MA571 Homework 8

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Problem 8.1 (Munkres §46, Ex. 6)

Show that the compact-open topology, $\mathcal{C}(X,Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular. [*Hint:* If $\overline{U} \subset V$, then $\overline{S(C,U)} \subset S(C,V)$.]

Proof. We shall first prove the following obvious fact:

Lemma 16. *Suppose C is a finite subset of X. Then C is compact.*

Proof of lemma. Put $C = \{x_1, ..., x_n\}$. Let $\mathcal{A} = \{U_\alpha\}$ be an open cover of C. Then, for every $x_i \in C$ there is at least one $V_i \in \mathcal{A}$ containing x_i. ♣

■

Problem 8.2 (Munkres §46, Ex. 7)

Show that if Y is locally compact Hausdorff, then composition of maps

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$$

is continuous, provided the compact-open topology is used throughout. [*Hint:* If $g \circ f \in S(C, U)$, find V such that $f(C) \subset V$ and $g\big(\overline{V}\big) \subset U$.]

Proof. ∎

Problem 8.3 (Munkres §46, Ex. 8)

Let $\mathcal{C}'(X,Y)$ denote the set $\mathcal{C}(X,Y)$ in some topology $\mathcal{T}$. Show that if the evaluation map

$$e\colon X \times \mathcal{C}'(X,Y) \longrightarrow Y$$

is continuous, then $\mathcal{T}$ contains the compact-open topology. [*Hint:* The induced map $E\colon \mathcal{C}'(X,Y) \to \mathcal{C}(X,Y)$ is continuous.]

Proof. ∎

Problem 8.4 ((A))

Definition 1. Definition. If X is a locally compact Hausdorff space then the space Y given by Theorem 29.1 is called the *one-point compactification* of X.

Let X be a compact Hausdorff space and let W be an open subset of X (so W is locally compact by Corollary 29.3) with $W \neq X$. Prove that the one-point compactification of W is homeomorphic to the quotient space $X/(X - W)$.

Proof. ■

Problem 8.5 ((B))

Let X be a compact Hausdorff space, let Y be a topological space, and let $p\colon X \to Y$ be a closed surjective continuous map. Prove that Y is Hausdorff. [*Hint:* one ingredient in the proof is p. 171 # 5.]

Note: combining this with HW 4 Problem E and HW 6 Problem A gives a necessary and sufficient condition for a quotient of a compact Hausdorff space to be Hausdorff.

Proof. ■

PROBLEM 8.6 ((C))

Let $S^2 \subset \mathbf{R}^3$ be the subspace

$$\{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}.$$

Prove that S^2 is a 2-manifold. (The definition of m-manifold, where m is a positive whole number, is given at the top of page 225.)

Proof. ■

Problem 8.7 ((D))

Prove that the union of the x and y-axes in $\mathbf{R}^2$ is not a 1-manifold.

Proof. ∎