MA571 Problem Set 2

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Problem 2.1 (Munkres §17, p. 100, 2)

Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. Let C denote the closure of A in X then, by Theorem 17.4, $A = \bar{A} = C \cap Y$ is the closure of A in Y. Thus, A is closed in X since it is the intersection of two closed subsets of X. ■

Problem 2.2 (Munkres §17, p. 100, 3)

Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Proof. Since A is closed in X and B is closed in Y their complements $X \setminus A$ and $Y \setminus B$ are open in X and Y, respectively. Thus $(X \setminus A) \times (B \setminus Y)$ are basic sets in the product topology on $X \times Y$. Hence

$$(X \times Y) \setminus ((X \setminus A) \times (Y \setminus B)) =$$

■

Problem 2.3 (Munkres §17, p. 101, 6(b))

Let A, B and A_α denote subsets of a space X. Prove the following:

(b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof. ∎

Problem 2.4 (Munkres §17, p. 101, 6(c))

Let A, B and A_α denote subsets of a space X. Prove the following:

(b) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$.

Proof. ∎

Problem 2.5 (Munkres §17, p. 101, 7)

Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \bar{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so x must belong to the closure of some A_α. Therefore, $x \in \bigcup \bar{A}_\alpha$.

Critique. ■

Problem 2.6 (Munkres §17, p. 101, 9)

Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}.$$

Proof. ∎

Problem 2.7 (Munkres §17, p. 101, 10)

Show that every order topology is Hausdorff.

Proof. ∎

Problem 2.8 (Munkres §17, p. 101, 13)

Show that X is Hausdorff if and only if the *diagonal* $\Delta = \{ x \times x \mid x \in X \}$ is closed in $X \times X$.

Proof. ∎

Problem 2.9 (Munkres §18, p. 111, 4)

Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f\colon X \to X \times Y$ and $g\colon Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

Proof. ∎

Problem 2.10 (Munkres §18, p. 111-112, 8(a,b))

Let Y be an ordered set in the order topology. Let $f, g\colon X \to Y$ be continuous.

(a) Show that the set $\{\, x \mid f(x) \leq g(x) \,\}$ is closed in X.
(b) Let $h\colon X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [*Hint:* Use the pasting lemma.]

Proof. ■

Problem 2.11

Given: X is a topological space with open sets $U_1, ..., U_n$ such that $\bar{U}_i = X$ for all i. Prove that the closure of $U_1 \cap \cdots \cap U_n$ is X.

Proof. ■