MA52300 FALL 2016

Final Exam Practice Problems

1. Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary. Show that the problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u + \alpha \frac{\partial u}{\partial \nu} &= g \quad \text{on } \partial\Omega \end{aligned}$$

   has at most one solution in $C^2(\Omega) \cap C(\overline{\Omega})$ if $\alpha > 0$. Here ν is the outward normal on $\partial\Omega$ and f, g assumed to be smooth.

2. Let g be a continuous function with compact support in $\mathbb{R}^n$. Write the formula for the bounded solution of

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{for } x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) &= g(x) \quad \text{for } x \in \mathbb{R}^n. \end{aligned}$$

   Prove that

$$\lim_{t\to\infty} u(x,t) = 0,$$

   where the convergence is uniform in $x \in \mathbb{R}^n$.

3. Find an explicit solution to the problem

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) &= e^{3x} \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

4. Find a formula for the solution of

$$u_{tt} - u_{xx} + u = 0 \quad \text{in } \mathbb{R} \times (0,\infty)$$

   such that

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

   where $f,\ g \in C_0^\infty(\mathbb{R})$

   Hint: Method I: Use Hadamard's method of descent. Namely, find $h(y)$ such that $v(x,y,t) := h(y)u(x,t)$ solves

$$v_{tt} - (v_{xx} + v_{yy}) = 0.$$

   Method II: Use Fourier transform.

5. Let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ satisfy

$$\begin{aligned} u_{tt} - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), &\quad u_t(x,0) = h(x). \end{aligned}$$

   Show that if both g and h are radial, then so is $u(\cdot,t)$ for any $t > 0$. (Recall that a function f is called radial if $f(x) = f(|x|)$).

6. Find the value of the solution u of the initial value problem

$$\begin{aligned} u_{tt} - \Delta u &= 0 \quad \text{for } x \in \mathbb{R}^3,\ t > 0 \\ u(x,0) = 0, &\quad u_t(x,0) = \psi(x), \end{aligned}$$

where

$$\psi(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}$$

at a point (x,t) such that $|x| + t < a$.

7. Let u be a nonzero harmonic function in $B(0,R) := \{x \in \mathbb{R}^n : |x| < R\}$. Define

$$E(r) := \fint_{\partial B(0,r)} u^2(y) d\sigma_y.$$

Show that $\log E(r)$ is a convex function of $\log r$, i.e.

$$E\left(\sqrt{ab}\right)^2 \leq E(a)E(b), \quad a, b > 0,$$

for any $0 < a \leq c < R$.

Hint. Use the representation of u as a uniformly convergent series

$$u(x) = \sum_{k=0}^{\infty} p_k(x), \quad |x| < R,$$

where $p_k(x)$ is a homogeneous harmonic polynomial of order k.

8. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution of the following Cauchy problem,

$$\begin{aligned} u_{tt} - \Delta u &= e^{-t} f(x), \quad x \in \mathbb{R}^3, t > 0 \\ u(x,0) = u_t(x,0) &= 0, \quad x \in \mathbb{R}^3. \end{aligned}$$

Verify that the integral representation reduces to the obvious solution $u = e^{-t} + t - 1$ when $f(x) = 1$.

9. Let $f(x) = e^{-|x|^2}$, $x \in \mathbb{R}^n$. Find $f * f$.

Hint. Use either the heat equation or the Fourier transform.

10. Recall that a solution to the heat equation

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{aligned}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dt,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|z|^2}{4t}}.$$

Assume that g is continuous and compactly supported. Show that there exists a $C > 0$ such that

$$|Du(x,t)| \le \frac{C}{\sqrt{t}}\, \|g\|_{L^\infty}.$$