Math 535 - General Topology
Fall 2012
Homework 5 Solutions

Problem 1. Let X be a *first-countable* topological space, $(x_n)_{n\in\mathbb{N}}$ a sequence in X, and $y \in X$ a cluster point of this sequence. Show that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to y.

Solution. Since X is first-countable, the point y has a countable neighborhood basis $V_1 \supseteq V_2 \supseteq \ldots$.

Since y is a cluster point of (x_n), there is an index $n_1 \geq 1$ satisfying $x_{n_1} \in V_1$. Then there is an index $n_2 > n_1$ satisfying $x_{n_2} \in V_2$. Repeating the argument, one can find an index $n_k > n_{k-1}$ satisfying $x_{n_k} \in V_k$ for all $k \in \mathbb{N}$.

We claim that the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges to y. Let V be a neighborhood of y. By definition of neighborhood basis, there is some $m \in \mathbb{N}$ satisfying $V_m \subseteq V$. For all index $k \geq m$, we have

$$x_{n_k} \in V_k \subseteq V_m \subseteq V$$

which proves convergence $x_{n_k} \to y$. $\square$

Remark. This shows that a first-countable compact space is always sequentially compact.

Problem 2. Consider the discrete space with two elements $\{0,1\}$ and consider the space

$$X := \{0,1\}^{\mathcal{P}(\mathbb{N})} \cong \prod_{S \in \mathcal{P}(\mathbb{N})} \{0,1\}$$

with the product topology. Here $\mathcal{P}(\mathbb{N})$ denotes the power set of $\mathbb{N}$, i.e. $\mathcal{P}(\mathbb{N}) = \{S \mid S \subseteq \mathbb{N}\}$ is the set of all subsets of $\mathbb{N}$.

One can view X as the set of all functions $f \colon \mathcal{P}(\mathbb{N}) \to \{0,1\}$. In this viewpoint, the canonical projection $p_S \colon X \twoheadrightarrow \{0,1\}$ corresponds to evaluation at S, i.e. sending the function $f \in X$ to $f(S) \in \{0,1\}$.

For each $n \in \mathbb{N}$, consider the element $f_n \in X$ whose components are

$$f_n(S) = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S. \end{cases}$$

In other words, f_n is the function "Is n in there?"

Show that the sequence $(f_n)_{n \in \mathbb{N}}$ in X has no convergent subsequence.

Solution. Given any subsequence $(f_{n_k})_{k \in \mathbb{N}}$, we will show that it does not converge by finding a subset $S \subseteq \mathbb{N}$ such that the sequence $(f_{n_k}(S))_{k \in \mathbb{N}}$ in $\{0,1\}$ does not converge. Indeed, the projection $p_S \colon X \twoheadrightarrow \{0,1\}$ is continuous and therefore sends convergent sequences to convergent sequences.

Take the set $S = \{n_2, n_4, n_6, \ldots\} = \{n_{2k} \mid k \in \mathbb{N}\} \subset \mathbb{N}$. Then the sequence $p_S(f_{n_k}) = f_{n_k}(S)$ is $(0,1,0,1,0,1,\ldots)$, more precisely

$$f_{n_k}(S) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

because the numbers n_k are all distinct.

The sequence $(f_{n_k}(S))_{k \in \mathbb{N}}$ in $\{0,1\}$ does not converge, since the only convergent sequences in the discrete space $\{0,1\}$ are the sequences that are eventually constant. $\square$

Remark. By Tychonoff's theorem, X is compact, so that the sequence $(f_n)_{n \in \mathbb{N}}$ has a cluster point, even though it has no convergent subsequence. Thus X is an example of compact space which is not sequentially compact.

Problem 3. Let X be a topological space and consider the set of all continuous functions on X with values in $[0,1]$

$$C := \{f\colon X \to [0,1] \mid f \text{ is continuous}\}.$$

Consider the set $[0,1]^C \cong \prod_{f\in C}[0,1]$ of all functions from C to $[0,1]$, endowed with the product topology. Consider the map

$$\begin{aligned}\varphi\colon X &\to [0,1]^C \\ x &\mapsto (f(x))_{f\in C}\end{aligned}$$

so that $\varphi(x)$ is "evaluation at x".

a. Show that φ is continuous.

Solution. A map into a product is continuous if and only if all of its components are continuous. For any index $f \in C$, the f^{th} component of φ is $p_f \circ \varphi = f$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rr, bend right, "p_f\circ\varphi=f"'] & {[0,1]^C} \arrow[r, two heads, "p_f"] & {[0,1]} \end{tikzcd}$$

which is continuous, by definition of C. $\square$

b. Show that the closure of the image $\overline{\varphi(X)} \subset [0,1]^C$ is a compact Hausdorff space. (Feel free to assume the axiom of choice!)

Solution. Since $[0,1]$ is Hausdorff, the product $[0,1]^C$ is Hausdorff, and so is the subspace $\overline{\varphi(X)} \subset [0,1]^C$.

Since $[0,1]$ is compact, the product $[0,1]^C$ is compact, by Tychonoff's theorem. Since $\overline{\varphi(X)}$ is closed in $[0,1]^C$, it is itself compact. $\square$

c. Show that φ is injective if and only if points of X can be separated by functions, i.e. for any distinct points $x, y \in X$, there is a *continuous* function $f\colon X \to [0,1]$ satisfying $f(x) = 0$ and $f(y) = 1$.

This property is sometimes called **functionally Hausdorff** or **completely Hausdorff**.

Solution. Consider the equivalent statements

$\varphi\colon X \to [0,1]^C$ is injective.

$\Leftrightarrow$ For any distinct points $x, y \in X$, we have $\varphi(x) \neq \varphi(y)$, i.e. there is some index $f \in C$ such that the f^{th} components $\varphi(x)_f = f(x)$ and $\varphi(y)_f = f(y)$ are distinct.

$\Leftrightarrow$ For any distinct points $x, y \in X$, there is a continuous function $f\colon X \to [0,1]$ satisfying $f(x) \neq f(y)$ (WLOG $f(x) = 0$ and $f(y) = 1$, rescaling the function if needed). $\square$

d. While we're at it, show that a functionally Hausdorff space is always Hausdorff.

Solution. Let $x, y \in X$ be distinct points. Pick a continuous function $f\colon X \to [0,1]$ that separates x and y, say $f(x) = 0$ and $f(y) = 1$. Then $f^{-1}[0, \frac{1}{3})$ and $f^{-1}(\frac{2}{3}, 1]$ are open in X (since f is continuous), disjoint, and satisfy $x \in f^{-1}[0, \frac{1}{3})$ and $y \in f^{-1}(\frac{2}{3}, 1]$. $\square$

Problem 4. (Willard Exercise 17F.1) A topological space X is **countably compact** if every *countable* open cover of X admits a finite subcover. (In particular, compact always implies countably compact, but not the other way around in general.)

Show that X is countably compact if and only if every sequence in X has a cluster point.

Hint: Recall that compactness can be described in terms of closed sets. Countable compactness has a very similar description in terms of closed sets, which could be useful here.

Solution. ($\Rightarrow$) Assume X is countably compact and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Consider the n-tail of the sequence

$$T_n := \{x_k \mid k \geq n\} \subseteq X$$

and note that the collection of all tails $\{T_n\}_{n\in\mathbb{N}}$ has the finite intersection property, therefore so does the collection of their closures $\{\overline{T_n}\}_{n\in\mathbb{N}}$. Since X is countably compact, the intersection $\bigcap_{n\in\mathbb{N}} \overline{T_n} \neq \emptyset$ contains a point y.

We claim that y is a cluster point of the sequence. Let V be a neighborhood of y and $N \in \mathbb{N}$ any index. The condition $y \in \overline{T_N}$ guarantees that there is some $x_n \in T_N \cap V$, i.e. some point $x_n \in V$ of the sequence, with index $n \geq N$.

($\Leftarrow$) Let $\{C_n\}_{n\in\mathbb{N}}$ be a countable collection of closed subsets of X with the finite intersection property. For every $n \in \mathbb{N}$, pick a point

$$x_n \in \bigcap_{i=1}^{n} C_i \neq \emptyset$$

and let $y \in X$ be a cluster point of the sequence $(x_n)_{n\in\mathbb{N}}$.

We claim $y \in \bigcap_{n\in\mathbb{N}} C_n$. Let V be a neighborhood of y. Since y is a cluster point of the sequence, for all index $n \in \mathbb{N}$, there is some index $m \geq n$ satisfying $x_m \in V$. The inequality $m \geq n$ guarantees

$$x_m \in \bigcap_{i=1}^{m} C_i \subseteq C_n$$

which yields $x_m \in V \cap C_n \neq \emptyset$. We conclude $y \in \overline{C_n} = C_n$, which holds for all $n \in \mathbb{N}$, i.e. $y \in \bigcap_{n\in\mathbb{N}} C_n$. $\square$

Problem 5. (Munkres Exercise 27.2) Let X be a metric space and $A \subseteq X$ a subset. Define the **ϵ-neighborhood** of A as the set

$$B_\epsilon(A) := \{x \in X \mid d(x, A) < \epsilon\}.$$

a. Show that the ϵ-neighborhood of A is

$$B_\epsilon(A) = \bigcup_{a \in A} B_\epsilon(a)$$

i.e. the union of all open balls of radius ϵ around points $a \in A$.

Solution. ($\supseteq$) Let $a \in A$. For any $x \in B_\epsilon(a)$, we have

$$d(x, a) < \epsilon \Rightarrow d(x, A) \leq d(x, a) < \epsilon$$

which implies $x \in B_\epsilon(A)$. Since this inclusion $B_\epsilon(a) \subseteq B_\epsilon(A)$ holds for all $a \in A$, we obtain $\bigcup_{a \in A} B_\epsilon(a) \subseteq B_\epsilon(A)$.

($\subseteq$) Let $x \in B_\epsilon(A)$, which means $d(x, A) < \epsilon$. By definition of $d(x, A) = \inf_{a \in A} d(x, a)$ there is some $\epsilon_0 < \epsilon$ and $a_0 \in A$ satisfying

$$d(x, a_0) = \epsilon_0 < \epsilon.$$

This implies $x \in B_\epsilon(a_0) \subseteq \bigcup_{a \in A} B_\epsilon(a)$. $\square$

b. Assume A is *compact* and let $U \subseteq X$ be an open set containing A. Show that some ϵ-neighborhood of A is contained in U, i.e. $B_\epsilon(A) \subseteq U$ for some $\epsilon > 0$.

Solution. Since U is open, for every $a \in A$, there is some radius $\epsilon(a) > 0$ satisfying $B_{\epsilon(a)}(a) \subseteq U$. Consider the open cover

$$A \subseteq \bigcup_{a \in A} B_{\frac{\epsilon(a)}{2}}(a)$$

of A. Since A is compact, there is an open subcover

$$A \subseteq B_{\frac{\epsilon(a_1)}{2}}(a_1) \cup \ldots \cup B_{\frac{\epsilon(a_n)}{2}}(a_n).$$

Take $\epsilon := \min\{\frac{\epsilon(a_1)}{2}, \ldots \frac{\epsilon(a_n)}{2}\}$. For any $a \in A$ and $x \in B_\epsilon(a)$, the point a is in one of the open balls $B_{\frac{\epsilon(a_k)}{2}}(a_k)$ and we obtain

$$d(x, a_k) \leq d(x, a) + d(a, a_k) < \epsilon + \frac{\epsilon(a_k)}{2} \leq \frac{\epsilon(a_k)}{2} + \frac{\epsilon(a_k)}{2} = \epsilon(a_k)$$

which implies $x \in B_{\epsilon(a_k)}(a_k) \subseteq U$.

Since x was arbitrary, the inclusion $B_\epsilon(a) \subseteq U$ holds. Since a was arbitrary, the inclusion $B_\epsilon(A) \subseteq U$ holds, using part (a). $\square$

Alternate solution. In the case $U = X$, any $\epsilon > 0$ works. In the case $U \neq X$, the complement U^c is a non-empty closed subset, disjoint from A. Consider the function $f\colon A \to \mathbb{R}$ defined by

$$f(a) = d(a, U^c).$$

Then f is continuous (c.f. HW 6 Problem 3a). Since A is compact, f is bounded and achieves its bounds, in particular its lower bound $f(a_0) = m \geq 0$. However, we have $m > 0$, since f is strictly positive on A:

$$f(a) = 0 = d(a, U^c) \Rightarrow a \in \overline{U^c} = U^c$$

but A and U^c are disjoint, i.e. $A \cap U^c = \emptyset$.

We claim $B_m(A) \subseteq U$, in other words $B_m(A) \cap U^c = \emptyset$. Any point $x \in U^c$ satisfies

$$d(a, x) \geq d(a, U^c) = f(a) \geq m$$

for all $a \in A$, which implies $d(x, A) \geq m$, i.e. $x \notin B_m(A)$. $\square$

Problem 6. In all parts of this problem, let $f\colon X \to Y$ be a *uniformly* continuous map between *metric* spaces.

a. Show that f sends Cauchy sequences to Cauchy sequences. In other words, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, show that $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

Solution. Given $\epsilon > 0$, we want to find $N \in \mathbb{N}$ so that $d(f(x_m), f(x_n)) < \epsilon$ for all $m, n \geq N$.

Since f is uniformly continuous, there is a $\delta > 0$ guaranteeing $d(f(x), f(x')) < \epsilon$ whenever $d(x, x') < \delta$.

Since the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, there is some $N \in \mathbb{N}$ guaranteeing $d(x_m, x_n) < \delta$ for all $m, n \geq N$.

Putting these two facts together, we conclude $d(f(x_m), f(x_n)) < \epsilon$ for all $m, n \geq N$. □

b. Assuming moreover that f is a homeomorphism and Y is complete, show that X is complete.

Solution. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Then $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, since f is uniformly continuous. Since Y is complete, this sequence converges: $f(x_n) \to y \in Y$.

Since the inverse $f^{-1}\colon Y \to X$ is continuous, it sends convergent sequences to convergent sequences, and we obtain

$$f^{-1}(f(x_n)) \to f^{-1}(y)$$

i.e. $x_n \to f^{-1}(y)$. □

c. Find an example where f is a homeomorphism and X is complete, but Y is *not* complete. (Don't forget to show that your example f is uniformly continuous.)

Solution. Consider $f = \arctan\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then f is continuous, and its inverse $\tan\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is also continuous, i.e. f is a homeomorphism.

Moreover, f is everywhere differentiable and its derivative $f'(x) = \frac{1}{1+x^2}$ is bounded by $|f'(x)| \leq 1$. Therefore f is Lipschitz continuous, and in particular uniformly continuous.

Finally, note that $\mathbb{R}$ is complete, whereas the open interval $Y := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is not, since the sequence $y_n = \frac{\pi}{2} - \frac{1}{n}$ is Cauchy but does not converge in Y. □

Remark. Part (b) implies that if two metric spaces are uniformly equivalent, then one is complete if and only if the other is complete. In other words, completeness depends on more than just the topology, but at most on the uniform type.