MA572 Hatcher Notes

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1 Homology

A summary of Hatcher's homology section from his *Algebraic Topology* book.

1.1 Simplicial and Signular Homology

Skip all this nonsense. I need to catch up.

1.2 Computations and Applications

Degree

For a map $f\colon S^n \to S^n$ with $n > 0$, the induced map $f_*\colon H_n(S^n) \to H_n(S^n)$ is a homomorphism from an infinite cyclic group to itself and so must be of the form $f_*(\alpha) = df(\alpha)$ for some integer d depending only on f. This integer is called the *degree* of f and is denoted by $\deg f$. Here are some basic properties of the degree

(1) $\deg \mathrm{id}_{S^n} = 1$ since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{H_n(S^n)}$.

(2) $\deg f = 0$ if f is not injective. For if we choose a point $x_0 \in S^n \setminus f(S^n)$ then f can be factored as a composition $S^n \to S^n \setminus \{x_0\} \hookrightarrow S^n$ and $H_n(S^n \setminus \{x_0\}) = 0$ since $S^n \setminus \{x_0\}$ is contractible.

(3) If $f \simeq g$ then $\deg f = \deg g$ since $f_* = g_*$. The converse statement, that if $\deg f = \deg g$, is a fundamental theorem of Hopf from around 1925 which we prove in Corollary 4.25.

(4) $\deg fg = \deg f \deg g$, since $(f \circ g)_* = f_* \circ g_*$. As a consequence, $\deg f = \pm 1$ if f is a homotopy equivalence since $f \circ g \simeq \mathrm{id}_{S^n}$ implies that $\deg f \deg g = \deg \mathrm{id}_{S^n} = 1$.

(5) $\deg f = -1$ if f is a reflection of S^n, fixing the points in some subsphere $S^{n-1} \subset S^n$ and interchanging the two complementary hemispheres. For we can give S^n a Δ-complex structure with these two hemispheres as its two n-simplices Δ_1^n and Δ_2^n, and the n-chain $\Delta_1^n - \Delta_2^n$ represents a generator of $H_n(S^n)$ as we saw in Example 2.23, so the reflection interchanging Δ_1^n and Δ_2^n sends this generator to its negative.

(6) The antipodal map $a\colon S^n \to S^n$, $x \mapsto -x$, has degree $(-1)^{n+1}$ since it is the composition of $n+1$ reflections, each changing the sign of one coordinate in $\mathbf{R}^{n+1}$.

(7) If $f\colon S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$. For if $f(x) \neq x$ for any $x \in S^n$, then the line segment from $f(x)$ to $-x$, defined by $t \mapsto (1-t)f(x) - tx$ for $0 \le t \le 1$, does not pass through the origin. Hence if f has no fixed points, the formula $f_t(x) \coloneqq [(1-t)f(x) - tx]/\|(1-t)f(x) - tx\|$ defines a homotopy from f to the antipodal map. Note that the antipodal map has no fixed points, so the fact that maps without fixed points are homotopic to the antipodal point is sort of a converse statement.

Theorem 1 (2.8)**.** *S^n has a continuous field of nonzero tangent vectors if and only if n is odd.*

Proposition (2.29)**.** *$\mathbf{Z}/2\mathbf{Z}$ is the only nontrivial group that can act freely on S^n if n is even.*

Recall that the action of a group G on a space X is a homomorphism from G to the group $\mathrm{Homeo}(X)$ of homeomorphisms $X \to X$, and the action is free if the homeomorphism corresponding to each nontrivial element of G has no fixed points. In the case of S^n, the antipodal map $x \mapsto -x$ generates a free action of $\mathbf{Z}/2\mathbf{Z}$.

Next we describe a technique for computing degrees which can be applied to most maps that arise in practice. Suppose $f\colon S^n \to S^n$, $n > 0$, has the property that for some point $y \in S^n$, the preimage $f^{-1}(y)$ consists of only finitely many points, say $x_1, \dots, x_m$. Let $U_1, \dots, U_m$ be disjoint neighborhoods of these points, mapped by f into a neighborhood V of y. Then $f(U_i \smallsetminus \{x_i\}) \subset V \smallsetminus \{y_i\}$ for each i, and we have a commutative diagram.

$$
\begin{tikzcd}
 & H_n(U_i, U_i \smallsetminus \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V \smallsetminus \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \smallsetminus \{x_i\}) & H_n(S^n, S^n \smallsetminus \{f^{-1}(y)\}) \arrow[l, "p_i"] \arrow[r, "f_*"] & H_n(S^n, S^n \smallsetminus \{y\}) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\cong"] \arrow[u, "j"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}
$$