

## MA 519: Homework 1

Carlos Salinas

August 30, 2016



**PROBLEM 1.1 (HANDOUT 1, # 5 [FELLER VOL. 1])**

A closet contains five pairs of shoes. If four shoes are selected at random, what is the probability that there is at least one complete pair among the four?

**Solution.** ▶ Let  $\Omega$  denote the sample space and  $A$  denote the event that at least 1 complete pair of shoes is among the 4. We can reduce the problem of finding  $P(A)$  into finding the probabilities of the mutually exclusive events

$$A_1 := \{ \text{exactly 1 pair is among the 4} \}$$

and

$$A_2 := \{ \text{exactly 2 pairs are among the 4} \}.$$

since  $A = A_1 \cup A_2$ , and using the additivity of  $P$ ,

$$P(A) = P(A_1) + P(A_2).$$

(To keep the problem short, we will not show that  $A_1 \cap A_2 = \emptyset$  and  $A = A_1 \cup A_2$ .)

First, let us count the number of sample points in  $\Omega$ : since the closet contains 5 pairs of shoes it contains a total of 10 choose out of which we are selecting 4. Hence, the number of sample points is

$$\#\Omega = \binom{10}{4} = \frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{4 \cdot 3 \cdot 2 \cdot 6!} = 10 \cdot 3 \cdot 7 = 210. \quad (1.1)$$

Now we count the sample points in  $A_1$  and  $A_2$ : counting the points in  $A_2$  is immediate since we are not taking into consideration the order in which we select the pair

$$\#A_2 = \binom{5}{2} = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3!}{2 \cdot 3!} = 5 \cdot 2 = 10. \quad (1.2)$$

Counting the points in  $A_1$  is not much harder: first, we observe that there are 5 pairs to choose from and for the remaining two shoes we must choose one shoe (either a right or a left) from the remaining 4 pairs which leaves  $7 - 1 = 6$  other shoes to choose from; i.e. the number of sample points in  $A_1$  is

$$5 \cdot 4 \cdot 6 = 120. \quad (1.3)$$

Taking the results of (1.1), (1.2) and (1.3), the probability that there is at least one complete pair among the four is

$$P(A) = P(A_1) + P(A_2) = \frac{120}{210} + \frac{10}{210} = \frac{130}{210} \approx 0.6190.$$

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**PROBLEM 1.2 (HANDOUT 1, # 7 [FELLER VOL. 1])**

A gene consists of 10 subunits, each of which is normal or mutant. For a particular cell, there are 3 mutant and 7 normal subunits. Before the cell divides into 2 daughter cells, the gene duplicates. The corresponding gene of cell 1 consists of 10 subunits chosen from the 6 mutant and 14 normal units. Cell 2 gets the rest. What is the probability that one of the cells consists of all normal subunits.

**Solution.** ► We shall employ the same strategy as that of Problem 1.1. Let  $A$  denote the event that one of the cells contains all normal units. Then, like Problem 1.1, we can reduce the problem of finding the probability of  $A$  to finding the probability of

$$A_1 := \{ \text{cell 1 consists of all normal subunits} \}$$

and

$$A_2 := \{ \text{cell 1 contains 6 mutant cells} \}$$

and taking their sum.

Now, let us count the number of points in our sample space  $\Omega$ . Assuming the configuration of the subunits in a gene does not matter, we have

$$\#\Omega = \binom{20}{10} = 184756 \quad (1.4)$$

sample points.

Now we count the number of points in  $A_1$  and  $A_2$  these are: for  $A_1$  we choose 10 subunits from among the 14 normal subunits giving us

$$\#A_1 = \binom{14}{10} = 1001 \quad (1.5)$$

sample points. For  $A_2$ , we must choose all 6 mutant subunits leaving 4 choices from among the 14 normal subunits giving us

$$\#A_2 = \binom{14}{4} = 1001. \quad (1.6)$$

Thus, we have

$$P(A) = P(A_1) + P(A_2) = \frac{1001}{184756} + \frac{1001}{184756} \approx 0.01083.$$

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**PROBLEM 1.3 (HANDOUT 1, # 9 [FELLER VOL. 1])**

From a sample of size  $n$ ,  $r$  elements are sampled at random. Find the probability that none of the  $N$  prespecified elements are included in the sample, if sampling is

- (a) with replacement;
- (b) without replacement.

Compute it for  $r = N = 10$ ,  $n = 100$ .

**Solution.** ► For part (a), with replacement, the number of points in the sample space  $\Omega_a$  is given by the expression

$$\#\Omega_a = \binom{n+r-1}{r}.$$

Let  $A_a$  be the event that none of the  $N$  prespecified elements appear (with  $N \leq r$ ). Now to find  $P(A_a)$ , we count the sample points in  $A_a$  these are: there are  $N$  elements to avoid so  $n - N$  elements to choose from with replacement. This gives us

$$\#A_a = \binom{(n-N)+r-1}{r}.$$

Thus, the probability of  $A_a$  happening is

$$P(A_a) = \frac{\binom{(n-N)+r-1}{r}}{\binom{n+r-1}{r}} = \frac{(n-1) \cdots ((n-1)-N+1)}{(n+r-1) \cdots ((n+r-1)-N+1)}. \quad (1.7)$$

For part (b), without replacement, the number of points in the sample space  $\Omega_b$  is given by the expression

$$\#\Omega_b = \binom{n}{r}.$$

Let  $A_b$  be the event that none of the  $N$  prespecified elements appear (with  $N \leq r$ ). Again, to find  $P(A_b)$  we need only count the sample points in  $A_b$ : there are  $N$  elements to avoid so  $n - N$  elements to choose from without replacement. Hence,

$$\#A_b = \binom{n-N}{r}.$$

Thus, the probability of  $A_b$  happening is

$$P(A_b) = \frac{\binom{n-N}{r}}{\binom{n}{r}} = \frac{(n-1) \cdots (n-N)}{(n+r-1) \cdots (n+r-N)}. \quad (1.8)$$

Lastly, we compute, using Eqs. (??) and (??), we compute the probabilities in (a) and (b) with  $r = N = 10$  and  $n = 100$ . These are:

$$P(A_a) = \frac{99 \cdots 90}{109 \cdots 100} \approx 0.3654,$$

and

$$P(A_b) = \frac{90 \cdots 81}{100 \cdots 91} \approx 0.3305.$$

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**PROBLEM 1.4 (HANDOUT 1, # 11 [TEXT 1.3])**

A telephone number consists of ten digits, of which the first digit is one of  $1, 2, \dots, 9$  and the others can be  $0, 1, 2, \dots, 9$ . What is the probability that 0 appears at most once in a telephone number, if all the digits are chosen completely at random?

**Solution.** ► Let  $\Omega$  be the sample space and let  $A$  be the event that 0 appears at most once in a telephone number if all the digits are chosen completely at random. First, let us count the number of elements in the sample space, this is

$$\#\Omega = 9 \cdot 10^9$$

where the first digit is taken from among  $1, 2, \dots, 9$  and the remaining 9 out of  $0, 1, 2, \dots, 9$ . Assuming randomness (*i.e.* that every sample point is equally likely), it suffices to count the sample points in the event. We do this by decomposing  $A$  into the union of mutually exclusive events

$$A_i = \{ \text{telephone numbers with exactly one 0 in the } i\text{-th position} \}.$$

The number of sample points in  $A_i$  is

$$\#A_i = 9 \cdot 9^8$$

since we must choose 8 digits of the number from among  $1, \dots, 9$  digits (with repetition). Thus,

$$P(A) = P(A_1) + \dots + P(A_9) = \frac{9 \cdot 9 \cdot 9^8}{9 \cdot 10^9} = \left(\frac{9}{10}\right)^9 \approx 0.3874.$$

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**PROBLEM 1.5 (HANDOUT 1, # 12 [TEXT 1.6])**

Events  $A$ ,  $B$  and  $C$  are defined in a sample space  $\Omega$ . Find expressions for the following probabilities in terms of  $P(A)$ ,  $P(B)$ ,  $P(C)$ ,  $P(AB)$ ,  $P(AC)$ ,  $P(BC)$  and  $P(ABC)$ ; here  $AB$  means  $A \cap B$ , etc.:

- (a) the probability that exactly two of  $A$ ,  $B$ ,  $C$  occur;
- (b) the probability that exactly one of these events occur;
- (c) the probability that none of these events occur.

**Solution.** ► These are all easy consequences of the inclusion-exclusion formula. For part (a) we have is  $AB + AC + BC - ABC$

$$P(AB + AC + BC) = P(AB) + P(AC) + P(BC) - 2P(ABC).$$

For part (b) we have

$$P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

Lastly, for part (c) we have

$$P(\Omega) - P(A) - P(B) - P(C) + P(AB) + P(AC) + P(BC) - 2P(ABC).$$

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**PROBLEM 1.6 (HANDOUT 1, # 13 [TEXT 1.8])**

Mrs. Jones predicts that if it rains tomorrow it is bound to rain the day after tomorrow. She also thinks that the chance of rain tomorrow is  $1/2$  and that the chance of rain the day after tomorrow is  $1/3$ . Are these subjective probabilities consistent with the axioms and theorems of probability?

**Solution.** ► The probabilities seem to be consistent with the axioms and theorems of probability. If the event  $A$  it will rain tomorrow implies that  $B$  it will rain the day after tomorrow then  $P(B) \leq P(A)$ . ◀



**PROBLEM 1.7 (HANDOUT 1, # 16)**

Consider a particular player, say North, in a Bridge game. Let  $X$  be the number of aces in his hand. Find the distribution of  $X$ .

**Solution.** ►

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**PROBLEM 1.8 (HANDOUT 1, # 20)**

If 100 balls are distributed completely at random into 100 cells, find the expected value of the number of empty cells.

Replace 100 by  $n$  and derive the general expression. Now approximate it as  $n$  tends to  $\infty$ .

**Solution.** ►

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