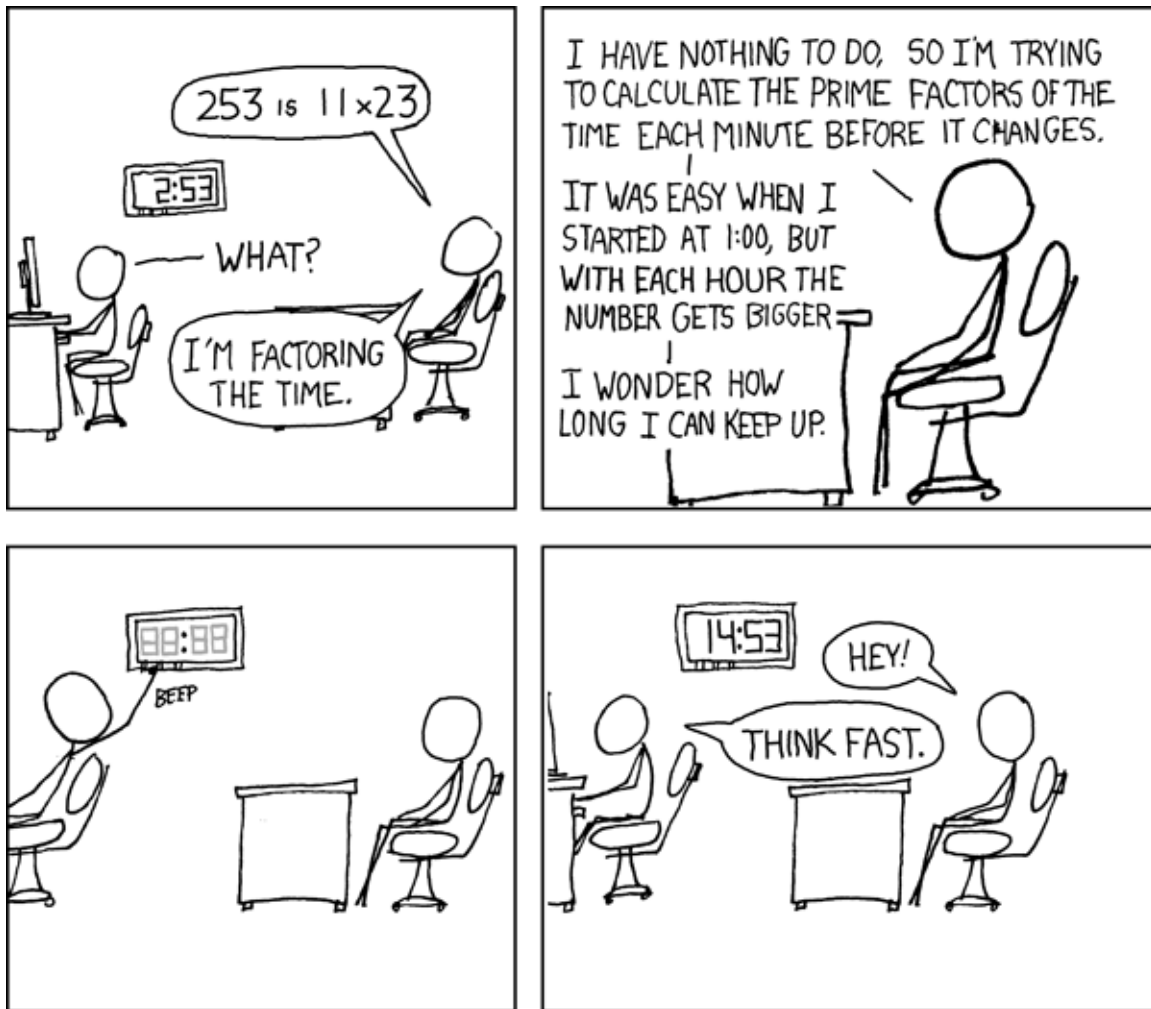

TUTORIAL 8



1 Pollard's ρ for the factorization problem

In this exercise we develop a variant of the Pollard's ρ method for factoring N . We assume that $p|N$ is the smallest (but still large for brute-force search) divisor of N . Pollard's ρ algorithm is an heuristic method: we assume that a certain deterministic sequence behaves like a truly random one.

The idea is to find two distinct $x, x' \in \mathbb{Z}_N$, s.t. $x - x' = 0 \pmod p$. The tuple (x, x') defines a *collision*. To find a collision efficiently, we define a random walk on \mathbb{Z}_N as

$$f(x) = x^2 + a \pmod N, \quad a \in \mathbb{Z}_N$$

and consider a sequence x_0, x_1, x_2, \dots such that $x_i = f(x_{i-1})$ (we fix some initial x_0 to be a random element from \mathbb{Z}_N).

1. Since f takes values in a finite set, the sequence $(x_i)_i$ should eventually repeat itself. Show that you can expect to find a collision after $\mathcal{O}(\sqrt{p})$ steps. (*Hint*: recall Birthday Paradox.) You should also be able to determine the constant in front of \sqrt{p} .
2. Describe a Pollard's ρ algorithm for factoring having the running time of order $\tilde{\mathcal{O}}(\sqrt{p})$.
3. Explain why the following choices for $f(x)$ are bad:
 - $f(x) = ax + b \pmod{N}$, $a, b \in \mathbb{Z}_N$,
 - $f(x) = x^2 \pmod{N}$,
 - $f(x) = x^2 - 2 \pmod{N}$.

2 Modular roots and factoring

The first goal of this exercise is to design an efficient algorithm to compute square roots in the group \mathbb{Z}_N . This problem is closely related to the one of factoring N . As a first step we study the Tonelli-Shanks algorithm to compute square roots modulo a prime p . In the next tutorial, we will extend it to the non-prime moduli.

The Euler criterion states that, for any odd prime p and any $a \in \mathbb{Z}_p^\times$, we have

$$a^{(p-1)/2} \equiv \begin{cases} 1 \pmod{p}, & \text{if } a \text{ is a square modulo } p \\ -1 \pmod{p}, & \text{if } a \text{ is not a square modulo } p. \end{cases}$$

1. Assume $p \equiv 3 \pmod{4}$ and let a be a square modulo p . Give an algorithm of binary complexity $\mathcal{O}(\log^3 p)$ to compute a square root of a .
2. We now assume that $p \equiv 1 \pmod{4}$, and write $p = 2^v m + 1$ with v maximal and m odd. Give a probabilistic algorithm that, given p , return $c \in \mathbb{Z}_p^\times$ which is not a square. What is its bit complexity? Show that c^m generates the (unique) subgroup of order 2^v in \mathbb{Z}_p^\times .
3. Let a be a square modulo p , and c be the output of the previous algorithm. Show that a^m belongs to the subgroup generated by c^m . Next, show how that computing a square root of a modulo p amounts to computing a discrete logarithm.
4. **Pohlig-Hellman's trick:** Let G be a cyclic group of order p^k , where $k \geq 1$. For a given generator g and $h = g^x$, show that we can compute x amounts to computations of discrete logarithms in a group of order p . (*Hint*: write x in base p .)
5. Deduce an algorithm to compute square roots modulo p .
6. (**Bonus**) Prove Euler's criterion.

3 Why not to choose primes close to \sqrt{N} for RSA

Assume one of the RSA primes is close to \sqrt{N} , more precisely,

$$|q - \sqrt{N}| < \sqrt[4]{N}.$$

Show how to factor N in time $\text{poly}(\log N)$. *Hint*. You might want to use the following fact: for $N = pq$, $N = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2$. Note that the first summand is $\approx \sqrt{N}$, while the second one is small.