Math 535 - General Topology
Fall 2012
Homework 1 Solutions

Definition. Let V be a (real or complex) vector space. A **norm** on V is a function $\|\cdot\|\colon V \to \mathbb{R}$ satisfying:

1. Positivity: $\|x\| \geq 0$ for all $x \in V$ and moreover $\|x\| = 0$ holds if and only if $x = 0$.
2. Homogeneity: $\|\alpha x\| = |\alpha| \|x\|$ for any scalar α and $x \in V$.
3. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

A **normed vector space** is the data $(V, \|\cdot\|)$ of a vector space V equipped with a norm $\|\cdot\|$.

Problem 1. Let $(V, \|\cdot\|)$ be a normed vector space. Define a function $d\colon V \times V \to \mathbb{R}$ by

$$d(x, y) := \|x - y\|.$$

Show that d is a metric on V, called the metric **induced** by the norm $\|\cdot\|$.

Solution. We check the three properties of a metric.

1. Positivity:
$$d(x, y) = \|x - y\| \geq 0 \text{ for all } x, y \in V,$$
$$\begin{aligned} d(x, y) = 0 &\Leftrightarrow \|x - y\| = 0 \\ &\Leftrightarrow x - y = 0 \\ &\Leftrightarrow x = y. \end{aligned}$$

2. Symmetry:
$$\begin{aligned} d(y, x) &= \|y - x\| \\ &= \|(-1)(x - y)\| \\ &= |-1| \|x - y\| \\ &= \|x - y\| \\ &= d(x, y). \end{aligned}$$

3. Triangle inequality:
$$\begin{aligned} d(x, y) &= \|x - y\| \\ &= \|x - z + z - y\| \\ &\leq \|x - z\| + \|z - y\| \\ &= d(x, z) + d(z, y). \quad \square \end{aligned}$$

$\boxed{\textbf{Problem 2.}}$ Denote by $\|\cdot\|_2$ the standard (Euclidean) norm on $\mathbb{R}^n$, defined by

$$\|x\|_2 := \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

Now consider the function $\|\cdot\|_1 \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|.$$

$\boxed{\textbf{a.}}$ Show that $\|\cdot\|_1$ is a norm on $\mathbb{R}^n$.

Solution. We check the three properties of a norm.

1. Positivity:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \geq 0 \text{ for all } x \in \mathbb{R}^n,$$

$$\begin{aligned}
\|x\|_1 = 0 \Leftrightarrow \sum_{i=1}^{n} |x_i| &= 0 \\
\Leftrightarrow |x_i| &= 0 \text{ for } 1 \leq i \leq n \\
\Leftrightarrow x_i &= 0 \text{ for } 1 \leq i \leq n \\
\Leftrightarrow x &= 0.
\end{aligned}$$

2. Homogeneity:

$$\begin{aligned}
\|\alpha x\|_1 &= \sum_{i=1}^{n} |\alpha x_i| \\
&= \sum_{i=1}^{n} |\alpha||x_i| \\
&= |\alpha| \sum_{i=1}^{n} |x_i| \\
&= |\alpha|\|x\|_1.
\end{aligned}$$

3. Triangle inequality:

$$\begin{aligned}
\|x + y\|_1 &= \sum_{i=1}^{n} |x_i + y_i| \\
&\leq \sum_{i=1}^{n} (|x_i| + |y_i|) \\
&= \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| \\
&= \|x\|_1 + \|y\|_1. \quad \square
\end{aligned}$$

Remark. The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are special cases of the so-called p-norm, for any real number $p \geq 1$ or $p = \infty$. See:

`http://en.wikipedia.org/wiki/Lp_spaces#The_p-norm_in_finite_dimensions`.

$\boxed{\textbf{b.}}$ Find constants $C, D > 0$ satisfying

$$\|x\|_2 \leq C\|x\|_1$$
$$\|x\|_1 \leq D\|x\|_2$$

for all $x \in \mathbb{R}^n$.

Solution. Claim: $\|x\|_2 \leq \|x\|_1$ for all $x \in \mathbb{R}^n$. In other words, we can take the constant $C = 1$.

Proof 1. The claim is equivalent to

$$\begin{aligned} \|x\|_2^2 &\leq \|x\|_1^2 \\ \sum_{i=1}^n x_i^2 &\leq \left(\sum_{i=1}^n |x_i|\right)^2 \\ \sum_{i=1}^n x_i^2 &\leq \sum_{i=1}^n x_i^2 + \sum_{i \neq j} |x_i||x_j| \\ 0 &\leq \sum_{i \neq j} |x_i||x_j| \end{aligned}$$

which holds for all $x \in \mathbb{R}^n$. $\square$

Proof 2. Write e_i for the standard basis vector $e_i = (0, \ldots, 0, \overbrace{1}^{i^{\text{th}}}, 0, \ldots, 0)$. Then we have

$$\begin{aligned} \|x\|_2 &= \|\sum_{i=1}^n x_i e_i\|_2 \\ &\leq \sum_{i=1}^n \|x_i e_i\|_2 \\ &= \sum_{i=1}^n |x_i| \|e_i\|_2 \\ &= \sum_{i=1}^n |x_i| \\ &= \|x\|_1. \quad \square \end{aligned}$$

For the bound $\|x\|_1 \leq D\|x\|_2$, here are two solutions.

Solution 1: Crude bound. Noting $|x_i| \leq \|x\|_2$, we obtain

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i| \\ &\leq \sum_{i=1}^{n} \|x\|_2 \\ &= n\|x\|_2 \end{aligned}$$

so that we can take the constant $D = n$. □

Solution 2: Better bound. The 1-norm can be expressed as a dot product

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i| \\ &= \sum_{i=1}^{n} \operatorname{sign}(x_i) x_i \\ &= s \cdot x \end{aligned}$$

where $s \in \mathbb{R}^n$ is the vector with entries ± 1 given by $s_i = \operatorname{sign}(x_i)$. Let's say $\operatorname{sign}(0) = +1$ by convention.

The Cauchy-Schwarz inequality yields

$$\begin{aligned} \|x\|_1 &= s \cdot x \\ &\leq \|s\|_2 \|x\|_2 \\ &= \sqrt{n}\, \|x\|_2 \end{aligned}$$

so that we can take the constant $D = \sqrt{n}$. □

Remark. In fact, this is the best possible bound, because equality is achieved by the vector $u = (1, 1, \ldots, 1)$. Indeed, we have $\|u\|_1 = n$ whereas $\|u\|_2 = \sqrt{n}$.

Definition. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are **equivalent** if they can be compared as in Problem 2b.

Definition. Two metrics d_1 and d_2 on a set X are **topologically equivalent** if for every $x \in X$ and $\epsilon > 0$, there is a $\delta > 0$ satisfying

$$d_1(x, y) < \delta \Rightarrow d_2(x, y) < \epsilon$$

$$d_2(x, y) < \delta \Rightarrow d_1(x, y) < \epsilon.$$

In other words, the identity function $(X, d_1) \to (X, d_2)$ is a homeomorphism.

$\boxed{\textbf{Problem 3.}}$ Show that equivalent norms on a vector space V induce topologically equivalent metrics.

Solution. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V, and let $C, D > 0$ be constants satisfying

$$\begin{aligned} \|x\|_2 &\leq C\|x\|_1 \\ \|x\|_1 &\leq D\|x\|_2 \end{aligned}$$

for all $x \in V$. Thus the induced metrics satisfy

$$\begin{aligned} d_2(x, y) &= \|x - y\|_2 \\ &\leq C\|x - y\|_1 \\ &= Cd_1(x, y) \end{aligned}$$

and likewise $d_1(x, y) \leq Dd_2(x, y)$ for all $x, y \in V$.

Take $K := \max\{C, D\}$. Let $x \in V$ and $\epsilon > 0$ be given. Pick $\delta := \frac{\epsilon}{K}$ (which in this case can be chosen uniformly, independently of x). Then we have

$$\begin{aligned} d_1(x, y) < \delta \Rightarrow d_2(x, y) &\leq Cd_1(x, y) \\ &< C\delta \\ &\leq K\delta \\ &= \epsilon \end{aligned}$$

and likewise

$$\begin{aligned} d_2(x, y) < \delta \Rightarrow d_1(x, y) &\leq Dd_2(x, y) \\ &< D\delta \\ &\leq K\delta \\ &= \epsilon. \quad \square \end{aligned}$$

$\boxed{\textbf{Problem 4.}}$ (Bredon Prop. I.1.3) Show that topologically equivalent metrics induce the same topology (which explains the terminology). In other words, if d_1 and d_2 are topologically equivalent metrics on X, then a subset $U \subseteq X$ is open with respect to d_1 if and only if it is open with respect to d_2.

Solution. By symmetry of the situation, it suffices to show one side of the equivalence. Let $U \subseteq X$ be a subset which is open with respect to d_1. We want to show that U is open with respect to d_2.

Let $x \in U$. Because U is open with respect to d_1, there is an ϵ-ball (in the d_1 metric) around x entirely contained in U, i.e. $B^1_\epsilon(x) \subseteq U$, for some $\epsilon > 0$.

Because d_2 is topologically equivalent to d_1, there is some $\delta > 0$ satisfying $B^2_\delta(x) \subseteq B^1_\epsilon(x)$. In particular we have

$$B^2_\delta(x) \subseteq B^1_\epsilon(x) \subseteq U$$

so that U is open with respect to d_2. $\square$

Definition. Let $(X, \mathcal{T})$ be a topological space. A subset $C \subseteq X$ is **closed** (with respect to $\mathcal{T}$) if its complement $C^c := X \setminus C$ is open (with respect to $\mathcal{T}$).

Problem 5. Show that the collection of closed subsets of X satisfies the following properties.

1. The empty subset $\emptyset$ and X itself are closed.
2. An arbitrary intersection of closed subsets is closed: C_α closed for all α implies $\bigcap_\alpha C_\alpha$ is closed.
3. A finite union of closed subsets is closed: C, C' closed implies $C \cup C'$ is closed.

Solution.

1. The empty set $\emptyset$ is closed because its complement $\emptyset^c = X$ is open.

   The entire set X is closed because its complement $X^c = \emptyset$ is open.
2. Let C_α be closed for all α and consider the intersection $\bigcap_\alpha C_\alpha$. Its complement
$$\left(\bigcap_\alpha C_\alpha\right)^c = \bigcup_\alpha C_\alpha^c$$
   is a union of open sets, therefore open. Thus $\bigcap_\alpha C_\alpha$ is closed.
3. Let C, C' be closed and consider the union $C \cup C'$. Its complement
$$(C \cup C')^c = C^c \cap C'^c$$
   is a finite intersection of open sets, therefore open. Thus $C \cup C'$ is closed. $\square$

Remark. In fact, a collection of subsets satisfies these properties if and **only if** their complements form a topology. Moreover, open subsets and closed subsets determine each other.

Upshot: One might as well define a topology via a collection of "closed subsets" satisfying the three properties above. Their complements then form the topology in question.

$\boxed{\textbf{Problem 6.}}$ Let X be a set. Consider the collection of **cofinite** subsets of X together with the empty subset:

$$\mathcal{T}_{\text{cofin}} := \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}.$$

$\boxed{\textbf{a.}}$ Show that $\mathcal{T}_{\text{cofin}}$ is a topology on X, called the **cofinite topology**.

Solution. By problem 5, it suffices to check that the collection of subsets

$$\{F \subseteq X \mid F \text{ is finite}\} \cup \{X\}.$$

satisfies the axioms of closed subsets.

1. The empty set $\emptyset$ is finite, and therefore belongs to the collection, whereas X belongs to the collection by definition.
2. Let C_α be a family of subsets that are either finite or all of X. Then the intersection $\bigcap_\alpha C_\alpha$ is finite (if at least one C_α is finite) or all of X (if all C_α are X).
3. Let C, C' be subsets that are either finite or all of X. Then the union $C \cup C'$ is finite (if both C and C' are finite) or all of X (if at least one of C or C' is X). $\square$

$\boxed{\textbf{b.}}$ Assuming X is infinite, show that the cofinite topology on X cannot be induced by a metric on X.

Solution. Let $B_r(x)$ be an open ball not containing all of X. If the complement $B_r(x)^c$ is infinite, then we are done: we have found an open subset which is not cofinite.

If the complement $B_r(x)^c$ is finite, then pick a point $y \in B_r(x)^c$ and choose a radius $\epsilon > 0$ small enough so that the open ball $B_\epsilon(y)$ does not intersect $B_r(x)$; any value $\epsilon \leq d(x,y) - r$ will do. Then $B_\epsilon(y)$ is finite, and thus its complement $B_\epsilon(y)^c$ is infinite, since X is infinite. We are done: we have found an open subset $B_\epsilon(y)$ which is not cofinite. $\square$

Slightly different solution. If X is infinite, then:

- Any cofinite subset of X is infinite;
- Any cofinite subset and any infinite subset must intersect.

In particular, any two non-empty open subsets of X intersect.

However, in any metric space containing at least two points, we can find non-empty open subsets that do not intersect. Indeed, pick distinct points x and y, and take small enough open balls $B_r(x)$ and $B_r(y)$ around them; any radius $r \leq \frac{d(x,y)}{2}$ will guarantee $B_r(x) \cap B_r(y) = \emptyset$.

Therefore the cofinite topology on X cannot be induced by a metric. $\square$

Remark. In a few lectures, we will say that such a topology is not Hausdorff, hence not metrizable.

Definition. Let X be a set.

- The **discrete** topology on X is the one where all subsets are open:

$$\mathcal{T}_{\text{disc}} = \mathcal{P}(X) = \{U \subseteq X\}.$$

- The **anti-discrete** (or **trivial**) topology on X is the one where only the empty subset and X itself are open:

$$\mathcal{T}_{\text{anti}} = \{\emptyset, X\}.$$

Problem 7. Let D be a discrete topological space and A an anti-discrete topological space.

a. Describe all continuous maps $f\colon D \to X$, where X is an arbitrary topological space.

Solution. The condition that $f^{-1}(V)$ be open in D for any open $V \subseteq X$ is automatically satisfied, since every subset of D is open. Thus *every function* $f\colon D \to X$ is continuous.

Remark. We will come back to the question of mapping *into* a discrete space when discussing the notion of connectedness.

b. Describe all continuous maps $f\colon X \to A$, where X is an arbitrary topological space.

Solution. For any function $f\colon X \to A$, we have $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(A) = X$, both of which are open in X. Since $\emptyset$ and A are the only open subsets of A, *every function* $f\colon X \to A$ is continuous.

c. Describe all continuous maps $f\colon A \to X$, where X is a metric space.

Solution. Let $f\colon A \to X$ be a continuous. We claim that f is a *constant* function.

Pick some $a \in A$ and look at its value $x := f(a) \in X$. For any other point $y \in X$, pick an open subset $V \subseteq X$ satisfying $x \in V$ but $y \notin V$. Since f is continuous, the preimage $f^{-1}(V)$ is open in A. The condition $f(a) = x \in V$ yields $a \in f^{-1}(V)$ so that $f^{-1}(V)$ is non-empty and must therefore be all of A, the only non-empty open subset of A.

Now the condition $f(A) \subseteq V$ implies that f never takes the value $y \notin V$. Since y was an arbitrary point distinct from x, we conclude that f is the constant function with value x. $\square$

Remark. The proof still holds whenever X is a T_1 space, a property that will be discussed in a few lectures.

Problem 8. Let $f\colon X \to Y$ be a function between topological spaces, and let $x \in X$.

a. Show that the following conditions (defining continuity of f at x) are equivalent.

1. For all neighborhood N of $f(x)$, there is a neighborhood M of x such that $f(M) \subseteq N$.
2. For all open neighborhood V of $f(x)$, there is an open neighborhood U of x such that $f(U) \subseteq V$.
3. For all neighborhood N of $f(x)$, the preimage $f^{-1}(N)$ is a neighborhood of x.

Solution. ($1 \Rightarrow 2$) Let V be an open neighborhood of $f(x)$. Since V is in particular a neighborhood of $f(x)$, the assumption (1) guarantees that there is a neighborhood M of x such that $f(M) \subseteq V$. Let U be an open of X satisfying $x \in U \subseteq M$. Then U is an open neighborhood of x satisfying $f(U) \subseteq f(M) \subseteq V$.

($2 \Rightarrow 3$) Let N be a neighborhood of $f(x)$. Let V be an open of Y satisfying $f(x) \in V \subseteq N$. Then we have $x \in f^{-1}(V) \subseteq f^{-1}(N)$. By the assumption (2), $f^{-1}(V)$ is an open neighborhood of x, so that $f^{-1}(N)$ is a neighborhood of x.

($3 \Rightarrow 1$) Let N be a neighborhood of $f(x)$ and take $M := f^{-1}(N)$. By the assumption (3), $f^{-1}(N)$ is a neighborhood of x, and moreover it satisfies $f\left(f^{-1}(N)\right) \subseteq N$. $\square$

b. Find an example of function $f\colon X \to Y$ between *metric* spaces which is continuous at a point $x \in X$, but there is an open neighborhood V of $f(x)$ such that the preimage $f^{-1}(V)$ is *not* an open neighborhood of x.

Upshot: The description "preimage of open is open" is really about global continuity, not pointwise continuity (or even local continuity).

Solution. Consider the "step" function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 43 \\ 1 & \text{if } x > 43. \end{cases}$$

Then f is continuous at $x = 0$ (in fact everywhere except at 43). However, take the open neighborhood $(-\frac{1}{2}, \frac{1}{2})$ of $f(0) = 0$. Its preimage under f is

$$f^{-1}\left((-\frac{1}{2}, \frac{1}{2})\right) = (-\infty, 43]$$

which is not open in $\mathbb{R}$. $\square$

$\boxed{\textbf{Problem 9.}}$ Let X be a topological space and $\mathcal{B}$ a collection of open subsets of X.

$\boxed{\textbf{a.}}$ Show that $\mathcal{B}$ is a basis for the topology of X if and only if for every open subset $U \subseteq X$ and $x \in U$, there is a $B \in \mathcal{B}$ satisfying $x \in B \subseteq U$.

Solution. ($\Rightarrow$) Assume $\mathcal{B}$ is a basis for the topology. Let $U \subseteq X$ be open (WLOG non-empty) and $x \in U$. Because $\mathcal{B}$ is a basis, U can be written as a union $U = \bigcup_\alpha B_\alpha$ for some family of subsets $B_\alpha \in \mathcal{B}$. Thus x is in at least one of those subsets B_{α_x}, yielding $x \in B_{\alpha_x} \subseteq U$.

($\Leftarrow$) To show that $\mathcal{B}$ is a basis, there are two things to check.

1) Every union of members of $\mathcal{B}$ is open in X. This is automatic, because each $B \in \mathcal{B}$ was assumed to be open.

2) Let $U \subseteq X$ be open (WLOG non-empty). We want to show that U is a union of subsets in the collection $\mathcal{B}$. By assumption on $\mathcal{B}$, for each $x \in U$, there is some $B_x \in \mathcal{B}$ satisfying $x \in B_x \subseteq U$. Thus we have $U = \bigcup_{x \in U} B_x$. $\square$

$\boxed{\textbf{b.}}$ Assuming X is a metric space, show that the collection of open balls

$$\mathcal{B} = \{B_{\frac{1}{n}}(x) \mid x \in X, n \in \mathbb{N}\}$$

is a basis for the topology of X.

Solution. We use the criterion from part (a).

Let $U \subseteq X$ be open and $x \in U$. Then there is some radius $r > 0$ such that the open ball of radius r centered at x is contained within U, i.e. $B_r(x) \subseteq U$.

Pick n large enough so that $\frac{1}{n} \leq r$. Then we have $x \in B_{\frac{1}{n}}(x) \subseteq B_r(x) \subseteq U$. $\square$

Problem 10. Let X be a set and $\mathcal{S}$ a collection of subsets of X.

a. Show that the collection

$$\mathcal{T} := \left\{ \bigcup_{\alpha} \bigcap_{i=1}^{n_\alpha} S_{\alpha,i} \mid S_{\alpha,i} \in \mathcal{S} \right\}$$

of (arbitrary) unions of finite intersections of members of $\mathcal{S}$ is a topology on X.

Solution. We check the properties of a topology.

1. Because unions indexed by the empty family are allowed, the empty set $\emptyset = \bigcup_{\emptyset} S_\alpha$ is in $\mathcal{T}$.

   Because intersections indexed by the empty family are allowed, the entire set $X = \bigcap_{\emptyset} S_\alpha$ is in $\mathcal{T}$.

2. Finite intersections of members of $\mathcal{T}$ are in $\mathcal{T}$. Let $U = \bigcup_\alpha \bigcap_{i=1}^{n_\alpha} S_{\alpha,i}$ and $V = \bigcup_\beta \bigcap_{j=1}^{n_\beta} S_{\beta,j}$ be members of $\mathcal{T}$. Their intersection is

$$\begin{aligned} U \cap V &= \left(\bigcup_{\alpha} \bigcap_{i=1}^{n_\alpha} S_{\alpha,i} \right) \cap \left(\bigcup_{\beta} \bigcap_{j=1}^{n_\beta} S_{\beta,j} \right) \\ &= \bigcup_{\alpha,\beta} \left(\bigcap_{i=1}^{n_\alpha} S_{\alpha,i} \cap \bigcap_{j=1}^{n_\beta} S_{\beta,j} \right) \end{aligned}$$

   which is in $\mathcal{T}$ since each $S_{\alpha,i}$ and $S_{\beta,j}$ is in $\mathcal{S}$.

3. Arbitrary unions of members of $\mathcal{T}$ are in $\mathcal{T}$. Let $\{U^\gamma = \bigcup_\alpha \bigcap_{i=1}^{n_\alpha^{(\gamma)}} S_{\alpha,i}^\gamma\}_\gamma$ be a family of members of $\mathcal{T}$. Then their union is

$$\begin{aligned} \bigcup_{\gamma} U^\gamma &= \bigcup_{\gamma} \left(\bigcup_{\alpha} \bigcap_{i=1}^{n_\alpha^{(\gamma)}} S_{\alpha,i}^\gamma \right) \\ &= \bigcup_{\gamma,\alpha} \left(\bigcap_{i=1}^{n_\alpha^{(\gamma)}} S_{\alpha,i}^\gamma \right) \end{aligned}$$

   which is in $\mathcal{T}$. $\square$

$\boxed{\textbf{b.}}$ Show that $\mathcal{T}$ is the topology $\mathcal{T}_{\mathcal{S}}$ generated by $\mathcal{S}$. In other words: $\mathcal{T}$ contains $\mathcal{S}$ and any other topology $\mathcal{T}'$ containing $\mathcal{S}$ must satisfy $\mathcal{T} \leq \mathcal{T}'$.

Solution.

1. $\mathcal{T}$ contains $\mathcal{S}$: Any $S \in \mathcal{S}$ can be viewed as the union of one set which is the intersection of one set, namely S itself, which is in $\mathcal{S}$. Therefore we have $\mathcal{S} \subseteq \mathcal{T}$.

2. Let $\mathcal{T}'$ be a topology containing $\mathcal{S}$. For any family of subsets $S_{\alpha,i} \in \mathcal{S}$, the finite intersection $\bigcap_{i=1}^{n_\alpha} S_{\alpha,i}$ is in $\mathcal{T}'$, since $\mathcal{T}'$ is a topology. Moreover, the union
$$\bigcup_{\alpha} \bigcap_{i=1}^{n_\alpha} S_{\alpha,i}$$
is also in $\mathcal{T}'$, since T' is a topology. Thus we have $\mathcal{T} \subseteq \mathcal{T}'$, as claimed. $\square$