MA 166 Review Sheet Solutions

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Students, I have typed up the solutions to some of the exercises in the review sheet with some commentary for your benefit. It's possible that I will not get to every last problem on the review sheet, but it may be worth while for you to peruse these solutions.

Problem (26)**.** Which of the following series converge?

$$\text{(i)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}} \qquad \text{(ii)} \sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \qquad \text{(iii)} \sum_{n=1}^{\infty} \frac{4}{3}\left(\frac{1}{2}\right)^n.$$

Solution. Kevin, you said that for (i) and (iii) it is easy to verify that they converge right? So let's focus on (ii) here. I claim that you can show the series

$$\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

converges by applying the ratio test. Here's how you do it. Just to get the notation straight, set $a_n \coloneqq n!/(1 \cdot 3 \cdot 5 \cdots (2n-1))$. Then, for the ratio test, we need to check that the limit of the quotient $|a_{n+1}/a_n|$ converges to some number which is less than 1. Let's show this:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!/(1 \cdot 3 \cdot 5 \cdots (2(n+1)-1))}{n!/(1 \cdot 3 \cdot 5 \cdots (2n-1))}\right|$$

since the terms are all positive, we can remove the absolute value signs

$$= \frac{(n+1)!/(1 \cdot 3 \cdot 5 \cdots (2(n+1)-1))}{n!/(1 \cdot 3 \cdot 5 \cdots (2n-1))}$$

rearrange it (remember that $1/(1/x) = x$, i.e., the inverse of the inverse of something is the original thing you started with, for example if you invert 2 you get $1/2$, if you invert $1/2$ you get 2, some people still get confused about this; I just want to make sure you are on board with what I am about to do)

$$= \frac{(n+1)!(1 \cdot 3 \cdot 5 \cdots (2n-1))}{n!(1 \cdot 3 \cdot 5 \cdots (2(n+1)-1))}$$

remember that $n!$ is defined recursively for example $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, but what is $4 \cdot 3 \cdot 2 \cdot 1$? It's nothing other than $4!$, so $5! = 5 \cdot 4!$. In particular, $(n+1)! = (n+1) \cdot n!$ so we have

$$= \frac{(n+1) \cdot n!(1 \cdot 3 \cdot 5 \cdots (2n-1))}{n!(1 \cdot 3 \cdot 5 \cdots (2(n+1)-1))}$$

the $n!$ in the denominator then cancels the $n!$ in the numerator, and we have

$$= \frac{(n+1)(1 \cdot 3 \cdot 5 \cdots (2n-1))}{(1 \cdot 3 \cdot 5 \cdots (2(n+1)-1))}$$

Now, see those 1's, 3's, and 5's in the numerator and denominator? We are going to cancel them. But when do we stop? Well, if you see the pattern, we are taking all of the odd numbers up to $2n$. Now, let's rewrite the equation we got above by expanding the $2(n+1)-1$ to $2n+2-1 = 2n+1$

$$= \frac{(n+1)(1 \cdot 3 \cdot 5 \cdots (2n-1))}{(1 \cdot 3 \cdot 5 \cdots (2n+1))}$$

What is the odd number preceding $2n+1$? Well, if we remove 1 from $2n+1$, we get $2n$ which is even, so we must take 2 from $2n+1$ to get the odd number just before $2n+1$, this is $2n+1-2 = 2n-1$. This gives us

$$= \frac{(n+1)(1 \cdot 3 \cdot 5 \cdots (2n-1))}{(1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1))}$$

Now, it gets easy. See how we have $1 \cdot 3 \cdot 5 \cdots (2n-1)$ on the top and the bottom? We can cancel these and we have

$$= \frac{n+1}{2n+1}$$

What is the limit of this sequence as $n \to \infty$? Use l'Hôpital's rule; replace n by x and differentiate

$$= \frac{x+1}{2x+1}$$

Now, by l'Hôpital's rule

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{x+1}{2x+1} \\
&= \lim_{x\to\infty} \frac{(x+1)'}{(2x+1)'} \\
&= \lim_{x\to\infty} \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Thus, the limit is $1/2 < 1$ so the series

$$\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

converges. ∎