

# MA 519: Homework 5

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## PROBLEM 5.1 (HANDOUT 7, # 6(D, F))

Find the variance of the following random variables

- (d)  $X = \#$  of tosses of a fair coin necessary to obtain a head for the first time.
  - (f)  $X = \#$  matches observed in random sitting of 4 husbands and their wives in opposite sides of a linear table.
- This is an example of the *matching problem*.

*SOLUTION.* Recall that the variance of a random variable can be computed as

$$\text{Var}(X) = E(X^2) - E(X)^2.$$

For part (d), let  $X$  be as above. First, note that  $X$  takes every value on  $\mathbb{N}$ . Thus, its PMF is

$$p(n) = P(X = n) = \frac{1}{2^n}$$

and its expectation the value of the series

$$E(X) = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Using a little bit of analysis we can find the value of  $E(X)$ , e.g., by considering the function  $f(x) := \sum_{n=1}^{\infty} nx^{n-1}$ , taking its indefinite integral, and noting that it is a geometric series sans the first term. Concretely,

$$\int f(x) dx = \sum_{n=1}^{\infty} x^n = -1 + \sum_{n=0}^{\infty} x^n,$$

which, for  $|x| < 1$ , converges to the value  $x/(1-x)$ . Taking the derivative of this, we have  $1/(1-x)^2$ . Thus,

$$\begin{aligned} E(X) &= \sum_{n=1}^{\infty} \frac{n}{2^n} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \\ &= \frac{1/2}{(1 - (1/2))^2} \\ &= 2. \end{aligned}$$

This is the mean of  $X$ .

Next we must compute the mean of  $X^2$ . We have already computed the PMF of  $X$  hence,

$$E(X^2) = \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

To find the limit of this series, we can use a similar method to the one in the last paragraph. That is, consider the function  $g(x) := \sum_{n=1}^{\infty} n^2 x^{n-1}$ . Taking its integral, we have

$$xG(x) = \int g(x) dx = \sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1}$$

and repeat this on  $G$ , giving us

$$\int G(x) dx = \sum_{n=1}^{\infty} x^n = -1 + \sum_{n=0}^{\infty} x^n = \frac{x}{1-x}.$$

Tracing back our steps,

$$\int g(x) = \frac{x}{(1-x)^2}$$

so

$$g(x) = \frac{1-x^2}{(1-x)^4}.$$

Thus,

$$\begin{aligned} E(X) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \\ &= \frac{(1/2)(1 - (1/2)^2)}{(1 - (1/2))^4} \\ &= 6. \end{aligned}$$

Putting all of this together, the variance is

$$\boxed{\text{Var}(X) = 6 - (2)^2 = 2.}$$

For part (f), again, we let  $X$  be as above. The PMF of  $X$  is given by

$$P(X = n) =$$

■

## PROBLEM 5.2 (HANDOUT 7, # 8)

*(Nonexistence of variance).*

- (a) Show that for a suitable positive constant  $c$ , the function  $p(x) = c/x^3$ ,  $x = 1, \dots$ , is a valid probability mass function (PMF).
- (b) Show that in this case, the expectation of the underlying random variable exists, but the variance does not!

*SOLUTION.* For part (a), note that  $p(x)$  given above satisfies the requirements to be a probability mass function. First, set  $1/c = \sum_{x=1}^{\infty} 1/x^3$ , and note that indeed  $c$  is well defined (because the relevant series converges, by the  $p$ -test.)

This means that  $1 = \sum_{x=1}^{\infty} c/x^3 = \sum p(x)$ , by definition. Moreover, because  $p(x) = c/x^3 > 0$  for all  $x$  in our domain,  $p(x) \in [0, 1]$ . That is,  $p$  is a valid probability mass function.

Set  $X$  equal to the random variable described by  $p$ . Next, note that

$$\begin{aligned} E(X) &= \sum_{n=1}^{\infty} n \frac{c}{n^3} \\ &= \sum_{n=1}^{\infty} \frac{c}{n^2} \end{aligned}$$

which converges (and thus exists), again by the  $p$ -test.

However,

$$\begin{aligned} E(X^2) &= \sum_{n=1}^{\infty} n^2 \frac{c}{n^3} \\ &= \sum_{n=1}^{\infty} \frac{c}{n} \end{aligned}$$

which does not converge, again by the  $p$ -test. That is, the variance  $E(X^2) - E(X)^2$  does not exist. ■

## PROBLEM 5.3 (HANDOUT 7, # 9)

In a box, there are 2 black and 4 white balls. These are drawn out one by one at random (without replacement).

- (a) Let  $X$  be the draw at which the first black ball comes out. Find the mean and the variance of  $X$ .  
 (b) Let  $X$  be the draw at which the second black ball comes out. Find the mean\* the variance of  $X$ .

*SOLUTION.* For part (a), we must first find the PMF of  $X$ . This we do explicitly,

$$\begin{aligned} P(X=1) &= \frac{2}{6} = \frac{1}{3}, & P(X=2) &= \frac{2}{5} \cdot \frac{4}{6} = \frac{4}{15}, \\ P(X=3) &= \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} = \frac{1}{5}, & P(X=4) &= \frac{2}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} = \frac{2}{15}, \\ P(X=5) &= 1 \cdot \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} = \frac{1}{15}. \end{aligned}$$

Thus,

$$\begin{aligned} E(X) &= 1 \cdot \frac{1}{3} + 2 \cdot \frac{4}{15} + 3 \cdot \frac{1}{5} + 4 \cdot \frac{2}{15} + 5 \cdot \frac{1}{15} \\ &= \frac{7}{3} \\ &= 2.333. \end{aligned}$$

Similarly, we have

$$\begin{aligned} E(X^2) &= 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{4}{15} + 3^2 \cdot \frac{1}{5} + 4^2 \cdot \frac{2}{15} + 5^2 \cdot \frac{1}{15} \\ &= 7. \end{aligned}$$

Hence,

$$\text{Var}(X) = 7 - \left(\frac{7}{3}\right)^2 \approx 1.556.$$

For part (b) we have a similar setup. We compute the PMF of  $X$  explicitly

$$\begin{aligned} P(X=2) &= \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{15}, & P(X=3) &= \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} + \frac{2}{6} \cdot \frac{4}{5} \cdot \frac{1}{4} = \frac{2}{15}, \\ P(X=4) &= 3 \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{3}{15}, & P(X=5) &= 4 \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{4}{15}, \\ P(X=6) &= 5 \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{2}{2} \cdot 1 = \frac{5}{15}. \end{aligned}$$

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\*What is a meman? How do you pronounce meman? Is it mee-man or muh-man?

Thus,

$$\begin{aligned} E(X) &= \frac{2 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4 + 6 \cdot 5}{15} \\ &= \frac{14}{3} \\ &\approx 4.667. \end{aligned}$$

Similarly,

$$\begin{aligned} E(X^2) &= \frac{2^2 + 3^2 \cdot 2 + 4^2 \cdot 3 + 5^2 \cdot 4 + 6^2 \cdot 5}{15} \\ &= \frac{70}{3} \\ &\approx 23.333. \end{aligned}$$

Thus,

$$\text{Var}(X) \approx \frac{70}{3} - \left(\frac{14}{3}\right)^2 \approx 1.556.$$

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## PROBLEM 5.4 (HANDOUT 7, # 10)

Suppose  $X$  has a *discrete uniform distribution* on the set  $\{1, \dots, N\}$ .

Find formulas for the mean and the variance of  $X$ .

*SOLUTION.* First, we find the mean:

$$\begin{aligned} E(X) &= \sum_{n=1}^N n \frac{1}{N} \\ &= \frac{1}{N} \frac{N(N+1)}{2} \\ &= \frac{(N+1)}{2} \end{aligned}$$

Next, we find the variance:

$$\begin{aligned} E(X^2) - E(X)^2 &= \sum_{n=1}^N n^2 \frac{1}{N} - \left[ \frac{(N+1)}{2} \right]^2 \\ &= \frac{N^2}{3} + \frac{N}{2} + \frac{1}{6} - \left[ \frac{(N+1)}{2} \right]^2 \\ &= \frac{N^2 - 1}{12} \end{aligned}$$

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## PROBLEM 5.5 (HANDOUT 7, # 11)

(Be Original) Give an example of a random variable with mean 1 and variance 100.

SOLUTION. Let  $X$  be the random variable whose PMF is given by

$$\begin{aligned}P(X = -10 - 1) &= 0.5 \\P(X = 10 - 1) &= 0.5 \\P(X \neq \pm\sqrt{10} - 1) &= 0\end{aligned}$$

(Note that those expressions are very easy to simplify ( $-10-1=-11$ ,  $10-1=9$ ), but leaving them in that form makes the arithmetic more obvious.)

Then we see that the mean of  $X$  is given by

$$\begin{aligned}E(X) &= 0.5(-10 - 1 + 10 - 1) \\&= 1\end{aligned}$$

and the variance of  $X$  is given by

$$\begin{aligned}E((X - E(X))^2) &= E((X - 1)^2) \\&= 0.5(10^2 + (-10)^2) \\&= 0.5(10^2 + (-10)^2) \\&= 100\end{aligned}$$

so that  $X$  is such a random variable as described in the problem. ■

## PROBLEM 5.6 (HANDOUT 7, # 13)

(*Be Original*). Suppose a random variable  $X$  has the property that its second and fourth moment are both 1.

What can you say about the nature of  $X$ ?

*SOLUTION.* Suppose that the second and fourth moment of  $X$  are both 1. By Hölder's inequality,

$$E(X^4) \leq E(X^2)^{1/2} E(X^2)^{1/2}.$$

But both the left-hand side and the right-hand side are equal to 1. ■

## PROBLEM 5.7 (HANDOUT 7, # 14)

(Be Original). One of the following inequalities is true in general for all nonnegative random variables. Identify which one!

$$\begin{aligned} E(X)E(X^4) &\geq E(X^2)E(X^3); \\ E(X)E(X^4) &\leq E(X^2)E(X^2). \end{aligned}$$

SOLUTION. ■

## PROBLEM 5.8 (HANDOUT 7, # 15)

Suppose  $X$  is the number of heads obtained in 4 tosses of a fair coin.

Find the expected value of the weird function

$$\log\left(2 + \sin\left(\frac{\pi}{4}x\right)\right).$$

*SOLUTION.* First, note that

$$\begin{aligned} P(X=0) &= \frac{1}{16}, & P(X=1) &= \frac{4}{16}, \\ P(X=2) &= \frac{6}{16}, & P(X=3) &= \frac{4}{16}, \\ P(X=4) &= \frac{1}{16}. \end{aligned}$$

Thus, computing the expected value of the function, we get

$$\begin{aligned} E\left[\log\left(2 + \sin\left(\frac{\pi}{4}X\right)\right)\right] &= \sum_{x=0}^4 p(X=x) \log\left(2 + \sin\left(\frac{\pi}{4}x\right)\right) \\ &= \frac{1}{16} \left[ \log(2) + 4 \log\left(2 + \frac{\sqrt{2}}{2}\right) + 6 \log(2+1) + 4 \log\left(2 + \frac{\sqrt{2}}{2}\right) + \log(2) \right] \\ &= \frac{1}{16} \left[ 2 \log(2) + 8 \log\left(2 + \frac{\sqrt{2}}{2}\right) + 6 \log(3) \right] \\ &\approx 0.9966. \end{aligned}$$

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## PROBLEM 5.9 (HANDOUT 7, # 16)

In a sequence of Bernoulli trials let  $X$  be the length of the run (of either successes or failures) started by the first trial.

- (a) Find the distribution of  $X$ ,  $E(X)$ ,  $\text{Var}(X)$ .

*SOLUTION.*

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## PROBLEM 5.10 (HANDOUT 7, # 17)

A man with  $n$  keys wants to open his door and tries the keys independently and at random. Find the mean and variance of the number of trials

- (a) if unsuccessful keys are not eliminated from further selections;
- (b) if they are.

(Assume that only one key fits the door. The exact distributions are given in II, 7, but are not required for the present problem.)

*SOLUTION.*

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