

MA 26500-215 Quiz 8

July 22, 2016

1. Let $\mathcal{P}_2(\mathbb{R})$ be the set of all polynomials of degree less than or equal to 2 with coefficients in \mathbb{R} , i.e., if $p(t) = at^2 + bt + c$ is a polynomial in $\mathcal{P}_2(\mathbb{R})$, then $a, b, c \in \mathbb{R}$.

- (a) (4 points) Show that the set $\mathcal{P}_2(\mathbb{R})$ is closed under addition and multiplication by scalars. What is a zero for this set?

Solution: Take $p(t) = a_1t^2 + b_1t + c_1$, $q(t) = a_2t^2 + b_2t + c_2$ in $\mathcal{P}_2(\mathbb{R})$ and $c \in \mathbb{R}$, then

$$\begin{aligned} p(t) + q(t) &= a_1t^2 + b_1t + c_1 + a_2t^2 + b_2t + c_2 & c(pt) &= c(a_1t^2 + b_1t + c_1) \\ &= (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2) & &= ca_1 + cb_1t + cc_1. \end{aligned}$$

More generally, we can show that $\mathcal{P}_2(\mathbb{R})$ satisfies all 8 of the vector space axioms; but they are all trivial calculations that come down to basically these two facts that $\mathcal{P}_2(\mathbb{R})$ is closed under addition and multiplication by scalars.

- (b) (4 points) The set $\mathcal{P}_2(\mathbb{R})$ is in fact a vector space. Find a basis for $\mathcal{P}_2(\mathbb{R})$.

Solution: The basis I was looking for was $\{1, t, t^2\}$. If your basis had three linearly independent elements, that should be enough.

- (c) (12 points) Define an inner product $\langle -, - \rangle: \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\langle p(t), q(t) \rangle \mapsto \int_0^1 p(t)q(t) dt.$$

Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that $\langle p, q \rangle = p(1/2)$ for every $p \in \mathcal{P}_2(\mathbb{R})$. [HINT: You should start by looking at the basis you found in part (b). If you chose a nice basis t^2 should be in your basis. Now for a general $q(t) = at^2 + bt + c \in \mathcal{P}_2(\mathbb{R})$ we have

$$\langle t^2, p(t) \rangle = \int_0^1 t^2 q(t) dt = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Can you come up with enough equations to solve for the unknowns a, b, c ?

Solution: Let $p(t) = at^2 + bt + c$. Using the basis $\{1, t, t^2\}$ we have

$$\begin{aligned}\int_0^1 at^2 + bt + c \, dt &= \frac{a}{3} + \frac{b}{2} + c \\ &= 1 \\ \int_0^1 t(at^2 + bt + c) \, dt &= \int_0^1 at^3 + bt^2 + ct \, dt \\ &= \frac{a}{4} + \frac{b}{3} + \frac{c}{2} \\ &= \frac{1}{2} \\ \int_0^1 t^2(at^2 + bt + c) \, dt &= \int_0^1 at^4 + bt^3 + ct^2 \, dt \\ &= \frac{a}{5} + \frac{b}{4} + \frac{c}{3} \\ &= \frac{1}{4}\end{aligned}$$

Now, we can write the system above as

$$A = \left[\begin{array}{ccc|c} 1/3 & 1/2 & 1 & 1 \\ 1/4 & 1/3 & 1/2 & 1/2 \\ 1/5 & 1/4 & 1/3 & 1/4 \end{array} \right].$$

Putting A in row reduced echelon form, we have

$$A_{\text{rref}} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & -3/2 \end{array} \right].$$

Thus, the polynomial $q(t) = -15t^2 + 15t - 3/2$.