Q1

(i) $\oint_c 1/z \, dz = 0$

Ans:- The given function is non-analytic at $z=0$. Therefore Cauchy-Integral Theorem will not be applicable for those closed contours C which enclose this singular point 0. Cauchy-Integral theorem will be applicable for those contours C which does not enclose this singular point 0.

(ii) $\oint \frac{\cos z}{z^6 - z^2} dz = 0$

Ans:- For the function to follow Cauchy-Integral theorem, it must be analytic in the simply connected domain. It can be seen that given function is non-analytic where denominator $(z^6 - z^2) = 0$.

Now we find values of z for which denominator is zero.

$$z^6 - z^2 = 0$$

$$z^2(z^4 - 1) = 0$$

$$z^2 = 0, \quad z^4 - 1 = 0$$

$$z = 0, \quad z^4 = 1$$

$$\sqrt[4]{z^4} = \sqrt[4]{1}$$

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As 1 on the Right hand side is a complex number, therefore by taking its fourth root we get 4 values of z.

Let $z' = 1$

$$\therefore \quad r = 1$$

$$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$$\therefore \sqrt[4]{z_k^4} = \sqrt[4]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

$$z_k = \sqrt[4]{1}\left(\cos\left(\frac{2\pi k}{4}\right) + i\sin\left(\frac{2\pi k}{4}\right)\right)$$

$$z_k = \cos\left(\frac{\pi k}{2}\right) + i\sin\left(\frac{\pi k}{2}\right) \quad \text{where } k = 0, 1, 2, 3$$

Now for $k = 0$; $z_0 = \cos(0) + i\sin(0)$

$$\boxed{z_0 = 1 + 0i}$$

$k = 1$; $z_1 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

$$\boxed{z_1 = 0 + i}$$

$k = 2$; $z_2 = \cos(\pi) + i\sin(\pi)$

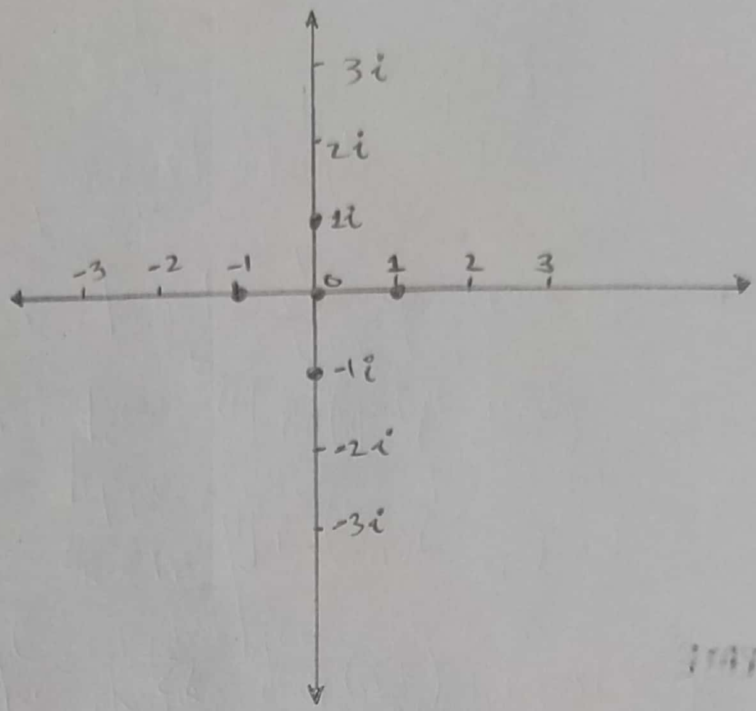
$$\boxed{z_2 = -1 + 0i}$$

$k = 3$; $z_3 = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$

$$\boxed{z_3 = 0 - 1i}$$

Hence the four roots of $z^4 - 1 = 0$ are $\boxed{1}$, $\boxed{-1}$, $\boxed{i}$, $\boxed{-i}$ ~~and~~ $\boxed{\cancel{0}}$.

So, the function is non-analytic at $z = 0, 1, -1, i,$ and $-i$



(iii) $\oint \frac{e}{z^2+9}\, dz = 0$

Ans:. For the function to follow Cauchy-Integral theorem, it must be analytic in the simply connected domain. It can be seen that the given function is non-analytic where denominator $(z^2+9) =$ Hence we find those values of z for which denominator is zero.

$$z^2+9=0$$

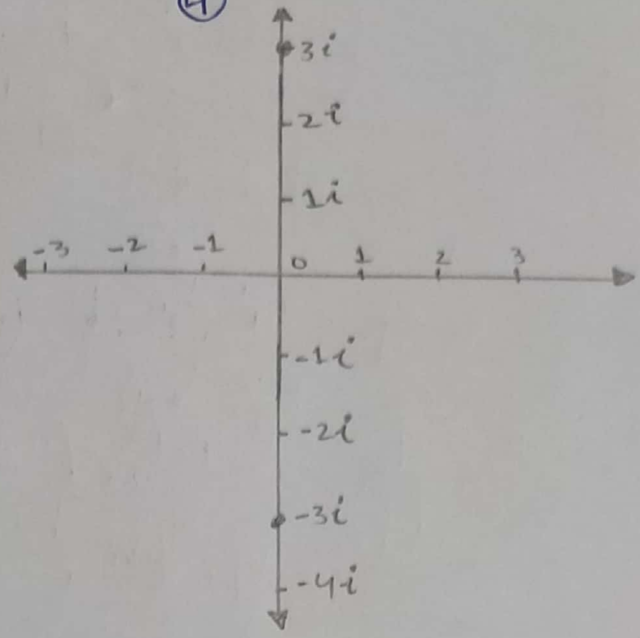
$$\sqrt{z^2} = \sqrt{-9}$$

$$z = \sqrt{-1 \times 3^2}$$

$$z = \pm i \cdot 3$$

$$z = \pm 3i$$

The function is non-analytic at $z=3i$ or $z=-3i$.
Therefore, for any pole which doesn't enclose $z=\pm 3i$
Cauchy Integral theorem will be applicable.



Q2: INTEGRATE

$$\int_c Re\, z\, dz \qquad C: y = x^2 \qquad [0, 1+i]$$

Ans:- Let $f(z) = Re\, z$. then $f(z)$ is non-analytic because real valued function are non-analytic. So, we can't use 1^{st} Evaluation Method. We must use 2^{nd} Evaluation Method to evaluate this integral.
For 2^{nd} Evaluation Method, we need to find $z(t)$ to represent the path of integration in parametric form.

$$\therefore\ z(t) = u(t) + v(t)\, i$$

$$z(t) = t + t^2 i \quad \text{for } [0, 1]$$

$$z^{\circ}(t) = 1 + 2t\, i$$

By 2^{nd} Evaluation Method.

$$\oint_c Re\, z\, dz = \int_0^1 f(z(t)) \cdot z^{\cdot}(t)\, dt$$

Putting values

$$\oint Re\, z\, dz = \int_0^1 (t)(1+2t\,i)\,dt$$

$$'' \quad = \int_0^1 (t + 2t^2 i)\,dt$$

$$'' \quad = \left[\frac{t^2}{2} + \frac{2t^3}{3}\right]_0^1$$

$$'' \quad = \left(\frac{(1)^2}{2} + \frac{2(1)^3}{3}i\right) - \left(\frac{(0)^2}{2} + \frac{2(0)^3}{3}i\right)$$

$$\boxed{\oint Re\, z\, dz = \frac{1}{2} + \frac{2}{3}i}$$

Q3: Ans:- The upper bound for the absolute value of the integral is given by:

$$\left|\oint_C Re\, z\, dz\right| \le ML \quad —(i)$$

L is the length of C which is a parabolic curve while M is the maximum value of given function.

$$M = f(1+i)$$

$$M = Re(1+i)$$

$$\boxed{M = 1}$$

As L is a curve, so we can calculate it's length by using the length of a curve formula:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$a=0,\ b=1$ and $y=x^2$

$$\therefore L=\int_0^1\sqrt{1+\left(\frac{d(x^2)}{dx}\right)^2}\,dx$$

$$L=\int_0^1\sqrt{1+(2x)^2}\,dx$$

$$L=\int_0^1\sqrt{1+4x^2}\,dx$$

By ~~calculating~~ evaluating this definit integral,

we get $\boxed{L=1.48}$

Now the absolute value of closed integral in problem 2 is

$$\left|\oint \mathrm{Re}\,z\,dz\right|=\left|\tfrac{1}{2}+\tfrac{2}{3}i\right|$$

$$\left|\oint \mathrm{Re}\,z\,dz\right|=\sqrt{(1/2)^2+(2/3)^2}$$

$$\boxed{\left|\oint \mathrm{Re}\,z\,dz\right|=0.83} \text{ — (ii)}$$

~~But~~ The upper bound of given integral is

$$(i)\Rightarrow \left|\oint \mathrm{Re}\,z\,dz\right|\le(1)(1.48)$$

$$\boxed{\left|\oint \mathrm{Re}\,z\,dz\right|\le 1.48}$$

This is true as

$$\left|\oint \mathrm{Re}\,z\,dz\right|=0.83 \text{ from (ii)}$$