MA571 Homework 13

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Problem 13.1 (Munkres §58, Ex.9(a,b,c))

We define the *degree* of a continuous map $h\colon S^1 \to S^1$ as follows:

Let b_0 be the point $(0,1)$ of S^1; choose a generator γ for the infinite cyclic group $\pi_1(S^1, b_0)$. If x_0 is any point of S^1, choose a path α in S^1 from b_0 to x_0 and define $\gamma(x_0) \coloneqq \hat{\alpha}(\gamma)$. Then $\gamma(x_0)$ generates $\pi_1(S^1, x_0)$. The element $\gamma(x_0)$ is independent of the choice of the path α, since the fundamental group of S^1 is Abelian.

Now given $h\colon S^1 \to S^1$, choose $x_0 \in S^1$ and let $h(x_0) = x_1$. Consider the homomorphism

$$h_*\colon \pi_1(S^1, x_0) \longrightarrow \pi_1(S^1, x_1).$$

Since both groups are infinite cyclic, we have

$$h_*(\gamma(x_0)) = d \cdot \gamma(x_1) \tag{*}$$

for some integer d, if the group is written additively. The integer d is called the *degree* of h and is denoted by $\deg h$.

The degree of h is independent of the choice of the generator γ; choosing the other generator woul merely change the sign of both sides of (*).

(e) Show that if $h, k\colon S^1 \to S^1$ have the same degree, they are homotopic.

Proof. ■

Problem 13.2 (Munkres §69, Ex. 1)

Check the details of Example 1.

Proof. The following is the statement of Example 1 as found in the book:

Examples 1. Consider the group P of bijections of the set $\{0, 1, 2\}$ with itself. For $i = 1, 2$, define an element π_1 of P by setting $\pi_i(i) = i - 1$ and $\pi_i(i - 1) = i$ and $\pi_i(j) = j$ otherwise. Then π_i generates a subgroup G_i of P of order 2. The group $G_1$and G_2 generate P, as you can check. But P is not their free product. The reduced words (π_1, π_2, π_1) and (π_2, π_1, π_2), for instance, represent the same element of P.

■

Problem 13.3 (Munkres §69, Ex. 2(a,b,c))

Let $G = G_1 * G_2$, where G_1 and G_2 are nontrivial groups.

(a) Show G is not Abelian.

(b) If $x \in G$, define the *length* of x to be the length of the unique reduced word in the elements of G_1 and G_2 that represents x. Show that if x has even length (at least 2), then x does not have finite order. Show that if x has odd length (at least 3), then x is conjugate to an element of shorter length.

(c) Show that the only elements of G that have finite order are the elements of G_1 and G_2 that have finite order, and their conjugates.

Proof. ■

Problem 13.4 (Munkres §69, Ex. 3)

Let $G = G_1 * G_2$. Given $c \in G$, let cG_1c^{-1} denrote the set of all elements of the form cxc^{-1}, for $x \in G_1$. It is a subgroup of G; show that the intersection with G_2 is the identity alone.

Proof. ■

Problem 13.5 (A)

Let $q\colon S^2 \to P^2$ be the quotient map, where P^2 is the projective plane. Let $x_0 = q(1,0,0)$ and let

$$f(s) = q(\cos(\pi s), \sin(\pi s), 0)$$

for $0 \le s \le 1$. Then $f\colon I \to P^2$ is a loop at x_0. Prove that $[f] * [f] = [e_{x_0}]$.

Proof. ■

Problem 13.6 (B)

Let Y be the following subset of $\mathbb{R}^2$: $Y = \{\, (s,t) \in I \times I \mid s \in \{0,1\} \text{ or } t \in \{0,1\} \,\}$ (that is, Y is the boundary of the square $I \times I$). Give Y the equivalence relation $\sim$ that identifies the top and the bottom edges and the left and the right edges: specifically, $\sim$ is the equivalence relation associated to the partition of Y into the following sets:

- for each $s \notin \{0,1\}$, the set $\{(s,0),(s,1)\}$,
- for each $t \notin \{0,1\}$, the set $\{(t,0),(t,1)\}$,
- the set $\{0,1\} \times \{0,1\}$.

Prove that $Y/\sim$ is a wedge of two circles.

Proof. ■

Problem 13.7 (Optional Problem)

Let B^2 denote the unit disk $\{\,(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}$ and let S^1 denote the unit circle. Let $\mathbf{a}\in B^2 - S^1$. In this problem we will show that there is a homeomorphism $h\colon B^2\to B^2$a which takes $(0,0)$ to $\mathbf{a}$ and fixes S^1.

(i) Let $h\colon B^2\to B^2$ be the function defined as follows: note that every point in B^2 is of the form

Proof. ■