

MA 523: Homework 2

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Problem 2.1

Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the noncharacteristic boundary condition when Γ is flat near x^0 since $\nu(x^0) = (0, \dots, 0, 1)$ giving us

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

To show (2.1), we will straighten the boundary near x^0 and apply the noncharacteristic boundary conditions. Write $y = \Phi(x)$ where we set

$$\begin{cases} \Phi^1(x) = x_1, \\ \vdots \\ \Phi^n(x) = x_n - \varphi(x_1, \dots, x_{n-1}), \end{cases}$$

with $\varphi(x_1^0, \dots, x_{n-1}^0) = x_n^0$. Then the image of Γ under this transformation is flat near $y^0 = \Phi(x^0)$ and we can apply the noncharacteristic boundary conditions on the PDE

$$F_{p_n}(p^0, z^0, x^0) \neq 0.$$

Now all we need to do is find ◀

Problem 2.2

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x .

Solution. ►

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Problem 2.3

Show that the function $u(x, t)$ defined by $t \geq 0$ by

$$u(x, t) = \begin{cases} -\frac{2}{3} \left(t + \sqrt{3x + t^2} \right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burger's equation*).

Solution. ►

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