

# MA 523: Homework 5

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## PROBLEM 5.1

Prove that Laplace's equation  $\Delta u = 0$  is rotation invariant; that is, if  $O$  is an orthogonal  $n \times n$  matrix and we define  $v(x) := u(Ox)$ ,  $x \in \mathbb{R}^n$ , then  $\Delta v = 0$ .

*SOLUTION.* Let

$$O = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be an orthogonal  $n \times n$  matrix. We will show that  $\Delta v = 0$ , where  $v(x) = u(Ox)$ .

First, let us compute the gradient of  $v$ ,

$$\begin{aligned} Dv(x) &= Du(Ox) \\ &= Du(a_{11}x_1 + \cdots + a_{1n}x_n, \dots, a_{n1}x_1 + \cdots + a_{nn}x_n) \\ &= \left( \sum_{j=1}^n a_{j1}u_{x_j}, \dots, \sum_{j=1}^n a_{jn}u_{x_j} \right) \\ &= O^T Du(x). \end{aligned}$$

Lastly, we compute the divergence of  $Dv$ ,

$$\begin{aligned} \Delta v(x) &= \operatorname{div} Dv(x) \\ &= \operatorname{div} \left( \sum_{j=1}^n a_{j1}u_{x_j}, \dots, \sum_{j=1}^n a_{jn}u_{x_j} \right). \end{aligned}$$

Here the partial derivatives become unwieldy so we will first examine the partial  $\frac{\partial}{\partial x_1}$  of the first term and proceed from there. In this case,

$$\begin{aligned} \frac{\partial}{\partial x_1} \sum_{j=1}^n a_{j1}u_{x_j} &= a_{11}(u_{x_1})_{x_1} + a_{21}(u_{x_2})_{x_1} + \cdots + a_{n1}(u_{x_n})_{x_1} \\ &= a_{11}(a_{11}u_{x_1x_1} + a_{21}u_{x_1x_2} + \cdots + a_{n1}u_{x_1x_n}) \\ &\quad + \cdots + a_{n1}(a_{11}u_{x_1x_n} + a_{21}u_{x_2x_n} + \cdots + a_{n1}u_{x_nx_n}) \\ &= a_{11}^2 u_{x_1x_1} + 2a_{11}a_{21}u_{x_1x_2} + 2a_{11}a_{31}u_{x_1x_3} + \cdots + a_{21}^2 u_{x_2x_2} \\ &\quad + \cdots + a_{k1}^2 u_{x_kx_k} + \cdots + a_{n1}^2 u_{x_nx_n}. \end{aligned}$$

Similarly, taking the  $k^{\text{th}}$  partial of the  $k^{\text{th}}$  entry of  $Dv$ , we have

$$\begin{aligned} \frac{\partial}{\partial x_k} \sum_{j=1}^n a_{jk}u_{x_j} &= a_{1k}(a_{1k}u_{x_1x_1} + \cdots + a_{nk}u_{x_1x_n}) \\ &\quad + \cdots + a_{nk}(a_{1k}u_{x_1x_n} + \cdots + a_{nk}u_{x_nx_n}) \\ &= a_{1k}^2 u_{x_1x_1} + a_{2k}^2 u_{x_2x_2} + \cdots + a_{kk}^2 u_{x_kx_k} \\ &\quad + \cdots + a_{nk}^2 u_{x_nx_n} + \{\text{mixed terms}\}. \end{aligned} \tag{5.1}$$

Now, since  $O$  is orthogonal, we have

$$\begin{aligned}
 OO^T &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \\
 &= \begin{bmatrix} a_{11}^2 + \cdots + a_{1n}^2 & a_{11}a_{21} + \cdots + a_{1n}a_{2n} & \cdots & a_{11}a_{n1} + \cdots + a_{1n}a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} + \cdots + a_{nn}a_{1n} & a_{n1}a_{21} + \cdots + a_{nn}a_{2n} & \cdots & a_{n1}^2 + \cdots + a_{nn}^2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.
 \end{aligned}$$

We can sum up the results of our calculation as

$$\begin{cases} \text{(a)} & \sum_{j=1}^n a_{kj}a_{\ell j} = \sum_{j=1}^n a_{kj}^2 = 1 & \text{if } k = \ell, \\ \text{(b)} & \sum_{j=1}^n a_{kj}a_{\ell j} = 0 & \text{if } k \neq \ell. \end{cases} \quad (5.2)$$

for  $1 \leq k, \ell \leq n$ .

Now, going back to (5.1), we have

$$\begin{aligned}
 \operatorname{div} Dv &= \sum_{k=1}^n \left[ \frac{\partial}{\partial x_k} \sum_{j=1}^n a_{jk} u_{x_j} \right] \\
 &= (a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2) u_{x_1 x_1} + (a_{12}^2 + a_{22}^2 + \cdots + a_{2n}^2) u_{x_2 x_2} \\
 &\quad + \cdots + (a_{1n}^2 + \cdots + a_{nn}^2) u_{x_n x_n} + \{\text{mixed terms}\} \\
 &= u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} \\
 &= 0,
 \end{aligned}$$

as desired.

All that is left to show is that the mixed terms in the expression above actually have coefficients of the form in (5.2) (b). A little routine calculation shows that indeed, the mixed terms have the form. Here is the first mixed term

$$2(a_{11}a_{21} + a_{12}a_{22} + \cdots + a_{1n}a_{2n})u_{x_1 x_2} = 0.$$

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## PROBLEM 5.2

Let  $n = 2$  and  $U$  be the halfplane  $\{x_2 > 0\}$ . Prove that

$$\sup_U u = \sup_{\partial U} u$$

for  $u \in C^2(U) \cap C(\bar{U})$  which are harmonic in  $U$  under the additional assumption that  $u$  is bounded from above in  $\bar{U}$ . (The additional assumption is needed to exclude examples like  $u = x_2$ .)

[Hint: Take for  $\varepsilon > 0$  the harmonic function

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region  $\{x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0\}$  with large  $a$ . Let  $\varepsilon \rightarrow 0$ .]

*SOLUTION.* Consider the harmonic function

$$u_\varepsilon(x_1, x_2) := u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Set  $U_a := \{x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0\}$ .

First, we note that  $u_\varepsilon \uparrow u$  as  $\varepsilon \rightarrow 0$  pointwise, i.e., given  $\eta > 0$ , for

$$0 < \varepsilon(x_1, x_2) < \eta / \ln \sqrt{x_1^2 + (x_2 + 1)^2},$$

we have

$$\begin{aligned} |u_\varepsilon(x_1, x_2) - u(x_1, x_2)| &= \left| u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} - u(x_1, x_2) \right| \\ &= \left| \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} \right| \\ &= \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} \\ &< \eta, \end{aligned}$$

for any  $(x_1, x_2) \in U_a$ .

Moreover, a simple calculation shows that  $u_\varepsilon$  is in fact harmonic. By the linearity the Laplacian, it suffices to show that the Laplacian of

$$v_\varepsilon(x) := \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}$$

is 0. First, we calculate the partial derivatives  $\frac{\partial^2}{\partial x_1 \partial x_1}$  and  $\frac{\partial^2}{\partial x_2 \partial x_2}$

$$\begin{aligned} \frac{(v_\varepsilon)_{x_1}}{\varepsilon} &= -\frac{x_1}{x_1^2 + (x_2 + 1)^2} & \frac{(v_\varepsilon)_{x_2}}{\varepsilon} &= -\frac{(x_2 + 1)}{x_1^2 + (x_2 + 1)^2} \\ \frac{(v_\varepsilon)_{x_1 x_1}}{\varepsilon} &= -\frac{-x_1^2 + (x_2 + 1)^2}{(x_1^2 + (x_2 + 1)^2)^2} & \frac{(v_\varepsilon)_{x_2 x_2}}{\varepsilon} &= -\frac{x_1^2 - (x_2 + 1)^2}{(x_1^2 + (x_2 + 1)^2)^2}. \end{aligned}$$

Thus,

$$\Delta u_\varepsilon = \Delta u + \Delta v_\varepsilon = \Delta v_\varepsilon = \varepsilon \left( -\frac{-x_1^2 + (x_2 + 1)^2}{(x_1^2 + (x_2 + 1)^2)^2} - \frac{x_1^2 - (x_2 + 1)^2}{(x_1^2 + (x_2 + 1)^2)^2} \right) = 0.$$

Now, observe that, for any  $a$ , by the maximum principle, we have

$$\max_{\bar{U}_a} u_\varepsilon = \max_{\partial U_a} u_\varepsilon$$

for any  $a$ . Let us choose  $a$  large enough so

$$\sup_{\partial U_a} u_\varepsilon = \sup_{\partial U_a} u - \varepsilon \ln a \leq \sup_{\partial U} u$$

or

$$\sup_{\partial U_a} u_\varepsilon \leq \sup_{\partial U} u + \varepsilon \ln a.$$

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## PROBLEM 5.3

Let  $U \subset \mathbb{R}^n$  be an open set. We say  $v \in C^2(U)$  is subharmonic if

$$-\Delta v \leq 0 \quad \text{in } U.$$

- (a) Let  $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}$  be smooth and convex. Assume  $u^1, \dots, u^m$  are harmonic in  $U$  and

$$v := \varphi(u_1, \dots, u_m).$$

Prove  $v$  is subharmonic.

[Hint: Convexity for a smooth function  $\varphi(z)$  is equivalent to  $\sum_{j,k=1}^m \varphi_{z_j, z_k}(z) \xi_j \xi_k \geq 0$  for any  $\xi \in \mathbb{R}^m$ .]

- (b) Prove  $v := |Du|^2$  is subharmonic, whenever  $u$  is harmonic. (Assume that harmonic functions are  $C^\infty$ .)

*SOLUTION.* For part (a), by the chain rule, we have

$$v_{x_i} = \varphi_{u_1} u_{x_i}^1 + \dots + \varphi_{u_m} u_{x_i}^m.$$

Taking another partial, we have

$$\begin{aligned} v_{x_i x_i} &= (v_{x_i})_{x_i} \\ &= \frac{\partial}{\partial x_i} (\varphi_{u_1} u_{x_i}^1 + \dots + \varphi_{u_m} u_{x_i}^m) \\ &= \varphi_{u_1} u_{x_i x_i}^1 + \dots + \varphi_{u_m} u_{x_i x_i}^m \\ &\quad + (\varphi_{u_1 u_1} u_{x_i}^1 + \dots + \varphi_{u_1 u_m} u_{x_i}^m) u_{x_i}^1 \\ &\quad + \dots + (\varphi_{u_1 u_m} u_{x_i}^1 + \dots + \varphi_{u_m u_m} u_{x_i}^m) u_{x_i}^m. \end{aligned} \tag{5.3}$$

Now, taking the sum

$$\begin{aligned} \sum_{i=1}^n v_{x_i x_i} &= \sum_{i=1}^n \sum_{j=1}^m \varphi_{u_j} u_{x_i x_i}^j \\ &= \sum_{j=1}^m \sum_{i=1}^n \varphi_{u_j} u_{x_i x_i}^j \\ &= \sum_{j=1}^m (\varphi_{u_j} u_{x_1 x_1}^j + \dots + \varphi_{u_j} u_{x_n x_n}^j) \\ &= \sum_{j=1}^m \varphi_{u_j} (u_{x_1 x_1}^j + \dots + u_{x_n x_n}^j) \\ &= 0, \end{aligned}$$

since  $\Delta u^j = 0$  for all  $j$ .

What about the remaining terms in (5.3)? These terms can be written in the form

$$\sum_{j,k=1}^m \varphi_{u_j u_k}(u) \xi_j \xi_k,$$

where  $\xi_i = (u_{x_i}^1, \dots, u_{x_i}^m)(x_1, \dots, x_n) \in \mathbb{R}^m$  for any  $(x_1, \dots, x_n) \in \mathbb{R}^n$ . Since  $\varphi$  is convex, by assumption, this quantity is greater than or equal to 0.

Thus,  $\Delta v \geq 0$  so  $v$  is subharmonic.

For part (b), we have

$$v = |Du|^2 = u_{x_1}^2 + \dots + u_{x_n}^2.$$

Taking the partial derivative with respect to  $x_i$ , we have

$$\begin{aligned} v_{x_i} &= \frac{\partial}{\partial x_i} (u_{x_1}^2 + \dots + u_{x_n}^2) \\ &= 2u_{x_1} u_{x_1 x_i} + \dots + 2u_{x_n} u_{x_i x_n}, \end{aligned}$$

and again

$$\begin{aligned} v_{x_i x_i} &= (v_{x_i})_{x_i} \\ &= \frac{\partial}{\partial x_i} (2u_{x_1} u_{x_1 x_i} + \dots + 2u_{x_n} u_{x_i x_n}) \\ &= 2u_{x_1} u_{x_1 x_i x_i} + 2u_{x_1 x_i}^2 + \dots + 2u_{x_n} u_{x_i x_i x_n} + 2u_{x_i x_n}^2 \\ &= 2 \sum_{j=1}^n (u_{x_j} u_{x_j x_i x_i} + u_{x_j x_i}^2). \end{aligned}$$

Then

$$\begin{aligned} \frac{\Delta v}{2} &= \sum_{i,j=1}^n (u_{x_j} u_{x_j x_i x_i} + u_{x_j x_i}^2) \\ &= \sum_{i,j=1}^n u_{x_j} u_{x_j x_i x_i} + \sum_{i,j=1}^n u_{x_j x_i}^2, \end{aligned}$$

splitting the second term into the sum

$$\begin{aligned} &= \sum_{i,j=1}^n u_{x_j} u_{x_j x_i x_i} + \sum_{1 \leq j < i \leq n} u_{x_j x_i}^2 \\ &\quad + \sum_{1 \leq i < j \leq n} u_{x_j x_i}^2 + \sum_{1 \leq i = j \leq n} u_{x_i x_i}^2, \end{aligned}$$

where the last term is 0 since  $u$  is harmonic, giving us

$$\begin{aligned} &= \sum_{i,j=1}^n u_{x_j} u_{x_j x_i x_i} + \sum_{1 \leq j < i \leq n} u_{x_j x_i}^2 + \sum_{1 \leq i < j \leq n} u_{x_j x_i}^2 \\ &= \sum_{i,j=1}^n u_{x_j} u_{x_j x_i x_i} + 2 \sum_{1 \leq j < i \leq n} u_{x_j x_i}^2, \end{aligned}$$



here  $\sum_{j=1}^n u_{x_i x_j x_j} = \Delta u_{x_i} = 0$  since the derivatives of harmonic functions are harmonic, so

$$\begin{aligned} &= \sum_{j=1}^n u_{x_j} (\Delta u_{x_j}) + 2 \sum_{1 \leq j < i \leq n} u_{x_j x_i}^2 \\ &= 2 \sum_{1 \leq j < i \leq n} u_{x_j x_i}^2 \\ &\geq 0, \end{aligned}$$

as desired. That is,  $\Delta v \geq 0$  so  $v$  is subharmonic. ■