MA 523: Homework 3

Carlos Salinas

September 19, 2016

Problem 3.1

Consider the initial value problem

$$u_t = \sin u_x; \qquad u(x,0) = \frac{\pi}{4}x.$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

Solution. First, write $\mathbf{u} \coloneqq (u, u_x, u_t) = (u, u_x, \sin u_x)$. By the Cauchy–Kovalevskaya theorem, ■

Problem 3.2

Consider the Cauchy problem for $u(x, y)$

$$\begin{aligned} u_y &= a(x, y, u)u_x + b(x, y, u) \\ u(x, 0) &= 0 \end{aligned}$$

Let a and b be analytic functions of their arguments. Assume that $D^\alpha a(0, 0, 0) \geq 0$ and $D^\alpha b(0, 0, 0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

(a) Show that $D^\beta u(0, 0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $D^\beta u(0, 0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$. (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in β_2)

Solution. ■

Problem 3.3

(Kovalevskaya's example) Show that the line $\{t = 0\}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution of the heat equation in $\mathbf{R} \times \mathbf{R}$, with $u = 1/(1 + x^2)$ on $\{t = 0\}$. (*Hint:* Assume there is an analytic solution, compute its coefficients, and show that the resulting power series diverges in any neighborhood of $(0, 0)$.)

Solution. ■