MA544: Qual Problems

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1.1 Exam 1 Prep

Problem 1.1. Let $E \subset \mathbf{R}^n$ be a measurable set, $r \in \mathbf{R}$ and define the set $rE = \{\, r\mathbf{x} \mid \mathbf{x} \in E \,\}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Define a linear map $T\colon \mathbf{R}^n \to \mathbf{R}^n$ by $\mathbf{x} \mapsto r\mathbf{x}$. Using the standard basis for $\mathbf{R}^n$, this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \tag{1}$$

which has determinant $\det T = r^n$. By 3.35, we have $|E| = |T(E)| = r^n |E| = |rE|$. ∎

Problem 1.2. Let $\{E_k\}$, $k \in \mathbf{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k\to\infty} E_k \right| \leq \liminf_{k\to\infty} |E_k|.$$

Proof. If the $\underline{\lim}|E_k| = \infty$ the inequality holds trivially. Hence, we may, without loss of generality, assume that $\underline{\lim}|E_k| < \infty$. By 3.20, the set $\underline{\lim}\, E_k$ is measurable and we have

$$\left| \underline{\lim}_{k\to\infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \tag{2}$$

where $F_k := \bigcap_{n=k}^{\infty} E_n$. Now, note that the collection of sets $F'_k := \bigcup_{\ell=1}^{k} F_\ell$ forms an increasing sequence of measurable sets $F'_k \nearrow F'$, where $F' = \bigcup_{k=1}^{\infty} F_k = \underline{\lim}\, E_k$. Then, by 3.26 (i), we have

$$\lim_{k\to\infty} |F'_k| = |F'| = \left| \underline{\lim}_{k\to\infty} E_k \right|. \tag{3}$$

Hence, it suffices to show that $|F'_k| \leq |E_k|$ for all k, but this follows by monotonicity of the outer measure, 3.3, since $F'_k \subset E_k$. Thus, we have the desired inequality

$$\left| \underline{\lim}_{k\to\infty} E_k \right| \leq \underline{\lim}_{k\to\infty} |E_k|. \tag{4}$$

∎

Problem 1.3. Consider the function

$$F(x) := \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) := \{\, \mathbf{y} \in \mathbf{R}^n \mid |\mathbf{y}| < r \,\}$. Prove that F is monotonic increasing and continuous.

Proof. That F is increasing is immediate from the monotonicity of the outer measure since for $x < x'$ we have $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$ so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \leq |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that F is continuous, we will prove the following lemma

Lemma 1. *For any $x > 0$, $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$.*

Proof of lemma. If $\mathbf{y} \in xB(\mathbf{0}, 1)$ then $\mathbf{y} = x\mathbf{y}'$ for $\mathbf{y}' \in B(\mathbf{0}, 1)$. Thus, $|\mathbf{y}'| = |\mathbf{y}|/x < 1$ so $|\mathbf{y}| < x$ implies that $\mathbf{y} \in B(\mathbf{0}, x)$. Hence, we have the containment $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$.

On the other hand, if $\mathbf{y} \in B(\mathbf{0}, x)$ then $|\mathbf{y}| < x$ so $|\mathbf{y}/x| < 1$. Hence, $\mathbf{y}/x \in B(\mathbf{0}, 1)$ so $x(\mathbf{y}/x) = \mathbf{y} \in B(\mathbf{0}, x)$. Thus, $B(\mathbf{0}, x) \subset xB(\mathbf{0},)$ and equality holds. ♣

In light of Lemma 1 and 3.35, for $x > 0$, we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n |B(\mathbf{0}, 1)|. \tag{5}$$

It is clear that F is continuous on the interval $[0, \infty)$ since F is a polynomial in x. ■

Problem 1.4. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. From the topological definition of continuity, f is continuous at $x \in C$ if and only if for every neighborhood U of $f(x)$, the preimage $f^{-1}(U)$ is a neighborhood of x. Now, ■

Let $x \in C$. Then, by the definition of continuity, for every natural number $n > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f(x) - f(x')| < \frac{1}{2n}. \tag{6}$$

Let $x'', x' \in B(x, \delta)$. Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x)''| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \tag{7}$$

In view of these estimates, define the set

$$A_n := \left\{ x \in \mathbf{R} \;\middle|\; \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \tag{8}$$

Good Lord, that was a long definition! We claim that $C = \bigcap_{n=1}^\infty A_n$ and that A_n is open for all n.

First, let us show that $C = \bigcap_{n=1}^\infty A_n$. Let $x \in C$. Then for every $n > 0$, there exists $\delta > 0$ such that $|x-x'| < \delta$ implies $|f(x)-f(x')| < 1/n$. Thus, $x \in A_n$ for all n so $x \in \bigcap A_n$. On the other hand, if $x \in \bigcap A_n$ for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$.

Fix $\varepsilon > 0$. By the Archimedean principle, there exists $N > 0$ such that $\varepsilon > 1/N$. Then, since $x \in A_N$ it follows that for some $\delta' > 0$, $|x - x'| < \delta'$ implies $|f(x) - f(x')| < 1/N < \varepsilon$. Thus, $x \in C$ and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$.

Lastly, we show that A_n is open. Let $x \in A_n$. Then there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. In particular, this means that $B(x, \delta) \subset A_n$ for any $x' \in B(x, \delta)$ satisfies $|f(x) - f(x')| < 1/n$. Thus, A_n is open and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$ is a G_δ set.

Problem 1.5. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function. Is it true that if the sets $\{\, f = r \,\}$ are measurable for all $r \in \mathbf{R}$, then f is measurable?

Proof. No. Recall that, by definition, or 4.1, f is measurable if and only if $\{\, f > a \,\}$ for all $a \in \mathbf{R}$. ■

Problem 1.6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbf{R}$. Prove that the set $\{\, x \mid \lim_{k\to\infty} f_k(x) \text{ exists} \,\}$ is measurable.

Proof. The idea here should be to rewrite

$$E \coloneqq \left\{ x \;\middle|\; \lim_{k\to\infty} f_k(x) \text{ exists} \right\} \tag{9}$$

as a countable union/intersection of measurable sets. Let $x \in E$. By the Cauchy criterion, for every $N > 0$ there exists a positive integer M such that $m, n \geq M$ implies $|f_n(x) - f_m(x)| < 1/N$. With this in mind, define

$$E_N \coloneqq \left\{ x \;\middle|\; \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \tag{10}$$

Then, like for Problem 1.4, it is not too hard to see that the E_n's are open and that $E = \bigcap_{n=1}^{\infty} E_n$. Thus, E is a G_δ set and therefore measurable. ■

Problem 1.7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, one has $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. Suppose $f\colon [a, b] \to \mathbf{R}$ is absolutely continuous. Then for fixed $\varepsilon = 1$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, we have $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Let $\Gamma \coloneqq \{x_k\}_{k=1}^{N}$ be a partition of $[a, b]$ into closed intervals such that $x_{k+1} - x_k < \delta$, then by absolute continuity we have

$$\begin{aligned} V[f; \Gamma] &= \sum_{k=1}^{N} |f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \tag{11}$$

Thus, $f \in \mathrm{BV}[a, b]$. ■

Problem 1.8. Let f be a continuous function from $[a,b]$ into $\mathbf{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, i.e., $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f \, \mathrm{d}\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a,b) \\ -f(a) & \text{if } c = a \\ f(a) & \text{if } c = b \end{cases}.$$

Proof. ■

2 Exam 1

2.1 Exam 2 Prep

Problem 2.1. Define for $\mathbf{x} \in \mathbf{R}^n$,

$$f(\mathbf{x}) \coloneqq \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B(0, \varepsilon)$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbf{R}^n \smallsetminus B_\varepsilon(\mathbf{0})} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. What does it mean for a measurable function f to be integrable over a set $E \subset \mathbf{R}^n$, i.e., that f belong to $L^1(E)$? It means that

$$\int_E f(\mathbf{x}) \, \mathrm{d}\mathbf{x} < \infty,$$

or equivalently that the integral of the absolute value of f be finite.

Put $E \coloneqq \mathbf{R}^n \smallsetminus B_\varepsilon(\mathbf{0})$ and $E_i \coloneqq \mathbf{R} \smallsetminus B_\varepsilon(0)$ for $i = 1, ..., n$. Now, suppose f is given as in the statement of the problem. It is enough to prove the inequality

$$\int_E f(\mathbf{x}) \, \mathrm{d}\mathbf{x} < \frac{C}{\varepsilon} \tag{12}$$

to show that f belongs to $L^1(E)$. Hence, we proceed in this spirit. First, let us jot down some estimates on the integral. For any $\mathbf{x} \in B_\varepsilon(\mathbf{0})$, $|\mathbf{x}| < \varepsilon$ so the integral

$$\int_{B_\varepsilon(\mathbf{0})} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_{B_\varepsilon(\mathbf{0})} \frac{\mathrm{d}\mathbf{x}}{|\mathbf{x}|^{n+1}} \geq \int_{B_\varepsilon(\mathbf{0})} \frac{\mathrm{d}\mathbf{x}}{\varepsilon^{n+1}} = \frac{\operatorname{Vol} B_\varepsilon(\mathbf{0})}{\varepsilon^{n+1}} \tag{13}$$

for every $\varepsilon > 0$, hence it diverges.

Now, taking the integral of f directly we have

$$\begin{aligned}
\int_E f(\mathbf{x}) \, \mathrm{d}\mathbf{x} &= \int_E \frac{\mathrm{d}\mathbf{x}}{|\mathbf{x}|^{n+1}} \\
&= \int_E \frac{\mathrm{d}\mathbf{x}}{\left(\sqrt{{x_1}^2 + \cdots + {x_n}^2}\right)^{n+1}} \\
&= \int \cdots \int_{E'} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{\left(\sqrt{{x_1}^2 + \cdots + {x_n}^2}\right)^{n+1}} \\
&= \int \cdots \int_{E_1} \left[\int_{E_1} \frac{\mathrm{d}x_1}{\left(\sqrt{{x_1}^2 + \cdots + {x_n}^2}\right)^{n+1}} \right] \mathrm{d}x_2 \cdots \mathrm{d}x_n
\end{aligned}$$

make the substitution, $\tan\theta = x_1/\sqrt{{x_2}^2 + \cdots + {x_n}^2}$ and put $E_\theta \coloneqq (-\pi/2, \arctan(-\varepsilon)) \cup (\arctan\varepsilon, \pi/2)$

$$
\begin{aligned}
&= \int_{E_n} \cdots \int_{E_2} \left[\int_{E_\theta} \frac{\cos^{n+1}\theta}{({x_2}^2 + \cdots + {x_n}^2)^{(n+1)/2}} ({x_2}^2 + \cdots + {x_n}^2)^{1/2} \sec^2\theta \,\mathrm{d}\theta \right] \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \int_{E_n} \cdots \int_{E_2} \left[\int_{E_\theta} \frac{\cos^{n-1}\theta}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} \,\mathrm{d}\theta \right] \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \int_{E_n} \cdots \int_{E_2} \frac{\mathrm{d}x_2 \cdots \mathrm{d}x_n}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1}\theta \,\mathrm{d}\,\mathrm{d}\theta \right]
\end{aligned}
$$

where

$$\int_{E_\theta} \cos^{n-1}\theta \,\mathrm{d}\,\mathrm{d}\theta < \infty.$$

Proceeding in this fashion, we arrive at the desired inequality.

Here is the approach taken by Prof. Danielli: Using spherical coordinates $(x_1, ..., x_n) \mapsto \left(\sqrt{{x_1}^2 + \cdots + {x_n}^2}, \vec{\theta}\right)$ ■

Problem 2.2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbf{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbf{R}^n} f = \lim_{k\to\infty} \int_{\mathbf{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbf{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbf{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. ■

Problem 2.3. Assume that E is a measurable set of $\mathbf{R}^n$, with $\lambda(E) < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} \lambda(\{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}) < \infty.$$

Proof. ■

Problem 2.4. Suppose that E is a measurable subset of $\mathbf{R}^n$, with $\lambda(E) < \infty$. If f and g are measurable functions on E, define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, g) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. ■

Problem 2.5. Define the *gamma function* $\Gamma\colon [0,\infty) \to \mathbf{R}$ by

$$\Gamma(y) \coloneqq \int_0^\infty e^{-u}u^{y-1}\,\mathrm{d}u,$$

and the *beta function* $\beta\colon [0,\infty)\times[0,\infty) \to \mathbf{R}$ by

$$\beta(x,y) \coloneqq \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L([0,\infty))$ for all $y \in [0,\infty)$.

(b) Show that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. ■

Problem 2.6. Let $f \in L^1(\mathbf{R}^n)$ and for $\mathbf{h} \in \mathbf{R}^n$ define $f_{\mathbf{h}}\colon \mathbf{R}^n \to \mathbf{R}$ be $f_{\mathbf{h}}(x) \coloneqq f(\mathbf{x}-\mathbf{h})$. Prove that

$$\lim_{\mathbf{h}\to\mathbf{0}} \int_{\mathbf{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. ■

Problem 2.7. (a) If $f_k, g_k, f, g \in L^1(\mathbf{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbf{R}^n$, $|f_k| \le g_k$ and

$$\int_{\mathbf{R}^n} g_k \to \int_{\mathbf{R}^n} g,$$

prove that

$$\int_{\mathbf{R}^n} f_k \to \int_{\mathbf{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L^1(\mathbf{R}^n)$ and $f_k \to f$ a.e. in $\mathbf{R}^n$, then

$$\int_{\mathbf{R}^n} |f_k - f| \to 0 \qquad \text{as} \qquad k \to \infty$$

if and only if

$$\int_{\mathbf{R}^n} |f_k| \to \int_{\mathbf{R}^n} |f| \qquad \text{as} \qquad k \to \infty.$$

Proof. ■