# MA 544: Homework 3

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## Problem 3.1 (Wheeden & Zygmund §3, Ex. 5)

Construct a subset of $[0, 1]$ in the same manner as the Cantor set, except that at the $k$th stage each interval removed has length $\delta 3^{-k}$, $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no interval.

*Proof.* Let $0 < \delta < 1$ be given. We begin by removing the open set $\left(\frac{\delta}{3}, 1 - \frac{\delta}{3}\right)$ from the closed interval $[0, 1]$. This leaves us with two closed subsets of $[0, 1]$, the sets $I_1^1 \coloneqq \left[0, \frac{\delta}{3}\right]$ and $I_2^1 \coloneqq \left[1 - \frac{\delta}{3}, 1\right]$. Define $C_1 \coloneqq I_1^1 \cup I_1^2$. Continue this process ad infinitum, e.g., remove the open interval $\left(\frac{\delta}{9}, \frac{\delta}{3} - \frac{\delta}{9}\right)$ from $\left[0, \frac{\delta}{3}\right]$ and the open interval $\left(1 - \frac{\delta}{3} + \frac{\delta}{9}, 1 - \frac{\delta}{9}\right)$ and so on, letting $C_k$ be the union of the remaining closed intervals. ■

## Problem 3.2 (Wheeden & Zygmund §3, Ex. 7)

Prove (3.15).

*Proof.*

**Lemma** (Wheeden & Zygmund (3.15))**.** *If* $\{I_k\}_k^N$ *is a finite collection of nonoverlapping intervals, then* $\bigcup I_k$ *is measurable and* $|\bigcup I_k| = \sum |I_k|$.

■

## Problem 3.3 (Wheeden & Zygmund §3, Ex. 8)

Show that the Borel algebra $\mathcal{B}$ in $\mathbf{R}^n$ is the smallest $\sigma$-algebra containing the closed sets in $\mathbf{R}^n$.

*Proof.* ■

## Problem 3.4 (Wheeden & Zygmund §3, Ex. 9)

If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ (and also $\liminf E_k$) has measure zero.

*Proof.* ■

## Problem 3.5 (Wheeden & Zygmund §3, Ex. 10)

If $E_1$ and $E_2$ are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

*Proof.* ■