Lemma. Let $S \subset \mathbb{R}$ be bounded above and nonempty. Let $c = \sup S$.

i) If $x > c$ then x is not in S.

ii) If $x < c$ then there is a y in S with $x < y$.

The proof is easy from the definition of sup.

Theorem $\mathbb{R}$ is connected.

Proof. Suppose not, and let U, V be a separation.

Let $a \in U, b \in V$, and assume $a < b$.

Let $S = (-\infty, b] \cap U$

Let $c = \sup(S)$, which exists because S is nonempty and bounded above.

Case 1: Suppose $c \in V$. Then there exist d_1, d_2 with $c \in (d_1, d_2) \subset V$. By part (ii) of the Lemma there is a y in S with $d_1 < y$, and we also have $y \leq c < d_2$. But then $y \in (d_1, d_2) \subset V$, contradicting the fact that $S \subset U$.

Case 2: Suppose $c \in U$. Then there exist e_1, e_2 with $c \in (e_1, e_2) \subset U$. Note that $c < b$ (because $c \leq b$ and $b \notin U$). Thus there is a w with $c < w < \min(e_2, b)$. Then $w \in (-\infty, b] \cap U = S$, contradicting part (i) of the Lemma.

QED