MA557 Homework 10

Carlos Salinas

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Problem 10.1

Let $\varphi\colon R \to S$ be a homomorphism of rings, ${}^a\varphi\colon \operatorname{Spec} S \to \operatorname{Spec} R$ the induced map of the spectra, and $\mathfrak{p} \in \operatorname{Spec} R$. Show that the fiber $({}^a\varphi)^{-1}(\mathfrak{p})$ is homeomorphic to $\operatorname{Spec}(S \otimes_R k(\mathfrak{p}))$.

Proof. This is demonstrated in Matsumura following Theorem 7.2. We try to provide some of the details here. From the definition of the pre-image we have that

$$ {}^a\varphi^{-1}(\mathfrak{p}) = \{\, \mathfrak{q} \in \operatorname{Spec} S \mid\ \mathfrak{q} \cap R = \mathfrak{p}\}. $$

Now, define a ring homomorphism $f\colon S \to S \otimes k(\mathfrak{p})$ by $f(s) \coloneqq s \otimes 1$. Then, since $k(\mathfrak{p}) = (R/\mathfrak{p}) \otimes R_{\mathfrak{p}}$ we have that

$$ S \otimes_R ((R/\mathfrak{p}) \otimes_R R_{\mathfrak{p}}) \cong (S/\mathfrak{p}S)_{\mathfrak{p}} = (S/\mathfrak{p}S)_{\varphi(\mathfrak{p})}. $$

Thus, ${}^a f\colon \operatorname{Spec}(S \otimes k(\mathfrak{p})) \to S$ has image

$$ \operatorname{im} {}^a f = \{\, \mathfrak{q} \in \operatorname{Spec} S\ \mathfrak{q} \supset \mathfrak{p}S \text{ and } \mathfrak{q} \cap f(R - \mathfrak{p}) = \emptyset \,\} = \{\, \mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{q} \cap R = \mathfrak{p} \,\}, $$

which is exactly ${}^a\varphi^{-1}(\mathfrak{p})$ ■

Problem 10.2

Let $R \subset S$ be an integral extension of rings with S a Noetherian ring, and let $\mathfrak{p} \in \operatorname{Spec} R$. Show that there are only finitely many primes in S lying over $\mathfrak{p}$.

Proof. ■

Problem 10.3

Let $\varphi\colon R \to S$ be a homeomorphism of rings with S a Noetherian ring. Show that the following are equivalent:

(i) φ satisfies going up.

(ii) ${}^a\varphi\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is a closed map.

(iii) for every $\mathfrak{q} \in \operatorname{Spec} S$, the induced map $\operatorname{Spec}(S/\mathfrak{q}) \to \operatorname{Spec}(R/\mathfrak{q} \cap S)$ is surjective.

Proof. ■

Problem 10.4

Let $R \subset S$ be an integral extension of domains with R normal, $K = \operatorname{Quot} R$, $\alpha \in S$, $X^n + a_1 X^{n-1} + \cdots + a_n$ the minimal polynomial of α over K (recall $a_i \in R$). Show that for any R-ideal I, $\alpha \in \sqrt{IS}$ if and only if $a_i \in \sqrt{I}$ for $1 \leq i \leq n$.

Proof. ■

Problem 10.5

Let k be a field and $R = k[X_1, ..., X_n]$ a k-algebra. Show that the following are equivalent:

(i) R is a domain with $\dim R = n - 1$

(ii) $R \cong k[X_1, ..., X_n]/(f)$, where $k[X_1, ..., X_n]$ is a polynomial ring and f is an irreducible polynomial.

Proof. ■