

## MA 26500-215 Quiz 6

July 28, 2016

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \\ 0 & -1 & -1 \end{bmatrix}. \quad (\star)$$

- (a) (12 points) Recall that the **nullspace** of an  $m \times n$  matrix  $A$  is the set of vectors  $\mathbf{x}$  in  $\mathbb{R}^m$  such that  $A\mathbf{x} = \mathbf{0}$ . This subset spans a subspace of  $\mathbb{R}^m$ . Give a description of the nullspace of the matrix  $(\star)$  by writing down basis for the nullspace.

[HINT: You should begin by putting the matrix in rref.]

**Solution:** We know from Kolman and Hill that elementary row operations do not change the nullspace of a matrix. Therefore, our first step should be to find the row-reduced echelon form  $A$ ; call it  $A_{\text{rref}}$ . After doing some calculations to the side, we get that

$$A_{\text{rref}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now,  $\mathbf{x} = (x_1, x_2, x_3, x_4)$  is in the nullspace of  $A_{\text{rref}}$  if  $A_{\text{rref}}\mathbf{x} = (0, 0, 0, 0)$ . When does this happen? Well

$$A_{\text{rref}}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so  $x_1 = x_2 = x_3 = 0$ . This forces  $\mathbf{x}$  to be  $(0, 0, 0, 0)$  so the nullspace of  $A_{\text{rref}}$  (which is the same as the nullspace of  $A$ ) is  $\{\mathbf{0}\}$ , hence, it has no basis.

- (b) (8 points) The **range** or **columnspace** of an  $m \times n$  matrix  $A$  is the set of vectors  $\mathbf{y}$  in  $\mathbb{R}^n$  that are, in some sense, “hit” by vectors  $\mathbf{x}$  in  $\mathbb{R}^n$  by the matrix  $A$ , i.e.,  $\mathbf{y} = A\mathbf{x}$  for some  $\mathbf{x}$ . Using your calculations from above (the hint), write down a basis for the range of  $(\star)$ .

**Solution:** Using the row reduced echelon of the matrix above, we see that

$$A_{\text{rref}}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence, the set  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$  is a basis for the range. **Note** this is not a basis for  $\mathbb{R}^4$  which is a 4-dimensional vector space.