PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Xxxx XXXX, Pages 000–000
S 0002-9939(XX)0000-0

MULTIPLICATIVE STRUCTURE OF KAUFFMAN BRACKET SKEIN MODULE QUANTIZATIONS

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(Communicated by)

Abstract. We describe, for a few small examples, the Kauffman bracket skein algebra of a surface crossed with an interval. If the surface is a punctured torus the result is a quantization of the symmetric algebra in three variables (and an algebra closely related to a cyclic quantization of $U(\mathfrak{so}_3)$. For a torus without boundary we obtain a quantization of "the symmetric homologies" of a torus (equivalently, the coordinate ring of the $SL_2(\mathbb{C})$-character variety of $\mathbb{Z} \oplus \mathbb{Z}$). Presentations are also given for the four punctured sphere and twice punctured torus. We conclude with an investigation of central elements and zero divisors.

1. Introduction

The *Kauffman bracket skein module*, $\mathcal{S}_{2,\infty}(M)$, is defined as follows [4, 6]: Assume that M is an oriented 3-manifold, that $\mathcal{L}_{fr}$ denotes unoriented framed links in M (including the empty link), and that R is any commutative ring with unit. With A invertible in R, let $S_{2,\infty}$ be the submodule of $R\mathcal{L}_{fr}$ generated by all expressions of the form

$$\times - A \asymp - A^{-1} \,)(\quad \text{and} \quad \bigcirc + A^2 + A^{-2}.$$

The diagrams in each relation indicate framed links that can be isotoped to identical embeddings except within the neighborhood shown, where framing is vertical. Set $\mathcal{S}_{2,\infty}(M; R, A) = R\mathcal{L}_{fr}/S_{2,\infty}$. Notation is shortened for two special cases: $\mathcal{S}_{2,\infty}(M) = \mathcal{S}_{2,\infty}(M; \mathbb{Z}[A^{\pm 1}], A)$ and $\mathcal{S}(M) = \mathcal{S}_{2,\infty}(M; \mathbb{Z}, -1)$.

The following standard results about skein modules will be useful later on. Only for (1) and (6) are the proofs more than elementary exercises. These two may be found in [9] and [6].

Proposition 1.1.

1. (Universal coefficients property). Let $r : R \to R'$ be a homomorphism of rings (commutative with 1), making R' into an R module. The identity map on $\mathcal{L}_{fr}$ induces an isomorphism of R' (and R) modules:

$$\mathcal{S}_{2,\infty}(M; R', r(A)) \cong \mathcal{S}_{2,\infty}(M; R, A) \otimes_R R'.$$

1991 *Mathematics Subject Classification.* 57M99.
Supported by an NSF postdoctoral fellowship.
Supported by USAF grant 1-443964-22502 while visiting the Mathematics Department, U.C. Berkeley.

2. An embedding of 3-manifolds $f : M \to N$ induces the homomorphism of skein modules $f_* : \mathcal{S}_{2,\infty}(M; R, A) \to \mathcal{S}_{2,\infty}(N; R, A)$. This leads to a functor from the category of 3-manifolds and embeddings to the category of R modules. If N is obtained from M by adding 2- and 3-handles, then f_* is an epimorphism.
3. If $M = F \times I$ for an oriented surface F and an interval I, then $\mathcal{S}_{2,\infty}(M; R, A)$ is an R algebra with $\emptyset$ as a unit element and $L_1 \cdot L_2$ defined by placing L_1 above L_2. This multiplication depends on the product structure of M, so we use the notation $\mathcal{S}_{2,\infty}(F; R, A)$. An embedding of oriented surfaces $f : F \to F'$ induces the homomorphism of skein algebras $f_* : \mathcal{S}_{2,\infty}(F; R, A) \to \mathcal{S}_{2,\infty}(F'; R, A)$. This leads to a functor from the category of surfaces and embeddings to the category of R algebras.
4. If $A = -1$ then, for any M, $\mathcal{S}_{2,\infty}(M, R, -1)$ is an R algebra: $L_1 \cdot L_2$ is defined to be the disjoint union of links. The algebra depends only on $\pi_1(M)$. In particular, if $f : M \to N$ is a homotopy equivalence then $f_* : \mathcal{S}_{2,\infty}(M, R, -1) \to \mathcal{S}_{2,\infty}(N, R, -1)$ is an isomorphism of algebras.
5. The skein module $\mathcal{S}_{2,\infty}(F; R, A)$ has a basis consisting of links on F without contractible components (but including an empty link).
6. The skein algebra $\mathcal{S}_{2,\infty}($$; R, A)$ is isomorphic to $R[x, y, z]$ where the indeterminates are the boundary curves.

Let $F_{g,n}$ denote an oriented surface of genus g with n boundary components. We will compute $\mathcal{S}_{2,\infty}(F_{g,n}; R, A)$ for $(g, n) \in \{(1, 0), (1, 1), (1, 2), (0, 4)\}$. One can intuit a presentation from existing computations of the $SL_2(\mathbb{C})$-character ring of a free group [3] and its relationship with $\mathcal{S}_{2,\infty}(F_{g,n}; \mathbb{C}, -1)$ [2, 8]. Verification is then a matter of confirming relations and constructing a basis. Often we will check linear independence by specializing to a simpler module:

Lemma 1.2. *Let (u) be a principal ideal in a Noetherian ring R, and let $\pi : R \to R/(u)$ be the natural epimorphism. Suppose $\{v_i\}$ is a subset of a torsion free R module S. If $\{v_i \otimes 1\}$ are linearly independent in the $R/(u)$ module $S \otimes_R R/(u)$, then $\{v_i\}$ are linearly independent in S.*

Proof. Suppose $\sum a_i v_i = 0$ with some $a_j \neq 0$. Since R is Noetherian, there is a maximal k such that u^k divides each a_i. Let $a_i = u^k a_i'$. No torsion in S implies $\sum a_i' v_i = 0$, so $\sum (a_i' + (u))(v_i \otimes 1) = 0$ in $S \otimes_R R/(u)$. Thus each $a_i' \in (u)$, contradicting maximality of k. $\square$

In particular, one sees that a linearly independent set of links in $\mathcal{S}(M)$ is also linearly independent in $\mathcal{S}_{2,\infty}(M)$.

2. Skein algebras of $F_{1,1}$ and $F_{1,0}$

For the rest of this paper let us agree on the shorthand nomenclature *curve* for a simple closed curve (up to isotopy) in a surface F, and for the knot in $F \times I$ created by framing it vertically. Suppose that x_1 and x_2 are curves in $F_{1,1}$ that intersect once. Applying a skein relation to the single crossing of the link $x_1 x_2$ resolves it into

$$x_1 x_2 = A x_3 + A^{-1} z, \tag{2.1}$$

where z and x_3 are the unique curves that meet both x_1 and x_2 once. For a unit $u \in R$ and a and b in an R algebra, let $[a, b]_u$ denote the deformed commutator $uab - u^{-1}ba$. Also, let $\delta = A^2 - A^{-2}$. We use $R\langle \{g\} \mid \{r\} \rangle$ to denote the free R

algebra in non-commuting variables $\{g\}$ modulo the ideal generated by $\{r\}$. (Often the relations r will be written as equations.)

Theorem 2.1. *With curves x_i as above, $\mathcal{S}_{2,\infty}(F_{1,1}; R, A)$ is presented as*

$$R\langle x_1, x_2, x_3 \mid [x_i, x_{i+1}]_A = \delta x_{i+2}\rangle,$$

where $i = 1, 2, 3$ and subscripts are interpreted modulo three.

Proof. The skein algebra is generated by curves in the surface (Proposition 1(5)), which, except for the boundary, may be identified with slopes in $\mathbb{Q} \cup \{\frac{1}{0}\}$. These in turn can be organized as the vertices in a tessellation of the upper half space model of $\mathbb{H}^2$ by ideal triangles—their sides are geodesics connecting slopes of curves that cross once. The product of two curves that meet once resolves, with invertible coefficients, into two other curves that meet each of the original pair in a single point, e.g. Equation (1). Hence, one can express the vertices of any triangle in terms of an adjacent one. It follows that every non-boundary parallel curve is generated by $\{x_i\}$. The boundary, ∂, is obtained from the resolution

$$zx_3 = A^2x_1^2 + A^{-2}x_2^2 - A^2 - A^{-2} + \partial, \tag{2.2}$$

where z is as is Equation (1).

The commutation relations follow easily by resolving the links x_ix_j. Any other relation among x_1, x_2 and x_3 reduces, using the commutators, to an expression in $\{x_1^i x_2^j x_3^k\}$. It suffices, then, to show that this set is a basis. Since $\mathcal{S}_{2,\infty}(F_{1,1}; R, A)$ is free, the universal coefficients property allows us to restrict to $\mathcal{S}_{2,\infty}(F_{1,1})$. By Lemma 1, linear independence may be checked in $\mathcal{S}(F_{1,1})$, where it follows from Proposition 1(6). $\square$

The presentation in Theorem 1 is best described as a *cyclic deformation* of $R[x_1, x_2, x_3]$. There is a cyclic deformation of $U(\mathfrak{so}_3)$ (Zachos terms it the cyclically symmetric Fairle rotation [5, 10]) given by $U_A(\mathfrak{so}_3) = R\langle y_1, y_2, y_3 \mid [y_i, y_{i+1}]_A = y_{i+2}\rangle$. This is related to the skein algebra as follows.

Corollary 2.2. There is a well defined map from $\mathcal{S}_{2,\infty}(F_{1,1}; R, A)$ to $U_A(\mathfrak{so}_3)$ sending x_i to δy_i. It is injective if and only if δ is not a zero divisor and surjective if and only if δ is invertible.

This is particularly interesting if $R = \mathbb{C}$, for it says there are two families of algebras parameterized by A—indistinguishable at generic A but having different fibers at $A = \pm 1$, where one family has $U(\mathfrak{so}_3)$ and the other has $\mathbb{C}[x_1, x_2, x_3]$.

It has also been observed that, generically, the algebra $\mathcal{S}_{2,\infty}(F_{1,1} \times I; R, A)$ cannot be generated by fewer than three elements. It is clear, however, that one of the generators may be eliminated if δ is invertible.

Finally, we obtain a presentation of the skein algebra of a torus by adjoining one relation to the presentation in Theorem 1.

Theorem 2.3. *$\mathcal{S}_{2,\infty}(F_{1,0}; R, A)$ is the quotient of $\mathcal{S}_{2,\infty}(F_{1,1}; R, A)$ by the principle ideal $(A^2x_1^2 + A^{-2}x_2^2 + A^2x_3^2 - Ax_1x_2x_3 - 2A^2 - 2A^{-2})$.*

Proof. Embedding $F_{1,1}$ into $F_{1,0}$ maps $\mathcal{S}_{2,\infty}(F_{1,1}; R, A)$ onto $\mathcal{S}_{2,\infty}(F_{1,0}; R, A)$, forcing $\partial = -(A^2 + A^{-2})$. Any relation that is not a multiple of $\partial + A^2 + A^{-2}$ would imply a non-trivial relation among the standard basis elements in $\mathcal{S}_{2,\infty}(F_{1,0}; R, A)$. Therefore, we need only express ∂ in the generators x_1, x_2, x_3. This is accomplished by eliminating z from Equations (1) and (2). $\square$

3. The Skein Algebra of $F_{0,4}$

Paralleling Section 2, chose distinct, non-boundary parallel curves x_1 and x_2 on $F_{0,4}$ that intersect in two points. Then compute

$$x_1x_2 = A^2x_3 + A^{-2}z + \text{boundary curves,} \tag{3.1}$$

where x_3 and z are the unique curves whose minimal intersection with each of x_1 and x_2 is a pair of points. The similarity with Equation (1) should call to mind the triangular tessellation of $\mathbb{H}^2$ from the previous section. The vertices can be matched to the non-boundary parallel curves on $F_{0,4}$ so that each side of a triangle joins curves that meet in two points. Moreover, Equation (3) holds with the variables replaced by the vertices of any adjacent pair of triangles, provided the product term is the pair of shared vertices.

We can extend R to $\bar{R}$ inside $\mathcal{S}_{2,\infty}(F_{0,4};R,A)$ by adjoining the boundary components of $F_{0,4}$. Since these are central, $\mathcal{S}_{2,\infty}(F_{0,4};R,A)$ is an algebra over the commutative ring $\bar{R}$. Links in the surface with no trivial or boundary parallel components form and $\bar{R}$-basis.

Suppose that x_1 separates the boundary curves a_1 and a_2 from the boundary curves a_3 and a_4. Set $p_1 = a_1a_2 + a_3a_4$. Define p_2 and p_3 similarly. Let $q = a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2$. One may check, by direct resolution or by eliminating z from a resolution of zx_3, that

$$\begin{aligned} A^2x_1x_2x_3 = {} & A^4x_1^2 + A^{-4}x_2^2 + A^4x_3^2 + A^2p_1x_1 + A^{-2}p_2x_2 + A^2p_3x_3 \\ & + q - (A^2 + A^{-2})^2. \end{aligned} \tag{3.2}$$

The similarity with the relation in $\mathcal{S}_{2,\infty}(F_{1,0};R,A)$, along with the $\mathbb{H}^2$ parameterization of curves in $F_{1,0}$ and $F_{0,4}$, suggests that one can pass between their skein algebras by sending A to A^2 while requiring the boundary curves to satisfy[1]

$$p_i = 0 \quad \text{and} \quad q + \delta^2 = 0. \tag{3.3}$$

In any R there is at least one solution given by $a_1 = a_2 = a_3 = -a_4 = A + A^{-1}$. Let J be an ideal in $\mathcal{S}_{2,\infty}(F_{0,4};R,A)$ generated by a solution of Equation (5).

Theorem 3.1. *With notation as above, $\mathcal{S}_{2,\infty}(F_{0,4};R,A)$ is presented as*

$$\bar{R}\langle x_1, x_2, x_3 \mid [x_i, x_{i+1}]_{A^2} = (A^4 - A^{-4})x_{i+2} - \delta p_{i+2}, \text{Equation(4)}\rangle.$$

Proof. The curves x_i generate as in the proof of Theorem 1. Commutators and Equation (4) are a matter of direct computation. The relations imply that the set $\{x_1^ix_2^jx_3^k \mid ijk = 0\}$ spans the module, so we need only show they are linearly independent. As before, it suffices to consider the case $(R,A) = (\mathbb{Z},-1)$. Here, formal identification of variables gives an isomorphism $\mathcal{S}(F_{1,0}) \cong \mathcal{S}(F_{0,4})/J$. One may check that $\{x_1^ix_2^jx_3^k \mid ijk = 0\}$ is a basis for $\mathcal{S}(F_{1,0})$. In particular, the same set of monomials is linearly independent in $\mathcal{S}(F_{0,4})/J$. Finally, Lemma 1 (used four times) proves they are linearly independent in $\mathcal{S}(F_{0,4})$. □

Corollary 3.2. One obtains a presentation of $\mathcal{S}_{2,\infty}(F_{0,4};R,A)$ over R from the one in Theorem 3 by adding generators $a_1, \ldots, a_4$, and by adding commutation relations to make them central.

[1] This was originally motivated by the double branched cover of the torus over the sphere, with four branch points.

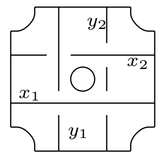

FIGURE 1.

FIGURE 2.

Proof. The set $\{x_1^i x_2^j x_3^k a_1^p a_2^q a_3^r a_4^s \mid pqrs = 0\}$ is seen to be a basis by working over $(R, A) = (\mathbb{Z}, -1)$. □

4. THE SKEIN ALGEBRA OF $F_{1,2}$

This algebra is harder to describe because there are far too many simple closed curves on the surface. However, since $F_{1,2}$ and $F_{0,4}$ are homotopy equivalent, we are able to guess and verify a presentation of $\mathcal{S}_{2,\infty}(F_{1,2}; R, A)$.

Choose curves x_1, x_2, y_1 and y_2 on $F_{1,2}$ as indicated in Figure 1. (Edges are identified top-to-bottom and left-to-right.) Let a denote the boundary curve in the center of the figure and define $\bar{R} = R[a] \subset \mathcal{S}_{2,\infty}(F_{1,2}; R, A)$. Define curves z_1 and z_2 by the resolutions $x_1 y_1 = A z_1 + A^{-1} w_1$, and $y_1 x_2 = A z_2 + A^{-1} w_2$.

There is a homotopy equivalence between $F_{1,2}$ and $F_{0,4}$ given by retracting either surface to the spine in Figure 2 and then identifying it with the spine in the other surface. The induced map turns Equation (4) into a relation in $\mathcal{S}(F_{1,2})$. By experimenting with coefficients we discovered the following relation in $\mathcal{S}_{2,\infty}(F_{0,4}; R, A)$.

$$\begin{aligned}(4.1)\qquad A^2 a z_2 z_1 &= A^2 a^2 + A^{-2} z_2^2 + A^6 z_1^2 + (y_1 y_2 + A^4 x_1 x_2) a \\ &\quad - (A^{-1} x_1 y_2 + A^{-1} x_2 y_1) z_2 - (A x_2 y_2 + A^5 x_1 y_1) z_1 \\ &\quad + x_2 y_1 x_1 y_2 + A^6 x_1^2 + A^2 x_2^2 + A^2 y_1^2 + A^{-2} y_2^2 - A^2 (A^2 + A^{-2})^2\end{aligned}$$

Let $C(t_1, t_2, t_3)$ denote the set of cyclic commutators, $\{[t_i, t_{i+1}]_A = \delta t_{i+2} \mid i = 1, 2, 3\}$, where subscripts are taken modulo three.

Theorem 4.1. *$\mathcal{S}_{2,\infty}(F_{2,1}; R, A)$ is presented*[2] *as*

$$\begin{aligned}&\bar{R}\langle x_i, y_i, z_i \mid [x_1, x_2] = 0, [y_1, y_2] = 0, [z_1, z_2] = \delta(x_2 y_2 - x_1 y_1), \\ &\qquad C(x_1, y_1, z_1), C(x_2, y_2, z_1), C(z_2, y_1, x_2), C(z_2, y_2, x_1), \text{Equation}(6)\rangle.\end{aligned}$$

Proof. We present over $\bar{R}$ only to save the trouble of writing out trivial commutators; in the proof we work R-linearly. Given a handle decomposition of $F_{1,2}$, there is a generating set for $\mathcal{S}_{2,\infty}(F_{1,2}; R, A)$ given in [1]. It is possible to choose the decomposition so that this set is $\{x_1, x_2, y_1, y_2, z_1, w_2, a\}$. Since $w_2 = A y_1 x_2 - A^2 z_2$ we can replace it with z_2.

[2] A slightly different presentation was quoted in the survey article [7]

Relations are checked by brute force. Equation (6) and commutators indicate that $\{a^i z_2^j z_1^k x_2^p y_1^q y_2^r x_1^s \mid pqrs = 0\}$ is a spanning set. Thus we need only show it is a basis in the case $(R, A) = (\mathbb{Z}, -1)$. This follows from the homotopy equivalence in Figure 2 and the proof of Corollary 3. $\square$

5. Central elements and zero divisors

One can prove that $\mathcal{S}_{2,\infty}(F)$ has no zero divisors and that its center is the subalgebra generated by boundary curves: All that is needed is a system for describing the standard basis so that the product of two basic elements is conveniently expressed in this language. In practice, one assigns a complexity to basis elements so that lower order terms can be ignored. In this section we will explicitly lay out these ingredients for $F_{1,0}$, $F_{1,1}$ and $F_{0,4}$.

5.1. **Toral examples.** Links in $F_{1,0}$ (with no trivial components) are denoted by points in $\mathbb{Z} \times \mathbb{Z}$, taking a point and its negative to represent the same link. The link $v = (p, q)$ is $\gcd(p, q)$ copies of the slope p/q curve. The complexity of (p, q) is $p^2 + q^2$. The same conventions apply to $F_{1,1}$, treating the boundary curve as a scalar. The notation $|v \cap w|$ means the minimal geometric intersection number of v and w as links.

Lemma 5.1. *Choose v and w in $\mathbb{Z} \times \mathbb{Z}$. The products vw and wv are expressed in the standard basis using only links in the parallelogram spanned by $\{\pm(v \pm w)\}$. The two distinct corners occur with coefficients $A^{\pm|v \cap w|}$.*

Proof. For $v = (n, 0)$, induct on n. The arbitrary case transforms to this one using an element of $PSL_2(\mathbb{Z})$. $\square$

Theorem 5.2. *If A is not a root of unity and R has no zero divisors, then the center of $\mathcal{S}_{2,\infty}(F_{1,0}; R, A))$ is R and the center of $\mathcal{S}_{2,\infty}(F_{1,1}; R, A))$ is $R[\partial]$.*

Proof. Let $\alpha = \sum r_i v_i$ with each $r_i \neq 0$ in R and each v_i a distinct, non-empty link in $F_{1,0}$. Choose w that is not parallel to any v_i. Suppose u is a maximal complexity link occurring in the resolution of $w\alpha - \alpha w$ into the standard basis. The resolution of each $r_i(wv_i - v_i w)$ lies in a distinct parallelogram, so u must be a vertex that is contained in exactly one of them. It cannot, therefore, cancel with any other terms of the resolution. Since its coefficient is $\pm r_i(A^{|v_i \cap w|} - A^{-|v_i \cap w|})$, α is not central. The same proof works for $F_{1,1}$, treating ∂ as a scalar. $\square$

For zero divisors we will not be able to separate the resolution into distinct parallelograms. Instead we use an elementary fact about vectors in $\mathbb{R}^n$.

Lemma 5.3. *Suppose v_1, v_2, w_1 and w_2 are vectors in $\mathbb{R}^n$ with $v_1 + w_1 = v_2 + w_2$. If $v_1 \neq v_2$ then one of $v_1 + w_2$ or $v_2 + w_1$ is longer than $v_1 + w_1$.*

Proof. Rotate so that $v_1 + w_1$ lies in the first coordinate and assume that each $||v_i + w_j|| \leq ||v_1 + w_1||$. Use both conditions to force the first coordinates of v_1 and v_2 agree, as well as those of w_1 and w_2. Agreement in the other coordinates follows easily. $\square$

Theorem 5.4. *$\mathcal{S}_{2,\infty}(F_{1,0}; R, A)$ and $\mathcal{S}_{2,\infty}(F_{1,1}; R, A)$ have zero divisors only if R does.*

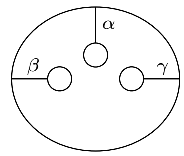

FIGURE 3.

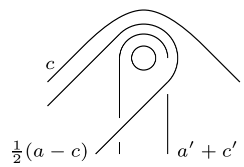

FIGURE 4.

Proof. Choose $\alpha = \sum r_i v_i \in \mathcal{S}_{2,\infty}(F_{1,0}; R, A)$ with each v_i distinct and each $r_i \neq 0$. Similarly, choose $\beta = \sum s_j w_j$. Suppose that u has maximal complexity in the resolution of $\alpha\beta = \sum_{i,j} r_i s_j (v_i w_j)$. Reordering if necessary, we may assume that u is one of $v_1 \pm w_1$. If u appears in the resolution of any $v_i w_j$ other than $v_1 w_1$, then Lemma 3 forces a term of either $v_1 w_j$ or $v_i w_1$ to have greater complexity. Thus $\alpha\beta$ expressed in the standard basis contains the non-zero term $r_1 s_1 A^{\pm|v_1 \cap w_1|} u$. The proof for $F_{1,1}$ is similar, noting that $R[\partial]$ has no zero divisors. $\square$

5.2. **The planar example.** Decorate $F_{0,4}$ with arcs α, β and γ as in Figure 3. A link in $F_{0,4}$ with no trivial components and no component parallel to an inner boundary curve defines a point $(a, b, c) \in \mathbb{N}^3$ as follows. Isotope it to meet $\alpha \cup \beta \cup \gamma$ minimally. The number of arcs joining α to β is a; the number from β to γ is b, and from γ to α, c. A point corresponds to a link if its coordinates are congruent modulo two. There are two useful complexities for (a, b, c): $a+b+c$ and $a^2+b^2+c^2$. Throughout this section we assume the three inner boundary components are scalars in $\mathcal{S}_{2,\infty}(F_{0,4}; R, A)$. Note that the extended scalars have zero divisors only if R does.

Lemma 5.5. *Using either complexity,*

$$(a, b, c)(a', b', c') = A^{(ab'-ba'+bc'-cb'+ca'-ac')/2}(a + a', b + b', c + c') + \mathcal{O},$$

where $\mathcal{O}$ denotes lower complexity terms.

Proof. Assume for now that $a > c$. Position the link so that near α it looks like Figure 4, in which a number next to a strand indicates so many parallel copies. If every crossing in Figure 4 is smoothed in the vertical direction the resulting coefficient is $A^{(a-c)(a'+c')/2}$. If $c < a$ the link should be isotoped to look like a left-to-right reflection of Figure 4, but vertical resolution of every crossing yields the same coefficient. In either case, a resolution including even one horizontal smoothing allows an isotopy that removes two points of intersection with α (possibly by creating a scalar boundary component.)

Similarly position $(a, b, c)(a', b', c')$ near β and γ. The rest of the link can be drawn with no more crossings. Therefore the complete resolution contains a term $A^{(ab'-ba'+bc'-cb'+ca'-ac')/2}(a + a', b + b', c + c')$. Moreover, any other link in the resolution will be some (x, y, z) with $x \leq a$, $y \leq b$, $z \leq c$, and at least one inequality strict. $\square$

Theorem 5.6. *If R has no zero divisors and A is not a root of unity then the center of $\mathcal{S}_{2,\infty}(F_{0,4}; R, A)$ is the subalgebra generated by boundary components.*

Proof. We use the the complexity $a + b + c$ for (a, b, c). Suppose α is a central element. Write it as $\sum r_i(a_i, b_i, c_i) + \beta$, where each (a_i, b_i, c_i) is distinct but all have the same complexity, and β consists of lower complexity terms. Lemma 4 implies $0 = [\alpha, (2, 0, 0)] = \sum r_i(A^{c_i - b_i} - A^{b_i - c_i})(a_i + 2, b_i, c_i) + \mathcal{O}$, so $b_i = c_i$. Similar computations with $(0, 2, 0)$ and $(0, 0, 2)$ show that each (a_i, b_i, c_i) is a power of the outer boundary curve. That means β is central, so we can repeat the same argument for it. Continuing in this fashion, we see that every term of α is a scalar times a power of the outer boundary component. □

Theorem 5.7. *If R has no zero divisors then neither does $\mathcal{S}_{2,\infty}(F_{0,4}; R, A)$.*

Proof. We use the complexity $a^2 + b^2 + c^2$ for (a, b, c). Choose $\alpha = \sum r_i(a_i, b_i, c_i)$ with $r_i \neq 0$ and each (a_i, b_i, c_i) distinct. Similarly choose $\beta = \sum s_j(a'_j, b'_j, c'_j)$. Reordering if necessary, we may assume $(a_1 + a'_1, b_1 + b'_1, c_1 + c'_1)$ is a maximal complexity link in the resolution of $\alpha\beta$. By Lemma 4, it occurs in the resolution of $r_1 s_1(a_1, b_1, c_1)(a'_1, b'_1, c'_1)$ with non-zero coefficient. By Lemma 3 it does not occur in any other $r_i s_j(a_i, b_i, c_i)(a'_j, b'_j, c'_j)$. Hence, $\alpha\beta \neq 0$. □

The central element theorems can fail at roots of unity. When $A^2 = 1$ any $\mathcal{S}_{2,\infty}(F; R, A)$ is commutative. When $A^4 = 1$, $\mathcal{S}_{2,\infty}(F_{0,4}; R, A)$ is still commutative. The toral examples are not, but the square of every curve is central.

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