# 15.3

Problem Set 15.3

(1) $\sum_{n=2}^{\infty} \frac{n(n-1)}{3^n}(z-2i)^n$ ...(1)

(a) Compare the following series with (1)

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ ...(2)

obtain $a_n = \frac{n(n-1)}{3^n}$, $z_0 = 2i$

According to Cauchy's-Hadamard formula

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$a_n = \frac{n(n-1)}{3^n} \qquad a_{n+1} = \frac{(n+1)(n+1-1)}{3^{n+1}} = \frac{n(n+1)}{3^{n+1}}$$

Therefore, the radius of convergence of series (1) is

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{n(n-1)}{3^n}}{\frac{n(n+1)}{3^{n+1}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n(n-1)\,3^{n+1}}{n(n+1)\,3^n}\right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{n}(n-1)\,\cancel{3^n}\,3^1}{\cancel{n}(n+1)\,\cancel{3^n}} \right|$$

$$= 3 \lim_{n\to\infty} \left| \frac{(n-1)}{(n+1)} \right|$$

$$= 3 \lim_{n\to\infty} \left| \frac{\cancel{n}\left(1-\frac{1}{n}\right)}{\cancel{n}\left(1+\frac{1}{n}\right)} \right|$$

$$= 3 \lim_{n\to\infty} \left| \frac{\left(1-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)} \right|$$

$$= 3(1)$$

$$\boxed{R = 3}$$

The series converges in the open disk $|z-2i| < 3$ of radius 2 and center $2i$.
Hence, the radius of convergence of the series (1) by Cauchy-Hadamard formula is $\boxed{3}$.

(b) (Theorem 3)

Differentiate eq.(1) with respect to z, obtained as

$$\left( \sum_{n=2}^{\infty} \frac{n(n-1)}{3^n} (z-2i)^n \right)' = \sum_{n=2}^{\infty} \frac{n^2(n-1)}{3^n} (z-2i)^{n-1} \quad \cdots (3)$$

So, the Cauchy-Hadamard formula becomes

$$R = \lim_{n\to\infty} \frac{n|a_n|}{(n+1)|a_{n+1}|}$$

$$a_n = \frac{n(n-1)}{3^n} \qquad a_{n+1} = \frac{n(n+1)}{3^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n\left|\frac{n(n-1)}{3^n}\right|}{(n+1)\frac{n(n+1)}{3^{n+1}}}$$

$$= \lim_{n\to\infty} \left| \frac{(n)\{n(n-1)\}\, 3^{n+1}}{\{n(n+1)\}(n+1)\, 3^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2(n-1)\, \cancel{3^n} \cdot 3^1}{n(n+1)^2\, \cancel{3^n}} \right|$$

$$= \lim_{n\to\infty} 3 \left| \frac{\cancel{n^2 \cdot n}\left(1-\frac{1}{n}\right)}{\cancel{n \cdot n^2}\left(1+\frac{1}{n}\right)^2} \right|$$

$$= 3 \lim_{n\to\infty} \left| \frac{\left(1-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2} \right|$$

$$= 3(1)$$

$$\boxed{= 3}$$

Hence, the radius of convergence of series (1) by term wise differentiation (theorem 3) is $\boxed{3}$

(9) $\sum_{n=0}^{\infty} \left[ \binom{n+k}{k} \right]^{-1} x^{n+k}$

(a)

Here $a_n = \left[ \frac{(n+k)!}{(k)!(n)!} \right]^{-1}$

$$a_n = \frac{k!n!}{(n+k)!}$$

Using Cauchy-Hadamard formula

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Here $a_n = \frac{k!n!}{(n+k)!}$ $a_{n+1} = \frac{k!(n+1)!}{(n+k+1)!}$

by above formula.

$$R = \lim_{n\to\infty} \left| \frac{\frac{k!n!}{(n+k)!}}{\frac{k!(n+1)!}{(n+k+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(k!n!)(n+k+1)!}{(n+k)!\,k!(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(\cancel{K!}\cancel{n!})(n+K+1)(n+K)!}{(n+K)!\ \cancel{K!}(n+1)\cancel{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+K+1}{n+1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1+K}{n+1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n+1} + \frac{K}{n+1} \right|$$

$$= \lim_{n\to\infty} \left| 1 + \frac{K}{n+1} \right|$$

$$\boxed{R = 1}$$

(b). (Theorem 3)

Power Series is $\sum_{n=0}^{\infty} \left[\binom{n+K}{K}\right]^{-1} z^{n+K}$

This series can be written as

$$z^K + \binom{1+K}{K}^{-1} z^{1+K} + \binom{2+K}{K}^{-1} z^{2+K} + \cdots$$

Differentiate it.

$$Kz^{K-1} + (1+K)\binom{1+K}{K}^{-1} z^K + (2+K)\binom{2+K}{K}^{-1} z^{1+K} \cdots$$

Then Series becomes

$$\sum_{n=0}^{\infty} \left[\binom{n+K}{K}\right]^{-1} (n+K) z^{n+K-1}$$

Now

$$a_n = \left(\frac{(n+K)!}{(K)!(n)!}\right)^{-1} (n+K)$$

$$= \frac{(n+K)\, K!\, n!}{(n+K)!}$$

and

$$a_{n+1} = \frac{(n+K+1)\, K!\, (n+1)!}{(n+K+1)!}$$

Thus.

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{(n+K)\, K!\, n!}{(n+K)!}}{\frac{(n+K+1)\, K!\, (n+1)!}{(n+K+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+K)\, K!\, n!\, (n+K+1)!}{(n+K)!\, (n+K+1)\, K!\, (n+1)!} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(n+k)\cancel{k!}\,\cancel{n!}\,\cancel{(n+k+1)}\,\cancel{(n+k)!}}{\cancel{(n+k)!}\,\cancel{(n+k+1)}\,\cancel{k!}\,(n+1)\,\cancel{n!}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{n+k}{n+1} \right|$$

$$= \lim_{n \to \infty} \left| \frac{\cancel{n}\left(1+\frac{k}{n}\right)}{\cancel{n}\left(1+\frac{1}{n}\right)} \right|$$

$$\boxed{R = 1}$$

<u>Hence, Radius of Convergence is 1.</u>