MA 572: Homework 1

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Problem 1.1 (Hatcher §2.1, Ex. 11)

Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Proof. Suppose that A is a retract of X. Then there exists a continuous map $r\colon X \to A$ such that $r(X) = A$ and $r \,|\, A = \mathrm{id}_A$. Let $i\colon A \hookrightarrow X$ denote the inclusion map and $i_*\colon H_n(A) \to H_n(X)$ denote the induced homomorphism on the homology groups of A and X; do the same for r, $r_*\colon H_n(X) \to H_n(X)$. Then $r \circ i = \mathrm{id}_A$ which induces the endomorphism $(r \circ i)_* = r_* \circ i_* = \mathrm{id}_{H_n(A)}$ on $H_n(A)$. Thus, the inclusion map i_* is injective (since it has a left inverse). ∎

Problem 1.2 (Hatcher §2.1, Ex. 12)

Show that chain homotopy of chain maps is an equivalence relation.

Proof. Let X and Y be topological spaces and $f, g, h \colon X \to Y$ be continuous maps. Then $f_\#, g_\#, h_\# \colon C_n(X) \to C_n(Y)$ denote the induced chain maps. We show that chain homotopy of chain maps is an equivalence relation:

(i) Let P be the 0 homomorphsim. Then, we have

$$\partial 0 + 0\partial = 0 = f_\# - f_\#.$$

Thus, $f_\#$ is chain homotopic to itself.

(ii) Suppose $f_\#$ is chain homotopic to $g_\#$. Then there exist a homomorphism $P \colon C_n(X) \to C_{n+1}(Y)$ such hat $\partial P + P\partial = g_\# - f_\#$. Put $Q \coloneqq -P$. Then, we have

$$\partial(-P) + (-P)\partial = -(\partial P + P\partial) = -(g_\# - f_\#) = f_\# - g_\#.$$

Thus, $g_\#$ is chain homotopic to $f_\#$.

(iii) Suppose that $f_\#$ is chain homotopic to $g_\#$ and $g_\#$ is chain homotopic to $h_\#$. Then there exists homomorphism $P \colon C_n(X) \to C_{n+1}(Y)$ and a homomorphism $Q \colon C_n(X) \to C_{n+1}(Y)$ such that $\partial P + P\partial = g_\# - f_\#$ and $\partial Q + Q\partial = h_\# - g_\#$. Put $R \coloneqq P + Q$. Then, we have

$$\begin{aligned}
\partial(P+Q) + (P+Q)\partial &= \partial P + \partial Q + P\partial + Q\partial \\
&= (\partial Q + Q\partial) + (\partial P + P\partial) \\
&= (h_\# - g_\#) + (g_\# - f_\#) \\
&= h_\# - f_\#.
\end{aligned}$$

Thus, $f_\#$ is chain homotopic to $h_\#$.

We conclude that 'chain homotopy' is an equivalence relation. ■