

## MA 26500-215 Quiz 7

July 13, 2016

1. For the following problems write **T** for true, **F** for false. You do not need to justify your answers.
- (a) (3 points) For all  $m \times n$  matrices  $A$  and  $B$ ,  $\text{nullity}(A + B) = \text{nullity } A + \text{nullity } B$ .
  - (b) (3 points) For all  $n \times n$  matrices  $A$  and  $B$ ,  $\text{nullity}(AB) = (\text{nullity } A)(\text{nullity } B)$ .
  - (c) (3 points) For all  $n \times n$  matrices  $A$  and  $B$ , where  $A$  is an elementary matrix,  $\text{nullity}(AB) = \text{nullity } B$ .
  - (d) (3 points) If  $\mathbf{x}_p$  is a solution to the system  $A\mathbf{x} = \mathbf{b}$ , then  $\mathbf{y} + \mathbf{x}_p$  is also a solution to  $A\mathbf{x} = \mathbf{b}$  for any  $\mathbf{y} \in \text{Nullspace } A$ .

**Solution:** The answers for part (a), (b), (c) and (d) are **F**, **F**, **T** and **T** respectively. For part (c) and (d) you should refer to Kolman and Hill (particularly Ch. 4.7 on *Homogeneous Systems*).

To see that (a) is false consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The nullity of  $A$  and  $B$  is both 0, but

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which has nullity 2 and  $0 + 0$  is by no means equal to 2.

To see that (b) is false, consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then nullity of  $A$  is 1 whereas the nullity of  $B$  is 0, but

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which has nullity 1. Again,  $0 \cdot 1$  is not equal to 1.

2. (8 points) Prove that if  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  are in  $\mathbb{R}^3$  and  $\mathbf{u}$  is orthogonal to both  $\mathbf{v}$  and  $\mathbf{w}$ , then  $\mathbf{u}$  is orthogonal to every vector in  $\text{span}\{\mathbf{v}, \mathbf{w}\}$ .

[Hint: What does it mean for a vector  $\mathbf{x}$  to be in  $\text{span}\{\mathbf{v}, \mathbf{w}\}$  and what does it mean for two vectors to be orthogonal?]

**Solution:** Starting from the top. We know that  $\mathbf{u} \cdot \mathbf{v} = 0$  and  $\mathbf{u} \cdot \mathbf{w} = 0$ . Now, what does it mean for  $\mathbf{x}$  to be in  $\text{span}\{\mathbf{v}, \mathbf{w}\}$ ? It means that there exists scalars  $a_1, a_2 \in \mathbb{R}$  (both can possibly be 0) such that  $\mathbf{x} = a_1\mathbf{v} + a_2\mathbf{w}$ . Thus

$$\begin{aligned}\mathbf{u} \cdot \mathbf{x} &= \mathbf{u} \cdot (a_1\mathbf{v} + a_2\mathbf{w}) \\ &= \mathbf{u} \cdot (a_1\mathbf{v}) + \mathbf{u} \cdot (a_2\mathbf{w}) \\ &= a_1(\mathbf{u} \cdot \mathbf{v}) + a_2(\mathbf{u} \cdot \mathbf{w}) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

so  $\mathbf{u}$  is orthogonal to  $\mathbf{x}$ . Since the choice of  $\mathbf{x}$  was arbitrary, we conclude that  $\mathbf{u}$  is orthogonal to every vector in  $\text{span}\{\mathbf{v}, \mathbf{w}\}$ .