Math 527 - Homotopy Theory Spring 2013 Homework 2 Solutions

Problem 1. Show that the reduced suspension $\Sigma X = X \wedge S^1$ of any pointed space X is a homotopy cogroup object in \mathbf{Top}_* , with structure maps coming from those of S^1 (c.f. Homework 1 Problem 3).

Solution. The functor $X \wedge -: \mathbf{Top}_* \to \mathbf{Top}_*$ preserves finite coproducts (by Homework 1 Problem 4), including the initial object (by the homeomorphism $* \xrightarrow{\cong} X \wedge *$). Therefore, applying $X \wedge -$ to the stucture maps of S^1 endows $X \wedge S^1$ with structure maps of the correct type. They satisfy the requisite equations up to pointed homotopy, because the functor $X \wedge -$ sends pointed-homotopic maps to pointed-homotopic maps (by Problem 3a.)

Problem 2. Show that a pointed homotopy between two pointed maps $X \to Y$ is the same as a pointed map

$$X \wedge (I_+) \rightarrow Y$$

where $(-)_+$ denotes the disjoint basepoint construction.

Solution. Given the homeomorphism

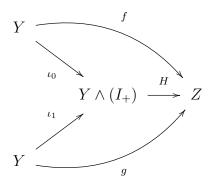
$$X \wedge (I_+) \cong X \times I/x_0 \times I$$

a (continuous) map $H: X \wedge (I_+) \to Y$ is the same as a (continuous) map $H: X \times I \to Y$ which is constant on the subset $x_0 \times I$. Thus, a *pointed* map $H: X \wedge (I_+) \to Y$ is the same as a map $H: X \times I \to Y$ with constant value y_0 on the subset $x_0 \times I$, i.e. a pointed homotopy between two pointed maps $X \to Y$.

Problem 3. Let X be a pointed space.

a. Show that the functor $X \wedge -: \mathbf{Top}_* \to \mathbf{Top}_*$ sends pointed-homotopic maps to pointed-homotopic maps.

Solution. Recall that two pointed maps $f, g: Y \to Z$ are pointed-homotopic if there exists a map H making the diagram



commute, where $\iota_0: Y \cong Y \wedge S^0 \to Y \wedge (I_+)$ denotes the inclusion at $0 \in I$ and likewise for ι_1 . Note that ι_0 is the map obtained by applying the functor $Y \wedge -$ to the pointed map

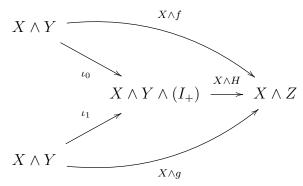
$$S^0 \cong \{0\}_+ \hookrightarrow I_+.$$

By associativity of the smash product, applying the functor $X \wedge -$ yields

$$X \wedge Y \to X \wedge (Y \wedge I_+) \cong (X \wedge Y) \wedge I_+$$

which is still the inclusion at 0.

Therefore, applying $X \wedge -$ to the diagram above yields, up to natural isomorphism, the commutative diagram



and thus a pointed homotopy from $X \wedge f$ to $X \wedge g$.

b. Show that the pointed map "inclusion at 0"

$$X \to X \land (I_+)$$
$$x \mapsto [x, 0]$$

is a pointed homotopy equivalence.

Solution. Note that the inclusion $\{0\} \hookrightarrow I$ is a homotopy equivalence. Since the disjoint base-point functor $(-)_+$ sends homotopic maps to pointed-homotopic maps, the inclusion $\{0\}_+ \hookrightarrow I_+$ is a pointed homotopy equivalence. By part (a), applying the functor $X \land -$ yields a pointed homotopy equivalence $X \to X \land (I_+)$.