# MA 519: Homework 4

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## Problem 4.1 (Handout 5, # 2)

In an urn, there are 12 balls. 4 of these are white. Three players: $A$, $B$, and $C$, take turns drawing a ball from the urn, in the alphabetical order. The first player to draw a white ball is the winner. Find the respective winning probabilities: assume that at each trial, the ball drawn in the trial before is put back into the urn (i.e., selection *with replacement*).

*Solution.* ■

## Problem 4.2 (Handout 5, # 8)

Consider $n$ families with 4 children each. How large must $n$ be to have a 90% probability that at least 3 of the $n$ families are all girl families?

*Solution.* ■

## Problem 4.3 (Handout 5, # 10)

(*Yahtzee*). In Yahtzee, five fair dice are rolled. Find the probability of getting a Full House, which is three rolls of one number and two rolls of another, in Yahtzee.

*Solution.* ∎

## Problem 4.4 (Handout 5, # 12)

The probability that a coin will show all heads or all tails when tossed four times is 0.25. What is the probability that it will show two heads and two tails?

*Solution.* ∎

## Problem 4.5 (Handout 5, # 13)

Let the events $A_1, A_2, \ldots, A_n$ be independent and $P(A_k) = p_k$. Find the probability $p$ that none of the events occurs.

*Solution.* ■

## Problem 4.6 (Handout 6, # 5)

Suppose a fair die is rolled twice and suppose $X$ is the absolute value of the difference of the two rolls. Find the PMF and the CDF of $X$ and plot the CDF. Find a median of $X$; is the median unique?

*Solution.* ∎

# Problem 4.7 (Handout 6, # 7)

Find a discrete random variable $X$ such that $E(X) = E(X^3) = 0$; $E(X^2) = E(X^4) = 1$.

*Solution.* Set $\Omega = \{0, 1\}$ and define a random variable $X \colon \Omega \to \mathbf{R}$ by $X(0) = -1$, $X(1) = 1$ as well as a probability $P(0) = P(1) = 1/2$. Then

$$E[X] = -1(1/2) + 1(1/2) = 0 = (-1)^3(1/2) + 1^3(1/2) = E[X^3],$$

whereas

$$E[X^2] = (-1)^2(1/2) + 1^2(1/2) = 1 = (-1)^4(1/2) + 1^4(1/2) = E[X^4],$$

as desired. ■

## Problem 4.8 (Handout 6, # 9)

(*Runs*). Suppose a fair die is rolled $n$ times. By using the indicator variable method, find the expected number of times that a six is followed by at least two other sixes. Now compute the value when $n = 100$.

*Solution.* Write $\Omega$ as the union events

$$A_i = \big\{ \text{roll of the die came up } i \big\}, \quad 1 \le n \le 6.$$

Note that the $A_n$ are not necessarily disjoint events. Now, define a random variable $X \colon \Omega \to \mathbf{R}$ by setting $X(\omega) = \sum_{n=1}^{6} \chi_{A_n}(x)$ where $\chi_{A_n}(x) = 1$ if $x \in A_n$, and $\chi_{A_n}(x) = 0$ otherwise.

Assuming every sample point is equally likely,

$$P(\omega) = \frac{1}{\#\Omega} = \frac{1}{6^n}$$

for every $\omega \in \Omega$. Thus, ■

## Problem 4.9 (Handout 6, # 10)

(*Birthdays*). For a group of $n$ people find the expected number of days of the year which are birthdays of exactly $k$ people. (Assume 365 days and that all arrangements are equally probable.)

*Solution.* ■

## Problem 4.10 (Handout 6, # 11)

(*Continuation*). Find the expected number of multiple birthdays. How large should $n$ be to make this expectation exceed 1?

*Solution.* ■

## Problem 4.11 (Handout 6, # 12)

(*The blood-testing problem*). A large number, $N$, of people are subject to a blood test. This can be administered in two ways, (i) Each person can be tested separately. In this case $N$ tests are required, (ii) The blood samples of $k$ people can be pooled and analyzed together. If the test is negative, this one test suffices for the $k$ people. If the test is positive, each of the $k$ persons must be tested separately, and in all $k+1$ tests are required for the $k$ people. Assume the probability $p$ that the test is positive is the same for all people and that people are stochastically independent.

(b) What is the expected value of the number, $X$, of tests necessary under plan (ii)?
(c) Find an equation for the value of $k$ which will minimize the expected number of tests under the second plan. (Do not try numerical solutions.)

*Solution.* ■

## Problem 4.12 (Handout 6, # 13)

(*Sample structure*). A population consists of $r$ (classes whose sizes are in the proportion $p_1 : p_2 : \cdots : p_r$. A random sample of size $n$ is taken with replacement. Find the expected number of classes not represented in the sample.

*Solution.* ■