MA571 Homework 14

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Problem 14.1 (Munkres §74, Ex. 6)

If $n > 1$, show that the fundamental group of the n-fold torus is not Abelian. [*Hint:* Let G be a free group on the set $\{\alpha_1, \beta_1, ..., \alpha_n, \beta_n\}$; let F be the free group on the set $\{\gamma, \delta\}$. Consider the homomorphism of G onto F that sends α_1 and β_1 to γ and all other α_i and β_i to δ.]

Proof. Let $\mathbf{T}^n$ denote the n-fold torus and let us fix a base-point $x_0 \in \mathbf{T}^n$. By Theorem 74.3, the fundamental group of $\mathbf{T}^n$, $\pi_1(\mathbf{T}^n, x_0)$, is isomorphic to the quotient of the free group G on the set $\{\alpha_1, \beta_1, ..., \alpha_n, \beta_n\}$, by the least normal subgroup N containing

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1}.$$

Now we proceed by the hint. Let F be the free group on the set $\{\gamma, \delta\}$. We define a homomorphism $\varphi\colon G \to F$ by the rule $\alpha_1 \mapsto \gamma$, $\beta_1 \mapsto \gamma$ and $\alpha_i \mapsto \delta$ and $\beta_i \mapsto \delta$ for all $i \neq 1$. By Lemma 69.1, φ determines a homomorphism $G \to F$. Moreover, note that φ is surjective since each generator is mapped onto by an element of G so by the 1st isomorphism theorem, the induced map $\bar{\varphi}\colon G/\ker\varphi \to F$ is an isomorphism. Now, we claim that it suffices to show that $\ker\varphi > N$ ■

Problem 14.2 (Munkres §76, Ex. 1)

Calculate $H_1(\mathbf{P}^2\#\mathbf{T})$. Assuming that the list of compact surfaces given in Theorem 75.5 is a complete list, to which of these surfaces is $\mathbf{P}^2\#\mathbf{T}$ homeomorphic?

Proof. ■

Problem 14.3 (Munkres §76, Ex. 2)

If $\mathbf{K}$ is the Klein bottle, calculate $H_1(\mathbf{K})$ directly.

Proof. ■

Problem 14.4 (Munkres §76, Ex. 3(a,b,c))

Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $acadbcb^{-1}d$.

(a) Check that all vertices of P are mapped to the same point of the quotient space X by the pasting map.

(b) Calculate $H_1(X)$.

(c) Assuming X is homeomorphic to one of the surfaces given in Theorem 75.5 (which it is), which surface is it ?

Proof. ■

Problem 14.5 (A)

Define P^n to be the space $\mathbf{S}^n/\!\sim$ where $z \sim z'$ if and only if $z = z'$ or $z = -z'$. Use the Seifert–van Kampen Theorem to calculate $\pi_1(\mathbf{P}^n)$. (Hint: induction starting from the case $n = 2$ that was done in class.)

Proof. ∎

Problem 14.6 (B)

A topological space X is called *homogeneous* if for every pair of points $x, y \in X$ there is a homeomorphism $\varphi\colon X \to X$ with $\varphi(x) = y$. Prove that every connected 2-manifold is homogeneous. (Hint: use the optional problem from the previous assignment.)

Proof. ∎

PROBLEM 14.7 (OPTIONAL PROBLEM)

(i) Let $x \subset \mathbf{R}^3$ be the cylinder

$$\left\{ (x, y, z) \;\middle|\; x^2 + y^2 = \tfrac{1}{\sqrt{2}} \text{ and } |z| \leq \tfrac{1}{\sqrt{2}} \right\}$$

and let $f\colon X \to \mathbf{R}^3$ be the map

$$f(x, y, z) = \left(2^{1/4} x\sqrt{1 - z^2}, 2^{1/4}, y\sqrt{1 - z^2}, z\right).$$

Prove that f is a homeomorphism from X to the subspace

$$Y = \mathbf{S}^2 \cap \left\{ (x, y, z) \;\middle|\; |z| \leq \tfrac{1}{\sqrt{2}} \right\}.$$

(ii) Prove that the Möbius band is homeomorphic to P^2 with an open disk removed (think of $\mathbf{P}^2$ as $\mathbf{S}^2/\!\sim$ and use part (i)).

Proof. ■