MA52300 Fall 2016

Midterm Exam Practice Problems

1. Solve $u_{x_1}^2 + x_2 u_{x_2} = u$ with initial conditions $u(x,1) = \frac{x^2}{4} + 1$.

2. Find the maximal $t_0 > 0$ for which the (classical) solution of the Cauchy problem

$$uu_x + u_t = 0, \quad u(x,0) = e^{-x^2/2},$$

exists in $\mathbb{R} \times [0,t)$; i.e., find the first time $t = t_0$ when the shock develops.

3. If ρ_0 denotes the maximum density of cars on a highway (i.e., under bumper-to-bumper conditions), then a reasonable model for traffic is density ρ is given by

$$\rho_t + (F(\rho))_x = 0, \quad F(\rho) = c\rho(1 - \rho/\rho_0),$$

where $c > 0$ is a constant (free speed of highway). Suppose the initial density is

$$\rho(x,0) = \begin{cases} \frac{1}{2}\rho_0 & x < 0 \\ \rho_0 & x > 0. \end{cases}$$

Find the shock curve and describe the weak solution. (Interpret your result for the traffic flow.)

4. Find the characteristics of the second order equation

$$u_{xx} - 2\cos x\, u_{xy} - (3 + \sin^2 x)u_{yy} - yu_y = 0.$$

and transform it to the canonical form.

5. Let $Lu := u_{xx} - 4u_{yy} + \sin(y + 2x)\, u_x = 0$.
   (a) Consider the level curve $\Gamma = \{(x,y) : \phi(x,y) = C\}$ where $|D\phi| \neq 0$ on Γ. Define what it means for Γ to be characteristic with respect to L at a point $(x_0, y_0) \in \Gamma$.
   (b) Find the points at which the curve $x^2 + y^2 = 5$ is characteristic.
   (c) Is it true that every smooth simple closed curve Γ in $\mathbb{R}^2$ has at least one point at which it is characteristic with respect to L?

6. Consider the second order equation

$$Lu := u_{xx} - 2xu_{xy} + x^2 u_{yy} - 2u_y = 0.$$

   (a) Find the characteristics curves of $Lu = 0$. What is the type of this equation?
   (b) Find the points on the line $\Gamma := \{(x,y) \in \mathbb{R}^2 : x + y = 1\}$ at which Γ is characteristic with respect to $Lu = 0$.

7. Solve the initial boundary value problem for the equation $u_{tt} = u_{xx}$ in $\{x > 0, t > 0\}$ satisfying

$$u(x,0) = \sin^2 x, \quad u_t(x,0) = \sin x, \quad u(0,t) = 0.$$

8. Consider the initial/boundary value problem

$$\begin{aligned} &u_{tt} - u_{xx} = 0, && 0 < x < \pi, t > 0 \\ &u(x,0) = x,\ u_t(x,0) = 0, && 0 < x < \pi \\ &u_x(0,t) = 0,\ u_x(\pi,t) = 0, && t > 0 \end{aligned}$$

   (a) Find a weak solution of the problem.

   (b) Is the solution unique? Continuous? C^1?

9. Let B_1^+ denote the open half-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$. Assume $u \in C(\overline{B}_1^+)$ is harmonic in B_1^+ with $u = 0$ on $\partial B_1^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

   for $x \in B_1$. Prove v is harmonic in B_1.

10. Let u and v be harmonic functions in the unit ball $B_1 \subset \mathbb{R}^n$. What can you conclude about u and v

    (a) if $D^\alpha u(0) = D^\alpha v(0)$ for every multi-index α?

    (b) if $u(x) \leq v(x)$ for every $x \in B_1$ and $u(0) = v(0)$?

    Justify your answer in each of the cases.

11. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$$

    is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial (i.e. $f(y) = f(|y|)$) and supported in $B_R := \{|x| < R\}$, then

$$u(x) = c\,\Phi(x), \quad \text{for any} \quad x \in \mathbb{R}^n \setminus B_R,$$

    where $c = \int_{\mathbb{R}^n} f(y) dy$.

    Hint: Use polar (spherical) coordinates and apply the mean value property for harmonic functions.