MA166: Recitation 11

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1 Homework

1.1 This Week's Summary

Here's a summary of the material that was (presumably) covered this week. From Stewart, §11.8 to §11.10, we have

§11.8: Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \tag{1}$$

where x is a variable and the c_n's are constant called the *coefficients* of the series. For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^x + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

Theorem 1. *For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:*

(i) *The series converges only when $x = a$.*

(ii) *The series converges for all x.*

(iii) *There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.*

The number R in case (iii) is called the *radius of convergence* of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The *interval of convergence* of a power series is the interval that consists of all values of x for which the series converges. In

§11.9: Representation of Functions as Power Series

We start with an equation that we've seen before

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1. \tag{2}$$

Theorem 2. *If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$ then the function f defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(ii) $\int f(x)dx = C + c_0(x-a) + c_1(x-a)^2/2 + c_2(x-a)^3/3 + \cdots = C + \sum_{n=0}^{\infty}(x-a)^{n+1}/(n+1)$.

The radii of convergence of the power series in Equations (i) and (ii) are both R.

§11.10: Taylor and Maclaurin Series

Theorem 3. *If f has a power series representation (expansion) at a, that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots. \end{aligned} \tag{3}$$

The series in (3) is called the *Taylor series of the function f at a* (or *about a* or *centered at a*). For the special case

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots.$$

This case arises frequently enough that it is given the special name *Maclaurin series*.

Theorem 4. *If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree Taylor polynomial of f at a and*

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Theorem 5 (Taylor's inequality)**.** *If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d.$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for all } x. \tag{4}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x. \tag{5}$$

Function	Taylor series	Radius of convergence
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$R = 1$
e^x	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$R = \infty$
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	$R = \infty$
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	$R = \infty$
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	$R = 1$
$\ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$	$R = 1$
$(1+x)^k$	$\sum_{n=0}^{\infty} \dbinom{k}{n} x^n$	$R = 1$

Table 1.1: Table of Taylor series.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x. \tag{6}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x. \tag{7}$$

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the *binomial coefficients.*

Theorem 6 (The binomial series)**.** *If k is any real number and $|x| < 1$, then*

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

Table of Taylor series:

1.2 WebAssign Homework

Solutions to selected problems

Homework 28

Problem 1 (WebAssign HW 28, # 1)**.** Find a power series representation for the function.

$$f(x) = \frac{1}{9+x}.$$

Determine the interval of convergence.

Problem 2 (WebAssign HW 28, # 2)**.** Find a power series representation for the function.

$$f(x) = \frac{8}{9-x}.$$

Determine the interval of convergence.

Problem 3 (WebAssign HW 28, # 3)**.** Find a power series representation for the function.

$$f(x) = \frac{x}{4+x^2}.$$

Determine the interval of convergence.

Problem 4 (WebAssign HW 28, # 4)**.** Find a power series representation for the function.

$$f(x) = \frac{10}{x^2-2x-24}.$$

Determine the interval of convergence.

Problem 5 (WebAssign HW 28, # 5)**.** Find a power series representation for the function.

$$f(x) = \frac{1}{(7+x)^2}.$$

Determine the interval of convergence.

Problem 6 (WebAssign HW 28, # 6)**.** Find a power series representation for the function.

$$f(x) = \ln(3-x).$$

Determine the interval of convergence.

Problem 7 (WebAssign HW 28, # 7)**.** Evaluate the indefinite integral as a power series.

$$\int \frac{t}{1-t^{11}}\,dt.$$

What is the radius of convergence R?

Problem 8 (WebAssign HW 28, # 8)**.** Use a power series to approximate the definite integral, I, to six decimal places.

$$\int_0^{0.1} \frac{1}{1+x^6}\,dx.$$

Homework 29

Problem 9 (WebAssign HW 29, # 1)**.** Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

$$f(x) = \sin\left(\frac{\pi x}{3}\right).$$

Find the associated radius of convergence R.

Problem 10 (WebAssign HW 29, # 2)**.** Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

$$f(x) = e^{-5x}.$$

Find the associated radius of convergence R.

Problem 11 (WebAssign HW 29, # 3)**.** Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

$$f(x) = x^4 - 4x^2 + 2, \qquad a = 2.$$

Find the associated radius of convergence R.

Problem 12 (WebAssign HW 29, # 4)**.** Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

$$f(x) = \ln x, \qquad a = 6.$$

Find the associated radius of convergence R.

Problem 13 (WebAssign HW 29, # 5)**.** Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

$$f(x) = \frac{10}{x}, \qquad a = -2.$$

Find the associated radius of convergence R.

Homework 30

Problem 14 (WebAssign HW 30, # 1)**.** Use the Maclaurin series in the table (it's somewhere in the book, I'll put a link here 1.1) to obtain the Maclaurin series for the given function.

$$f(x) = 6e^x + e^{4x}.$$

Problem 15 (WebAssign HW 30, # 2)**.** Use the Maclaurin series in the table to obtain the Maclaurin series for the given function.

$$f(x) = 4x\cos\left(\frac{x^2}{9}\right).$$

Problem 16 (WebAssign HW 30, # 3)**.** Use the Maclaurin series in the table to obtain the Maclaurin series for the given function.

$$f(x) = 9\sin^2 x.$$

[*Hint*: Use $\sin^2 x = (1 - \cos 2x)/2$.]

Problem 17 (WebAssign HW 30, # 4)**.** Use series to approximate the definite integral I to within the indicated accuracy.

$$I = \int_0^{0.5} x^4 e^{-x^2}\,dx$$

($|\text{error}| < 0.001$).

Problem 18 (WebAssign HW 30, # 5)**.** Use series to evaluate the limit.

$$\lim_{x \to 0} \frac{1 - \cos 3x}{1 + 3x - e^{3x}}.$$

Problem 19 (WebAssign HW 30, # 6)**.** Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for the function.

$$y = e^{-x^2} \cos x.$$

2 Exam 3: Problems