Q5: (a) $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-2i)^n$

Ans:- Given series can be obtained by differentiating $\sum_{n=1}^{\infty} (z-2i)^n/3^n$ two times and then multiplying $(z-2i)^2$ with each term.

Radius of Convergence of $\sum_{n=0}^{\infty} \frac{(z-2i)^n}{3^n}$ is the same as the given series. (By theorem (3))

Here $a_n = 1/3^n$

Now by Cauchy-Hadamard formula,

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{1/3^{n+1}}{1/3^n} \right| = \lim_{n\to\infty} \left| \frac{3^n}{3^{n+1}} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| 3^{\cancel{n}-\cancel{n}-1} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| 3^{-1} \right|$$

$$\frac{1}{R} = 3^{-1} = \frac{1}{3}$$

$$\boxed{R = 3}$$

ⓑ $\sum_{n=0}^{\infty} \frac{n}{2^n}(z-2i)^{2n}$

Given series can be obtained by differentiating

$\sum_{n=0}^{\infty} \frac{(z-2i)^{2n}}{2^n}$ one time and then multiplying

$\frac{(z-2i)}{2}$ with each term.

Radius of convergence of $\sum_{n=0}^{\infty} \frac{(z-2i)^{2n}}{2^n}$ will be same as the given series (By theorem B)).

Now by Cauchy-Hadamard formula.

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{1/2^{n+1}}{1/2^n} \right| = \lim_{n\to\infty} \left| \frac{2^n}{2^{n+1}} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n}\cdot 2}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{1}{2}$$

$$\frac{1}{R} = \frac{1}{2}$$

$$\boxed{R = 2}$$

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(c) $\sum_{n=1}^{\infty} \frac{3^n(n)(n+1)(z-1)^{2n}}{5^n}$

Ans:- Given Series can be written in simpler terms as $\sum_{n=1}^{\infty} \frac{3^n}{5^n}(z-1)^{2n}$. Now $a_n = \frac{3^n}{5^n}$

$\therefore$ the radius of convergence of given series will be same as the simpler series. (By theorem 3).

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{3^{n+1}/5^{n+1}}{3^n/5^n} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left(\frac{3^{n+1}}{5^{n+1}} \times \frac{5^n}{3^n} \right)$$

$$\frac{1}{R} = \lim_{n\to\infty} \left(\frac{3^{n-n+1}}{5^{n-n+1}} \right)$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{3}{5}$$

$$\boxed{R = \frac{5}{3}}$$

Q6: (a) $\sin z$, $\pi/2$

Ans:- $z_0 = \pi/2$

Taylor Expansion of given series is:

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)(z-z_0)^2}{2!} + \frac{f'''(z_0)(z-z_0)^3}{3!} + \cdots$$

$$\cdots \frac{f^{(2n)}(z_0).(z-z_0)^{2n}}{(2n)!} \cdots\cdots \text{(i)}$$

We are taking even ~~terms~~ ~~be~~ nth derivatives because odd numbered derivatives are zero for $\sin z$.

$$\boxed{f(z_0) = \sin(\tfrac{\pi}{2}) = 1}$$

$f'(z) = \cos z \Rightarrow \boxed{f'(z_0) = 0}$

$f''(z) = -\sin z \Rightarrow \boxed{f''(z_0) = -1}$

$f'''(z) = -\cos z \Rightarrow \boxed{f'''(z_0) = 0}$

$f^{(iv)}(z) = \sin z \Rightarrow \boxed{f^{(iv)}(z_0) = 1}$

$\vdots$

$f^{(2n)}(z_0) = (-1)^n$

Putting values in (i)

$$f(z) = 1 + \frac{0}{1!}(z-\frac{\pi}{2}) - \frac{1}{2!}(z-\pi/2)^2 + \frac{0}{3!}(z-\frac{\pi}{2})^3 + \frac{1}{4!}(z-\frac{\pi}{2})^4$$

$$\cdots \frac{(-1)^n (z-\pi/2)^{2n}}{(2n)!} \cdots\cdots$$

$$\sin z = \sum_{n=1}^{\infty} \frac{(-1)^n (z-\pi/2)^{2n}}{(2n)!}$$

Now $a_n = \frac{(-1)^n}{(2n)!}$

By Cauchy-Hadamard Formula, we have

$$\frac{1}{R} = \lim_{n\to\infty} \left| \sqrt[n]{\frac{(-1)^n}{(2n)!}} \right|$$

$$R = \frac{1}{\lim_{n\to\infty} \frac{(+1)}{((2n)!)^{1/n}}}$$

$$R = \frac{1}{\lim_{n\to\infty} \left(\frac{1}{(2n)!}\right)^{1/n}}$$

$$R = \frac{1}{\left(\frac{1}{(2(\infty))!}\right)^{1/\infty}}$$

$$R = \frac{1}{0}$$

$$\boxed{R = \infty}$$

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(b) $\frac{1}{1-z}$, i

Ans:- $z_0 = i$

Taylor Expansion of given series is:

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)(z-z_0)^2}{2!} + \frac{f'''(z_0)(z-z_0)^3}{3!} \cdots$$

$$\cdots \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} \cdots\cdots \quad \text{--- (i)}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot f^{(n)}(z_0) \cdot (z-z_0)^n \quad \text{--- (ii)}$$

$$\boxed{f(z_0) = \frac{1}{1-i}}$$

$$f'(z) = -1(1-z)^{-2}(-1)$$

$$f'(z) = \frac{1}{(1-z)^2} = \frac{1!}{(1-z)^2} \Rightarrow \boxed{f'(z_0) = \frac{1}{(1-i)^2}}$$

$$f''(z) = -2(1-z)^{-3} \cdot (-1) \Rightarrow \boxed{f''(z_0) = \frac{2}{(1-i)^3}}$$

$$f''(z) = \frac{2}{(1-z)^3} = \frac{2!}{(1-z)^3}$$

$$f'''(z) = 2(-3)(1-z)^{-4} \cdot (-1)$$

$$f'''(z) = \frac{6}{(1-z)^4} = \frac{3!}{(1-z)^4} \Rightarrow \boxed{f'''(z_0) = \frac{3!}{(1-i)^4}}$$

The nth derivative of f(z) will be:

$$f^n(z) = \frac{n!}{(1-z)^{n+1}} \Rightarrow \boxed{f^n(z_0) = \frac{n!}{(1-i)^{n+1}}}$$

Putting values in (i) we get the taylor expansion of given series.

$$\frac{1}{1-z} = \frac{1}{1-i} + \frac{\frac{\cancel{1}}{(1-i)^2}}{\cancel{1!}}(z-i) + \frac{\frac{\cancel{2!}}{(1-i)^3}}{\cancel{2!}}\cdot(z-i)^2 + \frac{\frac{\cancel{3!}}{(1-i)^4}}{\cancel{3!}}\cdot(z-i)^3 \cdots$$

$$\cdots \frac{\frac{\cancel{n!}}{(1-i)^{n+1}}}{\cancel{n!}}(z-i)^n \cdots$$

$$\frac{1}{1-z} = \frac{1}{1-i} + \frac{(z-i)}{(1-i)^2} + \frac{(z-i)^2}{(1-i)^3} + \frac{(z-i)^3}{(1-i)^4} \cdots \frac{(z-i)^n}{(1-i)^{n+1}} \cdots$$

Now to find the radius of convergence, first of all we write this series in it's standard form.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot f^n(z_0)\cdot(z-z_0)^n$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{1}{\cancel{n!}}\cdot\frac{\cancel{n!}}{(1-z)^{n+1}}(z-z_0)^n$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{1}{(1-z)^{n+1}}\cdot(z-z_0)^n$$

Now $a_n = \frac{1}{(1-i)^{n+1}}$

By Cauchy-Hadamard Formula,

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{\frac{1}{(1-i)^{n+2}}}{\frac{1}{(1-i)^{n+1}}} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{(1-i)^{n+1}}{(1-i)^{n+2}} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{\cancel{(1-i)^{n}} \cdot \cancel{(1-i)^{1}}}{\cancel{(1-i)^{n}} \cdot \cancel{(1-i)^{1}} \cdot (1-i)^{1}} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{1}{1-i} \right| = \lim_{n\to\infty} \frac{|1|}{|1-i|}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{1}{\sqrt{(1)^2+(-1)^2}}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{1}{\sqrt{2}}$$

$$\frac{1}{R} = \frac{1}{\sqrt{2}}$$

$$\boxed{R = \sqrt{2}}$$

© $Ln((1-z))$, i

Ans:- $z_0 = i$

Taylor Expansion of given series is:

~~Ln((1-z)) = ln((1-i))~~

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 +$$

$$\frac{f'''(z_0)}{3!}(z-z_0)^3 + \frac{f^{(iv)}(z_0)}{4!}(z-z_0)^4 + \text{-----}$$

$$\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \text{-----} \quad \text{—} (i)$$

OR

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot f^{(n)}(z_0)(z-z_0)^n \quad \text{—} (ii)$$

$$\boxed{f(z_0) = Ln(1-i)}$$

$$f'(z) = \frac{-1}{(1-z)^1} = -1(1-z)^{-1} \Rightarrow \boxed{f'(z_0) = \frac{-1}{1-i}}$$

$$f''(z) = +1(1-z)^{-2}(-1) = -1 \cdot \frac{1}{(1-z)^2} \Rightarrow \boxed{f''(z_0) = \frac{-1}{(1-i)^2}}$$

$$f'''(z) = +2(1-z)^{-3}(-1) = -1 \cdot \frac{2\times1}{(1-z)^3} \Rightarrow \boxed{f'''(z_0) = -1 \cdot \frac{2!}{(1-i)^3}}$$

$$f^{(iv)}(z) = 6(1-z)^{-4}(-1) = \frac{3\times2\times1}{(1-z)^4}(-1) \Rightarrow \boxed{f^{(iv)}(z_0) = -1 \cdot \frac{3!}{(1-i)^4}}$$

The nth derivative of $f(z)$ will be:

$$f^{(n)}(z) = \frac{(-1)(n-1)!}{(1-z)^n} \Rightarrow f^{(n)}(z_0) = \frac{(-1)(n-1)!}{(1-i)^n}$$

Putting values in (i) we get the taylor Expansion of given series.

$$\ln((1-z)) = \ln(1-i) + \frac{\frac{-1}{1-i}}{1!}(z-i) + \frac{\frac{-1}{(1-i)^2}}{2!}(z-i)^2 +$$

$$\frac{\frac{-1\times 2!}{(1-i)^3}}{3!}\cdot(z-i)^3 + \frac{\frac{-1\times 3!}{(1-i)^4}}{4!}\cdot(z-i)^4 \ldots\ldots$$

$$\ldots \frac{\frac{(-1)(n-1)!}{(1-i)^n}}{n!}\cdot(z-i)^n \ldots\ldots$$

Now to find the radius of convergence, first of all we write this series in it's Standard form.

$$\ln((1-z)) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{(-1)(n-1)!}{(1-i)^n}\cdot(z-i)^n$$

$$\ln((1-z)) = \sum_{n=0}^{\infty} \frac{(-1)\cancel{(n-1)!}}{n\cancel{(n-1)!}(1-i)^n}\cdot(z-i)^n$$

$$\ln((1-z)) = \sum^{\infty} \frac{(-1)}{n(1-i)^n}\cdot(z-i)^n$$

Now $a_n = \dfrac{(-1)}{n(1-i)^n}$

By Cauchy-Hadamard formula.

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{\dfrac{-1}{(n+1)(1-i)^{n+1}}}{\dfrac{-1}{n(1-i)^n}} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{\cancel{-}n(1-i)^n}{\cancel{-}(n+1)(1-i)^{n+1}} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{n(1-i)^{n-n}}{(n+1)(1-i)} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{n}{(n+1)(1-i)} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|n|}{|(n+1)|\,|1-i|}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{n}{(n+1)\sqrt{2}}$$

$$\frac{1}{R} = \sqrt{2}\cdot \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+1/n)}$$

$$\frac{1}{R} = \frac{1}{\sqrt{2}}\cdot \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{\sqrt{2}} \times \frac{1}{1+0} = \frac{1}{\sqrt{2}}$$

$\boxed{R = \sqrt{2}}$

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