and thus a Fraïssé limit for the class of finite linear orders.

We can start to connect Fraïssé limits to Fraïssé classes:

Theorem 2.2.3. *If $\mathcal{K}$ is a class of finite structures in a relational language with a Fraïssé limit $\mathcal{M}$, then $\mathcal{K}$ is a Fraïssé class.*

Proof. Because $\mathcal{K} = \mathsf{Age}(\mathcal{M})$, it must satisfy **EC**, **HP**, and **JEP**, so we just need to check **AP**. Suppose A, B_1, B_2 are isomorphic to finite substructures of $\mathcal{M}$, and $f_i : A \to B_i$ are embeddings. We can assume (up to an isomorphism of everything involved) that A, B_1, B_2 are actual substructures of $\mathcal{M}$, and that f_1 is the inclusion map of A into B_1, but we can't simultaneously assume that A is a substructure of B_2, or that f_2 is an inclusion.
We can view f_2 as an isomorphism from A to its image, a substructure of B_2, and by ultrahomogeneity, we can let $h : M \to M$ be an automorphism extending f_2. Then $h^{-1} \circ f_2$ is the inclusion map of A into M, and its image is contained in $h^{-1}(B_2)$. We now let C be a finite substructure containing both B_1 and $h^{-1}(B_2)$. We then let $g_1 : B_1 \hookrightarrow C$ be the inclusion map, and let $g_2 : B_2 \hookrightarrow C$ be the inclusion map ι composed with the restriction of h^{-1} to B_2. Then $g_1 \circ f_1$ is the inclusion map of A into C, while in $g_2 \circ f_2$, the restriction of h^{-1} to B_2 cancels with

f_2 to form another inclusion map, giving us the same map in the end. □

Meanwhile, to check that something is a Fraïssé limit, we can check homogeneity step-by-step, in a back-and-forth procedure that may look familiar:

Lemma 2.2.4. *A countable structure $\mathcal{M}$ is ultrahomogeneous if and only if the following holds: For any isomorphism $f : A \to B$ of finite substructures of $\mathcal{M}$, and any finite substructure $A \subseteq C$, the isomorphism f extends to an isomorphism $g : C \to D$ of finite substructures.*

Proof. If $\mathcal{M}$ is ultrahomogeneous, then given $f : A \to B$ and C, we simply let $h : M \to M$ extend f, and then restrict h to C.
For the main direction, we're actually going to prove something a smidgen more general:

Lemma 2.2.5. *Suppose $\mathcal{M}, \mathcal{N}$ are countable structures such that for every isomorphism $f : A \to B$ of finite substructures $A \subset M, B \subset N$, and any finite substructure $A \subseteq C \subset M$, the isomorphism f extends to an isomorphism $g : C \to D \leq N$ of finite substructures, and the same holds with $\mathcal{M}, \mathcal{N}$ switched.*
Then f extends to an isomorphism $h : M \to N$.

All we need for ultrahomogeneity is to apply this to $N = M$.
Now assume $\mathcal{M}, \mathcal{N}$ has this property, and let $f : A \to B$ be an isomorphism of finite substructures. We can construct an isomorphism $h : M \to N$ extending f recursively, as the union of isomorphisms $f_k : A_k \to B_k$, each of which extends the last, where $\bigcup_k A_k = M$ and $\bigcup_k B_k = N$. Enumerate $M = \{m_0, m_1, \dots\}, N = \{n_0, n_1, \dots\}$. We start with $f_0 = f$. Then we recurse a bit differently in even and odd steps. Assuming we have defined $f_{2k} : A_{2k} \to B_{2k}$, we want to make sure that A_{2k+1} contains m_k, which in the long run, will ensure that $\bigcup_k A_k = M$. If $m_k \in A_{2k}$, we don't need to do anything. If not, then let A_{2k+1} be a finite substructure with $A_{2k} \cup \{m_k\} \subseteq A'$. By our assumption, we can then extend to an isomorphism $f : A_{2k+1} \to B_{2k+1}$. Now assuming we have $f_{2k+1} : A_{2k+1} \to B_{2k+1}$ defined, let's make sure that B_{2k+2} contains n_k, ensuring that $\bigcup_k B_k = N$. We pick B_{2k+2} containing $B_{2k+1} \cup \{n_k\}$, then we extend f_{2k+1}^{-1} to an isomorphism $B_{2k+2} \to A_{2k+2}$, whose inverse extends f_{2k+1}. □

2.3 Random Graphs

Let's advance our other example. We had another Fraïssé class, the class of finite graphs. Does this have a Fraïssé limit? If we have no idea how to build one, we may as well try to build it at random. Let's just take a countably infinite set of vertices, and for each pair of vertices, we'll flip a coin to determine whether there should be an edge.

This generates a random countably infinite graph - what's its age? For any finite graph A with n vertices, the probability of a given n vertices forming an isomorphic graph is going to be positive. In fact, it's at least $2^{-\binom{n}{2}}$, because that's the probability of each precise configuration of edges and non-edges. Just call it $p > 0$.

Then we can bound the probability that *no* set of n vertices in the infinite random graph is isomorphic to A. To simplify things, let's split the vertices into an infinite sequence of disjoint sets of n vertices. Then let E_k be the event that the kth set is isomorphic to A - this is p. Then the probability that *none* of these events happen is $\lim_{k\to\infty}(1-p)^k = 0$, so with probability 1, at least one of these events happens, so A is in the age.

Taking the intersection over countably many graphs, we see that the probability of *every* finite graph being in the age is also 1.

But what's the probability of ultrahomogeneity? By (Lemma 2.2.4), to have homogeneity, we just need to check that for every isomorphism between finite subgraphs, and every extension of one to a larger finite subgraph, the isomorphism extends. There are countably many choices of $f : A \to B$ and $C \geq A$, so let's find the probability of each one extending. We just need to find $|C \setminus A|$ vertices that have the same relationship to the vertices of B, and to each other, as the vertices of $C \setminus A$ have to A. The probability of this happening, for any ordered list of $|C \setminus A|$ vertices, is positive, as it is 2^{-n}, where n is the number of edges we need. As before, the probability of this positive probability failing for each of an infinite sequence of independent ordered lists is 0, so with probability 1, we can extend this map. The probability that each of these countably many probability-1 properties holds is 1, so with probabilistic certainty, we have generated a Fraïssé limit, completely at random.

2.4 Uniqueness

We know that $(\mathbb{Q}, <)$ is a Fraïssé limit of the class of finite linear orders, but what about other countable structures with that age?

We can rule out $(\mathbb{N}, <)$, as any automorphism will leave 0 as the left endpoint This means that the substructure $\{0\}$ can't be mapped by an automorphism to any of the isomorphic substructures $\{n\}$. In general, any Fraïssé limit of this class must not have a left endpoint, or for that matter, a right endpoint.

We can also rule out $(\mathbb{Z}, <)$, as any automorphism will leave consecutive elements consecutive. This means that the substructure $\{0, 1\}$ can't be mapped by an automorphism to the isomorphic substructure $\{0, 2\}$.

We can extend that argument. Let $\mathcal{M}$ is a Fraïssé limit of this class with elements $a < b$. As there is no right endpoint, let $a < b < c$. As $\{a, b\}$ and $\{a, c\}$ are isomorphic, there must be an automorphism $h : M \to M$ with $h(a) = a$ and $h(c) = b$. It must thus send b to some $h(b)$ with $a = h(a) < h(b) < h(c) = b$, showing that $\mathcal{M}$ is dense.

We have thus shown that any Fraïssé limit of this class is a countable model of DLO - we know there's only one of these up to isomorphism. We will now provide another proof of this which works much more generally.

Theorem 2.4.1 (Uniqueness of Fraïssé limits)**.** *If $\mathcal{M}, \mathcal{N}$ are both Fraïssé limits of a class $\mathcal{K}$, then they are isomorphic.*
In fact, if $f : A \to B$ is an isomorphism between a finite substructure of $\mathcal{M}$ and a finite substructure of $\mathcal{N}$, then there is an isomorphism extending f.

Proof. By (Lemma 2.2.5) and symmetry, we only need to check that if $C \supset A$ is a finite substructure of $\mathcal{M}$, then f extends to C.
We know by amalgamation that there is some $D \in \mathcal{K}$, with embeddings $g_B : B \to D$ and $g_C : C \to D$, such that $g_B \circ f$ equals the composition of g_C with the inclusion map, which means g_C extends $g_B \circ f$. We can shrink D to be the range of g_C, in which case g_C is an isomorphism. This would be perfect, with g_C extending f, if $B \subseteq D \subset N$. All we know is that D is isomorphic to a substructure of N - but we may as well assume it actually is one, because composing with that isomorphism won't change anything so far. Then g_B, restricted to its image, is an isomorphism $B \to g_B(B) \subseteq D$, which must extend to an automorphism $h : N \to N$. Thus the substructure we're looking for is $B \subseteq h^{-1}(D) \subset N$, as $h^{-1} \circ g_B$ is just the inclusion map, and then $h^{-1} \circ g_C$, appropriately restricted, gives an isomorphism $C \to h^{-1}(D)$ extending $h^{-1} \circ g_B \circ f$, and thus f. □

To the category theory enthusiasts in the audience, note that this uniqueness is *not* uniqueness up to unique isomorphism. Fraïssé theory is otherwise very categorical, but this provides an unusual wrinkle in categorical presentations of this topic. If you're interested in giving a presentation on the categorical aspects of Fraïssé theory later in the semester, I have some recommended papers on the course website.

2.5 Examples

Example 2.5.1. The class of finite equivalence relations (in the language $\{E\}$ of one binary relation symbol for equivalence) is Fraïssé, and its Fraïssé limit is the equivalence relation with $\aleph_0$ many $\aleph_0$-sized classes.

Example 2.5.2. The class of finite triangle-free graphs has a Fraïssé limit, one of a family of *Henson graphs*.
One can construct this by ordering the random graph in a particular way, and removing the third vertex of every triangle that appears. (We will see a general Fraïssé-theoretic construction shortly.)

2.5.1 Now With Function Symbols

Most of the construction we've done so far works with function symbols also, although we'll have to reckon with the fact that not every subset is a substructure. For full generality, we'd have to work with *finitely generated* rather than finite substructures, but here are some examples that don't need that:

Example 2.5.3. The class of finite fields of characteristic p is Fraïssé, and its Fraïssé limit is the algebraic closure of F_p.

Example 2.5.4. If K is a finite field, then the class of finite-dimensional K-vector spaces is Fraïssé, and the $\aleph_0$-dimensional K-vector space is its Fraïssé limit.

Example 2.5.5. The class of finite boolean algebras (in the language $\{\top, \bot, \wedge, \vee, \cdot^c\}$) is Fraïssé, and it has a Fraïssé limit, the unique countable atomless boolean algebra.

Example 2.5.6. The classes of finite groups and finite abelian p-groups are also Fraïssé, although their limits are harder to introduce.

Example 2.5.7. If we generalize vastly to allow *continuous logic*, there are quite a few more familiar examples, such as

- the Urysohn space is the Fraïssé limit of finite metric spaces
- the separable Hilbert space is the Fraïssé limit of finite-dimensional/Euclidean Hilbert spaces
- $[0, 1]$ with Lebesgue measure is the Fraïssé limit of finite probability algebras
- various interesting Banach spaces are also Fraïssé limits

Take note that some things will be more complicated - only some of these have $\aleph_0$-categorical theories.

2.6 Existence

Now let's actually construct Fraïssé limits in general.

Theorem 2.6.1 (Existence of Fraïssé limits)**.** *If $\mathcal{K}$ is a Fraïssé class, then it has a Fraïssé limit.*

Proof. We will prove this assuming a relational language, just for simplicity.
We will construct this as a direct limit (union) of a chain of structures in $\mathcal{K}$. If we have a chain

$$D_0 \hookrightarrow D_1 \hookrightarrow D_2 \hookrightarrow \dots$$

of structures, and we think of each embedding as inclusion, then it is easy to check that $\bigcup_n D_n$ is also a structure in a sensible way.

Lemma 2.6.2. *In this construction,* $\mathsf{Age}(\bigcup_n D_n) = \bigcup_n \mathsf{Age}(D_n)$.

Proof. Any finite subset of $\bigcup_n D_n$ must already be contained in some D_n, and is thus already a finite substructure of some element of the chain. ☐

Thus to get the correct age, we only need to make sure that each element of $\mathcal{K}$ embeds into some element of our chain.
To get ultrahomogeneity, we need to make sure that for each D_n, and all substructures $A, B, C \subseteq D_n$ with $A \subseteq C$ and $f : A \to B$ an isomorphism, f can be extended to an isomorphism from C to some other substructure of the direct limit. For this, we will need to make sure that for some $m > n$, f can be extended to an isomorphism from C to another substructure of D_m.
To do all this, use essential countability to enumerate a representative of each isomorphism type in $\mathcal{K}$ as $A_0, A_1, A_2, \dots$. Then for each structure $D \in \mathcal{K}$, enumerate the triples (f, A, C) where $A \subseteq C$ are substructures and f is an isomorphism from A to another substructure. as $T(D)$.
Now fix a bijection $\pi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that if $\pi(n) = (j, k)$, we always have $j \leq n$.
Let $D_0 \in \mathcal{K}$. For our recursive construction, assume we have defined the chain through D_n. Then to define D_{n+1} extending D_n, let $(j, k) = \pi(n+1)$. We will make sure that A_n embeds into D_{n+1}, and that if (f, A, C) is the kth triple in $T(D_j)$, then f extends to an isomorphism from C into another substructure of D_{n+1}.
To do this, we first use JEP to find E into which both D_n and A_n embed, and for notational purposes, assume $D_n \subseteq E$. Then f embeds A into D_n, and thus into E, while inclusion embeds A into C. By AP, we can find embed C and E into another structure F making this square commute, and we may assume that the map from E to F is inclusion. We then let $D_{n+1} = F$. ☐

2.7 Axiomatizability and Categoricity

So, how do we get from all of this to the theory DLO being complete?

Lemma 2.7.1. *Any countable model of* DLO *is a Fraïssé limit for the class of finite linear orders.*

Proof. Any infinite linear order has this age, so it suffices to check ultrahomogeneity.
Suppose $\mathcal{M} \vDash \mathsf{DLO}$, $f : A \to B$ is an isomorphism of finite substructures, and $C \supseteq A$ is a finite substructure. Then we can extend the isomorphism one element at a time. To add $c \notin A$ to the domain A, we first see how it compares to A in the order. It must be in some interval with endpoints in A, either between consecutive elements, or above or below all elements of A. In each of these cases, by density, we can find an element d that fits in the corresponding interval of B by density, and by sending $c \mapsto d$, we can extend the isomorphism. □

Recall that if κ is an infinite cardinal, a theory is *κ-categorical* when it has precisely one model of cardinality κ up to isomorphism.

Theorem 2.7.2. DLO *is $\aleph_0$-categorical.*

Proof. Any two models of cardinality $\aleph_0$ are Fraïssé limits for the same class, and are thus isomorphic. □

This also gives us completeness.

Theorem 2.7.3 (Łoś-Vaught Test)**.** *If an $\mathcal{L}$-theory T is κ-categorical, and $\kappa \geq \aleph_0, |\mathcal{L}|$, then T is complete.*

Proof. Let $\mathcal{M}, \mathcal{N} \vDash T$ be two models. We can then find, by Löwenheim-Skolem, two models $\mathcal{M}', \mathcal{N}' \vDash T$ with $\mathcal{M}' \equiv \mathcal{M}, \mathcal{N}' \equiv \mathcal{N}$, of size κ. These must be isomorphic, and thus elementarily equivalent, so

$$\mathcal{M} \equiv \mathcal{M}' \equiv \mathcal{N}' \equiv \mathcal{N}.$$

□

What we can generalize from this is that if there is a theory whose countable models are precisely the Fraïssé limits of $\mathcal{K}$, then that theory is $\aleph_0$-categorical and thus complete.

We will now examine when this is possible.

Lemma 2.7.4. *Let $\mathcal{L}$ be a finite language, and $A = \{a_1, \ldots, a_n\}$ a finite $\mathcal{L}$-structure. Then there is a formula $\phi(\bar{x})$ with $\bar{x} = (x_1, \ldots, x_n)$ such that in any $\mathcal{L}$-structure $\mathcal{M}$, if $\bar{m} \in M^n$, then $\mathcal{M} \vDash \phi(\bar{m})$ if and only if the map $a_i \mapsto m_i$ defines an isomorphism from A to the set $\{m_1, \ldots, m_n\}$, which is a substructure.*
Furthermore, ϕ doesn't use any quantifiers.

Proof. We need to construct $\phi(\bar{x})$ to check that each of the finitely many symbols of $\mathcal{L}$ is respected. We can write down a formula checking this for each symbol separately, and then take a conjunction.
For a relation symbol $R(x_1, \ldots, x_k)$, we will build a large conjunction. For each of the n^k tuples $(i_1, \ldots, i_k) \in [n]^k$, if $A \vDash R(a_{i_1}, \ldots, a_{i_k})$, we include the literal $R(x_{i_1}, \ldots, x_{i_k})$, and if not, we include the literal $\neg R(x_{i_1}, \ldots, x_{i_k})$.
For a function symbol $f(x_1, \ldots, x_k)$, we still build a large conjunction. For each of the n^k tuples $(i_1, \ldots, i_k) \in [n]^k$, there is a unique $j \in [n]$ such that if $A \vDash f(a_{i_1}, \ldots, a_{i_k}) = a_j$, so we include the literal $f(x_{i_1}, \ldots, x_{i_k}) = x_j$. □

Theorem 2.7.5. *Let $\mathcal{L}$ be a finite language, and let $\mathcal{K}$ be a Fraïssé class of finite $\mathcal{L}$-structures that is* uniformly locally finite*: there is a function $F : \mathbb{N} \to \mathbb{N}$ such that for any n and any subset A of a structure $C \in \mathcal{K}$, there is a substructure B of C with $A \subseteq B \subseteq C$ and $|B| \leq F(|A|)$.*
Then there is a theory T whose models are precisely Fraïssé limits of $\mathcal{K}$.

Proof. First, we will want to limit the age of a model of T. One consequence of uniform local finiteness is that for each n, there is a finite subclass of $\mathcal{K}$ (say, those of size at most $F(n)$), which we call $\mathcal{K}_n$, such that for any n elements in a structure in $\mathcal{K}$, or with age a subsclass of $\mathcal{K}$, those n elements must belong to a substructure isomorphic to one in $\mathcal{K}_n$.
We can then add to T the sentences, for each n,

$$\forall \bar{x}, \bigvee_{A \in \mathcal{K}} \exists \bar{y} \phi_A(\bar{x}, \bar{y}),$$

where $|\bar{x}| = n$, and $\phi_A(\bar{x}, \bar{y})$ is the formula indicating that the elements of $\bar{x}, \bar{y}$ enumerate a structure isomorphic to A.
Any finite tuple of elements in a model of T will now have to be contained in a substructure isomorphic to one of $\mathcal{K}$, ensuring that the age of this model is a subclass of $\mathcal{K}$.
To make sure the age is *all* of $\mathcal{K}$, we can add the sentences

$$\exists \bar{x}, \phi_A(\bar{x})$$

for each $A \in \mathcal{K}$.
Now we wish to ensure ultrahomogeneity. We need to show that if $A \subseteq C \in \mathcal{K}$, then any substructure isomorphic to A extends to one isomorphic to C. We can do this with the sentence

$$\forall \bar{x}, \phi_A(\bar{x}) \to \exists \bar{y}, \phi_C(\bar{x}, \bar{y}).$$

□

Corollary 2.7.6. *Each of the following Fraïssé limits is the unique countable model of a complete, $\aleph_0$-categorical theory:*

- *the countable random graph*
- *the $\aleph_0$-dimensional K-vector space whenever K is a finite field*
- *the countable atomless boolean algebra.*

We should also note that the *quantifiers* $(\forall, \exists)$ appeared in every sentence of these theories in about the same way. Every single sentence can be written (with perhaps a bit of rewriting effort) as $\forall \bar{x}, \exists \bar{y}, \phi(\bar{x}, \bar{y})$, where ϕ has no quantifiers at all.