COMP 3105 – Assignment 1 Report
– Fall 2025 –

Due: Sunday September 28, 2025 23:59.

Group 51
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Question 1: (7.5%) Linear Regression

(a) (1%) L_2 Regression
Please see A1codes.py for implementation.

(b) (3%) L_∞ Regression
Here we are going to solve the L_∞ loss regression problem

$$\mathbf{w} = \text{argmin}_{\mathbf{w}\in\mathbb{R}^d} = ||X\mathbf{w} - \mathbf{y}||$$

Recall that this optimization can be expressed as a linear programming problem with the joint paramters $\begin{bmatrix} \underset{d\times 1}{\mathbf{w}} \\ \underset{1\times 1}{\delta} \end{bmatrix} \in \mathbb{R}^{d+1}$ as follows

$$\begin{gathered} \min_{\mathbf{w},\delta}\delta \\ \text{s.t.} \\ \delta \geq 0 \iff -\delta \leq 0 \\ X\mathbf{w} - \mathbf{y} \preceq \delta \cdot \mathbf{1}_n \iff X\mathbf{w} - \delta \cdot \mathbf{1}_n \preceq \mathbf{y} \\ \mathbf{y} - X\mathbf{w} \preceq \delta \cdot \mathbf{1}_n \iff -X\mathbf{w} - \delta \cdot \mathbf{1}_n \preceq -\mathbf{y} \end{gathered}$$

In the following answers, we will convert the optimization to a form that is solvable by the **cxvopt** linear programming (LP) solver, which solves the following form of LP

$$\begin{gathered} \min_{\mathbf{u}}\mathbf{c}^T\mathbf{u} \\ \text{s.t. } G\mathbf{u} \preceq \mathbf{h} \end{gathered}$$

Let the unknown variables be $\mathbf{u} = \begin{bmatrix} \underset{d\times 1}{\mathbf{w}} \\ \underset{1\times 1}{\delta} \end{bmatrix} \in \mathbb{R}^{d+1}$

For the constraints, since we have three sets of constraints, the matrix G and $\mathbf{h}$ can be decomposed into three parts

$$G \cdot \mathbf{u} = \begin{bmatrix} \underset{1\times(d+1)}{G^{(1)}} \\ \underset{n\times(d+1)}{G^{(2)}} \\ \underset{n\times(d+1)}{G^{(3)}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w} \\ \delta \end{bmatrix} \preceq \begin{bmatrix} \underset{1\times 1}{\mathbf{h}^{(1)}} \\ \underset{d\times 1}{\mathbf{h}^{(2)}} \\ \underset{d\times 1}{\mathbf{h}^{(3)}} \end{bmatrix}$$

(b.1) (0.25%) For the objective function, we want $\mathbf{c}^T\mathbf{u} = \delta$. What should $\mathbf{c} \in \mathbb{R}^{d+1}$ be?
Recall that $\mathbf{u} = \begin{bmatrix} \mathbf{w} \\ \delta \end{bmatrix}$
For the objective function to be δ we have:

$$\mathbf{c}^T\mathbf{u} = [c_1, c_2, \ldots c_d, c_{d+1}] \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \\ \delta \end{bmatrix} = c_1 w_1 + c_2 w_2 + \ldots + w_d u_d + c_{d+1}\delta$$

Now let $c_1 = c_2 = \ldots = c_d = 0$
And $c_{d+1} = 1$
This gives us:
$0w_1 + 0w_2 + \ldots 0w_d + \delta$
$= \delta.$

Thus $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Where $c_1 = c_2 = \ldots = c_d = 0$ and $c_{d+1} = 1$.

(b.2) (0.25%) We want $G^{(1)}\mathbf{u} \preceq \mathbf{h}^{(1)} \iff \delta \geq 0$. What should $G^{(1)} \in \mathbb{R}^{1\times(d+1)}$ and $h^{(1)} \in \mathbb{R}$ be?
Recall the first constraint: $-\delta \leq 0$
We want

$$\underset{1\times(d+1)}{G^{(1)}} \cdot \mathbf{u} \preceq \underset{n\times 1}{\mathbf{h}^{(1)}}$$

We will now map our constraint into this form.

$$\begin{bmatrix} \underset{1\times d}{G^{(11)}} & \underset{1\times 1}{G^{(12)}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w} \\ \delta \end{bmatrix} \preceq \underset{n\times 1}{\mathbf{h}^{(1)}} \iff -\delta \leq 0$$
$$\underset{1\times d}{G^{(11)}} \cdot \mathbf{w} + \underset{1\times 1}{G^{(12)}} \cdot \delta = -\delta \leq 0 = \underset{n\times 1}{\mathbf{h}^{(1)}}$$

So,
$\underset{1\times d}{G^{(11)}} = \mathbf{0}_{1\times d}$
$\underset{1\times 1}{G^{(12)}} = -1$
$\underset{1\times(d+1)}{G^{(1)}} = \begin{bmatrix} \mathbf{0}_{1\times d} & -1 \end{bmatrix}$
$\underset{n\times 1}{\mathbf{h}^{(1)}} = \mathbf{1}_{n\times 1}$

(b.3) (0.25%) We want $G^{(2)}\mathbf{u} \preceq \mathbf{h}^{(2)} \iff X\mathbf{w} - \mathbf{y} \preceq \delta \cdot \mathbf{1_n}$. What should $G^{(2)} \in \mathbb{R}^{n\times(d+1)}$ and $h^{(2)} \in \mathbb{R}^n$ be?
Recall our second constraint is
$X\mathbf{w} - \mathbf{y} \preceq \delta \cdot \mathbf{1}_n \iff X\mathbf{w} - \delta \cdot \mathbf{1}_n \preceq \mathbf{y}$
We want

$$\underset{n\times(d+1)}{G^{(2)}} \cdot \mathbf{u} \preceq \underset{n\times 1}{\mathbf{h}^{(2)}}$$

We will now map our constraint into this form.

$$\begin{bmatrix} \underset{n\times d}{G^{(21)}} & \underset{n\times 1}{G^{(22)}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w} \\ \delta \end{bmatrix} \preceq \underset{n\times 1}{\mathbf{h}^{(2)}} \iff X\mathbf{w} - \delta \cdot \mathbf{1}_n \preceq \mathbf{y}$$
$$\underset{n\times d}{G^{(21)}} \cdot \mathbf{w} + \underset{n\times 1}{G^{(22)}} \cdot \delta = X\mathbf{w} - \delta \cdot \mathbf{1}_n \preceq \mathbf{y} = \underset{n\times 1}{\mathbf{h}^{(1)}}$$

So,
$\underset{n\times d}{G^{(21)}} = X$
$\underset{n\times 1}{G^{(22)}} = -\mathbf{1}_{n\times 1}$
$\underset{n\times (d+1)}{G^{(2)}} = \begin{bmatrix} X & -\mathbf{1}_{n\times 1} \end{bmatrix}$
$\underset{n\times 1}{h^{(2)}} = \mathbf{y}$

(b.4) (0.25%) We want $G^{(3)}\mathbf{u} \preceq \mathbf{h}^{(3)} \iff \mathbf{y} - X\mathbf{w} \preceq \delta \cdot \mathbf{1_n}$. What should $G^{(3)} \in \mathbb{R}^{n\times(d+1)}$ and $h^{(3)} \in \mathbb{R}^n$ be?
Recall our third constraint is
$\mathbf{y} - X\mathbf{w} \preceq \delta \cdot \mathbf{1}_n \iff -X\mathbf{w} - \delta \cdot \mathbf{1}_n \preceq -\mathbf{y}$
We want

$$\underset{n\times (d+1)}{G^{(3)}} \cdot \mathbf{u} \preceq \underset{n\times 1}{\mathbf{h}^{(3)}}$$

We will now map our constraint into this form.

$$\begin{bmatrix} \underset{n\times d}{G^{(31)}} & \underset{n\times 1}{G^{(32)}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w} \\ \delta \end{bmatrix} \preceq \underset{n\times 1}{\mathbf{h}^{(3)}} \iff -X\mathbf{w} - \delta \cdot \mathbf{1}_n \preceq -\mathbf{y}$$
$$\underset{n\times d}{G^{(31)}} \cdot \mathbf{w} + \underset{n\times 1}{G^{(32)}} \cdot \delta = -X\mathbf{w} - \delta \cdot \mathbf{1}_n \preceq -\mathbf{y} = \underset{n\times 1}{\mathbf{h}^{(3)}}$$

So,
$\underset{n\times d}{G^{(31)}} = -X$
$\underset{n\times 1}{G^{(32)}} = -\mathbf{1}_{n\times 1}$
$\underset{n\times (d+1)}{G^{(3)}} = \begin{bmatrix} -X & -\mathbf{1}_{n\times 1} \end{bmatrix}$
$\underset{n\times 1}{h^{(3)}} = -\mathbf{y}$

(b.5) (2%) Based on your derivations in (b), implement a Python function

$$\mathbf{w} = \text{minimizeLinf}(X, y)$$

that returns a $d \times 1$ vector of weights/parameters $\mathbf{w}$ corresponding to the solution of minimum L_∞ loss.

$$\mathbf{w} = \text{argmin}_{w\in\mathbb{R}^d} ||X\mathbf{w} - \mathbf{y}||$$

Based on our derivations in (b), we result in

$$\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{d+1}$$

$$\mathbf{u} = \begin{bmatrix} \mathbf{w} \\ \delta \end{bmatrix} \in \mathbb{R}^{d+1}$$

$$G = \begin{bmatrix} \mathbf{0}_{1\times d} & -1 & \\ X & --\mathbf{1}_n - X & --\mathbf{1}_n \end{bmatrix} \in \mathbb{R}^{(2n+1)\times(d+1)}$$

$$\mathbf{h} = \begin{bmatrix} 0 \\ \mathbf{y} \\ -\mathbf{y} \end{bmatrix} \in \mathbb{R}^{2n+1}$$

$$\text{s.t.}$$

$$\underset{(2n+1)\times(d+1)}{G} \cdot \underset{(d+1)\times 1}{\mathbf{u}} \preceq \underset{(2n+1)\times 1}{\mathbf{h}}$$

Please see A1codes.py for full implementation.