MA553: Spring 2016 Homework

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1 Homework 1

Problem 1.1. Let G be a group, $a \in G$ an element of finite order m, and n a positive integer. Prove that

$$|a^n| = \frac{m}{\gcd(m, n)}.$$

Proof. ∎

Problem 1.2. Let G be a group, and let a, b be elements of finite order m, n respectively. Show that if $ba = ab$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$, then $|ab| = \operatorname{lcm}(m, n)$.

Proof. ∎

Problem 1.3. Let G be a group and H, K normal subgroups with $H \cap K = \{e\}$. Show that

(a) $hk = kh$ for every $h \in H$, $k \in K$.

(b) HK is a subgroup of G with $HK \cong H \times K$.

Proof. ∎

Problem 1.4. Show that A_4 has no subgroup of order 6 (although $6 \mid 12 = |A_4|$).

Proof. ∎

2 Homework 2

Problem 2.1. Let G be the group of order $2^3 \cdot 3$, $n \geq 2$. Show that G has a normal 2-subgroup $\neq \{e\}$.

Proof. ∎

Problem 2.2. Let G be a group of order p^2q, p and q primes. Show that the Sylow p-Sylow subgroup or the q-Sylow subgroup of G is normal in G.

Proof. ∎

Problem 2.3. Let G be a subgroup of order pqr, $p < q < r$ primes. Show that the r-Sylow subgroup of G is normal in G.

Proof. ∎

Problem 2.4. Let G be a group of order n and let $\varphi \colon G \to S_n$ be given by the action of G on G via translation.

(a) For $a \in G$ determine the number and the lengths of the disjoint cycles of the permutation $\phi(a)$.

(b) Show that $\varphi(G) \not\subset A_n$ if and only if n is even and G has a cyclic 2-Sylow subgroup.

(c) If $n = 2m$, m odd, show that G has a subgroup of index 2.

Proof. ∎

Problem 2.5. Show that the only simple groups $\neq \{e\}$ of order < 60 are the groups of prime order.

Proof. ∎

2.1 Homework 3

Problem 2.6. Let G be a finite group, p a prime number, N the intersection of all p-Sylow subgroups of G. Show that N is a normal p-subgroup of G and that every normal p-subgroup of G is contained in N.

Proof. ∎

Problem 2.7. Let G be a group of order 231 and let H be an 11-Sylow subgroup of G. Show that $H \subset Z(G)$.

Proof. ∎

Problem 2.8. Let $G = \{e, a_1, a_2, a_3\}$ be a non-cyclic group of order 4 and define $\varphi\colon S_3 \to \mathrm{Aut}(G)$ by $\varphi(\sigma)(e) = e$ and $\varphi(\sigma)(a_1) = a_{\sigma(i)}$. Show that φ is well-defined and an isomorphism of groups.

Proof. ∎

Problem 2.9. Determine all groups of order 18.

Proof. ∎

3 Homework 5

Problem 3.1. Find all composition series and the composition factors of D_6.

Proof. ∎

Problem 3.2. Let T be the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of all upper triangular invertible matrices. Show that T is solvable.

Proof. ∎

Problem 3.3. Let $p \in \mathbb{Z}$ be a prime number. Show:

(a) $(p-1)! \equiv -1 \mod p$.

(b) If $p \equiv 1 \mod 4$ then $x^2 \equiv -1 \mod p$ for some $x \in \mathbb{Z}$.

Proof. ∎

Problem 3.4. (a) Show that the following are equivalent for an odd prime number $p \in \mathbb{Z}$:

(i) $p \equiv 1 \mod 4$.

(ii) $p = a^2 + b^2$ for some a, b in $\mathbb{Z}$.

(iii) p is not prime in $\mathbb{Z}[i]$.

(b) Determine all prime ideals of $\mathbb{Z}[i]$.

Proof. ∎

4 Homework 6

Problem 4.1. Let R be a domain. Show that R is a UFD if and only if every nonzero nonunit in R is a product of irreducible elemnets and the intersection of any two principal ideals is again principal.

Proof. ∎

Problem 4.2. Let R be a PID and p a prime ideal of $R[X]$. Show that p is principal or $p = (a, f)$ for some $a \in R$ and some monic $f \in R[X]$.

Proof. ∎

Problem 4.3. Let k be a field and $n \geq 1$. Show that $Z^n + Y^3 + X^2 \in k(X,Y)[Z]$ is irreducible.

Proof. ∎

Problem 4.4. Let k be a field of characteristic zero and $n \geq 1$, $m \geq 2$. Show that ${X_1}^n + \cdots + {X_m}^n - 1 \in k[X_1, ..., X_m]$ is irreducible.

Proof. ∎

Problem 4.5. Show that $X^{3^n} + 2 \in \mathbb{Q}(i)[X]$ is irreducible.

Proof. ∎

5 Homework 7

Problem 5.1. Let $k \subset K$ and $k \subset L$ be finite field extensions contained in some field. Show that:

(a) $[KL : L] \leq [K : k]$.

(b) $[KL : k] \leq [K : k][L : k]$.

(c) $K \cap L = k$ if equality holds in (b).

Proof. ■

Problem 5.2. Let k be a field of characteristic $\neq 2$ and a, b elements of k so that a, b, ab are not squares in k. Show that $\left[k\left(\sqrt{a}, \sqrt{b}\right) : k\right] = 4$.

Proof. ■

Problem 5.3. Let R be a UFD, but not a field, and write $K = \mathrm{Quot}(R)$. Show that $[\bar{K} : k] = \infty$.

Proof. ■

Problem 5.4. Let $k \in K$ be an algebraic field extension. Show that every k-homomorphism $\delta \colon K \to K$ is an isomorphism.

Proof. ■

Problem 5.5. Let K be the splitting field of $X^6 - 4$ over $\mathbb{Q}$. Determine K and $[K : \mathbb{Q}]$.

Proof. ■