

## MA571 Homework 9

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**PROBLEM 9.1 (MUNKRES §46, EX. 6)**

Show that the compact-open topology,  $\mathcal{C}(X, Y)$  is Hausdorff if  $Y$  is Hausdorff, and regular if  $Y$  is regular. [Hint: If  $\overline{U} \subset V$ , then  $\overline{S(C, U)} \subset S(C, V)$ .]

*Proof.* Suppose that  $Y$  is regular. We shall proceed by the hint and Lemma 31.1(b). Consider the subbasis element  $S(C, U)$ . Since  $Y$  is regular, there exists a neighborhood  $V \supset U$  such that  $V \supset \overline{U}$ . Let  $f \in \overline{S(C, U)}$ . Then, we claim that  $f \in S(C, V)$ . For suppose not, then there exists an element  $x_0 \in C$  such that  $f(x_0) \notin V$ . Then, since  $\overline{U} \subset V$ , by hypothesis,  $f(x_0) \notin \overline{U}$ . Consider the subbasic neighborhood  $S(\{x_0\}, Y - \overline{U})$  of  $f$ . Then,  $S(\{x_0\}, Y - \overline{U}) \cap S(C, U)$  is nonempty. Let  $g$  be in the aforementioned intersection. Then  $g(x_0) \in g(C) \subset U$ , but  $g(x_0) \in Y - \overline{U}$ . This is a contradiction. Thus,  $\overline{S(C, U)} \subset S(C, V)$ .

Now, let  $f \in \mathcal{C}(X, Y)$  and let  $V = \bigcap_{i=1}^n S(C_i, V_i)$  be a basic neighborhood of  $f$ . Then, since  $Y$  is regular, for every  $y = f(x_i) \in f(C_i)$  there exists an open neighborhood  $U_{x_i}$  such that  $\overline{U_{x_i}} \subset V_i$ . These  $U_{x_i}$ 's form an open cover of  $f(C_i)$  which is compact by Theorem 25.6 so there exists a finite collection of them, say  $\{U_{x_{i,j}}\}_{j=1}^{n_i}$  that covers  $f(C_i)$ . Let  $U_i = \bigcup_{j=1}^{n_i} U_{x_{i,j}}$ . Then  $\overline{U_i} = \bigcup_{j=1}^{n_i} \overline{U_{x_{i,j}}} \subset V_i$  by induction on Problem 2.2 (Munkres §17, Ex. 6(b)). Let  $U = \bigcap_{i=1}^n S(C_i, U_i)$ . We claim that  $U$  is the desired neighborhood of  $f$  that, by Theorem 31.1(b), shows that  $\mathcal{C}(X, Y)$  is regular. Let us verify this. First, note that  $f \in U$  since  $f(C_i) \subset U_i$  for all  $i$  so  $U$  is indeed a neighborhood of  $f$ . Moreover, by the hint, we have that  $\overline{S(C_i, U_i)} \subset S(C_i, V_i)$  since  $\overline{U_i} \subset V_i$ . Then  $\overline{U} \subset \bigcap_{i=1}^n \overline{S(C_i, U_i)} \subset V$  by Lemma B. It follows, by Theorem 31.1(b), that  $\mathcal{C}(X, Y)$  is regular. ■

**PROBLEM 9.2 (MUNKRES §46, EX. 9(A,B,C))**

Here is a (unexpected) application of Theorem 46.11 to quotient maps. (Compare Exercise 11 of §29.)

**Theorem.** *If  $p: A \rightarrow B$  is a quotient map and  $X$  is locally compact Hausdorff, then  $(\text{id}_X, p): X \times A \rightarrow X \times B$  is a quotient map.*

*Proof.* (a) Let  $Y$  be the quotient space induced by  $(\text{id}_X, p)$ ; let  $q: X \times A \rightarrow Y$  be the quotient map. Show there is a bijective continuous map  $f: Y \rightarrow X \times B$  such that  $f \circ q = (\text{id}_X, p)$ .

(b) Let  $g = f^{-1}$ . Let  $G: B \rightarrow \mathcal{C}(X, Y)$  and  $Q: A \rightarrow \mathcal{C}(X, Y)$  be the maps induced by  $g$  and  $q$ , respectively. Show that  $Q = G \circ p$ .

(c) Show that  $Q$  is continuous; conclude that  $G$  is continuous, so that  $g$  is continuous.

*Actual proof.* (a) First, let us note that  $Y$  has the same underlying set as the Cartesian product  $X \times B$  with the topology induced by the map  $(\text{id}_X, p)$ . Moreover, the quotient map  $q$  is (as a map between sets) equivalent to  $(\text{id}_X, p)$ . Thus, the natural bijective map to consider between  $Y$  and  $X \times B$  is the identity map  $f = \text{id}_{X \times B} = (\text{id}_X, \text{id}_B)$ . It is clear that  $f$  is bijective. To see that  $f$  is continuous we note that since the composition

$$(f \circ q)(x, a) = (\text{id}_X, \text{id}_B) \circ (\text{id}_X, p)(x, a) = (\text{id}_X(a), \text{id}_B(p(a))) = (x, p(a)) = (\text{id}_X, p)(x, a)$$

and the map  $(\text{id}_X, p)$  is continuous by Theorem 18.4, it follows by Theorem Q.2 that  $f$  is continuous.

(b) Recall, from the definition given on Munkres §46, p. 287, that the induced map  $G$  (respectively  $Q$ ) are defined by the equation  $(G(b))(x) = (x, b)$  (respectively  $(Q(a))(x) = (x, p(a))$ ). Then we have that the composition

$$(G \circ p)(a) = G(p(a)) = (G(p(a)))(x) = (x, p(a)) = (Q(a))(x) = Q(a)$$

as desired.

(c) By Theorem 46.11, since  $q$  is continuous with respect to the quotient topology on  $Y$ , it follows that the induced map  $Q$  is continuous. Additionally, since  $Q$  is equal to the composition  $G \circ p$  by part (b) so by Theorem Q.2  $G$  is continuous. Since  $X$  is locally compact Hausdorff, it follows by Theorem 46.11 that the map  $g$  is continuous. ■

**PROBLEM 9.3 (MUNKRES §51, EX. 1)**

Show that if  $h, h': X \rightarrow Y$  are homotopic and  $k, k': Y \rightarrow Z$  are homotopic, then  $k \circ h$  and  $k' \circ h'$  are homotopic.

*Proof.* Let  $H: X \times I \rightarrow Y$  and  $K: Y \times I \rightarrow Z$  denote the homotopies from  $h$  to  $h'$  and  $k$  to  $k'$ , respectively. Then, we claim that the map  $L(x, t) = K(H(x, t), t)$  is a homotopy from  $k \circ h$  to  $k' \circ h'$ . First, we check that  $L$  starts and ends where we want it to, i.e.,  $L(x, 0) = K(H(x, 0), 0) = k(h(x))$  and  $L(x, 1) = K(H(x, 1), 1) = k'(h'(x))$ . Lastly, we must assure ourselves that  $L$  is in fact continuous. But this last claim follows from the fact that  $L$  can be expressed as the composition  $K \circ (h_t, t)$  where  $h_t$  denotes the continuous map  $H(x, t)$  at time  $t$ . Since  $K$  is (by assumption) continuous and  $(h_t, t)$  are continuous by Theorem 18.4, it follows by Theorem 18.2(a) that  $L$  is continuous. Thus,  $k \circ h \simeq k' \circ h'$  as desired. ■

**PROBLEM 9.4 (MUNKRES §51, EX. 2)**

Given spaces  $X$  and  $Y$ , let  $[X, Y]$  denote the homotopy classes of maps of  $X$  into  $Y$

- (a) Let  $I = [0, 1]$ . Show that for any  $X$ , the set  $[X, I]$  has a single element.
- (b) Show that if  $Y$  is path connected, the set  $[I, Y]$  has a single element.

*Proof.* (a) Let  $f, g: X \rightarrow I$  be arbitrary continuous maps. Then we claim that the straight line homotopy  $H(x, t) = (1 - t)f(x) + tg(x)$  gives a homotopy from  $f$  to  $g$ . Note that the image of  $H(x, t)$  stays in the interval  $I$  since  $(1 - t)f(x) + tg(x) \leq (1 - t) + t = 1$  for all  $x$  and for all  $t$ . Lastly, note that by Theorem 25.1  $H$  is continuous since it is the sum of a product of continuous functions. Hence,  $f \simeq g$ . Since  $f$  and  $g$  were arbitrary, it follows that  $[X, I]$  consists of a single equivalence class.

(b) Note that if  $f, g: I \rightarrow Y$  are constant maps, say  $f(x) = x_0$  and  $g(x) = x_1$  for all  $x \in I$ , then the path  $p: I \rightarrow Y$  where  $p(0) = x_0$  and  $p(1) = x_1$  defines a homotopy  $H(x, t) = p(t)$ . This map is clearly continuous since for any open neighborhood  $U$  of  $Y$ , since  $p$  is continuous, by Theorem 18.1(4) there exists a neighborhood  $V \subset I$  such that  $p(V) \subset U$  so  $H(I \times V) = p(V) \subset U$  implies  $H$  is continuous by Theorem 18.1(4). Therefore, it suffices to show that given a continuous map  $f: I \rightarrow Y$ ,  $f$  is nullhomotopic. Let  $H(x, t)$  be the map  $f((t - 1)x)$ . The map  $(t - 1)x$  is continuous by Theorem 25.1 so the composition  $f \circ ((t - 1)x)$  is continuous by Theorem 18.2(c). Then, observing that  $H(x, 0) = f(x)$  and  $H(x, 1) = f(0)$ ,  $H(x, t)$  gives a homotopy from  $f$  to  $f(0)$ . It follows by Lemma 51.1 that given any  $f, g: I \rightarrow Y$  continuous maps  $f \simeq g$  by transitivity of homotopy. ■

**PROBLEM 9.5 (MUNKRES §51, EX. 3(A,B,C,))**

A space  $X$  is said to be *contractible* if the identity map  $\text{id}_X: X \rightarrow X$  is nullhomotopic.

- (a) Show that  $I$  and  $\mathbf{R}$  are contractible.
- (b) Show that a contractible space is path connected.
- (c) Show that if  $Y$  is contractible, then for any  $X$ , the set  $[X, Y]$  has a single element.

*Proof.* (a) It is clear that  $\text{id}_I: I \rightarrow I$  is nullhomotopic, say to the constant map 0, via the homotopy  $H(x, t) = (1 - t)x$ . Note that  $H(x, 0) = x = \text{id}_I(x)$  and  $H(x, 1) = 0$  and  $H(x, t)$  is continuous since  $(1 - t)x$  is continuous by Theorem 25.1.<sup>1</sup> In the case of  $\mathbf{R}$  the previous map  $H(x, t)$  also works to show that  $\text{id}_{\mathbf{R}}$  is nullhomotopic since  $H(x, 0) = x = \text{id}_{\mathbf{R}}$  and  $H(x, 1) = 0$  and  $H(x, t)$  is continuous by Theorem 25.1.

(b) Suppose that  $X$  is contractible. Then there exists a homotopy  $H(x, t)$  with  $H(x, 0) = x$  and  $H(x, 1) = x_0$  for some point  $x_0 \in X$ . Now, let  $x_1, x_2 \in X$ . Then the map  $p_1(t) = H(x_1, t)$  and  $p_2(t) = H(x_2, t)$  are path homotopies from  $x_1$  to  $x_0$  and  $x_2$  to  $x_0$ . It follows by the fact that  $\simeq_p$  is an equivalence relation that  $x_1 \simeq_p x_2$ .

(c) Since  $Y$  is contractible there exist a homotopy  $H(y, t)$  with  $H(y, 0) = y$  and  $H(y, 1) = y_0$  for some fixed  $y_0 \in Y$ . Therefore, it suffices to show that an arbitrary continuous map  $f: X \rightarrow Y$  is nullhomotopic. Consider the map  $K(x, t) = H(f(x), t)$ . This map is continuous since it is the composition  $H \circ (f, \text{id}_I)$ . Moreover,  $K(x, 0) = \text{id}_Y(f(x)) = f(x)$  and  $K(x, 1) = e_{y_0}(f(x)) = y_0$ . Thus,  $f$  is nullhomotopic and it follows that  $[X, Y]$  has a single element (all maps are null homotopic and  $Y$  is path connected by part (b)). ■

<sup>1</sup>More generally, we showed that products, sums and quotients (when they are defined) of maps from a metric space  $(X, d)$  to  $\mathbf{R}$  (or a subspace of  $\mathbf{R}$  by Theorem 18.2(d)) for that matter, are continuous.