The fundamental group of S^1

Please let me know about any misprints you notice.

Let x_0 denote the point $(1,0)$ in S^1. Our goal is to prove:

Theorem A. There is an isomorphism

$$W : \pi_1(S^1, x_0) \xrightarrow{\cong} \mathbb{Z}.$$

which takes the class of the path $f_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ to n.

As a tool, we will use the map

$$p : \mathbb{R} \to S^1$$

defined by

$$p(u) = (\cos 2\pi u, \sin 2\pi u).$$

We need to know three things about the map p:

Proposition B. For every path $f : I \to S^1$ with $f(0) = x_0$ there is a path $\phi : I \to \mathbb{R}$ with $f = p \circ \phi$ and $\phi(0) = 0$.

Proposition C. Let A be a connected space and let $a \in A$. If two continuous functions $\alpha, \beta : A \to \mathbb{R}$ have the property that $\alpha(a) = \beta(a)$ and $p \circ \alpha = p \circ \beta$ then $\alpha = \beta$.

In particular, for a given f, any two paths $I \to \mathbb{R}$ with the two properties given in Proposition B must be equal. We can therefore write $\tilde{f}$ for the unique path given by Proposition B.

Proposition D. For every continuous $H : I \times I \to S^1$ there is a continuous $\Phi : I \times I \to \mathbb{R}$ with $H = p \circ \Phi$ and $\Phi(0,0) = 0$.

(Φ is uniquely determined by the two properties in Proposition D, but we won't need this.)

Proofs of B and D will be given below; I will ask you to prove C on the homework.

Now we can begin the process of defining $W : \pi_1(S^1, x_0) \xrightarrow{\cong} \mathbb{Z}$. First note that $p^{-1}(x_0) = \mathbb{Z}$ (by trigonometry). Given a loop $f : I \to S^1$ with $f(0) = f(1) = x_0$, Proposition B gives a path $\tilde{f} : I \to \mathbb{R}$ with $p \circ \tilde{f} = f$ and $\tilde{f}(0) = 0$. Then $\tilde{f}(1)$ is in $p^{-1}(x_0) = \mathbb{Z}$. Define

$$w(f) = \tilde{f}(1).$$

Lemma E. If f and g are loops at $(1,0)$ with $f \simeq_p g$ then $w(f) = w(g)$.

Proof. Let H be a path-homotopy from f to g. Let Φ be the map given by Proposition D.

Step 1. The path $I \to \mathbb{R}$ which takes s to $\Phi(s,0)$ has the two properties in Proposition B, so it's the path $\tilde{f}$. In particular, $\Phi(1,0) = \tilde{f}(1)$.

Step 2. The path which takes t to $\Phi(0,t)$ is the constant path e_0 in $\mathbb{R}$. (This follows easily from Proposition C.) In particular, $\Phi(0,1) = 0$.

Step 3. Step 2 shows that the path $I \to \mathbb{R}$ which takes s to $\Phi(s,1)$ has the two properties which uniquely determind $\tilde{g}$, so it is $\tilde{g}$. In particular, $\Phi(1,1) = \tilde{g}(1)$.

Step 4. Let $u = \tilde{f}(1)$. The path which takes t to $\Phi(1,t)$ is the constant path e_u in $\mathbb{R}$. (This follows easily from Proposition C.) In particular, $\Phi(1,1) = \tilde{f}(1)$.

Step 5. $\tilde{g}(1) = \tilde{f}(1)$. (This is immediate from Steps 3 and 4.) $\square$

Finally, we can define $W : \pi_1(S^1, x_0) \xrightarrow{\cong} \mathbb{Z}$ by $W([f]) = w(f)$. This is well-defined by Lemma E.

On the homework I will ask you to show the following, which completes the proof of Theorem A:

Proposition F. (i) W takes the class of the path $f_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ to n (and therefore W is onto).

(ii) W is 1–1.

(iii) W is a homomorphism.

Proof of Proposition B

We need a fact about the map p, which I will ask you to prove on the homework:

Lemma G. For each $a \in \mathbb{R}$, the map

$$p_a : (a, a+1) \to S^1 - \{p(a)\}$$

given by $p_a(u) = p(u)$ is a homeomorphism.

For each $a \in \mathbb{R}$, let $U_a \subset S^1$ denote the open set $S^1 - \{p(a)\}$.

Now let f be a path in S^1. The sets $f^{-1}(U_a)$ are an open cover of I, so by the Lebesgue Lemma (Lemma 27.5 in Munkres) there is a $\delta > 0$ such that every set $S \subset I$ with diameter $< \delta$ is contained in some $f^{-1}(U_a)$. Now fix an n with $\frac{1}{n} < \delta$; then each interval $[\frac{i-1}{n}, \frac{i}{n}]$ is contained in some $f^{-1}(U_a)$.

We claim by (finite!) induction on i that for each i with $0 \le i \le n$ there is a continuous $g_i : [0, \frac{i}{n}] \to \mathbb{R}$ with $p \circ g_i = f|_{[0,\frac{i}{n}]}$ and $g_i(0) = 0$ (then we can let $\tilde{f}$ be g_n).

To start the induction, let $g_0 : [0,0] \to \mathbb{R}$ take 0 to 0.

Now suppose g_i exists for some $i < n$. Choose an a with $[\frac{i}{n}, \frac{i+1}{n}] \subset f^{-1}(U_a)$. Then $p_a^{-1}(f(\frac{i}{n}))$ and $g_i(\frac{i}{n})$ are both in $p^{-1}(f(\frac{i}{n}))$, so by trigonometry they must differ by an element of $\mathbb{Z}$; that is, there is a $k \in \mathbb{Z}$ with

$$(*) \qquad g_i(\tfrac{i}{n}) = p_a^{-1}(f(\tfrac{i}{n})) + k.$$

Now define

$$g_{i+1}(s) = \begin{cases} g_i(s) & \text{if } s \le \frac{i}{n}, \\ p_a^{-1}(f(s)) + k & \text{if } s \ge \frac{i}{n}, \end{cases}$$

which is well-defined by $(*)$ and hence continuous by the Pasting Lemma. Then $g_{i+1}(0) = 0$ and (using trigonmetry) $p \circ g_{i+1} = f|_{[0,\frac{i+1}{n}]}$. $\square$

Proof of Proposition D

The proof uses ideas similar to the proof of Proposition B.

Let $H : I \times I \to S^1$ be continuous. The sets $H^{-1}(U_a)$ are an open cover of $I \times I$, so by the Lebesgue Lemma there is a $\delta > 0$ such that every set $S \subset I \times I$ with diameter $< \delta$ is contained in some $H^{-1}(U_a)$. Now fix an n with $\frac{1}{n} < \delta$; then each set $[\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}]$ with $0 \le i < n$ and $0 \le j < n$ is contained in some $H^{-1}(U_a)$.

We claim by (finite) induction on j that for each j with $0 \le j \le n$ there is a continuous $K_j : I \times [0, \frac{j}{n}] \to \mathbb{R}$ with $p \circ K_j = H|_{I \times [0, \frac{j}{n}]}$ and $K_j(0,0) = 0$ (then we can let Φ be K_n).

To start the induction, define $f(s) = H(s, 0)$, and let $K_0(s, 0) = \tilde{f}(s)$, where $\tilde{f}$ is the map given by Proposition B.

Now *fix* a j with $0 \le j < n$ and suppose K_j exists. For $0 \le i \le n$ let S_i denote the set

$$(I \times [0, \tfrac{j}{n}]) \cup ([0, \tfrac{i}{n}] \times [\tfrac{j}{n}, \tfrac{j+1}{n}]).$$

We claim by a (second!) finite induction that for each i with $0 \le i \le n$ there is a continuous

$$L_i : S_i \to \mathbb{R}$$

such that $p \circ L_i = H|_{S_i}$ and $L_i(0,0) = 0$ (then we can let K_{j+1} be L_n).

To start the (second) induction, choose an a with $\{0\} \times [\frac{j}{n}, \frac{j+1}{n}] \subset H^{-1}(U_a)$. Then $p_a^{-1}(H(0, \frac{j}{n}))$ and $K_j(0, \frac{j}{n})$ are both in $p^{-1}(H(0, \frac{j}{n}))$, so there is a $k \in \mathbb{Z}$ with

$$(**) \qquad K_j(0, \tfrac{j}{n}) = p_a^{-1}(H(0, \tfrac{j}{n})) + k.$$

Now define $L_0 : S_0 \to \mathbb{R}$ by

$$L_0(s,t) = \begin{cases} K_j(s,t) & \text{if } t \le \frac{j}{n}, \\ p_a^{-1}(H(0,t)) + k & \text{if } s = 0 \text{ and } t \in [\frac{j}{n}, \frac{j+1}{n}]. \end{cases}$$

This is well-defined by $(**)$ and hence continuous by the Pasting Lemma. Then $L_0(0,0) = 0$ and (using trigonometry) $p \circ L_0 = H|_{S_0}$.

Now suppose that L_i exists for some $i < n$. Choose an a with $[\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}] \subset H^{-1}(U_a)$. Then $p_a^{-1}(H(\frac{i}{n}, \frac{j}{n}))$ and $L_i(\frac{i}{n}, \frac{j}{n})$ are both in $p^{-1}(H(\frac{i}{n}, \frac{j}{n}))$, so there is a $k \in \mathbb{Z}$ with

$$(***) \qquad L_i(\tfrac{i}{n}, \tfrac{j}{n}) = p_a^{-1}(H(\tfrac{i}{n}, \tfrac{j}{n})) + k.$$

Now define $L_{i+1} : S_{i+1} \to \mathbb{R}$ by

$$L_{i+1}(s,t) = \begin{cases} L_i(s,t) & \text{if } s \le \frac{i}{n} \text{ or } t \le \frac{j}{n}, \\ p_a^{-1}(H(s,t)) + k & \text{if } (s,t) \in [\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}]. \end{cases}$$

To see that L_{i+1} is well-defined we need to know that $L_i = p_a^{-1} \circ H + k$ on the set

$$A = \{\tfrac{i}{n}\} \times [\tfrac{j}{n}, \tfrac{j+1}{n}] \cup [\tfrac{i}{n}, \tfrac{i+1}{n}] \times \{\tfrac{j}{n}\}.$$

But A is connected by Theorem 23.3, so the fact we need follows from $(***)$ and Proposition C (with $a = (\frac{i}{n}, \frac{j}{n})$). Now L_{i+1} is continuous by the Pasting Lemma and we have $L_{i+1}(0,0) = 0$ and (using trigonometry) $p \circ L_{i+1} = H|_{S_{i+1}}$. $\square$