Exercises in Basic Mathematics

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Definition 1. Given any ideal $\mathfrak{a} \subset k[X_1, \dots, X_q]$, define $\mathcal{V}_k(\mathfrak{a})$ by

$$\mathcal{V}_k(\mathfrak{a}) := \{\, \mathbf{x} \in \mathbb{A}^q : \text{for every } f \in \mathfrak{a},\ f(\mathbf{x}) = 0 \,\}.$$

We call $\mathcal{V}_k(\mathfrak{a})$ the *set of Ω-points of the affine k-variety determined by* $\mathfrak{a}$. With a slight abuse of language, we call $\mathcal{V}_k(\mathfrak{a})$ the *affine k-variety determined by* $\mathfrak{a}$. Similarly, given by any ideal $\mathfrak{a} \subset \bar{k}[X_1, \dots, X_q]$, defined by $\mathcal{V}_{\bar{k}}(\mathfrak{a})$ by

$$\mathcal{V}_{\bar{k}}(\mathfrak{a}) := \{\, \mathbf{x} \in \mathbb{A}^q : \text{for every } f \in \mathfrak{a},\ f(\mathbf{x}) = 0 \,\}.$$

We call $\mathcal{V}_{\bar{k}}(\mathfrak{a})$ the *set of Ω-points of the (geometric) affine $\bar{k}$-variety determined by* $\mathfrak{a}$, or for short, the *(geometric) affine variety determined by* $\mathfrak{a}$.

To ease the notation, we usually drop the subscript k or $\bar{k}$ and simply write $\mathcal{V}$.

If A is a (commutative) ring (with unit 1), recall that the *radical*, $\sqrt{\mathfrak{b}}$, of an ideal, $\mathfrak{a} \subset A$, is defined by

$$\sqrt{\mathfrak{a}} := \{\, a \in A : \text{there exists } n \geq 1,\ a^n \in \mathfrak{a} \,\}.$$

A *radical ideal* is an ideal, $\mathfrak{a}$, such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

The following properties are easily verified. We state them for $\mathcal{V}_k$, but they also hold for $\mathcal{V}_{\bar{k}}$:

$$\begin{aligned}
&\mathcal{V}(0) = \mathbb{A}^n,\ \mathcal{V}(A) = \emptyset \\
&\mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{V}(\mathfrak{a}\mathfrak{b}) = \mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) \\
&\mathfrak{a} \subset \mathfrak{b} \text{ implies that } \mathcal{V}(\mathfrak{b}) \subset \mathcal{V}(\mathfrak{b}) \\
&\mathcal{V}\Big(\textstyle\sum_\alpha \mathfrak{a}_\alpha\Big) = \textstyle\bigcap_\alpha \mathcal{V}(\mathfrak{a}_\alpha) \\
&\mathcal{V}(\sqrt{\mathfrak{a}}) = \mathcal{V}(\mathfrak{a})
\end{aligned}$$

From the relations above, it follows that the sets $\mathcal{V}(\mathfrak{a})$ can be taken as closed subsets of $\mathbb{A}^q$, and we obtain a topology on $\mathbb{A}^q$. This is the *k-topology on $\mathbb{A}^q$.* If we consider ideals in $\bar{k}[X_1, \ldots, X_q]$ (i.e., sets of the form $\mathcal{V}_{\bar{k}}(\mathfrak{a})$), we obtain the *Zariski topology on $\mathbb{A}^q$.*

The Zariski topology is not necessarily Hausdorff (except when $\mathcal{V}(\mathfrak{a})$ consits of a finite set of points.)

Let us see that $\mathbb{A}^q$ is not Hausdorff in the Zariski topology. Let $P, Q \in \mathbb{A}^q$, with $P \neq Q$. The line $\overleftrightarrow{PQ}$ is isomorphic to $\mathbb{A}^1$. Thus, it is enough to show that $\mathbb{A}^1$ is not Hausdorff. Consider any ideal $\mathfrak{a} \subset \bar{k}[X]$. Then, $\mathfrak{a}$ is a principal ideal, and thus

$$\mathfrak{a} = (f)$$

for some polynomial f, which shows that $\mathcal{V}(\mathfrak{a}) = \mathcal{V}(f)$ is a finite set. As a consequence, the closed sets of $\mathbb{A}^1$ (other than $\mathbb{A}^1$) are finite. Then, the union of two closed sets (distinct from $\mathbb{A}^1$) is also finite, and thus distinct from $\mathbb{A}^1$.

The topology on $\mathbb{A}^q$ is not the product topology on $\prod_{i=1}^{q} \mathbb{A}^1$.

For example, when $n = 2$, the closed set in $\mathbb{A}^1 \times \mathbb{A}^1$ are those sets consisting of finitely many horizontal and vertical lines, and intersections of such sets. However

$$X^2 + Y^2 - 1 = 0$$

defines a closed set in $\mathbb{A}^2$ not of the previous form.

To go backwards from subsets of $\mathbb{A}^q$ to ideals, we make the following definition.

Definition 2. Given any subset $S \subset \mathbb{A}^q$, define $\mathcal{I}_k(S)$ and $\mathcal{I}_{\bar{k}}(S)$ by

$$\mathcal{I}_k(S) := \left\{ f \in k[X_1, \ldots, X_q] : \text{for every } s \in S,\ f(s) = 0 \right\}$$

and

$$\mathcal{I}_{\bar{k}}(S) := \left\{ f \in \bar{k}[X_1, \ldots, X_q] : \text{for every } s \in S,\ f(s) = 0 \right\}$$

The following properties are easily shown (following our conventions, they are stated for $\mathcal{I}_k$, but they are easily shown for $\mathcal{I}_{\bar{k}}$).

CHAPTER 4

Differential Geometry Exercises

4.1 The Matrix Exponential; Some Matrix Lie Groups

The Exponential Map

The Lie Groups $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, $\mathfrak{o}(n)$, $\mathrm{SO}(n)$, the Lie Algebras $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{o}(n)$, $\mathfrak{so}(n)$, and the Exponential Map

The notation $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{so}(n)$ is rather strange and deserves some explanation. The groups $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, $\mathfrak{o}(n)$, and $\mathrm{SO}(n)$ are more than just groups. They are also topological groups, which means that they are topological spaces (viewed as subspaces of $\mathbb{R}^{n^2}$) and that the multiplication and the inverse operations are smooth. In fact, they are smooth real manifolds. Such objects are called *Lie groups*. The real vector spaces $\mathfrak{sl}(n)$ and $\mathfrak{so}(n)$ are what are called *Lie algebras*. However, we have not defined the algebra structure on $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{so}(n)$ yet. The algebra structure is called the *Lie bracket*, which is defined as

$$[A,B] = AB - BA.$$

Lie algebras are associated with Lie groups. What is going on is that the Lie algebra of a Lie group is its tangent space at the identity, i.e., the space of all tangent vectors at the identity. In some sense, the Lie algebra achieves a linearization of the Lie group. The exponential map is a map from the Lie algebra to the Lie group, for example,

$$\exp\colon \mathfrak{so}(n) \longrightarrow \mathrm{SO}(n)$$

and

$$\exp\colon \mathfrak{sl}(n,\mathbb{R}) \longrightarrow \mathrm{SL}(n,\mathbb{R}).$$

The exponential map often allows a parametrization of the Lie group elements by simpler objects, the Lie algebra elements.

One might ask, "What happened to the Lie algebras $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{o}(n)$ associated with the Lie group $\mathrm{GL}(n,\mathbb{R})$ and $\mathfrak{o}(n)$?" We will see later that $\mathfrak{gl}(n,\mathbb{R})$ is the set of all real $n \times n$ matrices and that $\mathfrak{o}(n) = \mathfrak{so}(n)$.

The properties of the exponential map play an important role in the study of Lie groups. For example, it is clear that the map

$$\exp\colon \mathfrak{gl}(n,\mathbb{R}) \longrightarrow \mathrm{GL}(n,\mathbb{R})$$

is well-defined, but since $\det(e^A) = e^{\operatorname{tr} A}$, every matrix of the form e^A has a positive determinant and exp is not surjective. However, this map is not surjective. However, as we will see very soon, the map

$$\exp\colon \mathfrak{so}(n) \longrightarrow \mathrm{SO}(n)$$

is well-defined and surjective. The map

$$\exp\colon \mathfrak{o}(n) \longrightarrow \mathrm{O}(n)$$

is well-defined, but it is not surjective, since there are matrices in $\mathrm{O}(n)$ with the determinant -1.

The situation for matrices over the field of complex numbers $\mathbb{C}$ is quite different.

We now show the fundamental relationship between $\mathrm{SO}(n)$ and $\mathfrak{so}(n)$

Theorem 1. *The exponential map*

$$\exp\colon \mathfrak{so}(n) \longrightarrow \mathrm{SO}(n)$$

is well-defined and surjective.

Proof. First, we show that if A is a skew-symmetric matrix then e^A is a rotation matrix. For this, we quickly check that

$$\left(e^A\right)^{\top} = e^{A^{\top}}$$

This is a consequence of the definition $e^A = \sum_{p\geq 0} A^p/(p!\,)$ as an absolutely convergent series, the observation that $\left(A^p\right)^{\top} = \left(A^{\top}\right)^p$, and the linear of the transpose map, i.e., $(A+B)^{\top} = A^{\top} + B^{\top}$. Then, since $A^{\top} = -A$, we get

$$\left(e^A\right)^{\top} e^A = e^{-A} e^A = e^{-A+A} = e^0 = I,$$

and similarly,

$$e^A\left(e^A\right)^{\top} = I$$

showing that e^A is orthogonal. Also,

$$\det e^A = e^{\operatorname{tr} A}$$

and since A is real skew-symmetric, its diagonal entries are 0, i.e., $\operatorname{tr} A = 0$, and so $\det e^A = +1$. ∎