MA571 Problem Set 7

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Problem 7.1 (Munkres §26, Ex. 8)

Theorem. *Let $f\colon X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if he* graph *of f,*

$$G_f = \{\, (x, f(x)) \mid x \in X \,\},$$

is closed in $x \times Y$.

[*Hint:* If G_f is closed and V is a neighborhood of $f(x_0)$, then the intersection of G_f and $X \times (Y - V)$ is closed. Apply Exercise 7.]

Proof. ■

Problem 7.2 (Munkres §26, Ex. 9)

Generalize the tube lemma as follows:

Theorem. *Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that*

$$A \times B \subset U \times V \subset N.$$

Proof. ■

Problem 7.3 (Munkres §26, Ex. 12)

Theorem. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then*

$$Y = \bigcap_{A \in \mathcal{A}} A.$$

Proof. ■

Problem 7.4 (Munkres §27, Ex. 2(b,d))

Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ε-neighborhood of A is contained in U.

Proof. ■

Problem 7.5 (Munkres §27, Ex. 5)

Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X. [*Hint:* Imitate the proof of Theorem 27.7.]

This is a special case of the *Baire category theorem*, which we shall study in Chapter 8.

Proof. ■

Problem 7.6 (Munkres §28, Ex. 2(a))

Let $\{X_\alpha\}$ be a nindexed family of nonempty spaces.

(a) Show that if $\prod X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α.

Proof. ∎

Problem 7.7 (Munkres §28, Ex. 10)

Show that if X is a Hausdorff space that is locally compact at the point x, then for each neighborhood U of x, there is a neighborhood V of x such that V is compact and $\overline{V} \subset U$.

Proof. ∎

Problem 7.8

Proof. ■

Problem 7.9 (A)

Let S^1 denote the circle
$$S^1 = \left\{\, (x, y) \in \mathbf{R}^2 \;\middle|\; x^2 + y^2 = 1 \,\right\}$$
and let B^2 denote the closed disk
$$B^2 = \left\{\, (x, y) \in \mathbf{R}^2 \;\middle|\; x^2 + y^2 \leq 1 \,\right\}.$$
Prove that the quotient space $(S^1 \times [0, 1])/(S^1 \times 0)$ (see HW #4 for the notation) is homeomorphic to B^2.

Proof. ∎