MA571: Qual Preparation

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Chapter 1

Gepner

1.1 Gepner's homework

Homework 1

1.2 McClure's Quals

McClure: Summer 2006

Exercise 1.1. Let X be a connected space and let $f: X \to Y$ be a function which is continuous and onto. Prove that Y is connected. (This is a theorem in Munkres—prove it from the definitions).

Solution. ► We will show that the only separation of Y is the trivial one.

Let C, D be a separation of Y. Then, C and D are open and $Y = C \cup D$. Since f is continuous $f^{-1}(C)$ and $f^{-1}(D)$ are open in X and

$$X = f^{-1}(Y) = f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D).$$

Since X is connected we have $f^{-1}(C) = \varnothing, f^{-1}(D) = X$ or $f^{-1}(C) = X, f^{-1}(D) = \varnothing$; we may, without loss of generality, assume that $f^{-1}(C) = \varnothing$ and $f^{-1}(D) = X$. But since f is onto, from elementary set theory, we have $f(f^{-1}(C)) = C$ and $f(f^{-1}(D)) = D$. Thus, it must be the case that $C = \varnothing$ and $D = Y$. It follows that the only separation of Y is the trivial separation and so Y is connected. ◄

Exercise 1.2. Let X be the Cartesian product $\prod_{i=1}^{\infty} \mathbb{R}$ with the *box topology* (recall that a basis for this topology consists of all sets of the form $\prod_{i=1}^{\infty} U_i$, where U_i is open in $\mathbb{R}$. Let $f: \mathbb{R} \to X$ be the function which takes t to $(t, t, \ldots)$. Prove that f is not continuous.

Solution. ► We show that for some neighborhood U of $\mathbf{0}$, $f^{-1}(U)$ is not open in $\mathbb{R}$ with the standard topology. Consider the set

$$U = \prod_{n=1}^{\infty} U_n$$

where $U_n = (-1/n, 1/n)$. Since each U_n is open in $\mathbb{R}$, U is open in the box topology. Moreover, $\mathbf{0} \in U$ since $0 \in U_n$ for all n. Therefore, U is a neighborhood of $\mathbf{0}$. Now, consider the preimage of U under f, $f^{-1}(U)$. We claim that $f^{-1}(U) = \{0\}$. We have already seen that $0 \in f^{-1}(U)$ so $\{0\} \subseteq f^{-1}(U)$. It remains to show that 0 is the only element in $f^{-1}(U)$. Let $x \in f^{-1}(U)$. We may, without loss of generality, assume that $x > 0$. Then, by the Archimedean property of $\mathbb{R}$, there exist an natural number N such that $1/N < x$. Then, for every $n \geq N$, $x \notin U_n$. This is a contradiction. Therefore $x \leq 0$. A similar argument shows that $x \geq 0$. Thus, $x = 0$. Since the preimage of $f^{-1}(U) = \{0\}$ is not open in $\mathbb{R}$, it follows that f is not continuous. ◄

Exercise 1.3. Let Y be a topological space. Let X be a set and let $f: X \to Y$ be a function. Give X the topology in which the open sets are the empty set and the sets $f^{-1}(V)$ with V open in Y (you do not have to verify that this is a topology). Let $a \in X$ and let B be a closed set in X not containing a. Prove that $f(a)$ is not in the closure of $f(B)$.

Solution. ► Seeking a contradiction, suppose that $f(a) \in \overline{f(B)}$. Then, for every neighborhood V of $f(a)$, $V \cap f(B)$ is nonempty. Let $y \in V \cap f(B)$. Then, there exist $x \in B$ such that $f(x) = y$. Moreover, $f^{-1}(V)$ is, by definition, open in X and is a neighborhood of a with $x \in f^{-1}(V) \cap B$. Since every neighborhood of a the preimage of some open subset of Y, it follows that $a \in \overline{B}$. This is a contradiction. Therefore, it must be the case that $f(a) \notin \overline{f(B)}$. ◄

For the next two problems, let P be the Cartesian product $\prod_{i=1}^{\infty}\{0, 1\}$ with the usual Cartesian product topology. (Note that $\{0, 1\}$ is a set with two points, it is not an interval.)

Exercise 1.4. Prove that every function from the Cantor set C to P which is one-to-one, onto and continuous is a homeomorphism.

Solution. ► First, note that P is Hausdorff since $\{0, 1\}$ is a subspace of $\mathbb{R}$ which is Hausdorff and the product of Hausdorff spaces equipped with the product topology is again Hausdorff.

Moreover, since $C = \bigcap_{n=1}^{\infty} A_n$ and each A_n is the disjoint union of a finite collection of closed intervals in $[0, 1]$, C is closed since the intersection of a collection of closed sets is again closed. Then C is compact since it is a closed subspace of $[0, 1]$, the latter a compact subspace of $\mathbb{R}$.

It follows from a theorem in chapter 23 that if $f\colon C \to P$ is one-to-one, onto and continuous it is a homeomorphism since C is compact and P is Hausdorff. ◄

Exercise 1.5.

(a) Give a clear and specific description of a function from the Cantor set to P which is one-to-one and onto. You do not have to prove that your function is one-to-one and onto.

(b) Prove that the function you described in part (a) is continuous. (If it isn't continuous, go back and choose a different function that is).

Solution. ► For part (a) we take the following claim will be taken for granted: The set C consists of all term series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

where $a_n \in \{0, 2\}$. Define a map $f\colon C \to P$ by

$$f\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) = (a_1/2, a_2/2, \ldots).$$

For part (b), to see that f is continuous note that $\pi_n \circ f = a_n/2$ is multiplication on $\mathbb{R}$ which is continuous on $\mathbb{R}$, hence it is continuous on the subspace $\{0, 1\}$ of $\mathbb{R}$. By the universal mapping property of the product topology (see the diagram below)

$$\begin{tikzcd} & P \arrow[d, "\pi_n"] \\ C \arrow[ur, "f"] \arrow[r, "\pi_n \circ f"'] & \{0,1\} \end{tikzcd}$$

since every projection of f, $\pi_n \circ f$ is continuous, it follows that f is continuous. ◄

Exercise 1.6. Let X and Y be topological spaces, let $x_0 \in X$, $y_0 \in Y$ and let $f\colon X \to Y$ be a continuous function which takes x_0 to y_0.

Is the following statement true? If f is one-to-one then $f_*\colon \pi(X, x_0) \to \pi_1(Y, y_0)$ is one-to-one. Prove or give a counterexample (and if you give a counterexample, justify it). You may use anything in Munkres's book.

Solution. ► We provide a counter example. Let B^2 be the open ball of radius 1 cenntered at $(0,0)$, S^1 the circle of radius 1 centered at $(0,0)$ and fix $x_0 = (0,1)$ on S^2 Then, the canonical inclusion $\iota\colon S^1 \to B^2$ is injective. However, $\pi_1(S^1, x_0) \cong \mathbb{Z}$ whereas $\pi_1(B^2, x_0) \cong \{0\}$ and no map $f : \mathbb{Z} \to \{0\}$ is injective. ◄

Exercise 1.7. Let S^2 be the 2-sphere, that is, the following subspace of $\mathbb{R}^3$: the set

$$\left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \right\}.$$

Let x_0 be the point $(0, 0, 1)$ of S^2.

Use the Seifert–van Kampen theorem to prove that $\pi_1(S^2, x_0)$ is the trivial group. You may use either of the two versions of the Seifert–van Kampen theorem given in Munkres's book. You will not get credit for any other method.

Solution. ► Let N and S denote the points $(0, 0, 1)$ and $(0, 0, -1)$, respectively. Then the sets

$$U = S^2 \setminus \{N\} \qquad \text{and} \qquad V = S^2 \setminus \{S\}$$

are open in S^2 and both contain the point x_0. Now since $U, V \approx \mathbb{R}$ via the stereographic projections $\pi_N : N \to \mathbb{R}^2$, $\pi_S : S \to \mathbb{R}^2$ we have

$$\pi_1(U, x_0) \cong \pi_1(V, x_0) \cong \pi_1(\mathbb{R}^2, \pi_N(x_0)) \cong \{0\}.$$

Thus, by the Seifert–van Kampen theorem,

$$\pi_1(S^2, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{N'} \cong \{0\}$$

since $\{0\} * \{0\} \cong \{0\}$ and any quotient of the trivial subgroup is trivial. ◄

McClure: Winter 2008

Exercise 1.1. Let X be a topological space with a countable basis. Prove that every open cover of X has a countable subcover.

Solution. ► Suppose $\mathcal{B}$ is a countable basis for X. Let $\{B_n : n \in \mathbb{N}\}$ be an enumeration of $\mathcal{B}$ and let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ an open cover of X. To every B_n in $\mathcal{B}$ associate an element U_n such that $B_n \subseteq U_n$ if such an n exists; otherwise, associate to B_n the empty set $\varnothing$. We claim that the subcollection $\mathcal{U}' = \{U_n : n \in \mathbb{N}\}$ is a countable subcover of $\mathcal{U}$.

The countability of $\mathcal{U}'$ follows from the countability of $\mathcal{B}$. Let $f : \mathcal{B} \to \mathcal{U}$ be the function which assigns B_n to U_n. Then, by definition, $\mathcal{U}' = f(\mathcal{B})$ and, restricting the codomain of f to $\mathcal{U}'$, we have $f' : \mathcal{B} \to \mathcal{U}'$ is surjective. Since $\mathcal{B}$ is countable, there exists a surjection $g : \mathbb{N} \to \mathcal{B}$. Then, the map $f' \circ g : \mathbb{N} \to \mathcal{U}'$ is surjective. It follows that $\mathcal{U}'$ is at most countable.

It remains to show that $\mathcal{U}'$ covers X. Suppose $x \notin U_n$ for any $n \in \mathbb{N}$. Then, since $\mathcal{U}$ is a cover of X, there exists an open set $U_\alpha \in \mathcal{U}$ such that $x \in U_\alpha$. Moreover, since $\mathcal{B}$ is a basis for X, and U_α is open, there exists a basis element $B_m \in \mathcal{B}$ containing x contained in U_α. It follows that there is some nonempty open set U_m in the cover $\mathcal{U}$ containing B_m which implies that $f(B_m) \neq \varnothing$. This contradicts the assumption that no $x \notin U_n$ for any $n \in \mathbb{N}$. Thus, $\mathcal{U}'$ covers X.

It follows that every cover $\mathcal{U}$ of X admits a finite subcover $\mathcal{U}'$ of $\mathcal{U}$. ◄

Exercise 1.2. Let X be a compact space and suppose there is a finite family of continuous functions $f_i : X \to \mathbb{R}$, $i = 1, \dots, n$, with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. Prove that X is homeomorphic to a subspace of $\mathbb{R}^n$.

Solution. ► Consider the map $f : X \to \mathbb{R}^n$ given by

$$f(x) = \big(f_1(x), \dots, f_n(x)\big).$$

By a theorem from Munkres regarding continuous bijective maps between compact and Hausdorff spaces, it suffices to show that the restriction $f' : X \to f(X)$ of the codomain of f to its image $f(X)$ is a bijection.

First, from Munkres, we know that the restriction of the codomain of a continuous function to a subspace is again continuous. Therefore, $f' : X \to f(X)$ is continuous.

Moreover, since $\mathbb{R}^n$ is Hausdorff and $f(X) \subseteq \mathbb{R}^n$ is a subspace, $f(X)$ is Hausdorff.

Lastly, we show that $f' : X \to f(X)$ is bijective. It is clear that f' is surjective since we have restricted the codomain of f to its image. To see that f' is injective let $x, y \in X$ and suppose that $f'(x) = f'(y)$. Then

$$\big(f_1(x), \dots, f(x)\big) = \big(f_1(y), \dots, f(y)\big).$$

This implies that $x = y$ for otherwise, there is a $x \neq y$ in X such that $f_i(x) = f_i(y)$ for all $1 \leq i \leq n$; which contradicts our assumption.

It follows that $f' : X \to f(X)$ is a continuous bijection from a compact to a Hausdorff space and thus, is a homeomorphism. ◄

Exercise 1.3. Let X be any topological space and let Y be a Hausdorff space. Let $f, g : X \to Y$ be continuous functions. Prove that the set $\{\, x \in X : f(x) = g(x) \,\}$ is closed.

Solution. ► Recall that a topological space Y is Hausdorff if and only if the diagonal $\Delta = \{ (y, y) : y \in Y \}$ is closed in $Y \times Y$ with the product topology. Define a map $F \colon X \longrightarrow Y \times Y$ by

$$F(x) = \big(f(x), g(x)\big).$$

Then, by the u.m.p of the product topology, F is continuous because

$$(\pi_1 \circ F)(x) = f(x), \quad (\pi_2 \circ F)(x) = g(x)$$

are continuous; where $\pi_1, \pi_2 \colon Y \times Y \longrightarrow Y$ are the standard projections $(y_1, y_2) \mapsto y_1$ and $(y_1, y_2) \mapsto y_2$, respectively. We claim that

$$\{ x \in X : f(x) = g(x) \} = F^{-1}(\Delta).$$

Let $x \in \{ x \in X : f(x) = g(x) \}$ then $f(x) = g(x)$ so $F(x) \in \Delta$. Thus, $x \in F^{-1}(\Delta)$. On the other hand, if $x \in F^{-1}(\Delta)$ then $F(x) = (f(x), g(x)) = (y, y)$ for some $y \in Y$. Thus, $f(x) = g(x)$ so $x \in \{ x \in X : f(x) = g(x) \}$.

It follows that since

$$\{ x \in X : f(x) = g(x) \} = F^{-1}(\Delta),$$

F is continuous and Δ is closed in $Y \times Y$, the set

$$\{ x \in X : f(x) = g(x) \}$$

is closed in X. ◄

Exercise 1.4. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0s and 1s.

For each $n \geq 1$, let $y_n \in Y$ be the element $(1, \dots, 1, 0, \dots)$, with n 1s at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1s. Prove that the set $\{y_n\}_{n \in \mathbb{N}} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

Solution. ► Let $A = \{y_n\}_{n \in \mathbb{N}} \cup \{y\}$. Assuming Y is given the discrete topology, we must show that for every $x \in A$, for every neighborhood U of x, $U \cap A \neq \varnothing$. By one of Prof. McClure's lemmas, it suffices to show that this holds for basic open sets.

We break the proof into two cases (1) $x \neq y$ and (2) $x = y$.

For (1), let $U = \prod_{n \in \mathbb{N}} U_n$ be a basic neighborhood of x. Since $x \neq y$, the first N entries of x consists of all 1. Therefore, the first N U_n must contain $\{1\}$ and the last U_n, $n \geq N$, must contain $\{0\}$. Since a U is a basic open, $U_n = X$ for infinitely many U_n. Let N' be the smallest integer such that $U_n = X$ for all $n \geq N'$. Then U contains the element $(1, \dots, 1, 0, \dots)$ with N' 1 at the beginning. This is an element of A. Thus, $U \cap A \neq \varnothing$.

For (2), let $U = \prod_{n \in \mathbb{N}} U_n$ be a basic open set containing $x = y$. Let N be the smallest integer such that $U_n = X$ for all $n \geq N$. Then U contains the element $(1, \dots, 1, 0, \dots)$ with N 1 in the beginning. This is an element of A, thus $U \cap A \neq \varnothing$.

Since the latter arguments holds for any basic neighborhood U for any $x \in A$, it follows that A is closed. ◄

Exercise 1.5. Let X be a connected space. Let $\mathcal{U}$ be an open covering of X and let U be a nonempty set in $\mathcal{U}$. Say that a set V in $\mathcal{U}$ is *reachable from U* if there is a sequence

$$U = U_1, U_2, \dots, U_n = V$$

of sets in $\mathcal{U}$ such that $U_i \cap U_{i+1} \neq \varnothing$ for each i from 1 to $n-1$. Prove that every nonempty V in $\mathcal{U}$ is reachable from U.

Solution. ► Fix U in the cover $\mathcal{U}$. Seeking a contradiction, suppose U' is not reachable from U. Let

$$\mathscr{A} = \{ V \in \mathcal{U} : V \text{ is reachable from } U \},$$
$$\mathscr{B} = \{ V \in \mathcal{U} : V \text{ is not reachable from } U \}.$$

Then, $\mathscr{A}$ is nonempty since it contains U and $\mathscr{B}$ is nonempty since it contains U'.

We claim that $C = \bigcup_{V \in \mathscr{A}} V$ and $D = \bigcup_{V \in \mathscr{B}} V$ form a separation of X. First, C and D are open since they are the (arbitrary) union of open sets. Next, we must show that $C \cup D = X$. Let $x \in X$. Then $x \in V$ for some $V \in \mathcal{U}$ since $\mathcal{U}$ covers X. Thus, either V is reachable from U or V is not reachable from U. In the former case, $x \in C$ and in the latter case $x \in D$. Thus, $C \cup D \supseteq X$ and $C \cup D = X$; it is clear that both $C, D \subseteq X$ so the union $C \cup D \subseteq X$.

Thus, C, D forms a separation of X. This contradicts the assumption that X is connected. Therefore, it must be the case that every set V is reachable from U for any $U \in \mathcal{U}$. ◄

Exercise 1.6. Let $p\colon E \to B$ be a covering map. Suppose that points are closed in B. Let $A \subseteq E$ be compact. Prove that for every $b \in B$ the set $A \cap p^{-1}\{b\}$ is finite.

Solution. ► Fix a point $b \in B$. Then the singleton set $\{b\}$ is closed in B and since $p\colon E \to B$ is a covering map, it is continuous so $p^{-1}\{b\}$ is closed in A. By some theorem from Munkres dealing with the subspace topology, $A \cap p^{-1}\{b\}$ is a closed subset of A in the subspace topology. Thus, $A \cap p^{-1}\{b\}$ is compact in A. Now, since p is a covering map, there exists an evenly covered neighborhood V of b, i.e., a neighborhood such that $p^{-1}(V)$ is the disjoint union of open sets $\bigsqcup_{\alpha \in A} U_\alpha$ such that the restriction $p_\alpha = p\restriction_{U_\alpha}$ yields a homeomorphism $U_\alpha \approx_{p_\alpha} V$. Since $p^{-1}\{b\} \subset p^{-1}(V)$, the $\bigsqcup_{\alpha \in A} U_\alpha$ form an open cover of $A \cap p^{-1}\{b\}$. Since $A \cap p^{-1}\{b\}$ is compact, only finitely many of the U_α, say $\{U_1, \dots, U_N\}$, cover $A \cap p^{-1}\{b\}$. Thus, $A \cap p^{-1}\{b\} \subseteq \bigcup_{n=1}^{N} U_n$. Since each $U_n \approx V$, each $U_n \cap p^{-1}\{b\}$ consists of a single point. Thus, the number of points in $A \cap p^{-1}\{b\}$ is less than or equal to N. ◄

Exercise 1.7. Let $p\colon E \to B$ be a covering map. Let Y be a path-connected space and let y_0 be a point in Y. Let $h, k\colon Y \to E$ be continuous functions with $h(y_0) = k(y_0)$ and $p \circ h = p \circ k$. Prove that h and k are the same function.

Solution. ► Since the image of a path-connected space under a continuous map is again path-connected and $h(y_0) = k(y_0)$, $h(Y)$ and $k(Y)$ both lie in the same path component of E. Now, suppose that $h \neq k$. Then there exist $y_1 \in Y$ such that $h(y_1) \neq k(y_1)$. ◄

McClure: Summer 2008

Exercise 1.1. Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Prove that

$$f(\bar{A}) \subseteq \overline{f(A)}$$

for all subsets A of X.

Solution. ► This is another characterization of continuity. Let $x \in \bar{A}$. Then for every neighborhood U of x, $U \cap A \neq \varnothing$. In particular, if U is an neighborhood of $f(x)$, $f^{-1}(V)$ is a neighborhood of x so $f^{-1}(V) \cap A \neq \varnothing$. Thus, $V \cap f(A) \neq \varnothing$ so $f(x) \in \overline{f(A)}$. It follows that $f(\bar{A}) \subseteq \overline{f(A)}$. ◄

Exercise 1.2. Suppose that X is connected and every point of X has a path-connected open neighborhood. Prove that X is path-connected.

Solution. ► Let $x, y \in X$ and suppose there does not exists a path from x to y. Hence, the sets

$$\begin{aligned} C &= \left\{\, y \in X : \text{there exists a path } p\colon I \to X \text{ from } x \text{ to } y \,\right\} \\ D &= \left\{\, y \in X : \text{there does not exists a path } p\colon I \to X \text{ from } x \text{ to } y \,\right\} \end{aligned}$$

are nonempty. We claim that both C and D form a separation of X. First we must show that these sets are open and that their union X. The latter is immediate as every point in X is either connected to x by a path or it is not, i.e., $C = X \setminus D$, and at least one of these sets is nonempty, namely, C. To see that C is open let $y \in C$. Then there exists a path-connected open neighborhood U of y. If $U \cap D \neq \varnothing$ and z is a point in that intersection, there exists a path $q\colon I \to X$ from y to z. But there exists a path $p\colon I \to X$ from x to y so $q * p\colon I \to Xs$ is a path from x to z, which is a contradiction. Thus, $U \subset C$ and C must be open. Consequently, a similar argument tells us that C is closed since if $y \in \bar{C}$, $U \cap C \neq \varnothing$ for every neighborhood of y. In particular, for that special path-connected neighborhood U of y, there is some $z \in C \cap U$. If $y \in D$, then there exist a path $p\colon I \to X$ from x to y, namely, the composite of the path from x to z and from z to y. This yields a contradiction so $y \in C$. Thus, C is closed so $X \setminus D$ is open. It follows that C and D form a separation of X. But X is connected ◄

Exercise 1.3. Let X be a topological space and let $f, g\colon X \to [0,1]$ be a continuous function. Suppose that X is connected and f is onto. Prove that there must be a point $x \in X$ with $f(x) = g(x)$.

Solution. ► Seeking a contradiction, suppose $f(x) \neq g(x)$ for any $x \in X$. Consider the map $h\colon X \to I \times I$ given by $h(x) = \big(f(x), g(x)\big)$. This map is continuous by the u.m.p. of the product topology since each projection $\pi_1 \circ h = f, \pi_2 \circ h = g$ is continuous. Since $f(x) \neq g(x)$, $h(x) \subseteq I \times I \setminus \Delta$ for otherwise, there is a point $x \in X$ such that $h(x) = \big(f(x), g(x)\big) \in \Delta$ which implies $f(x) = g(x)$. Now, we note that the set $(I \times I) \setminus \Delta$ consists of two connected components,

$$\begin{aligned} C &= \left\{\, x \in X : f(x) > g(x) \,\right\} \\ D &= \left\{\, x \in X : f(x) < g(x) \,\right\}. \end{aligned}$$

Thus, since X is connected, $h(X) \subseteq C$ or $h(X) \subseteq D$. In the former case, since f is onto, $f(x) = 0$ for some $x \in X$ so $g(x) < 0$, which contradicts the assumption that the image of X under g lies in I. Similarly, if

$h(X) \subseteq D$, then since f is onto, $f(x) = 1$ for some $x \in X$, which implies $g(x) > 1$. Again, this is nonsense. Therefore, the image of X under h must contain a point in the diagonal Δ, i.e., $f(x) = g(x)$ for some $x \in X$. ◄

Exercise 1.4. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0s and 1s. Let A be the subset of Y consisting of sequences with only a finite number of 1s. Is A closed? Prove or disprove.

Solution. ► We shall disprove this by showing that the sequence consisting of infinitely many 1s is in the closure of this set. It suffices to show that for every basic neighborhood U of $\mathbf{1}$, $U \cap A \neq \emptyset$. Let U be a basic open neighborhood of $\mathbf{1}$. Then $U = \prod_{i \in \mathbb{N}} U_i$ where only finitely many of then $U_i \neq \{0, 1\}$. Let U_N be the last such U_i. Then, for $1 \leq i \leq N$, $U_i \supseteq \{1\}$ otherwise $\mathbf{1}$ is not in the set U. Let $\mathbf{x}$ be the sequence consisting of N 1s and 0s for the remainder of the sequence. Then $\mathbf{x} \in A$ since it contains finitely many 1s and $\mathbf{x} \in U$ since the first N terms of the sequence $\mathbf{x}$ are in one of the U_i and the last terms in the sequence are in $\{0, 1\}$. Since this can be done for any basic neighborhood of $\mathbf{1}$, it follows that $\mathbf{1}$ is contained in the closure of A. In particular, A is not closed. ◄

Exercise 1.5. Prove the Tube Lemma: given topological spaces X and Y with Y compact, a point $x_0 \in X$, and an open set N of $X \times Y$ containing $\{x_0\} \times Y$, prove that there is an open set W of X containing x_0 with $W \times Y \subseteq N$.

Solution. ► Let $f \colon Y \to X \times Y$ be the map $f(y) = (x_0, y)$ for a fixed $x_0 \in X$. This map is continuous since the projections $\pi_1 \circ f = x_0$ and $\pi_2 \circ f = \mathrm{id}_Y$ are continuous. Thus, since the image of a compact space under a continuous map is compact, the set

$$f(Y) = \{x_0\} \times Y$$

is compact. For each point $(x_0, y) \in \{x_0\} \times Y$, take an basic neighborhood $U_y \times V_y \subseteq N$; we can do this since N is open. Then the collection $\left\{U_y \times V_y : y \in Y\right\}$ forms an open cover of $\{x_0\} \times Y$ since the sets $U_y \times V_y$ are open and for any $(x_0, y) \in Y$, $(x_0, y) \in U_y \times V_y$ so

$$\bigcup_{y \in Y} U_y \times V_y \supseteq \{x_0\} \times Y.$$

Now, since $\{x_0\} \times Y$ is compact, there exists a finite subcover, say $\left\{U_1 \times V_1, \ldots, U_i \times V_i\right\}$ of $\left\{U_y \times V_y : y \in Y\right\}$ that covers $\{x_0\} \times Y$. Now, since the projection map is open

$$W = \bigcap_{i=1}^{N} \pi_1\left(U_i \times V_i\right) = \bigcap_{i=1}^{N} U_i$$

is open in X and it contains x_0 since each U_i does. We claim that $W \times Y \subseteq N$. In fact, $W \times Y \subseteq \bigcup_{i=1}^{N} U_i \times V_i$. Let $(x, y) \in W$ and $(x_0, y) \in \{x_0\}, Y$ containing the same y point. Then since $\left\{U_1 \times V_1, \ldots, U_i \times V_i\right\}$ covers $\{x_0\} \times Y$, some $U_i \times V_i$ contains (x_0, y) so $y \in V_i$. But, $x \in U_i$ for all $1 \leq i \leq N$. Thus, $(x, y) \in U_i \times V_i$. Thus,

$$W \times Y \subseteq \bigcup_{i=1}^{N} U_i \times V_i \subseteq N,$$

as desired. ◄

Exercise 1.6. Let X and Y be topological spaces and let $f: X \longrightarrow Y$ be a continuous function. Let $x_0 \in X$ and let $y_0 = f(x_0)$. Find an example in which f is onto but $f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is not onto. Prove that your example really has this property. You may use any fact from Munkres.

Solution. ► Borrowing from complex analysis, let $X = \mathbb{C}$, $Y = \mathbb{C} \setminus \{0\}$ and $x = 1$. Then the map $f: X \longrightarrow Y$ given by $z \mapsto \exp z$ is continuous and onto, but $Y \simeq S^1$ so $\pi_1(Y) \cong \mathbb{Z}$ whereas $\pi_1(X) \cong \{0\}$ since X is contractible. ◄

Exercise 1.7. Prove that every continuous map from S^2 to S^1 is homotopic to a constant map (*Hint*: use covering spaces). You may use any fact from Munkres.

Solution. ► Let $p: \mathbb{R} \longrightarrow S^1$ be the exponential covering map, i.e., $p(t) = \exp(2\pi i t)$ and let $p_0: I \longrightarrow S^1$ be its restriction to the unit interval. Then $[p_0]$ is a generator of $\pi_1(S^1)$ since p_0 is a loop in S^1 whose lift to $\mathbb{R}$ begins at 0 and ends at 1.

Now, fix $x_0 \in S^2$ and let $y_0 = f(x_0)$. Then, since $\pi_1(S^2, x_0) \cong \{0\}$, the induced homomorphism $f_*: \pi_1(S^2, x_0) \longrightarrow \pi_1(S^1, y_0)$ must be the trivial homomorphism. Thus, the map

Unfinished! ◄

McClure: Winter 2011

Exercise 1.1. Let A be a subset of a topological space X and let B be a subset of A. Prove that $\bar{A} \setminus \bar{B} \subseteq \overline{A \setminus B}$.

Solution. ► Let $x \in \bar{A} \setminus \bar{B}$, then x is in the closure of A, but not in the closure of B, i.e., there exists a neighborhood U of x such that $U \cap B = \varnothing$ and $U \cap A \neq \varnothing$. In particular, the latter shows that $A \setminus B \neq \varnothing$ and so $U \cap A \setminus B \neq \varnothing$. This is true for every neighborhood of $x \in \bar{A} \setminus \bar{B}$. Thus, $x \in \overline{A \setminus B}$. ◄

Exercise 1.2. Let G be a topological group (that is, a group with a topology for which the group operations are continuous) and let H be a subgroup of G. Suppose that G is connected, that H is a normal subgroup of G, and that the subspace topology on H is discrete. Prove that $gh = hg$ for every $g \in G$, $h \in G$.

Solution. ► Fix $h \in H$ and consider the map $f_h \colon H \to H$ given by $f_h(g) = ghg^{-1}$. Since H is normal in G, $ghg^{-1} \in H$ and since multiplication is continuous in G and f_h is the composition $g \mapsto gh \mapsto ghg^{-1}$, f_h is continuous. Now, since H has the discrete topology, it is totally disconnected, i.e., singleton sets $\{h\}$ are the only connected components of H and since G is connected, $f_h(G) = \{h'\}$ for some $h' \in H$. Since $ehe^{-1} = h$, $h' = h$. Since we can do this for any $h \in H$, it follows that $gh = hg$ for all $g \in G$, $h \in H$. ◄

Exercise 1.3. Let X be the space with two points and the discrete topology. Let $Y = \prod_{n=1}^{\infty} X$, with the product topology. What are the connected components of Y? Prove that your answer is correct.

Solution. ► Let $X = \{a, b\}$. The connected components of Y are precisely the singleton sets, i.e., the sets consisting of a single sequence as and bs. First, note that if C is a component of Y, then $\pi_n(C)$ is either contained in $\{a\}$ or $\{b\}$ since $\{a\}$ and $\{b\}$ are the components of X. Suppose, without loss of generality, that $\pi_n(C) = \{a\}$. Then C must consist of those sequences of a and b which have a as their nth term. Proceeding in this fashion, we see that C must be a sequence of a, b and not a collection of these. ◄

Exercise 1.4. Let X and Y be topological spaces. Let $x_0 \in X$ and let C be a compact subset of Y. Let N be an open set in $X \times Y$ containing $\{x_0\} \times C$. Prove that there is an open set U containing x_0 and an open set V containing C such that $U \times V \subseteq N$.

Solution. ► Same as the proof of the tube lemma. ◄

Exercise 1.5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is connected. Prove that Y is connected.

Solution. ► Suppose that $X \simeq Y$. Then there exists a continuous maps $f \colon X \to Y$ and $g \colon Y \to X$ such that the composition $g \circ f \colon X \to X$ is homotopic to id_X and $f \circ g \colon Y \to Y$ is homotopic to id_Y. Seeking a contradiction, suppose C, D is a separation of Y. Then $f^{-1}(C)$ and $f^{-1}(D)$ are open and disjoint in X; that these sets are open follows from continuity, that they are disjoint: suppose not, then $x \in f^{-1}(C) \cap f^{-1}(D)$ so $f(x) \in C \cap D$, but C, D is a separation of Y (in particular, $C \cap D = \varnothing$). It follows that either $f^{-1}(C)$ or $f^{-1}(D)$ is empty. Assume the latter. Then $f \circ g(Y) \subseteq V$. But since $f \circ g \simeq \mathrm{id}_Y$ there exists a homotopy $H \colon I \times Y \to Y$ such that $H(y, 0) = f \circ g(y)$ and $H(y, 1) = y$. Fix $y \in Y$, then the map $p_y = H(y, \cdot)$ is a path from y to $f \circ g(y)$. It follows that $y \in V$. Thus, $V = Y$ and $U = \varnothing$ so Y is connected. ◄

Exercise 1.6. Let $p\colon E \to B$ be a covering map. Let $e_0 \in E$ and $b_0 \in B$ with $p(e_0) = b_0$. Let Y be simply connected (in particular, Y is path-connected). Let $y_0 \in Y$. Let $f : Y \to B$ be continuous, with $f(y_0) = b_0$. Prove that the following function $g\colon Y \to E$ is well-defined: Given $y \in Y$, choose a path γ from y_0 to y; let β be the lift of $f \circ \gamma$ to E starting at e_0; now define $g(y) = \beta(1)$. You may use the fact (without proving it) that the lift of a path homotopy is again a path homotopy.

Solution. ► Let γ_1 and γ_2 be two paths beginning at y_0 and ending at y. Now, lift $f \circ \gamma_1$ to β_1 beginning at e_0 and ending at $e_1 = \beta_1(1)$ and lift $f \circ \bar{\gamma}_2$ to β_2 beginning at e_1 and ending at $e_2 = \beta_2(1)$. Then $\beta_1 * \beta_2$ is a lifting of the loop $f \circ (\gamma_1 * \bar{\gamma}_2)$. But by hypothesis

$$\pi_1\big(\pi_1(Y, y_0)\big) \subseteq p_*\big(\pi_1(E, e_0)\big)$$

so $\big[f \circ (\gamma_1 * \bar{\gamma}_2)\big]$ is in $p_*\big(\pi_1(E, e_0)\big)$. By Theorem 54.6, $\beta_1 * \beta_2$ is a loop in E so we must have $e_2 = e_0$. Thus, lifting of $f \circ \gamma_1$ and $f \circ \gamma_2$ must begin and end at the same point so g is well defined. ◄

Exercise 1.7. Let S^2 be the 2-sphere, that is, the following subspace of $\mathbb{R}^3$:

$$\left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \right\}.$$

Let x_0 be the point $(0, 0, 1) \in S^2$.

Use the Seifert–van Kampen theorem to prove that $\pi_1(S^2, x_0)$ is the trivial group. You may use either of the two versions of the Seifert–van Kampen theorem given in Munkres's book. You will not get credit for any other method.

Solution. ► ◄

McClure: Winter 2012

Exercise 1.1. Let X be a topological space. Recall that a subset of X is *dense* if its closure is X. Prove that the intersection of two dense open sets is dense.

Solution. ► Suppose U and V are open dense subsets of X. We will show that $U \cap V$ is dense in X, i.e., $\overline{U \cap V} = X$. To that end, we will show that for any point $x \in X$, for any neighborhood W of x, $W \cap (U \cap V) \neq \varnothing$. Therefore, let $x \in X$. Let W be a neighborhood of x. Then, since U is dense in X, $W \cap U \neq \varnothing$. Let $y \in W \cap U$. Then, since U and V are open, $U \cap V$ is open so $U \cap V$ is a neighborhood of y. Moreover, since V is dense in X, $(W \cap U) \cap V \neq \varnothing$. Now, since intersection is associative, $(W \cap U) \cap V = W \cap (U \cap V) \neq \varnothing$. Thus, $x \in \overline{U \cap V}$ and we have $\overline{U \cap V} = X$ as desired. ◄

Exercise 1.2. Let X be a set with two elements $\{a, b\}$. Give X the *indiscrete* topology. Give $X \times \mathbb{R}$ the product topology. Let $A \subseteq X \times \mathbb{R}$ be $(\{a\} \times [0, 1]) \cup (\{b\} \times (0, 1))$. Prove that A is compact.

You may use the fact that a set is compact if every covering by *basic* open sets has a finite subcovering.

Solution. ► Let $\mathcal{U}$ be an open cover of A by basic open sets. Then each $U \in \mathcal{U}$ is of the form $\{a, b\} \times V$ where V is an open subset of $\mathbb{R}$. Then, the V's, i.e., $\pi_2(U)$ where $\pi_2 \colon X \times \mathbb{R} \to \mathbb{R}$ is an open map by previous work, form open cover of $[0, 1]$ (since $\bigcup_{U \in \mathcal{U}} U \supseteq A$, we must have $\bigcup_{U \in \mathcal{U}} \pi_2(U) \supseteq [0, 1]$. Now, since $[0, 1]$ is compact in X there is a finite collection of the V's, say $\{V_1, \ldots, V_n\}$, that cover $[0, 1]$. Call U_i the element of $\mathcal{U}$ such that $\pi_2(U_i) = V_i$. Then the U_i's form a finite subcover of A. Thus, A is compact. ◄

Exercise 1.3. Let B^2 be the disk

$$\left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \right\}.$$

Let S^1 be the circle

$$\left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \right\}.$$

Prove that there is an equivalence relation $\sim$ such that B^2 is homeomorphic to $(S^1 \times [0, 1])/\sim$. As port of your solution explain how you are using one or more properties of the quotient topology.

Solution. ► Such an equivalence relation is called the cone of S^1. We define it as follows, let $(x, y, z), (x', y', z') \in S^1 \times [0, 1]$ then we say $(x, y, z) \sim (x', y', z')$ if and only if $(x, y) = (x', y')$ or $z = z' = 0$. We shall take it on faith that $\sim$ is in fact an equivalence relation (we may return to this if time permits).

By the UMP of the quotient space, we need to find a continuous surjection $f \colon S^1 \times [0, 1] \to B^2$ that preserves the equivalence relation $\sim$. So consider the map $f(x, y, z) = (zx, zy)$. This map is continuous by Theorem 18.4 since $\pi_1 \circ f(x, y, z) = zx$ is multiplication on $\mathbb{R}$ and similarly for $\pi_2 \circ f(x, y, z)$. Moreover, this map preserves the equivalence relation: let $(x, y, z) \sim (x', y', z')$ then $(x, y, z) = (x', y', z')$ in which case

$$f(x, y, z) = (zx, zy) = (z'x', z'y') = f(x', y', z')$$

or $z = z' = 0$ so

$$f(x, y, 0) = (0 \cdot x, 0 \cdot y) = (0, 0) = (0 \cdot x', 0 \cdot y') = f(x', y', 0).$$

In either case, we have $f(x, y, z) = f(x', y', z')$. Thus, by the UMP of the quotient space, the induced map $f' \colon (S^1 \times [0, 1])/\sim \to B^2$ is continuous.

Now, since $S^1 \times [0,1]$ is closed and bounded, by Heine–Borel, $S^1 \times [0,1]$ is a compact subset of $\mathbb{R}^3$. Therefore, $(S^1 \times [0,1])/\sim$ is compact. Since $B^2 \subseteq \mathbb{R}^2$ is Hausdorff, it suffices to show, by Theorem 26.6, that f is bijective.

It is eassy to see that f is surjective since for any point $(x, y) \neq (0,0)$ in B^2, $\sqrt{x^2+y^2} \leq 1$ so letting $z = \sqrt{x^2+y^2}$, $x' = x/\sqrt{x^2+y^2}$, and $y' = y/\sqrt{x^2+y^2}$ we have

$$f(x', y', z) = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) = (x, y).$$

And, trivially, if $(x, y) = 0$, we have $\varphi(x, y, 0) = 0$ for any $(x, y) \in S^1$.

To see that it is injective, merely note that, by the definition of f, $f(x, y, z) = f(x', y', z')$ if and only if $(x, y, z) = (x', y', z')$ or $z = z' = 0$ which precisely means that $(x, y, z) \sim (x', y', z')$. Thus, f is injective.

It follows that $(S^1 \times [0,1])/\sim \approx B^2$. ◄

Exercise 1.4. Let X be a set with 2 elements $\{a, b\}$. Give X the *discrete* topology. Let Y be any topological space. Recall that $\mathscr{C}(X, Y)$ denotes the set of continuous functions from X to Y, with the compact-open topology. Prove that $\mathscr{C}(X, Y)$ is homeomorphic to $Y \times Y$ (with the product topology).

Solution. ► Consider the map $F\colon \mathscr{C}(X, Y) \to Y \times Y$ given by $F(f) = (f(a), f(b))$. This map is continuous by Theorem 18.4, since $\pi_1(F)$ and $\pi_2(F)$ are, respectively, the evaluation of f at a and the evaluation of f and b, both of which are continuous because under the compact-open topology. This map is clearly surjective since for any $(y_1, y_2) \in Y \times Y$ we may define the function $f(a) = y_1$ and $f(b) = y_2$ which is continuous sinced X has the discrete topology. Moreover, F is injective since if $(f(a), f(b)) = (g(a), g(b))$ then $f(x) = g(x)$ for all $x \in X$ hence, $f = g$. Therefore, to show that F is a homeomorphism, it suffices to show that F is an open map.

Now it suffices to find a continuous inverse. For any $(y_1, y_2) \in Y \times Y$, define the map $g\colon Y \times Y \to \mathscr{C}(X, Y)$.

$$g(y_1, y_2) = f(y) = \begin{cases} a & \text{if } y = y_1 \\ b & \text{if } y = y_2. \end{cases}$$

◄

Exercise 1.5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is path-connected. Prove that Y is path-connected.

Solution. ► ◄

Exercise 1.6. Suppose that X is a wedge of two circles: that is, X is a Hausdorff space which is a union of two subspaces A_1 and A_2 such that A_1 and A_2 are each homeomorphic to S^1 and $A_1 \cap A_2$ is a single point p.

Use the Seifert–van Kampen theorem to calculate $\pi_1(X, p)$. You should state what deformation retractions you are using, but you do not have to give formulas for them.

Solution. ► ◄

Exercise 1.7. Let $p\colon E \to B$ be a covering map. Let A be a connected space and let $a \in A$. Prove that if two continuous functions $\alpha, \beta\colon A \to E$ have a property that $\alpha(a) = \beta(a)$ and $p \circ \alpha = p \circ \beta$ then $\alpha = \beta$.

For partial credit, you may assume that p is the standard covering map from $\mathbb{R}$ to S^1.

Solution. ► ◄

Here's an extra problem I felt like doing since I thought it might be on the exam:

Problem.

Theorem (Munkres, Theorem 18.4)**.** *Let $f\colon A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Then f is continuous if and only if $f_1\colon A \to X$ and $f_2\colon A \to Y$ are continuous.*

Solution. ► Let $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ be projections onto the 1st and 2nd factors, respectively. These maps are continuous and open by previous work. Now, for every $a \in A$ we have

$$\pi_1(f(a)) = f_1(a) \qquad \text{and} \qquad \pi_2(f(a)) = f_2(a).$$

Therefore, if f is continuous, then f_1 and f_2 are the composites of the continuous functions above therefore, are continuous.

Conversely, suppose that f_1 and f_2 are continuous. By Lemma C, it suffices to show that for each basic open set $U \times V \subseteq X \times Y$, the preimage $f^{-1}(U \times V)$ is open. But $a \in f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$, if and only if $f_1(a) \in U$ and $f_2(a) \in V$. Thus, $f^{-1}(U \times V) = f^{-1}(U) \cap f^{-1}(V)$ which is open in A since U is open in X and V is open in Y and f_1, f_2 are continuous. ◄

McClure: Winter 2014

Exercise 1.1. Let X be a topological space, let A be a subset of X, and let U be an open subset of X. Prove that $U \cap \bar{A} \subseteq \overline{U \cap A}$.

Solution. ► The solution is simple and we have shown this before in the August 2014 quals, it goes as follows: If $U \cap \bar{A} = \varnothing$, there is nothing to show. Let $x \in U \cap \bar{A}$. Then $x \in U$ and $x \in \bar{A}$. Since $x \in U$ and U is open, by Lemma C, there exists a neighborhood V of x such that $V \subseteq U$; in particular, note that $V \cap U \neq \varnothing$. But $x \in \bar{A}$ so $V \cap A \neq \varnothing$. Thus, $V \cap (U \cap A) \neq \varnothing$. Thus, $x \in \overline{U \cap A}$. ◄

Exercise 1.2. Let $\sim$ be an equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. Prove that the quotient space $\mathbb{R}^2/\sim$ is compact but not Hausdorff.

Solution. ► To show that $\mathbb{R}^2/\sim$ is compact, we need to show that for every open covering $\mathscr{A}$ of $\mathbb{R}^2/\sim$, there is a finite subcover $\mathscr{A}' \subseteq \mathscr{A}$. Let $q\colon \mathbb{R}^2 \to \mathbb{R}^2/\sim$ denote the quotient map. Then, since q is continuous and onto $\mathbb{R}^2/\sim$, the set $\left\{q^{-1}(A_\alpha)\right\}_{A_\alpha \in \mathscr{A}}$ is an open cover of $\mathbb{R}^2$. In particular, there exists at least one A_α such that $q^{-1}(A_\alpha)$ is a neighborhood of $(0, 0)$. By Lemma C, there exists a basic open neighborhood, i.e., an open ball $B((0, 0), \varepsilon) \subseteq q^{-1}(A_\alpha)$ for $\varepsilon > 0$. Now, for any point $[(x, y)] \in \mathbb{R}^2$ pick a representative $(x, y) \in \mathbb{R}^2$. Then, by the Archimedean principle, there exists a positive real numbers $t', t'' > 0$ such that $t'x < \sqrt{\varepsilon}$ and $t''y < \sqrt{\varepsilon}$. Define $t = \min\{t', t''\}$. Then $tx < \sqrt{\varepsilon}$ and $ty < \varepsilon$. Thus, $(tx, ty) \in A_\alpha$ (since $t^2x^2 + t^2y^2 < \varepsilon$. Since we can do this for any point $[x] \in \mathbb{R}^2/\sim$, it follows that $A_\alpha \supset \mathbb{R}^2/\sim$. Thus, $\mathscr{A}' = \{A_\alpha\}$ is a finite subset of $\mathscr{A}$ which covers $\mathbb{R}^2/\sim$. Thus, $\mathbb{R}^2/\sim$ is compact.

To show that $\mathbb{R}^2/\sim$ is not compact, we will employ a very similar strategy, that is, we will show that every neighborhood of the point $[0, 0] \in \mathbb{R}^2/\sim$, contains every point $[x, y] \in \mathbb{R}^2/\sim$. Let $[x, y] \in \mathbb{R}^2/\sim$ and let U be a neighborhood of $[0, 0]$. Then $q^{-1}(U)$ is an open neighborhood of $(0, 0)$, i.e., there exists an open ball $B((0, 0), \varepsilon) \subseteq q^{-1}(U)$. But as we have just shown, for sufficiently small values of $t > 0$, $(tx, ty) \in B((0, 0), \varepsilon) \subseteq q^{-1}(U)$. Thus, $[x, y] \in U$. In particular, for any open neighborhood V of $[x, y]$, $V \cap U \neq \varnothing$. Thus, $\mathbb{R}^2/\sim$ is not Hausdorff. ◄

Exercise 1.3. Let X and Y be topological spaces. Let $x_0 \in X$ and let C be a compact subset of Y. Let N be an open set in $X \times Y$ containing $\{x_0\} \times C$. Prove that there is an open set U containing x_0 and an open set V containing C such that $U \times V \subseteq N$.

Solution. ► This is a classical theorem called the tube lemma. We shall prove first in the style of Munkres and second in the style of McClure (if I can find the solution or somehow reconstruct it).

Let X, Y, x_0, N, and C be as above. Note that since C is compact and the injection $\iota_{x_0}\colon X \hookrightarrow X \times Y$ given by $\iota_{x_0}(y) = (x_0, y)$ is continuous by Theorem 18.4 (since its components, i.e., projections to X and Y, are continuous these are $\pi_1(\iota_{x_0})(x) = x_0$ and $\pi_1(\iota_{x_0})(y) = y$ a constant map and identity map, respectively) so the image of C under ι_{x_0}, $\{x_0\} \times C$, is compact by Theorem 23.5. For every point $x \in \{x_0\} \times C$, let $U_x \times V_x$ be a basic open neighborhood of x contained in N (this can be arranged by Lemma C). Then the collection $\mathscr{A} = \{U_x \times V_x\}_{x \in \{x_0\} \times Y}$ forms an open covering of $\{x_0\} \times C$. Thus, there exists a finite subcover $\left\{U_{x_i} \times V_{x_i}\right\}_{i=1}^{n}$ of $\mathscr{A}$.

Define $W = U_{x_1} \cap \cdots \cap U_{x_n}$. This set is clearly open since it is a finite intersection of open sets and contains x_0 since every $U_{x_i} \times V_{x_i}$ intersects $\{x_0\} \times Y$. Define $W' = \pi_2(N) \cap Y$. This set is open since it is a finite intersection of open sets in Y. The $W \times W' \subseteq N$. This is clear since every point $(x, y) \in W \times W'$ is in N

($x \in W \subseteq U_{x_i}$ for all i which in turn is a subset of $\pi_1(N)$ and $y \in W' = \pi_2(N)$. Lastly, $W \times W' \supset \{x_0\} \times C$ since $x_0 \in W$ and $W' = \pi_1(N) \supset C$. Thus, $W \times W' \subseteq N$ containing $\{x_0\} \times C$ as desired. ◄

Exercise 1.4. Let X be a locally compact Hausdorff space and let A be a subset with the property that $A \cap K$ is closed for every compact K. Prove that A is closed.

Solution. ► Here's an idea: It suffices to show that A contains all of its limit points. Let $\{x_n : n \in \mathbb{N}\}$ with $x_n \in A$ be a sequence of points converging to $x \in X$. Then the set

$$K = \left\{ x_n : n \in \mathbb{N} \right\} \cup \{x\}$$

is compact since for every neighborhood U of $\{x\}$, there exists an index $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in U$ for all $n \geq N$; in particular, given a cover $\mathcal{U}$ of K, choose $U_N \in \mathcal{U}$ containing x, then there exists an index $N \in \mathbb{N}$ such that $x_n \in U_N$ for all $n \geq N$ and since $\mathcal{U}$ covers K, pick $U_1, \ldots, U_{N_1}$ containing $x_1, \ldots, x_{N-1}$. Thus, K is compact. Now, since $A \cap K$ is closed, it must contain every limit point; in particular, $x \in A \cap K$ so $x \in A$. Thus, A is closed. ◄

Exercise 1.5. Let X and Y be path-connected and let $h\colon X \to Y$ be a continuous function which induces the trivial homomorphism of fundamental groups. Let $x_0, x_1 \in X$ and let f and g be paths from x_0 to x_1. Prove that $h \circ f$ and $h \circ g$ are homotopic.

Solution. ► Consider the path-product $\gamma = f * \bar{g}$. γ is a loop based at x_0 since $\gamma(0) = f(0) = x_0$ and $\gamma(1) = \bar{g}(2-1) = \bar{g}(1) = x_0$. Thus, $[\gamma] \in \pi_1(X, x_0)$. Now, since $h_*\colon \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$ induces the trivial homomorphism, i.e., $h(\gamma) \simeq_p e_{x_0}$, there exists a homotopy $H\colon [0,1] \times [0,1] \to Y$ such that $H(s, 0) = h \circ \gamma(s)$ and $H(s, 1) = e_{x_0}(s)$. Now, since Y is path-connected, there exists a path $\delta\colon [0,1] \to Y$ from $h(x_0)$ to $h(x_1)$. ◄

Exercise 1.6. Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $aabbcdc^{-1}d^{-1}$.

(i) Calculate $H_1(X)$.
(ii) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?

Solution. ► ◄

Exercise 1.7. Let $p\colon E \to B$ be a covering map with B locally connected, and let $x \in B$. Prove that x has a neighborhood W with the following property: for every connected component C of $p^{-1}(W)$, the map $p\colon C \to W$ is a homeomorphism.

Solution. ► ◄

McClure: Summer 2014

Exercise 1.1. Let X be a topological space, let A be a subset of X, and let U be an open subset of X. Prove that $U \cap \bar{A} \subseteq \overline{U \cap A}$.

Solution. ► Let $x \in U \cap \bar{A}$. Then $x \in U$ and $x \in \bar{A}$. This means that, since U is open, by Lemma C there exist an open neighborhood V of x such that $V \subseteq U$. Moreover, since $x \in \bar{A}$, $V' \cap A \neq \varnothing$ for every open neighborhood V' of x. In particular, $V \cap A \neq \varnothing$. Thus, we have $V \cap U \neq \varnothing$ and $V \cap A \neq \varnothing$ so $V \cap (U \cap A) \neq \varnothing$. ◄

Exercise 1.2. Let X be the following subspace of $\mathbb{R}^2$:

$$((0,1] \times [0,1]) \cup ([2,3) \times [0,1]).$$

Let $\sim$ be the equivalence relation on X with $(1,t) \sim (2,t)$ (that is $(s,t) \sim (s',t') \iff (s,t) = (s',t')$ or $t = t'$ and $\{s, s'\} = \{1,2\}$; you do *not* have to prove that this is an equivalence relation). Prove that $X/{\sim}$ is homeomorphic to $(0,2) \times [0,1]$. (*Hint*: construct maps in both directions).

Solution. ► We shall proceed by the hint. Let $q\colon X \to X/{\sim}$ denote the quotinet map. Then, for $(x,y) \in X$, we define the map

We shall proceed by the hint. Let $q\colon X \to X/{\sim}$ denote the quotient map. Then, for $x \in X$, we define the map

$$h(s,t) = \begin{cases} (s,t) & \text{if } (s,t) \in (0,1] \times [0,1] \\ (s-1,t) & \text{if } (s,t) \in (2,3] \times [0,1]) \end{cases}$$

from $X \to (0,2) \times [0,1]$.

By the UMP of the quotient space (Theorem Q.3), if we can show that h is continuous and preserves the equivalence relation, the induced map on the quotient space, $h'\colon X/{\sim} \to (0,2) \times [0,1]$ will be continuous. To that end, we will use the pasting lemma. First, note that $(0,1] \times [0,1]$ and $[2,3) \times [0,1]$ are closed subsets of X since $(0,1] \times [0,1]$ is the complement of $((1,\infty) \times (-2,2)) \cap X$ which is open in X (since X inherits its topology from $\mathbb{R}^2$, similarly, $[2,3) \times [0,1]$ is closed in X since it is the complement of $((-\infty,2) \times (-2,2)) \cap X$ which is open in X for the same reasons. It is clear that the maps $x \mapsto x$ and $x \mapsto x-1$ are continuous onto their image, since the latter is nothing more than the inclusion map and the former is nothing more than subtraction, which is continuous by Theorem 21.5. Thus, by the pasting lemma, h is continuous.

Now we show that h does in fact preserve the equivalence relation. Suppose $(s,t) \sim (s',t')$. Then either $(s,t) = (s',t')$ or $t = t'$ and $s, s' \in \{1,2\}$. In the former case, we have $h(s,t) = h(s',t')$ (whether $(s,t), (s',t') \in (0,1] \times [0,1]$ or its complement). In the latter case, we may, without loss of generality, assume that $(s,t) = (1,t)$ and $(s',t') = (2,t)$. Then $h(s,t) = (1,t) = (2-1,t) = h(s',t')$. Thus, by Theorem Q.3, the induced map $h'\colon X/{\sim} \to (0,2) \times [0,1]$ is continuous. Moreover, the map is bijective with inverse

$$(h')^{-1} = \begin{cases} [s,t] & \text{if } x \in (0,1] \\ [s+1,t] & \text{if } x \in [1,2) \end{cases}.$$

This is clearly an inverse as

$$h' \circ (h')^{-1} = \mathrm{id}_{X/\sim}$$

and

$$(h')^{-1} \circ h' = \mathrm{id}_{(0,2)\times[0,1]}\,.$$

Thus, by Theorem 26.6, h' is a homeomorphism. ◄

Exercise 1.3. Prove that there is an equivalence relation $\sim$ on the interval $[0, 1]$ such that $[0, 1]/\sim$ is homeomorphic to $[0, 1] \times [0, 1]$. As part of your solution *explain* how you are using one or more properties of the quotient topology.

Solution. ► First, it suffices to find a continuous surjective map $f\colon [0, 1] \to [0, 1] \times [0, 1]$ and quotient out by the preimage of every point $x \in [0, 1] \times [0, 1]$. These maps are hard to describe in general, but they exists (take for example a space-filling curve). Next, note that if C is a closed subset of $[0, 1]$ then it is compact so $f(C)$ is compact. But since $[0, 1] \times [0, 1]$ is compact Hausdorff, then $f(C) \subseteq [0, 1] \times [0, 1]$ will be closed. It follows by that f will be a Munkres quotient map, so by Theorem Q.4, $f'\colon [0, 1]/\sim \to [0, 1] \times [0, 1]$ is a homeomorphism for some equivalence relation $\sim$ on $[0, 1]$. ◄

Exercise 1.4. Let D be the closed unit disk in $\mathbb{R}^2$, that is, the set

$$\left\{ (x, y) : x^2 + y^2 \leq 1 \right\}.$$

Let E be the open unit disk

$$\left\{ (x, y) : x^2 + y^2 < 1 \right\}.$$

Let X be the one-point compactification of E, and let $f\colon D \to X$ be the map defined by

$$f(x, y) = \begin{cases} (x, y) & \text{if } x^2 + y^2 < 1 \\ \infty & \text{if } x^2 + y^2 = 1. \end{cases}$$

Prove that f is continuous.

Solution. ► By the section on the one-point-compactification, it suffices to check two cases of open sets (1) all sets U open in E, and (2) all sets of the form $U = X \setminus C$ containing the point at infinity, ∞, where C is compact. In the first case, it is clear that f is continuous since it is just the inclusion map and is in fact bijective on E. For the second case, suppose that U is a neighborhood of ∞. Then $Y - U$ is a compact subset of E, hence closed since X is a compact Hausdorff space. But since f is bijective, continuous on E, then $f^{-1}(X - U)$ is a closed subset of E. Thus, by theorem 18.2, f is continuous. ◄

Exercise 1.5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is path-connected. Prove that Y is path-connected.

Solution. ► First we prove the following important result:

Lemma. *Path-connectedness is a topological property, i.e., if X is path-connected and $f\colon X \to Y$ is a continuous map then, $f(X)$ is path connected.*

Proof. Since X is path-connected, for any pair of points $x, x' \in X$ there exists a continuous map $p\colon [0, 1] \to X$ such that $p(0) = x$ and $p(1) = x'$. Since composition of continuous maps is continuous, $f \circ p\colon [0, 1] \to Y$ is a path from $f(x)$ to $f(x')$. Since this property holds for any $y \in f(X)$, it follows that $f(X)$ is path-connected. □

Now, suppose that X is homotopy-equivalent to Y. Then there exists continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Now, since X is path-connected, by the lemma above we have

$f(X)$ is path-connected. Thus, it suffices to show that for every point $y \in Y$ there exists a path $p\colon [0,1] \to Y$ from y to some point $y' \in f(X)$. Now, since $f \circ g \simeq \mathrm{id}_Y$, there exists a homotopy, say $H\colon Y \times [0,1] \to Y$ such that $H(s,0) = f \circ g(s)$ and $H(s,1) = s$. Consider the evaluation $H_y = H(y,t) \circ H(y,t)$ where the map $(y,t)\colon [0,1] \to Y \times [0,1]$ is the imbedding of $[0,1]$ at y (which is continuous by Theorem 18.4) thus, H_y is continuous. Moreover, $H_y(0) = f \circ g(y) \in f(Y)$ and $H_y(1) = \mathrm{id}_Y(y) = y$ so H_y is a path from y to a point $f \circ g(y)$ in $f(X)$. Since we can do this for any point $y \in Y$, it follows, since path-connectedness is an equivalence relation, that Y is path-connected. ◄

Exercise 1.6. Let a and b denote the points $(-1,0)$ and $(1,0)$ in $\mathbb{R}^2$. Let x_0 denote the origin $(0,0)$. Use the Seifert–van Kampen theorem to calculate $\pi_1\left(\mathbb{R}^2 \setminus \{a,b\}, x_0\right)$. You may not use any other method.

Solution. ► We'll use Theorem 70.2's version of the Seifert–van Kampen theorem. Define

$$U = \left(-\infty, \tfrac{1}{2}\right) \times \mathbb{R} \quad \text{and} \quad V = \left(-\tfrac{1}{2}, \infty\right) \times \mathbb{R}.$$

Then $U \cap V = (-1/2, 1/2) \times \mathbb{R}$ is clearly path-connected since it is a convex set. Moreover, note that $U \simeq \mathbb{R}^2 \setminus \{x_0\}$ and $V \simeq \mathbb{R}^2 \setminus \{x_0\}$ (in the case of U, first consider the homeomorphism $(x,y) \mapsto (x+1, y)$ which sends a to $(0,0)$ and then the homotopy $(x,y) \mapsto \frac{1}{t}(x, ys)$.

Once we have established the above, since the fundamental group of a space is invariant under homotopy-equivalence, $\pi_1(U, x_0) \cong \pi_1\left(\mathbb{R}^2 \setminus \{x_0\}, y_0\right) \cong \mathbb{Z}$ for some arbitrary $y_0 \neq x_0$ and similarly $\pi_1(V, x_0) \cong \mathbb{Z}$. Thus, by the classical version of the Seifert–van Kampen theorem

$$\pi_1\left(\mathbb{R}^2 \setminus \{a,b\}, x_0\right) \cong \frac{\mathbb{Z} * \mathbb{Z}}{N}$$

where N is the least normal subgroup ◄

Exercise 1.7. Let $p\colon E \to B$ be a covering map with B locally connected, and let $x \in B$. Prove that x has a neighborhood W with the following property: for every connected component C of $p^{-1}(W)$, the map $p\colon C \to W$ is a homeomorphism.

Solution. ► Let U be an evenly covered neighborhood of x. Then $p^{-1}(U) = \bigcup_\alpha V_\alpha$ where the V_α are open in E and $V_\alpha \cap V_\beta = \varnothing$ whenever $\alpha \neq \beta$. For any α, let C be a connected component of $p^{-1}(U)$ containing $p^{-1}(x) \cap V_\alpha$ (the latter is a one point set since $p|_{V_\alpha}$ is a bijection). Then $C \subseteq V_\alpha$ for at most one such α for otherwise $C \cap V_\beta \neq \varnothing$ for some $\beta \neq \alpha$, so $C \cap V_\beta$ and $C \cap V_\alpha$ form a separation of (note that $C \setminus (C \cap V_\beta) = C \cap V_\alpha$ and vice-versa thus, $C \cap V_\beta$ and $C \cap V_\alpha$ are open and closed in the subspace topology on C, conversely) by Lemma 23.1.

Thus, $p(C) \subseteq U$ is connected by Theorem 23.5. Moreover, since $V_\alpha \supset C$ is homeomorphic to U by the restriction $p|_{V_\alpha}$, $p(C)$ is a connected component of U as the following lemma shows

Lemma. *Suppose C is a connected component of X and $h\colon X \to Y$ is a homeomorphism. Then $h(C)$ is a connected component of Y.*

> *Solution of lemma.* Let C be a connected component of X. By theorem 23.5, $h(C)$ is a connected subset of Y, moreover, is open. By Theorem 25.1, $h(C)$ is contained in a connected component of Y, say D. Hence, we must show that $D \subseteq h(C)$. Now, since h is a homeomorphism, $h^{-1}(D)$ is

a connected subset of X, by Theorem 23.5, so is contained in only one component of X. But $h^{-1}(D) \cap C \neq \varnothing$ so $h^{-1}(D) \subseteq C$. Thus, since h is a set-bijection, $D \subseteq h(C)$. □

so by Theorem 25.3, $p(C)$ is open in B since B is locally connected. Thus, the restriction $p|_C$ is a homeomorphism onto its image $W = p(C)$, by Lemma A, which is a neighborhood of x. ◀