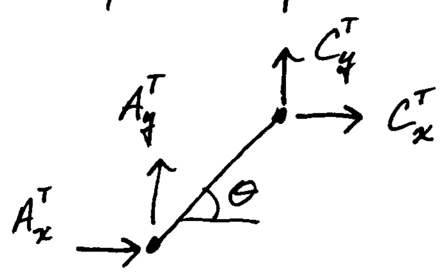
We could proceed to solve truss problems in this way, and we would always obtain correct answers. However, there are more direct ways to calculate and assemble stiffness matrices.

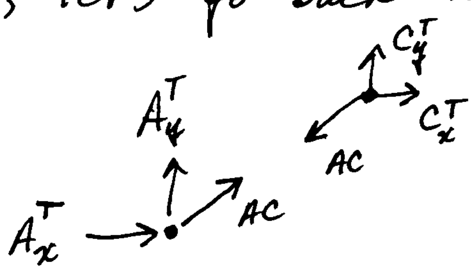
Let's go back and consider how the element AC contributed to the stiffness matrix.

First, let's isolate bar AC and draw a free body diagram of it.



A_x^T, A_y^T, C_x^T and C_y^T are the total forces applied to element AC. These forces include the external forces applied to the nodes plus the forces applied to the nodes by the other elements.

Now, let's go back to nodal equilibrium.



$$\left.\begin{aligned} A:\quad & AC\cos\theta = -A_x^T \\ & AC\sin\theta = -A_y^T \\ C:\quad & -AC\cos\theta = -C_x^T \\ & -AC\sin\theta = -C_y^T \end{aligned}\right\}\text{Equilibrium}$$

Material Constitutive Law : $AC = EA\,\varepsilon_{AC}$

Kinematic Compatibility: $\varepsilon_{AC} = \dfrac{u_C - u_A}{L_{AC}}\cos\theta + \dfrac{v_C - v_A}{L_{AC}}\sin\theta$

$$\therefore\quad \frac{EA}{L_{AC}}\left[(u_C-u_A)\cos^2\theta + (v_C-v_A)\cos\theta\sin\theta\right] = -A_x^T$$

$$\frac{EA}{L_{AC}}\left[(u_C-u_A)\cos\theta\sin\theta + (v_C-v_A)\sin^2\theta\right] = -A_y^T$$

$$\frac{EA}{L_{AC}}\left[-(u_C-u_A)\cos^2\theta - (v_C-v_A)\cos\theta\sin\theta\right] = -C_x^T$$

$$\frac{EA}{L_{AC}}\left[-(u_C-u_A)\cos\theta\sin\theta - (v_C-v_A)\sin^2\theta\right] = -C_y^T$$

$$\underbrace{\frac{EA}{L_{AC}}\left[\begin{array}{cc|cc} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ \hline -\cos^2\theta & -\cos\theta\sin\theta & +\cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{array}\right]}_{\text{Element Stiffness Matrix}} \begin{pmatrix} u_A \\ v_A \\ u_C \\ v_C \end{pmatrix} = \begin{pmatrix} A_x^T \\ A_y^T \\ C_x^T \\ C_y^T \end{pmatrix}$$

Now, let's look at each bar:

AB:
$$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{pmatrix} u_A \\ v_A \\ u_B \\ v_B \end{pmatrix} = \begin{pmatrix} A_x^{AB} \\ A_y^{AB} \\ B_x^{AB} \\ B_y^{AB} \end{pmatrix}$$

AC:
$$\frac{EA}{\sqrt{2}L}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{pmatrix} u_A \\ v_A \\ u_C \\ v_C \end{pmatrix} = \begin{pmatrix} A_x^{AC} \\ A_y^{AC} \\ C_x^{AC} \\ C_y^{AC} \end{pmatrix}$$

AD:
$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} u_A \\ v_A \\ u_D \\ v_D \end{pmatrix} = \begin{pmatrix} A_x^{AD} \\ A_y^{AD} \\ D_x^{AD} \\ D_y^{AD} \end{pmatrix}$$

BC:
$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} u_B \\ v_B \\ u_C \\ v_C \end{pmatrix} = \begin{pmatrix} B_x^{BC} \\ B_y^{BC} \\ C_x^{BC} \\ C_y^{BC} \end{pmatrix}$$

BD:
$$\frac{EA}{\sqrt{2}L}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{pmatrix} u_B \\ v_B \\ u_D \\ v_D \end{pmatrix} = \begin{pmatrix} B_x^{BD} \\ B_y^{BD} \\ D_x^{BD} \\ D_y^{BD} \end{pmatrix}$$

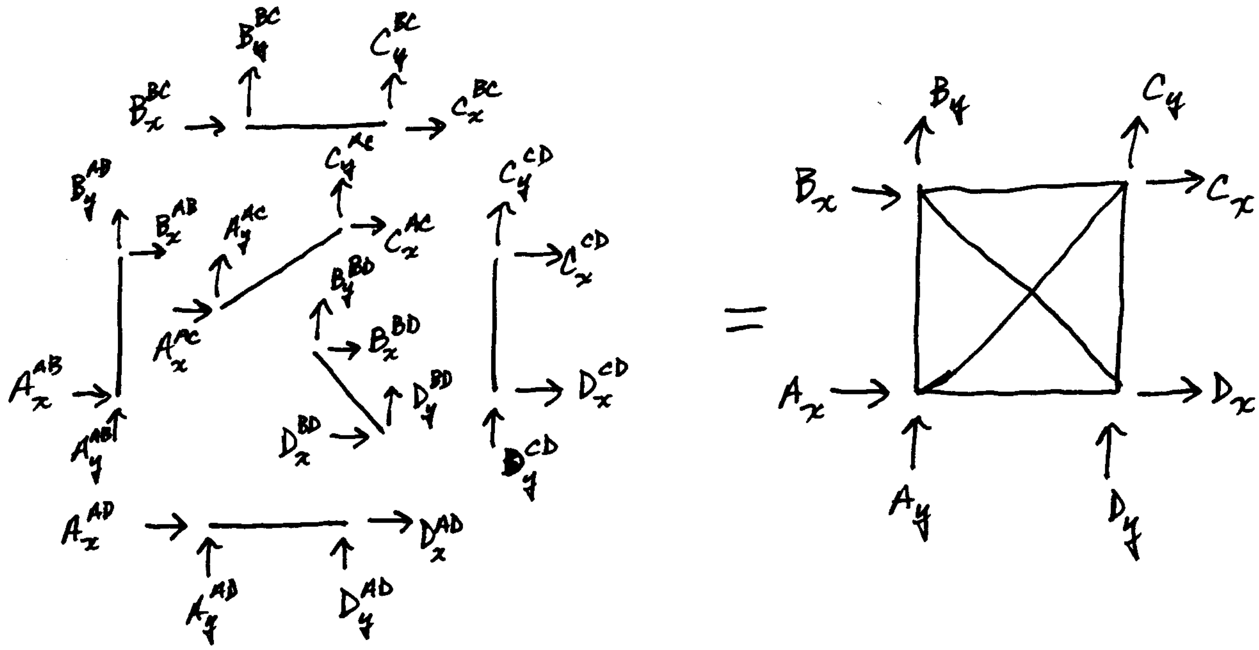
CD:
$$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{pmatrix} u_C \\ v_C \\ u_D \\ v_D \end{pmatrix} = \begin{pmatrix} C_x^{CD} \\ C_y^{CD} \\ D_x^{CD} \\ D_y^{CD} \end{pmatrix}$$

Now that we have the element stiffness matrices, we need to assemble them into one <u>global</u> stiffness matrix.

$$\frac{EA}{L}\begin{bmatrix}
\frac{1}{2\sqrt{2}}+1 & \frac{1}{2\sqrt{2}} & 0 & 0 & \frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & -1 & 0 \\
\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & -1 & \frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 & 0 \\
0 & 0 & 1+\frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & -1 & 0 & \frac{-1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
0 & -1 & \frac{-1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & 0 & \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} \\
\frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & -1 & 0 & \frac{1}{2\sqrt{2}}+1 & \frac{1}{2\sqrt{2}} & 0 & 0 \\
\frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}+1 & 0 & -1 \\
-1 & 0 & \frac{-1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & 0 & 1+\frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} \\
0 & 0 & \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 & -1 & \frac{-1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}+1
\end{bmatrix}\begin{pmatrix} u_A \\ v_A \\ u_B \\ v_B \\ u_C \\ v_C \\ u_D \\ v_D \end{pmatrix}$$

$$= \begin{pmatrix}
A_x^{AB} + A_x^{AC} + A_x^{AD} \\
A_y^{AB} + A_y^{AC} + A_y^{AD} \\
B_x^{AB} + B_x^{BC} + B_x^{BD} \\
B_y^{AB} + B_y^{BC} + B_y^{BD} \\
C_x^{AC} + C_x^{BC} + C_x^{CD} \\
C_y^{AC} + C_y^{BC} + C_y^{CD} \\
D_x^{AD} + D_x^{BD} + D_x^{CD} \\
D_y^{AD} + D_y^{BD} + D_y^{CD}
\end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ B_x \\ B_y \\ C_x \\ C_y \\ D_x \\ D_y \end{pmatrix}$$

External applied forces only because all internal forces cancel



When we "sum up" all of the individual FBDs, we must ultimately "recover" the FBD of the entire structure.

Now that we have the unreduced form for $[K](u) = (F)$, we go through the same procedures that we have described previously to apply the constraints on (u), and reduce $[K]$ and (F), and then solve for the unknown displacements.

** Once we have all of the u's we can compute all of the strains in the bars as,

$$\varepsilon_{AB} = \frac{u_B - u_A}{L_{AB}} \cos\theta + \frac{v_B - v_A}{L_{AB}} \sin\theta$$

and the forces in the bars $AB = EA\,\varepsilon_{AB}$

Computer Procedures for Solving Truss Problems

Preprocessing:
1) Get nodal positions
2) Get element connections & properties
3) Get applied nodal forces
4) Get applied/specified nodal displacements
5) Create data structure to "locate" the degrees of freedom

Assembly:
6) Compute element stiffness matrix for each element based on its geometry and properties
7) "Assemble" the element stiffness matrices into the global stiffness
8) Assemble the nodal force vector
9) Reduce global stiffness and force ** or use penalty method to apply displacement constraints.

Solve:
10) Solve $[K](u)=(F)$
$\rightarrow (u)=[K]^{-1}(F)$

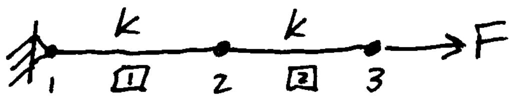
This is usually done with a "black box" linear algebra routine. Though you could write your own.

Post-processing: 11) Determine element strains from nodal displacements

12) Determine element forces/stresses from element strains

Penalty Method

Let's look at the following problem in 1-D.



Element $\boxed{1}$: $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} F_1^{\boxed{1}} \\ F_2^{\boxed{1}} \end{pmatrix}$

Element $\boxed{2}$: $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} F_2^{\boxed{2}} \\ F_3^{\boxed{2}} \end{pmatrix}$

Global : $\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ 0 \\ F \end{pmatrix}$

Reduced : $\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$

$$[K]^{-1}(F) = \frac{1}{k^2}\begin{bmatrix} k & k \\ k & 2k \end{bmatrix}\begin{pmatrix} 0 \\ F \end{pmatrix} = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

$$\frac{1}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} 0 \\ F \end{pmatrix} = \begin{pmatrix} F/k \\ 2F/k \end{pmatrix}$$

$$u_1 = 0 \quad , \quad u_2 = \frac{F}{k} \quad , \quad u_3 = 2F/k$$

$$F_{12} = k(u_2 - u_1) = F \quad \checkmark$$

$$F_{23} = k(u_3 - u_2) = F \quad \checkmark$$

This reduction step requires us to rearrange the matrices and vectors we are trying to solve. Conceptually this is fine, but this requires careful implementation in codes. Essentially it requires a reordering of the degree of freedom numbers such that all known dofs are placed at the end of the array. This requires some special book-keeping.

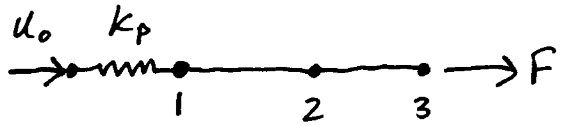
Instead we can use a penalty method.

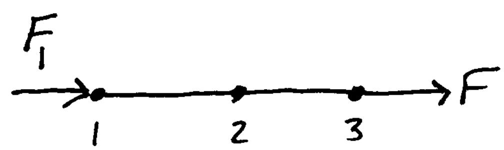
For these equations our goal is to get a solution where $u_1 = u_0$ (for our case $u_0 = 0$ but let's keep it more general for now).

The equation for F_1 that we eliminated was

$$k u_1 - k u_2 = F_1$$

Let's add a penalty spring to the left of node 1.





$$\rightarrow F_1 = k_p (u_0 - u_1)$$

$$\rightarrow k u_1 - k u_2 = k_p (u_0 - u_1)$$

$$(k + k_p) u_1 - k u_2 = k_p u_0$$

Recall that our goal is to have $u_1 = u_0$

$$F_1 = k_p (u_0 - u_1) \rightarrow u_1 = u_0 - F_1 / k_p$$

So, as long as F_1 is finite (which should be the case based on equilibrium) we can enforce our constraint exactly as $k_p \rightarrow \infty$ and approximately as $k_p \gg k$.

So our equations become:

$$\begin{bmatrix} k+k_p & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} k_p u_o \\ 0 \\ F \end{pmatrix}$$

Solutions: (with $u_o = 0$)

$k_p/k \downarrow$	$u_1 \cdot k/F$	$u_2 \cdot k/F$	$u_3 \cdot k/F$
1	1	2	3
10	$1/10$	$11/10$	$21/10$
100	$1/100$	$101/100$	$201/100$
1000	$1/1000$	$1001/1000$	$2001/1000$

So, implemented the penalty method does not require any special book-keeping. We simply add k_p to the entry on the diagonal of the global stiffness and set $F_1 = k_p u_o$ in the force vector.

Note: If the displacement at u_2 was specified, you should think of the extra penalty spring connection as: