Topic: 2.5 (Page-149) MARKOV CHAINS

Definition: A Markov chain or Markov Process is a Process in which the Probability of the system being in a Particular state in a given observation period depends only on its state at the immediately preceding observation period.

Explanation of Markov Process and Transition Probability:-
Suppose the system has n Possible states. For each $i=1,2,\dots,n$, $j=1,2,\dots,n$, let t_{ij} be the Probability that if the system is in state j at a certain observation Period, it will be in state i at the next observation Period; t_{ij} is called a transition Probability. Since t_{ij} is a Probability, so its value lies b/w 0 and 1
That is, $0 \le t_{ij} \le 1 \quad (1 \le i, j \le n)$
Also, if the system is in state j at a certain observation period, then it must be in one of the n states (it may remain in state j again) at the next observation Period. Thus we have

$$\boxed{t_{1j} + t_{2j} + \dots + t_{nj} = 1 \text{ --- } (1)}$$

As the sum of the Probabilities of all the states/events in a system is 1.

It is convenient to arrange the transition Probabilities as the $n \times n$ matrix $T = [t_{ij}]$ which is called the transition matrix of the Markov Process which is defined as.
Transition matrix (or Markov matrix / stochastic matrix / Probability matrix):- The transition matrix of the Markov Process / Markov chain denoted by

$$T = [t_{ij}] = \begin{bmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & & & \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{bmatrix}$$

is a square matrix whose all entries are nonnegative (being representing Probabilities of events/states in the system) and the sum of the entries in each column is 1 (as mentioned in (1))

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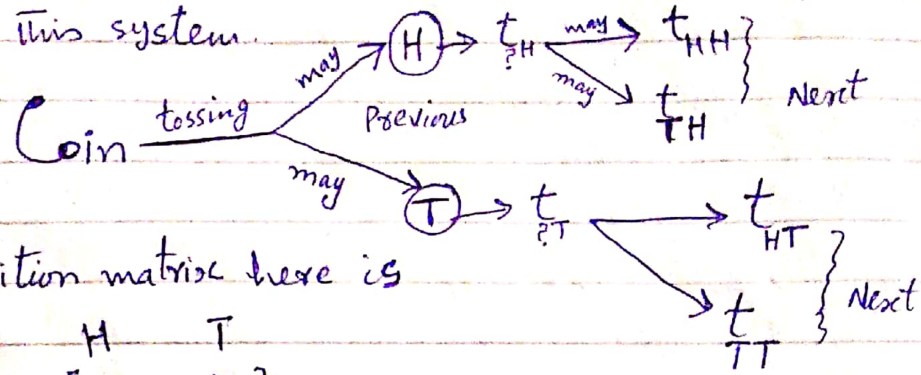
EXS-1) i) To toss a coin, There exist two states/events in This system.



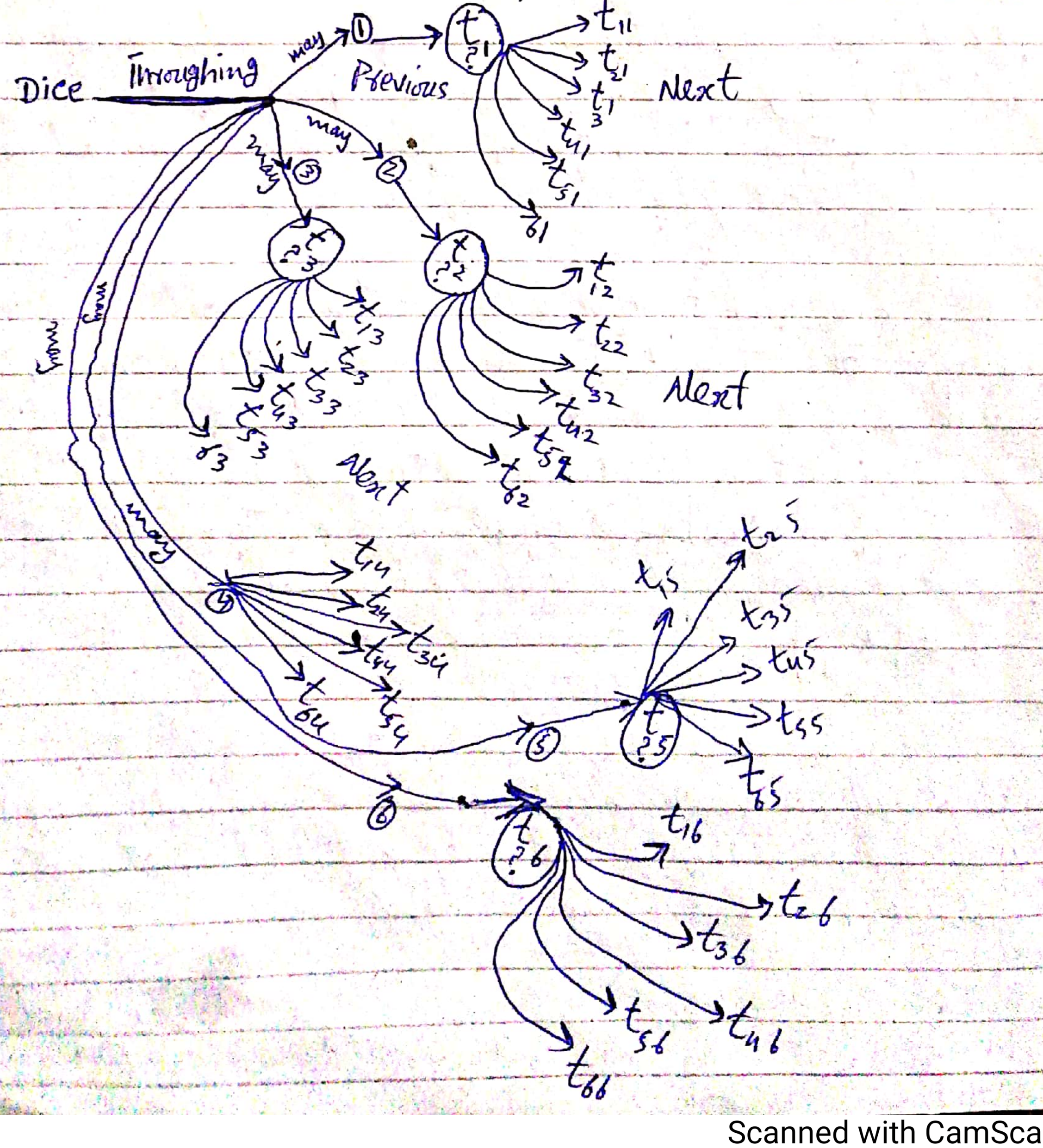
Transition matrix here is

$$T = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} H & T \\ \end{array}$$

$$T = \left[\begin{array}{cc} t_{HH} & t_{HT} \\ t_{TH} & t_{TT} \end{array}\right]\begin{array}{c} H \\ T \end{array}$$

, size of T is 2x2 (As There are two states)

ii) To Through a dice, here in This system/experiment, There are six states/events i.e ①, ②, ③, ④, ⑤, ⑥



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The transition matrix, here is given as

$$T = \begin{array}{c} \begin{array}{cccccc} ① & ② & ③ & ④ & ⑤ & ⑥ \end{array} \\ \left[\begin{array}{cccccc} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} & t_{36} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} & t_{46} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} & t_{56} \\ t_{61} & t_{62} & t_{63} & t_{64} & t_{65} & t_{66} \end{array}\right] \end{array} \begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \end{array}$$

State vector :– The state vector of the Markov Process/chain at the observation period K is defined as

$$X^{(K)} = \begin{bmatrix} p_1^{(K)} \\ p_2^{(K)} \\ \vdots \\ p_n^{(K)} \end{bmatrix} , \quad K \geq 0$$

where $p_j^{(K)}$ is the Probability that the system is in state j at the observation period K. The state vector $X^{(0)}$, at the observation Period 0, is called the initial state vector.

Theorem 2.4 (Page-151) :– If T is the transition matrix of the Markov Process, then the state vector $X^{(K+1)}$, at the $(K+1)$th observation period, can be determined from the state vector $X^{(K)}$, at the Kth observation period, as

$$X^{(K+1)} = TX^{(K)} \quad \text{——— ①}$$

For $K=0$, ① $\Rightarrow$ $X^{(1)} = TX^{(0)}$

For $K=1$, ① $\Rightarrow$ $X^{(2)} = TX^{(1)} = T(TX^{(0)}) = T^2 X^{(0)}$

For $K=2$, ① $\Rightarrow$ $X^{(3)} = TX^{(2)} = T(T^2 X^{(0)}) = T^3 X^{(0)}$

and in general $X^{(n)} = T^n X^{(0)}$

Steady-state vector: If T is the transition matrix for a Markov Process and U is a state vector at some observation Period, then U is called the steady-state vector if

$$TU = U$$

In this case we say that the Markov Process reaches the equilibrium.

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EX3 (Page-151):- Suppose that the wheather in a certain city is either rainy or dry. As a result of extensive record keeping it has been determined that the Probability of a rainy day following a dry day is $\frac{1}{3}$, and the Probability of a rainy day following a rainy day is $\frac{1}{2}$.

Suppose that when we begin our observations (day 0), it is dry.

i) what is the initial state vector of the Markov Process?

ii) construct the transition matrix for the given Markov Process.

iii) compute $X^{(1)}$, $X^{(2)}$, $X^{(3)}$, $X^{(4)}$ and $X^{(5)}$ to three decimal Places.

iv) Find the steady state vector for the Markov Process if it exist.

or Does the steady-state vector for the Markov Process exist?

or Does/Can the Markov Process reach equilibrium?

or what is the behavior of the system in the long run?

Solution:— (i) The initial state vector for the given Markov Process is as
$$X^{(0)} = \begin{array}{c} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}\begin{array}{c} D \\ R \end{array}$$

(ii) The Transition Matrix / Markov Matrix / stochastic Matrix / Probability Matrix for the given Markov Process is as

$$T = \begin{array}{c} \quad D \qquad R \\ \begin{bmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix} \end{array}\begin{array}{c} \\ D \\ R \end{array} = \begin{array}{c} \quad D \qquad R \\ \begin{bmatrix} 0.67 & 0.5 \\ 0.33 & 0.5 \end{bmatrix} \end{array}\begin{array}{c} \\ D \\ R \end{array}$$

(iii) As by Theorem 2.4, mentioned above, we have
$$X^{(K+1)} = T X^{(K)} \quad \text{—— (1)}$$

For $K=0$, (1) $\Rightarrow$ $X^{(1)} = T X^{(0)} = \begin{bmatrix} 0.67 & 0.5 \\ 0.33 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$

For $K=1$, (1) $\Rightarrow$ $X^{(2)} = T X^{(1)} = \begin{bmatrix} 0.67 & 0.5 \\ 0.33 & 0.5 \end{bmatrix}\begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix} = \begin{bmatrix} 0.614 \\ 0.386 \end{bmatrix}$

For $K=2$, (1) $\Rightarrow$ $X^{(3)} = T X^{(2)} = \begin{bmatrix} 0.67 & 0.5 \\ 0.33 & 0.5 \end{bmatrix}\begin{bmatrix} 0.614 \\ 0.386 \end{bmatrix} = \begin{bmatrix} 0.604 \\ 0.396 \end{bmatrix}$

For $K=3$, (1) $\Rightarrow$ $X^{(4)} = T X^{(3)} = \begin{bmatrix} 0.67 & 0.5 \\ 0.33 & 0.5 \end{bmatrix}\begin{bmatrix} 0.604 \\ 0.396 \end{bmatrix} = \begin{bmatrix} 0.603 \\ 0.397 \end{bmatrix}$

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For $K=4$, ① $\Rightarrow$ $X^{(5)} = TX^{(4)} = \begin{bmatrix} 0.67 & 0.5 \\ 0.33 & 0.5 \end{bmatrix} \begin{bmatrix} 0.603 \\ 0.397 \end{bmatrix} = \begin{bmatrix} 0.603 \\ 0.397 \end{bmatrix}$

(iv) The steady-state vector exists which is as

$$X^{(4)} = \begin{bmatrix} 0.63 \\ 0.397 \end{bmatrix}, \text{ as } TX^{(4)} = X^{(5)} = \begin{bmatrix} 0.63 \\ 0.397 \end{bmatrix} = X^{(4)}$$

, that is, $X^{(4)}$ satisfies the definition as

$$TU = U$$

2nd-method:–

In this case, we can say that the Markov Process reaches equilibrium. Here we can use the definition to find the steady-state vector or to see whether the Markov Process reaches equilibrium.

Suppose U is the steady-state vector, then

$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times1}$, as $T_{2\times2}$, then by definition, we have

$$TU = U \Rightarrow U = TU \Rightarrow U - TU = 0$$

$$\Rightarrow (I_2 - T)U = 0 \Rightarrow \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.67 & 0.5 \\ 0.33 & 0.5 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0.33 & -0.5 \\ -0.33 & 0.5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.33u_1 - 0.5u_2 \\ -0.33u_1 + 0.5u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{array}{l} 0.33u_1 - 0.5u_2 = 0 \\ -0.33u_1 + 0.5u_2 = 0 \end{array} \right\} \Rightarrow 0.33u_1 - 0.5u_2 = 0$$

$\Rightarrow u_1 = \frac{0.5}{0.33} u_2 \Rightarrow u_1 = \frac{50}{33} r$, where $u_2 = r$,

$0 \le r \le 1$. Also $u_1 + u_2 = 1$ ⊛, as U the steady-state vector, being the probability vector.

⊛ $\Rightarrow \frac{50}{33} r + r = 1 \Rightarrow \frac{85r}{33} = 1 \Rightarrow r = \frac{33}{85} = 0.397 = u_2$

Again ⊛ $\Rightarrow u_1 = 1 - u_2 \Rightarrow u_1 = 1 - 0.397 \Rightarrow u_1 = .603$

Hence $U = \begin{bmatrix} 0.603 \\ 0.397 \end{bmatrix}$ is the desired steady-state vector.

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Probability vector:– The vector $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ is called a Probability vector if $u_i \geq 0$ $(1 \leq i \leq n)$ and $u_1 + u_2 + \cdots + u_n = 1$ i.e $\sum_{i=1}^{n} u_i = 1$

exs: The vectors $\begin{bmatrix} 1/2 \\ 1/4 \\ 1/4 \end{bmatrix}$ and $\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$ are Probability vectors

but $\begin{bmatrix} 1/5 \\ 1/5 \\ 2/5 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \end{bmatrix}$ are not Probability vectors.

Note-1:– Not every Markov Process reaches equilibrium. That is, not the steady-state vector for every Markov Process exists. ex: Let $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be a transition matrix of a Markov Process and $X^{(0)} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ be the initial state vector, Then

$$X^{(1)} = T X^{(0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$X^{(2)} = T X^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$X^{(3)} = T X^{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$X^{(4)} = T X^{(3)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

From above we conclude that the give Markov Process does not reach equilibrium i.e the state vectors do not convert to a fixed vector (steady-state vector). In this case we say that the state vectors oscillate between the vectors $\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$ and $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ and do not converge to a fixed vector.

Note-2:– The steady-state vector u is the unique Probability vector satisfying the matrix equation

$$Tu = u$$

, where T is the transition matrix for the Markov Process.

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Regular Transition Matrix:– A transition matrix T of a Markov process is called regular if all the entries in some power of T are positive. A Markov process is called regular if its transition matrix is regular.

Ex(2) The transition matrix

$$T = \begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}$$ is regular as

$$T^2 = \begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}\begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix} = \begin{bmatrix} 0.04+0.8 & 0.2+0 \\ 0.16+0 & 0.8+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.84 & 0.2 \\ 0.16 & 0.8 \end{bmatrix}$$, that is, in 2nd power of T every entry is +ve, though in in the 1st power not all entries are +ve, as we see the 2nd entry in the 2nd row is 0, which is not +ve.

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certain observation period, it will be in state i at the next observation period; t_{ij} is called a **transition probability**. Moreover, t_{ij} applies to every time period; that is, it does not change with time.

Since t_{ij} is a probability, we must have

$$0 \leq t_{ij} \leq 1 \quad (1 \leq i, j \leq n).$$

Also, if the system is in state j at a certain observation period, then it must be in one of the n states (it may remain in state j) at the next observation period. Thus we have

$$t_{1j} + t_{2j} + \cdots + t_{nj} = 1. \tag{1}$$

It is convenient to arrange the transition probabilities as the $n \times n$ matrix $T = [t_{ij}]$, which is called the **transition matrix** of the Markov chain. Other names for a transition matrix are **Markov matrix, stochastic matrix,** and **probability matrix.** We see that the entries in each column of T are nonnegative and, from Equation (1), add up to 1.

EXAMPLE 1 Suppose that the weather in a certain city is either rainy or dry. As a result of extensive record keeping it has been determined that the probability of a rainy day following a dry day is $\frac{1}{3}$, and the probability of a rainy day following a rainy day is $\frac{1}{2}$. Let state D be a dry day and state R be a rainy day. Then the transition matrix of this Markov chain is

$$T = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \text{D} & \text{R} \\ \left[\begin{array}{cc} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{array}\right] \end{array}\begin{array}{l} \\ \text{D} \\ \text{R} \end{array}$$

■

Example 9 in Section 1.4 portrays a situation that is similar to that in Example 2, which follows.

EXAMPLE 2 A market research organization is studying a large group of coffee buyers who buy a can of coffee each week. It is found that 50% of those presently using brand A will again buy brand A next week, 25% will switch to brand B, and 25% will switch to another brand. Of those using brand B now, 30% will again buy brand B next week, 60% will switch to brand A, and 10% will switch to another brand. Of those using another brand now, 30% will again buy another brand next week, 40% will switch to brand A, and 30% will switch to brand B. Let states A, B, and D denote brand A, brand B, and another brand, respectively. The probability that a person presently using brand A will switch to brand B is 0.25, the probability that a person presently using brand B will again buy brand B is 0.3, and so on. Thus the transition matrix of this Markov chain is

$$T = \begin{array}{ccc} \text{A} & \text{B} & \text{D} \\ \left[\begin{array}{ccc} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{array}\right] \end{array}\begin{array}{l} \\ \text{A} \\ \text{B} \\ \text{D} \end{array}$$

■

We shall now use the transition matrix of the Markov process to determine the probability of the system being in any of the n states at future times.

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From the fourth day on, the state vector of the system is always the same,

$$\begin{bmatrix} 0.603 \\ 0.397 \end{bmatrix}.$$

This means that from the fourth day on, it is dry about 60% of the time, and it rains about 40% of the time. ■

EXAMPLE 4 Consider Example 2 again. Suppose that when our survey begins, we find that brand A has 20% of the market, brand B has 20% of the market, and the other brands have 60% of the market. Thus

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}.$$

The state vector after the first week is

$$\mathbf{x}^{(1)} = T\mathbf{x}^{(0)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.4600 \\ 0.2900 \\ 0.2500 \end{bmatrix}.$$

Similarly,

$$\mathbf{x}^{(2)} = T\mathbf{x}^{(1)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.4600 \\ 0.2900 \\ 0.2500 \end{bmatrix} = \begin{bmatrix} 0.5040 \\ 0.2770 \\ 0.2190 \end{bmatrix}$$

$$\mathbf{x}^{(3)} = T\mathbf{x}^{(2)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.5040 \\ 0.2770 \\ 0.2190 \end{bmatrix} = \begin{bmatrix} 0.5058 \\ 0.2748 \\ 0.2194 \end{bmatrix}$$

$$\mathbf{x}^{(4)} = T\mathbf{x}^{(3)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.5058 \\ 0.2748 \\ 0.2194 \end{bmatrix} = \begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix}$$

$$\mathbf{x}^{(5)} = T\mathbf{x}^{(4)} = \begin{bmatrix} 0.50 & 0.60 & 0.40 \\ 0.25 & 0.30 & 0.30 \\ 0.25 & 0.10 & 0.30 \end{bmatrix} \begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix} = \begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix}.$$

Thus as n increases, the state vectors approach the fixed vector

$$\begin{bmatrix} 0.5055 \\ 0.2747 \\ 0.2198 \end{bmatrix}.$$

This means that in the long run, brand A will command about 51% of the market, brand B will retain about 27%, and the other brands will command about 22%. ■

In the last two examples we have seen that as the number of observation periods increases, the state vectors converge to a fixed vector. In this case we say that the Markov process has reached equilibrium. The fixed vector is called the **steady-state vector**. Markov processes are generally used to determine the behavior of a system in the long run; for example, the share of the market that a certain manufacturer can expect to retain on a somewhat permanent basis. Thus the question of whether or not a Markov process reaches equilibrium is of paramount importance. The following example shows that not every Markov process reaches equilibrium.

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2.5 Exercises

1. Which of the following can be transition matrices of a Markov process?

(a) $\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$ (b) $\begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.8 & 0.5 & 0.7 \\ 0.0 & 0.2 & 0.2 \end{bmatrix}$

(c) $\begin{bmatrix} 0.55 & 0.33 \\ 0.45 & 0.67 \end{bmatrix}$ (d) $\begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.2 & 0.0 & 0.8 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$

2. Which of the following are probability vectors?

(a) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{4} \\ \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{1}{10} \\ \frac{2}{10} \end{bmatrix}$

In Exercises 3 and 4, determine a value of each missing entry, denoted by □, so that the matrix will be a transition matrix of a Markov chain. In some cases there may be more than one correct answer.

3. $\begin{bmatrix} \square & 0.4 & 0.3 \\ 0.3 & \square & 0.5 \\ \square & 0.2 & \square \end{bmatrix}$ 4. $\begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.3 & \square & 0.5 \\ \square & \square & \square \end{bmatrix}$

5. Consider the transition matrix

$$T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}.$$

(a) If $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, compute $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ to three decimal places.

(b) Show that T is regular and find its steady-state vector.

6. Consider the transition matrix

$$T = \begin{bmatrix} 0 & 0.2 & 0.0 \\ 0 & 0.3 & 0.3 \\ 1 & 0.5 & 0.7 \end{bmatrix}.$$

(a) If

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

compute $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$, and $\mathbf{x}^{(4)}$ to three decimal places.

(b) Show that T is regular and find its steady-state vector.

7. Which of the following transition matrices are regular?

(a) $\begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{4} & \frac{3}{5} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{2}{5} & \frac{1}{2} \end{bmatrix}$

8. Show that each of the following transition matrices reaches a state of equilibrium.

(a) $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{3} & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{4} \end{bmatrix}$ (d) $\begin{bmatrix} 0.3 & 0.1 & 0.4 \\ 0.2 & 0.4 & 0.0 \\ 0.5 & 0.5 & 0.6 \end{bmatrix}$

9. Let

$$T = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

(a) Show that T is not regular.

(b) Show that $T^n\mathbf{x} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for any probability vector $\mathbf{x}$.

Thus a Markov chain may have a unique steady-state vector even though its transition matrix is not regular.

10. Find the steady-state vector of each of the following regular matrices.

(a) $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.9 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{2}{3} \\ \frac{3}{4} & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0.4 & 0.0 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.5 & 0.6 \end{bmatrix}$

11. (*Psychology*) A behavioral psychologist places a rat each day in a cage with two doors, A and B. The rat can go through door A, where it receives an electric shock, or through door B, where it receives some food. A record is made of the door through which the rat passes. At the start of the experiment, on a Monday, the rat is equally likely to go through door A as through door B. After going through door A, and receiving a shock, the probability of going through the same door on the next day is 0.3. After going through door B, and receiving food, the probability of going through the same door on the next day is 0.6.

(a) Write the transition matrix for the Markov process.

(b) What is the probability of the rat going through door A on Thursday (the third day after starting the experiment)?

(c) What is the steady-state vector?

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12. ***(Business)*** The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine, while others do not. From this mailing list, 60% of those who already subscribe will subscribe again, while 25% of those who do not now subscribe will subscribe.

(a) Write the transition matrix for this Markov process.

(b) On the last letter it was found that 40% of those receiving it, ordered a subscription. What percentage of those receiving the current letter can be expected to order a subscription?

13. ***(Sociology)*** A study has determined that the occupation of a boy, as an adult, depends upon the occupation of his father and is given by the following transition matrix, where P = professional, F = farmer, and L = laborer.

$$\text{Son's occupation}\quad \begin{array}{c} \\ P \\ F \\ L \end{array} \overset{\displaystyle\text{Father's occupation}}{\begin{array}{c} \begin{array}{ccc} P & F & L \end{array} \\ \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} \end{array}}$$

1.M 0.7, .02, 0.69

Thus the probability that the son of a professional will also be a professional is 0.8, and so on.

(a) What is the probability that the grandchild of a professional will also be a professional?

(b) In the long run, what proportion of the population will be farmers?

14. ***(Genetics)*** Consider a plant that can have red flowers (R), pink flowers (P), or white flowers (W), depending upon the genotypes RR, RW, and WW. When we cross each of these genotypes with a genotype RW, we obtain the transition matrix

$$\text{Flowers of offspring plant}\quad \begin{array}{c} \\ R \\ P \\ W \end{array} \overset{\displaystyle\text{Flowers of parent plant}}{\begin{array}{c} \begin{array}{ccc} R & P & W \end{array} \\ \begin{bmatrix} 0.5 & 0.25 & 0.0 \\ 0.5 & 0.50 & 0.5 \\ 0.0 & 0.25 & 0.5 \end{bmatrix} \end{array}}.$$

Suppose that each successive generation is produced by crossing only with plants of RW genotype. When the process reaches equilibrium, what percentage of the plants will have red, pink, or white flowers?

15. ***(Mass Transit)*** A new mass transit system has just gone into operation. The transit authority has made studies that predict the percentage of commuters who will change to mass transit (M) or continue driving their automobile (A). The following transition matrix has been obtained:

$$\text{Next year}\quad \begin{array}{c} \\ M \\ A \end{array} \overset{\displaystyle\text{This year}}{\begin{array}{c} \begin{array}{cc} M & A \end{array} \\ \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \end{array}}.$$

Suppose that the population of the area remains constant, and that initially 30% of the commuters use mass transit and 70% use their automobiles.

(a) What percentage of the commuters will be using the mass transit system after 1 year? After 2 years?

(b) What percentage of the commuters will be using the mass transit system in the long run?

Theoretical Exercise

T.1. Is the transpose of a transition matrix of a Markov chain also a transition matrix of a Markov chain? Explain.

MATLAB Exercises

The computation of the sequence of vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ as in Examples 3 and 4 can easily be done using MATLAB *commands. Once the transition matrix T and initial state vector $\mathbf{x}^{(0)}$ are entered into* MATLAB, *the state vector for the kth observation period is obtained from the* MATLAB *command*

```
T^k*x
```

ML.1. Use MATLAB to verify the computations of state vectors in Example 3 for periods 1 through 5.

ML.2. In Example 4, if the initial state is changed to

$$\begin{bmatrix} 0.1 \\ 0.3 \\ 0.6 \end{bmatrix},$$

determine $\mathbf{x}^{(5)}$.

ML.3. In MATLAB, enter **help sum** and determine the action of command **sum** on an $m \times n$ matrix. Use command **sum** to determine which of the following are Markov matrices.

(a) $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$

(b) $\begin{bmatrix} 0.5 & 0.6 & 0.7 \\ 0.3 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.0 \end{bmatrix}$

(c) $\begin{bmatrix} 0.66 & 0.25 & 0.125 \\ 0.33 & 0.25 & 0.625 \\ 0.00 & 0.50 & 0.250 \end{bmatrix}$

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Exercise 2.5

Q.1 (a) $T = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$ is not transition matrix

Reason: The Ist condition of definition is satisfied i.e all the entries are nonnegative but the 2nd condition is not satisfied, that is, $0.3+0.4 \neq 1$ and $0.7+0.6 \neq 1$.

b) $T = \begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.8 & 0.5 & 0.7 \\ 0.0 & 0.2 & 0.2 \end{bmatrix}$ is a transition matrix as both the conditions of definition are satisfied, that is, all the entries are nonnegative and the sum of entries in each colum is 1.

Q2 (a) $u = \begin{bmatrix} 1/2 \\ 1/3 \\ 2/3 \end{bmatrix}$ is not probability vector

Reason: The Ist condition of definition is satisfied, that is, all entries are nonnegative but the 2nd condition is not satisfied, that is, $1/2 + 1/3 + 2/3 \neq 1$.

Q3 $\begin{bmatrix} \square & 0.4 & 0.3 \\ 0.3 & \square & 0.5 \\ \square & 0.2 & \square \end{bmatrix} \Longrightarrow \begin{bmatrix} \boxed{0.1} & 0.4 & 0.3 \\ 0.3 & \boxed{0.4} & 0.5 \\ \boxed{0.6} & 0.2 & \boxed{0.2} \end{bmatrix}$, which is transition.

Q5 — Q6: similar to EX-3/iii on Page-4 and EX-2 on Page-7

Q7: similar to EX-2 on Page-7.

Q8: similar to EX-3/iv on Page-4

Q9: Similar to EX-2 on Page-7.

Q10: Similar to 2nd method of Part-iv in EX-3 on Page-5.

Q11: Here the initial state vector is $X^{(0)} = \begin{array}{c} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \end{array}\begin{array}{c} A \\ B \end{array}$

(a) $T = \begin{array}{cc} A & B \end{array}$

$\quad \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}\begin{array}{l} A \longrightarrow \text{electric shock is attached} \\ B \longrightarrow \text{some food is kept.} \end{array}$

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2.5

b) The probability of the rat going through door A on Thursday is calculated as

0-day	Monday as the experiment is started from this day		
1-day	Tuesday	1st day	$X^{(1)}$
2-day	wednesday	2nd day	$X^{(2)}$
3-day	Thursday	3rd day	$X^{(3)}$

$$X^{(0)} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\begin{matrix} A \\ B \end{matrix}$$

$$X^{(1)} = T X^{(0)} = \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.15+0.20 \\ 0.35+0.30 \end{bmatrix} = \begin{bmatrix} 0.35 \\ 0.65 \end{bmatrix}\begin{matrix} A \\ B \end{matrix}$$

$$X^{(2)} = T X^{(1)} = \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}\begin{bmatrix} 0.35 \\ 0.65 \end{bmatrix} = \begin{bmatrix} 0.105+0.260 \\ 0.245+0.390 \end{bmatrix} = \begin{bmatrix} 0.365 \\ 0.635 \end{bmatrix}\begin{matrix} A \\ B \end{matrix}$$

$$X^{(3)} = T X^{(2)} = \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}\begin{bmatrix} 0.365 \\ 0.635 \end{bmatrix} = \begin{bmatrix} 0.1095+0.2540 \\ 0.2555+0.3810 \end{bmatrix} = \begin{bmatrix} 0.3635 \\ 0.6365 \end{bmatrix}\begin{matrix} A \\ B \end{matrix}$$

Thus the probability of the rat going through door A on Thursday is 0.3635

c) Let $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be the steady-state vector of the Markov process, Then by definition, we have

$$TU = U \Rightarrow U = TU \Rightarrow U - TU = 0$$

$$\Rightarrow (I_2 - T)U = 0 \Rightarrow \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0.7 & -0.4 \\ -0.7 & 0.4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 0.7u_1 - 0.4u_2 = 0 \\ -0.7u_1 + 0.4u_2 = 0 \end{matrix}$$

$$\Rightarrow 0.7u_1 - 0.4u_2 = 0 \Rightarrow u_1 = 4/7\, u_2 \Rightarrow \boxed{u_1 = \frac{4}{7}r}$$

where $u_2 = r$, $0 \le r \le 1$. Also $u_1 + u_2 = 1$, (*) as U is the probability vector (every state vector/steady-state vector is the probability vector). (*) $\Rightarrow 4/7\,r + r = 1 \Rightarrow \boxed{r = 7/11 = u_2}$

$u_1 = 4/7\,(7/11) \Rightarrow u_1 = 4/11$. Hence the steady-state vector is $U = \begin{bmatrix} 4/11 \\ 7/11 \end{bmatrix} = \begin{bmatrix} 0.364 \\ 0.636 \end{bmatrix}$

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Q12: similar to Q11 2.5.

Q13 P = Professional; F = Farmer, and L = Laborer

$$\begin{array}{ccc} & & \text{Father's occupation} \\ & & \begin{array}{ccc} P & F & L \end{array} \\ \text{Son's occupation} & \begin{array}{c} P \\ F \\ L \end{array} & \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} \end{array}$$

Thus the Probability that the son of a Professional will also be a Professional is 0.8 and so on.

a) The Probability that the grandchild of a Professional will also be a Professional is given by

$$X^{(1)} = TX^{(0)} = \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.64+0.03+0.02 \\ 0.08+0.05+0.02 \\ 0.08+0.02+0.06 \end{bmatrix}$$

$$= \begin{bmatrix} 0.69 \\ 0.15 \\ 0.16 \end{bmatrix} \begin{matrix} P \\ F \\ L \end{matrix}$$

Hence the Probability that the grandchild of a Professional will also be a Professional = 0.69 Ans