Fall 2016 Notes

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Contents

Chapter 1

Probability

We will devote this chapter to the material that is covered in MA 51900 (discrete probability) as it was covered in DasGupta's class. We will, for the most part, reference Feller's *An introduction to probability theory and its applications, Volume 1* [4] (especially for the discrete noncalculus portion of the class) and DasGupta's own book *Fundamentals of Probability: A First Course* [2].

1.1 Counting

Chapter 2

Introduction to Partial Differential Equations

Here we summarize some important points about PDEs. The material is mostly taken from Evans's *Partial Differential Equations* [3] with occasional detours to Strauss's *Partial Differential Equations: An Introduction* [9]. We will be following Dr. Petrosyan's Course Log which can be found here `https://www.math.purdue.edu/~arshak/F16/MA523/courselog/`, i.e., summarizing the appropriate chapters from [3].

2.1 First-Order PDEs

The transport equation

In this section, we consider the simplest first-order PDE, the *transport equation* with constant coefficients, i.e., the PDE

$$u_t + b \cdot Du = 0 \qquad \text{in } \mathbb{R} \times (0, \infty), \tag{2.1}$$

where b is a fixed vector in $\mathbb{R}^n$, and $u \colon \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is the solution to the PDE. Our task is to find solutions u which satisfy the equation (2.1).

To address this task, let us suppose for a moment that we have a (smooth) solution u and try to compute it using the PDE (2.1). First, note that (2.1) asserts that the directional derivative $D_{(b,1)}u = 0$. Fix a point $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and define

$$z(s) := u(x + sb, t + s)$$

for $s \in \mathbb{R}$. Then

$$\dot{z}(s) = Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) = 0.$$

Thus, z is a constant function of s and, consequently for each (x, t), u is constant on the line through (x, t) with direction $(b, 1) \in \mathbb{R}^{n+1}$. Hence if we know the value of u at any point on each such line, we know its value everywhere in $\mathbb{R}^n \times (0, \infty)$.

Initial-value problem

Let's now look at the transport equation with *initial conditions*

$$\begin{cases} u_t + b \cdot Du = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{ t = 0 \}. \end{cases} \tag{2.2}$$

Here $b \in \mathbb{R}^n$ and $g \colon \mathbb{R}^n \to \mathbb{R}$ are known, and u is the unknown. Given (x, t), the line through (x, t) with direction $(b, 1)$ is represented parametrically by $(x + sb, t + s)$ for $s \in \mathbb{R}$. This line hits the plane $\Gamma := \mathbb{R}^n \times \{ t = 0 \}$ when $s = -t$, at the point $(x - tb, 0)$. Since u is constant on the line and $u(x - tb, 0) = g(x - tb)$, we deduce

$$u(x, t) = g(x - tb) \tag{2.3}$$

for $x \in \mathbb{R}^n$, $t \geq 0$. So if (2.2) has a sufficiently regular solution u (at least C^1), it must certainly be given by (2.3).

2.2 Characteristics

We now turn our attention to a very important method for solving first-order PDEs, the method of *characteristics*.

Derivation of characteristic ODEs

Consider the first-order (possibly non-linear) PDE

$$F(Du, u, x) = 0 \quad \text{in } U, \tag{2.4}$$

subject to the *boundary condition*

$$u = g \quad \text{on } \Gamma, \tag{2.5}$$

where $\Gamma \subset \partial U$ and $g \colon \Gamma \to \mathbb{R}$ are known. We shall assume, for simplicity, that F and g are smooth.

We now develop the *method of characteristics* to solve (2.4), (2.5) by converting the PDE into a system of ODEs. We proceed as follows: Suppose u solves (2.4), (2.5) and fix a point $x \in U$. We would like to calculate $u(x)$ by finding some curve lying within U, connecting x with a point $x^0 \in \Gamma$ and along which we can compute u. Since (2.5) says $u = g$ on Γ, we know the value of u at one end x^0. We hope to be able to calculate u all along the curve, and so in particular at x.

Finding the characteristic curve

But how do we choose a path in U so all of this will work? Suppose the curve is described parametrically by the function $\mathbf{x}(x) = (x^1(s), \ldots, x^n(s))$, the parameter s lying in some subinterval $I \subset \mathbb{R}$. Assuming u is a C^2 solution of (2.4), we define also

$$z(s) := u(\mathbf{x}(s)). \tag{2.6}$$

In addition, set

$$\mathbf{p}(s) := Du(\mathbf{x}(s)); \tag{2.7}$$

that is, $\mathbf{p}(s) = (p^1(s), \ldots, p^n(s))$, where

$$p^i(s) = u_{xi}(\mathbf{x}(s)), \qquad 1 \leq i \leq n.. \tag{2.8}$$

So $z(\,\cdot\,)$ gives us the values of u along the curve and $\mathbf{p}(\,\cdot\,)$ records the values of gradient Du. We must choose a function $\mathbf{x}(\,\cdot\,)$ that will allow us to compute $z(\,\cdot\,)$ and $\mathbf{p}(\,\cdot\,)$.

Differentiating (2.8), we have

$$\dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}(\mathbf{x}(s))\dot{x}^j(s). \tag{2.9}$$

But this expression is not too promising since it involves second order derivatives of u which we do not know (in fact, our solution need not be so regular as to have second order derivatives). On the other hand, if we differentiate (2.4) with respect to x_i, we have

$$\sum_{j=1}^{n} F_{p_j}(Du,u,x)u_{x_j x_i} + F_z(Du,u,x)u_{x_i} + F_{x_i}(Du,u,x) = 0. \tag{2.10}$$

We can use this identity to get rid of the second order derivatives in (2.12), provided we first set

$$\dot{x}^j(s) = F_{p_j}(\mathbf{p}(s),z(s),\mathbf{x}(s)), \qquad 1 \le j \le n. \tag{2.11}$$

Assuming (2.11) holds, we evaluate (2.10) at $x = \mathbf{x}(s)$, thereby obtaining from (2.6) and (2.7) the identity

$$\sum_{j=1}^{n} F_{p_j}(\mathbf{p}(s),z(s),\mathbf{x}(s))u_{x_i x_j}(\mathbf{x}(s)) + F_z(\mathbf{p}(s),z(s),\mathbf{x}(s))p^i(s) + F_{x_i}(\mathbf{p}(s),z(s),\mathbf{x}(s)) = 0$$

Finally, we differentiate (2.6) to give us

$$\begin{aligned} \dot{z}(s) &= \sum_{j=1}^{n} u_{x_j}(\mathbf{x}(s))\dot{x}^j(s) \\ &= \sum_{j=1}^{n} p^j(s)F_{p_j}(\mathbf{p}(s),z(s),\mathbf{x}(s)), \end{aligned} \tag{2.12}$$

the second equality holding by (2.8) and (2.9).

We summarize our results by rewriting equations (2.11), (2.11), and (2.12) as

$$\begin{cases} \text{(a) } \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s),z(s),\mathbf{x}(s)) - D_z F(\mathbf{p}(s),z(s),\mathbf{x}(s))\mathbf{p}(s), \\ \text{(b) } \dot{z}(s) = D_p F(\mathbf{p}(s),z(s),\mathbf{x}(s)) \cdot \mathbf{p}(s), \\ \text{(c) } \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s),z(s),\mathbf{x}(s)). \end{cases} \tag{2.13}$$

Furthermore,

$$F(\mathbf{p}(s),z(s),\mathbf{x}(s)) = 0. \tag{2.14}$$

These identities hold for $s \in I$.

The system (2.13) of $2n+1$ first order ODEs comprises the *characteristic equotions/ODEs* of the nonlinear first-order PDE (2.4). The functions $\mathbf{p}(\,\cdot\,)$, $\mathbf{x}(\,\cdot\,)$ are called *characteristics* and $\mathbf{x}(\,\cdot\,)$ is called the *projected characteristic* (it is the projection of the full characteristics $(\mathbf{p},z,\mathbf{x}) \subset \mathbb{R}^{2n+1}$ onto the physical region $U \subset \mathbb{R}^n$).

Theorem 2.1 (Structure of characteristic ODEs). *Let $u \in C^2(U)$ solve the nonlinear, first-order partial differential equation* (2.4) *in U. Assume $\mathbf{x}(\cdot)$ solves the ODE* (2.13)(c)*, where $\mathbf{p}(\cdot) = Du(\mathbf{x}(\cdot))$, $z(\,\cdot\,) = u(\mathbf{x}(\,\cdot\,))$. Then $\mathbf{p}(\,\cdot\,)$ solves the ODE* (2.13)(a) *and $z(\,\cdot\,)$ solves the ODE* (2.13)(b)*, for those s such that $\mathbf{x}(s) \in U$.*

We still need to discover appropriate initial conditions for (2.13) to be useful. We do that in the following section.

Examples

But before we move on, we look at some examples to show you how to use (2.13) to find solutions to (2.4).

The linear case

Suppose (2.4) is linear, i.e., has the form

$$F(D, u, x) = \mathbf{x} \cdot Du(x) + c(x)u(x) = 0, \qquad x \in U. \tag{2.15}$$

Then, rewriting (2.15) in terms of p, z, and x, we have$F(p, z, x) = \mathbf{b}(x) \cdot p + c(x)z$, so

$$D_p F = \mathbf{b}(x)$$

so (2.13)(c) becomes

$$\dot{\mathbf{x}}(s) = \mathbf{x}(\mathbf{x}(s)), \tag{2.16}$$

an ODE involving only the function $\mathbf{x}(\,\cdot\,)$. Furthermore (2.13)(b) becomes

$$\dot{z}(s) = \mathbf{b}(\mathbf{x}(s)) \cdot \mathbf{p}(s). \tag{2.17}$$

Then equation (2.14) simplifies (2.17), yielding

$$\dot{z}(s) = -c(\mathbf{x}(s))z(s).$$

This ODE is linear in $z(\,\cdot\,)$, noce we know the function $\mathbf{x}(\,\cdot\,)$ by solving (2.16). In summary, we have

$$\begin{cases} \text{(a) } \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)), \\ \text{(b) } \dot{z}(s) = -c(\mathbf{x}(s))z(s). \end{cases} \tag{2.18}$$

Example 2.2. Let's now look at a simple example to see how to use (2.18) to solve a PDE. Consider the PDE

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } U, \\ \qquad\qquad\quad u = g & \text{on } \Gamma, \end{cases} \tag{$*$}$$

where $U = \{\, x_1 > 0, x_2 > 0 \,\}$ and $\Gamma = \{\, x_1 > 0, x_2 = 0 \,\} \subset \partial U$. The PDE $(*)$ is of the form (2.15) with $\mathbf{b} = (-x_2, x_1)$ and $c = -1$. Thus the equations (2.18) read

$$\begin{cases} \dot{x}^1 = -x^2, \quad \dot{x}^2 = x^1, \\ \ \dot{z} = z. \end{cases} \tag{$**$}$$

Solving this system of ODEs we have

$$\begin{cases} x^1(s) = x^0 \cos s, & x^2(s) = x^0 \sin x, \\ z(s) = z^0 e^s = g(x^0)e^s, \end{cases}$$

where $x^0 \geq 0$, $0 \leq s \leq \pi/2$. Now, fix $(x_1, x_2) \in U$. Select $s > 0$, $x^0 > 0$ so that $(x_1, x_2) = (x^1(s), x^2(s)) = (x^0 \cos s, x^0 \sin s)$ and solve for x^0, in this case, $x^0 = \sqrt{x_1^2 + x_2^2}$, $s = \arctan(x_2/x_1)$, and therefore

$$\begin{aligned} u(x) &= u(x^1(s), x^2(s)) \\ &= z(s) \\ &= g(x^0)e^s \\ &= g\left(\sqrt{x_1^2 + x_2^2}\right)e^{\arctan(x_2/x_1)}. \end{aligned}$$

The quasilinear case

Let's look at the quasilinear case now, i.e., (2.4) with the form

$$F(Du, u, x) = \mathbf{b}(x, u(x)) \cdot Du(x) + c(x, u(x)) = 0. \tag{2.19}$$

In this circumstance $F(p, z, x) = \mathbf{b}(x, z) \cdot p + c(x, z)$, whence

$$D_p F = \mathbf{b}(x, z).$$

Hence equation (2.13)(c) reads

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s), z(s)), \tag{2.20}$$

an ODE involving only the function $\mathbf{x}$. Furthermore, (2.13)(c) becomes

$$\dot{z}(s) = \mathbf{b}(\mathbf{x}(s)) \cdot \mathbf{p}(s), \tag{2.21}$$

which, after applying (2.14), turns into

$$\dot{z}(s) = -c(\mathbf{x}(s))z(s). \tag{2.22}$$

In summary, we have

$$\begin{cases} \text{(a) } \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)), \\ \text{(b) } \dot{z}(s) = -c(\mathbf{x}(s))z(s). \end{cases} \tag{2.23}$$

We will see later that the equation for $\mathbf{p}(\cdot)$ is in fact not needed (at least in the linear and quasilinear cases).

Example 2.3. Let's look at an example of a quasilinear pde. Consider the PDE

$$\begin{cases} u_{x_2} + u_{x_1} = u & \text{in } U, \\ u = g & \text{on } \Gamma. \end{cases} \tag{$*$}$$

Here $U = \{ x_2 > 0 \}$ and $\Gamma = \{ x_2 = 0 \} = \partial U$ with $\mathbf{b} = (1,1)$ and $c = -z^2$. Thus, the equations (2.23) yield

$$\begin{cases} \dot{x}^1 = 1, \quad \dot{x}^2 = 1, \\ \dot{z} = z^2. \end{cases}$$

Consequently

$$\begin{cases} x^1(s) = x^0 + s, & x^2(s) = s, \\ z(s) = \dfrac{z^0}{1 - sz^0} = \dfrac{g(x^0)}{1 - sg(x^0)}, \end{cases}$$

where $x^0 \in \mathbb{R}$, $s \geq 0$, provided the denominator is not zero.

Chapter 3

Algebraic Geometry

A summary to a course on an introduction to sheaf cohomology. We will mostly reference Donu's notes available here `https://www.math.purdue.edu/~dvb/classroom.html`, but also cite Ravi Vakil's *Fundamentals of Algebraic Geometry* [10] available here `https://math216.wordpress.com/`.

3.1 The statement of de Rham's theorem

These are almost verbatim Arapura's notes on the de Rham Complex and cohomology.

Before doing anything fancy, let's start at the beginning. Let $U \subseteq \mathbb{R}^3$ be an open set. In calculus class, we learn about operations

$$\{\,\text{functions}\,\} \xrightarrow{\nabla} \{\,\text{vector fields}\,\} \xrightarrow{\nabla\times} \{\,\text{vector fields}\,\} \xrightarrow{\nabla\cdot} \{\,\text{functions}\,\}$$

such that $(\nabla\times)(\nabla) = 0$ and $(\nabla\cdot)(\nabla\times) = 0$. This is a prototype for a *complex*. An obvious question: does $\nabla \times v = 0$ imply that v is a gradient? Answer: sometimes yes (e.g. if $U = \mathbb{R}^3$) and sometimes no (e.g. if $U = \mathbb{R}^3$ minus a line). To quantify the failure we introduce the first de Rham cohomology

$$H^1_{\mathrm{dR}}(U) = \frac{\{\, v \text{ a vector field on } U : \nabla \times v = 0 \,\}}{\{\, \nabla f \,\}}.$$

Contrary to first appearances, for reasonable U this is finite dimensional and computable. This follows from the de Rham's theorem, which we now explain. First, let's generalize this to an open set $U \subset \mathbb{R}^n$. Once $n > 3$ vector calculus is useless, but there is a good replacement. A differential form of degree p, or p-form, is an expression

$$\alpha = \sum f_{i_1,\dots,i_p}(x_1,\dots,x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

such that the x_i are coordinates, the f are C^∞ functions, $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ are symbols where $\wedge$ is an anticommutative product. Let $\mathcal{E}^p(U)$ denote the vector space of p-forms. Define the exterior derivative by

$$d\alpha = \sum \sum_j \frac{\partial f_{i_1,\dots,i_p}}{\partial x_j}\, dx_j \wedge \cdots \wedge dx_{i_p}.$$

This is a $(p+1)$-form.

Lemma 3.1. $d^2 = 0$.

Proof. We prove it for $p = 0$. In this case, we have

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$
$$d(df) = \sum_{i,j} \sum \frac{\partial^2}{\partial x_j \partial x_i} dx_j \wedge dx_i.$$

Using anticommutativity, we can rewrite this as

$$\sum_{j<i} \left(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dx_j \wedge dx_i = 0.$$

∎

A cochain complex is a collection of Abelian groups M^i and homomorphisms $d\colon M^i \to M^{i+1}$ such that $d^2 = 0$. We define the p^{th} cohomology of this by

$$H^p_{\text{dR}}(M^\bullet, d) = \frac{\operatorname{Ker} d\colon M^p \to M^{p+1}}{\operatorname{Im} d\colon M^{p-1} \to M^p}.$$

So we have an example of a complex $(\mathcal{E}^\bullet(U), d)$ called the de Rham complex of U. It's cohomology is the de Rham cohomology $H^p_{\text{dR}}(U) = H^p\big(\mathcal{E}^\bullet(U), d\big)$. Here is a basic computation.

Theorem 3.2 (Poincaré's lemma)**.**

$$H^p_{\text{dR}}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \textit{if } p = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We show this for $n \leq 2$. We first treat the case $n = 1$. Clearly $H^p_{\text{dR}}(\mathbb{R})$ consists of constant functions. If $\alpha = f(x)\,dx$, then

$$d\left(\int_0^x f(t)\,dt \right) = \alpha.$$

There are no p-forms for $p > 1$.

Next, we treat $n = 2$ which contains all of the ideas of the general case. Let x, y be coordinates. We define some operators

$$\begin{tikzcd} \mathcal{E}^\bullet(\mathbb{R}^2) \arrow[r, bend left, "s*"] & \mathcal{E}^\bullet(\mathbb{R}), \arrow[l, bend left, "\pi^*"] \end{tikzcd}$$

where π^* is the pullback along the projection $\mathbb{R}^2 \to \mathbb{R}$. It takes a form in x and treats it as a form in x, y. The pullback along the zero section s^* sets y and dy to zero. Note that $s^* \circ \pi^*$ is the identity. Although $\pi^* \circ s^*$ is not the identity, we will show that it induces the identity on cohomology. This will show that $H^*_{\text{dR}}(\mathbb{R}^2) \cong H^*_{\text{dR}}(\mathbb{R})$, which is all we need. This involves a new concept. We introduce an operator $H\colon \mathcal{E}^p(\mathbb{R}^2) \to \mathcal{E}^{p-1}(\mathbb{R}^2)$ of degree -1 called a *homotopy*. It integrates y as follows:

$$\begin{aligned} H\big(f(x,y)\big) &= 0 \\ H\big(f(x,y)\,dx\big) &= 0 \\ H\big(f(x,y)\,dy\big) &= \int_0^y f(x,t)\,dt \\ H\big(f(x,y)\,dx \wedge dy\big) &= \left[\int_0^y f(x,t)\,dt\right] dx. \end{aligned}$$

A computation using nothing more than the fundamental theorem of calculus shows that

$$1 - \pi^* s^* = \pm(Hd - dH).$$

This implies that the left side induces 0 on $H^*_{\text{dR}}(\mathbb{R}^2)$, or equivalently $\pi^* \circ s^*$ acts like the identity on cohomology. ■

Before describing de Rham's theorem, we have to say what's happening at the othe rend. The standard n dimensional simplex, or n-simplex, $\Delta^n \subset \mathbb{R}^{n+1}$ is the convex hull of the unit vectors $(1,0,\dots,0), (0,1,0,\dots,0), \dots$. The convex hull of the subset of these is called a face. This is homeomorphic to a simplex of smaller dimension. Omitting all but the i^{th} vertex is called the i^{th} face of Δ^n. We have a standard homeomorphism

$$\delta_i\colon \Delta^{n-1} \to i^{\text{th}} \text{ face of } \Delta^n.$$

A geometric simplicial complex is given by a collection of simplices glued along faces. Historically, the first cohomology theory was defined for simplicial complexes. A bit later singular cohomology was developed, which is an arbitrary topological space X. A (real/complex) singular p-cochain α is an integer (real/complex) valued function on the set of all continuous maps $f\colon \Delta^p \to X$. It might help to think of $\alpha(f)$ as a combinatorial integral $\int_f \alpha$. Let $S^p(X)$ ($S^p(X,\mathbb{R})$, $S^p(X,\mathbb{C})$) denote the group of these cochains. Define $\delta\colon S^p(X) \to S^{p+1}(X)$ by

$$\delta(\alpha)(f) = \sum (-1)^i \alpha(f \circ \delta_i).$$

Lemma 3.3.

$$\delta^2 = 0.$$

For $p = 0$. Let $\alpha \in S^0$. Fix $f\colon \Delta^2 \to X$. Label the restriction of f to the vertices by $0, 1, 2$ and faces $01, 02, 12$. Then

$$\begin{aligned} \delta^2(f) &= \delta\alpha(12) - \delta\alpha(02) + \delta\alpha(01) \\ &= \alpha(1) - \alpha(2) - \alpha(0) + \alpha(2) + \alpha(0) - \alpha(1) \\ &= 0. \end{aligned}$$

■

Thus we have a complex. Singular cohomology is defined by $H^p(X,\mathbb{Z}) = H^p(S^\bullet(X), \delta)$, and similarly for real or complex valued singular cohomology. These groups are highly computable.

Theorem 3.4 (de Rham). *If $X \subset \mathbb{R}^n$ is open, or more generally a manifold, then $H^p_{\mathrm{dR}}(X, \mathbb{R}) \cong H^p(X, \mathbb{R})$ for all p.*

We will give a proof of this later on as an easy application of sheaf theory. Sheaf methods will help obtain parallel theorems.

Theorem 3.5 (Holomorphic de Rham). *If $X \subset \mathbb{C}^n$ is a complex manifold, then $H^p(X, \mathbb{C})$ can be computed using algebraic differential forms.*

The last theorem is due to Grothendieck. The proof is a lot harder, so we'll try to give the proof by the end of the semester, but there's no guarantee.

3.2 A crash course in homological algebra

By the 1940s techniques from algebraic topology began to be applied to pure algebra, giving rise to a new subject. To begin with, recall that a category $\mathscr{C}$ consists of a set or class of objects (e.g., sets, groups, topological spaces) and morphisms (e.g., functions, homomorphisms, continuous maps) between pairs of objects $\mathrm{Hom}_{\mathscr{C}}(A, B)$. We require an identity $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ for each object A, and associative composition law.

In this section, we will focus on one particular example. Let R be an associative (but possibly noncommutative ring) with identity 1, and let R-**Mod** be the category of left R-modules and homomorphisms. We write $\mathrm{Hom}_R(\,\cdot\,,\,\cdot\,)$ for the morphisms. It is worth noting that $\mathbb{Z}$-**Mod** is the category of Abelian groups. These categories have the following features:

1. $\mathrm{Hom}_R(\,\cdot\,,\,\cdot\,)$ is an Abelian group, and composition is distributive.
2. There is a zero object 0 such that $\mathrm{Hom}_R(0, M) = \mathrm{Hom}_R(M, 0) = 0$.
3. Every pair of objects A, B has a direct sum $A \oplus b$ characterized by certain universal properties.
4. Morphisms have kernels and images, characterized by the appropriate universal properties.

We will encounter other categories satisfying these conditions later on. Such categories are called Abelian. We have been a bit vague about the precise axioms; se Weibel's Homological Algebra for this.

Diagram Chasing

A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called *exact* if $\mathrm{Ker}\, g = \mathrm{Im}\, f$. A useful skill in this business is to be able to prove things by diagram chasing.

Exercise 3.1. Given a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0,
\end{array}$$

show that g is an isomorphism if f and h are isomorphisms.

Solution. ■

Theorem 3.6 (Snake lemma)**.** *If*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h \xrightarrow{\partial} \operatorname{Coker} f \longrightarrow \operatorname{Coker} g \longrightarrow \operatorname{Coker} h.$$

3.3 Hom Functors

A (*covariant*) *functor* F from one category to another is a function taking objects to objects and morphisms to morphisms such that if $f\colon A \to B$ then $F(f)\colon F(A) \to F(B)$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, and $F(f \circ g) = F(f) \circ F(g)$. A contravariant functor reverses direction in the sense that $F(f)\colon F(B) \to F(A)$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, and $F(f \circ g) = F(g) \circ F(f)$. Here are two basic examples: If $M \in R$-**Mod**, then $F(\,\cdot\,) = \mathrm{Hom}_R(M, \,\cdot\,)$ is a covariant functor from R-**Mod** to $\mathbb{Z}$-**Mod**.

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
M \arrow[u, "g"] \arrow[ur, dashed, "F(f)=f\circ g"'] &
\end{tikzcd}$$

When R is commutative, $F(\,\cdot\,)$ is naturally an R-module, but not otherwise. Similarly, $\mathrm{Hom}_R(\,\cdot\,, M)$ is a contravariant functor from R-**Mod** to $\mathbb{Z}$-**Mod** (or R-**Mod**) when R is commutative).

Lemma 3.7. *Suppose that*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact. Then

(a)
$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C),$$

(b)
$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M)$$

are both exact.

The proof is straight forward and will be omitted.

Exercise 3.2. Prove the lemma.

Exercise 3.3. Prove that

$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C),$$

and

$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M)$$

are exact when the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is split exact. This means that there exists a map $s\colon C \to B$, called a splitting, such that $p \circ s = \mathrm{id}_C$.

A (contravariant) functor is called exact if it preserves exact sequences. The lemma says that the Hom functors have the weaker property left exactness. They are not exact, in general:

Example 3.8. Let $R = \mathbb{Z}$, $M = \mathbb{Z}/2$. Note that $\mathrm{Hom}(M, \mathbb{Z}) = 0$ and $\mathrm{Hom}(M, M) = \mathbb{Z}/2$. So $\mathrm{Hom}(M, \cdot\,)$ applied to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

yields the sequence

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2.$$

The last map is certainly not onto.

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