

MA 519: Homework 9

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PROBLEM 9.1 (HANDOUT 13, # 7)

Let X have a *double exponential* density $f(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$, $-\infty < x < \infty$, $\sigma > 0$.

- Show that all moments exist for this distribution.
- However, show that the MGF exists only for restricted values. Identify them and find a formula.

SOLUTION. For part (a), we show that $E(X^n) < \infty$ for all $n \in \mathbb{N}$. By direct calculation, we have

$$\begin{aligned} E(X^n) &= \int_{-\infty}^{\infty} x^n f(x) dx \\ &= \int_{-\infty}^{\infty} \frac{x^n}{2\sigma} e^{-\frac{|x|}{\sigma}} dx \\ &= \underbrace{\int_{-\infty}^0 \frac{x^n}{2\sigma} e^{\frac{x}{\sigma}} dx}_L + \int_0^{\infty} \frac{x^n}{2\sigma} e^{-\frac{x}{\sigma}} dx, \end{aligned}$$

making the substitution $x \mapsto -y$ to L and relabeling y to x again, we see that

$$\begin{aligned} &= \int_{-\infty}^0 \frac{x^n + (-1)x^n}{2\sigma} e^{-\frac{x}{\sigma}} dx \\ &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ I := \int_0^{\infty} \frac{x^n}{\sigma} e^{-\frac{x}{\sigma}} dx & \text{if } n = 2k \text{ is even.} \end{cases} \end{aligned}$$

To evaluate I we use integration by parts recursively to arrive at

$$\begin{aligned} I &= \int_{-\infty}^0 \frac{x^n}{\sigma} e^{-\frac{x}{\sigma}} \\ &= (-0 + 0) + \int_0^{\infty} n\sigma x^{n-1} e^{-\frac{x}{\sigma}} dx \\ &= (-0 + 0) + (-0 + 0) + \int_0^{\infty} n(n-1)\sigma^2 x^{n-2} e^{-\frac{x}{\sigma}} dx \\ &\vdots \\ &= (-0 - 0) + \cdots + (-0 + 0) + (-0 + n!\sigma^n) \\ &= n!\sigma^n. \end{aligned}$$

Therefore, $E(X^n) < \infty$ for all $n \in \mathbb{N}$, i.e., all moments of this distribution exist.

For part (b), the MGF associated to f is

$$m(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n E(X^n)}{n!} = 1 + \sum_{k=1}^{\infty} t^{2k} \sigma^{2k}. \quad (9.1)$$

This is a geometric series and so converges for all $-\frac{1}{\sigma} < t < \frac{1}{\sigma}$, in which case (9.1) becomes

$$m(t) = \frac{1}{1 - t^2 \sigma^2}.$$

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PROBLEM 9.2 (HANDOUT 13, # 10)

Suppose X has Cauchy distribution as in # 6. Which of the following functions have finite expectation

$$X; \quad -X; \quad |x|; \quad \frac{1}{X}; \quad \sin X; \quad \ln |X|; \quad e^X; \quad e^{-|X|}?$$

SOLUTION. Recall from Handout 13, # 6, that X is the horizontal distance along the wall when a flashlight is aimed at random with angle $\Theta \sim \text{Uniform}[-\pi, \pi]$. First, let us find the PDF of X . Let y be the vertical distance from the light source to the wall. Then, by elementary trigonometry, we have the following relation between X and Θ :

$$X = y \tan \Theta. \tag{9.2}$$

Using (9.2), we can figure out the CDF of X and consequently its PDF:

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= P(y \tan \Theta \leq x) \\ &= P\left(\Theta \leq \tan^{-1}\left(\frac{x}{y}\right)\right) \\ &= P\left(-\frac{\pi}{2} \leq \Theta \leq \tan^{-1}\left(\frac{x}{y}\right)\right) \\ &= \\ &= \frac{1}{2\pi} \int_{-\pi}^{\tan^{-1}(\frac{x}{y})} 1 \, d\theta \\ &= \frac{1}{2\pi} \left[\tan^{-1}\left(\frac{x}{y}\right) + \pi \right] \\ &= \frac{1}{2\pi} \tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}, \end{aligned}$$

therefore

$$\begin{aligned} f(x) &= \frac{dF(x)}{dx} \\ &= \end{aligned}$$

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PROBLEM 9.3 (HANDOUT 13, # 16)

Give an example of each of the following phenomena:

- (a) A continuous random variable taking values in $[0, 1]$ with equal mean and median.
- (b) A continuous random variable taking values in $[0, 1]$ with mean equal to twice the median.
- (c) A continuous random variable for which the mean does not exist.
- (d) A continuous random variable for which the mean exists, but the variance does not exist.
- (e) A continuous random variable with a PDF that is not differentiable at zero.
- (f) a positive continuous random variable for which the mode is zero, but the mean does not exist.
- (g) A continuous random variable for which all moments exist.
- (h) A continuous random variable with median equal to zero, and 25th and 75th percentiles equal to 1.
- (i) A continuous random variable X with mean equal to median equal to mode equal to zero, and $E(\sin X) = 0$.

SOLUTION. First, note that $[0, 1]$ is a probability space under the standard Lebesgue measure on \mathbb{R} . Therefore, it makes sense to consider $X: [0, 1] \rightarrow \mathbb{R}$ random variables.

For part (a), consider the random variable $X: [0, 1] \rightarrow \mathbb{R}$ defined by $x \mapsto x$ with $X \sim \text{Uniform}[0, 1]$. Then the mean is

$$\mu = \int_{-\infty}^{\infty} xf(x) dx = \int_0^1 x dx = \frac{1}{2}$$

and the median is

$$m = \inf\{x : F(x) = x \geq 0.5\} = \frac{1}{2}.$$

For part (b), consider again the random variable $X(x) = x$ for $x \in [0, 1]$, but this time let

$$f(x) = \begin{cases} & , \\ & . \end{cases}$$

be the PDF of X . Then the mean is ■

PROBLEM 9.4 (HANDOUT 13, # 17)

An exponential random variable with mean 4 is known to be larger than 6. What is the probability that it is larger than 8?

SOLUTION.

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PROBLEM 9.5 (HANDOUT 13, # 18)

(Sum of Gammas). Suppose X, Y are independent random variables, and $X \sim \Gamma(\alpha, \lambda)$, $Y \sim \Gamma(\beta, \lambda)$. Find the distribution of $X + Y$ by using moment-generating functions.

SOLUTION. ■

PROBLEM 9.6 (HANDOUT 13, # 19)

(*Product of Chi Squares*). Suppose X_1, X_2, \dots, X_n are independent chi square variables, with $X_i \sim \chi_{m_i}^2$. Find the mean and variance of $\prod_{i=1}^n X_i$.

SOLUTION. ■

PROBLEM 9.7 (HANDOUT 13, # 20)

Let $Z \sim \text{Normal}(0, 1)$. Find

$$P\left(0.5 < \left|Z - \frac{1}{2}\right| < 1.5\right); \quad P\left(\frac{e^Z}{1 + e^Z} > \frac{3}{4}\right); \quad P(\Phi(Z) < 0.5).$$

SOLUTION.

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PROBLEM 9.8 (HANDOUT 13, # 21)

Let $Z \sim \text{Normal}(0, 1)$. Find the density of $\frac{1}{Z}$. Is the density bounded?

SOLUTION. ■

PROBLEM 9.9 (HANDOUT 13, # 22)

The 25th and the 75th percentile of a normally distributed random variable are -1 and 1 . What is the probability that the random variable is between -2 and 2 ?

SOLUTION.

