Algebraic Structures & Fields

Algebraic Structure

- An algebraic structure consists of:
  - a nonempty set A,
  - a collection of operations on A (typically binary operations such as addition and multiplication), and
  - a finite set of identities (axioms), that these operations must satisfy, i.e.:
    - Commutativity: x * y = y * x, for operation ‘*’, and for every x,y in the algebraic structure
    - Associativity: (x * y) * z = x * (y * z), for operation ‘*’, and for every x,y in the algebraic structure
    - Identity element: A *binary* operation ‘*’ has an identity element if there is an element *e* such that: x * *e* = x and *e* * x = x
    - Inverse element: given a binary operation * that has an identity element *e*, and element x is “invertible” if it has an inverse element; if there exist an element inv(x), such that:

      inv(x) * x = e and x * inv(x) = e.

Relational Algebra (RA)

- RA is a theory that uses **algebraic structures** with a well-founded semantics for modelling data and defining queries on it.
- In mathematics, an algebra (over a field) is a vector space equipped with a bilinear product. Hence:
  - It is an algebraic structure consisting of a set together with operations of:
    - Multiplication
    - Addition
    - Scalar multiplication

    By elements of a field and satisfying the axioms implied by “vector space” and “bilinear”.
- A “Field” is a set on which addition, subtraction, multiplication, and division are defined and behave as the corresponding operations on rational and real numbers do.

Field (formal definition)

Formally, a field is a set F together with two binary operations on F called *addition* and *multiplication*.[1] A binary operation on F is a mapping $F \times F \to F$, that is, a correspondence that associates with each ordered pair of elements of F a uniquely determined element of F.[2][3] The result of the addition of a and b is called the sum of a and b, and is denoted $a + b$. Similarly, the result of the multiplication of a and b is called the product of a and b, and is denoted ab or $a \cdot b$. These operations are required to satisfy the following properties, referred to as *field axioms* (in these axioms, a, b, and c are arbitrary elements of the field F):

- Associativity of addition and multiplication: $a + (b + c) = (a + b) + c$, and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- Commutativity of addition and multiplication: $a + b = b + a$, and $a \cdot b = b \cdot a$.
- Additive and multiplicative identity: there exist two different elements 0 and 1 in F such that $a + 0 = a$ and $a \cdot 1 = a$.
- Additive inverses: for every a in F, there exists an element in F, denoted $-a$, called the *additive inverse* of a, such that $a + (-a) = 0$.
- Multiplicative inverses: for every $a \neq 0$ in F, there exists an element in F, denoted by a^{-1} or $1/a$, called the *multiplicative inverse* of a, such that $a \cdot a^{-1} = 1$.
- Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

This may be summarized by saying: a field has two operations, called addition and multiplication; it is an abelian group under addition with 0 as the additive identity; the nonzero elements are an abelian group under multiplication with 1 as the multiplicative identity; and multiplication distributes over addition.

Even more summarized: a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible under multiplication.