

MA544: Qual Problems

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MA 544 (Spring 2016)

1.1 Exam 1 Prep

Problem 1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{r\mathbf{x} : \mathbf{x} \in E\}$. Prove that rE is measurable, and that $|rE| = |r|^n|E|$.

Proof. Define a map $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T\mathbf{x} := r\mathbf{x}$. Note that T is *Lipschitz continuous* since for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the equality

$$|T\mathbf{x} - T\mathbf{y}| = |r\mathbf{x} - r\mathbf{y}| = |r||\mathbf{x} - \mathbf{y}| \quad (1)$$

is satisfied. By Theorem 3.33 from [2, §3.3, p.55], the image of E under T is measurable. Moreover, by Theorem 3.35 [2, §3.3, p. 56], since T is linear, it follows that $|T(E)| = |\det T||E|$ where $\det T = |r|^n$. Lastly, we note that the image of E under T is precisely the set rE so that $|T(E)| = |rE| = |r|^n|E|$, as was to be shown. ■

Problem 2. Let $\{E_k\}$, $k \in \mathbb{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k \rightarrow \infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|.$$

Proof. Following the style of [2, §1.1, p. 2], particularly, the sets defined after the introduction of equation (1.1), set

$$V_k := \bigcap_{\ell=k}^{\infty} E_{\ell}. \quad (2)$$

Note that the collection of sets $\{V_k\}$ forms an increasing sequence, that is, if $\mathbf{x} \in V_k$ then, by (2), \mathbf{x} is in the intersection $E_k \cap \left(\bigcap_{\ell=k+1}^{\infty} E_{\ell} \right)$, but, by (2), $\bigcap_{\ell=k+1}^{\infty} E_{\ell} = V_{k+1}$ thus, \mathbf{x} is in V_{k+1} so $V_{k+1} \supset V_k$. Hence, we have $V_k \nearrow \liminf E_k$.

Now, consider the sequence $\{|V_k|\}$ formed by the Lebesgue measure of the V_k . By Theorem 3.26 from [2, §3.3, p. 51], since $V_k \nearrow \liminf E_k$,

$$\lim_{k \rightarrow \infty} |V_k| = \lim_{k \rightarrow \infty} \left| \bigcap_{\ell=k}^{\infty} E_{\ell} \right| = \left| \liminf_{k \rightarrow \infty} E_k \right|. \quad (3)$$

But note that, by the monotonicity of the Lebesgue measure, we have

$$\left| \bigcap_{\ell=k}^{\infty} E_{\ell} \right| \leq |E_k|, \quad (4)$$

so, by properties of the \liminf , in particular, by Theorem 19(v) from [1, §1.5, p. 23], we have

$$\limsup_{k \rightarrow \infty} |V_k| \leq \liminf_{k \rightarrow \infty} |E_k|. \quad (5)$$

Hence, by (3) and Proposition 19 (iv), since the sequence $\{|V_k|\}$ converges and is equal to the measure of $\liminf E_k$, by (5), we have

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k| \quad (6)$$

as was to be shown. ■

Problem 3. Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r\}$. Prove that F is monotonic increasing and continuous.

Proof. Define the linear map $T: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T(r)\mathbf{x} := r\mathbf{x}$. We claim that $B(\mathbf{0}, r) = T(r, B(\mathbf{0}, 1))$. To reduce notation, set $B_1 := B(\mathbf{0}, 1)$ and $B_r := B(\mathbf{0}, r)$.

Proof of claim. \subset : Let $\mathbf{x} \in B_r$. Then $|\mathbf{x}| < r$ so $|\mathbf{x}|/r < 1$. Thus, $|\mathbf{x}|/r \in B_1$ so it is in the image of B_1 under the map $T(r, -)$.

\supset : On the other hand, suppose $\mathbf{x} \in T(r, B_1)$. Then $\mathbf{x} = r\mathbf{y}$ for some $\mathbf{y} \in B_1$. Then, since $|\mathbf{y}| < 1$, $|\mathbf{x}| = r|\mathbf{y}| < r$ so $\mathbf{x} \in B_r$. ♣

From the claim, we see that $F(x) = |T(x, B(\mathbf{0}, 1))|$ which, by Problem 1, is nothing more than the polynomial $|B_1|x^n$. It is clear, from this equivalence, that F is monotonically increasing: Take $x, y \in [0, \infty)$ such that $x < y$, then $x^n < y^n$ so

$$F(x) = |B_1|x^n < |B_1|y^n = F(y). \quad (7)$$

Thus, F is monotonically increasing.

In the argument above, since $F(x) = |B_1|x^n$ is a polynomial in $[0, \infty)$ (and polynomials are continuous on \mathbb{R}) F is continuous on $[0, \infty)$. ■

Problem 4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ .

Proof. From the topological definition of continuity, f is continuous at $x \in C$ if and only if for every neighborhood U of $f(x)$, the preimage $f^{-1}(U)$ is a neighborhood of x . Now, ■

Let $x \in C$. Then, by the definition of continuity, for every natural number $n > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f(x) - f(x')| < \frac{1}{2n}. \quad (8)$$

Let $x'', x' \in B(x, \delta)$. Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x'')| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \quad (9)$$

In view of these estimates, define the set

$$A_n = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \quad (10)$$

Good Lord, that was a long definition! We claim that $C = \bigcap_{n=1}^{\infty} A_n$ and that A_n is open for all n .

First, let us show that $C = \bigcap_{n=1}^{\infty} A_n$. Let $x \in C$. Then for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Thus, $x \in A_n$ for all n so $x \in \bigcap A_n$. On the other hand, if $x \in \bigcap A_n$ for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Fix $\varepsilon > 0$. By the Archimedean principle, there exists $N > 0$ such that $\varepsilon > 1/N$. Then, since $x \in A_N$ it follows that for some $\delta' > 0$, $|x - x'| < \delta'$ implies $|f(x) - f(x')| < 1/N < \varepsilon$. Thus, $x \in C$ and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$.

Lastly, we show that A_n is open. Let $x \in A_n$. Then there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. In particular, this means that $B(x, \delta) \subset A_n$ for any $x' \in B(x, \delta)$ satisfies $|f(x) - f(x')| < 1/n$. Thus, A_n is open and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$ is a G_δ set.

Problem 5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Is it true that if the sets $\{f = r\}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Proof. No. Recall that, by definition, or 4.1, f is measurable if and only if $\{f > a\}$ for all $a \in \mathbb{R}$. ■

Problem 6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on \mathbb{R} . Prove that the set $\{x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists}\}$ is measurable.

Proof. The idea here should be to rewrite

$$E = \left\{ x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists} \right\} \quad (11)$$

as a countable union/intersection of measurable sets. Let $x \in E$. By the Cauchy criterion, for every $N > 0$ there exists a positive integer M such that $m, n \geq M$ implies $|f_n(x) - f_m(x)| < 1/N$. With this in mind, define

$$E_N = \left\{ x : \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \quad (12)$$

Then, like for Problem 1.4, it is not too hard to see that the E_n 's are open and that $E = \bigcap_{n=1}^{\infty} E_n$. Thus, E is a G_δ set and therefore measurable. ■

Problem 7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. Suppose $f: [a, b] \rightarrow \mathbb{R}$ is absolutely continuous. Then for fixed $\varepsilon = 1$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, we have $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Let $\Gamma = \{x_k\}_{k=1}^N$ be a partition of $[a, b]$ into closed intervals such that $x_{k+1} - x_k < \delta$, then by absolute continuity we have

$$V[f; \Gamma] = \sum_{k=1}^N |f(x_{k+1}) - f(x_k)| < 1. \quad (13)$$

Thus, f is b.v. on $[a, b]$. ■

Problem 8. Let f be a continuous function from $[a, b]$ into \mathbb{R} . Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, that is, $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}.$$

Proof. ■

1.2 Exam 1

1.3 Exam 2 Prep

Problem 1. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B_\varepsilon(\mathbf{0})$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x}) d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is (Lebesgue) integrable over a subset E of \mathbb{R}^n (or, alternatively, f belongs to $L(E)$) if

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty.$$

Put $E = \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$. Then, to show that f belongs to $L(E)$ it suffices to prove the inequality

$$\int_E f(\mathbf{x}) d\mathbf{x} < \frac{C}{\varepsilon} \quad (1)$$

for some appropriate constant C . We proceed by directly computing the Lebesgue integral of f and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x}) d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{(x_1^2 + \cdots + x_n^2)^{(n+1)/2}} \end{aligned}$$

let E_i denote the projection of E onto its i -th coordinate and make the trigonometric substitution $x_1 = \sqrt{x_2^2 + \cdots + x_n^2} \tan \theta$, $dx_1 = \sqrt{x_2^2 + \cdots + x_n^2} \sec^2 \theta d\theta$ with $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$ giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[\frac{\cos^{n-1} \theta}{(x_2^2 + \cdots + x_n^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{(x_2^2 + \cdots + x_n^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1} \theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1} \theta d\theta < \infty. \quad (2)$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned} \int \cdots \int_E f(\mathbf{x}) d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{x_n^2} \\ &= 2C' \int_\varepsilon^\infty \frac{dx_n}{x_n^2} \\ &= \frac{C}{\varepsilon} \end{aligned} \tag{3}$$

as desired. ■

Problem 2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on \mathbb{R}^n , and assume that f_k converges pointwise almost everywhere to a function f . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets E of \mathbb{R}^n . Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$.

Proof. This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit f of $\{f_k\}$ must be nonnegative a.e. in \mathbb{R}^n . For assume otherwise. Then there exists a collection of points \mathbf{x} in \mathbb{R}^n of nonzero \mathbb{R}^n -Lebesgue measure such that $f(\mathbf{x}) < 0$. But $f_k(\mathbf{x}) \geq 0$ for all $k \in \mathbb{N}$. Set $0 < \varepsilon < |f(\mathbf{x})|$ then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{4}$$

for all k which contradicts our assumption that $f_k \rightarrow f$ a.e. on \mathbb{R}^n . Therefore, the set of points $\mathbf{x} \in \mathbb{R}^n$ where $f(\mathbf{x}) < 0$ must have measure zero.

Now, based on pointwise convergence a.e. to f , given $\varepsilon > 0$ for a.e. $\mathbf{x} \in \mathbb{R}^n$ we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{5}$$

for sufficiently large k ; say k greater than or equal to some index $N \in \mathbb{N}$. Moreover, we are given convergence in $L(\mathbb{R}^n)$ of f_k to f

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f < \infty. \tag{6}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{7}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{8}$$

for all $k \in \mathbb{N}$. By Theorem 5.5(ii), f and the f_k 's are finite a.e. in \mathbb{R}^n so for some sufficiently large real number M , $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in \mathbb{R}^n$. In particular, for any measurable subset E of \mathbb{R}^n , $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in E$ so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \rightarrow \int_E f < \infty. \quad (9)$$

However, if f does not belong to $L(\mathbb{R}^n)$, i.e., its integral over \mathbb{R}^n is infinity, there is no guarantee that f will be finite a.e. in \mathbb{R}^n . This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset E of \mathbb{R}^n . Let us demonstrate this with an example. Consider the sequence of functions ■

Problem 3. Assume that E is a measurable set of \mathbb{R}^n , with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of \mathbb{R}^n , then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \quad (10)$$

Set $E_k = \{\mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k\}$ and $F_k = \{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}$. Note the following properties about the sets we have just defined: first, the E_k 's are pairwise disjoint and the F_k 's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k 's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \quad (11)$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k , we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \quad (12)$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \quad (13)$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k , F_k , and F_{k+1} are all finite. Hence, (13) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \quad (14)$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned} \tag{16}$$

Thus, from (15) and (16)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \tag{17}$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ■

Problem 4. Suppose that E is a measurable subset of \mathbb{R}^n , with $|E| < \infty$. If f and g are measurable functions on E , define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, f) \rightarrow 0$ as $k \rightarrow \infty$ if and only if f_k converges to f as $k \rightarrow \infty$.

Proof. \implies : First note that ρ is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that $\rho(f_k, f) \rightarrow 0$ as $k \rightarrow \infty$. Then, given $\varepsilon > 0$ there exist an

sufficiently large index N such that for every $k \geq N$ we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \quad (18)$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in E which happens if $|f_k - f| = 0$ a.e. in E .

\Leftarrow : Suppose that $f_k \rightarrow f$ as $k \rightarrow \infty$.

I don't know how to solve this. This is the intended solution:

\Rightarrow : Given $\varepsilon > 0$, $\rho(f_k, f) \rightarrow 0$ implies that

$$\int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \rightarrow 0.$$

Observe that the function $\Phi: \mathbb{R}^+ \rightarrow \mathbb{R}$ given by $\Phi(x) = x/(1+x)$ is increasing on \mathbb{R}^+ and $0 < \Phi(x) < 1$, hence

$$\begin{aligned} \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1 + \varepsilon} |\{x \in E : |f_k(x) - f(x)| > \varepsilon\}|. \end{aligned}$$

Therefore,

$$|\{x \in E : |f_k(x) - f(x)| > \varepsilon\}| \leq \frac{1 + \varepsilon}{\varepsilon} \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \rightarrow 0$$

as $k \rightarrow \infty$.

\Leftarrow : Conversely, given $\delta > 0$, we have

$$\begin{aligned} \rho(f_k, f) &= \int_{\{x \in E : |f_k(x) - f(x)| > \delta\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\quad + \int_{\{x \in E : |f_k(x) - f(x)| \leq \delta\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\leq |\{x \in E : |f_k(x) - f(x)| > \delta\}| + \frac{\delta}{1 + \delta} |E|. \end{aligned}$$

Since $|E| < \infty$ and $\delta/(1+\delta) \searrow 0$, then for any $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\frac{\delta'}{1 + \delta'} |E| < \frac{\varepsilon}{2}.$$

If $f_k \rightarrow f$ as $k \rightarrow \infty$ in measure, then for the above δ' there is an index $N > 0$ such that $k \geq N$ implies

$$|\{x \in E : |f_k(x) - f(x)| > \delta'\}| < \frac{\varepsilon}{2}.$$

Therefore, $f_k \rightarrow f$ in measure implies $\rho(f_k, f) \rightarrow 0$ as $k \rightarrow \infty$. ■

Problem 5. Define the *gamma function* $\Gamma: \mathbb{R}^+ \rightarrow \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function* $\beta: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u} u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. (a) Fix $y \in \mathbb{R}^+$. Then we must show that $\Gamma(y) < \infty$. First, since $(0, 1)$ and $[1, \infty)$ are disjoint measurable subsets of \mathbb{R} , by Theorem 5.7 we can split the integral $\Gamma(y)$ into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u} u^{y-1} du}_{I_1} + \underbrace{\int_1^\infty e^{-u} u^{y-1} du}_{I_2}. \quad (19)$$

We will show, separately, that I_1 and I_2 are finite.

To see that I_1 is finite, note that

$$\begin{aligned} e^{-u} u^{y-1} &= e^{-u} e^{(y-1) \log u} \\ &= e^{-u+(y-1) \log u} \\ &\leq e^{(y-1) \log u} \\ &= u^{y-1} \end{aligned} \quad (20)$$

since $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u} u^{y-1} du \\ &\leq \int_0^1 u^{y-1} du \\ &= \left[\frac{u^y}{y} \right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \quad (21)$$

To see that I_2 is finite, note that

$$e \quad (22)$$

Intended solution:

(b) ■

Problem 6. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}}: \mathbb{R}^n \rightarrow \mathbb{R}$ be $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{23}$$

■

Problem 7. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \rightarrow f$ and $g_k \rightarrow g$ a.e. in \mathbb{R}^n , $|f_k| \leq g_k$ and

$$\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \rightarrow f$ a.e. in \mathbb{R}^n , then

$$\int_{\mathbb{R}^n} |f_k - f| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \rightarrow \int_{\mathbb{R}^n} |f| \quad \text{as} \quad k \rightarrow \infty.$$

Proof. (a) Since $f_k \rightarrow f$ and $g_k \rightarrow g$ a.e. and $|f_k| \leq g_k$, then by Fatou's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} (g - f) &= \int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} g_k - f_k \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_k - f_k, \\ \int_{\mathbb{R}^n} g + f &= \int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} g_k + f_k \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_k + f_k. \end{aligned}$$

Since $f_k, g_k, f, g \in L(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g$, then using the similar argument as problem 2, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f &\geq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k, \\ \int_{\mathbb{R}^n} f &\leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k. \end{aligned}$$

Therefore, $\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f$.

(b) \implies : This direction is obvious by the inequality

$$\left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \leq \int_{\mathbb{R}^n} ||f_k| - |f|| \leq \int_{\mathbb{R}^n} |f_k - f|.$$

\Leftarrow : Let $g_k = |f_k| + |f|$ and $g = 2|f|$. Since $f_k, f \in L(\mathbb{R}^n)$ and $f_k \rightarrow f$ a.e., then $g_k, g \in L(\mathbb{R}^n)$ and $g_k \rightarrow g$ a.e. in \mathbb{R}^n . By the assumption, $\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g$.

Let $\tilde{f}_k = |f_k - f|$. Then $\tilde{f}_k \rightarrow 0$ a.e. in \mathbb{R}^n and $\tilde{f}_k \leq g_k$. Applying part (a) to \tilde{f}_k we have

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

■

1.4 Midterm 2

Problem 1. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Proof. Recall that $f \in L(\mathbb{R}^n)$ if and only if $|f| \in L(\mathbb{R}^n)$. Let $B_k = B(\mathbf{0}, k)$ for $k \in \mathbb{N}$ and χ_{B_k} be the indicator function associated with B_k . Then, the sequence of maps $\{|f_k|\}$ defined $f_k = f\chi_{B_k}$ converge pointwise to $|f|$. Since $|f| \in L(\mathbb{R}^n)$, by the monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbb{R}^n} |f|. \quad (1)$$

But this means, exactly, that for every $\varepsilon > 0$ there exists sufficiently large $N \in \mathbb{N}$ such that

$$\begin{aligned} \varepsilon &> \left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \\ &= - \int_{\mathbb{R}^n} |f_k| + \int_{\mathbb{R}^n} |f| \\ &= - \int_{\mathbb{R}^n} |f| + \int_{\mathbb{R}^n} |f| \\ &= - \int_{B_k} |f| + \int_{\mathbb{R}^n} |f| \\ &= \int_{\mathbb{R}^n \setminus B_k} |f| \end{aligned} \quad (2)$$

as desired. ■

Problem 2. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E , such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. First, since the E_j 's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \quad (3)$$

Let χ_{E_j} be the characteristic function of the subset E_j of E and define $f_j = f\chi_{E_j}$ for $j \in \mathbb{N}$. Note that, since both f and χ_{E_j} are measurable on E , f_j is measurable on E and $\sum_{j=1}^{\infty} f_j = f$. Moreover, since $E_j \subset E$, by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \setminus E_j} f = \int_E f_j + \int_{E \setminus E_j} f. \quad (4)$$

Hence, because the E_j 's are disjoint $(E \setminus E_k) \setminus E_\ell = (E \setminus E_\ell) \setminus E_k$ so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \quad (5)$$

as desired. ■

Problem 3. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exists $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbb{N}$. Assume $|E| < \infty$, and $f_k(x) \rightarrow f(x)$ as $k \rightarrow \infty$ for a.e. $x \in E$. Show that

$$\lim_{k \rightarrow \infty} \int_E f_k = \int_E f.$$

(Hint: Use Egorov's theorem.)

Proof. Let $\varepsilon > 0$ be given. Then, by the hypothesis, there exists $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon \quad (6)$$

for all $k \in \mathbb{N}$. By Egorov's theorem, there exists a closed subset F of E such that $|E \setminus F| < \delta$ and $f_k \rightarrow f$ uniformly on F . Then, by the uniform convergence theorem,

$$\int_F f_k \rightarrow \int_F f \quad (7)$$

as $k \rightarrow \infty$. But by hypothesis, we have

$$\int_{E \setminus F} |f_k| < \varepsilon. \quad (8)$$

Letting $\varepsilon \rightarrow 0$, we achieved the desired convergence. ■

Problem 4. Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t) dt$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Proof. By Lusin's theorem, there exists a closed subset F of I with $|I \setminus F| < \varepsilon$ such that the restriction of f to $F = I \setminus E$ is continuous. Now, since F is closed in I and I is compact, it follows that F is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials $\{p_k\}$ such that $p_k \rightarrow f$ uniformly on F . Then, by the uniform convergence theorem, we have

$$\int_F p_k \rightarrow \int_F f \quad (9)$$

so

$$\begin{aligned}\int_F \left[\int_x^1 t^{-1} p_k(t) dt \right] dx &= \int_F \left[\int_x^1 at^{-1} + q_k(t) dt \right] dx \\ &= \int_F q'_k(x) - a \log(x) dx \\ &< \infty\end{aligned}\tag{10}$$

for all k and converges uniformly to g so $g \in L(I)$. I don't know how to show that in fact $\int_I g = \int_I f$. Perhaps you show that the places where they differ is a set of measure zero. ■

1.5 Final Practice

Problem 1. Suppose $f \in L^1(\mathbb{R})$ and that x is a point in the Lebesgue set of f . For $r > 0$, let

$$A(r) := \frac{1}{r} \int_{B(0,r)} |f(x-y) - f(x)| dy.$$

Show that:

- (a) $A(r)$ is a continuous function of r , and $A(r) \rightarrow 0$ as $r \rightarrow 0$;
- (b) there exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

Proof. ■

Problem 2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \leq n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \rightarrow 0$$

if and only if

$$\|f_k\|_p \rightarrow \|f\|_p$$

as $k \rightarrow \infty$.

Proof. ■

Problem 3. Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

- (a) Prove that $f * g \in C(\mathbb{R}^n)$.
- (b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

Proof. ■

Problem 4. Let $f \in L(\mathbb{R})$, and let $F(t) := \int_{\mathbb{R}} f(x) \cos(tx) dx$.

- (a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
- (b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

Proof. ■

Problem 5. Let f be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular. Show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh$$

for all functions φ continuous on $[a, b]$.

Proof. ■

MA 544 Past Quals

2.1 Danielli: Winter 2012

Problem 1. Let $f(x, y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each x , $f(x, y)$ is an integrable function of y , and $\partial f(x, y)/\partial x$ is a bounded function of (x, y) . Prove that $\partial f(x, y)/\partial x$ is a measurable function of y for each x and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial f(x, y)}{\partial x} dy.$$

Proof. ■

Problem 2. Let f be a function of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular, show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh.$$

Hint: A function h is said to be singular if $h' = 0$.

Proof. ■

Problem 3. Let $E \subset \mathbb{R}$ be a measurable set, and let K be a measurable function on $E \times E$. Assume that there exists a positive constant C such that

$$\int_E K(x, y) dx \leq C \tag{1}$$

for a.e. $y \in E$, and

$$\int_E K(x, y) dy \leq C \tag{2}$$

for a.e. $x \in E$.

Let $1 < p < \infty$, $f \in L^p(E)$, and define

$$T_f(x) := \int_E K(x, y) f(y) dy.$$

(a) Prove that $T_f \in L^p(E)$ and

$$\|T_f\|_p \leq C \|f\|_p. \quad (3)$$

(b) Is (3) still valid if $p = 1$ or ∞ ? If so, are assumptions (1) and (2) needed?

Proof. ■

Problem 4. Let f be a nonnegative measurable function on $[0, 1]$ satisfying

$$|\{x \in [0, 1] : f(x) > \alpha\}| < \frac{1}{1 + \alpha^2} \quad (4)$$

for $\alpha > 0$.

(a) Determine values of $p \in [1, \infty)$ for which $f \in L^p[0, 1]$.

(b) If p_0 is the minimum value of p for which p may fail to be in L^p , give an example of a function which satisfies (4), but which is not in $L^{p_0}[0, 1]$.

Proof. ■

2.2 Danielli: Summer 2011

Problem 1. Let $f \in L^1(\mathbb{R})$, and let $F(t) := \int_{\mathbb{R}} f(x) \cos(tx) dx$.

- (a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
- (b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

Hint: Start by proving the statement for $f = \chi_{[a,b]}$.

Proof. ■

Problem 2. (a) Suppose that $f_k, f \in L^2(E)$, with E a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E f g \tag{1}$$

as $k \rightarrow \infty$ for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \rightarrow \|f\|_2$ show that f_k converges to f in L^2 , i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as $k \rightarrow \infty$.

- (b) Provide an example of a sequence f_k in L^2 and a function f in L^2 satisfying (1), but such that f_k does *not* converge to f in L^2 .

Proof. ■

Problem 3. A bounded function f is said to be of bounded variation on \mathbb{R} if it is of bounded variation on any finite subinterval $[a, b]$, and moreover $A := \sup_{a,b} V[a, b; f] < \infty$. Here, $V[a, b; f]$ denotes the total variation of f over the interval $[a, b]$. Show that:

- (a) $\int_{\mathbb{R}} |f(x+h) - f(x)| dx \leq A|h|$ for all $h \in \mathbb{R}$.

Hint: For $h > 0$, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)| dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)| dx.$$

- (b) $\left| \int_{\mathbb{R}} f(x) \varphi'(x) dx \right| \leq A$, where φ is any function of class C^1 , of bounded variation, compactly supported, with $\sup_{x \in \mathbb{R}} |\varphi(x)| \leq 1$.

Proof. ■

Problem 4. (a) Prove the *generalized Hölder's inequality*: Assume $1 \leq p \leq \infty$, $j = 1, \dots, n$, with $\sum_{j=1}^{\infty} 1/p_j = 1/r \leq 1$. If E is a measurable set and $f_j \in L^{p_j}(E)$ for $j = 1, \dots, n$, then $\prod_{j=1}^n f_j \in L^r(E)$ and

$$\|f_1 \cdots f_n\|_r \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if $1 \leq p, q, r \leq \infty$, with $1/p + 1/q = 1/r + 1$, $f \in L^p(\mathbb{R})$, and $g \in L^q(\mathbb{R})$, then

$$|(f * g)(x)| \leq \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q dy.$$

(Recall that $(f * g)(x) := \int f(y)g(x-y) dy$.)

(c) Prove *Young's convolution theorem*: Assume that p, q, r, f , and g are as in part (b). Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof.

■

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