Recall the lemma that was used in the proof that $\mathbb{R}$ is connected:

Lemma. Let $S \subset \mathbb{R}$ be bounded above and nonempty. Let $c = \sup S$.

i) If $x > c$ then x is not in S.

ii) If $x < c$ then there is a y in S with $x < y$.

Theorem Let $a, b \in \mathbb{R}$. Then $[a, b]$ is compact.

Proof. Let $\{A_\alpha\}$ be an open cover of $[a, b]$.

Let $S = \{x \in [a, b] \,|\, [a, s]$ is covered by finitely many of the $A_\alpha\}$. Let $c = \sup S$, which exists since S is nonempty and bounded above.

There is an α_0 with $c \in A_{\alpha_0}$. There is an open set U of $\mathbb{R}$ such that $A_{\alpha_0} = U \cap [a, b]$, and there are $d_1, d_2 \in \mathbb{R}$ with $c \in (d_1, d_2) \subset U$. By part (ii) of the lemma, there is an $s \in S$ with $d_1 < s$, and we also have $s \leq c < d_2$. Then $[a, c] = [a, s] \cup [s, c]$ is covered by finitely many A_α, since $s \in S$ and $[s, c] \subset (d_1, d_2) \cap [a, b] \subset A_{\alpha_0}$.

We claim that $c = b$ (and then we're done).

To prove the claim, if $c < b$ then there is an e with $c < e < \min(d_2, b)$, and then $[c, e] \subset (d_1, d_2) \cap [a, b] \subset A_{\alpha_0}$, so $[a, e] = [a, c] \cup [c, e]$ is covered by finitely many A_α. But then $e \in S$, which contradicts part (i) of the lemma since $e > c$.

QED