

MA 519: Homework 4

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PROBLEM 4.1 (HANDOUT 5, # 2)

In an urn, there are 12 balls. 4 of these are white. Three players: A , B , and C , take turns drawing a ball from the urn, in the alphabetical order. The first player to draw a white ball is the winner. Find the respective winning probabilities: assume that at each trial, the ball drawn in the trial before is put back into the urn (i.e., selection *with replacement*).

SOLUTION. Denote the events that player A wins, player B wins, and player C wins by A , B , and C respectively.

Note that A wins if some multiple of 3 losses occurs, followed by a win. Also, B wins if some multiple of three losses occurs, followed by a loss, then a win. Last, C wins if some multiple of three losses occurs, followed by two losses then a win.

A win occurs with probability $1/3$ each time, and a loss occurs with probability $2/3$ each time. Thus, because each draw is independent,

$$P(A) = \sum_{i=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{3i}, \quad P(B) = \sum_{i=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{3i+1}, \quad P(C) = \sum_{i=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{3i+2}.$$

That is,

$$P(A) = \sum_{i=0}^{\infty} \frac{9}{19} \approx 0.47, \quad P(B) = \sum_{i=0}^{\infty} \frac{6}{19} \approx 0.32, \quad P(C) = \sum_{i=0}^{\infty} \frac{4}{19} \approx 0.21.$$

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PROBLEM 4.2 (HANDOUT 5, # 8)

Consider n families with 4 children each. How large must n be to have a 90% probability that at least 3 of the n families are all girl families?

SOLUTION. The probability that a family has all girls is $(0.5)^4$.

In n families, the probability that at least 3 are all-girl is $1 - P(0) - P(1) - P(2)$, where $P(m)$ is the probability that exactly m families are all-girl.

Note that

$$\begin{aligned} P(0) &= (1 - (0.5)^4)^n \\ P(1) &= \binom{n}{1} (0.5)^4 (1 - (0.5)^4)^{n-1} \\ P(2) &= \binom{n}{2} (0.5)^8 (1 - (0.5)^4)^{n-2} \end{aligned}$$

(we choose a number of families to be all-girl, and then find the probability that that family is all-girl, while all of the rest are not all-girl).

So that the probability that at least 3 are all girl is

$$1 - (1 - (0.5)^4)^n - n(0.5)^4 (1 - (0.5)^4)^{n-1} - n(n-1)(0.5)^8 (1 - (0.5)^4)^{n-2}.$$

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PROBLEM 4.3 (HANDOUT 5, # 10)

(*Yahtzee*). In Yahtzee, five fair dice are rolled. Find the probability of getting a Full House, which is three rolls of one number and two rolls of another, in Yahtzee.

SOLUTION. There are 30 different kinds of full house (6 different three of a kinds, and 5 different kinds of different two of a kind).

The probability of rolling a specific kind of full house is

$$\binom{5}{3} \left(\frac{1}{6}\right)^3 \left(\frac{1}{6}\right)^2$$

(Choose 3 dice to be the three of a kind, have them all rolled the specific number, have the other two dice rolled the other specific number.)

So the probability of rolling some kind of full house is

$$30 \binom{5}{3} \left(\frac{1}{6}\right)^3 \left(\frac{1}{6}\right)^2 = \frac{25}{648} \approx 0.039.$$

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PROBLEM 4.4 (HANDOUT 5, # 12)

The probability that a coin will show all heads or all tails when tossed four times is 0.25. What is the probability that it will show two heads and two tails?

SOLUTION. Let H denote the event that, on a given coin toss, that coin is heads, and let T denote the event that, on a given coin toss, that coin is tails. Then, because each coin toss is independent, this says that

$$P(H)^4 + P(T)^4 = \frac{1}{4}.$$

Moreover, because this is a coin,

$$P(H) + P(T) = 1.$$

So,

$$\begin{aligned} P(H)^4 + (1 - P(H))^4 &= \frac{1}{4} \\ 2P(H)^4 - 4P(H)^3 + 6P(H)^2 - 4P(H) + 1 &= \frac{1}{4} \end{aligned}$$

We can approximate a solution to the above by $P(H) \approx 0.299$ or by $P(H) \approx 0.701$.

Now, the probability that two coin flips of four are heads is

$$\binom{4}{2} P(H)^2 P(T)^2 \approx 0.263.$$

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PROBLEM 4.5 (HANDOUT 5, # 13)

Let the events A_1, A_2, \dots, A_n be independent and $P(A_k) = p_k$. Find the probability p that none of the events occurs.

SOLUTION. Let the events A_1, A_2, \dots, A_n be independent. Then the events $A_1^\sim, A_2^\sim, \dots, A_n^\sim$ are independent. (Where A denotes the event that A does *not* happen, i.e., $A^\sim = \Omega \setminus A$). The events A_i^\sim each have probability $1 - p_i$ of occurring. Thus, the probability, p , that none of the events occur is

$$p = \prod_{k=1}^n (1 - p_k)$$

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PROBLEM 4.6 (HANDOUT 6, # 5)

Suppose a fair die is rolled twice and suppose X is the absolute value of the difference of the two rolls. Find the PMF and the CDF of X and plot the CDF. Find a median of X ; is the median unique?

SOLUTION. First, we compute the probability mass function. Note that X is an integer between 0 and 5, so computing the probabilities that X is each of 0 through 5 suffices to describe the PMF of X .

We calculate the probabilities, by counting.

$$\begin{aligned} P(0) &= \frac{6}{36} = \frac{1}{6} \\ P(1) &= \frac{10}{36} = \frac{5}{18} \\ P(2) &= \frac{8}{36} = \frac{2}{9} \\ P(3) &= \frac{6}{36} = \frac{1}{6} \\ P(4) &= \frac{4}{36} = \frac{1}{9} \\ P(5) &= \frac{2}{36} = \frac{1}{18} \end{aligned}$$

We calculate the CDF by summing.

$$\begin{aligned} \text{CDF}(0) &= \frac{6}{36} = \frac{1}{6} \\ \text{CDF}(1) &= \frac{16}{36} = \frac{4}{9} \\ \text{CDF}(2) &= \frac{24}{36} = \frac{2}{3} \\ \text{CDF}(3) &= \frac{30}{36} = \frac{5}{6} \\ \text{CDF}(4) &= \frac{34}{36} = \frac{17}{18} \\ \text{CDF}(5) &= \frac{36}{36} = 1 \end{aligned}$$

The median is 2. ■

PROBLEM 4.7 (HANDOUT 6, # 7)

Find a discrete random variable X such that $E(X) = E(X^3) = 0$; $E(X^2) = E(X^4) = 1$.

SOLUTION. Set $\Omega = \{0, 1\}$ and define a random variable $X: \Omega \rightarrow \mathbb{R}$ by $X(0) = -1$, $X(1) = 1$ as well as a probability $P(0) = P(1) = 1/2$. Then

$$E(X) = -1(1/2) + 1(1/2) = 0 = (-1)^3(1/2) + 1^3(1/2) = E(X^3),$$

whereas

$$E(X^2) = (-1)^2(1/2) + 1^2(1/2) = 1 = (-1)^4(1/2) + 1^4(1/2) = E(X^4),$$

as desired. ■

PROBLEM 4.8 (HANDOUT 6, # 9)

(Runs). Suppose a fair die is rolled n times. By using the indicator variable method, find the expected number of times that a six is followed by at least two other sixes. Now compute the value when $n = 100$.

SOLUTION. Let Ω denote the sample space and let A denote the event that in a sequence of n rolls of a die, a subsequence of sixes of length at least three occurs. Define a random variables $X_i : \Omega \rightarrow \mathbb{R}$, $1 \leq i \leq n-2$, as the indicator variable that there are three or more consecutive 6s starting at the i^{th} roll. Then,

$$E(X_1 + \cdots + X_{n-2}) = E(X_1) + \cdots + E(X_{n-2}) = p_1 + \cdots + p_{n-2}.$$

We need to find these p_i . There are 6^n points in the sample space Ω . Starting at the i^{th} place, we can have exactly 6^{n-3} ways of choosing the remaining terms in the sequence. Thus,

$$p_i = \frac{6^{n-3}}{6^n} = \frac{1}{6^3}.$$

Then,

$$E(X_1 + \cdots + X_{n-2}) = (n-2) \left(\frac{1}{6^3} \right).$$

For $n = 100$, this value is

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PROBLEM 4.9 (HANDOUT 6, # 10)

(*Birthdays*). For a group of n people find the expected number of days of the year which are birthdays of exactly k people. (Assume 365 days and that all arrangements are equally probable.)

SOLUTION. Let Ω denote the sample space and let A denote the event that exactly k people share a birthday. Define a random variable $X_i: \Omega \rightarrow \mathbb{R}$, $1 \leq i \leq 365$, as the indicator variable that the i^{th} day of the year is the birthday of exactly k people. Then,

$$E(X_1 + \cdots + X_{365}) = E(X_1) + \cdots + E(X_{365}) = 365p,$$

where p is the probability that exactly k people have a given day of the year as their birthday.

The latter probability is not difficult to compute. There are exactly $\binom{n}{k}$ ways to choose k people having the 1st of January as their birthday and $(365 - 1)^{n-k} = 364^{n-k}$ choices for we can assign to the remaining $n - k$ people and of course, the sample space has cardinality $\#\Omega = 365^n$. Thus, the probability that exactly k people have the 1st of January as their birthday is

$$P = \frac{\binom{n}{k} 364^{n-k}}{365^n}.$$

Thus,

$$E[X] = 365 \left(\frac{\binom{n}{k} 364^{n-k}}{365^n} \right) = \frac{\binom{n}{k} 364^{n-k}}{365^{n-1}}.$$

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PROBLEM 4.10 (HANDOUT 6, # 11)

(Continuation). Find the expected number of multiple birthdays. How large should n be to make this expectation exceed 1?

SOLUTION. For any given person, the probability that the other $n-1$ people do not share a birthday with them is

$$\frac{\binom{365}{1}(365-1)^{n-1}}{365^n} = \frac{365 \cdot 364^{n-1}}{365^n} = \left(\frac{364}{365}\right)^{n-1}.$$

Thus, the probability that any given person shares a birthday with somebody else is

$$p = 1 - \left(\frac{364}{365}\right)^{n-1}.$$

Now, define a random variable $X_i: \Omega \rightarrow \mathbb{R}$, $1 \leq i \leq n$, the indicator random variable that the i^{th} person shares a birthday. Then

$$\begin{aligned} E(X_1 + \cdots + X_n) &= E(X_1) + \cdots + E(X_1) \\ &= np \\ &= n \left(1 - \left(\frac{364}{365}\right)^{n-1} \right). \end{aligned}$$

For the expectation to exceed 1, we must have

$$\begin{aligned} 1 &< n \left(1 - \left(\frac{364}{365}\right)^{n-1} \right) \\ \frac{1}{n} &< 1 - \left(\frac{364}{365}\right)^{n-1} \\ \frac{365^n}{n} &< 365^n - 364^n. \end{aligned}$$

Experimentally, this number seems to be 20. ■

PROBLEM 4.11 (HANDOUT 6, # 12)

(*The blood-testing problem*). A large number, N , of people are subject to a blood test. This can be administered in two ways, (i) Each person can be tested separately. In this case N tests are required, (ii) The blood samples of k people can be pooled and analyzed together. If the test is negative, this one test suffices for the k people. If the test is positive, each of the k persons must be tested separately, and in all $k + 1$ tests are required for the k people. Assume the probability p that the test is positive is the same for all people and that people are stochastically independent.

- (b) What is the expected value of the number, X , of tests necessary under plan (ii)?
- (c) Find an equation for the value of k which will minimize the expected number of tests under the second plan. (Do not try numerical solutions.)

SOLUTION. ■

PROBLEM 4.12 (HANDOUT 6, # 13)

(*Sample structure*). A population consists of r (classes whose sizes are in the proportion $p_1 : p_2 : \cdots : p_r$). A random sample of size n is taken with replacement. Find the expected number of classes not represented in the sample.

SOLUTION. Let A_i be the event that the i^{th} class is not represented in the sample of size n . Let A be the number of classes not represented in the sample of size n . Let A_i also represent the number of times that the i^{th} class is not represented. Then

$$\begin{aligned} E(A) &= \sum_{i=1}^r E(A_i) \\ &= \sum_{i=1}^r P(A_i) \\ &= \sum_{i=1}^r (1 - p_i)^n \end{aligned}$$

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