3

Principles of Money–Time Relationships

The objective of this chapter is to provide an understanding of the *return to capital* in the form of interest (or profit) and to illustrate how basic equivalence calculations are made with respect to the cost of capital in engineering economy studies. The following topics are discussed in this chapter:

- Return to capital
- Origins of interest
- Simple interest
- Compound interest
- The concept of equivalence
- Cash flow diagrams/tables
- Interest formulas
- Arithmetic sequences of cash flows
- Geometric sequences of cash flows
- Nominal versus effective interest rates
- Interest rates that vary with time
- Continuous compounding

3.1 Introduction

The term *capital* refers to wealth in the form of money or property that is cap of being used to produce more wealth. The majority of engineering econ studies involve commitment of capital for extended periods of time, so the e of time must be considered. In this regard, it is recognized that a dollar to is worth more than a dollar 1 or more years from now because of the inte (or profit) it can earn. Therefore, money has a *time value*.

3.2 Why Consider Return to Capital?

Capital in the form of money for the people, machines, materials, energy, other things needed in the operation of an organization may be classified two basic categories. *Equity capital* is that owned by the individuals who h invested their money or property in a business project or venture in the h of receiving a profit. *Debt capital*, often called *borrowed capital*, is obtained fr lenders (e.g., through the sale of bonds) for investment. In return, the lend receive interest from the borrowers.

Normally, the lenders do not receive any other benefits that may accrue fr the investment of capital borrowed. They are not owners of the organization do not participate as fully in the risks of the project or venture as the own do. Thus, their fixed return on the capital loaned, in the form of interest more assured (has less risk) than the receipt of profit by the owners of equ capital. If the project or venture is successful, the return (profit) to the own of equity capital can be substantially more than the interest received by lend of capital; however, the owners could lose some or all of their money inves while the lenders still could receive all the interest owed plus repayment of money borrowed by the firm.

There are fundamental reasons why return to capital in the form of in est and profit is an essential ingredient of engineering economy studies. F interest and profit pay the providers of capital for forgoing its use during time the capital is being used. The fact that the supplier can realize a ret on capital acts as an *incentive* to accumulate capital by savings, thus postpo immediate consumption in favor of creating wealth in the future. Second, in est and profit are payments for the *risk* the investor takes in permitting ano person, or an organization, to use his or her capital.

In typical situations, investors must decide whether the expected ret on their capital is sufficient to justify buying into a proposed project or v ture. If capital is invested in a project, investors would expect, as a minimu to receive a return at least equal to the amount they have sacrificed by not us it in some other available opportunity of comparable risk. This interest or profit available from an alternative investment is called the *opportunity cost* of using capital in the proposed undertaking. Thus, whether borrowed capital or equity capital is involved, there is a cost for the capital employed in the sense that the project and venture must provide a return sufficient to be financially attractive to suppliers of money or property.

In summary, whenever capital is required in engineering and other business projects and ventures, it is essential that proper consideration be given to its cost. The remainder of this chapter deals with time value of money principles, which are vitally important to the proper evaluation of engineering projects that form the foundation of a firm's competitiveness, and hence to its very survival.

3.3 The Origins of Interest

Like taxes, interest has existed from earliest recorded human history. Records reveal its existence in Babylon in 2000 B.C.. In the earliest instances interest was paid in money for the use of grain or other commodities that were borrowed; it was also paid in the form of grain or other goods. Many of the existing interest practices stem from early customs in the borrowing and repayment of grain and other crops.

History also reveals that the idea of interest became so well established that a firm of international bankers existed in 575 B.C., with home offices in Babylon. The firm's income was derived from the high interest rates it charged for the use of its money for financing international trade.

Throughout early recorded history, typical annual rates of interest on loans of money were in the neighborhood of 6 to 25%, although legally sanctioned rates as high as 40% were permitted in some instances. The charging of exorbitant interest rates on a loan was termed *usury,* and prohibition of usury is found in the Bible (see Exodus 22: 21–27).

During the Middle Ages, interest taking on loans of money was generally outlawed on scriptural grounds. In 1536 the Protestant theory of usury was established by John Calvin, and it refuted the notion that interest was unlawful. Consequently, interest taking again became viewed as an essential and legal part of doing business. Eventually, published interest tables became available to the public.

3.4 Simple Interest

When the total interest earned or charged is directly proportional to the initial amount of the loan (principal), the interest rate, and the number of interest periods for which the principal is committed, the interest and interest rate are said to be *simple.* Simple interest is not used frequently in commercial practice in modern times.

When simple interest is applicable, the total interest, $\underline{I}$, earned or paid may be computed in the formula

$$\underline{I} = (P)(N)(i) \tag{3-1}$$

where P = principal amount lent or borrowed
N = number of interest periods (e.g., years)
i = interest rate per interest period.

The total amount repaid at the end of N interest periods is $P + \underline{I}$. Thus, if \$1,000 is loaned for 3 years at a simple interest rate of 10% per annum, the interest earned will be

$$\underline{I} = \$1{,}000 \times 0.10 \times 3 = \$300$$

The total amount owed at the end of 3 years would be \$1,000 + \$300 = \$1,300. Notice that the cumulative amount of interest owed is a linear function of time until the interest is repaid (usually not until the end of period N).

3.5 Compound Interest

Whenever the interest charge for any interest *period* (a year, for example) is based on the remaining principal amount plus any accumulated interest charges up to the beginning of that period, the interest is said to be *compound*. The effect of compounding of interest can be seen in the following table for \$1,000 loaned for three periods at an interest rate of 10% compounded each period.

Period	(1) *Amount Owed at Beginning of Period*	(2) = (1) × 10% *Interest Amount for Period*	(3) = (1) + (2) *Amount Owed at End of Period*
1	\$1,000.00	\$100.00	\$1,100.00
2	\$1,100.00	\$110.00	\$1,210.00
3	\$1,210.00	\$121.00	\$1,331.00

As you can see, a total of \$1,331.00 would be due for repayment at the end of the third period. If the length of a period is 1 year, the \$1,331.00 at the end of three periods (years) can be compared with the \$1,300.00 given earlier for the same problem with simple interest. A graphical comparison of simple interest and compound interest is given in Figure 3-1. The difference is due to the effect of *compounding,* which essentially is the calculation of interest on previously earned interest. This difference would be much greater for larger amounts of money, higher interest rates, or greater numbers of years. Thus, simple interest does consider the time value of money but does not involve compounding of interest. Compound interest is much more common in practice than simple interest and is used throughout the remainder of this book.

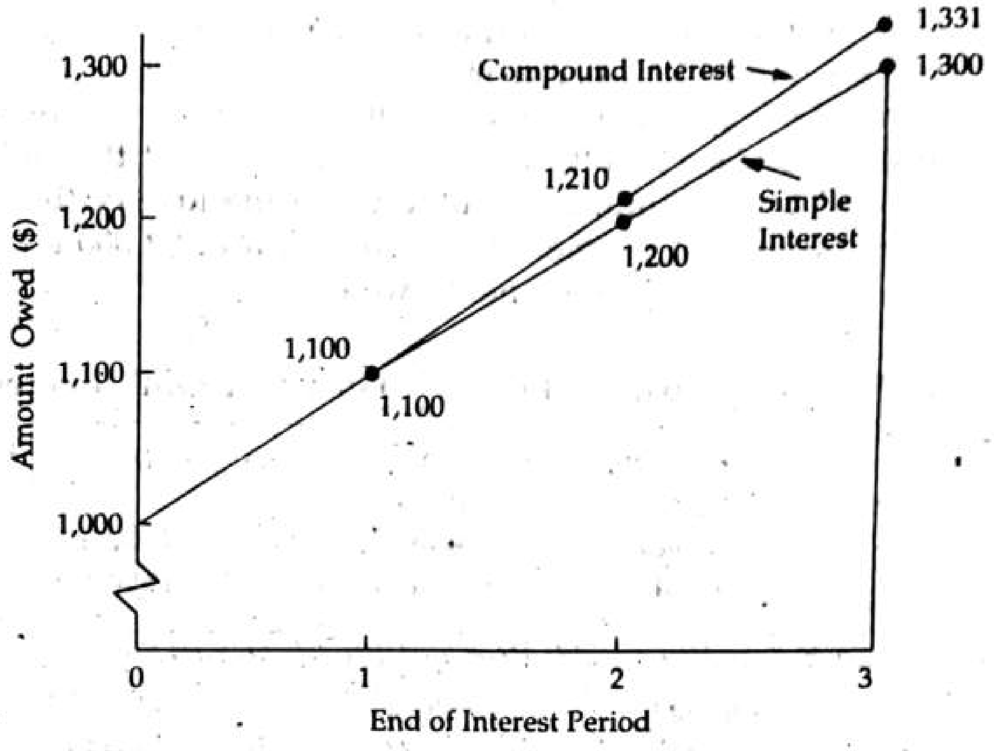


FIGURE 3-1 Illustration of Simple Versus Compound Interest

3.6 The Concept of Equivalence

Alternatives should be compared as far as possible when they produce similar results, serve the same purpose, or accomplish the same function. This is not always possible in some types of economy studies, as we shall see later, but now our attention is directed at answering the question: How can alternatives for providing the same service or accomplishing the same function be compared when interest is involved over extended periods of time? Thus, we should consider the comparison of alternative options, or proposals, by reducing them to an *equivalent basis* that is dependent on (1) the interest rate, (2) the amounts of money involved, (3) the timing of the monetary receipts and/or disbursements, and (4) the manner in which the interest, or profit, on invested capital is repaid and the initial capital recovered.

To understand better the mechanics of interest and to expand on the notion of economic equivalence, we consider a situation in which we borrow $8,000 and agree to repay it in 4 years at an annual interest rate of 10%. There are many plans by which the principal of this loan (i.e., $8,000) and the interest on it can be repaid. For simplicity, we have selected four plans to demonstrate the idea of economic equivalence. Here *equivalence* means that all four plans are equally desirable to the borrower. Thus, any one is tradeable for one of the others. In each the interest rate is 10% and the original amount borrowed is $8,000; thus differences among the plans rest with items (3) and (4) above. The four plans are shown in Table 3-1, and it will soon be apparent that all are *equivalent* at an interest rate of 10% per year.

In plan 1, \$2,000 of the loan principal is repaid at the end of each of years 1 through 4. As a result, the interest we repay at the end of a particular year is affected by how much we still owe on the loan at the beginning of that year. Our end-of-year payment is just the sum of \$2,000 and interest computed on the beginning-of-year amount owed.

TABLE 3-1 Four Plans for Repayment of \$8,000 in 4 Years with Interest at 10%

(1) Year	(2) *Amount Owed at Beginning of Year*	(3) = 10% × (2) *Interest Accrued for Year*	(4) = (2) + (3) *Total Money Owed at End of Year*	(5) *Principal Payment*	(6) = (3) + (5) *Total End-of-Year Payment*
Plan 1: At End of Each Year Pay \$2,000 Principal Plus Interest Due					
1	\$ 8,000	\$ 800	\$8,800	\$2,000	\$ 2,800
2	6,000	600	6,600	2,000	2,600
3	4,000	400	4,400	2,000	2,400
4	2,000	200	2,200	2,000	2,200
	20,000 \$-yr	\$2,000 (total interest)		\$8,000	\$10,000 (total amount repaid)
Plan 2: Pay Interest Due at End of Each Year and Principal at End of 4 Years					
1	\$ 8,000	\$ 800	\$8,800	\$ 0	\$ 800
2	8,000	800	8,800	0	800
3	8,000	800	8,800	0	800
4	8,000	800	8,800	8,000	8,800
	32,000 \$-yr	\$3,200 (total interest)		\$8,000	\$11,200 (total amount repaid)
Plan 3: Pay in Four Equal End-of-Year Payments					A
1	\$ 8,000 M	\$ 800	\$8,800	\$1,724	\$ 2,524
2	6,276 M-L	628 I	6,904	L=A-I 1,896	2,524
3	4,380	438	4,818	2,086	2,524
4	2,294	230	2,524	2,294	2,524
	20,960 \$-yr	\$2,096 (total interest)		\$8,000	\$10,096 (total amount repaid)
Plan 4: Pay Principal and Interest in One Payment at End of 4 Years					
1	\$ 8,000	\$ 800	\$ 8,800	\$ 0	\$ 0
2	8,800	880	9,680	0	0
3	9,680	968	10,648	0	0
4	10,648	1,065	11,713	8,000	11,713
	37,130 \$-yr	\$3,713 (total interest)		\$8,000	\$11,713 (total amount repaid)

When total dollar-years are calculated for each plan and divided into total interest paid over the 4 years (the sum of column 3), the ratio is found to be constant:

Plan	Area Under Curve (Dollar-Years)	Total Interest Paid	Ratio of Total Interest to Dollar-Years
1	\$20,000	\$2,000	0.10
2	32,000	3,200	0:10
3	20,960	2,096	0.10
4	37,130	3,713	0.10

Because the ratio is constant at 0.10 for all plans, we can deduce that all repayment methods considered in Table 3-1 are equivalent even though each involves a different total end-of-year payment in column 6. Dissimilar dollar-years of borrowing, by itself, does not necessarily allow one to conclude that different loan repayment plans *are* or *are not* equivalent. In summary, equivalence is established when total interest paid, divided by dollar-years of borrowing, is a constant ratio among financing plans (i.e., alternatives).

One last important point to emphasize is that the loan repayment plans of Table 3-1 are equivalent only at an interest rate of 10%. If these plans are evaluated with methods presented later in this chapter at interest rates other than 10%, one plan can be identified that is superior to the other three. For instance, when \$8,000 has been lent at 10% interest and subsequently the cost of borrowed money increases to 15%, the *lender* would prefer plan 1 in order to recover his or her funds quickly so that they might be reinvested at higher interest rates.

3.7 Notation and Cash Flow Diagrams/ Tables

The following notation is utilized in formulas for compound interest calculations:

i = effective interest rate per interest period
N = number of compounding periods
P = present sum of money; the *equivalent* worth of one or more cash flows at a reference point in time called the present
F = future sum of money; the *equivalent* worth of one or more cash flows at a reference point in time called the future
A = end-of-period cash flows (or *equivalent* end-of-period values) in a uniform series continuing for a specified number of periods, starting at the end of the first period and continuing through the last period

The use of cash flow (time) diagrams and/or tables is strongly recommended for situations in which the analyst needs to clarify or visualize what is involved

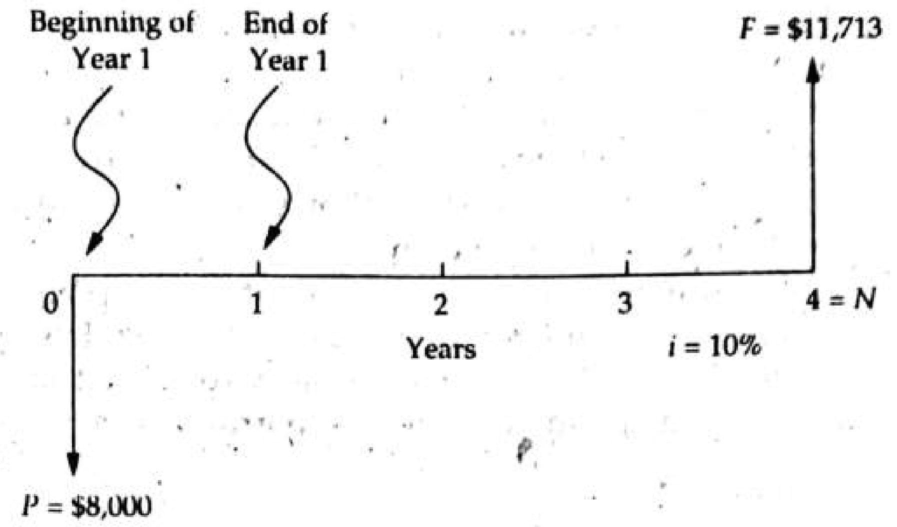


FIGURE 3-2 Cash Flow Diagram for Plan 4 of Table 3-1 (Lender's Viewpoint)

when flows of money occur at various times. In summary, the difference between total cash inflows (receipts) and cash outflows (expenditures) for a specified period of time (e.g., 1 year) is the net cash flow for the period. As discussed in Chapter 2, cash flows are important in engineering economy because they form the basis for evaluating alternatives. Indeed, the usefulness of this cash flow diagram for economic analysis problems is analogous to that of the free-body diagram for mechanics problems.

Figure 3-2 shows a cash flow diagram for plan 4 of Table 3-1, and Figure 3-3 depicts the net cash flow of plan 3. These two figures also illustrate the definition of the above symbols and their placement on a cash flow diagram. Notice that all cash flows have been placed at the end of the year to correspond with the convention used in Table 3-1. In addition, a viewpoint has been specified.

The cash flow diagram employs several conventions:

1. The horizontal line is a *time scale*, with progression of time moving from left to right. The period (e.g., year, quarter, month) labels are applied to intervals of time rather than to points on the time scale. Note, for example,

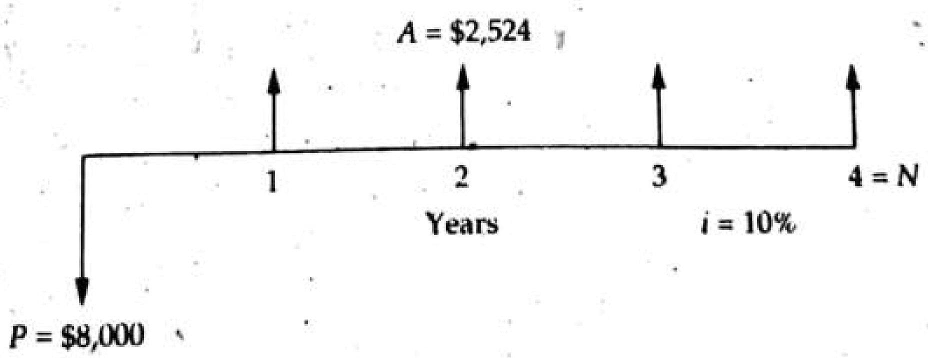


FIGURE 3-3 Cash Flow Diagram for Plan 3 of Table 3-1 (Lender's Viewpoint)

that the end of period 2 is coincident with the beginning of period 3. When the end-of-period cash flow convention is used, period numbers are placed at the end of each time interval.

2. The arrows signify cash flows and are placed at the end of the period. If a distinction needs to be made, downward arrows represent expenses (negative cash flows or cash outflows) and upward arrows represent receipts (positive cash flows or cash inflows).
3. The cash flow diagram is dependent on the point of view. For example, the situations shown in Figures 3-2 and 3-3 were based on cash flow as seen by the lender. If the directions of all arrows had been reversed, the problem would be diagrammed from the borrower's viewpoint.

EXAMPLE 3-1

Before evaluating the economic merits of a proposed investment, the XYZ Corporation insists that its engineers develop a cash flow diagram of the proposal. An investment of $10,000 can be made that will produce uniform annual revenue of $5,310 for 5 years and then have a positive salvage value of $2,000 at the end of year 5. Annual expenses will be $3,000 at the end of each year for operating and maintaining the project. Draw a cash flow diagram for the 5-year life of the project.

Solution As shown in Figure 3-4, the initial investment of $10,000 and annual expenses of $3,000 are cash outflows, while annual revenues and the salvage value are cash inflows.

Notice that the beginning of a given year is the end of the preceding year. For example, the beginning of year 2 is the end of year 1. ■

Example 3-2 presents a situation in which cash flows are represented in tabular form to facilitate the analysis of plans/designs.

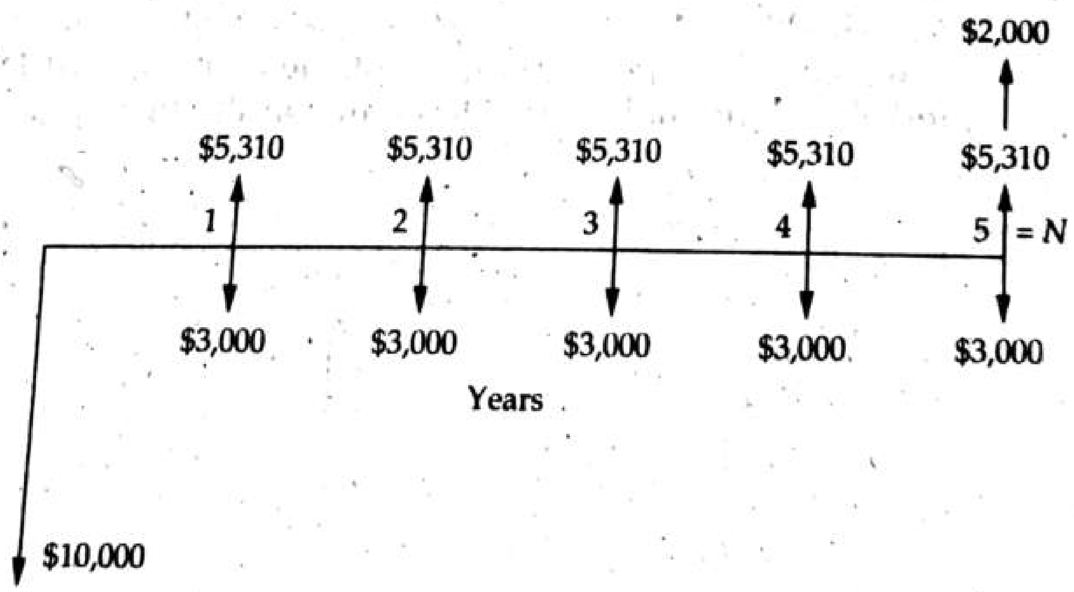


FIGURE 3-4 Cash Flow Diagram for Example 3-1

EXAMPLE 3-2

In the renovation of a small office building, two feasible alternatives for upgrading the heating, ventilation, and air conditioning (HVAC) system have been identified. The costs are

Alternative A *Rebuild (overhaul) the existing HVAC system*

- Equipment, labor, and materials to upgrade $18,000
- Annual cost of electricity 32,000
- Annual maintenance 2,400

Alternative B *Install a new HVAC system that utilizes existing ductwork*

- Equipment, labor, and materials to install $60,000
- Annual cost of electricity 9,000
- Annual maintenance 16,000
- Replacement of a major component 4 years hence .. 9,400

At the end of 8 years, the estimated salvage value for Alternative A is $2,000, and for Alternative B it is $8,000. Assume that both alternatives will provide comparable service (comfort) over an 8-year time period, and assume that the major component replaced in Alternative B will have no salvage value at the end of year 8. (1) Use a cash flow table and end-of-year convention to tabulate the net cash flows for both alternatives. (2) Determine the annual net cash flow difference between the alternatives (B−A). (3) Compute the cumulative difference through the end of year 8. (The cumulative difference is the sum of differences, B − A, from year 0 through year 8.)

Solution The cash flow table for this example is given as Table 3-2. Based on these results, several points can be made: (1) doing nothing is not an option—either A or B must be selected; (2) even though positive and negative cash

TABLE 3-2 Cash Flow for Example 3-2

End-of-Year	*Alternative A Net Cash Flow*	*Alternative B Net Cash Flow*	*Difference (B − A)*	*Cumulative Difference*
0 (now)	−$ 18,000	−$ 60,000	−$42,000	−$42,000
1	− 34,400	− 25,000	9,400	− 32,600
2	− 34,400	− 25,000	9,400	− 23,200
3	− 34,400	− 25,000	9,400	− 13,800
4	− 34,400	− 34,400	0	− 13,800
5	− 34,400	− 25,000	9,400	− 4,400
6	− 34,400	− 25,000	9,400	5,000
7	− 34,400	− 25,000	9,400	14,400
8	− 34,400 + 2,000	− 25,000 + 8,000	15,400	29,800
Total	−$291,200	−$261,400		

flows are included in the table, on balance we are investigating two "cost-only" alternatives; (3) a decision between the two alternatives can be made just as easily on the *difference* in cash flows (i.e., on the avoidable difference) as it can on the stand-alone net cash flows for Alternatives A and B; (4) Alternative B has cash flows identical to those of Alternative A *except for* the differences shown in the table, so if the avoidable difference can "pay its own way," Alternative B is the recommended choice; (5) cash flow changes caused by inflation or other suspected influences could have easily been inserted into the table and included in the analysis; and (6) it takes 6 *years* for the extra $42,000 investment in Alternative B to generate sufficient cumulative savings in annual expenses to justify the higher investment (this ignores the time value of money). So, which alternative is better? We'll be able to answer this question later when we consider the time value of money in arriving at recommendations concerning choices between alternatives.

It should be apparent that a cash flow table serves to clarify the timing of cash flows, the assumptions that are being made, and the data that are available. A cash flow table is often useful when the complexity of a situation makes it difficult to show all cash flow amounts on a diagram.

The remainder of Chapter 3 deals with the development and illustration of equivalence (time value of money) principles for assessing the economic attractiveness of investments such as those proposed in Examples 3-1 and 3-2.

3.8 Interest Formulas Relating Present and Future Equivalent Values of Single Cash Flows

Figure 3-5 shows a cash flow diagram involving a present single sum, P, and a future single sum, F, separated by N periods with interest at i% per period. Throughout this chapter a dashed arrow, such as that shown in Figure 3-5, indicates the quantity to be determined. Two formulas relating a given P and its unknown equivalent F are provided in Equations 3-2 and 3-3.

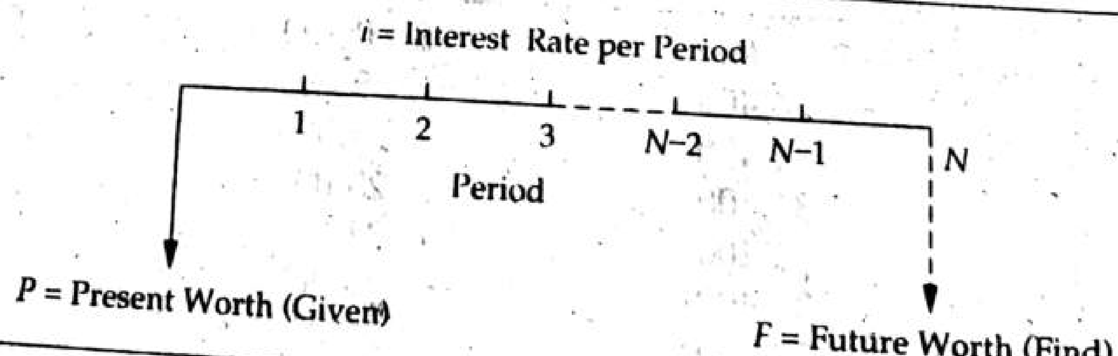


FIGURE 3-5 General Cash Flow Diagram Relating Present Worth and Future Worth of Single Payments

3.8.1 Finding F When Given P

If an amount of P dollars exists at a point in time and $i\%$ is the interest (profit or growth) rate per period, the amount will grow to a future amount of $P + Pi = P(1 + i)$ by the end of one period; by the end of two periods, the amount will grow to $P(1+i)(1+i) = P(1+i)^2$; by the end of three periods, the amount will grow to $P(1 + i)^2(1 + i) = P(1 + i)^3$; and by the end of N periods the amount will grow to

$$F = P(1 + i)^N \tag{3-2}$$

EXAMPLE 3-3

Suppose that you borrow \$8,000 now, with the promise to repay the loan principal plus accumulated interest in 4 years at $i = 10\%$ per year. How much would you owe at the end of 4 years?

Solution

Year	Amount Owed at Start of Year	Interest Owed for Each Year	Amount Owed at End of Year	Total End-of-Year Payment
1	P = \$ 8,000	iP = \$ 800	$P(1+i)$ = \$ 8,800	0
2	$P(1+i)$ = \$ 8,800	$iP(1+i)$ = \$ 880	$P(1+i)^2$ = \$ 9,680	0
3	$P(1+i)^2$ = \$ 9,680	$iP(1+i)^2$ = \$ 968	$P(1+i)^3$ = \$10,648	0
4	$P(1+i)^3$ = \$10,648	$iP(1+i)^3$ = \$1,065	$P(1+i)^4$ = \$11,713	F = \$11,713

In general, we see that $F = P(1 + i)^N$, and the total amount to be repaid is \$11,713. This further illustrates plan 4 in Table 3-1 in terms of notation that we shall be using throughout this book. The compounding effect in this example is shown in Figure 3-6. ∎

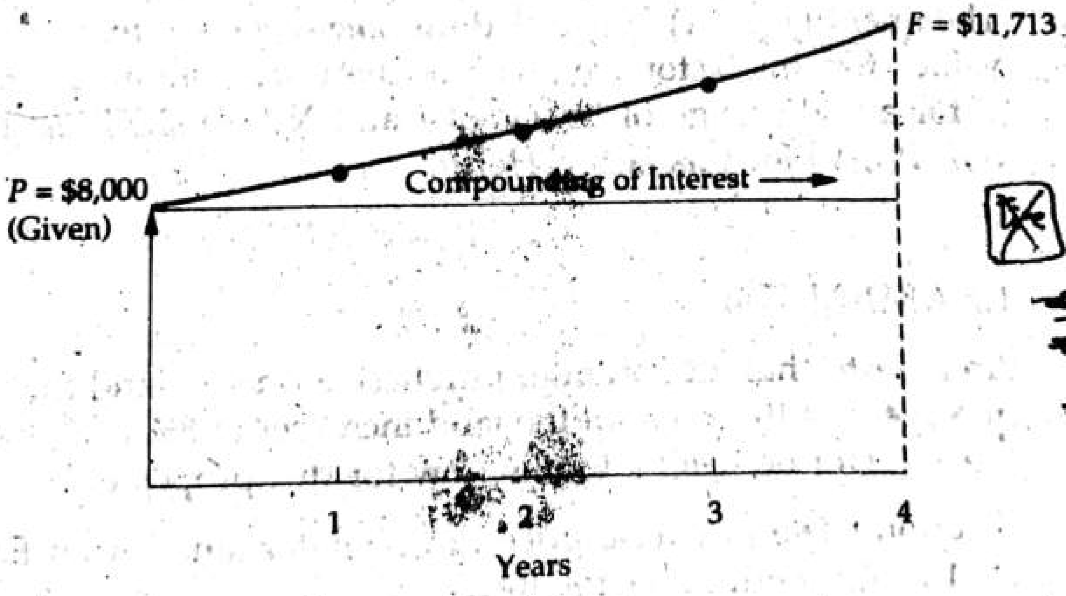


FIGURE 3-6 Illustration of Compounding Effect

Mind

The quantity $(1 + i)^N$ is commonly called the *single payment compound amount factor*. Numerical values for this factor are given in the second column from the left in the tables of Appendix C for a wide range of values of i and N. In this book we shall use the functional symbol $(F/P, i\%, N)$ for $(1 + i)^N$. Hence Equation 3-2 can be expressed as

$$F = P(F/P, i\%, N) \tag{3-3}$$

where the factor in parentheses is read "find F given P at $i\%$ interest per period for N interest periods." Note that the sequence of F and P in F/P is the same as in the initial part of Equation 3-3, where the unknown quantity, F, is placed on the left-hand side of the equation. This sequencing of letters is true of all functional symbols used in this book and makes them easy to remember.

Another example of finding F when given P, together with a cash flow diagram and solution, is given in Table 3-3. Note in Table 3-3 that for each of the six common discrete compound interest circumstances covered, two problem statements are given—(a) in borrowing–lending terminology and (b) in equivalence terminology—but they both represent the same cash flow situation. Indeed, there are generally many ways in which a given cash flow situation can be expressed.

In general, a good way to interpret a relationship such as Equation 3-3 is that the calculated amount, F, at the point in time at which it occurs, *is equivalent to* (i.e., can be traded for) the known value, P, at the point in time at which it occurs, for the given interest or profit rate, i.

3.8.2 Finding P When Given F

From Equation 3-2, $F = P(1 + i)^N$. Solving this for P gives the relationship

$$P = F\left(\frac{1}{1+i}\right)^N = F(1 + i)^{-N} \tag{3-4}$$

The quantity $(1 + i)^{-N}$ is called the *single payment present worth factor*. Numerical values for this factor are given in the third column of the tables in Appendix C for a wide range of values of i and N. We shall use the functional symbol $(P/F, i\%, N)$ for this factor. Hence

$$P = F(P/F, i\%, N) \tag{3-5}$$

EXAMPLE 3-4

An investor has an option to purchase a tract of land that will be worth $10,000 in 6 years. If the value of the land increases at 8% each year, how much should the investor be willing to pay now for this property?

Solution The purchase price can be determined from Equation 3-5 and Table C-11 in Appendix C as follows:

$$P = \$10{,}000(P/F, 8\%, 6)$$
$$P = \$10{,}000(0.6302)$$
$$= \$6{,}302$$

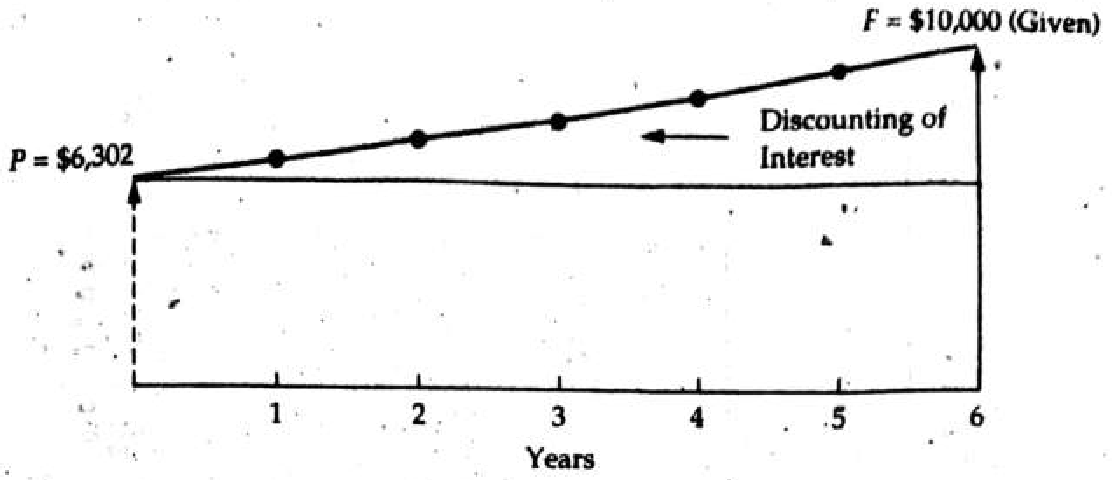


FIGURE 3-7 Illustration of Discounting Effect

The discounting effect in this example is illustrated in Figure 3-7. ■

Another example of this type of problem, together with a cash flow diagram and solution, is given in Table 3-3.

3.9 Interest Formulas Relating a Uniform Series (Annuity) to Its Present and Future Equivalent Value

Figure 3-8 shows a general cash flow diagram involving a series of uniform receipts, each of amount A, occurring at the end of each period for N periods with interest at i% per period. Such a uniform series is often called an *annuity*. It should be noted that the formulas and tables to be presented are derived such that A occurs at the end of each period, and thus:

1. P (present worth, PW) occurs one interest period before the first A (uniform payment).

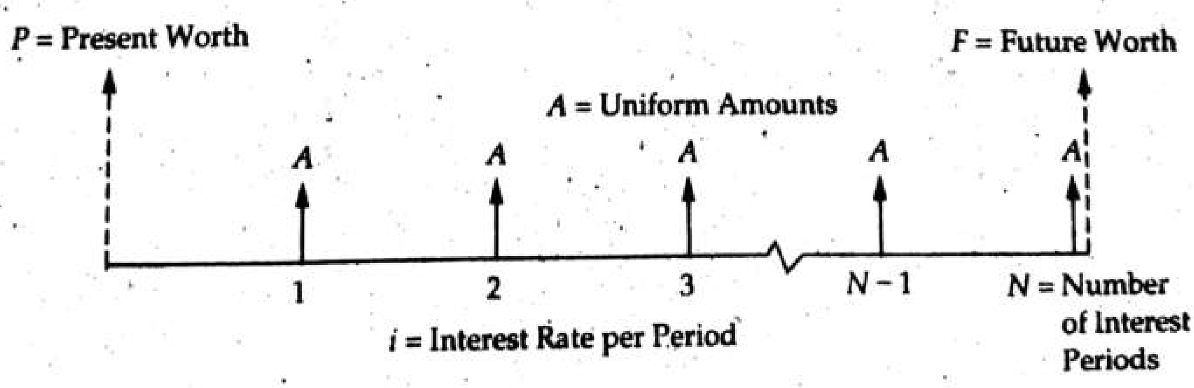
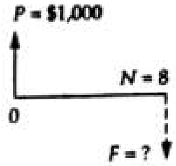
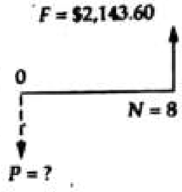
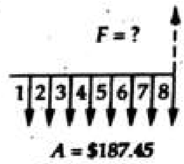
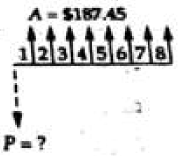
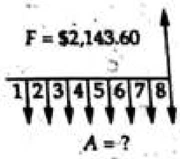
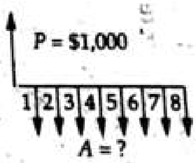


FIGURE 3-8 General Cash Flow Diagram Relating Uniform Series (Ordinary Annuity) to Its PW and FW

TABLE 3-3 Discrete Cash Flow Examples Illustrating Equivalence

Example Problems (all using an interest rate of i = 10% compounded annually)

To Find:	*Given:*	*(a) In Borrowing–Lending Terminology:*	*(b) In Equivalence Terminology:*	*Cash Flow Diagram[a]*	*Solution*
For single cash flows:					
F	P	A firm borrows \$1,000 for 8 years. How much must it repay in a lump sum at the end of the eighth year?	What is the equivalent worth at the end of 8 years of \$1,000 at the beginning of those 8 years?		$F = (F/P, 10\%, 8)$ $= \$1{,}000(2.1436)$ $= \$2{,}143.60$
P	F	A firm wishes to have \$2,143.60 8 years from now. What amount should be deposited now to provide for it?	What is the equivalent present worth of \$2,143.60 received 8 years from now?		$P = (P/F, 10\%, 8)$ $= \$2{,}143.60(0.4665)$ $= \$1{,}000.00$
For uniform series:					
F	A	If eight annual deposits of \$187.45 each are placed in an account, how much money has accumulated immediately after the last deposit?	What amount at the end of the eighth year is equivalent to eight end-of-year payments of \$187.45 each?		$F = A(F/A, 10\%, 8)$ $= \$187.45(11.4359)$ $= \$2{,}143.60$
P	A	How much should be deposited in a fund now to provide for eight end-of-year withdrawals of \$187.45 each?	What is the equivalent present worth of eight end-of-year payments of \$187.45 each?		$P = A(P/A, 10\%, 8)$ $= \$187.45(5.3349)$ $= \$1{,}000.00$
A	F	What uniform annual amount should be deposited each year in order to accumulate \$2,143.60 at the time of the eighth annual deposit?	What uniform payment at the end of 8 successive years is equivalent to \$2,143.60 at the end of the eighth year?		$A = F(A/F, 10\%, 8)$ $= \$2{,}143.60(0.0874)$ $= \$187.45$
A	P	What is the size of eight equal annual payments to repay a loan of \$1,000? The first payment is due 1 year after receiving the loan.	What uniform payment at the end of 8 successive years is equivalent to \$1,000 at the beginning of the first year?		$A = P(A/P, 10\%, 8)$ $= \$1{,}000(0.18745)$ $= \$187.45$

[a]The cash flow diagram represents the example as stated in borrowing-lending terminology.

2. F (future worth, FW) occurs at the same time as the last A, and N periods after P.
3. A (annual worth, AW) occurs at the end of periods 1 through N, inclusive.

The timing relationship for P, A, and F can be observed in Figure 3-8. Four formulas relating A to F and P will be developed.

3.9.1 Finding F When Given A

If a cash flow in the amount of A dollars occurs at the end of each period for N periods and $i\%$ is the interest (profit or growth) rate per period, the future worth, F, at the end of the Nth period is obtained by summing the future worths of each of the cash flows. Thus,

$$\begin{aligned} F &= A(F/P, i\%, N-1) + A(F/P, i\%, N-2) + A(F/P, i\%, N-3) + \cdots \\ &\quad + A(F/P, i\%, 1) + A(F/P, i\%, 0) \\ &= A[(1+i)^{N-1} + (1+i)^{N-2} + (1+i)^{N-3} + \cdots + (1+i)^1 + (1+i)^0] \end{aligned}$$

The bracketed terms comprise a geometric sequence having a common ratio of $(1+i)^{-1}$. Recall that the sum of the first N terms (S_N) of a geometric sequence is

$$S_N = \frac{a_1 - ba_N}{1-b} \qquad (b \neq 1)$$

where a_1 is the first term in the sequence, a_N is the last term, and b is the common ratio. If we let $b = (1+i)^{-1}$, $a_1 = (1+i)^{N-1}$, and $a_N = (1+i)^0$, then

$$F = A\left[\frac{(1+i)^{N-1} - \frac{1}{(1+i)}}{1 - \frac{1}{(1+i)}}\right]$$

which reduces to

$$F = A\left[\frac{(1+i)^N - 1}{i}\right] \qquad (3\text{-}6)$$

The quantity $\{[(1+i)^N - 1]/i\}$ is called the *uniform series compound amount factor*. Numerical values for this factor are given in the fourth column of the tables in Appendix C for a wide range of values of i and N. We shall use the functional symbol $(F/A, i\%, N)$ for this factor. Hence Equation 3-6 can be expressed as

$$F = A(F/A, i\%, N) \qquad (3\text{-}7)$$

Examples of this type of problem are provided here and in Table 3-3.

EXAMPLE 3-5

A woman wishes to have $100,000 in her retirement savings plan after working for 25 years. She will accomplish this by depositing A dollars each year in a savings account that earns 6% per year. How much must she save each year?

Solution The annual deposit required to accumulate $100,000 at 6% annual interest is

$$\begin{aligned} A &= \$100{,}000(A/F, 6\%, 25) \\ &= \$100{,}000(0.0182) \\ &= \$1{,}820.00 \end{aligned}$$

■

9.2 Finding P When Given A

From Equation 3-2, $F = P(1+i)^N$. Substituting for F in Equation 3-6, one determines that

$$P(1+i)^N = A\left[\frac{(1+i)^N - 1}{i}\right]$$

Dividing both sides by $(1+i)^N$,

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] \tag{3-8}$$

Thus, Equation 3-8 is the relation for finding the equivalent present worth (as of the beginning of the first period) of a uniform series of end-of-period cash flows of amount A for N periods. The quantity in brackets is called the *uniform series present worth factor.* Numerical values for this factor are given in the fifth column of the tables in Appendix C for a wide range of values of i and N. We shall use the functional symbol $(P/A, i\%, N)$ for this factor. Hence

$$P = A(P/A, i\%, N) \tag{3-9}$$

EXAMPLE 3-6

(a) If a certain machine undergoes a major overhaul, its output can be increased by 20%—which translates into extra cash flow of $20,000 at the end of each year for 5 years. If i = 15% per year, how much can we afford to invest to overhaul this machine?

Solution The increase in cash flow is $20,000 per year, and it continues for 5 years at 15% annual interest. The upper limit on what we can afford to spend is

$$\begin{aligned} P &= \$20{,}000(P/A, 15\%, 5) \\ &= \$20{,}000(3.3522) \\ &= \$67{,}044 \end{aligned}$$

?

■

(b) Suppose that your rich uncle has $1,000,000 that he wishes to distribute to his heirs at the rate of $100,000 per year. If the $1,000,000 is deposited in a bank account that earns 6% effective interest each year, how many years will it take to completely deplete the account? How long will it take if the account earns 8% interest instead of 6%?

Solution Solve for N in the following equation: $\$1{,}000{,}000 = \$100{,}000(P/A, 6\%, N)$; $N = 15.7$ years. When the interest rate is increased to 8%, it will take 20.9 years to bring the account balance to zero, which is found by solving this equation: $10 = (P/A, 8\%, N)$.

3.9.3 Finding *A* When Given *F*

Taking Equation 3-6 and solving for A, one finds that

$$A = F\left[\frac{i}{(1+i)^N - 1}\right] \tag{3-10}$$

Thus, Equation 3-10 is the relation for finding the amount, A, of a uniform series of cash flows occurring at the end of N interest periods that would be equivalent to (have the same value as) its future worth, F, occurring at the end of the last period. The quantity in brackets is called the *sinking fund factor*. Numerical values for this factor are given in the sixth column of the tables in Appendix C for a wide range of values of i and N. We shall use the functional symbol $(A/F, i\%, N)$ for this factor. Hence

$$A = F(A/F, i\%, N) \tag{3-11}$$

An example of this type of problem, together with a cash flow diagram and solution, is given in Table 3-3.

3.9.4 Finding *A* When Given *P*

Taking Equation 3-8 and solving for A, one finds that

$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right] \tag{3-12}$$

Thus, Equation 3-12 is the relation for finding the amount, A, of a uniform series of cash flows occurring at the end of each of N interest periods that would be equivalent to, or could be traded for, the present worth, P, occurring at the beginning of the first period. The quantity in brackets is called the *capital recovery factor*.* Numerical values for this factor are given in the seventh column of the tables in Appendix C for a wide range of values of i and N. We shall use the functional symbol $(A/P, i\%, N)$ for this factor. Hence

$$A = P(A/P, i\%, N) \tag{3-13}$$

An example that utilizes the equivalence between a present lump-sum amount and a series of equal uniform annual amounts starting at the end

*The capital recovery factor is more conveniently expressed as $i/[1 - (1+i)^{-N}]$ for computation with a hand-held calculator.

of year 1 and continuing through year 4 is provided in Table 3-1 as plan 3. Equation 3-13 yields the equivalent value of A that repays the \$8,000 loan plus 10% interest per year over 4 years:

$$A = \$8{,}000(A/P, 10\%, 4) = \$8{,}000(0.3155) = \$2{,}524$$

The entries in columns 3 and 5 of plan 3 in Table 3-1 can now be better understood. Interest owed at the end of year 1 equals \$8,000(0.10), and therefore the principal repaid out of the total end-of-year payment of \$2,524 is the difference, \$1,724. At the beginning of year 2, the amount of principal owed is \$8,000 − \$1,724 = \$6,276. Interest owed at the end of year 2 is \$6,276(0.10) ≅ \$628, and the principal repaid at that time is \$2,524 − \$628 = \$1,896. The remaining entries in plan 3 are obtained by performing these calculations for years 3 and 4.

A graphical summary of plan 3 is given in Figure 3-9. Here it can be seen that 10% interest is being paid on the beginning-of-year amount owed and that year-end payments of \$2,524, consisting of interest and principal, bring the amount owed to \$0 at the end of the fourth year. (The exact value of A is \$2,523.77 and produces an exact value of \$0 at the end of 4 years.) It is important to note that all the uniform series interest factors in Table 3-3 involve the same concept as the one illustrated in Figure 3-9.

Another example of a problem where we desire to compute an equivalent value for A, from a given value of P and a known interest rate and number of compounding periods, is given in Table 3-3.

For an annual interest rate of 10%, the reader should now be convinced from Table 3-3 that \$1,000 at the beginning of year 1 is equivalent to the \$187.45 at the end of years 1–8, which is then equivalent to \$2,143.60 at the end of year 8.

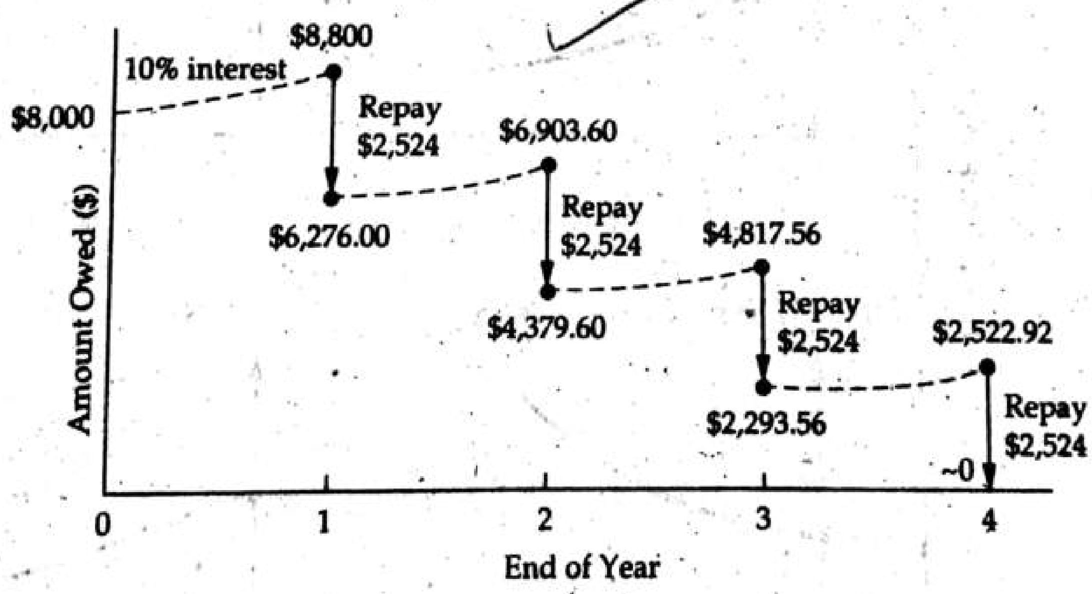


FIGURE 3-9 Relationship of Cash Flows for Plan 3 of Table 3-1 to Repayment of the \$8,000 Loan Principal

3.9.5 Interest Factor Relationships: Summary

This section is summarized by presenting equations and graphs of the relationships between an annuity and its present and future equivalent value.

Equations:

$$(A/P, i\%, N) = \frac{1}{(P/A, i\%, N)} \tag{3-14}$$

$$(A/F, i\%, N) = \frac{1}{(F/A, i\%, N)} \tag{3-15}$$

$$(F/A, i\%, N) = (P/A, i\%, N)(F/P, i\%, N) \tag{3-16}$$

$$(P/A, i\%, N) = \sum_{k=1}^{N}(P/F, i\%, k) \tag{3-17}$$

$$(F/A, i\%, N) = \sum_{k=1}^{N}(F/P, i\%, N-k) \tag{3-18}$$

$$(A/F, i\%, N) = (A/P, i\%, N) - i \tag{3-19}$$

Graphs:

(For a fixed value of N, these graphs help to visualize the preceding equations):

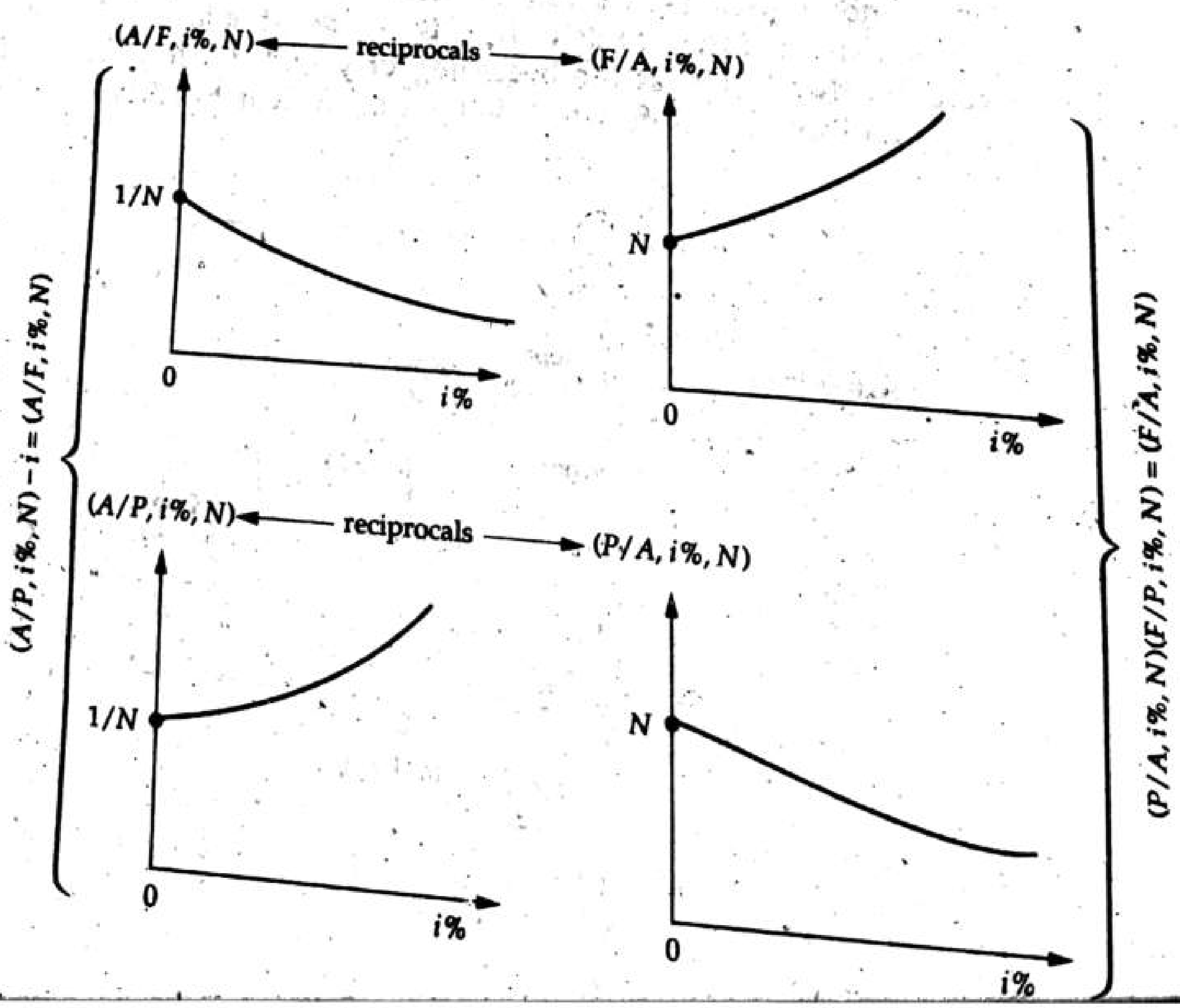


TABLE 3-4 Discrete Compounding Interest Factors and Symbols[a]

To Find:	Given:	Factor by Which to Multiply "Given"[a]	Factor Name	Factor Functional Symbol[b]
For single cash flows:				
F	P	$(1+i)^N$	Single payment compound amount	$(F/P, i\%, N)$
P	F	$\dfrac{1}{(1+i)^N}$	Single payment present worth	$(P/F, i\%, N)$
For uniform series (annuities):				
F	A	$\dfrac{(1+i)^N-1}{i}$	Uniform series compound amount	$(F/A, i\%, N)$
P	A	$\dfrac{(1+i)^N-1}{i(1+i)^N}$	Uniform series present worth	$(P/A, i\%, N)$
A	F	$\dfrac{i}{(1+i)^N-1}$	Sinking fund	$(A/F, i\%, N)$
A	P	$\dfrac{i(1+i)^N}{(1+i)^N-1}$	Capital recovery	$(A/P, i\%, N)$

[a] i, effective interest rate per interest period; N, number of interest periods; A, uniform series amount (occurs at the end of each interest period); F, future worth; P, present worth.

[b] The functional symbol system is used throughout this book.

3.10 Interest Formulas for Discrete Compounding and Discrete Cash Flows

Table 3-4 provides a summary of the six most common discrete compound interest factors, utilizing notation of the preceding sections. The formulas are for *discrete compounding*, which means that the interest is compounded at the end of each finite-length period, such as a month or a year. Furthermore, the formulas also assume discrete (i.e., lump-sum) cash flows spaced at the end of equal time intervals on a cash flow diagram. Discrete compound interest factors are given in Appendix C, where the assumption is made that i remains constant during the N compounding periods.

3.11 Deferred Annuities (Uniform Series)

All annuities (uniform series) discussed to this point involve the first cash flow being made at the end of the first period, and they are called *ordinary annuities*. If the cash flow does not begin until some later date, the annuity is known as ife cycle cost 2 is our assignment

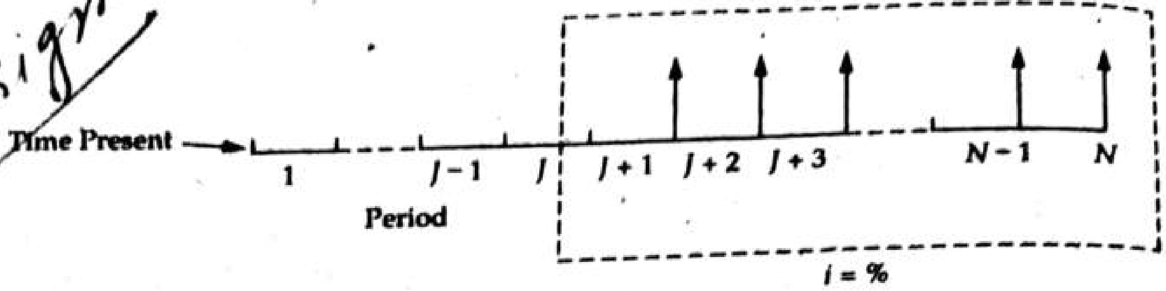


FIGURE 3-10 General Cash Flow Representation of a Deferred Annuity (Uniform Series)

a *deferred annuity*. If the annuity is deferred J periods ($J < N$), the situation is as portrayed in Figure 3-10, in which the entire framed ordinary annuity has been moved forward from "time present," or "time 0," by J periods. It must be remembered that in an annuity deferred for J periods the first payment is made at the end of period ($J + 1$), assuming that all periods involved are equal in length.

The present worth at the end of period J of an annuity with cash flows of amount A is, from Equation 3-9, $A(P/A, i\%, N - J)$. The present worth of the single amount $A(P/A, i\%, N - J)$ as of time 0 will then be

$$A(P/A, i\%, N - J)(P/F, i\%, J)$$

EXAMPLE 3-7

To illustrate the preceding discussion, suppose that a father, on the day his son is born, wishes to determine what lump amount would have to be paid into an account bearing interest of 12% compounded annually to provide payments of \$2,000 on each of the son's 18th, 19th, 20th, and 21st birthdays.

Solution The problem is represented in Figure 3-11. One should first recognize that an ordinary annuity of four payments of \$2,000 each is involved, and that the present worth of this annuity occurs at the 17th birthday when a (P/A,

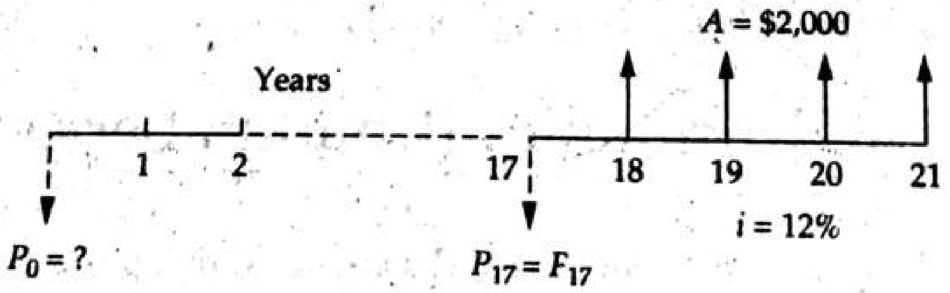


FIGURE 3-11 Cash Flow Diagram of the Deferred Annuity Problem in Example 3-7

$i\%, N - J)$ factor is utilized. In this problem, $N = 21$ and $J = 17$. It is often helpful to use a subscript with P or F to denote the point in time. Hence

$$P_{17} = A(P/A, 12\%, 4) = \$2{,}000(3.0373) = \$6{,}074.60$$

Note the dashed arrow in Figure 3-11, denoting P_{17}. Now that P_{17} is known, the next step is to calculate P_0. With respect to P_0, P_{17} is a future worth, and hence it could also be denoted F_{17}. Money at a given point in time, such as the end of period 17, is the same regardless of whether it is called a present worth or a future worth. Hence

$$P_0 = F_{17}(P/F, 12\%, 17) = \$6{,}074.60(0.1456) = \$884.46$$

which is the amount that the father would have to deposit. ■

EXAMPLE 3-8

As an addition to the problem in Example 3-7, suppose that it is desired to determine the equivalent worth of the four \$2,000 payments as of the son's 24th birthday. This could mean that four payments were never withdrawn or possibly that the son took them and immediately redeposited them in an account also earning interest at 12% compounded annually. Using our subscript system, we desire to calculate F_{24} as shown in Figure 3-12.

Solution One way to work this is to calculate

$$F_{21} = A(F/A, 12\%, 4) = \$2{,}000(4.7793) = \$9{,}558.60$$

To determine F_{24}, F_{21} can now be denoted P_{21}, and

$$F_{24} = P_{21}(F/P, 12\%, 3) = \$9{,}558.60(1.4049) = \$13{,}428.88$$

Another, quicker way to work the problem is to recognize that $P_{17} = \$6{,}074.60$ and $P_0 = \$884.46$ are each equivalent to the four \$2,000 payments. Hence one can find F_{24} directly, given P_{17} or P_0. Using P_0, we obtain

$$F_{24} = P_0(F/P, 12\%, 24) = \$884.46(15.1786) = \$13{,}424.86$$

which closely approximates the previous answer. The two numbers differ by \$4.02, which can be attributed to round-off error in the interest factors. ■

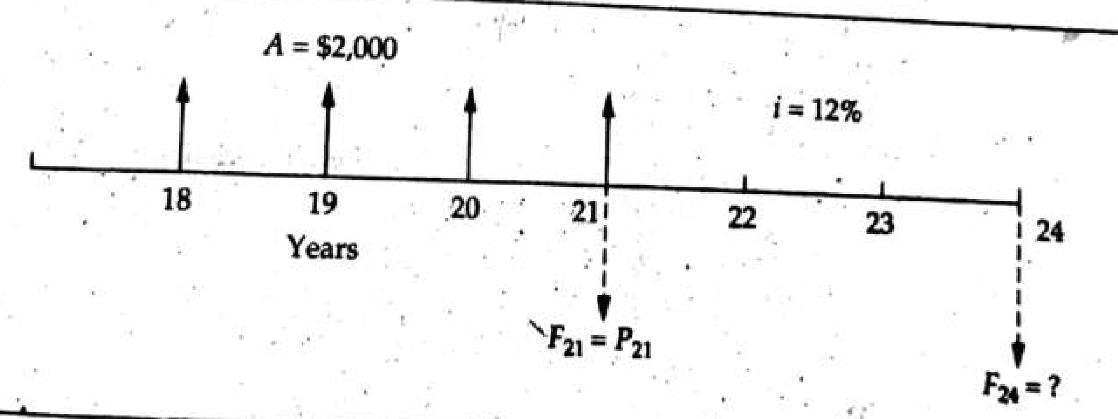


FIGURE 3-12 Cash Flow Diagram for the Deferred Annuity Problem in Example 3-8

3.12 Uniform Series with Beginning-of-Period Cash Flows

It should be noted that up to this point all the interest formulas and correspond-ing tabled values for uniform series have assumed *end-of-period* cash flows. These same tables can be used for cases in which beginning-of-period cash flows exist merely by remembering that:

1. P (present worth) occurs one interest period before the first A (uniform series amount).
2. F (future worth) occurs at the same time as the last A and N periods after P.

On a beginning-of-period cash flow diagram, all cash flows that occur during a time period are placed at the point designated as the beginning of the period.

EXAMPLE 3-9

Figure 3-13a is a cash flow diagram depicting a uniform series of five beginning-of-period cash flows of $100 each. Thus, the first cash flow is at the beginning

$\left(1 + \frac{R}{M}\right)^M - 1$

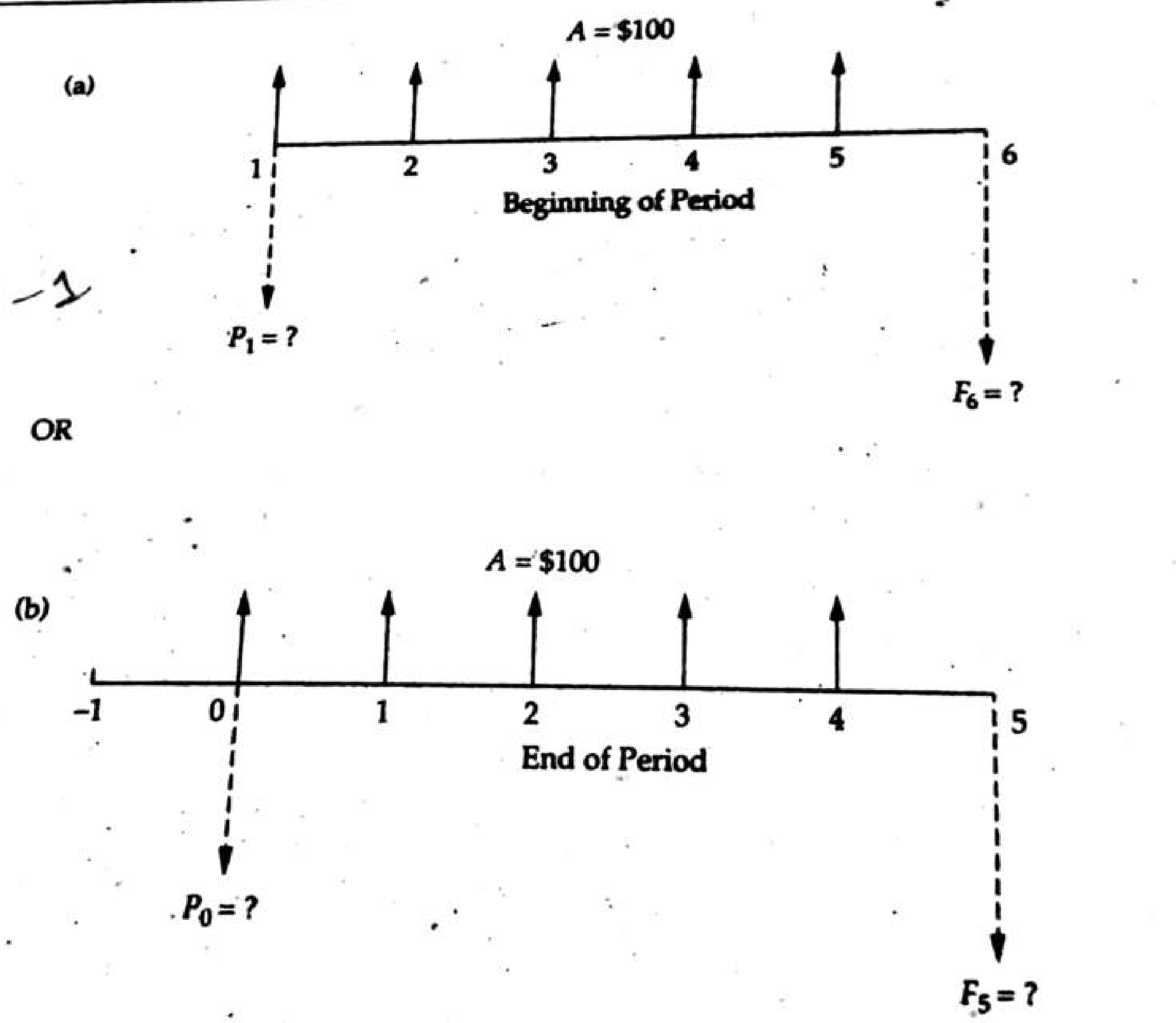


FIGURE 3-13 Cash Flow Diagram of Uniform Series at (a) Beginning of Period and Corresponding Diagram at (b) End of Period

of the first period and the fifth is at the beginning of the fifth period. (The corresponding end-of-period cash flow diagram is shown in Figure 3-13b). If the interest rate is 10%, it is desired to find (a) the present worth of the uniform series at the beginning of the first period and (b) the future worth at the *end* of the fifth period.

Solution

(a) There are several ways to work these types of problems. To find the present worth, P_0, in Figure 13-3b, one way is first to calculate

$$P_{-1} = A(P/A, 10\%, 5) = \$100(3.7908) = \$379.08$$

This P is at time -1 because that is one period before the first A cash flow. Also note that the interest factor is for five periods because there were five cash flows. Next, the problem is to find P_0 given P_{-1}. In this case P_0 becomes a future worth and could be denoted F_0. Hence

$$P_0 = F_0 = P_{-1}(F/P, 10\%, 1) = \$379.08(1.100) = \$416.99$$

Alternatively, one could directly utilize the Appendix C interest factors to determine the present worth in Figure 3-13a in this manner:

$$\begin{aligned} P_1 &= \$100 + \$100(P/A, 10\%, 4) \\ &= 100 + 100(3.1699) \\ &= \$416.99 \end{aligned}$$

(b) To find F_6 in Figure 3-13a, the first logical step is to calculate

$$F_5 = A(F/A, 10\%, 5) = \$100(6.1051) = \$610.51$$

Note that the F is at time 5 because that is at the same time as the last A cash flow. Also note that the interest factor is again for five periods, corresponding to the number of cash flows. Next, the problem is to find F_6 given F_5. In this case F_5 becomes a present worth and can be denoted P_5. Hence

$$F_6 = P_5(F/P, 10\%, 1) = \$610.51(1.10) = \$671.56$$

A convenient way to determine F_5 in Figure 3-13b would be to start with \$379.08 as of time -1, or \$416.99 as of time 0, and to calculate the future worth at time 5: $F_5 = \$416.99(F/P, 10\%, 5) = \$671.56$. ■

3.13 Equivalence Calculations Involving Multiple Interest Formulas

The reader should now be comfortable with equivalence problems that involve discrete compounding of interest and discrete cash flows. All compounding of interest takes place once per time period (e.g., a year), and to this point cash flows also occur once per time period. This section provides two examples involving two or more equivalence calculations to solve for an unknown quantity. End-of-year cash flow convention is used.

EXAMPLE 3.10

Figure 3-14 depicts an example problem with a series of year-end cash flows extending over 8 years. The amounts are $100 for the first year, $200 for the second year, $500 for the third year, and $400 for each year from the fourth through the eighth. These could represent something like the expected maintenance expenditures for a certain piece of equipment or payments into a fund. Note that the payments are shown at the end of each year, which is a standard assumption for this book and for economic analyses in general unless one

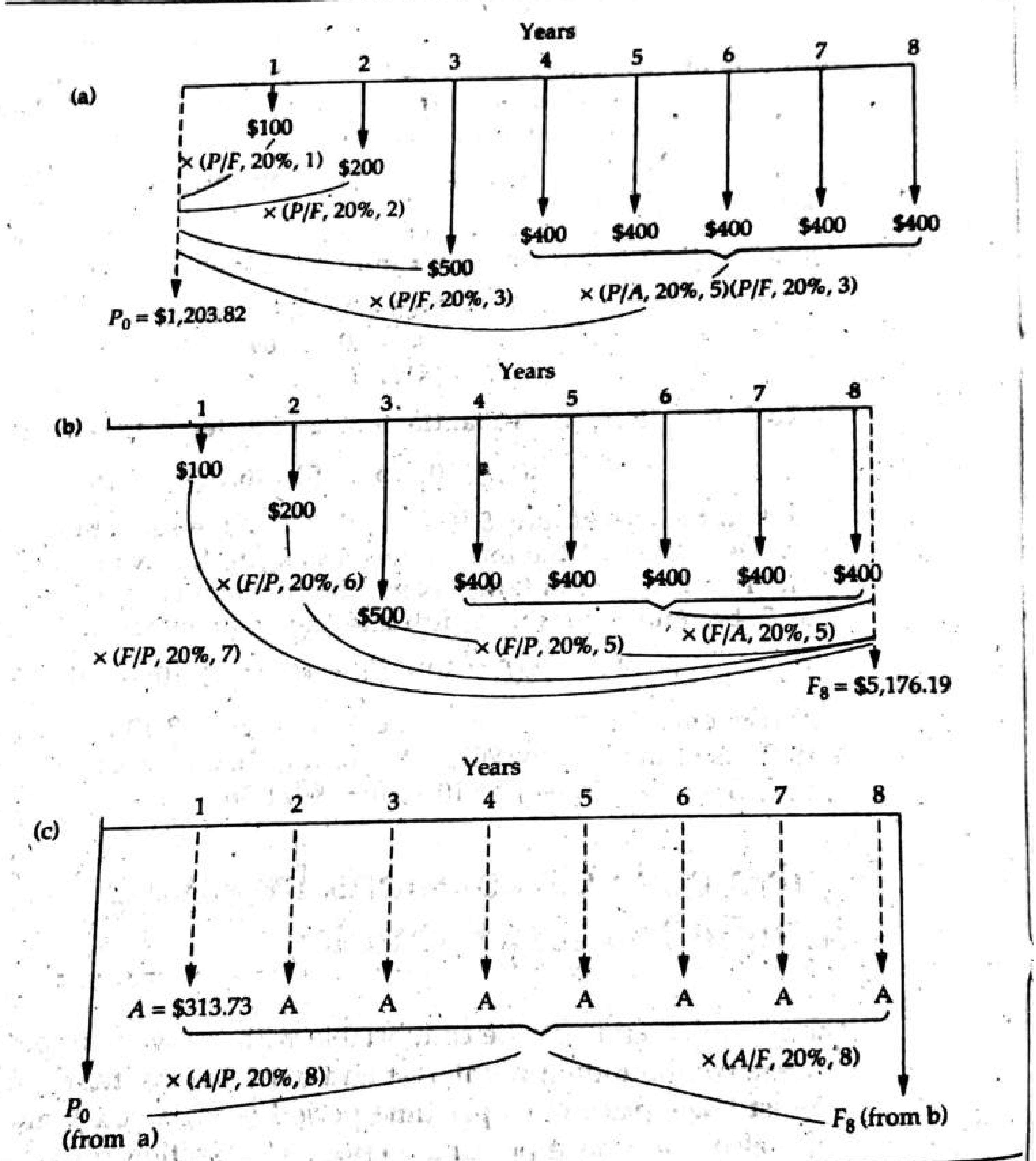


FIGURE 3-14 Example 3-10 for Calculating the Equivalent PW, FW, and AW

has information to the contrary. It is desired to find the equivalent (a) present worth (PW), P_0; (b) future worth (FW), F_8; and (c) annual worth (AW), A, of these cash flows if the annual interest rate is 20%.

Solution

(a) To find the equivalent PW, P_0, one needs to sum the worth of all payments as of the beginning of the first year (time 0). The required movements of money through time are shown graphically in Figure 3-14a.

$$
\begin{aligned}
P_0 &= F_1(P/F, 20\%, 1) & &= \$100(0.8333) & &= \$\ 83.33\\
&\quad + F_2(P/F, 20\%, 2) & &\quad + \$200(0.6944) & &\quad + 138.88\\
&\quad + F_3(P/F, 20\%, 3) & &\quad + \$500(0.5787) & &\quad + 289.35\\
&\quad + A(P/A, 20\%, 5) \times (P/F, 20\%, 3) & &\quad + \$400(2.9900) \times (0.5787) & &\quad + \underline{692.26}\\
& & & & &\quad \$1{,}203.82
\end{aligned}
$$

(b) To find the equivalent FW, F_8, one can sum the worth of all payments as of the end of the eighth year (time 8). Figure 3-14b indicates these movements of money through time. However, since the equivalent PW is already known to be \$1,203.82, one can calculate directly

$$F_8 = P_0(F/P, 20\%, 8) = \$1{,}203.82(4.2998) = \$5{,}176.19$$

(c) The equivalent AW of the irregular cash flows can be calculated directly from either P_0 or F_8 as follows:

$$A = P_0(A/P, 20\%, 8) = \$1{,}203.82(0.2606) = \$313.73$$

or

$$A = F_8(A/F, 20\%, 8) = \$5{,}176.19(0.0606) = \$313.73$$

The computation of A from P_0 and F_8 is shown in Figure 3-14c. Thus, one finds that the irregular series of payments shown in Figure 3-14 is equivalent to \$1,203.82 at time 0, \$5,176.19 at time 8, or a uniform series of \$313.73 at the end of each of the 8 years. ■

EXAMPLE 3-11

Transform the cash flows on the left-hand side of Figure 3-15 to their equivalent cash flows on the right-hand side. That is, take the left-hand quantities as givens and determine the unknown value of Q in terms of H in Figure 3-15. The interest rate is 10% per year.

Solution If all cash flows on the left are discounted to year 0, we have $P_0 = 2H(P/A, 10\%, 4) + H(P/A, 10\%, 3)(P/F, 10\%, 5) = 7.8839H$. When cash flows on the right are also discounted to year 0, we can solve for Q in terms of H. [Notice that Q at the end of year (EOY) 2 is positive, Q at EOY 7 is negative, and the two Q values must be equal in amount.]

$$7.8839H = Q(P/F, 10\%, 2) - Q(P/F, 10\%, 7)$$

or

$$Q = 25.172H$$

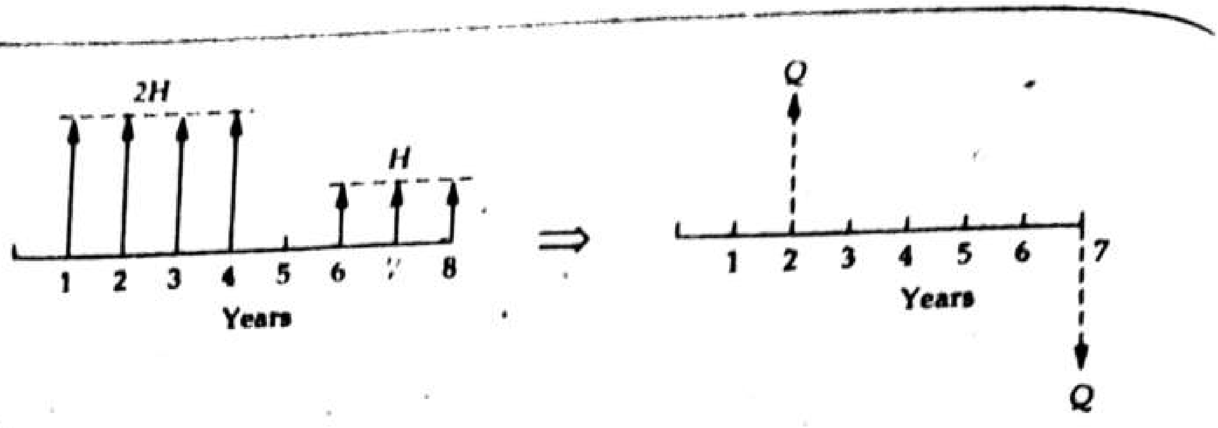


FIGURE 3-15 Cash Flow Diagrams for Example 3-11

3.14 Interest Formulas Relating a Uniform Gradient of Cash Flows to Its Annual and Present Worths

Some economic analysis problems involve receipts or expenses that are projected to ~~increase or decrease~~ by a uniform *amount* each period, thus constituting an arithmetic sequence of cash flows. For example, maintenance and repair expenses on specific equipment may increase by a relatively constant amount each period. This situation can be modeled with a uniform gradient, as demonstrated here.

Figure 3-16 is a cash flow diagram of a sequence of end-of-period cash flows increasing by a constant amount, G, in each period. The G is known as the

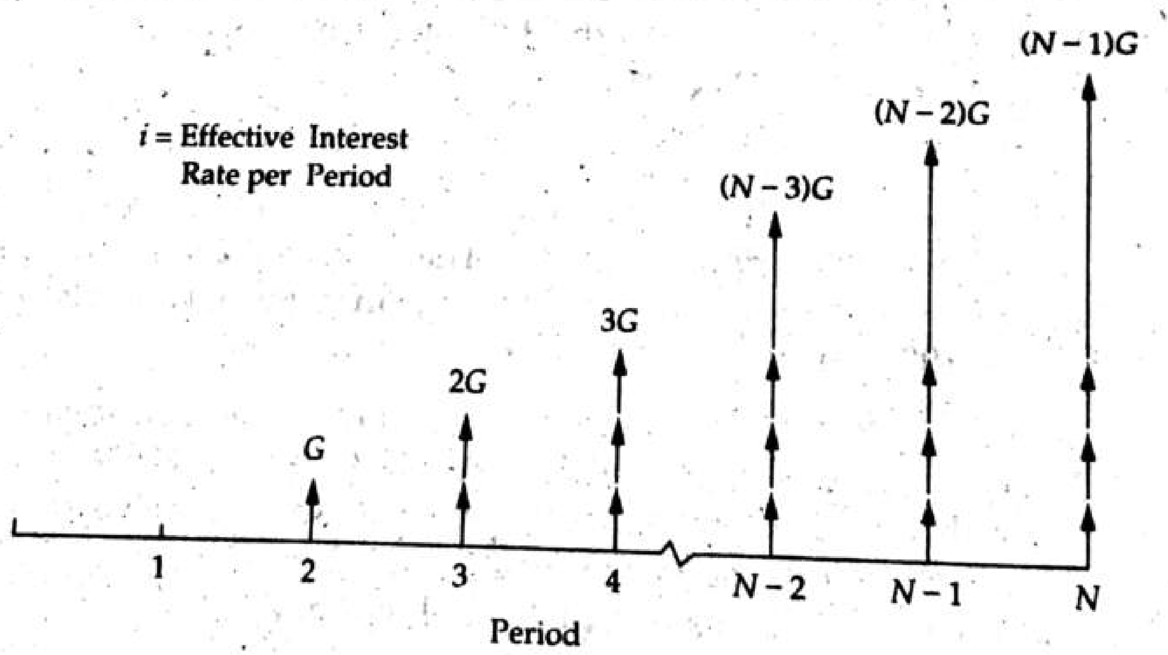


FIGURE 3-16 Cash Flow Diagram for a Uniform Gradient Increasing by G Dollars per Period

uniform gradient amount. Note that the timing of cash flows on which the derived formulas and tabled values are based is as follows:

End of Period	*Payments*
1	0
2	G
3	2G
.	.
.	.
.	.
$N-1$	$(N-2)G$
N	$(N-1)G$

500 (1.07)

3.14.1 Finding F When Given G

The future worth, F, of the arithmetic sequence of cash flows shown in Figure 3-16 is

$$F = G(F/A,\, i\%, N-1) + G(F/A,\, i\%, N-2) + \cdots + G(F/A,\, i\%, 2) + G(F/A,\, i\%, 1)$$

or

$$\begin{aligned} F &= G\left[\frac{(1+i)^{N-1}-1}{i} + \frac{(1+i)^{N-2}-1}{i} + \cdots + \frac{(1+i)^2-1}{i} + \frac{(1+i)^1-1}{i}\right] \\ &= \frac{G}{i}[(1+i)^{N-1} + (1+i)^{N-2} + \cdots + (1+i)^2 + (1+i)^1 + 1] - \frac{NG}{i} \\ &= \frac{G}{i}\left[\sum_{k=0}^{N-1}(1+i)^k\right] - \frac{NG}{i} \\ &= \frac{G}{i}(F/A,\, i\%, N) - \frac{NG}{i} \end{aligned} \tag{3-20}$$

3.14.2 Finding A When Given G

From Equation 3-20, it is easy to develop an expression for A as follows:

$$\begin{aligned} A &= F(A/F, i, N) \\ &= \left[\frac{G}{i}(F/A, i, N) - \frac{NG}{i}\right](A/F, i, N) \\ &= \frac{G}{i} - \frac{NG}{i}(A/F, i, N) \end{aligned}$$

$$= \frac{G}{i} - \frac{NG}{i}\left[\frac{i}{(1+i)^N - 1}\right]$$

$$= G\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right] \quad (3\text{-}21)$$

The term in braces in Equation 3-21 is called the *gradient to uniform series conversion factor.* Numerical values for this factor are given in Table C-22 of Appendix C for a wide range of i and N values. We shall use the functional symbol $(A/G, i\%, N)$ for this factor. Thus

$$A = G(A/G, i\%, N) \quad (3\text{-}22)$$

3.14.3 Finding P When Given G

We may now utilize Equation 3-21 to establish the equivalence between P and G:

$$P = A(P/A, i\%, N)$$

$$= G\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right]\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$$

$$= G\left[\frac{(1+i)^N - 1 - Ni}{i^2(1+i)^N}\right]$$

$$= G\left\{\frac{1}{i}\left[\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right]\right\} \quad (3\text{-}23)$$

The term in braces in Equation 3-23 is called the *gradient to present worth conversion factor.* It can also be expressed as $(1/i)[(P/A, i\%, N) - N(P/F, i\%, N)]$. Numerical values for this factor are given in Table C-21 of Appendix C for a wide assortment of i and N values. We shall use the functional symbol $(P/G, i\%, N)$ for this factor. Hence

$$P = G(P/G, i\%, N) \quad (3\text{-}24)$$

3.14.4 Computations Using G

Be sure to notice that the direct use of gradient conversion factors applies when there is no cash flow at the end of period 1, as in Example 3-12. There may be an A amount at the end of period 1, but it is treated separately, as illustrated in Examples 3-13 and 3-14. A major advantage of using gradient conversion factors (i.e., computational time savings) is realized when N becomes large.

EXAMPLE 3-12

As an example of the straightforward use of the gradient conversion factors, suppose that certain end-of-year cash flows are expected to be \$1,000 for the *second* year, \$2,000 for the third year, and \$3,000 for the fourth year, and that if interest is 15% per year, it is desired to find the equivalent (a) present worth at the beginning of the first year, and (b) uniform annual worth at the end of each of the 4 years.

Solution It can be observed that this schedule of cash flows fits the model of the arithmetic gradient formulas with $G = \$1{,}000$ and $N = 4$ (see Figure 3-16). Note that there is no cash flow at the end of the first period.

(a) The present worth can be calculated as

$$P_0 = G(P/G, 15\%, 4) = \$1{,}000(3.79) = \$3{,}790$$

(b) The uniform AW can be calculated from Equation 3-22 as

$$A = G(A/G, 15\%, 4) = \$1{,}000(1.3263) = \$1{,}326.30$$

Of course, once the present worth is known, the uniform AW can be calculated as

$$A = P_0(A/P, 15\%, 4) = \$3{,}790(0.3503) = \$1{,}326.30$$

■

EXAMPLE 3-13

As a further example of the use of arithmetic gradient formulas, suppose that one has payments as follows:

End of Year	*Payment*
1	\$5,000
2	6,000
3	7,000
4	8,000

and that one wishes to calculate their equivalent present worth at $i = 15\%$ using arithmetic gradient interest formulas.

Solution The schedule of payments is depicted in the top diagram of Figure 3-17. The bottom two diagrams of Figure 3-17 show how the original schedule can be broken into two separate sets of payments, a uniform series of \$5,000 payments plus an arithmetic gradient of \$1,000 that fits the general gradient model for which factors are tabled. The summed PWs of these two separate sets of payments equal the PW of the original problem. Thus, using the symbols shown in Figure 3-17, we have

$$\begin{aligned} P_{0T} &= P_{0A} + P_{0G} \\ &= A(P/A, 15\%, 4) + G(P/G, 15\%, 4) \\ &= \$5{,}000(2.8550) + \$1{,}000(3.79) = \$14{,}275 + 3{,}790 = \$18{,}065 \end{aligned}$$

The uniform AW of the original payments could be calculated with the aid of Equation 3-22 as follows:

$$\begin{aligned} A_T &= A + A_G \\ &= \$5{,}000 + \$1{,}000(A/G, 15\%, 4) = \$6{,}326.30 \end{aligned}$$

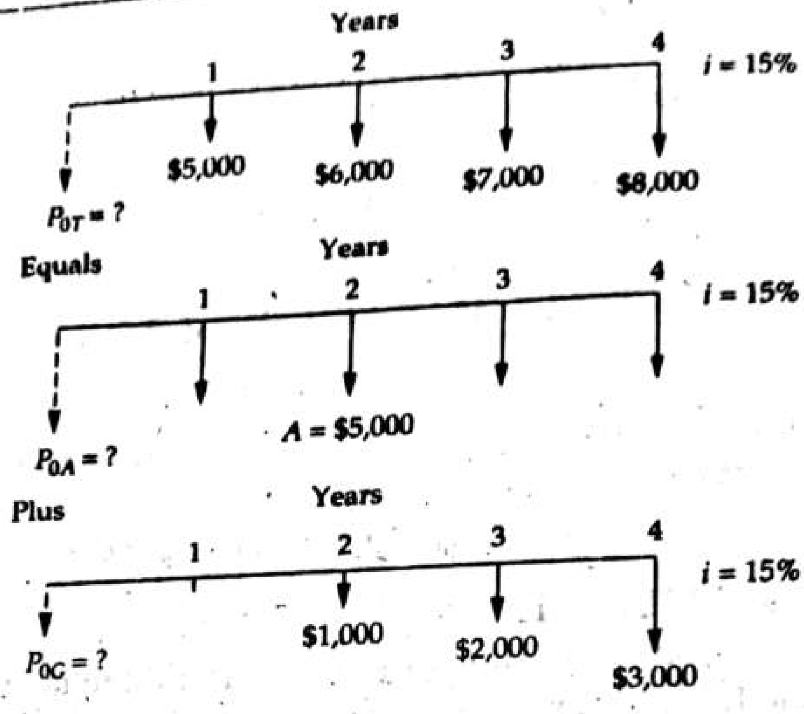


FIGURE 3-17 Example 3-13 Involving an Increasing Arithmetic Gradient

A_T is equivalent to P_{0T} because $\$6,326.30(P/A, 15\%, 4) = \$18,061$, which is the same value obtained previously (subject to round-off error).

EXAMPLE 3-14

For another example of the use of arithmetic gradient formulas, suppose that one has payments that are timed in exact reverse of the payments depicted in Example 3-13. The top diagram of Figure 3-18 shows these payments as follows:

End of Year	*Payment*
1	$8,000
2	7,000
3	6,000
4	5,000

Calculate the equivalent present worth at $i = 15\%$ using arithmetic gradient interest factors.

Solution The bottom two diagrams of Figure 3-18 show how these payments can be broken into two separate sets of payments. It must be remembered that the arithmetic gradient formulas and tables provided are for increasing gradients only. Hence one must subtract an *increasing* gradient of payments that did not occur. Thus

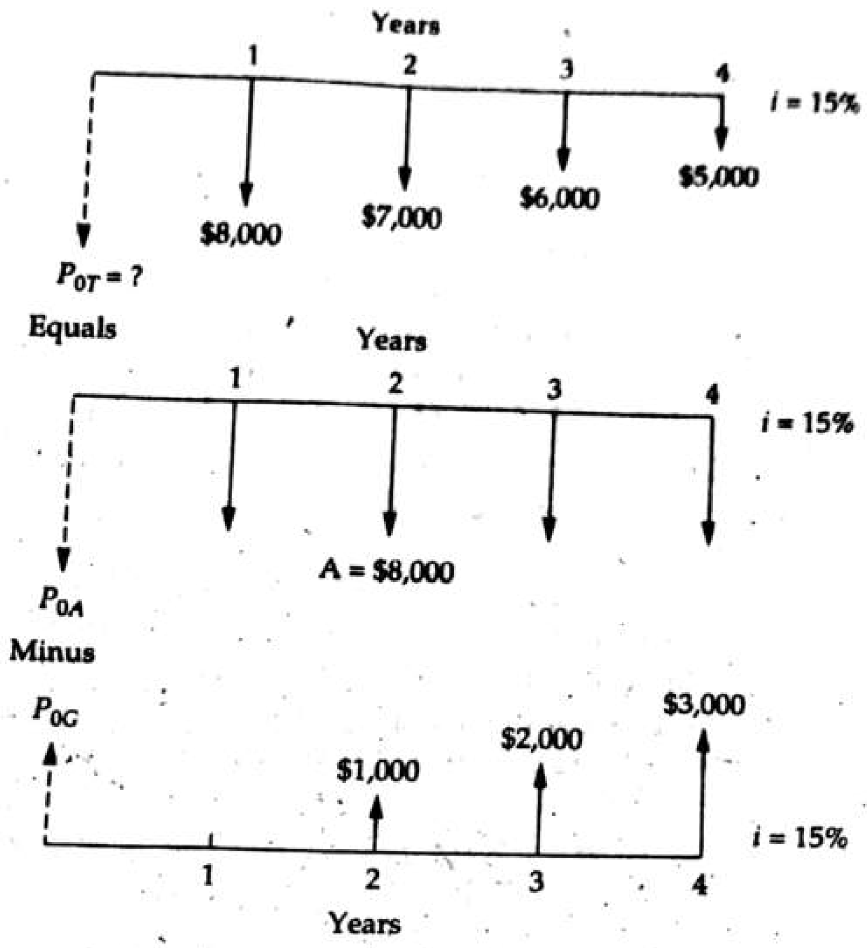


FIGURE 3-18 Example 3-14 Involving a Decreasing Arithmetic Gradient

$$
\begin{aligned}
P_{0T} &= P_{0A} - P_{0G} \\
&= A(P/A, 15\%, 4) - G(P/G, 15\%, 4) \\
&= \$8{,}000(2.8550) - \$1{,}000(3.79) \\
&= \$22{,}840 - \$3{,}790 = \$19{,}050
\end{aligned}
$$

Again, the uniform annual worth of the original decreasing series of payments can be calculated by the same rationale:

$$
\begin{aligned}
A &= A - A_G \\
&= \$8{,}000 - \$1{,}000(A/G, 15\%, 4) \\
&= \$6{,}673.70
\end{aligned}
$$

■

Note from Examples 3-13 and 3-14 that the PW of $18,065 for an increasing arithmetic gradient series of payments is different from the PW of $19,050 for an arithmetic gradient of payments of like amounts but reversed timing. This difference would be even greater for higher interest rates and gradient payments and exemplifies the marked effect of timing of cash flows on equivalent worths.

3.15 Interest Formulas Relating a Geometric Sequence of Cash Flows to Its Present and Annual Worths

Some economic equivalence problems involve projected cash flow patterns that are increasing at an average *rate*, $\bar{f}$, each period. A fixed amount of a commodity that inflates in price at a constant rate each year is a typical situation that can be modeled with a geometric sequence of cash flows. The resultant end-of-period cash flow pattern is referred to as a *geometric gradient series* and has the general appearance shown in Figure 3-19. Notice that *the initial cash flow in this series*, A_1, *occurs at the end of period 1* and that $A_k = (A_{k-1})(1+\bar{f}), 2 \le k \le N$. The Nth term in this geometric sequence is $A_N = A_1(1+\bar{f})^{N-1}$, and the common ratio throughout the sequence is $(A_k - A_{k-1})/A_{k-1} = \bar{f}$. Be sure to notice that $\bar{f}$ can be positive *or* negative.

Each term in Figure 3-19 could be discounted, or compounded, at interest rate i per period to obtain a value of P or F, respectively. However, this becomes quite tedious for large N, so it is convenient to have a single equation instead.

To develop a compact expression for P at interest rate i per period for the cash flows of Figure 3-19, consider the following summation:

$$P = \sum_{k=1}^{N} A_k(1+i)^{-k} = \sum_{k=1}^{N} A_1(1+\bar{f})^{k-1}(1+i)^{-k}$$

or

$$P = \frac{A_1}{1+\bar{f}} \sum_{k=1}^{N} \left(\frac{1+\bar{f}}{1+i}\right)^k \tag{3-25}$$

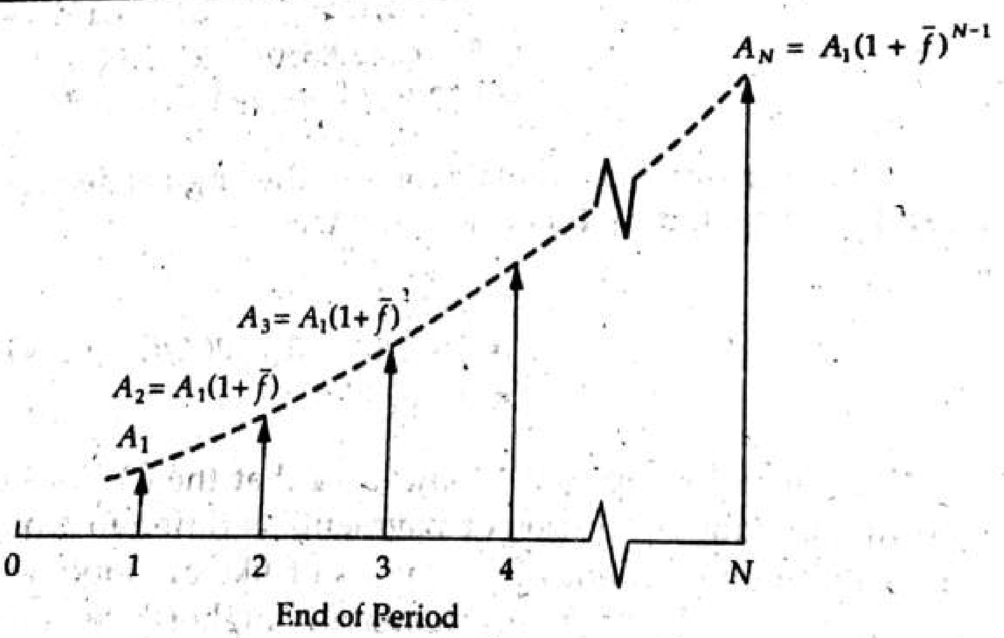


FIGURE 3-19 Cash Flow Diagram for a Geometric Sequence of Cash Flows Increasing at a Constant Rate of $\bar{f}$ per Period

When $i \neq \bar{f}$, we can simplify Equation 3-25 by defining a "convenience rate," i_{CR}, as follows:

$$i_{CR} = \frac{1+i}{1+\bar{f}} - 1 \tag{3-26}$$

The convenience rate can also be written as $i_{CR} = (i - \bar{f})/(1+\bar{f})$. In the situation where $i \neq \bar{f}$, Equation 3-25 can thus be rewritten as

$$\begin{aligned} P &= \frac{A_1}{1+\bar{f}} \sum_{k=1}^{N} \left(\frac{1+i}{1+\bar{f}}\right)^{-k} \\ &= \frac{A_1}{1+\bar{f}} \sum_{k=1}^{N} (1+i_{CR})^{-k} \\ &= \frac{A_1}{1+\bar{f}} (P/A, i_{CR}\%, N)^{\dagger} \end{aligned} \tag{3-27}$$

Equation 3-27 makes use of the fact that

$$(P/A, i_{CR}\%, N) = \sum_{k=1}^{N} (1+i_{CR})^{-k} = \sum_{k=1}^{N} (P/F, i_{CR}\%, k)$$

When $i = \bar{f}$ and $i_{CR} = 0$, Equation 3-27 reduces to

$$P = \frac{A_1}{1+\bar{f}} (P/A, 0\%, N) = \frac{NA_1}{1+\bar{f}} \tag{3-28}$$

The interested reader can verify Equation 3-28 by applying L'Hôpital's Rule to the $(P/A, i_{CR}\%, N)$ factor in Equation 3-27 and taking the limit as $i_{CR} \rightarrow 0$.

Values of i_{CR} used in connection with Equation 3-20 are typically not included in the tables in Appendix C. Because i_{CR} is usually a noninteger interest rate, resorting to the definition of a $(P/A, i_{CR}\%, N)$ factor (see Table 3-4) and substituting terms into it is a satisfactory way to obtain values of these interest factors.

The end-of-period uniform annual equivalent, A, of a geometric gradient series can be determined from Equation 3-27 (or Equation 3-28) as follows:

$$A = P(A/P, i\%, N) \tag{3-29}$$

The year 0 "base" of this annuity, which increases at a constant rate of $\bar{f}$% per period, is A_0 and equals

$$A_0 = P(A/P, i_{CR}\%, N) \tag{3-30}$$

The difference between A and A_0 can be seen in Figure 3-20. Finally, the future equivalent of this geometric gradient series is simply

$$F = P(F/P, i\%, N) \tag{3-31}$$

† When $\bar{f}$ exceeds i, i_{CR} is negative and the above summation is valid only when N is finite.

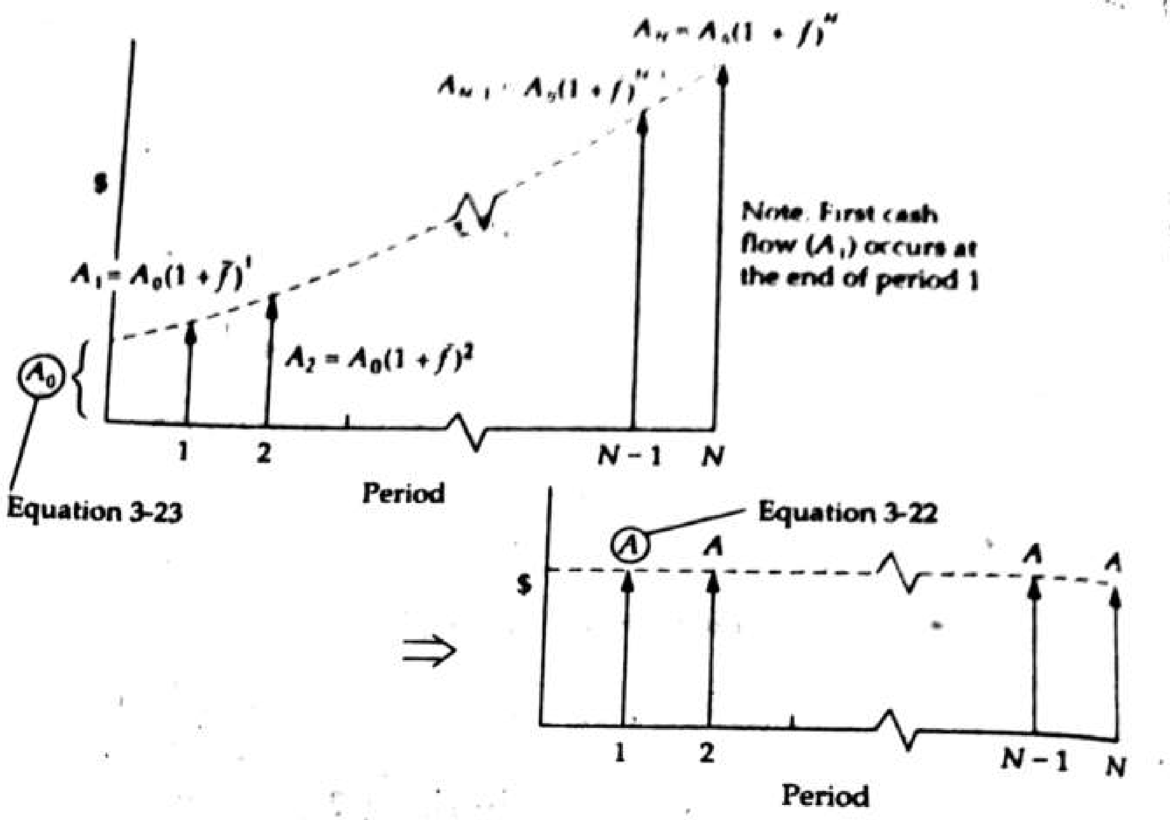


FIGURE 3-20 Graphical Interpretation of A and A_0 Terms in a Geometric Gradient Series when $\bar{f} > 0$

Additional discussion of geometric sequences of cash flows is provided in Chapter 9 (Section 9.4), which deals with inflation and price changes.

EXAMPLE 3-15

Consider the end-of-year geometric sequence of cash flows in Figure 3-21 and determine the P, A, A_0, and F equivalent values. The rate of increase is 20% per year after the first year, and the annual interest rate is 25%.

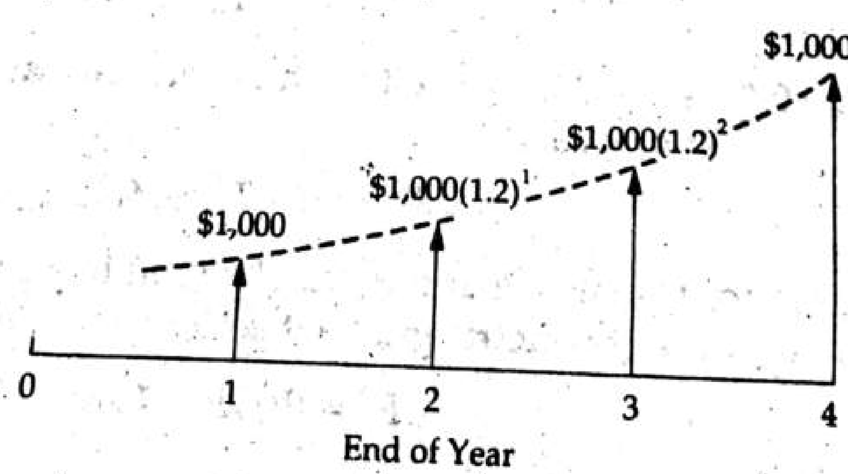


FIGURE 3-21 Cash Flow Diagram for Example 3-15

Solution

$$P = \frac{\$1{,}000}{1.2}\left(P/A, \frac{25\% - 20\%}{1.20}, 4\right) = \$833.33(P/A, 4.167\%, 4)$$
$$= \$833.33\left[\frac{(1.04167)^4 - 1}{0.04167(1.04167)^4}\right]$$
$$= \$833.33(3.6157) = \$3{,}013.08$$
$$A = \$3{,}013.08(A/P, 25\%, 4) = \$1{,}275.86$$
$$A_0 = \$3{,}013.08(A/P, 4.167\%, 4)$$
$$= \$3{,}103.08\left[\frac{0.04167(1.04167)^4}{(1.04167)^4 - 1}\right] = \$833.34$$
$$F = \$3{,}013.08(F/P, 25\%, 4) = \$7{,}356.15$$

■

EXAMPLE 3-16

Suppose that the geometric gradient in Example 3-15 begins with \$1,000 at the end of year 1 and *decreases* by 20% per year after the first year. Determine P A, A_0, and F under this condition.

Solution The value of $\bar{f}$ is -20% in this case and $i_{CR} = \frac{1+i}{1-\bar{f}} - 1 = 1.25/0.80 - 1 = 0.5625$, or 56.25% per year. The desired quantities are as follows:

$$P = \frac{\$1{,}000}{0.8}(P/A, 56.25\%, 4) = \$1{,}250(1.4795)$$
$$= \$1{,}849.38$$
$$A = \$1{,}849.38(A/P, 25\%, 4) = \$783.03$$
$$A_0 = \$1{,}849.38(A/P, 56.25\%, 4) = \$1{,}250.00.$$
$$F = \$1{,}849.38(F/P, 25\%, 4) = \$4{,}515.08$$

■

3.16 Nominal and Effective Interest Rates

Very often the interest period, or time between successive compounding, is less than 1 year. It has become customary to quote interest rates on an annual basis, followed by the compounding period if different from 1 year in length. For example, if the interest rate is 6% per interest period and the interest period is 6 months, it is customary to speak of this rate as "12% compounded semiannually." The basic annual rate of interest is known as the *nominal rate*, 12% in this case. A nominal interest rate is represented by *r*. The actual annual rate on the principal is not 12% but something greater, because of the compounding that occurs twice during the year.

Thus, the frequency at which a nominal interest rate is compounded each year can have a pronounced effect on the dollar amount of interest earned. For instance, consider \$1,000 to be invested for 3 years at a nominal rate of 12% compounded semiannually. The interest earned during the first year would be as follows:

First 6 months:

$$Pi = \$1{,}000 \times 0.06 = \$60$$

Total principal and interest at beginning of the second period:

$$P + Pi = \$1{,}000 + \$60 = \$1{,}060$$

Interest earned during second 6 months:

$$\$1{,}060 \times 0.06 = \$63.60$$

Total interest earned during the year:

$$\$60.00 + \$63.60 = \$123.60$$

Effective annual interest rate:

$$\frac{\$123.60}{\$1{,}000} \times 100 = \boxed{12.36\%}$$

If this process is repeated for years 2 and 3, the *accumulated* (compounded) *amount of interest* can be plotted as in Figure 3-22. Suppose that the same $1,000 had been invested at 12% compounded *monthly*. The accumulated interest over 3 years that results from semiannual and monthly compounding is shown in Figure 3-23.

The actual or exact rate of interest earned on the principal during 1 year is known as the *effective rate*. It should be noted that effective interest rates are always expressed on an annual basis unless specifically stated otherwise. In this text the effective interest rate per year is customarily designated by i and the nominal interest rate per year by r. In engineering economy studies in which

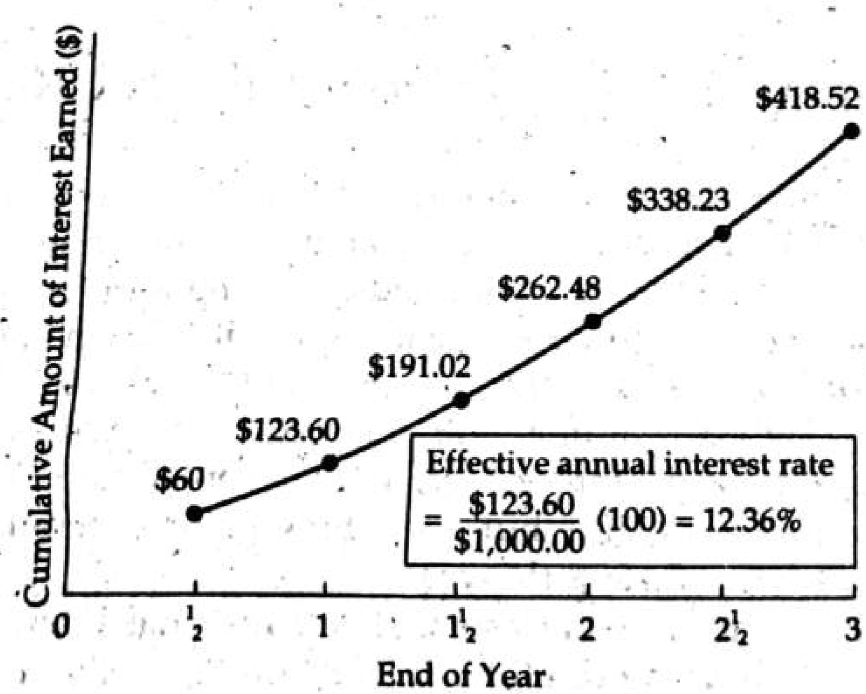


FIGURE 3-22 $1,000 Compounded at a Semiannual Frequency (r = 12%)

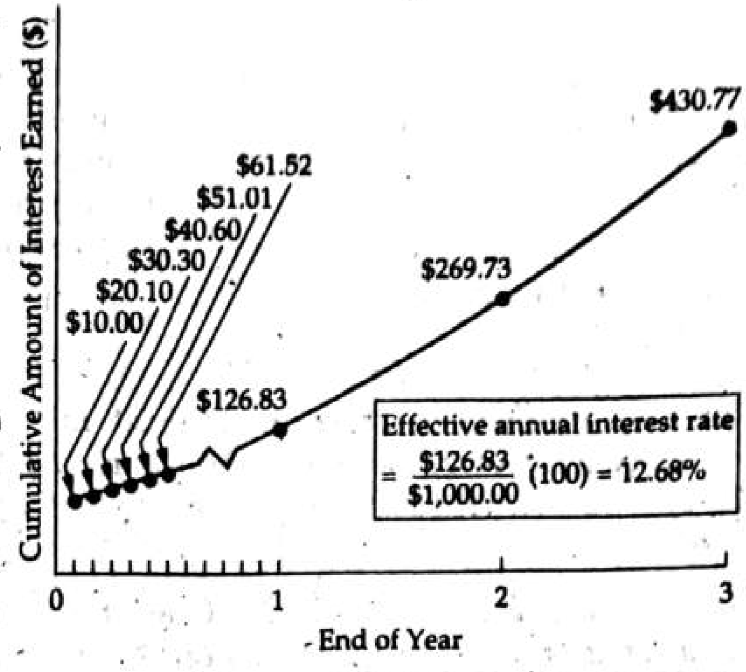


FIGURE 3-23 $1,000 Compounded at a Monthly Frequency ($r = 12\%$)

compounding is annual, $i = r$. The relationship between effective interest, i, and nominal interest, r, is

$$
\begin{aligned}
i &= (1 + r/M)^M - 1 \\
&= (F/P, r/M, M) - 1
\end{aligned}
\tag{3-32}
$$

where M is the number of compounding periods per year. It is now clear from Equation 3-32 why $i > r$ when $M > 1$.

The effective rate of interest is useful for describing the compounding effect of interest earned on interest in 1 year. Table 3-5 shows effective rates for various nominal rates and compounding periods.

E 3-5 Effective Interest Rates for Various Nominal Rates and pounding Frequencies

ounding ency	Number of Compounding Periods per Year, M	Effective Rate (%) for Nominal Rate of 6%	8%	10%	12%	15%	24%
ually	1	6.00	8.00	10.00	12.00	15.00	24.00
iannually	2	6.09	8.16	10.25	12.36	15.56	25.44
rterly	4	6.14	8.24	10.38	12.55	15.87	26.25
onthly	6	6.15	8.27	10.43	12.62	15.97	26.53
thly	12	6.17	8.30	10.47	12.68	16.08	26.82
y	365	6.18	8.33	10.52	12.75	16.18	27.11

Interestingly, the federal truth in lending law now requires a statement regarding the annual percentage rate (APR) being charged in contracts involving borrowed money. The APR is a nominal interest rate and *does not* account for compounding that may occur, or be appropriate, during a year. Before this legislation was passed by Congress in 1969, creditors had no obligation to explain how interest charges were determined or what the true cost of money on a loan was. As a result, borrowers were generally unable to compute their APR and compare different financing plans.

EXAMPLE 3-17

A credit card company charges an interest rate of 1.375% per month on the unpaid balance of all accounts. The annual interest rate, they claim, is 12(1.375%) = 16.5%. What is the effective rate of interest per year being charged by the company?

Solution Interest tables in Appendix C are based on time periods that may be annual, quarterly, monthly, and so on. Because we have no 1.375% tables (or 16.5% tables), Equation 3-32 must be used to compute the effective rate of interest in this example:

$$i = \left(1 + \frac{0.165}{12}\right)^{12} - 1$$
$$= 0.1781, \text{ or } 17.81\%/\text{year}$$

Note that $r = 12(1.375\%) = 16.5\%$, which is the APR. In general, it is true that $r = M(r/M)$, as seen in Example 3-17, where r/M is the interest rate per period.

3.17 Interest Problems with Compounding More Often Than Once per Year

3.17.1 Single Amounts

If a nominal interest rate is quoted and the number of compounding periods per year and number of years are known, any problem involving future, annual, or present equivalent values can be calculated by straightforward use of Equations 3-3 and 3-32, respectively.

EXAMPLE 3-18

Suppose that a \$100 lump-sum amount is invested for 10 years at 6% compounded quarterly. How much is it worth at the end of the tenth year?

Solution There are four compounding periods per year, or a total of 4 × 10 = 40 periods. The interest rate per interest period is 6%/4 = 1.5%. When the values are used in Equation 3-3, one finds that

$$F = P(F/P, 1.5\%, 40) = \$100.00(1.814) = \$181.40$$

Alternatively, the effective interest rate from Equation 3-32 is 6.14%. Therefore, $F = \$100.00(1.0614)^{10} = \$181.40$ ■

17.2 Uniform Series and Gradient Series

When there is more than one compounded interest period per year, the formulas and tables for uniform series and gradient series can be used *as long as* there is a cash flow at the end of each interest period, as shown in Figures 3-8 and 3-16 for uniform series and gradient series, respectively.

EXAMPLE 3-19

Suppose that one has a beginning indebtedness of \$10,000, which is to be repaid in equal end-of-month installments for 5 years with nominal interest of 12% compounded monthly. What is the amount of each payment?

Solution The number of installment payments is $5 \times 12 = 60$, and the interest rate per month is $12\%/12 = 1\%$. When these values are used in Equation 3-13, one finds that

$$A = P(A/P, 1\%, 60) = \$10{,}000(0.0222) = \$222$$

Notice that there is a cash flow at the end of each month (interest period) in this example. ■

EXAMPLE 3-20

Certain operating expenditures are expected to be 0 at the end of the first 6 months, to be \$1,000 at the end of the second 6 months, and to increase by \$1,000 at the end of each 6-month period thereafter for a total of 4 years. It is desired to find the equivalent uniform payment at the end of each of the eight 6-month periods if nominal interest is 20% compounded semiannually.

Solution A cash flow diagram is given in Figure 3–24, and the solution is

$$A = G(A/G, 10\%, 8) = \$1{,}000(3.0045) = \$3{,}004.50$$

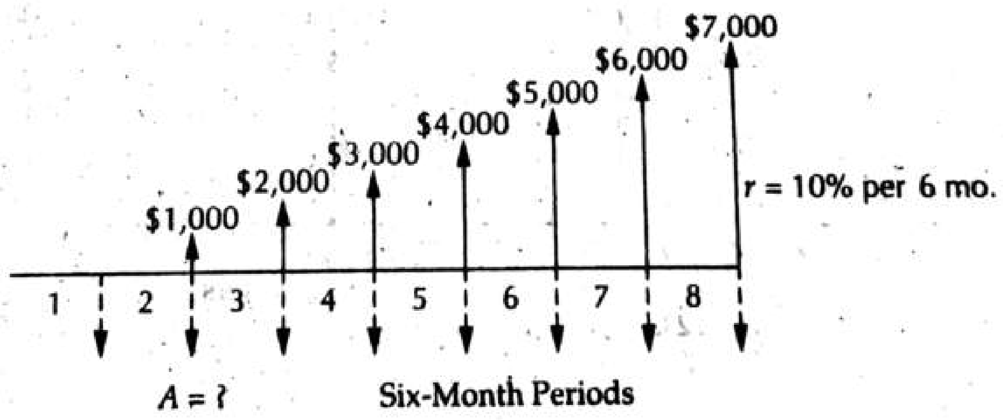


FIGURE 3-24 Arithmetic Gradient with Compounding More Often Than Once per Year in Example 3-20

Here nominal interest every 6 months is 10%, and cash flows occur every 6 months.

3.18 Interest Problems with Cash Flows Less Often Than Compounding Periods

In general, if i is the effective interest rate per interest period and there is a uniform cash flow, X, at the *end* of each Kth interest period ($K > 1$), then the equivalent payment, A, at the end of each interest period is

$$A = X(A/F, i\%, K) \tag{3-33}$$

By similar reasoning, if i is the effective interest rate per interest period and there is a uniform payment, X, at the *beginning* of each Kth interest period, then the equivalent payment, A, at the end of each interest period is

$$A = X(A/P, i\%, K) \tag{3-34}$$

EXAMPLE 3-21

Suppose that there exists a series of 10 end-of-year payments of $1,000 each and it is desired to compute the equivalent worth of those payments as of the end of the tenth year if interest is 12% compounded quarterly. The problem is depicted in Figure 3-25.

Solution Interest is 12%/4 = 3% per quarter, but the uniform series cash flows do not occur at the end of each quarter. In such cases, one can make special adaptations to fit the interest formulas to the tables provided. To solve this type of problem, an equivalent cash flow is computed for the time interval that corresponds to the stated compounding frequency, *or* an effective interest rate must be determined for the interval of time separating cash flows.

One useful adaptation procedure is to take the number of compounding periods over which a cash flow occurs and convert the cash flow into its equivalent

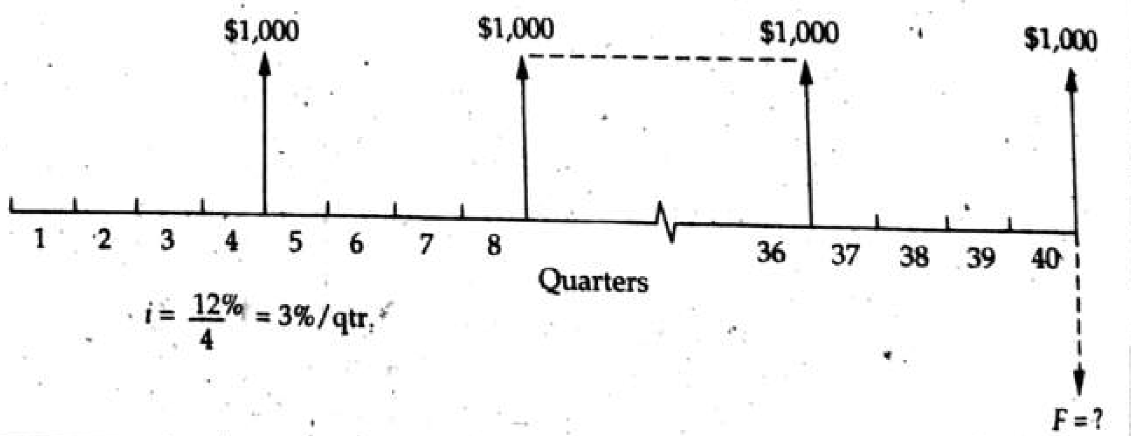


FIGURE 3-25 Uniform Series with Cash Flows Less Often Than Compounding Periods in Example 3-21

uniform end-of-period series. The upper cash flow diagram in Figure 3-26 shows this approach applied to the first year (four interest periods) in the example of Figure 3-25. The uniform end-of-quarter payment, equivalent to \$1,000 at the end of the year with interest at 3% per quarter, can be calculated by using Equation 3-33:

$$A = F(A/F, 3\%, 4) = \$1{,}000(0.2390) = \$239$$

Thus, \$239 at the end of each quarter is equivalent to \$1,000 at the end of each year. This is true not only for the first year but also for each of the 10 years under consideration. Hence the original series of 10 end-of-year payments of \$1,000 each can be converted to a problem involving 40 end-of-quarter payments of \$239 each, as shown in the lower cash flow diagram of Figure 3-26.

The equivalent worth at the end of the 10th year (40th quarter) may then be computed as

$$F = A(F/A, 3\%, 40) = \$239(75.4012) = \$18{,}021$$

A second procedure for handling cash flows occurring less often than compounding periods is to find the exact interest rate for each time period *separating* cash flows and then to straightforwardly apply the interest formulas and tables for the exact interest rate. For Example 3-21, interest is 3% per quarter and payments occur each year. Hence, the interest rate to be found is the exact rate each year, or the *effective rate* per year. The effective rate per year that corresponds to 3% per quarter (12% nominal) can be found from Equation 3-32.

$$\left(1 + \frac{0.12}{4}\right)^4 - 1 = (F/P, 3\%, 4) - 1 = 0.1255$$

Hence, the original problem in Figure 3-25 can now be expressed as shown in Figure 3-27. The future worth of this series can then be found as

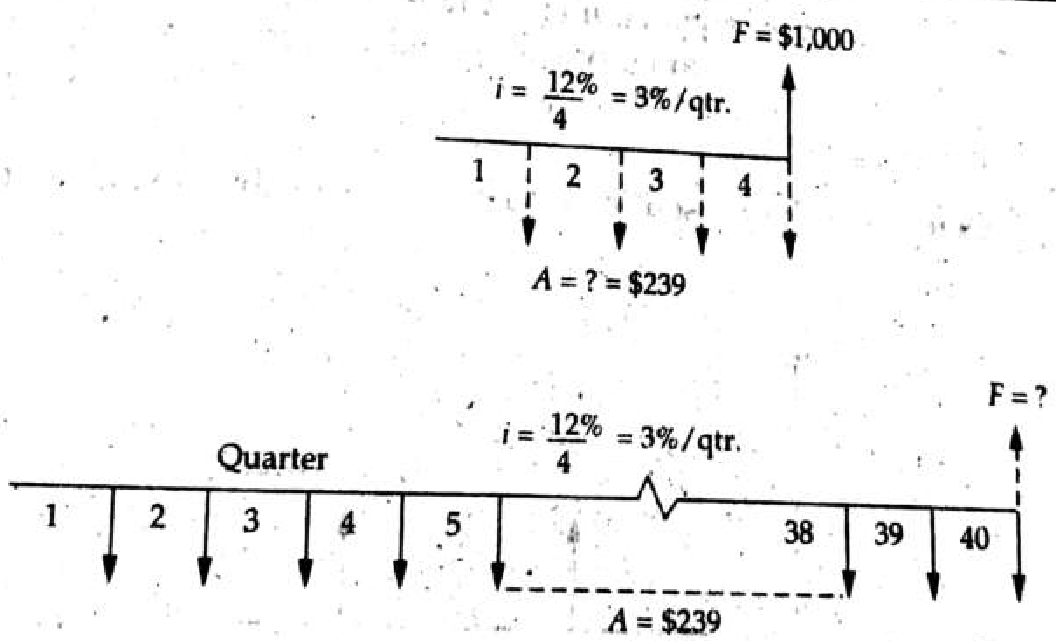


FIGURE 3-26 First Adaptation to Solve Example 3-21

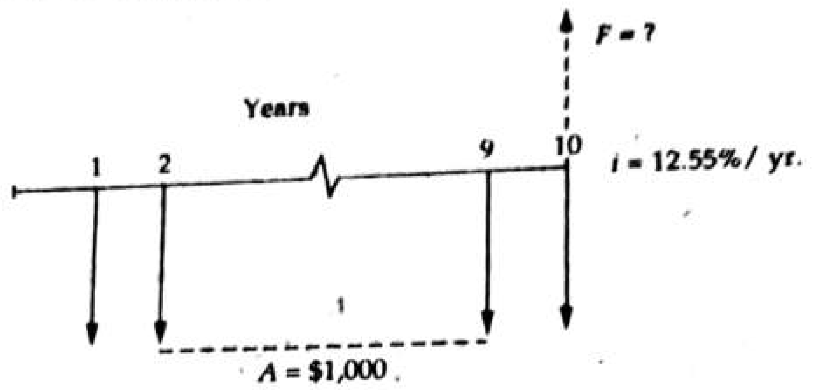


FIGURE 3-27 Second Adaptation to Solve Example 3-21

$$F = A(F/A, 12.55\%, 10) = \$1{,}000(F/A, 12.55\%, 10) = \$18{,}022$$

Because interest factors are not commonly tabled for $i = 12.55\%$, one must compute the $(F/A, 12.55\%, 10)$ factor by substituting $i = 0.1255$ and $N = 10$ into its algebraic equivalent, $[(1 + i)^N - 1]/i$.

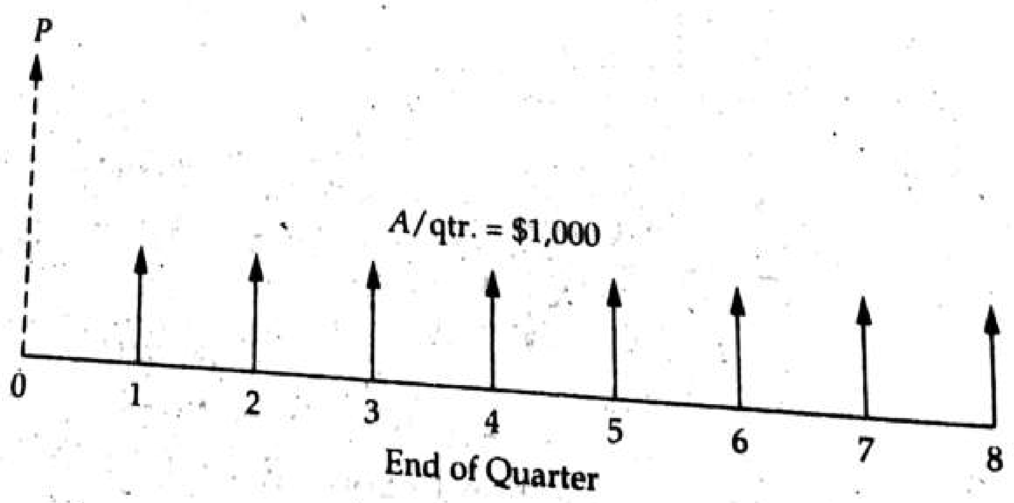
The second procedure just illustrated is probably the more popular way to deal with problems in which cash flows occur every K ($K > 1$) compounding periods. By using the second procedure, the basic question becomes: "How do we find an effective interest rate for the fixed time interval (K compounding periods) separating the cash flows?" We now formalize this procedure by using a modified version of Equation 3-32 to determine an effective interest rate per K compounding periods:

$$i(\text{per } K \text{ compounding periods}) = (1 + r/M)^K - 1 \tag{3-35}$$

where K = number of compounding periods per fixed time interval separating cash flows
r = nominal interest rate per year
M = number of compounding periods per year

EXAMPLE 3-22

Determine the present equivalent worth of this cash flow diagram:



The nominal interest rate is 15% per year compounded monthly. Cash flows occur every 3 months (once per quarter).

Solution By using Equation 3-35, we determine the effective interest rate per quarter: i/qtr. $= (1 + \frac{0.15}{12})^3 - 1 = (1.0125)^3 - 1 = 0.038$, or 3.8%. Then $P = \$1{,}000 \times (P/A, 3.8\%, 8) = \$6{,}788.70$. Note that the second procedure can also be easily used with nonuniform cash flows (e.g., gradients). ■

Interest Rates that Vary with Time

When the interest rate on a loan can vary with, for example, the Federal Reserve Board's discount rate, it is necessary to take this into account when determining the future worth of the loan. It is becoming common to see interest-rate "escalation riders" on some types of loans. Example 3-23 demonstrates how this situation is treated.

EXAMPLE 3-23

A person has made an arrangement to borrow \$1,000 now and another \$1,000 2 years hence. The entire obligation is to be repaid at the end of 4 years. If the projected interest rates in years 1, 2, 3, and 4 are 10%, 12%, 12%, and 14%, respectively, how much will be repaid as a lump-sum amount at the end of 4 years?

Solution This problem can be solved by compounding the amount owed at the beginning of each year by the interest rate that applies to each individual year and repeating this process over the 4 years to obtain the total future equivalent value:

$$F_1 = \$1{,}000(F/P, 10\%, 1) = \$1{,}100$$
$$F_2 = \$1{,}100(F/P, 12\%, 1) = \$1{,}232$$
$$F_3 = (\$1{,}232 + \$1{,}000)(F/P, 12\%, 1) = \$2{,}500$$
$$F_4 = \$2{,}500(F/P, 14\%, 1) = \$2{,}850$$

■

To obtain the present worth of a series of future amounts, a procedure similar to the preceding one would be utilized with a sequence of single-payment present worth factors. In general, the present equivalent value of a cash flow occurring at the end of period N can be computed with Equation 3-36, where i_k is the interest rate for the kth period:

$$P = \frac{F_N}{\prod_{k=1}^{N}(1 + i_k)} \tag{3-36}$$

For instance, if $F_4 = \$1{,}000$ and $i_1 = 10\%$, $i_2 = 12\%$, $i_3 = 13\%$, and $i_4 = 10\%$

$$\begin{aligned} P &= \$1{,}000(P/F, 10\%, 1)(P/F, 12\%, 1)(P/F, 13\%, 1)(P/F, 10\%, 1) \\ &= \$1{,}000[(0.9091)(0.8929)(0.8850)(0.9091)] = \$653 \end{aligned}$$

3.20 Interest Formulas for Continuous Compounding and Discrete Cash Flows

In most business transactions and economy studies, interest is compou at the end of discrete periods of time and, as has been discussed previo cash flows are assumed to occur in discrete amounts at the beginning or of such periods. *This practice will be used throughout the remaining chapters of book.* However, it is evident that in most enterprises cash is flowing in and in an almost continuous stream. Because cash, whenever available, can usu be used profitably, this situation creates opportunities for very frequent co pounding of the interest earned. So that this condition can be dealt with w continuously compounded interest rates are available, the concepts of con uous compounding and continuous cash flow are sometimes used in econo studies. Actually, the effects of these procedures compared to those of disc compounding are rather small in most cases.

Continuous compounding assumes that cash flows occur at discrete inter (e.g., once per year) but that compounding is continuous throughout the in val. For example, with a nominal rate of interest per year of r, if the interes compounded M times per year, at the end of 1 year one unit of principal w amount to $[1 + (r/M)]^M$. Letting $M/r = p$, the foregoing expression becom

$$\left[1 + \frac{1}{p}\right]^{rp} = \left[\left(1 + \frac{1}{p}\right)^{p}\right]^{r} \qquad (3\text{-}$$

Because

$$\lim_{p \to \infty}\left(1 + \frac{1}{p}\right)^{p} = e^{1} = 2.71828\ldots,$$

Equation 3-37 can be written as e^r. Consequently, the *continuously compoun compound amount factor (single cash flow)* at $r\%$ nominal interest for N year e^{rN}. Using our functional notation, we express this as

$$(F/P, \underline{r}\%, N) = e^{rN} \qquad (3\text{-}$$

Note that the symbol $\underline{r}$ is directly comparable to that used for discrete co pounding and discrete cash flows ($i\%$) except that $\underline{r}\%$ is used to denote nominal rate and the use of continuous compounding.

Since e^{rN} for continuous compounding corresponds to $(1 + i)^N$ for disc compounding, e^r is equal to $(1 + i)$. Hence, we may correctly conclude tha

$$i = e^r - 1 \qquad (3\text{-}$$

By using this relationship, the corresponding values of (P/F), (F/A), and (P for continuous compounding may be obtained from Equations 3-5, 3-7, 3-9, respectively, by substitution of $e^r - 1$ for i in these equations. Thus, continuous compounding and discrete cash flows,

$$(P/F, \underline{r}\%, N) = \frac{1}{e^{rN}} = e^{-rN} \tag{3-40}$$

$$(F/A, \underline{r}\%, N) = \frac{e^{rN} - 1}{e^{r} - 1} \tag{3-41}$$

$$(P/A, \underline{r}\%, N) = \frac{1 - e^{-rN}}{e^{r} - 1} = \frac{e^{rN} - 1}{e^{rN}(e^{r} - 1)} \tag{3-42}$$

Values for $(A/P, \underline{r}\%, N)$, and $(A/F, \underline{r}\%, N)$ may be derived through their inverse relationships to $(P/A, \underline{r}\%, N)$ and $(F/A, \underline{r}\%, N)$, respectively. Numerous continuous compounding, discrete cash flow interest factors, and their uses are summarized in Table 3-6.

Because continuous compounding is used infrequently in this text, detailed values for $(A/F, \underline{r}\%, N)$ and $(A/P, \underline{r}\%, N)$ are not given in Appendix D. However,

TABLE 3-6 Continuous Compounding and Discrete Cash Flows: Interest Factors and Symbols[a]

To Find:	*Given:*	*Factor by Which to Multiply "Given"*	*Factor Name*	*Factor Functional Symbol*
For single cash flows:				
F	P	e^{rN}	Continuous compounding compound amount (single cash flow)	$(F/P, \underline{r}\%, N)$
P	F	e^{-rN}	Continuous compounding present worth (single cash flow)	$(P/F, \underline{r}\%, N)$
For uniform series (annuities):				
F	A	$\frac{e^{rN} - 1}{e^{r} - 1}$	Continuous compounding compound amount (uniform series)	$(F/A, \underline{r}\%, N)$
P	A	$\frac{e^{rN} - 1}{e^{rN}(e^{r} - 1)}$	Continuous compounding present worth (uniform series)	$(P/A, \underline{r}\%, N)$
A	F	$\frac{e^{r} - 1}{e^{rN} - 1}$	Continuous compounding sinking fund	$(A/F, \underline{r}\%, N)$
A	P	$\frac{e^{rN}(e^{r} - 1)}{e^{rN} - 1}$	Continuous compounding capital recovery	$(A/P, \underline{r}\%, N)$

[a] $\underline{r}$, nominal annual interest rate, compounded continuously; N, number of periods (years); A, uniform series amount (occurs at the end of each year); F, future worth; P, present worth.

the tables in Appendix D do provide values of $(F/P, \underline{r}\%, N)$, $(P/F, \underline{r}\%, N)$, $\underline{r}\%, N)$, and $(P/A, \underline{r}\%, N)$ for a limited number of interest rates.

It is important to note that tables of interest and annuity factors for conti compounding are tabulated in terms of nominal annual rates of interest.

EXAMPLE 3-24

Suppose that one has a present amount of $1,000 and desires to determine equivalent uniform end-of-year payments could be obtained from it for 10 if interest is 20% compounded continuously ($M = \infty$).

Solution. Here we utilize this formulation:

$$A = P(A/P, \underline{r}\%, N)$$

Since the (A/P) factor is not tabled for continuous compounding, we subs its inverse (P/A), which is tabled in Appendix D. Thus

$$A = P \times \frac{1}{(P/A, \underline{20}\%, 10)} = \$1,000 \times \frac{1}{3.9054} = \$256$$

It is interesting to note that the answer to the same problem, except discrete annual compounding ($M = 1$), is

$$\begin{aligned} A &= P(A/P, 20\%, 10) \\ &= \$1,000(0.2385) = \$239 \end{aligned}$$

EXAMPLE 3-25

A person needs $12,000 immediately as a down payment on a new ho Suppose that he can borrow this money from his insurance company. He be required to repay the loan in equal payments, made every 6 months the next 8 years. The nominal interest rate being charged is 7% compoun continuously. What is the amount of each payment?

Solution The nominal interest rate per 6 months is 3.5%. Thus, A ea months is $\$12,000(A/P, \underline{r} = 3.5\%, 16)$. By substituting terms in Equation and then using its inverse, we determine the value of A per 6 months t $997, that is

$$A = \$12,000\left[\frac{1}{(P/A, \underline{r} = 3.5\%, 16)}\right] = \frac{\$12,000}{12.038} = \$997$$

3.21 Interest Formulas for Continuous Compounding and Continuous Cash Flows

Continuous flow of funds means a series of cash flows occurring at infin imally short intervals of time; this corresponds to an annuity having an finite number of short periods. This model could apply to companies hav receipts and expenses that occur frequently during each working day. In such cases the interest normally is compounded continuously. If the nominal interest rate per year is $\underline{r}$ and there are p payments per year, which amount to a total of one unit per year, then, by using Equation 3-8, for 1 year the PW at the beginning of the year is

$$P = \frac{1}{p}\left\{\frac{[1+(r/p)]^p - 1}{r/p[1+(r/p)]^p}\right\} = \frac{[1+(r/p)]^p - 1}{r[1+(r/p)]^p} \tag{3-43}$$

The limit of $[1+(r/p)]^p$ as p approaches infinity is e^r. By letting the PW of one unit per year, flowing continuously and with continuous compounding of interest, be called the *continuous compounding present worth factor (continuous, uniform cash flow over one period)*, one finds

$$(P/\bar{A}, \underline{r}\%, 1) = \frac{e^r - 1}{re^r} \tag{3-44}$$

where $\bar{A}$ is the amount flowing uniformly and continuously over 1 year (here \$1).

For $\bar{A}$ flowing each year over N years, as depicted in Figure 3-28,

$$(P/\bar{A}, \underline{r}\%, N) = \frac{e^{rN} - 1}{re^{rN}} \tag{3-45}$$

which is the *continuous compounding PW factor (continuous, uniform cash flows)*.

Equation 3-44 can also be written

$$(P/\bar{A}, \underline{r}\%, 1) = e^{-r}\left[\frac{e^r - 1}{r}\right] = (P/F, \underline{r}\%, 1)\left[\frac{e^r - 1}{r}\right]$$

Because the PW of \$1 per year, flowing continuously with continuous compounding of interest, is $(P/F, \underline{r}\%, 1)(e^r - 1)/r$, it follows that $(e^r - 1)/r$ must also be the compound amount of \$1 per year, flowing continuously with continuous compounding of interest. Consequently, the *continuous compounding compound amount factor (continuous, uniform cash flow over 1 year)* is

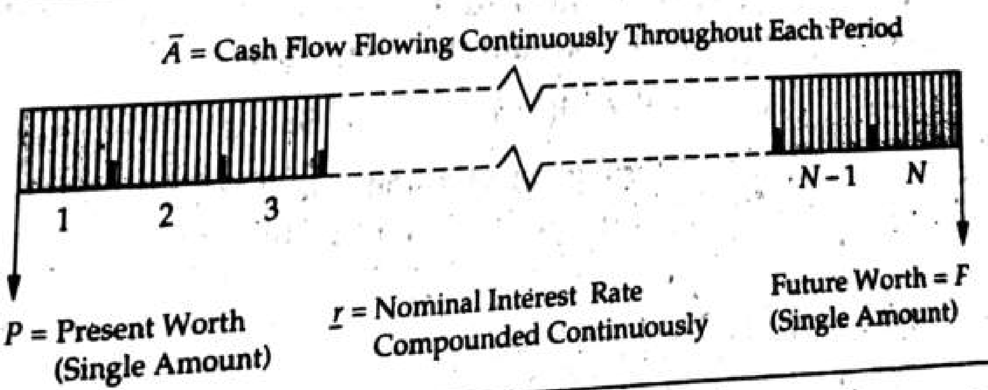


FIGURE 3-28 General Cash Flow Diagram for Continuous Compounding, Continuous Cash Flows

$$(F/\bar{A}, \underline{r}\%, 1) = \frac{e^r - 1}{r} \tag{3-46}$$

For N years,

$$(F/\bar{A}, \underline{r}\%, N) = \frac{e^{rN} - 1}{r} \tag{3-47}$$

Equation 3-47 can also be developed by integration in this manner:

$$F = \bar{A}\int_0^N e^{rt}\,dt = \bar{A}\left(\frac{1}{r}\right)\int_0^N re^{rt}\,dt$$

or

$$F = \frac{\bar{A}}{r}(e^{rt})\Big|_0^N = \bar{A}\left[\frac{e^{rN} - 1}{r}\right]$$

This is the *continuous compounding compound amount factor (continuous uniform cash flows for N years).*

Values of $(P/\bar{A}, \underline{r}\%, N)$ and $(F/\bar{A}, \underline{r}\%, N)$ are given in the tables in Appendix D for various interest rates. Values for $(\bar{A}/P, \underline{r}\%, N)$ and $(\bar{A}/F, \underline{r}\%, N)$ can be readily obtained through their inverse relationship to $(P/\bar{A}, \underline{r}\%, N)$ and $(F/\bar{A}, \underline{r}\%, N)$, respectively. A summary of these factors and their use is given in Table 3-7.

EXAMPLE 3-26

What will be the FW at the end of 5 years of a uniform, continuous cash flow, at the rate of \$500 per year for 5 years, with interest compounded continuously at the nominal annual rate of 8%?

Solution

$$F = \bar{A}(F/\bar{A}, \underline{8}\%, 5) = \$500 \times 6.1478 = \$3,074$$

It is interesting to note that if this cash flow had been in year-end amounts of \$500 with discrete annual compounding of $i = 8\%$, the FW would have been

$$F = A(F/A, 8\%, 5) = \$500 \times 5.8666 = \$2,933$$

If the year-end payments had occurred with 8% nominal interest compounded continuously, the FW would then have been

$$F = A(F/A, \underline{8}\%, 5) = \$500 \times 5.9052 = \$2,953$$

It is clear that for a given A amount and continuous compounding of a given nominal interest rate, continuous funds flow produces the largest-valued future worth.

EXAMPLE 3-27

What is the future worth of \$10,000 per year that flows continuously for 8 1/2 years if the nominal interest is 10% per year? Continuous compounding is utilized.

TABLE 3-7 Continuous Compounding Continuous Uniform Cash Flows: Interest Factors and Symbols[a]

To Find:	*Given:*	*Factor by Which to Multiply "Given"*	*Factor Name*	*Factor Functional Symbol*
F	$\bar{A}$	$\frac{e^{rN}-1}{r}$	Continuous compounding compound amount (continuous, uniform cash flows)	$(F/\bar{A}, \underline{r}\%, N)$
P	$\bar{A}$	$\frac{e^{rN}-1}{re^{rN}}$	Continuous compounding present worth (continuous, uniform cash flows)	$(P/\bar{A}, \underline{r}\%, N)$
$\bar{A}$	F	$\frac{r}{e^{rN}-1}$	Continuous compounding sinking-fund (continuous, uniform cash flows)	$(\bar{A}/F, \underline{r}\%, N)$
$\bar{A}$	P	$\frac{re^{rN}}{e^{rN}-1}$	Continuous compounding capital recovery (continuous, uniform cash flows)	$(\bar{A}/P, \underline{r}\%, N)$

[a] $\underline{r}$, nominal annual interest rate, compounded continuously; N, number of periods (years); $\bar{A}$, amount of money flowing continuously and uniformly during each period; F, future worth; P, present worth.

Solution There are seventeen 6-month periods in 8 1/2 years, and the $\underline{r}$ per 6 months is 5%. The $\bar{A}$ every 6 months is \$5,000, so F = \$5,000$(F/\bar{A}, \underline{5}\%, 17)$ = \$133,964.50. This formulation is utilized to enable us to find an interest factor having an integer-valued N. The same answer could have been obtained by resorting to the definition of the $(F/\bar{A}, \underline{r}\%, N)$ factor given in Table 3-7 with N = 8.5 years:

$$F = \$10{,}000\left[\frac{e^{0.10(8.5)}-1}{0.10}\right]$$

$$= \$133{,}964.50$$

3.22 Summary

Chapter 3 has presented the fundamental time value of money relationships that are utilized throughout the remainder of this book. Considerable emphasis has been placed on the notion of economic equivalence, whether the relevant cash flows and interest rates are discrete or continuous. Students should feel comfortable with the material in this chapter before embarking on their journey through subsequent chapters. Important abbreviations and notation in Chapter 3 are listed in Appendix B, which will serve as a handy reference in your use of this book.

3.23 References

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3.24 Problems

The numbers in parentheses () that follow these problems indicate the section from which each problem is taken. (Refer to Appendix I of this book for answers to even-numbered problems.)

3-1. What lump-sum amount of interest will be paid on a $10,000 loan that was made on August 1, 1993, and repaid on November 1, 1999, with ordinary simple interest at 12% per year? (3.4)

3-2. Draw a cash flow diagram for $10,500 being loaned out at an interest rate of 15% per annum over a period of 6 years. How much simple interest would be repaid as a lump-sum amount at the end of the sixth year? (3.4, 3.7)

3-3. What is the future worth of $1,000 invested at 6% simple interest for ($2\frac{1}{2}$) years?
a. $1,157 **b.** $1,188 **c.** $1,184
d. $1,175 **e.** $1,150

3-4. How much interest is *payable each year* on a loan of $2,000 if the interest rate is 10% per year when half of the loan principal will be repaid as a lump sum at the end of 3 years and the other *half* will be repaid in one lump-sum amount at the end of 6 years? How much interest will be paid over the 6-year period? (3.6)

3-5. In Problem 3-4, if the interest had not been paid each year but had been allowed to compound, how much interest would be due to the lender as a lump sum at the end of the sixth year? How much extra interest is being paid here (as compared to Problem 3-4) and what is the reason for the difference? (3.6)

3-6.

a. Suppose that in plan 1 of Table 3-1 $4,000 in principal is to be repaid at the end of year

2 and 4 only. How much total interest would have been paid by the end of year 4? (3.6)

b. Rework plan 3 of Table 3-1 when an annual interest rate of 8% is being charged on the loan. How much *principal* is now being repaid in the third year's total end-of-year payment? How much total interest has been paid by the end of the fourth year? (3.6)

3-7.

a. In view of the information here, determine the value of each "?" in the following table. (3.6)

Loan Principal = \$10,000
Interest Rate = 8%/yr.
Duration of Loan = 3 yrs.

EOY k	*Interest Paid*	*Principal Repayment*
1	\$800	?
2	\$553.60	\$3,326.40
3	?	?

b. What is the amount of principal owed at the *beginning* of year 3?

c. Why is the total interest paid in (a) different from $\$10,000(1.08)^3 - \$10,000 \simeq \$2,597$ that would be repaid according to plan 4 in Table 3-1?

3-8. Compute the value of P_0 for each of the two alternatives discussed in Example 3-2. The annual interest rate is 10%. Which alternative should be selected, assuming that doing nothing is not an option? (3.9)

3-9. What amount would need to be paid each January 1 if at the end of 15 years (15 payments) you desired the amount of \$10,000? Annual interest is 5%. (Note: The last payment will coincide with the time of the \$10,000 balance.) (3.9)

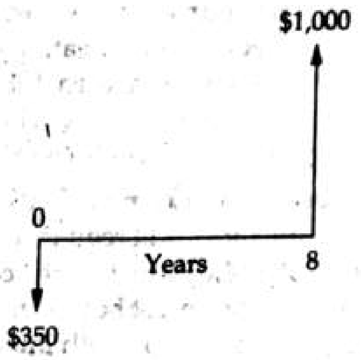
3-10. A future amount, F, is equivalent to \$1,500 now when 8 years separates the amounts and the annual interest is 12%. What is the value of F? (3.9)

3-11. A present obligation of \$20,000 is to be repaid in equal uniform annual amounts, each of which includes repayment of the debt (principal) and interest on the debt, over a period of 5 years. If the interest rate per year is 10%, what is the amount of the annual repayment? (3.9)

3-12. Suppose that the \$20,000 in Problem 3-11 is to be repaid at a rate of \$4,000 per year plus the interest that is owed and based on the beginning-of-year unpaid principal. Compute the total amount of interest repaid in this situation and compare it with that of Problem 3-11. Why are the two amounts different? (3.6)

3-13. A person desires to accumulate \$2,500 over a period of 7 years so that a cash payment can be made for a new roof on a summer cottage. To have this amount when it is needed, annual payments will be made to a savings account that earns 8% annual interest per year. How much must each annual payment be? Draw a cash flow diagram. (3.9)

3-14. You have just learned that ABC Corporation has an investment opportunity that costs \$350 and 8 years later pays a lump-sum amount of \$1,000. The cash flow diagram looks like this:



What interest rate would be earned on this investment? Calculate your answer to the nearest 0.1 of 1% (3.9)

3-15. It is estimated that a copper mine will produce 10,000 tons of ore during the coming year. Production is expected to increase by 5% per year thereafter in each of the following 6 years. Profit per ton will be \$12 for years 1–7.

a. Draw a cash flow diagram for this copper mine operation. (3.7)

b. If the company can earn 12% per year on its capital, what is the future equivalent worth of the copper mine's cash flows at the end of year 7? (3.9)

3-16. Mrs. Green has just purchased a new car for \$12,000. She makes a down payment of 30% of the negotiated price and then makes payments of \$303.68 per month thereafter for 36 months.

Furthermore, she believes the car can be sold for \$3,500 at the end of 3 years. Draw a cash flow diagram of this situation from Mrs. Green's viewpoint. (3.7)

3-17. If \$25,000 is deposited now into a savings account that earns 12% per year, what uniform annual amount could be withdrawn at the end of each year for 10 years so that nothing would be left in the account after the tenth withdrawal? (3.9)

3-18. It is estimated that a certain piece of equipment can save \$6,000 per year in labor and materials costs. The equipment has an expected life of 5 years and no salvage value. If the company must earn a 20% annual return on such investments, how much could be justified now for the purchase of this piece of equipment? Draw a cash flow diagram. (3.9)

3-19. Suppose that installation of Low-Loss thermal windows in your area is expected to save \$150 a year on your home heating bill for the next 18 years. If you can earn 8% a year on other investments, how much could you afford to spend now for these windows? (3.9)

3-20. A proposed product modification to avoid production difficulties will require an immediate expenditure of \$14,000 to modify certain dies. What annual savings must be realized to recover this expenditure in 4 years with interest at 10%? (3.9)

3-21. You can buy a machine for \$100,000 that will produce a net income, after operating expenses, of \$10,000 per year. If you plan to keep the machine for 4 years, what must the salvage (resale) value be at the end of four years to justify the investment? You must make a 15% annual return on your investment. (3.9)

3-22. Consider the cash flow diagram below and answer (a)–(d). (3.9)

a. If P = \$1,000, A = \$200, and $i\%$ = 12%, then N = ?

b. If P = \$1,000, A = \$200, and N = 10 years, then i = ?

c. If A = \$200, $i\%$ = 12%, and N = 5 years, then P = ?

d. If P = \$1,000, $i\%$ = 12%, and N = 5 years, then A = ?

3-23. Use the rule of 72 to determine how long it takes to accumulate \$2,000 in a savings account when P = \$1,000 and i = 10% per annum. (3.9)

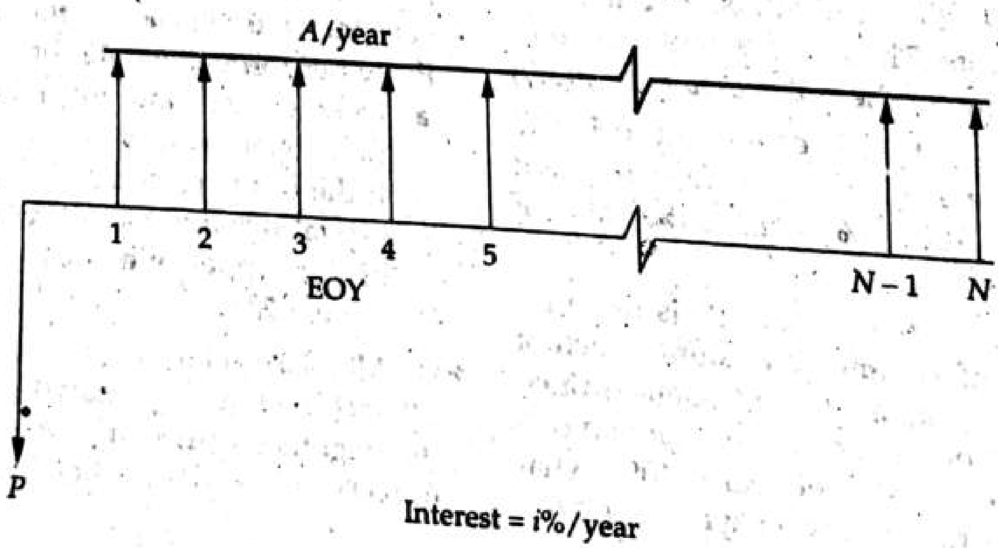
> Rule of 72: The time required to double the value of a lump-sum investment that is allowed to compound is approximately
>
> 72 ÷ annual interest rate (as a %)

3-24.

a. Show that the following relationship is true: $(A/P, i, N) = i/[1-(P/F, i, N)]$. (3.10)

b. Show that $(A/G, 0\%, N) = (N-1)/N$. (3.14)

3-25. What must be the prospective saving in money 6 years hence to justify a present investment of \$2,250? Use interest at 12% compounded annually. (3.9)



(Pr. 3-22)

3-26. Suppose that \$10,000 is borrowed now at 15% interest per annum. A partial repayment of \$3,000 is made 4 years from now. The amount that will remain to be paid then is most nearly: (3.9)
a. \$7,000 b. \$8,050 c. \$8,500
d. \$13,000 e. \$14,490

3-27. How much should be deposited each year for 12 years if you wish to withdraw \$309 each year for four years, beginning at the end of the 15th year? Let i = 8% per year. (3.11)

3-28. Suppose that you have \$10,000 cash today and can invest it at 10% compound interest each year. How many years will it take you to become a millionaire? (3.9)

3-29. Equal end-of-year payments of \$263.80 each are being made on a \$1,000 loan at 10% effective interest per year. (3.6, 3.10)
a. How many payments are required to repay the entire loan?
b. Immediately after the second payment, what lump-sum amount would completely pay off the loan?

3-30. Maintenance costs for a new bridge with an expected 50-year life are estimated to be \$1,000 each year for the first 5 years, followed by a \$10,000 expenditure in the fifteenth year and a \$10,000 expenditure in year 30. If i = 10% per year, what is the equivalent uniform annual cost over the entire 50-year-period? (3.13)

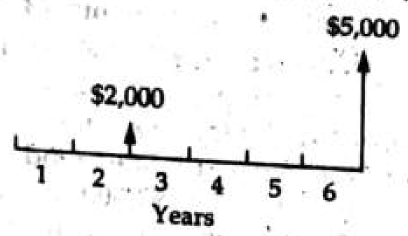
3-31. In 1971 first-class postage for a 1-ounce envelope was \$0.10. In 1991, a first-class stamp for the same envelope cost \$0.29. What compounded *annual* increase in the cost of first-class postage was experienced during the 20 years? (3.9)

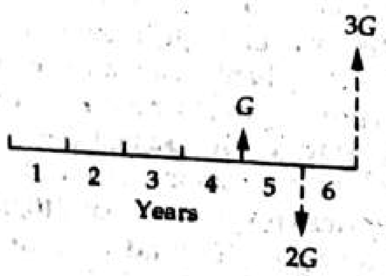
3-32. You purchase special equipment that reduces defects by \$10,000 per year on item A. This item is sold on contract for the next 5 years. After the contract expires, the special equipment will save approximately \$2,000 per year for 5 years. You assume that the machine has no salvage value at the end of 10 years. How much can you afford to pay for this equipment now if you require a 25% return on your investment? All cash flows are end-of-year amounts. (3.14)

3-33. John Q wants his estate to be worth \$65,000 at the end of 10 years. His net worth is now zero. He can accumulate the desired \$65,000 by depositing \$3,887 at the end of each year for the next 10 years. At what interest rate must his deposits be invested? Give an answer to the nearest tenth of a percent (3.10)

3-34. What lump sum of money must be deposited into a bank account at the present time so that \$500 per month can be withdrawn for 5 years, with the first withdrawal scheduled for 6 years from today? The interest rate is 1% per month. (*Hint:* Monthly withdrawals begin at the end of month 72.) (3.11)

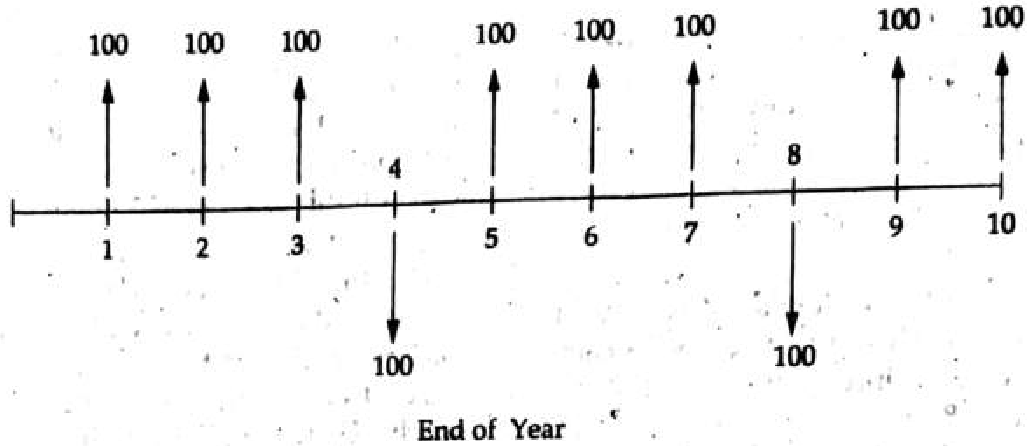
3-35. Solve for the value of G in the figure below so that the top cash flow diagram is equivalent to the bottom one. Let i = 10% per year. (3.13)





3-36. An individual is borrowing \$100,000 at 8% compounded annually. The loan is to be repaid in equal annual payments over 30 years. However, just after the eighth payment is made, the lender allows the borrower to double the annual payment. The borrower agrees to this increased payment. If the lender is still charging 8%, compounded annually, on the unpaid balance of the loan, what is the balance still owed just after the twelfth payment is made? (3.13)

3-37. A woman arranges to repay a \$1,000 bank loan in 10 equal payments at a 10% effective annual interest rate. Immediately after her third payment she borrows another \$500, also at 10%. When she borrows the \$500, she talks the banker into letting her repay the remaining



(Pr. 3-40)

debt of the first loan and the entire amount of the second loan in 12 equal annual payments. The first of these 12 payments would be made 1 year after she receives the $500. Compute the amount of each of the 12 payments. (3.13)

3-38. Suppose that you have an opportunity to invest in a fund that pays 12% interest compounded annually. Today, you invest $10,000 into this fund. Three years later (end of year, or EOY, 3), you borrow $5,000 from a local bank at 10% effective annual interest and invest it in the fund. Two years later (EOY 5), you withdraw enough money from the fund to repay the bank loan and all interest due on it. Three years from this withdrawal (EOY 8) you start taking $2,000 per year out of the fund. After 5 more years (EOY 12), you have withdrawn your original $10,000. The amount remaining in the fund is earned interest. *How much remains?* (*Hint:* Draw a cash flow diagram.) (3.13)

3-39. A loan of $4,000 is to be repaid over a period of 8 years. During the first 4 years, exactly half of the *loan principal* is to be repaid (along with accumulated compound interest) by a uniform series of payments of A_1 dollars per year. The other half of the loan principal is to be repaid over 4 years with accumulated interest by a uniform series of payments of A_2 dollars per year. If $i = 12\%$ per year, what are A_1 and A_2? (3.13)

3-40. Determine the present equivalent value at time 0 in the cash flow diagram above when $i = 10\%$ per year. Try to minimize the number of interest factors you use. (3.13)

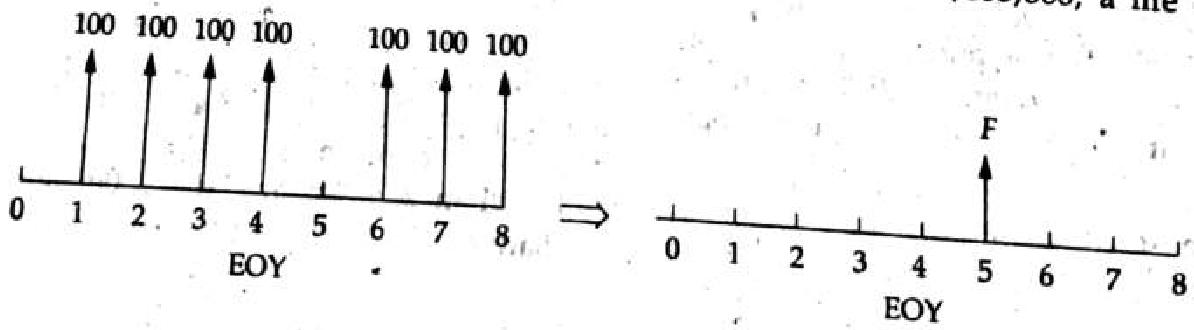
3-41. Transform the cash flows on the left-hand side of the diagram below to their equivalent amount, F, shown on the right-hand side. The effective rate is 10% per year. (3.13)

3-42. Determine the value of W on the right-hand side of the diagram on the top of page 121 that makes the two cash flow diagrams equivalent when $i = 15\%$ per year. (3.13)

3-43. Determine the value of Z on the left-hand side of the cash flow diagram near the middle of page 121 that establishes equivalence with the right-hand side. The nominal interest rate is 12%, compounded quarterly. (3.13)

3-44. Repairs to an existing warehouse can be made immediately for $10,000. If instead the repair work is postponed for 4 years, the estimated cost will be $25,000. The useful life of the warehouse is not affected by when the repairs are made. Assume that there are no extra costs incurred by postponing the work. If the company requires a 20% annual return on its investment in this situation, when should the repairs be made? (3.9)

3-45. A certain fluidized-bed combustion vessel has a first cost of $100,000, a life of 10 years,



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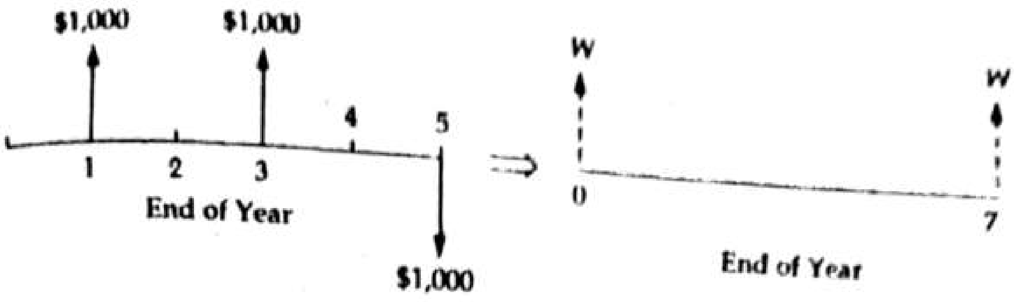

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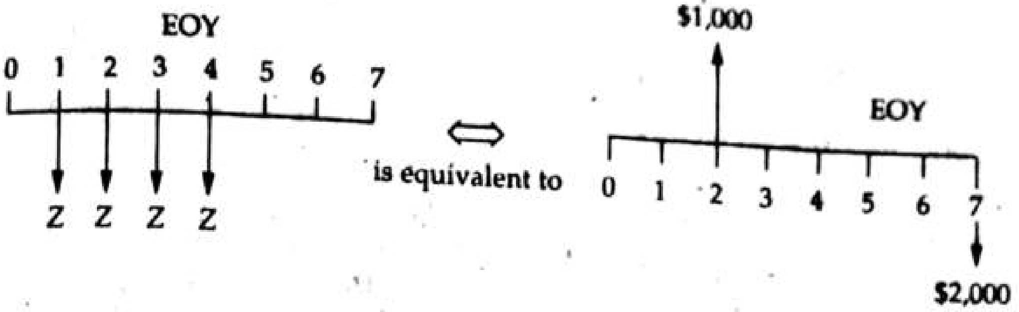

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and negligible salvage value. Annual costs of materials, maintenance, and electric power for the vessel are expected to total $8,000. A major relining of the combustion vessel will occur during the fifth year at a cost of $20,000; during this year, the vessel will *not* be in service. If the interest rate is 15% per year, what is the lump-sum equivalent cost of this project at the present time (t = 0)? Assume that a beginning-of-year cash flow convention is being utilized. (3.12)

46. Suppose that $400 is deposited at the beginning of each year into a bank account that pays interest annually (i = 10%). If 12 payments are made into the account, how much would be accumulated in this fund by the end of the twelfth year? (3.12)

47. An expenditure of $20,000 is made to modify a materials-handling system in a small job shop. This modification will result in first-year savings of $2,000, second-year savings of $4,000, and savings of $5,000 per year thereafter. How many years must the system last if a 25% return on investment is required? The system is tailor-made for this job shop and has no salvage value at any time. Use beginning-of-year cash flow convention to solve this problem. (3.12)

3-48. Determine the present equivalent and annual equivalent value of the cash flow pattern shown below when i = 8% per year. (3.14)

3-49. Find the uniform annual amount that is equivalent to a gradient series in which the first year's payment is $500, the second year's payment is $600, the third year's payment is $700, and so on, and there is a total of 10 payments. The annual interest rate is 8%. (3.14)

3-50. Suppose that annual income from a rental is expected to start at $1,200 per year and decrease at a uniform rate of about $40 each year for the 15-year expected life of the property. The investment cost is $7,000 and i is 10% per year. Is this a good investment? (3.14)

3-51. For a repayment schedule that starts at the end of year 3 at $Z and proceeds for years 4 through 10 as $2Z, $3Z..., what is the value of Z, if the principal of this loan is $10,000 and the interest rate is 10% compounded annually? (3.14)

3-52. If $10,000 now is equivalent to 4Z at the end of year 2, 3Z at the end of year 3, 2Z at the end of year 4, and Z at the end of year 5, what is the value of Z when i = 10% per year? (3.14)

End of Year	0	1	2	3	4	5	6	7
Amount ($)	−1,500	+500	+500	+500	+400	+300	+200	+100

3-53. Refer to the cash flow diagram on page 123 and solve for the unknown quantity in parts (a) through (d). (3.14)

a. If $F = \$10,000$, $G = \$600$, and $N = 6$, then $i = ?$

b. If $F = \$10,000$, $G = \$600$, and $i = 5\%$, then $N = ?$

c. If $G = \$1,000$, $N = 12$, and $i = 10\%$, then $F = ?$

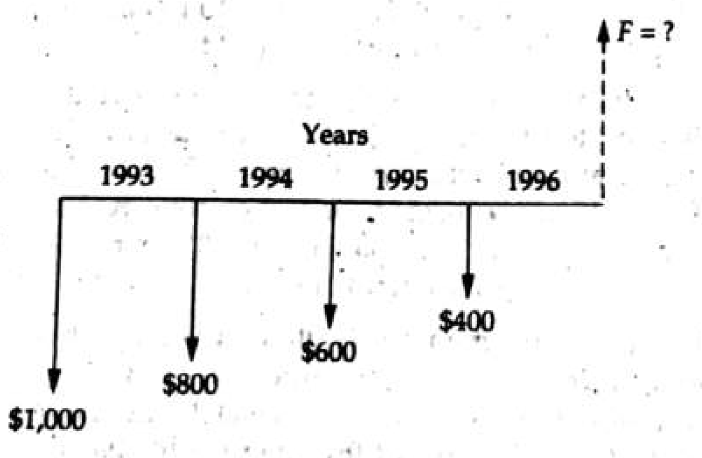
d. If $F = \$8,000$, $N = 6$, and $i = 10\%$, then $G = ?$

3-54. Solve for P_0 in the cash flow diagram on page 123, by using only two interest factors. The interest rate is 11% per year. (3.14)

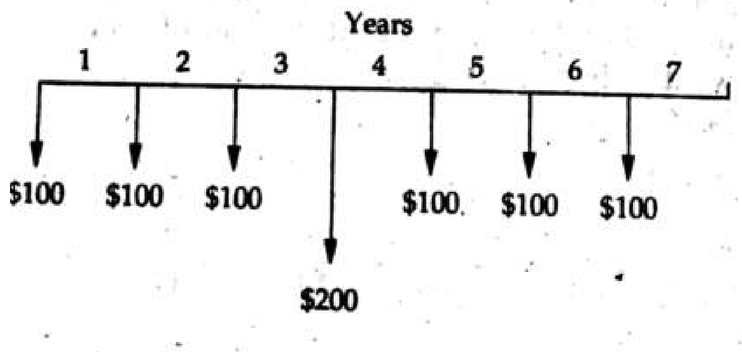
3-55. On page 123, what is the value of K on the left-hand cash flow diagram that is equivalent to the right-hand cash flow diagram? Let $i = 15\%$ per year. (3.14)

3-56. For the cash flow diagram on page 123, complete the equivalence equation. (3.14)
$P_0 = \$100(P/A, 10\%, 4) +$ ______ (This can be completed with one more term)

3-57. Calculate the future worth at the end of 1996, at 8% compounded annually, of this savings account: (3.14)



3-58. Convert the cash flow pattern below to a uniform series of end-of-year costs over a 7-year-period. Let $i = 12\%$. (3.12)

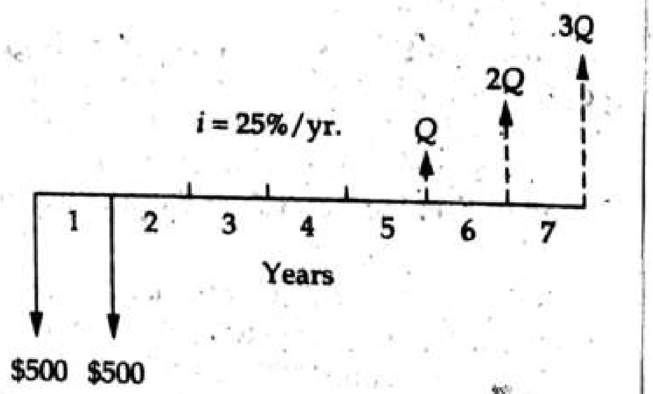


3-59. Suppose that the parents of a young child decide to make annual payments into a savings account, with the first payment being made on the child's fifth birthday and the last payment being made on the fifteenth birthday. Then starting on the child's eighteenth birthday, the withdrawals shown on page 124 (top) will be made. If the effective annual interest rate is 10% during this period of time, what is the amount in the account in years 5 through 15? (3.14)

3-60. Find the value of the unknown quantity in the cash flow diagram on page 124. Let $i = 12\%$ per year. Use an arithmetic gradient factor in your solution. (3.14)

3-61. The heat loss through the exterior walls of a certain poultry processing plant is estimated to cost the owner \$3,000 next year. A salesman from Superfiber Insulation, Inc., has told you, the plant engineer, that he can reduce the heat loss by 80% with the installation of \$15,000 worth of Superfiber now. If the cost of heat loss rises by \$200 per year (gradient) after the next year and the owner plans to keep the present building for 10 more years, what would you recommend if the cost of money is 12% per year? (3.14)

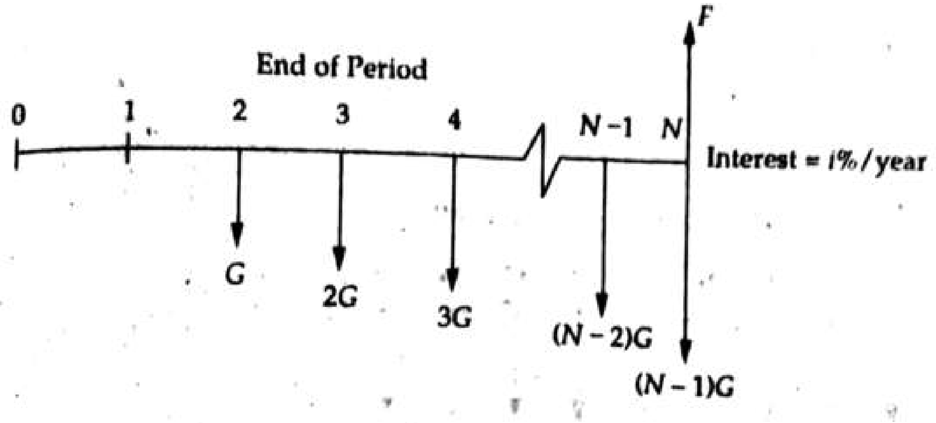
3-62. Find the equivalent value of Q in this cash flow diagram: (3.14)



3-63. What value of N comes closest to making the left-hand cash flow diagram on page 124 equivalent to the one on the right? Let $i = 15\%$ per year. (3.14)

3-64. Find the value of B on the left-hand diagram (page 124) that makes the two cash flow diagrams equivalent at $i = 10\%$/year. (3.14)

3-65. You are the manager of a large crude oil refinery. As part of the refining process, a certain heat exchanger (operated at high temperature



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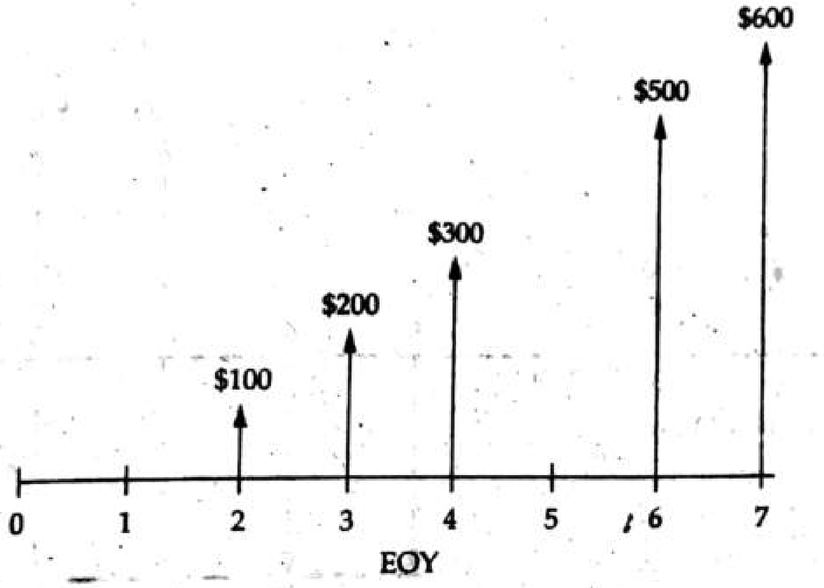

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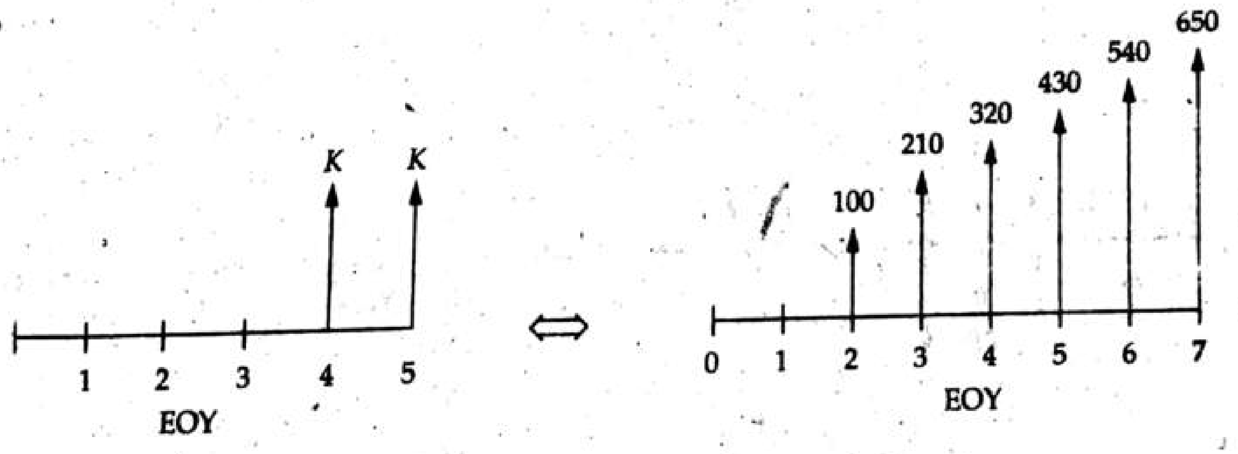

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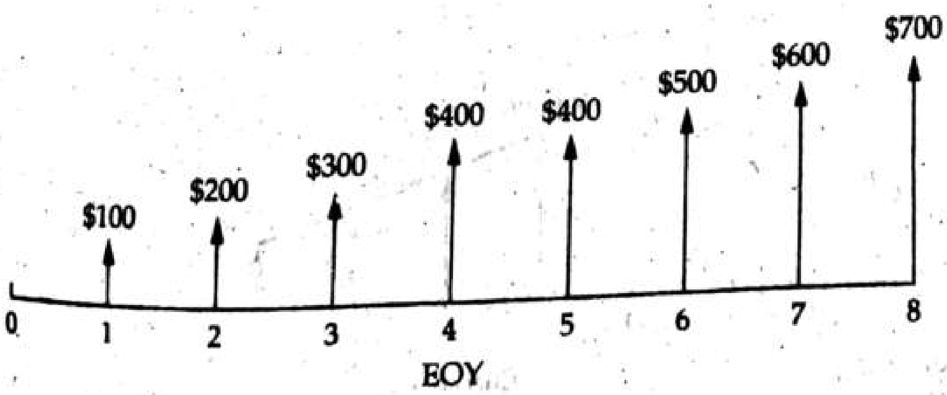
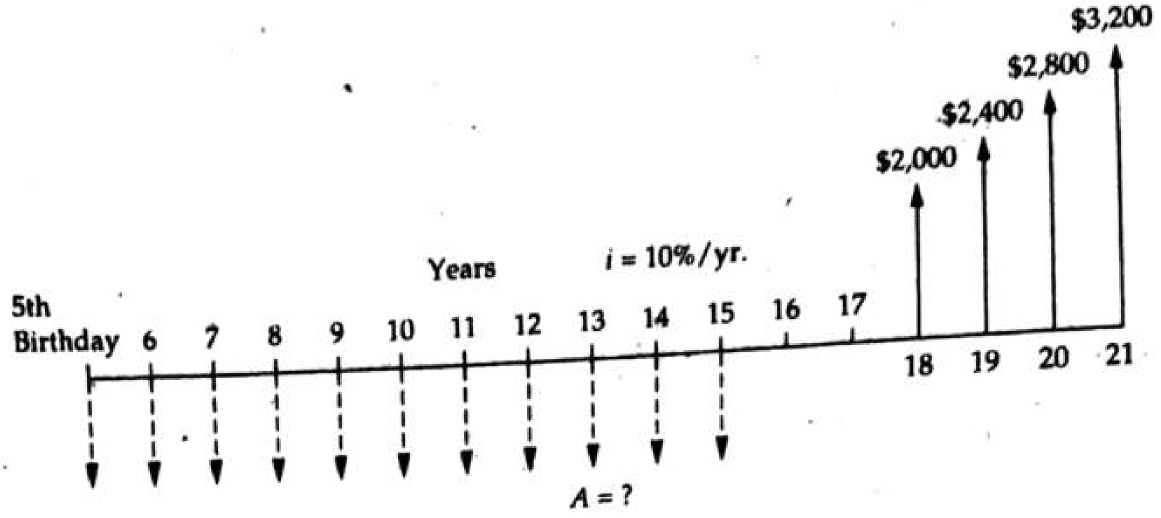
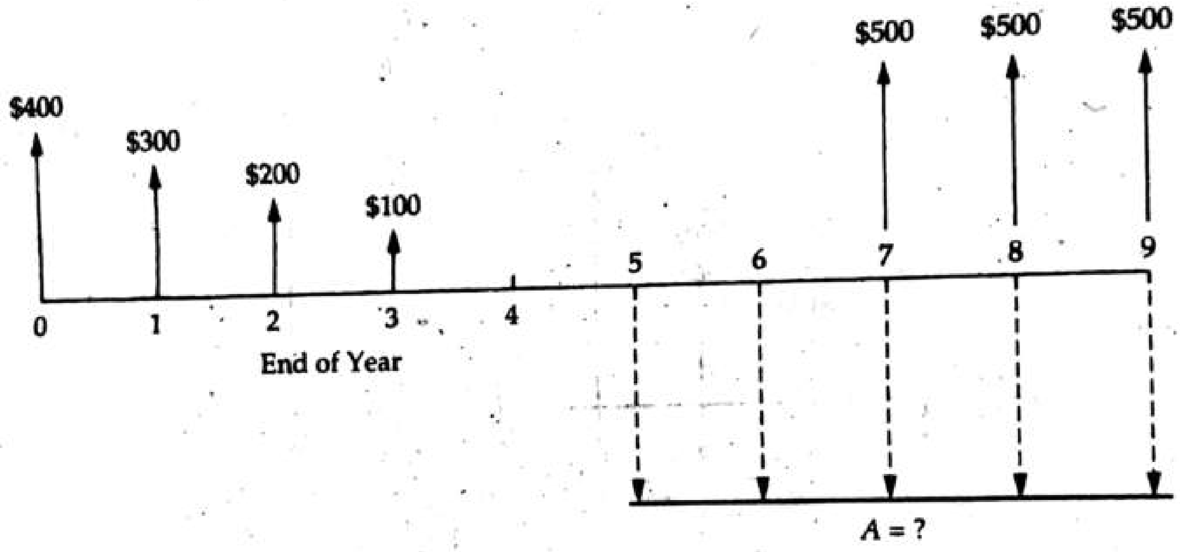


Note: Cash flow at EOY 4 and 5 is $400 (Pr. 3-56)



(Pr. 3-59)



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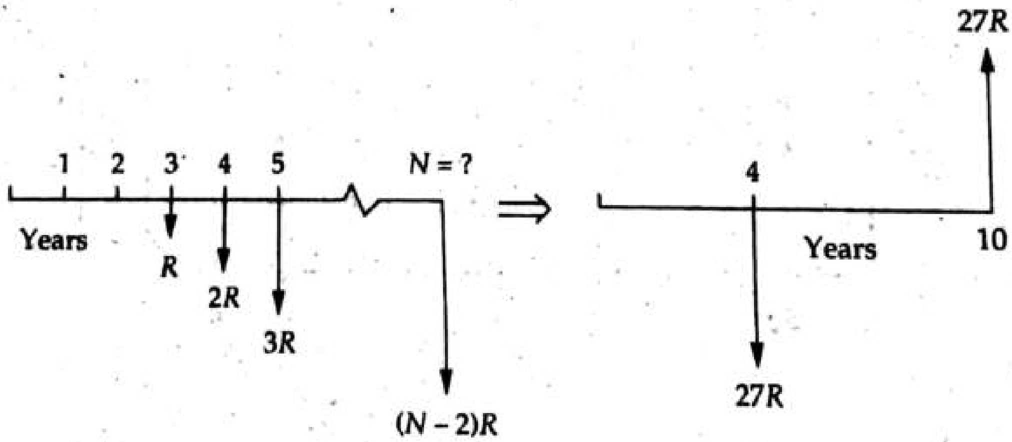

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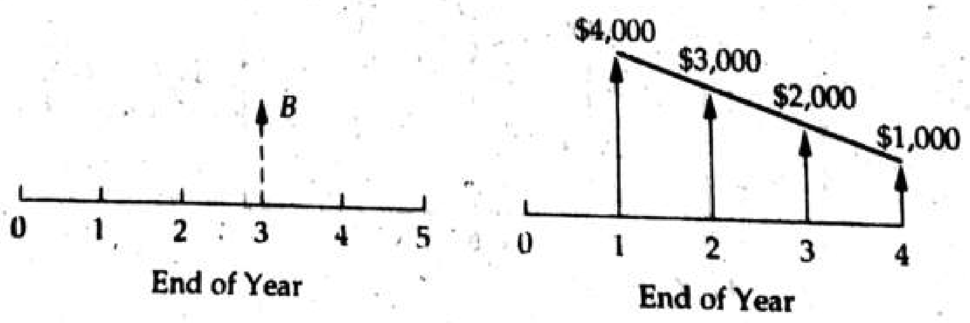

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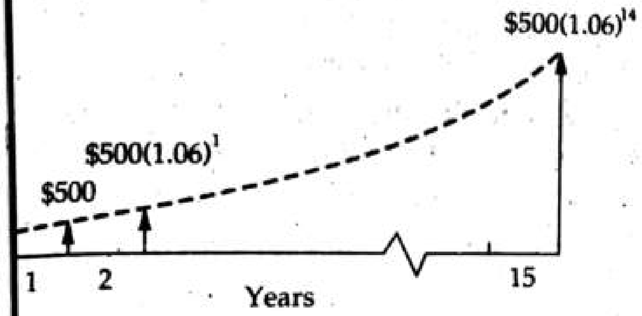
and with abrasive material flowing through it) must be replaced every year. The replacement and downtime cost in the first year is $75,000. It is expected to increase due to inflation at a rate of 8% per year for 5 years, at which time this particular heat exchanger will no longer be needed. If the company's cost of capital is 18% per year, how much could you afford to spend for a higher-quality heat exchanger so that these annual replacement and downtime costs could be eliminated? (3.15)

6. A geometric gradient that increases at $\bar{f}$ = % per year for 15 years is shown in the following diagram. The annual interest rate is 12%. What is the present equivalent value of this gradient? (3.15)



In a geometric sequence of annual cash flows starting at end of year *zero*, the value of is $1,304.35 (which *is* a cash flow). The value the last term in the series, A_{10}, is $5,276,82. hat is the equivalent value of A for years 1–10? t i = 20% per year. (3.15)

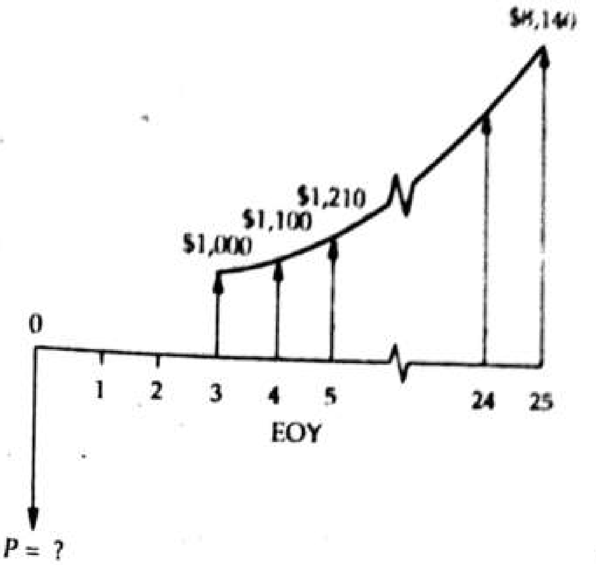
An electronic device is available that will duce this year's labor costs by $10,000. The uipment is expected to last for 8 years. If la- costs increase at an average rate of 7% per r and the interest rate is 15% per year, an- er these questions:

What is the maximum amount that we could justify spending for the device?

What is the uniform annual equivalent value (A) of labor costs over the 8-year period?

What annual year 0 amount (A_0) that inflates at 7% per year is equivalent to the answer in part (a)? (3.15)

Determine the present worth (at time 0) the following geometric sequence of cash vs. Let i = 15.5% per year and $\bar{f}$ = 10%. 5)



3-70. Rework problem 3-69 when the cash flow at the end of year 3 is $8,140, and cash flows at the end of years 4 through 25 *decrease* by 10% per year (i.e., $\bar{f}$ = −10% per year). (3.15)

3-71. For the cash flow diagram at the top of page 126, solve for X such that the cash receipt in year 0 is equivalent to the cash outflows in years 1 through 6. (3.15)

3-72. An end-of-year (EOY) geometric gradient lasts for 8 years, whose initial value at EOY = 1 is $5,000 and $\bar{f}$ = 6.04%. Find the equivalent uniform gradient amount (G) over the same time period if the initial value of the uniform gradient at EOY 1 is $4,000. Answer the following questions in determining the value of the gradient amount, G. The interest rate is 8% nominal, compounded semiannually. (3.14, 3.15)

a. What is i_{CR}?

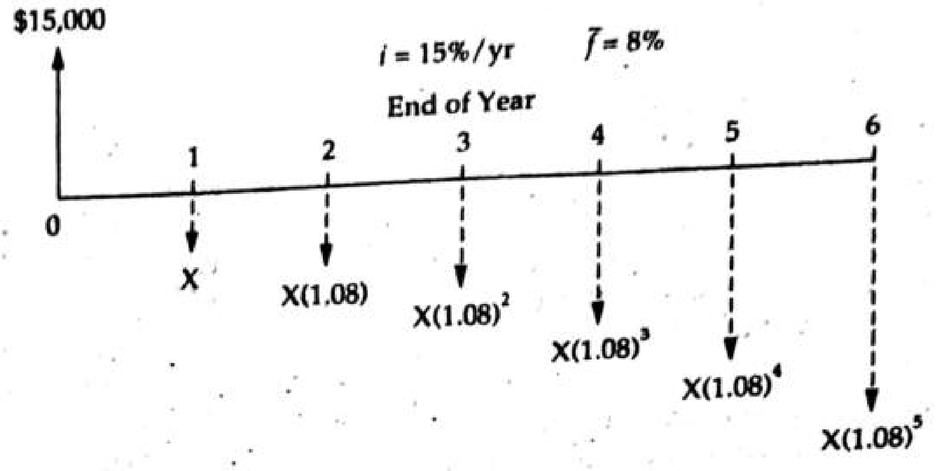
b. What is P_0 for the geometric gradient?

c. What is P_0 of the uniform (arithmetic) gradient?

d. What is the value of G?

3-73. *Set up an expression* for the unknown quantity, Z, in the cash flow diagram on the bottom of page 126. (3.14, 3.15)

3-74. Peggy Sue was left $50,000 by her uncle. She has decided to put it into a savings account for the next year or so. She finds that there are three different interest rates at savings institutions: 5⅜% compounded annually,



(Pr. 3-71)

5¼% compounded quarterly, and 5⅛% compounded continuously. She wishes to select the savings institution that will give her the highest return on her money. What interest rate should be selected? (3.16)

3-75. Compute the effective annual interest rate in each of these situations: (3.16)

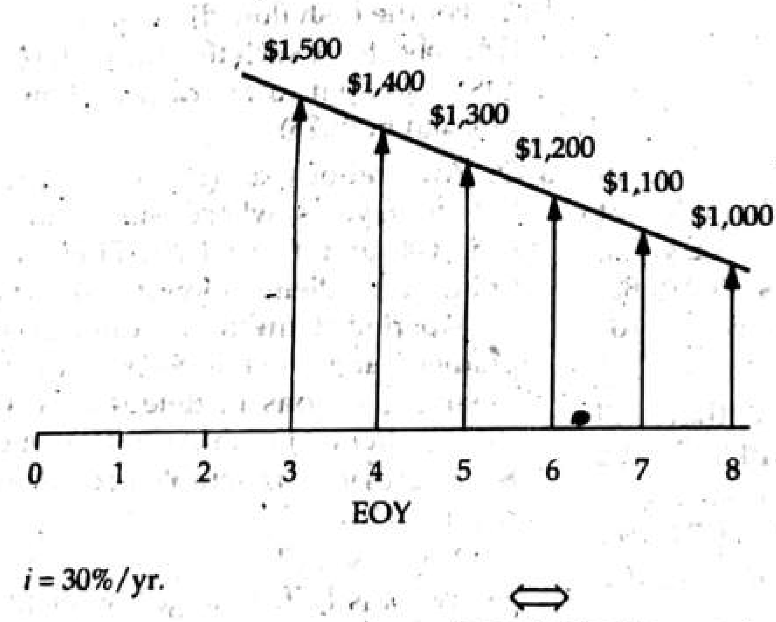
a. 10% nominal interest, compounded semiannually.

b. 10% compounded quarterly.

c. 10% compounded continuously.

d. 10% compounded weekly.

3-76. Sixty monthly deposits are made into an account paying 6% nominal interest compounded



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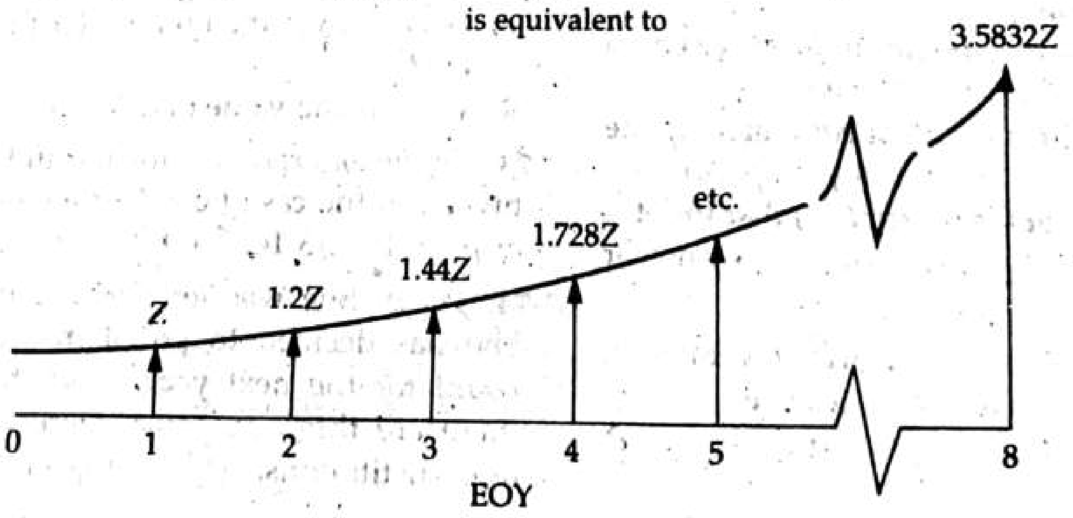


monthly. If the objective of these deposits is to accumulate $100,000 by the end of the fifth year, what is the amount of each deposit? (3.17)
a. $1,930 b. $1,478 c. $1,667
d. $1,430 e. $1,695

3-77.
a. What extra semiannual expenditure for 5 years would be justified for the maintenance of a machine in order to avoid an overhaul costing $3,000 at the end of 5 years? Assume nominal interest at 8%, compounded semiannually. (3.17)
b. What is the annual equivalent worth of $125,000 now when 18% nominal interest per year is compounded monthly? Let N = 15 years. (3.17)

3-78.
a. What equal monthly payments will repay an original loan of $1,000 in 6 months at a nominal rate of 6% compounded monthly? What is the effective annual rate?
b. For part (a), what is the effective quarterly interest rate? (3.16)

3-79. Determine the current amount of money that must be invested at 6% nominal interest, compounded monthly, to provide an annuity of $10,000 (per year) for 6 years, starting 8 years from now. The interest rate remains constant over this entire period of time. (3.17)

3-80. Find the present equivalent value of the following series of payments: $100 at the end of each month for 125 months at a nominal rate of 15% compounded monthly. (3.17)

3-81. Determine the present equivalent value for $400 paid every three months over a period of seven years in each of these situations: (3.18)
a. The interest rate is 12%, compounded annually.
b. The interest rate is 12%, compounded quarterly.
c. The interest rate is 12%, compounded weekly.

3-82. Suppose that you have just borrowed $7,500 at 10% nominal interest compounded quarterly. What is the total lump-sum, compounded amount to be paid by you at the end of a 6-year loan period? (3.17)

3-83. How many deposits of $100 each must you make at the end of each month if you desire to accumulate $3,350 for a new home entertainment center? Your savings account pays 9% interest compounded monthly. (3.17)

3-84. You have used your credit card to purchase automobile tires for $340. Unable to make payments for 7 months, you then write a letter of apology and enclose a check to pay your bill in full. The credit card company's nominal interest rate is 18% compounded monthly. For what amount should you write the check? (3.17)

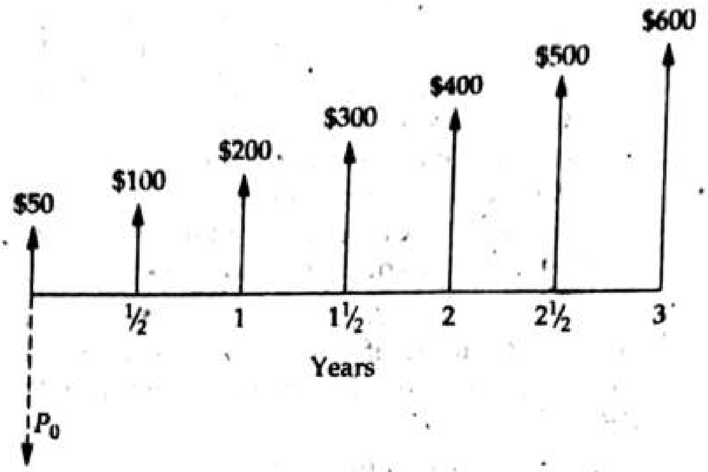
3-85. How long does it take a given amount of money to triple if the money is invested at a nominal rate of 15%, compounded monthly? (3.17)

3-86. A local bank offers a "Vacation Made Easy" plan as follows. Each participant in the plan deposits an amount of money, A, at the end of each week for 50 weeks, with no interest being paid by the bank. Then the bank makes the 51st and 52nd payments and returns to each participant a grand total of $52A$ at the end of week 52 that is used to pay for the vacation. What is the true interest rate per year being earned by participants in this plan? Assume that an opportunity exists for weekly compounding if participants elect another investment plan elsewhere. (3.18)

3-87.
a. A certain savings and loan association advertises that it pays 8% interest, compounded quarterly. What is the *effective* interest rate per annum? If you deposit $5,000 now and plan to withdraw it in 3 years, how much would your account be worth at that time? (3.17)
b. If instead you decide to deposit $800 every year for 3 years, how much could be withdrawn at the end of the third year? Suppose that, instead, you deposit $400 every six months for 3 years. What would the accumulated amount be? (3.18)

3-88. The effective annual interest rate, i, has been determined to be 26.82%, compounded monthly. Calculate how much can be spent now to avoid future computer software maintenance expenses of $1,000 *per quarter* for the next 5 years. (3.18)

3-89. If the nominal interest rate is 10% and compounding is semiannual, what is the present equivalent value of the receipts in the following diagram? (3.17)



3-90. What is the monthly payment on a loan of $15,000 for 5 years at a nominal rate of interest of 9% compounded monthly? (3.17)
a. $214 b. $250 c. $312 d. $324 e. $381

3-91. The effective annual interest rate, i, is stated to be 19.2%. What is the nominal interest rate per year, r, if continuous compounding is being used? (3.20)

3-92. Find the value of A that is equivalent to the gradient shown in the diagram below if the nominal interest rate is 12% compounded monthly. (3.18)

3-93. Suppose that you have a money market certificate earning an average annual rate of interest, which varies over time as follows:

Year k	1	2	3	4	5
i_k	14%	12%	10%	10%	12%

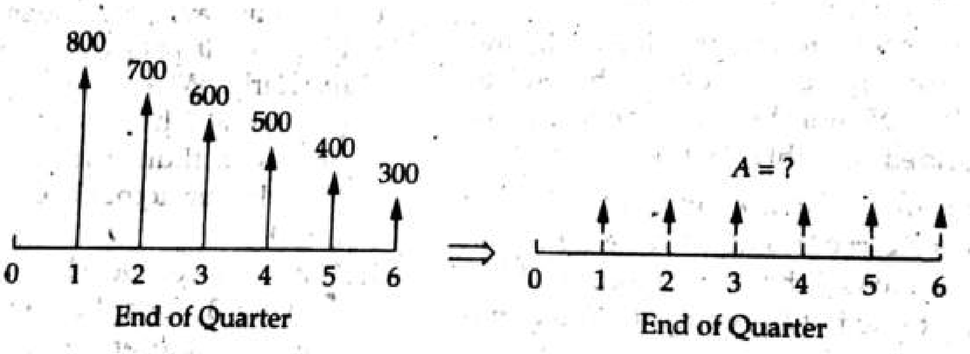
If you invest $5,000 in this certificate at the beginning of year 1 and do not add or withdraw any money for 5 years, what is the value of the certificate at the end of the fifth year? (3.19)

3-94. Determine the present equivalent value of the cash flow diagram at the bottom of this page when the annual interest rate, i_k, varies as indicated. (3.19)

3-95. What is the value of F_4 in the cash flow diagram at the top of page 129? (3.19)

3-96. Indicate whether the following statements are true (T) or False (F).

a. T F Interest is money paid for the use of equity capital.
b. T F $(A/F, i\%, N) = (A/P, i\%, N) + i$
c. T F Simple interest ignores the time value of money principle.
d. T F Cash flow diagrams are analagous to free body diagrams for mechanics problems.



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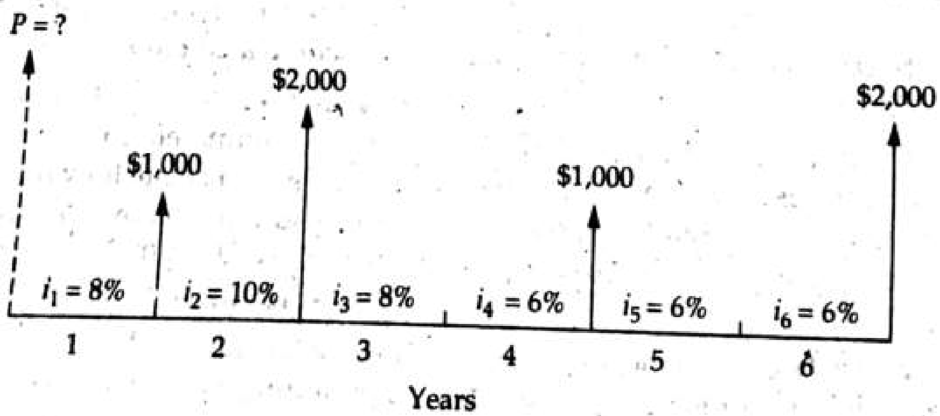

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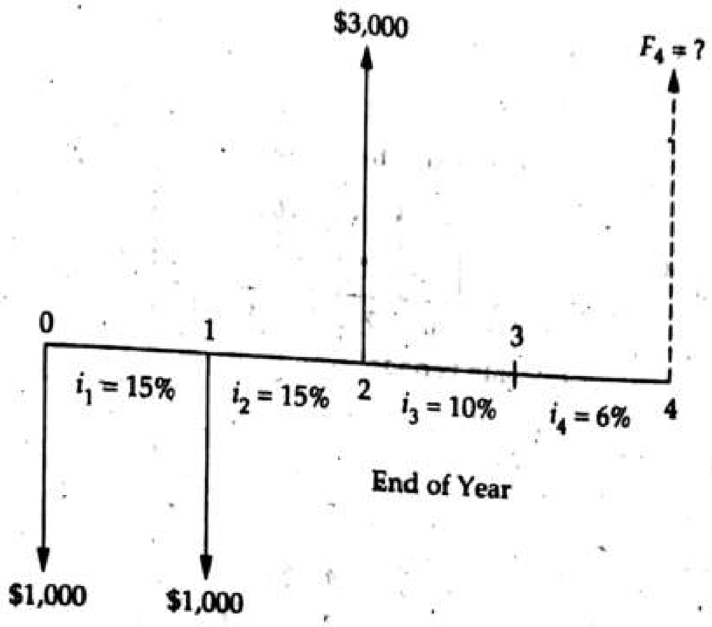


(Pr. 3-95)

e. T F $1,791 10 years from now is equivalent to $900 now if the interest rate equals 8% per year.

f. T F It is always true that $i > r$ when $M \geq 2$.

g. T F Suppose that a lump-sum of $1,000 is invested at $\underline{r} = 15\%$ for 8 years. The future worth is greater for daily compounding than it is for continuous compounding.

h. T F For a fixed amount, F dollars, that is received at EOY N, the "A equivalent" increases as the interest rate increases.

i. T F For a specified value of F at EOY N, P at time 0 will be larger for $\underline{r} = 10\%$ per year than it will be for $r = 10\%$ per year, compounded monthly.

j. T F *Profit* may be broadly defined as money paid to the owners of equity capital.

3-97. If a nominal interest rate of 8% is compounded continuously, determine the unknown quantity in each of the following situations. (3.20)

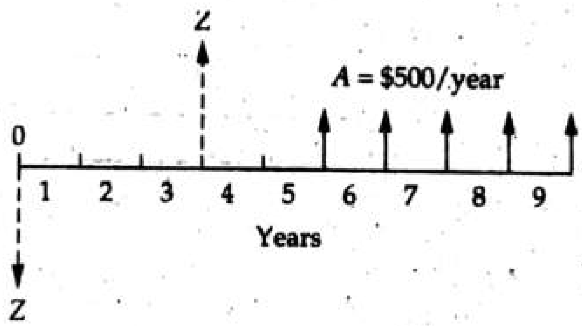
a. What uniform end-of-year amount for 10 years is equivalent to $8,000 at the end of year 10?

b. What is the present equivalent value of $1,000 per year for 12 years?

c. What is the future worth at the end of the sixth year of $243 payments every 6 months during the 6 years? The first payment occurs 6 months from the present and the last occurs at the end of the sixth year.

d. Find the equivalent lump-sum amount at the end of year 9 when $P_0 = \$1,000$ and a nominal interest rate of 8% is compounded continuously.

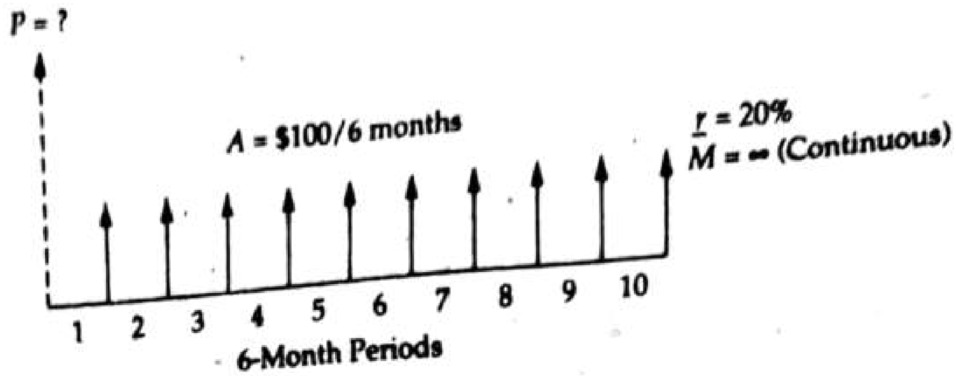
3-98. Find the value of the unknown quantity Z in the following diagram when $\underline{r} = 10\%$ compounded continuously. (3.20)



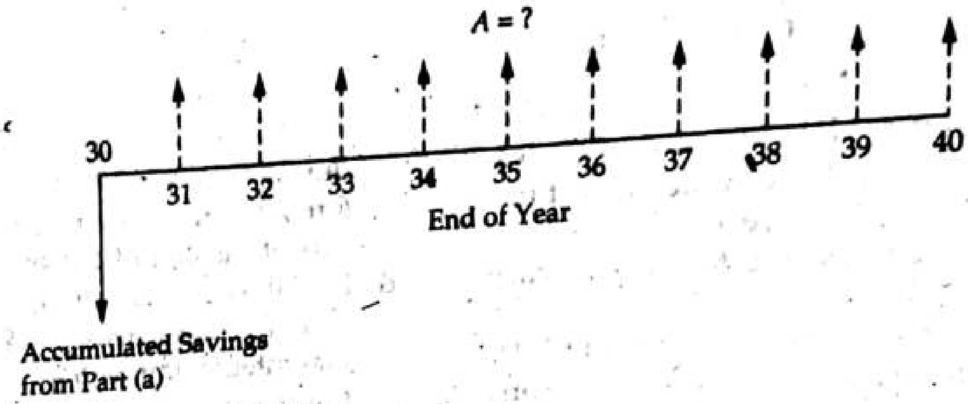
3-99. A man deposited $2,000 in a savings account when his son was born. The nominal interest rate was 8% per year, compounded continuously. On the son's eighteenth birthday, the accumulated sum is withdrawn from the account. How much will this accumulated amount be? (3.20)

3-100. Find the value of P in the cash flow diagram on the top of page 130. (3.20)

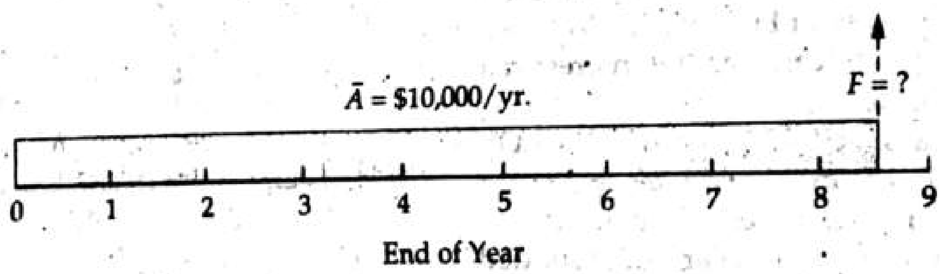
3-101. Your rich uncle has just offered to make you wealthy! For every dollar you save in an insured, continuously compounded, bank



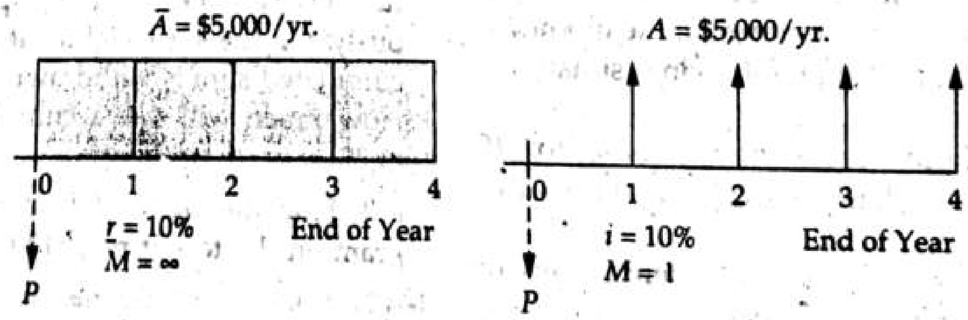
(Pr. 3-100)



(Pr. 3-104)



(Pr. 3-105)



(Pr. 3-108)

account during the next 10 years, he will give you a dollar to match it. Because your modest income permits you to save $3,000 per year for each of the next 10 years, your uncle will be willing to give you $30,000 at the end of the tenth year. If you desire a *total* of $70,000 10 years from now, what annual interest rate would you have to earn on your insured bank account to make your goal possible?

3-102. A person needs $12,000 immediately as a down payment on a new home. Suppose that he can borrow this money from his company credit union. He will be required to repay the loan in equal payments *made every 6 months* over the next 8 years. The annual interest rate being charged is 10% compounded continuously. What is the amount of each payment? (3.20)

3-103.
a. What is the present worth of a uniform series of annual payments of $3,500 each for 5 years if the interest rate, compounded continuously, is 10%?
b. The amount of $7,000 is invested in a certificate of deposit (CD) and will be worth $16,000 in 9 years. What is the continuously compounded nominal interest rate for this CD? (3.20)

3-104.
a. Many persons prepare for retirement by making monthly contributions to a savings program. Suppose that $100 is set aside each month and invested in a savings account that pays 12% interest each year, compounded continuously. Determine the accumulated savings in this account at the end of 30 years.
b. In part (a), suppose that an annuity will be withdrawn from savings that have been accumulated at the end of year 30. The annuity will extend from the end of year 31 to the end of year 40. What is the value of this annuity if the interest rate and compounding frequency in part (a) do not change? Refer to the diagram on page 130. (3.20)

3-105.
a. What is the future worth of a continuous funds flow amounting to $10,500 per year when $\underline{r}$ = 15%, $M = \infty$, and N = 12 years?
b. If the nominal interest rate is 10% per year, continuously compounded, what is the future value of $10,000 per year flowing continuously for 8.5 years? See the cash flow diagram on page 130.
c. Let $\bar{A}$ = $7,859 per year with $\underline{r}$ = 15%, $M = \infty$. How many years will it take to have $1 million in this account? (3.21)

3-106. For how many years must an investment of $63,000 provide a continuous flow of funds at the rate of $16,000 per year so that a nominal interest rate of 10%, continuously compounded, will be earned? (3.21)

3-107. What is the present value of the following continuous funds flow situations?
a. $1,000,000 per year for 4 years at 10% compounded continuously.
b. $6,000 per year for 10 years at 8% compounded annually.
c. $500 per quarter for 6.75 years at 12% compounded continuously. (3.21)

3-108. What is the difference in present equivalent worths for the two cash flow diagrams shown at the bottom of page 130? (3.21)