PHYS630 Problem Set 1

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August 28, 2015

Problem 1.1

Consider the transmission line described in Jackson's problem 1.7. Suppose the conductors carry charges $\pm Q$.

(a) Find the electric field at any point of the plane passing through the axes of the conductors. (Please neglect the end effects, i.e., assume the length of the line to be effectively infinite.)
(b) Using your result from (a), verify the formula given in the problem for the capacitance. Why is the formula listed as approximate?

Solution. (a) From Gauss's law, the electric field due to one conductor is

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r} \tag{1}$$

where Q is the charge per unit length and r is the perpendicular distance from the point of interest, say P, to the conductor. Along the perpendicular line joining the two conductors, the field due to to the two conductors are in the same direction. Therefore, the total field along this line is

$$\vec{E} = \frac{Q}{2\pi\varepsilon_0 r_1} + \frac{Q}{2\pi\varepsilon_0 r_2} \tag{2}$$

where r_1 and r_2 are the perpendicular distances from the point P to each conductor. The electric field $\vec{E}$ points from $+Q$ to $-Q$ and the potential difference is given by the expression

$$\begin{aligned} V &= \Phi_1 + \Phi_2 \\ &= -\int \vec{E}\cdot \mathrm{d}\vec{\ell} \\ &= \frac{Q}{2\pi\varepsilon_0}\left(\int_{a_1}^{d-a_2} \frac{1}{r_1}\,\mathrm{d}\,r_1 + \int_{a_2}^{a_1-d} \frac{1}{r_2}\,\mathrm{d}\,r_2\right) \\ &= \frac{Q}{2\pi\varepsilon_0}\log((d-a_1)(d-a_2)/a_1a_2). \end{aligned}$$

Thus the capacitance is

$$\begin{aligned} C &= Q/V \\ &= \frac{2\pi\varepsilon_0}{\log\big(d^2/\sqrt{a_1a_2}^2\big)} \\ &\approx \frac{Q}{2\pi\varepsilon_0}\log\left(\frac{d^2}{\sqrt{a_1a_2}^2}\right) \\ &= \frac{\pi\varepsilon_0}{\log(d/a)}. \end{aligned}$$

(b) ■

Problem 1.2

An uncharged conducting sphere of a radius a is placed in the electric field produced by some large distant conductors (basically, the plates of a large parallel capacitor). Without the sphere, the field was uniform (i.e., independent of the location) and equal to $\mathbf{E}_0 = E_0\hat{\mathbf{z}}$. We will be looking for the field $\mathbf{E}(x, y, z)$ in the presence of the sphere.

Look for a solution to the Laplace equation outside the sphere in the form

$$\varphi(r, \theta) = -E_0 z + \sum_{\ell=1}^{\infty} \frac{c_\ell P_\ell(\cos\theta)}{r^{\ell+1}},$$

where r, θ, φ are spherical coordinates (there is no dependence on φ), $z = r\cos\theta$, P_ℓ are the Legendre polynomials, and c_ℓ are the expansion coefficients to be found.

(a) Use the boundary condition at the surface of the plane to find all c_ℓ.
(b) Find the Cartesian components of the electric field $\vec{E}$.
(c) Find the change in the electrostatic energy caused by the presence of the sphere.

Solution. (a)

(b)

(c) ■