MA598: Lie Groups

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CHAPTER 1

Prologue

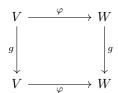
This summer, we will be making our way through Knapp's *Lie Groups Beyond an Introduction* [2] although, I (the writer of these notes) will occasionally refer to [1] for examples.

1.1 Representation of Finite Groups

Definitions

A representation of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho \colon G \to \operatorname{GL}(V)$; we say that such a map ρ gives V the structure of a G-module. When there is little ambiguity about the map ρ we will call V itself as a representation of G; in this vein, we suppose the symbol ρ and write gv for $\rho(g)(v)$. The dimension of V is sometimes called the degree of ρ .

A map φ between two representations V and W of G is a vector space map $\varphi\colon V\to W$ such that



commutes for every $g \in G$. (We will call this a G-linear map when we want to distinguish it from an arbitrary linear map between the vector spaces V and W). We can then define $\operatorname{Ker} \varphi$, $\operatorname{Im} \varphi$, and $\operatorname{Coker} \varphi$, which are also G-modules.

A subrepresentation of a representation V is a vector subspace W of V which is invariant under G. A representation V is called *irreducible* if there is no proper nonzero invariant subspace W of V.

If V and W are representations, so are $V \oplus W$ and $V \otimes W$ with $g(v \otimes w) := gv \otimes gw$. Moreover, the nth tensor power $\bigotimes^n V$, the exterior power $\bigwedge^n V$ and symmetric powers $\operatorname{Sym}^n V$ are subrepresentations of it. The dual $V^* = \operatorname{Hom}(V, \mathbb{C})$ of V is also a representation, though not in the most

obvious way: We want the two representations of G with respect to the natural pairing between V and V^* , so that if $\rho \colon G \to \operatorname{GL}(V)$ is a representation and $\rho^* \colon G \to \operatorname{GL}(V)$ is its dual, then we have

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle \tag{1}$$

for all $g \in G$, $v \in V$, and $v^* \in V^*$. This in turn forces us to define the dual representation by

$$\rho^*(g) := {}^{\operatorname{t}} \rho(g^{-1}) \colon V^* \longrightarrow V^*$$

for all $g \in G$.

Exercise 1. Let us verify that (1) is satisfied by the definition of ρ^* .

Proof. With ρ^* as defined above, choose $g \in G$, $v \in V$ and $v^* \in V$. Then, we have

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle = \langle \rangle$$

Having defined the dual representation of the tensor product of two representations, it is likewise the case that if V and W are representations, then $\operatorname{Hom}(V,W)$ is also a representation, via the identification $\operatorname{Hom}(V,W) = V^* \otimes W$. Unraveling this, if we view an element of $\operatorname{Hom}(V,W)$ as a linear map φ from V to W, we have

$$(g\varphi)(v)=g\varphi(g^{-1}v)$$

for all $v \in V$. In other words, the definition is such that the diagram

$$\begin{array}{ccc}
V & \xrightarrow{\varphi} & W \\
\downarrow^g & & \downarrow^g \\
V & \xrightarrow{g\varphi} & W
\end{array}$$

commutes. Note that the dual representation is, in turn, a special case of this: When $W=\mathbb{C}$ is the trivial representation, i.e., gw=w for all $w\in\mathbb{C}$, this makes V^* into a G-module, with $g\varphi(v)=\varphi(g^{-1}v)$, i.e., $g\varphi={}^{\mathrm{t}}(g^{-1})$.

Exercise 2. We verify that in general the vector space of G-linear maps between two representations V and W of G is just the subspace $\operatorname{Hom}(V,W)^G$ of elements of $\operatorname{Hom}(V,G)$ fixed under the action of G. We will often denote this space by $\operatorname{Hom}_G(V,W)$.

Proof.

We have taken the identification $\operatorname{Hom}(V,W)=V^*\otimes W$ as the definition of the representation $\operatorname{Hom}(V,W)$. More generally, the usual identities for vector spaces are also true for representations, e.g.,

$$\begin{split} V \otimes (U \oplus W) &= (V \otimes U) \oplus (V \otimes W) \\ \bigwedge^k (V \oplus W) &= \bigoplus_{a+b=k} \bigwedge^a V \otimes \bigwedge^b W \\ \bigwedge^k V^* &= \left(\bigwedge^k V\right)^* \end{split}$$

Bibliography

- [1] B. Hall. Lie Groups, Lie Algebras, and Representations: An Elementary Introduction. Graduate Texts in Mathematics. Springer, 2003.
- $[2]\;$ A.W. Knapp. Lie Groups Beyond an Introduction. Progress in Mathematics. Birkhäuser Boston, 2002.