Probability Methods in Engineering

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Lecture 2

Randomness

- What is Randomness?
  - Chaos
  - Uncertainty
  - Doubt

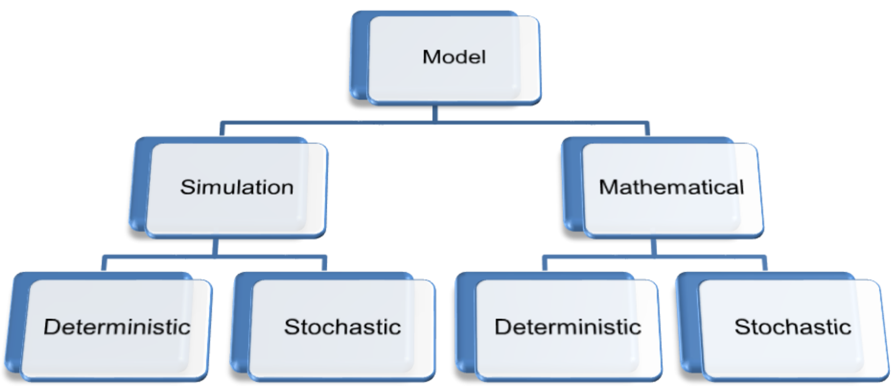
- Humans desire some level of 'certainty'
- Examples
  - Solar system
  - Weather forecast at Chitral Airport
  - Traffic situation on University road

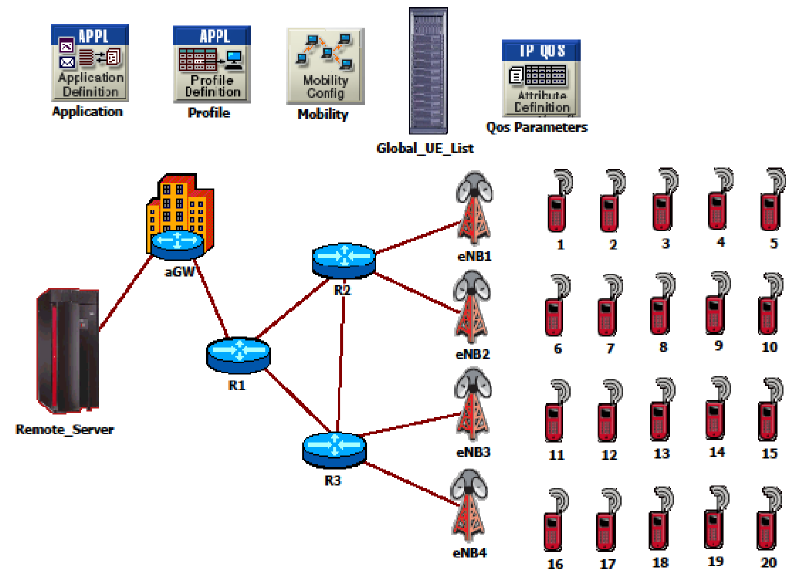
- Engineers **quantify** 'certainty'

Source: Computer Vision Research Group, *CIIT Lahore*, Pakistan

Model of a Physical System

- **Model**: Approximate representation of physical situation
  - **Mathematical model**: Set of assumptions about how system works
    - **Deterministic model**: Offers repeatability of results, (e.g. Ohm's Laws)
    - **Stochastic model**: Characterizes randomness and uncertainty
  - **Simulation model**: Imitation of real system
    - **Deterministic model**: No random component involved, (e.g. chemical reaction)
    - **Stochastic model**: Must have random input component





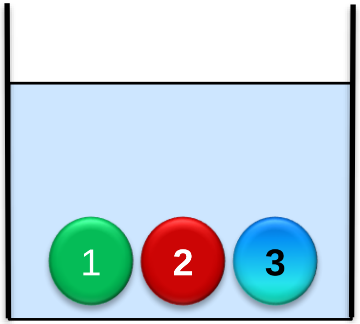
Source: S. N. K. Marwat, PhD Thesis, *University of Bremen, Germany*

Random Experiment

- **Random Experiment**: The result varies in random manner
- **Sample Space**: Set of all possible experiment results
- **Outcome**: A single element of sample space
- **Event**: A subset of sample space

- Example: An urn containing three balls, one is drawn
  - How probable it is that a ball withdrawn at random is labeled '1'?
  - Can you quantify this 'chance'?
  - Everyone of you should be able to write the sample space for this experiment!

$$S = \{ \qquad\qquad\qquad \}$$



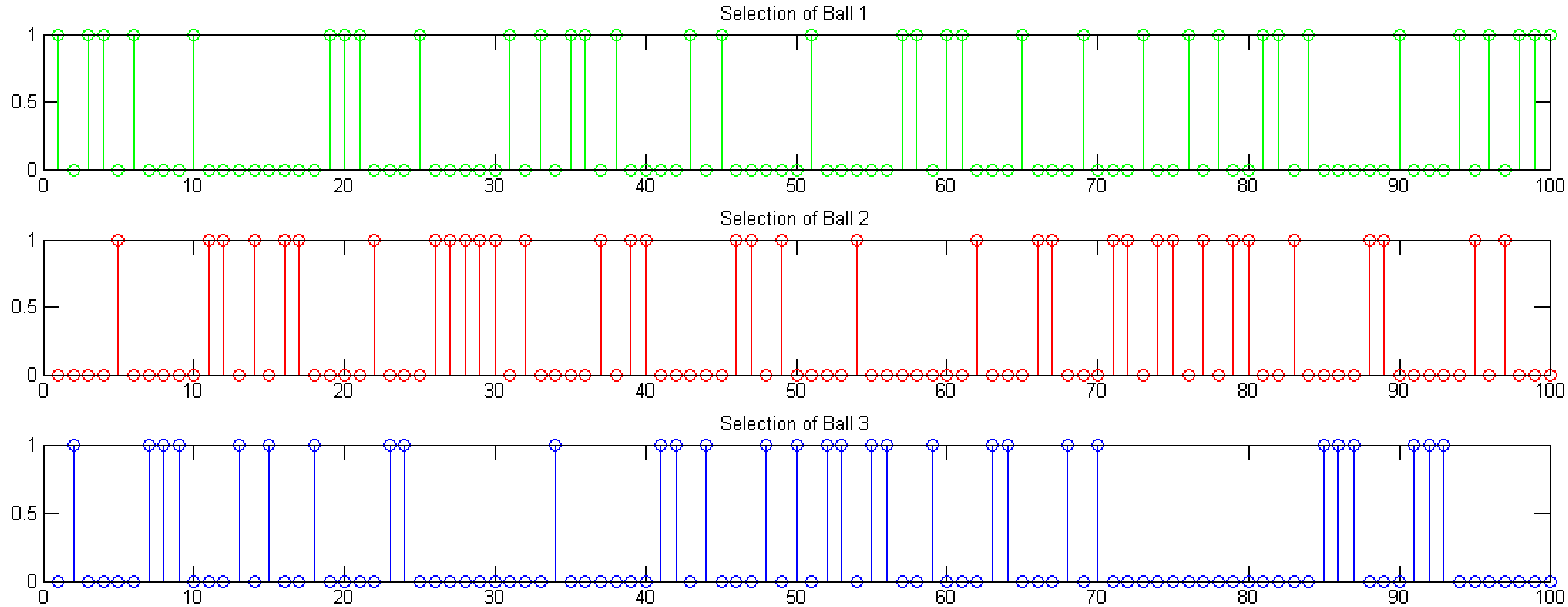
Random Experiment (cont.)

- Some more questions to answer
  - Is withdrawing all the three balls equi-probable (or is any ball more likely to be drawn)?
  - If '1' means 'sure occurrence' and '0' means 'no chance of occurrence', what number can be given to the chance of getting 'ball 1'?
  - What is the chance of withdrawing an odd-numbered (or even-numbered) ball?

Let the nature answer this

Random Experiment (cont.)

- Take a ball from the urn
- Record the outcome
- Put it back in the urn
- Do the experiment 'n' times



Relative Frequency

Relative frequency:

The number of times an event occurs is called a frequency. **Relative frequency** is an experimental one, but not a theoretical one. Since it is an experimental one, it is possible to obtain different relative frequencies when we repeat the experiments. To calculate the frequency we need

- Frequency count for the total population
- Frequency count for a subgroup of the population

Random Experiment (cont.)

- Number of times k^{th} outcome occurred (or **frequency** of k) in a total of n trails

$$N_k(n)$$

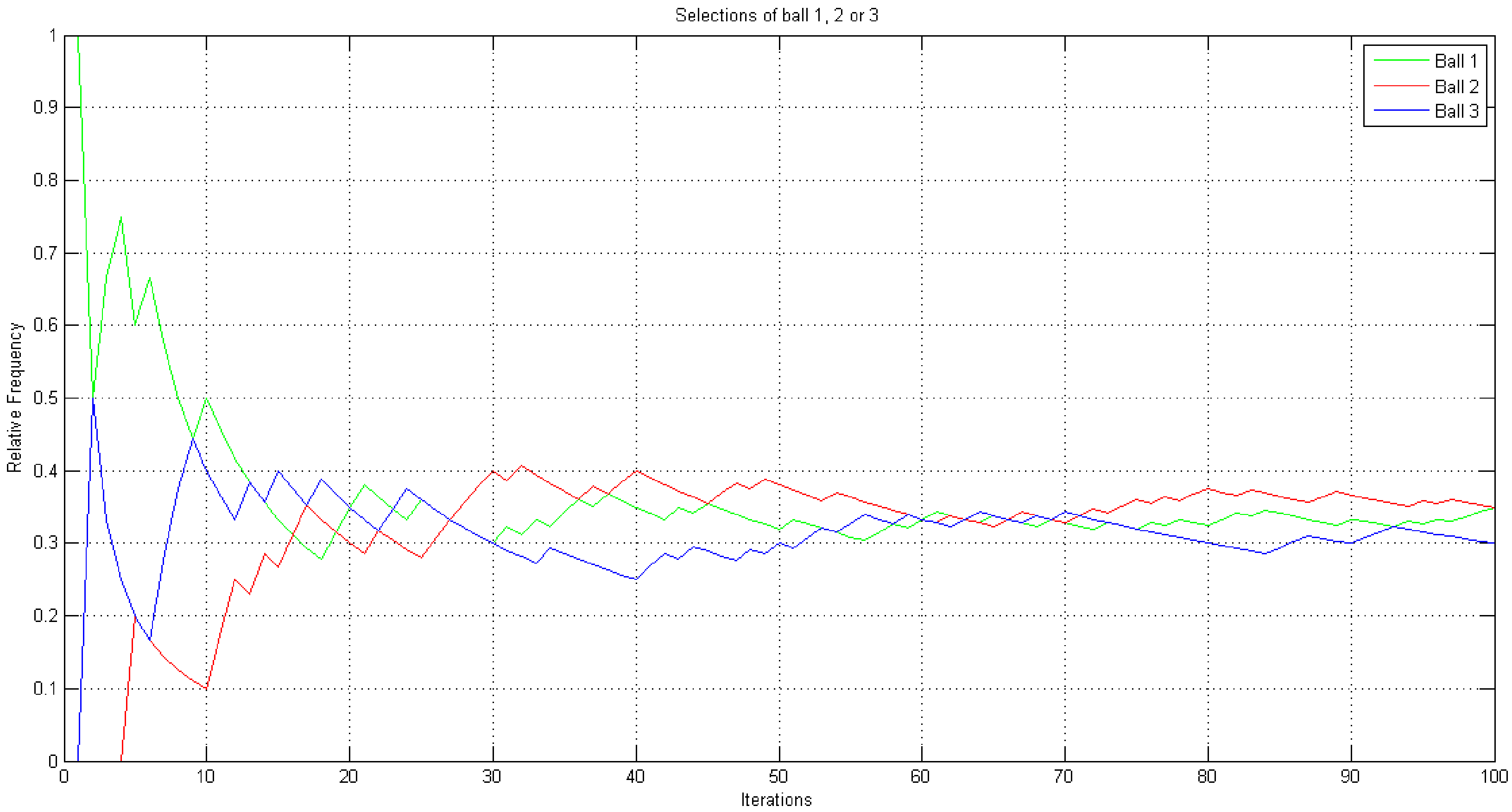
- The **relative frequency** of k^{th} outcome

$$f_k(n) = \frac{N_k(n)}{n}$$

- **Relative frequency** can be defined as the number of times an event occurs divided by the total number of events occurring in a given scenario.
- The relative frequency formula is given as:
  Relative Frequency = Subgroup frequency/ Total frequency.

Random Experiment (cont.)

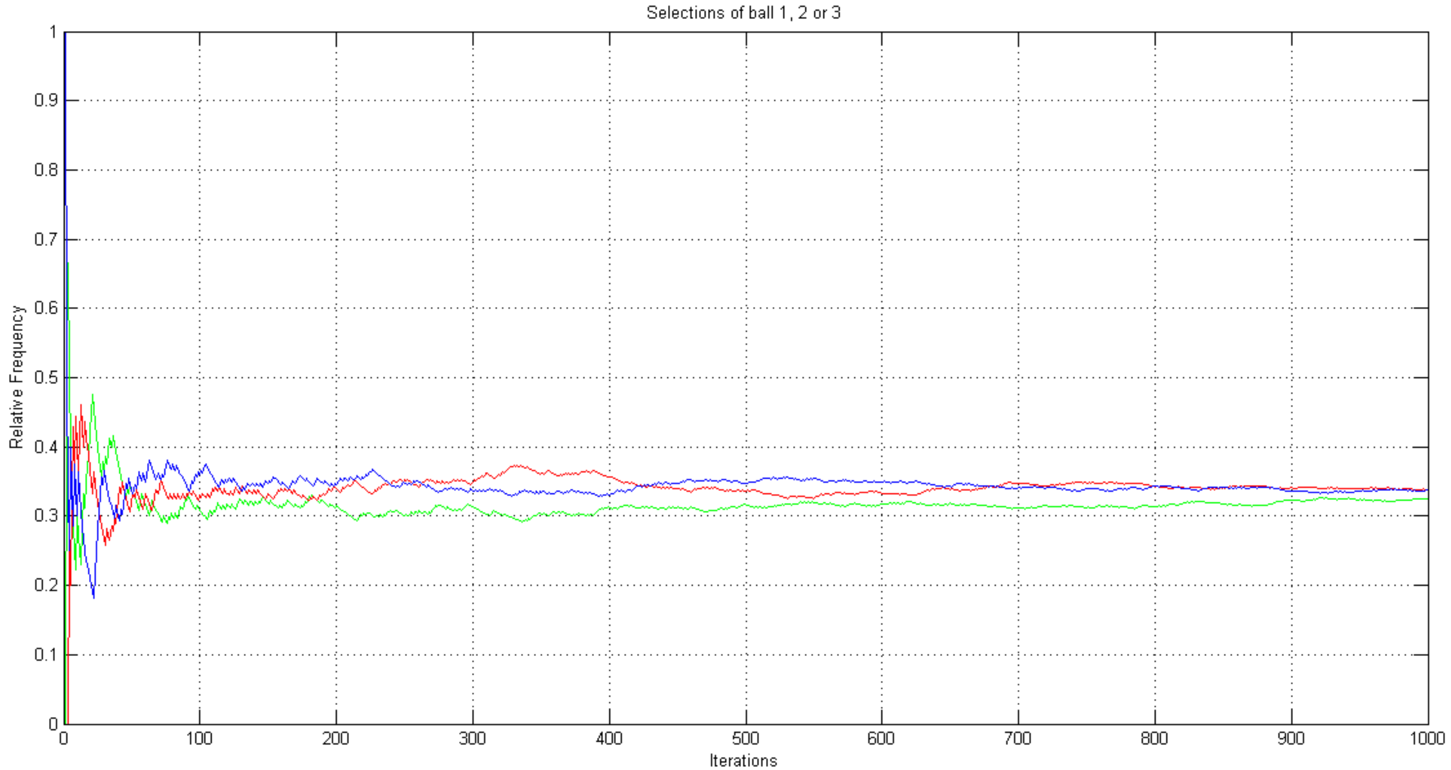
- Statistical Regularity
  - Averages obtained in long sequences yield same value



Random Experiment (cont.)

- **Probability** defined by *von Mises* as 'limiting case of relative frequency'

$$\lim_{n\to\infty} f_k(n) = \lim_{n\to\infty} \frac{N_k(n)}{n} = p_k$$



Properties of Relative Frequency

- Number of occurrences of an outcome in n trials
  - A number between zero and n

$$\boxed{0 \leq N_k(n) \leq n}$$

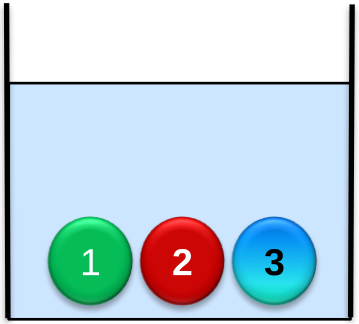
- Relative frequencies are
  - A number between zero and one
  - Divide the above equation by n to get

$$\boxed{0 \leq f_k(n) \leq 1}$$

Properties of Relative Frequency

- Number of occurrences of an outcome in n trials
  - A number between zero and n
  - Also called frequency

$$\boxed{0 \le N_k(n) \le n}$$



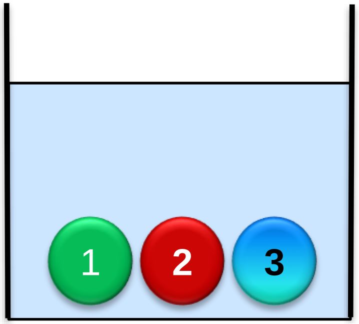
- Relative frequencies are
  - A number between zero and one
  - Divide the above equation by n to get

$$\boxed{0 \le f_k(n) \le 1}$$

Properties of Relative Frequency (cont.)

- Sum of number of occurrences of all possible outcomes
  - Must be n
  - n, sum all frequencies

$$\boxed{\sum_{k=1}^{K} N_k(n) = n}$$



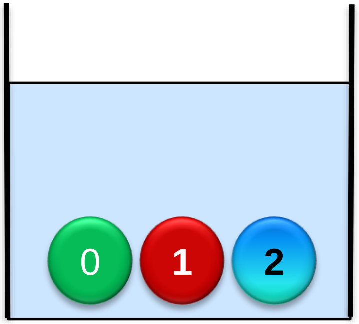
- Sum of all relative frequencies
  - Must be 1

$$\boxed{\sum_{k=1}^{K} f_k(n) = 1}$$

Properties of Relative Frequency (cont.)

- E is an event that the outcome is even

$$f_E(n) = f_0(n) + f_2(n)$$



- If C is an event such that, "either A **or** B occurs" but simultaneous occurrence not possible

$$\boxed{f_C(n) = f_A(n) + f_B(n)}$$

Relative Frequency

- **Example 1: A die is tossed 40 times and lands 6 times on the number 4. What is the relative frequency of observing the die land on the number 4?**

Relative Frequency

- **Example 2: A coin is tossed 20 times and lands 15 time on heads. What is the relative frequency of observing the coin land on heads?**

Axioms of Probability

- Axiom
  - Universally accepted principle or rule
- If A and B are two events from a sample space S
- In modern theory of probability, probability of event A

$$\boxed{0 \leq P[A] \leq 1}$$

- Probability of S

$$\boxed{P[S] = 1}$$

- If A and B cannot occur simultaneously

$$\boxed{P[A \text{ or } B] = P[A] + P[B]}$$

Problems

- Consider the following three random experiments:
  - Experiment 1: Toss a coin.
  - Experiment 2: Roll a die.
  - Experiment 3: Select a ball at random from an urn containing balls numbered 0 to 9.

a) Specify the sample space of each experiment.
b) Find the relative frequency of each outcome in each of the above experiments in a large number of repetitions of the experiment. Explain your answer.

Problems (cont.)

- Explain how the following experiments are equivalent to random urn experiments:
  a) Flip a fair coin twice.
  b) Toss a pair of fair dice.
  c) Draw two cards from a deck of 52 distinct cards, with replacement after the first draw; without replacement after the first draw.

Problems (cont.)

- Explain under what conditions the following experiments are equivalent to a random coin toss. What is the probability of heads in the experiment?
  a) Observe a pixel (dot) in a scanned black-and-white document.
  b) Receive a binary signal in a communication system.
  c) Test whether a device is working.
  d) Determine whether your friend Joe is online.
  e) Determine whether a bit error has occurred in a transmission over a noisy communication channel.

Problems (cont.)

- An urn contains three electronically labelled balls with labels 00, 01, 10. Lisa, Homer, and Bart are asked to characterize the random experiment that involves selecting a ball at random and reading the label. Lisa's label reader works fine; Homer's label reader has the most significant digit stuck at 1; Bart's label reader's least significant digit is stuck at 0.
  a) What is the sample space determined by Lisa, Homer, and Bart?
  b) What are the relative frequencies observed by Lisa, Homer, and Bart in a large number of repetitions of the experiment?

Problems (cont.)

- A random experiment has sample space $S = \{1, 2, 3, 4\}$ with probabilities $p_1 = 1/2$, $p_2 = 1/4$, $p_3 = 1/8$ and $p_4 = 1/8$.
  a) Describe how this random experiment can be simulated using tosses of a fair coin.
  b) Describe how this random experiment can be simulated using an urn experiment.
  c) Describe how this experiment can be simulated using a deck of 52 distinct cards.

Problems (cont.)

- A random experiment consists of selecting two balls in succession from an urn containing two black balls and one white ball.
  a) Specify the sample space for this experiment.
  b) Suppose that the experiment is modified so that the ball is immediately put back into the urn after the first selection. What is the sample space now?
  c) What is the relative frequency of the outcome (white, white) in a large number of repetitions of the experiment in part a? In part b?
  d) Does the outcome of the second draw from the urn depend in any way on the outcome of the first draw in either of these experiments?

Problems (cont.)

- Let A be an event associated with outcomes of a random experiment, and let the event B be defined as "event A does not occur." Show that

$$f_B(n) = 1 - f_A(n)$$

Problems (cont.)

- Let A, B, and C be events that cannot occur simultaneously as pairs or triplets, and let D be the event "A or B or C occurs." Show that

$$f_D(n) = f_A(n) + f_B(n) + f_C(n)$$

Problems (cont.)

Write the sample space S for the following random experiments.

a. We toss a coin until we see two consecutive tails. We record the total number of coin tosses.

b. A bag contains 4 balls: one is red, one is blue, one is white, and one is green. We choose two distinct balls and record their color in order.

c. A customer arrives at a bank and waits in the line. We observe T, which is the total time (in hours) that the customer waits in the line. The bank has a strict policy that no customer waits more than 20 minutes under any circumstances.