MA557 Problem Set 5

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Problem 5.1

For I an R-ideal consider the multiplicatively closed set $S = 1 + I$. Show that

(a) $\widetilde{S} = R \smallsetminus \bigcup \mathfrak{m}$, where the union is taken over all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R) \cap V(I)$.
(b) $\mathfrak{m}\text{-Spec}(R/I)$ are homeomorphic.

Proof. ∎

Problem 5.2

Show that the following are equivalent for a ring R:

(a) there exist rings $R_1 \neq 0$ and $R_2 \neq 0$ so that $R \cong R_1 \times R_2$;
(b) there exist an idempotent $e \in R$ with $e \neq 0$ and $e \neq 1$;
(c) $\mathrm{Spec}(R)$ is disconnected.

Proof. ■

Problem 5.3

A topological space is called *Noetherian* if the set of closed sets satisfies the dcc. Show that if a ring R is Noetherian then so is $\operatorname{Spec}(R)$, but that the converse does not hold.

Proof. ■

Problem 5.4

A nonempty closed subset V of a topological space is called *irreducible* if $V = V_1 \cup V_2$, V_1 and V_2 closed subset, implies $V_1 = V$ or $V_2 = V$.

(a) Show that in a Noetherian topological space every nonempty closed subset is a finite union of irreducible closed subsets.
(b) Show that $V(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec}(R)$, are exactly the irreducible closed subsets of $\operatorname{Spec}(R)$.

Proof. ∎

Problem 5.5

Show that a Noetherian ring has only finitely many minimal prime ideals.

Proof. ∎

Problem 5.6

Show that a nonzero ring has at least one minimal prime ideal.

Proof. ■