MA598: Lie Groups

Carlos Salinas

June 30, 2016

Contents

Chapter 1

Introduction

In any algebra textbook, the study of group theory is usually concerned about the theory of finite (or at least finitely generated) groups. However, most groups which appear as groups of symmetries of various geometric objects are not finite: for example, the group $\mathrm{SO}_3 \mathbb{R}$ of all rotations of three-dimensional space is not finite and is not even finitely generated. Thus, much of the material learned in a basic algebra course does not apply here; for exampel, it is not clear, whether, say, the set of all morphisms between such groups can be explicitly described.

The theory of Lie groups answers these questions by replacing the notion of a finitely generated group by that of a Lie group — a group which at the same time is a finite dimensional manifold. It turns out that in many ways such groups can be described and studied as easily as finitely generated groups — or even easier. The key role is played by the notion of a Lie algebra, the tangent space to G at the identity. It turns out that the group operation on G defines a certain skew-symmetric bilinear form on $\mathfrak{g} = T_1 G$; axiomatizing the properties of this operation gives a definition of the Lie algebra.

The fundamental result of the theory of Lie groups is that many properties of Lie groups are completely determined by the properties of corresponding Lie algebras. For example, the set of morphisms between two (connected and simply connected) Lie groups is the same as the set of morphisms between the corresponding Lie algebras; thus, describing them is essentially reduced to a linear algebra problem.

Similarly, Lie algebras also provide a key to the study of the structure of Lie groups and their representations. In particular, this allows one to get a complete classification of Lie groups (semisimple and more generally, reductive Lie groups; this includes all compact Lie groups and all classical Lie groups such as $\mathrm{SO}_n \mathbb{R}$) in terms of relatively simple geometric objects so-called root systems. This result is considered by many mathematicians to be one of the most beautiful achievements in all mathematics.

To conclude this introduction, we will give a simple example which shows that Lie groups naturally appear as groups of symmetries of various objects — and how one can use the theory of Lie grousp and Lie algebras to make use of these symmetries.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Define the Laplace operator $\Delta \colon C^\infty(S^2) \to C^\infty(S^2)$ by $\Delta_{\mathrm{sph}} f = \Delta(\tilde{f})\big|_{S^2}$, where $\tilde{f}$ is the result of extending f to $\mathbb{R}^3 \setminus \{(0,0,0)\}$ (constant along each ray), and Δ is the usual Laplace operator in $\mathbb{R}^3$. It is easy to see that Δ_{sph} is a second order differential operator on the sphere; one can write explicit formulas for it in the spherical coordinates, but they are not particularly nice.

For many applications, it is important to know that the eigenvalues and eigenfunctions of Δ_{sph}. In particular, this problem arises in quantum mechanics: the eigenvalues are related to to the energy levels of a hydrogen atom in quantum mechanical description. Unfortunately, trying to find the eigenfunctions by brute force gives a second-order differential equation which is very difficult to solve.

However, it is easy to notice that this problem has some symmetry — namely, the group $\mathrm{SO}_3\,\mathbb{R}$ acting on the sphere by rotations. How can one us this symmetry?

If we had just one symmetry, given by some rotation $R\colon S^2 \to S^2$, we could consider its action on the space of complex-valued functions $C^\infty(S^2, \mathbb{C})$. If we could diagonalize this operator, this would help us study Δ_{sph}: it is a general result of linear algebra that if A and B are two linear operators, and A is diagonalizable, then B must preserve eigenspaces for A. Applying this to the pair R, $\Delta_{textsph}$, we get that Δ_{sph} preserves eigenspaces for R, so we can diagonalize Δ_{sph} independently in each of the eigenspaces.

HOwever, this will not solve the problem: for each individual rotation R, the eigenspace will still be too large (in fact, infinite-dimensional), so diagonalizing Δ_{sph} in each of them is not very easy either. This is not surprising: after all, we only used one of many symmetries. Can we use all of the rotations in $\mathrm{SO}_3\,\mathbb{R}$ simultaneously?

This however presents two problems

- $\mathrm{SO}_3\,\mathbb{R}$ is not a finitely generated group, so apparently we will need to use infinitely (in fact, uncountably) many different symmetries and diagonalize each of them.
- $\mathrm{SO}_3\,\mathbb{R}$ is not commutative, so different operators from $\mathrm{SO}_3\,\mathbb{R}$ cannot be diagonalized simultaneously.

The goal of the theory of Lie groups is to give tools to deal with these (and similar) problems. IN short, the answer to the first problem is that $\mathrm{SO}_3\,\mathbb{R}$ is in a certain sense finitely generated — namely, it is generated by three generators, infinitesimal rotations around the x-, y-, z-axes.

The answer to the second problem is that instead of decomposing the $C^\infty(S^2, \mathbb{C})$ into adirect sum of common eigenspaces for operators $R \in \mathrm{SO}_3\,\mathbb{R}$, we need to decompose it into irreducible representations of $\mathrm{SO}_3\,\mathbb{R}$. In order to do this, we need to develop the theory of representations of $\mathrm{SO}_3\,\mathbb{R}$. WE will do this an complete the analysis of this example in a couple of sections.

Chapter 2

Lie Groups: Basic Definitions

2.1 Differential geometry review

This book assumes that the reader is familiar with basic notions of differential geometry. For the reader's convenience, in this section, we briefly remind you of some of the definitions and fix notation for further use.

Unless otherwise specified, all manifolds considered will be real smooth manifolds. All manifolds we will consider will have at most countably many connected components.

For a manifold M and a point $p \in M$, we denote by T_pM the tangent space to M at point p, and by TM the tangent bundle to M. The space of vector fields on M (i.e, global sections of TM) is denoted by $\operatorname{Vect} M$. For a mrphism $f\colon M \to N$ and a point $p \in M$, $q = f(p)$, $df\colon T_pM \to T_qN$ is the corresponding map of map of tangent spaces.

Recall that a morphism $f\colon M \to N$ is called an *immersion* if $\operatorname{rk} df = \dim M$ for every point $p \in M$; in this case, one can choose local coordinates in a neighborhood of p and in a neighborhood of q such that f is given by $f(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$.

An *immersed submanifold* in a manifold M is a subset $N \subset M$ with a structure of a manifold (not necessarily the one inherited from M) such that the inclusion map $\iota\colon N \hookrightarrow M$ is an immersion. Note that the manifold structure on N is part of the data: in general it is not unique. However, it is usually suppressed in the notation. Note also that for any point $p \in N$, the tangent space to N is naturally a subspace of the tangent space to M at p, i.e., $T_pN \subset T_pM$.

An *embedded submanifold* $N \subset M$ is an immersed submanifold such that the inclusion map $\iota\colon N \hookrightarrow M$ is a homeomorphism. In this case the smooth structure on N is uniquely determined by the smooth structure on M.

Following Spivak, we will use the word submanifold for *embedded submanifolds.*

All of the notions above have complex analogs, in which manifolds are replaced by complex analytic manifolds and smooth maps by holomorphic maps.

2.2 Lie groups, subgroups and cosets

Definition 1. A Lie group is a set G with two structures: G is a group and G is a manifold. These structures agree in the following sense: the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are smooth maps.

A morphism of Lie groups is a smooth map which also preserves the group operation: $f(gh) = f(g)f(h)$, $f(1) = 1$. We will use the standard notation $\operatorname{Im} f$ and $\operatorname{Ker} f$ for the image and the kernel of the morphism.

The word real is used to distinguish these Lie groups from complex Lie groups. However, it is frequently omitted.

One can also consider other classes of manifolds: C^1, C^2, analytic (denoted C^ω). It turns out that all of them are equivalent: every C^0 Lie group has a unique analytic structure. This is a higly non-trivial result (it was one of Hilber's 20 problems), and we are not going to prove it. Proof of a weaker result, that C^2 implies analyticity, is much easier.

In a similar way, one defines complex Lie groups.

Definition 2. A complex Lie group is a set G with two structures: G is a group and G is an analytic manifold. These structures agree in the following sense: multiplication map $G \times G \to G$ and inversion map $G \to G$ are analytic maps.

A moprhism of complex Lie groups is an analytic map which also preserves the group operation, $f(gh) = f(g)f(h)$, $f(1) = 1$.

Through out this book, we try to treat both real and complex cases simultaneously. Thus, most theorems in this book apply both to real and complex Lie groups.

When talking about complex Lie groups, submanifolds will mean complex analytic submanifold, tangent spaces will be considered as complex vector spaces, all morphisms between manifolds will be assumed holomorphic, etc.

Examples 1. The following are examples of Lie groups

(1) $\mathbb{R}^n$, with the group operation given by addition

(2) $\mathbb{R}^\times = (\mathbb{R} \setminus \{0\}, \cdot)$

$\mathbb{R}^+ = (\{\, x \in \mathbb{R} : x > 0 \,\}, +)$

(3) $S^1 = (\{\, z \in \mathbb{C} : |z| = 1 \,\}, \cdot)$

(4) $\mathrm{GL}(n, \mathbb{R} \subset \mathbb{R}^{n^2}$. Many of the groups we will consider are subgroups of $\mathrm{GL}_n\mathbb{R}$ or $\mathrm{GL}_n\mathbb{C}$.

(5) $SU(2) = \{\, A \in \mathrm{GL}(2, \mathbb{C}) : AA^\intercal = I, \det A = 1 \,\}$. Indeed, one can easily see that

$$\mathrm{SU}_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

Writing $a = x_1 + \mathrm{i}x_2$, $b = x_3 + \mathrm{i}x_4$, $x_i \in \mathbb{R}$, we see that $\mathrm{SU}(2)$ is diffeomorphic to $S^3 = \{\, (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = 1 \,\}$.

(6) In fact, all usual groups of linear algebra, such as $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{U}(n)$, $SO(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(2n)$ are all Lie groups.

Note that this definition of a Lie group does not require G to be connected. Thus, any finite group is a 0-dimensional Lie group. Since the theory of finite groups is complicated enough, it makes sense to separate the finite part. It can be done as follows.

Theorem 1. *Let G be a real or complex Lie group. Denote by G^0 the connected component of the identity. Then G^0 is a normal subgroup of G and is a Lie group itself (real or complex, respectively). The quotient group G/G^0 is discrete.*

Proof. We need to show that G^0 is closed under the operations of multiplication and inversion. Since the image of a connected topological space under a continuous map is connected, the inversion map i must take G^0 to one component of G, that which contains $i(1) = 1$, namely G^0. In a similar way one shows that G^0 is closed under multiplication.

To check that this is a normal subgroup, we must show that if $g \in G$ and $h \in G^0$, then $ghg^{-1} \in G^0$. Conjugation by g is continuous and thus will take G^0 to some connected component of G; since it fixes 1, this component is G^0.

The fact that the quotient is discrete is obvious. ◀

This theorem mostly reduces the study of arbitrary Lie groups to the study of finite groups and connected Lie groups. In fact, one can go further and reduce the study of connected Lie groups to the study of connected simply-connected Lie groups.

Theorem 2. *If G is a connected Lie group, then its universal cover $\tilde{G}$ has a canonical structure of a Lie group such that the covering map $p\colon \tilde{G} \to G$ is a marphism of a Lie groups whose kernel is isomorphic to the fundamental group of G;* $\operatorname{Ker} p = \pi_1(G)$ *as a group. Moreover, in this case* $\operatorname{Ker} p$ *is a discrete central subgroup in $\tilde{G}$.*

Proof. The proof follows from the following general result of topology: if M, N are manifolds (or, more generally, nice enough topological spaces), then any continuous mapping $f\colon M \to N$ can be lifted to a map of universal covers $\tilde{f}\colon \tilde{M} \to \tilde{N}$. Moreover, if we choose $m \in M$, $n \in N$ such that $f(m) = n$ and choose liftings $\tilde{m} \in M$, $\tilde{n} \in \tilde{N}$ such that $p(\tilde{m}) = m$, $p(\tilde{n}) = n$, there is a unique lifting $\tilde{f}$ of f such that $\tilde{f}(\tilde{m}) = \tilde{n}$.

Now let us choose some element $\tilde{1} \in \tilde{G}$ such that $p(\tilde{1}) = 1$ in G. Then, by the above theorem, there is a unique map $\tilde{\imath}\colon \tilde{G} \to \tilde{G}$ which lifts the inversion map $i\colon G \to G$ and satisfies $\tilde{\imath}(\tilde{1}) = \tilde{1}$. In a similar way, one constructs the multiplication map $\tilde{G} \times \tilde{G} \to \tilde{G}$.

Finally, the fact that $\operatorname{Ker} p$ is central follows from the results of Exercise 2.2. (Whatever that exercise may be.) ◀

Definition 3. A closed Lie subgroup H of a Lie group G is a subgroup which is also a submanifold.

Theorem 3. (1) *Any closed LIe subgroup is closed in G.*

(2) *Any closed subgroup of a Lie goup is a closed real Lie subgroup.*

Proof. The proof o fthe first part is given in Exercise 2.1. The second part is much harder and will not be proved here. Thep roof uses the technique of Lie algebras and can be found, for example, in 10, Corollary 1.10.7. ◀

Corollary 4.

(1) *If G is a connected Lee group and U is a neighborhood of 1, then U generates G.*

(2) *Let $f\colon G_1 \to G_2$ be a morphism of Lie groups, with G_2 connected, such that $df\colon T_1G_1 \to T_2G_2$ is surjective. Then f is surjective.*

Proof. (1) Let H be the subgroup generated by U. Then H is open in G: for any element $h \in H$, the set $h \cdot U$ is a neighborhoof of h in G. Since it is an open subset of a manifold, it is a submanifold, so H is a closed Lie subgroup. Therefore, by Theorem 2.9 it is closed and is nonempty, so $H = G$.

(2) Given the assumptions, the inverse function theorem says that f is surjective onto some neighborhood U of $1 \in G_2$. Since an image of a group morphism is a subgroup, and U generates G_2, f is surjective. ◀

As in the theory of discrete groups, given a closed Lie subgroup $H < G$, we can define the notion of cosets and define the coset space G/H as the set of equivalence classes. The following theorem shows that the coset space is actually a manifold.

Theorem 5.

(1) *Let G be a Lie group of dimension n and $H < G$ a closed Lie subgroup of dimension k. Then the coset space G/H has a natural structure of a manifold of dimension $n - k$ such that the canonical map $p\colon G \to G/H$ is a fiber bundle, with fiber diffeomorphic to H. The tangent space at $\tilde{1} = p(1)$ is given by $T_{\tilde{1}}G/H = T_1G/T_1H$.*

(2) *If H is a normal closed Lie subgroup then G/H has a canonical structure of a Lie group.*

Proof. Denote by $p\colon G \to G/H$ the canonical map. Let $g \in G$ and $\bar{g} = p(g) \in G/H$. Then the set $g \cdot H$ is a submanifold in G as it is an image of H under diffeomorphism $x \mapsto gx$. Choose a submanifold $M \subset G$ such that $g \in M$ and M is transversal to the manifold gH, i.e., $T_gG = T_gH \oplus T_gM$. Let $U \subset M$ be a sufficiently small neighborhood of g in M. Then the set $UH = \{\, uh : u \in U, h \in H \,\}$ is open in G (which easily follows from the inverse function theorem applied to the map $U \times H \to G$). Consider $\bar{U} = p(U)$; since $p^{-1}(\bar{U}) = UH$ is open, $\bar{U}$ is an open neighborhood of $\bar{g}$ in G/H and the map $U \to \bar{U}$ is a homeomorphism. This gives a local chart for G/H and at the same time shows that $G \to G/H$ is a fiber bundle with fiber H. Now we just need to show that the transition functions between such charts are smooth and that the smooth structure does not depend on the choice of g, M.

This argument also shows that the kernel of the projection $\pi\colon T_gG \to T_{\bar{g}}G/H$ is equal to T_ggH. In particular, for $g = 1$ this gives us an isomorphism $T_{\tilde{1}}G/H \simeq T_1G/T_1H$. ◀

Corollary 6. *Let H be a closed Lie subgroup of a Lie group G.*

(1) *If H is connected, then the set of connected components $\pi_0G = \pi_0G/H$. In particular, if H, G/H are connected, then so is G.*

(2) *If G, H are connected, then there is an exact sequence of fundamental groups*

$$\pi_2G/H \longrightarrow \pi_1H \longrightarrow \pi_1G/H \longrightarrow 1.$$

This corollary follows from more general long exact sequence of homotopy groups associated with any fiber bundle.

2.3 Analytic subgroups and the homomorphism theorem

For many purposes, the notion of a closed Lie subgroup introduced above is too restrictive. For example

Examples 2. Let $G_1 = \mathbb{R}$, $G_2 = T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Define the map $f\colon G_1 \to G_2$ by $f(t) = (t \mod \mathbb{Z}, \alpha t \mod \mathbb{Z})$, where α is some fixed irrational number. Then it is well-known that the image of this map is everywhere dense in T^2 (it is sometimes called the *irrational winding* on the torus)..

It is therefore useful to introduce a more general notion of a subgroup. Recall the definition of immersed submanifold.

Definition 4. A *Lie subgroup* in a Lie group $H < G$ is an immersed submanifold which is also a subgroup.

It is easy to see that in such a situation H is itself a Lie group and the inclusion map $\iota\colon H \hookrightarrow G$ is a morphism of Lie groups.

Clearly, every closed Lie subgroup is a Lie subgroup, but the converse is not true as the example above demonstrated.

With this new notion of a subgroup we can formulate an analog of the standard homomorphism theorems.

Theorem 7. *Let $f\colon G_1 \to G_2$ be a morphism of Lie groups. Then $H = \operatorname{Ker} f$ is a normal closed Lie subgroup in G_1, and f gives rise to an injective morphism $G_1/H \to G_2$, which is an immersion; thus, $\operatorname{Im} f$ is a Lie subgroup in G_2. If $\operatorname{Im} f$ is a submanifold, then it is a closed lie subgroup in G_2 and f gives an isomorphism of Lie gruops $G_1/H \simeq \operatorname{Im} f$.*

With this new notation of of a subgroup we can formulate an analog of the standard hobmomorphism theorems.

Theorem 8. *Let $f\colon G_1 \to G_2$ be a morphism of Lie groups. Then $H = \operatorname{Ker} f$ is a normal closed Lie subgroup in G_1, and f gives rise to an injective morphism $G_1/H \to G_2$, which is an immersion; thus, $\operatorname{Im} f$ is a Lie subgroup of G_2. If $\operatorname{Im} f$ is a submanifold then it is a closed Lie subgroup in G_2 and f gives an isomorphism of Lie groups $G_1/H \simeq \operatorname{Im} f$.*

The easiest way to prove this theorem is by using the theory of Lie algebras which we develeop in the next chapter.

2.4 Action of Lie groups on manifolds and representations

The primary reason why Lie groups are so frequently used is that they usually appear as symmetry groups of various geometric objects. In this section, we will show several examples.

Definition 5. An action of a real Lie group G on a manifold M is an assignment to each $g \in G$ of a diffeomorphism $\rho(g) \in \mathrm{d}^M$ such that $\rho(1) = \mathrm{id}$, $\rho(gh) = \rho(g)\rho(h)$ and such that the map

$$\begin{aligned} G \times M &\longrightarrow M \\ (g, m) &\longmapsto \rho(g)m \end{aligned}$$

is a smooth map.

A holomorphic action of a complex Lie group G on a complex manifold M is an assignment to each $g \in G$ an invertible holomorphic map $\rho(g) \in \mathrm{d}^M$ such that $\rho(1) = \mathrm{id}$, $\rho(gh) = \rho(g)\rho(h)$ and such that the map

$$\begin{aligned} G \times M &\longrightarrow M \\ (g, m) &\longmapsto \rho(g)m \end{aligned}$$

is holomorphic.

Examples 3.

(1) The group $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$.

(2) The group $\mathrm{O}(n)$ acts on the sphere $S^{n-1} \subset \mathbb{R}^n$. The group $\mathrm{U}(n)$ acts on the sphere $S^{2n-1} \subset \mathbb{C}^n$.

Closely related with the notion of a group action on a manifold is the notion of a representation.

Definition 6. A representation of a Lie group G is a vector space V together with a group morphism $\rho\colon G \to \operatorname{End} V$. If V is finite-dimensional, we require that ρ be smooth (respectively, analytic), so it is a morphism of Lie groups. A morphism between two representations V, W of the same group G is a linear map $f\colon V \to W$ which commutes with the group action: $f \circ \rho_v(g) = \rho_w(g) \circ f$.

In other words, we assigne to every $g \in G$ a linear map $\rho(g)\colon V \to V$ so that $\rho(g)\rho(h) = \rho(gh)$.

We will frequently use the shorter notation $g \cdot m$, $g \cdot v$ instead of $\rho(g)(m)$ in the cases where there is no ambiguity about the representation being used.

Any action of the group G on a manifold M gives rise to several representations of G on various vector spaces associated with M:

(1) Representation of G on the infinite-dimensional space of functions $C^\infty(M)$ or the space of holomorphic functions $C^\omega(M)$ defined by

$$(\rho(g) \circ f)(m) = f(g^{-1} \cdot m).$$

(note that we need g^{-1} rather than g to satisfy $\rho(g)\rho(h) = \rho(gh)$).

(2) Representation of G on the infinite-dimensional space of vector fields $\vec{M}$ defined by

$$(\rho(g)(v))(m) = dg(v(g^{-1} \cdot m)).$$

In a similar way, we defined the action of G on the spaces of differential forms and other types of tensor fields on M.

(3) Assume that $m \in M$ is a fixed-point: $g \cdot m = m$ for all $g \in G$. Then we have an action of G on the tangent space $T_m M$ given by $\rho(g) = dg\colon T_m M \to T_m M$, and similarly for the spaces $T_m^* M$, $\bigwedge^k T_m^* M$.

2.5 Orbits and homogeneous spaces

Let G be a Lie group acting on a manifold M. Then for every point $m \in M$ we defined its *orbit* by $\mathcal{O}_m = Gm = \{ g \cdot m : g \in G \}$ and stabilizer

$$G_m = \{ g \in G : g \cdot m = m \}.$$

Theorem 9. *Let M be a manifold with an action of a Lie group G. Then for any $m \in M$ the stabilizer G_m is closed is a closed Lie subgroup in G, and $g \mapsto g \cdot m$ is an injective immersion $G/G_m \hookrightarrow M$ whose image coincides with the orbit $\mathcal{O}_m$.*

Proof. The fact that the orbit is in bijection with G/G_m is obvious. For the proof of the fact that G_m is a closed Lie subgroup, we could just refer to Theorem 2.9. However, this would not help proving that $G/G_m \to M$ is an immersion. Both of these statements are esiest proved using the technique of Lie algebras. ◀

Corollary 10. *The orbit $\mathcal{O}_m$ is an immersed submanifold in M, with tangeth space $T_m\mathcal{O}_m \simeq T_1G/T_1G_m$. If $\mathcal{O}_m$ is a submanifold, then $g \mapsto g \cdot m$ is a diffeomorphism $G/G_m \rightleftarrows \mathcal{O}_m$.*

An important special case is when the action of G is transitive, i.e., when there is only one orbit.

Definition 7. A G-homogeneous space is a manifold with a transitive action of G.

As an immediate corollary of Corollary 2.21, we see that each homogeneous space is diffeomorphic to a coset space G/H. Combining Theorem 2.11, we get the following result.

Corollary 11. *Let M be a G-homogeneous space and choose $m \in M$. Then the map $G \to M$ given by $g \mapsto g \cdot m$ is a fiber bundle over M with fiber G_m.*

(1) Consider the action of $\mathrm{SO}_n\,\mathbb{R}$ on the sphere $S^{n-1} \subset \mathbb{R}^n$. It is transitive then S^{n-1} is a homogeneous space, so we have the fiber bundle

$$\begin{tikzcd} \mathrm{SO}_{n-1}\mathbb{R} \arrow[r] & \mathrm{SO}_n\mathbb{R} \arrow[d] \\ & S^{n-1}. \end{tikzcd}$$

(2) Consider the action of SU_n on the sphere $S^{2n-1} \subset \mathbb{C}^n$. Then it is a homogeneous space, so we have a fiber bundle

$$\begin{tikzcd} \mathrm{SU}_{n-1}\mathbb{R} \arrow[r] & \mathrm{SU}_n\mathbb{R} \arrow[d] \\ & S^{2n-1}. \end{tikzcd}$$

In fact, action of G can be used to define smooth structure on a set. Indeed, if M is a set (with no predetermined smooth structure) with a transitive action of a Lie group G, then M is in bijection with G/H, $H = \operatorname{Stab}_G m$ and thus, by Theorem 2.11 M has a canonical structure of a manifold of dimension $\dim G - \dim H$.

Examples 4. Define a *flag* in $\mathbb{R}^n$ to be a sequence of subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n, \qquad \dim V_i = i.$$

Let $\mathcal{F}_n(\mathbb{R})$ be the set of all flags in $\mathbb{R}^n$. It turns out that $\mathcal{F}_n(\mathbb{R})$ has a canonical structure of a smooth manifold which is called the *flag manifold*. The easiest way to define it is to note that we have an obvious action of the group $\mathrm{GL}_n\mathbb{R}$ on $\mathcal{F}_n(\mathbb{R})$. This action is transitive: by a change of basis, any flag can be identified with the standard flag

$$V^{\mathrm{std}} = (\{0\} \subset \mathrm{span}(e_1) \subset \mathrm{span}(e_1, e_2) \subset \cdots \subset \mathrm{span}(e_1, \ldots, e_n) = \mathbb{R})$$

where $\langle e_1, \ldots, e_n\rangle$ is the standard basis for $\mathbb{R}^n$. Thus, $\mathcal{F}_n(\mathbb{R})$ can be identified with the coset space $\mathrm{GL}(n,\mathbb{R})/B_n\mathbb{R}$ where $B_n\mathbb{R} = \mathrm{Stab}\, V^{\mathrm{std}}$ is the group of all invertible upper-triangular matrices. Therefore, $\mathcal{F}_n$ is a manifold of dimension $n^2 - n(n+1)/2 = n(n-1)/2$.

Finally, we should say a few words about taking the quotient by the action of a group. In many cases when we have an action of a group G no a manifold M one would like to consider the quotient space, i.e., the set of all G-orbits. This set is commonly denoted by M/G. It has a canonical quotient space topology. However, this space can be vary singular, even if G is a Lie group; for example, it can be a non-Hausdorff space. For example, for the group $G = \mathrm{GL}(n,\mathbb{C})$ acting on the set of all $n \times n$ matrices by conjugation the set of orbits is described by the Jordan canonical form. However, it is well-know that by a small perturbation, any matrix can be made diagonalizable. Thus, if X i sa diagonalizable matrix and Y is a non-diagonalizable matrix with the same eigenvalues as X, then any neighborhood of the orbit of Y contains points from the orbit of X.

There are several ways of dealing with this problem. one of them is to impose additional requirements on the acction, for example assuming that he action is proper. In this case it can be shown that M/G is indeed a Hausdorff topologircal space, and under some additional conditions, it is actually a manifold. Another approach, usually called geometric invariant theory, is based on using the methods of algebraic geometry. Both of these methods go beyond the scope of this book.

2.5.1 Left, right, and adjoint actions

Important examples of group actions are the following actions of a group G on itself:

Left action: $L_g\colon G \to G$ defined by $L_g(h) = gh$;
Right action: $R_g\colon G \to G$ defined by $R_g(h) = hg^{-1}$;
Adjoint action: $\mathrm{Ad}(g)\colon G \to G$ defined by $A_g(h) = ghg^{-1}$.

One easily sees that he left and right actions are transitive; in fact, each of them is simply transitive. It is also easy to see that the left and right actions commute and that $\mathrm{Ad}(g) = L_g R_g$.

Each of these actions also defines the action of G on the spaces of functions, vector fields, forms, etc. on G. For simplicity, for a tangent vector $v \in T_m G$, we will frequently write just $g \cdot v \in T_{gm}G$ instead of the technically more accurate but cumbersome notation $d(L_g)(v)$. Similarly, we will write $v \cdot g$ for $d(R_{g^{-1}})(v)$. This is justified by Exercise 2.6, where it is shown that for matrix groups this notation agrees with usual multiplication of matrices.

Since the adjoint action preserves the identity element $1 \in G$, it also defines an action of G on the (finite-dimensional) space T_1G. Slightly abusing the notation, we will denote this action also by

$$\mathrm{Ad}\, g\colon T_1G \longrightarrow T_1G.$$

Definition 8. A vector field $v \in \mathrm{Vect}(G)$ is *left-invariant* if $g \cdot v = v$ for every $g \in G$ and right-invariant if $v \cdot g = v$ for every $g \in G$. A vector field is called bi-invariant if it is both left- and right-invariant.

In a similar way, one defines left-, right-invariant differential froms and other tensors.

Theorem 12. *The map $v \mapsto v(1)$ (where 1 is the identity element of the group) defines an isomorphism of the vector space of left-invariant vector fields on G with the vector space T_1G, and similarly for right-invariant vectors spaces.*

Proof. It suffices to prove that every $x \in T_1G$ can be uniquely extended to a left-invariant vector field on G. Let us define the extension by $v(g) = g \cdot x$ in T_gG. Then one easily sees that a so defined vector field is left-invariant, and $v(1) = x$. This proves existence of an extension; uniqueness is obvious. ◀

Theorem 13. *The map $v \mapsto v(1)$ defines an isomorphism of the vector space of bi-invariant vector fields on G with the vector space of invariants of the adjoint action:*

$$(T_1G)^{\mathrm{Ad}\,G} = \{\, x \in T_1G : \mathrm{Ad}_g(x) = x \textit{ for all } g \in G \,\}.$$

2.5.2 Classical groups

In this section, we discuss the so-called classical groups, or various subgroups of the general linear group which are frequently used in linear algebra. Traditionally, the name "classical groups" is applied t othe following groups

- $\mathrm{GL}(n, \mathbb{K})$
- $\mathrm{SL}(n, \mathbb{K})$
- $(n, \mathbb{K})$
- $\mathrm{SO}(n, \mathbb{K})$ and more the general groups $\mathrm{SO}(p, q, \mathbb{K})$
- $\mathrm{Sp}(2n, \mathbb{K}) = \{\, A \colon \mathbb{K}^{2n} \to \mathbb{K}^{2n} : \omega(Ax, Ay) = \omega(x, y) \,\}$. Here $\omega(x, y)$ is a skew-symmetric bilinear form $\sum_{i=1}^n x_i y_{i+n} - y_i x_{i+n}$ which, up to a change of basis, is the unique non-degenerate skew-symmetric bilinear form on $\mathbb{K}^{2n}$. Equivalently, one can write $\omega(x, y) = (Jx, y)$ where $\langle -, - \rangle$ is the standard symmetric bilinear form on $\mathbb{K}^{2n}$ and

$$J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}.$$

- $U(n)$
- $\mathrm{SU}(n)$
- Group of unitary quaternionic trasformations $\mathrm{Sp}(2n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{SU}(2n)$. Another description of this group, which explains its relation to the quaternions is given in Exercise 2.15.

We have already shown that $\mathrm{GL}(n)$ and $\mathrm{SU}(2)$ are Lie groups. In this little section, we will show that each of the classical groups listed above are in fact Lie groups and we will find their dimensions.

A straightforward approach, based on implicit function theorem is hard: for example, $\mathrm{SO}(n, \mathbb{K})$ is defined by n^2 equations in $\mathbb{K}^{n^2}$, and finding the rank of this system is not an easy task. We could just refer to the theorem about closed subgroups; this would prove that each of them is a Lie group, but wuold give us no other information — not even the dimension of G. Thus, we will need another approach.

Our approach is based on the use of exponential map. Recall that for matrices the exponential map is defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

It is well-known that this power series converges and defines an analytic map $\mathfrak{gl}(n, \mathbb{K})$ is the set of all $n \times n$ matrices. In a similar way, we define the logarithmic map by

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

So defined log is an analytic map defined in a neighborhood of $1 \in \mathfrak{gl}(n, \mathbb{K})$.

The following theorem summarizes properties of the exponential and logarithmic maps. Most of the properties are the same as for numbers; however, there are also some differences due to the fact that multiplication of matrices is not commutative. All the statements of this theorem apply equally in real and complex cases.

Theorem 14.

(1) $\log(\exp(x)) = x$; $\exp(\log(X)) = X$ *whenever they are defined.*

(2) $\exp(x) = 1 + x + \cdots$. *This means* $\exp(0) = 1$ *and* $\exp(0) = I$.

(3) *If* $xy = yx$ *then* $\exp(x + y) = \exp(x)\exp(y)$. *If* $XY = YX$ *then* $\log(XY) = \log(X) + \log(Y)$ *in some neighborhood of the identity. In particular, for any* $x \in \mathfrak{gl}(n, \mathbb{K})$, $\exp(x)\exp(-x) = 1$, *so* $\exp x \in \mathrm{GL}(n, \mathbb{K})$.

(4) *For fixed* $x \in \mathfrak{gl}(n, \mathbb{K})$, *consider the map* $\mathbb{K} \to \mathrm{GL}(n, \mathbb{K})$ *given by* $t \mapsto \exp(tx)$. *Then* $\exp((t + s)x) = \exp(tx)\exp(sx)$. *In other words, this map is a morphism of Lie groups.*

(5) *The exponential map agrees with change of basis and transposition:*

$$\exp(AxA^{-1}) = A\exp(x)A^{-1}, \qquad \exp(x^t) = (\exp(x))^t.$$

Note that group morphisms $\mathbf{F} \to G$ are frequently called *one-parameter subgroups* in G. Thus, we cane reformulate part (4) of the theorem saying that $\exp(tx)$ is a one-parameter subgroup in $\mathrm{GL}(n, \mathbb{K})$.

But how does this help us to study various matrix groups? The key idea is that the logarithmic map identifies some neighborhood of the identity in $\mathrm{GL}(n, \mathbb{K})$ with some neighborhood of 0 in the vector space $\mathfrak{gl}(n, \mathbb{K})$. It turns out that it also does the same for all the classical groups.

Theorem 15. *For each classical group* $G \subset \mathrm{GL}(n, \mathbb{K})$ *there exists a vector space* $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{K})$ *such that for some neighborhood* U *of* 1 *in* $\mathrm{GL}(n, \mathbb{K})$ *and some neighborhood* V *of* 0 *in* $\mathfrak{gl}(n, \mathbb{K})$ *the following maps are mutually inverse*

$$\begin{tikzcd} (U \cap G) \arrow[r, bend left, "\log"] & (V \cap \mathfrak{g}) \arrow[l, bend left, "\exp"] \end{tikzcd}$$

Before proving this theorem, note that it immediately implies the following important corollary.

Corollary 16. *Each classical group is a Lie group with tangent space at the identity* $T_1 G = \mathfrak{g}$ *and* $\dim G = \dim \mathfrak{g}$*. The groups* $\mathrm{U}(n)$*,* $\mathrm{SU}(n)$*,* $\mathrm{Sp}(2n)$ *are real Lie groups; the groups* $\mathrm{GL}(n, \mathbb{K})$*,* $\mathrm{SL}(n, \mathbb{K})$*,* $\mathrm{SO}(n, \mathbb{K})$*,* $\mathrm{O}(n, \mathbb{K})$*,* $\mathrm{Sp}(2n, \mathbb{K})$ *are real Lie groups for* $\mathbb{K} = \mathbb{R}$ *and complex for* $\mathbb{K} = \mathbb{C}$*.*

Let us prove this corollary first because it is very easy. Indeed, theorem 2.30 shows that near 1, G is identified with an open set in a vector space. So it is immediate that near 1, G is locally a submanifold in $\mathrm{GL}(n, \mathbb{K})$. If $g \in G$ then $g \cdot U$ is a neighborhood of g in $\mathrm{GL}(n, \mathbb{K})$ and $(g \cdot U) \cap G = g \cdot (U \cap G)$ is a neighborhood of g in G; thus, G is a submanifold in a neighborhood of g.

For the second part, consider the differential of the exponential map $d\exp \colon T_0\mathfrak{g} \to T_1 G$. Since $\mathfrak{g}$ is a vector space, $T_0\mathfrak{g} = \mathfrak{g}$, and since $\exp(x) = 1 + x + \cdots$, the derivative is the identity; thus $T_0\mathfrak{g} = \mathfrak{g} = T_1 G$.