MA 544: Homework 4

Carlos Salinas

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Problem 4.1 (Wheeden & Zygmund §3, Ex. 12)

If E_1 and E_2 are measurable sets in $\mathbf{R}^1$, show $E_1 \times E_2$ is a measurable subset of $\mathbf{R}^2$ and $|E_1 \times E_2| = |E_1||E_2|$. (Interpret $0 \cdot \infty$ as 0.) [Hint: Use a characterization of measurability.]

Proof. By (3.28) (i) we may write E_1 and E_2 as the set difference $H_1 \setminus Z_1$ and $H_2 \setminus Z_2$, respectively, where H_1 and H_2 are G_δ and Z_1 and Z_2 are measure zero. Now, by elementary set theory, the Cartesian product $E_1 \times E_2$ can then be written as

$$E_1 \times E_2 = (H_1 \setminus Z_1) \times (H_2 \setminus Z_2) = \underbrace{(H_1 \times H_2)}_{H} \setminus \underbrace{(Z_1 \times H_2 \setminus H_1 \times Z_2 \setminus Z_1 \times Z_2)}_{Z} \tag{1}$$

Hence, we win by (3.28) (i) if we can show that the Cartesian product of two G_δ sets is an G_δ set and if the Cartesian product of a measurable set with a set of measure zero is measure zero.

First, we prove the former, since the argument to be made is little more than elementary set theory.

Lemma 1. *The Cartesian product of G_δ sets is again G_δ.*

Proof of lemma 1. Let G_1 and G_2 be G_δ. Write $G_1 = \bigcap G_k'$ and $G_2 = \bigcap G_k''$ where the G_k''s and the G_k'''s are open subsets of $\mathbf{R}$. Then, $G_k' \times G_\ell''$ are open subsets of $\mathbf{R}^2$ by the definition of the product topology. Moreover, $G_k' \times G_\ell'' \subset G_1 \times G_2$ hence, $\bigcap_{k,\ell} G_k' \times G_\ell'' \subset G_1 \times G_2$. Thus, it suffices to show that $\bigcap_{k,\ell} G_k' \times G_\ell'' \supset G_1 \times G_2$. Let $(x, y) \in G_1 \times G_2$. Then $x \in G_1$ and $y \in G_2$. But since $G_1 = \bigcap G_k'$ and $G_2 = \bigcap G_k''$ then $x \in G_k'$ and $x \in G_\ell''$ for some k, ℓ. In other words, $(x, y) \in G_k' \times G_\ell''$ so (x, y) is in the intersection $\bigcap_{k,\ell} G_k' \times G_\ell''$. Hence, we have $G_1 \times G_2 = \bigcap_{k,\ell} G_k' \times G_\ell''$. We conclude that if G_1 and G_2 are G_δ, then so is their Cartesian product $G_1 \times G_2$. ♣

Lemma 2. *Let E be measurable and Z be measure zero. Then $E \times Z$ is measure zero.*

Proof of lemma 2. Let E be a measurable set with $|E| < \infty$ and Z a set of measure zero. Then, for every $\varepsilon > 0$ there exists a countable collection of intervals $\{I_k\}$ containing Z such that $\sum \operatorname{vol}(I_k) < \varepsilon$. Similarly, we can find a collection $\{I_k'\}$ of intervals containing E such that $\sum \operatorname{vol}(I_k') < |E| + \varepsilon$. Then, $\{I_k' \times I_\ell\}$ is a countable collection of 2-intervals containing $E \times Z$ with

$$\begin{aligned}
\sum_{k,\ell} \operatorname{vol}(I_k' \times I_\ell) &= \sum_{k,\ell} \operatorname{vol}(I_k') \operatorname{vol}(I_\ell) \\
&= \sum_k \sum_\ell \operatorname{vol}(I_k') \operatorname{vol}(I_\ell) \\
&= \left(\sum_k \operatorname{vol}(I_k') \right) \left(\sum_\ell \operatorname{vol}(I_\ell) \right) \\
&= (|E| + \varepsilon)\varepsilon
\end{aligned}$$

Letting $\varepsilon \to 0$, we have $E \times Z$ is measure zero. If $|E| = \infty$, partition E into disjoint finite measure subsets of $\mathbf{R}$ by taking the following intersection

$$E_k = E \cap (B(0, k) \setminus B(0, k-1))$$

for $k \in \mathbf{N}$.[1] By our previous argument, $E_k \times Z$ is measure zero so $\{E_k \times Z\}$ is a cover of $E \times Z$

[1] In fact, it might be quicker from now on to quote the fact that $\mathbf{R}^n$ is σ-finite.

hence, by (3.24), we have

$$\begin{aligned}|E \times Z| &= \left|\left(\bigcup_k E_k\right) \times Z\right| \\ &= \left|\bigcup_k E_k \times Z\right| \\ &= \sum_k |E_k \times Z| \\ &= 0.\end{aligned}$$

Thus, $E \times Z$ is measure zero. ♣

By lemma 1 and 2, $E_1 \times E_2$ is measurable with $|E_1 \times E_2| = |H_1 \times H_2|$. It's left to show is that $|H_1 \times H_2| = |H_1||H_2|$.

Lemma 3. *If G_1 and G_2 are G_δ then $|G_1 \times G_2| = |G_1||G_2|$.*

But before that, we need to prove the above for the case where G_1 and G_2 are open sets.

Lemma 4. *If G_1 and G_2 are open then $|G_1 \times G_2| = |G_1||G_2|$.*

Proof of lemma 4. Let G_1 and G_2 be open with $|G_1|, |G_2| < \infty$. By (1.11), we may write G_1 and G_2 as the countable intersection of a collection of nonoverlapping closed intervals $\{I_k\}$ and $\{I'_k\}$, respectively. Therefore, we have

$$|G_1| = \sum_k \operatorname{vol}(I_k) \quad \text{and} \quad |G_2| = \sum_k \operatorname{vol}(I'_k).$$

Moreover the collection $\{I_k \times I'_\ell\}$ is a cover of $G_1 \times G_2$ of nonoverlapping closed 2-intervals,[2] so by (3.2) we have

$$\begin{aligned}|G_1 \times G_2| &= \sum_{k,\ell} \operatorname{vol}(I_k \times I'_\ell) \\ &= \sum_{k,\ell} \operatorname{vol}(I_k) \operatorname{vol}(I'_\ell) \\ &= \left(\sum_k \operatorname{vol}(I_k)\right)(\operatorname{vol}(I'_\ell)) \\ &= |G_1||G_2|\end{aligned}$$

♣

[2]They are closed because of elementary topology: the Cartesian product of two closed sets is again closed in the product topology; and they are nonoverlapping because if $(x, y) \in I_k \times I'_\ell \cap I_{k'} \times I_{\ell'} \neq \emptyset$ then $x \in I_k \cap I_{k'}$ and $y \in I_\ell \cap I_{\ell'}$ a contradiction.

Proof of lemma 3. Now that we have the result of lemma 4 we may easily proceed to the countable case. Let G_1 and G_2 be G_δ. Then by lemma 1 $G_1 \times G_2$ is G_δ and we may write $G_1 \times G_2$ as the intersection of a countable collection of open sets $\{G'_k\}$. In particular, if $\{G'_k\}$ is a collection of open sets covering $G_1 \times G_2$ that intersects to $G_1 \times G_2$ then the collection $\{G''_k\}$, where $G''_k \coloneqq \bigcap_{\ell=1}^k G'_\ell$, also intersects to G_1 and has the property that $G''_{k+1} \subset G''_k$. Thus, we may as well assume that $\{H_k\}$ is decreasing so, by (3.26), we have

$$|G_1 \times G_2| = \lim_{k\to\infty} |H_k|,$$

but H_k is open in the product topology so $H_k = H'_k \times H''_k$ for open subsets $H'_k, H''_k \subset \mathbf{R}$, giving us

$$= \lim_{k\to\infty} |H_k \times H''_k|,$$

which, by lemma 4, is just

$$\begin{aligned} &= \lim_{k\to\infty} |H'_k||H''_k| \\ &= |E_1||E_2|, \end{aligned}$$

since $H'_k \supset E_1$ and $H''_k \supset E_2$ are open so $\bigcap H'_k \supset E_1$ and $\bigcap H''_k \supset E_2$ and their outer measure approach the outer measure of E_1 and E_2 as $k \to \infty$. ♣

Putting together our results, by equation 1, lemma 2, and lemma 3, we can express $E_1 \times E_2$ as a G_δ set H minus a set of measure zero Z and its measure is

$$|E_1 \times E_2| = |H_1||H_2| = |E_1||E_2|,$$

as desired. ■

Problem 4.2 (Wheeden & Zygmund §3, Ex. 13)

Motivated by (3.7), define the *inner measure* of E by $|E|_i = \sup|F|$, where the supremum is taken over all closed subsets F of E. Show that

(i) $|E|_i \leq |E|_e$, and

(ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$.

[Use (3.22).]

Proof. (i) If the outer measure of E is infinite, the inequality holds trivially. Suppose $|E|_e < \infty$. Since closed sets are measurable and their outer measure is equal to their Lebesgue measure, then we may replace $|F|$ by $|F|_e$ to mirror the definition of the outer-measure and, by the monotonicity of the outer measure, we have

$$|F| = |F|_e \leq |E|_e. \tag{2}$$

Taking the supremum on both sides of (2), we obtain the desired inequality

$$|E|_i \leq |E|_e. \tag{3}$$

(ii) $\Longrightarrow$ Suppose E is measurable with $|E| < \infty$. By (3.22), given $\varepsilon > 0$, there exists a closed set $F \subset E$ such that $|E \setminus F|_e < \varepsilon$. Since F is measurable, by (3.31), we have

$$|E \setminus F|_e = |E|_e - |F|. \tag{4}$$

But E is also measurable, so equation (4) becomes

$$|E \setminus F|_e + |F| = |E| < \varepsilon + |F|. \tag{5}$$

Taking the supremum of (5) over all F, we gave

$$|E|_e = |E| \leq |F| + \varepsilon = |E|_i + \varepsilon$$

for all $\varepsilon > 0$. By equation (3), we achieve equality of the inner and outer measure, i.e., $|E|_i = |E|_e$.

$\Longleftarrow$ Conversely, suppose that $|E|_i = |E|_e$. Then, given $\varepsilon > 0$, by the definition of outer measure, there exists an open set $G \supset E$ and, by the definition of inner measure, closed set $F \subset E$ such that

$$|G| - |E|_e < \frac{\varepsilon}{2} \qquad \text{and} \qquad |E|_i - |F| = |E|_e - |F| < \frac{\varepsilon}{2}. \tag{6}$$

Then

$$|E \setminus F|_e < |G \setminus F|_e = |G|_e - |G \cap F|_e = |G|_e - |F|_e < 2\Big(\frac{\varepsilon}{2}\Big) = \varepsilon.$$

So by (3.22) E is measurable. ■

Problem 4.3 (Wheeden & Zygmund §3, Ex. 15)

If E is measurable and A is any subset of E, show that $|E| = |A|_i + |E \setminus A|_e$. (See Exercise 13 for the definition of $|A|_i$.)

Proof. If A is measurable, by our previous problem, $|A|_e = |A| = |A|_i$ so by (3.31), we have

$$|E \setminus A|_e = |E|_e - |A| = |E| - |A|_i,$$

so $|E| = |A|_i + |E \setminus A|_e$.

If A is not measurable and $|E| < \infty$ then $|A|_e$ and $|E \setminus A|_e < \infty$ since $E \setminus A \subset E$. Hence, we can subtract the quantity $|E \setminus A|_e$ from $|E|_e = |E|$ and we get

$$\begin{aligned} |E| - |E \setminus A|_e &= |E| - \inf\{\, |G| : G \supset E \setminus A \text{ is open}\,\} \\ &= |E| - \inf\{\, |G| : G \supset E \setminus A \text{ is open}\,\} \\ &= |E| - \inf\{\, |G| : E \supset G \supset E \setminus A \text{ is open}\,\} \\ &= |E| - \inf\{\, |E| - |F| : E \setminus G \subset A,\, G \supset E \setminus A \text{ is open}\,\} \end{aligned}$$

pulling the $|E|$ inside of the infimum we get

$$\begin{aligned} &= -\inf\{\, -|F| : F \subset A, \text{ where } F \coloneqq E \setminus G \text{ and } G \supset A \text{ is open}\,\} \\ &= \sup\{\, |F| : F \subset A, \text{ where } F \coloneqq E \setminus G \text{ and } G \supset A \text{ is open}\,\} \\ &= |A|_i, \end{aligned}$$

so $|E| = |A|_i + |E \setminus A|_e$ as desired.

More generally, if E is measurable we can establish the following lower bound on $|E|$: By definition of the inner measure, for every $F \subset A$ closed we have

$$|E| = |F| + |E \setminus F| \geq |F| + |E \setminus A|_e, \tag{7}$$

which follows from (3.31), since $F \subset A \subset E$ is measurable, $E \setminus A \subset E \setminus F$, and the monotonicity of outer measure. Taking the supremum on both sides of equation (7) over all closed $F \subset A$ we achieve the desired lower bound

$$|E| \geq |A|_i + |E \setminus A|_e. \tag{8}$$

Now, let E be measurable. Then, we have

$$\begin{aligned} |A|_i + |E \setminus A|_e &\geq \sup\{\, |F \cap A|_i + |F \setminus A|_e : F \subset E \text{ is closed}, |F| < \infty\,\} \\ &= \sup\{\, |F| : F \subset E \text{ closed}, |F| < \infty\,\} \\ &= |E|_i \end{aligned}$$

which by our last problem is just

$$= |E|, \tag{9}$$

since E is measurable. Having bounded $|E|$ by equations (8) and (9) we achieve the equality

$$|E| = |A|_i + |E \setminus A|_e. \qquad \blacksquare$$