

# MA544: Qual Preparation

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# 1 MA 544 Spring 2016

This is material from the course MA 544 as it was taught in the spring of 2016.

## 1.1 Homework

These exercises were assigned from Wheeden and Zygmund's *Measure and Integral*, therefore, most of the theorems I reference will be from [5]. Other resources include [1] and [2]. For more elementary results, I cite [3]. Unless otherwise stated, whenever we quote a result, e.g., Theorem 1.1, it is understood to come from Wheeden and Zygmund's *Measure and Integral*.

Throughout these notes

$\mathbb{R}$	is the set of real numbers
$\mathbb{R}^+$	is the set of positive real numbers, that is, $x \in \mathbb{R}$ with $x \geq 0$
$\mathbb{C}$	is the set of complex numbers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{Z}$	is the set of the integers
$\mathbb{Z}^+$	is the set of positive integers, that is, $x \in \mathbb{Z}$ with $x \geq 0$
$\mathbb{N}$	is the set of the natural numbers $1, 2, \dots$
$A \setminus B$	is the set difference of $A$ and $B$ , that is, the complement of $A \cap B$ in $A$
$m^*(E)$	the outer measure of $E$
$m_*(E)$	the inner measure of $E$
$m(E)$	the Lebesgue measure of $E$
$\ -\ $	the standard Euclidean norm on $\mathbb{R}^n$
$f \asymp g$	means $f$ is asymptotically equivalent to $g$ , that is, $\lim_{x \rightarrow \infty} g(x)/f(x) = 1$

### 1.1.1 Homework 1

**Problem 1** (Wheeden & Zygmund Ch. 2, Ex. 1). Let  $f(x) = x \sin(1/x)$  for  $0 < x \leq 1$  and  $f(0) = 0$ . Show that  $f$  is bounded and continuous on  $[0, 1]$ , but that  $V[f; 0, 1] = \infty$ .

**Solution.** ► Let  $f$  equal  $x \sin(1/x)$ . We will show that  $f$  is bounded and continuous on  $[0, 1]$ , but that it is not of bounded variation on  $[0, 1]$ .

First we will show that  $f$  is bounded. Note that both  $|x|$  and  $|\sin(1/x)|$  are bounded by 1 on the interval  $[0, 1]$ . Since  $|f| = |x| |\sin(1/x)|$ , it follows that  $|f| \leq 1$  on  $[0, 1]$ . Thus,  $f$  is bounded on  $[0, 1]$ .

Next we show that  $f$  is continuous. It is easy to show that  $f$  is continuous on the subinterval  $(0, 1]$  since both  $|x|$  and  $\sin(1/x)$  are continuous on that interval and we know that the product of continuous functions is continuous. To see that  $f$  is continuous at 0 we must show that  $f(x^+) = f(0)$ ; that is, the limit of  $f$  as  $x$  approaches 0 from the right is  $f(0)$  which by definition is 0. To this end, it suffices to take a (monotonically decreasing) sequence  $x_n \searrow 0$  and show that the limit of the sequence  $\{f(x_n)\}_{n=1}^\infty$  is 0. Let  $\varepsilon > 0$  be given then, since  $x_n$  converges to 0 there exists an index  $N$  such that  $|0 - x_n| < \varepsilon$  whenever  $n \geq N$ . Since  $|f(x_n)| \leq |x_n|$  on  $[0, 1]$ , the following inequality holds

$$\begin{aligned} |0 - f(x_n)| &= |0 - x_n \sin(1/x_n)| \\ &\leq |x_n| \\ &< \varepsilon. \end{aligned}$$

Thus,  $f$  is continuous at 0 and it converges to 0.

Despite the nice properties that  $f$  seemingly possesses,  $f$  is not b.v. on  $[0, 1]$ . To show that  $f$  is not b.v. on  $[0, 1]$  we must show that for any positive real number  $M$  there exists some partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of  $[0, 1]$  such that the sum associated to  $\Gamma$

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| > M.$$

Let  $N$  be the smallest integer greater than  $M$  and let  $n$  be the smallest integer greater than or equal to  $N/2$ . Then the partition  $\Gamma = \{x_0 = 1 < x_1 < \cdots < x_{n+1} = 1\}$  where  $x_i = 2/((3 + (n - i))\pi)$  for  $1 \leq i \leq N$ . Then we have the inequality

$$\begin{aligned} S_\Gamma &= \sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=2}^n |f(x_i) - f(x_{i-1})| + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &= N + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &> M. \end{aligned}$$

Thus,  $f$  is not b.v. on  $[0, 1]$ . ◀

**Problem 2** (Wheeden & Zygmund Ch. 2, Ex. 2). Prove theorem (2.1).

**Solution.** ► Recall the statement of Theorem 2.1:

- (a) If  $f$  is of bounded variation on  $[a, b]$ , then  $f$  is bounded on  $[a, b]$ .
- (b) Let  $f$  and  $g$  be of bounded variation on  $[a, b]$ . Then  $cf$  (for any real constant  $c$ ),  $f + g$ , and  $fg$  are of bounded variation on  $[a, b]$ . Moreover,  $f/g$  is of bounded variation on  $[a, b]$  if there exists an  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for  $x \in [a, b]$ .

We shall prove these in alphabetical order:

For part (a) we shall proceed by contradiction. First, without loss of generality, we may assume that  $f(a) = 0$  since the function the variation of  $g(x) = f(x) - f(a)$  is equal to the variation of  $f$  and  $g(a) = 0$ . Suppose that  $f$  is b.v. on  $[a, b]$  with variation  $V = V[f; a, b]$ , but that  $f$  is unbounded on  $[a, b]$ ; that is, given a positive real number  $M$  there exists a point  $x$  in  $[a, b]$  such that  $|f(x)| > M$ . In particular, there exists  $x \in [a, b]$  such that  $|f(x)| > V$ . Hence, for any  $x \in [a, b]$  by the triangle inequality we have

$$\begin{aligned}
 V &< |f(x)| \\
 &= |f(x) - f(a) + f(a)| \\
 &\leq |f(x) - f(a)| + |f(a)| \\
 &\leq V.
 \end{aligned}$$

This is a contradiction. Therefore, it must be the case that if  $f$  is b.v. on  $[a, b]$  then  $f$  is bounded on  $[a, b]$ .

We break part (b) into three sections. Suppose  $f$  and  $g$  are b.v. on  $[a, b]$  with variation  $V$  and  $V'$ , respectively. We will show that (i)  $cf$ ; (ii)  $f + g$ ; and (iii)  $fg$  are b.v. on  $[a, b]$ . Moreover, we show that (iv)  $f/g$  is b.v. on  $[a, b]$  if there exists  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for all  $x \in [a, b]$ .

For part (i) above let  $c$  be a real number. Given a partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of  $[a, b]$ , we have

$$\begin{aligned}
 S_\Gamma &= \sum_{i=1}^n |cf(x_i) - cf(x_{i-1})| \\
 &= \sum_{i=1}^n |c| |f(x_i) - f(x_{i-1})| \\
 &= |c| \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \\
 &\leq |c|V
 \end{aligned}$$

since  $V$  is the supremum of the sums of the form  $\sum_{i=1}^m |f(x_i) - f(x_{i-1})|$  over all partitions of  $[a, b]$ . Thus,  $V[cf; a, b] \leq |c|V$  so  $cf$  is b.v. on  $[a, b]$ .

For part (ii) given a partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of the interval  $[a, b]$ , by the triangle

inequality we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^n |(f(x_i) + g(x_i)) - (f(x_{i-1}) + g(x_{i-1}))| \\
&= \sum_{i=1}^n |(f(x_i) - f(x_{i-1})) + (g(x_i) - g(x_{i-1}))| \\
&\leq \sum_{i=1}^n |f(x_i) - f(x_{i-1})| + \sum_{i=1}^n |g(x_i) - g(x_{i-1})| \\
&\leq V + V'.
\end{aligned}$$

Thus,  $f + g$  is b.v. on  $[a, b]$

For part (iii) since  $f$  and  $g$  are b.v. on  $[a, b]$  by part (a)  $f$  and  $g$  are bounded on  $[a, b]$  by, say,  $M$  and  $N$ , respectively. Now, given a partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of  $[a, b]$ , by the triangle inequality we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1}) \\
&\quad + f(x_i)g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^n |(f(x_i)g(x_i) - f(x_i)g(x_{i-1})) \\
&\quad - (f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1}))| \\
&\leq \sum_{i=1}^n |f(x_i)g(x_i) - f(x_i)g(x_{i-1})| \\
&\quad + \sum_{i=1}^n |f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^n |f(x_i)||g(x_i) - g(x_{i-1})| + \sum_{i=1}^n |g(x_{i-1})||f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^n M|g(x_i) - g(x_{i-1})| + \sum_{i=1}^n N|f(x_i) - f(x_{i-1})| \\
&\leq MV' + NV.
\end{aligned}$$

Thus,  $fg$  is b.v. on  $[a, b]$ .

Finally, for part (iv) suppose there exists  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for all  $x \in [a, b]$ . Then, given

a partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of  $[a, b]$ , largely by the triangle inequality, we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^n |f(x_i)/g(x_i) - f(x_{i-1})/g(x_{i-1})| \\
&= \sum_{i=1}^n \left| \frac{f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)}{g(x_i)g(x_{i-1})} \right| \\
&\leq \frac{1}{\varepsilon^2} \sum_{i=1}^n |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^n |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1}) \\
&\quad - (f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1}))| \\
&\leq \frac{1}{\varepsilon^2} \sum_{i=1}^n |g(x_{i-1})||f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} \sum_{i=1}^n |f(x_{i-1})||g(x_i) - g(x_{i-1})| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^n M_g |f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} \sum_{i=1}^n M_f |g(x_i) - g(x_{i-1})| \\
&= \frac{1}{\varepsilon^2} M_g \sum_{i=1}^n |f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} M_f \sum_{i=1}^n |g(x_i) - g(x_{i-1})| \\
&\leq \frac{1}{\varepsilon^2} (NV + MV')
\end{aligned}$$

where, as above,  $f$  is bounded by  $M$  and  $g$  is bounded by  $N$ . Thus,  $f/g$  is b.v. on  $[a, b]$ .

This concludes the proof of Theorem 2.1. ◀

**Problem 3** (Wheeden & Zygmund Ch. 2, Ex. 3). If  $[a', b']$  is a subinterval of  $[a, b]$  show that  $P[a', b'] \leq P[a, b]$  and  $N[a', b'] \leq N[a, b]$ .

**Solution.** ▶ We will prove this by digging in to the definition of  $N$  and  $P$ . Recall that given a partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of the interval  $[a, b]$ ,  $P$  and  $N$  are defined to be the supremum over the sum of the positive and, respectively, the sum negative terms of  $S_\Gamma$ ; that is,  $P$  and  $N$  are the supremum over every partition  $\Gamma$  of  $[a, b]$  of

$$P_\Gamma = \sum_{i=1}^n [f(x_i) - f(x_{i-1})]^+ \quad \text{and} \quad N_\Gamma = \sum_{i=1}^n [f(x_i) - f(x_{i-1})]^-.$$

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation on  $[a, b]$  and let  $[a', b']$  be a subinterval of  $[a, b]$ . Without loss of generality, we may assume that  $[a', b']$  is strictly contained in  $[a, b]$ ; that is,  $a' \neq a$  and  $b' \neq b$ . We aim to show that  $P[a', b'] \leq P[a, b]$  and  $N[a', b'] \leq N[a, b]$ . Since the argument for  $N$  is similar to that of  $P$ , we will omit it here for the sake of brevity. Now, consider the closure of the complement of  $[a', b']$  in  $[a, b]$ ,  $\overline{[a, b] \setminus [a', b']} = [a, a'] \cup [b', b]$ . Since  $[a, a']$ ,  $[a', b']$

and  $[b', b]$  are close intervals we may take partitions

$$\begin{aligned}\Gamma_a &= \{x_0 < x_1 < \cdots < x_\ell\}, \\ \Gamma_{ab} &= \{x_\ell < x_{\ell+1} < \cdots < x_m\}\end{aligned}$$

and

$$\Gamma_b = \{x_m < x_{m+1} < \cdots < x_n\}$$

of  $[a, a']$ ,  $[a', b']$  and  $[b', b]$ , respectively and extend this to a partition

$$\Gamma = \{x_0 < x_1 < \cdots < x_\ell < x_{\ell+1} < \cdots < x_m < x_{m+1} < \cdots < x_n\}$$

of  $[a, b]$ . Then, by the definition of  $N$  we have the string of inequalities

$$\begin{aligned}P_\Gamma &= \sum_{i=1}^n [f(x_i) - f(x_{i-1})]^+ \\ &= \sum_{i=1}^{\ell} [f(x_i) - f(x_{i-1})]^+ \\ &\quad + \sum_{i=\ell+1}^m [f(x_i) - f(x_{i-1})]^+ \\ &\quad + \sum_{i=m+1}^n [f(x_i) - f(x_{i-1})]^+ \\ &= P_{\Gamma_{ab}} + P_{\Gamma_a} + P_{\Gamma_b} \\ &\leq P[a, b].\end{aligned}$$

Taking the supremum on the left, we have

$$P[a, a'] + P[a', b'] + P[b', b] \leq P[a, b].$$

Since  $P$  is strictly positive, it must be the case that  $P[a', b'] \leq P[a, b]$ . ◀

**Problem 4** (Wheeden & Zygmund Ch. 2, Ex. 11). Show that  $\int_a^b f d\varphi$  exists if and only if given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|R_\Gamma - R_{\Gamma'}| < \varepsilon$  if  $|\Gamma|, |\Gamma'| < \delta$ .

**Solution.** ▶ One direction is straightforward. Namely  $\Leftarrow$ : suppose that given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|R_\Gamma - R_{\Gamma'}| < \varepsilon$  whenever  $|\Gamma|$  and  $|\Gamma'|$  are less than  $\delta$ . Let  $\{\Gamma_n\}_{n=1}^\infty$  be a decreasing sequence of partitions (by which we mean  $\Gamma_n \subseteq \Gamma_{n+1}$ ) of  $[a, b]$  such that  $|\Gamma_n| \rightarrow 0$ . Then, by convergence, there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies  $|\Gamma_n| < \delta$ . Then, for  $n, m \geq N$ , we have

$$|R_{\Gamma_n} - R_{\Gamma_m}| < \varepsilon.$$

Thus, by the Cauchy criterion for convergence, the sequence  $\{R_{\Gamma_n}\}_{n=0}^\infty$  converges and its limit is by definition the Riemann–Stieltjes integral  $\int_a^b f d\varphi$ .

On the other hand  $\implies$  : suppose that  $I = \int_a^b f \, d\varphi$  exists. Then given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|I - R_\Gamma| < \varepsilon/2$  whenever  $|\Gamma| < \delta$ . Let  $\Gamma$  and  $\Gamma'$  be two partitions of  $[a, b]$  with norm  $|\Gamma|, |\Gamma'| < \delta$ . Then we have

$$\begin{aligned} |R_\Gamma - R_{\Gamma'}| &= |R_\Gamma - I - (R_{\Gamma'} - I)| \\ &\leq |R_\Gamma - I| + |R_{\Gamma'} - I| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus,  $I$  satisfies the Cauchy condition. ◀

**Problem 5** (Wheeden & Zygmund Ch. 2, Ex. 13). Prove theorem (2.16).

**Solution.** ▶ Recall the statement of Theorem 2.16:

- (i) If  $\int_a^b f \, d\varphi$  exists, then so do  $\int_a^b cf \, d\varphi$  and  $\int_a^b f \, d(c\varphi)$  for any constant  $c$ , and

$$\int_a^b cf \, d\varphi = \int_a^b f \, d(c\varphi) = c \int_a^b f \, d\varphi.$$

- (ii) If  $\int_a^b f_1 \, d\varphi$  and  $\int_a^b f_2 \, d\varphi$  both exist, so does  $\int_a^b (f_1 + f_2) \, d\varphi$ , and

$$\int_a^b (f_1 + f_2) \, d\varphi = \int_a^b f_1 \, d\varphi + \int_a^b f_2 \, d\varphi.$$

- (iii) If  $\int_a^b f \, d\varphi_1$  and  $\int_a^b f \, d\varphi_2$  both exist, so does  $\int_a^b f \, d(\varphi_1 + \varphi_2)$ , and

$$\int_a^b f \, d(\varphi_1 + \varphi_2) = \int_a^b f \, d\varphi_1 + \int_a^b f \, d\varphi_2.$$

We prove this in (Roman) numerical order.

For (i) suppose that  $I = \int_a^b f \, d\varphi$  exists. Then, given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|I - R_\Gamma| < \varepsilon/|c|$  whenever  $\Gamma$  is a partition of  $[a, b]$  with  $|\Gamma| < \delta$ . We claim that  $\int_a^b cf \, d\varphi = |c|I$ . Let  $\Gamma = \{x_0 < x_1 < \dots < x_n\}$  be a partition  $[a, b]$  with  $|\Gamma| < \delta$ . Then the Riemann–Stieltjes sums  $R'_\Gamma$  of the pair  $(cf, \varphi)$  associated to  $\Gamma$  give us the chain of inequalities

$$\begin{aligned} ||c|I - R'_\Gamma| &= \left| |c|I - \sum_{i=1}^n cf(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c| \left| \sum_{i=1}^n f(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c| |I - R_\Gamma| \\ &< |c| \frac{\varepsilon}{|c|} \\ &= \varepsilon. \end{aligned}$$



Thus,  $\int_a^b cf \, d\varphi$  is Riemann–Stieltjes integrable and its integral is equal to  $|c|I$ . A similar argument shows that  $\int_a^b f \, d(c\varphi)$  is Riemann–Stieltjes integrable with integral  $|c|I$ .

For (ii) let  $I_1 = \int_a^b f_1 \, d\varphi$  and  $I_2 = \int_a^b f_2 \, d\varphi$ . Then, we claim that  $I = \int_a^b (f_1 + f_2) \, d\varphi$  exists and that  $I = I_1 + I_2$ . Since both  $I_1$  and  $I_2$  exist, given  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$|I_1 - R_\Gamma^1| < \frac{\varepsilon}{2} \quad \text{and} \quad |I_2 - R_\Gamma^2| < \frac{\varepsilon}{2}$$

whenever  $|\Gamma| < \delta$ . Let  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  be a partition of  $[a, b]$  with  $|\Gamma| < \delta$ . Then the Riemann–Stieltjes sums  $R_\Gamma$  of the pair  $(f_1 + f_2, \varphi)$  associated to  $\Gamma$  give is the following chain of inequalities

$$\begin{aligned} |(I_1 + I_2) - R_\Gamma| &= \left| (I_1 + I_2) - \sum_{i=1}^n (f_1(\xi_i) + f_2(\xi_i))[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= \left| I_1 - \sum_{i=1}^n f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right. \\ &\quad \left. + I_2 - \sum_{i=1}^n f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &\leq \left| I_1 - \sum_{i=1}^n f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &\quad + \left| I_2 - \sum_{i=1}^n f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |I_1 - R_\Gamma^1| + |I_2 - R_\Gamma^2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus,  $I$  exists and it is equal to the sum  $I_1 + I_2$ .

Part (iii) is similar to part (ii) in the above equation except that instead of splitting the sum at  $f_1 + f_2$  part, we split it at  $\varphi_1 + \varphi_2$  part. ◀

### 1.1.2 Homework 2

**Problem 1.** Show that the boundary of any interval has outer measure zero.

**Solution.** ► Let  $I := \prod_{i=1}^n I_i$  be a closed interval in  $\mathbb{R}^n$  and let  $J$  be the boundary of  $I$ . We must show that given  $\varepsilon > 0$  there exists a countable collection of intervals  $\{I_n\}_{n \in J}$  covering  $J$  such that

$$\sum_{n \in J} \text{vol}(I_n) < \varepsilon.$$

First, note that we can write  $J$  as the union  $\bigcup_{i=1}^n J_i$  where

$$J_i := [a_1, b_1] \times \cdots \times \{a_i\} \times \cdots \times [a_n, b_n] \cup [a_1, b_1] \times \cdots \times \{b_i\} \times \cdots \times [a_n, b_n].$$

Since the countable union of null sets has measure zero, it suffices to show that the set

$$[a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$$

has measure zero. Consider the collection  $\{I_\varepsilon\}$  consisting of the single interval

$$I_\varepsilon := [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \left[ a_n - \frac{\varepsilon}{2 \prod_{i=1}^{n-1} (b_i - a_i)}, a_n + \frac{\varepsilon}{2 \prod_{i=1}^{n-1} (b_i - a_i)} \right].$$

It is clear that  $I_\varepsilon \supseteq J$ . Now, computing the volume of this interval, we have

$$\begin{aligned} \text{vol}(I_\varepsilon) &= \prod_{i=1}^{n-1} (b_i - a_i) \left[ a_n + \frac{\varepsilon}{2 \prod_{i=1}^{n-1} (b_i - a_i)} - \left( a_n - \frac{\varepsilon}{2 \prod_{i=1}^{n-1} (b_i - a_i)} \right) \right] \\ &= \left[ \prod_{i=1}^{n-1} (b_i - a_i) \right] \frac{\varepsilon}{\prod_{i=1}^{n-1} (b_i - a_i)} \\ &= \varepsilon. \end{aligned}$$

Thus,  $J$  has measure zero. ◀

**Problem 2.** Show that a set consisting of a single point has outer measure zero.

**Solution.** ► Let  $\{a\}$  be the set consisting of a single point  $a \in \mathbb{R}$ . Then we must show that given  $\varepsilon > 0$  there exists a countable collection of intervals  $\{I_n\}$  such that

$$\sum_{n \in J} m(I_n) < \varepsilon.$$

Consider the collection  $\{I_\varepsilon\}$  consisting of the single interval

$$I_\varepsilon := \left[ a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2} \right].$$

It is clear that  $\{a\} \subseteq I_\varepsilon$ . Moreover,

$$\begin{aligned} \text{vol}(I_\varepsilon) &= a + \frac{\varepsilon}{2} - \left( a - \frac{1}{\varepsilon} \right) \\ &= \varepsilon. \end{aligned}$$

Thus,  $\{a\}$  has measure zero. ◀

### 1.1.3 Homework 3

**Problem 1** (Wheeden & Zygmund Ch. 3, Ex. 5). Construct a subset of  $[0, 1]$  in the same manner as the Cantor set, except that at the  $k$ th stage each interval removed has length  $\delta 3^{-k}$ ,  $0 < \delta < 1$ . Show that the resulting set is perfect, has measure  $1 - \delta$ , and contains no interval.

**Solution.** ► We construct the prescribed subset as follows: take the open interval  $(1/2 - \delta/6, 1/2 + \delta/6)$  and remove it from the closed interval  $[0, 1]$  the result is a union of two disjoint closed intervals

$$E_{1,1} = \left[0, \frac{1}{2} - \frac{1}{6}\delta\right], \quad E_{1,2} = \left[\frac{1}{2} + \frac{1}{6}\delta, 1\right],$$

whose union we call  $E_1$ ; this marks the first step in the construction of this Cantor-like set. Next, we remove the set

$$\left(\frac{1}{4} - \frac{5}{36}\delta, \frac{1}{4} + \frac{1}{36}\delta\right) \cup \left(\frac{3}{4} + \frac{\delta}{36}, \frac{3}{4} + \frac{5}{36}\delta\right)$$

from the set  $E_1$  which yields  $E_2$  the union of the four closed intervals

$$\begin{aligned} E_{2,1} &= \left[0, \frac{1}{4} - \frac{5}{36}\delta\right], & E_{2,2} &= \left[\frac{1}{4} + \frac{1}{36}\delta, \frac{1}{2} - \frac{1}{6}\delta\right], \\ E_{2,3} &= \left[\frac{1}{2} + \frac{1}{6}\delta, \frac{3}{4} + \frac{\delta}{36}\right], & E_{2,4} &= \left[\frac{3}{4} + \frac{5}{36}\delta, 1\right]. \end{aligned}$$

In the  $n$ th step of the construction, we remove an open interval of length  $3^{-n}\delta$  from the center of each interval  $E_{n-1,i}$  yielding  $E_n$  which is the union of  $2^n$  intervals  $E_{n,i}$  of length  $2^{-n} - \delta 2^{-n} \sum_{i=1}^n 2^{i-1} 3^{-i}$ . Let  $E$  be the intersection  $\bigcap_{i=1}^{\infty} E_i$ . This concludes our construction.

Next we show that  $E$  is perfect, has measure  $1 - \delta$  and contains no interval.

To see that  $E$  is perfect, we must show that  $E$  is closed and that and dense in itself. The set  $E$  is closed because it is the (arbitrary) intersection of closed intervals. To see that  $E$  is dense in itself, we must show that for every  $\varepsilon > 0$ , for every  $x \in E$ , the intersection  $(B(x, \varepsilon) \cap E) \setminus \{x\}$  is nonempty. Let  $\varepsilon > 0$  and  $x \in E$  be given. Then, since  $x \in E$ ,  $x \in E_n$  for every  $n$ . Thus,  $x$  is in some closed interval  $E_{n,i} \subseteq E_n$ . Let  $N$  be the smallest integer such that the length of  $E_{N,i} = [a, b]$  is less than  $\varepsilon$ . Then,  $a, b \in E$  and  $a, b \in B(x, \varepsilon)$  and  $x$  cannot be equal to both  $a$  and  $b$ . Thus,  $(E \cap B(x, \varepsilon)) \setminus \{x\} \neq \emptyset$ . It follows that  $E$  is a perfect set.

To see that the measure of  $E$  is  $1 - \delta$  by Theorem 3.26 (ii) since  $m(E_1) = 1 - \delta/3 < \infty$  and

$E_n \searrow E$  we have

$$\begin{aligned}
m(E) &= m\left(\bigcap_{i=1}^{\infty} E_i\right) \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n m(E_i) \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^{2^n} \left[ \frac{1}{2^n} - \frac{\delta}{2^n} \sum_{i=1}^n \frac{2^{i-1}}{3^i} \right] \\
&= \lim_{n \rightarrow \infty} \left[ 1 - \delta \sum_{i=1}^n \frac{2^{i-1}}{3^i} \right] \\
&= \lim_{n \rightarrow \infty} \left[ 1 - \frac{\delta}{3} \sum_{i=1}^n \left(\frac{2}{3}\right)^{i-1} \right]
\end{aligned}$$

letting  $j = i - 1$ , we can rewrite the series above as the geometric series

$$\begin{aligned}
&= 1 - \frac{\delta}{3} \lim_{n \rightarrow \infty} \sum_{j=0}^n \left(\frac{2}{3}\right)^j \\
&= 1 - \delta,
\end{aligned}$$

as desired.

Lastly, we must show that  $E$  contains no interval. Seeking a contradiction, suppose that  $E$  contains an interval  $I = [a, b]$  of length  $b - a$ . Then, since  $I \subseteq E$ ,  $I \subseteq E_n$  for all  $n$  so, since  $I$  is connected, it must be contained in one of the  $E_{n,i}$  for all  $n$ . Let  $N$  be the smallest integer such that  $m(E_{N,i}) < b - a$  and  $E_{N,i} = [c, d]$  contains  $I$ . Then, since  $I \subseteq E_{N,i}$ , both  $a$  and  $b$  are points in  $I$ ,  $|b - a| \leq |d - c| = m(E_{N,i})$ . This is a contradiction. Thus, it must be the case that  $E$  contains no interval.  $\blacktriangleleft$

**Problem 2** (Wheeden & Zygmund Ch. 3, Ex. 7). Prove (3.15).

**Solution.**  $\blacktriangleright$  Here is the statement of the lemma:

*If  $\{I_k\}_{k=1}^N$  is a finite collection of nonoverlapping intervals, then  $\bigcup_{k=1}^N I_k$  is measurable and  $m\left(\bigcup_{k=1}^N I_k\right) = \sum_{k=1}^N m(I_k)$ .*

By Theorem 3.12, the union  $\bigcup_{n=1}^N I_n$  is measurable. Hence, it remains to show that  $m\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n)$ .

We take the approach of extending the argument provided in Theorem 3.2. As in Theorem 3.2, we note that, since  $\{I_n\}_{n=1}^N$  covers the union  $\bigcup_{n=1}^N I_n$ , then

$$m\left(\bigcup_{n=1}^N I_n\right) \leq \sigma\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n).$$

On the other hand, note that  $I_n$  is the union  $I_n^\circ \cup \partial I_n$  of its interior and its boundary. In the previous homework, we showed that the boundary of an interval has measure zero. Hence, we have

$$m(I_n^\circ) \leq m(I_n) \leq m(I_n^\circ) + m(\partial I_n) = m(I_n^\circ)$$

so  $m(I_n) = m(I_n^\circ)$ . Now, note that

$$m\left(\bigcup_{n=1}^N I_n^\circ\right) = \sum_{n=1}^N m(I_n^\circ) = \sum_{n=1}^N m(I_n).$$

Hence, we have

$$\begin{aligned} \sum_{n=1}^N m(I_n) &= m\left(\bigcup_{n=1}^N I_n^\circ\right) \\ &\leq m\left(\bigcup_{n=1}^N I_n\right) \\ &\leq \sum_{n=1}^N m(I_n). \end{aligned}$$

Thus, equality  $m\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n)$  holds. ◀

**Problem 3** (Wheeden & Zygmund Ch. 3, Ex. 8). Show that the Borel algebra  $\mathcal{B}$  in  $\mathbb{R}^n$  is the smallest  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ .

**Solution.** ▶ Since  $\mathcal{B}$  is the smallest  $\sigma$ -algebra containing all of the open sets of  $\mathbb{R}^n$ , it contains all of the closed sets of  $\mathbb{R}^n$ . Now, suppose that  $\mathcal{B}'$  is another  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ . Then,  $\mathcal{B}' \subseteq \mathcal{B}$  since  $\mathcal{B}$  contains all of the closed sets in  $\mathbb{R}^n$ . However, since  $\mathcal{B}'$  is a  $\sigma$ -algebra, it contains all of the open sets in  $\mathbb{R}^n$ , so  $\mathcal{B}' \subseteq \mathcal{B}$  since  $\mathcal{B}$  is the smallest  $\sigma$ -algebra containing the open sets in  $\mathbb{R}^n$ . Thus,  $\mathcal{B}' = \mathcal{B}$ . ◀

**Problem 4** (Wheeden & Zygmund Ch. 3, Ex. 9). If  $\{E_k\}_{k=1}^\infty$  is a sequence of sets with  $\sum m^*(E_k) < \infty$ , show that  $\limsup E_k$  (and also  $\liminf E_k$ ) has measure zero.

**Solution.** ▶ First, since  $\{E_n\}_{n=1}^\infty$  is a sequence of sets with

$$\sum_{i=1}^{\infty} m^*(E_i) < \infty$$

for every  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies

$$\sum_{i=n}^{\infty} m^*(E_i) < \varepsilon.$$

Let's put this aside for now.

Define  $E = \limsup_{n \rightarrow \infty} E_n$  and  $E'_n = \bigcup_{i=n}^{\infty} E_i$ . It is easy to see that  $\{E'_n\}_{n=1}^{\infty}$  is a decreasing sequence of sets whose intersection  $\bigcap_{n=1}^{\infty} E'_n$  is the limit supremum  $E$ . By the monotonicity of the outer measure, we have

$$m^*(E) \leq m^*(E'_n)$$

for all  $n \in \mathbb{N}$ . On the other hand,

$$m^*(E'_n) \leq \sum_{i=n}^{\infty} m^*(E_i) < \varepsilon$$

for every  $\varepsilon$ . Letting  $\varepsilon$  go to 0 we have  $m^*(E) = 0$ .

Lastly, we note that  $E' = \liminf_{n \rightarrow \infty} E_n$  is a subset of  $\limsup_{n \rightarrow \infty} E_n$ , so that  $m^*(E') = 0$ . ◀

**Problem 5** (Wheeden & Zygmund Ch. 3, Ex. 10). If  $E_1$  and  $E_2$  are measurable, show that  $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$ .

**Solution.** ▶ We may, without loss of generality, assume that  $m(E_1), m(E_2) < \infty$  for otherwise there is nothing to show as equality holds trivially.

Now, by Carathéodory's theorem we have the following characterization of measurability: a set  $E$  is measurable if and only if for every set  $A$  we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E).$$

Therefore, the following equalities hold

$$\begin{aligned} m(E_1) &= m(E_1 \cap E_2) + m(E_1 \setminus E_2) \\ m(E_2) &= m(E_1 \cap E_2) + m(E_2 \setminus E_1). \end{aligned}$$

Moreover, from elementary set theory we have

$$(E_1 \cup E_2) \setminus E_2 = E_1 \setminus (E_1 \cap E_2),$$

$E_1 \subseteq E_1 \cup E_2$  and  $E_1 \cap E_2 \subseteq E_1$  so

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

as desired. ◀

### 1.1.4 Homework 4

**Problem 1** (Wheeden & Zygmund Ch. 3, Ex. 12). If  $E_1$  and  $E_2$  are measurable sets in  $\mathbb{R}^1$ , show  $E_1 \times E_2$  is a measurable subset of  $\mathbb{R}^2$  and  $m(E_1 \times E_2) = m(E_1)m(E_2)$ . (Interpret  $0 \cdot \infty$  as 0.) [Hint: Use a characterization of measurability.]

**Solution.** ► The proof of this result is rather long and we shall omit it for now as I gain nothing from retracing my steps on this one. ◀

**Problem 2** (Wheeden & Zygmund Ch. 3, Ex. 13). Motivated by (3.7), define the *inner measure* of  $E$  by  $m_*(E) = \sup m(F)$ , where the supremum is taken over all closed subsets  $F$  of  $E$ . Show that

- (i)  $m_*(E) \leq m^*(E)$ , and
- (ii) if  $m^*(E) < \infty$ , then  $E$  is measurable if and only if  $m_*(E) = m^*(E)$ .

[Use (3.22).]

**Solution.** ► First we show part (i). If  $m^*(E) = \infty$ , the inequality holds trivially. Suppose that  $m^*(E) < \infty$ . Then, since  $F$  is closed, it is measurable and  $m(F) = m^*(F)$ . Moreover,  $F \subseteq E$  so by the monotonicity of the outer measure,

$$m(F) = m^*(F) < m^*(E).$$

Taking the supremum over all  $F$  on the left, we have

$$m_*(E) = \sup_{F \subseteq E} m(F) < m^*(E)$$

as we set out to show.

Next we show part (ii). Let  $E \subseteq \mathbb{R}^n$  with  $m^*(E) < \infty$ .  $\implies$  Suppose that  $E$  is measurable. Then, by Lemma 3.22, there exists a closed set  $F \subseteq E$  such that  $m^*(E \setminus F) < \varepsilon$ . Since closed sets are measurable, by Corollary 3.31, we have

$$m^*(E \setminus F) = m(E) - m(F) < \varepsilon$$

so

$$m(E) < m(F) + \varepsilon.$$

Letting  $\varepsilon$  go to 0, we have

$$m(E) \leq m(F);$$

and taking the supremum on the right

$$m(E) \leq m_*(E).$$

But, by part (i),  $m_*(E) \leq m^*(E) = m(E)$ . Thus,  $m_*(E) = m^*(E)$  as was to be shown.

$\Leftarrow$  On the other hand, suppose that  $m_*(E) = m^*(E)$ . Then, given  $\varepsilon > 0$  there exists an open set  $G$  containing  $E$  and a closed set  $F$  contained in  $E$  such that

$$\begin{aligned} m(G) - m^*(E) &< \frac{\varepsilon}{2} \\ m_*(E) - m(F) &< \frac{\varepsilon}{2}. \end{aligned}$$

Then

$$\begin{aligned}
m^*(E \setminus F) &< m^*(G \setminus F) \\
&= m^*(G) - m^*(G \cap F) \\
&= m^*(G) - m^*(F) \\
&< \frac{\varepsilon}{2} + m^*(E) - \left(m^*(E) - \frac{\varepsilon}{2}\right) \\
&= \varepsilon.
\end{aligned}$$

Thus, by Lemma 3.22,  $E$  is measurable. ◀

**Problem 3** (Wheeden & Zygmund Ch. 3, Ex. 15). If  $E$  is measurable and  $A$  is any subset of  $E$ , show that  $m(E) = m_*(A) + m^*(E \setminus A)$ . (See Exercise 13 for the definition of  $m_*(A)$ .)

**Solution.** ▶ Suppose  $A \subseteq E$ . If  $A$  is measurable, by Problem 2, the outer and inner measure of  $A$  agree; symbolically, we have  $m(A) = m^*(A) = m_*(A)$ . Thus, we have

$$m^*(E \setminus A) = m^*(E) - m^*(A) = m^*(E) - m_*(A).$$

If  $A$  is not measurable and  $m(E) < \infty$ , then we must have  $m^*(A), m^*(E \setminus A) < \infty$  by the monotonicity of the outer measure; since both  $A$  and  $E \setminus A$  are subsets of  $E$ . Hence, we may, without any ambiguity, subtract the quantity  $m^*(E \setminus A)$  from  $m(E)$  and we have

$$\begin{aligned}
m(E) - m^*(E \setminus A) &= m(E) - \inf\{m(G) : E \setminus A \subseteq G \text{ and } G \text{ is open}\} \\
&= m(E) - \inf\{m(G) : E \setminus A \subseteq G \subseteq E \text{ and } G \text{ is open}\} \\
&=
\end{aligned}$$
◀



### 1.1.5 Homework 5

**Problem 1** (Wheeden & Zygmund Ch. 3, Ex. 14). Show that the conclusion of part (ii) of Exercise 13 is false if  $m^*(E) = \infty$ .

**Solution.** ► Part (ii) of Exercise 13 is part (ii) of Problem 2 from the last section (Homework 4). In that problem we showed that if the outer measure of  $E$  is finite, then  $E$  is measurable if and only if its outer and inner measure agree. Here we construct a counter example to this when the outer measure of  $E$  is  $\infty$ ; that is, we show that there exists a set  $E$  with  $m^*(E) = \infty$  such that  $m^*(E) \neq m_*(E)$ . So, which set shall it be? Since we are unoriginal, we will pull an example from Wheeden and Zygmund itself.

Let  $V \subseteq [0, 1]$  be Vitali's unmeasurable (Theorem 3.38) and consider the union  $E = V \cup (2, \infty)$ . It is clear that the inner and outer measure of  $E$  are both  $\infty$ . However,  $E$  itself must be unmeasurable for otherwise  $E \cap [0, 1] = V$  is measurable. ◀

**Problem 2** (Wheeden & Zygmund Ch. 3, Ex. 16). Prove (3.34).

**Solution.** ► We must prove Equation 3.34; that is, if  $P$  is a parallelepiped

$$m(P) = \text{vol}(P).$$

We may, without loss of generality, assume that one of the vertices of  $P$  is  $\mathbf{0}$ . Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be a set of vectors such that

$$P = \left\{ x \in \mathbb{R}^n : x = \sum_{k=1}^n t_k \mathbf{e}_k, 0 \leq t_k \leq 1 \right\}.$$

By definition, the measure of  $P$  is

$$m(P) = \inf_{\mathcal{S}} \left[ \sum_{I_n \in \mathcal{S}} \text{vol}(I_n) \right]$$

where  $\mathcal{S}$  is a cover of  $P$  by intervals. Take the set of

*Remarks.* Literally nobody cares about this problem. I don't remember how to do it, but it must have been painful if I can't figure it out now, even. ◀

**Problem 3** (Wheeden & Zygmund Ch. 3, Ex. 18). Prove that outer measure is *translation invariant*; that is, if  $E_h = \{x + h : x \in E\}$  is the translate of  $E$  by  $h$ ,  $h \in \mathbb{R}^n$ , show that  $m^*(E_h) = m^*(E)$ . If  $E$  is measurable, show that  $E_h$  is also measurable. [This fact was used in proving (3.37).]

**Solution.** ► Let  $E \subseteq \mathbb{R}^n$  and  $h \in \mathbb{R}^n$  and define the set  $E_h$  to be the set  $E_h = \{x + h : x \in E\}$ . We will show that the outer measure of  $E$  is preserved under such translations. But first, let us point out that  $E_h$  is nothing more than the image of  $E$  under the linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$

given by  $x \mapsto x + h$ . By Theorem 3.35, such a map preserves measurability of sets and for any measurable set  $E' \subseteq \mathbb{R}^n$ ,  $m(T(E')) = (\det T)m(E') = m(E')$  (since  $\det T = 1$ ). Now, by Theorem 3.6, for every  $\varepsilon > 0$ , there exist an open set  $G \supseteq E$  such that  $m^*(G) \leq m^*(E) + \varepsilon$ . Consider the image of  $G$  under  $T$ ,  $T(G)$  is an open set containing  $E_h$  so  $m^*(G) \geq m^*(E)$  and

$$m^*(T(G)) = m^*(G) < m^*(E) + \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$ , we achieve the inequality

$$m^*(E_h) \leq m^*(E).$$

To get the other inequality, take the map  $T^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  which takes  $x \mapsto x - h$ ; this sends  $E_h$  to  $E$  and the same argument shows that

$$m^*(E) \leq m^*(E_h).$$

Thus, we have  $m^*(E) = m^*(E_h)$ , as was to be shown. ◀

**Problem 4** (Wheeden & Zygmund Ch. 4, Ex. 1). Prove corollary (4.2) and theorem (4.8)

**Solution.** ▶ The corollary and theorem in question are:

*If  $f$  is measurable, then  $\{f > -\infty\}$ ,  $\{f < +\infty\}$ ,  $\{f = +\infty\}$ ,  $\{a \leq f \leq b\}$ ,  $\{f = a\}$ , etc., are all measurable. Moreover  $f$  is measurable if and only if  $\{a < f < +\infty\}$  is measurable for every finite  $a$ .*

and

*If  $f$  is measurable and  $\lambda$  is any real number, then  $f + \lambda$  and  $\lambda f$  are measurable.*

Their proofs are quite simple. For the corollary: Suppose  $f: E \rightarrow \mathbb{R}$  is a measurable function. By Theorem 4.1,  $f$  is measurable if and only if for every finite  $\alpha \in \mathbb{R}$ , the sets

$$\begin{aligned} \{x \in E : f(x) \geq \alpha\} \\ \{x \in E : f(x) < \alpha\} \\ \{x \in E : f(x) \leq \alpha\} \end{aligned}$$

are measurable. Since measurable sets form a  $\sigma$ -algebra on  $\mathbb{R}^n$ , we know that the countable union and intersection of measurable sets is measurable. Thus,

$$\begin{aligned} \{x \in E : f(x) > -\infty\} &= \bigcup_{\alpha \in \mathbb{Z}} \{x \in E : f(x) > \alpha\} \\ \{x \in E : f(x) = \infty\} &= \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\} \\ \{x \in E : f(x) < \infty\} &= \bigcup_{\alpha \in \mathbb{Z}} \{x \in E : f(x) < \alpha\} \end{aligned}$$

are easily seen to be measurable.

Showing that  $\{x \in E : f(x) = \alpha\}$  and  $\{x \in E : \alpha < f(x) < \beta\}$  are measurable requires some clever (but not too clever) intersection/union of the sets we get from Theorem 4.1.

For the theorem: Suppose  $f$  is measurable and  $\lambda$  is a constant. By Theorem 4.1, for any finite  $\alpha \in \mathbb{R}$  we have

$$\{x \in E : f(x) > \alpha - \lambda\}$$

so

$$\{x \in E : f(x) + \lambda > \alpha\}$$

is measurable. Thus,  $f + \lambda$  is measurable. Similarly, for  $\lambda \neq 0$ , taking the set

$$\{x \in E : f(x) > \alpha/\lambda\} = \{x \in E : \lambda f(x) > \alpha\}$$

shows that  $\lambda f$  is measurable; otherwise, if  $\lambda = 0$ ,  $\lambda f = 0$  is constant and hence is continuous which in turn implies that it is measurable. ◀

**Problem 5** (Wheeden & Zygmund Ch. 4, Ex. 2). Let  $f$  be a simple function, taking its distinct values on disjoint sets  $E_1, \dots, E_N$ . Show that  $f$  is measurable if and only if  $E_1, \dots, E_N$  are measurable.

**Solution.** ▶  $\implies$  Suppose that  $f$  is measurable. Then, by Corollary 4.2, the sets of the form  $\{f = \alpha_n\} = E_n$  are measurable. So the sets  $E_n$  are measurable.

$\impliedby$  On the other hand, suppose that the sets  $E_n$  are measurable. Then,  $\chi_{E_n}$  is measurable so by Theorem 4.8,  $f$  is measurable since it is the sum

$$f = \sum_{n=1}^N \alpha_{E_n}.$$

◀

### 1.1.6 Homework 6

**Problem 1** (Wheeden & Zygmund Ch. 4, Ex. 4). Let  $f$  be defined and measurable in  $\mathbb{R}^n$ . If  $T$  is a nonsingular linear transformation of  $\mathbb{R}^n$ , show that  $f(T(x))$  is measurable. [If  $E_1 = \{x : f(x) > a\}$  and  $E_2 = \{x : f(T(x)) > a\}$ , show  $E_2 = T^{-1}(E_1)$ .]

**Solution.** ► Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a measurable function and  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation. Then, we show that the composition  $f \circ T$  is measurable. Fix a finite  $\alpha \in \mathbb{R}$  and let

$$\begin{aligned} E_1 &= \{x : f(x) > \alpha\} \\ E_2 &= \{x : f(T(x)) > \alpha\}. \end{aligned}$$

Then, by Theorem 3.35, it suffices to show that  $E_2 = T^{-1}(E_1)$  since  $T^{-1}$  is a nonsingular linear transformation so it sends measurable sets to measurable sets. But this equality is obvious: Suppose  $x \in E_2$ ; then  $f(T(x)) > \alpha$  so, because  $T$  is nonsingular and therefore bijective, clearly  $x \in T^{-1}(E_1)$  so  $E_2 \subseteq T^{-1}(E_1)$ . On the other hand, if  $x \in T^{-1}(E_1)$  then  $x$  is a point in  $E$  such that  $f(T(x)) > \alpha$  so  $x \in E_2$ . Thus,  $E_2 = T^{-1}(E_1)$  and consequently,  $f \circ T$  is a measurable function. ◀

**Problem 2** (Wheeden & Zygmund Ch. 4, Ex. 7). Let  $f$  be usc and less than  $\infty$  on a compact set  $E$ . Show that  $f$  is bounded above on  $E$ . Show also that  $f$  assumes its maximum on  $E$ , i.e., that there exists  $x_0 \in E$  such that  $f(x_0) \geq f(x)$  for all  $x \in E$ .

**Solution.** ► First we show that  $f$  is bounded. Suppose that  $f$  is u.s.c. on  $E$ . Then, by Theorem 4.14 (i), sets of the form  $\{x \in E : f(x) < \alpha\}$  are relatively open. Let  $\mathcal{G} = \{G_\alpha\}_{\alpha \in \mathbb{Z}}$  where  $G_\alpha = \{x \in E : f(x) < \alpha\}$ . Then  $\mathcal{G}$  forms an open cover of  $E$  and since  $E$  is compact there exists a finite subset  $\{G_{\alpha_n}\}_{n=1}^N$  for some finite subset  $\{\alpha_1, \dots, \alpha_N\}$  of  $\mathbb{Z}$ . Let  $\alpha = \max\{\alpha_1, \dots, \alpha_N\}$ . Then,  $f(x) < \alpha$  for all  $x \in E$  so  $f$  is bounded above by  $\alpha$ .

Next, we show that  $f$  in fact assumes its maximum (locally) on  $E$  by using only topological properties of  $f$ . Since sets of the form  $\{x \in E : f(x) \geq \alpha\}$  are relatively closed, by Theorem 4.14 (i), for fixed  $x \in E$  the sets  $F_x = \{y \in E : f(y) \geq f(x)\}$  are relatively closed. Consider the collection  $\{F_x\}_{x \in E}$  of closed subsets of  $E$ . First, note that each of these sets is nonempty since  $f(x) \geq f(x)$  so  $x \in F_x$  for every  $x \in E$ . Now, let  $\{x_n\}_{n=1}^N \subseteq E$  and consider the collection  $\{F_{x_n}\}_{n=1}^N$ . Then  $\bigcap_{n=1}^N F_{x_n} \neq \emptyset$  since for  $x$  the point in  $\{x_1, \dots, x_N\}$  such that  $f(x) = \min\{f(x_1), \dots, f(x_N)\}$ ,  $x \in F_{x_n}$  for all  $1 \leq n \leq N$ . Thus, by the finite intersection property, the intersection  $F = \bigcap_{x \in E} F_x$  is nonempty. Let  $y \in \bigcap_{x \in E} F_x$ , then  $f(y) \geq f(x)$  for all  $x \in E$  so  $f$  achieves its maximum (locally) on  $E$ . ◀

**Problem 3** (Wheeden & Zygmund Ch. 4, Ex. 8).

- Let  $f$  and  $g$  be two functions which are u.s.c. at  $x_0$ . Show that  $f + g$  is u.s.c. at  $x_0$ . Is  $f - g$  u.s.c. at  $x_0$ ? When is  $fg$  u.s.c. at  $x_0$ ?
- If  $\{f_k\}$  is a sequence of functions are u.s.c. at  $x_0$ , show that  $\inf f_k(x)$  is u.s.c. at  $x_0$ .
- If  $\{f_k\}$  is a sequence of functions which are u.s.c. at  $x_0$  and which converge uniformly near  $x_0$ , show that  $\lim f_k$  is u.s.c. at  $x_0$ .

**Solution.** ► We prove these in alphabetical order (a)  $\rightarrow$  (b)  $\rightarrow$  (c).

For (a), suppose that  $f$  and  $g$  are u.s.c. at  $x_0$ . Then given  $M > f(x_0), g(x_0)$  there exists  $\delta_1, \delta_2 > 0$  such that  $f(x), g(x) < M/2$  for all  $|x_1 - x_0| < \delta_1, |x_2 - x_0| < \delta_2$ , respectively. Let  $\delta$  be the minimum of  $\{\delta_1, \delta_2\}$ . Then for any  $x$  such that  $|x - x_0| < \delta$ , we have

$$\begin{aligned} |f(x) + g(x) - (f(x_0) + g(x_0))| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \frac{M}{2} + \frac{M}{2} \\ &= M. \end{aligned}$$

Thus,  $f + g$  is u.s.c.

For that second little part of (a), the one that asks “Is  $f - g$  u.s.c. at  $x_0$ ?” we provide a counterexample. In fact, the following is enough of a counterexample: Take  $f = 0$  (which is continuous everywhere) and  $g$  any function that is u.s.c., but not continuous, at  $x_0$  then  $f - g = -g$  is l.s.c. at  $x_0$ . Another counterexample is provided by the equations  $u_1$  and  $u_2$  from Ch. 4 of Wheeden and Zygmund: Fix an  $x_0 \in \mathbb{R}$  and define

$$u_1(x) = \begin{cases} 0 & \text{if } x < x_0, \\ 1 & \text{if } x \geq x_0, \end{cases} \quad u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

Then

$$u_1(x) - u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

is not u.s.c. at  $x_0$  since being u.s.c. at  $x_0$  implies that for  $1/2 > f(x_0) = 0$  there exists  $\delta > 0$  such that  $f(x) < 1/2$  for all  $x \in (x_0 - \delta, x_0 + \delta)$ . But for any  $x' > x_0$  in  $(x_0 - \delta, x_0 + \delta)$ ,  $u(x') = 1 > 1/2$  which contradicts the assumption that  $u$  is u.s.c. at  $x_0$ .

For (b), suppose  $\{f_n\}_{n=1}^\infty$  is a sequence of functions that are u.s.c. at  $x_0$ . Then

$$\limsup_{\substack{x \rightarrow x_0 \\ x \in E}} f_n(x) \leq f_n(x_0)$$

for all  $n \in \mathbb{N}$ . We must show that

$$\limsup_{\substack{x \rightarrow x_0 \\ x \in E}} [\inf f_n(x)] \leq \inf f_n(x_0).$$

◀

### 1.1.7 Homework 7

**Problem 1** (Wheeden & Zygmund Ch. 4, Ex. 9).

- (a) Show that the limit of a decreasing (increasing) sequence of functions u.s.c. (l.s.c.) at  $x_0$  is u.s.c. (l.s.c.) at  $x_0$ . In particular, the limit of a decreasing (increasing) sequence of functions continuous at  $x_0$  is u.s.c. (l.s.c.) at  $x_0$ .
- (b) Let  $f$  be u.s.c. and less than  $\infty$  on  $[a, b]$ . Show that there exists continuous  $f_k$  on  $[a, b]$  such that  $f_k \downarrow f$ .

**Solution.** ► For part (a) we may as well assume that  $f \geq 0$  for all  $x$ . Let  $\{f_n\}$ ,  $n \in \mathbb{N}$ , be a sequence of decreasing functions with limit  $f$  which are u.s.c. at  $x_0$ . Then, for every  $n \in \mathbb{N}$ , for every sequence  $x \rightarrow x_0$ ,

$$\limsup_{x \rightarrow x_0} f_n(x) \leq f_n(x_0).$$

Now, we claim that  $f(x) \leq f_n(x)$  for every  $x$  and every  $n \in \mathbb{N}$ .

*Proof of claim.* Suppose  $f(x) > f_{N_1}(x)$  for some  $x$ ,  $N_1 \in \mathbb{N}$ . Then there exists a real number  $\varepsilon > 0$  such that  $0 < \varepsilon < |f(x) - f_{N_1}(x)|$  (we may, for example, take  $\varepsilon$  to be in  $\mathbb{Q}$  which is dense in  $\mathbb{R}$ ). Then, since  $f_n \downarrow f$ , there exists an index  $N_1 \in \mathbb{N}$  such that

$$|f(x) - f_{N_1}(x)| < \varepsilon.$$

However, since the sequence  $f_n$  decreases to  $f$ , for  $n \geq \max\{N_1, N_2\}$ ,  $f_n(x) \leq f_{N_1}(x)$  so

$$|f(x) - f_n(x)| > |f(x) - f_{N_1}(x)| > \varepsilon.$$

This is a contradiction. □

Having established this, for every sequence  $x \rightarrow x_0$ , we have

$$\limsup_{x \rightarrow x_0} f(x) \leq \limsup_{x \rightarrow x_0} f_n(x) \leq f_n(x_0).$$

Letting  $n \rightarrow \infty$ ,

$$\limsup_{x \rightarrow x_0} f(x) \leq \lim_{n \rightarrow \infty} f_n(x_0) = f(x_0).$$

For part (b) suppose  $f: [a, b] \rightarrow \mathbb{R}$  is u.s.c. on  $[a, b]$  and  $f(x) < \infty$  for all  $x \in [a, b]$ . For a fixed  $x \in [a, b]$ ,  $f$  is u.s.c. at  $x$  if for every  $\varepsilon > 0$ , there exists a neighborhood  $B(x, \delta)$  such that  $f(y) < f(x) + \varepsilon$ . Now, let  $\varepsilon = 1/n$ . Then, for each  $x \in [a, b]$ , there exists a neighborhood  $B(x, \delta_x)$  such that  $f(y) < f(x) + \varepsilon$  for  $y \in B(x, \delta_x)$ .

The following post on the Mathematics StackExchange contains a solution to part (b) of this problem.

First, we claim that  $f(x) \neq \infty$  for any  $x \in [a, b]$ , it must be bounded.

*Proof of claim.* By Theorem 4.14 (a), sets of the form  $\{x \in [a, b] : f(x) < a\}$  is relatively open for all finite  $a$ . Define

$$E_n = \{x \in [a, b] : f(x) < n\}.$$

Then, the collection  $\mathcal{E} = \{E_n\}$ ,  $n \in \mathbb{N}$ , is an open cover of  $[a, b]$ . Since  $[a, b]$  is compact, there exists a finite subcover  $\{E_{n_1}, \dots, E_{n_m}\}$  of  $\mathcal{E}$ . Letting  $M = \max\{n_1, \dots, n_m\}$ , we have  $f < M$  for all  $x \in [a, b]$ . Thus,  $f$  is bounded on  $[a, b]$ .  $\square$

Now that we have established that  $f$  is bounded on  $[a, b]$  by, say,  $M$  then  $\sup_{x \in [a, b]} f \leq M$ . Define

$$f_n(x) = \sup_{y \in [a, b]} [f(y) - n|x - y|].$$

◀

**Problem 2** (Wheeden & Zygmund Ch. 4, Ex. 11). Let  $f$  be defined on  $\mathbb{R}^n$  and let  $B(x)$  denote the open ball  $\{y : |x - y| < r\}$  with center  $x$  and fixed radius  $r$ . Show that the function  $g(x) = \sup\{f(y) : y \in B(x)\}$  is l.s.c. and the function  $h(x) = \inf\{f(y) : y \in B(x)\}$  is u.s.c. on  $\mathbb{R}^n$ . Is the same true for the closed ball  $\{y : |x - y| \leq r\}$ ?

**Solution.** ▶ Note that, by properties of the infimum/supremum for any set of real numbers  $S \subset \mathbb{R}$ ,

$$\sup S = -\inf(-S)$$

where  $-S = \{-s : s \in S\}$ . Thus,

$$\begin{aligned} g(x) &= -\inf\{-f(y) : y \in B(x, r)\} \\ &= \sup\{f(y) : y \in B(x, r)\}. \end{aligned}$$

Letting  $f' = -f$ , it suffices to show that  $g'(x) = \inf\{f'(y) : y \in B(x, r)\}$  is u.s.c. since for any u.s.c. function  $f$ ,  $-f$  is l.s.c. Therefore, we show that  $h$  is u.s.c.

To see that  $h$  is u.s.c., let  $M > h(x_0)$ . Then we must show that there exists a neighborhood  $B(x_0, \delta)$  such that  $M > h(x)$  for every  $x \in B(x_0, \delta)$ . Since  $h(x_0)$  is the infimum of  $f(x)$  over all  $x \in B(x_0, r)$ , given  $\varepsilon > 0$  there exists  $x \in B(x_0, r)$  such that  $f(x) < h(x_0) + \varepsilon < M$ . Define  $\delta = (r - |x - x_0|)/2$ . Then we claim that for any  $x \in B(x_0, \delta)$ ,

$$g(x) < M.$$

*Proof of claim.* Let  $x \in B(x_0, \delta)$ . Then  $y \in B(x_0, \delta)$  since

$$\begin{aligned} |x - y| &= |x - x_0 - (y - x_0)| \\ &\leq |x - x_0| + |y - x_0| \\ &= (r - |y - x_0|)/2 + |y - x_0| \\ &= r/2 + |y - x_0|/2 \\ &< r. \end{aligned}$$

Thus,

$$g(x) \leq f(y) < g(x_0) + \varepsilon < M.$$

$\square$

It follows that  $g$  is u.s.c. ◀

**Problem 3** (Wheeden & Zygmund Ch. 4, Ex. 15). Let  $\{f_k\}$  be a sequence of measurable functions defined on a measurable set  $E$  with  $m(E) < \infty$ . If  $|f_k(x)| \leq M_x < \infty$  for all  $k$  for each  $x \in E$ , show that given  $\varepsilon > 0$ , there is closed  $F \subseteq E$  and finite  $M$  such that  $m(E \setminus F) < \varepsilon$  and  $|f_k(x)| \leq M$  for all  $x \in F$ .

**Solution.** ► Suppose that  $\{f_n\}$  is a function such that ◀

**Problem 4** (Wheeden & Zygmund Ch. 4, Ex. 18). If  $f$  is measurable on  $E$ , define  $\omega_f(a) = |\{f > a\}|$  for  $-\infty < a < \infty$ . If  $f_k \uparrow f$ , show that  $\omega_{f_k} \uparrow \omega_f$ . If  $f_k \rightarrow f$ , show that  $\omega_{f_k} \rightarrow \omega_f$  at each point of continuity of  $\omega_f$ . [For the second part, show that if  $f_k \rightarrow f$ , then  $\limsup_{k \rightarrow \infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$  and  $\liminf_{k \rightarrow \infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$  for every  $\varepsilon > 0$ .]

**Solution.** ► ◀

**Problem 5** (Wheeden & Zygmund Ch. 5, Ex. 1). If  $f$  is a simple measurable function (not necessarily positive) taking values  $a_j$  on  $E_j$ ,  $j = 1, \dots, N$ , show that  $\int_E f = \sum_{j=1}^N a_j m(E_j)$ . [Use (5.24)].

**Solution.** ► ◀

**Problem 6** (Wheeden & Zygmund Ch. 5, Ex. 3). Let  $\{f_k\}$  be a sequence of nonnegative measurable functions defined on  $E$ . If  $f_k \rightarrow f$  and  $f_k \leq f$  a.e. on  $E$ , show that  $\int_E f_k \rightarrow \int_E f$ .

**Solution.** ► ◀



### 1.1.8 Homework 8

**Problem 1** (Wheeden & Zygmund Ch. 5, Ex. 2). Show that the conclusion of (5.32) are not true without the assumption that  $\varphi \in L(E)$ . [In part (ii), for example, take  $f_k = \chi_{(k,\infty)} \cdot$ ]

**Solution.** ►

**Problem 2** (Wheeden & Zygmund Ch. 5, Ex. 4). If  $f \in L(0, 1)$ , show that  $x^k f(x) \in L(0, 1)$  for  $k = 1, 2, \dots$ , and  $\int_0^1 x^k f(x) dx \rightarrow 0$ .

**Solution.** ►

**Problem 3** (Wheeden & Zygmund Ch. 5, Ex. 6). Let  $f(x, y)$ ,  $0 \leq x, y \leq 1$ , satisfy the following conditions: for each  $x$ ,  $f(x, y)$  is an integrable function of  $y$ , and  $\partial f(x, y)/\partial x$  is a bounded function of  $(x, y)$ . Show that  $\partial f(x, y)/\partial x$  is a measurable function of  $y$  for each  $x$  and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial}{\partial x} f(x, y) dy.$$

**Solution.** ►

**Problem 4** (Wheeden & Zygmund Ch. 5, Ex. 7). Give an example of an  $f$  that is not integrable, but whose improper Riemann integral exists and is finite.

**Solution.** ►

**Problem 5** (Wheeden & Zygmund Ch. 5, Ex. 21). If  $\int_A f = 0$  for every measurable subset  $A$  of a measurable set  $E$ , show that  $f = 0$  a.e. in  $E$ .

**Solution.** ►

**Problem 6** (Wheeden & Zygmund Ch. 6, Ex. 10). Let  $V_n$  be the volume of the unit ball in  $\mathbb{R}^n$ . Show by using Fubini's theorem that

$$V_n = 2V_{n-1} \int_0^1 (1-t^2)^{(n-1)/2} dt.$$

(We also observe that by setting  $w = t^2$ , the integral is a multiple of a classical  $\beta$ -function and so can be expressed in terms of the  $\Gamma$ -function:  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ ,  $s > 0$ .)

**Solution.** ►

**Problem 7** (Wheeden & Zygmund Ch. 6, Ex. 11). Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}.$$

(For  $n = 1$ , write  $\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy$  and use polar. For  $n > 1$ , use the formula  $e^{-|x|^2} = e^{-x_1^2} \cdots e^{-x_n^2}$  and Fubini's theorem to reduce the case  $n = 1$ .)

**Solution.** ►

◀

### 1.1.9 Homework 9

**Problem 1** (Wheeden & Zygmund Ch. 6, Ex. 1).

- (a) Let  $E$  be a measurable subset of  $\mathbb{R}^2$  such that for almost every  $x \in \mathbb{R}$ ,  $\{y : (x, y) \in E\}$  has  $\mathbb{R}$ -measure zero. Show that  $E$  has measure zero and that for almost every  $y \in \mathbb{R}$ ,  $\{x : (x, y) \in E\}$  has measure zero.
- (b) Let  $f(x, y)$  be nonnegative and measurable in  $\mathbb{R}^2$ . Suppose that for almost every  $x \in \mathbb{R}$ ,  $f(x, y)$  is finite for almost every  $y$ . Show that for almost  $y \in \mathbb{R}$ ,  $f(x, y)$  is finite for almost every  $x$ .

**Solution.** ►

**Problem 2** (Wheeden & Zygmund Ch. 6, Ex. 3). Let  $f$  be measurable and finite a.e. on  $[0, 1]$ . If  $f(x) - f(y)$  is integrable over the square  $0 \leq x \leq 1, 0 \leq y \leq 1$ , show that  $f \in L[0, 1]$ .

**Solution.** ►

**Problem 3** (Wheeden & Zygmund Ch. 6, Ex. 4). Let  $f$  be measurable and periodic with period 1:  $f(t+1) = f(t)$ . Suppose there is a finite  $c$  such that

$$\int_0^1 |f(a+t) - f(b+t)| dt \leq c$$

for all  $a$  and  $b$ . Show that  $f \in L[0, 1]$ . (Set  $a = x, b = -x$ , integrate with respect to  $x$ , and make the change of variables  $\xi = x + t, \eta = -x + t$ .)

**Solution.** ►

**Problem 4** (Wheeden & Zygmund Ch. 6, Ex. 6). For  $f \in L(\mathbb{R})$ , define the *Fourier transform*  $\hat{f}$  of  $f$  by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt$$

for  $x \in \mathbb{R}$ . (For complex-valued function  $F = F_0 + iF_1$  whose real and imaginary parts  $F_0$  and  $F_1$  are integrable, we define  $\int F = \int F_0 + i \int F_1$ .) Show that if  $f$  and  $g$  belong to  $L(\mathbb{R})$ , then

$$\widehat{(f * g)}(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

**Solution.** ►

**Problem 5** (Wheeden & Zygmund Ch. 6, Ex. 7). Let  $F$  be a closed subset of  $\mathbb{R}$  and let  $\delta(x) = \delta(x, F)$  be the corresponding distance function. If  $\lambda > 0$  and  $f$  is nonnegative and integrable over the complement of  $F$ , prove that the function

$$\int_{\mathbb{R}} \frac{\delta^\lambda(y) f(y)}{|x - y|^{1+\lambda}} dy$$

is integrable over  $F$  and so is finite a.e. in  $F$ . (In case  $f = \chi_{(a,b)}$ , this reduces to Theorem 6.17.)

**Solution.** ►



**Problem 6** (Wheeden & Zygmund Ch. 6, Ex. 9).

- (a) Show that  $M_\lambda(x; F) = +\infty$  if  $x \notin F$ ,  $\lambda > 0$ .
- (b) Let  $F = [c, d]$  be a closed subinterval of a bounded open interval  $(a, b) \subseteq \mathbb{R}$ , and let  $M_\alpha$  be the corresponding Marcinkiewicz integral,  $\lambda > 0$ . Show that  $M_\lambda$  is finite for every  $x \in (c, d)$  and that  $M_\lambda(c) = M_\lambda(d) = \infty$ . Show also that  $\int M_\lambda \leq \lambda^{-1}|G|$ , where  $G = (a, b) - [c, d]$ .

**Solution.** ►



### 1.1.10 Homework 10

**Problem 1** (Wheeden & Zygmund Ch. 7, Ex. 1). Let  $f$  be measurable in  $\mathbb{R}^n$  and different from zero in some set of positive measure. Show that there is a positive constant  $c$  such that  $f^*(x) \geq c\|x\|^{-n}$  for  $\|x\| \geq 1$ .

**Solution.** ► ◀

**Problem 2** (Wheeden & Zygmund Ch. 7, Ex. 2). Let  $\varphi(x), x \in \mathbb{R}^n$ , be a bounded measurable function such that  $\varphi(x) = 0$  for  $\|x\| \geq 1$  and  $\int \varphi = 1$ . For  $\varepsilon > 0$ , let  $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ . ( $\varphi_\varepsilon$  is called an *approximation to the identity*.) If  $f \in L(\mathbb{R}^n)$ , show that

$$\lim_{\varepsilon \rightarrow 0} (f * \varphi_\varepsilon)(x) = f(x)$$

in the Lebesgue set of  $f$ . (Note that  $\int \varphi_\varepsilon = 1$ ,  $\varepsilon > 0$ , so that

$$(f * \varphi_\varepsilon)(x) - f(x) = \int [f(x-y) - f(x)]\varphi_\varepsilon(y) \, dy.$$

Use Theorem 7.16.)

**Solution.** ► ◀

**Problem 3** (Wheeden & Zygmund Ch. 7, Ex. 6). Show that if  $\alpha > 0$ , then  $x^\alpha$  is absolutely continuous on every bounded subinterval of  $[0, \infty)$ .

**Solution.** ► ◀

**Problem 4** (Wheeden & Zygmund Ch. 7, Ex. 8). Prove the following converse of Theorem 7.31: If  $f$  is of bounded variation on  $[a, b]$ , and if the function  $V(x) = V[a, x]$  is absolutely continuous on  $[a, b]$ , then  $f$  is absolutely continuous on  $[a, b]$ .

**Solution.** ► ◀

**Problem 5** (Wheeden & Zygmund Ch. 7, Ex. 9). If  $f$  is of bounded variation on  $[a, b]$ , show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then  $f$  is absolutely continuous on  $[a, b]$ . (For the second part, use Theorems 2.2(ii) and 7.24 to show that  $V(x)$  is absolutely continuous and then use the result of Exercise 8).

**Solution.** ► ◀

**Problem 6** (Wheeden & Zygmund Ch. 7, Ex. 12). Use Jensen's inequality to prove that if  $a, b \geq 0$ ,  $p, q > 1$ ,  $(1/p) + (1/q) = 1$ , then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

More generally, show that

$$a_1 \cdots a_N = \sum_{j=1}^N \frac{a_j^{p_j}}{p_j},$$

where  $a_j \geq 0$ ,  $p_j > 1$ ,  $\sum_{j=1}^N (1/p_j) = 1$ . (Write  $a_j = e^{x_j/p_j}$  and use the convexity of  $e^x$ ).

**Solution.** ►

◀

**Problem 7** (Wheeden & Zygmund Ch. 7, Ex. 13). Prove Theorem 7.36.

**Solution.** ► Recall the statement of Theorem 7.36

- (i) If  $\varphi_1$  and  $\varphi_2$  are convex in  $(a, b)$ , then  $\varphi_1 + \varphi_2$  is convex in  $(a, b)$ .
- (ii) If  $\varphi$  is convex in  $(a, b)$  and  $c$  is a positive constant, then  $c\varphi$  is convex in  $(a, b)$ .
- (iii) If  $\varphi_k$ ,  $k = 1, 2, \dots$ , are convex in  $(a, b)$  and  $\varphi_k \rightarrow \varphi$  in  $(a, b)$ , then  $\varphi$  is convex in  $(a, b)$ .

◀

### 1.1.11 Homework 11

**Problem 1** (Wheeden & Zygmund Ch. 7, Ex. 11). Prove the following result concerning changes of variable. Let  $g(t)$  be monotone increasing and absolutely continuous on  $[\alpha, \beta]$  and let  $f$  be integrable on  $[a, b]$ ,  $a = g(\alpha)$ ,  $b = g(\beta)$ . Then  $f(g(t))g'(t)$  is measurable and integrable on  $[\alpha, \beta]$ , and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t) dt.$$

(Consider the case when  $f$  is the characteristic function of an interval, an open set, etc.)

**Solution.** ►

◀

**Problem 2** (Wheeden & Zygmund Ch. 7, Ex. 15). Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If  $\varphi(x) = \int_a^x f(t) dt + \varphi(a)$  in  $(a, b)$  and  $f$  is monotone increasing, then  $\varphi$  is convex in  $(a, b)$ . (Use Exercise 14.)

**Solution.** ►

◀

**Problem 3** (Wheeden & Zygmund Ch. 5, Ex. 8). Prove (5.49).

**Solution.** ►

◀

**Problem 4** (Wheeden & Zygmund Ch. 5, Ex. 11). For which  $p$  does  $1/x \in L^p(0, 1)$ ?  $L^p(1, \infty)$ ?  $L^p(0, \infty)$ ?

**Solution.** ►

◀

**Problem 5** (Wheeden & Zygmund Ch. 5, Ex. 12). Give an example of a bounded continuous  $f$  on  $(0, \infty)$  such that  $\lim_{x \rightarrow \infty} f(x) = 0$  but  $f \notin L^p(0, \infty)$  for any  $p > 0$ .

**Solution.** ►

◀

**Problem 6** (Wheeden & Zygmund Ch. 5, Ex. 17). If  $f \geq 0$  and  $\omega(\alpha) \leq c(1 + \alpha)^p$  for all  $\alpha > 0$ , show that  $f \in L^r$ ,  $0 < r < p$ .

**Solution.** ►

◀

**Problem 7** (Wheeden & Zygmund Ch. 8, Thm. 8.3). If  $f, g \in L^p(E)$ ,  $p > 0$ , then  $f + g \in L^p(E)$  and  $cf \in L^p(E)$  for any constant  $c$ .

**Solution.** ►

◀

### 1.1.12 Homework 12

**Problem 1** (Wheeden & Zygmund Ch. 8, Ex. 2). Prove the converse of Hölder's inequality for  $p = 1$  and  $\infty$ . Show also that for  $1 \leq p \leq \infty$ , a real-valued measurable  $f$  belongs to  $L^p(E)$  if  $fg \in L^1(E)$  for every  $g \in L^{p'}(E)$ ,  $1/p + 1/p' = 1$ . The negation is also of interest: if  $f \in L^p(E)$  then there exists  $g \in L^{p'}(E)$  such that  $fg \notin L^1(E)$ . (To verify the negation, construct  $g$  of the form  $\sum a_k g_k$  satisfying  $\int_E fg_k \rightarrow \infty$ .)

**Solution.** ► ◀

**Problem 2** (Wheeden & Zygmund Ch. 8, Ex. 3). Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when  $p < 1$ .

**Solution.** ► ◀

**Problem 3** (Wheeden & Zygmund Ch. 8, Ex. 4). Let  $f$  and  $g$  be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let  $1 < p < \infty$ . Prove that equality holds in the inequality  $|\int fg| \leq \|f\|_p \|g\|_{p'}$  if and only if  $fg$  has constant sign a.e. and  $|f|^p$  is a multiple of  $|g|^{p'}$  a.e.

If  $\|f + g\|_p = \|f\|_p + \|g\|_p$  and  $g \neq 0$  in Minkowski's inequality, show that  $f$  is a multiple of  $g$ .

Find analogues of these results for the spaces  $\ell^p$ .

**Solution.** ► ◀

**Problem 4** (Wheeden & Zygmund Ch. 8, Ex. 5). For  $0 < p \leq \infty$  and  $0 < |E| < \infty$ , define

$$N_p[f] = \left( \frac{1}{|E|} \int_E |f|^p \right)^{1/p},$$

where  $N_\infty[f]$  means  $\|f\|_\infty$ . Prove that if  $p_1 < p_2$ , then  $N_{p_1}[f] \leq N_{p_2}[f]$ . Prove also that if  $1 \leq p \leq \infty$ , then  $N_p[f + g] \leq N_p[f] + N_p[g]$ ,  $(1/|E|) \int_E |fg| \leq N_p[f] N_{p'}[g]$ ,  $1/p + 1/p' = 1$ , and  $\lim_{p \rightarrow \infty} N_p[f] = \|f\|_\infty$ . Thus,  $N_p$  behaves like  $\|\cdot\|_p$  but has the advantage of being monotone in  $p$ . Recall Exercise 28 of Chapter 5.

**Solution.** ► ◀

**Problem 5** (Wheeden & Zygmund Ch. 8, Ex. 6).

- (a) Let  $1 \leq p_i$ ,  $r \leq \infty$  and  $\sum_{i=1}^k 1/p_i = 1/r$ . Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$



- (b) Let  $1 \leq p < r < q \leq \infty$  and define  $\theta \in (0, 1)$  by  $1/r = \theta/p + (1 - \theta)/q$ . Prove the interpolation estimate

$$\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}.$$

In particular, if  $A = \max\{\|f\|_p, \|f\|_q\}$ , then  $\|f\|_r \leq A$ .

**Solution.** ►

◀

**Problem 6** (Wheeden & Zygmund Ch. 8, Ex. 9). If  $f$  is real-valued and measurable on  $E$ ,  $|E| > 0$ , define its essential infimum on  $E$  by

$$\operatorname{ess\,inf} f = \sup\{\alpha : |\{x \in E : f(x) < \alpha\}| = 0\}.$$

If  $f \geq 0$ , show that  $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup} 1/f)^{-1}$ .

**Solution.** ►

◀

**Problem 7** (Wheeden & Zygmund Ch. 8, Ex. 11). If  $f_k \rightarrow f$  in  $L^p$ ,  $1 \leq p < \infty$ ,  $g_k \rightarrow g$  pointwise, and  $\|g_k\|_\infty < M$  for all  $k$ , prove that  $f_k g_k \rightarrow f g$  in  $L^p$ .

**Solution.** ►

◀

## 2 Danielli

### 2.1 Danielli: Practice Exams Spring 2016

#### 2.1.1 Exam 1 Practice

**Problem 1.** Let  $E \subseteq \mathbb{R}^n$  be a measurable set,  $r \in \mathbb{R}$  and define the set  $rE = \{rx : x \in E\}$ . Prove that  $rE$  is measurable, and that  $|rE| = |r|^n|E|$ .

**Solution.** ► Define a map a linear map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $T(x) = rx$ . Since a the image of a measurable set  $E$  under linear map is measurable and  $m(T(E)) = |\det T|m(E) = |r|^n m(E)$ , it suffices to show that  $T(E) = rE$ .

Let  $y \in T(E)$  then  $y = rx$  for some  $x \in E$ . Thus,  $y \in rE$ . Let  $y \in rE$ . Then,  $y = rx = T(x)$  for some  $x \in E$ . Thus,  $y \in T(E)$ . It follows that  $m(rE) = |r|^n m(E)$ . ◀

**Problem 2.** Let  $\{E_n\}$ ,  $n \in \mathbb{N}$  be a collection of measurable sets. Define the set

$$\liminf_{n \rightarrow \infty} E_n = \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} E_k \right).$$

Show that

$$m\left(\liminf_{n \rightarrow \infty} E_n\right) \leq \liminf_{n \rightarrow \infty} m(E_n).$$

**Solution.** ► Here's a quick and dirty way of proving this: let  $\chi_{E_n}$  be the characteristic function of  $E_n$ . Then, by Fatou's lemma,

$$\int \liminf_{n \rightarrow \infty} \chi_{E_n}(x) \, dx \leq \liminf_{n \rightarrow \infty} \int \chi_{E_n}(x) \, dx. \quad (1)$$

By definition of the characteristic function, it is easy to see that the right hand-side of the Equation (1) is

$$\liminf_{k \rightarrow \infty} m(E_k).$$

But what about the left-hand side of (1)? We claim that

$$\liminf_{n \rightarrow \infty} \chi_{E_n} = \chi_E$$

where  $E = \liminf_{n \rightarrow \infty} E_n$ .

*Proof of claim.* Suppose  $x \in E$ . We must show that  $\liminf_{n \rightarrow \infty} \chi_{E_n}(x) = 1$ . By definition

$$\liminf_{n \rightarrow \infty} \chi_{E_n} = \lim_{n \rightarrow \infty} \left[ \inf_{k \geq n} \chi_{E_k} \right].$$

Now  $x \in E$  if and only if  $x \in \bigcap_{k=N}^{\infty} E_k$  for some  $N \in \mathbb{N}$ . Then for  $k \geq N$

$$\inf_{k \geq n} \chi_{E_k}(x) = 1$$

so  $\liminf_{n \rightarrow \infty} \chi_{E_n}(x) = 1$ .

On the other hand, if  $x \notin E$  then  $x \notin \bigcap_{k=n}^{\infty} E_k$  for all  $n \in \mathbb{N}$ . Thus, for all  $n \in \mathbb{N}$ ,

$$\inf_{k \geq n} \chi_{E_k}(x) = 0$$

so  $\liminf_{n \rightarrow \infty} \chi_{E_k} = 0$ . ■

Having established this equivalence, we have

$$m\left(\liminf_{n \rightarrow \infty} E_n\right) = \int \liminf_{n \rightarrow \infty} \chi_{E_n}(x) dx \leq \liminf_{n \rightarrow \infty} \int \chi_{E_n}(x) dx = \liminf_{n \rightarrow \infty} m(E_n).$$

Using more elementary methods, define

$$V_n = \bigcap_{k=n}^{\infty} E_k.$$

The sets  $\{V_n\}$ ,  $n \in \mathbb{N}$ , are measurable since they are the countable intersection of measurable subsets. ◀

**Problem 3.** Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here  $B(\mathbf{0}, r) = \{y \in \mathbb{R}^n : |y| < r\}$ . Prove that  $F$  is monotonic increasing and continuous.

**Solution.** ▶ Define the linear map  $T: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $T(r)x = rx$ . We claim that  $B(\mathbf{0}, r) = T(r, B(\mathbf{0}, 1))$ . To reduce notation, set  $B_1 = B(\mathbf{0}, 1)$  and  $B_r = B(\mathbf{0}, r)$ .

*Proof of claim.* Let  $x \in B_r$ . Then  $|x| < r$  so  $|x|/r < 1$ . Thus,  $|x|/r \in B_1$  so it is in the image of  $B_1$  under the map  $T(r, \cdot)$ .

On the other hand, suppose  $x \in T(r, B_1)$ . Then  $x = ry$  for some  $y \in B_1$ . Then, since  $|y| < 1$ ,  $|x| = r|y| < r$  so  $x \in B_r$ . ■

From the claim, we see that  $F(x) = |T(x, B(\mathbf{0}, 1))|$  which, by Problem 1, is nothing more than the polynomial  $|B_1|x^n$ . It is clear, from this equivalence, that  $F$  is monotonically increasing: Take  $x, y \in [0, \infty)$  such that  $x < y$ , then  $x^n < y^n$  so

$$F(x) = |B_1|x^n < |B_1|y^n = F(y). \tag{2}$$

Thus,  $F$  is monotonically increasing.

In the argument above, since  $F(x) = |B_1|x^n$  is a polynomial in  $[0, \infty)$  (and polynomials are continuous on  $\mathbb{R}$ )  $F$  is continuous on  $[0, \infty)$ . ◀

**Problem 4.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $C$  be the set of all points at which  $f$  is continuous. Show that  $C$  is a set of type  $G_\delta$ .

**Solution.** ► (Without much motivation) let us consider the collection of sets  $\{E_k\}$  defined by

$$E_k = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } y, z \in B(x, \delta) \text{ implies } |f(y) - f(z)| < \frac{1}{k} \right\}. \quad (3)$$

We claim that  $C = \bigcap_{k=1}^{\infty} E_k$  and that each  $E_k$  is open.

**Proof of claim.** ► First, we demonstrate equality.  $\subseteq$ : Suppose  $x \in C$ . Then, by the definition of continuity, for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $y \in B(x, \delta)$  implies  $|f(x) - f(y)| < \delta$ . In particular, for every  $k$ , there exists  $\delta > 0$  such that for  $y \in B(x, \delta)$  the inequality  $|f(x) - f(y)| < 1/k$  holds. Thus,  $x$  is in  $\bigcap_{k=1}^{\infty} E_k$ .

$\supseteq$ : On the other hand, suppose that  $x \in \bigcap_{k=1}^{\infty} E_k$ . Then, given  $\varepsilon > 0$ , by the Archimedean property, there exists a positive integer  $N$  such that  $1/N < \varepsilon$ . Then, since  $x \in \bigcap_{k=1}^{\infty} E_k$ ,  $x \in E_N$  so

$$|f(x) - f(y)| < \frac{1}{N} < \varepsilon. \quad (4)$$

Thus,  $x$  is in  $C$  and  $C = \bigcap_{k=1}^{\infty} E_k$ .

All that remains to be shown is that the  $E_k$  are open. But this is clear by the way we defined  $E_k$  in (3): Let  $x \in E_k$ , then there exists  $\delta > 0$  such that for any  $y, z \in B(x, \delta)$ ,  $|f(y) - f(z)| < 1/k$ ; Let  $x' \in B(x, \delta)$  and set  $\delta' = \min\{|(x + \delta) - x'|, |(x - \delta) - x'|\}$ . Then, since  $B(x', \delta') \subseteq B(x, \delta)$ , for every  $y, z \in B(x', \delta')$ , we have  $|f(y) - f(z)| < 1/k$ . Hence,  $x' \in E_k$  for any  $x' \in B(x, \delta)$  so  $B(x, \delta) \subseteq E_k$ . ◀

Since  $C$  can be expressed as the countable intersection of open sets  $E_k$ , it follows that  $C$  is a  $G_\delta$  set. ◀

**Problem 5.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Is it true that if the sets  $\{f = r\}$  are measurable for all  $r \in \mathbb{R}$ , then  $f$  is measurable?

**Solution.** ► If  $\{f = r\}$  are measurable for all  $r \in \mathbb{R}$ , it is not necessarily the case that  $f$  is measurable. Consider the following construction: Let  $E \subseteq (0, 1)$  be an unmeasurable set.\* Define a map  $f: \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{R} \setminus ((0, 1) \setminus E), \\ x + 1 & \text{if } x \in (0, 1) \setminus E. \end{cases} \quad (5)$$

By the definition, it is clear that  $\{f = r\}$  is measurable and  $|\{f = r\}| = 0$  since  $\{f = r\}$  contains at most two elements. However, the set  $\{0 < f < 1\} = E$  is not measurable. Thus,  $f$  is not measurable. ◀

**Problem 6.** Let  $\{f_k\}_{k=1}^{\infty}$  be a sequence of measurable functions on  $\mathbb{R}$ . Prove that the set  $\{x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists}\}$  is measurable.

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\*It's construction does not concern us. The interested reader such direct their refer to Theorem 3.38 from [5, Ch. 3, p. 57-58] or Theorem 17 from [2, Ch. 2§7, p. 48].

**Solution.** ► By Theorem 4.12 from [5, Ch. 4, p. 67],  $\liminf_{k \rightarrow \infty} f_k$  and  $\limsup_{k \rightarrow \infty} f_k$  are measurable. By Theorem 4.7 from [5, Ch. 4, p. 66]

$$\left\{ \liminf_{k \rightarrow \infty} f_k < \limsup_{k \rightarrow \infty} f_k \right\} \quad (6)$$

is measurable. Since

$$\left\{ \lim_{k \rightarrow \infty} f_k \text{ exists} \right\} = \left\{ \limsup_{k \rightarrow \infty} f_k = \liminf_{k \rightarrow \infty} f_k \right\} = \mathbb{R} \setminus \left\{ \liminf_{k \rightarrow \infty} f_k < \limsup_{k \rightarrow \infty} f_k \right\}, \quad (7)$$

by Theorem 3.17 from [5, Ch. 3, p. 48], the set  $\{\lim_{k \rightarrow \infty} f_k \text{ exists}\}$  is measurable. ◀

**Problem 7.** A real valued function  $f$  on an interval  $[a, b]$  is said to be *absolutely continuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Show that an absolutely continuous function on  $[a, b]$  is of bounded variation on  $[a, b]$ .

**Solution.** ► Suppose  $f$  is absolutely continuous on  $[a, b]$ . Let  $\varepsilon = 1$ . Then, there exists  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < 1$ . Let  $N = \lceil (b-a)/\delta \rceil$ , that is,  $N$  is the smallest integer greater than  $(b-a)/\delta$ , and consider the partition  $\Gamma = \{x_k\}$  where  $x_k = a + k(b-a)/N$ , for  $k = 0, \dots, N$ . Then  $x_k - x_{k-1} < (b-a)/N < \delta$  so, by Theorem 2.2(i) from [5, Ch. 2, p. 19], we have  $V[f; x_{k-1}, x_k] < 1$  for  $k = 0, \dots, N$ . It follows by Theorem 2.2(ii) that

$$V[f; a, b] = \sum_{k=1}^N V[f; x_{k-1}, x_k] < N. \quad (8)$$

Thus,  $f$  is b.v. on  $[a, b]$ . ◀

**Problem 8.** Let  $f$  be a continuous function from  $[a, b]$  into  $\mathbb{R}$ . Let  $\chi_{\{c\}}$  be the characteristic function of a singleton  $\{c\}$ , that is,  $\chi_{\{c\}}(x) = 0$  if  $x \neq c$  and  $\chi_{\{c\}}(c) = 1$ . Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b), \\ -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

**Solution.** ► The result follows quite easily from more sophisticated measure theoretic arguments. At this point, however, such language has not been discussed so we shall prove this using nothing but the definition of the Riemann–Stieltjes integral and properties thereof.

Let us consider each case  $c \in (a, b)$ ,  $c = a$ , and  $c = b$  separately.

Recall that the given a partition  $\Gamma = \{x_0, \dots, x_m\}$  of  $[a, b]$ , the Riemann–Stieltjes sum of  $f$  with respect to  $\varphi$  is

$$R_\Gamma = \sum_{k=1}^m f(\xi_k) [\varphi(x_k) - \varphi(x_{k-1})]. \quad (9)$$

The Riemann–Stieltjes integral is defined as the limit

$$\int_a^b f \, d\varphi = \lim_{|\Gamma| \rightarrow 0} R_\Gamma \quad (10)$$

if it exists.

Suppose  $c \in (a, b)$ . Then, for any partition  $\Gamma$  of  $[a, b]$ , either  $c \in \Gamma$  or  $c \notin \Gamma$ . In the latter case,  $R_\Gamma = 0$ . In the former case  $c$  is one of the  $x_k$ , say  $c = x_\ell$  for  $0 < \ell < m$ . Then

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^m f(\xi_k) [\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= 0 + \cdots + 0 + f(\xi_{\ell-1}) - f(\xi_\ell) + 0 + \cdots + 0 \\ &= f(\xi_{\ell-1}) - f(\xi_\ell). \end{aligned} \quad (11)$$

Since  $f$  is continuous, given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|\xi_\ell - \xi_{\ell-1}| < \delta$  implies  $|f(\xi_\ell) - f(\xi_{\ell-1})| < \varepsilon$ . It follows that the quantity in (11) approaches 0 as  $|\Gamma|$  approaches 0. Therefore,  $\int_a^b f \, d\chi_{\{c\}} = 0$ .

Suppose  $c = a$ . Then, since any partition  $\Gamma$  of  $[a, b]$  must contain the point  $a$ , we have

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^m f(\xi_k) [\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= f(\xi_1) [\chi_{\{c\}}(x_1) - \chi_{\{c\}}(x_0)] + f(\xi_2) [\chi_{\{c\}}(x_2) - \chi_{\{c\}}(x_1)] \\ &\quad + \cdots + f(\xi_m) [\chi_{\{c\}}(x_m) - \chi_{\{c\}}(x_{m-1})] \\ &= -f(\xi_1) + 0 + \cdots + 0 \\ &= -f(\xi_1) \end{aligned} \quad (12)$$

Taking the limit as  $|\Gamma| \rightarrow 0$ ,  $\xi_1 \rightarrow a$  so, by continuity of  $f$ ,  $f(\xi_1) \rightarrow f(a)$ . Thus,  $\int_a^b f \, d\chi_{\{c\}} = -f(a)$ .

A similar argument to the one above shows that, if  $c = b$ , the Riemann–Stieltjes integral  $\int_a^b f \, d\chi_{\{c\}} = f(b)$ . ◀

### 2.1.2 Exam 1

#### Problem 1.

Solution. ►

◀

#### Problem 2.

Solution. ►

◀

#### Problem 3.

(i) Show that if  $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ , then there exists a constant  $C$  such that  $|B_r| = Cr^n$ .

(Hint: Think of  $B_r$  as  $\{rx : x \in B_1\}$ .)

(ii) Let  $E \subseteq \mathbb{R}^n$  be a measurable set and let  $\varphi_E : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined  $\varphi_E(x) = |E \cap B_{|x|}|$ . Use part (i) to prove that  $\varphi_E$  is continuous.

**Solution.** ► (i) To prove this result, we use the map constructed in Problem 1 of the review sheet for Exam 1, the map  $T : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Set  $T_r : \mathbb{R}^n \rightarrow \mathbb{R}^n$  to be  $T_r = T(r)$ . Then, we claim  $B_r = T_r(B_1)$  and  $|B_r| = |T_r(B_1)|$ , which, as we saw in Problem 1 of the review sheet, has measure  $|B_1|r^n$ . Setting  $C = |B_1|$ , we have  $|B_r| = C|r|^n$  as desired.

(ii) To prove that  $\varphi_E$  is continuous, we provide an  $(\varepsilon, \delta)$ -argument. Let  $\varepsilon > 0$  be given. We must show that there exists  $\delta > 0$  such that  $y \in B(x, \delta)$  implies

$$|\varphi_E(x) - \varphi_E(y)| < \varepsilon. \quad (1)$$

First, note that since  $x \mapsto |x|$  is continuous and polynomials  $p : \mathbb{R} \rightarrow \mathbb{R}$  are continuous, then the composition  $x \mapsto |x|^n$  is continuous. Therefore, there exists  $\delta > 0$  such that  $y \in B(x, \delta)$  implies

$$||x|^n - |y|^n| < \frac{\varepsilon}{C}, \quad (2)$$

where  $C = |B_1|$ .

Now, let  $x \in \mathbb{R}^n$  and  $y \in B(x, \delta)$  as above. Then, by (2) we have

$$\begin{aligned} |\varphi_E(x) - \varphi_E(y)| &= \left| |E \cap B_{|x|}| - |E \cap B_{|y|}| \right| \\ &\leq \left| |B_{|x|}| - |B_{|y|}| \right| \\ &= C||x|^n - |y|^n| \\ &\leq C \left[ \frac{\varepsilon}{C} \right] \\ &= \varepsilon. \end{aligned} \quad (3)$$

It follows that  $\varphi_E$  is continuous.

◀

**Problem 4.** Assume that  $f: [a, b] \rightarrow \mathbb{R}$  is of bounded variation on  $[a, b]$ . Prove that  $f$  is measurable.

**Solution.** ► By Jordan's theorem (Corollary 2.7 from [5, Ch. 2, p. 21]), the function  $f$  is of bounded variation on  $[a, b]$  if and only if it can be written as the difference  $f_1 - f_2$  of two bounded functions  $f_1$  and  $f_2$  that are monotone increasing on  $[a, b]$ . Then,  $f_1$  and  $f_2$  are continuous a.e. on  $[a, b]$  and hence, are measurable. ◀



### 2.1.3 Exam 2 Practice Problems

**Problem 1.** Define for  $x \in \mathbb{R}^n$ ,

$$f(x) = \begin{cases} |x|^{-(n+1)} & \text{if } x \neq \mathbf{0}, \\ 0 & \text{if } x = \mathbf{0}. \end{cases}$$

Prove that  $f$  is integrable outside any ball  $B(\mathbf{0}, \varepsilon)$ , and that there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n \setminus B(\mathbf{0}, \varepsilon)} f(x) \, dx \leq \frac{C}{\varepsilon}.$$

**Solution.** ► Recall that a real-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is Lebesgueintegrable on a subset  $E$  of  $\mathbb{R}^n$  if

$$\int_E f(x) \, dx < \infty. \quad (1)$$

Let  $f$  be as given in the statement of the problem and set  $B_\varepsilon = B(\mathbf{0}, \varepsilon)$ . Consider the change of variables to *hyperspherical coordinates*  $(x_1, \dots, x_n) \mapsto (r, \Theta)$  where  $\Theta = (\theta_1, \dots, \theta_{n-1})$ .<sup>†</sup> By Theorem 7.26(iii) from [4, Ch. 7, p. 123], we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\varepsilon} f(x) \, dx &= \int_{\mathbb{R}^n \setminus B_\varepsilon} f(x) \, dx \\ &= \int_{\mathbb{R}^n \setminus B_\varepsilon} \frac{1}{|x|^{n+1}} \, dx. \\ &= \int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}} \, dr dV, \end{aligned} \quad (2)$$

where  $S_r^{n-1}$  is the  $(n-1)$ -sphere centered at  $\mathbf{0}$  with radius  $r$ , that is, the subset  $\{x \in \mathbb{R}^n : |x| = r\}$  of  $\mathbb{R}^n$  and  $dV$  is the *volume element* of  $S_r^{n-1}$ . Since  $1/|r|^{n+1}$  is nonnegative, by Tonelli's theorem the iterated integrals in (2) may be exchange, that is,

$$\int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}} \, dr dV = \int_\varepsilon^\infty \left( \int_{S_r^{n-1}} 1 \, dV \right) \frac{1}{|r|^{n+1}} \, dr. \quad (3)$$

Now, note that from Problem 1 of the review sheet for Exam 1, we have

$$\int_{S_r^{n-1}} 1 \, dV = |S_r^{n-1}|_{\mathbb{R}^{n-1}} = |S^{n-1}|_{\mathbb{R}^{n-1}} |r|^{n-1}. \quad (4)$$

Set  $C = |S^{n-1}|_{\mathbb{R}^{n-1}}$ . Putting equations (2), (3), and (4) together, we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\varepsilon} f(x) \, dx &= \int_\varepsilon^\infty C |r|^{n-1} \frac{1}{|r|^{n+1}} \, dr \\ &= \int_\varepsilon^\infty \frac{C}{|r|^2} \, dr \\ &= \lim_{x \rightarrow \infty} \left[ -\frac{C}{x} - \left( -\frac{C}{\varepsilon} \right) \right] \\ &= \frac{C}{\varepsilon}, \end{aligned} \quad (5)$$

---

<sup>†</sup>The explicit construction of the map  $(x_1, \dots, x_n) \mapsto (r, \Theta)$  is of no concern to us for now. What is important is that it exists.

as was to be shown. ◀

**Problem 2.** Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $\mathbb{R}^n$ , and assume that  $f_k$  converges pointwise almost everywhere to a function  $f$ . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets  $E$  of  $\mathbb{R}^n$ . Moreover, show that this is not necessarily true if  $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$ .

**Solution.** ▶ Let  $E \subseteq \mathbb{R}^n$  be a measurable subset of  $\mathbb{R}^n$ . Then, since  $f_k \rightarrow f$  pointwise a.e. on  $\mathbb{R}^n$ , then  $f_k \rightarrow f$  pointwise a.e. on  $E$  and  $\mathbb{R}^n \setminus E$ . To prove that the limit of the sequence of integrals  $\{\int_E f_k\}$  exist and is equal to  $\int_E f$ , it suffices to prove that

$$\int_E f \leq \liminf_{k \rightarrow \infty} \int_E f_k \leq \limsup_{k \rightarrow \infty} \int_E f_k \leq \int_E f. \quad (6)$$

The lower bound in (6) follows from an application of Fatou's lemma:

$$\int_E f = \int_E \liminf_{k \rightarrow \infty} f \leq \liminf_{k \rightarrow \infty} \int_E f_k. \quad (7)$$

Also by Fatou's lemma, we have

$$\int_{\mathbb{R}^n \setminus E} f = \int_{\mathbb{R}^n \setminus E} \liminf_{k \rightarrow \infty} f \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus E} f_k. \quad (8)$$

Now, since  $f \in L^1(\mathbb{R}^n)$ , by equation (8) and properties of the  $\liminf$  and  $\limsup$ <sup>‡</sup> we have

$$\begin{aligned} \int_E f &= \int_{\mathbb{R}^n} f - \int_{\mathbb{R}^n \setminus E} f \geq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} f - \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus E} f_k \\ &\geq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k - \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus E} f_k \\ &= \limsup_{k \rightarrow \infty} \left[ \int_{\mathbb{R}^n} f_k - \int_{\mathbb{R}^n \setminus E} f_k \right] \\ &= \limsup_{k \rightarrow \infty} \int_E f_k. \end{aligned} \quad (9)$$

By equations (7) and (9) it follows that  $\lim_{k \rightarrow \infty} \int_E f_k$  exists and is equal to  $\int_E f$ .

---

<sup>‡</sup>Namely, for any sequence of positive real numbers  $\{a_k\}$  the inequality  $\liminf a_k \leq \limsup a_k$  holds

To see that the result need not be true if  $\int_E f = \infty$ , consider the following example: Let  $f_k : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f_k(x) = \begin{cases} k^2/2 & \text{if } x \in (-1/k, 1/k), \\ 1 & \text{otherwise} \end{cases} \quad (10)$$

and  $f = 1$ .

It is easy to see that  $f_k \rightarrow f$  a.e. in  $\mathbb{R}$  and that both  $\int_{\mathbb{R}} f = \infty$  and  $\lim_{k \rightarrow \infty} \int_{\mathbb{R}} f_k = \infty$ . However, if  $E = (-1, 1)$  then  $\int_E f = 1$ , but  $\lim_{k \rightarrow \infty} \int_E f_k = \infty$ .  $\blacktriangleleft$

**Problem 3.** Assume that  $E$  is a measurable set of  $\mathbb{R}^n$ , with  $|E| < \infty$ . Prove that a nonnegative function  $f$  defined on  $E$  is integrable if and only if

$$\sum_{k=0}^{\infty} |\{x \in E : f(x) \geq k\}| < \infty.$$

**Solution.**  $\blacktriangleright$  If  $f$  is integrable over a measurable subset  $E$  of  $\mathbb{R}^n$ , then

$$\int_E f(x) dx < \infty. \quad (11)$$

Set  $E_k = \{x \in E : k+1 > f(x) \geq k\}$  and  $F_k = \{x \in E : f(x) \geq k\}$ . Note the following properties about the sets we have just defined: first, the  $E_k$ 's are pairwise disjoint and the  $F_k$ 's are nested in the following way  $F_{k+1} \subseteq F_k$ ; second,  $E = \bigcup_{k=1}^{\infty} E_k$  and  $E_k = F_k \setminus F_{k+1}$ . By Theorem 3.23, since the  $E_k$ 's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \quad (12)$$

Now, since  $k\chi_{E_k}(x) \leq f(x) \leq (k+1)\chi_{E_k}(x)$  on  $E_k$ , we have

$$k|E_k| \leq \int_{E_k} f(x) dx \leq (k+1)|E_k|. \quad (13)$$

Then we have the following upper and lower estimates on the integral of  $f$  over  $E$

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(x) dx \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \quad (14)$$

But note that  $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$  by Corollary 3.25 since the measures of  $E_k$ ,  $F_k$ , and  $F_{k+1}$  are all finite. Hence, (14) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(x) dx \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \quad (15)$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned} \tag{17}$$

Thus, from (16) and (17)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(x) dx \leq \sum_{k=0}^{\infty} |F_k| \tag{18}$$

so the integral  $\int_E f$  converges if and only if the sum  $\sum_{k=0}^{\infty} |F_k|$  converges. ◀

**Problem 4.** Suppose that  $E$  is a measurable subset of  $\mathbb{R}^n$ , with  $|E| < \infty$ . If  $f$  and  $g$  are measurable functions on  $E$ , define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$  if and only if  $f_k$  converges to  $f$  as  $k \rightarrow \infty$ .

**Solution.** ▶ ◀

**Problem 5.** Define the *gamma function*  $\Gamma: \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function*  $\beta: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

- (a) Prove that the definition of the gamma function is well-posed, i.e., the function  $u \mapsto e^{-u} u^{y-1}$  is in  $L(\mathbb{R}^+)$  for all  $y \in \mathbb{R}^+$ .
- (b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

**Solution.** ►

◀

**Problem 6.** Let  $f \in L(\mathbb{R}^n)$  and for  $\mathbf{h} \in \mathbb{R}^n$  define  $f_{\mathbf{h}}: \mathbb{R}^n \rightarrow \mathbb{R}$  be  $f_{\mathbf{h}}(x) = f(x - \mathbf{h})$ . Prove that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

**Solution.** ►

◀

**Problem 7.** (a) If  $f_k, g_k, f, g \in L(\mathbb{R}^n)$ ,  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ ,  $|f_k| \leq g_k$  and

$$\int_{\mathbb{R}^n} g_k \longrightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f.$$

- (b) Using part (a) show that if  $f_k, f \in L(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e. in  $\mathbb{R}^n$ , then

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \quad \text{as } k \rightarrow \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \longrightarrow \int_{\mathbb{R}^n} |f| \quad \text{as } k \rightarrow \infty.$$

**Solution.** ► (a)  $\implies$  (b): Assume part (a) then  $\implies$  if

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \tag{19}$$

as  $k \rightarrow \infty$ , we have

(b):

◀

### 2.1.4 Exam 2 (2010)

**Problem 1.** Suppose  $f \in L^1(\mathbb{R}^n)$ . Show that for every  $\varepsilon > 0$  there exists a ball  $B$ , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

*Hint:* Use the monotone convergence theorem.

**Solution.** ►

**Problem 2.** (a) Prove the following generalization of *Chebyshev's inequality*: Let  $0 < p < \infty$  and  $E \subseteq \mathbb{R}^n$  be measurable. assume that  $|f|^p \in L^1(E)$ . Then

$$|\{x \in E : f(x) > \alpha\}| \leq \frac{1}{\alpha^p} \int_{\{f > \alpha\}} f^p,$$

for  $\alpha > 0$ .

(b) Let  $p$ ,  $E$ , and  $f$  be as in part (a). In addition, assume that  $\{f_k\}$  is a sequence such that  $\int_E |f_k - f|^p \rightarrow 0$  as  $k \rightarrow \infty$ . Show that  $f_k \rightarrow f$  in measure on  $E$ .

Recall that  $f_k \rightarrow f$  in measure on  $E$  if and only if for every  $\varepsilon > 0$

$$\lim_{k \rightarrow \infty} |\{x \in E : |f_k(x) - f(x)| > \varepsilon\}| = 0.$$

**Solution.** ►

**Problem 3.** Let  $f \in L^1(\mathbb{R})$ , and define

$$F(\xi) = \int_{\mathbb{R}} f(x) \cos(2\pi x \xi) dx.$$

Prove that  $F$  is continuous and bounded on  $\mathbb{R}$ .

**Solution.** ►

**Problem 4.** Use repeated integration techniques to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}.$$

*Hint:* Start from the case  $n = 1$  by using the polar coordinates in

$$\left[ \int_{\mathbb{R}} e^{-x^2} dx \right]^2 = \left[ \int_{\mathbb{R}} e^{-x^2} dx \right] \left[ \int_{\mathbb{R}} e^{-y^2} dy \right]$$

**Solution.** ►



**Problem 5.**

**Solution.** ►



### 2.1.5 Exam 2

**Problem 1.** Assume that  $f \in L(\mathbb{R}^n)$ . Show that for every  $\varepsilon > 0$  there exists a ball  $B$ , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

**Solution.** ►

◀

**Problem 2.** Let  $f \in L(E)$ , and let  $\{E_j\}$  be a countable collection of pairwise disjoint measurable subsets of  $E$ , such that  $E = \bigcup_{j=1}^{\infty} E_j$ . Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

**Solution.** ►

◀

**Problem 3.** Let  $\{f_k\}$  be a family in  $L(E)$  satisfying the following property: For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|A| < \delta$  implies

$$\int_A |f_k| < \varepsilon$$

for all  $k \in \mathbb{N}$ . Assume  $|E| < \infty$ , and  $f_k(x) \rightarrow f(x)$  as  $k \rightarrow \infty$  for a.e.  $x \in E$ . Show that

$$\lim_{k \rightarrow \infty} \int_E f_k = \int_E f.$$

(Hint: Use Egorov's theorem.)

**Solution.** ►

◀

**Problem 4.** Let  $I = [0, 1]$ ,  $f \in L(I)$ , and define  $g(x) = \int_x^1 t^{-1} f(t) dt$  for  $x \in I$ . Prove that  $g \in L(I)$  and

$$\int_I g = \int_I f.$$

**Solution.** ►

◀



### 2.1.6 Final Exam Practice Problems

**Problem 1.** Suppose  $f \in L^1(\mathbb{R}^n)$  and that  $x$  is a point in the Lebesgue set of  $f$ . For  $r > 0$ , let

$$A(r) = \frac{1}{|r|^n} \int_{B(0,r)} |f(x-y) - f(x)| \, dy.$$

Show that:

- (a)  $A(r)$  is a continuous function of  $r$ , and  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ ;
- (b) there exists a constant  $M > 0$  such that  $A(r) \leq M$  for all  $r > 0$ .

**Solution.** ► (a) Without loss of generality, we may assume  $r < s$ . Then, we want to show that as  $r \rightarrow s$ , the quantity

$$|A(s) - A(r)| \rightarrow 0.$$

Set  $F(y) = |f(x-y) - f(x)|$  and consider said quantity

$$\begin{aligned} |A(s) - A(r)| &= \left| \frac{1}{|s|^n} \int_{B_s} F(y) \, dy - \frac{1}{|r|^n} \int_{B_r} F(y) \, dy \right| \\ &= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y) \, dy + \frac{1}{|s|^n} \int_{B_r} F(y) \, dy - \frac{1}{|r|^n} \int_{B_r} F(y) \, dy \right| \\ &= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y) \, dy + \left( \frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(y) \, dy \right| \\ &\leq \underbrace{\frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y) \, dy}_{I_1} + \underbrace{\left( \frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(y) \, dy}_{I_2}. \end{aligned}$$

Hence, we must show that the quantities  $I_1, I_2 \rightarrow 0$  as  $r \rightarrow s$ .

To see that  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ , note that  $x$  is a point of the Lebesgue set of  $f$  and that

$$0 = \lim_{B_r \searrow x} \frac{1}{|B_1||r|^n} \int_{B_r} |f(y) - f(x)| \, dy = \frac{1}{|B_1|} \lim_{B_r \searrow x} \frac{1}{|r|^n} \int_{B_r} |f(t) - f(x)| \, dt = \lim_{r \rightarrow 0} A(r).$$

by making the change of variables  $t = x - y$ .

(b) ◀

**Problem 2.** Let  $E \subseteq \mathbb{R}^n$  be a measurable set,  $1 \leq n < \infty$ . Assume  $\{f_k\}$  is a sequence in  $L^p(E)$  converging pointwise a.e. on  $E$  to a function  $f \in L^p(E)$ . Prove that

$$\|f_k - f\|_p \rightarrow 0$$

if and only if

$$\|f_k\|_p \rightarrow \|f\|_p$$

as  $k \rightarrow \infty$ .

**Solution.** ►



**Problem 3.** Let  $1 < p < \infty$ ,  $f \in L^p(E)$ ,  $g \in L^{p'}(E)$ .

- (a) Prove that  $f * g \in C(\mathbb{R}^n)$ .
- (b) Does this conclusion continue to be valid when  $p = 1$  and  $p = \infty$ ?

**Solution.** ►



**Problem 4.** Let  $f \in L(\mathbb{R})$ , and let  $F(t) = \int_{\mathbb{R}} f(x) \cos(tx) dx$ .

- (a) Prove that  $F(t)$  is continuous for  $t \in \mathbb{R}$ .
- (b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

**Solution.** ►



**Problem 5.** Let  $f$  be of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular. Show that

$$\int_a^b \varphi \, df = \int_a^b \varphi f' \, dx + \int_a^b \varphi \, dh$$

for all functions  $\varphi$  continuous on  $[a, b]$ .

**Solution.** ►



### 2.1.7 Final Exam 2010

**Problem 1.** Suppose that  $f \in L^1(\mathbb{R}^n)$ , and that  $x$  is a point in the Lebesgue set of  $f$ . For  $r > 0$ , let

$$A(r) = \frac{1}{r^n} \int_{B_r} |f(x-y) - f(x)| \, dy,$$

where  $B_r = B(\mathbf{0}, r)$ .

Show that

- (a)  $A(r)$  is a continuous function of  $r$ , and  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ .
- (b) There exists a constant  $M > 0$  such that  $A(r) \leq M$  for all  $r > 0$ .

**Solution.** ► (a)

(b) ◀

**Problem 2.** Let  $E \subseteq \mathbb{R}^n$  be a measurable set,  $1 \leq p < \infty$ . assume that  $\{f_k\}$  is a sequence in  $L^p(E)$  converging pointwise a.e. on  $E$  to a function  $f \in L^p(E)$ . Prove that

$$\|f_k - f\|_p \rightarrow 0 \iff \|f_k\|_p \rightarrow \|f\|_p$$

*Hint:* To prove one of the implications, you can use the following fact without proving it:

$$\left| \frac{a-b}{2} \right| \leq \frac{|a|^p + |b|^p}{2}$$

for all  $a, b \in \mathbb{R}$ .

**Solution.** ► ◀

**Problem 3.** Let  $0 < p < q < r \leq \infty$ ,  $E \subseteq \mathbb{R}^n$  be a measurable set. Show that each  $f \in L^q(E)$  is the sum of a function  $g \in L^p(E)$  and a function  $h \in L^r(E)$ .

**Solution.** ► ◀

**Problem 4.** Prove that  $f: [a, b] \rightarrow \mathbb{R}$  is Lipschitz continuous if and only if  $f$  is absolutely continuous and there exists a constant  $M > 0$  such that  $|f'| < M$  a.e. on  $[a, b]$ .

**Solution.** ► ◀

**Problem 5.** Let  $1 < p < \infty$ ,  $f \in L^p(\mathbb{R}^n)$ ,  $g \in L^{p'}(\mathbb{R}^n)$ .

- (a) Prove that  $f * g \in C(\mathbb{R}^n)$ .
- (b) Does this conclusion continue to be valid when  $p = 1$  or  $p = \infty$ ?

**Solution.** ► ◀

### 2.1.8 Final Exam

## 2.2 Danielli: Winter 2012

**Problem 1.** Let  $f(x, y)$ ,  $0 \leq x, y \leq 1$ , satisfy the following conditions: for each  $x$ ,  $f(x, y)$  is an integrable function of  $y$ , and  $\partial f(x, y)/\partial x$  is a bounded function of  $(x, y)$ . Prove that  $\partial f(x, y)/\partial x$  is a measurable function of  $y$  for each  $x$  and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial f(x, y)}{\partial x} dy.$$

**Solution.** ►

**Problem 2.** Let  $f$  be a function of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular, show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh.$$

*Hint:* A function  $h$  is said to be singular if  $h' = 0$ .

**Solution.** ►

**Problem 3.** Let  $E \subseteq \mathbf{R}$  be a measurable set, and let  $K$  be a measurable function on  $E \times E$ . Assume that there exists a positive constant  $C$  such that

$$\int_E K(x, y) dx \leq C \tag{1}$$

for a.e.  $y \in E$ , and

$$\int_E K(x, y) dy \leq C \tag{2}$$

for a.e.  $x \in E$ .

Let  $1 < p < \infty$ ,  $f \in L^p(E)$ , and define

$$T_f(x) = \int_E K(x, y) f(y) dy.$$

(a) Prove that  $T_f \in L^p(E)$  and

$$\|T_f\|_p \leq C \|f\|_p. \tag{3}$$

(b) Is (3) still valid if  $p = 1$  or  $\infty$ ? If so, are assumptions (1) and (2) needed?

**Solution.** ►

**Problem 4.** Let  $f$  be a nonnegative measurable function on  $[0, 1]$  satisfying

$$|\{x \in [0, 1] : f(x) > \alpha\}| < \frac{1}{1 + \alpha^2} \tag{4}$$

for  $\alpha > 0$ .

- (a) Determine values of  $p \in [1, \infty)$  for which  $f \in L^p[0, 1]$ .
- (b) If  $p_0$  is the minimum value of  $p$  for which  $p$  may fail to be in  $L^p$ , give an example of a function which satisfies (4), but which is not in  $L^{p_0}[0, 1]$ .

**Solution.** ►



## 2.3 Danielli: Summer 2011

**Problem 1.** Let  $f \in L^1(\mathbf{R})$ , and let  $F(t) = \int_{\mathbf{R}} f(x) \cos(tx) \, dx$ .

- (a) Prove that  $F(t)$  is continuous for  $t \in \mathbf{R}$ .
- (b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

*Hint:* Start by proving the statement for  $f = \chi_{[a,b]}$ .

**Solution.** ►

◀

**Problem 2.** (a) Suppose that  $f_k, f \in L^2(E)$ , with  $E$  a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E f g \tag{1}$$

as  $k \rightarrow \infty$  for all  $g \in L^2(E)$ . If, in addition,  $\|f_k\|_2 \rightarrow \|f\|_2$  show that  $f_k$  converges to  $f$  in  $L^2$ , i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as  $k \rightarrow \infty$ .

- (b) Provide an example of a sequence  $f_k$  in  $L^2$  and a function  $f$  in  $L^2$  satisfying (1), but such that  $f_k$  does *not* converge to  $f$  in  $L^2$ .

**Solution.** ►

◀

**Problem 3.** A bounded function  $f$  is said to be of bounded variation on  $\mathbf{R}$  if it is of bounded variation on any finite subinterval  $[a, b]$ , and moreover  $A = \sup_{a,b} V[a, b; f] < \infty$ . Here,  $V[a, b; f]$  denotes the total variation of  $f$  over the interval  $[a, b]$ . Show that:

- (a)  $\int_{\mathbf{R}} |f(x+h) - f(x)| \, dx \leq A|h|$  for all  $h \in \mathbf{R}$ .

*Hint:* For  $h > 0$ , write

$$\int_{\mathbf{R}} |f(x+h) - f(x)| \, dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)| \, dx.$$

- (b)  $\left| \int_{\mathbf{R}} f(x) \varphi'(x) \, dx \right| \leq A$ , where  $\varphi$  is any function of class  $C^1$ , of bounded variation, compactly supported, with  $\sup_{x \in \mathbf{R}} |\varphi(x)| \leq 1$ .

**Solution.** ►

◀

**Problem 4.** (a) Prove the *generalized Hölder's inequality*: Assume  $1 \leq p \leq \infty$ ,  $j = 1, \dots, n$ , with  $\sum_{j=1}^{\infty} 1/p_j = 1/r \leq 1$ . If  $E$  is a measurable set and  $f_j \in L^{p_j}(E)$  for  $j = 1, \dots, n$ , then  $\prod_{j=1}^n f_j \in L^r(E)$  and

$$\|f_1 \cdots f_n\|_r \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if  $1 \leq p, q, r \leq \infty$ , with  $1/p + 1/q = 1/r + 1$ ,  $f \in L^p(\mathbf{R})$ , and  $g \in L^q(\mathbf{R})$ , then

$$|(f * g)(x)| \leq \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q dy.$$

(Recall that  $(f * g)(x) = \int f(y)g(x-y) dy$ .)

(c) Prove *Young's convolution theorem*: Assume that  $p, q, r, f$ , and  $g$  are as in part (b). Then  $f * g \in L^r(\mathbf{R})$  and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

**Solution.** ►

◀



### 3 Bañuelos

#### 3.1 Bañuelos: Summer 2000

**Problem 1.** Let  $(X, \mathcal{F}, \mu)$  be a measure space and suppose  $\{f_n\}$  is a sequence of measurable functions with the property that for all  $n \geq 1$

$$\mu(\{x \in X : |f_n(x)| \geq \lambda\}) \leq C \exp(-\lambda^2/n)$$

for all  $\lambda > 0$ . (Here  $C$  is a constant independent of  $n$ .) Let  $n_k = 2^k$ . Prove that

$$\limsup_{k \rightarrow \infty} \frac{|f_{n_k}|}{\sqrt{n_k \log(\log(n_k))}} \leq 1 \quad \text{a.e.}$$

**Solution.** ► Suppose  $\{f_n\}_{n=1}^\infty$  is a sequence of measurable functions such that

$$\mu(\{x \in X : |f_n(x)| \geq \lambda\}) \leq C \exp(-\lambda^2/n) \quad (1)$$

for all  $\lambda$ . Now, consider the subsequence  $\{f_{2^k}\}_{k=1}^\infty$  of  $\{f_n\}_{n=1}^\infty$ . We aim to show that

$$\limsup_{k \rightarrow \infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} \leq 1$$

almost everywhere. To that end, it suffices to show that the set

$$E = \left\{ x \in X : \limsup_{k \rightarrow \infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} > 1 \right\}$$

has measure zero. Let  $x \in E$  then

$$\limsup_{k \rightarrow \infty} \frac{|f_{2^k}(x)|}{\sqrt{2^k \log(\log(2^k))}} > 1.$$

This means that there exists some subsequence  $\{k_m\}_{m=1}^\infty \subseteq \{k\}_{n=1}^\infty$  such that

$$\lim_{m \rightarrow \infty} \frac{|f_{2^{k_m}}(x)|}{\sqrt{2^{k_m} \log(\log(2^{k_m}))}} > 1.$$

This means that, for sufficiently large  $N$

$$|f_{2^{k_n}}(x)| > \sqrt{2^{k_n} \log(\log(2^{k_n}))}$$

for all  $n \geq N$ . But by Equation (1) we have

$$\begin{aligned} \mu\left(\left\{x \in X : \frac{|f_{2^{k_n}}(x)|}{\sqrt{2^{k_n} \log(\log(2^{k_n}))}} \geq 1\right\}\right) &\leq C \exp\left(-\left(\sqrt{2^{k_n} \log(\log(2^{k_n}))}\right)^2 / 2^{k_n}\right) \\ &= C \exp(-2^{k_n} \log(\log(2^{k_n}))/2^{k_n}) \\ &= C \exp(-\log(\log(2^{k_n}))) \\ &= C \exp(\log(1/\log(2^{k_n}))) \\ &= \frac{C}{\log(2^{k_n})}. \end{aligned} \quad (2)$$

Letting  $n \rightarrow \infty$ , we see that the measure of the set on the left-hand side of Equation (2) must go to 0 so  $\mu(E) = 0$ . ◀

**Problem 2.** Let  $(X, \mathcal{F}, \mu)$  be a finite measure space. Let  $f_n$  be a sequence of measurable functions with  $f_1 \in L^1(\mu)$  and with the property that

$$\mu(\{x \in X : |f_n(x)| > \lambda\}) \leq \mu(\{x \in X : |f_1(x)| > \lambda\})$$

for all  $n$  and all  $\lambda > 0$ . Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_X \left[ \max_{1 \leq j \leq n} |f_j| \right] d\mu = 0.$$

[Hint: You may use the fact that  $\|f\|_1 = \int_0^\infty \mu(\{|f(x)| > \lambda\}) d\lambda$ .]

**Solution.** ▶ Define  $g_n, h_n : \mathcal{F} \rightarrow [0, \infty]$  for  $n \in \mathbb{N}$  by

$$g_n(\lambda) = \mu(\{x \in X : |f_n(x)| > \lambda\}), \quad h_n(\lambda) = \mu\left(\left\{x \in X : \max_{1 \leq i \leq n} |f_i(x)| > \lambda\right\}\right).$$

Now, note that, by the monotonicity of  $\mu$ , we have

$$h_n(\lambda) \leq \sum_{i=1}^n g_n(\lambda) \leq n g_1(\lambda).$$

Thus,

$$\frac{h_n(\lambda)}{n} \leq g_1(\lambda).$$

Since  $\|f_1\|_1 = \int_0^\infty g_1(\lambda) d\lambda$ , by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \left[ \max_{1 \leq j \leq n} |f_j| \right] d\mu &= \lim_{n \rightarrow \infty} \int_X \frac{h_n(x)}{n} d\mu \\ &= \int_X \lim_{n \rightarrow \infty} \frac{h_n(x)}{n} d\mu \\ &\leq \int_X \lim_{n \rightarrow \infty} \frac{\mu(X)}{n} \\ &= 0 \end{aligned}$$

as we wanted to show. ◀

**Problem 3.**

- (i) Let  $(X, \mathcal{F}, \mu)$  be a finite measure space. Let  $\{f_n\}$  be a sequence of measurable functions. Prove that  $f_n \rightarrow f$  is measurable if and only if every subsequence  $\{f_{n_k}\}$  contains a further subsequence  $\{f_{n_{k_j}}\}$  that converges a.e. to  $f$ .

- (ii) Let  $(X, \mathcal{F}, \mu)$  be a finite measure space. Let  $F: \mathbb{R} \rightarrow \mathbb{R}$  be continuous and  $f_n \rightarrow f$  in measure. Prove that  $F(f_n) \rightarrow F(f)$  in measure. (You may assume, of course, that  $f_n$ ,  $F$ ,  $F(f_n)$ , and  $F(f)$  are all measurable.)

**Solution.** ► Recall that a sequence of measurable functions  $\{f_n\}$  converge in measure to a limit  $f$  if for every  $\varepsilon > 0$  the limit

$$\lim_{n \rightarrow \infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) = 0.$$

For part (i)  $\implies$  suppose that  $f_n \rightarrow f$  in measure. Then given  $\varepsilon > 0$  and  $\delta > 0$  there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies

$$\mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) < \delta.$$

In particular, given  $\varepsilon = k^{-1}$  and  $\delta = 2^{-k}$ , consider the countable collection of measurable sets  $\{E_k\}_{k=1}^\infty$  given by

$$E_k = \left\{ x \in X : |f(x) - f_{n_k}(x)| \geq \frac{1}{k} \right\},$$

where  $n_k \geq N(k)$  (which depends on our choice of  $k$ ) such that

$$\mu(E_k) < \frac{1}{2^k}.$$

Now, by the Borel–Cantelli lemma, since

$$\sum_{k=1}^{\infty} \mu(E_k) < \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty,$$

for almost every  $x \in X$ , there exists  $N_x \in \mathbb{N}$  such that  $x \notin E_k$  for  $k \geq N_x$ . This means that for  $k \geq N_x$ , we have

$$|f(x) - f_{n_k}(x)| < \frac{1}{k}.$$

Let  $\{f_{n_{k+1}}\}$  be the subsequence of  $\{f_{n_k}\}$ . Then

$$\lim_{k \rightarrow \infty} f_{n_{k+1}} = f$$

as desired.

$\Leftarrow$  On the other hand, suppose that every subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  contains a subsequence  $\{f_{n_{k_j}}\}$  that converges to  $f$ . Seeking a contradiction, suppose that given  $\varepsilon > 0$  there exists a subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  such that

$$M = \mu(\{x \in X : |f(x) - f_{n_k}(x)| \geq \varepsilon\}) > 0.$$

But by assumption there exists a subsequence  $\{f_{n_{k_j}}\}$  of  $\{f_{n_k}\}$  that converges almost everywhere to  $f$ . We claim that this implies that  $f_{n_{k_j}} \rightarrow f$  in measure.

*Proof of claim.* This is adapted from a proof in Royden, Proposition 3, Ch. 5.

First note that  $f$  is measurable since it is the pointwise limit almost everywhere of a sequence of measurable functions. Let  $\varepsilon, \delta > 0$  be given.

By Egorov's theorem, there is a measurable subset  $E \subseteq X$  with  $\mu(X \setminus E) < \delta$  such that  $f_n \rightarrow f$  uniformly on  $E$ . Thus, there is an index  $N$  such that  $n \geq N$  implies

$$|f_n(x) - f(x)| < \varepsilon$$

for all  $x \in E$ . Thus, for  $n \geq N$ ,

$$\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\} \subseteq X \setminus E$$

so

$$\mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) < \varepsilon.$$

Thus, we have

$$\lim_{n \rightarrow \infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) = 0,$$

i.e.,  $f_n \rightarrow f$  in measure. ■

Hence, since  $f_{n_{k_j}} \rightarrow f$  in measure, but  $M > 0$  we have a contradiction.

For (ii) since  $F$  is continuous given  $\varepsilon > 0$  there exist  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|F(x) - F(x')| < \varepsilon$ . By part (i),  $f_n \rightarrow f$  in measure if and only if every subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  contains a subsequence  $\{f_{n_{k_j}}\}$  that converges to  $f$  almost everywhere, i.e., given  $\delta > 0$  there exists an index  $N$  such that  $n_{k_j} \geq N$  implies

$$|f(x) - f_{n_{k_j}}(x)| < \delta$$

for almost every  $x \in X$ . Thus,

$$|F(f(x)) - F(f_{n_{k_j}}(x))| < \varepsilon$$

and we see that for every subsequence  $\{F \circ f_{n_k}\}$  of  $\{F \circ f_n\}$  we can find a subsequence  $\{F \circ f_{n_{k_j}}\}$  that converges almost everywhere to  $F \circ f$ . ◀

**Problem 4.** Let  $(X, \mathcal{F}, \mu)$  be a finite measure space and suppose  $f \in L^1(\mu)$  is nonnegative. Suppose  $1 < p < \infty$  and let  $1 < q < \infty$  be its conjugate exponent, i.e.,  $1/p + 1/q = 1$ . Suppose  $f$  has the property that

$$\int_E f \, d\mu \leq \mu(E)^{1/q}$$

for all measurable sets  $E$ . Prove that  $f \in L^r(\mu)$  for any  $1 \leq r < p$ .

[Hint: Consider  $\{x \in X : 2^n \leq f(x) < 2^{n+1}\}$ , if you like.]

**Solution.** ▶ By previous problems, we know that if  $\mu(X) < \infty$  and  $f \in L^p(X)$ , then  $f \in L^r(X)$  for  $1 \leq r < p$ , so it suffices to show that  $\|f\|_p < \infty$ .

Instead of following the hint, consider the set

$$E_t = \{x \in X : f(x) \geq t\}$$

and let

$$\omega(t) = \mu(E_t),$$

i.e., the distribution function of  $f$ . Then, we have

$$\int_0^\infty \omega(t) dt = \int_X f d\mu.$$

In particular, if we make the substitution  $\alpha = t^{1/p}$ ,  $d\alpha = t^{1/q}/p dt = \alpha^{p/q}/p dt$ , we have

$$\int_X f^r d\mu = \int_0^\infty p\alpha^{-p/q}\omega(\alpha) d\alpha.$$

Now, by Chebyshev's inequality, we have

$$t\omega(t) \leq \int_{E_t} f d\mu \leq \omega(t)^{1/q}$$

so

$$\omega(t) \leq t^{-p}.$$

Thus,

$$\int_X f^r d\mu = \int_0^\infty p\alpha^{-p/q}\omega(\alpha) d\alpha \leq \int_0^\infty p\alpha^{-p-p/q} d\alpha.$$

Since  $p + p/q > 1$ , the integral above is finite. Thus,  $f \in L^p(X)$  and we have  $f \in L^r(X)$  for all  $1 \leq r < p$ . ◀

**Problem 5.** Let  $f$  be a continuous function on  $[-1, 1]$ . Find

$$\lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|) dx.$$

**Solution.** ▶ To find the limit of the integral

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|) dx$$

we first make the following substitutions: Let  $y = nx$ ,  $dy = n dx$ . Then

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|) dx = \frac{1}{n} \int_{-1}^1 f(y/n)(1 - |y|) dy.$$

By the extreme value theorem, since  $f$  is continuous and  $[-1, 1]$  is compact  $f$  is bounded on  $[-1, 1]$  by, say  $M$ . Let  $g(y) = f(y/n)(1 - |y|)$ . Then  $g \in L^1(X)$  since  $\|g\|_1 = 2M$ . Thus, by the Lebesgue dominated convergence theorem, since

$$|f(y/n)(1 - |y|)| \leq M$$

on  $[-1, 1]$  and  $g \in L^1([-1, 1])$  it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|) dx &= \lim_{n \rightarrow \infty} \frac{1}{n} \int_{-1}^1 f(y/n)(1 - |y|) dy \\ &= \int_{-1}^1 \lim_{n \rightarrow \infty} \left[ \frac{f(y/n)(1 - |y|)}{n} \right] dy \\ &= \int_{-1}^1 \lim_{n \rightarrow \infty} \left[ \frac{f(y/n)}{n} - \frac{|y|}{n} \right] dy \\ &= 0. \end{aligned}$$

◀

**Problem 6.** Let  $(X, \mathcal{F}, \mu)$  be a measure space and suppose  $f \in L^p(\mu)$ ,  $1 \leq p < \infty$ . Suppose  $E_n$  is a sequence of measurable sets satisfying  $\mu(E_n) = 1/n$  for all  $n$ . Prove that

$$\lim_{n \rightarrow \infty} \left[ n^{(p-1)/p} \int_{E_n} |f| d\mu \right] = 0.$$

**Solution.** ▶ The result follows immediately by Hölder's inequality. Let  $C = \|f\|_p$ . Since  $f \in L^p(X)$ , then  $f \in L^p(E_n)$  for all  $n \in \mathbb{N}$ . Thus, by Hölder's inequality

$$\begin{aligned} \|f\|_{L^1(E_n)} &\leq \|f\|_{L^p(E_n)} \mu(E_n)^{1/q} \\ &\leq C \mu(E_n)^{1/q} \\ &= C \mu(E)^{p/(p-1)} \\ &= C n^{-p/(p-1)} \\ &= C n^{p/(1-p)}. \end{aligned}$$

Hence, the integral is bounded above by

$$\begin{aligned} 0 \leq n^{(p-1)/p} \int_{E_n} |f| d\mu &\leq C n^{(p-1)/p + p/(1-p)} \\ &= C n^{(2p-1)/(p(1-p))}. \end{aligned}$$

Since  $p > 1$ ,  $1 - p < 0$  and  $2p - 1 > 0$  so the exponent  $(2p - 1)/(p(1 - p)) < 0$ . Thus, as  $n \rightarrow \infty$

$$C n^{(2p-1)/(p(1-p))} \longrightarrow 0.$$

It follows that

$$\lim_{n \rightarrow \infty} \left[ n^{(p-1)/p} \int_{E_n} |f| d\mu \right] = 0.$$

◀

**Problem 7.** Let  $(X, \mathcal{M}, \mu)$  be a measure space and let  $\{g_n\}$  be a sequence of nonnegative measurable functions with the property that  $g_n \in L^1(\mu)$  for every  $n$  and  $g_n \rightarrow g$  in  $L^1(\mu)$ . Let  $\{f_n\}$  be another sequence of nonnegative measurable functions on  $(X, \mathcal{F}, \mu)$ .

(i) If  $f_n \leq g_n$  almost everywhere for every  $n$ , prove that

$$\limsup_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \int_X \limsup_{n \rightarrow \infty} f_n \, d\mu.$$

[Hint: Start by considering a subsequence  $\{f_{n_k}\}$  such that

$$\lim_{n_k \rightarrow \infty} \int_X f_{n_k} \, d\mu = \limsup_{n \rightarrow \infty} \int_X f_n \, d\mu$$

and let  $\{g_{n_{k_j}}\}$  be a subsequence of  $\{g_{n_k}\}$  such that  $g_{n_{k_j}} \rightarrow g$  almost everywhere.]

(ii) If  $f_n \rightarrow f$  almost everywhere and if  $f_n \leq g_n$  almost everywhere for all  $n$ , then  $\|f_n - f\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ .

**Solution.** ► Part (i) is a generalization of what is colloquially known as the reverse Fatou's lemma. Consider the sequence of measurable functions  $\{h_n\}$  where  $h_n = g_n - f_n$ . Note that  $h_n \geq 0$  for all  $x \in X$  since  $g_n \geq f$  for all  $x \in X$ . Thus, by Fatou's lemma, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_X g_n - f_n \, d\mu &\leq \int_X \liminf_{n \rightarrow \infty} g_n - f_n \, d\mu \\ \liminf_{n \rightarrow \infty} \left[ \int_X g_n \, d\mu - \int_X f_n \, d\mu \right] &\leq \int_X \left[ \liminf_{n \rightarrow \infty} g_n + \liminf_{n \rightarrow \infty} (-f_n) \right] \, d\mu \\ \liminf_{n \rightarrow \infty} \int_X g_n \, d\mu + \liminf_{n \rightarrow \infty} \left[ - \int_X f_n \, d\mu \right] &\leq \int_X \liminf_{n \rightarrow \infty} g_n + \int_X \liminf_{n \rightarrow \infty} (-f_n) \, d\mu \\ \liminf_{n \rightarrow \infty} \int_X g_n \, d\mu - \limsup_{n \rightarrow \infty} \int_X f_n \, d\mu &\leq \int_X \liminf_{n \rightarrow \infty} g_n - \int_X \limsup_{n \rightarrow \infty} f_n \, d\mu \\ \int_X g \, d\mu - \limsup_{n \rightarrow \infty} \int_X f_n \, d\mu &\leq \int_X \liminf_{n \rightarrow \infty} g_n - \int_X \limsup_{n \rightarrow \infty} f_n \, d\mu. \end{aligned}$$

Now, let  $\{f_{n_k}\}$  be a subsequence of  $\{f_n\}$  such that

$$\int_X f_{n_k} \, d\mu \longrightarrow \limsup_{n \rightarrow \infty} \int_X f_n \, d\mu.$$

Since  $g_n \rightarrow g$  in  $L^1(X)$ , for every subsequence  $\{g_{n_k}\}$  there exists a subsequence  $\{g_{n_{k_j}}\}$  that converges to  $g$ . ◀

**Problem 8.** Let  $f \in L^1(\mathbb{R})$ . Consider the function

$$F(x) = \int_{\mathbb{R}} \exp(ixt) f(t) \, dt.$$

- (i) Show that  $F \in L^\infty(\mathbb{R})$  and that  $F$  is continuous at every  $x \in \mathbb{R}$ . Moreover, if  $|t|^k f(t) \in L^\infty(\mathbb{R})$  for all  $k \geq 1$ , show that  $F$  is infinitely differentiable, i.e.,  $F \in C^\infty(\mathbb{R})$ .
- (ii) Suppose  $f$  is continuous as well as in  $L^1(\mathbb{R})$ . Show that  $\lim_{|x| \rightarrow \infty} F(x) = 0$ .

[Hint: Using  $\exp(-i\pi) = -1$ , write  $F(x) = (\int_{\mathbb{R}} (\exp(ixt) - \exp(ixt - i\pi)) f(t) \, dt) / 2$ .]

**Solution.** ► For part (a), we claim that

$$|F(x)| \leq \|f\|_1.$$

Since  $|\cdot|: \mathbb{C} \rightarrow \mathbb{R}$  is convex, by Jensen's inequality, we have

$$\begin{aligned} |F(x)| &= \left| \int_X \exp(ixt) f(t) \, dt \right| \\ &\leq \int_X |\exp(ixt) f(t)| \, dt \\ &\leq \int_X |f(t)| \, dt \\ &= \|f\|_1. \end{aligned}$$

Thus,  $\text{ess sup}_{x \in \mathbb{R}} F \leq \|f\|_1$  so  $F \in L^\infty(\mathbb{R})$ . To see that  $F$  is continuous, take let  $\varepsilon > 0$  be given. Consider the sequence of functions  $f_n = f \chi_{\{|t| \leq n\}}$ . Then  $f_n \rightarrow f$  and, by Lebesgue's dominated convergence theorem, there exists an index  $N$  such that for every  $n \geq N$  we have

$$\int_{\mathbb{R}} |f| \, dt - \int_{\mathbb{R}} |f_n| \, dt = \int_{\{|t| > n\}} |f| \, dt < \frac{\varepsilon}{4}.$$

Let  $\delta < \varepsilon/$

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### 3.2 Bañuelos: Summer 2000

**Problem 1.** For any two subsets  $A$  and  $B$  of  $\mathbb{R}$  define  $A + B = \{a + b : a \in A, b \in B\}$ .

- (i) Suppose  $A$  is closed and  $B$  is compact. Prove that  $A + B$  is closed.
- (ii) Give an example that shows that (i) may be false if we only assume that  $A$  and  $B$  are closed.

**Solution.** ► For (i), suppose  $x \in \overline{A + B}$ . Then there exists a sequence  $\{a_n + b_n\}$  that converges to  $x$ . ◀

**Problem 2.** Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is differentiable at every  $x \in [0, 1]$  where by differentiability at 0 and 1 we mean right and left differentiability, respectively. Prove that  $f'$  is continuous if and only if  $f$  is uniformly differentiable. That is, if and only if for all  $\varepsilon > 0$  there is an  $h_0 > 0$  such that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

whenever  $0 \leq x, x+h \leq 1, 0 < |h| < h_0$ .

**Solution.** ► ◀

**Problem 3.** Let  $(X, \mathcal{F}, \mu)$  be a measure space with  $\mu(X) = 1$  and let  $F_1, \dots, F_{17}$  be seventeen measurable subsets of  $X$  with  $\mu(F_j) = 1/4$  for every  $j$ .

- (i) Prove that five of these subsets must have an intersection of positive measure. That is, if  $E_1, \dots, E_k$  denotes the collection of all nonempty intersections of the  $F_j$  taken five at a time ( $k \leq 6188$ ), show that at least one of these sets must have positive measure.
- (ii) Is the conclusion in (i) true if we take sixteen sets instead of seventeen?

**Solution.** ► ◀

**Problem 4.** Let  $f_n : X \rightarrow [0, \infty)$  be a sequence of measurable functions on the measure space  $(X, \mathcal{F}, \mu)$ . Suppose there is a positive constant  $M$  such that the functions  $g_n(x) = f(x)\chi_{\{f_n \leq M\}}(x)$  satisfy  $\|g_n\|_1 \leq A/n^{4/3}$  and for which  $\mu(\{x \in X : f_n(x) > M\}) \leq B/n^{5/4}$ , where  $A$  and  $B$  are positive constants independent of  $n$ . Prove that

$$\sum_{n=1}^{\infty} f_n < \infty$$

almost everywhere.

**Solution.** ► ◀

**Problem 5.** Let  $\{g_n\}$  be a bounded sequence of functions on  $[0, 1]$  which is uniformly Lipschitz. That is there is a constant  $M$  (independent of  $n$ ) such that for all  $n$ ,  $|g_n(x) - g_n(y)| \leq M|x - y|$  for all  $x, y \in [0, 1]$  and  $|g_n(x)| \leq M$  for all  $x \in [0, 1]$ .

(i) Prove that for any  $0 \leq a \leq b \leq 1$ ,

$$\lim_{n \rightarrow \infty} \int_a^b g_n(x) \sin(2n\pi x) \, dx = 0.$$

(ii) Prove that for any  $f \in L^1[0, 1]$ ,

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) g_n(x) \sin(2n\pi x) \, dx = 0.$$

**Solution.** ►

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**Problem 6.** Let  $\{f_n\}$  be a sequence of nonnegative functions in  $L^1[0, 1]$  with the property that  $\int_0^1 f_n(t) \, dt = 1$  and  $\int_{1/n}^1 f_n(t) \, dt \leq 1/n$  for all  $n$ . Define  $h(x) = \sup_n f_n(x)$ . Prove that  $h \notin L^1[0, 1]$ .

**Solution.** ►

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### 3.3 Bañuelos: Winter 2007

**Problem 1.** Let  $f: [0, 1] \rightarrow \mathbb{R}$ .

- (i) Define what it means for  $f$  to be absolutely continuous.
- (ii) Define what it means for  $f$  to be of bounded variation.
- (iii) Let  $V(f; 0, x)$  be the total variation of  $f$  on  $[0, x]$ . Prove that if  $f$  is absolutely continuous on  $[0, 1]$  so is  $V(f; 0, x)$ .

**Solution.** ►

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**Problem 2.**

- (i) Suppose that  $f: [0, 1] \rightarrow \mathbb{R}$  is nondecreasing with  $f(0) = 0$  and  $f(1) = 1$ . For  $a > 0$ , let  $A$  be set of all  $x \in (0, 1)$  for which

$$\limsup_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} > a.$$

Prove that  $m^*(A) < 1/a$ , where  $m^*$  denotes the Lebesgue outer measure.

- (ii) Prove that there is no Lebesgue measurable set  $A$  in  $[0, 1]$  with the property that  $m(A \cap I) = m(I)/4$  for every interval  $I$ .

[Hint: Consider the function  $f(x) = \chi_A(x)$ .]

**Solution.** ►

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**Problem 3.** Let  $\{E_j\}_{j=1}^\infty$  be Lebesgue measurable sets in  $[0, 1]$  and let  $E = \bigcup_{j=1}^\infty E_j$  and suppose there is an  $\varepsilon > 0$  such that  $\sum_{j=1}^\infty m(E_j) \leq m(E) + \varepsilon$ .

- (i) Show that for all measurable sets  $A \subseteq [0, 1]$

$$\sum_{j=1}^\infty m(A \cap E_j) \leq m(A \cap E) + \varepsilon.$$

- (ii) Let  $A$  be the set of all  $x \in [0, 1]$  which are in at least two of  $E'_j$ . Prove that  $m(A) \leq \varepsilon$ .

**Solution.** ►

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**Problem 4.** Let  $(X, \mathcal{F}, \mu)$  be a finite measure space. Let  $f_n: X \rightarrow [0, \infty)$  be a sequence measurable functions and suppose that  $\|f_n\|_p \leq 1$ ,  $1 < p < \infty$ , and that  $f_n \rightarrow f$  almost everywhere. Prove

- (i)  $f \in L^p(\mu)$ .
- (ii)  $\|f_n - f\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ .

**Solution.** ►

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**Problem 5.**

**Solution.** ►



**Problem 6.**

**Solution.** ►



### 3.4 Bañuelos: Winter 2013

#### Problem 1.

(a)

- (i) Define almost uniform convergence on the measure space  $(X, \mathcal{F}, \mu)$ .
- (ii) Let  $f_n$  be a sequence of nonnegative measurable functions converging almost uniformly to the nonnegative function  $f$ . Prove that  $\sqrt{f_n}$  converges almost uniformly to  $\sqrt{f}$ .

(b)

- (i) Suppose  $f_n$  has the property that  $\int_X |f_n| d\mu \rightarrow 0$ .
- (ii) Does it follow that  $f_n \rightarrow 0$  almost everywhere? Justify your answer.
- (iii) Does it follow that  $f_n \rightarrow 0$  almost uniformly? Justify your answer.

**Solution.** ►

**Problem 2.** Let  $(X, \mathcal{F}, \mu)$  be a measure space and let  $1 \leq p \leq \infty$  and  $q$  be its conjugate exponent. Suppose  $f_n \rightarrow f$  in  $L^p$  and  $g_n \rightarrow g$  in  $L^q$ . Prove that  $f_n g_n \rightarrow fg$  in  $L^1$ .

**Solution.** ►

**Problem 3.** Let  $\{a_k\}$  be a sequence of positive numbers converging to infinity. Prove that the following limit exists

$$\lim_{k \rightarrow \infty} \int_0^\infty \frac{\exp(-x) \cos x}{a_k x^2 + (1/a_k)} dx$$

and find it. Make sure to justify all steps.

**Solution.** ►

**Problem 4.** Let  $(X, \mathcal{F}, \mu)$  be  $\sigma$ -finite and  $f$  be measurable such that for all  $\lambda > 0$

$$\mu(\{x \in X : |f(x)| > \lambda\}) \leq \frac{20}{\lambda^p}$$

where  $1 < p < \infty$ . Let  $q$  be the conjugate exponent of  $p$ . Prove that there is a constant  $C$  depending only on  $p$  such that

$$\int_E |f(x)| d\mu \leq C m(E)^{1/q},$$

for all measurable sets  $E$  with  $0 < \mu(E) < \infty$ . (The inequality holds trivially when  $\mu(E) = 0$  or  $\mu(E) = \infty$ .)

[Hint: Recall  $\int_E |f(x)| d\mu = \int_0^\infty \lambda d\lambda$  and “break it” at the right place!]

**Solution.** ►

**Problem 5.** Suppose  $f: [0, 1] \rightarrow \mathbb{R}$  is of bounded variation with  $V(f; 0, 1) = \alpha$ . For any  $\beta > \alpha$ , set

$$A = \left\{ x \in (0, 1) : \limsup_{h \rightarrow 0} \frac{|f(x+h) - f(x)|}{|h|} > \beta \right\}.$$

Prove that for any  $0 < p < 1$ ,  $m(A) \leq (\alpha/\beta)^p$ , where  $m$  denotes the Lebesgue measure.

**Solution.** ►

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**Problem 6.** Let  $f \in L^1(0, 1)$  and for  $x \in (0, 1)$ , define

$$h(x) = \int_x^1 \frac{f(t)}{t} dt.$$

- (i) Prove that  $h$  is continuous on  $(0, 1)$ .
- (ii) Show that

$$\int_0^1 h(t) dt = \int_0^1 f(t) dt.$$

**Solution.** ►

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