Fall 2016 Notes

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Contents

Chapter 1

Probability

We will devote this chapter to the material that is covered in MA 51900 (discrete probability) as it was covered in DasGupta's class. We will, for the most part, reference Feller's *An introduction to probability theory and its applications, Volume 1* [4] (especially for the discrete noncalculus portion of the class) and DasGupta's own book *Fundamentals of Probability: A First Course* [2].

1.1 Discrete Probability

The material in this chapter is mostly pulled from Sheldon Ross's *A First Course in Probability Theory* [8] with some examples from [2] and [4]. I find Ross's book to be better structured than the latter two.

Combinatorial Analysis

These are the main results from this section.

Theorem 1.1 (The basic principle of counting)**.** *Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.*

Theorem 1.2 (The generalized principle of counting)**.** *If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes for the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes for the third experiment; etc. ..., then there is a total of $n_1 n_2 \cdots n_r$ possible outcomes of the r experiments.*

Using notation as in [4], the number

$$(n)_r = n(n-1)\cdots(n-r+1)$$

represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{(n)_r}{r!} = \frac{n!}{(n-r)!r!}$$

for which we reserve the notation

$$\binom{n}{r}$$

read n choose r. (This is called a binomial coefficient since it appears in the binomial expansion $(a+b)^n$.)

A useful combinatorial identity on binomial coefficients is the following

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

for $1 \leq r \leq n$.

Theorem 1.3 (The binomial theorem)**.**

$$(a+b)^n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}.$$

Proof. We provide a combinatiorial proof of the theorem. Consider the product

$$(a_1 + b_1) \cdots (a_n + b_n).$$

Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Furthermore, each of the 2^n terms in the sum will contain as a factor either a_i or b_i for each $1 \leq i \leq n$. Now, how many of the 2^n terms in the sum will have k of the a_i and $n-k$ of the b_i as factors? As each term consisting of k of the a_i and $n-k$ of the b_i correspond to a choice of a group of k from the values $a_1, \ldots, a_n$, there are $\binom{n}{k}$ such terms. Thus, letting $a_i = a$, $b_i = b$, $1 \leq i \leq n$, we see that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

■

Chapter 2

Introduction to Partial Differential Equations

Here we summarize some important points about PDEs. The material is mostly taken from Evans's *Partial Differential Equations* [3] with occasional detours to Strauss's *Partial Differential Equations: An Introduction* [9]. We will be following Dr. Petrosyan's Course Log which can be found here `https://www.math.purdue.edu/~arshak/F16/MA523/courselog/`, i.e., summarizing the appropriate chapters from [3].

2.1 Introduction

Partial differential equations

Definition 2.1. An expression of the form

$$F\big(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\big) = 0, \quad x \in U, \tag{2.1}$$

is called a *kth-order partial differential equation* (*PDE*), where

$$F\colon \mathbf{R}^{n^k} \times \mathbf{R}^{n^{k-1}} \times \cdots \times \mathbf{R}^n \times U \longrightarrow \mathbf{R}$$

is given, and

$$u\colon U \longrightarrow \mathbf{R}$$

is the unknown.

Here are some more definitions,

Definition 2.2.

(i) The partial differential equation (2.1) is called *linear* if it has the form

$$\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u = f(x)$$

for given functions $a_\alpha(|\alpha| \le k)$, f. This linear PDE is *homogeneous* if $f = 0$.

(ii) The PDE (2.1) is *semilinear* if it has the form

$$\sum_{|\alpha|=k} a_\alpha D^\alpha u + a_0\big(D^{k-1}u, \ldots, Du, u, x\big) = 0.$$

(iii) The PDE (2.1) is *quasilinear* if it has the form

$$\sum_{|\alpha|=k} a_\alpha\big(D^{k-1}u, \ldots, Du, u, x\big) D^\alpha u + a_0\big(D^{k-1}u, \ldots, Du, u, x\big) = 0.$$

(iv) The PDE (2.1) is *fully nonlinear* if it depends upon the highest order derivatives.

A *system* of partial differential equations is, informally speaking, a collection of several PDEs for several unknown functions.

Definition 2.3. An expression of the form

$$\mathbf{F}\big(D^k\mathbf{u}(x), D^{k-1}\mathbf{u}(x), \ldots, D\mathbf{u}(x), \mathbf{u}(x), x\big) = 0, \quad x \in U, \tag{2.2}$$

is called a *kth-order system of PDEs*, where

$$\mathbf{F}\colon \mathbf{R}^{mn^k} \times \mathbf{R}^{mn^{k-1}} \times \cdots \times \mathbf{R}^{mn} \times \mathbf{R}^m \times U \longrightarrow \mathbf{R}^m$$

is given and

$$\mathbf{u}\colon U \longrightarrow \mathbf{R}^m, \quad \mathbf{u} = (u^1, \ldots, u^m)$$

is the unknown.

Remark 2.4. We haven't talked much about systems of PDEs and I suspect we will not do so very much in this course.

Examples

This is only a fraction of the PDEs listed in Evan's chapter.

Linear equations

1. Laplace's equation

$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0.$$

2. Helmholtz's (or eigenvalue) equation

$$-\Delta u = \lambda u.$$

3. Linear transport equation

$$u_t + \sum_{i=1}^n b^i u_{x_i} = 0.$$

4. Liouville's equation
$$u_t - \sum_{i=1}^{n}(b^i u)_{x_i} = 0.$$
5. Heat (or diffusion) equation
$$u_t - \Delta u = 0.$$
6. Wave equation
$$u_{tt} - \Delta u = 0.$$
7. Telegraph equation
$$u_{tt} + du_t - u_{xx} = 0.$$

Nonlinear equations

1. Eikonal equation
$$|Du| = 1.$$
2. Nonlinear Poisson equation
$$-\Delta u = f(u).$$
3. Inviscid Burgers' equation
$$u_t + uu_x = 0.$$

and so on.

2.2 The transport equation

We begin our study with one of the simplest PDEs, the *transport equation* with constant coefficients. This is the PDE
$$u_t + b \cdot Du = 0, \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{2.3}$$
where b is a fixed vector in $\mathbf{R}^n$, $b = (b_1, \ldots, b_n)$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ is a typical point in space, $t \geq 0$ denotes a typical time and $u\colon \mathbf{R} \times [0,\infty) \to \mathbf{R}$ is the unknown, $u = u(x,t)$. We write $Du = D_x u = (u_{x_1}, \ldots, u_{x_n})$ for the gradient of u with respect to the spatial variable x.

So, which functions solve (2.3)? Well, let us suppose for a moment that u is a smooth solution to the PDE and let us try to compute it. To do so, we first recognize that (2.3) asserts that a particular directional derivative of u vanishes, namely, $D_b u = 0$. We exploit this by fixing a point $(x,t) \in \mathbf{R}^n \times (0,\infty)$ and defining
$$z(s) := u(x + sb, t + s), \quad s \in \mathbf{R}.$$
Then we calculate
$$\begin{aligned} \dot{z}(s) &= Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) \\ &= 0, \end{aligned}$$
the second equality holding by (2.3). Thus, z is a constant function of s, and consequently for each (x,t), u is constant on the line through (x,t) with direction $(b,1) \in \mathbf{R}^{n+1}$. Hence, if we know the value of u at any point on ecah such line, we know its value everywhere in $\mathbf{R}^n \times (0,\infty)$.

2.3 Characteristics

Derivation of characteristic ODEs

Consider the nonlinear first-order PDE

$$F(Du, u, x) = 0 \quad \text{in } U, \tag{2.4}$$

subject now to the boundary condition

$$u = g \quad \text{on } \Gamma, \tag{2.5}$$

where $\Gamma \subseteq \partial U$ and $g \colon \Gamma \to \mathbf{R}$ are given. We hereafter suppose that F, g are smooth functions.

We now develop the method of *characteristics*, which solves (2.4) and (2.5) by converting the PDE into an appropriate system of ODEs. Suppose u solves the (2.4), (2.5) and fix any point $x \in U$. We would like to calculate $u(x)$ by finding some curve lying within U, connecting x with a point $x^0 \in \Gamma$ and along which we can compute u. Since (2.5) says $u = g$ on Γ, we know the value of u at the one end x^0. We hope then to be able to calculate u all along the curve, and so in particular at x.

Finding the characteristic ODEs

How can we choose the curve so all this will work? Let us suppose it is described parametrically by the function $\mathbf{x}(s) = \big(x^1(s), \ldots, x^n(s)\big)$, the parameter s lying in some subinterval of $\mathbf{R}$. Assuming u is a C^2 solution of (2.4), we define also

$$z(s) := u\big(\mathbf{x}(s)\big).$$

In addition, set

$$\mathbf{p}(s) := Du(\mathbf{x}(s));$$

that is, $\mathbf{p}(s) = \big(p^1(s), \ldots, p^n(s)\big)$, where

$$p^i(s) = u_{x_i}\big(\mathbf{x}(s)\big), \tag{2.6}$$

$1 \leq i \leq n$. So z gives the values of u along the curve and $\mathbf{p}$ records the values of the gradient Du. We must choose a function $\mathbf{x}$ in such a way that we can compute z and $\mathbf{p}$.

For this, first differentiate (2.6)

$$\dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}\big(\mathbf{x}(s)\big)\dot{x}^j(s)$$

This expression is not too promising, since it involves the second derivatives of u. On the other hand, we can also differentiate the PDE (2.4) with respect to x_i to get

$$\sum_{j=1}^{n} \frac{\partial}{\partial p_j} F(Du, u, x) u_{x_j x_i} + \frac{\partial}{\partial z} F(Du, u, x) u_{x_i} + \frac{\partial}{\partial x_i} F(Du, u, x) = 0.$$

We are able to employ this identity to get rid of the *dangerous* second derivative terms provided we first set

$$\dot{x}^j(s) = \frac{\partial}{\partial p_j} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big).$$

Assuming now that the above equation holds, we can evaluate the partials

$$\sum_{j=1}^{n} \frac{\partial}{\partial p_j} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) \\ + \frac{\partial}{\partial z} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) p^i(s) + \frac{\partial}{\partial x_i} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) = 0.$$

Substitute this expression and the previous one into the derivative for $\dot{p}^i$ and we get

$$\dot{p}^i(s) = \frac{\partial}{\partial x_i} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big)$$

Finally, we differentiate z to get

$$\dot{z}(s) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} u\big(\mathbf{x}(s)\big) \dot{x}^j(s) = \sum_{j=1}^{n} p^j(s) \frac{\partial}{\partial p_j} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big),$$

the second equality holding by –fuck this guy for numbering every expression–(5) and (8)–whatever they are.

We summarize by rewriting equations (8)–(10) in vector notation:

$$\begin{cases} \text{(a) } \dot{\mathbf{p}}(s) = -D_x F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) - D_z F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big)\mathbf{p}(s), \\ \text{(b) } \dot{z}(s) = D_p F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) \cdot \mathbf{p}(s), \\ \text{(c) } \dot{\mathbf{x}}(s) = D_p F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big). \end{cases} \tag{2.7}$$

This important system of $2n+1$ first-order ODEs comprises the *characteristic equations* of the nonlinear first-order PDE (2.4). The functions $\mathbf{p} = (p^1, \ldots, p^n)$, z, $\mathbf{x} = (x^1, \ldots, x^n)$ are a called the *characteristics*. We will sometimes refer to $\mathbf{x}$ as the *projected characteristics*: it is the projection of the full characteristics $(\mathbf{p}, z, \mathbf{x}) \subseteq \mathbf{R}^{2n+1}$ onto the physical region $U \subseteq \mathbf{R}^n$.

Theorem 2.5 (Structure of characteristic ODEs)**.** *Let $u \in C^2(U)$ solve the nonlinear, first-order partial differential equation* (2.4) *in U. Assume $\mathbf{x}$ solves the ODEs* (2.7)(c)*, where $\mathbf{p} = Du$, $z = u$. Then $\mathbf{p}$ solves the ODE* (2.7)(a) *and z solves the ODE* (2.7)(b)*, for those s such that $\mathbf{x} \in U$.*

Examples

F linear

Consider first the situation that (2.4) is linear and homogeneous, and thus has the form

$$F(Du, u, x) = \mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0, \qquad x \in U.$$

Then $F(p, z, x) = \mathbf{b}(x) \cdot p + c(x)z$, and so

$$D_p F = \mathbf{b}(x).$$

In this circumstance (2.7)(c) becomes

$$\mathbf{x}(s) = \mathbf{b}\big(\mathbf{x}(s)\big),$$

an ODE involving only the function $\mathbf{x}$. Furthermore (2.7)(b) becomes

$$\dot{z}(s) = \mathbf{b}\big(\mathbf{x}(s)\big) \cdot \mathbf{p}(s). \tag{2.8}$$

Since $\mathbf{p}(\cdot) = Du\big(\mathbf{x}(\cdot)\big)$, the PDE simplifies the above to

$$\dot{z}(s) = -c\big(\mathbf{x}(s)\big)z(s).$$

This ODE is linear in z, once we know the function $\mathbf{x}$ by solving its ODE.

In summary,

$$\begin{cases} \text{(a) } \dot{\mathbf{x}}(s) = \mathbf{b}\big(\mathbf{x}(s)\big) \\ \text{(b) } \dot{z}(s) = -c\big(\mathbf{x}(s)\big)z(s) \end{cases} \tag{2.9}$$

comprise the characteristic equations for the linear, first-order PDE (2.8).

Example 2.6. We demonstrate the utility of equations (2.9) by explicitly solving the problem

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } U \\ \hfill u = g & \text{on } \Gamma, \end{cases} \tag{2.10}$$

where U is the quadrant $\{\, x_1 > 0, x_2 > 0 \,\}$ and $\Gamma = \{\, x_1 > 0, x_2 = 0 \,\} \subseteq \partial U$. The PDE in (2.10) is of the form (2.8), for $\mathbf{b} = (-x_2, x_1)$ and $c = -1$. Thus the equations (2.9) read

$$\begin{cases} (x^1, x^2)(s) = (x^0 \cos s, x^0 \sin s) \\ \hfill z(s) = z^0 \mathrm{e}^s = g(x^0)\mathrm{e}^s, \end{cases} \tag{2.11}$$

where $x^0 \geq 0$, $0 \leq s \leq \pi/2$. Fix a point $(x_1, x_2) \in U$. We select $s > 0$, $x^0 > 0$ so that $(x_1, x_2) = \big(x^1(s), x^2(s)\big) = (x^0 \cos s, x^0 \sin s)$. That is, $x^0 = \big({x_1}^2 + {x_2}^2\big)^{1/2}$, $s = \arctan(x_2/x_1)$. Therefore,

$$\begin{aligned} u(x_1, x_2) &= u\big(x^1(s), x^2(s)\big) \\ &= z(s) \\ &= g(x^0)\mathrm{e}^s \\ &= g\big(({x_1}^2 + {x_2}^2)^{1/2}\big)\mathrm{e}^{\arctan(x_2/x_1)}. \end{aligned}$$

F quasilinear

The PDE (2.4) is quasilinear if it has the form

$$F(Du, u, x) = \mathbf{b}\big(x, u(x)\big) \cdot Du(x) + c\big(x, u(x)\big) = 0. \tag{2.12}$$

In this circumstance $F(p, z, x) = \mathbf{b}(x, z) \cdot p + c(x, z)$; whence

$$D_p F = \mathbf{b}(x, z).$$

Hence equation (2.9)(c) reads

$$\dot{\mathbf{x}} = \mathbf{b}\big(\mathbf{x}(s), z(s)\big),$$

and (2.9)(b) becomes

$$\begin{aligned}\dot{z}(s) &= \mathbf{b}\big(\mathbf{x}(s), z(s)\big) \cdot \mathbf{p}(s)\\ &= -c\big(\mathbf{x}(s), z(s)\big)\end{aligned}$$

by (2.12). Consequently

$$\begin{cases} \text{(a) } \dot{\mathbf{x}}(s) = \mathbf{b}\big(\mathbf{x}(s), z(s)\big), \\ \text{(b) } \dot{z}(s) = -c\big(\mathbf{x}(s), z(s)\big) \end{cases} \tag{2.13}$$

are the characteristic equations for the quasilinear first-order PDE.

Example 2.7. The characteristic ODEs (2.13) are in general difficult to solve, and so we work out in this example the simpler case of a boundary-value problem for a semilinear PDE:

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 \quad \text{in } U \\ \qquad\quad\;\; u = g \quad \text{on } \Gamma. \end{cases} \tag{2.14}$$

Now U is the half-space $\{\, x_2 > 0 \,\}$ and $\Gamma = \{\, x_2 = 0\, h\} = \partial U$. Here $\mathbf{b} = (1,1)$ and $c = -z^2$. Then (2.13) becomes

$$\begin{cases} (\dot{x}^1, \dot{x}^2) = 1 \\ \qquad\quad\; \dot{z} = z^2. \end{cases}$$

Consequently

$$\begin{cases} (x^1, x^2)(s) = (x^0 + s, s) \\ \qquad\quad\; z(s) = \dfrac{z^0}{1 - sz^0} \\ \qquad\qquad\quad\; = \dfrac{g(x^0)}{1 - sg(x^0)}, \end{cases}$$

Chapter 3

Algebraic Geometry

A summary to a course on an introduction to sheaf cohomology. We will mostly reference Donu's notes available here `https://www.math.purdue.edu/~dvb/classroom.html`, but also cite Ravi Vakil's *Fundamentals of Algebraic Geometry* [10] available here `https://math216.wordpress.com/`.

3.1 The statement of de Rham's theorem

These are almost verbatim Arapura's notes on the de Rham Complex and cohomology.

Before doing anything fancy, let's start at the beginning. Let $U \subseteq \mathbf{R}^3$ be an open set. In calculus class, we learn about operations

$$\{\text{ functions }\} \xrightarrow{\nabla} \{\text{ vector fields }\} \xrightarrow{\nabla\times} \{\text{ vector fields }\} \xrightarrow{\nabla\cdot} \{\text{ functions }\}$$

such that $(\nabla\times)(\nabla) = 0$ and $(\nabla\cdot)(\nabla\times) = 0$. This is a prototype for a *complex.* An obvious question: does $\nabla \times v = 0$ imply that v is a gradient? Answer: sometimes yes (e.g. if $U = \mathbf{R}^3$) and sometimes no (e.g. if $U = \mathbf{R}^3$ minus a line).

Chapter 4

Algebraic Topology

From my meetings with Mark. We reference Hatcher's *Algebraic Topology* [6] freely available here `https://www.math.cornell.edu/~hatcher/#ATI`.

4.1 Cohomology

Chapter 5

Group Theory and Differential Equations

This is a summary of Kuga's *Galois' Dream: Group Theory and Differential Equations* book.

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