# MA553: Qual Preparation

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## Contents

# 1 Ulrich

## 1.1 Ulrich: Winter 2002

**Problem 1.** Let $G$ be a group and $H$ a subgroup of finite index. Show that there exists a normal subgroup $N$ of $G$ of finite index with $N \subset H$.

**Solution.** ► ◄

**Problem 2.** Show that every group of order 992 ($= 2^5 \cdot 31$) is solvable.

**Solution.** ► ◄

**Problem 3.** Let $G$ be a group of order 56 with a normal 2-Sylow subgroup $Q$, and let $P$ be a 7-Sylow subgroup of $G$. Show that either $G \simeq P \times Q$ or $Q \simeq \mathbb{Z}/(2) \times \mathbb{Z}/(2) \times \mathbb{Z}/(2)$.

[*Hint*: $P$ acts on $Q \setminus \{e\}$ via conjugation. Show that this action is either trivial or transitive.]

**Solution.** ► ◄

**Problem 4.** Let $R$ be a commutative ring and $\mathrm{Rad}(R)$ the intersection of all maximal ideals of $R$.

(a) Let $a \in R$. Show that $a \in \mathrm{Rad}(R)$ if and only if $1 + ab$ is a unit for every $b \in R$.
(b) Let $R$ be a domain and $R[X]$ the polynomial ring over $R$. Deduce that $\mathrm{Rad}(R[X]) = 0$.

**Solution.** ► ◄

**Problem 5.** Let $R$ be a unique factorization domain and $P$ a prime ideal of $R[X]$ with $P \cap R = 0$.

(a) Let $n$ be the smallest possible degree of a nonzero polynomial in $P$. Show that $P$ contains a primitive polynomial $f$ of degree $n$.
(b) Show that $P$ is the principal ideal generated by $f$.

**Solution.** ► ◄

**Problem 6.** Let $k$ be a field of characteristic zero. assume that every polynomial in $k[X]$ of odd degree and every polynomial in $k[X]$ of degree two has a root in $k$. Show that $k$ is algebraically closed.

**Solution.** ► ◄

**Problem 7.** Let $k \subset K$ be a finite Galois extension with Galois group $\mathrm{Gal}(K/k)$, let $L$ be a field with $k \subset L \subset K$, and set $H = \{ \sigma \in \mathrm{Gal}(K/k) : \sigma(L) = L \}$.

(a) Show that $H$ is the normalizer of $\mathrm{Gal}(K/L)$ in $\mathrm{Gal}(K/k)$.
(b) Describe the group $H/\mathrm{Gal}(K/L)$ as an automorphism group.

**Solution.** ► ◄

## References


[1] Dummit, D., and Foote, R. *Abstract Algebra*. Wiley, 2004.

[2] Herstein, I. *Topics in algebra*. Xerox College Pub., 1975.

[3] Hungerford, T. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2003.

[4] Milne, J. S. Group theory (v3.13), 2013. Available at `www.jmilne.org/math/`.

[5] Milne, J. S. Fields and galois theory (v4.50), 2014. Available at `www.jmilne.org/math/`.