Math 527 - Homotopy Theory
Spring 2013
Homework 9 Solutions

Problem 1. Consider the Hopf map $\eta \colon S^3 \to S^2$.

a. Describe the cofiber $C(\eta)$. It is a familiar space.

Solution. Recall that $\eta \colon S^3 \twoheadrightarrow \mathbb{C}P^1$ was defined as the quotient by the action of $S^1 \subset \mathbb{C}^\times$ (via scalar multiplication) on the unit sphere $S^3 \subset \mathbb{C}^2 \setminus \{0\}$. Its cofiber $C(\eta)$ is obtained by attaching a 4-cell to $\mathbb{C}P^1$ using η as attaching map. This CW structure on $C(\eta)$ is the standard CW structure on $\mathbb{C}P^2$. $\square$

b. Consider the canonical comparison map $\varphi \colon F(\eta) \to \Omega C(\eta)$ from the homotopy fiber to the loop space of the cofiber. Find the lowest dimension k such that $\pi_k F(\eta)$ is not isomorphic to $\pi_k \Omega C(\eta)$ (and thus φ cannot possibly induce an isomorphism on π_k).

Solution. Since η is a fibration, its homotopy fiber $F(\eta)$ is homotopy equivalent to its strict fiber $\eta^{-1}(*) = S^1$. Thus the comparison map φ can be viewed as a map

$$\varphi \colon S^1 \to \Omega \mathbb{C}P^2.$$

Recall the natural isomorphism $\pi_i(\Omega X) \cong \pi_{i+1}(X)$ for all $i \geq 0$. To study the homotopy groups of $\mathbb{C}P^2$, note that the long exact sequence on homotopy of the fibration

$$S^1 \to S^5 \xrightarrow{q} \mathbb{C}P^2$$

yields the isomorphisms

$$\partial \colon \pi_2(\mathbb{C}P^2) \xrightarrow{\cong} \pi_1(S^1) \cong \mathbb{Z}$$
$$q_* \colon \pi_i(S^5) \xrightarrow{\cong} \pi_i(\mathbb{C}P^2)$$

for all $i \geq 3$. In particular, we obtain:

$$\pi_0(\Omega\mathbb{C}P^2) \cong \pi_1(\mathbb{C}P^2) = 0$$
$$\pi_1(\Omega\mathbb{C}P^2) \cong \pi_2(\mathbb{C}P^2) \cong \mathbb{Z}$$
$$\pi_2(\Omega\mathbb{C}P^2) \cong \pi_3(\mathbb{C}P^2) \cong \pi_3(S^5) = 0$$
$$\pi_3(\Omega\mathbb{C}P^2) \cong \pi_4(\mathbb{C}P^2) \cong \pi_4(S^5) = 0$$
$$\pi_4(\Omega\mathbb{C}P^2) \cong \pi_5(\mathbb{C}P^2) \cong \pi_5(S^5) \cong \mathbb{Z}.$$

Comparing with the homotopy groups of S^1, we conclude $\pi_i(S^1) \simeq \pi_i(\Omega\mathbb{C}P^2)$ for all $i < 4$ whereas $\pi_4(S^1) \not\simeq \pi_4(\Omega\mathbb{C}P^2)$. $\square$

Problem 2. Show that a (strictly) commutative diagram

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is homotopy k-Cartesian if and only if for all $x \in X$, the induced map on homotopy fibers

$$f''\colon F_x(g') \to F_{f(x)}(g)$$

over the respective basepoints $x \in X$ and $f(x) \in Z$ is k-connected. Here we have $k \geq 0$ or $k = \infty$.

Solution. Consider the comparison map φ from W to the homotopy pullback

$$\begin{tikzcd} W \arrow[dr, "\varphi"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "g'"'] & & \\ & X \times^h_Z Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z. \end{tikzcd}$$

Because the homotopy pullback square induces a homotopy equivalence on homotopy fibers $F(p_X) \xrightarrow{\simeq} F(g)$, the condition on homotopy fibers in the statement is equivalent to the following: The map on homotopy fibers $\varphi''\colon F(g') \to F(p_X)$ induced by the diagram

$$\begin{tikzcd} W \arrow[r, "\varphi"] \arrow[d, "g'"'] & X \times^h_Z Y \arrow[d, "p_X"] \\ X \arrow[r, "\mathrm{id}"'] & X \end{tikzcd}$$

is k-connected. For any basepoint $x \in X$ (which we omit from the notation), consider the diagram

$$\begin{tikzcd} F(g') \arrow[r, "\varphi''"] \arrow[d] & F(p_X) \arrow[d] \\ W \arrow[r, "\varphi"] \arrow[d, "g'"'] & X \times^h_Z Y \arrow[d, "p_X"] \\ X \arrow[r, "\mathrm{id}"'] & X \end{tikzcd} \tag{1}$$

where both columns are fiber sequences. Now consider the induced map of long exact sequences on homotopy

$$\begin{tikzcd}
\dots \arrow[r] & \cdot \arrow[r] \arrow[d, "\pi_{i+1}(\varphi)"] & \cdot \arrow[r] \arrow[d, "\simeq"] & \cdot \arrow[r] \arrow[d, "\pi_i(\varphi'')"] & \cdot \arrow[r] \arrow[d, "\pi_i(\varphi)"] & \cdot \arrow[r] \arrow[d, "\simeq"] & \cdot \arrow[r] \arrow[d, "\pi_{i-1}(\varphi'')"] & \dots \\
\dots \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \dots
\end{tikzcd}$$

Recall the four-lemma for epimorphisms, schematically summarized here:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, "\therefore \text{epi}"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, hook] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

This implies the following:

1. If $\pi_i(\varphi)$ is an epimorphism, then $\pi_i(\varphi'')$ is an epimorphism.
2. If $\pi_i(\varphi'')$ is an epimorphism and $\pi_{i-1}(\varphi'')$ is a monomorphism, then $\pi_i(\varphi)$ is an epimorphism.

Now recall the four-lemma for monomorphisms, schematically summarized here:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, hook] & \cdot \arrow[r] \arrow[d, "\therefore \text{mono}"] & \cdot \arrow[d, hook] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

This implies the following:

1. If $\pi_i(\varphi)$ is an monomorphism and $\pi_{i+1}(\varphi)$ is an epimorphism, then $\pi_i(\varphi'')$ is a monomorphism.
2. If $\pi_i(\varphi'')$ is a monomorphism, then $\pi_i(\varphi)$ is a monomorphism.

From both statements 1., we deduce: If φ is k-connected, then φ'' is k-connected.

From both statements 2., we deduce: If φ'' is k-connected, then φ is k-connected.

Therefore, φ is k-connected if and only if φ'' is k-connected. $\square$

Alternate solution. Taking horizontal fibers of the two bottom rows in (1) yields the diagram

$$\begin{tikzcd}
 & F(g') \arrow[r, "\varphi''"] \arrow[d] & F(p_X) \arrow[d] \\
F(\varphi) \arrow[r] \arrow[d, "g''"'] & W \arrow[r, "\varphi"] \arrow[d, "g'"'] & X \times^h_Z Y \arrow[d, "p_X"] \\
* \simeq F(\mathrm{id}) \arrow[r] & X \arrow[r, "\mathrm{id}"'] & X.
\end{tikzcd}$$

It is a fact (c.f. Strom, Theorem 8.57) that filling in the top left corner of the 3-by-3 diagram using either the vertical or the horizontal homotopy fiber will yield equivalent results:

$$\begin{tikzcd}
F(g'') \simeq F(\varphi'') \arrow[r] \arrow[d] & F(g') \arrow[r, "\varphi''"] \arrow[d] & F(p_X) \arrow[d] \\
F(\varphi) \arrow[r] \arrow[d, "g''"'] & W \arrow[r, "\varphi"] \arrow[d, "g'"'] & X \times^h_Z Y \arrow[d, "p_X"] \\
* \simeq F(\mathrm{id}) \arrow[r] & X \arrow[r, "\mathrm{id}"'] & X
\end{tikzcd}$$

from which we deduce the equivalence $F(\varphi'') \simeq F(g'') \simeq F(\varphi)$.

To conclude, consider the equivalent conditions:

$$\begin{aligned}
&\varphi \text{ is } k\text{-connected} \\
\Leftrightarrow\ & F(\varphi) \text{ is } (k-1)\text{-connected} \\
\Leftrightarrow\ & F(\varphi'') \text{ is } (k-1)\text{-connected} \\
\Leftrightarrow\ & \varphi'' \text{ is } k\text{-connected.} \quad \square
\end{aligned}$$

Problem 3. Let $f\colon X \to Y$ be an n-connected map between spaces, and assume X is m-connected.

a. Using Blakers-Massey, show that the canonical comparison map

$$\varphi\colon F(f) \to \Omega C(f)$$

from the homotopy fiber to the loop space of the cofiber of f is $(m+n)$-connected.

Solution. Consider the homotopy pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\text{ho} \ulcorner"] & Y \arrow[d] \\ * \arrow[r] & C(f). \end{tikzcd}$$

Since X is m-connected, the map $X \to *$ is $(m+1)$-connected. By Blakers-Massey, the diagram is $(m+1+n-1 = m+n)$-Cartesian. By Problem 2, the map induced on (horizontal) homotopy fibers $F(f) \to \Omega C(f)$ is $(m+n)$-connected. □

b. Looking back (c.f. Problem 1) at the example of the Hopf map $\eta\colon S^3 \to S^2$, conclude that:

- The connectivity estimate $m+n$ in part (a) cannot be improved in general;
- The map $\varphi_\eta\colon F(\eta) \to \Omega C(\eta)$ does in fact induce isomorphisms on homotopy groups below the least dimension k satisfying $\pi_k F(\eta) \not\cong \pi_k \Omega C(\eta)$.

Solution. The map $\eta\colon S^3 \to S^2$ is 1-connected, and its source S^3 is 2-connected. By part (a), the comparison map

$$\varphi\colon F(\eta) \to \Omega C(\eta)$$

is 3-connected. In particular, φ induces isomorphisms on homotopy groups π_i for $i < 4$. In the cases $i = 0$, 2, and 3, this is automatic since both groups are trivial. In the remaining case $i = 1$, the fact that φ is 3-connected guarantees that it induces an isomorphism on π_1.

However, φ cannot be 4-connected, because the induced map on π_4

$$0 = \pi_4(S^1) \xrightarrow{\varphi_*} \pi_4(\Omega \mathbb{C}P^2) \cong \mathbb{Z}$$

cannot be surjective. □