MA 519: Homework 3

Carlos Salinas, Max Jeter

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Problem 3.1 (Handout 3, # 3)

n sticks are broken into one short and one long part. The $2n$ parts are then randomly paired up to form n new sticks. Find the probability that

(a) the parts are joined in their original order, i.e., the new sticks are the same as the old sticks;
(b) each long part is paired up with a short part.

Solution. For part (a): let A_i denote the event that at the ith we choose a pair of sticks we get pair that joins up to one of the original sticks. We are after the probability of A the event that all the parts are joined in their original order, i.e., $A = \bigcap_{i=1}^n A_i$. First, we find $P(A_i)$. There are $2n$ ways of choosing a part of a stick and, once we have made that choice, only one way of choosing the original complement to it. The probability of making this choice on our first try is

$$P(A_1) = \frac{2n}{2n(2n-1)} = \frac{1}{2n-1}.$$

Now, assuming that the event of choosing a part of a stick and its original complement is independent from our other choices, we have $2n-2$ choices for our next stick and only one way to chose its complement. Therefore, the probability of making this choice is

$$P(A_2) = \frac{2n-2}{(2n-2)(2n-3)} = \frac{1}{2n-3}.$$

Proceeding in this way we see that at the ith step, the probability of choosing a two broken sticks that make up an original stick is

$$P(A_i) = \frac{1}{2(n-i+1)-1}.$$

Thus, by the hierarchical multiplicative formula the probability that the sticks are paired in their original order is

$$P(A) = \left(\frac{1}{2n-1}\right)\left(\frac{1}{2n-3}\right)\cdots\left(\frac{1}{3}\right)\left(\frac{1}{1}\right).$$

For part (b): let A_i denote the event that at the ith step we pair a long stick with a short stick. Then, to find the probability of $A = \bigcap_{i=1}^n A_i$, we first find the probabilities of the A_i. There are $2n$ ways to choose the first stick and n ways to chose either a long or a short part. Thus, the probability of choosing a long and a short part on the first try is

$$P(A_1) = \frac{2n(n)}{2n(2n-1)} = \frac{n}{2n-1}.$$

As in part (a), at the ith step, the probability of choosing a long and a short stick together is

$$P(A_i) = \frac{n-i+1}{n} 2(n-i+1) - 1.$$

Thus, the probability of event A, that each long and short stick is paired together, is

$$P(A) = \left(\frac{n}{2n-1}\right)\left(\frac{n-1}{2n-3}\right)\cdots\left(\frac{2}{3}\right)\left(\frac{1}{1}\right).$$

■

Problem 3.2 (Handout 3, # 5)

In a town, there are 3 plumbers. On a certain day, 4 residents need a plumber and they each call one plumber at random.

(a) What is the probability that all the calls go to one plumber (not necessarily a specific one)?
(b) What is the expected value of the number of plumbers who get a call?

Solution. Label the plumbers as a, b, and c. Let A be the event that plumber a gets every call, let B be the event that plumber b gets every call, and let C be the event that plumber c gets every call.

For (a): let A be the event that plumber a gets every call, let B be the event that plumber b gets every call, and let C be the event that plumber c gets every call. Then $P(A) = P(B) = P(C) = (1/3)^4$. (The probability of plumber a getting called by the first resident is $1/3$. The residents call independently, the probability that all 4 residents call him is $(1/3)^4$). Because all events are disjoint, $P(A \cup B \cup C) = 1/3^4 + 1/3^4 + 1/3^4$. That is, the probability that some plumber gets every call is $(1/3)^3 = 1/27 \approx 0.0370$.

For (b): let X be the number of plumbers who get called. As above,

$$P(X = 1) = \frac{1}{27}.$$

Similar to the above,

$$P(X = 2) = 3\left(\left(\frac{2}{3}\right)^4 - \frac{2}{3^4}\right)$$

(The probability of a resident calling either plumber a or plumber b is $2/$). The probability of every resident calling one of plumber a or b is $(2/3)^4$. The probability that plumber a was the only plumber called is $(1/3)^4$. The probability that plumber b was the only plumber called is $(1/3)^4$. Thus, the probability that plumber a and plumber b have both been called without calling plumber c is $(2/3)^4 - 2/3^4$. The probability that two plumbers have been called is the sum of the probabilities that exactly plumbers a and b have been called, exactly plumbers b and c have been called, and exactly plumbers a and c have been called; that is, it is as given above.) Lastly,

$$P(X = 3) = 1 - P(X = 1) - P(X = 2)$$

so that

$$\begin{aligned}
E(X) &= P(X = 1) + 2P(X = 2) + 3P(X = 3) \\
&= \frac{1}{27} + 6\left(\left(\frac{2}{3}\right)^4 - \frac{2}{3^4}\right) + 3\left(1 - \frac{1}{27} - 3\left(\left(\frac{2}{3}\right)^4 - \frac{2}{3^4}\right)\right) \\
&= \frac{1}{27} + 6\frac{2^4}{3^4} - \frac{12}{3^4} + 3 - \frac{1}{9} - 9\left(\left(\frac{2}{3}\right)^4 - \frac{2}{3^4}\right) \\
&= \frac{1}{27} + 6\frac{16}{81} - \frac{12}{81} + 3 - \frac{1}{9} - \frac{16}{9} - \frac{2}{9} \\
&= \frac{53}{27} \\
&\approx 1.9630.
\end{aligned}$$

That is, the expected number of plumbers that get called is 53/27, which is slightly less than two. ■

Problem 3.3 (Handout 4, # 7)

(*Polygraphs*). Polygraphs are routinely administered to job applicants for sensitive government positions. Suppose someone actually lying fails the polygraph 90% of the time. But someone telling the truth also fails the polygraph 15% of the time. If a polygraph indicates that an applicant is lying, what is the probability that he is in fact telling the truth? Assume a general prior probability p that the person is telling the truth.

Solution. Let T denote the event that the person is telling the truth. Set $P(T) = p$. Let F denote the event that the person fails the polygraph. Let L denote the event that the person has lied. Then $P(F|L) = 0.9$, and $P(F|T) = 0.15$.

By Bayes' theorem,

$$P(T|F) = \frac{P(F|T)P(T)}{P(F|T)P(T) + P(F|L)P(L)}$$

which reduces to

$$P(T|F) = \frac{0.15p}{0.15p + 0.9 - 0.9p} = \frac{0.15p}{0.9 - 0.75p}$$

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Problem 3.4 (Handout 4, # 8)

In a bolt factory machines A, B, C manufacture, respectively, 25, 35, and 40 per cent of the total. Of their output 5, 4, and 2 per cent are defective bolts. A bolt is drawn at random from the produce and is found defective. What are the probabilities that it was manufactured by machines A, B, C?

Solution. Let D denote the event in which the random bolt we have drawn was defective. Let A, B, and C denote the events in which we have drawn our bolts from machines A, B, and C (respectively). Then

$$\begin{aligned} P(D|A) &= 0.05 \\ P(D|B) &= 0.04 \\ P(D|C) &= 0.02 \\ P(A) &= 0.25 \\ P(B) &= 0.35 \\ P(C) &= 0.4 \end{aligned}$$

Which means that, by Bayes' theorem,

$$\begin{aligned} P(A|D) &= \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} \\ &= \frac{0.05 \cdot 0.25}{0.05 \cdot 0.25 + 0.04 \cdot 0.35 + 0.02 \cdot 0.4} \\ &= \frac{0.0125}{0.0125 + 0.014 + 0.008} \\ &= \frac{0.0125}{0.0345} \\ &\approx 0.3623, \\ P(B|D) &= \frac{0.014}{0.0125 + 0.014 + 0.008} \\ &= \frac{0.014}{0.0345} \\ &\approx 0.4058, \\ P(C|D) &= \frac{0.008}{0.0125 + 0.014 + 0.008} \\ &= \frac{0.008}{0.0345} \\ &\approx 0.2319. \end{aligned}$$

That is, there's about a 36 percent chance that our defective bolt came from A, a 41 percent chance it came from B, and a 23 percent chance it came from C. ■

Problem 3.5 (Handout 4, # 9)

Suppose that 5 men out of 100 and 25 women out of 10 000 are colorblind. A colorblind person is chosen at random. What is the probability of his being male? (Assume males and females to be in equal numbers.)

Solution. Let C be the event that a randomly chosen person is color blind. Let M be the event that the randomly chosen person was male, and let F be the event that the randomly chosen person was female. Then

$$P(C|M) = 0.05$$
$$P(C|F)0.0025$$

So, by Bayes' theorem,

$$\begin{aligned} P(M|C) &= \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|F)P(F)} \\ &= \frac{0.05}{0.05 + 0.0025} \\ &= \frac{0.05}{0.0525} \\ &\approx 0.9524. \end{aligned}$$

That is, there is about a 95 percent chance that the randomly chosen color blind person was male. ■

Problem 3.6 (Handout 4, # 10)

(*Bridge.*) In a Bridge party West has no ace. What probability should he attribute to the event of his partner having

(a) no ace,
(b) two or more aces?

Verify the result by a direct argument.

Solution. For part (a): let A be the event of West's partner having an ace and B be the event that West has no ace. We can compute the probability $P(A|B)$ directly. Since West has no aces, there are $52 - 13 = 39$ cards, 4 of which are aces. Then, the probability that West's partner has no aces is

$$\begin{aligned} P(A|B) &= \binom{39-4}{13} \Big/ \binom{39}{13} \\ &= \frac{1\,476\,337\,800}{8\,122\,425\,444} \\ &\approx 0.1818. \end{aligned}$$

For part (b): let A denote the event that West's partner has two or more aces and let B remain as above. Then, by the inclusion-exclusion principle we need only find the probability that West's partner has exactly one ace. This is,

$$\begin{aligned} \binom{38}{12} \Big/ \binom{39}{13} &= \frac{2\,707\,475\,148}{8\,122\,425\,444} \\ &= \frac{1}{3} \\ &\approx 0.3333. \end{aligned}$$

Then, the probability that West's partner has two or more aces is

$$\begin{aligned} P(A|B) &= (1 - 0.1818) - 0.3333 \\ &\approx 0.4849. \end{aligned}$$

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Problem 3.7 (Handout 4, # 12)

A true-false question will be posed to a couple on a game show. The husband and the wife each has a probability p of picking the correct answer. Should they decide to let one of the answer the question, or decide that they will give the common answer if they agree and toss a coin to pick the answer if they disagree?

Solution. The probability that the couple wins if they let one of them answer the question is p (that is, it is the probability that that person gets it right.)

The probability that the couple wins if they give a common answer if they agree and toss a coin to pick the answer if they disagree is the sum of the probabilities that they agree on the correct answer and the probability that they guess differently and then win on the coin flip.

The probability that they agree on the correct answer is p^2.

The probability that they guess differently is $2(1-p)p$; that is, this is the probability that the husband is wrong and the wife is right plus the probability that the wife is wrong and the husband is right.

The probability that they win, given that they've guessed differently, is $1/2$. So, the probability that they win because of the coin flip after they guess differently is $(1-p)p$.

So, the probability that they win in this fashion is $p^2+(1-p)p = p$. That is, from the perspective of the game, it does not matter which course of action they take. They should do whichever thing is most likely to make them the least mad at each other if they lose. ■

Problem 3.8 (Handout 4, # 13)

An urn containing 5 balls has been filled up by taking 5 balls at random from a second urn which originally had 5 black and 5 white balls. A ball is chosen at random from the first urn and is found to be black. What is the probability of drawing a white ball if a second ball is chosen from among the remaining 4 balls in the first urn?

Solution. We shall use the total probability formula to figure out the probabilities in question. Let B denote the event that that on our second drawing we draw a white ball. First, we must find a suitable partition $A_1, \ldots, A_n$ of Ω such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, for $1 \leq i, j \leq n$, i.e., the events A_i are mutually exclusive. Consider the following partition of Ω,

$$A_i = \big\{ \text{exactly } i \text{ white balls are put into the second urn} \big\}.$$

The events A_i, $1 \leq i \leq 4$, are clearly mutually exclusive (if we have exactly i white balls in the urn, we cannot simultaneously have j white balls in the urn for $i \neq j$). Therefore, to find $P(A)$, we need only find the probabilities $P(B|A_i)$ and $P(A_i)$.

The probabilities of A_i are easy to calculate: there are $\binom{10}{5} = 252$ ways to chose 5 balls from the urn containing the 5 white and 5 black balls, and the number of ways of choosing exactly i black balls are $\binom{5}{5-i}$. Thus,

$$P(A_i) = \frac{\binom{5}{i}}{252} = \binom{5}{i}^2 \Big/ 252.$$

The probabilities of B given A_i are also easy to calculate

$$P(B|A_i) = \frac{i}{4},$$

since we have removed one ball and it was not white and there are i white balls remaining.

Thus, by the total probability formula,

$$\begin{aligned} P(B) &= \left(\frac{1}{4}\right)\left(\frac{25}{252}\right) + \left(\frac{2}{4}\right)\left(\frac{100}{252}\right) \\ &\quad + \left(\frac{3}{4}\right)\left(\frac{100}{252}\right) + \left(\frac{4}{4}\right)\left(\frac{25}{252}\right) \\ &\approx 0.6200. \end{aligned}$$

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Problem 3.9 (Handout 4, # 15)

Events A, B, C have probabilities p_1, p_2, p_3. Given that exactly two of the three events occured, the probability that C occured is greater than 1/2 if and only if ... (write down the necessary and sufficient condition).

Solution. For convenience, let D denote the event that two of A, B, C occured. By Bayes' theorem, we have

$$\begin{aligned} P(C|D) &= \frac{P(D|C)P(C)}{P(D)} \\ &= \frac{P(D|C)P(C)}{P(D|C)P(C) + P(D|\neg C)P(\neg C)} \\ &= \frac{P\big((A \cup B) \smallsetminus (A \cap B)\big)p_3}{P\big((A \cup B) \smallsetminus (A \cap B)\big)p_3 + P\big((A \cup B) \smallsetminus C\big)(1-p_3)}, \end{aligned}$$

but by the inclusion-exclusion principle, $P\big((A \cup B) \smallsetminus (A \cap B)\big) = P(A) + P(B) - 2P(A \cap B)$ so, letting $x = P(A \cap B)$, the above becomes

$$\begin{aligned} &= \frac{(p_1 + p_2 - 2x)p_3}{(p_1 + p_2 - 2x)p_3 + x(1-p_3)} \\ &= \frac{p_1p_3 + p_2p_3 - 2xp_3}{p_1p_3 + p_2p_3 + x - 3xp_3} \\ &\geq \frac{1}{2} \end{aligned}$$

if and only if

$$2(p_1p_3 + p_2p_3 - 2xp_3) \geq p_1p_3 + p_2p_3 + x - 3xp_3,$$

that is,

$$p_1p_3 + p_2p_3 \geq x + xp_3.$$

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Problem 3.10 (Handout 5, # 1)

There are five coins on a desk: 2 are double-headed, 2 are double-tailed, and 1 is a normal coin.
One of the coins is selected at random and tossed. It shows heads.
What is the probability that the other side of this coin is a tail?

Solution. Let A denote the event that the other side of a coin is tail and let B denote the event that after picking a coin at random and tossing it, it comes up heads. By Bayes' theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)};$$

where $P(B|A) = 1/3$ since there are 3 coins whose backside is tail, but only one of which can come up heads; $P(A) = 3/5$ since 3 out of the 5 coins have a tail; and $P(B) = 1/2$ since $1 + 4 = 5$ of the 10 faces are heads. Thus,

$$P(A|B) = \frac{(1/3)(3/5)}{1/2} = \frac{2}{5} = 0.4.$$

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Problem 3.11 (Handout 5, # 2)

(*Genetic testing*). There is a 50-50 chance that the Queen carries the gene for hemophilia. If she does, then each Prince has a 50-50 chance of carrying it. Three Princesses were recently tested and found to be non-carriers. Find the following probabilities:

(a) that the Queen is a carrier;
(b) that the fourth Princess is a carrier.

Solution. For part (a): let A denote the event that the Queen has hæmophilia and let B be the event that three princesses were tested and found to be non-carriers. Then, by Bayes' theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

By assumption, $P(A) = 1/2$. We cannot calculate $P(B)$ directly but, by the total probability formula,

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|\neg A)P(\neg A) \\ &= \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) \\ &= \frac{9}{16}. \end{aligned}$$

Thus,

$$P(A|B) = \frac{(1/8)(1/2)}{9/16} = \frac{1}{9} \approx 0.1111.$$

For part (b): let A denote the event that the fourth princess is a carrier and B remain as above, i.e., the event that three princesses were found to be non-carriers. By Bayes' theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

If the fourth princess is a carrier then the Queen is a carrier. Thus, $P(B) = 9/16$ and $P(B|A) = 1/8$ as above. Now, $P(A) = P(A|C)P(C) + P(A|\neg C)P(\neg C)$ where C denotes the event that the Queen is a carrier. Thus,

$$P(A) = \left(\frac{1}{2}\right)\frac{1}{2} = \frac{1}{4}.$$

Thus,

$$P(A|B) = \frac{(1/8)(1/4)}{9/16} = \frac{1}{18} \approx 0.0556.$$

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Problem 3.12 (Handout 5, # 4)

(*Is Johnny in Jail*). Johnny and you are roommates. You are a terrific student and spend Friday evenings drowned in books. Johnny always goes out on Friday evenings. 40% of the times, he goes out with his girlfriend, and 60% of the times he goes to a bar. If he goes out with his girlfriend, 30% of the times he is just too lazy to come back and spends the night at hers. If he goes to a bar, 40% of the times he gets mad at the person sitting on his right, beats him up, and goes to jail.
On one Saturday morning, you wake up to see Johnny is missing. Where is Johnny?

Solution. Let A denote the event that Johnny is in jail and B denote the event that Johnny is missing. Then, by Bayes' theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

The probability $P(B|A) = 1$ clearly. Let C denote the event that Johnny went to the bar. Then, by the total probability formula,

$$\begin{aligned} P(A) &= P(A|C)P(C) + P(A|\neg C)P(\neg C) \\ &= (0.4)(0.6) + 0 \\ &= 0.24. \end{aligned}$$

Also by the total probability formula,

$$\begin{aligned} P(B) &= P(B|C)P(C) + P(B|\neg C)P(\neg C) \\ &= (0.4)(0.6) + (0.3)(0.4) \\ &= 0.36. \end{aligned}$$

Thus,

$$P(A|B) = \frac{0.24}{0.36} = \frac{2}{3} \approx 0.6667.$$

■