MA 572: Homework 2

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beginproblem[Hatcher §2.1, Ex. 16]

(a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.

(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof. (a) $\implies$ Suppose that the relative 0th homology of X with respect to A, $H_0(X, A)$, is trivial. Let $\{X_\alpha\}$ be the set of path-components of X. We aim to show that $A \cap X_\alpha \neq \emptyset$ for all α. Let $i\colon A \hookrightarrow X$ denote the canonical inclusion map $A \subset X$. Now, the map i can be extended to a chain map between chain complexes which, by proposition 2.9, induces a homomorphism $i_*\colon H_n(A) \to H_n(X)$ between the homology groups of A and X. Similarly, the map $j\colon C_n(X) \to C_n(X, A)$ induces a map $j_*\colon H_n(X) \to H_n(X, A)$ so, by theorem 2.16, we have a long exact sequence

$$\cdots \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0. \tag{1}$$

In particular, the short exact sequence

$$0 \xrightarrow{0} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0. \tag{2}$$

But $H_0(X, A) = 0$ so the map $j_* = 0$. By short exactness of (**??**) we have $\operatorname{im} i_* = \ker j_* = H_0(X)$, so i_* is surjective.

(b) ∎

Problem 2.1 (Hatcher §2.1, Ex. 17)

(a) Compute the homology groups $H_n(X, A)$ when X is $\mathbf{S}^2$ or $\mathbf{S}^1 \times \mathbf{S}^1$ and A is a finite set of points in X.

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown. [What are X/A and X/B?]

Proof. (a) Since A is a finite collection of points in $\mathbf{S}^2$, let us enumerate the set A via $\{a_1, ..., a_n\}$ and denote by A_k the subset $\{a_1, ..., a_k\}$ of A, where $k \leq n$. Now, by the generalization of theorem 2.16 to triples, we have the long exact sequence

$$\cdots \longrightarrow H_m(A_n, A_{n-1}) \longrightarrow H_m(\mathbf{S}^2, A_{n-1}) \longrightarrow H_m(\mathbf{S}^2, A_n) \longrightarrow H_{m-1}(A_n, A_{n-1}) \longrightarrow \cdots . \tag{3}$$

Exactness of (**??**) tells us that for $m \geq 2$ we have $H(\mathbf{S}^2, A_{n-1}) \cong H(\mathbf{S}^2, A_n)$ since

$$H_m(A_n, A_{n-1}) = 0 \longrightarrow H_m(\mathbf{S}^2, A_{n-1}) \longrightarrow H_m(\mathbf{S}^2, A_n) \longrightarrow 0 = H_{m-1}(A_n, A_{n-1})$$

is exact. Evidently, $H_m(A_n, A_{n-1}) = 0$ for $m > 1$.[1]

(b) ■

Problem 2.2

Proof. ■

[1] I will prove this if time permits.