MA598: Lie Groups

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CHAPTER 1

Prologue

This summer, we will be making our way through Knapp's *Lie Groups Beyond an Introduction* [2] although, I (the writer of these notes) will occasionally refer to [1] for examples.

1.1 Representation of Finite Groups

Definitions

A *representation* of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$; we say that such a map ρ *gives* V *the structure of a* G*-module.* When there is little ambiguity about the map ρ we will call V itself as a representation of G; in this vein, we supress the symbol ρ and write gv for $\rho(g)(v)$. The dimension of V is sometimes called the *degree* of ρ.

A map φ between two representations V and W of G is a vector space map $\varphi\colon V \to W$ such that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

commutes for every $g \in G$. (We will call this a G*-linear map* when we want to distinguish it from an arbitrary linear map between the vector spaces V and W). We can then define $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$, and $\operatorname{Coker}\varphi$, which are also G-modules.

A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under G. A representation V is called *irreducible* if there is no proper nonzero invariant subspace W of V.

If V and W are representations, so are $V \oplus W$ and $V \otimes W$ with $g(v \otimes w) := gv \otimes gw$. Moreover, the nth tensor power $\bigotimes^n V$, the exterior power $\bigwedge^n V$ and symmetric powers $\mathrm{Sym}^n V$ are subrepresentations of it. The dual $V^* = \mathrm{Hom}(V, \mathbb{C})$ of V is also a representation, though not in the most obvious way: We want the two representations of G with respect to the natural pairing between V and V^*, so that if $\rho\colon G \to \mathrm{GL}(V)$ is a representation and $\rho^*\colon G \to \mathrm{GL}(V)$ is its dual, then we have

$$\langle \rho^*(g)(v^*), \rho(g)(v)\rangle = \langle v^*, v\rangle \tag{1.1}$$

for all $g \in G$, $v \in V$, and $v^* \in V^*$. This in turn forces us to define the dual representation by

$$\rho^*(g) := {}^{\mathrm{t}}\rho(g^{-1})\colon V^* \longrightarrow V^*$$

for all $g \in G$.

Exercise 1. Let us verify that (1.1) is satisfied by the definition of ρ^*.

Proof. With ρ^* as defined above, choose $g \in G$, $v \in V$ and $v^* \in V$. Then, we have

$$\begin{aligned}\langle \rho^*(g)(v^*), \rho(g)(v)\rangle = \langle v^*, v\rangle &= \langle\rangle \\ &= \end{aligned}$$

■

Having defined the dual representation of the tensor product of two representations, it is likewise the case that if V and W are representations, then $\mathrm{Hom}(V, W)$ is also a representation, via the identification $\mathrm{Hom}(V, W) = V^* \otimes W$. Unraveling this, if we view an element of $\mathrm{Hom}(V, W)$ as a linear map φ from V to W, we have

$$(g\varphi)(v) = g\varphi(g^{-1}v)$$

for all $v \in V$. In other words, the definition is such that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "g\varphi"'] & W \end{tikzcd}$$

commutes. Note that the dual representation is, in turn, a special case of this: When $W = \mathbb{C}$ is the trivial representation, i.e., $gw = w$ for all $w \in \mathbb{C}$, this makes V^* into a G-module, with $g\varphi(v) = \varphi(g^{-1}v)$, i.e., $g\varphi = {}^{\mathrm{t}}(g^{-1})$.

Exercise 2. We verify that in general the vector space of G-linear maps between two representations V and W of G is just the subspace $\mathrm{Hom}(V, W)^G$ of elements of $\mathrm{Hom}(V, G)$ fixed under the action of G. We will often denote this space by $\mathrm{Hom}_G(V, W)$.

Proof. ■

We have taken the identification $\mathrm{Hom}(V,W) = V^* \otimes W$ as the definition of the representation $\mathrm{Hom}(V,W)$. More generally, the usual identities for vector spaces are also true for representations, e.g.,

$$V \otimes (U \oplus W) = (V \otimes U) \oplus (V \otimes W)$$
$$\bigwedge^k (V \oplus W) = \bigoplus_{a+b=k} \bigwedge^a V \otimes \bigwedge^b W$$
$$\bigwedge^k V^* = \left(\bigwedge^k V\right)^*$$

If X is any finite set and G acts on the left on X, i.e., $G \to \mathrm{Aut}(X)$ is a homomorphism to the premutation group of X, there is an associated permutation representation: Let V be the vector space with basis $\{\mathbf{e}_x : x \in X\}$, and let G act on V by

$$g \cdot \sum a_x \mathbf{e}_x := \sum a_x \mathbf{e}_{gx}.$$

The regular representation, denoted R_G or simply R, corresponds to the left action of G on itself. Alternatively, R is the space of complex-valued functions on G, where an element $g \in G$ acts on a function α by $(g\alpha)(h) = \alpha(g^{-1}h)$.

Complete Reducibility; Schur's Lemma

Before we begin classifying the representations of a finite group G we should try to simplify life by restricting our search somewhat. Specifically, we have seen that representations of G can be built up out of other representations by linear algebraic operations, most simply by taking the direct sum. We should focus, then, on representations that are "atomic" with respect to this operation, i.e., that cannot be expressed as a direct sum of others; the usual term for such a representation is *indecomposable*. Happily, this situation is as nice as it could possibly be: A representation is atomic in this sense if and only if it is irreducible (i.e., contains no proper subrepresentations); and every representation is the direct sum of irreducibles, in a suitable sense uniquely so. The key to all this is

Proposition 1. *If W is a subrepresentation of a representation V of a finite group G, then there is an elementary invariant subspace W' of V, so that $V = W \oplus W'$.*

Proof. There are two ways of showing this. One can introduce a positive definite Hermitian inner product H on V which is preserved by each $g \in G$ (i.e., such that $H(g\mathbf{v}, g\mathbf{w}) = H(\mathbf{v}, \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V$, $g \in G$). Indeed, if H_0 is any Hermitian product on V, one gets such an H by averaging over G:

$$H(\mathbf{v}, \mathbf{w}) := \sum_{g \in G} H_0(g\mathbf{v}, g\mathbf{w}). \tag{1.2}$$

Then the perpendicular subspace $W^\perp$ is complementary to W in V. Alternatively, we can simply choose an arbitrary subspace U complementary to W, let $\pi_0 \colon V \to W$ be the projection given by the direct sum decomposition $V = W \oplus U$, and average the map π_0 over G: i.e., take

$$\pi(\mathbf{v}) := \sum_{g \in G} g(\pi_0(g^{-1}\mathbf{v})). \tag{1.3}$$

this will then be a G-linear map from V onto W, which is multiplication by $|G|$ on W; its kernel will, therefore, be a subspace of V invariant under G and complemnetary to W. ∎

Corollary 2. *Any representation is a direct sum of irreducible representations.*

This property is called complete reducibility, or semisimplicity. We will see that, for continuous representations, the circle S^1, or any compact group, has this property; integration over the group (with respect to an invariant measure on the group) plays the role of averaging in the above proof. The (additive) group $\mathbb{R}$ does not have this property: The representation

$$a \longmapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

leaves the x axis fixed, but there is no complementary subspace. We will see other Lie groups such as $\mathrm{SL}_n(\mathbb{C})$ that are semisimple in this sense. Note also that this argument would fail if the vector space V was over a field of finite characteristic since it might then be the case that $\pi(\mathbf{v}) = \mathbf{0}$ for for $\mathbf{v} \in W$. The failure, of complete reducibility is one of is one of the things that makes the subject of modular representations, or representations on vector spaces over finite fields, so tricky.

The extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

Theorem 3 (Schur's lemma)**.** *If V and W are irreducible representations of G and $\varphi\colon V \to W$ is a G-module homomorphism, the*

(a) *Either φ is an isomorphism, or $\varphi = \mathbf{0}$.*
(b) *If $V = W$, then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$, I being the identity.*

Proof. The first claim follows from the fact that $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ are invariant subspaces. For the second, since $\mathbb{C}$ is algebraically closed, φ must have an eigenvalue λ, i.e., for some $\lambda \in \mathbb{C}$, $\varphi - \lambda I$ has a nonzero kernel. By Theorem 1, we must have $\varphi - \lambda I = \mathbf{0}$ so $\varphi = \lambda I$. ∎

We can summarize what we have shown thus far in

Proposition 4. *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.

Proof. It follows from Schur's lemma that if W is another representation of G, with decomposition $W = \bigoplus W_j^{\oplus b_j}$, and $\varphi\colon V \to W$ is a map of representations, then φ must map the factor $V_i^{\oplus a_i}$ into that factor $W_j^{\oplus b_j}$ for which $W_j \simeq V_i$; when applied to the identity map of V to V, the stated uniqueness follows. ∎

Bibliography

[1] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Graduate Texts in Mathematics. Springer, 2003.

[2] A.W. Knapp. *Lie Groups Beyond an Introduction.* Progress in Mathematics. Birkhäuser Boston, 2002.