MA557 Problem Set 1

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Problem 1.1

Show that $\mathrm{rad}(R[X]) = \mathrm{nil}(R[X])$.

Proof. We will first prove the following results (which can be found in Dummit and Foote, §7.3, p. 33):

Lemma 1. *Let $f = a_n X^n + \cdots + a_0 \in R[X]$. Then*

(a) f is nilpotent in $R[X]$ if and only if $a_0, a_1, \ldots, a_n$ are nilpotent elements of R;
(b) f is a unit in $R[X]$ if and only if a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent in R.

Proof of lemma. (a) $\Longleftarrow$: Suppose that $a_0, \ldots, a_n$ are nilpotent. Then $a_0, \ldots, a_n \in \mathrm{nil}(R)$, hence $f \in \mathrm{nil}(R) \subset \mathrm{nil}(R[X])$. $\Longrightarrow$: Conversely, if $f^k = 0$ for some positive integer k, then $(a_n X^n)^k = 0$, so $a_n x^n \in \mathrm{nil}(R[X])$ so $f - a_n X^n \in \mathrm{nil}(R[X])$, in particular $a_n \in \mathrm{nil}(R[X])$. By induction on n, $a_0, \ldots, a_n \in \mathrm{nil}(R[X])$.

(b) $\Longleftarrow$: Suppose a_0 is unit and $a_1, \ldots, a_n$ are nilpotent. Then, by (a), $f - a_0 = a_n X^n + \cdots + a_1 X$ is nilpotent so $f - a_0 \in \mathrm{rad}(R[X])$. By Proposition 1.13, f is a unit. $\Longrightarrow$: On the other hand, if f is a unit, there exist $g = b_m X^m + \cdots + b_0$ in $R[X]$ with $fg = 1$. Now, let $\mathfrak{p}$ be an arbitrary prime ideal. Since f is a unit in $R[X]$, $\bar{f} = \bar{a}_n X^n + \cdots + \bar{a}_0$ is a unit in $R[X]/\mathfrak{p}$. But since $R[X]/\mathfrak{p}$ is an integral domain and $\bar{f}$ is a unit, $\deg \bar{f} = 0$ so $\bar{a}_i = 0$ for every $i \in \{1, \ldots, n\}$. Since $\mathfrak{p}$ was chosen arbitrarily, ♦

By definition $\mathrm{rad}(R)$ is the intersection of every maximal (hence prime) ideal of R so, by Theorem 1.12, $\mathrm{rad}(R) \supset \mathrm{nil}(R)$. To see the reverse containment let $f = a_n X^n + \cdots + a_0$ be in $\mathrm{rad}(R[X])$. By Proposition 1.13, $1 + fg$ is a unit for every $g \in R[X]$. In particular, $1 + fX$ is a unit, so by Lemma 1(b), $a_0, \ldots, a_n$ are nilpotent so $f \in \mathrm{nil}(R[X])$. ■

Problem 1.2

Let I and J be R-ideals. Show that

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

Proof. $\sqrt{IJ} = \sqrt{I \cap J}$: By contradiction, suppose that there exists some prime ideal $\mathfrak{p} \supset IJ$, but $\mathfrak{p} \not\supset I \cap J$. Then there exists some element $x \in I \cap J$ with $x \notin \mathfrak{p}$. However, $x^2 \in IJ$. This contradicts the primality of $\mathfrak{p}$. Hence, if $\mathfrak{p}$ is a prime ideal containing IJ, it must also contain $I \cap J$ so $\sqrt{IJ} = \sqrt{I \cap J}$.

$\sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$: Let $x \in \sqrt{I} \cap \sqrt{J}$. Then $x^n \in I$ for some $n > 0$ and $x^m \in J$ for some $m > 0$. Then $x^{n+m} \in IJ$ so $x \in \sqrt{IJ}$. Hence $\sqrt{IJ} \supset \sqrt{I} \cap \sqrt{J}$. To see the reverse containment note that, by above, since $\sqrt{IJ} = \sqrt{I \cap J}$, then $x \in \sqrt{IJ}$ implies $x^n \in J$ an $x^n \in J$ for some $n > 0$, hence $x \in \sqrt{I} \cap \sqrt{J}$ so $\sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$.

By transitivity of "=", it follows that $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. ∎

Problem 1.3

Let S be a subset of a ring R. Show that the following are equivalent:

(i) $R \setminus S$ is a union of prime ideals.
(ii) $1 \in S$, and for any elements x, y of R, $x \in S$ and $y \in S$ if and only if $xy \in S$.

Proof. (ii) $\implies$ (i): Suppose that S is a saturated multiplicative subset of R. Then $S \supset R^\times$ so every element of $R \setminus S$ is a non-unit. By Corollary 1.5, for every $x \in R \setminus S$, there exists a maximal ideal $\mathfrak{m} \supset (x)$. Hence

$$R \setminus S = \bigcup_{\mathfrak{m} \supset (x)} \mathfrak{m},$$

in particular $R \setminus S$ is a union of prime ideals.

(i) $\implies$ (ii): Suppose that $R \setminus S$ is a union of prime ideals. Then, it is clear that $R^\times \subset S$ so $1 \in S$. Now $x, y \in S$ if and only if $x, y \notin R \setminus S$ if and only if $xy \notin \mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset R \setminus S$. Hence, S is a saturated multiplicative subset of R, i.e., satisfies the conditions given in (ii). ∎

Problem 1.4

Show that the set of all zero divisors in a ring is a union of prime ideals.

Proof. By Problem 1.3, it suffices to show that the complement of the set of all zero-divisors, call it Z, of a ring R is a saturated multiplicative subset. It is clear that $R \setminus Z \supset R^\times$ (since, if $u \in R^\times$, $ub = 0$ if and only if $b = 0$: $\implies$ is easily seen since $u^{-1}ub = 1 \cdot b = 0$ so $b = 0$; the converse is immediate). Now, xy in R is a zero-divisor if and only if x or y are zero-divisors, hence (by taking the negation of this statement) $xy \in R \setminus Z$ implies $x, y \in R \setminus Z$. Thus, $R \setminus Z$ is a saturated multiplicative subset of R. ■

Problem 1.5

Let $\varphi\colon R \to S$ be a surjective homomorphism of rings.

(a) Show that $\varphi(\mathrm{rad}(R)) \subset \mathrm{rad}(S)$, but that equality does not hold in general.
(b) Show that $\varphi(\mathrm{rad}(R)) = \mathrm{rad}(S)$ if R is semilocal.

Proof. (a) The containment $\varphi(\mathrm{rad}(R)) \subset \mathrm{rad}(S)$ follows easily from Proposition 1.13: $x \in \mathrm{rad}(R)$ if and only if $1 + xy$ is a unit for every $y \in R$. Then

$$\begin{aligned}\varphi(1+xy) &= \varphi(1) + \varphi(xy)\\ &= \varphi(1) + \varphi(x)\varphi(y)\\ & 1 + \varphi(x)\varphi(y).\end{aligned}$$

Since φ is surjective, $1 + \varphi(x)s$ is a unit for every $s \in S$ so $\varphi(x) \in \mathrm{rad}(S)$.

To see that equality does not, in general, hold take R and S to be the rings $\mathbf{Z}$ and $\mathbf{Z}/(pq)$ for p and q primes in $\mathbf{Z}$. Then, the canonical projection $\pi\colon \mathbf{Z} \to \mathbf{Z}/(pq)$ is a surjection. Since R is an integral domain, $\mathrm{rad}(R) = 0$, but $\mathrm{rad}(S) = (p) \cap (q) = (pq) \neq \varphi(0) = 0$.

(b) $\varphi(\mathrm{rad}(R)) \subset \mathrm{rad}(S)$ by above. Suppose R is semilocal with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Then, by Corollary 1.15, $\mathrm{rad}(R) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Now, by the Homeomorphism Theorem, $S \cong R/\ker\varphi$ so, by Proposition 1.2, the maximal ideals of S are in 1-1 correspondence with the maximal ideals of R that contain $\ker\varphi$. Thus, it suffices to show that $\mathfrak{m}_i \supset \ker\varphi$ for all i. We will proceed by contradiction. Without loss of generality, suppose that $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$, $0 < m < n$, do not contain $\ker\varphi$. Then, $y \in \prod_{i=1}^m \varphi(\mathfrak{m}_i)$ so

$$y = \sum_i \varphi(x_{i1}) \cdots \varphi(x_{im}) = \varphi\left(\sum_i x_{i1} \cdots x_{im}\right)$$

for $x_{ij} \in \varphi(\mathfrak{m}_j)$. Now, for $i > n$, $\mathfrak{m}_i$ and $\ker\varphi$ are comaximal so $x + x_0 = 1$ for some $x \in \mathfrak{m}_i$, $x_0 \in \ker\varphi$. ∎

Problem 1.6

An element $e \in R$is called *idempotent* if $e^2 = e$. Show that in a local ring, 0 and 1 are the only idempotents.

Proof. Suppose R is a local ring with maximal ideal $\mathfrak{m}$. Suppose, by contradiction, that there exists some $e \in R$, $e \neq 0$ or 1, with $e^2 = e$. Then $e^2 - e = e(e-1) = 0$ so e and $e-1$ are zero-divisors, in particular, e and $e-1$ are non-units and hence contained in the maximal ideal $\mathfrak{m}$. But then $e - (e-1) = 1 \in \mathfrak{m}$. This contradicts the maximality of $\mathfrak{m}$. ■

Problem 1.7

Let I be an R-ideal. Show that I is finitely generated and $I^2 = I$ if and only if $I = Re$ with e idempotent.

Proof. Recall Nakayama's lemma (Theorem 2.2). If M is an finitely generated R-module and I is an R-ideal. Then $M = IM$ if and only if $aM = 0$ for some $a \in 1 + I$. ∎