MA 544: Homework 3

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Problem 3.1 (Wheeden & Zygmund §3, Ex. 5)

Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage each interval removed has length $\delta 3^{-k}$, $0<\delta<1$. Show that the resulting set is perfect, has measure $1-\delta$, and contains no interval.

Proof. Let $1>\delta>0$ be given. Subdivide the interval $[0,1]$ into thirds, $\left[1,\frac{1}{3}\right]$, $\left[\frac{1}{3},\frac{2}{3}\right]$, and $\left[\frac{2}{3},1\right]$ and remove the open subset $\left(\frac{1}{2}-\delta,\frac{1}{2}+\delta\right)$ from the middle third $\left[\frac{1}{3},\frac{2}{3}\right]$. Repeat this process ad infinitum. ■

Problem 3.2 (Wheeden & Zygmund §3, Ex. 7)

Prove (3.15).

Proof.

Lemma (Wheeden & Zygmund (3.15)). *If $\{I_k\}_k^N$ is a finite collection of nonoverlapping intervals, then $\bigcup I_k$ is measurable and $|\bigcup I_k| = \sum |I_k|$.*

■

Problem 3.3 (Wheeden & Zygmund §3, Ex. 9)

If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ (and also $\liminf E_k$) has measure zero.

Proof. ∎

Problem 3.4 (Wheeden & Zygmund §3, Ex. 12)

If E_1 and E_2 are measurable subsets of $\mathbf{R}^1$, show that $E_1 \times E_2$ is measurable subset of $\mathbf{R}^2$ and $|E_1 \times E_2|_e = |E_1||E_2|_e$. (Interpret $0 \cdot \infty$ as 0) [Hint: Use a characterization of measurability.]

Proof. ■

Problem 3.5 (Wheeden & Zygmund §3, Ex. 13)

Motivated by (3.7), define the *inner measure of* E to by $|E|_i \coloneqq \sup|F|$, where the supremum is taken over all closed subsets F of E. Show that

(i) $|E|_i \leq |E|_e$, and

(ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_e = |E|_i$. [Use (3.22).]

Proof. ■

Problem 3.6 (Wheeden & Zygmund §3, Ex. 14)

Show that the conclusion of part (ii) of Exercise 13 (Problem) is false if $|E|_e = +\infty$.

Proof. ■

Problem 3.7 (Wheeden & Zygmund §3, Ex. 8)

Show that the Borel σ-algebra $\mathcal{B}$ in $\mathbf{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbf{R}^n$.

Proof. ■

Problem 3.8 (Wheeden & Zygmund §3, Ex. 10)

If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

Proof. ■

Problem 3.9 (Wheeden & Zygmund §3, Ex. 15)

If E is measurable and A is any subset of E, show that $|E| = |A|_i + |E \setminus A|_e$. [See Exercise 13 for the definition of $|A|_i$.]

Proof. ■

Problem 3.10 (Wheeden & Zygmund §3, Ex. 16)

Prove (3.34).

Proof.

Lemma. $|P| = v(P)$.

∎

Problem 3.11 (Wheeden & Zygmund §3, Ex. 18)

Prove that outer measure is *translation invariant*; that is, if $E_{\mathbf{h}} := \{ \mathbf{x} + \mathbf{h} \mid \mathbf{x} \in E \}$ is the translate of E by $\mathbf{h}$, $\mathbf{h} \in \mathbf{R}^n$, show that $|E_{\mathbf{h}}|_e = |E|_e$. If E is measurable, show that $E_{\mathbf{h}}$ is also measurable. [This fact was used in proving (3.37).]

Proof. ■