

# MA557 Homework 10

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### PROBLEM 10.1

Let  $\varphi: R \rightarrow S$  be a homomorphism of rings,  ${}^a\varphi: \operatorname{Spec} S \rightarrow \operatorname{Spec} R$  the induced map of the spectra, and  $\mathfrak{p} \in \operatorname{Spec} R$ . Show that the fiber  $({}^a\varphi)^{-1}(\mathfrak{p})$  is homeomorphic to  $\operatorname{Spec}(S \otimes_R k(\mathfrak{p}))$ .

*Proof.* This is demonstrated (to some extent) by Matsumura, following Theorem 7.2 on p. 47, we shall attempt to supply the missing details here. Recall, from the definition of the pre-image, that

$$({}^a\varphi)^{-1}(\mathfrak{p}) = \{ \mathfrak{q} \in \operatorname{Spec} S \mid \mathfrak{q} \cap R = \mathfrak{p} \}.$$

Define a ring homomorphism  $\psi: S \rightarrow S \otimes k(\mathfrak{p})$  via  $\psi(s) := s \otimes 1$ . We claim that the induced map on the spectra, i.e., the map  ${}^a\psi: \operatorname{Spec}(S \otimes k(\mathfrak{p})) \rightarrow \operatorname{Spec} S$ , is a homeomorphism onto its image and that  $\operatorname{im} {}^a\psi = ({}^a\varphi)^{-1}$ . First we will show the latter, that is,

$$\begin{aligned} \operatorname{im} {}^a\psi &= \{ \mathfrak{q} \in \operatorname{Spec} S \mid \mathfrak{q} = \mathfrak{r} \cap S \text{ for } \mathfrak{r} \in \operatorname{Spec}(S \otimes k(\mathfrak{p})) \} \\ &= \end{aligned}$$

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### PROBLEM 10.2

Let  $R \subset S$  be an integral extension of rings with  $S$  a Noetherian ring, and let  $\mathfrak{p} \in \operatorname{Spec} R$ . Show that there are only finitely many primes in  $S$  lying over  $\mathfrak{p}$ .

*Proof.* This is the same as Exercise 9.3 from Matsumura, where he suggests the following approach: Taking a prime ideal  $\mathfrak{p} \in \operatorname{Spec} R$  and localizing at  $\mathfrak{p}$  we may assume that  $\mathfrak{p}$  is maximal. ■

### PROBLEM 10.3

Let  $\varphi: R \rightarrow S$  be a homeomorphism of rings with  $S$  a Noetherian ring. Show that the following are equivalent:

- (i)  $\varphi$  satisfies going up.
- (ii)  ${}^a\varphi: \operatorname{Spec} S \rightarrow \operatorname{Spec} R$  is a closed map.
- (iii) for every  $\mathfrak{q} \in \operatorname{Spec} S$ , the induced map  $\operatorname{Spec}(S/\mathfrak{q}) \rightarrow \operatorname{Spec}(R/\mathfrak{q} \cap S)$  is surjective.

*Proof.*

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**PROBLEM 10.4**

Let  $R \subset S$  be an integral extension of domains with  $R$  normal,  $K = \text{Quot } R$ ,  $\alpha \in S$ ,  $X^n + a_1X^{n-1} + \cdots + a_n$  the minimal polynomial of  $\alpha$  over  $K$  (recall  $a_i \in R$ ). Show that for any  $R$ -ideal  $I$ ,  $\alpha \in \sqrt{IS}$  if and only if  $a_i \in \sqrt{I}$  for  $1 \leq i \leq n$ .

*Proof.*

■

### PROBLEM 10.5

Let  $k$  be a field and  $R = k[X_1, \dots, X_n]$  a  $k$ -algebra. Show that the following are equivalent:

- (i)  $R$  is a domain with  $\dim R = n - 1$
- (ii)  $R \cong k[X_1, \dots, X_n]/(f)$ , where  $k[X_1, \dots, X_n]$  is a polynomial ring and  $f$  is an irreducible polynomial.

*Proof.*

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