MA 519: Homework 12

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Problem 12.1 (Handout 15, # 10)

Consider the experiment of picking one word at random from the sentence

All is well in the newell family

Let X be the length of the word selected and Y the number of Ls in it. Find in a tabular form the joint PMF of (X, Y), their marginal PMFs, means, and variances, and the correlation between X and Y.

Solution. The joint PMF of (X, Y) is given by

$Y \backslash X$	2	3	4	5	6
0	$\frac{2}{7}$	$\frac{1}{7}$	0	0	0
1	0	0	0	0	$\frac{1}{7}$
2	0	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	0

The marginal PMF of X is thus given by

$$f_X(x) = \begin{cases} \frac{2}{7} & \text{for } x = 2, 3, \\ \frac{1}{7} & \text{for } x = 4, 5, 6 \end{cases}$$

and the marginal PMF of Y is given by

$$f_Y(x) = \begin{cases} \frac{3}{7} & \text{for } x = 0, 2, \\ \frac{1}{7} & \text{for } x = 1. \end{cases}$$

So the mean and variance of X and Y are

$$\begin{aligned} \mu_X &= \frac{4+6+4+5+6}{7} & \mu_Y &= 1, \\ &= \frac{25}{7}, \\ \mathrm{Var}(X) &= \frac{8+18+16+25+36}{7} - \left(\frac{25}{7}\right) & \mathrm{Var}(Y) &= \frac{1+12}{7} - 1 \\ &= \frac{96}{49}, & &= \frac{6}{7}. \end{aligned}$$

Lastly, the correlation between X and Y is

$$\rho_{X,Y} = \frac{5}{\sqrt{\frac{576}{7}}} \approx 0.551.$$

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Problem 12.2 (Handout 15, # 11)

Consider the joint PMF $p(x, y) = cxy$, $1 \leq x \leq 3$, $1 \leq y \leq 3$.

(a) Find the normalizing constant c.
(b) Are X and Y independent? Prove your claim.
(c) Find the expectations of X, Y, and XY.

Solution. *Remark:* Note that below parts (a), (b), and (c) are out of order.

For part (a): The normalizing constant is $c = \frac{1}{36}$; this is because

$$\sum_{x,y=(1,1)}^{(3,3)} cxy = 36c$$

For part (c): First,

$$E(X) = E(Y) = \sum_{x=1}^{3} x^2(1 + 2 + 3)c = 6c \sum_{x=1}^{3} x^2 = \frac{7}{3}$$

and

$$E(XY) = \sum_{(x,y)=(1,1)}^{(3,3)} cx^2y^2 = \frac{49}{9}$$

For part (b): We see that X and Y are independent; $E(XY) = E(X)E(Y)$. ∎

Problem 12.3 (Handout 15, # 12)

A fair die is rolled twice. Let X be the maximum and Y the minimum of the two rolls. By using the joint PMF of X and Y worked out in the text, find the PMF of $\frac{X}{Y}$, and hence the mean of $\frac{X}{Y}$.

Solution. The PMF of $\frac{X}{Y}$ is given by

$$f_{\frac{X}{Y}}(x) = \begin{cases} \frac{1}{6} & \text{for } x = 1, 2, \\ \frac{1}{9} & \text{for } x = \frac{3}{2}, 3, \\ \frac{1}{18} & \text{for} x = \frac{5}{2}, 4, 5, 6, \frac{5}{3}, \frac{4}{3}, \frac{5}{4}, \frac{5}{6}. \end{cases}$$

So that the mean is

$$\mu_{\frac{X}{Y}} = \frac{487}{216} \approx 2.255$$

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Problem 12.4 (Handout 15, # 13)

Two random variables have the joint PMF $p(x, x+1) = \frac{1}{n+1}$, $x = 0, \ldots, n$. Answer the following question with as little calculation as possible.

(a) Are X and Y independent?
(b) What is the variance of $Y - X$?
(c) What is $\mathrm{Var}(Y \mid X = 1)$?

Solution. For part (a): No. The probability that $Y = 2$ given that $X = 1$ is 1, but the probability that $Y = 2$ is $\frac{1}{n+1}$.

For part (b): $\mathrm{Var}(Y - X) = 0$, because $Y - X$ is constant; it is always 1.

For part (c): $\mathrm{Var}(Y \mid X = 1) = 0$, because $Y = 2$ if $X = 1$. ∎

Problem 12.5 (Handout 15, # 14)

(Binomial Conditional Distribution). Suppose X and Y are independent random variables, and $X \sim \mathrm{Bin}(m, p)$, $Y \sim \mathrm{Bin}(n, p)$. Show that the conditional distribution of X given by $X + Y = t$ is a hypergeometric distribution; identify the parameters of this hypergeometric distribution.

Solution. First, let us find the PMF of X given $X + Y = t$:

$$\begin{aligned} P(X = x \mid X + Y = t) &= \frac{P\big(\{ X = x \} \cap \{ X + Y = t \}\big)}{P(X + Y = t)} \\ &= \frac{P(Y = t - x)}{P(X + Y = t)} \\ &= \frac{\binom{n}{x}\binom{m}{t-x} p^t (1-p)^{m+n-t}}{\binom{m+n}{t} p^t (1-p)^{m+n-t}} \\ &= \frac{\binom{n}{x}\binom{m}{t-x}}{\binom{m+n}{t}}. \end{aligned}$$

This distribution is precisely $\mathrm{Hypergeo}(t, m, n + m)$. ∎

Problem 12.6 (Handout 15, # 15)

Suppose a fair die is rolled twice. Let X and Y be the two rolls. Find the following with as little calculation as possible.

(a) $E(X+Y \mid Y=y)$.
(b) $E(XY \mid Y=y)$.
(c) $\mathrm{Var}(X^2Y \mid Y=y)$.
(d) $\rho_{X+Y,X-Y}$.

Solution. For part (a):

$$E(X+Y \mid Y=y) = E(X \mid Y=y) + E(Y \mid Y=y) = 3.5 + y.$$

For part (b):

$$E(XY \mid Y=y) = E(X \mid Y=y)E(Y \mid Y=y) = 3.5y.$$

For part (c):

$$\mathrm{Var}(X^2Y \mid Y=y) = E((X^2Y)^2 \mid Y=y) - E(X^2Y \mid Y=y)^2 = c^2\left(\frac{91}{6} - 3.5\right).$$

For part (d):

$$\begin{aligned}\mathrm{Cov}(X+Y, X-Y) &= E((X+Y)(X-Y)) - E(X+Y)E(X-Y)\\ &= E(X)E(X) - E(Y)E(Y) - E(X)E(X) + E(Y)E(Y)\\ &= 0,\end{aligned}$$

so $\rho_{X+Y,X-Y} = 0$. ■

Problem 12.7 (Handout 15, # 16)

(A Standard Deviation Inequality). Let X and Y be two random variables. Show that $\sigma_{X+Y} \leq \sigma_X + \sigma_Y$.

Solution. Suppose σ_X and σ_Y exist and are finite. We want to show

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y;$$

this is the same as showing that

$$\begin{aligned} \sigma_{X+Y}^2 &\leq \sigma_X + \sigma_Y^2 + 2\sigma_X\sigma_Y \\ \operatorname{Var}(X+Y) &\leq \operatorname{Var}(X) + \operatorname{Var}(Y) + 2[\operatorname{Var}(X)\operatorname{Var}(Y)]^{\frac{1}{2}}. \end{aligned}$$

First, let us expand $\operatorname{Var}(X+Y)$ using the definition of variance, we have

$$\begin{aligned} \operatorname{Var}(X+Y) &= E\big((X+Y)^2\big) - E(X+Y)^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 \\ &= \big(E(X^2) - E(X)^2\big) + \big(E(Y^2) - E(Y)^2\big) + 2[E(XY) - E(X)E(Y)] \\ &= \operatorname{Var}(X) + \operatorname{Var}(Y) + 2[E(XY) - E(X)E(Y)]. \end{aligned}$$

Therefore, it suffices to show that

$$E(XY) - E(X)E(Y) \leq [\operatorname{Var}(X)\operatorname{Var}(Y)]^{\frac{1}{2}},$$

or, rewritten using covariance,

$$\operatorname{Cov}(X,Y)^2 \leq \operatorname{Var}(X)\operatorname{Var}(Y).$$

By the Cauchy–Schwartz inequality, we have

$$\begin{aligned} \operatorname{Cov}(X,Y)^2 &= E\big[(X - E(X))(Y - E(Y))\big]^2 \\ &\leq E\big[(X - E(X))^2\big]E\big[(Y - E(Y))^2\big] \\ &= \operatorname{Var}(X)\operatorname{Var}(Y). \end{aligned}$$

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Problem 12.8 (Handout 15, # 17)

Seven balls are distributed randomly in seven cells. Let X_k be the number of cells containing exactly k balls. Using the probabilities tabulated in II, 5, write down the joint distribution of X_2, X_3.

Solution. The tabled referenced in this problem is on p. 40 of Feller. Let us write down a table of our own for the joint distribution of (X_2, X_3):

$X_3 \backslash X_2$	0	1	2	3
0	0.048	0.156	0.321	0.107
1	0.109	0.214	0.027	0
2	0.018	0	0	0

.

Let us do a sanity check by summing over all of the entries in the table above

$$0.048 + 0.156 + 0.321 + 0.107 + 0.109 + 0.214 + 0.027 + 0 + 0.018 + 0 + 0 + 0 \approx 1. \qquad \blacksquare$$

Problem 12.9 (Handout 15, # 18)

Two ideal dice are thrown. Let X be the score on the first die and Y be the larger of two scores.

(a) Write down the joint distribution of X and Y.
(b) Find the means, the variances, and the covariance.

Solution. For part (a): The random variable X takes on integer values between zero and six and so does Y. Moreover, the dependence of Y on X tells us that $P(\{X = k\} \cap \{Y = \ell\}) = 0$ if $\ell < k$; this allows us to fill in a significant portion of the joint distribution table:

$Y \backslash X$	1	2	3	4	5	6
1	$\frac{1}{36}$	0	0	0	0	0
2	$\frac{1}{36}$	$\frac{2}{36}$	0	0	0	0
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{3}{36}$	0	0	0
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{4}{36}$	0	0
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	0
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{6}{36}$

.

(One can easily verify that the sum of the entries in this table do in fact add up to one.)

For part (b): We can recover the individual PMFs for X and Y using the table in part (a) and so recover the mean and variance. These are

$$\begin{aligned}
E(X) &= \frac{6}{36} + 2\left(\frac{6}{36}\right) + 3\left(\frac{6}{36}\right) + 4\left(\frac{6}{36}\right) + 5\left(\frac{6}{36}\right) + 6\left(\frac{6}{36}\right) \\
&= 3.5, \\
E(X) &= 1^2\left(\frac{6}{36}\right) + 2^2\left(\frac{6}{36}\right) + 3^2\left(\frac{6}{36}\right) + 4^2\left(\frac{6}{36}\right) + 5^2\left(\frac{6}{36}\right) + 6^2\left(\frac{6}{36}\right) \\
&\approx 15.167, \\
\mathrm{Var}(X) &\approx 2.917,
\end{aligned}$$

and

$$\begin{aligned}
E(Y) &= \frac{1}{36} + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right) \\
&\approx 4.472, \\
E(Y^2) &= 1^2\left(\frac{1}{36}\right) + 2^2\left(\frac{3}{36}\right) + 3^2\left(\frac{5}{36}\right) + 4^2\left(\frac{7}{36}\right) + 5^2\left(\frac{9}{36}\right) + 6^2\left(\frac{11}{36}\right) \\
&\approx 21.972, \\
\mathrm{Var}(Y) &\approx 1.971,
\end{aligned}$$

and lastly (after a long calculation which we omit) the covariance is

$$\mathrm{Cov}(X, Y) \approx 2.061.$$

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Problem 12.10 (Handout 15, # 19)

Let X_1 and X_2 be independent and have the common geometric distribution $\{q^k p\}$ (as in problem 4). Show without calculations that the *conditional distribution of* X_1 *given* $X_1 + X_2$ *is uniform*, that is,

$$P(X_1 = k \,|\, X_1 + X_2 = n) = \frac{1}{n+1}, \quad k = 0, \ldots, n. \tag{12.1}$$

Solution. By definition of conditional probability, we have

$$\begin{aligned} P(X_1 = k \,|\, X_1 + X_2 = n) &= \frac{P(\{\, X_1 = k \,\} \cap \{\, X_1 + X_2 = n \,\})}{P(X_1 = k)} \\ &= \frac{P(X_2 = n - k)}{P(X_1 + X_2 = n)} \\ &= \frac{q^{n-k} p}{q^{n-k} p(n+1)} \\ &= \frac{1}{n+1}. \end{aligned}$$

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Problem 12.11 (Handout 15, # 20)

If two random variables X and Y assume only two values each, and if $\mathrm{Cov}(X, Y) = 0$, then X and Y are independent.

Solution. We show that the joint PDF of (X, Y) is

$$f_{X,Y}(x, y) = f_X(x) f_Y(y).$$

Suppose X assumes the values $\{a, b\}$ and Y assumes the values $\{c, d\}$ where, without loss of generality, we may assume $a < b$ and $c < d$; however, we may have $a = c$, $b = c$, $a = d$, etc. Let p_a, p_b, p_c, and p_d be the probabilities associated to a, b, c, and d, respectively. Then, we have

$$p_a + p_b = 1, \quad p_c + p_d = 1,$$

and more significantly

$$\begin{aligned} \mathrm{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ E(XY) &= (ap_a + bp_b)(cp_c + dp_d) \\ \sum_{\substack{x \in \{a,b\}, \\ y \in \{c,d\}}} xy f_{X,Y}(x, y) &= (ap_a + bp_b)(cp_c + dp_d) \\ \begin{matrix} acf_{X,Y}(a, c) + adf_{X,Y}(a, d) \\ + bcf_{X,Y}(b, c) + bdf_{X,Y}(b, d) \end{matrix} &= \begin{matrix} acp_a p_c + adp_a p_d \\ + bcp_b p_c + bdp_b p_d. \end{matrix} \end{aligned}$$

A term by term comparison shows that we must have

$$f(x, y) = xyp_x p_y$$

for $x \in \{a, b\}$, $y \in \{c, d\}$. Thus, $f_{X,Y}(x, y) = f_X(x) f_Y(y)$; i.e., X and Y are independent. ■