MA52300 Fall 2016

Homework Assignment 8

Due Wed, Nov 9

1. Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t), \quad \text{where} \quad \Phi(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

is positive for $|x| < 1$, $t > 0$.

Hint: Show that u satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{aligned} u &= 0 \quad \text{on} \quad \{|x| = 1\} \times \{t \geq 0\} \\ u &= \delta_0 \quad \text{on} \quad \{|x| \leq 1\} \times \{t = 0\} \end{aligned}$$

Then, carefully apply maximum (minimum) principle in a domain $\{|x| \leq 1\} \times \{\epsilon \leq t \leq T\}$ for small $\epsilon > 0$ and large $T > 0$ and pass to the limit as $\epsilon \to 0+$ and $T \to \infty$.

2. (Tikhonov's example) Let

$$g(t) := \begin{cases} \exp\left(-t^{-2}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}.$$

Then $g \in C^\infty(\mathbb{R})$ and we define

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that $u(x,t)$ solves the heat equation in $\mathbb{R} \times (0,\infty)$ with the initial condition $u(x,0) = 0$, $x \in \mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

3. Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax)\, e^{-x^2} dx \qquad (a > 0).$$

Hint: Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.