Instructor: Tatsunari Watanabe **Name:**____________________.
TA: Carlos Salinas

MA 26500-215 Quiz 9

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1. Let $\mathcal{P}_2$ be the set of all polynomials of degree less than or equal to 2. We define an inner product on $\mathcal{P}_2$ by

$$\langle p(t), q(t)\rangle = \int_{-1}^{1} p(t)q(t)\,\mathrm{d}t \tag{$\bigstar$}$$

for polynomials $p(t), q(t) \in \mathcal{P}_2$.

(a) (12 points) The set $\{1, t, t^2\}$ is a basis for $\mathcal{P}_2$. Use the Gram–Schmidt process to find an orthonormal basis for $\mathcal{P}_2$ using the inner product ($\bigstar$).

Solution: Following the general Gram–Schmidt process, define

$$\begin{aligned}
u_1(t) &= 1 \\
u_2(t) &= t - \left[\frac{\langle 1, t\rangle}{\langle 1, 1\rangle}\right]1 \\
&= t - \left[\frac{\int_{-1}^{1} t\,\mathrm{d}t}{\int_{-1}^{1} 1\,\mathrm{d}t}\right]1 \\
&= t - \left[\frac{1^2 - ((-1)^2)}{2}\right]1 \\
&= t \\
u_3(t) &= t^2 - \left[\frac{\langle t, t^2\rangle}{\langle t, t\rangle}\right]t - \left[\frac{\langle 1, t\rangle}{\langle 1, 1\rangle}\right]1 \\
&= t^2 - \left[\frac{\int_{-1}^{1} t^3\,\mathrm{d}t}{\int_{-1}^{1} t^2\,\mathrm{d}t}\right]t - \left[\frac{\int_{-1}^{1} t^2\,\mathrm{d}t}{\int_{-1}^{1} 1\,\mathrm{d}t}\right]1 \\
&= t^2 - \left[\frac{1/4(1)^4 - ((1/4)(-1)^4)}{2/3}\right]t - \left[\frac{1/3(1)^3 - ((1/3)(-1)^3)}{2}\right]1 \\
&= t^2 - \frac{1}{3}
\end{aligned}$$

(b) (8 points) Find an orthonormal basis for $\mathcal{P}_2$. [*Hint:* Use the normal basis you found in part (b).]

Solution: Using $\{u_1(t), u_2(t), u_3(t)\}$ we have

$$\begin{aligned}
\frac{u_1(t)}{\|u_1(t)\|} &= \frac{1}{\sqrt{\int_{-1}^{1} 1 \,\mathrm{d}t}} \\
&= \frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2}. \\
\frac{u_2(t)}{\|u_2(t)\|} &= \frac{t}{\sqrt{\int_{-1}^{1} t^2 \,\mathrm{d}t}} \\
&= \frac{t}{\sqrt{2/3}} \\
&= \sqrt{\frac{3}{2}} t \\
\frac{u_3(t)}{\|u_3(t)\|} &= \frac{t^2 - 1/3}{\int_{-1}^{1} (t^2 - 1/3)(t^2 - 1/3) \,\mathrm{d}t} \\
&= \sqrt{\frac{45}{8}} \left(t^2 - \frac{1}{3} \right) \\
&= \frac{1}{2} \sqrt{\frac{45}{2}} \left(t^2 - \frac{1}{3} \right).
\end{aligned}$$