

# MA 519: Homework 11

Max Jeter, Carlos Salinas

November 10, 2016



## PROBLEM 11.1 (DASGUPTA 7.2 (A), (B), (C), (D), (E))

- (a) Suppose  $E|X_n - c|^\alpha \rightarrow 0$ , where  $0 < \alpha < 1$ . Does  $X_n$  necessarily converge in probability to  $c$ ?
- (b) Suppose  $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0, 1)$ . Under what condition on  $a_n$  can we conclude that  $X_n \xrightarrow{\mathcal{P}} \theta$ ?
- (c)  $o_p(1) + O_p(1) = ?$
- (d)  $o_p(1)O_p(1) = ?$
- (e)  $o_p(1) + o_p(1)O_p(1) = ?$

*SOLUTION.* First let us tackle part (a) of this problem. We suspect that the statement “ $X_n$  converges in probability to  $c$ ” is false but first let us attempt to prove this to see where the problem in the implication may lie. Fix  $0 < \alpha < 1$  and suppose

$$E(|X_n - c|^\alpha) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (11.1)$$

Set  $\varphi(x) := x - c$  and let  $g_n$  denote the PDF of  $X_n - c$ . By the Jacobian formula

$$g_n(x) = \frac{f_n(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))} = f_n(x + c)$$

where we denote by  $f_n$  the PDF of  $X_n$ . Then equation (11.1) now reads

$$\left( \int_{-\infty}^{\infty} x f_n(x + c) dx \right)^\alpha \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Fix  $\varepsilon > 0$ . Now observe, by Markov's inequality, that

$$P(|X_n - c| > \varepsilon) \leq \frac{E(|X_n - c|^\alpha)}{\varepsilon^\alpha}$$

For part (c), suppose  $\{a_n\}$  and  $\{b_n\}$  are sequences such that  $a_n = o_p(1)$  and  $b_n = O_p(1)$ , then for the sequence  $\{c_n := a_n + b_n\}$  the most we can expect is  $c_n = O_p(1)$ . Indeed, we know that if a sequence is  $o_p(1)$  then it is also  $O_p(1)$  therefore there exists  $K_1$  and  $K_2$  such that  $|a_n| \leq K_1$ ,  $|b_n| \leq K_2$  for all  $n \geq 1$ . Therefore,  $|c_n| \leq K_1 + K_2$  for all  $n \geq 1$ . ■

## PROBLEM 11.2 (DASGUPTA 7.3 [MONTE CARLO])

Consider the purely mathematical problem of finding a definite integral  $\int f(x) dx$  for some (possibly complicated) function  $f(x)$ . Show that the SLLN provides a method for approximately finding the value of the integral by using appropriate averages  $\frac{1}{n} \sum_{k=1}^n f(X_k)$ .

Numerical analysts call this Monte Carlo integration.

*SOLUTION.*

■

## PROBLEM 11.3 (DASGUPTA 7.4 (A), (B))

Suppose  $X_1, \dots$ , are i.i.d. and that  $E(X_1) = \mu \neq 0$ ,  $\text{Var}(X_1) = \sigma^2 < \infty$ . Let  $S_{m,p} = \sum_{k=1}^m X_k^p$ ,  $m \geq 1$ ,  $p = 1, 2$ .

- (a) Identify with proof the almost sure limit of  $S_{m,1}/S_{n,1}$  for fixed  $m$ , and  $n \rightarrow \infty$ .
- (b) Identify with proof the almost sure limit of  $S_{n-m,1}/S_{n,1}$  for fixed  $m$ , and  $n \rightarrow \infty$ .

SOLUTION. ■

## PROBLEM 11.4 (DASGUPTA 7.5 (A))

Let  $A_n$ ,  $n \geq 1$ ,  $A$  be events with respect to a common sample space  $\Omega$ .

- (a) Prove that  $I_{A_n} \xrightarrow{\mathcal{L}} I_A$  if and only if  $P(A_n) \rightarrow P(A)$ .

SOLUTION. ■

## PROBLEM 11.5 (DASGUPTA 7.11 [SAMPLE MAXIMUM])

Let  $X_k$ ,  $k \geq 1$ , be an i.i.d. sequence, and  $X_{(n)}$  the maximum of  $X_1, \dots, X_n$ . Let  $\xi(F) = \sup\{x : F(x) < 1\}$ , where  $F$  is the common CDF of the  $X_k$ . Prove that  $X_{(n)} \xrightarrow{\text{a.s.}} \xi(F)$ .

SOLUTION. ■

## PROBLEM 11.6 (DASGUPTA 7.14 (A))

Suppose  $X_k$  are i.i.d. standard Cauchy. Show that

- (a)  $P(|X_n| > n \text{ infinitely often}) = 1$ .

*SOLUTION.*

■



PROBLEM 11.7 (DASGUPTA 7.16 [COUPON COLLECTION])

Cereal boxes contain independently and with equal probability exactly one of  $n$  different celebrity pictures. Someone having the entire set of  $n$  pictures can cash them in for money. Let  $W_n$  be the minimum number of cereal boxes one would need to purchase to own a complete set of the pictures. Find a sequence  $a_n$  such that  $W_n/a_n \xrightarrow{\mathcal{P}} 1$ .  
(*Hint:* Approximate the mean of  $W_n$ .)

SOLUTION. ■

## PROBLEM 11.8 (DASGUPTA 7.17)

Let  $X \sim \text{Bin}(n, p)$ . Show that  $(X_n/n)^2$  and  $X_n(X_n - 1)/(n(n - 1))$  both converging in probability to  $p^2$ . Do they converge almost surely?

SOLUTION. ■

## PROBLEM 11.9 (DASGUPTA 7.21)

Let  $X_1, X_2, \dots$ , be i.i.d.  $U[0, 1]$ . Let

$$G_n = (X_1 \cdots X_n)^{1/n}.$$

Find  $c$  such that  $G_n \xrightarrow{\mathcal{P}} c$ .

*SOLUTION.*

■

## PROBLEM 11.10 (DASGUPTA 7.30 [CONCEPTUAL])

Suppose  $X_n \xrightarrow{\mathcal{L}} X$ , and also  $Y_n \xrightarrow{\mathcal{L}} X$ . Does this mean that  $X_n - Y_n$  converge in distribution to (the point mass at) zero?

*SOLUTION.*

■

## PROBLEM 11.11 (DASGUPTA 7.31 (A))

- (a) Suppose  $a_n(X_n - \theta) \rightarrow N(0, \tau^2)$ ; what can be said about the limiting distribution of  $|X_n|$ , when  $\theta \neq 0$ ,  $\theta = 0$ ?

SOLUTION. ■