

MA 519: Homework 13

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PROBLEM 13.1 (HANDOUT 17, # 16)

Suppose $X \sim \text{Exp}(1)$, $Y \sim U[0, 1]$, and X, Y are independent.

- (a) Find the density of $X + Y$.
- (b) Find the density of XY .

SOLUTION. For part (a): Since X and Y are independent, the distribution of $X + Y$ is given by the convolution

$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_X(x-y)f_Y(y) dy,$$

where

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad f_Y(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, a straight forward calculation gives us

$$\begin{aligned} f_{X+Y}(x) &= \int_{-\infty}^{\infty} \chi_{[0,\infty)}(x-y)e^{-(x-y)}\chi_{[0,1]}(y) dy \\ &= e^{-x} \int_{-\infty}^{\infty} e^y \chi_{[0,\infty)}(x-y)\chi_{[0,1]}(y) dy \\ &= \begin{cases} 0 & \text{for } x < 0, \\ 1 - e^{-x} & \text{for } 0 \leq x \leq 1, \\ (e-1)e^{-x} & \text{for } x > 1. \end{cases} \end{aligned}$$

Now let us run a sanity check by demonstrating that $\int_{-\infty}^{\infty} f_{X+Y}(x) dx = 1$,

$$\begin{aligned} \int_{-\infty}^{\infty} f_{X+Y}(x) dx &= \int_0^1 [1 - e^{-x}] dx + (e-1) \int_1^{\infty} e^{-x} dx \\ &= [1 + e^{-1} - 1 - 0] + (e-1)[0 - (-e^{-1})] \\ &= e^{-1} + 1 - e^{-1} \\ &= 1. \end{aligned}$$

For part (b): Let us find the CDF of XY . By a direct computation

$$\begin{aligned} P(XY \leq z) &= \iint_{\{(x,y):xy \leq z\}} f_X(x)f_Y(y) dx dy \\ &= \iint_{\{(x,y):xy \leq z\}} e^{-x}\chi_{[0,\infty)}(x)\chi_{[0,1]}(y) dx dy \\ &= \int_0^1 \int_0^{y/z} e^{-x} dx dy \\ &= \int_0^1 1 - e^{-z/y} dy \end{aligned}$$

so

$$f_{XY}(z) = \frac{d}{dz} \int_0^1 1 - e^{-z/y} dy = \int_0^1 \frac{e^{-z/y}}{y} dy. \quad \blacksquare$$

PROBLEM 13.2 (HANDOUT 17, # 18)

Two points A, B are chosen at random from the unit circle. Find the probability that the circle centered at A with radius AB is fully contained within the original unit circle.

SOLUTION. The probability that a circle centered at A with radius AB is contained in the original circle is zero. What the professor means is “two points A, B are chosen at random from *inside* the unit circle”. We can think of choosing A as choosing a random variable $0 < R < 1$ representing the distance of A from the origin and we ask what is the probability that the point B lands inside the circle of radius $1 - R$ centered at A .

First, let us find the distribution for the radius R . We can find the CDF of R as the ratio of the area of the circle centered at the origin with radius x and the unit circle; i.e.,

$$P(R \leq x) = \frac{\pi x^2}{\pi \cdot 1^2} = x^2 \quad \text{for } 0 < x < 1.$$

Thus the PDF of R is

$$f_R(x) = 2x \quad \text{for } 0 < x < 1.$$

Then $A = R\Theta$ where $\Theta \sim U(0, 2\pi)$.

Now, the probability we are after is

$$P(B \in \{x : |x - A| < 1 - R\}).$$

To find this probability we use a bit of calculus

$$\begin{aligned} P(B \in \{x : |x - A| < 1 - R\} | A = x') &= \frac{1}{\pi} \int_{\{x : |x - x'| < 1 - R\}} \chi_{\{|x|=1\}}(y) dy \\ &= \end{aligned}$$

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PROBLEM 13.3 (HANDOUT 17, # 19)

Let X, Y be i.i.d. $U[0, 1]$ random variables. Find the correlation between $\max\{X, Y\}$ and $\min\{X, Y\}$.

SOLUTION. First, let us find the CDF of $W := \max\{X, Y\}$ and $Z := \min\{X, Y\}$. These are

$$\begin{aligned} P(W \leq x) &= P(\max\{X, Y\} \leq x) \\ &= P(X \leq x \text{ and } Y \leq x) \\ &= P(X \leq x)P(Y \leq x) \\ &= \begin{cases} x^2 & \text{for } 0 \leq x \leq 1, \\ 1 & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

and

$$\begin{aligned} P(Z \leq x) &= 1 - P(Z \geq x) \\ &= 1 - P(\min\{X, Y\} \geq x) \\ &= 1 - P(X \geq x \text{ and } Y \geq x) \\ &= 1 - P(X \geq x)P(Y \geq x) \\ &= \begin{cases} 1 - (1 - x)^2 & \text{for } 0 \leq x \leq 1, \\ 1 & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, their corresponding PDFs are

$$f_W(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}, \quad f_Z(x) = \begin{cases} 2(1 - x) & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

With this data we can now compute the mean of W and Z and find the correlation between W and Z . First, let us find the first and second moments of W, Z .

The first moment of W and Z are

$$\begin{aligned} E(W) &= \int_0^1 2x^2 dx \\ &= \frac{2}{3}, \end{aligned}$$

and

$$\begin{aligned} E(Z) &= \int_0^1 2x(1 - x) dx \\ &= \frac{1}{3}. \end{aligned}$$

The second moment of W and Z are

$$\begin{aligned} E(W) &= \int_0^1 2x^3 dx \\ &= \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} E(Z) &= \int_0^1 2x^2(1-x) dx \\ &= \frac{2}{3} - \frac{1}{2} \\ &= \frac{1}{6}. \end{aligned}$$

Thus, the variances are

$$\begin{aligned} \text{Var}(W) &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{1}{18} \end{aligned}$$

and

$$\begin{aligned} \text{Var}(Z) &= \frac{1}{6} - \frac{1}{9} \\ &= \frac{1}{18}. \end{aligned}$$

Thus,

$$\sigma_W = \sigma_Z = \sqrt{1/18}.$$

Lastly, we must find the covariance of W and Z . That is,

$$\begin{aligned} \text{Cov}(W, Z) &= E(WZ) - E(W)E(Z) \\ &= E(\max\{X, Y\} \min\{X, Y\}) - E(W)E(Z) \\ &= E(XY) - E(W)E(Z) \end{aligned}$$

since X and Y are independent

$$\begin{aligned} &= E(X)E(Y) - E(W)E(Z) \\ &= \frac{1}{4} - \frac{2}{9} \\ &= \frac{1}{36}. \end{aligned}$$

At last we come to the correlation of W and Z ,

$$\rho_{WZ} = \frac{\text{Cov}(W, Z)}{\sigma_W \sigma_Z} = \frac{1/36}{1/18} = \frac{1}{2}.$$

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