MA 519: Homework 9

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Problem 9.1 (Handout 13, # 7)

Let X have a *double exponential* density $f(x) = \frac{1}{2\sigma}\mathrm{e}^{-\frac{|x|}{\sigma}}$, $-\infty < x < \infty$, $\sigma > 0$.

(a) Show that all moments exist for this distribution.
(b) However, show that the MGF exists only for restricted values. Identify them and find a formula.

Solution. For part (a), we show that $E(X^n) < \infty$ for all $n \in \mathbb{N}$. By direct calculation, we have

$$\begin{aligned} E(X^n) &= \int_{-\infty}^{\infty} x^n f(x)\,dx \\ &= \int_{-\infty}^{\infty} \frac{x^n}{2\sigma}\mathrm{e}^{-\frac{|x|}{\sigma}}\,dx \\ &= \underbrace{\int_{-\infty}^{0} \frac{x^n}{2\sigma}\mathrm{e}^{\frac{x}{\sigma}}\,dx}_{L} + \int_{0}^{\infty} \frac{x^n}{2\sigma}\mathrm{e}^{-\frac{x}{\sigma}}\,dx \end{aligned}$$

making the substitution $x \mapsto -y$ to L and relabeling y to x again, we see that

$$\begin{aligned} &= \int_{-\infty}^{0} \frac{x^n + (-1)x^n}{2\sigma}\mathrm{e}^{-\frac{x}{\sigma}}\,dx \\ &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ I := \int_0^\infty \frac{x^n}{\sigma}\mathrm{e}^{-\frac{x}{\sigma}}\,dx & \text{if } n = 2k \text{ is even.} \end{cases} \end{aligned}$$

To evaluate I we use integration by parts recursively to arrive at

$$\begin{aligned} I &= \int_{-\infty}^{0} \frac{x^n}{\sigma}\mathrm{e}^{-\frac{x}{\sigma}} \\ &= (-0+0) + \int_0^\infty n\sigma x^{n-1}\mathrm{e}^{-\frac{x}{\sigma}}\,dx \\ &= (-0+0) + (-0+0) + \int_0^\infty n(n-1)\sigma^2 x^{n-1}\mathrm{e}^{-\frac{x}{\sigma}}\,dx \qquad (9.1) \\ &\vdots \\ &= (-0-0) + \cdots + (-0+0) + (-0+n!\sigma^n) \\ &= n!\sigma^n. \end{aligned}$$

Therefore, $E(X^n) < \infty$ for all $n \in \mathbb{N}$, i.e., all moments of this distribution exist.

For part (b), the MGF associated to f is

$$m(t) = \sum_{i=0}^{\infty} \frac{t^i E(X^i)}{i!} = \sum_{i=0}^{\infty} t^i\sigma^i.$$

This is a geometric series and so converges for all $-\frac{1}{\sigma} < t < \frac{1}{\sigma}$. ■

Problem 9.2 (Handout 13, # 16)

Give an example of each of the following phenomena:

(a) A continuous random variable taking values in $[0, 1]$ with equal mean and median.
(b) A continuous random variable taking values in $[0, 1]$ with mean equal to twice the median.
(c) A continuous random variable for which the mean does not exist.
(d) A continuous random variable for which the mean exists, but the variance does not exist.
(e) A continuous random variable with a PDF that is not differentiable at zero.
(f) a positive continuous random variable for which the mode is zero, but the mean does not exist.
(g) A continuous random variable for which all moments exist.
(h) A continuous random variable with median equal to zero, and 25^{th} and 75^{th} percentiles equal to 1.
(i) A continuous random variable X with mean equal to median equal to mode equal to zero, and $E(\sin X) = 0$.

Solution. ∎

Problem 9.3 (Handout 13, # 17)

An exponential random variable with mean 4 is known to be larger than 6. What is the probability that it is larger than 8?

Solution. ■

Problem 9.4 (Handout 13, # 18)

(Sum of Gammas). Suppose X, Y are independent random variables, and $X \sim G(\alpha, \lambda)$, $Y \sim G(\beta, \lambda)$. Find the distribution of $X + Y$ by using moment-generating functions.

Solution. ■

Problem 9.5 (Handout 13, # 19)

(Product of Chi Squares). Suppose $X_1, X_2, \ldots, X_n$ are independent chi square variables, with $X_i \sim \chi^2_{m_i}$. Find the mean and variance of $\prod_{i=1}^n X_i$.

Solution. ■

Problem 9.6 (Handout 13, # 20)

Let $Z \sim N(0,1)$. Find

$$P\big(0.5 < |Z - \tfrac{1}{2}| < 1.5\big); \quad P\Big(\tfrac{e^Z}{1+e^Z} > \tfrac{3}{4}\Big); \quad P(\Phi(Z) < 0.5).$$

Solution. ■

PROBLEM 9.7 (HANDOUT 13, # 21)

Let $Z \sim N(0,1)$. Find the density of $\frac{1}{Z}$. Is the density bounded?

SOLUTION. ∎

Problem 9.8 (Handout 13, # 22)

The 25^{th} and the 75^{th} percentile of a normally distributed random variable are -1 and 1. What is the probability that the random variable is between -2 and 2?

Solution. ■