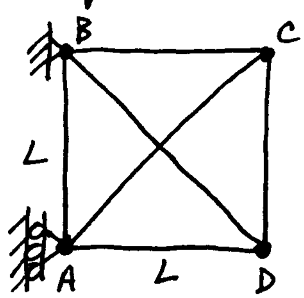
Structural Analysis

1) Equilibrium

2) Geometry (Kinematics)

3) Material behavior (constitutive law)

Example

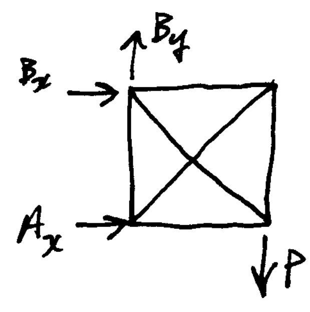


6 struts, no joint at center

Determine the forces/stresses in each strut.

E, A are identical for each strut.

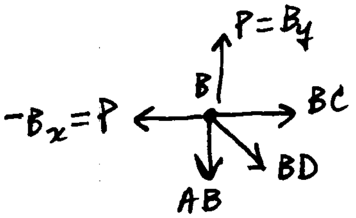
Equilibrium:

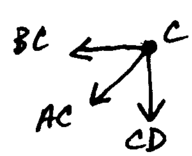


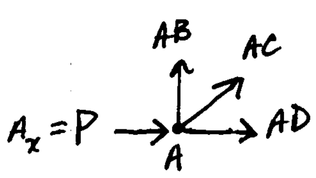
$\Sigma F_y = 0 \rightarrow B_y = P$

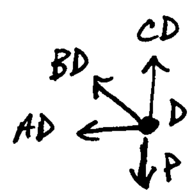
$\Sigma M_z^A = 0 \rightarrow B_x = -P$

$\Sigma F_x = 0 \rightarrow A_x = -B_x = P$









2 equilibrium equations for each joint $\rightarrow$ 8 equations

However, 3 are redundant b/c we used them for A_x, B_x, B_y.

$\rightarrow$ 5 equations, 6 unknowns

$$
\begin{array}{lllllllll}
 & & AB & AC & AD & BC & BD & CD & \\
A) & \Sigma F_x: & & \frac{\sqrt{2}}{2}AC & +AD & & & & = -P \\
 & \Sigma F_y: & AB & +\frac{\sqrt{2}}{2}AC & & & & & = 0 \\
B) & \Sigma F_x: & & & & BC & +\frac{\sqrt{2}}{2}BD & & = P \\
 & \Sigma F_y: & -AB & & & & -\frac{\sqrt{2}}{2}BD & & = -P \\
C) & \Sigma F_x: & & -\frac{\sqrt{2}}{2}AC & & -BC & & & = 0 \\
 & \Sigma F_y: & & -\frac{\sqrt{2}}{2}AC & & & & -CD & = 0 \\
D) & \Sigma F_x: & & & -AD & & -\frac{\sqrt{2}}{2}BD & & = 0 \\
 & \Sigma F_y: & & & & & \frac{\sqrt{2}}{2}BD & +CD & = P
\end{array}
$$

ASIDE

Show 3 redundant equations.

Σ all F_x equations $\rightarrow$ $0 = 0$

Σ all F_y equations $\rightarrow$ $0 = 0$

at this point you can't choose to sum all equations because these previous two demonstrations already cover this

$-X_B - X_C + Y_C + Y_D \rightarrow 0 = 0$ (consider moments about A)

Matrix form:

$$
\underbrace{\begin{bmatrix}
0 & \frac{\sqrt{2}}{2} & 1 & 0 & 0 & 0 \\
1 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{\sqrt{2}}{2} & 0 \\
-1 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 \\
0 & -\frac{\sqrt{2}}{2} & 0 & -1 & 0 & 0 \\
0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & -\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 1
\end{bmatrix}}_{[M]}
\begin{pmatrix} AB \\ AC \\ AD \\ BC \\ BD \\ CD \end{pmatrix}
=
\begin{pmatrix} -P \\ 0 \\ P \\ -P \\ 0 \\ 0 \\ 0 \\ P \end{pmatrix}
=
\begin{pmatrix} -A_x \\ 0 \\ -B_x \\ -B_y \\ 0 \\ 0 \\ 0 \\ P \end{pmatrix}
$$

We certainly can't go any farther with equilibrium.

What else can we relate to these internal forces?

Well we were told that we had a Young's modulus E and cross-sectional area A for each bar, so they must be elastic.

So for each bar we have $\sigma = E\varepsilon$
where $\sigma = F/A \rightarrow F = EA\varepsilon$

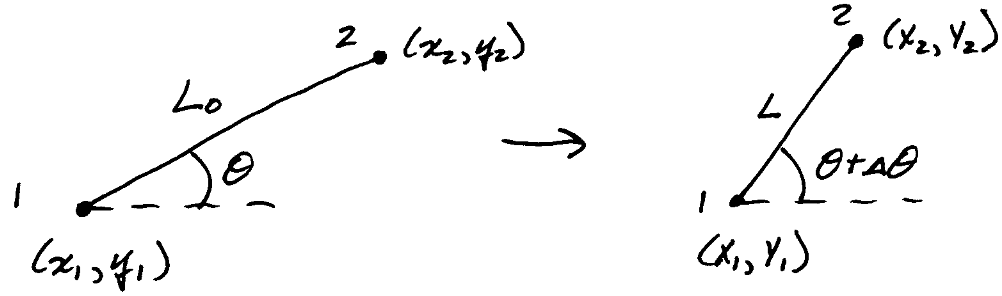
$$\begin{pmatrix} AB \\ AC \\ AD \\ BC \\ BD \\ CD \end{pmatrix} = EA \begin{pmatrix} \varepsilon_{AB} \\ \varepsilon_{AC} \\ \varepsilon_{AD} \\ \varepsilon_{BC} \\ \varepsilon_{BD} \\ \varepsilon_{CD} \end{pmatrix}$$

This has not helped us much as far as number of equations vs number of unknowns is concerned because we have simply replaced the unknown forces with unknown strains.

Now what?

We still have to consider kinematics.

Consider an arbitrarily oriented strut



What is ε ? $\quad \varepsilon = \dfrac{L - L_0}{L_0} = \dfrac{L}{L_0} - 1$

Some geometry:

$$x_2 = x_1 + L_0 \cos\theta$$
$$y_2 = y_1 + L_0 \sin\theta$$

$$X_2 = X_1 + L \cos(\theta + \Delta\theta)$$
$$Y_2 = Y_1 + L \sin(\theta + \Delta\theta)$$

Displacements :

$$X_1 = x_1 + u_1$$
$$Y_1 = y_1 + v_1$$

$$X_2 = x_2 + u_2$$
$$Y_2 = y_2 + v_2$$

$$X_2 = x_2 + u_2 = X_1 + L\cos(\theta + \Delta\theta)$$
$$\cancel{x_1} + L_0 \cos\theta + u_2 = \cancel{x_1} + u_1 + L\cos(\theta + \Delta\theta)$$

$$Y_2 = y_2 + v_2 = Y_1 + L\sin(\theta + \Delta\theta)$$
$$\cancel{y_1} + L_0 \sin\theta + v_2 = \cancel{y_1} + u_1 + L\sin(\theta + \Delta\theta)$$

$X_2 \cos\theta$: $L_0 \cos^2\theta + (u_2 - u_1)\cos\theta = L\cos\theta\cos(\theta+\Delta\theta)$

$Y_2 \sin\theta$: $L_0 \sin^2\theta + (v_2 - v_1)\sin\theta = L\sin\theta\sin(\theta+\Delta\theta)$

Add these together, divide by L_0 and rearrange

$$\frac{u_2 - u_1}{L_0}\cos\theta + \frac{v_2 - v_1}{L_0}\sin\theta = \underbrace{\frac{L}{L_0}\left[\cos\theta\cos(\theta+\Delta\theta) + \sin\theta\sin(\theta+\Delta\theta)\right] - 1}$$

This almost looks like ε.

Note that $\Delta\theta$ depends on u_1, u_2, v_1, v_2 in a non-linear fashion.

Many structures that are stiff and anchored to prevent rigid body motions have small $\Delta\theta$.

$$\cos(\theta+\Delta\theta) = \cos\theta\cos\Delta\theta - \sin\theta\sin\Delta\theta \simeq \cos\theta - \Delta\theta\sin\theta$$
$$\sin(\theta+\Delta\theta) = \sin\theta\cos\Delta\theta + \cos\theta\sin\Delta\theta \simeq \underbrace{\sin\theta + \Delta\theta\cos\theta}$$

To first order in $\Delta\theta$.

$\rightarrow \cos\theta\cos(\theta+\Delta\theta) + \sin\theta\sin(\theta+\Delta\theta)$

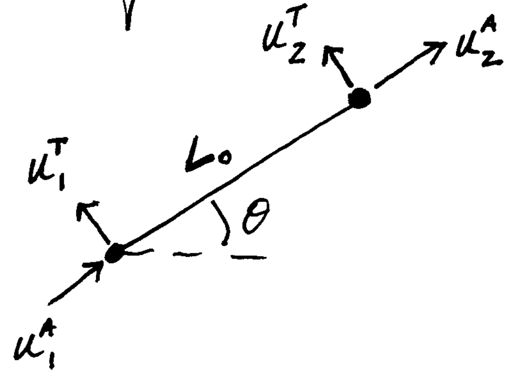
$\simeq \cos^2\theta - \Delta\theta\,\cancel{\sin\theta}\cos\theta + \sin^2\theta + \Delta\theta\,\cancel{\sin\theta}\cos\theta$

$= 1 + O(\Delta\theta^2)$

So to 1st order in $\Delta\theta$ (small rotations) we have

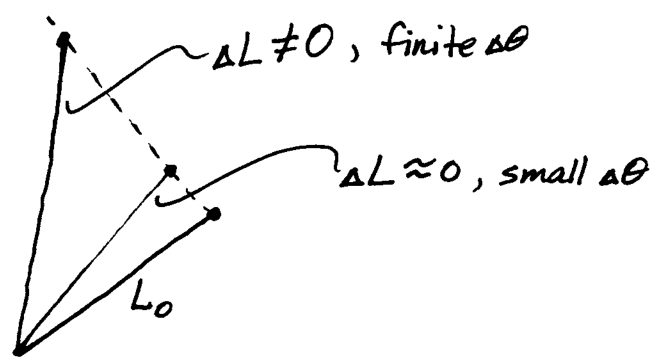
$$\boxed{\varepsilon = \frac{u_2 - u_1}{L_0}\cos\theta + \frac{v_2 - v_1}{L_0}\sin\theta}$$

Another approach to <u>linearized</u> kinematics assuming small rotations from the outset.



$u^A \equiv$ axial displacement
$u^T \equiv$ transverse displacement

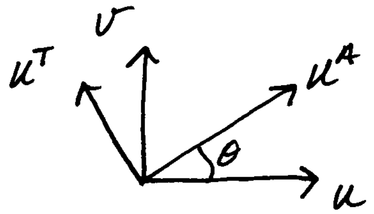
Notice that u_1^T and u_2^T either translate the strut (for the $u_1^T = u_2^T$ mode) or rotate the strut (for the $u_1^T = -u_2^T$ mode). Hence the u^T's do not cause a strain in the bar. Also note that this is only true for small differences in u_1^T and u_2^T



So, for small rotations of the strut, the change in length of the strut is $u_2^A - u_1^A$ and the strain is

$$\varepsilon = \frac{u_2^A - u_1^A}{L_0}$$

Next we need the u^A's in terms of u and v



$$u^A = u\cos\theta + v\sin\theta$$

$$u^T = -u\sin\theta + v\cos\theta$$

$$\therefore \quad \boxed{\varepsilon = \frac{u_2 - u_1}{L_0}\cos\theta + \frac{v_2 - v_1}{L_0}\sin\theta}$$

So we have recovered our previous result. ✓

OK, now let's use this in the analysis of our structure.

$$\varepsilon_{AB} = \frac{v_B - v_A}{L}$$

$$\varepsilon_{AC} = \frac{\frac{\sqrt{2}}{2}(u_C - u_A)}{\sqrt{2}L} + \frac{\frac{\sqrt{2}}{2}(v_C - v_A)}{\sqrt{2}L} = \frac{u_C - u_A}{2L} + \frac{v_C - v_A}{2L}$$

$$\varepsilon_{AD} = \frac{u_D - u_A}{L}$$

$$\varepsilon_{BC} = \frac{u_C - u_B}{L}$$

$$\varepsilon_{BD} = \frac{\frac{\sqrt{2}}{2}(u_D - u_B)}{\sqrt{2}L} - \frac{\frac{\sqrt{2}}{2}(v_D - v_B)}{\sqrt{2}L} = \frac{u_D - u_B}{2L} - \frac{v_D - v_B}{2L}$$

$$\varepsilon_{CD} = \frac{v_C - v_D}{L}$$

$$
\begin{pmatrix} \varepsilon_{AB} \\ \varepsilon_{AC} \\ \varepsilon_{AD} \\ \varepsilon_{BC} \\ \varepsilon_{BD} \\ \varepsilon_{CD} \end{pmatrix}
=
\underbrace{\begin{bmatrix}
0 & -\frac{1}{L} & 0 & \frac{1}{L} & 0 & 0 & 0 & 0 \\
-\frac{1}{2L} & -\frac{1}{2L} & 0 & 0 & \frac{1}{2L} & \frac{1}{2L} & 0 & 0 \\
-\frac{1}{L} & 0 & 0 & 0 & 0 & 0 & \frac{1}{L} & 0 \\
0 & 0 & -\frac{1}{L} & 0 & \frac{1}{L} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2L} & \frac{1}{2L} & 0 & 0 & \frac{1}{2L} & -\frac{1}{2L} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{L} & 0 & -\frac{1}{L}
\end{bmatrix}}_{[N]}
\begin{pmatrix} u_A \\ v_A \\ u_B \\ v_B \\ u_C \\ v_C \\ u_D \\ v_D \end{pmatrix}
$$

$$
\therefore \quad
\begin{pmatrix} -A_x \\ 0 \\ -B_x \\ -B_y \\ 0 \\ 0 \\ 0 \\ P \end{pmatrix}
=
\underbrace{\underbrace{[M]}_{8\times 6} \overbrace{EA}^{\text{scalars}} \underbrace{[N]}_{6\times 8}}_{\substack{8\times 8 \\ -[K]}}
\begin{pmatrix} u_A \\ v_A \\ u_B \\ v_B \\ u_C \\ v_C \\ u_D \\ v_D \end{pmatrix}
$$

$$
-[K] = -\frac{EA}{L}
\begin{bmatrix}
1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -1 & 0 \\
\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & -1 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\
0 & 0 & 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -1 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
0 & -1 & -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -1 & 0 & 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & 0 \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & -1 \\
-1 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & -1 & -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}}
\end{bmatrix}
$$

If we had not previously analyzed the entire structure and solved for A_x, B_x and B_y, this matrix equation would represent 8 equations for 8 unknown displacements and the 3 unknown reaction forces.

However we also know that some of the displacements at the supports are fixed.

Specifically, $u_A = 0$, $u_B = 0$ and $v_B = 0$.

We can now solve the remaining equations for v_A, u_C, v_C, u_D and v_D!

$$v_A = -\frac{5+3\sqrt{2}}{4(1+\sqrt{2})^2} PL/AE$$

$$u_C = \frac{5+3\sqrt{2}}{4(1+\sqrt{2})^2} PL/AE \qquad v_C = -\frac{5+3\sqrt{2}}{2(1+\sqrt{2})} PL/AE$$

$$u_D = -\frac{7+5\sqrt{2}}{4(1+\sqrt{2})^2} PL/AE \qquad v_D = -\frac{27+19\sqrt{2}}{4(1+\sqrt{2})^2} PL/AE$$

$$
\mathbf{Mm} = \begin{pmatrix}
0 & \frac{\sqrt{2}}{2} & 1 & 0 & 0 & 0 \\
1 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{\sqrt{2}}{2} & 0 \\
-1 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 \\
0 & -\frac{\sqrt{2}}{2} & 0 & -1 & 0 & 0 \\
0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & -\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 1
\end{pmatrix};
$$

$$
\mathbf{Nn} = \begin{pmatrix}
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{pmatrix};
$$

← Notice I factored out $\frac{1}{L}$.

```
K = -Mm.Nn;
MatrixForm[K]
```

← Notice that I changed signs here. I also factored out EA.

$$
\begin{pmatrix}
1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -1 & 0 \\
\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & -1 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\
0 & 0 & 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -1 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
0 & -1 & -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -1 & 0 & 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & 0 \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & -1 \\
-1 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & 0 & 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
0 & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & -1 & -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}}
\end{pmatrix}
$$

$$F = \begin{pmatrix} Ax \\ 0 \\ Bx \\ By \\ 0 \\ 0 \\ 0 \\ -P \end{pmatrix};$$

← Sign change here
to compensate for the
previous one.

$$U = \begin{pmatrix} uA \\ vA \\ uB \\ vB \\ uC \\ vC \\ uD \\ vD \end{pmatrix};$$

$$\text{Kreduced} = \begin{pmatrix} 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & -1 \\ 0 & 0 & 0 & 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -1 & -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} \end{pmatrix};$$

$$\text{Freduced} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -P \end{pmatrix};$$

$$\text{Ureduced} = \begin{pmatrix} vA \\ uC \\ vC \\ uD \\ vD \end{pmatrix};$$

```
Solve[Kreduced.Ureduced == Freduced, {vA, uC, vC, uD, vD}]
```

$$\left\{\left\{uD \to -\frac{7P + 5\sqrt{2}\,P}{4\left(1+\sqrt{2}\right)^2},\ vD \to -\frac{27P + 19\sqrt{2}\,P}{4\left(1+\sqrt{2}\right)^2},\right.\right.$$

$$\left.\left. vA \to -\frac{5P + 3\sqrt{2}\,P}{4\left(1+\sqrt{2}\right)^2},\ uC \to -\frac{-5P - 3\sqrt{2}\,P}{4\left(1+\sqrt{2}\right)^2},\ vC \to -\frac{5P + 3\sqrt{2}\,P}{2\left(1+\sqrt{2}\right)}\right\}\right\}$$

* Note that I factored out E, A and L.
I replace them in the notes.
$P \to PL/AE$