MA 523: Homework 3

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Problem 3.1

Consider the initial value problem

$$u_t = \sin u_x; \qquad u(x,0) = \frac{\pi}{4}x.$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the taylor series of the solution about the origin.

Solution. The initial value problem certainly satisfies the assumptions of the Cauchy–Kovalevskaya theorem, that is, setting $\mathbf{u} := (u, u_x, u_t, t)$, the $\mathbf{b}$ are all identically 0, and $\mathbf{c}(\mathbf{u}, x) = \sin u_x(x,t)$ is analytic. Next we show that the Taylor series of u at $(0,0)$,

$$u(x,t) = \sum_{\alpha,\beta} \frac{a_{\alpha,\beta}}{\alpha!\beta!} x^\alpha t^\beta$$

is a solution to our PDE.

First, we must compute the coefficients $a_{\alpha,\beta}$. To this end, we must find the partial derivatives $u_{\alpha,\beta}$ and potentially, relations among them which will help us to find these coefficients. Naïvely listing the partials with respect to t and x, we have

$$\begin{aligned}
u(0,0) &= 0\\
u_x(0,0) &= \frac{\pi}{4}\\
u_t(0,0) &= \sin u_x(0,0) = \frac{\sqrt{2}}{2}\\
u_{xx}(0,0) &= 0\\
u_{tx}(0,0) &= 0\\
u_{tt}(0,0) &= -\cos\big(u_x(0,0)\big)u_{xt}(0,0) = 0\\
u_{xxx}(0,0) &= 0\\
u_{ttx}(0,0) &= 0,
\end{aligned}$$

etc. Thus,

$$u = \frac{\pi}{4}x + \frac{\sqrt{2}}{2}t. \tag{3.1}$$

Plugging in Eq. (3.1) into our PDE, we have

$$u_t - \sin u_x = \frac{\sqrt{2}}{2} - \sin(\pi/4) = 0,$$

as desired. ∎

Problem 3.2

Consider the Cauchy problem for $u(x, y)$

$$\begin{aligned} u_y &= a(x, y, u)u_x + b(x, y, u) \\ u(x, 0) &= 0 \end{aligned}$$

let a and b be analytic functions of their arguments. Assume that $d^\alpha a(0, 0, 0) \geq 0$ and $d^\alpha b(0, 0, 0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $d^\alpha f = f$.)

(a) Show that $d^\beta u(0, 0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $d^\beta u(0, 0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$. (*hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in β_2)

Solution. Write

$$a(x, y, u) = \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma} x^\alpha y^\beta u^\gamma, \qquad b(x, y, u) = \sum_{\alpha,\beta,\gamma} b_{\alpha,\beta,\gamma} x^\alpha y^\beta u^\gamma$$

where the right-hand side of the expressions above converge to the left-hand side for $|x|+|y|+|u| < r$ for some sufficiently small r.

For part (a) we show this explicitly by considering all cases. The case $\beta = (0, 0)$ is obvious as are the cases $\beta = (0, 1)$ and $\beta = (1, 0)$ since $u_x(0, 0) = 0$ and

$$\begin{aligned} u_y(0, 0) &= a\big(0, 0, u(0, 0)\big)u_x(0, 0) + b\big(0, 0, u(0, 0)\big) \\ &= a(0, 0, 0)u_x(0, 0) + b(0, 0, 0) \\ &= b(0, 0, 0) \\ &\geq 0 \end{aligned}$$

since b is a series of strictly positive numbers. For $\beta = (2, 0)$, we have $u_{xx}(0, 0) = 0$. For $\beta = (1, 1)$, we have

$$\begin{aligned} u_{xy}(0, 0) &= a\big(0, 0, u(0, 0)\big)u_{xx}(0, 0) + \frac{\partial}{\partial x} a\big(0, 0, u(0, 0)\big)u_x(0, 0) + \frac{\partial}{\partial x} b\big(0, 0, u(0, 0)\big) \\ &= \frac{\partial}{\partial x} b(0, 0, 0) \\ &\geq 0. \end{aligned}$$

For $\beta = (0, 2)$, we have

$$\begin{aligned} u_{yy}(0, 0) &= a\big(0, 0, u(0, 0)\big)u_{xy}(0, 0) + \frac{\partial}{\partial y} a\big(0, 0, u(0, 0)\big)u_x(0, 0) + \frac{\partial}{\partial y} b\big(0, 0, u(0, 0)\big) \\ &= a(0, 0, 0)\frac{\partial}{\partial y} b(0, 0, 0) + \frac{\partial}{\partial y} b(0, 0, 0) \\ &\geq 0 \end{aligned}$$

since the latter is a sum of positive numbers.

For part (b), in the proof of the Cauchy–Kovalevskaya theorem, for $\beta_2 = 0$, we have

$$d^\beta u(0,0) = 0$$

since u is constant on the hypersurface $\{\, y = 0 \,\}$. In particular, $d^\beta u(0,0) \geq 0$.

Now, suppose $d^\beta u(0,0) \geq 0$ for all $\beta_2 \leq n-1$. Then, for $\beta = (m,n)$, we have

$$\begin{aligned} d^\beta u(0,0) &= d^{(m,n-1)} u_y(0,0) \\ &= d^{(m,n-1)} \big(a u_x + b\big)(0,0) \\ &= \end{aligned}$$

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Problem 3.3

(Kovalevskaya's example) show that the line $\{ t = 0 \}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution u of the heat equation in $\mathbf{R} \times \mathbf{R}$, with $u = 1/(1+x^2)$ on $\{t = 0\}$. (*Hint:* assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of $(0,0)$.)

Solution. First we show that the line $\gamma := \{ t = 0 \}$ is characteristic for the heat equation. With $\boldsymbol{\nu} = (1,0)$ the normal to the line γ, the noncharacteristic condition reads

$$\sum_{|\alpha|=2} a_\alpha \boldsymbol{\nu}^\alpha \neq 0.$$

However,

$$\sum_{|\alpha|=2} a_\alpha \boldsymbol{\nu}^\alpha = 1 \cdot 1 + a_{0,2} \cdot 0 = 1 \neq 0.$$

Thus, γ is characteristic for $u_t = u_{xx}$.

Next suppose u is an analytic solution to the heat equation with

$$u(x,t) = \sum_{m,n} \frac{a_{m,n}}{m!n!} x^m t^n$$

on $\mathbf{R} \times \mathbf{R}$.

Let us compute the coefficients $a_{m,n}$ near $(0,0)$. From the PDE, we have the relation

$$\begin{aligned} a_{m,n} &= d^{(m,n)} u(0,0) \\ &= d^{(m,n-1)} u_t(0,0) \\ &= d^{(m,n-1)} u_{xx}(0,0) \\ &= d^{(m+2,n-1)} u(0,0) \\ &= a_{m+2,n-1}. \end{aligned} \tag{3.2}$$

Form the boundary condition, we have

$$u(x,0) = \sum_{k=1}^{\infty} (-1)^k x^{2k} \tag{3.3}$$

for a sufficiently small neighborhood about $(0,0)$, where the right-hand side is given taylor series of $1/(1+x^2)$. Taking the m^{th} x-partial derivative at $(0,0)$, with the help of Eq. (3.3) we find the coefficients

$$a_{m,0} = \begin{cases} 0 & \text{if } m = 2k+1 \text{ is odd} \\ (-1)^k (2k)! & \text{if } m = 2k \text{ is even.} \end{cases} \tag{3.4}$$

Putting all of this information together, we deduce that

$$a_{2m+1,n} = 0$$

for all m, n and, recursively,

$$a_{2m,n} = a_{2m+2,n-1} = \cdots = a_{2(m+n),0} = (-1)^{m+n}\big(2(m+n)\big)!.$$

From this we see that the coefficients of the form $a_{2n,n}$ grow very quickly, that is,

$$\begin{aligned}\frac{a_{2n,n}}{(2n)!n!} &= (-1)^{2n}\frac{\big(2(n+n)\big)!}{(2n)!n!}\\ &= \frac{(4n)!}{(2n)!n!}\end{aligned}$$

which, by Stirling's formula, is asymptotically equal to

$$\begin{aligned}&\asymp \frac{\sqrt{2\pi n}(4n/\mathrm{e})^{4n}}{\sqrt{4\pi n}(2n/\mathrm{e})^{2n}\sqrt{2\pi n}(n/\mathrm{e})^{n}}\\ &= \frac{\sqrt{2\pi n}(4n/\mathrm{e})^{4n}}{\sqrt{8\pi n^2}(2n/\mathrm{e})^{2n}(n/\mathrm{e})^{n}}\\ &= \frac{\big(\sqrt{\pi/n}\big)4^{4n}}{2\cdot 2^{2n}}\left(\frac{n}{\mathrm{e}}\right)^{4n-3n-n}\\ &= \frac{\sqrt{\pi/n}}{2}\left(\frac{16}{2}\right)^{2n}\left(\frac{n}{\mathrm{e}}\right)^{n}\\ &= \big(\sqrt{\pi/n}\big)2^{6n-1}\left(\frac{n}{\mathrm{e}}\right)^{n}\\ &= \alpha\beta_n n^{n+1/2}\end{aligned}$$

which approaches ∞ as $n \to \infty$. This shows that for $x, t > 0$, the terms $a_{2n,n}$ grow arbitrarily large; taking $X = \min\{x, t\}$, we have

$$\begin{aligned}\frac{a_{2n,n}}{(2n)!n!}x^{2n}t^{n} &\asymp \alpha\beta_n n^{n+1/2}x^{2n}t^{n}\\ &\geq \alpha\beta_n n^{n+1/2}X^{3n}.\end{aligned}$$

If $X \geq 1$, these terms clearly grow arbitrarily large so suppose that $X < 1$. Then we can write $X = 1/Y$ for some $Y > 1$ and we have

$$\alpha\beta_n\frac{n^{n+1/2}}{Y^{3n}} \geq M$$

if and only if

$$\begin{aligned}\frac{\alpha\beta_n n^{n+1/2}}{M} &\geq Y^{3n}\\ \log_Y\left(\frac{\alpha\beta_n n^{n+1/2}}{M}\right) &\geq 3n.\end{aligned}$$

(I'm sure this can be achieved somehow).

This shows that if such a solution exists, it has radius of convergence equal to 0 and hence, is not a solution for the PDE on $\mathbf{R} \times \mathbf{R}$. ■