T.2. Let $f: R^n \to R^m$ be a matrix transformation defined by $f(\mathbf{u}) = A\mathbf{u}$, where A is an $m \times n$ matrix. Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n such that $f(\mathbf{u}) = \mathbf{0}$ and $f(\mathbf{v}) = \mathbf{0}$, where

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

then $f(c\mathbf{u} + d\mathbf{v}) = \mathbf{0}$ for any real numbers c and d.

T.3. (a) Let $O: R^n \to R^m$ be the matrix transformation defined by $O(\mathbf{u}) = O\mathbf{u}$, where O is the $m \times n$ zero matrix. Show that $O(\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u}$ in R^n.

(b) Let $I: R^n \to R^n$ be the matrix transformation defined by $I(\mathbf{u}) = I_n\mathbf{u}$, where I_n is the identity matrix (see Section 1.4). Show that $I(\mathbf{u}) = \mathbf{u}$ for all $\mathbf{u}$ in R^n.

1.6 SOLUTIONS OF LINEAR SYSTEMS OF EQUATIONS

In this section we shall systematize the familiar method of elimination of unknowns (discussed in Section 1.1) for the solution of linear systems and thus obtain a useful method for solving such systems. This method starts with the augmented matrix of the given linear system and obtains a matrix of a certain form. This new matrix represents a linear system that has exactly the same solutions as the given system but is easier to solve. For example, if

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & 3 & 6 \end{array}\right]$$

represents the augmented matrix of a linear system, then the solution is easily found from the corresponding equations

$$\begin{aligned} x_1 \quad\quad + 2x_4 &= 4 \\ x_2 \quad - x_4 &= -5 \\ x_3 + 3x_4 &= 6. \end{aligned}$$

The task of this section is to manipulate the augmented matrix representing a given linear system into a form from which the solution can easily be found.

DEFINITION An $m \times n$ matrix A is said to be in **reduced row echelon form** if it satisfies the following properties:

(a) All zero rows, if there are any, appear at the bottom of the matrix.

(b) The first nonzero entry from the left of a nonzero row is a 1. This entry is called a **leading one** of its row.

(c) For each nonzero row, the leading one appears to the right and below any leading one's in preceding rows.

(d) If a column contains a leading one, then all other entries in that column are zero.

A matrix in reduced row echelon form appears as a staircase ("echelon") pattern of leading ones descending from the upper left corner of the matrix.

An $m \times n$ matrix satisfying properties (a), (b), and (c) is said to be in **row echelon form**.

EXAMPLE 1 The following are matrices in reduced row echelon form since they satisfy properties (a), (b), (c), and (d):

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The following matrices are not in reduced row echelon form. (Why not?)

$$D = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ ■

EXAMPLE 2 The following are matrices in row echelon form:

$$H = \begin{bmatrix} 1 & 5 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$J = \begin{bmatrix} 0 & 0 & 1 & 3 & 5 & 7 & 9 \\ 0 & 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ ■

A useful property of matrices in reduced row echelon form (see Exercise T.9) is that if A is an $n \times n$ matrix in reduced row echelon form not equal to I_n, then A has a row consisting entirely of zeros.

We shall now turn to the discussion of how to transform a given matrix to a matrix in reduced row echelon form.

DEFINITION An **elementary row operation** on an $m \times n$ matrix $A = [a_{ij}]$ is any of the following operations:

(a) Interchange rows r and s of A. That is, replace $a_{r1}, a_{r2}, \ldots, a_{rn}$ by $a_{s1}, a_{s2}, \ldots, a_{sn}$ and $a_{s1}, a_{s2}, \ldots, a_{sn}$ by $a_{r1}, a_{r2}, \ldots, a_{rn}$.

(b) Multiply row r of A by $c \neq 0$. That is, replace $a_{r1}, a_{r2}, \ldots, a_{rn}$ by $ca_{r1}, ca_{r2}, \ldots, ca_{rn}$.

(c) Add d times row r of A to row s of A, $r \neq s$. That is, replace $a_{s1}, a_{s2}, \ldots, a_{sn}$ by $a_{s1} + da_{r1}, a_{s2} + da_{r2}, \ldots, a_{sn} + da_{rn}$.

Observe that when a matrix is viewed as the augmented matrix of a linear system, the elementary row operations are equivalent, respectively, to interchanging two equations, multiplying an equation by a nonzero constant, and adding a multiple of one equation to another equation.

EXAMPLE 3 Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}.$$

Interchanging rows 1 and 3 of A, we obtain

$$B = \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

~~Multiplying the~~ third row of A by $\frac{1}{3}$, we obtain

$$C = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}.$$

Adding (-2) times row 2 of A to row 3 of A, we obtain

$$D = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}.$$

Observe that in obtaining D from A, row 2 of A *did not change.* ■

DEFINITION An $m \times n$ matrix A is said to be **row equivalent** to an $m \times n$ matrix B if B can be obtained by applying a finite sequence of elementary row operations to the matrix A.

EXAMPLE 4 Let

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -2 & 2 & 3 \end{bmatrix}.$$

If we add 2 times row 3 of A to its second row, we obtain

$$B = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & -3 & 7 & 8 \\ 1 & -2 & 2 & 3 \end{bmatrix},$$

so B is row equivalent to A.

Interchanging rows 2 and 3 of B, we obtain

$$C = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & -2 & 2 & 3 \\ 4 & -3 & 7 & 8 \end{bmatrix},$$

so C is row equivalent to B and also row equivalent to A.

Multiplying row 1 of C by 2, we obtain

$$D = \begin{bmatrix} 2 & 4 & 8 & 6 \\ 1 & -2 & 2 & 3 \\ 4 & -3 & 7 & 8 \end{bmatrix},$$

so D is row equivalent to C. It then follows that D is row equivalent to A, since we obtained D by applying three successive elementary row operations to A. ■

It is not difficult to show (Exercise T.2) that

1. every matrix is row equivalent to itself;
2. if A is row equivalent to B, then B is row equivalent to A; and
3. if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

In view of 2, both statements, "A is row equivalent to B" and "B is row equivalent to A," can be replaced by "A and B are row equivalent."

THEOREM 1.5 *Every $m \times n$ matrix is row equivalent to a matrix in row echelon form.*

We shall illustrate the proof of the theorem by giving the steps that must be carried out on a specific matrix A to obtain a matrix in row echelon form that is row equivalent to A. We use the following example to illustrate the steps involved.

EXAMPLE 5 Let

$$A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}.$$

The procedure for transforming a matrix to row echelon form follows.

Procedure / **Example**

Step 1. Find the first (counting from left to right) column in A not all of whose entries are zero. This column is called the **pivotal column**.

$$A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

Pivotal column of A

Step 2. Identify the first (counting from top to bottom) nonzero entry in the pivotal column. This element is called the **pivot**, which we circle in A.

$$A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

Pivot

Step 3. Interchange, if necessary, the first row with the row where the pivot occurs so that the pivot is now in the first row. Call the new matrix A_1.

$$A_1 = \begin{bmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

The first and third rows of A were interchanged.

Step 4. **Multiply the first row of A_1 by the reciprocal of the pivot. Thus the entry in the first row and pivotal column (where the pivot was located) is now a 1. Call the new matrix A_2.**

$$A_2 = \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

The first row of A_1 was multiplied by $\frac{1}{2}$.

Step 5. **Add appropriate multiples of the first row of A_2 to all other rows to make all entries in the pivotal column, except the entry where the pivot was located, equal to zero. Thus all entries in the pivotal column and rows 2, 3, ..., m are zero. Call the new matrix A_3.**

$$A_3 = \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

(-2) times the first row of A_2 was added to its fourth row.

Step 6. **Identify B as the $(m-1) \times n$ submatrix of A_3 obtained by ignoring or covering the first row of A_3. Repeat Steps 1 through 5 on B.**

$$\begin{array}{c} \begin{matrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \end{matrix} \\ B = \begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \end{array}$$

Pivotal column of B Pivot

$$\begin{array}{c} \begin{matrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \end{matrix} \\ B_1 = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \end{array}$$

The first and second rows of B were interchanged.

$$\begin{array}{c} \begin{matrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \end{matrix} \\ B_2 = \begin{bmatrix} 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \end{array}$$

The first row of B_1 was multiplied by $\frac{1}{2}$.

$$\begin{array}{c} \begin{matrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \end{matrix} \\ B_3 = \begin{bmatrix} 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \end{array}$$

2 times the first row of B_2 was added to its third row.

Step 7. Identify C as the $(m-2) \times n$ submatrix of B_3 obtained by ignoring or covering the first row of B_3. Repeat Steps 1 through 5 on C.

$$\begin{array}{ccccc} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}$$

$$C = \begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

Pivotal column of C — Pivot

$$\begin{array}{ccccc} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

No rows of C had to be interchanged. The first row of C was multiplied by $\frac{1}{2}$.

$$\begin{array}{ccccc} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}$$

$$C_3 = \begin{bmatrix} 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(-2) times the first row of C_2 was added to its second row.

Step 8. Identify D as the $(m-3) \times n$ submatrix of C_3 obtained by ignoring or covering the first row of C_3. We now try to repeat Steps 1–5 on D. However, because there is no pivotal row in D, we are finished. The matrix, denoted by H, consisting of D and the shaded rows above D is in row echelon form.

$$\begin{array}{ccccc} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \end{array}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ■

Remark When doing hand computations, it is sometimes possible to avoid fractions by suitably modifying the steps in the procedure.

EXAMPLE 6 Let

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}.$$

To find a matrix in row echelon form that is row equivalent to A, we modify the foregoing procedure to avoid fractions and proceed as follows.

Add (-1) times row 1 to row 2 of A to obtain

$$A_1 = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}.$$

Interchange rows 1 and 2 of A_1 to obtain

$$A_2 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}.$$

Jordan** **reduction**; the method where $[C \mid \mathbf{d}]$ is in row echelon form is called **Gaussian elimination**. Strictly speaking, the alternate Gauss–Jordan reduction method described in the Remark above is not as efficient as that used in Examples 5 and 6. In actual practice, neither Gauss–Jordan reduction nor Gaussian elimination is used as much as the method involving the LU-factorization of A that is presented in Section 1.8. However, Gauss–Jordan reduction and Gaussian elimination are fine for small problems, and we use the former frequently in this book.

The Gauss–Jordan reduction procedure for solving the linear system $A\mathbf{x} = \mathbf{b}$ is as follows.

Step 1. Form the augmented matrix $[A \mid \mathbf{b}]$.

Step 2. Obtain the reduced row echelon form $[C \mid \mathbf{d}]$ of the augmented matrix $[A \mid \mathbf{b}]$ by using elementary row operations.

Step 3. For each nonzero row of the matrix $[C \mid \mathbf{d}]$, solve the corresponding equation for the unknown associated with the leading one in that row. The rows consisting entirely of zeros can be ignored, because the corresponding equation will be satisfied for any values of the unknowns.

The Gaussian elimination procedure for solving the linear system $A\mathbf{x} = \mathbf{b}$ is as follows.

Step 1. Form the augmented matrix $[A \mid \mathbf{b}]$.

Step 2. Obtain a row echelon form $[C \mid \mathbf{d}]$ of the augmented matrix $[A \mid \mathbf{b}]$ by using elementary row operations.

Step 3. Solve the linear system corresponding to $[C \mid \mathbf{d}]$ by **back substitution** (illustrated in Example 11). The rows consisting entirely of zeros can be ignored, because the corresponding equation will be satisfied for any values of the unknowns.

The following examples illustrate the Gauss–Jordan reduction procedure.

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publications include important contributions in number theory, mathematical astronomy, mathematical geography, statistics, differential geometry, and magnetism. His diaries and private notes contain many other discoveries that he never published.

An austere, conservative man who had few friends and whose private life was generally unhappy, he was very concerned that proper credit be given for scientific discoveries. When he relied on the results of others, he was careful to acknowledge them; and when others independently discovered results in his private notes, he was quick to claim priority.

In his research he used a method of calculation that later generations generalized to row reduction of matrices and named in his honor although the method was used in China almost 2000 years earlier.

**Wilhelm Jordan (1842–1899) was born in southern Germany. He attended college in Stuttgart and in 1868 became full professor of geodesy at the technical college in Karlsruhe, Germany. He participated in surveying several regions of Germany. Jordan was a prolific writer whose major work, *Handbuch der Vermessungskunde* (*Handbook of Geodesy*), was translated into French, Italian, and Russian. He was considered a superb writer and an excellent teacher. Unfortunately, the Gauss–Jordan reduction method has been widely attributed to Camille Jordan (1838–1922), a well-known French mathematician. Moreover, it seems that the method was also discovered independently at the same time by B. I. Clasen, a priest who lived in Luxembourg. This biographical sketch is based on an excellent article: S. C. Althoen and R. McLaughlin, "Gauss–Jordan reduction: A Brief History," *MAA Monthly*, 94 (1987), 130–142.

EXAMPLE 8 Solve the linear system

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - y + z &= 8 \\ 3x - z &= 3 \end{aligned} \qquad (1)$$

by Gauss–Jordan reduction.

Solution ***Step 1.*** The augmented matrix of this linear system is

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right].$$

Step 2. We now transform the matrix in Step 1 to reduced row echelon form as follows:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$$

R_2-2R_1
$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right]$$
(-2) times the first row was added to its second row. (-3) times the first row was added to its third row.

$-1/5$
$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right]$$
The second row was multiplied by $\left(-\frac{1}{5}\right)$.

R_3+6R_2
$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right]$$
6 times the second row was added to its third row.

$-\frac{1}{4}R_3$
$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
The third row was multiplied by $\left(-\frac{1}{4}\right)$.

R_2-R_3
$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
(-1) times the third row was added to its first row.

R_1-3R_3
$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
(-3) times the third row was added to its first row.

R_1-2R_2
$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
(-2) times the second row was added to its first row.

Thus, the augmented matrix is row equivalent to the matrix

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad (2)$$

in reduced row echelon form.

Step 3. The linear system represented by (2) is

$$\begin{aligned} x \qquad &= \ \ 2 \\ y \quad &= -1 \\ z &= \ \ 3 \end{aligned}$$

so that the unique solution to the given linear system (1) is

$$\begin{aligned} x &= \ \ 2 \\ y &= -1 \\ z &= \ \ 3. \end{aligned}$$

EXAMPLE 9 Solve the linear system

$$\begin{aligned} x + \ y + 2z - 5w &= \ \ \ 3 \\ 2x + 5y - \ z - 9w &= \ -3 \\ 2x + \ y - \ z + 3w &= -11 \\ x - 3y + 2z + 7w &= \ -5 \end{aligned} \tag{3}$$

by Gauss–Jordan reduction.

Solution *Step 1.* The augmented matrix of this linear system is

$$\left[\begin{array}{rrrr|r} 1 & 1 & 2 & -5 & 3 \\ 2 & 5 & -1 & -9 & -3 \\ 2 & 1 & -1 & 3 & -11 \\ 1 & -3 & 2 & 7 & -5 \end{array}\right].$$

Step 2. The augmented matrix is row equivalent to the matrix (verify)

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{4}$$

which is in reduced row echelon form.

Step 3. The linear system represented by (4) is

$$\begin{aligned} x \qquad + 2w &= -5 \\ y \quad - 3w &= \ \ 2 \\ z - 2w &= \ \ 3. \end{aligned}$$

The row in (4) consisting entirely of zeros has been ignored.

Solving each equation for the unknown that corresponds to the leading entry in each row of (4), we obtain

$$\begin{aligned} x &= -5 - 2w \\ y &= \ \ 2 + 3w \\ z &= \ \ 3 + 2w. \end{aligned}$$

Thus, if we let $w = r$, any real number, then a solution to the linear system (3) is

$$\begin{aligned} x &= -5 - 2r \\ y &= \ \ 2 + 3r \\ z &= \ \ 3 + 2r \\ w &= \ \ r. \end{aligned} \tag{5}$$

Because r can be assigned any real number in (5), the given linear system (3) has infinitely many solutions. ■

EXAMPLE 10 Solve the linear system

$$\begin{aligned} x_1 + 2x_2 - 3x_4 + x_5 &= 2 \\ x_1 + 2x_2 + x_3 - 3x_4 + x_5 + 2x_6 &= 3 \\ x_1 + 2x_2 - 3x_4 + 2x_5 + x_6 &= 4 \\ 3x_1 + 6x_2 + x_3 - 9x_4 + 4x_5 + 3x_6 &= 9 \end{aligned} \tag{6}$$

by Gauss–Jordan reduction.

Solution *Step 1.* The augmented matrix of this linear system is

$$\left[\begin{array}{cccccc:c} 1 & 2 & 0 & -3 & 1 & 0 & 2 \\ 1 & 2 & 1 & -3 & 1 & 2 & 3 \\ 1 & 2 & 0 & -3 & 2 & 1 & 4 \\ 3 & 6 & 1 & -9 & 4 & 3 & 9 \end{array}\right].$$

Step 2. The augmented matrix is row equivalent to the matrix (verify)

$$\left[\begin{array}{cccccc:c} 1 & 2 & 0 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{7}$$

Step 3. The linear system represented by (7) is

$$\begin{aligned} x_1 + 2x_2 - 3x_4 - x_6 &= 0 \\ x_3 + 2x_6 &= 1 \\ x_5 + x_6 &= 2. \end{aligned}$$

Solving each equation for the unknown that corresponds to the leading entry in each row of (7), we obtain

$$\begin{aligned} x_1 &= x_6 + 3x_4 - 2x_2 \\ x_3 &= 1 - 2x_6 \\ x_5 &= 2 - x_6. \end{aligned}$$

Letting $x_6 = r$, $x_4 = s$, and $x_2 = t$, a solution to the linear system (6) is

$$\begin{aligned} x_1 &= r + 3s - 2t \\ x_2 &= t \\ x_3 &= 1 - 2r \\ x_4 &= s \\ x_5 &= 2 - r \\ x_6 &= r, \end{aligned} \tag{8}$$

where r, s, and t are any real numbers. Thus (8) is the solution to the given linear system (6). Since r, s, and t can be assigned any real numbers, the given linear system (6) has infinitely many solutions. ■

The following example illustrates the Gaussian elimination procedure and back substitution.

EXAMPLE 11 Solve the linear system given in Example 8 by Gaussian elimination.

Solution *Step 1.* The augmented matrix of this linear system is

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right].$$

Step 2. A row echelon form of the augmented matrix is (verify)

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

This augmented matrix corresponds to the equivalent linear system

$$\begin{aligned} x + 2y + 3z &= 9 \\ y + z &= 2 \\ z &= 3. \end{aligned}$$

Step 3. The process of back substitution starts with the equation $z = 3$. We next substitute this value of z in the preceding equation $y + z = 2$ and solve for y, obtaining $y = 2 - z = 2 - 3 = -1$. Finally, we substitute the values just obtained for y and z in the first equation $x + 2y + 3z = 9$ and solve for x, obtaining $x = 9 - 2y - 3z = 9 + 2 - 9 = 2$. Thus, the solution is $x = 2$, $y = -1$, and $z = 3$. ∎

EXAMPLE 12 Solve the linear system

$$\begin{aligned} x + 2y + 3z + 4w &= 5 \\ x + 3y + 5z + 7w &= 11 \\ x - z - 2w &= -6 \end{aligned} \tag{9}$$

by Gauss–Jordan reduction.

Solution *Step 1.* The augmented matrix of this linear system is

$$\left[\begin{array}{rrrr|r} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -6 \end{array}\right].$$

Step 2. The augmented matrix is row equivalent to the matrix (verify)

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{10}$$

Step 3. The last equation of the linear system represented by (10) is

$$0x + 0y + 0z + 0w = 1,$$

which has no solution for any x, y, z, and w. Consequently, the given linear system (9) has no solution. ∎

The last example is characteristic of the way in which we recognize that a linear system has no solution. That is, a linear system $A\mathbf{x} = \mathbf{b}$ in n unknowns has no solution if and only if its augmented matrix is row equivalent to a matrix in reduced row echelon form or row echelon form, which has a row whose first n elements are zero and whose $(n + 1)$st element is 1 (Exercise T.4).

The linear systems of Examples 8, 9, and 10 each had at least one solution, while the system in Example 12 had no solution. Linear systems with at least one solution are called **consistent**, and linear systems with no solutions are called **inconsistent**. Every inconsistent linear system when in reduced row echelon form or row echelon form results in the situation illustrated in Example 12.

Remarks

1. As we perform elementary row operations, we may encounter a row of the augmented matrix being transformed to reduced row echelon form whose first n entries are zero and whose $(n + 1)$st entry is not zero. In this case, we can stop our computations and conclude that the given linear system is inconsistent.
2. Sometimes we need to solve k linear systems

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \ldots, \quad A\mathbf{x} = \mathbf{b}_k,$$

all having the same $m \times n$ coefficient matrix A. Instead of solving each linear system separately, we proceed as follows. Form the $m \times (n + k)$ augmented matrix

$$\left[A \mid \mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_k\right].$$

The reduced row echelon form

$$\left[C \mid \mathbf{d}_1 \quad \mathbf{d}_2 \quad \cdots \quad \mathbf{d}_k\right]$$

of this matrix corresponds to the linear systems

$$C\mathbf{x} = \mathbf{d}_1, \quad C\mathbf{x} = \mathbf{d}_2, \ldots, \quad C\mathbf{x} = \mathbf{d}_k,$$

which have the same solutions as the corresponding given linear systems. This approach will be useful in Section 6.7. Exercises 35 and 36 ask you to explore this technique.

HOMOGENEOUS SYSTEMS

A linear system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{11}$$

is called a **homogeneous system**. We can also write (11) in matrix form as

$$A\mathbf{x} = \mathbf{0}. \tag{12}$$

The solution

$$x_1 = x_2 = \cdots = x_n = 0$$

to the homogeneous system (12) is called the **trivial solution**. A solution $x_1, x_2, \ldots, x_n$ to a homogeneous system in which not all the x_i are zero is called a **nontrivial solution**. We see that a homogeneous system is always consistent, since it always has the trivial solution.

EXAMPLE 13 Consider the homogeneous system

$$\begin{aligned} x + 2y + 3z &= 0 \\ -x + 3y + 2z &= 0 \\ 2x + y - 2z &= 0. \end{aligned} \tag{13}$$

The augmented matrix of this system,

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ -1 & 3 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right],$$

is row equivalent to (verify)

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

which is in reduced row echelon form. Hence the solution to (13) is

$$x = y = z = 0,$$

which means that the given homogeneous system (13) has only the trivial solution. ■

EXAMPLE 14 Consider the homogeneous system

$$\begin{aligned} x + y + z + w &= 0 \\ x \qquad\quad + w &= 0 \\ x + 2y + z \qquad &= 0. \end{aligned} \tag{14}$$

The augmented matrix of this system,

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{array}\right],$$

is row equivalent to (verify)

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right],$$

which is in reduced row echelon form. Hence the solution to (14) is

$$\begin{aligned} x &= -r \\ y &= r \\ z &= -r \\ w &= r, \end{aligned}$$

where r is any real number. For example, if we let $r = 2$, then

$$x = -2, \quad y = 2, \quad z = -2, \quad w = 2$$

is a nontrivial solution to this homogeneous system. That is,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(Verify by computing the matrix product on the left side.) Hence, this linear system has infinitely many solutions. ■

Example 14 shows that a homogeneous system may have a nontrivial solution. The following theorem tells of one case when this occurs.

THEOREM 1.8 *A homogeneous system of m equations in n unknowns always has a nontrivial solution if $m < n$, that is, if the number of unknowns exceeds the number of equations.*

Proof Let C be the reduced row echelon form of A. Then the homogeneous systems $A\mathbf{x} = \mathbf{0}$ and $C\mathbf{x} = \mathbf{0}$ are equivalent. If we let r be the number of nonzero rows of C, then $r \leq m$. If $m < n$, we conclude that $r < n$. We are then solving r equations in n unknowns and can solve for r unknowns in terms of the remaining $n - r$ unknowns, the latter being free to take on any real numbers. Thus, by letting one of these $n - r$ unknowns be nonzero, we obtain a nontrivial solution to $C\mathbf{x} = \mathbf{0}$ and thus to $A\mathbf{x} = \mathbf{0}$. ■

We shall also use Theorem 1.8 in the following equivalent form: If A is $m \times n$ and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $m \geq n$.

The following result is important in the study of differential equations. (See Section 9.2.)

Let $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$, be a consistent linear system. If $\mathbf{x}_p$ is a particular solution to the given nonhomogeneous system and $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_p + \mathbf{x}_h$ is a solution to the given system $A\mathbf{x} = \mathbf{b}$. Moreover, every solution $\mathbf{x}$ to the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution to the given nonhomogeneous system and $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. For a proof, see Exercise T.13.

EXAMPLE 15 Consider the linear system given in Example 9. A solution to this linear system was given by

$$\begin{aligned} x &= -5 - 2r \\ y &= 2 + 3r \\ z &= 3 + 2r \\ w &= r, \end{aligned}$$

where r is any real number. If we let

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix},$$

then the solution can be expressed as

$$\mathbf{x} = \begin{bmatrix} -5 - 2r \\ 2 + 3r \\ 3 + 2r \\ r \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2r \\ 3r \\ 2r \\ r \end{bmatrix}.$$

Let

$$\mathbf{x}_p = \begin{bmatrix} -5 \\ 2 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_h = \begin{bmatrix} -2r \\ 3r \\ 2r \\ r \end{bmatrix}.$$

Then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$. Moreover, $\mathbf{x}_p$ is a particular solution to the given system and $\mathbf{x}_h$ is a solution to the associated homogeneous system [verify that $A\mathbf{x}_p = \mathbf{b}$ and $A\mathbf{x}_h = \mathbf{0}$, where A is the coefficient matrix in Example 9 and $\mathbf{b}$ is the right side of Equation (3)]. ■

Remark Homogeneous systems are special and they will play a key role in later chapters in this book.

POLYNOMIAL INTERPOLATION

Suppose we are given the n distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Can we find a polynomial of degree $n-1$ or less that "interpolates" the data, that is, passes through the n points? Thus, the polynomial we seek has the form

$$y = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0.$$

The n given points can be used to obtain an $n \times n$ linear system whose unknowns are $a_0, a_1, \ldots, a_{n-1}$. It can be shown that this linear system has a unique solution. Thus, there is a unique interpolating polynomial.

We consider the case where $n = 3$ in detail. Here we are given the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, where $x_1 \neq x_2$, $x_1 \neq x_3$, and $x_2 \neq x_3$, and seek the polynomial

$$y = a_2x^2 + a_1x + a_0. \tag{15}$$

Substituting the given points in (15), we obtain the linear system

$$\begin{aligned} a_2x_1^2 + a_1x_1 + a_0 &= y_1 \\ a_2x_2^2 + a_1x_2 + a_0 &= y_2 \\ a_2x_3^2 + a_1x_3 + a_0 &= y_3. \end{aligned} \tag{16}$$

We show in Section 3.2 that the linear system (16) has a unique solution. Thus, there is a unique interpolating quadratic polynomial. In general, there is a unique interpolating polynomial of degree $n-1$ passing through n given points.

EXAMPLE 16 Find the quadratic polynomial that interpolates the points $(1, 3)$, $(2, 4)$, $(3, 7)$.

Solution Setting up the linear system (16) we have

$$\begin{aligned} a_2 + a_1 + a_0 &= 3 \\ 4a_2 + 2a_1 + a_0 &= 4 \\ 9a_2 + 3a_1 + a_0 &= 7 \end{aligned}$$

whose solution is (verify)

$$a_2 = 1, \qquad a_1 = -2, \qquad a_0 = 4.$$

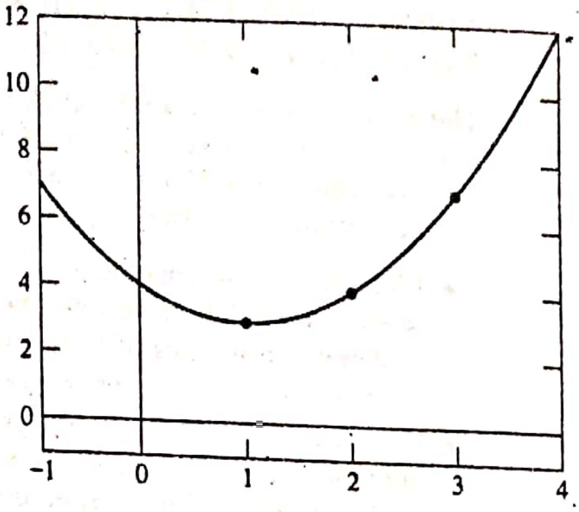
Hence, the quadratic interpolating polynomial is

$$y = x^2 - 2x + 4.$$

Its graph, shown in Figure 1.17, passes through the three given points. ■

Section 2.4, Electrical Circuits, Section 2.5, Markov Chains, and Chapter 11, Linear Programming, which can be studied at this time, use material from this section.

Figure 1.17 ►



TEMPERATURE DISTRIBUTION

A simple model for estimating the temperature distribution on a square plate gives rise to a linear system of equations. To construct the appropriate linear system, we use the following information. The square plate is perfectly insulated on its top and bottom so that the only heat flow is through the plate itself. The four edges are held at various temperatures. To estimate the temperature at an interior point on the plate, we use the rule that it is the average of the temperatures at its four compass point neighbors, to the west, north, east, and south.

EXAMPLE 17 Estimate the temperatures T_i, $i = 1, 2, 3, 4$, at the four equispaced interior points on the plate shown in Figure 1.18.

Solution We now construct the linear system to estimate the temperatures. The points at which we need the temperatures of the plate for this model are indicated in Figure 1.18 by dots. Using our averaging rule, we obtain the equations

$$
\begin{aligned}
T_1 &= \frac{60 + 100 + T_2 + T_3}{4} &\quad\text{or}\quad& 4T_1 - T_2 - T_3 = 160\\
T_2 &= \frac{T_1 + 100 + 40 + T_4}{4} &\quad\text{or}\quad& -T_1 + 4T_2 - T_4 = 140\\
T_3 &= \frac{60 + T_1 + T_4 + 0}{4} &\quad\text{or}\quad& -T_1 + 4T_3 - T_4 = 60\\
T_4 &= \frac{T_3 + T_2 + 40 + 0}{4} &\quad\text{or}\quad& -T_2 - T_3 + 4T_4 = 40.
\end{aligned}
$$

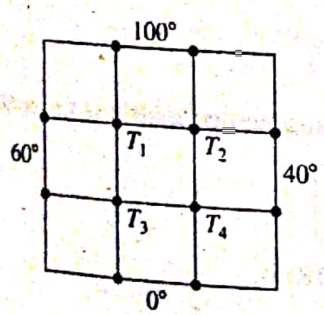


Figure 1.18 ▲

The augmented matrix for this linear system is (verify)

$$
\left[A \,\vdots\, \mathbf{b}\right] = \left[\begin{array}{rrrr|r} 4 & -1 & -1 & 0 & 160\\ -1 & 4 & 0 & -1 & 140\\ -1 & 0 & 4 & -1 & 60\\ 0 & -1 & -1 & 4 & 40 \end{array}\right].
$$

Using Gaussian elimination or Gauss–Jordan reduction, we obtain the unique solution (verify)

$$
T_1 = 65^\circ,\quad T_2 = 60^\circ,\quad T_3 = 40^\circ,\quad \text{and}\quad T_4 = 35^\circ. \quad ■
$$

Key Terms

Reduced row echelon form	Row echelon form of a matrix	Homogeneous system
Leading one	Gauss–Jordan reduction	Trivial solution
Row echelon form	Gaussian elimination	Nontrivial solution
Elementary row operation	Back substitution	Bit linear systems
Row equivalent	Consistent linear system	
Reduced row echelon form of a matrix	Inconsistent linear system	

1.6 Exercises

In Exercises 1 through 8, determine whether the given matrix is in reduced row echelon form, row echelon form, or neither.

1. $\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & -2 & 3 \end{bmatrix}$

9. Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 1 & 4 \\ 4 & 2 & 2 \\ 5 & -1 & 5 \end{bmatrix}.$$

Find the matrices obtained by performing the following elementary row operations on A.

(a) Interchanging the second and fourth rows

(b) Multiplying the third row by 3

(c) Adding (-3) times the first row to the fourth row

10. Let

$$A = \begin{bmatrix} 2 & 0 & 4 & 2 \\ 3 & -2 & 5 & 6 \\ -1 & 3 & 1 & 1 \end{bmatrix}.$$

Find the matrices obtained by performing the following elementary row operations on A.

(a) Interchanging the second and third rows

(b) Multiplying the second row by (-4)

(c) Adding 2 times the third row to the first row

11. Find three matrices that are row equivalent to

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 5 & 2 & -3 & 4 \end{bmatrix}.$$

12. Find three matrices that are row equivalent to

$$\begin{bmatrix} 4 & 3 & 7 & 5 \\ -1 & 2 & -1 & 3 \\ 2 & 0 & 1 & 4 \end{bmatrix}.$$

In Exercises 13 through 16, find a row echelon form of the given matrix.

13. $\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 26/7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -2 & 0 & 2 \\ 2 & -3 & -1 & 5 \\ 1 & 3 & 2 & 5 \\ 1 & 1 & 0 & 2 \\ 2 & -6 & -2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$

16. $\begin{bmatrix} 2 & -1 & 0 & 1 & 4 \\ 1 & -2 & 1 & 4 & -3 \\ 5 & -4 & 1 & 6 & 5 \\ -7 & 8 & -3 & -14 & 1 \end{bmatrix}$

17. For each of the matrices in Exercises 13 through 16, find the reduced row echelon form of the given matrix.

18. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix}.$$

In each part, determine whether $\mathbf{x}$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$.

(a) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; $\mathbf{b} = \mathbf{0}$ (b) $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $\mathbf{b} = \mathbf{0}$

(c) $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$

(d) $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

19. Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 0 & 2 \\ -1 & 2 & 1 & 3 \end{bmatrix}.$$

In each part, determine whether $\mathbf{x}$ is a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

(a) $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 5 \\ 2 \end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(c) $\mathbf{x} = \begin{bmatrix} 1 \\ -\frac{3}{5} \\ 1 \\ \frac{2}{5} \end{bmatrix}$ (d) $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

In Exercises 20 through 22, find all solutions to the given linear system.

0,2!

2)

20. (a)
$$\begin{aligned} x + y + 2z &= -1 \\ x - 2y + z &= -5 \\ 3x + y + z &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x + y + 3z + 2w &= 7 \\ 2x - y + 4w &= 8 \\ 3y + 6z &= 8 \end{aligned}$$

(c)
$$\begin{aligned} x + 2y - 4z &= 3 \\ x - 2y + 3z &= -1 \\ 2x + 3y - z &= 5 \\ 4x + 3y - 2z &= 7 \\ 5x + 2y - 6z &= 7 \end{aligned}$$

(d)
$$\begin{aligned} x + y + z &= 0 \\ x + z &= 0 \\ 2x + y - 2z &= 0 \\ x + 5y + 5z &= 0 \end{aligned}$$

like example 8

21. (a)
$$\begin{aligned} x + y + 2z + 3w &= 13 \\ x - 2y + z + w &= 8 \\ 3x + y + z - w &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z &= 1 \\ x + y - 2z &= 3 \\ 2x + y + z &= 2 \end{aligned}$$

(c)
$$\begin{aligned} 2x + y + z - 2w &= 1 \\ 3x - 2y + z - 6w &= -2 \\ x + y - z - w &= -1 \\ 6x + z - 9w &= -2 \\ 5x - y + 2z - 8w &= 3 \end{aligned}$$

(d)
$$\begin{aligned} x + 2y + 3z - w &= 0 \\ 2x + y - z + w &= 3 \\ x - y + w &= -2 \end{aligned}$$

22. (a)
$$\begin{aligned} 2x - y + z &= 3 \\ x + 3y + z &= 4 \\ -5x - 2z &= -5 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z + w &= 6 \\ 2x + y - z &= 3 \\ 3x + y + 2w &= 6 \end{aligned}$$

(c)
$$\begin{aligned} 2x - y + z &= 3 \\ 3x + y - 2z &= -2 \\ x - y + z &= 7 \\ x + 5y + 7z &= 13 \\ x - 7y - 5z &= 12 \end{aligned}$$

(d)
$$\begin{aligned} x + 2y - z &= 0 \\ 2x + y + z &= 0 \\ 5x + 7y + z &= 0 \end{aligned}$$

In Exercises 23 through 26, find all values of a for which the resulting linear system has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

23.
$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

24.
$$\begin{aligned} x + y + z &= 2 \\ 2x + 3y + 2z &= 5 \\ 2x + 3y + (a^2 - 1)z &= a + 1 \end{aligned}$$

25.
$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

26.
$$\begin{aligned} x + y &= 3 \\ x + (a^2 - 8)y &= a \end{aligned}$$

In Exercises 27 through 30, solve the linear system with the given augmented matrix.

27. (a) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right]$.

28. (a) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 5 & 7 & 9 & 0 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 2 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ 1 & 2 & 3 & 11 \\ 2 & 1 & 4 & 12 \end{array}\right]$.

29. (a) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 8 \\ 1 & 3 & 0 & 1 & 7 \\ 1 & 0 & 2 & 1 & 3 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & -1 & -3 & 5 \\ 3 & 0 & 1 & 2 \\ 3 & -3 & 0 & 7 \end{array}\right]$

30. (a) $\left[\begin{array}{ccc|c} 4 & 2 & -1 & 5 \\ 3 & 3 & 6 & 1 \\ 5 & 1 & -8 & 8 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 1 & 1 & 3 & -3 & 0 \\ 0 & 2 & 1 & -3 & 3 \\ 1 & 0 & 2 & -1 & -1 \end{array}\right]$.

31. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find x, y, z so that $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$.

32. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find x, y, z so that $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$.

33. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find an equation relating a, b, and c so that we can always compute values of x, y, and z for which

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

34. Let $f: R^3 \to R^3$ be the matrix transformation defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find an equation relating a, b, and c so that we can always compute values of x, y, and z for which

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

In Exercises 35 and 36, solve the linear systems $A\mathbf{x} = \mathbf{b}_1$ *and* $A\mathbf{x} = \mathbf{b}_2$ *separately and then by obtaining the reduced row echelon form of the augmented matrix* $\left[A \;\vdots\; \mathbf{b}_1 \;\; \mathbf{b}_2\right]$. *Compare your answers.*

35. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

36. $A = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & -1 \\ 4 & -2 & 3 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 3 \\ -7 \\ 12 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -4 \\ 6 \\ -10 \end{bmatrix}$

In Exercises 37 and 38, let

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}.$$

37. Find a nontrivial solution to the homogeneous system $(-4I_3 - A)\mathbf{x} = \mathbf{0}$.*

38. Find a nontrivial solution to the homogeneous system $(2I_3 - A)\mathbf{x} = \mathbf{0}$.*

39. Find an equation relating a, b, and c so that the linear system

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 3y + 3z &= b \\ 5x + 9y - 6z &= c \end{aligned}$$

is consistent for any values of a, b, and c that satisfy that equation.

40. Find an equation relating a, b, and c so that the linear system

$$\begin{aligned} 2x + 2y + 3z &= a \\ 3x - y + 5z &= b \\ x - 3y + 2z &= c \end{aligned}$$

is consistent for any values of a, b, and c that satisfy that equation.

*This type of problem will play a key role in Chapter 8.

41. Find a 2×1 matrix $\mathbf{x}$ with entries not all zero such that $A\mathbf{x} = 4\mathbf{x}$, where

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}.$$

[*Hint:* Rewrite the matrix equation $A\mathbf{x} = 4\mathbf{x}$ as $4\mathbf{x} - A\mathbf{x} = (4I_2 - A)\mathbf{x} = \mathbf{0}$ and solve the homogeneous system.]

42. Find a 2×1 matrix $\mathbf{x}$ with entries not all zero such that $A\mathbf{x} = 3\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

43. Find a 3×1 matrix with entries not all zero such that $A\mathbf{x} = 3\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

44. Find a 3×1 matrix $\mathbf{x}$ with entries not all zero such that $A\mathbf{x} = 1\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

In Exercises 45 and 46, solve the given linear system and write the solution $\mathbf{x}$ as $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution to the given system and $\mathbf{x}_h$ is a solution to the associated homogeneous system.

45.
$$\begin{aligned} x + 2y - z - 2w &= 2 \\ 2x + y - 2z + 3w &= 2 \\ x + 2y + 3z + 4w &= 5 \\ 4x + 5y - 4z - w &= 6 \end{aligned}$$

46.
$$\begin{aligned} x - y - 2z + 3w &= 4 \\ 3x + 2y - z + 2w &= 5 \\ -y - 7z + 9w &= -2 \end{aligned}$$

In Exercises 47 and 48, find the quadratic polynomial that interpolates the given points.

47. $(1, 2), (3, 3), (5, 8)$

48. $(1, 5), (2, 12), (3, 44)$

In Exercises 49 and 50, find the cubic polynomial that interpolates the given points.

49. $(-1, -6), (1, 0), (2, 8), (3, 34)$

50. $(-2, 2), (-1, 2), (1, 2), (2, 10)$

51. A furniture manufacturer makes chairs, coffee tables, and dining-room tables. Each chair requires 10 minutes of sanding, 6 minutes of staining, and 12 minutes of varnishing. Each coffee table requires 12 minutes of sanding, 8 minutes of staining, and 12 minutes of varnishing. Each dining-room table requires 15 minutes of sanding, 12 minutes of staining, and 18 minutes of varnishing. The sanding bench is available 16 hours per week, the staining bench 11 hours per week, and the varnishing bench 18 hours per week. How many (per week) of each type of furniture should be made so that the benches are fully utilized?

52. A book publisher publishes a potential best seller in three different bindings: paperback, book club, and deluxe. Each paperback book requires 1 minute for sewing and 2 minutes for gluing. Each book club book requires 2 minutes for sewing and 4 minutes for gluing. Each deluxe book requires 3 minutes for sewing and 5 minutes for gluing. If the sewing plant is available 6 hours per day and the gluing plant is available 11 hours per day, how many books of each type can be produced per day so that the plants are fully utilized?

53. (***Calculus Required***) Construct a linear system of equations to determine a quadratic polynomial

$$p(x) = ax^2 + bx + c$$

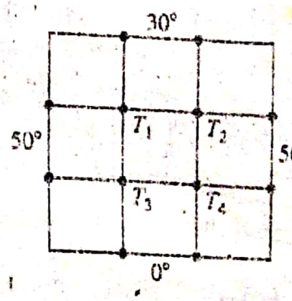
that satisfies the conditions $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$, where $f(x) = e^{2x}$.

54. (***Calculus Required***) Construct a linear system of equations to determine a quadratic polynomial

$$p(x) = ax^2 + bx + c$$

that satisfies the conditions $p(1) = f(1)$, $p'(1) = f'(1)$, and $p''(1) = f''(1)$, where $f(x) = xe^{x-1}$.

55. Determine the temperatures at the interior points T_i, $i = 1, 2, 3, 4$ for the plate shown in the figure. (See Example 17.)



In Exercises 56 through 59, solve the bit linear systems.

56. (a)
$$\begin{aligned} x + y + z &= 0 \\ y + z &= 1 \\ x + y &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z &= 1 \\ x + z &= 0 \\ y + z &= 1 \end{aligned}$$

57. (a)
$$\begin{aligned} x + y + w &= 0 \\ x + z + w &= 1 \\ y + z + w &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y &= 0 \\ x + y + z &= 1 \\ x + y + z + w &= 0 \end{aligned}$$

58. (a)
$$\begin{aligned} x + y + z &= 1 \\ y + z + w &= 1 \\ x + w &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z &= 0 \\ y + z + w &= 0 \\ x + w &= 0 \end{aligned}$$

*This type of problem will play a key role in Chapter 8.

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Exercise 1.6 ①

Q18 — Q19 : → $\frac{e}{Q_{18}}$: let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$

$$AX = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+2+2 \\ 1+1+4 \\ -2+1-4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix} = b \quad \text{i.e.}$$

$AX = b \Rightarrow X = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is a solution to the linear system $AX = b$

$Q_{20} - Q_{22}$: Find all solutions by "using Gauss-Jordan Reduction Method" to the given linear system

solution: Similar to Ex-8 (P-71) to Ex-10 (P-73)

($Q_{23} - Q_{26}$) Q_{23}:

$$\left.\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2-5)z &= a \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & a^2-5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ a \end{bmatrix}}_{b}$$

Augmented matrix: $[A \,\vdots\, b] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right] \to (*)$

i) The resulting linear system has no solution if $a = -2$
as for $a = -2$, 3rd row $\Rightarrow 0x + 0y + 0z = -4$ which is not satisfied for any choice of x, y and z.

ii) For $a \neq \pm 2$, the resulting linear system has a unique solution

As for $a = 1$, $(*) \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -1 \end{array}\right] \Rightarrow \left.\begin{aligned} x + y - z &= 2 \\ y + 2z &= 1 \\ -3z &= -1 \end{aligned}\right\}$ $-3z = -1 \Rightarrow z = 1/3$

$y + 2z = 1 \Rightarrow y = 1 - 2/3 \Rightarrow y = 1/3$ and $x + y - z = 2 \Rightarrow x = 2$

Hence $x = 2, y = 1/3, z = 1/3$ is the unique solution.

(iii) For $a = 2$, the resulting linear system has infinitely many solutions.

Q_{26}:

$$\left.\begin{array}{l} x + y = 3 \\ x + (a^2-8)y = a \end{array}\right\} \Rightarrow \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & a^2-8 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{b}{\begin{bmatrix} 3 \\ a \end{bmatrix}}$$

$$[A \vdots b] = \left[\begin{array}{cc:c} 1 & 1 & 3 \\ 1 & a^2-8 & a \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc:c} 1 & 1 & 3 \\ 0 & a^2-9 & a-3 \end{array}\right]$$

i) For $a = -3$, The resulting linear system has no solution.

ii) For $a \neq \pm 3$, The resulting linear system has a unique solution

iii) For $a = 3$, The resulting linear system has infinitely many solutions.

Explain each part of the above by Q_{23}.

Q_{27} — Q_{30}: Solve the linear system with the given augmented matrix.

a) / Q_{27}:

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \text{ by } R_3} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3]{R_1 + R_3} \left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & 3 \end{array}\right] \xrightarrow[(-1)R_3]{R_1 - R_2} \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right] \Rightarrow \begin{array}{l} x = -1 \\ y = 4 \\ z = -3 \end{array}$$ is the unique solution.

Q_{31} — Q_{32}: Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be the matrix transformation defined by

Q_{32} $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ — ①. Find x, y, z so that $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ — ②

Ⓢ From ① and ② mentioned above, we have

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{b}{\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}} \Rightarrow AX = b, \text{ linear system in matrix form}$$

$$[A \vdots b] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ -3 & -2 & -1 & 2 \\ -2 & 0 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left.\begin{array}{l} x - z = -2 \\ y + 2z = 2 \end{array}\right\} \Rightarrow$$

$\boxed{\text{i) } R_2 + 3R_1 \text{ and } R_3 + 2R_1 \quad \text{ii) } \tfrac{1}{4}R_2 \wedge R_3 - 4R_2 \quad \text{iii) } R_1 - 2R_2}$

$$\begin{array}{l} x = -2 + r \\ y = 2 - 2r \\ z = r \end{array}, \ r \in \mathbb{R}$$ is required solution which is the case of infinitely many solutions having dimension 1.

Exercise 1.6 3

Q33 — Q34: Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

Q34: $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ — ① and $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ — ②

Find ~~a relation~~ an equation relating a, b and c so that we can always compute values of x, y and z.

③ From ① and ②, we have

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{d} \Rightarrow AX = d$$

$$[A \vdots d] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ -3 & -2 & -1 & b \\ -2 & 0 & 2 & c \end{array}\right] \xrightarrow[R_3+2R_1]{R_2+3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 4 & 8 & b+3a \\ 0 & 4 & 8 & c+2a \end{array}\right] \overset{R_3-R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 4 & 8 & b+3a \\ 0 & 0 & 0 & c-b-a \end{array}\right]$$

The solution exists if $c-b-a=0 \Rightarrow a+b-c=0$ which is the desired equation relating a, b and c so that we can always compute value of x, y and z.

Q35 — Q36: $AX = b_1$ and $AX = b_2$, $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$, $b_2 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

Q35: $$[A \vdots b_1] = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 3 & -8 \end{array}\right] \overset{R_2-2R_1}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 5 & -10 \end{array}\right] \overset{\frac{1}{5}R_2}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -2 \end{array}\right] \overset{R_1+R_2}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -2 \end{array}\right]$$

$\Rightarrow \boxed{x=-1, y=-2}$ is the solution

$$[A \vdots b_2] = \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 2 & 3 & -5 \end{array}\right] \overset{R_2-2R_1}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 5 & -15 \end{array}\right] \overset{\frac{1}{5}R_2}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -3 \end{array}\right] \overset{R_1+R_2}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

$\Rightarrow \boxed{x=2, y=-3}$ is the solution.

Also $$[A \vdots b_1 \ b_2] = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 5 \\ 2 & 3 & -8 & -5 \end{array}\right] \overset{R_2-2R_1}{\sim} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 5 \\ 0 & 5 & -10 & -15 \end{array}\right] \overset{\frac{1}{5}R_2}{\sim} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 5 \\ 0 & 1 & -2 & -3 \end{array}\right]$$

$$\overset{R_1+R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & -3 \end{array}\right] \Rightarrow \begin{matrix} x=-1 \\ y=-2 \end{matrix}$$, solution of $Ax=b_1$ and $\begin{matrix} x=2 \\ y=-3 \end{matrix}$, solution of $Ax=b_2$. By comparing, both the methods give the same solution.

Exercise 1.6

Q37—Q38: Find a nontrivial solution to the homogenous system

Q38) $(2I_3 - A)X = 0$, where $A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$

$$\Rightarrow \left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & -5 \\ -1 & 1 & -1 \\ 0 & -1 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 6 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & -1 & 6 & 0 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left.\begin{array}{l} x - 5z = 0 \\ y - 6z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = -5r \\ y = 6r \\ z = r \end{array}, r \in \mathbb{R}$$

is the desired solution which is the case of infinitely many solutions. For $r = 1$, $\boxed{x = -5, y = 6, z = 1}$ is the required nontrivial solution to the given homogeneous system.

Q39—Q40:- Similar to Q34 after converting into matrix form.

Q41—Q44: Find a 2×1 matrix X with entries not all zero such that $AX = 4X$, where $A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}$

(S) $AX = 4X \Rightarrow 4X = AX \Rightarrow 4X - AX = 0$

$\Rightarrow (4I_2 - A)X = 0$ By Q37—Q38

Q45—Q46: Similar to example-15 (Page-77

Exercise 1.6 5

Q45 — Q46: Similar to Example-15 (Page-77)

Q47 — Q50: similar to example-16 (Page-78)

Q51: Let x, y and z be the number of chairs, coffee tables and dining-room tables, then the time required for those furnitures in sanding, staining and varnishing is given in the following table

	chair	coffee table	dining-room table	
sanding	$10x$	$12y$	$15z$	$\rightarrow 16\times 60 = 960$ minutes
staining	$6x$	$8y$	$12z$	$\rightarrow 11\times 60 = 660$ minutes
varnishing	$12x$	$12y$	$18z$	$\rightarrow 18\times 60 = 1080$ minutes

Thus we have the following linear system for the given problem

$$10x + 12y + 15z = 960$$
$$6x + 8y + 12z = 660$$
$$12x + 12y + 18z = 1080$$

Solve the above system by Gauss-Jordan Reduction to find the values of x, y and z which will be the desired number of chairs, coffee tables and dining-room tables made per week.

Q52: is similar to question 51.

Q53: $P(x) = ax^2 + bx + c$ — (1) and $f(x) = e^{2x}$ — (2)

(1) $\Rightarrow P'(x) = 2ax + b \Rightarrow P''(x) = 2a$

(2) $\Rightarrow f'(x) = 2e^{2x} \Rightarrow f''(x) = 4e^{2x}$

Given that $P(0) = f(0) \Rightarrow \boxed{c = 1}$; $P'(0) = f'(0) \Rightarrow \boxed{b = 2}$; $P''(0) = f''(0)$ $\Rightarrow 2a = 4 \Rightarrow \boxed{a = 2}$. Thus $\boxed{P(x) = 2x^2 + 2x + 1}$ is the desired quadratic polynomial. Q54 is similar to Q53 and Q55 is similar to Ex-17 (Page-79)