

# MA571: Qual Preparation

Carlos Salinas

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# Chapter 1

## Gepner

### 1.1 Gepner's homework

#### Homework 1

**Exercise 1.1.** Let  $\{X_i : i \in I\}$  be an  $I$ -indexed family of topological spaces. Show that the Cartesian product

$$X = \prod_{i \in I} X_i,$$

equipped with the product topology, has the property that for each  $i \in I$  the projection  $\pi_i : X \rightarrow X_i$  is continuous, and moreover, that  $X$  has the following universal property: for any other topological space  $Y$ , the function

$$\mathrm{Hom}_{\mathbf{Top}}(Y, X) \longrightarrow \prod_{i \in I} \mathrm{Hom}_{\mathbf{Top}}(Y, X_i),$$

induced by the projections  $\pi_i : X \rightarrow X_i$ , is a bijection.

**Solution.** ■

**Exercise 1.2.** Let  $X$  be the set equipped with a topology and let  $\{\mathcal{U}_i : i \in I\}$  a family of topologies on  $X$ . Show that

$$\mathcal{U} = \bigcap_{i \in I} \mathcal{U}_i$$

is a topology on  $X$ . Show that if  $\mathcal{B}$  is a basis for a topology on  $X$ , then the topology  $\mathcal{U}$  on  $X$  generated by  $\mathcal{B}$  is the intersection of all topologies on  $X$  which contain  $\mathcal{B}$ , and that this holds even if we only require that  $\mathcal{B}$  be a subbasis.

**Solution.** ■

**Exercise 1.3.** A topological space  $X$  is said to be *Hausdorff* if, for every pair of points  $x_0, x_1 \in X$  with  $x_0 \neq x_1$ , there exists open subsets  $U_0, U_1$  of  $X$  such that  $x_0 \in U_0$ ,  $x_1 \in U_1$ , and  $U_0 \cap U_1 = \emptyset$ . Show that a topological space  $X$  is Hausdorff if and only if the diagonal inclusion  $X \rightarrow X \times X$  is closed.

**Solution.** ■

**Exercise 1.4.** Let  $X$  be a topological space and let  $Y \subseteq X$  be a subset of  $X$ . Show that if  $Y$  is equipped with the subspace topology then the inclusion function  $\iota: Y \rightarrow X$  is continuous. Show that if there exists a continuous function  $q: X \rightarrow Y$  such that  $q \circ \iota = \text{id}_Y$  then  $q$  is a quotient map (that is,  $Y$  is also a quotient topology). Give an example of such a situation.

**Solution.** ■

**Exercise 1.5.** A *topological group* is a group  $G$  with a topology  $\mathcal{U}$  such that the multiplication  $\mu: G \times G \rightarrow G$  and inversion  $\iota: G \rightarrow G$  are continuous (it is standard to also assume that the topology  $\mathcal{U}$  on  $G$  is Hausdorff, which we shall do). Let  $H$  be a subgroup of  $G$ , and let  $G/H$  denote the quotient of  $G$  by the action of  $H$ , equipped with the quotient topology. Show that  $G/H$  is a homogeneous space and that the quotient map  $q: G \rightarrow G/H$  is open. If, moreover,  $H$  is a closed subset of  $G$ , show that  $G/H$  has the property that points are closed. Finally, show that if  $H$  is a normal subgroup of  $G$ , then  $G/H$  is a topological group. (Optional: is it Hausdorff?)

**Solution.** ■

## 1.2 Homework 2