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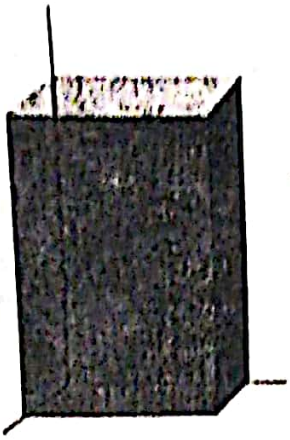
CHAPTER 6



REAL VECTOR SPACES

6.1 VECTOR SPACES

We have already defined R^n and examined some of its basic properties in Theorem 4.2. We must now study the fundamental structure of R^n. In many applications in mathematics, the sciences, and engineering, the notion of a vector space arises. This idea is merely a carefully constructed generalization of R^n. In studying the properties and structure of a vector space, we can study not only R^n, in particular, but many other important vector spaces. In this section we define the notion of a vector space in general and in later sections we study their structure.

DEFINITION 1*

vector space:-

The collection of vector is called vector space

A real vector space is a set of elements V together with two operations $\oplus$ and $\odot$ satisfying the following properties:

(α) If **u** and **v** are any elements of V, then $\mathbf{u} \oplus \mathbf{v}$ is in V (i.e., V is closed under the operation $\oplus$).

(a) $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$, for **u** and **v** in V.

(b) $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$, for **u**, **v**, and **w** in V.

(c) There is an element **0** in V such that

$$\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}, \quad \text{for all } \mathbf{u} \text{ in } V.$$

(d) For each **u** in V, there is an element $-\mathbf{u}$ in V such that

$$\mathbf{u} \oplus -\mathbf{u} = \mathbf{0}.$$

(β) If **u** is any element of V and c is any real number, then $c \odot \mathbf{u}$ is in V (i.e., V is closed under the operation $\odot$).

(e) $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$, for all real numbers c and all **u** and **v** in V.

(f) $(c + d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$, for all real numbers c and d, and all **u** in V.

(g) $c \odot (d \odot \mathbf{u}) = (cd) \odot \mathbf{u}$, for all real numbers c and d and all **u** in V.

(h) $1 \odot \mathbf{u} = \mathbf{u}$, for all **u** in V.

The elements of V are called **vectors**; the real numbers are called **scalars**. The operation $\oplus$ is called **vector addition**; the operation $\odot$ is called **scalar**

*Although the definitions in this book are not numbered, *this* definition is numbered because it will be referred to a number of times in this chapter.

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multiplication. The vector **0** in property (c) is called a **zero vector.** The vector $-\mathbf{u}$ in property (d) is called a **negative** of $\mathbf{u}$. It can be shown (see Exercises T.5 and T.6) that the vectors **0** and $-\mathbf{u}$ are unique.

Property (α) is called the **closure property** for $\oplus$ and property (β) is called the **closure property** for $\odot$. We also say that V is **closed** under the operations of vector addition, $\oplus$, and scalar multiplication, $\odot$.

If we allow the scalars in Definition 1 to be complex numbers, we obtain a **complex vector space.** More generally, the scalars can be members of a field F,[†] and we obtain a vector space over F. Such spaces are important in many applications in mathematics and the physical sciences. We provide a brief introduction to complex vector spaces in Appendix A. Although most of our attention in this book will be focused on real vector spaces, we now take a brief look at a vector space over the field consisting of the bits 0 and 1, with the operations of binary addition and binary multiplication. In this case, we take the set of vectors V to be B^n, the set of bit n-vectors. With the operations of addition of bit n-vectors using binary addition and scalar multiplication with bit scalars, all the properties listed in Definition 1 are valid. [Note that in Section 1.4 we remarked that Theorem 1.1 and Theorem 1.3 (a)–(c) are valid for bit matrices, hence for B^n; properties (α), (β), and (h) of Definition 1 are also valid.] (See Exercises T.7–T.9.) Hence B^n is a vector space.

EXAMPLE 1 Consider the set R^n together with the operations of vector addition and scalar multiplication as defined in Section 4.2. Theorem 4.2 in Section 4.2 established the fact that R^n is a vector space under the operations of addition and scalar multiplication of n-vectors. ■

EXAMPLE 2 Consider the set V of all ordered triples of real numbers of the form $(x, y, 0)$ and define the operations $\oplus$ and $\odot$ by

$$(x, y, 0) \oplus (x', y', 0) = (x + x', y + y', 0)$$
$$c \odot (x, y, 0) = (cx, cy, 0).$$

It is then not difficult to show (Exercise 7) that V is a vector space, since it satisfies all the properties of Definition 1. ■

EXAMPLE 3 Consider the set V of all ordered triples of real numbers (x, y, z) and define the operations $\oplus$ and $\odot$ by

$$(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$$
$$c \odot (x, y, z) = (cx, y, z).$$

It is then easy to verify (Exercise 8) that properties (α), (β), (a), (b), (c), (d), and (e) of Definition 1 hold. Here $\mathbf{0} = (0, 0, 0)$ and the negative of the vector (x, y, z) is the vector $(-x, -y, -z)$. For example, to verify property (e) we proceed as follows. First,

$$c \odot [(x, y, z) \oplus (x', y', z')] = c \odot (x + x', y + y', z + z')$$
$$= (c(x + x'), y + y', z + z').$$

[†]A field is an algebraic structure enjoying the arithmetic properties shared by the real, complex, and rational numbers. Fields are studied in detail in an abstract algebra course.

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Also,

$$\begin{aligned} c \odot (x, y, z) \oplus c \odot (x', y', z') &= (cx, y, z) \oplus (cx', y', z') \\ &= (cx + cx', y + y', z + z') \\ &= (c(x + x'), y + y', z + z'). \end{aligned}$$

However, we now show that property (f) fails to hold. Thus

$$(c + d) \odot (x, y, z) = ((c + d)x, y, z).$$

On the other hand,

$$\begin{aligned} c \odot (x, y, z) \oplus d \odot (x, y, z) &= (cx, y, z) \oplus (dx, y, z) \\ &= (cx + dx, y + y, z + z) \\ &= ((c + d)x, 2y, 2z). \end{aligned}$$

Thus V is not a vector space under the prescribed operations. Incidentally, properties (g) and (h) *do* hold for this example. ■

EXAMPLE 4 Consider the set M_{23} of all 2×3 matrices under the usual operations of matrix addition and scalar multiplication. In Section 1.4 (Theorems 1.1 and 1.3) we have established that the properties in Definition 1 hold, thereby making M_{23} into a vector space. Similarly, the set of all $m \times n$ matrices under the usual operations of matrix addition and scalar multiplication is a vector space. This vector space will be denoted by M_{mn}. ■

EXAMPLE 5 Let $F[a, b]$ be the set of all real-valued functions that are defined on the interval $[a, b]$. If f and g are in V, we define $f \oplus g$ by

$$(f \oplus g)(t) = f(t) + g(t).$$

If f is in $F[a, b]$ and c is a scalar, we define $c \odot f$ by

$$(c \odot f)(t) = cf(t).$$

Then $F[a, b]$ is a vector space (Exercise 9). Similarly, the set of all real-valued functions defined for all real numbers denoted by $F(-\infty, \infty)$ is a vector space. ■

Another source of examples of vector spaces will be sets of polynomials; therefore, we recall some well-known facts about such functions. A *polynomial* (in t) is a function that is expressible as

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0, \tag{1}$$

where n is an integer ≥ 0 and the coefficients $a_0, a_1, \ldots, a_n$ are real numbers.

EXAMPLE 6 The following functions are polynomials:

$$\begin{aligned} p_1(t) &= 3t^4 - 2t^2 + 5t - 1 \\ p_2(t) &= 2t + 1 \\ p_3(t) &= 4. \end{aligned}$$

The following functions are not polynomials (explain why):

$$f_4(t) = 2\sqrt{t} - 6 \quad \text{and} \quad f_5(t) = \frac{1}{t^2} - 2t + 1.$$ ■

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The polynomial $p(t)$ in (1) is said to have **degree** n if $a_n \neq 0$. Thus the degree of a polynomial is the highest power having a nonzero coefficient.

EXAMPLE 7 The polynomials defined in Example 6 have the following degrees:

$$p_1(t)\text{: degree 4}$$
$$p_2(t)\text{: degree 1}$$
$$p_3(t)\text{: degree 0.} \quad \blacksquare$$

The **zero polynomial** is defined as

$$0t^n + 0t^{n-1} + \cdots + 0t + 0.$$

Note that, by definition, the zero polynomial has no degree.

We now let P_n be the set of all polynomials of degree $\leq n$ together with the zero polynomial. Thus $2t^2 - 3t + 5$, $2t + 1$, and 1 are a members of P_2.

EXAMPLE 8 If

$$p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

and

$$q(t) = b_n t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0,$$

we define $p(t) \oplus q(t)$ as

$$p(t) \oplus q(t) = (a_n + b_n)t^n + (a_{n-1} + b_{n-1})t^{n-1} + \cdots + (a_1 + b_1)t + (a_0 + b_0)$$

(i.e., add coefficients of like-power terms). If c is a scalar, we also define $c \odot p(t)$ as

$$c \odot p(t) = (ca_n)t^n + (ca_{n-1})t^{n-1} + \cdots + (ca_1)t + (ca_0)$$

(i.e., multiply each coefficient by c). We now show that P_n is a vector space.

Let $p(t)$ and $q(t)$ defined previously be elements of P_n; that is, they are polynomials of degree $\leq n$ or the zero polynomial. Then the preceding definitions of the operations $\oplus$ and $\odot$ show that $p(t) \oplus q(t)$ and $c \odot p(t)$, for any scalar c, are polynomials of degree $\leq n$ or the zero polynomial. That is, $p(t) \oplus q(t)$ and $c \odot p(t)$ are in P_n so that (α) and (β) in Definition 1 hold. To verify property (a), we observe that

$$q(t) \oplus p(t) = (b_n + a_n)t^n + (b_{n-1} + a_{n-1})t^{n-1} + \cdots + (b_1 + a_1)t + (a_0 + b_0),$$

and since $a_i + b_i = b_i + a_i$ holds for the real numbers, we conclude that $p(t) \oplus q(t) = q(t) \oplus p(t)$. Similarly, we verify property (b). The zero polynomial is the element $\mathbf{0}$ needed in property (c). If $p(t)$ is as given previously, then its negative, $-p(t)$, is

$$-a_n t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0.$$

We shall now verify property (f) and leave the verification of the remaining properties to the reader. Thus

$$\begin{aligned}
(c + d) \odot p(t) &= (c + d)a_n t^n + (c + d)a_{n-1}t^{n-1} + \cdots + (c + d)a_1 t \\
&\quad + (c + d)a_0 \\
&= ca_n t^n + da_n t^n + ca_{n-1}t^{n-1} + da_{n-1}t^{n-1} + \cdots + ca_1 t \\
&\quad + da_1 t + ca_0 + da_0 \\
&= c(a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0) \\
&\quad + d(a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0) \\
&= c \odot p(t) \oplus d \odot p(t).
\end{aligned} \quad \blacksquare$$

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6.1 Exercises

In Exercises 1 through 4, determine whether the given set V is closed under the operations ⊕ and ⊙.

1. V is the set of all ordered pairs of real numbers (x, y), where $x > 0$ and $y > 0$;
$$(x, y) \oplus (x', y') = (x + x', y + y')$$
and
$$c \odot (x, y) = (cx, cy).$$

2. V is the set of all ordered triples of real numbers of the form $(0, y, z)$;
$$(0, y, z) \oplus (0, y', z') = (0, y + y', z + z')$$
and
$$c \odot (0, y, z) = (0, 0, cz).$$

3. V is the set of all polynomials of the form $at^2 + bt + c$, where a, b, and c are real numbers with $b = a + 1$;
$$(a_1t^2 + b_1t + c_1) \oplus (a_2t^2 + b_2t + c_2) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$$
and
$$r \odot (at^2 + bt + c) = (ra)t^2 + (rb)t + rc.$$

4. V is the set of all 2×2 matrices
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
where $a = d$; ⊕ is matrix addition and ⊙ is scalar multiplication.

5. Verify in detail that R^2 is a vector space.

6. Verify in detail that R^3 is a vector space.

7. Verify that the set in Example 2 is a vector space.

8. Verify that all the properties of Definition 1, except property (f), hold for the set in Example 3.

9. Show that the set in Example 5 is a vector space.

10. Show that the space P of all polynomials is a vector space.

In Exercises 11 through 17, determine whether the given set together with the given operations is a vector space. If it is not a vector space, list the properties of Definition 1 that fail to hold.

11. The set of all ordered triples of real numbers (x, y, z) with the operations
$$(x, y, z) \oplus (x', y', z') = (x', y + y', z')$$
and
$$c \odot (x, y, z) = (cx, cy, cz)$$

12. The set of all ordered triples of real numbers (x, y, z) with the operations
$$(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$$
and
$$c \odot (x, y, z) = (x, 1, z)$$

13. The set of all ordered triples of real numbers of the form $(0, 0, z)$ with the operations
$$(0, 0, z) \oplus (0, 0, z') = (0, 0, z + z')$$
and
$$c \odot (0, 0, z) = (0, 0, cz)$$

14. The set of all real numbers with the usual operations of addition and multiplication

15. The set of all ordered pairs of real numbers (x, y), where $x \leq 0$, with the usual operations in R^2

16. The set of all ordered pairs of real numbers (x, y) with the operations $(x, y) \oplus (x', y') = (x + x', y + y')$ and $c \odot (x, y) = (0, 0)$

17. The set of all positive real numbers $\mathbf{u}$ with the operations $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ and $c \odot \mathbf{u} = \mathbf{u}^c$

18. Let V be the set of all real numbers; define ⊕ by $\mathbf{u} \oplus \mathbf{v} = 2\mathbf{u} - \mathbf{v}$ and ⊙ by $c \odot \mathbf{u} = c\mathbf{u}$. Is V a vector space?

19. Let V be the set consisting of a single element $\mathbf{0}$. Let $\mathbf{0} \oplus \mathbf{0} = \mathbf{0}$ and $c \odot \mathbf{0} = \mathbf{0}$. Show that V is a vector space.

20. (a) If V is a vector space that has a nonzero vector, how many vectors are in V?

(b) Describe all vector spaces having a finite number of vectors.

Theoretical Exercises

In Exercises T.1 through T.4, establish the indicated result for a real vector space V.

T.1. Show that $c\mathbf{0} = \mathbf{0}$ for every scalar c.

T.2. Show that $-(-\mathbf{u}) = \mathbf{u}$.

T.3. Show that if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

T.4. Show that if $\mathbf{u} \neq \mathbf{0}$ and $a\mathbf{u} = b\mathbf{u}$, then $a = b$.

T.5. Show that a vector space has only one zero vector.

T.6. Show that a vector $\mathbf{u}$ in a vector space has only one negative $-\mathbf{u}$.

T.7. Show that B^n is closed under binary addition of bit n-vectors.

T.8. Show that B^n is closed under scalar multiplication by bits 0 and 1.

T.9. Show that property (h) is valid for all vectors in B^n.

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Exercise 6.1

Q1. $V = \{(x, y) \mid x > 0 \text{ and } y > 0\}$ with the operations defined by $(x, y) \oplus (x', y') = (x + x', y + y')$ and $c \odot (x, y) = (cx, cy)$

clearly V is closed under $\oplus$, as (x, y) and $(x', y') \in V \Rightarrow$ $x, x', y, y' > 0 \Rightarrow x + x' > 0$ and $y + y' > 0 \Rightarrow (x + x', y + y') \in V$.

But for $c \in \mathbb{R}$ and $(x, y) \in V \Rightarrow x, y > 0 \Rightarrow cx \leq 0$ and $cy \leq 0$ for $c \leq 0$

$\Rightarrow c \odot (x, y) = (cx, cy) \notin V \Rightarrow$ V is not closed under $\odot$.

Q2. $V = \{(0, y, z) \mid y, z \in \mathbb{R}\}$ with the operations defined by $(0, y, z) \oplus (0, y', z') = (0, y + y', z + z')$ and $c \odot (0, y, z) = (0, 0, cz)$

V is closed under $\oplus$, as $(0, y + y', z + z') \in V$

Also V is closed under $\odot$, as $c \odot (0, y, z) = (0, 0, cz) \in V$,

$\Rightarrow c \in \mathbb{R}$ and $z \in \mathbb{R} \Rightarrow cz \in \mathbb{R}$.

Q3 $V = \{at^2 + bt + c \mid a, b, c \in \mathbb{R} \text{ with } b = a + 1\}$ with the operations

$u \oplus v = (a_1t^2 + b_1t + c_1) \oplus (a_2t^2 + b_2t + c_2) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$

and $r \odot (at^2 + bt + c) = (ra)t^2 + (rb)t + (rc)$

V is not closed under $\oplus$, as for $u = a_1t^2 + b_1t + c_1$, $v = a_2t^2 + a_2t + c_2 \in V$

$\Rightarrow b_1 = a_1 + 1$ and $b_2 = a_2 + 1$ but $b_1 + b_2 = (a_1 + 1) + (a_2 + 1) = (a_1 + a_2) + 2$

$\neq (a_1 + a_2) + 1$.

Also V is not closed under $\odot$ as for $r \in \mathbb{R}$ and $u = at^2 + bt + c$ $\in V \Rightarrow$ $b = a + 1$ but $rb = r(a + 1) = ra + r \neq ra + 1$.

Hence V is neither closed under $\oplus$ nor closed under $\odot$.

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Exercise 6.1

Q5 — Q10 : By examples mentioned on pages 2, 3, 4

$\boxed{Q_{11} \text{ ——— } Q_{17}}$ Q12: The set of all ordered triples of real numbers (x, y, z) with the operations $(x, y, z) \oplus (x', y', z') = (x+x', y+y', z+z')$ and $c \odot (x, y, z) = ~~\text{.....}~~ (x, 1, z)$

③ $V = \{ (x, y, z) / x, y, z \in \mathbb{R} \}$ with the operations mentioned above.

To determine whether V under the given operations is a vector space, we need to verify all the ten conditions of definition.

(α) For any $u = (x, y, z)$ and $v = (x', y', z') \in V$, we have

$u \oplus v = (x+x', y+y', z+z') \in V$ as $x, y, z, x', y', z' \in \mathbb{R} \Rightarrow x+x', y+y', z+z' \in \mathbb{R}$

$\Rightarrow$ (α) is satisfied.

(a) For any $u = (x, y, z), v = (x', y', z') \in V$, we have

$u \oplus v = (x+x', y+y', z+z') = (x'+x, y'+y, z'+z)$

$= (x', y', z') \oplus (x, y, z) = v \oplus u \Rightarrow$ (a) is satisfied

(b) For any $u = (x, y, z), v = (x', y', z')$ and $w = (x'', y'', z'') \in V$,

we have $(u \oplus v) \oplus w = (x+x', y+y', z+z') \oplus (x'', y'', z'')$

$= ((x+x')+x'', (y+y')+y'', (z+z')+z'')$

$= (x+(x'+x''), y+(y'+y''), z+(z'+z''))$

$= (x, y, z) \oplus (x'+x'', y'+y'', z'+z'')$

$= (x, y, z) \oplus ((x', y', z') \oplus (x'', y'', z''))$

$= u \oplus (v \oplus w) \Rightarrow$ (b) is satisfied.

(c) For any $u = (x, y, z)$ in V, there exists $0 = (0, 0, 0)$ in V such that $u \oplus 0 = (x, y, z) \oplus (0, 0, 0) = (x+0, y+0, z+0)$

$= (x, y, z) = u$. Similarly $0 \oplus u = u$

$\Rightarrow$ (c) is satisfied i.e the identity w.r.t $\oplus$ exists in V.

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Exercise 6.1

(d) For any $u = (x, y, z)$ in V, there exists $-u = (-x, -y, -z)$ in V such that $u \oplus (-u) = (x-x, y-y, z-z) = (0, 0, 0) = 0 \in V$

Similarly $-u \oplus u = 0 \Rightarrow$ (d) is satisfied

(β) For any $c \in \mathbb{R}$ and any $u = (x, y, z)$ in V, we have $c \odot u = c \odot (x, y, z) = (x, 1, y) \in V \Rightarrow$ (β) is also satisfied.

(e) For any $c \in \mathbb{R}$ and any $u = (x, y, z), v = (x', y', z') \in V$,

$c \odot (u \oplus v) = c \odot (x + x', y + y', z + z') = (x + x', 1, z + z')$

Also $c \odot u \oplus c \odot v = (x, 1, z) \oplus (x', 1, z') = (x + x', 2, z + z')$

$\Rightarrow c \odot (u \oplus v) \neq c \odot u \oplus c \odot v \Rightarrow$ (e) is not satisfied

Hence V is not a vector space

(f) For any $c, d \in \mathbb{R}$ and $u = (x, y, z) \in V$, we have

$(c+d) \odot u = (c+d) \odot (x, y, z) = (x, 1, z)$

but $c \odot u \oplus d \odot u = c \odot (x, y, z) \oplus d \odot (x, y, z)$

$= (x, 1, z) \oplus (x, 1, z)$

$= (2x, 2, 2z)$

$\Rightarrow (c+d) \odot u \neq c \odot u \oplus d \odot u \Rightarrow$ (f) is not satisfied.

(g) For any $c, d \in \mathbb{R}$ and any $u = (x, y, z) \in V$, we have

$c \odot (d \odot u) = c \odot (x, 1, z) = (x, 1, z)$

Also $(cd) \odot u = (x, 1, z) \Rightarrow c \odot (d \odot u) = (cd) \odot u$

$\Rightarrow$ (g) is satisfied.

(h) For any $u = (x, y, z)$ in V, we have

$1 \odot u = 1 \odot (x, y, z) = (x, 1, z) \neq u = (x, y, z) \Rightarrow$ (h) is not satisfied. Hence the list contains the properties that do not satisfy, is as follows

$\{ (e), (f), (h) \}$

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Exercise 6.1

Q16. The set of all ordered pairs of real numbers (x,y) with the operations $(x,y)\oplus(x',y') = (x+x', y+y')$ and $c\odot(x,y)=(0,0)$

(5) Let $V=\{(x,y) / x,y \in \mathbb{R}\}$. V is a vector space with the operations given above if 10 conditions of definition are satisfied.

(α) For any $u=(x,y)$ and $v=(x',y')$ in V, we have $u\oplus v = (x+x', y+y')$ is in V, as $x,x' \in \mathbb{R} \Rightarrow x+x' \in \mathbb{R}$ and $y,y' \in \mathbb{R}$ $\Rightarrow y+y' \in \mathbb{R} \Rightarrow (x+x', y+y') \in V \Rightarrow (\alpha)$ is satisfied.

(a) For any $u=(x,y)$, $v=(x',y') \in V \Rightarrow u\oplus v = (x+x', y+y')$ $= (x'+x, y'+y) = (x',y')\oplus(x,y) = v\oplus u$ i.e $u\oplus v = v\oplus u \Rightarrow$ (a) is satisfied.

(b) For any $u=(x,y)$, $v=(x',y')$ and $w=(x'',y'')$ in V, we have,
$(u\oplus v)\oplus(w) = (x+x', y+y')\oplus(x'',y'') = \left((x+x')+x'', (y+y')+y''\right)$
$= \left(x+(x'+x''), y+(y'+y'')\right) = (x,y)\oplus\left((x',y')\oplus(x'',y'')\right)$
$= u\oplus(v\oplus w)$ i.e $(u\oplus v)\oplus w = u\oplus(v\oplus w) \Rightarrow$ (b) is satisfied.

(c) For any $u=(x,y)$ in V, There exists $0=(0,0)$ in V such that $u\oplus 0 = (x,y)\oplus(0,0) = (x+0, y+0) = (x,y) = u$
Similarly $0\oplus u = u \Rightarrow$ (c) is satisfied.

(d) For any $u=(x,y)$ in V, there exists $-u=(-x,-y)$ in V such that $u\oplus(-u) = (x,y)\oplus(-x,-y) = (x-x, y-y) = (0,0) = 0$
$\Rightarrow u\oplus(-u) = 0$. Similarly $-u\oplus u = 0 \Rightarrow$ (d) is satisfied.

(β) For any $c\in\mathbb{R}$ and any $u=(x,y)$ in V, we have $c\odot u = c\odot(x,y) = (0,0)$ $\in V$ i.e V is closed under $\odot \Rightarrow (\beta)$ is satisfied.

(e) For any $c\in\mathbb{R}$ and $u=(x,y)$, $v=(x',y')$ in V, we have
$c\odot(u\oplus v) = c\odot(x+x', y+y') = (0,0)$. Also $c\odot u\oplus c\odot v =$
$(0,0)\oplus(0,0) = (0,0) \Rightarrow c\odot(u\oplus v) = c\odot u + c\odot v \Rightarrow$
(e) is satisfied.

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Exercise 6.1

(f) For any $c, d \in \mathbb{R}$ and any $u = (x, y) \in V$, we have
$(c+d) \odot u = (c+d) \odot (x, y) = (0, 0)$. Also
$c \odot u \oplus d \odot u = (0,0) \oplus (0,0) = (0,0)$
$\Rightarrow (c+d) \odot u = c \odot u \oplus d \odot u \Rightarrow$ (f) is satisfied.

(g) For any $c, d \in \mathbb{R}$ and any $u = (x, y)$ in V, we have
$c \odot (d \odot u) = c \odot (0,0) = (0,0)$. Also $(cd) \odot u = (0,0)$
$\Rightarrow c \odot (d \odot u) = (cd) \odot u \Rightarrow$ (g) is satisfied.

(h) For any $u = (x, y)$ and $1 \in \mathbb{R}$, we have
$1 \odot u = 1 \odot (x, y) = (0,0) \neq u \Rightarrow$ (h) is not satisfied. Hence V is not a vector space.
The list contains the properties that do not satisfy is as

$\{ (h) \}$.

Similarly, we can do the rest of the questions.

Subspace: Let V be a vector space and W be a nonempty subset of V. If W is a vector space with respect to the operations in V, then W is called a subspace of V.

$W \subseteq V$
W satisfies all the 10 conditions w.r.t $\oplus$ and $\odot$, being the operations of V.

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Theorem 6.2: Let V be a vector space with operations $\oplus$ and $\odot$ and let W be a nonempty subset of V.
Then W is a subspace of V if and only if the following conditions hold

(α) If u and v are any vectors in W, then $u \oplus v$ is in W
(β) If c is any real number and u is any vector in W, then $c \odot u$ is in W.

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(11)

Thus let

$$\mathbf{w}_1 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{w}_2 = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$$

be vectors in W. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2) = (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2,$$

which is in W. Also, if c is a scalar, then

$$c\mathbf{w}_1 = (ca_1)\mathbf{v}_1 + (ca_2)\mathbf{v}_2$$

is in W. Hence W is a subspace of V. ■

The construction carried out in Example 10 for two vectors can easily be performed for more than two vectors. We now give a formal definition.

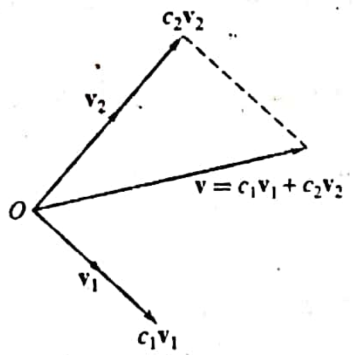
DEFINITION Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V. A vector $\mathbf{v}$ in V is called a **linear combination of** $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

for some real numbers $c_1, c_2, \ldots, c_k$. (See also Section 1.3.)

In Figure 6.4 we show the vector $\mathbf{v}$ in R^2 or R^3 as a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Figure 6.4 ► Linear combination of two vectors



EXAMPLE 11 In R^3 let

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (1, 0, 2), \quad \text{and} \quad \mathbf{v}_3 = (1, 1, 0).$$

The vector

$$\mathbf{v} = (2, 1, 5)$$

is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ if we can find real numbers c_1, c_2, and c_3 so that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}.$$

Substituting for $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we have

$$c_1(1, 2, 1) + c_2(1, 0, 2) + c_3(1, 1, 0) = (2, 1, 5).$$

Combining terms on the left and equating corresponding entries leads to the linear system (verify)

$$\Rightarrow (c_1+c_2+c_3,\ 2c_1+c_3,\ c_1+2c_2) = (2,1,5) \Rightarrow$$

$$\begin{aligned} c_1 + c_2 + c_3 &= 2 \\ 2c_1 + c_3 &= 1 \\ c_1 + 2c_2 &= 5. \end{aligned}$$

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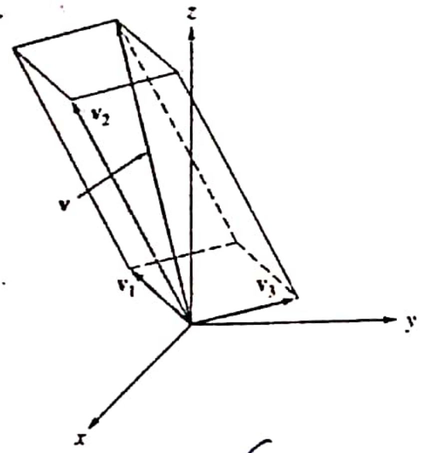
Solving this linear system by the methods of Chapter 1 gives (verify) $c_1 = 1$, $c_2 = 2$, and $c_3 = -1$, which means that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Thus

$$\mathbf{v} = \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3.$$

Figure 6.5 shows $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. ■

Figure 6.5 ▶

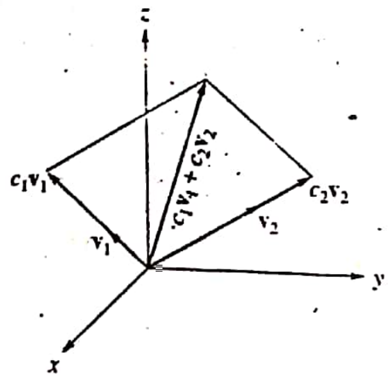


DEFINITION If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the set of all vectors in V that are linear combinations of the vectors in S is denoted by

$$\text{span } S \quad \text{or} \quad \text{span } \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

In Figure 6.6 we show a portion of span $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are noncollinear vectors in R^3; span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane that passes through the origin and contains the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Figure 6.6 ▶



EXAMPLE 12 Consider the set S of 2×3 matrices given by

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Then span S is the set in M_{23} consisting of all vectors of the form

$$a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}, \quad \text{where } a, b, c, \text{ and } d \text{ are real numbers.}$$

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That is, span S is the subset of M_{23} consisting of all matrices of the form

$$\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix},$$

where a, b, c, and d are real numbers. ■

THEOREM 6.3 *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space V. Then span S is a subspace of V.*

Proof See Exercise T.4. ■

EXAMPLE 13 In P_2 let

$$\mathbf{v}_1 = 2t^2 + t + 2, \quad \mathbf{v}_2 = t^2 - 2t, \quad \mathbf{v}_3 = 5t^2 - 5t + 2, \quad \mathbf{v}_4 = -t^2 - 3t - 2.$$

Determine if the vector

$$\mathbf{u} = t^2 + t + 2$$

belongs to span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Solution If we can find scalars c_1, c_2, c_3, and c_4 so that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{u},$$

then $\mathbf{u}$ belongs to span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Substituting for $\mathbf{u}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, we have

$$\begin{aligned} c_1(2t^2 + t + 2) + c_2(t^2 - 2t) + c_3(5t^2 - 5t + 2) + c_4(-t^2 - 3t - 2) \\ = t^2 + t + 2 \end{aligned}$$

or

$$\begin{aligned} (2c_1 + c_2 + 5c_3 - c_4)t^2 + (c_1 - 2c_2 - 5c_3 - 3c_4)t + (2c_1 + 2c_3 - 2c_4) \\ = t^2 + t + 2. \end{aligned}$$

Now two polynomials agree for all values of t only if the coefficients of respective powers of t agree. Thus we get the linear system

$$\begin{aligned} 2c_1 + c_2 + 5c_3 - c_4 &= 1 \\ c_1 - 2c_2 - 5c_3 - 3c_4 &= 1 \\ 2c_1 \qquad + 2c_3 - 2c_4 &= 2. \end{aligned}$$

To investigate whether or not this system of linear equations is consistent, we form the augmented matrix and transform it to reduced row echelon form, obtaining (verify)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

which indicates that the system is inconsistent, that is, it has no solution. Hence $\mathbf{u}$ does not belong to span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. ■

Remark In general, to determine if a specific vector $\mathbf{v}$ belongs to span S, we investigate the consistency of an appropriate linear system.

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This linear combination can be written as the matrix product (verify)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

which has the form of a linear system. Forming the corresponding augmented matrix and transforming it to reduced row echelon form, we obtain (verify)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

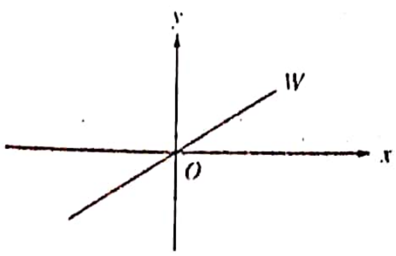
Hence the linear system is consistent with $c_1 = 0$, $c_2 = 1$, and $c_3 = 1$. Thus $\mathbf{u}$ is in span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. ■

Key Terms

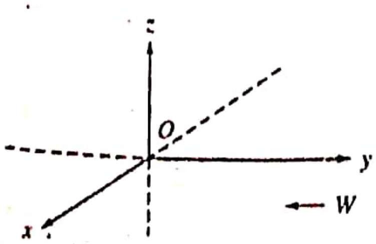
Subspace
Zero subspace
Closure property
Solution space
Linear combination

6.2 Exercises

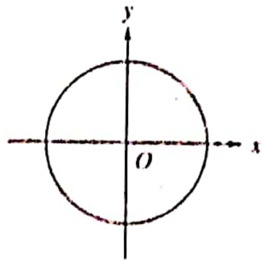
1. The set W consisting of all the points in R^2 of the form (x, x) is a straight line. Is W a subspace of R^2? Explain.



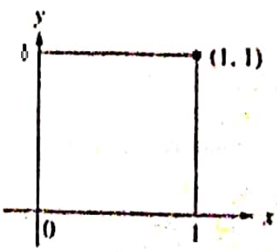
2. Let W be the set of all points in R^3 that lie in the xy-plane. Is W a subspace of R^3? Explain.



3. Consider the circle in the xy-plane centered at the origin whose equation is $x^2 + y^2 = 1$. Let W be the set of all vectors whose tail is at the origin and whose head is a point inside or on the circle. Is W a subspace of R^2? Explain.



4. Consider the unit square shown in the accompanying figure. Let W be the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $0 \le x \le 1$, $0 \le y \le 1$. That is, W is the set of all vectors whose tail is at the origin and whose head is a point inside or on the square. Is W a subspace of R^2? Explain.



5. Which of the following subsets of R^3 are subspaces of R^3? The set of all vectors of the form
(a) $(a, b, 2)$
(b) (a, b, c), where $c = a + b$
(c) (a, b, c), where $c > 0$

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6. Which of the following subsets of R^3 are subspaces of R^3? The set of all vectors of the form

(a) (a, b, c), where $a = c = 0$

(b) (a, b, c), where $a = -c$

(c) (a, b, c), where $b = 2a + 1$

7. Which of the following subsets of R^4 are subspaces of R^4? The set of all vectors of the form

(a) (a, b, c, d), where $a - b = 2$

(b) (a, b, c, d), where $c = a + 2b$ and $d = a - 3b$

(c) (a, b, c, d), where $a = 0$ and $b = -d$

8. Which of the following subsets of R^4 are subspaces of R^4? The set of all vectors of the form

(a) (a, b, c, d), where $a = b = 0$

(b) (a, b, c, d), where $a = 1, b = 0$, and $c + d = 1$

(c) (a, b, c, d), where $a > 0$ and $b < 0$

9. Which of the following subsets of P_2 are subspaces? The set of all polynomials of the form

(a) $a_2t^2 + a_1t + a_0$, where $a_0 = 0$

✓(b) $a_2t^2 + a_1t + a_0$, where $a_0 = 2$

(c) $a_2t^2 + a_1t + a_0$, where $a_2 + a_1 = a_0$

10. Which of the following subsets of P_2 are subspaces? The set of all polynomials of the form

(a) $a_2t^2 + a_1t + a_0$, where $a_1 = 0$ and $a_0 = 0$

(b) $a_2t^2 + a_1t + a_0$, where $a_1 = 2a_0$

(c) $a_2t^2 + a_1t + a_0$, where $a_2 + a_1 + a_0 = 2$

11. (a) Show that P_2 is a subspace of P_3.

(b) Show that P_n is a subspace of P_{n+1}.

12. Show that P_n is a subspace of P.

13. Show that P is a subspace of the vector space defined in Example 5 of Section 6.1.

14. Let $\mathbf{u} = (1, 2, -3)$ and $\mathbf{v} = (-2, 3, 0)$ be two vectors in R^3 and let W be the subset of R^3 consisting of all vectors of the form $a\mathbf{u} + b\mathbf{v}$, where a and b are any real numbers. Give an argument to show that W is a subspace of R^3.

15. Let $\mathbf{u} = (2, 0, 3, -4)$ and $\mathbf{v} = (4, 2, -5, 1)$ be two vectors in R^4 and let W be the subset of R^4 consisting of all vectors of the form $a\mathbf{u} + b\mathbf{v}$, where a and b are any real numbers. Give an argument to show that W is a subspace of R^4.

16. Which of the following subsets of the vector space M_{23} defined in Example 4 of Section 6.1 are subspaces? The set of all matrices of the form

(a) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $b = a + c$

(b) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $c > 0$

(c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = -2c$ and $f = 2e + d$

17. Which of the following subsets of the vector space M_{23} defined in Example 4 of Section 6.1 are subspaces? The set of all matrices of the form

(a) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = 2c + 1$

(b) $\begin{bmatrix} 0 & 1 & a \\ b & c & 0 \end{bmatrix}$

(c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a + c = 0$ and $b + d + f = 0$

18. Which of the following subsets of the vector space M_{nn} are subspaces?

(a) The set of all $n \times n$ symmetric matrices

(b) The set of all $n \times n$ nonsingular matrices

(c) The set of all $n \times n$ diagonal matrices

19. Which of the following subsets of the vector space M_{nn} are subspaces?

(a) The set of all $n \times n$ singular matrices

(b) The set of all $n \times n$ upper triangular matrices

(c) The set of all $n \times n$ matrices whose determinant is 1

20. (*Calculus Required*) Which of the following subsets are subspaces of the vector space $C(-\infty, \infty)$ defined in Example 8?

(a) All nonnegative functions

(b) All constant functions

(c) All functions f such that $f(0) = 0$

(d) All functions f such that $f(0) = 5$

(e) All differentiable functions.

21. (*Calculus Required*) Which of the following subsets of the vector space $C(-\infty, \infty)$ defined in Example 8 are subspaces?

(a) All integrable functions

(b) All bounded functions

(c) All functions that are integrable on $[a, b]$

(d) All functions that are bounded on $[a, b]$

22. (*Calculus Required*) Consider the differential equation

$$y'' - y' + 2y = 0.$$

A solution is a real-valued function f satisfying the equation. Let V be the set of all solutions to the given differential equation; define $\oplus$ and $\odot$ as in Example 5 in Section 6.1. Show that V is a subspace of the vector space of all real-valued functions defined on $(-\infty, \infty)$. (See also Section 9.2.)

23. Determine which of the following subsets of R^2 are subspaces.

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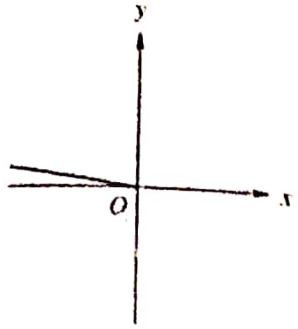
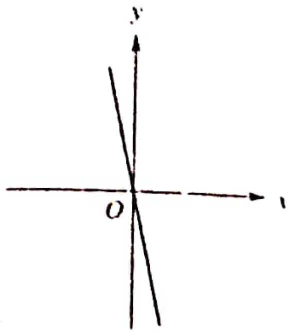
(16)

(a)

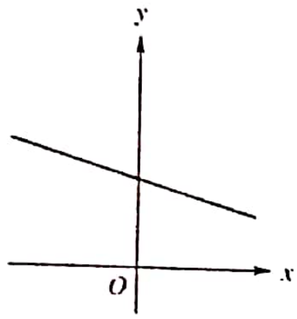


(b)

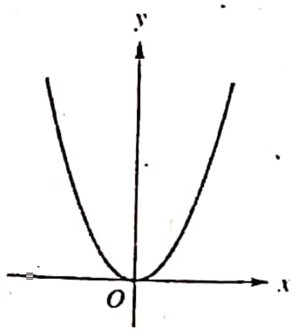


24. Determine which of the following subsets of R^2 are subspaces.

(a)



(b)



25. In each part, determine whether the given vector $\mathbf{v}$ belongs to span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = (1, 0, 0, 1), \qquad \mathbf{v}_2 = (1, -1, 0, 0),$$

and

$$\mathbf{v}_3 = (0, 1, 2, 1).$$

(a) $\mathbf{v} = (-1, 4, 2, 2)$ (b) $\mathbf{v} = (1, 2, 0, 1)$

(c) $\mathbf{v} = (-1, 1, 4, 3)$ (d) $\mathbf{v} = (0, 1, 1, 0)$

26. Which of the following vectors are linear combinations of

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}?$$

(a) $\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & -1 \\ 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

27. In each part, determine whether the given vector $p(t)$ belongs to span $\{p_1(t), p_2(t), p_3(t)\}$, where

$$p_1(t) = t^2 - t,$$
$$p_2(t) = t^2 - 2t + 1,$$
$$p_3(t) = -t^2 + 1.$$

(a) $p(t) = 3t^2 - 3t + 1$ (b) $p(t) = t^2 - t + 1$

(c) $p(t) = t + 1$ (d) $p(t) = 2t^2 - t - 1$

Exercises 28 through 33 use bit matrices.

28. Let $V = B^3$. Determine if

$$W = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a subspace of V.

29. Let $V = B^3$. Determine if

$$W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a subspace of V.

30. Let $V = B^4$. Determine if W, the set of all vectors in V with first entry zero, is a subspace of V.

31. Let $V = B^4$. Determine if W, the set of all vectors in V with second entry one, is a subspace of V.

32. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Determine if $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ belongs to span S.

33. Let

$$S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Determine if $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ belongs to span S.

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(4) (20) (17)

Exercise 6.2

Q10/b. Let $W = \{a_2t^2 + a_1t + a_0 \,/\, a_2, a_1, a_0 \in \mathbb{R} \text{ and } a_1 = 2a_0\} \subset P_2$

where P_2, is a vector space of all polynomials of degree 2 or less than 2 together with zero polynomial.

(α) For any $u = a_2t^2 + a_1t + a_0$ and $v = a_2't^2 + a_1't + a_0'$ in W

$\Rightarrow a_1 = 2a_0$ and $a_1' = 2a_0'$

Now $u \oplus v = (a_2 + a_2')t^2 + (a_1 + a_1')t + (a_0 + a_0') \in W$, as

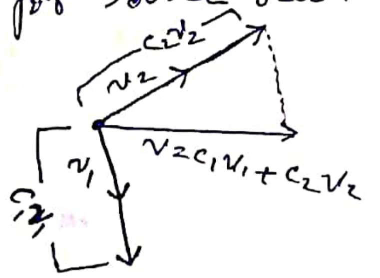
$a_1 + a_1' = 2a_0 + 2a_0' = 2(a_0 + a_0') \Rightarrow (\alpha)$ is satisfied

(β) For any $c \in \mathbb{R}$ and $u = a_2t^2 + a_1t + a_0 \in W$, then $a_1 = 2a_0$

Now $c \odot u = (ca_2)t^2 + (ca_1)t + (ca_0) \in W$, as

$ca_1 = ca_0 \Rightarrow (\beta)$ is satisfied. Hence W is a subspace of P_2.

Linear combination of vectors: Let $v_1, v_2, \dots v_k$ be vectors in a vector space V. A vector v in V is called a linear combination of $v_1, v_2, \dots v_k$ if $v = c_1v_1 + c_2v_2 + \dots + c_kv_k$ for some real numbers $c_1, c_2, \dots c_k$.



see Ex11 (P-283)

P-283 + 284 + 287 + 288 + 289

6.2

Q25 — Q27 on Ex13 (P-283).

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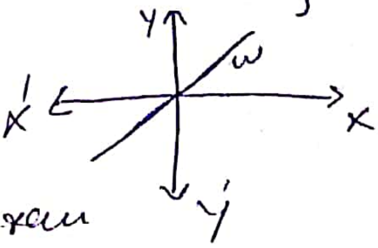
Exercise 6.2

Q1 — Q10: Q.1 : $W = \{(x,x) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$

Clearly W is a nonempty subset of $\mathbb{R}^2$ satisfying (α) and (β) conditions of theorem 6.2 i.e



(α) For any $u=(x,x)$ and $v=(x',x')$ in W, we have
$u \oplus v\ (x+x', x+x') \in W$

(β) For any $c \in \mathbb{R}$ and any $u=(x,x)$ in W, we have
$c \odot u = (cx, cx) \in W$

Q5/(a): Let $W = \{(a,b,2) \mid a,b \in \mathbb{R}\} \subset \mathbb{R}^3$. Then W is a subspace of $\mathbb{R}^3$ if (α) and (β) conditions of Theorem 6.2 are satisfied.

(α) For any $u=(a,b,2)$ and $v=(a',b',2)$ in W, we have
$u \oplus v = (a+a', b+b', 4) \notin W \Rightarrow$ (α) is not satisfied.
Hence W is not a subspace of $\mathbb{R}^3$.

Q5/(b) Let $W = \{(a,b,c) \mid a,b,c \in \mathbb{R}$ and $c = a+b\} \subset \mathbb{R}^3$

(α) For any $u=(a,b,c)$, $v=(a',b',c')$ in W, then $c=a+b$ and $c'=a'+b'$ and we have
$u \oplus v = (a+a', b+b', c+c') = (a+a', b+b', a+b+a'+b')$
$= (a+a', b+b', (a+a')+(b+b')) \in W$, as
$c+c' = (a+a')+(b+b') \Rightarrow$ (α) is satisfied

(β) For any $r \in \mathbb{R}$ and $u=(a,b,c)$ in $W \Rightarrow c=a+b$, we have
$r \odot w = (ra, rb, rc) \in W$, as $rc = r(a+b) = ra+rb \Rightarrow$ (β) is satisfied. Hence W here is a subspace of $\mathbb{R}^3$.

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ML.5. Use MATLAB to determine if $\mathbf{v}$ is a linear combination of the members of set S. If it is, express $\mathbf{v}$ in terms of the members of S.

(a) $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

$$= \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

(b) $S = \{p_1(t), p_2(t), p_3(t)\}$
$= \{2t^2 - t + 1, t^2 - 2, t - 1\}$
$\mathbf{v} = p(t) = 4t^2 + t - 5$

ML.6. In each part, determine whether $\mathbf{v}$ belongs to span S, where

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 1, 0, 1), (1, -1, 0, 1), (0, 1, 2, 1)\}.$$

(a) $\mathbf{v} = (2, 3, 2, 3)$

(b) $\mathbf{v} = (2, -3, -2, 3)$

(c) $\mathbf{v} = (0, 1, 2, 3)$

ML.7. In each part, determine whether $p(t)$ belongs to span S, where

$$S = \{p_1(t), p_2(t), p_3(t)\} = \{t - 1, t + 1, t^2 + t + 1\}.$$

(a) $p(t) = t^2 + 2t + 4$

(b) $p(t) = 2t^2 + t - 2$

(c) $p(t) = -2t^2 + 1$

6.3 LINEAR INDEPENDENCE

Thus far we have defined a mathematical system called a real vector space and noted some of its properties. We further observe that the only real vector space having a finite number of vectors in it is the vector space whose only vector is $\mathbf{0}$, for if $\mathbf{v} \neq \mathbf{0}$ is in a vector space V, then by Exercise T.4 in Section 6.1, $c\mathbf{v} \neq c'\mathbf{v}$, where c and c' are distinct real numbers, and so V has infinitely many vectors in it. However, in this section and the following one we show that most vector spaces V studied here have a set composed of a finite number of vectors that completely describe V; that is, we can write every vector in V as a linear combination of the vectors in this set. It should be noted that, in general, there is more than one such set describing V. We now turn to a formulation of these ideas.

DEFINITION The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to **span** V if every vector in V is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Moreover, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, then we also say that the set S **spans** V, or that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ **spans** V, or that V is **spanned** by S, or in the language of Section 6.2, span $S = V$.

> The procedure to check if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span the vector space V is as follows.
>
> ***Step 1.*** Choose an arbitrary vector $\mathbf{v}$ in V.
>
> ***Step 2.*** Determine if $\mathbf{v}$ is a linear combination of the given vectors. If it is, then the given vectors span V. If it is not, they do not span V.

Again, in Step 2, we investigate the consistency of a linear system, but this time for a right side that represents an arbitrary vector in a vector space V.

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EXAMPLE 1 Let V be the vector space R^3 and let

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (1, 0, 2), \quad \text{and} \quad \mathbf{v}_3 = (1, 1, 0).$$

Do $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span V?

Solution *Step 1.* Let $\mathbf{v} = (a, b, c)$ be any vector in R^3, where a, b, and c are arbitrary real numbers.

Step 2. We must find out whether there are constants c_1, c_2, and c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}.$$

This leads to the linear system (verify)

$$\begin{aligned} c_1 + c_2 + c_3 &= a \\ 2c_1 + c_3 &= b \\ c_1 + 2c_2 &= c. \end{aligned}$$

A solution is (verify)

$$c_1 = \frac{-2a + 2b + c}{3}, \qquad c_2 = \frac{a - b + c}{3}, \qquad c_3 = \frac{4a - b - 2c}{3}.$$

Since we have obtained a solution for every choice of a, b, and c, we conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span R^3. This is equivalent to saying that span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = R^3$. ∎

EXAMPLE 2 Show that

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans the subspace of M_{22} consisting of all symmetric matrices.

Solution *Step 1.* An arbitrary symmetric matrix has the form

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

where a, b, and c are any real numbers.

Step 2. We must find constants d_1, d_2, and d_3 such that

$$d_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

which leads to a linear system whose solution is (verify)

$$d_1 = a, \qquad d_2 = b, \qquad d_3 = c.$$

Since we have found a solution for every choice of a, b, and c, we conclude that S spans the given subspace. ∎

EXAMPLE 3 Let V be the vector space P_2. Let $S = \{p_1(t), p_2(t)\}$, where $p_1(t) = t^2 + 2t + 1$ and $p_2(t) = t^2 + 2$. Does S span P_2?

Solution *Step 1.* Let $p(t) = at^2 + bt + c$ be any polynomial in P_2, where a, b, and c are any real numbers.

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homogeneous system, we find that the reduced row echelon form of the augmented matrix is (verify)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is then given by

$$\begin{aligned} x_1 &= -r - 2s \\ x_2 &= r \\ x_3 &= s \\ x_4 &= s, \end{aligned}$$

where r and s are any real numbers. In matrix form we have that any member of the solution space is given by

$$\mathbf{x} = r\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ span the solution space. ■

LINEAR INDEPENDENCE

DEFINITION The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to be **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}. \tag{1}$$

Otherwise, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are called **linearly independent**. That is, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if whenever $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, we must have

$$c_1 = c_2 = \cdots = c_k = 0.$$

That is, the *only* linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ that yields the zero vector is that in which all the coefficients are zero. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, then we also say that the set S is **linearly dependent** or **linearly independent** if the vectors have the corresponding property defined previously.

It should be emphasized that for any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, Equation (1) always holds if we choose all the scalars $c_1, c_2, \ldots, c_k$ equal to zero. The important point in this definition is whether or not it is possible to satisfy (1) with at least one of the scalars different from zero.

The procedure to determine if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent or linearly independent is as follows.

Step 1. Form Equation (1), which leads to a homogeneous system.

Step 2. If the homogeneous system obtained in Step 1 has only the trivial solution, then the given vectors are linearly independent; if it has a nontrivial solution, then the vectors are linearly dependent.

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EXAMPLE 15 Let V be the vector space B^3 and let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Do $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span V?

Solution Let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be any vector in B^3, where a, b, and c are any of the bits 0 or 1. We must determine if there are scalars c_1, c_2, and c_3 (which are bits 0 or 1) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}.$$

This leads to the linear system

$$\begin{aligned} c_1 + c_2 &= a \\ c_1 + c_3 &= b \\ c_2 + c_3 &= c. \end{aligned}$$

We form the augmented matrix and obtain its reduced row echelon form (verify):

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a+a+b \\ 0 & 1 & 1 & a+b \\ 0 & 0 & 0 & a+b+c \end{array}\right].$$

The system is inconsistent if the choice of bits for a, b, and c are such that $a + b + c \neq 0$. For example, if $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then the system is inconsistent; hence $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span V. ■

EXAMPLE 16 Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in Example 15 linearly independent?

Solution Setting up Equation (1), we are led to the homogeneous linear system

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + c_3 &= 0 \\ c_2 + c_3 &= 0. \end{aligned}$$

The reduced row echelon form of this system is (verify)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $c_1 = -c_3$ and $c_2 = -c_3$, where c_3 is either 0 or 1. Choosing $c_3 = 1$, we find a nontrivial solution $c_1 = c_2 = c_3 = 1$ (verify). Thus $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent. ■

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6.3 Exercises

1. Which of the following vectors span R^2?
   (a) $(1, 2), (-1, 1)$
   (b) $(0, 0), (1, 1), (-2, -2)$
   (c) $(1, 3), (2, -3), (0, 2)$
   (d) $(2, 4), (-1, 2)$

2. Which of the following sets of vectors span R^3?
   (a) $\{(1, -1, 2), (0, 1, 1)\}$
   (b) $\{(1, 2, -1), (6, 3, 0), (4, -1, 2), (2, -5, 4)\}$
   (c) $\{(2, 2, 3), (-1, -2, 1), (0, 1, 0)\}$
   (d) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$

3. Which of the following vectors span R^4?
   (a) $(1, 0, 0, 1), (0, 1, 0, 0), (1, 1, 1, 1), (1, 1, 1, 0)$
   (b) $(1, 2, 1, 0), (1, 1, -1, 0), (0, 0, 0, 1)$
   (c) $(6, 4, -2, 4), (2, 0, 0, 1), (3, 2, -1, 2), (5, 6, -3, 2), (0, 4, -2, -1)$
   (d) $(1, 1, 0, 0), (1, 2, -1, 1), (0, 0, 1, 1), (2, 1, 2, 1)$

4. Which of the following sets of polynomials span P_2?
   (a) $\{t^2 + 1, t^2 + t, t + 1\}$
   (b) $\{t^2 + 1, t - 1, t^2 + t\}$
   (c) $\{t^2 + 2, 2t^2 - t + 1, t + 2, t^2 + t + 4\}$
   (d) $\{t^2 + 2t - 1, t^2 - 1\}$

5. Do the polynomials $t^3 + 2t + 1$, $t^2 - t + 2$, $t^3 + 2$, $-t^3 + t^2 - 5t + 2$ span P_3?

6. Find a set of vectors spanning the solution space of $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

7. Find a set of vectors spanning the null space of

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}.$$

8. Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

belong to the solution space of $A\mathbf{x} = \mathbf{0}$. Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent?

9. Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 0 \end{bmatrix}$$

belong to the null space of A. Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent?

10. Which of the following sets of vectors in R^3 are linearly dependent? For those that are, express one vector as a linear combination of the rest.
    (a) $\{(1, 2, -1), (3, 2, 5)\}$
    (b) $\{(4, 2, 1), (2, 6, -5), (1, -2, 3)\}$
    (c) $\{(1, 1, 0), (0, 2, 3), (1, 2, 3), (3, 6, 6)\}$
    (d) $\{(1, 2, 3), (1, 1, 1), (1, 0, 1)\}$

11. Consider the vector space R^4. Follow the directions of Exercise 10.
    (a) $\{(1, 1, 2, 1), (1, 0, 0, 2), (4, 6, 8, 6), (0, 3, 2, 1)\}$
    (b) $\{(1, -2, 3, -1), (-2, 4, -6, 2)\}$
    (c) $\{(1, 1, 1, 1), (2, 3, 1, 2), (3, 1, 2, 1), (2, 2, 1, 1)\}$
    (d) $\{(4, 2, -1, 3), (6, 5, -5, 1), (2, -1, 3, 5)\}$

12. Consider the vector space P_2. Follow the directions of Exercise 10.
    (a) $\{t^2 + 1, t - 2, t + 3\}$
    (b) $\{2t^2 + 1, t^2 + 3, t\}$
    (c) $\{3t + 1, 3t^2 + 1, 2t^2 + t + 1\}$
    (d) $\{t^2 - 4, 5t^2 - 5t - 6, 3t^2 - 5t + 2\}$

13. Consider the vector space M_{22}. Follow the directions of Exercise 10.

(a) $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\right\}$

14. Let V be the vector space of all real-valued continuous functions. Follow the directions of Exercise 10.
    (a) $\{\cos t, \sin t, e^t\}$
    (b) $\{t, e^t, \sin t\}$
    (c) $\{t^2, t, e^t\}$
    (d) $\{\cos^2 t, \sin^2 t, \cos 2t\}$

15. For what values of c are the vectors $(-1, 0, -1)$, $(2, 1, 2)$, and $(1, 1, c)$ in R^3 linearly dependent?

16. For what values of λ are the vectors $t + 3$ and $2t + \lambda^2 + 2$ in P_1 linearly dependent?

17. Determine if the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ span B^3.

18. Determine if the vectors $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ span B^3.

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Exercise 6.3

Q2/c Let $S = \{(2,2,3), (-1,-2,1), (0,1,0)\} \subseteq \mathbb{R}^3$

S spans $\mathbb{R}^3$ if every vector in $\mathbb{R}^3$ is a linear combination of the vectors in S. Let $v = (a,b,c)$ be an arbitrary vector in S, then there exist constants c_1, c_2 and c_3 such that

$$c_1(2,2,3) + c_2(-1,-2,1) + c_3(0,1,0) = (a,b,c)$$

$$\Rightarrow (2c_1, 2c_1, 3c_1) + (-c_2, -2c_2, c_2) + (0, c_3, 0) = (a,b,c)$$

$$\Rightarrow (2c_1 - c_2,\ 2c_1 - 2c_2 + c_3,\ 3c_1 + c_2) = (a,b,c)$$

$$\Rightarrow \left.\begin{aligned} 2c_1 - c_2 &= a \quad -(1) \\ 2c_1 - 2c_2 + c_3 &= b \quad -(2) \\ 3c_1 + c_2 &= c \quad -(3) \end{aligned}\right\} \rightarrow (I)$$

$$(1) + (3) \Rightarrow 5c_1 = a + c \Rightarrow \boxed{c_1 = \frac{a+c}{5}}$$

$$(1) \Rightarrow c_2 = -a + 2c_1 \Rightarrow c_2 = -a + 2\left(\frac{a+c}{5}\right)$$

$$\Rightarrow \boxed{c_2 = \frac{-3a+2c}{5}}$$

$$(2) \Rightarrow c_3 = b - 2c_1 + 2c_2 \Rightarrow c_3 = b - 2\left(\frac{a+c}{5}\right) + 2\left(\frac{-3a+2c}{5}\right)$$

$\Rightarrow \boxed{c_3 = \dfrac{-8a+2c+5b}{5}}$. Since c_1, c_2 and c_3 are welldefined real numbers i.e system (I) is consistent. Hence v is the linear combination of the vectors in S but v is arbitrary, it means that every vector in $\mathbb{R}^3$ is a linear combination of the vectors in S. Hence S spans $\mathbb{R}^3$.

Exercise 6.3

Q4/a Let $S=\{t^2+1, t^2+t, t+1\} \subseteq P_2$, where P_2 is the vector space of all polynomials of degree 2 or less than 2 together with zero polynomial.
S spans P_2 if every polynomial in P_2 is the linear combination of the polynomials in S. Let $v=at^2+bt+c$ be an arbitrary polynomial, then there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1(t^2+1)+c_2(t^2+t)+c_3(t+1)=at^2+bt+c$$

$$\Rightarrow (c_1+c_2)t^2+(c_2+c_3)t+(c_1+c_3)=at^2+bt+c$$

$$\Rightarrow \left.\begin{aligned} c_1+c_2 &= a \quad -(1)\\ c_2+c_3 &= b \quad -(2)\\ c_1+c_3 &= c \quad -(3)\end{aligned}\right\} \Rightarrow (I)$$

$(1)-(3) \Rightarrow c_2-c_3=a-c \quad -(4)$

$(2)+(4) \Rightarrow 2c_2=a+b-c \Rightarrow \boxed{c_2=\frac{a+b-c}{2}}$

$(1) \Rightarrow c_1=a-c_2 \Rightarrow c_1=a-\frac{a+b-c}{2} \Rightarrow \boxed{c_1=\frac{a-b+c}{2}}$

$(2) \Rightarrow c_3=b-c_2 \Rightarrow c_3=b-\frac{a+b-c}{2} \Rightarrow \boxed{c_3=\frac{-a+b+c}{2}}$

As c_1, c_2 and c_3 are welldefined real numbers i.e system (I) is consistent, it means that v is the linear combination of the polynomials in S but v is arbitrary, showing that every polynomial in P_2 is a linear combination of the polynomials in S. Hence S spans P_2.

Q6 Find a set of vectors spanning the solution space of $AX=0$, where

$$A=\begin{bmatrix}1&0&1&0\\1&2&3&1\\2&1&3&1\\1&1&2&1\end{bmatrix}$$

(S) The augmented matrix of $AX=0$ is as

$$[A\,\vdots\,b]=[A\,\vdots\,0]=\left[\begin{array}{cccc|c}1&0&1&0&0\\1&2&3&1&0\\2&1&3&1&0\\1&1&2&1&0\end{array}\right]\underset{R_4-R_1}{\overset{R_2-R_1,\ R_3-2R_1}{\sim}}\left[\begin{array}{cccc|c}1&0&1&0&0\\0&2&2&1&0\\0&1&1&1&0\\0&1&1&1&0\end{array}\right]$$

$$\overset{R_2\text{ by }R_4}{\sim}\left[\begin{array}{cccc|c}1&0&1&0&0\\0&1&1&1&0\\0&1&1&1&0\\0&2&2&1&0\end{array}\right]\underset{R_4-2R_3}{\overset{R_3-R_2}{\sim}}\left[\begin{array}{cccc|c}1&0&1&0&0\\0&1&1&1&0\\0&0&0&0&0\\0&0&0&-1&0\end{array}\right]$$

$$\overset{R_3\text{ by }R_4}{\sim}\left[\begin{array}{cccc|c}1&0&1&0&0\\0&1&1&1&0\\0&0&0&-1&0\\0&0&0&0&0\end{array}\right]\overset{(-1)R_3}{\sim}\left[\begin{array}{cccc|c}1&0&1&0&0\\0&1&1&1&0\\0&0&0&1&0\\0&0&0&0&0\end{array}\right]$$

$$\overset{R_2-R_3}{\sim}\left[\begin{array}{cccc|c}1&0&1&0&0\\0&1&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{array}\right]\Rightarrow\left.\begin{array}{l}x+z=0\\y+z=0\\w=0\end{array}\right\}\Rightarrow\begin{array}{l}x=-z\\y=-z\\w=0\end{array}$$

$\Rightarrow$ $x=-r,\ y=-r,\ z=r,\ w=0$, where $r\in\mathbb{R}$

Thus $X=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}-r\\-r\\r\\0\end{bmatrix}$, $r\in\mathbb{R}$, is the solution which is the case of infinitely many solutions. solution space $\left\{\begin{bmatrix}-r\\-r\\r\\0\end{bmatrix}\Big/ r\in\mathbb{R}\right\}$

The set of vectors spanning the solution space of $AX=0$ is given by

$\left\{\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}\right\}$. The dimension of solution space of $AX=0$ is 1, as the number of vectors in the basis set is 1.

Exercise 6.2

Explanation of Q4/a: As $c_2 = \frac{a+b-c}{2}$, $c_1 = \frac{a-b+c}{2}$

$c_3 = \frac{-a+b+c}{2}$. since $v = -3t^2+2t+5 \in P_2$

How v is the L.C of $v_1 = t^2+1$, $v_2 = t^2+t$, $v_3 = t+1$

~~As in V ea~~ compare $v = -3t^2+2t+5$ with

at^2+bt+c, we get $a=-3$, $b=2$, $c=5$, then

$$c_1 = \frac{a-b+c}{2} = \frac{-3-2+5}{2} = 0$$

$$c_2 = \frac{a+b-c}{2} = \frac{-3+2-5}{2} = -\frac{6}{2} = -3$$

$$c_3 = \frac{-a+b+c}{2} = \frac{3+2+5}{2} = \frac{10}{2} = 5$$

Thus $v = -3t^2+2t+5 = 0v_1 - 3v_2 + 5v_3$

As $0v_1 - 3v_2 + 5v_3 = 0(t^2+1) - 3(t^2+t) + 5(t+1)$

$= -3t^2+2t+5 = v$.

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Exercise 6.3

Q8 — Q9 : similar to Ex 7 + Ex 8 on Page 295.

Q10/a) suppose $c_1(1,2,-1) + c_2(3,2,5) = 0$

$\Rightarrow (c_1, 2c_1, -c_1) + (3c_2, 2c_2, 5c_2) = 0$

$\Rightarrow (c_1 + 3c_2, 2c_1 + 2c_2, -c_1 + 5c_2) = 0$

$\left.\begin{aligned} \Rightarrow c_1 + 3c_2 &= 0 \quad (1) \\ 2c_1 + 2c_2 &= 0 \quad (2) \\ -c_1 + 5c_2 &= 0 \quad (3) \end{aligned}\right\} \Rightarrow$ (1)+(3) $\Rightarrow 8c_2 = 0 \Rightarrow c_2 = 0$

(1) $\Rightarrow c_1 = 0$

(2) $\Rightarrow 0 = 0$ satisfied

$\Rightarrow c_1 = c_2 = 0$. Hence the given set is linearly independent

Q11 — Q14 : similar to Q10.

Q15 — Q16 : Q16 : For what values of λ are the vectors $t+3$ and $2t+\lambda^2+2$ in P_1 linearly dependent?.

(S) P_1 : is the set of all polynomials of degree one or less than one together with zero polynomial.

suppose $c_1(t+3) + c_2(2t+\lambda^2+2) = 0$

$(c_1 + 2c_2)t + 3c_1 + c_2(\lambda^2+2) = 0$

$\left.\begin{aligned} \Rightarrow c_1 + 2c_2 &= 0 \quad (1) \\ 3c_1 + 2c_2 + c_2\lambda^2 &= 0 \quad (2) \end{aligned}\right\} \Rightarrow$ (2) − 3× (1) $\Rightarrow$ $-4c_2 + c_2\lambda^2 = 0$

$\Rightarrow (\lambda^2 - 4)c_2 = 0 \Rightarrow \lambda^2 - 4 = 0$ as $c_2 \neq 0$ for the given vectors to be dependent $\Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$.

Thus for $\lambda = \pm 2$, the given polynomials/vectors are linearly dependent.

Q2/c Let $S = \{(3,2,2), (-1,2,1), (0,1,0)\} \subset \mathbb{R}^3$

To see whether S is a basis for $\mathbb{R}^3$, we need to verify (a) and (b) conditions of definition.

(a) suppose S spans $\mathbb{R}^3$, then every vector in $\mathbb{R}^3$ is a linear combination of vectors in S. Let $v = (a,b,c)$ be an arbitrary vector in $\mathbb{R}^3$, then there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1(3,2,2) + c_2(-1,2,1) + c_3(0,1,0) = v = (a,b,c)$$

$$\Rightarrow (3c_1 - c_2,\ 2c_1 + 2c_2 + c_3,\ 2c_1 + c_2) = (a,b,c)$$

$$\Rightarrow \left.\begin{array}{ll} 3c_1 - c_2 = a & -(1) \\ 2c_1 + 2c_2 + c_3 = b & -(2) \\ 2c_1 + c_2 = c & -(3) \end{array}\right\} \longrightarrow (I)$$

$$(1) + (3) \Rightarrow 5c_1 = a + c \Rightarrow \boxed{c_1 = \frac{a+c}{5}}$$

$$(1) \Rightarrow c_2 = 3c_1 - a \Rightarrow c_2 = 3\left(\frac{a+c}{5}\right) - a$$

$$\Rightarrow \boxed{c_2 = \frac{-2a+3c}{5}}$$

$$(2) \Rightarrow c_3 = b - 2c_1 - 2c_2 \Rightarrow c_3 = b - 2\left(\frac{a+c}{5}\right) - 2\left(\frac{-2a+3c}{5}\right)$$

$$\Rightarrow \boxed{c_3 = \frac{2a+5b-8c}{5}}$$

. since c_1, c_2 and c_3 are welldefined real numbers, it means system(I) is consistent, showing v is the linear combination of the vectors in S but v is arbitrary, it means that every vector in $\mathbb{R}^3$ is a linear combination of vectors in S.
Hence S spans $\mathbb{R}^3$.

(b) suppose $c_1(3,2,2)+c_2(-1,2,1)+c_3(0,1,0)=0$

$\Rightarrow (3c_1-c_2,\ 2c_1+2c_2+c_3,\ 2c_1+c_2)=0$

$\Rightarrow 3c_1-c_2=0$ — (1)

$2c_1+2c_2+c_3=0$ — (2)

$2c_1+c_2=0$ — (3)

(1) + (3) $\Rightarrow 5c_1=0 \Rightarrow \boxed{c_1=0}$

(1) $\Rightarrow \boxed{c_2=0}$, (3) $\Rightarrow \boxed{c_3=0}$

i.e $c_1=c_2=c_3=0 \Rightarrow S$ is linearly independent.

As both the conditions of definition are satisfied.

Hence $S=\{(3,2,2),(-1,2,1),(0,1,0)\}$ is a basis for $\mathbb{R}^3$.

Q1 — Q3 : similar to Q2/c

Q4 — Q5 — Q10 : similar to ex 3 Page 304

Q17 — Q18 :- Find a basis for the given subspaces of $\mathbb{R}^3$ and $\mathbb{R}^4$

Q18/c All vectors of the form (a,b,c) where $a-b+5c=0$

(S) Let $W=\{(a,b,c)/\ a-b+5c=0 \Rightarrow a=b-5c\} \subset \mathbb{R}^3$

~~Therefore~~ To find the basis of the sub space W of $\mathbb{R}^3$, we consider an arbitrary vector in W as

$v=(a,b,c)=(b-5c,\ b,\ c)=(b,b,0)+(-5c,0,c)$

$=b(1,1,0)+c(-5,0,1) \Rightarrow v$ is the linear combination of $v_1=(1,1,0)$ and $v_2=(-5,0,1)$ but v is arbitrary it means that every vector in W is the linear combination of v_1 and v_2 i.e $S=\{v_1,v_2\}=\{(1,1,0),(-5,0,1)\}$ spans W

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ML.2. Find a spanning set of the solution space of $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & -1 & 5 & 7 \\ 0 & 2 & -2 & -2 \end{bmatrix}.$$

6.4 BASIS AND DIMENSION

In this section we continue our study of the structure of a vector space V by determining a smallest set of vectors in V that completely describes V.

BASIS

DEFINITION The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to form a **basis** for V if (a) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span V and (b) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Remark If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a basis for a vector space V, then they must be nonzero (see Example 12 in Section 6.3) and distinct and so we write them as a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

EXAMPLE 1 The vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ form a basis for R^2, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for R^3 and, in general, the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ form a basis for R^n. Each of these sets of vectors is called the **natural basis** or **standard basis** for R^2, R^3, and R^n, respectively. ■

EXAMPLE 2 Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where $\mathbf{v}_1 = (1, 0, 1, 0)$, $\mathbf{v}_2 = (0, 1, -1, 2)$, $\mathbf{v}_3 = (0, 2, 2, 1)$, and $\mathbf{v}_4 = (1, 0, 0, 1)$ is a basis for R^4.

Solution To show that S is linearly independent, we form the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

and solve for c_1, c_2, c_3, and c_4. Substituting for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, we obtain the linear system (verify)

$$\begin{aligned} c_1 \qquad\qquad\qquad + c_4 &= 0 \\ c_2 + 2c_3 \qquad &= 0 \\ c_1 - \quad c_2 + 2c_3 \qquad &= 0 \\ 2c_2 + \quad c_3 + c_4 &= 0, \end{aligned}$$

which has as its only solution $c_1 = c_2 = c_3 = c_4 = 0$ (verify), showing that S is linearly independent. Observe that the coefficient matrix of the preceding linear system consists of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ written in column form.

To show that S spans R^4, we let $\mathbf{v} = (a, b, c, d)$ be any vector in R^4. We now seek constants k_1, k_2, k_3, and k_4 such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{v}.$$

Substituting for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}$, we find a solution (verify) for k_1, k_2, k_3, and k_4 to the resulting linear system for any a, b, c, and d. Hence S spans R^4 and is a basis for R^4. ■

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EXAMPLE 3 Show that the set $S = \{t^2+1, t-1, 2t+2\}$ is a basis for the vector space P_2.

Solution We must show that S spans V and is linearly independent. To show that it spans V, we take any vector in V, that is, a polynomial at^2+bt+c, and must find constants a_1, a_2, and a_3 such that

$$\begin{aligned} at^2+bt+c &= a_1(t^2+1)+a_2(t-1)+a_3(2t+2) \\ &= a_1t^2+(a_2+2a_3)t+(a_1-a_2+2a_3). \end{aligned}$$

Since two polynomials agree for all values of t only if the coefficients of respective powers of t agree, we get the linear system

$$\begin{aligned} a_1 \qquad\qquad &= a \\ a_2+2a_3 &= b \\ a_1-a_2+2a_3 &= c. \end{aligned}$$

Solving, we have

$$a_1 = a, \quad a_2 = \frac{a+b-c}{2}, \quad a_3 = \frac{c+b-a}{4}.$$

Hence S spans V.

To illustrate this result, suppose that we are given the vector $2t^2+6t+13$. Here $a=2$, $b=6$, and $c=13$. Substituting in the foregoing expressions for a_1, a_2, and a_3, we find that

$$a_1 = 2, \quad a_2 = -\tfrac{5}{2}, \quad a_3 = \tfrac{17}{4}.$$

Hence

$$2t^2+6t+13 = 2(t^2+1) - \tfrac{5}{2}(t-1) + \tfrac{17}{4}(2t+2).$$

To show that S is linearly independent, we form

$$a_1(t^2+1)+a_2(t-1)+a_3(2t+2) = 0.$$

Then

$$a_1t^2+(a_2+2a_3)t+(a_1-a_2+2a_3) = 0.$$

Again, this can hold for all values of t only if

$$\begin{aligned} a_1 \qquad\qquad &= 0 \\ a_2+2a_3 &= 0 \\ a_1-a_2+2a_3 &= 0. \end{aligned}$$

The only solution to this homogeneous system is $a_1 = a_2 = a_3 = 0$, which implies that S is linearly independent. Thus S is a basis for P_2. ∎

The set of vectors $\{t^n, t^{n-1}, \ldots, t, 1\}$ forms a basis for the vector space P_n called the **natural basis** or **standard basis** for P_n. It has already been shown in Example 5 of Section 6.3 that this set of vectors spans P_n. Linear independence is left as an exercise (Exercise T.15).

EXAMPLE 4 Find a basis for the subspace V of P_2, consisting of all vectors of the form at^2+bt+c, where $c = a-b$.

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Proof Let $T_1 = \{\mathbf{w}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Since S spans V, so does T_1. Since $\mathbf{w}_1$ is a linear combination of the vectors in S, we find that T_1 is linearly dependent. Then, by Theorem 6.4, some $\mathbf{v}_j$ is a linear combination of the preceding vectors in T_1. Delete that particular vector $\mathbf{v}_j$.

Let $S_1 = \{\mathbf{w}_1, \mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\}$. Note that S_1 spans V. Next, let $T_2 = \{\mathbf{w}_2, \mathbf{w}_1, \mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\}$. Then T_2 is linearly dependent and some vector in T_2 is a linear combination of the preceding vectors in T_2. Since T is linearly independent, this vector cannot be $\mathbf{w}_1$, so it is $\mathbf{v}_i$, $i \neq j$. Repeat this process over and over. If the $\mathbf{v}$ vectors are all eliminated before we can run out of $\mathbf{w}$ vectors, then the resulting set of $\mathbf{w}$ vectors, a subset of T, is linearly dependent, which implies that T is also linearly dependent. Since we have reached a contradiction, we conclude that the number r of $\mathbf{w}$ vectors must be no greater than the number n of $\mathbf{v}$ vectors. That is, $r \leq n$. ∎

COROLLARY 6.1 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ are bases for a vector space, then $n = m$.*

Proof Since T is a linearly independent set of vectors, Theorem 6.7 implies that $m \leq n$. Similarly, $n \leq m$ because S is linearly independent. Hence $n = m$. ∎

Thus, although a vector space has many bases, we have just shown that for a particular vector space V, all bases have the same number of vectors. We can then make the following definition.

DIMENSION

DEFINITION The **dimension** of a nonzero vector space V is the number of vectors in a basis for V. We often write **dim** V for the dimension of V. Since the set $\{\mathbf{0}\}$ is linearly dependent, it is natural to say that the vector space $\{\mathbf{0}\}$ has dimension zero.

EXAMPLE 6 The dimension of R^2 is 2; the dimension of R^3 is 3; and in general, the dimension of R^n is n. ∎

EXAMPLE 7 The dimension of P_2 is 3; the dimension of P_3 is 4; and in general, the dimension of P_n is $n + 1$. ∎

It can be shown that all finite-dimensional vector spaces of the same dimension differ only in the nature of their elements; their algebraic properties are identical.

It can also be shown that if V is a finite-dimensional vector space, then every nonzero subspace W of V has a finite basis and $\dim W \leq \dim V$ (Exercise T.2).

EXAMPLE 8 The subspace W of R^4 considered in Example 5 has dimension 2. ∎

We might also consider the subspaces of R^2 [recall that R^2 can be viewed as the (x, y)-plane]. First, we have $\{\mathbf{0}\}$ and R^2, the trivial subspaces of dimension 0 and 2, respectively. Now the subspace V of R^2 spanned by a vector $\mathbf{v} \neq \mathbf{0}$ is a one-dimensional subspace of R^2; V is represented by a line through the origin. Thus the subspaces of R^2 are $\{\mathbf{0}\}$, R^2, and all the lines through the

(34)

We form the augmented matrix and use row opertions: Add row 1 to row 4, add row 2 to row 3, and add row 2 to row 4, to obtain the equivalent augmented matrix (verify)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & b+c \\ 0 & 0 & 0 & 0 & a+b+d \end{array}\right].$$

It follows that this system is inconsistent if the choice of bits a, b, and d is such that $a+b+d \neq 0$. For example, if

$$\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

then the system is inconsistent; hence S does not span B^4 and is not a basis for B^4. ∎

Key Terms

Basis
Natural (or standard) basis
Finite-dimensional vector space
Infinite-dimensional vector space
Dimension

Q13(a), Q15 Quiz # 5, Q17, Q8 Quiz# 6

6.4 Exercises

1. Which of the following sets of vectors are bases for R^2?
   (a) $\{(1,3),(1,-1)\}$
   (b) $\{(0,0),(1,2),(2,4)\}$
   (c) $\{(1,2),(2,-3),(3,2)\}$
   (d) $\{(1,3),(-2,6)\}$

2. Which of the following sets of vectors are bases for R^3?
   (a) $\{(1,2,0),(0,1,-1)\}$
   (b) $\{(1,1,-1),(2,3,4),(4,1,-1),(0,1,-1)\}$
   (c) $\{(3,2,2),(-1,2,1),(0,1,0)\}$
   (d) $\{(1,0,0),(0,2,-1),(3,4,1),(0,1,0)\}$

3. Which of the following sets of vectors are bases for R^4?
   (a) $\{(1,0,0,1),(0,1,0,0),(1,1,1,1),(0,1,1,1)\}$
   (b) $\{(1,-1,0,2),(3,-1,2,1),(1,0,0,1)\}$
   (c) $\{(-2,4,6,4),(0,1,2,0),(-1,2,3,2),(-3,2,5,6),(-2,-1,0,4)\}$
   (d) $\{(0,0,1,1),(-1,1,1,2),(1,1,0,0),(2,1,2,1)\}$

4. Which of the following sets of vectors are bases for P_2?
   (a) $\{-t^2+t+2,\ 2t^2+2t+3,\ 4t^2-1\}$
   (b) $\{t^2+2t-1,\ 2t^2+3t-2\}$
   (c) $\{t^2+1,\ 3t^2+2t,\ 3t^2+2t+1,\ 6t^2+6t+3\}$
   (d) $\{3t^2+2t+1,\ t^2+t+1,\ t^2+1\}$

5. Which of the following sets of vectors are bases for P_3?
   (a) $\{t^3+2t^2+3t,\ 2t^3+1,\ 6t^3+8t^2+6t+4,\ t^3+2t^2+t+1\}$
   (b) $\{t^3+t^2+1,\ t^3-1,\ t^3+t^2+t\}$
   (c) $\{t^3+t^2+t+1,\ t^3+2t^2+t+3,\ 2t^3+t^2+3t+2,\ t^3+t^2+2t+2\}$
   (d) $\{t^3-t,\ t^3+t^2+1,\ t-1\}$

6. Show that the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22}.

In Exercises 7 and 8, determine which of the given subsets forms a basis for R^3. Express the vector $(2,1,3)$ as a linear combination of the vectors in each subset that is a basis.

7. (a) $\{(1,1,1),(1,2,3),(0,1,0)\}$
   (b) $\{(1,2,3),(2,1,3),(0,0,0)\}$

8. (a) $\{(2,1,3),(1,2,1),(1,1,4),(1,5,1)\}$
   (b) $\{(1,1,2),(2,2,0),(3,4,-1)\}$

(35)

In Exercises 9 and 10, determine which of the given subsets form a basis for P_2. Express $5t^2 - 3t + 8$ as a linear combination of the vectors in each subset that is a basis.

9. (a) $\{t^2 + t, t - 1, t + 1\}$
   (b) $\{t^2 + 1, t - 1\}$

10. (a) $\{t^2 + t, \ldots t^2 + 1\}$
   (b) $\{t^2 + 1, t^2 - t + 1\}$

11. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = (1, 2, 2), \quad \mathbf{v}_2 = (3, 2, 1),$$
$$\mathbf{v}_3 = (11, 10, 7), \text{ and } \mathbf{v}_4 = (7, 6, 4).$$

Find a basis for the subspace $W = \text{span } S$ of R^3. W...

12. Let $S = \ldots$

$$\mathbf{v}_1 = (1, 1, 0, -1), \quad \mathbf{v}_2 = (0, 1, 2, 1),$$
$$\ldots = (1, 0, 1, -1), \quad \mathbf{v} = (1, 1, -6, -\ldots$$

$W = \text{span} \ldots$

13. Consider the following subset of P_3:

$$S = \{t^3 + t^2 - 2t + 1, t^2 + 1, t^3 - 2t,$$
$$2t^3 + 3t^2 - 4t + 3\}.$$

Find a basis for the subspace $W = \text{span } S$. What is $\dim W$?

14. Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}.$$

... basis for the subspace $W = \text{span } S$ of M_{22}.

15. ... ion of M_{23}?
Generalize ...

16. Consider the following subset of the vec...
...ued functions

... cos 2t}.

Find a basis for the subspace ... dim W?

In Exercises 17 and 18, find a basis for the given subspaces of R^3 and R^4.

17. (a) All vectors of the form (a, b, c), where $b = a + c$
   (b) All vectors of the form (a, b, c), where $b = a$
   (c) All vectors of the form (a, b, c), where $2a + b - c = 0$

18. (a) All vectors of the form (a, b, c), where $a = 0$
   (b) All vectors of the form $(a + c, a - b, b + c, -a + b)$
   (c) All vectors of the form (a, b, c), where $a - b + 5c = 0$

In Exercises 19 and 20, find the dimensions of the given subspaces of R^4.

19. (a) All vectors of the form (a, b, c, d), where $d = a + b$
   (b) All vectors of the form (a, b, c, d), where $c = a - b$ and $d = a + b$

20. (a) All vectors of the form (a, b, c, d), where $a = b$
   (b) All vectors of the form $(a + c, -a + b, -b - c, a + b + 2c)$

21. Find a basis for the subspace of P_2 consisting of all vectors of the form $at^2 + bt + c$, where $c = 2a - 3b$.

22. Find a basis for the subspace of P_3 consisting of all vectors of the form $at^3 + bt^2 + ct + d$, where $a = b$ and $c = d$.

23. Find the dimensions of the subspaces of R^2 spanned by the vectors in Exercise 1.

24. Find the dimensions of the subspaces of R^3 spanned by the vectors in Exercise 2.

25. Find the dimensions of the subspaces of R^4 spanned by the vectors in Exercise 3.

26. Find the dimension of the subspace of P_2 consisting of all vectors of the form $at^2 + bt + c$, where $c = b - 2a$.

27. Find the dimension of the subspace of P_3 consisting of all vectors of the form $at^3 + bt^2 + ct + d$, where $b = 3a - 5d$ and $c = d + 4a$.

28. Find a basis for R^3 that includes the vectors
   (a) $(1, 0, 2)$
   (b) $(1, 0, 2)$ and $(0, 1, 3)$

29. Find a basis for R^4 that includes the vectors $(1, 0, 1, 0)$ and $(0, 1, -1, 0)$.

30. Find all values of a for which $\{(a^2, 0, 1), (0, a, 2), (\ldots)\}$ is a basis for R^3.

31. Find a basis for the subspace ... consisting of all symmetric matrices.

32. Find a basis for the subspace of M_{33} consisting of all diagonal matrices.

33. Give an example of ... ensional subspace of R^4.

34. Give an example of a two-dimensional subspace of P_3.

In Exercises 35 and 36, find a basis for the given plane.

35. $2x - 3y + 4z = 0$. 36. $x + y - 3z = 0$.

37. Determine if the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for R^3.

Also $c_1 v_1 + c_2 v_2 = 0 \Rightarrow c_1(1,1,0) + c_2(-5,0,1) = 0$
$\Rightarrow (c_1 - 5c_2, c_1, c_2) = 0 \Rightarrow \left.\begin{array}{l} c_1 - 5c_2 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$
$\Rightarrow$ v_1 and v_2 are linearly independent i.e
S is linearly independent.
As S spans W and S is linearly independent.
Hence $S = \{(1,1,0), (-5,0,1)\}$ is a basis for
$W = \{(a,b,c) \mid a - b + 5c = 0\} = \{(a,b,c) \mid a = b - 5c\}$.

Q19 – Q20: Find the dimensions of the given subspaces of $\mathbb{R}^4$

Q19/b All vectors of the form (a,b,c,d) where
$c = a - b$ and $d = a + b$

(S) Let $W = \{(a,b,c,d) / c = a - b \text{ and } d = a + b\} \subset \mathbb{R}^4$
W here is subspace of $\mathbb{R}^4$, we need to find its dimension
As by definition dimension of a vector space is the number
of vectors in the basis of the vector space, so we will
first find the basis for the given subspace W of $\mathbb{R}^4$.
Let $v = (a,b,c,d) = (a, b, a-b, a+b) = (a,0,a,a) + (0,b,-b,b)$
$= a(1,0,1,1) + b(0,1,-1,1) \Rightarrow v$ is the linear combination
of $v_1 = (1,0,1,1)$ and $v_2 = (0,1,-1,1)$. But v is arbitrary vector
in W, it means that every vector in W is the linear
combination of v_1 and v_2 i.e $S = \{v_1, v_2\} = \{(1,0,1,1), (0,1,-1,1)\}$
spans W. Now we need to show that S is linearly

independent. suppose $c_1v_1 + c_2v_2 = 0$ for some $c_1, c_2 \in \mathbb{R}$

$\Rightarrow c_1(1,0,1,1) + c_2(0,1,-1,1) = 0$

$\Rightarrow (c_1, c_2, c_1 - c_2, c_1 + c_2) = (0,0,0,0)$

$\Rightarrow c_1 = 0, c_2 = 0, c_1 - c_2 = 0, c_1 + c_2 = 0$

$\Rightarrow c_1 = c_2 = 0 \Rightarrow S$ is linearly independent

As S spans W and S is linearly independent.

Hence $S = \{(1,0,1,1), (0,1,-1,1)\}$ is a basis for W.

The dimension of W is 2, as there are 2 vectors in the basis S of W.

Q21, similar to Q17 + Q18

Q22 — Q27: similar to Q19 + Q20.

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