

# MA 166: Quiz 3

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You have **15 minutes** to complete this quiz. You may work in groups, but you are not allowed to use any other resources.

**Problem 1.** Let  $\mathbf{u} = \langle 6, 3, 1 \rangle$ ,  $\mathbf{v} = \langle 0, 1, 2 \rangle$ , and  $\mathbf{w} = \langle 4, -2, 5 \rangle$ .

- (i) Find the scalar projection  $\text{comp}_{\mathbf{v}} \mathbf{w}$ .
- (ii) Find the projection  $\text{proj}_{\mathbf{u}} \mathbf{v}$ .
- (iii) Find the cross product  $\mathbf{v} \times \mathbf{w}$ .
- (iv) What is a vector orthogonal to  $\mathbf{v}$  and  $\mathbf{w}$ ?
- (v) Find the scalar triple product  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ .
- (vi) Are the vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  coplanar?

**Problem 2.** Find the area enclosed by the regions

- (i)  $y = x^3$ , and  $y = |x|$ .
- (ii)  $y = e^x$ ,  $y = e^2x$ , and  $x = \ln 2$ .

## Solutions

*Solution to Problem 1.* (i) Recall the formula for the scalar projection of the vector  $\mathbf{b}$  onto  $\mathbf{a}$

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}. \quad (1)$$

All we need to do for this problem is to substitute  $\mathbf{v}$  for  $\mathbf{a}$  and  $\mathbf{w}$  for  $\mathbf{b}$  in equation (1) and we have

$$\begin{aligned} \text{comp}_{\mathbf{v}} \mathbf{w} &= \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|} \\ &= \frac{\langle 0, 1, 2 \rangle \cdot \langle 4, -2, 5 \rangle}{|\langle 0, 1, 2 \rangle|} \\ &= \frac{0 \cdot 4 + 1(-2) + 2 \cdot 5}{\sqrt{0^2 + 1^2 + 2^2}} \\ &= \boxed{\frac{8}{\sqrt{5}}}. \end{aligned}$$

(ii) The equation for the projection of  $\mathbf{b}$  onto  $\mathbf{a}$  is given by

$$\text{proj}_{\mathbf{a}} \mathbf{b} = (\text{comp}_{\mathbf{a}} \mathbf{b}) \frac{\mathbf{a}}{|\mathbf{a}|} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a}. \quad (2)$$

Then substituting  $\mathbf{u}$  for  $\mathbf{a}$  and  $\mathbf{v}$  for  $\mathbf{b}$ , by equation 2, we have

$$\begin{aligned} \text{proj}_{\mathbf{u}} \mathbf{v} &= \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \right) \mathbf{u} \\ &= \left( \frac{\langle 6, 3, 1 \rangle \cdot \langle 0, 1, 2 \rangle}{|\langle 6, 3, 1 \rangle|^2} \right) \langle 6, 3, 1 \rangle \\ &= \frac{6 \cdot 0 + 3 \cdot 1 + 1 \cdot 2}{(6^2 + 3^2 + 1)^2} \langle 6, 3, 1 \rangle \\ &= \frac{3 + 2}{36 + 9 + 1} \langle 6, 3, 1 \rangle \\ &= \boxed{\frac{5}{46} \langle 6, 3, 1 \rangle}. \end{aligned}$$

(iii) The cross product of  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  is

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle. \quad (3)$$

hen substituting  $\mathbf{v}$  for  $\mathbf{a}$  and  $\mathbf{w}$  for  $\mathbf{b}$ , by equation 3, we have

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \langle 1 \cdot 5 - 2(-2), 2 \cdot 4 - 0 \cdot 5, 0(-2) - 1 \cdot 4 \rangle \\ &= \boxed{\langle 9, 8, -4 \rangle}. \end{aligned}$$

Of course, I would never try to memorize that horrible formula, but instead write the vectors as the rows of a matrix like so

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix}$$

and taking the determinant like so

$$\begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \mathbf{i} + \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix} \mathbf{j} + \begin{bmatrix} 0 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{k},$$

and again

$$(1 \cdot 5 - 2(-2))\mathbf{i} + (2 \cdot 4 - 0 \cdot 5)\mathbf{j} + (0(-2) - 1 \cdot 4)\mathbf{k} = \boxed{9\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}}.$$

(iv) Remember that the cross product of  $\mathbf{a}$  and  $\mathbf{b}$  has the property that it is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ . In the case of  $\mathbf{v}$  and  $\mathbf{w}$  we can demonstrate that the the cross product  $\mathbf{v} \times \mathbf{w}$  is in fact orthogonal to  $\mathbf{v}$  and  $\mathbf{w}$ :

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= \langle 0, 1, 2 \rangle \cdot \langle 9, 8, -4 \rangle & \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= \langle 4, -2, 5 \rangle \cdot \langle 9, 8, -4 \rangle \\ &= 0 \cdot 9 + 1 \cdot 8 + 2(-4) & &= 4 \cdot 9 + (-2)8 + 5(-4) \\ &= 8 - 8 & &= 36 - 16 - 20 \\ &= 0 & &= 0. \end{aligned}$$

(v) Let's take our value for the cross product of  $\mathbf{v}$  with  $\mathbf{w}$  from part (iii) and dot it with  $\mathbf{u}$  this gives us

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \langle 6, 3, 1 \rangle \cdot \langle 9, 8, -4 \rangle \\ &= 6 \cdot 9 + 3 \cdot 8 + 1(-4) \\ &= 74.\end{aligned}$$

(vi) Remember, three vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are coplanar if they all lie on a plane. This means that if we can find a vector which is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$  then it will be orthogonal to  $\mathbf{c}$ . For the case of  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ , from the previous problem we have  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 72 \neq 0$  so  $\mathbf{u}$  is not orthogonal to  $\mathbf{v} \times \mathbf{w}$  so it cannot lie in the same plane as  $\mathbf{v}$  and  $\mathbf{w}$  ■

*Solution to Problem 2.* ((i)) A great way to begin this problem is by plotting the curves  $y = |x|$  and  $y = x^3$ . As you can see from Figure 1 the curves  $y = |x|$  and  $y = x^3$  intersect at the points  $x = 0$  and  $x = 1$ . Moreover, we see that on this interval,  $0 \leq x \leq 1$ ,  $y = x^3$  is always smaller than  $y = |x|$  so evaluating the integral

$$\int_0^1 ||x| - x^3| \, dx$$

comes down to computing the integral

$$\int_0^1 |x| - x^3 \, dx.$$

Now, remember the definition of the absolute value: If  $f(x)$  then its absolute value is the piecewise defined function

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0. \end{cases} \quad (4)$$

Since  $x$  is  $\geq 0$  on  $0 \leq x \leq 1$ ,  $y = |x| = x$  on the interval  $0 \leq x \leq 1$  so

$$\begin{aligned} \int_0^1 |x| - x^3 \, dx &= \int_0^1 x - x^3 \, dx \\ &= \left. \frac{x^2}{2} - \frac{x^4}{4} \right|_0^1 \\ &= \frac{1^2}{2} - \frac{1^4}{4} - \left( \frac{0^2}{2} - \frac{0^4}{4} \right) \\ &= \boxed{\frac{1}{4}}. \end{aligned}$$

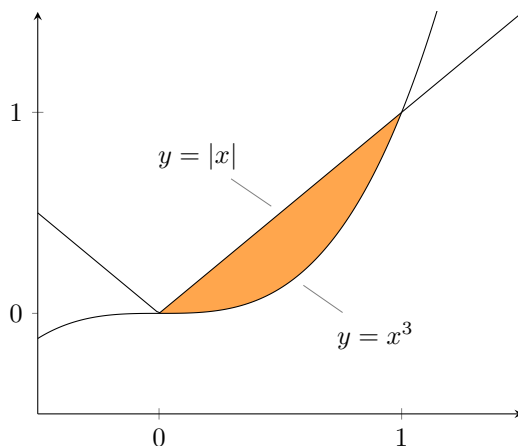


Figure 1: The area enclosed by the curves  $y = |x|$  and  $y = x^3$  at their points of intersection.

((ii)) As before, it helps to graph these curves (see Figure 2). From the figure we see that  $y = e^{2x}$  and  $e^x$  intersect at  $x = 0$  where  $e^{2 \cdot 0} = 1 = e^0$  and they both intersect the vertical line  $x = \ln 2$  at, well, obviously at  $x = \ln 2$ . So our integral will be

$$\int_0^{\ln 2} |e^{2x} - e^x| \, dx.$$

Like before, note that  $e^{2x} > e^x$  for any  $0 \leq x \leq \ln 2$  so we can forget about the

absolute value and evaluate the integral

$$\begin{aligned}
 \int_0^{\ln 2} e^{2x} - e^x \, dx &= \left. \frac{e^{2x}}{2} - e^x \right|_0^{\ln 2} \\
 &= \frac{e^{2\ln 2}}{2} - e^{\ln 2} - \left( \frac{e^{2 \cdot 0}}{2} - e^0 \right) \\
 &= \frac{e^{\ln 2^2}}{2} - 2 - \left( \frac{1}{2} - 1 \right) \\
 &= \frac{2^2}{2} - 2 + \frac{1}{2} \\
 &= 2 - 2 + \frac{1}{2} \\
 &= \boxed{\frac{1}{2}}.
 \end{aligned}$$

■

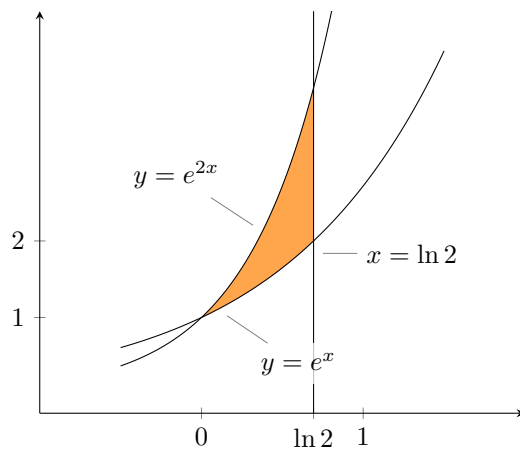


Figure 2: The area enclosed by the curves  $y = e^x$ ,  $y = e^{2x}$ , and  $x = \ln 2$  at their points of intersection.