Math 535 - General Topology
Fall 2012
Homework 13 Solutions

Note: In this problem set, function spaces are endowed with the compact-open topology unless otherwise noted.

Problem 1. Let X be a compact topological space, and (Y, d) a metric space. Consider the uniform metric

$$d(f,g) := \sup_{x \in X} d\left(f(x), g(x)\right)$$

on the set of continuous maps $C(X,Y)$.

Show that the topology on $C(X,Y)$ induced by the uniform metric is the compact-open topology.

Solution. Denote respectively by $\mathcal{T}_{co}$ and $\mathcal{T}_{met}$ the compact-open topology and the uniform metric topology on $C(X,Y)$.

$(\mathcal{T}_{met} \subseteq \mathcal{T}_{co})$ It suffices to show that for any $f \in C(X,Y)$ and $\epsilon > 0$, the open ball $B(f,\epsilon) \subseteq C(X,Y)$ is a neighborhood of f in the compact-open topology.

Cover Y by open balls of radius $\frac{\epsilon}{3}$ and pull this open cover back to X via f:

$$\begin{aligned} X &= f^{-1}(Y) \\ &= f^{-1}\left(\bigcup_{y \in Y} B(y, \frac{\epsilon}{3})\right) \\ &= \bigcup_{y \in Y} f^{-1}\left(B(y, \frac{\epsilon}{3})\right). \end{aligned}$$

Since X is compact, there is a finite subcover

$$\begin{aligned} X &= f^{-1}\left(B(y_1, \frac{\epsilon}{3})\right) \cup \ldots \cup f^{-1}\left(B(y_n, \frac{\epsilon}{3})\right) \\ &=: U_1 \cup \ldots \cup U_n. \end{aligned}$$

The closures $\overline{U_i} \subseteq X$ are compact, since they are closed in X which is compact. Moreover they satisfy:

$$\begin{aligned} f(\overline{U_i}) &\subseteq \overline{f(U_i)} \text{ since } f \text{ is continuous} \\ &= \overline{f\left(f^{-1}\left(B(y_i, \frac{\epsilon}{3})\right)\right)} \\ &\subseteq \overline{B(y_i, \frac{\epsilon}{3})} \\ &\subseteq B^{cl}(y_i, \frac{\epsilon}{3}) := \{y \in Y \mid d(y_i, y) \leq \frac{\epsilon}{3}\} \\ &\subseteq B(y_i, \frac{\epsilon}{2}). \end{aligned}$$

Therefore f satisfies

$$f \in \bigcap_{i=1}^{n} V(\overline{U_i}, B(y_i, \frac{\epsilon}{2})) =: N$$

where the latter subset $N \subseteq C(X,Y)$ is open in the compact-open topology.

Claim: $N \subseteq B(f, \epsilon)$**.** Let $g \in N$ and take any point $x \in X$. Since the U_i cover X, we have $x \in U_i$ for some index $1 \leq i \leq n$. The values $f(x)$ and $g(x)$ satisfy:

$$f(x) \in f(U_i) \subseteq B(y_i, \frac{\epsilon}{3})$$
$$g(x) \in g(U_i) \subseteq g(\overline{U_i}) \subseteq B(y_i, \frac{\epsilon}{2}).$$

By the triangle inequality, we obtain:

$$\begin{aligned} d(f(x), g(x)) &\leq d(f(x), y_i) + d(y_i, g(x)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{2} \\ &= \frac{5\epsilon}{6} \end{aligned}$$

and since $x \in X$ was arbitrary, we conclude $d(f,g) \leq \frac{5\epsilon}{6} < \epsilon$.

[Note that the condition $d(f(x), g(x)) < \epsilon$ for all $x \in X$ was enough to guarantee $d(f,g) < \epsilon$, since the supremum $\sup_{x \in X} d(f(x), g(x))$ is achieved at a point $x_0 \in X$.]

($\mathcal{T}_{co} \subseteq \mathcal{T}_{met}$**)** Take a subbasic open $V(K,U) \subseteq C(X,Y)$ and $f \in V(X,U)$. We want to show that $V(K,U)$ is metrically open, i.e. find a radius ϵ satisfying $B(f, \epsilon) \subseteq V(K,U)$.

The subset $f(K) \subseteq Y$ is compact, and moreover it satisfies $f(K) \subseteq U$ by assumption. Hence there is a number $\epsilon > 0$ such that the ϵ-neighborhood of $f(K)$ is contained in U, i.e. $B(f(K), \epsilon) \subseteq U$ (c.f. Homework 5 Problem 5b).

Claim: $B(f, \epsilon) \subseteq V(K,U)$**.** Let $g \in B(f, \epsilon)$. For any $x \in K$, we have:

$$g(x) \in B(f(x), \epsilon) \subseteq B(f(K), \epsilon) \subseteq U$$

so that the map $g \colon X \to Y$ satisfies $g(K) \subseteq U$, i.e. $g \in V(K,U)$. $\square$

Problem 2. Let X and Y be topological spaces. Let $f, g\colon X \to Y$ be two continuous maps. Show that a homotopy from f to g induces a (continuous) path from f to g in the space of continuous maps $C(X, Y)$.

More precisely, let $F(X, Y)$ denote the *set of all functions* from X to Y. There is a natural bijection of sets:

$$\varphi\colon F(X \times [0,1], Y) \xrightarrow{\cong} F\left([0,1], F(X,Y)\right)$$

sending a function $H\colon X \times [0,1] \to Y$ to the function $\varphi(H)\colon [0,1] \to F(X,Y)$ defined by $\varphi(H)(t) = H(-,t) =: h_t$.

Your task is to show that if a function $H\colon X \times [0,1] \to Y$ is continuous, then the following two conditions hold:

1. $h_t\colon X \to Y$ is continuous for all $t \in [0,1]$;
2. The corresponding function $\varphi(H)\colon [0,1] \to C(X,Y)$ is continuous.

Solution.

1. For $t \in [0,1]$, the "slice inclusion" map $\iota_t\colon X \to X \times [0,1]$ defined by

$$\iota_t(x) = (x,t)$$

is an embedding. Therefore the map $h_t\colon X \to Y$ is continuous, since it is the composite $h_t = H \circ \iota_t$, as illustrated here:

$$\begin{tikzcd} X \arrow[r, "\iota_t"] \arrow[rr, bend right, "h_t"'] & X \times [0,1] \arrow[r, "H"] & Y. \end{tikzcd}$$

2. Let $V(K,U) \subseteq C(X,Y)$ be a subbasic open subset, with $K \subseteq X$ compact and $U \subseteq Y$ open. We want to show that the preimage $\varphi(H)^{-1}\left(V(K,U)\right) \subseteq [0,1]$ is open in $[0,1]$. This preimage is:

$$\begin{aligned} \varphi(H)^{-1}\left(V(K,U)\right) &= \{t \in [0,1] \mid h_t \in V(K,U)\} \\ &= \{t \in [0,1] \mid h_t(K) \subseteq U\} \\ &= \{t \in [0,1] \mid H(k,t) \in U \ \text{ for all } k \in K\} \\ &= \{t \in [0,1] \mid K \times \{t\} \subseteq H^{-1}(U)\}. \end{aligned}$$

Since $H\colon X \times [0,1] \to Y$ is continuous, $H^{-1}(U)$ is open in $X \times [0,1]$. If the inclusion $K \times \{t\} \subseteq H^{-1}(U)$ holds, then since K and $\{t\}$ are compact, there exist open subsets $W \subseteq X$ and $J \subseteq [0,1]$ satisfying

$$K \times \{t\} \subseteq W \times J \subseteq H^{-1}(U)$$

by the tube lemma. In particular, every $t' \in J$ satisfies $K \times \{t'\} \subseteq W \times \{t'\} \subseteq H^{-1}(U)$. Therefore the inclusion

$$J \subseteq \varphi(H)^{-1}\left(V(K,U)\right)$$

holds, so that $\varphi(H)^{-1}\left(V(K,U)\right)$ is open in $[0,1]$. $\square$

Remark. If X is *locally compact Hausdorff*, then the converse holds as well: the two conditions guarantee that $H \colon X \times [0,1] \to Y$ is continuous. In that case, a homotopy from f to g is really the same as a path from f to g in the function space $C(X, Y)$.

Problem 3.

a. Let X and Y be topological spaces, where Y is Hausdorff. Show that $C(X,Y)$ is Hausdorff.

Solution. Let $f, g \in C(X,Y)$ be distinct elements, i.e. there is a point $x \in X$ where $f(x) \neq g(x)$. Since Y is Hausdorff, the points $f(x)$ and $g(x)$ in Y can be separated by neighborhoods $U, U' \subseteq Y$, satisfying $f(x) \in U$, $g(x) \in U'$, and $U \cap U' = \emptyset$. Then we have $f \in V(\{x\}, U)$ and $g \in V(\{x\}, U')$, where both subsets are open in $C(X,Y)$ since the singleton $\{x\} \subseteq X$ is compact. Moreover, their intersection is:

$$\begin{aligned} V(\{x\},U) \cap V(\{x\},U') &= V(\{x\}, U \cap U') \\ &= V(\{x\}, \emptyset) \\ &= \emptyset \end{aligned}$$

so that f and g can be separated by neighborhoods in $C(X,Y)$. $\square$

b. Assume there exists a topological space X such that $C(X,Y)$ is Hausdorff. Show that Y is Hausdorff.

Solution. Let $y, y' \in Y$ be distinct points. Consider the constant functions $f, f' \colon X \to Y$ at y and y' respectively, i.e. $f \equiv y$ and $f' \equiv y'$. Note that f and f' are continuous, i.e. elements of $C(X,Y)$.

Since $C(X,Y)$ is Hausdorff, the distinct elements f and f' can be separated by basic open neighborhoods

$$\begin{aligned} f &\in N = V(K_1, U_1) \cap \ldots \cap V(K_n, U_n) \\ f' &\in N' = V(K'_1, U'_1) \cap \ldots \cap V(K'_k, U'_k) \end{aligned}$$

in $C(X,Y)$. The condition $f \in V(K_i, U_i)$ means $f(K_i) \subseteq U_i$, or equivalently $y \in U_i$ since f is the constant function $f \equiv y$. Likewise, we have $y' \in U'_i$ for all i. Therefore we obtain neighborhoods of y and y' in Y:

$$\begin{aligned} y &\in U := U_1 \cap \ldots \cap U_n \\ y' &\in U' := U'_1 \cap \ldots \cap U'_k. \end{aligned}$$

Claim: The neighborhoods U and U' are disjoint. For any point $y_0 \in U \cap U'$, the constant function $g \colon X \to Y$ with value y_0 is continuous and satisfies

$$\begin{aligned} g &\in V(K_1, U \cap U') \cap \ldots \cap V(K_n, U \cap U') \\ &\subseteq V(K_1, U) \cap \ldots \cap V(K_n, U) \\ &= V(K_1, U_1) \cap \ldots \cap V(K_n, U_n) = N \end{aligned}$$

and likewise $g \in N'$. This implies $g \in N \cap N' = \emptyset$, and therefore $U \cap U' = \emptyset$. $\square$

Problem 4.

a. Let X, Y, and Z be topological spaces. Let $g\colon Y \to Z$ be a continuous map. Show that the induced map "postcomposition by g"

$$\begin{aligned} g_*\colon C(X,Y) &\to C(X,Z) \\ f &\mapsto g_*(f) = g \circ f \end{aligned}$$

is continuous.

Solution. It suffices to show that the preimage of a subbasic open is open.

Consider the subbasic open subset $V(K,U) \subseteq C(X,Z)$, where $K \subseteq X$ is compact and $U \subseteq Z$ is open. Its preimage in $C(X,Y)$ is

$$\begin{aligned} (g_*)^{-1}V(K,U) &= \{f \in C(X,Y) \mid g \circ f \in V(K,U)\} \\ &= \{f \in C(X,Y) \mid g(f(K)) \subseteq U\} \\ &= \{f \in C(X,Y) \mid f(K) \subseteq g^{-1}(U)\} \\ &= V(K, g^{-1}(U)) \end{aligned}$$

which is open in $C(X,Y)$ since $K \subseteq X$ is compact and $g^{-1}(U) \subseteq Y$ is open in Y. $\square$

Remark. Alternate proof when X is locally compact Hausdorff.

It suffices to show that the corresponding map

$$\widetilde{g_*}\colon C(X,Y) \times X \to Z$$

is continuous. But this map is the composite $g \circ e$ where $e\colon C(X,Y) \times X \to Y$ is the evaluation map:

$$\begin{tikzcd} C(X,Y) \times X \arrow[r, "\widetilde{g_*}"] \arrow[d, "e"'] & Z. \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

Since X is locally compact Hausdorff, the evaluation map e is continuous, and so is the composite $\widetilde{g_*} = g \circ e$. $\square$

b. Let W, X, and Y be topological spaces. Let $d\colon W \to X$ be a continuous map. Show that the induced map "precomposition by d"

$$\begin{aligned} d^*\colon C(X,Y) &\to C(W,Y) \\ f &\mapsto d^*(f) = f \circ d \end{aligned}$$

is continuous.

Solution. Consider the subbasic open subset $V(K,U) \subseteq C(W,Y)$, where $K \subseteq W$ is compact and $U \subseteq Y$ is open. Its preimage in $C(X,Y)$ is

$$\begin{aligned} (d^*)^{-1}V(K,U) &= \{f \in C(X,Y) \mid f \circ d \in V(K,U)\} \\ &= \{f \in C(X,Y) \mid f(d(K)) \subseteq U\} \\ &= V(d(K),U) \end{aligned}$$

which is open in $C(X,Y)$ since $d(K) \subseteq X$ is compact and $U \subseteq Y$ is open in Y. □

Remark. Alternate proof when X is locally compact Hausdorff.

It suffices to show that the corresponding map

$$\widetilde{d^*}\colon C(X,Y) \times W \to Y$$

is continuous. But this map is the composite $e \circ (\mathrm{id} \times d)$ where $e\colon C(X,Y) \times X \to Y$ is the evaluation map:

$$\begin{tikzcd} C(X,Y) \times W \arrow[r, "\widetilde{d^*}"] \arrow[d, "\mathrm{id}\times d"'] & Y. \\ C(X,Y) \times X \arrow[ur, "e"'] & \end{tikzcd}$$

Since X is locally compact Hausdorff, the evaluation map e is continuous, and so is the composite $\widetilde{d^*} = e \circ (\mathrm{id} \times d)$. □

Problem 5. Let X and Y be topological spaces, where X is *Hausdorff*. Let $\mathcal{S}$ be a subbasis for the topology of Y. Show that the collection

$$\{V(K,S) \mid K \subseteq X \text{ compact, } S \in \mathcal{S}\}$$

is a subbasis for the compact-open topology on $C(X,Y)$.

The notation above is $V(K,S) = \{f \in C(X,Y) \mid f(K) \subseteq S\}$.

Solution. Let $\mathcal{T}$ be the topology on $C(X,Y)$ generated by the collection above. We want to show the equality $\mathcal{T} = \mathcal{T}_{co}$, where the latter denotes the compact-open topology.

($\mathcal{T} \subseteq \mathcal{T}_{co}$) Since $S \in \mathcal{S}$ is open in Y, the subsets $V(K,S) \subseteq C(X,Y)$ are open in the compact-open topology.

($\mathcal{T}_{co} \subseteq \mathcal{T}$) It suffices to show that the generating open subsets $V(K,U)$ are in $\mathcal{T}$, where $K \subseteq X$ is compact and $U \subseteq Y$ is open.

First, let $\mathcal{B}$ denote the collection of finite intersections of members of $\mathcal{S}$, so that $\mathcal{B}$ is a basis for the topology on Y. The equality

$$\begin{aligned}
V(K,S_1) \cap V(K,S_2) &= \{f \in C(X,Y) \mid f(K) \subseteq S_1 \text{ and } f(K) \subseteq S_2\} \\
&= \{f \in C(X,Y) \mid f(K) \subseteq S_1 \cap S_2\} \\
&= V(K, S_1 \cap S_2)
\end{aligned}$$

shows that subsets of the form $V(K,B) \subseteq C(X,Y)$ are in $\mathcal{T}$, where $K \subseteq X$ is compact and $B \in \mathcal{B}$.

Let $f \in V(K,U)$, i.e. f satisfies $f(K) \subseteq U$. We want to find a $\mathcal{T}$-neighborhood of f contained in $V(K,U)$. Since $\mathcal{B}$ is a basis for the topology on Y, the open subset $U \subseteq Y$ can be written as a union

$$U = \bigcup_{\alpha \in A} B_\alpha$$

with $B_\alpha \in \mathcal{B}$. Therefore K is covered by the preimages:

$$\begin{aligned}
K &\subseteq f^{-1}(U) \\
&= f^{-1}\left(\bigcup_{\alpha \in A} B_\alpha\right) \\
&= \bigcup_{\alpha \in A} f^{-1}(B_\alpha)
\end{aligned}$$

where each $f^{-1}(B_\alpha) \subseteq X$ is open in X. Since X is Hausdorff, there exist compact subsets $K_1, \ldots, K_n \subseteq X$ satisfying

$$K = K_1 \cup \ldots \cup K_n$$

and $K_i \subseteq f^{-1}(B_{\alpha_i})$ for some indices α_i (c.f. Homework 8 Problem 4b).

The condition $f(K_i) \subseteq B_{\alpha_i}$ says $f \in V(K_i, B_{\alpha_i})$ for all $i = 1, \ldots, n$. The conditions $K = K_1 \cup \ldots \cup K_n$ and $B_{\alpha_i} \subseteq U$ imply:

$$\begin{aligned} V(K_1, B_{\alpha_1}) \cap \ldots \cap V(K_n, B_{\alpha_n}) &\subseteq V(K_1, U) \cap \ldots \cap V(K_n, U) \\ &= V(K_1 \cup \ldots \cup K_n, U) \\ &= V(K, U). \end{aligned}$$

Therefore $V(K_1, B_{\alpha_1}) \cap \ldots \cap V(K_n, B_{\alpha_n})$ is a $\mathcal{T}$-neighborhood of f contained in $V(K, U)$. $\square$

Problem 6. Consider the real line $\mathbb{R}$ and the rationals $\mathbb{Q}$ with their standard (metric) topology. Consider the evaluation map

$$e\colon \mathbb{Q} \times C(\mathbb{Q},\mathbb{R}) \to \mathbb{R}.$$

Let $f\colon \mathbb{Q} \to \mathbb{R}$ be a constant function (say, $f \equiv 0$), and let $q \in \mathbb{Q}$. Show that the evaluation map e is *not* continuous at $(q,f) \in \mathbb{Q} \times C(\mathbb{Q},\mathbb{R})$.

Hint: You may want to use the fact that all compact subsets of $\mathbb{Q}$ have empty interior (c.f. Homework 7 Problem 5), and the fact that $\mathbb{Q}$ is completely regular (since it is normal).

Solution. Take $\epsilon = 1$ and consider the open ball $B_1(0)$ of radius 1 centered at $e(q,f) = f(q) = 0 \in \mathbb{R}$. We will show that on every neighborhood of (q,f) in $\mathbb{Q} \times C(\mathbb{Q},\mathbb{R})$, the evaluation map e takes values larger than 1, proving discontinuity of e at (q,f).

Since "open boxes" form a basis of the product topology on $\mathbb{Q} \times C(\mathbb{Q},\mathbb{R})$, it suffices to consider a neighborhood of (q,f) of the form $V \times N$ where $V \subseteq \mathbb{Q}$ is an open neighborhood of $q \in \mathbb{Q}$ and $N \subseteq C(\mathbb{Q},\mathbb{R})$ is a basic open neighborhood of $f \in C(\mathbb{Q},\mathbb{R})$, of the form

$$N = V(K_1,U_1) \cap \ldots \cap V(K_n,U_n)$$

for some compact subsets $K_i \subset \mathbb{Q}$ and open subsets $U_i \subseteq \mathbb{R}$.

The condition $f \in N$ can be restated as the following equivalent conditions:

$$\begin{aligned} f \in N &\Leftrightarrow \; f \in V(K_i,U_i) \text{ for all } i \\ &\Leftrightarrow \; f(K_i) \subseteq U_i \text{ for all } i \\ &\Leftrightarrow \; 0 \in U_i \text{ for all } i. \end{aligned}$$

Therefore any continuous function $g\colon \mathbb{Q} \to \mathbb{R}$ that vanishes on $K := K_1 \cup \ldots \cup K_n$, i.e. satisfying $g|_K \equiv 0$, automatically satisfies $g \in N$.

Moreover, K is a finite union of compact subsets of $\mathbb{Q}$, thus itself compact. Therefore $K \subset \mathbb{Q}$ has empty interior, which implies $V \nsubseteq K$ since $V \subseteq \mathbb{Q}$ is open in $\mathbb{Q}$. Pick a rational $q' \in V \setminus K$.

Since K is closed in $\mathbb{Q}$ (being compact in the Hausdorff space $\mathbb{Q}$), and $\mathbb{Q}$ is completely regular, there exists a continuous function

$$g\colon \mathbb{Q} \to [0,75] \subset \mathbb{R}$$

satisfying $g|_K \equiv 0$ and $g(q') = 75$. By construction, the pair (q',g) is sufficiently close to (q,f):

$$(q',g) \in V \times N$$

and satisfies $e(q',g) = g(q') = 75 > 1$. $\square$