MA 571: Homework # 13 due Wednesday December 2.

Please do:

p. 367 # 9(e) (Hint. First reduce to the case where h and k take b_0 to b_0. Then recall that the function $f : I \to S^1$ which takes s to $(\cos 2\pi s, \sin 2\pi s)$ is a representative for a generator of $\pi_1(S^1, b_0)$. Finally, notice that $f \times i_I : I \times I \to S^1 \times I$ is a quotient map).

p. 421 # 1, 2(abc), 3 (For # 2 and # 3, use the paragraph in the middle of page 418. Also, in the last sentence of 2(b), "odd length" should be "odd length > 1").

A) Let

$$q : S^2 \to P^2$$

be the quotient map, where P^2 is the projective plane. Let $x_0 = q(1, 0, 0)$ and let

$$f(s) = q(\cos \pi s, \sin \pi s, 0)$$

for $0 \le s \le 1$. (Note: the use of π in this formula instead of 2π is not a misprint.) Then $f : I \to P^2$ is a loop at x_0. **Prove** that $[f] * [f] = [e_{x_0}]$.

B) Let Y be the following subset of $\mathbb{R}^2$: $Y = \{(s,t) \in [0,1] \times [0,1] \mid s \in \{0,1\} \text{ or } t \in \{0,1\}\}$ (that is, Y is the boundary of the square $[0,1] \times [0,1]$). Give Y the equivalence relation $\sim$ that identifies the top and bottom edges and the left and right edges: specifically, $\sim$ is the equivalence relation associated to the partition of Y into the following sets:

$$\text{for each } s \notin \{0,1\}, \text{ the set } \{(s,0),(s,1)\},$$

$$\text{for each } t \notin \{0,1\}, \text{ the set } \{(0,t),(1,t)\},$$

$$\text{the set } \{0,1\} \times \{0,1\}.$$

Prove that $Y/\sim$ is a wedge of two circles (see the top of page 434 for the definition).

Optional problem (This problem will be used in the next assignment to show that for a 2-manifold there is a homeomorphism taking any point to any other point.) Let B^2 denote the unit disk $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$ and let S^1 denote the unit circle. Let $\mathbf{a} \in B^2 - S^1$. In this problem we will show that there is a homeomorphism $h : B^2 \to B^2$ which takes $(0,0)$ to $\mathbf{a}$ and fixes S^1.

i) Let $h : B^2 \to B^2$ be the function defined as follows: note that every point in B^2 has the form $t\mathbf{y}$ for some $\mathbf{y} \in S^1$ and $t \in [0,1]$, and define $h(t\mathbf{y}) = (1-t)\mathbf{a} + t\mathbf{y}$. Prove that this is well-defined, continuous, and lands in B^2. (Hint: to show continuity, you can give a more explicit formula or you can use a quotient map.)

ii) Show that $h(0,0) = \mathbf{a}$ and that h fixes S^1.

iii) Prove that h is one-to-one. (Hint: first use the dot product and the quadratic formula to show that if $\mathbf{u}$, $\mathbf{v}$ are vectors with $|\mathbf{u}| < 1$ then there is a unique positive s with $|\mathbf{u} + s\mathbf{v}| = 1$; geometrically this just says that any ray that starts inside the unit circle has exactly one point on the unit circle.)

iv) Prove that h is onto. (Hint: if $|\mathbf{u}| < 1$, $|\mathbf{u}+\mathbf{v}| \le 1$, and $|\mathbf{u}+s\mathbf{v}| = 1$ with s positive, show that $s \ge 1$).

v) Prove that h is a homeomorphism.