MA571 Homework 13

Carlos Salinas

November 30, 2015

Problem 13.1 (Munkres §68, Ex. 1)

Check the details of Example 1.

Proof. The following is the statement of Example 1 as found in the book:

Examples 1. Consider the group P of bijections of the set $\{0, 1, 2\}$ with itself. For $i = 1, 2$, define an element π_1 of P by setting $\pi_i(i) = i - 1$ and $\pi_i(i-1) = i$ and $\pi_i(j) = j$ otherwise. Then π_i generates a subgroup G_i of P of order 2. The group G_1 and G_2 generate P, as you can check. But P is not their free product. The reduced words (π_1, π_2, π_1) and (π_2, π_1, π_2), for instance, represent the same element of P.

We need to check two claims (i) that G_1 and G_2, as defined above, generate P and (ii) that $P \neq G_1 * G_2$, i.e., show that $(\pi_1, \pi_2, \pi_1) = (\pi_2, \pi_1, \pi_2)$. Let us deal with (i) first. We show that $\langle G_1, G_2 \rangle = P$. Our strategy is the following, by the pigeon-hole principle, it suffices to show that $\langle G_1, G_2 \rangle \subset P$ and that $|\langle G_1, G_2 \rangle| = |P|$. Since $G_1, G_2 < P$, i.e., G_1 and G_2 are subgroups of P, the group generated by G_1 and G_2 will be a subgroup of P hence, $\langle G_1, G_2 \rangle \subset P$. The group P is a well-known group, namely (up to group isomorphism) S_3, and we shall not waste time any time showing that $|P| = |\{0, 1, 2\}| = 3! = 6$, but instead we proceed to showing that $|\langle G_1, G_2 \rangle| = 6$. From the definitions of G_1 and G_2, we have at least 3 in $\langle G_1, G_2 \rangle$, these are the elements 1, π_1 and π_2 (the latter two have order 2, e.g.,

$$\pi_i^2(j) = \pi_i \left(\begin{cases} i-1 & \text{if } j = i \\ i & \text{if } j = i-1 \\ j & \text{otherwise} \end{cases} \right) = \begin{cases} i & \text{if } j = i \\ i-1 & \text{if } j = i-1 \\ j & \text{otherwise} \end{cases}$$

which is the identity on $\{0, 1, 2\}$.) So the elements $1, \pi_1, \pi_2, \pi_1\pi_2, \pi_2\pi_1, \pi_1\pi_2\pi_1 \in \langle G_1, G_2 \rangle$ and all finite strings $\pi_1\pi_2 \cdots \pi_i$, $\pi_2\pi_1 \cdots \pi_i$ for that matter. But as a consequence of Lagrange's theorem, the size of $\langle G_1, G_2 \rangle$ must not exceed the size of P so that we are done when we show that the elements $\pi_1\pi_2$, $\pi_2\pi_1$ and $\pi_1\pi_2\pi_1$ are distinct elements. First, observe that

$$\pi_2\pi_1(j) = \pi_2 \left(\begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ 2 & \text{if } j = 2 \end{cases} \right) = \begin{cases} 2 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} \qquad \pi_1\pi_2(j) = \pi_1 \left(\begin{cases} 0 & \text{if } j = 0 \\ 2 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} \right) = \begin{cases} 1 & \text{if } j = 0 \\ 2 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

and, using the computations above,

$$\pi_1\pi_2\pi_1(j) = \pi_1 \left(\begin{cases} 2 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} \right) = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}.$$

Note that none of these elements are equivalent to any of 1, π_1 or π_2 and are certainly not equal to each other. Moreover, there are six of these elements and there are no more elements in P since $|P| = 6$. Thus, $\langle G_1, G_2 \rangle = P$.

Lastly, we show that $P \neq G_1 * G_2$ since

$$(\pi_1, \pi_2, \pi_1) = \pi_1\pi_2\pi_1(j) = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

and

$$(\pi_2, \pi_1, \pi_2) = \pi_2\pi_1\pi_2(j) = \pi_1\left(\begin{cases} 1 & \text{if } j = 0 \\ 2 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}\right) = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

would imply that $(\pi_1, \pi_2, \pi_1) = (\pi_2, \pi_1, \pi_2)$ in the free product $G_1 * G_2$, but $\pi_1 \neq \pi_2$. ■

Problem 13.2 (Munkres §68, Ex. 2(a,b,c))

Let $G = G_1 * G_2$, where G_1 and G_2 are nontrivial groups.

(a) Show G is not Abelian.

(b) If $x \in G$, define the *length* of x to be the length of the unique reduced word in the elements of G_1 and G_2 that represents x. Show that if x has even length (at least 2), then x does not have finite order. Show that if x has odd length (at least 3), then x is conjugate to an element of shorter length.

(c) Show that the only elements of G that have finite order are the elements of G_1 and G_2 that have finite order, and their conjugates.

Proof. (i) Suppose G is Abelian. Take an element $x \in G_1$ and $y \in G_2$. Then $(x, y) = (y, x)$. By the definition of a free product (Munkres §68, pp. 413-414) this implies that the word $(x^{-1}, y^{-1}, x, y) = 1$ which implies that $y^{-1}x = 1$, but $y^{-1} \notin G_1$.

(ii) Let $x \in G$ be a word of even length. Then $x = (y_1, y_2, ..., y_{2k})$ for $k \in \mathbf{N}$ where the right hand-side is irreducible, i.e., either $y_i \in G_1$ if $2 \mid i$ and $y_j \in G_2$ if $2 \nmid j$ or vice-versa since two consecutive "letters" in a word must be from distinct groups or else we can reduce the word further. Then $x^2 = (y_1, y_2, ..., y_{2k}, y_1, y_2, ..., y_{2k})$ is again irreducible since $y_{2k} \in G_1$ and $y_1 \in G_2$ or vice-versa. It follows by induction that $x^n \neq 1$ for any finite positive integer n.

Now, suppose that $x \in G$ has odd length. Then $x = (y_1, y_2, ..., y_{2k+1})$ for $k \in \mathbf{N}$ where the right hand-side is irreducible. Without loss of generality, we may assume that $y_1, y_{2k+1} \in G_1$. Then, setting $y'_{2k+1} \coloneqq y_{2k+1}y_1$, we have

$$y_1^{-1}xy_1 = y_1^{-1}(y_1, y_2, ..., y_{2k+1})y_1 = (y_2, y_3, ..., y_{2k+1}y_1) = (y_2, y_3, ..., y'_{2k+1})$$

which has length $2k$. Thus, x is conjugate to a word of shorter length.

(iii) Suppose that $x \in G$ has finite order. By part (i) the length of x cannot be even. Moreover, if x is of finite order, i.e., if $x^n = 1$ for some positive integer n, and y is conjugate to x, i.e., there exist $g \in G$ such that $y = g^{-1}xg$, then

$$y^n = (g^{-1}xg)^n = (g^{-1}xg)(g^{-1}xg) \cdots (g^{-1}xg) = g^{-1}x^n g = 1$$

so y is of finite order. It remains to show that if x has finite order then x is a conjugate of an element y of G_i, where $i = 1, 2$. Let $2k + 1$ be the length of x. By part (ii), x is conjugate to an element y' of shorter length. Since x has finite order y has finite order so by part (i) y' must be of odd length. If y' is of length 1 we are done. If not, then y' is conjugate to a word y'' of shorter length with finite order. Since the length of x is finite, this process must terminate at a word y of length 1 with finite order. ∎

Problem 13.3 (Munkres §68, Ex. 3)

Let $G = G_1 * G_2$. Given $c \in G$, let cG_1c^{-1} denote the set of all elements of the form cxc^{-1}, for $x \in G_1$. It is a subgroup of G; show that the intersection with G_2 is the identity alone.

Proof. Let $y \in cG_1c^{-1} \cap G_2$, then $y = cxc^{-1} \in G_2$ for some word x in G_1. Hence, we have that $c = ycx^{-1}$. Let us deal with the trivial case first. If $c = 1$ then, since G is the free product of G_1 and G_2, we have $1 \cdot G_1 \cdot 1^{-1} = G_1$ so $(1 \cdot G_1 \cdot 1^{-1}) \cap G_2 = G_1 \cap G_2 = 1$. Now, suppose that $c \neq 1$, say c is represented by the reduced word $(y_1, ..., y_k)$ for $k \in \mathbf{N}$. Then we show that for the following cases (i) $y_1, y_k \in G_i$, (ii)$y_1 \in G_1$ and $y_2 \in G_2$ or (iii) $y_1 \in G_2$ and $y_2 \in G_1$, the intersection $cG_1c^{-1} \cap G_2 = 1$.

In the first case, we have $c = ycx^{-1}$ so c is represented by $(y_1, ..., y_k)$ and $(y, y_1, ..., y_k, x^{-1})$, where the latter is unreduced. Reducing the word $(y, y_1, ..., y_k, x^{-1})$ we have $(y, y_1, ..., y_k x^{-1})$ if $y_1, y_2 \in G_1$ or $(yy_1, ..., y_k, x^{-1})$ if $y_1, y_2 \in G_2$. Without loss of generality, we assume that $y_1, y_2 \in G_2$ as the argument for $y_1, y_2 \in G_1$ is similar. Then, $(yy_1, ..., y_k, x^{-1}) = (y_1, ..., y_k)$. Since the left-hand side is the unique representation of c and has length k while the right-hand side has length $k + 1$ and both words are reduced, it must be that $x^{-1} = 1$ and consequently $y = 1$.

In the second case, we have the two representations of c by the reduced words $(y_1, ..., y_k)$ and $()$ ■

Problem 13.4 (A)

(i) Do the case of p. 367 # 9(e) where h and k take b_0 to b_0. (The proof is similar to the proof of Lemma 55.3, (3) $\implies$ (1), that I gave in class).

(ii) Let G be a path-connected topological group and let $a \in G$. Prove that the map $\varphi\colon G \to G$ defined by $\varphi(g) := ag$ is homotopic to the identity map.

(iii) Use part (ii) to complete the proof of p. 367 # 9(e).

Proof. (i) Let $x_0 \in S^1$. Set $d := \deg h$ and suppose that $\deg h = \deg k$. Then the induced map on the fundamental group, i.e., $h_*\colon \pi_1(S^1, x_0) \to \pi_1(S^1, h(x_0))$ and $k_*\colon \pi_1(S^1, x_0) \to \pi_1(S^1, k(x_0))$, are equivalent since

$$h_*(\gamma(x_0)) = d \cdot \gamma(h(x_0)) = k_*(\gamma(x_0)),$$

that is, they send the generator of $\pi_1(S^1, x_0)$ to d times the generator of $\pi_1(S^1, h(x_0))$. Define $p(s) := (\cos(2\pi s), \sin(2\pi s))$. From the notes about "The fundamental group of S^1," we know that p is a quotient map so, by a previous problem (Prob. 9.2 [Munkres §46, Ex. 9]), the map $(p, \mathrm{id}_I)\colon I \times I \to S^1 \times I$ is a quotient map. Therefore, the diagram below commutes

$$\begin{tikzcd} I \times I \arrow[r, "H"] \arrow[d, "{(p,\mathrm{id}_I)}"'] & S^1 \\ S^1 \times I \arrow[ur, "h\circ\pi_1"'] & \end{tikzcd}$$

so, by Theorem Q.2, the map $H\colon I \times I \to S^1$ is continuous since the map $h \circ \pi_1$, where $\pi_1\colon S^1 \times I \to S^1$ is the canonical projection $\pi_1(x, s) = x$ (Theorem 18.4), is a composition of continuous maps (Theorem 18.2(c)).

(ii)

(iii) Recall the statement of Ex. 9 on p. 367: Show that if $h, k\colon S^1 \to S^1$ have the same degree, they are homotopic. ∎

Problem 13.5 (B)

Let $q\colon S^2 \to P^2$ be the quotient map, where P^2 is the projective plane. Let $x_0 = q(1,0,0)$ and let

$$f(s) = q(\cos(\pi s), \sin(\pi s), 0)$$

for $0 \le s \le 1$. Then $f\colon I \to P^2$ is a loop at x_0. Prove that $[f] * [f] = [e_{x_0}]$.

Proof. ■

Problem 13.6 (C)

Let Y be the following subset of $\mathbf{R}^2$: $Y = \{ (s,t) \in I \times I \mid s \in \{0,1\} \text{ or } t \in \{0,1\} \}$ (that is, Y is the boundary of the square $I \times I$). Give Y the equivalence relation $\sim$ that identifies the top and the bottom edges and the left and the right edges: specifically, $\sim$ is the equivalence relation associated to the partition of Y into the following sets:

- for each $s \notin \{0,1\}$, the set $\{(s,0),(s,1)\}$,
- for each $t \notin \{0,1\}$, the set $\{(t,0),(t,1)\}$,
- the set $\{0,1\} \times \{0,1\}$.

Prove that $Y/\sim$ is a wedge of two circles.

Proof. ■

Problem 13.7 (Optional Problem)

Let B^2 denote the unit disk $\{ (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1 \}$ and let S^1 denote the unit circle. Let $\mathbf{a} \in B^2 - S^1$. In this problem we will show that there is a homeomorphism $h \colon B^2 \to B^2$a which takes $(0, 0)$ to $\mathbf{a}$ and fixes S^1.

(i) Let $h \colon B^2 \to B^2$ be the function defined as follows: note that every point in B^2 is of the form

Proof. ■