Spatial Gaussian Process Regression for Bayesian Optical Flow Estimation

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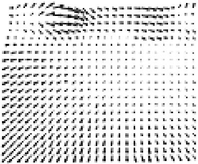
Motion in Image Sequences

First frame

$I(x, y, t)$



$\mathbf{f}(x, y, t)$

Second frame

$I(x, y, t + \delta t)$

Given two sequential images $I : \mathcal{D} \times [0, T] \rightarrow \mathcal{R}$, i.e.

$$I(x, y, t) \text{ and } I(x, y, t + \delta t)$$

(for fixed t and δt), the optical flow at time t is a vector field

$$\boldsymbol{f}(x, y, t) = (u(x, y, t), v(x, y, t))$$

that transforms one image into the next:

$$I\,(x + u(x, y, t), y + v(x, y, t), t + \delta t) \simeq I(x, y, t)$$

Barron, J. L., Fleet, D. J., & Beauchemin, S. S. (1994). Performance of Optical Flow Techniques. *International Journal of Computer Vision, 12*(1), 43-77.

Lucas-Kanade Method

Brightness constancy assumption:

$$\frac{d}{dt} I(x(t), y(t), t) = \frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \frac{\partial I}{\partial t} = 0, \quad u := \frac{dx}{dt}, v := \frac{dy}{dt}$$

Underdetermined, so assume same $(u(x,y), v(x,y))$ $\forall (x,y) \in W$:

$$\underbrace{\begin{bmatrix} I_x(x_1, y_1) & I_y(x_1, y_1) \\ \vdots & \vdots \\ I_x(x_n, y_n) & I_y(x_n, y_n) \end{bmatrix}}_{\boldsymbol{\nabla I}^T} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\boldsymbol{f}} = \underbrace{\begin{bmatrix} -I_t(x_1, y_1) \\ \vdots \\ -I_t(x_n, y_n) \end{bmatrix}}_{-\boldsymbol{I_t}}$$

Overdetermined: least squares solution is $\boldsymbol{v} = \left(\boldsymbol{A}^T \boldsymbol{A}\right)^{-1} \boldsymbol{A}^T \boldsymbol{b}$:

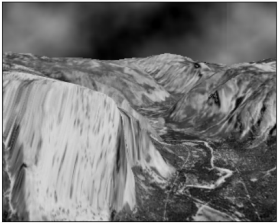
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum I_x(x_i, y_i)^2 & \sum I_x(x_i, y_i) I_y(x_i, y_i) \\ \sum I_x(x_i, y_i) I_y(x_i, y_i) & \sum I_y(x_i, y_i)^2 \end{bmatrix}^{-1} \begin{bmatrix} -\sum I_x(x_i, y_i) I_t(x_i, y_i) \\ -\sum I_y(x_i, y_i) I_t(x_i, y_i) \end{bmatrix}$$

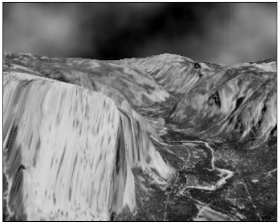
Optional weighting of pixels via diagonal matrix $\boldsymbol{W}$:

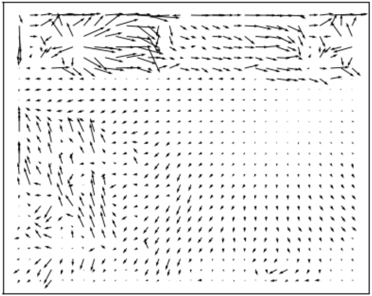
$$\boldsymbol{v} = \left(\boldsymbol{A}^T \boldsymbol{W} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^T \boldsymbol{W} \boldsymbol{b}$$

Lucas, B. D., & Kanade, T. (1981). An Iterative Image Registration Technique with an Application to Stereo Vision. *7th International Joint Conference on Artificial Intelligence*, *2*, 674-679.

Lucas-Kanade Results







Barron, J. L., Fleet, D. J., & Beauchemin, S. S. (1994). Performance of Optical Flow Techniques. *International Journal of Computer Vision, 12*(1), 43-77.

Probabilistic Motivation

Classical methods produce deterministic point estimates.
Uncertainty is inherent in optical flow estimation, due to:

- Image noise
- Brightness changes
- Low contrast regions
- Object occlusion
- Aperture problem
- Incompatible motions in localized regions

Quantification of confidence is desirable for safety-critical aplications, e.g.:

- Autonomous navigation
- Computer-integrated surgery
- Real-time surveillance

Wang-Orquiza Method

Suppose $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}$, $\boldsymbol{A}$ fixed, $\boldsymbol{x}$ fixed unknown, $\boldsymbol{y}$ observed, $\boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{H})$:

$$\hat{\boldsymbol{x}}_{\mathrm{MLE}}(\boldsymbol{y}) = \arg\max_{\boldsymbol{x}} f_{\boldsymbol{y}}(\boldsymbol{y}|\boldsymbol{x}) = \arg\max_{\boldsymbol{x}} f_{\boldsymbol{\eta}}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}) = \boldsymbol{\Sigma}_{\hat{\boldsymbol{x}}} \boldsymbol{A}^T \boldsymbol{H}^{-1} \boldsymbol{y}$$

$$\boldsymbol{\Sigma}_{\hat{\boldsymbol{x}}} = \mathrm{Cov}[\hat{\boldsymbol{x}}_{\mathrm{MLE}}(\boldsymbol{y}) - \boldsymbol{x}] = (\boldsymbol{A}^T \boldsymbol{H}^{-1} \boldsymbol{A})^{-1}$$

Same assumptions as Lucas-Kanade, plus additive Gaussian noise:

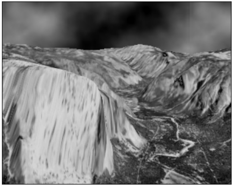
$$\boldsymbol{I_t} = -\boldsymbol{\nabla I}^T \boldsymbol{f} + \boldsymbol{\eta} \qquad \boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{H})$$

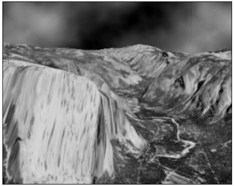
Results above give:

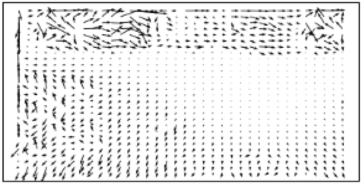
$$\hat{\boldsymbol{f}} = -\boldsymbol{\Sigma}_{\hat{\boldsymbol{f}}} \boldsymbol{\nabla I} \boldsymbol{H}^{-1} \boldsymbol{I}_t \qquad \boldsymbol{\Sigma}_{\hat{\boldsymbol{f}}} = (\boldsymbol{\nabla I} \boldsymbol{H}^{-1} \boldsymbol{\nabla I}^T)^{-1}$$

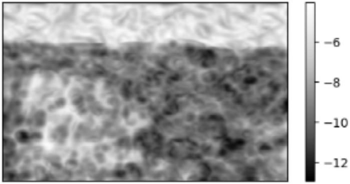
Poor texture around $\boldsymbol{x} \implies$ small $\nabla \boldsymbol{I} \implies$ large $\boldsymbol{\Sigma}_{\hat{\boldsymbol{f}}}$.

Wang-Orquiza Results









Spatial Gaussian Processes

2-dimensional Gaussian process (distribution over functions of 2D inputs):

$$\boldsymbol{f}(\cdot) = \begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} \sim \mathcal{GP}(\boldsymbol{m}(\cdot), \boldsymbol{k}(\cdot, \cdot'))$$

Example mean & covariance kernel (affine & RBF):

$$\boldsymbol{m}(\cdot) = \boldsymbol{A}(\cdot) + \boldsymbol{b}, \;\; \boldsymbol{k}(\cdot, \cdot') = \exp\left(-\frac{\| \cdot - \cdot' \|^2}{2\lambda^2}\right) \boldsymbol{\Sigma}$$

Joint distribution of function values at finite subset of points:

$$\begin{bmatrix} \boldsymbol{f}(\boldsymbol{x}_1) \\ \vdots \\ \boldsymbol{f}(\boldsymbol{x}_n) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{m}(\boldsymbol{x}_1) \\ \ddots \\ \boldsymbol{m}(\boldsymbol{x}_n) \end{bmatrix}, \begin{bmatrix} \boldsymbol{k}(\boldsymbol{x}_1, \boldsymbol{x}_1) & \cdots & \boldsymbol{k}(\boldsymbol{x}_1, \boldsymbol{x}_n) \\ \vdots & \ddots & \vdots \\ \boldsymbol{k}(\boldsymbol{x}_n, \boldsymbol{x}_1) & \cdots & \boldsymbol{k}(\boldsymbol{x}_n, \boldsymbol{x}_n) \end{bmatrix} \right)$$

Equivalently:

$$\boldsymbol{F} := \begin{bmatrix} \boldsymbol{U}(\boldsymbol{X}) \\ \boldsymbol{V}(\boldsymbol{X}) \end{bmatrix} \sim \mathcal{N}\left(\boldsymbol{\mu} := \begin{bmatrix} \boldsymbol{\mu}_u \\ \boldsymbol{\mu}_v \end{bmatrix}, \boldsymbol{K} := \begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{uv} \\ \boldsymbol{K}_{vu} & \boldsymbol{K}_{vv} \end{bmatrix} \right)$$

Gaussian Process Regression

Through Wang-Orquiza method, we obtain noisy observations $\tilde{\boldsymbol{F}}$:

$$\tilde{\boldsymbol{f}}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{\eta}(\boldsymbol{x}),\ \ \boldsymbol{H}(\boldsymbol{x}) \sim \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}}\right)$$

The noisy distribution is:

$$\tilde{\boldsymbol{F}} \middle| \boldsymbol{F} \sim \mathcal{N}\left(\boldsymbol{F}, \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}}\right),\ \ \boldsymbol{F} \sim \mathcal{N}\left(\boldsymbol{\mu}, \boldsymbol{K}\right) \implies \tilde{\boldsymbol{F}} \sim \mathcal{N}\left(\boldsymbol{\mu}, \boldsymbol{K} + \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}}\right)$$

We want to estimate true optical flows: we form the joint distribution.

$$\begin{bmatrix} \boldsymbol{F} \\ \tilde{\boldsymbol{F}} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu} \end{bmatrix}, \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K} \\ \boldsymbol{K} & \boldsymbol{K} + \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}} \end{bmatrix}\right)$$

We derive the conditional distribution (posterior):

$$\boldsymbol{F}|\tilde{\boldsymbol{F}} \sim \mathcal{N}\left(\boldsymbol{\mu} + \boldsymbol{K}\left(\boldsymbol{K} + \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}}\right)^{-1}\left(\tilde{\boldsymbol{F}} - \boldsymbol{\mu}\right), \boldsymbol{K} - \boldsymbol{K}\left(\boldsymbol{K} + \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}}\right)^{-1}\boldsymbol{K}\right)$$

Mean serves as estimate, covariance as uncertainty.

Parameter Fitting

Mean/covariance parameters (e.g. lengthscale λ^2) are optimized by maximizing the marginal likelihood of observations $\tilde{\boldsymbol{F}}$. This likelihood is:

$$p\left(\tilde{\boldsymbol{F}}\right) \propto \frac{1}{\sqrt{\det\left(\boldsymbol{K} + \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}}\right)}} \exp\left(-\frac{1}{2}\left(\tilde{\boldsymbol{F}} - \boldsymbol{\mu}\right)^T \left(\boldsymbol{K} + \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}}\boldsymbol{I}\right)^{-1}\left(\tilde{\boldsymbol{F}} - \boldsymbol{\mu}\right)\right)$$

In practice we minimize negative log-likelihood via gradient descent:

$$-\log p\left(\tilde{\boldsymbol{F}}\right) \propto \frac{1}{2}\left(\tilde{\boldsymbol{F}} - \boldsymbol{\mu}\right)^T \left(\boldsymbol{K} + \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}}\right)^{-1}\left(\tilde{\boldsymbol{F}} - \boldsymbol{\mu}\right) + \frac{1}{2}\det\left(\boldsymbol{K} + \boldsymbol{\Sigma}_{\tilde{\boldsymbol{f}}}\right)$$

We minimize this numerically via gradient descent.

Parameter Fitting (cont.)

Let φ be a parameter of $\boldsymbol{m}$, $\boldsymbol{y} = \tilde{\boldsymbol{F}} - \boldsymbol{\mu}$, $\boldsymbol{K}_y = \boldsymbol{K} + \sigma_\eta^2 \boldsymbol{I}$:

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{y}^T}{\partial \varphi}\right)\boldsymbol{K}_y^{-1}\boldsymbol{y} + \boldsymbol{y}^T\boldsymbol{K}_y\left(\frac{\partial \boldsymbol{y}}{\partial \varphi}\right)\right] = \frac{1}{2}\left(2\boldsymbol{y}^T\boldsymbol{K}_y\left(\frac{\partial \boldsymbol{y}}{\partial \varphi}\right)\right) = -\boldsymbol{y}^T\boldsymbol{K}_y\left(\frac{\partial \boldsymbol{m}}{\partial \varphi}\right)$$

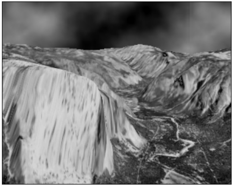
Let θ be a parameter of $\boldsymbol{k}$:

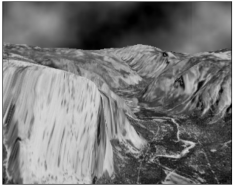
$$\frac{\partial \mathcal{L}}{\partial \theta} = \boldsymbol{y}^T\left(\frac{\partial \boldsymbol{K}_y^{-1}}{\partial \theta}\right)\boldsymbol{y} + \frac{\partial |\boldsymbol{K}_y|}{\partial \theta} = -\frac{1}{2}\boldsymbol{y}^T\boldsymbol{K}^{-1}\left(\frac{\partial \boldsymbol{K}_y}{\partial \theta}\right)\boldsymbol{K}^{-1}\boldsymbol{y} + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{K}_y^{-1}\left(\frac{\partial \boldsymbol{K}_y}{\partial \theta}\right)\right)$$

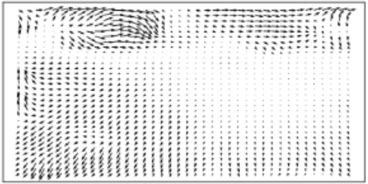
RBF parameter λ^2:

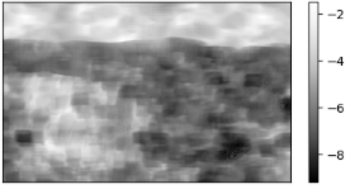
$$\frac{\partial \boldsymbol{K}_y}{\partial \lambda} = \boldsymbol{K} \odot \frac{\|\boldsymbol{x} - \boldsymbol{x}'\|}{\lambda^3}$$

GP Regression Results









Discussion

Our maximum-likelihood method produced an initial estimate and uncertainty.

Our Gaussian-process Bayesian method provided updated these (posterior), after enforcing a global smoothness constraint (Gaussian process prior).

Possible next steps:

- Obtain better observations: FlowNet, RAFT, etc.
- Experiment with more complex covariance kernels.
- Perform segmentation, smooth each segment independently.
- Extend to spatiotemporal regression (3D, multiple frames).