MA166: Solutions to Homework 8-10

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1 Homework 8 Solutions

Problem 1.1 (WebAssign, HW8, 1). A variable force of $2x^{-2}$ pounds moves an object along a straight line when it is x feet from the origin. Calculate the work done in moving the object from $x = 1$ ft to $x = 15$ ft. (Round your answer to two decimal places.)

Solution. Recall the formula for the work done by a fore as a function of distance? It is given by

$$W = \int_1^1 5F(x)\,\mathrm{d}x = 2\int_1^{15} x^{-2}\,\mathrm{d}x = -2\int_1^{15} \frac{1}{x} = -\frac{2}{x}\Big|_1^{15} = -\frac{2}{15} + \frac{2}{1} = \frac{28}{15}.$$

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Problem 1.2 (WebAssign, HW8, 2). Shown is the graph of a force function (in Newtons) that increases to its maximum value and then remains constant. How much work W is done by the force in moving an object a distance of $24\,\mathrm{m}$?
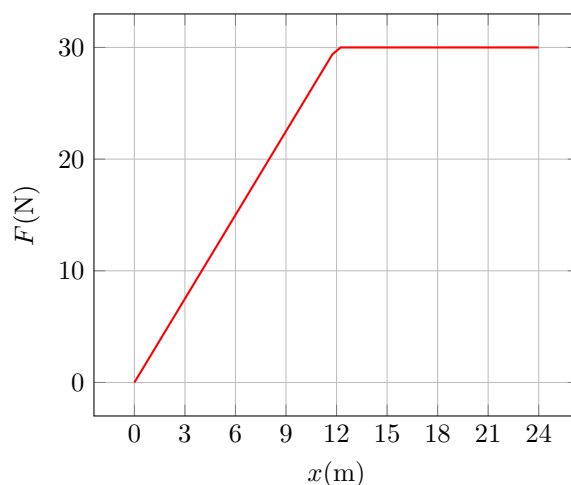

Figure 1.1: Graph of the force F with respect to distance.

Solution. Remember that the work done by a force $F(x)$ over a distance $a \leq x \leq b$ is given by the formula

$$W = \int_a^b F(x)\,\mathrm{d}x. \tag{1}$$

Since the graph in Figure (1.1) is piecewise, all we need to do is break up our interval $0 \leq x \leq 24$ into the part where y is a diagonal line and where y is a horizontal line, compute the work done on each interval, W_1 and W_2, and add them up to get the total work $W = W_1 + W_2$. First, note that over $0 \leq x \leq 12$, $F(x) = \frac{30}{12}x = \frac{5}{2}x$ so the work done from $0 \leq x \leq 12$ is

$$W_1 = \int_0^{12} \frac{5}{2}x\,\mathrm{d}x = \frac{5}{4}x^2\Big|_0^{12} = \frac{5}{4}(12)^2 = 180\,\mathrm{J}.$$

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Next, we see that $F(x) = 30$ is constant for all $12 \leq x \leq 24$ so we have

$$W_2 = \int_{12}^{24} F(x) \, dx = \int_{12}^{24} 30 \, dx = 30x|_{12}^{24} = 30 \cdot 24 - 30 \cdot 12 = 30 \cdot 12 = 360 \, \text{J}.$$

Hence, the total work done by $F(x)$ over $0 \leq x \leq 24$ is

$$\boxed{W = W_1 + W_2 = 180 \, \text{J} + 360 \, \text{J} = 540 \, \text{J}.}$$

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Problem 1.3 (WebAssign, HW8, 3)**.** A force of 6 lb is required to hold a spring stretched 8 in beyond its natural length. How much work W is done in stretching it from its natural length to 14 in beyond its natural length?

Solution. Recall Hooke's law for the force required to stretch a spring a distance x beyond its natural length

$$F(x) = kx. \tag{2}$$

Now, we are given a force of 6 lb and a distance of 8 in. Using this information, we can figure out what the coefficient k must be:

$$k = F/x = 6/8 \text{lb/in}.$$

Using the Equation (1) for work, we have that the work done on a spring by stretching it from x_1 to x_2 is

$$W(x_1, x_2) = \int_{x_1}^{x_2} F(x) \, dx = \int_{x_1}^{x_2} kx \, dx = \frac{1}{2} kx^2 \Big|_{x_1}^{x_2} = \frac{1}{2} k \left(x_1{}^2 - x_2{}^2 \right) \tag{3}$$

so, plugging in 14, into our equation $W(x')$ above we get

$$W(14) = \frac{1}{2} \cdot \frac{6}{8} \cdot 14^2 = 7 \cdot \frac{6}{8} = \boxed{\frac{49}{8}} \text{ ft-lb.}$$

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Problem 1.4 (WebAssign, HW8, 4)**.** If the work required to stretch a spring 3 ft beyond its natural length is 9 ft-lb, how much work is needed to stretch it 9 in beyond its natural length?

Solution. We employ the same idea as the one we used to calculate the work on the last problem. First, we find the coefficient k, wit the help of Equation (3) this value will be

$$k = \frac{2W}{x^2} = \frac{2 \cdot 9}{9} = 2 \text{ lb/ft}.$$

Now, convert our 9 in to ft we get 3/4 ft. Lastly, applying Equation (3) on 3/4 ft we get

$$W(3/4) = \frac{1}{2} \cdot 2 \cdot (3/4)^2 = \boxed{\frac{9}{16}} \text{ in-lb.}$$

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Problem 1.5 (WebAssign, HW8, 5). An aquarium 6 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use $9.8 \, \text{m/s}^2$ for g and the fact that the density of water is $1000 \, \text{kg/m}^3$.)

- Show how to approximate the required work by a Riemann sum. (Let x be the height in meters below the top of the tank. Enter x_i^* as x_i.)

- Express the work as an integral.

Solution. Let's place the origin at the top of the tank. Then, the expression for the work required to pump out half the water from our tank will be given approximately by

$$W(x) \approx \lim_{n \to \infty} \sum_{i=1}^{n} 9.8 \cdot 1000 \cdot 6 \cdot x_i \Delta x = \boxed{\sum_{i=1}^{n} 58800 x_i^* \Delta x.}$$

Now we compute the integral from $0 \le x \le 1/2$. We have

$$W = 58800 \int_0^{1/2} x \, dx = 5800 \left(\frac{1}{2} x^2 \Big|_0^{1/2} \right) = 29400 \big((1/2)^2 - 0 \big) = \boxed{7350 \, \text{J.}}$$

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Problem 1.6 (WebAssign, HW8, 6). A tank is full of water (see Figure 1.2). Find the work W required to pump the water out of the spout. (Use $9.8 \, \text{m/s}^2$ for g. Use $1000 \, \text{kg/m}^3$ as the weight density of water. Assume that $a = 4 \, \text{m}$, $b = 4 \, \text{m}$, $c = 12 \, \text{m}$, and $d = 2 \, \text{m}$.)
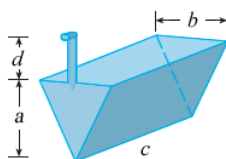

Figure 1.2: A sketch of the tank.

Solution. The work required to pump the water out of the spout of a tank with the dimensions above is given by the

$$W = \lim_{n \to \infty} \sum_{i=0}^{n} F(x_i) \Delta x$$

where $0 \, \text{m} \le x \le 8 \, \text{m}$. Now, the first thing we need to do is to find the way the force changes as we move up the tank. Let x be the vertical distance from the bottom of the tank and y be the horizontal distance from. Then, the work needed to lift the slice slice of volume ΔV that is a distance $(4+2) - x$ from the spout is

$$W(x) \approx g\rho(6 - x)\Delta V, \tag{4}$$

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where ρ is the density of the water and g is gravity. Now, how does our little slice of volume change as we move up the tank? If y is the side of the right-triangle made by going up a distance x up the tank, then

$$\frac{x}{y} = \frac{4}{4} = 1$$

since the triangles are similar. Hence, $y = 2x$ so the volume change will be

$$V(x) \approx 12 \cdot \frac{1}{2}(2x)\Delta x = 12x\Delta x(6 - x),$$

this is just the area of a triangle $x\Delta x$ (base times height) times the depth $12\,\mathrm{m}$. Hence, the, plugging in the last equation into our equation for the force (Equation (4)) will be

$$W(x) \approx g\rho(6 - x)(12x\Delta x) = 117600(6 - x)x\Delta x$$

Last but not least, we compute the limit as $n \to \infty$,

$$W = \lim_{n\to\infty} \sum_{i=0}^{n} W(x_i)\Delta x$$

this is just the integral

$$= 117600 \int_0^4 (6 - x)x \, \mathrm{d}x$$

$$= 117600 \int_0^4 6x - x^2 \, \mathrm{d}x$$

$$= 117600 \left(3x^2 - \frac{1}{3}x^3 \Big|_0^4 \right)$$

$$= 117600 \left(3 \cdot 4^2 - \frac{4^3}{3} - (0 - 0) \right)$$

$$= 117600 \left(\frac{3 \cdot 3 \cdot 4^2 - 4^3}{3} \right)$$

$$= 117600 \left(4^2 \cdot \frac{9 - 4}{3} \right)$$

$$= 117600 \left(4^2 \cdot \frac{5}{3} \right)$$

$$= \boxed{3\,136\,000 \,\mathrm{J.}}$$

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Problem 1.7 (WebAssign, HW8, 7)**.** Consider the given function and the given interval

$$f(x) = 10\sin x - 5\sin 2x, \quad [0, \pi]$$

(a) Find the average value f_{ave} of f on the given interval.

(b) Find c such that $f_{\mathrm{ave}} = f(c)$.

(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

Solution. (a) Recall the definition of the average of a function over an interval $[a, b]$, it is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f \, \mathrm{d}x. \tag{5}$$

Now, all we need to do is calculate the integral of our f and divide that value by $\pi - 0$, i.e.,

$$
\begin{aligned}
f_{\text{ave}} &= \frac{1}{\pi} \int_0^\pi f(x) \, \mathrm{d}x \\
&= \frac{1}{\pi} \int_0^\pi 10 \sin x - 5 \sin 2x \, \mathrm{d}x \\
&= \frac{1}{\pi} \left(-10 \cos x + \frac{5}{2} \cos 2x \Big|_0^\pi \right) \\
&= \frac{1}{\pi} \left(-10 \cos \pi + \frac{5}{2} \cos 2\pi - \left(-10 \cos 0 + \frac{5}{2} \cos 0 \right) \right) \\
&= \frac{1}{\pi} \left(10 + \frac{5}{2} + 10 - \frac{5}{2} \right) \\
&= \frac{1}{\pi} \cdot 20 \\
&= \boxed{\frac{20}{\pi}}.
\end{aligned}
$$

(b) For this part all we need to do is set the equation $f(x) = 10 \sin x - 5 \sin 2x$ equal to $20/\pi$ and solve for x:

$$10 \sin x - 5 \sin 2x = 5(2 \sin x - \sin 2x) = \frac{20}{\pi}$$

so

$$2 \sin x - \sin 2x = \frac{4}{\pi}.$$

I can't think of any way to solve this than plugging it into your calculator and having your calculator approximate the solution (your calculator probably uses Newton's method, which you will learn about later in your career). The values are about $\underline{c \approx 1.238 \text{ and } 2.808}$.

(c) Sketching the graph is easy and the box having the same area as will have height equal to $20/\pi$ and length π. Multiply these together and we get the area under the curve, which we computed to be 20. ☺

Problem 1.8 (WebAssign, HW8, 8). Find the numbers b such that the average value of $f(x) = 7 + 10x - 9x^2$ on the interval $[0, b]$ is equal to 8.

Solution. This exercise is easy-peasy :-). All you need to do is compute the average

$$f_{\text{ave}} = \frac{1}{b-0} \int_0^b 7 + 10x - 9x^2 \, \mathrm{d}x = 7 + 5b - 3b^2.$$

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Now, set the above equal to 8 and we have

$$7 + 5b - 3b^2 = 8$$

and solve for b. We get $\underline{b = (5 - \sqrt{13})/6 \text{ or } (5 + \sqrt{13})/6.}$ ☺

2 Homework 9 Solutions

Problem 2.1 (WebAssign HW 9, 1)**.** Evaluate the integral using integration by parts with the indicated choices of u and dv. (Use C for the constant of integration.)

$$\int 7x^2 \ln x \, dx; \qquad u = \ln x, \quad dv = 7x^2 \, dx.$$

Solution. Remember the formula for integration by parts? No? Well, here it is:

$$\int u \, dv = uv - \int v \, du. \tag{6}$$

Now, let us integrate the above expression making the substitutions which were given to us:

$$\int u \, dv = \int \ln 7x^2 \, dx$$
$$= \tfrac{7}{3}x^3 \ln x - \int \frac{1}{x} \cdot \tfrac{7}{3}x^3 \, dx$$
$$= \tfrac{7}{3}x^3 \ln x - \int \tfrac{7}{3}x^2 \, dx$$
$$= \boxed{\tfrac{7}{3}x^3 \ln x - \tfrac{7}{9}x^3 + C}$$

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Problem 2.2 (WebAssign HW 9, 2)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int 5x \cos 6x \, dx.$$

Solution. Since the derivative of $5x$ is just 5, let's use the substitution $u = 5x$ and $dv = \cos 6x \, dx$. Then, substituting this into Equation (6) we have:

$$\int u \, dv = \int 5x \cos 6x \, dx$$
$$= \tfrac{5}{6}x \sin 6x - \int \tfrac{5}{6} \sin 6x \, dx$$
$$= \boxed{\tfrac{25}{6}x \sin 6x + \tfrac{5}{36} \cos 6x + C.}$$

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Problem 2.3 (WebAssign HW 9, 3)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int \left(x^2 + 3x\right) \cos x \, dx.$$

Solution. Again, since the third derivative of $x^2 + 3x$ disappears, let's make make the substitution $u = x^2 + 3x$ and $dv = \cos x \, dx$. Then, we have

$$\int u \, dv = \int \left(x^2 + 3\right) \cos x \, dx$$

$$= \left(x^2 + 3x\right) \sin x - \int (2x + 3) \sin x \, dx. \tag{7}$$

Now what? That second integral looks pretty hard to integrate? We use integration by parts on that. Since the derivative of $2x$ is 2 let's make the substitution $u = 2x$ and $dv = \sin x \, dx$ and we have

$$\int (2x + 3) \sin x \, dx = -(2x + 3) \cos x - \int 2(-\cos x) \, dx = -(2x + 3) \cos x + 2 \sin x.$$

Substituting this back into Equation (7), we have

$$\int u \, dv = \left(x^2 + 3x\right) \sin x - \int 2x \sin x \, dx$$

$$= \left(x^2 + 3x\right) \sin x - \left(-(2x + 3) \cos x + 2 \sin x\right)$$

$$= \left(x^2 + 3x\right) \sin x + (2x + 3) \cos x - 2 \sin x + C$$

$$= \boxed{\left(x^2 + 3x - 2\right) \sin x + (2x + 3) \cos x + C.}$$

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Problem 2.4 (WebAssign HW 9, 4)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int \sin^{-1} x \, dx$$

Solution. If it were not for integration by parts, I wouldn't know how to even begin to integrate this one. Since we know the derivative of $\sin^{-1} x$, let's make the substitution $u = \sin^{-1} x$ and $dv = dx$. Then we have

$$\int u \, dv = \int \sin^{-1} x \, dx$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, dx.$$

Now, how in the world do we evaluate $\int \frac{x}{\sqrt{1-x^2}} \, dx$? Integration by parts? That sounds hard. Why not make a substitution? What substitution? Well, we have a power of x in the denominator of the integral

$$\int \frac{x}{\sqrt{1 - x^2}} \, dx,$$

and we have a square root of a x^2 in the denominator, why not $w = 1 - x^2$? If we do that, we have $dw = -2x \, dx$,i.e., $dx = dw/(-2x)$ and we have

$$\int \frac{x}{\sqrt{w}} \frac{dw}{-2x} = \int \frac{1}{-2\sqrt{w}} \, dw$$

$$= -\frac{1}{2} \int w^{-1/2} \, dw$$

$$= -\frac{1}{2} \left(\frac{w^{1/2}}{1/2} \right)$$

$$= w^{1/2}$$

$$= \sqrt{1 - x^2}.$$

Now, back to our original problem. Substituting what we just computed, we have

$$\int \sin^{-1} x \, dx = \boxed{x \sin^{-1} x - \sqrt{1 - x^2} + C.}$$

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Problem 2.5 (WebAssign HW 9, 5). Evaluate the integral.

$$\int_1^3 17r^3 \ln r \, dr.$$

Solution. Hopefully, you have caught on the pattern here. We make a choice of u and dv depending on which one we think is easier to differentiate or integrate. In this case, the integral of $\ln r$ is hard, but the integral of $17r^3$ is easy so let's make the substitution $u = \ln r$ and $dv = 17r^3 \, dx$ and we get

$$\int_1^3 u \, dv = \int_1^3 17r^3 \ln r \, dr$$

$$= \tfrac{17}{4} r^4 \ln r \Big|_1^3 - \int_1^3 \frac{17r^4}{4r} \, dr$$

$$= \tfrac{17}{4} r^4 \ln r - \tfrac{17}{16} r^4 \Big|_1^3$$

$$= \frac{4 \cdot 17r^4 \ln r - 17r^4}{16} \Big|_1^3$$

$$= \frac{17r^4 (4 \ln r - 1)}{16} \Big|_1^3$$

$$= \frac{17 \cdot 3^4 (4 \ln 3 - 1)}{16} - \frac{17 \cdot (4 \ln 1 - 1)}{16}$$

$$= \frac{1377(4 \ln 3 - 1)}{16} + \frac{17}{16}$$

$$= \frac{1377 \cdot 4 \ln 3 - 1377 + 17}{16}$$

$$= \tfrac{1377}{4} \ln 3 - \tfrac{1360}{16}$$

$$= \boxed{\tfrac{1377}{4} \ln 3 - 85.}$$

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Problem 2.6 (WebAssign HW 9, 6). Evaluate the integral.

$$\int_0^1 \frac{y}{e^{3y}} \, dy.$$

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Solution. By parts, let $u = y$ and $dv = e^{-3y}\,dy$. Then, we have

$$\int_0^1 \frac{y}{e^{3y}}\,dy = -\frac{ye^{-3y}}{3}\Big|_0^1 - \int -\frac{e^{-3y}}{3}\,dy$$

$$= -\frac{ye^{-3y}}{3}\Big|_0^1 + \int \frac{e^{-3y}}{3}\,dy$$

$$= -\frac{ye^{-3y}}{3} - \frac{e^{-3y}}{9}\Big|_0^1$$

$$= -\frac{3ye^{-3y} + e^{-3y}}{9}\Big|_0^1$$

$$= -\frac{3e^{-3} + e^{-3}}{9} + \frac{1}{9}$$

$$= \boxed{\frac{1 - 4e^{-3}}{9}.}$$

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Problem 2.7 (WebAssign HW 9, 7). Evaluate the integral.

$$\int_1^5 \frac{\ln^2 x}{x^3}\,dx.$$

Solution. First, make the substitution $u = \ln^2 x$ and $dv = dx/x^3$. Then, we have

$$\int u\,dv = \int_1^5 \frac{\ln^2 x}{x^3}\,dx$$

$$= -\frac{\ln^2 x}{2x^2}\Big|_1^5 - \int_1^5 -\frac{2\ln x}{2x^3}\,dx$$

$$= -\frac{\ln^2 x}{2x^2}\Big|_1^5 + \int_1^5 \frac{\ln x}{x^3}\,dx.$$

Then we integrate $\int_1^5 -\ln x/x^3\,dx$ by parts and we get

$$\int_1^5 \frac{\ln x}{x^3}\,dx = -\frac{\ln x}{2x^2}\Big|_1^5 + \int_1^5 \frac{1}{x}\frac{1}{2x^2}\,dx$$

$$= -\frac{\ln x}{2x^2}\Big|_1^5 + \int_1^5 \frac{1}{2x^3}\,dx$$

$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2}\Big|_1^5$$

$$= -\frac{2\ln x + 1}{4x^2}\Big|_1^5.$$

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So our integral is

$$-\frac{\ln^2 x}{2x^2} - \frac{2\ln x + 1}{4x^2}\Bigg|_1^5 = -\frac{2\ln^2 x + 2\ln x + 1}{4x^2}\Bigg|_1^5$$

$$= -\frac{2\ln^2 5 + 2\ln 5 + 1}{4 \cdot 5^2} + \frac{1}{4}$$

$$= -\frac{2\ln^2 5 + 2\ln 5 - 24}{4 \cdot 5^2}$$

$$= \boxed{\frac{6}{25} - \tfrac{1}{50}\ln^2 5 - \tfrac{1}{50}\ln 5.}$$

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Problem 2.8 (WebAssign HW 9, 8)**.** First make a substitution and then use integration by parts to evaluate the integral. (Use C for the constant of integration.)

$$\int 7\cos\sqrt{x}\,\mathrm{d}x.$$

Proof. What's the most obvious substitution? Perhaps $w = \sqrt{x}$. Then, we get

$$\mathrm{d}w = -\frac{\mathrm{d}x}{2\sqrt{x}} = \frac{\mathrm{d}x}{2w}.$$

So our integral turns into

$$\int 14w\cos w\,\mathrm{d}w$$

Then, by integration by parts, setting $u = 14w$, we get

$$\int u\,\mathrm{d}v = \int 14w\cos w\,\mathrm{d}w$$

$$= 14w\sin w - \int 14\sin w\,\mathrm{d}w$$

$$= 14w\sin w - \int 14\sin w\,\mathrm{d}w$$

$$= 14w\sin w + 14\cos w + C.$$

Substituting back, we get our answer

$$\boxed{14\sqrt{x}\sin\sqrt{x} + 14\cos\sqrt{x} + C.}$$

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Problem 2.9 (WebAssign HW 9, 9)**.** Use the method of cylindrical shells to find the volume V generated by rotating the region bounded by the given curves about the specified axis.

$$y = 4e^x, \quad y = 4e^{-x}, \quad x = 1; \qquad \text{about the } y\text{-axis.}$$

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Solution. I give up on trying to plot these, (it's too much work for one person). By the method of cylindrical shells, we want to find an expression for the cross-sectional area (which has the form of a cylinder) in terms of our variable perpendicular to the axis we are rotating about. In this case, we are rotating about the y-axis, so we must solve for the cross-sectional area in terms of x like so:

$$A(x) = 2\pi x(4e^x - 4e^{-x}) = 8\pi x(e^x - e^{-x}).$$

Then, we integrate the area between where the curves $4e^x$, $4e^{-x}$ and $x = 1$ intersect (if you plot the curves, this will be the area of the difference of e^x and e^{-x} from $0 \leq x \leq 1$). Thus, the volume is

$$
\begin{aligned}
V &= \int_0^1 8\pi x(e^x - e^{-x}) \, \mathrm{d}x \\
&= 8\pi \left(x(e^x + e^{-x}) \big|_0^1 - \int_0^1 e^x + e^{-x} \, \mathrm{d}x \right) \\
&= 8\pi \left(x(e^x + e^{-x}) \big|_0^1 - \int_0^1 e^x + e^{-x} \, \mathrm{d}x \right) \\
&= 8\pi \left(x(e^x + e^{-x}) - e^x + e^{-x} \big|_0^1 \right) \\
&= 8\pi \left(e + e^{-1} - e + e^{-1} - (0 - 1 + 1) \right) \\
&= \boxed{\frac{16\pi}{e}}.
\end{aligned}
$$

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3   Homework 10 Solutions

Problem 3.1 (WebAssign HW 10, 1)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int 2\sin^2 x \cos^3 x \, \mathrm{d}x$$

Solution. First, note that $\cos^2 x = 1 - \sin^2 x$ so the integral above can be written as

$$\int 2\sin^2 x \cos^3 x \, \mathrm{d}x = \int 2\sin^2 x \cos^2 x \cos x \, \mathrm{d}x$$

$$= \int 2\sin^2 x (1 - \sin^2 x) \cos x \, \mathrm{d}x.$$

Now, make the substitution $u = \sin x$. Then $\mathrm{d}u = \cos x \, \mathrm{d}x$ so our integral is now

$$\int 2u^2 (1 - u^2) \cos x \frac{\mathrm{d}u}{\cos x} = \int 2u^2 (1 - u^2) \, \mathrm{d}u$$

$$= \int 2u^2 - 2u^4 \, \mathrm{d}u$$

$$= \tfrac{2}{3} u^3 - \tfrac{2}{5} u^5 + C$$

$$= \boxed{\tfrac{2}{3} \sin^3 x - \tfrac{2}{5} \sin^5 x + C.}$$

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Problem 3.2 (WebAssign HW 10, 2)**.** Evaluate the integral

$$\int_0^{\pi/2} 5\cos^2 \theta \, \mathrm{d}\theta.$$

Solution. For this problem, we can use the double-angle identity, i.e.,

$$\cos 2\theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 x \tag{8}$$

Substituting Equation (8) into our integral, we have

$$\int_0^{\pi/2} 5\cos^2 \theta \, \mathrm{d}\theta = \int_0^{\pi/2} \tfrac{5}{2}(1 + \cos 2\theta) \, \mathrm{d}\theta$$

$$= \tfrac{5}{2} \int_0^{\pi/2} (1 + \cos 2\theta) \, \mathrm{d}\theta$$

$$= \tfrac{5}{2} \left(\theta + \tfrac{1}{2}\sin 2\theta \big|_0^{\pi/2} \right)$$

$$= \tfrac{5}{2} \left(\frac{\pi}{2} + 0 - (0 + 0) \right)$$

$$= \boxed{\frac{5\pi}{4}.}$$

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Problem 3.3 (WebAssign HW 10, 3)**.** Evaluate the integral.

$$\int_0^{\pi/2} 5\sin^2 x \cos^2 x \, \mathrm{d}x.$$

Solution. Again, we just use a trigonometric identity, specifically

$$\sin 2x = 2\sin x \cos x. \tag{9}$$

Using Equation (9) our integral turns into

$$\int_0^{\pi/2} 5\sin^2 x \cos^2 x \, \mathrm{d}x = 5\int_0^{\pi/2} \tfrac{1}{4}(4\sin^2 x \cos^2 x) \, \mathrm{d}x$$

$$= \frac{5}{4}\int_0^{\pi/2} (2\sin x \cos x)^2 \, \mathrm{d}x$$

$$= \frac{5}{4}\int_0^{\pi/2} (\sin 2x)^2 \, \mathrm{d}x$$

$$= \frac{5}{4}\int_0^{\pi/2} \sin^2 2x \, \mathrm{d}x$$

$$= \frac{5}{4}\int_0^{\pi/2} \sin^2 2x \, \mathrm{d}x$$

now we use this identity $2\sin^2 x = 1 - \cos 2x$ to get

$$= \frac{5}{4}\int_0^{\pi/2} \frac{1}{2}(1 - \cos 4x) \, \mathrm{d}x$$

$$= \frac{5}{8}\int_0^{\pi/2} 1 - \cos 4x \, \mathrm{d}x$$

$$= \frac{5}{8}\left(x - \tfrac{1}{4}\sin 4x \big|_0^{\pi/2}\right)$$

$$= \boxed{\tfrac{5\pi}{16}.}$$

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Problem 3.4 (WebAssign HW 10, 4)**.** Evaluate the integral. (Remember to use $\ln(|u|)$ where appropriate. Use C for constant of integration.)

$$\int 9\cos^2 x \tan^3 x \, \mathrm{d}x.$$

Solution. Let's make the substitution $u = \cos x$ so $\mathrm{d}u = -\sin x \, \mathrm{d}x$ and we have

$$\int 9\cos^2 x \tan^3 x \, \mathrm{d}x = 9\int \cos^2 x \tan^3 x \, \mathrm{d}x$$

$$= 9\int \frac{\cos^2 x \sin^3 x}{\cos^3 x} \, \mathrm{d}x$$

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$$= 9 \int \frac{\sin^3 x}{\cos x} \, dx$$

$$= 9 \int \frac{\sin x(1 - \cos^2 x)}{\cos x} \, dx$$

now, for the substitution,

$$= 9 \int \frac{\sin x(1 - u^2)}{u} \frac{du}{-\sin x}$$

$$= 9 \int \frac{u^2 - 1}{u} \, du$$

$$= 9 \int u - \frac{1}{u} \, du$$

$$= \tfrac{9}{2} u^2 - 9 \ln |u| + C$$

and back, we have

$$= \boxed{\tfrac{9}{2} \cos^2 x - 9 \ln |\cos x| + C.}$$

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Problem 3.5 (WebAssign HW 10, 5). Evaluate the integral. (Use C for the constant of integration.)

$$\int 5 \tan^2 x \, dx.$$

Solution. The easiest one so far. Recall the identity

$$\sec^2 x - \tan^2 x = 1. \tag{10}$$

Then, we gave

$$\int 5 \tan^2 x \, dx = 5 \int (\sec^2 x - 1) \, dx = \boxed{5 \tan x - 5x + C.}$$

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Problem 3.6 (WebAssign HW 10, 6).

$$\int 8 \left(\tan^2 x + \tan^4 x \right) dx.$$

Solution. We'll make the substitution $u = \tan x$, so $du = \sec^2 x \, dx$. Proceeding, we have

$$\int 8 \left(\tan^2 x + \tan^4 x \right) dx = 8 \int \tan^2 x(1 + \tan^2 x) \, dx$$

by Equation (10), we have

$$= 8 \int \tan^2 x \sec^2 x \, dx$$

$$= 8 \int u^2 \, du$$

$$= \tfrac{8}{3} u^3 + C$$

$$= \boxed{\tfrac{8}{3} \tan^3 x + C.}$$

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Problem 3.7 (WebAssign HW 10, 7). Evaluate the integral

$$\int_{\pi/6}^{\pi/2} \cot^2 x \,\mathrm{d}x$$

Solution. Remember that, like Equation (10), there is a cotangent-identity

$$1 + \cot^2 x = \csc^2 x. \tag{11}$$

With the help of this equation, we have

$$
\begin{aligned}
\int_{\pi/6}^{\pi/2} \cot^2 x \,\mathrm{d}x &= \int_{\pi/6}^{\pi/2} (\csc^2 - 1) \,\mathrm{d}x \\
&= -\cot x - x \Big|_{\pi/6}^{\pi/2} \\
&= 0 - \frac{\pi}{2} - \left(\sqrt{3} - \frac{\pi}{6}\right) \\
&= \boxed{\sqrt{3} - \frac{\pi}{3}.}
\end{aligned}
$$

Problem 3.8 (WebAssign HW 10, 8). Find the volume V obtained by rotating the region bounded by the given curves about the specified axis.

$$y = 9\sin x, \quad y = 0, \quad \frac{\pi}{2} \leq x \leq \pi; \qquad \text{about the } x\text{-axis.}$$

Solution. Since we are rotating about the x-axis, we need to find the cross-sectional area in terms of y in terms of x, hence we have

$$A(x) = \pi y^2 = 81\pi \sin^2 x.$$

Now, integrating from $\pi/2$ to π, we get

$$\int_{\pi/2}^{\pi} A(x) \,\mathrm{d}x = 81\pi \int_{\pi/2}^{\pi} \sin^2 x$$

using the double-angle identity for cosine, Equation (8), we get

$$
\begin{aligned}
&= 81\pi \int_{\pi/2}^{\pi} \tfrac{1}{2}(1 - \cos 2x) \,\mathrm{d}x \\
&= \tfrac{81\pi}{2} \left(x - \tfrac{1}{2}\sin 2x \Big|_{\pi/2}^{\pi} \right) \\
&= \tfrac{81\pi}{2} \left(\pi - 0 - (\tfrac{1}{2}\pi) \right) \\
&= \boxed{\frac{81\pi^2}{4}.}
\end{aligned}
$$