Math 535 - General Topology
Fall 2012
Homework 9 Solutions

Problem 1.

a. Let X be a topological space with finitely many connected components. Show that each connected component is open in X.

Solution. Let $X = X_1 \sqcup \ldots \sqcup X_n$ be the partition of X into connected components. Since the connected components of X are always closed in X, the complement of any component

$$X \setminus X_k = \bigsqcup_{\substack{1 \leq i \leq n \\ i \neq k}} X_i$$

is a finite union of closed subsets, hence closed in X. Therefore X_k is open in X. $\square$

b. Let X be a topological space and $\{U_i\}_{i \in I}$ a collection of open subsets of X such that X is the disjoint union $X = \bigsqcup_{i \in I} U_i$. Show that X is the coproduct $X = \coprod_{i \in I} U_i$.

In particular, this conclusion applies to the situation in part (a).

Solution. We want to show that a subset $U \subseteq X$ is open in X if and only if $U \cap U_i$ is open in U_i for all $i \in I$.

($\Rightarrow$) If U is open in X, then $U \cap U_i$ is open in U_i (subspace topology).

($\Leftarrow$) Assume $U \cap U_i$ is open in U_i. Then $U \cap U_i$ is open in X since U_i is open in X. Therefore

$$U = \bigsqcup_{i \in I} (U \cap U_i)$$

is a union of open subsets of X, thus open in X. $\square$

c. Find an example of *metrizable* space X with a connected component $C \subset X$ which is *not open* in X.

Solution. (c.f. HW 8 #6b) Take $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ viewed as a subspace of $\mathbb{R}$.

The singleton $\{0\}$ is a connected component of X. Indeed, every other point $\frac{1}{n} \neq 0$ is isolated in X and thus is its own connected component. In particular, each $\frac{1}{n}$ is not in the connected component of 0.

However, the singleton $\{0\}$ is not open in X. Indeed, any open ball around 0 contains infinitely many points of the form $\frac{1}{n}$. $\square$

Problem 2. Show that the n-dimensional sphere S^n is path-connected (for $n \geq 1$).

Solution. (c.f. HW 4 #7) Write the standard unit sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

as a union $S^n = D_u^n \cup D_l^n$ of the upper and lower hemispheres

$$\begin{aligned} D_u^n &= \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \text{ and } x_{n+1} \geq 0\} \\ D_l^n &= \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \text{ and } x_{n+1} \leq 0\}. \end{aligned}$$

Note that the intersection $D_u^n \cap D_l^n \cong S^{n-1}$ is non-empty (since $n \geq 1$): it is the equator of S^n. Therefore it suffices to show that both hemispheres are path-connected.

Both hemispheres are homeomorphic to the standard unit disc

$$D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}.$$

Writing $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$, consider the map $f_U \colon D^n \to D_u^n$ sending the disc to the upper hemisphere:

$$f_u(x) = (x, \sqrt{1 - \|x\|^2}).$$

Then f_u is a homeomorphism onto the upper hemisphere, with inverse the projection $p_{\mathbb{R}^n} \colon D_u^n \to D^n$ onto the first n coordinates. Likewise, the lower hemisphere D_l^n is homeomorphic to the standard unit disc D^n. Therefore, it suffices to show that D^n is path-connected.

$D_n \subset \mathbb{R}^n$ is in fact a convex subset, hence path-connected via straight line segments. Indeed, let $x, y \in D^n$ and consider the straight line segment from x to y, given by

$$\gamma(t) = (1-t)x + ty \, , t \in [0, 1].$$

For all $t \in [0, 1]$, the norm of $\gamma(t)$ is

$$\begin{aligned} \|\gamma(t)\| &= \|(1-t)x + ty\| \\ &\leq \|(1-t)x\| + \|ty\| \\ &= |1-t|\|x\| + |t|\|y\| \\ &\leq |1-t| + |t| \\ &= (1-t) + t \\ &= 1 \end{aligned}$$

which ensures $\gamma(t) \in D^n$. $\square$

Alternate solution. (c.f. HW 9 #5) Consider the map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \twoheadrightarrow S^n$ defined by

$$\pi(x) = \frac{x}{\|x\|}.$$

Then π is continuous, since $\|x\|$ is a continuous function of x which satisfies $\|x\| > 0$ for all $x \in \mathbb{R}^{n+1} \setminus \{0\}$.

Moreover π is surjective, since we have $\pi(x) = x$ for all $x \in S^n$. Therefore it suffices to show that $\mathbb{R}^{n+1} \setminus \{0\}$ is path-connected to ensure that $S^n = \pi(\mathbb{R}^{n+1} \setminus \{0\})$ is path-connected.

Let $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$ and consider the straight line segment γ in $\mathbb{R}^{n+1}$ from x to y.

If γ does not go through the origin 0, then it is a path in $\mathbb{R}^{n+1} \setminus \{0\}$ from x to y.

If γ does go through the origin, then pick a point $z \in \mathbb{R}^{n+1} \setminus \{0\}$ which is not on the line through x, 0, and y. This is possible since the dimension is $n + 1 \geq 2$. Then the unique line through x and z does not contain 0, and likewise the unique line through z and y does not contain 0.

Let α be the straight line segment from x to z and β be the straight line segment from z to y. Then α and β never go through 0, so that they are paths in $\mathbb{R}^{n+1} \setminus \{0\}$. Therefore their concatenation $\alpha * \beta$ is a path in $\mathbb{R}^{n+1} \setminus \{0\}$ from x to y. $\square$

Problem 3. Consider the "infinite ladder" $X \subset \mathbb{R}^2$ consisting of two vertical "sides" $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$ along with horizontal "rungs" $[0,1] \times \{\frac{1}{n}\}$ for all $n \in \mathbb{N}$ as well as $[0,1] \times \{0\}$. In other words, X is the union

$$X = (\{0,1\} \times \mathbb{R}) \cup \left([0,1] \times \left(\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \right) \right) \subset \mathbb{R}^2.$$

Show that X is path-connected, but **not** locally path-connected.

Solution. **X is path-connected.** The left side $\{0\} \times \mathbb{R} \cong \mathbb{R}$ is path-connected, as is each "rung" $[0,1] \times \{t\} \cong [0,1]$. Therefore the "comb"

$$Y = (\{0\} \times \mathbb{R}) \cup \left([0,1] \times \left(\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \right) \right)$$

is path-connected, since it is a union of path-connected subspaces that all intersect one of them, namely the left side $\{0\} \times \mathbb{R}$. It follows that X is path-connected, since it is the non-disjoint union

$$X = Y \cup (\{1\} \times \mathbb{R})$$

where both Y and the right side $\{1\} \times \mathbb{R}$ are path-connected.

X is not locally path-connected. Let us show that X is not locally path-connected at the point $x = (\frac{1}{2}, 0) \in X$. Consider the neighborhood

$$U := (0,1) \times \left(\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \right) \subset X$$

of x. Note that the path components of U are the individual "rungs" $(0,1) \times \{t\}$.

Let $V \subseteq U$ be a neighborhood of x, so that V contains an open ball $B_r(x)$ for some radius $r > 0$. Pick distinct integers $k_1, k_2 \in \mathbb{N}$ large enough so that $\frac{1}{k_i} < r$. Then the two points $a_i := (\frac{1}{2}, \frac{1}{k_i})$ satisfy

$$(\frac{1}{2}, \frac{1}{k_i}) \in B_r(x) \subseteq V \subseteq U.$$

Because their vertical coordinates $\frac{1}{k_1} \neq \frac{1}{k_2}$ are distinct, the points a_1 and a_2 are in different path components of U, i.e. there is no path in U from a_1 to a_2. In particular, there is no path in V from a_1 to a_2, hence V is not path-connected. $\square$

Problem 4. Let (X, d) be a metric space. Given points $x, y \in X$ and $\epsilon > 0$, an **ϵ-chain** from x to y in X is a finite sequence of points

$$x = z_0, z_1, \ldots, z_{n-1}, z_n = y$$

in X such that the distance from one to the next is less than ϵ, i.e. $d(z_{i-1}, z_i) < \epsilon$ for $1 \leq i \leq n$.

Show that if X is connected, then for all $\epsilon > 0$, any two points $x, y \in X$ can be connected by an ϵ-chain.

Solution. Let $\epsilon > 0$ and let $x \in X$. Let $U \subseteq X$ be the subset of all points that can be reached by an ϵ-chain starting at x. Note $x \in U$, because of the trivial chain $x = z_0$ (or the constant chain $x = z_0 = z_1$ in case the trivial chain is not allowed).

U **is open.** Let $y \in U$, i.e. there is an ϵ-chain $x = z_0, z_1, \ldots, z_n = y$. For any $y' \in B_\epsilon(y)$, the list $z_0, z_1, \ldots, z_n, z_{n+1} := y'$ is still an ϵ-chain in X, by the condition

$$d(z_n, z_{n+1}) = d(y, y') < \epsilon.$$

This ϵ-chain from $z_0 = x$ to $z_{n+1} = y'$ proves $y' \in U$, and thus $B_\epsilon(y) \subseteq U$.

U **is closed.** Let $y \in U^c$, i.e. there is no ϵ-chain from x to y For any $y' \in B_\epsilon(y)$, there is no ϵ-chain from x to y'. If there were such an ϵ-chain $z_0, z_1, \ldots, z_n = y'$, then the list $z_0, z_1, \ldots, z_n, z_{n+1} \ldots = y$ would still be an ϵ-chain in X, by the condition

$$d(z_n, z_{n+1}) = d(y', y) < \epsilon.$$

This proves $y' \in U^c$, and thus $B_\epsilon(y) \subseteq U^c$, so that U^c is open.

Since X is connected and U is a non-empty clopen subset of X, we conclude $U = X$, i.e. all points of X are connected to x by an ϵ-chain. $\square$

Problem 5. Let $p \in \mathbb{R}^n$. Show that $\mathbb{R}^n \setminus \{p\}$ is homotopy equivalent to the $(n-1)$-dimensional sphere S^{n-1}.

Solution. Translation $x \mapsto x + p$ is a homeomorphism $\mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^n \setminus \{p\}$, with inverse the translation $x \mapsto x - p$. Therefore it suffices to show the statement for $p = 0$, the origin in $\mathbb{R}^n$.

Consider the map $\pi\colon \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by

$$\pi(x) = \frac{x}{\|x\|}.$$

Then π is continuous, since $\|x\|$ is a continuous function of x which satisfies $\|x\| > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

We claim that π is a homotopy equivalence, with homotopy inverse the inclusion $i\colon S^{n-1} \hookrightarrow \mathbb{R}^n$.

For all $x \in S^{n-1}$, we have

$$\pi \circ i(x) = \pi(x) = \frac{x}{\|x\|} = \frac{x}{1} = x$$

which proves $\pi \circ i = \mathrm{id}_{S^{n-1}}$.

For the other composite $i \circ \pi$, consider the map

$$H\colon (\mathbb{R}^n \setminus \{0\}) \times [0,1] \to \mathbb{R}^n \setminus \{0\}$$

defined by

$$H(x,t) = (1-t)x + t\frac{x}{\|x\|} = ((1-t)\|x\| + t)\,\frac{x}{\|x\|}$$

which is well defined, i.e. never takes the value 0. Indeed, for all $x \in \mathbb{R}^n \setminus \{0\}$ and $t \in [0,1]$, consider the equivalent conditions

$$\begin{aligned} &((1-t)\|x\| + t)\,\frac{x}{\|x\|} = 0 \\ \Leftrightarrow\ &(1-t)\|x\| + t = 0 \ \text{ since } x \neq 0 \\ \Leftrightarrow\ &(1-t)\|x\| = 0 \text{ and } t = 0 \ \text{ since both terms are } \geq 0 \\ \Leftrightarrow\ &t = 1 \text{ and } t = 0 \end{aligned}$$

which never holds.

Moreover, H is continuous since sums and products of continuous functions are continuous, and $\|x\|$ is continuous and never zero on $\mathbb{R}^n \setminus \{0\}$.

At $t = 0$, we have $H(x,0) = x$, whereas at $t = 1$, we have $\frac{x}{\|x\|} = (i \circ \pi)(x)$. Therefore H is a homotopy between $\mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ and $i \circ \pi$. $\square$

Problem 6. Let X be a topological space and denote by $\pi_0(X)$ the set of path components of X.

a. Show that any continuous map $f\colon X \to Y$ induces a well-defined function

$$\pi_0(f)\colon \pi_0(X) \to \pi_0(Y).$$

Solution. Let $[x] \subseteq X$ denote the path component of a point $x \in X$. The induced function $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is given by

$$\pi_0(f)[x] = [f(x)].$$

To check that this is well defined, let $x, x' \in [x]$ be two representatives, so that there is a path $\gamma\colon [0,1] \to X$ from x to x'. Then $f \circ \gamma\colon [0,1] \to Y$ is a path from $f(\gamma(0)) = f(x)$ to $f(\gamma(1)) = f(x')$. Thus $f(x)$ and $f(x')$ are in the same path component of Y, i.e.

$$[f(x)] = [f(x')] \in \pi_0(Y)$$

and the formula for $\pi_0(f)$ is well defined. □

b. Show that the induced function $\pi_0(f)$ only depends on the homotopy class of f. In other words, if $f \simeq f'$ are homotopic maps, then $\pi_0(f) = \pi_0(f')$.

Solution. Let $H\colon X \times [0,1] \to Y$ be a homotopy from f to f'. We want to show that for any $[x] \in \pi_0(X)$, the equality $\pi_0(f)[x] = \pi_0(f')[x]$ holds. This can be rewritten as $[f(x)] = [f'(x)]$, i.e. there is a path in Y from $f(x)$ to $f(x')$.

Consider the path $\gamma\colon [0,1] \to Y$ defined by $\gamma(t) = H(x,t)$, which is indeed continuous since H is. Then the endpoints of γ are $\gamma(0) = H(x,0) = f(x)$ and $\gamma(1) = H(x,1) = f'(x)$. □

c. Show that a homotopy equivalence $f\colon X \xrightarrow{\simeq} Y$ induces a bijection $\pi_0(f)\colon \pi_0(X) \xrightarrow{\simeq} \pi_0(Y)$.

Solution. Let $g\colon Y \to X$ be a homotopy inverse of f, i.e. it satisfies $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. By part (b), we have

$$\pi_0(g \circ f) = \pi_0(\mathrm{id}_X)$$
$$\pi_0(f \circ g) = \pi_0(\mathrm{id}_Y).$$

Moreover, π_0 preserves composition:

$$\pi_0(g \circ f)[x] = [g(f(x))] = \pi_0(g)[f(x)] = \pi_0(g) \circ \pi_0(f)[x]$$

as well as identities:

$$\pi_0(\mathrm{id}_X)[x] = [\mathrm{id}_X(x)] = [x]$$

from which we obtain

$$\pi_0(g) \circ \pi_0(f) = \pi_0(g \circ f) = \pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$$
$$\pi_0(f) \circ \pi_0(g) = \pi_0(f \circ g) = \pi_0(\mathrm{id}_Y) = \mathrm{id}_{\pi_0(Y)}.$$

Therefore $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is an invertible function, or equivalently, a bijection. □

Remark. This proves in particular that path-connectedness is a homotopy invariant. Given homotopy equivalent spaces X and Y, then X is path-connected if and only if Y is path-connected.

Remark. We have shown that $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$ is a functor.