

## MA 544: Homework 10

Carlos Salinas

April 4, 2016



**PROBLEM 10.1 (WHEEDEN & ZYGMUND §7, EX. 1)**

Let  $f$  be measurable in  $\mathbf{R}^n$  and different from zero in some set of positive measure. Show that there is a positive constant  $c$  such that  $f^*(\mathbf{x}) \geq c\|\mathbf{x}\|^{-n}$  for  $\|\mathbf{x}\| \geq 1$ .

*Proof.*

■

**PROBLEM 10.2 (WHEEDEN & ZYGMUND §7, EX. 2)**

Let  $\varphi(\mathbf{x})$ ,  $\mathbf{x} \in \mathbf{R}^n$ , be a bounded measurable function such that  $\varphi(\mathbf{x}) = 0$  for  $\|\mathbf{x}\| \geq 1$  and  $\int \varphi = 1$ . For  $\varepsilon > 0$ , let  $\varphi_\varepsilon(\mathbf{x}) := \varepsilon^{-n} \varphi(\mathbf{x}/\varepsilon)$ . ( $\varphi_\varepsilon$  is called an *approximation to the identity*.) If  $f \in L^1(\mathbf{R}^n)$ , show that

$$\lim_{\varepsilon \rightarrow 0} (f * \varphi_\varepsilon)(x) = f(\mathbf{x})$$

in the Lebesgue set of  $f$ . (Note that  $\int \varphi_\varepsilon = 1$ ,  $\varepsilon > 0$ , so that

$$(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x}) = \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})] \varphi_\varepsilon(\mathbf{y}) \, d\mathbf{y}.$$

Use Theorem 7.16.)

*Proof.*

■

**PROBLEM 10.3 (WHEEDEN & ZYGMUND §7, EX. 6)**

Show that if  $\alpha > 0$ , then  $x^\alpha$  is absolutely continuous on every bounded subinterval of  $[0, \infty)$ .

*Proof.*

■

**PROBLEM 10.4 (WHEEDEN & ZYGMUND §7, EX. 8)**

Prove the following converse of Theorem 7.31: If  $f$  is of bounded variation on  $[a, b]$ , and if the function  $V(x) = V[a, x]$  is absolutely continuous on  $[a, b]$ , then  $f$  is absolutely continuous on  $[a, b]$ .

*Proof.*

■

**PROBLEM 10.5 (WHEEDEN & ZYGMUND §7, EX. 9)**

If  $f$  is of bounded variation on  $[a, b]$ , show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then  $f$  is absolutely continuous on  $[a, b]$ . (For the second part, use Theorems 2.2(ii) and 7.24 to show that  $V(x)$  is absolutely continuous and then use the result of Exercise 8).

*Proof.*

■

**PROBLEM 10.6 (WHEEDEN & ZYGMUND §7, EX. 11)**

Prove the following result concerning changes of variable. Let  $g(t)$  be monotone increasing and absolutely continuous on  $[\alpha, \beta]$  and let  $f$  be integrable on  $[a, b]$ ,  $a := g(\alpha)$ ,  $b := g(\beta)$ . Then  $f(g(t))g'(t)$  is measurable and integrable on  $[\alpha, \beta]$ , and

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(g(t))g'(t) \, dt.$$

(Consider the case when  $f$  is the characteristic function of an interval, an open set, etc.)

*Proof.*

■