MA 544: Homework 5

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Problem 5.1 (Wheeden & Zygmund §3, Ex. 14)

Show that the conclusion of part (ii) of Exercise 13 (Problem) is false if $|E|_e = +\infty$.

Proof. Let $V \subset [0,1]$ denote the Vitali set defined in 3.38 and consider the union $E := V \cup (2, \infty)$. It is clear that the inner and outer measure of E is ∞. However, E itself is unmeasurable since otherwise $E \cap [0,1] = V \cap [0,1] = V$ would be measurable. ■

Problem 5.2 (Wheeden & Zygmund §3, Ex. 16)

Prove (3.34).

Proof.

Lemma. $|P| = v(P)$.

Let $\{\mathbf{e}_k\}_{k=1}^n$ be a set of orthogonal vectors emenating from a point in $\mathbb{R}^n$. The closed parallelapiped corresponding to $\{\mathbf{e}_k\}_{k=1}^n$ is the set

$$P = \left\{ \mathbf{x} \;\middle|\; \mathbf{x} = \sum_{k=1}^n t_k \mathbf{e}_k,\ 0 \le t \le 1 \right\}. \tag{1}$$

■

Problem 5.3 (Wheeden & Zygmund §3, Ex. 18)

Prove that outer measure is *translation invariant*; that is, if $E_{\mathbf{h}} \coloneqq \{\, \mathbf{x} + \mathbf{h} \mid \mathbf{x} \in E \,\}$ is the translate of E by $\mathbf{h}$, $\mathbf{h} \in \mathbb{R}^n$, show that $|E_{\mathbf{h}}|_e = |E|_e$. If E is measurable, show that $E_{\mathbf{h}}$ is also measurable. [This fact was used in proving (3.37).]

Proof. First, note that $E_{\mathbf{h}} = T(E)$ where $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{h}$. By 3.36, we know that $|E_{\mathbf{h}}|_e \le \delta |E|_e = |E|_e$ since $\delta = |\det T| = 1$.[1] Therefore, to prove equality, it suffices to demonstrate the reverse inequality. First, let us observe that if $G \subset \mathbb{R}^n$ is open, then $G_{\mathbf{h}}$ is open so that if F is a G_δ set, then $F_{\mathbf{h}}$ is G_δ. ■

[1]This can be computed classically, or if we view the translation map T as the matrix

$$\begin{bmatrix} 1 & & \cdots & h_1 \\ & 1 & \cdots & h_2 \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

which clearly has determinant equal to 1, every other term in the sum $\sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) a_{1,\sigma(i)} \cdots a_{n,\sigma(n)}$ besides the diagonal having at least one $a_{i,j} = 0$.

Problem 5.4 (Wheeden & Zygmund §4, Ex. 1)

Prove corollary (4.2) and theorem (4.8)

Proof.

Corollary (Wheeden & Zygmund, 4.2). *If f is measurable, then $\{ f > -\infty \}$, $\{ f < +\infty \}$, $\{ f = +\infty \}$, $\{ a \leq f \leq b \}$, $\{ f = a \}$, etc., are all measurable. Moreover f is measurable if and only if $\{ a \leq f < +\infty \}$ is measurable for every finite a.*

Theorem (Wheeden & Zygmund, 4.8). *If f is measurable and λ is any real number, then $f + \lambda$ and λf are measurable.*

■

Problem 5.5 (Wheeden & Zygmund §4, Ex. 2)

Let f be a simple function, taking its distinct values on disjoint sets $E_1, ..., E_N$. Show that f is measurable if and only if $E_1, ..., E_N$ are measurable.

Proof. ∎