

# MA 519: Homework 9

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## PROBLEM 9.1 (HANDOUT 13, # 7)

Let  $X$  have a *double exponential* density  $f(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}$ ,  $-\infty < x < \infty$ ,  $\sigma > 0$ .

- (a) Show that all moments exist for this distribution.
- (b) However, show that the MGF exists only for restricted values. Identify them and find a formula.

*SOLUTION.* For part (a), we show that the moments  $m_n := E(X^n) < \infty$  for all  $n \in \mathbb{N}$ . By direct calculation, we have

$$\begin{aligned} m_n &= \int_{-\infty}^{\infty} x^n f(x) dx \\ &= \int_{-\infty}^{\infty} \frac{x^n}{2\sigma} e^{-\frac{|x|}{\sigma}} dx \\ &= \underbrace{\int_{-\infty}^0 \frac{x^n}{2\sigma} e^{\frac{x}{\sigma}} dx}_L + \int_0^{\infty} \frac{x^n}{2\sigma} e^{-\frac{x}{\sigma}} dx, \end{aligned}$$

making the substitution  $x \mapsto -y$  to  $L$  and relabeling  $y$  to  $x$  again, the above becomes

$$\begin{aligned} &= \int_0^{\infty} \frac{x^n + (-1)x^n}{2\sigma} e^{-\frac{x}{\sigma}} dx \\ &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ I := \int_0^{\infty} \frac{x^n}{\sigma} e^{-\frac{x}{\sigma}} dx & \text{if } n = 2k \text{ is even.} \end{cases} \end{aligned}$$

To evaluate  $I$  we use integration by parts recursively to arrive at

$$\begin{aligned} I &= \int_{-\infty}^0 \frac{x^n}{\sigma} e^{-\frac{x}{\sigma}} \\ &= (-0 + 0) + \int_0^{\infty} n\sigma x^{n-1} e^{-\frac{x}{\sigma}} dx \\ &= (-0 + 0) + (-0 + 0) + \int_0^{\infty} n(n-1)\sigma^2 x^{n-2} e^{-\frac{x}{\sigma}} dx \\ &\vdots \\ &= (-0 - 0) + \cdots + (-0 + 0) + (-0 + n!\sigma^n) \\ &= n!\sigma^n. \end{aligned}$$

Therefore,  $m_n < \infty$  for all  $n \in \mathbb{N}$ , i.e., all moments of this distribution exist.

Set  $m_0 := 1$ . Then for part (b), the MGF associated to  $f$  is

$$m(t) = \sum_{n=0}^{\infty} \frac{t^n m_n}{n!} = \sum_{k=1}^{\infty} t^{2k} \sigma^{2k}. \quad (9.1)$$

This is a geometric series and so converges for all  $-\frac{1}{\sigma} < t < \frac{1}{\sigma}$ , in which case (9.1) becomes

$$m(t) = \frac{1}{1 - t^2 \sigma^2}.$$

■

## PROBLEM 9.2 (HANDOUT 13, # 10)

Suppose  $X$  has Cauchy distribution as in # 6. Which of the following functions have finite expectation

$$X; \quad -X; \quad |X|; \quad \frac{1}{X}; \quad \sin X; \quad \ln |X|; \quad e^X; \quad e^{-|X|}?$$

*SOLUTION.* Suppose  $X \sim \text{Cauchy}(0, 1)$ . Then the PDF of  $X$  is given by

$$f(x) = \frac{1}{\pi(x^2 + 1)}. \quad (9.2)$$

We now find the expectations in question. For  $X$  itself, we have

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} dx, \end{aligned}$$

which by substitution with  $u = x^2 + 1$ ,  $du = 2x dx$ , evaluates to

$$= \lim_{x_1, x_2 \rightarrow \infty} \frac{2}{\pi} \ln \left( \frac{x_2^2 + 1}{x_1^2 + 1} \right).$$

Fix  $x_2$ . Then,

$$\lim_{x_1 \rightarrow \infty} \frac{2}{\pi} \ln \left( \frac{x_2^2 + 1}{x_1^2 + 1} \right) = -\infty.$$

Whereas, if we fix  $x_1$ ,

$$\lim_{x_2 \rightarrow \infty} \frac{2}{\pi} \ln \left( \frac{x_2^2 + 1}{x_1^2 + 1} \right) = \infty.$$

We conclude that the limit of this function as  $x_1, x_2 \rightarrow \infty$  does not exist. Thus,  $E(X)$  does not exist.

For  $-X$ , first we find the CDF:

$$\begin{aligned} F_{-X}(x) &= P(-X \leq x) \\ &= 1 - P(X \leq x) \\ &= 1 - \frac{1}{\pi} \tan^{-1}(x) - \frac{1}{2} \\ &= \frac{1}{2} - \frac{1}{\pi} \tan^{-1}(x). \end{aligned}$$

Thus,

$$f_{-X}(x) = \frac{dF_{-X}(x)}{dx} = -\frac{1}{\pi(x^2 + 1)}.$$

■

## PROBLEM 9.3 (HANDOUT 13, # 16)

Give an example of each of the following phenomena:

- (a) A continuous random variable taking values in  $[0, 1]$  with equal mean and median.
- (b) A continuous random variable taking values in  $[0, 1]$  with mean equal to twice the median.
- (c) A continuous random variable for which the mean does not exist.
- (d) A continuous random variable for which the mean exists, but the variance does not exist.
- (e) A continuous random variable with a PDF that is not differentiable at zero.
- (f) a positive continuous random variable for which the mode is zero, but the mean does not exist.
- (g) A continuous random variable for which all moments exist.
- (h) A continuous random variable with median equal to zero, and 25<sup>th</sup> and 75<sup>th</sup> percentiles equal to 1.
- (i) A continuous random variable  $X$  with mean equal to median equal to mode equal to zero, and  $E(\sin X) = 0$ .

*SOLUTION.* First, note that  $[0, 1]$  is a probability space under the standard Lebesgue measure on  $\mathbb{R}$ . Therefore, it makes sense to consider  $X: [0, 1] \rightarrow \mathbb{R}$  random variables.

For part (a), consider the random variable  $X: [0, 1] \rightarrow \mathbb{R}$  defined by  $x \mapsto x$  with  $X \sim \text{Uniform}[0, 1]$ . Then the mean is

$$\mu = \int_{-\infty}^{\infty} xf(x) dx = \int_0^1 x dx = \frac{1}{2}$$

and the median is

$$m = \inf\{x : F(x) = x \geq 0.5\} = \frac{1}{2}.$$

For part (b), consider again the random variable  $X(x) = x$  for  $x \in [0, 1]$ , but this time let

$$f(x) = \begin{cases} & , \\ & . \end{cases}$$

be the PDF of  $X$ . Then the mean is ■

## PROBLEM 9.4 (HANDOUT 13, # 17)

An exponential random variable with mean 4 is known to be larger than 6. What is the probability that it is larger than 8?

*SOLUTION.*

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## PROBLEM 9.5 (HANDOUT 13, # 18)

(Sum of Gammas). Suppose  $X, Y$  are independent random variables, and  $X \sim \Gamma(\alpha, \lambda)$ ,  $Y \sim \Gamma(\beta, \lambda)$ . Find the distribution of  $X + Y$  by using moment-generating functions.

SOLUTION. ■



## PROBLEM 9.6 (HANDOUT 13, # 19)

(*Product of Chi Squares*). Suppose  $X_1, X_2, \dots, X_n$  are independent chi square variables, with  $X_i \sim \chi_{m_i}^2$ . Find the mean and variance of  $\prod_{i=1}^n X_i$ .

SOLUTION. ■

## PROBLEM 9.7 (HANDOUT 13, # 20)

Let  $Z \sim \text{Normal}(0, 1)$ . Find

$$P\left(0.5 < \left|Z - \frac{1}{2}\right| < 1.5\right); \quad P\left(\frac{e^Z}{1 + e^Z} > \frac{3}{4}\right); \quad P(\Phi(Z) < 0.5).$$

*SOLUTION.*

■

## PROBLEM 9.8 (HANDOUT 13, # 21)

Let  $Z \sim \text{Normal}(0, 1)$ . Find the density of  $\frac{1}{Z}$ . Is the density bounded?

SOLUTION. ■

## PROBLEM 9.9 (HANDOUT 13, # 22)

The 25<sup>th</sup> and the 75<sup>th</sup> percentile of a normally distributed random variable are  $-1$  and  $1$ . What is the probability that the random variable is between  $-2$  and  $2$ ?

*SOLUTION.*

