

# MA553: Qual Preparation

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# 1 MA 553 Spring 2016

This is material from the course MA 533 as it was taught in the spring of 2016.

## 1.1 Homework

Most of the homework is Ulrich original (or as original as elementary exercises in abstract algebra can be). However, an excellent resource and one that I will often quote on these solutions is [3]. Other resources include [1] and (to a lesser extent) [2]. I may also cite Milne's *Group Theory*, *Field Theory*, and *Commutative Algebra: A Primer* notes, respectively, [4], [5], and (no reference for the last). Unless otherwise stated, whenever we quote a result, e.g., Theorem 1.1, it is understood to come from Hungerford's *Algebra*.

Throughout these notes

$\mathbb{R}$	is the set of real numbers
$\mathbb{C}$	is the set of complex numbers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{F}_q$	is the finite field of order $q = p^n$ for some prime $p$
$\mathbb{Z}$	is the set of the integers
$\mathbb{N}$	is the set of the natural numbers $1, 2, \dots$
$k$	is used to denote the base field with characteristic $\text{ch } k$
$K, E, L$	is used to denote field extensions over the base field $k$
$Z_n$	is the cyclic group of order $n$ not necessarily equal (but isomorphic) to $\mathbb{Z}/p\mathbb{Z}$
$S_n$	is the symmetric group on $\{1, \dots, n\}$
$A_n$	is the alternating group on $\{1, \dots, n\}$
$D_n$	is the dihedral group of order $n$
$A \setminus B$	is the set difference of $A$ and $B$ , that is, the complement of $A \cap B$ in $A$
$X \cong Y$	means $X$ and $Y$ are isomorphic as groups, rings, $R$ -modules, or fields

### 1.1.1 Homework 1

**Exercise 1.** Let  $G$  be a group,  $a \in G$  an element of finite order  $m$ , and  $n$  a positive integer. Prove that

$$\text{ord}(a^n) = \frac{m}{(m, n)}.$$

**Solution.** ▶ Let  $\ell$  denote the order of  $a^n$ . Then  $\ell$  is the minimal power of  $a^n$  such that  $(a^n)^\ell = e$ . Now, observe that

$$\begin{aligned} (a^n)^{m/(m, n)} &= a^{nm/(m, n)} \\ &= a^{mn/(m, n)} \\ &= (a^m)^{n/(m, n)} \\ &= e^{n/(m, n)} \\ &= e. \end{aligned}$$

Thus  $\ell \leq m/(m, n)$ .

On the other hand, by Theorem 3.4 (iv) since  $(a^n)^\ell = a^{n\ell} = e$  and the order of  $a$  is  $m$ ,  $m \mid n\ell$  or, equivalently,  $mk = n\ell$  for some  $k \in \mathbb{Z}^+$ . Now, since  $(m, n) \mid m$  and  $(m, n) \mid n$ , we can represent  $m$  and  $n$  as the products  $(m, n)m'$  and  $(m, n)n'$ , respectively. Now, note that  $m' = m/(m, n)$  so we must show that  $m' \leq \ell$ . Putting all of this together, we have  $mk$

$$mk = (m, n)m'k = (m, n)n'\ell = n\ell$$

so

$$m'k = n'\ell.$$

Thus  $m' \mid n'\ell$  so either  $m' \mid n'$  or  $m' \mid \ell$ . But since we factored the  $(m, n)$  from  $m$  and  $n$ , it follows that  $(m', n') = 1$  so  $m' \mid \ell$ . Therefore  $m' \leq \ell$  and equality holds, that is,  $\ell = m/(m, n)$ . ◀

**Exercise 2.** Let  $G$  be a group, and let  $a, b$  be elements of finite order  $m, n$  respectively. Show that if  $ba = ab$  and  $\langle a \rangle \cap \langle b \rangle = \{e\}$ , then  $\text{ord}(ab) = mn/(m, n)$ .

**Solution.** ▶ Let  $\ell$  denote the order of  $ab$ . Now, playing around with powers of  $ab$ , we have

$$\begin{aligned} (ab)^n &= a^n b^n \\ &= a^n \\ &\neq e \end{aligned}$$

since the order of  $a$  is  $m$  and  $n < m$ . Thus, by Problem 1,  $\text{ord}(a^n) = m/(m, n)$  so  $\text{ord}(ab) = mn/(m, n)$ . ◀

**Exercise 3.** Let  $G$  be a group and  $H, K$  normal subgroups with  $H \cap K = \{e\}$ . Show that

- (a)  $hk = kh$  for every  $h \in H, k \in K$ .
- (b)  $HK$  is a subgroup of  $G$  with  $HK \cong H \times K$ .

**Solution.** ► (a) Suppose that  $H$  and  $K$  are normal in  $G$ . Then, for every  $g \in G$ ,  $gh = hg$  and  $gk = kg$  for any  $h \in H$ ,  $k \in K$ . In particular, since  $H \subseteq G$ ,  $h \in G$  so  $hk = kh$ .

(b) Consider the subset  $HK$  of  $G$  consisting of all products  $hk$  where  $h \in H$ ,  $k \in K$ . First, we show that  $HK$  is closed under multiplication: Pick  $h_1k_1, h_2k_2 \in HK$  then  $h_1k_1h_2k_2 = h_1(k_1h_2)k_2 = h_1h_2(k_1k_2)$  is in  $HK$  since  $h_1h_2 \in H$ ,  $k_1k_2 \in K$ . Moreover, since  $e \in H$  and  $e \in K$ ,  $ee = e \in HK$ . Lastly, given  $hk \in HK$ ,  $hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} = kk^{-1} = e$  so  $HK$  is closed under taking inverses. Thus,  $HK$  is a subgroup of  $G$ .

To see that  $HK \cong H \times K$ , consider the map  $\varphi: HK \rightarrow (HK/K) \times (HK/H)$  given by  $\varphi(hk) = (\pi_K(h), \pi_H(k))$  where  $\pi_H: HK \rightarrow HK/H$  and  $\pi_K: HK \rightarrow HK/K$  are quotient maps. By the first (or second) isomorphism theorem,  $H \cong HK/H$  and  $K \cong HK/K$  so  $HK \cong H \times K$ . ◀

**Exercise 4.** Show that  $A_4$  has no subgroup of order 6 (although  $6 \mid 12 = \text{card } A_4$ ).

**Solution.** ► We proceed by contradiction. Suppose that  $A_4$  has a subgroup of order 6, call it  $H$ . Then, we claim that  $H$  must contain all elements  $\sigma^2$  where  $\sigma \in A$ .

*Proof of claim.* Since  $\text{card } H = 6$ ,  $(A_4 : H) = 2$  which implies that  $H$  must be a normal subgroup of  $A_4$ . Now, consider the collection of  $G/H$  of right-cosets of  $H$  in  $G$ . By Theorem 5.4,  $G/H$  is a group with order  $\text{card}(G/H) = 2$  so either  $\bar{\sigma} = \bar{e}$  or  $\bar{\sigma}^2 = \bar{e}$ . Thus,  $\sigma^2 \in H$ . ■

Thus,  $H$  must contain all of the squares in  $A_4$ . However, counting all of the elements in  $A_4$  and squaring them

$$\begin{array}{ll} (1)^2 = (1) & (1\ 2\ 3)^2 = (1\ 3\ 2) \\ (1\ 3\ 2)^2 = (1\ 2\ 3) & (1\ 2\ 4)^2 = (1\ 4\ 2) \\ (1\ 4\ 2)^2 = (1\ 2\ 4) & (1\ 3\ 4)^2 = (1\ 4\ 3) \\ (1\ 4\ 3)^2 = (1\ 3\ 4) & (2\ 3\ 4)^2 = (2\ 3\ 4) \\ (2\ 4\ 3)^2 = (2\ 4\ 3) & ((1\ 2)(3\ 4))^2 = (1) \\ ((1\ 3)(2\ 4))^2 = (1) & ((1\ 4)(2\ 3))^2 = (1) \end{array}$$

we see that there are a total of 9 squares (8 nontrivial ones) which exceeds the order of  $H$ . This is a contradiction therefore,  $G$  has no subgroup of order 6. ◀

### 1.1.2 Homework 2

**Exercise 1.** Let  $G$  be the group of order  $2^n \cdot 3$ ,  $n \geq 2$ . Show that  $G$  has a normal 2-subgroup  $\neq \{e\}$ .

**Solution.** ▶ Suppose  $\text{card } G = 2^n \cdot 3$ . By Sylow's theorem,  $G$  contains a 2-Sylow subgroup  $P$  of order  $\text{card } P = 2^n$ . If  $P$  is the unique 2-Sylow subgroup in  $G$ ,  $P \trianglelefteq G$ .

Otherwise, Sylow's theorem implies that  $\text{card}(\text{Syl}_2(G))$  must divide 3 and, since 3 is prime, must in fact equal 3. Then, each  $Q \in \text{Syl}_2(G)$  is conjugate to  $P$ . Enumerate the set  $\text{Syl}_2(G) = \{P, P', P''\}$  and let  $G$  act on  $\text{Syl}_2(G)$  by conjugation. This action gives rise to a homomorphism  $\varphi: G \rightarrow S_3$  given by the permutation representation of the action. This action is nontrivial since there exists elements  $g_1, g_2 \in G$  such that  $P' = g_1 P g_1^{-1}$  and  $P'' = g_2 P g_2^{-1}$  (which correspond to the permutations  $(1\ 2)$  and  $(1\ 3)$ ). By the first isomorphism theorem,  $\text{Ker } \varphi \trianglelefteq G$  and  $(G : \text{Ker } \varphi) \mid \text{card } S_3 = 6$ . But we observed that the image of  $G$  in  $S_3$  contains at least 3 permutations:  $(1\ 2)$ ,  $(1\ 3)$  and  $(1\ 2)(1\ 3) = (1\ 3\ 2)$ . Thus,  $(G : \text{Ker } \varphi) = 3$  or 6. In either case,  $\text{Ker } \varphi$  is a 2-subgroup of  $G$ . ◀

**Exercise 2.** Let  $G$  be a group of order  $p^2 q$ ,  $p$  and  $q$  primes. Show that the  $p$ -Sylow subgroup or the  $q$ -Sylow subgroup of  $G$  is normal in  $G$ .

**Solution.** ▶ Suppose  $\text{card } G = p^2 q$ . Assuming  $p < q$  there are 1 or  $p^2$   $q$ -Sylow subgroups. If there is 1  $q$ -Sylow subgroup  $Q$  then  $Q \trianglelefteq G$ . Otherwise, there are  $p^2$   $q$ -Sylow subgroups in  $G$  and, counting the total number of elements of order  $q$ , there are  $p^2(q - 1) = p^2 q - p^2$  remaining elements in  $G$  which leaves just enough room for 1  $p$ -Sylow subgroup  $P$  which implies that  $P \trianglelefteq G$ . Otherwise,  $p > q$  and we must be one 1  $p$ -Sylow subgroup  $P$  in  $G$  which implies  $P \trianglelefteq G$ . In each case, we either have a normal  $p$ -Sylow subgroup or a normal  $q$ -Sylow subgroup. ◀

**Exercise 3.** Let  $G$  be a subgroup of order  $pqr$ ,  $p < q < r$  primes. Show that the  $r$ -Sylow subgroup of  $G$  is normal in  $G$ .

**Solution.** ▶ By Sylow's theorem, we have 1 or  $pq$   $r$ -Sylow subgroups in  $G$ . In the former case, there is a unique  $r$ -Sylow subgroup  $R$  which implies  $R \trianglelefteq G$ . In the latter case, there are  $pq$   $r$ -Sylow subgroups in  $G$  and that implies that we have  $pq(r - 1) = pqr - pq$  elements of order  $r$ . That leaves room for exactly  $pq$  elements that do not have order  $r$ . Now we ask, what are the possible number of  $p$ - and  $q$ -Sylow subgroups? At minimum, we have 1  $p$ - and 1  $q$ -Sylow subgroups. This yields a total of

$$\begin{aligned} (p - 1) + (q - 1) + 1 &= p + q - 1 \\ &< pq \end{aligned}$$

which flows under the total number of elements to complete the size of the group. What is the next smallest possible number of  $p$ - and  $q$ -Sylow subgroups is  $r$ . In this case, we have

$$\begin{aligned} r(p - 1) + r(q - 1) + 1 &= rp - r + rq - r + 1 \\ &= r(p + q - 2) + 1 \\ &> pq \end{aligned}$$

since  $r > p$  and  $p + q - 2 > 2p - 2 > p$ . Thus, we cannot have  $pq$   $r$ -Sylow subgroups in  $G$ . It follows that there is only 1  $r$ -Sylow subgroup  $R$  in  $G$  and so  $R \trianglelefteq G$ . ◀

**Exercise 4.** Let  $G$  be a group of order  $n$  and let  $\varphi: G \rightarrow S_n$  be given by the action of  $G$  on  $G$  via translation.

- (a) For  $a \in G$  determine the number and the lengths of the disjoint cycles of the permutation  $\varphi(a)$ .
- (b) Show that  $\varphi(G) \not\subseteq A_n$  if and only if  $n$  is even and  $G$  has a cyclic 2-Sylow subgroup.
- (c) If  $n = 2m$ ,  $m$  odd, show that  $G$  has a subgroup of index 2.

**Solution.** ► For (a), let  $\{g_0 = e, g_1, \dots, g_{n-1}\}$  be an enumeration of  $G$ . Fix  $a = g_k$  in  $G$  for some  $0 \leq k \leq n-1$ . Then the action of  $G$  on itself by translation gives a homomorphism  $\varphi: G \rightarrow S_n$  which sends  $\{g_0, g_1, \dots, g_n\}$  to the set  $\{ag_0, ag_1, \dots, ag_n\}$ . If  $a$  is nontrivial, the latter set equals  $G$  so has no fixed point. This implies that every nontrivial  $a$  in  $G$  corresponds to an  $n$ -cycle in  $S_n$ . I don't know what he's talking about so I am just moving on.

For (b),

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**Exercise 5.** Show that the only simple groups  $\neq \{e\}$  of order  $< 60$  are the groups of prime order.

**Solution.** ► First, let us list all of the possible orders of groups with order less than 60, these orders are

4	6	8	9	10
12	14	15	16	18
20	21	22	24	25
26	27	28	30	32
33	34	35	36	38
39	40	42	44	45
46	48	49	50	51
52	54	55	56	58

These integers fall into one of the following categories  $n = p^2, pq, p^3, p^2q, pqr, p^4, p^3q, p^2q^2, p^5, p^4q$ ; here they are by type

$p^2$	$pq$	$p^3$	$p^2q$	$pqr$	$p^4$	$p^3q$	$p^2q^2$	$p^5$	$p^4q$
4	6	8	12	30	16	24	36	32	48
9	10	27	18	42		40			
25	14		20			56			
49	15		28						
	21		44						
	22		45						
	26		50						
	33		52						
	34								
	35								
	38								
	39								
	46								
	51								
	54								
	55								
	58								

All  $p$ -groups have a nontrivial center, so groups of orders corresponding to the  $p^2$ ,  $p^3$ ,  $p^4$  and  $p^5$  columns are not simple. Similarly, groups of order  $pq$  are not simple and we have just shown that groups of order  $p^2q$  and  $pqr$  are not simple.

Now we cover the following cases:

**Claim.**

- (a) If  $\text{card } G = p^n q$  for  $n \geq 2$ ,  $G$  contains a nontrivial normal subgroup.
- (b) If  $\text{card } G = p^2 q^2$ ,  $G$  contains a nontrivial normal subgroup.

*Proof of claim.* For (a), consider the  $p$ -Sylow subgroup  $P$  of  $G$ .

■

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### 1.1.3 Homework 3

**Exercise 1.** Let  $G$  be a finite group,  $p$  a prime number,  $N$  the intersection of all  $p$ -Sylow subgroups of  $G$ . Show that  $N$  is a normal  $p$ -subgroup of  $G$  and that every normal  $p$ -subgroup of  $G$  is contained in  $N$ .

**Solution.** ►

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**Exercise 2.** Let  $G$  be a group of order 231 and let  $H$  be an 11-Sylow subgroup of  $G$ . Show that  $H \subseteq Z(G)$ .

**Solution.** ►

◀

**Exercise 3.** Let  $G = \{e, a_1, a_2, a_3\}$  be a non-cyclic group of order 4 and define  $\varphi: S_3 \rightarrow \text{Aut}(G)$  by  $\varphi(\sigma)(e) = e$  and  $\varphi(\sigma)(a_i) = a_{\sigma(i)}$ . Show that  $\varphi$  is well-defined and an isomorphism of groups.

**Solution.** ►

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**Exercise 4.** Determine all groups of order 18.

**Solution.** ►

◀



#### 1.1.4 Homework 4

**Exercise 1.** Let  $p$  be a prime and let  $G$  be a nonAbelian group of order  $p^3$ . Show that  $G' = Z(G)$ .

**Solution.** ►

◀

**Exercise 2.** Let  $p$  be an odd prime and let  $G$  be a nonAbelian group of order  $p^3$  having an element of order  $p^2$ . Show that there exists an element  $b \notin \langle a \rangle$  of order  $p$ .

**Solution.** ►

◀

**Exercise 3.** Let  $p$  be an odd prime. Determine all groups of order  $p^3$ .

**Solution.** ►

◀

**Exercise 4.** Show that  $(S_n)' = A_n$ .

**Solution.** ►

◀

**Exercise 5.** Show that every group of order  $< 60$  is solvable.

**Solution.** ►

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**Exercise 6.** Show that every group of order 60 that is simple (or not solvable) is isomorphic to  $A_5$ .

**Solution.** ►

◀

### 1.1.5 Homework 5

**Exercise 1.** Find all composition series and the composition factors of  $D_6$ .

**Solution.** ► ◀

**Exercise 2.** Let  $T$  be the subgroup of  $GL(n, \mathbb{R})$  consisting of all upper triangular invertible matrices. Show that  $T$  is solvable.

**Solution.** ► ◀

**Exercise 3.** Let  $p \in \mathbb{Z}$  be a prime number. Show:

- (a)  $(p-1)! \equiv -1 \pmod{p}$ .
- (b) If  $p \equiv 1 \pmod{4}$  then  $x^2 \equiv -1 \pmod{p}$  for some  $x \in \mathbb{Z}$ .

**Solution.** ► ◀

**Exercise 4.**

(a) Show that the following are equivalent for an odd prime number  $p \in \mathbb{Z}$ :

- (i)  $p \equiv 1 \pmod{4}$ .
- (ii)  $p = a^2 + b^2$  for some  $a, b$  in  $\mathbb{Z}$ .
- (iii)  $p$  is not prime in  $\mathbb{Z}[i]$ .

(b) Determine all prime ideals of  $\mathbb{Z}[i]$ .

**Solution.** ► ◀

### 1.1.6 Homework 6

**Exercise 1.** Let  $R$  be a domain. Show that  $R$  is a u.f.d. if and only if every nonzero nonunit in  $R$  is a product of irreducible elements and the intersection of any two principal ideals is again principal.

**Solution.** ►

◀

**Exercise 2.** Let  $R$  be a p.i.d. and  $\mathfrak{p}$  a prime ideal of  $R[X]$ . Show that  $\mathfrak{p}$  is principal or  $\mathfrak{p} = (a, f)$  for some  $a \in R$  and some monic polynomial  $f \in R[X]$ .

**Solution.** ►

◀

**Exercise 3.** Let  $k$  be a field and  $n \geq 1$ . Show that  $Z^n + Y^3 + X^2 \in k(X, Y)[Z]$  is irreducible.

**Solution.** ►

◀

**Exercise 4.** Let  $k$  be a field of characteristic zero and  $n \geq 1, m \geq 2$ . Show that  $X_1^n + \cdots + X_m^n - 1 \in k[X_1, \dots, X_m]$  is irreducible.

**Solution.** ►

◀

**Exercise 5.** Show that  $X^{3^n} + 2 \in \mathbb{Q}(i)[X]$  is irreducible.

**Solution.** ►

◀

### 1.1.7 Homework 7

**Exercise 1.** Let  $k \subseteq K$  and  $k \subseteq L$  be finite field extensions contained in some field. Show that:

- (a)  $[KL : L] \leq [K : k]$ .
- (b)  $[KL : k] \leq [K : k][L : k]$ .
- (c)  $K \cap L = k$  if equality holds in (b).

**Solution.** ▶

◀

**Exercise 2.** Let  $k$  be a field of characteristic  $\neq 2$  and  $a, b$  elements of  $k$  so that  $a, b, ab$  are not squares in  $k$ . Show that  $[k(\sqrt{a}, \sqrt{b}) : k] = 4$ .

**Solution.** ▶

◀

**Exercise 3.** Let  $R$  be a u.f.d, but not a field, and write  $K = \text{Quot}(R)$ . Show that  $[\bar{K} : k] = \infty$ .

**Solution.** ▶

◀

**Exercise 4.** Let  $k \subseteq K$  be an algebraic field extension. Show that every  $k$ -homomorphism  $\delta : K \rightarrow K$  is an isomorphism.

**Solution.** ▶

◀

**Exercise 5.** Let  $K$  be the splitting field of  $X^6 - 4$  over  $\mathbb{Q}$ . Determine  $K$  and  $[K : \mathbb{Q}]$ .

**Solution.** ▶

◀

### 1.1.8 Homework 8

**Exercise 1.** Let  $k$  be a field,  $f \in k[X]$  is a polynomial of degree  $n \geq 1$ , and  $K$  the splitting field of  $f$  over  $k$ . Show that  $[K : k] \mid n!$ .

**Solution.** ► ◀

**Exercise 2.** Let  $k$  be a field and  $n \geq 0$ . Define a map  $\Delta_n : k[X] \rightarrow k[X]$  by  $\Delta_n(\sum a_i X^i) = \sum a_i \binom{i}{n} X^{i-n}$ . Show:

- (a)  $\Delta_n$  is  $k$ -linear, and for  $f, g$  in  $k[X]$ ,  $\Delta_n(fg) = \sum_{j=0}^n \Delta_j(f) \Delta_{n-j}(g)$ ;
- (b)  $f^{(n)} = n! \Delta_n(f)$ ;
- (c)  $f(X+a) = \sum \Delta_n(f)(a) X^n$ , where  $a \in k$ ;
- (d)  $a \in k$  is a root of  $f$  of multiplicity  $n$  if and only if  $\Delta_i(f)(a) = 0$  for  $0 \leq i \leq n-1$  and  $\Delta_n(f)(a) \neq 0$ .

**Solution.** ► ◀

**Exercise 3.** Let  $k \subseteq K$  be a finite field extension. Show that  $k$  is perfect if and only if  $K$  is perfect.

**Solution.** ► ◀

**Exercise 4.** Let  $K$  be the splitting field of  $X^p - X - 1$  over  $k = \mathbb{Z}/p\mathbb{Z}$ . Show that  $k \subseteq K$  is normal, separable, of degree  $p$ .

**Solution.** ► ◀

**Exercise 5.** Let  $k$  be a field of characteristic  $p > 0$ , and  $k(X, Y)$  the field of rational functions in two variables.

- (a) Show that  $[k(X, Y) : k(X^p, Y^p)] = p^2$ .
- (b) Show that the extension  $k(X^p, Y^p) \subseteq k(X, Y)$  is not simple.
- (c) Find infinitely many distinct fields  $L$  with  $k(X^p, Y^p) \subseteq L \subseteq k(X, Y)$ .

**Solution.** ► ◀

### 1.1.9 Homework 9

**Exercise 1.** Let  $k \subseteq K$  be a finite extension of fields of characteristic  $p > 0$ . Show that if  $p \nmid [K : k]$ , then  $k \subseteq K$  is separable.

**Solution.** ► ◀

**Exercise 2.** Let  $k \subseteq K$  be an algebraic extension of fields of characteristic  $p > 0$ , let  $L$  be an algebraically closed field containing  $K$ , and let  $\delta : k \rightarrow L$  be an embedding. Show that  $k \subseteq K$  is purely inseparable if and only if there exists exactly one embedding  $\tau : K \rightarrow L$  extending  $\delta$ .

**Solution.** ► ◀

**Exercise 3.** Let  $k \subseteq K = k(\alpha, \beta)$  be an algebraic extension of fields of characteristic  $p > 0$ , where  $\alpha$  is separable over  $k$  and  $\beta$  is purely inseparable over  $k$ . Show that  $K = k(\alpha + \beta)$ .

**Solution.** ► ◀

**Exercise 4.** Let  $f(X) \in \mathbb{F}_q[X]$  be irreducible. Show that  $f(X) \mid X^{q^n} - X$  if and only if  $\deg f(X) \mid n$ .

**Solution.** ► ◀

**Exercise 5.** Show that  $\text{Aut}_{\mathbb{F}_q}(\bar{\mathbb{F}}_q)$  is an infinite Abelian group which is torsionfree (i.e.,  $\delta^n = \text{id}$  implies  $\delta = \text{id}$  or  $n = 0$ ).

**Solution.** ► ◀

**Exercise 6.** Show that in a finite field, every element can be written as a sum of two perfect squares.

**Solution.** ► ◀

### 1.1.10 Homework 10

**Exercise 1.** Let  $k \subset K = k(\alpha)$  be a simple field extension, let  $G = \{\delta_1, \dots, \delta_n\}$  be a finite subgroup of  $\text{Aut}_k(K)$ , and write  $f(X) = \prod_{i=1}^n (X - \delta_i(\alpha)) = \sum_{i=0}^n a_i X^i$ . Show that  $f(X)$  is the minimal polynomial of  $\alpha$  over  $K^G$  and that  $K^G = k(a_0, \dots, a_{n-1})$ .

**Solution.** ► ◀

**Exercise 2.** Let  $k$  be a field,  $k(X)$  the field of rational functions, and  $u \in k(X) \setminus k$ . Write  $u = f/g$  with  $f$  and  $g$  relatively prime in  $k[X]$ . Show that  $[k(X) : k(u)] = \max\{\deg f, \deg g\}$ .

**Solution.** ► ◀

**Exercise 3.** Let  $k$  be a field and  $K = k(X)$  the field of rational functions. Show that for every  $\delta \in \text{Aut}_k(K)$ ,  $\delta(X) = (aX + b)/(cX + d)$  for some  $a, b, c, d$  in  $k$  with  $ad - bc \neq 0$ , and that conversely, every such rational function uniquely determines an automorphism  $\delta \in \text{Aut}_k(K)$ .

**Solution.** ► ◀

**Exercise 4.** With the notion of the previous problem let  $\delta \in \text{Aut}_k(K)$  and  $G = \langle \delta \rangle$ .

- (a) Assume  $\delta(X) = 1/(1 - X)$ . Show that  $|G| = 3$  and determine  $K^G$ .
- (b) Assume  $\text{ch } k = 0$  and  $\delta(X) = X + 1$ . Show that  $G$  is infinite and determine  $K^G$ .

**Solution.** ► ◀

**Exercise 5.** Let  $k \subset K$  be a finite Galois extension with  $G = \text{Gal}(K/k)$ , let  $L$  be a subfield of  $K$  containing  $k$  with  $H = \text{Gal}(K/L)$ , and let  $L'$  be the compositum in  $K$  of the fields  $\delta(L)$ ,  $\delta \in G$ . Show that:

- (a)  $L'$  is the unique smallest subfield of  $K$  that contains  $L$  and is Galois over  $k$ .
- (b)  $\text{Gal}(K/L') = \bigcap_{\delta \in G} \delta H \delta^{-1}$ .

**Solution.** ► ◀

### 1.1.11 Homework 11

**Exercise 1.** Show that every algebraic extension of a finite field is Galois and Abelian.

**Solution.** ▶ ◀

**Exercise 2.** Let  $k$  be a field of characteristic  $\neq 2$  and  $f(X) \in k[X]$  a cubic whose discriminant is a square. Show that  $f$  is either irreducible or a product of linear polynomials in  $k[X]$ .

**Solution.** ▶ ◀

**Exercise 3.** Let  $k$  be a field of characteristic  $\neq 2$ , and let  $f(X) = X^4 + aX^2 + b \in k[X]$  be irreducible with Galois group  $G$ . Show:

- (i) If  $b$  is a square in  $k$ , then  $G = H$ .
- (ii) If  $b$  is not a square in  $k$ , but  $b(a^2 - 4b)$  is, then  $G \cong C_4$ .
- (iii) If neither  $b$  nor  $b(a^2 - 4b)$  is a square in  $k$ , then  $G \cong D_4$ .

**Solution.** ▶ ◀

**Exercise 4.** Determine the Galois group of:

- (a)  $X^4 - 5$  over  $\mathbb{Q}$ , over  $\mathbb{Q}(\sqrt{5})$ , over  $\mathbb{Q}(\sqrt{-5})$ ;
- (b)  $X^3 - 10$  over  $\mathbb{Q}$ ;
- (c)  $X^4 - 4X^2 + 5$  over  $\mathbb{Q}$ ;
- (d)  $X^4 + 3X^3 + 3X - 2$  over  $\mathbb{Q}$ ;
- (e)  $X^4 + 2X^2 + X + 3$  over  $\mathbb{Q}$ .

**Solution.** ▶ ◀

**Exercise 5.** Let  $K$  be the splitting field of  $X^4 - X^2 - 1$  over  $\mathbb{Q}$ . Determine all intermediate fields  $L$ ,  $\mathbb{Q} \subseteq L \subseteq K$ . Which of these are Galois over  $\mathbb{Q}$ ?

**Solution.** ▶ ◀



### 1.1.12 Homework 12

**Exercise 1.** Prove that the resolvent cubic  $X^4 + aX^2 + bX + c$  is given by  $X^3 - aX^2 - 4cX + 4ac - b^2$ .

**Solution.** ►

**Exercise 2.** Show that the general polynomial  $g(Y) = Y^n + u_1Y^{n-1} + \cdots + u_n$  is irreducible in  $k(u_1, \dots, u_n)[Y]$ .

**Solution.** ►

**Exercise 3.** Let  $k$  be a field.

- (a) Compute the discriminant  $Y^3 - Y \in k[Y]$  and  $Y^3 - 1 \in k[Y]$ .
- (b) Show that the discriminant of the polynomial  $(Y - X_1)(Y - X_2)(Y - X_3)$  over  $k(X_1, X_2, X_3)$  is of the form
$$\lambda_1 s_1^4 + \lambda_2 s_1^4 s_2 + \lambda_3 s_1^3 s_3 + \lambda_4 s_1^2 s_2^2 + \lambda_5 s_1 s_2 s_3 + \lambda_6 s_2^3 + \lambda_7 s_3^2$$
with  $\lambda_i \in k$ .
- (c) From (b) and (a) conclude that the discriminant  $Y^3 + aY + b \in k[Y]$  is  $-4a^3 - 27b^2$ .

**Solution.** ►

**Exercise 4.** Let  $\Phi_n(X)$  be the  $n$ th cyclotomic polynomial over  $\mathbb{Q}$ .

- (a) Let  $n = p_1^{r_1} \cdots p_s^{r_s}$  with  $p_i$  distinct prime numbers and  $r_i > 0$ . Show that  $\Phi(X) = \Phi_{p_1 \cdots p_s}(X^{p_1^{r_1-1} \cdots p_s^{r_s-1}})$ .
- (b) For a prime number  $p$  with  $p \nmid n$  show that  $\Phi_{pn}(X) = \Phi_n(X^p)/\Phi_n(X)$ .

**Solution.** ►

### 1.1.13 Homework 13

**Exercise 1.** Let  $n \geq 3$  and  $\rho$  a primitive  $n$ th root of unity over  $\mathbb{Q}$ . Show that  $[\mathbb{Q}(\rho + \rho^{-1}) : \mathbb{Q}] = \varphi(n)/2$ .

**Solution.** ► ◀

**Exercise 2.** Let  $\rho$  be a primitive  $n$ th root of unity over  $\mathbb{Q}$ . Determine all  $n$  so that  $\mathbb{Q} \subseteq \mathbb{Q}(\rho)$  is cyclic.

**Solution.** ► ◀

**Exercise 3.** Let  $k \subseteq K$  be an extension of finite fields. Show that  $\text{norm}_k^K$  and  $\text{tr}_k^K$  are surjective maps from  $K$  to  $k$ .

**Solution.** ► ◀

**Exercise 4.** Let  $f(X) \in k[X]$  be a separable polynomial of degree  $n \geq 3$  with Galois group isomorphic to  $S_n$ , and let  $\alpha \in \bar{k}$  be a root of  $f(X)$ .

- (a) Show that  $f(X)$  is irreducible.
- (b) Show that  $\text{Aut}_k(k(\alpha)) = \{\text{id}\}$ .
- (c) Show that  $\alpha^n \notin k$  if  $n \geq 4$ .

**Solution.** ► ◀

**Exercise 5.** Let  $k \subseteq K$  be a Galois extension.

- (a) For  $k \subseteq L \subseteq K$  show that  $\text{Gal}(K/L)$  is solvable if  $\text{Gal}(K/k)$  is solvable.
- (b) For  $k \subseteq L \subseteq K$  with  $k \subseteq L$  normal show that  $\text{Gal}(L/k)$  and  $\text{Gal}(K/L)$  are solvable if and only if  $\text{Gal}(K/k)$  is solvable.
- (c) For  $k \subseteq L$  with  $K$  and  $L$  in a common field show that  $\text{Gal}(KL/L)$  is solvable if  $\text{Gal}(K/k)$  is solvable.

**Solution.** ► ◀

## 2 Ulrich

### 2.1 Ulrich: Winter 2002

**Exercise 1.** Let  $G$  be a group and  $H$  a subgroup of finite index. Show that there exists a normal subgroup  $N$  of  $G$  of finite index with  $N \subseteq H$ .

**Solution.** ▶ Suppose  $N < G$  with  $n = [G : N] < \infty$ . Let  $G$  act on  $H$  by translation. This action gives a homomorphism  $\varphi: G \rightarrow S_n$ . Then, by the first isomorphism theorem  $[G : \text{Ker } \varphi] \mid \text{card } S_n = n!$ . Thus,  $\text{Ker } \varphi$  is a normal subgroup of  $G$  with finite index. ◀

**Exercise 2.** Show that every group of order 992 ( $= 32 \cdot 31$ ) is solvable.

**Solution.** ▶ Suppose  $\text{card } G = 992 = 32 \cdot 31 = 2^5 \cdot 31$ . By Sylow's theorem,  $G$  has 1 or 32 31-Sylow subgroups. In the former case, this implies that there is a unique 31-Sylow subgroup  $P$  and therefore  $P \trianglelefteq G$ . Moreover, since  $\text{card}(G/P) = 2^3$ ,  $G/P$  is solvable since it is a  $p$ -group. Thus, both  $G/P$  and  $P$  are solvable (the latter since it is Abelian), which implies that  $G$  is solvable.

On the other hand, if  $G$  contains 32 31-Sylow subgroups, then there are exactly  $32 \cdot 31 - 32 \cdot 30 = 32$  elements not of order 31. This implies that there is exactly one 2-Sylow subgroup  $Q$  in  $G$ . Again, since  $\text{card } G/Q = 31$ ,  $G/Q$  is solvable and  $Q$  is solvable since it is a  $p$ -group. Thus,  $G$  is solvable.

In every case,  $G$  we see that is solvable. ◀

**Exercise 3.** Let  $G$  be a group of order 56 with a normal 2-Sylow subgroup  $Q$ , and let  $P$  be a 7-Sylow subgroup of  $G$ . Show that either  $G \cong P \times Q$  or  $Q \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2) \times \mathbb{Z}/(2)$ . [Hint:  $P$  acts on  $Q \setminus \{e\}$  via conjugation. Show that this action is either trivial or transitive.]

**Solution.** ▶ Suppose  $G$  is a group of order  $56 = 2^3 \cdot 7$  with a normal 2-Sylow subgroup  $Q$  and let  $P \in \text{Syl}_7(G)$ . ◀

**Exercise 4.** Let  $R$  be a commutative ring and  $\text{Rad}(R)$  the intersection of all maximal ideals of  $R$ .

(a) Let  $a \in R$ . Show that  $a \in \text{Rad}(R)$  if and only if  $1 + ab$  is a unit for every  $b \in R$ .

(b) Let  $R$  be a domain and  $R[X]$  the polynomial ring over  $R$ . Deduce that  $\text{Rad}(R[X]) = 0$ .

**Solution.** ▶ ◀

**Exercise 5.** Let  $R$  be a unique factorization domain and  $\mathfrak{p}$  a prime ideal of  $R[X]$  with  $\mathfrak{p} \cap R = 0$ .

(a) Let  $n$  be the smallest possible degree of a nonzero polynomial in  $\mathfrak{p}$ . Show that  $\mathfrak{p}$  contains a primitive polynomial  $f$  of degree  $n$ .

(b) Show that  $\mathfrak{p}$  is the principal ideal generated by  $f$ .

**Solution.** ▶ ◀

**Exercise 6.** Let  $k$  be a field of characteristic zero. Assume that every polynomial in  $k[X]$  of odd degree and every polynomial in  $k[X]$  of degree two has a root in  $k$ . Show that  $k$  is algebraically closed.

**Solution.** ►

◀

**Exercise 7.** Let  $k \subseteq K$  be a finite Galois extension with Galois group  $\text{Gal}(K/k)$ , let  $L$  be a field with  $k \subseteq L \subseteq K$ , and set  $H = \{ \sigma \in \text{Gal}(K/k) : \sigma(L) = L \}$ .

- (a) Show that  $H$  is the normalizer of  $\text{Gal}(K/L)$  in  $\text{Gal}(K/k)$ .
- (b) Describe the group  $H/\text{Gal}(K/L)$  as an automorphism group.

**Solution.** ►

◀

## 2.2 Ulrich: Summer 2006

**Exercise 1.** Let  $G$  be a group of order  $2n$ , where  $n$  is odd. Show that  $G$  has a subgroup of index 2. (*Hint:* embed  $G$  into  $S_{2n}$ ).

**Solution.** ▶ ◀

**Exercise 2.** Let  $G$  be a group of odd order and let  $H$  be a normal subgroup of order 5. Show that  $H$  is in the center of  $G$ .

**Solution.** ▶ ◀

**Exercise 3.** Show that up to isomorphism, there are at most two groups of order 147 having an element of order 49.

**Solution.** ▶ ◀

**Exercise 4.** Let  $R$  be a principal ideal domain and  $\mathfrak{m}$  a maximal ideal of the polynomial ring  $R[X]$  with  $\mathfrak{m} \cap R \neq \{0\}$ . Show that  $\mathfrak{m} = (p, f)$  for some prime element  $p$  of  $R$  and some monic irreducible polynomial  $f$  in  $R[X]$ .

**Solution.** ▶ ◀

**Exercise 5.** Let  $k \subseteq K$  be a normal extension of fields of characteristic  $p > 0$  with  $G = \text{Aut}_k(K)$ . Show that the extension  $k \subseteq K^G$  is purely inseparable.

**Solution.** ▶ ◀

**Exercise 6.** Let  $k \subseteq K_1$  and  $k \subseteq K_2$  be finite Galois extensions contained in a common field, and write  $K = K_1 K_2$ .

(a) Show that the extension  $k \subseteq K$  is finite Galois.

(b) Show that the Galois group  $\text{Gal}(K/k)$  is isomorphic to the subgroup  $H = \{(\sigma, \tau) : \sigma|_{K_1 \cap K_2} = \tau|_{K_1 \cap K_2}\}$  of  $\text{Gal}(K_1/k) \times \text{Gal}(K_2/k)$ .

**Solution.** ▶ ◀

**Exercise 7.** Let  $p$  be a prime number,  $\zeta \in \mathbb{C}$  a primitive  $p$ th root of unity and  $K = \mathbb{Q}(\zeta)$ . Determine those  $p$  for which  $K$  has a unique maximal proper subfield  $k \subsetneq K$ .

**Solution.** ▶ ◀

### 2.3 Ulrich: Summer 2009

**Exercise 1.** Let  $G$  be a group such that  $G/Z(G)$  is Abelian, and let  $H \neq \{e\}$  be a normal subgroup of  $G$ . Show that  $H \cap Z(G)$ . (*Hint:* Consider the commutator subgroup  $G'$  of  $G$ ).

**Solution.** ► ◀

**Exercise 2.** Let  $G$  be a group of order 150. Show that  $G$  has a normal subgroup of order 25. (*Hint:* You may want to show that  $G$  has a normal subgroup of order 5 or 25.)

**Solution.** ► ◀

**Exercise 3.** Show that up to isomorphism, there are at most three non-Abelian groups of order 70.

**Solution.** ► ◀

**Exercise 4.** Let  $R$  be a unique factorization domain with quotient field  $K$ , let  $K \subseteq L$  be a field extension, and let  $\alpha$  be an element of  $L$  that is algebraic over  $K$ . Consider the subring  $R[\alpha]$  of  $L$ . Find an ideal  $I$  of the polynomial ring  $R[X]$  so that  $R[\alpha] \cong R[X]/I$  (*Hint:* Consider the minimal polynomial of  $\alpha$  over  $K$ .)

**Solution.** ► ◀

**Exercise 5.** Let  $k$  be a field of characteristic  $p > 0$ , and let  $k \subseteq K$  be an algebraic field extension of finite inseparable degree.

- (a) Show that there exists  $e \in \mathbb{N}$  such that  $kK^{p^n} = kK^{p^e}$  for every  $n \geq e$ .
- (b) Show that the inseparable degree of  $k \subseteq K$  in  $[K : kK^{p^e}]$  for  $e$  as in (a).

**Solution.** ► ◀

**Exercise 6.** Let  $k$  be a field, let  $f[X] \in k[X]$  be a separable polynomial of degree  $n$  whose Galois group is isomorphic to  $S_n$ , and let  $\alpha$  be a root of  $f(X)$  in some algebraic closure  $\bar{k}$ .

- (a) Show that  $f(X)$  is irreducible.
- (b) Show that  $\text{Aut}_k(k(\alpha)) = \{\text{id}_k\}$  if  $n \geq 3$ .
- (c) Show that  $\alpha^n \notin k$  if  $n \geq 4$ .

**Solution.** ► ◀

**Exercise 7.** Determine the Galois group (up to isomorphism) of the polynomial  $f = X^4 - 4X^2 + 2$  over  $\mathbb{Q}$ . Find all intermediate fields between  $\mathbb{Q}$  and the splitting field of  $f$  over  $\mathbb{Q}$ .

**Solution.** ► ◀

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