Math 220A
Practice Final Exam Solutions - Fall 2001

1. (a) Suppose $S(t)$ is the solution operator associated with the homogeneous equation

$$(*)\begin{cases} u_t + au_x = 0 \\ u(x,0) = \phi(x). \end{cases}$$

In particular, assume the solution of (*) is given by $u(x,t) = S(t)\phi(x)$. Show that $v(x,t) = S(t)\phi(x) + \int_0^t S(t-s)f(x,s)\,ds$ solves the inhomogeneous problem

$$\begin{cases} u_t + au_x = f(x,t) \\ u(x,0) = 0. \end{cases}$$

Answer:

$$\begin{aligned}
[\partial_t + a\partial_x]v &= [\partial_t + a\partial_x]\left\{S(t)\phi(x) + \int_0^t S(t-s)f(x,s)\,ds\right\} \\
&= 0 + S(t-t)f(x,t) + \int_0^t [\partial_t + a\partial_x]S(t-s)f(x,s)\,ds \\
&= S(0)f(x,t) = f(x,t).
\end{aligned}$$

In addition,

$$v(x,0) = S(0)\phi(x) + \int_0^0 S(0-s)f(x,s)\,ds = \phi(x).$$

(b) Find the solution operator $S(t)$ for (*).

Answer: The solution of (*) is given by $u(x,t) = \phi(x-at)$. Therefore, the solution operator $S(t)$ is the operator such that

$$\boxed{S(t)\phi(x) = \phi(x-at).}$$

(c) Find a solution of the inhomogeneous initial-value problem

$$\begin{cases} u_t + au_x = f(x,t) \\ u(x,0) = \phi(x). \end{cases}$$

Answer: A solution is given by

$$\boxed{\begin{aligned} v(x,t) &= S(t)\phi + \int_0^t S(t-s)f(x,s)\,ds \\ &= \phi(x-at) + \int_0^t f(x-a(t-s),s)\,ds. \end{aligned}}$$

2. (a) Solve the following initial-value problem.

$$\begin{cases} u_x^2 u_t - 1 = 0 \\ u(x,0) = x. \end{cases}$$

Answer: Let

$$F(p,q,z,x,t) = p^2 q - 1.$$

The set of characteristic equations are given by

$$\begin{array}{ll} \frac{dx}{ds} = 2pq & x(r,0) = r \\ \frac{dt}{ds} = p^2 & t(r,0) = 0 \\ \frac{dz}{ds} = 3 & z(r,0) = r \\ \frac{dp}{ds} = 0 & p(r,0) = \psi_1(r) \\ \frac{dq}{ds} = 0 & q(r,0) = \psi_2(r) \end{array}$$

where ψ_1, ψ_2 satisfy

$$\begin{aligned} \phi'(r) &= \psi_1(r) \\ \psi_1^2 \psi_2 - 1 &= 0. \end{aligned}$$

Therefore,

$$\psi_1(r) = 1 = \psi_2(r).$$

Solving this system of ODEs, we have

$$\begin{aligned} p &= 1 \\ q &= 1 \\ x &= 2s + r \\ t &= s \\ z &= 3s + r. \end{aligned}$$

Solving for r, s, we find our solution is given by

$$\boxed{u(x,t) = z(r(x,t), s(x,t)) = x + t.}$$

(b) Consider the initial-value problem

$$\begin{cases} u_t + u_x = x \\ u(x,x) = 1. \end{cases}$$

Explain why there is no solution to this problem.
Answer: The projected characteristic curves for this PDE are given by

$$\begin{aligned} \frac{dt}{ds} &= 1 \\ \frac{dx}{ds} &= 1. \end{aligned}$$

Therefore, they are the lines $x-t=c$. Further, $du/ds = x$ along the characteristic curves. But we are prescribing initial data which is constant along the projected characteristics. Therefore, $du/ds \neq x$. Our initial data does not satisfy our equation.

3. (a) Find the general solution of

$$u_{tt} + 2u_{xt} - 3u_{xx} = 0.$$

Answer: Factoring as

$$(\partial_t - \partial_x)(\partial_t + 3\partial_x)u = 0,$$

then we make a change of variables by defining new coordinates ξ, η such that

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \partial_t - \partial_x \\ \frac{\partial}{\partial \eta} &= \partial_t + 3\partial_x. \end{aligned}$$

In particular, we let

$$\begin{aligned} \xi &= -\frac{1}{4}(x - 3t) \\ \eta &= \frac{1}{4}(x + t). \end{aligned}$$

Therefore, we have

$$u_{\xi\eta} = 0,$$

which implies

$$\boxed{u(x,t) = f(\xi(x,t)) + g(\eta(x,t)) = f(x - 3t) + g(x + t).}$$

(b) Find the solution of the initial-value problem,

$$\begin{cases} u_{tt} + 2u_{xt} - 3u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

Answer: The general solution is given by

$$u(x,t) = f(x - 3t) + g(x + t).$$

Therefore, the initial data implies we need

$$\begin{aligned} u(x,0) &= f(x) + g(x) = \phi(x) \\ u_t(x,0) &= -3f'(x) + g'(x) = \psi(x). \end{aligned}$$

Solving this system of equations, we have

$$\begin{aligned} f'(x) &= \frac{1}{4}[\phi'(x) - \psi(x)] \\ g'(x) &= \frac{1}{4}[3\phi'(x) + \psi(x)]. \end{aligned}$$

Integrating these equations, we conclude that the solution to our initial-value problem is given by

$$\boxed{u(x,t) = \frac{1}{4}[\phi(x-3t) + 3\phi(x+t)] + \frac{1}{4}\int_{x-3t}^{x+t} \psi(y)\,dy.}$$

4. Consider the initial-value problem

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} a & x \le 0 \\ a(1-x) & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}$$

where $a > 0$. Find the unique, weak solution which satifies the entropy condition.

Answer: The projected characteristics are given by

$$x(r) = \phi(r)t + r.$$

For $r < 0$, we have $x = at + r$. For $0 < r < 1$, we have $x = a(1-r)t + r$. For $r > 1$, we have $x = r$. We see these curves do not intersect until $t = 1/a$. Therefore, for $0 \le t \le 1/a$, our solution is well-defined, and the solution is constant along these projected characteristics. In particular, for $0 \le t \le 1/a$, our solution is given by

$$\boxed{u(x,t) = \begin{cases} a & x < at \\ a\left(\dfrac{1-x}{1-at}\right) & at < x < 1 \\ 0 & x > 1. \end{cases}}$$

For $t \ge 1/a$, the projected characteristics intersect. Therefore, we need to introduce a shock curve. The values of the solution to the left and right of the curve of discontinuity are given by $u^- = a$ and $u^+ = 0$. Our shock curve $x = \xi(t)$ must satisfy

$$\begin{aligned} \xi'(t) = \frac{[f(u)]}{[u]} &= \frac{\frac{1}{2}(u^-)^2 - \frac{1}{2}(u^+)^2}{u^- - u^+} \\ &= \frac{\frac{1}{2}a^2}{a} = \frac{1}{2}a. \end{aligned}$$

This curve $x = \xi(t)$ also contains the point $t = 1/a$, $x = 1$. Therefore, this curve is given by $(x-1) = \frac{1}{2}a\left(t - \frac{1}{a}\right)$. Therefore, for $t \geq 1/a$ our solution is given by

$$\boxed{u(x,t) = \begin{cases} a & x < \dfrac{1}{2}at + \dfrac{1}{2} \\ 0 & x > \dfrac{1}{2}at + \dfrac{1}{2}. \end{cases}}$$

5. Consider the initial-value problem

$$\begin{cases} u_{tt} + 2u_{xt} - 3u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

(a) Use energy methods to prove the value of the solution u at the point (x_0, t_0) depends at most on the values of the initial data in the interval $(x_0 - 3t_0, x_0 + t_0)$.

Answer: Define an energy for this problem by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} (u_t^2 + 3u_x^2)\,dx.$$

Now for a fixed t, define the energy over the interval $(x_0 - 3(t_0 - t), x_0 + (t_0 - t))$ as

$$e(t) = \frac{1}{2}\int_{x_0-3(t_0-t)}^{x_0+(t_0-t)} (u_t^2 + 3u_x^2)\,dx.$$

Suppose the initial data ϕ, ψ is zero in the interval $(x_0 - 3t_0, x_0 + t_0)$. We will show that the solution is zero in the triangle bounded by the lines $t = 0$, $x = x_0 - 3(t_0 - t)$ and $x = x_0 + (t_0 - t)$. We will do so by showing that $e'(t) \leq 0$ and then use the fact that $e(0) = 0$ and $e(t) \geq 0$ to conclude that $e(t) \equiv 0$ for all t such that $0 \leq t \leq t_0$. We proceed as follows.

$$\begin{aligned}
e'(t) &= -\frac{1}{2}[u_t^2 + 3u_x^2]|_{x=x_0+(t_0-t)} - \frac{3}{2}[u_t^2 + 3u_x^2]|_{x=x_0-3(t_0-t)} \\
&\quad + \frac{1}{2}\int_{x_0-3(t_0-t)}^{x_0+(t_0-t)} (2u_t u_{tt} + 6u_x u_{xt})\,dx \\
&= -\frac{1}{2}[u_t^2 + 3u_x^2]|_{x=x_0+(t_0-t)} - \frac{3}{2}[u_t^2 + 3u_x^2]|_{x=x_0-3(t_0-t)} \\
&\quad + \frac{1}{2}\int_{x_0-3(t_0-t)}^{x_0+(t_0-t)} (2u_t u_{tt} - 6u_{xx} u_t)\,dx + 3u_x u_t|_{x=x_0+(t_0-t)} - 3u_x u_t|_{x=x_0-3(t_0-t)} \\
&= -\frac{1}{2}[u_t^2 - 6u_x u_t + 3u_x^2]|_{x=x_0+(t_0-t)} - \frac{3}{2}[u_t^2 + 2u_x u_t + 3u_x^2]|_{x=x_0-3(t_0-t)} \\
&\quad - \int_{x_0-3(t_0-t)}^{x_0+(t_0-t)} (u_t^2)_x\,dx \\
&= -\frac{1}{2}[3u_t^2 - 6u_x u_t + 3u_x^2]|_{x=x_0+(t_0-t)} - \frac{3}{2}[\frac{1}{3}u_t^2 + 2u_x u_t + 3u_x^2]|_{x=x_0-3(t_0-t)} \\
&= -\frac{3}{2}[u_t - u_x]^2|_{x=x_0+(t_0-t)} - \frac{1}{2}[u_t^2 + 3u_x]^2|_{x=x_0-3(t_0-t)} \leq 0.
\end{aligned}$$

Therefore, $e'(t) \le 0$, which implies $u_t = 0 = u_x$ within the interval $(x_0 - 3(t_0 - t), x_0 + (t_0 - t))$. Therefore, $u \equiv C$ for some constant C. But, $u(x,0) \equiv 0$ in the interval $(x_0 - 3t_0, x_0 + t_0)$ implies $u \equiv 0$ in that interval.

(b) Use energy methods to prove uniqueness of solutions to this initial-value problem if the initial data has compact support.

Answer: We define the energy as

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t^2 + 3u_x^2)\,dx.$$

Now assume we have two solutions u, v with the same initial data. Let $w = u - v$. Therefore, w satisfies the initial-value problem with zero initial data. Now

$$\begin{aligned}
E'(t) &= \frac{1}{2}\int_{-\infty}^{\infty} (2w_t w_{tt} + 6w_x w_{xt})\,dx \\
&= \int_{-\infty}^{\infty} w_t w_{tt} - 3u_{xx} w_t\,dx + w_x w_t\big|_{x\to-\infty}^{x\to+\infty}. \\
&= -2\int_{-\infty}^{\infty} w_t w_{xt}\,dx \\
&= -\int_{-\infty}^{\infty} (w_t^2)_x\,dx = 0,
\end{aligned}$$

using the fact that if the initial data has compact support, then the solution has compact support. Therefore, $E'(t) = 0$. Therefore,

$$\int_{-\infty}^{\infty} (w_t^2 + 3w_x^2)\,dx = 0,$$

which implies $w_t = 0 = w_x$. Using the fact that $w(x,0) = 0$, we conclude that $w \equiv 0$, and, therefore, $u \equiv v$.

6. Consider the following eigenvalue problem.

$$\begin{cases} y'' + \lambda y = 0, & 0 < x < l \\ y'(0) + y(0) = 0 \\ y(l) = 0. \end{cases}$$

(a) Show the boundary conditions are symmetric.

Answer: First,

$$f'(l)g(l) - f(l)g'(l) = 0$$

for any functions f, g satisfying the boundary conditions, because $f(l) = 0 = g(l)$. Second,

$$f'(0)g(0) - f(0)g'(0) = -f(0)g(0) + f(0)g(0) = 0$$

for any functions satisfying the boundary conditions. Therefore, the boundary conditions are symmetric.

(b) State the definition of orthogonality of functions on $[0, l]$.

Answer: The functions f and g are orthogonal on $[0, l]$ if

$$\int_0^l f(x)g(x)\,dx = 0.$$

(c) Use the fact that the boundary conditions are symmetric to prove all eigenfunctions of this operator must be orthogonal.

Answer: *Note: I should say to prove that eigenfunctions corresponding to distinct eigenvalues are orthogonal. Eigenfunctions corresponding to the same eigenvalue can be chosen to be orthogonal using a Gram-Schmidt orthogonalization process.*

Let X_m, X_n be two eigenfunctions corresponding to *distinct* eigenvalues $\lambda_n \neq \lambda_m$. Therefore,

$$\begin{aligned}
\lambda_n \int_0^l X_n X_m\,dx &= -\int_0^l X_n'' X_m\,dx \\
&= \int_0^l X_n' X_m'\,dx - X_n' X_m|_{x=0}^{x=l} \\
&= -\int_0^l X_n X_m'' + (X_n X_m' - X_n' X_m)|_{x=0}^{x=l} \\
&= \lambda_m \int_0^l X_n X_m''\,dx,
\end{aligned}$$

using the fact that the boundary conditions are symmetric. Therefore,

$$(\lambda_n - \lambda_m)\int_0^l X_n X_m\,dx = 0.$$

But, $\lambda_n \neq \lambda_m$. Therefore,

$$\int_0^l X_n X_m\,dx = 0,$$

as claimed.

(d) Find all *positive* eigenvalues and their corresponding eigenfunctions. (Note: You may not be able to find an explicit formula for these eigenvalues.) Show graphically that there are an infinite number of positive eigenvalues $\{\lambda_n\}$ such that $\lambda_n \to +\infty$.

Answer: Look for positive eigenvalues $\lambda = \beta^2 > 0$. Therefore,

$$\begin{cases} Y'' + \beta^2 Y = 0 \\ Y'(0) + Y(0) = 0 \\ Y(l) = 0. \end{cases}$$

Now the general solution of this ODE is given by

$$Y(y) = C\cos(\beta y) + D\sin(\beta y).$$

Now $Y(0) = C$ and $Y'(0) = D\beta$. Therefore, the first boundary condition implies $C + D\beta = 0$. Further, the second boundary condition implies

$$Y(l) = C\cos(\beta l) + D\sin(\beta l) = 0.$$

Therefore, by the first condition, we need

$$-D\beta\cos(\beta l) + D\sin(\beta l) = 0.$$

We don't want $D = 0$. Therefore, we need

$$\sin(\beta l) = \beta\cos(\beta l),$$

or

$$\tan(\beta l) = \beta.$$

Therefore, the eigenvalues and corresponding eigenfunctions are given by

$$\boxed{\begin{array}{l} \lambda_n = \beta_n^2 \text{ where } \tan(\beta_n l) = \beta_n \\ Y_n(y) = -D_n\beta_n\cos(\beta_n y) + D\sin(\beta_n y). \end{array}}$$

7. Consider the following initial/boundary value problem,

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & 0 < x < l, t > 0 \\ u(x,0) = 0 & 0 < x < l \\ u_t(x,0) = 0 & 0 < x < l \\ u(0,t) = \sin t & \\ u(l,t) = 1. & \end{cases}$$

Define a function $\mathcal{U}(x,t)$ such that by letting $v(x,t) = u(x,t) - \mathcal{U}(x,t)$, then $v(x,t)$ will satisfy

$$\begin{cases} v_{tt} - 4v_{xx} = f(x,t) & 0 < x < l, t > 0 \\ v(x,0) = \phi(x) & 0 < x < l \\ v_t(x,0) = \psi(x) & 0 < x < l \\ v(0,t) = 0 = v(l,t) & t > 0 \end{cases}$$

for some functions $f(x,t), \phi(x)$ and $\psi(x)$, thus, reducing the problem with inhomogeneous boundary data to an inhomogeneous problem with Dirichlet boundary data. You do **not** need to solve the new inhomogeneous problem.

Answer: Let

$$\boxed{\mathcal{U}(x,t) = \frac{1}{l}((l-x)\sin t + x).}$$

8. Consider the initial-value problem for the wave equation in n dimensions,

$$\begin{cases} u_{tt} - \Delta u = 0 & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = \phi(x) & \\ u_t(x,0) = \psi(x) & \end{cases}$$

(a) If the initial data is supported in the annular region $\{a < |x| < b\}$, find where the solution is definitely zero in

i. $\mathbb{R}^2$

Answer:

$$\boxed{|x| + t < a \text{ and } |x| - t > b.}$$

ii. $\mathbb{R}^3$.

Answer:

$$\boxed{|x| + t < a \text{ and } |x| - t > b \text{ and } t - |x| > b.}$$

(b) Find the value of the solution u of the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & x \in \mathbb{R}^3, t \geq 0 \\ u(x,0) = 0 \\ u_t(x,0) = \psi(x) \end{cases}$$

where

$$\psi(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

at a point (x,t) such that $|x| + t < a$.

Answer: By Kirchoff's formula, the solution is given by

$$\frac{1}{4\pi t^2} \int_{\partial B(x,t)} t\psi(y)\, dS(y).$$

Now, $\psi(y) \equiv 1$ for $|y| < a$. Therefore, if $|x| + t < a$, then $\psi \equiv 1$. Therefore, the solution is given by

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t\, dS(y)$$

or

$$\boxed{u(x,t) = t.}$$

9. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \pi\}$. Solve the following initial/boundary value problem.

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + 1 & (x,y) \in \Omega, t > 0 \\ u(x,y,0) = \sin(x)\sin(2y) \\ u_t(x,y,0) = 0 \\ u(x,y,t) = 0 & (x,y) \in \partial\Omega. \end{cases}$$

Answer: First, we will solve the homogeneous problem. Then, we will use Duhamel's principle. Using separation of variables, we have

$$-\frac{T''}{T} = -\frac{X''}{X} - \frac{Y''}{Y} = \lambda,$$

which leads us to

$$-\frac{X''}{X} = \lambda + \frac{Y''}{Y} = \mu.$$

Now, first, we consider the eigenvalue problem

$$-X'' = \mu X \qquad 0 < x < \pi. \qquad\qquad X(0) = 0 = X(\pi).$$

The solutions of this eigenvalue problem are given by $\mu_n = n^2$, $X_n(x) = \sin(nx)$. Next, we solve

$$\begin{gathered} -\frac{Y''}{Y} = \lambda - \mu \qquad 0 < y < \pi \\ Y(0) = 0 = Y(\pi). \end{gathered}$$

The solutions of this eigenvalue problem are given by $\lambda - \mu = m^2$. Therefore, we conclude that $\lambda_{mn} = m^2 + n^2$ and $X_n(x)Y_m(y) = \sin(nx)\sin(ny)$. Solving our equation for T_{mn}, we have

$$T_{mn}(t) = A_{mn}\cos(\sqrt{\lambda_{mn}}t) + B_{mn}\sin(\sqrt{\lambda_{mn}}t).$$

Therefore, our solution has the form

$$u(x, y, t) = \sum_{m,n}[A_{mn}\cos(\sqrt{\lambda_{mn}}t) + B_{mn}\sin(\sqrt{\lambda_{mn}}t)]\sin(nx)\sin(my).$$

Now $u(x, y, 0) = \sin(x)\sin(2y)$ implies

$$A_{mn} = \begin{cases} 1 & n = 1, m = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Now $u_t(x, y, 0) = 0$ implies $B_{mn} = 0$. Therefore, the solution of the homogeneous problem is given by

$$u(x, y, t) = \cos(\sqrt{\lambda_{2,1}}t)\sin(x)\sin(2y) = \cos(\sqrt{5}t)\sin(x)\sin(2y).$$

Using Duhamel's principle, we conclude that the inhomogeneous part of the solution is given by

$$\sum_{m,n} B_{mn}\sin(\sqrt{\lambda_{mn}}(t - s))\sin(nx)\sin(my)$$

where

$$\sqrt{\lambda_{mn}}B_{mn} = \frac{\langle 1, \sin(nx)\sin(my)\rangle}{\langle \sin(nx)\sin(my), \sin(nx)\sin(my)\rangle} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \sin(nx)\sin(my)\,dx\,dy.$$

Therefore, our solution is given by

$$\boxed{u(x, y, t) = \cos(\sqrt{5}t)\sin(x)\sin(2y) + \int_0^t \sum_{m,n} B_{mn}(s)\sin(\sqrt{\lambda_{mn}}(t - s))\sin(nx)\sin(my)\,ds.}$$

where $B_{mn}(s)$ is defined above.

10. Use Green's Theorem to show that the value of the solution u at the point $(0, t_0)$ of the wave equation on the half-line with Neumann boundary conditions

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \infty, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u_x(0,t) = 0 \end{cases}$$

is given by

$$u(0,t_0) = \phi(ct_0) + \frac{1}{c}\int_0^{ct_0} \psi(y)\,dy + \frac{1}{c}\iint_\Delta f(y,s)\,dy\,ds$$

where Δ is the triangle in the xt-plane bounded by the lines $x = 0$, $t = 0$ and $x = c(t_0 - t)$.

Answer: *Note: This should be the inhomogeneous problem!* Integrating over Δ, we have

$$\iint_\Delta (u_{tt} - c^2 u_{xx})\,dx\,dt = \iint f(x,t)\,dx\,dt.$$

By Green's Theorem, we have

$$\begin{aligned} -\iint_\Delta [(c^2 u_x)_x - (u_t)_t]\,dx\,dt &= -\int_{\partial\Delta} [u_t\,dx + c^2 u_x\,dt] \\ &= -\int_{L_1} [u_t\,dx + c^2 u_x\,dt] - \int_{L_2} [u_t\,dx + c^2 u_x\,dt] \\ &\quad - \int_{L_3} [u_t\,dx + c^2 u_x\,dt], \end{aligned}$$

where L_1 is the line segment $t = 0$ from $x = 0$ to $x = ct_0$, L_2 is the line segment $x = c(t_0 - t)$ from $(ct_0, 0)$ to $(0, t_0)$ and L_3 is the line segment $x = 0$ from $(0, t_0)$ to $(0, 0)$.

Now

$$-\int_{L_1} [u_t\,dx + c^2 u_x\,dt] = -\int_0^{ct_0} u_t(x,0)\,dx = -\int_0^{ct_0} \psi(x)\,dx.$$

$$\begin{aligned} -\int_{L_2} [u_t\,dx + c^2 u_x\,dt] &= -\int_0^{t_0} [-cu_t(c(t_0-t),t) + c^2 u_x(c(t_0-t),t)]\,dt \\ &= c\int_0^{t_0} [u_t - cu_x]\,dt \\ &= c\int_0^{t_0} [u_t + u_x \frac{dx}{dt}]\,dt \\ &= c\int_0^{t_0} du \\ &= c[u(0,t_0) - u(x_0,0)] \\ &= cu(0,t_0) - c\phi(ct_0). \end{aligned}$$

Lastly,

$$-\int_{L_3} [u_t\,dx + c^2 u_x\,dt] = -\int_{t_0}^{0} c^2 u_x(0,t)\,dt = 0.$$

Therefore, we conclude that

$$cu(0,t_0) = c\phi(ct_0) + \int_0^{ct_0} \psi(x)\,dx + \iint_\Delta f(x,t)\,dx\,dt,$$

which implies

$$u(0,t_0) = \phi(ct_0) + \frac{1}{c}\int_0^{ct_0} \psi(x)\,dx + \frac{1}{c}\iint_\Delta f(x,t)\,dx\,dt,$$

as claimed.