MA553: Qual Problems

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1 Ulrich's MA 553 Problems for Spring '16

1.1 Homework 1

Problem 1.1. Let G be a group, $a \in G$ an element of finite order m, and n a positive integer. Prove that

$$|a^n| = \frac{m}{\gcd(m, n)}.$$

Proof. ∎

Problem 1.2. Let G be a group, and let a, b be elements of finite order m, n respectively. Show that if $ba = ab$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$, then $|ab| = \operatorname{lcm}(m, n)$.

Proof. ∎

Problem 1.3. Let G be a group and H, K normal subgroups with $H \cap K = \{e\}$. Show that

(a) $hk = kh$ for every $h \in H$, $k \in K$.

(b) HK is a subgroup of G with $HK \cong H \times K$.

Proof. ∎

Problem 1.4. Show that A_4 has no subgroup of order 6 (although $6 \mid 12 = |A_4|$).

Proof. ∎

1.2 Homework 2

Problem 1.5. Let G be the group of order $2^3 \cdot 3$, $n \geq 2$. Show that G has a normal 2-subgroup $\neq \{e\}$.

Proof. ■

Problem 1.6. Let G be a group of order $p^2 q$, p and q primes. Show that the Sylow p-Sylow subgroup or the q-Sylow subgroup of G is normal in G.

Proof. ■

Problem 1.7. Let G be a subgroup of order pqr, $p < q < r$ primes. Show that the r-Sylow subgroup of G is normal in G.

Proof. ■

Problem 1.8. Let G be a group of order n and let $\varphi \colon G \to S_n$ be given by the action of G on G via translation.

(a) For $a \in G$ determine the number and the lengths of the disjoint cycles of the permutation $\phi(a)$.

(b) Show that $\varphi(G) \not\subset A_n$ if and only if n is even and G has a cyclic 2-Sylow subgroup.

(c) If $n = 2m$, m odd, show that G has a subgroup of index 2.

Proof. ■

Problem 1.9. Show that the only simple groups $\neq \{e\}$ of order < 60 are the groups of prime order.

Proof. ■

1.3 Homework 3

Problem 1.10. Let G be a finite group, p a prime number, N the intersection of all p-Sylow subgroups of G. Show that N is a normal p-subgroup of G and that every normal p-subgroup of G is contained in N.

Proof. ■

Problem 1.11. Let G be a group of order 231 and let H be an 11-Sylow subgroup of G. Show that $H \subset Z(G)$.

Proof. ■

Problem 1.12. Let $G = \{e, a_1, a_2, a_3\}$ be a non-cyclic group of order 4 and define $\varphi \colon S_3 \to \operatorname{Aut}(G)$ by $\varphi(\sigma)(e) = e$ and $\varphi(\sigma)(a_1) = a_{\sigma(i)}$. Show that φ is well-defined and an isomorphism of groups.

Proof. ■

Problem 1.13. Determine all groups of order 18.

Proof. ■