# MA571 Homework 11

Carlos Salinas

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## Problem 11.1 (Munkres §53, Ex. 7(abcd))

Let $G$ be a topological group with operation $\cdot$ and identity element $x_0$. Let $\Omega(G, x_0)$ denote the set of all loops in $G$ based at $x_0$. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]

(d) Show that $\pi_1(G, x_0)$ is Abelian.

*Proof.* For part (a) we need to show that the operation (i) $\otimes$ is associtive, (ii) $\Omega(G, x_0)$ is closed under $\otimes$, (iii) $\Omega(G, x_0)$ contains an identity element $e$ and (iv) for every $f \in \Omega(G, x_0)$ there exists an element $f^{-1}$ such that $f \otimes f^{-1} = f^{-1} \otimes f = e$. We shall proceed in order: (i) is easy since the operation $\otimes$ is associative so for any triple $f, g, h \in \Omega(G, x_0)$ we have

$$(f \otimes g) \otimes h = (f(s) \cdot g(s)) \otimes h = (f(s) \cdot g(s)) \cdot h(s)$$

which one clearly sees, by asociativity of $\cdot$, is the same as $f \otimes (g \otimes h)$. (ii) Let $f, g \in \Omega(G, x_0)$ then, since $\cdot$ is continuous, the map $f \otimes g \colon I \to G$ is continuous and

$$(f \otimes g)(0) = f(0) \cdot g(0) = x_0 \cdot x_0 = f(1) \cdot g(1) = (f \otimes g)(1).$$

Thus, $f \otimes g \in \Omega(G, x_0)$. Next, for (iii) consider the constant loop $e_{x_0}(s)$. This map is clearly the identity on $\Omega(G, x_0)$ for if $f \in \Omega(G, x_0)$ then $e_{x_0} \otimes f = e_{x_0}(s) \cdot f(s) = x_0 \cdot f(s) = f(s)$ for all $s$; similarly for $f \otimes e_{x_0}$. Lastly, (iv) consider the map $f^{-1}(s) \coloneqq (f(s))^{-1} \colon I \to G$. This map is continuous since taking the inverse in $G$ is continuous and composition of continuous maps is continuous by Theorem 18.2(c). Lastly, note that $f^{-1}(0) = x_0^{-1} = x_0$ similarly for $f^{-1}(1)$. Thus, $f^{-1}$ is a loop and

$$f^{-1} \otimes f = (f(s))^{-1} \cdot f(s) = x_0 = f(s) \cdot (f(s))^{-1} = f \otimes f^{-1}$$

so $\Omega(G, x_0)$ is closed under inverses. Thus, $\Omega(G, x_0)$ is a group.

(b) The map $\otimes \colon \Omega(G, x_0) \times \Omega(G, x_0) \to \Omega(G, x_0)$ clearly induces a group operation on $\Pi_1(X, x_0)$ given by $[f] \otimes [g] = [f \otimes g]$. All we need to check is that this operation is in fact well defined on the equivalence class of loops based at $x_0$, i.e., if $f_1 \simeq_p f_2$ with path homotopy $H$ and $g_1 \simeq_p g_2$ with path homotopy $K$ we want that $f_1 \otimes g_1 \simeq_p f_2 \otimes g_2$. But this is immediate via the homotopy $L(s, t) \coloneqq H(s, t) \cdot K(s, t) \colon G \times I \to G$. This map is continuous since it can be realized as the sequence of compositions

$$(s, t) \longmapsto (H(s, t), K(s, t)) \longmapsto H(s, t) \cdot H(s, t)$$

where the intermediate step in the composition, i.e., the map from $I \times I \to G \times G$, is continuous by Theorem 18.4 since $H$ and $K$ are continuous and lastly $L(s,0) = H(s,0) \cdot K(s,0) = f_1(s) \cdot g_1(s) = f_1 \otimes g_1$ and $L(s,1) = H(s,1) \cdot K(s,1) = f_2(s) \cdot g_2(s) = f_2 \otimes g_2$. Thus, $f_1 \otimes g_1 \simeq_p f_2 \otimes g_2$. It follows that $\otimes$ is a well-defined binary operation on $\pi_1(X, x_0)$.

(c) Following the hint, we shall compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$. Recall that

$$f * e_{x_0} = \begin{cases} f(2s) & \text{for } s \in [0,1/2] \\ e_{x_0}(2s-1) & \text{for } s \in [1/2,1] \end{cases} \qquad \text{and} \qquad e_{x_0} * g = \begin{cases} e_{x_0}(2s) & \text{for } s \in [0,1/2] \\ g(2s-1) & \text{for } s \in [1/2,1] \end{cases}.$$

Then

$$\begin{aligned} (f * e_{x_0}) \otimes (e_{x_0} * g) &= (f * e_{x_0})(s) \cdot (e_{x_0} * g)(s) \\ &= \begin{cases} f(2s) \cdot e_{x_0}(2s) & \text{for } s \in [0,1/2] \\ e_{x_0}(2s-1) \cdot g(2s-1) & \text{for } s \in [1/2,1] \end{cases} \\ &= \begin{cases} f(2s) & \text{for } s \in [0,1/2] \\ g(2s-1) & \text{for } s \in [1/2,1] \end{cases} \\ &= f * g. \end{aligned}$$

Since $f * e_{x_0} \simeq_p f$ and $e_{x_0} * g \simeq_p g$, we have at last that

$$[f \otimes g] = [(f * e_{x_0}) \otimes (e_{x_0} * g)] = [f * g].$$

(d) Lastly, we show that $\pi_1(X, x_0)$ must be Abelian. It suffices to show that given a class of loop $[f]$ at $x_0$, the conjugacy class of $[f]$ consists of the singleton $\{[f]\}$. First, note that if $[g] \in \pi_1(X, x_0)$ then $[g^{-1}] \otimes [g] = [e_{x_0}] = [\bar{g}] * [g]$ so $[g^{-1}] = [g^{-1}] \otimes ([g] * [\bar{g}]) = [\bar{g}]$. Thus, we have

$$\begin{aligned} [\bar{g}] * [f] * [g] &= \big([g^{-1}] * [f]\big) * [g * e_{x_0}] \\ &= \big([g^{-1}] * [f]\big) \otimes [g * e_{x_0}] \\ &= \begin{cases} g^{-1}(2s) \cdot g(2s) & \text{for } s \in [0,1/2] \\ f(2s-1) \cdot e_{x_0}(2s-1) & \text{for } s \in [1/2,1] \end{cases} \\ &= [e_{x_0} * f] \\ &= [f]. \end{aligned}$$

It follows that $\pi_1(X, x_0)$ is Abelian. ■

## Problem 11.2 ((A))

Prove Proposition F from the note on the Fundamental Group of the Circle.

*Proof.* Recall proposition F:

**Proposition F.** (i) *$W$ takes the class of the path $f_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ to $n$ (and therefore $W$ is onto).*

(ii) *$W$ is one-to-one.*

(iii) *$W$ is a homomorphism.*

(i) Now, recall that $W\colon \pi_1(S^1, x_0) \to \mathbf{Z}$ defined by $W([f]) \coloneqq w(f)$ where $w(f) = \tilde{f}(1)$ where $\tilde{f}\colon I \to \mathbf{R}$ is the lift of $f$, i.e. $p \circ \tilde{f} = f$. Now, let $f_n$ be a path as above. Now, by Proposition C, since

$$f_n(s) = (\cos(2\pi ns), \sin(2\pi ns)) = (\cos(2\pi \tilde{f}_n(s)), \sin(2\pi \tilde{f}_n(s)))$$

and $\tilde{f}_n(0) = 0 = n \cdot 0$, by Proposition C, it follows that $f_n(s) = ns$. Thus, $\tilde{f}(1) = n$.

(ii) Suppose $f_1, f_2\colon I \to S^1$ and $\tilde{f}_1(1) = \tilde{f}_2(1)$

(iii) ■

## Problem 11.3 ((B))

Prove Lemma G from the note on the Fundamental Group of the Circle. (Hint: one way to do this is to use the fact, which you don't have to prove, that if $\sim$ is the equivalence relation on $[a, a+1]$ which identifies $a$ and $a+1$ then the restriction of $p$ induces a homeomorphism $[a, a+1]/\sim \to S^1$.)

*Proof.* Recall the statement of Lemma G:

**Lemma G.** *For each $a \in \mathbf{R}$, the map*

$$p_a \colon (a, a+1) \longrightarrow S^1 - p(a)$$

*given by $p_a(u) = p(u)$ is a homomorphism.*

We shall proceed by the hint. ∎

## Problem 11.4 ((C))

Show that for every point $x \in S^n$ the space $S^n - x$ is homeomorphic to $\mathbf{R}^n$. You may use the fact, shown in Step 1 of the proof of Theorem 59.3, that $S^n$ with the *north pole* removed is homeomorphic to $\mathbf{R}^n$. (Hint: linear algebra.)

*Proof.* ■

## Problem 11.5 ((D))

Show that every loop in $S^n$ which is not onto is path-homotopic to a constant path. (Hint: use Problem C).

*Proof.* ■

## Problem 11.6 ((E))

Let $X$ be a topological space and let $A \subset X$ be a deformation retract. In the space $X/A$, the set $A$ is a point (because it is an equivalence class). Show that this point is a deformation retract of $X/A$. (Hint: use p. 289 # 9.)

*Proof.* Let $H\colon X \times I \to X$ be a deformation retraction from $X$ to $A$, that is, $H(0, x) = \mathrm{id}_X$ and $H(1, x) = r(x)$ where $r\colon X \to A$ is a retraction of $X$ onto $A$ and $\iota\colon A \hookrightarrow X$ is the inclusion of $A$ into $X$. Let $p\colon X \to X/A$ be a quotient map. Now, we want to construct a deformation retraction $h\colon X/A \times I \to X/A$ from the quotient $X/A$ to $*$, which we shall use to denote the image of $A$ in $X/A$ under $p$, and what better candidate than the map induced by $p \circ H\colon X \times I \to X/A$ on the quotient $X/A \times I$ into $X/A$. Consider the map $(p, \mathrm{id}_I)\colon X \times I \to X/A \times I$. This map is a quotient map by Problem 9.2 (Munkres §46, x. 9). Moreover, the map $p \circ H$ preserves the equivalence relation on $X/A \times I$ since for any two representatives $(x_1, t)$ and $(x_2, t)$ of $[(x, t)]$ in $X/A \times I$, we have $H(x_1, t) = H(x_2, t)$ if $x \in X - A$ and $H(x_1, t) = H_2(x_2, t)$ so $p(H(x_1, t)) = p(H(x_2, t))$ and if $x_1, x_2 \in A$ then $H(x_1, t), H(x_2, t) \in A$ so $p(H(x_1, t)) = p(H(x_2, t))$. Thus, by Theorem Q.3 the map $h\colon X/A \times I \to X/A$ induced by $H$, i.e., the map defined by $h(x, t) \coloneqq [H(x, t)]$, is continuous and the diagram

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "{(p,\mathrm{id}_I)}"'] & X \arrow[d, "p"] \\ X/A \times I \arrow[r, "h"'] & X/A \end{tikzcd}$$

commutes. We claim that $h$ is a deformation retraction from $X/A$ to $*$. To that end, it suffices to show that $h(x, 0) = \mathrm{id}_{X/A}$ and, using suggestive notation, $h(x, 1) = \bar{r}$ where $\bar{r}\colon X/A \to *$ is a retraction of $X/A$ onto $A$ and $\bar{\iota}\colon * \hookrightarrow X/A$ is the inclusion of $*$ into $X/A$. The first is easy to verify since $h(x, 0) = [H(x, 0)] = [x] = \mathrm{id}_{X/A}$. Next, $h(x, 1) = [H(x, 1)] = [r(x)]$ and we claim that $\bar{r}(x) \coloneqq [r(x)]$ is a retraction of $X/A$ into $*$. The map $\bar{r}$ is continuous since $h$ is continuous (by Lemma 1 from Hw. #9 Munkres §18, Ex. 11) and $\bar{r}\colon X/A \to *$ since $r(x) \in A$ for every $x \in X$. It follows that $*$ is a deformation retract of $X/A$. ■