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MA 26500-215 Quiz 6

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1. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \\ 0 & -1 & -1 \end{bmatrix}. \tag{$\bigstar$}$$

(a) (12 points) Recall that the **nullspace** of an $m \times n$ matrix A is the set of vectors $\mathbf{x}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{0}$. This subset spans a subspace of $\mathbb{R}^m$. Give a description of the nullspace of the matrix ($\bigstar$) by writing down basis for the nullspace.
[Hint: You should begin by putting the matrix in rref.]

Solution: We know from Kolman and Hill that elementary row operations do not change the nullspace of a matrix. Therefore, our first step should be to find the row-reduced echelon form A; call it A_{rref}. After doing some calculations to the side, we get that

$$A_{\text{rref}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is in the nullspace of A_{rref} if $A_{\text{rref}}\mathbf{x} = (0,0,0,0)$. When does this happen? Well

$$A_{\text{rref}}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so $x_1 = x_2 = x_3 = 0$. This forces $\mathbf{x}$ to be $(0,0,0,0)$ so the nullspace of A_{rref} (which is the same as the nullspace of A) is $\{\mathbf{0}\}$, hence, it has no basis.

(b) (8 points) The **range** or **columnspace** of an $m \times n$ matrix A is the set of vectors $\mathbf{y}$ in $\mathbb{R}^n$ that are, in some sense, "hit" by vectors $\mathbf{x}$ in $\mathbb{R}^n$ by the matrix A, i.e., $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$. Using your calculations from above (the hint), write down a basis for the range of ($\bigstar$).

Solution: Using the row reduced echelon of the matrix above, we see that

$$A_{\text{rref}}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence, the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the range. **Note** this is not a basis for $\mathbb{R}^4$ which is a 4-dimensional vector space.