Math 535 - General Topology
Fall 2012
Homework 11 Solutions

Problem 1. Let X be a topological space.

a. Show that the following properties of a subset $A \subseteq X$ are equivalent.

1. The closure of A in X has empty interior: $\operatorname{int}(\overline{A}) = \emptyset$.
2. For all non-empty open subset $U \subseteq X$, there is a non-empty open subset $V \subseteq U$ satisfying $V \cap A = \emptyset$.

A subset $A \subseteq X$ satisfying these equivalent properties is called **nowhere dense** in X.

Solution. Recall that a subset $B \subseteq X$ is dense if and only if its complement has empty interior:

$$\overline{B} = X \Leftrightarrow \overline{B}^c = \emptyset = \operatorname{int}(B^c).$$

Now consider the following equivalent conditions.

$\overline{A}$ has empty interior.

$\Leftrightarrow$ $\overline{A}^c$ is dense. But note $\overline{A}^c = \operatorname{int}(A^c)$.

$\Leftrightarrow$ For all non-empty open subset $U \subseteq X$, we have $U \cap \operatorname{int}(A^c) \neq \emptyset$.

$\Leftrightarrow$ For all non-empty open subset $U \subseteq X$, there is a point $x \in U \cap A^c$ and an open neighborhood W of x satisfying $W \subseteq A^c$, in other words $W \cap A = \emptyset$.

$\Leftrightarrow$ (Taking $V = U \cap W$) For all non-empty open subset $U \subseteq X$, there is a non-empty open subset $V \subseteq U$ satisfying $V \cap A = \emptyset$. $\square$

b. Show that the following properties of the space X are equivalent.

1. Any countable intersection of open dense subsets is dense. In other words, if each $U_n \subseteq X$ is open and dense in X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.
2. Any countable union of closed subsets with empty interior has empty interior. In other words, if each $C_n \subseteq X$ is closed in X and satisfies $\operatorname{int}(C_n) = \emptyset$, then their union satisfies $\operatorname{int}\left(\bigcup_{n=1}^{\infty} C_n\right) = \emptyset$.

A space X satisfying these equivalent properties is called a **Baire space**.

Solution. Consider the following equivalent conditions.

If each $U_n \subseteq X$ is open and dense in X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

$\Leftrightarrow$ If each $U_n^c \subseteq X$ is closed and has empty interior in X, then $\left(\bigcap_{n=1}^{\infty} U_n\right)^c$ has empty interior in X.

$\Leftrightarrow$ (Taking $C_n := U_n^c$) If each $C_n \subseteq X$ is closed and has empty interior in X, then $\bigcup_{n=1}^{\infty} C_n$ has empty interior in X. $\square$

Definition. Let X be a topological space. A function $f\colon X \to \mathbb{R}$ is **lower semicontinuous** if for all $a \in \mathbb{R}$, the preimage $f^{-1}(a, +\infty)$ is open in X.

Equivalently: For all $x_0 \in X$ and $\epsilon > 0$, there is a neighborhood U of x_0 satisfying $f(x) > f(x_0) - \epsilon$ for all $x \in U$. This means that the values close to x_0 can "suddenly jump up" but not down.

Problem 2.

a. Let X be a topological space and $f\colon X \to \mathbb{R}$ a continuous real-valued function. Show that for every non-empty open subset $U \subseteq X$, there is a non-empty open subset $V \subseteq U$ on which f is bounded.

Solution. Pick a point $x \in U$. Since f is continuous at x, there is an open neighborhood W of x satisfying $f(W) \subseteq (f(x) - 1, f(x) + 1)$, in particular f is bounded on W. Now the subset $V := W \cap U$ is non-empty (since $x \in V$), open, and f is bounded on V. □

b. (Willard Exercise 25C) Let X be a Baire space and $f\colon X \to \mathbb{R}$ a lower semicontinuous function. Show that for every non-empty open subset $U \subseteq X$, there is a non-empty open subset $V \subseteq U$ on which f is bounded above.

Solution. Note that for all $a \in \mathbb{R}$, the preimage $f^{-1}(-\infty, a] = \left(f^{-1}(a, +\infty)\right)^c$ is closed in X.

Express X as the countable union

$$\begin{aligned} X &= f^{-1}(\mathbb{R}) \\ & f^{-1}\left(\bigcup_{n=1}^{\infty}(-\infty, n]\right) \\ & \bigcup_{n=1}^{\infty} f^{-1}(-\infty, n] \\ &=: \bigcup_{n=1}^{\infty} A_n \end{aligned}$$

of closed subsets, and likewise

$$U = \bigcup_{n=1}^{\infty}(A_n \cap U).$$

Since U is open (and non-empty) and X is Baire, U cannot be meager, so that for some $m \in \mathbb{N}$, $A_m \cap U$ is not nowhere dense. Let $W \subseteq X$ be a non-empty open subset satisfying

$$W \subseteq \overline{A_m \cap U} \subseteq \overline{A_m} \cap \overline{U} = A_m \cap \overline{U}.$$

Since W is open and satisfies $W \subseteq \overline{U}$, it also satisfies $W \cap U \neq \emptyset$. This subset $V := W \cap U$ is non-empty, open, and contained in A_m so that f is bounded above on V (by the upper bound m). □

Problem 3. Show that a topological space X is of second category in itself if and only if any countable intersection of open dense subsets of X is non-empty.

Solution. Consider the following equivalent conditions.

X is of second category in itself, i.e. for any countable collection of nowhere dense subsets $A_n \subseteq X$, we have $\bigcup_{n=1}^{\infty} A_n \neq X$.

$\Leftrightarrow$ For any countable collection of *closed* nowhere dense subsets $C_n \subseteq X$, we have $\bigcup_{n=1}^{\infty} C_n \neq X$. (This implies the previous condition since A being nowhere dense implies $\overline{A}$ being nowhere dense.)

$\Leftrightarrow$ (Taking $U_n = C_n^c$) For any countable collection of open dense subsets $U_n \subseteq X$, we have $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$. $\square$

Problem 4. (Uniform boundedness principle) (Willard Exercise 25D.5) (Munkres Exercise 48.10) (Bredon I.17.2)

Let X be a Baire space and $S \subseteq C(X, \mathbb{R})$ a collection of real-valued continuous functions on X which is pointwise bounded: for each $x \in X$, there is a bound $M_x \in \mathbb{R}$ satisfying

$$|f(x)| \leq M_x \text{ for all } f \in S.$$

Show that there is a non-empty open subset $U \subseteq X$ on which the collection S is uniformly bounded: there is a bound $M \in \mathbb{R}$ satisfying

$$|f(x)| \leq M \text{ for all } x \in U \text{ and all } f \in S.$$

Solution. For all $n \in \mathbb{N}$, consider the subset of X

$$\begin{aligned} C_n &= \{x \in X \mid |f(x)| \leq n \text{ for all } f \in S\} \\ &= \bigcap_{f \in S} \{x \in X \mid |f(x)| \leq n\} \\ &= \bigcap_{f \in S} f^{-1}[-n, n] \end{aligned}$$

which is closed in X since each $f \in S$ is continuous.

Pointwise boundedness of the collection S yields $x \in C_n$ whenever $n \geq M_x$, or equivalently

$$X = \bigcup_{n=1}^{\infty} C_n.$$

Since X is Baire, it is in particular of second category in itself, so that for some $m \in \mathbb{N}$, C_m is not nowhere dense. Let $U \subseteq X$ be a non-empty open subset satisfying $U \subseteq \overline{C_m} = C_m$. Then the bound $|f(x)| \leq m$ holds for all $x \in U$ and all $f \in S$. $\square$

Definition. Let X and Y be normed real vector spaces. A linear map $T\colon X \to Y$ is **bounded** if there exists a constant $C \in \mathbb{R}$ satisfying

$$\|Tx\| \le C\|x\|$$

for all $x \in X$.

By linearity, this condition is equivalent to the following number being finite:

$$\begin{aligned} \|T\| &:= \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|} \\ &= \sup_{\|x\|=1} \|Tx\| \\ &= \sup_{\|x\| \le 1} \|Tx\|. \end{aligned}$$

The number $\|T\| \in \mathbb{R} \cup \{\infty\}$ is called the **operator norm** of T.

Let

$$\mathcal{L}(X,Y) := \{T\colon X \to Y \mid T \text{ is linear and } \|T\| < \infty\}$$

denote the vector space of bounded linear maps from X to Y. It is a vector space under pointwise addition and scalar multiplication. One readily checks that the assignment $T \mapsto \|T\|$ is indeed a norm on $\mathcal{L}(X,Y)$.

Problem 5. Let $T\colon X \to Y$ be a linear map between normed real vector spaces. Show that the following are equivalent.

1. T is continuous (everywhere).
2. T is continuous at some point $x_0 \in X$.
3. T is continuous at $0 \in X$.
4. T is bounded.

Solution. $(1 \Rightarrow 2)$ X is non-empty since it contains $0 \in X$.

$(2 \Rightarrow 3)$ Let $\epsilon > 0$. By continuity of T at x_0, there is a $\delta > 0$ satisfying $TB_\delta(x_0) \subseteq B_\epsilon(Tx_0)$. For any $x \in B_\delta(0)$, we have

$$\begin{aligned} d(Tx, T(0)) &= \|Tx - 0\| \\ &= \|Tx\| \\ &= \|T(x_0 + x - x_0)\| \\ &= \|T(x_0 + x) - Tx_0\| \\ &= d(T(x_0 + x), Tx_0) \\ &< \epsilon \end{aligned}$$

so that T is continuous at 0.

$(3 \Rightarrow 4)$ Taking $\epsilon = 1$, since T is continuous at 0, there is a $\delta > 0$ satisfying $TB_\delta(0) \subseteq B_1(T(0)) = B_1(0)$. Thus for any x with $\|x\| < 1$, we have

$$\begin{aligned} \|Tx\| &= \|T\left(\frac{\delta}{\delta}x\right)\| \\ &= \|\frac{1}{\delta}T\left(\delta x\right)\| \\ &= \frac{1}{\delta}\|T(\delta x)\| \\ &< \frac{1}{\delta}(1) \\ &= \frac{1}{\delta} \end{aligned}$$

and linearity of T implies $\|Tx\| \leq \frac{1}{\delta}$ whenever $\|x\| \leq 1$. Therefore T is bounded:

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| \leq \frac{1}{\delta}.$$

($4 \Rightarrow 1$) If T has bound C, then T is Lipschitz continuous with Lipschitz constant C, hence continuous. For all $x, x' \in X$, we have

$$\begin{aligned} d(Tx, Tx') &= \|Tx - Tx'\| \\ &= \|T(x - x')\| \\ &\leq C\|x - x'\| \\ &= Cd(x, x'). \quad \square \end{aligned}$$

Problem 6. Consider the Banach space

$$l^\infty = \{x \in \mathbb{R}^{\mathbb{N}} \mid \|x\|_\infty < \infty\}$$

with the supremum norm $\|x\|_\infty = \sup_{i\in\mathbb{N}}|x_i|$. Consider the linear subspace of lists that are eventually zero:

$$X := \{x \in l^\infty \mid \exists\, N \in \mathbb{N} \text{ such that } x_i = 0 \text{ for all } i > N\} \subset l^\infty.$$

Consider the continuous linear maps $T_n \colon X \to \mathbb{R}$ defined by

$$T_n(x) = nx_n.$$

a. Show that the collection $\{T_n\}_{n\in\mathbb{N}}$ is pointwise bounded but not uniformly bounded.

Solution. **Pointwise bounded.** Let $x \in X$ and let $N \in \mathbb{N}$ be large enough so that $x_i = 0$ for all $i > N$. Then for all $n > N$, we have

$$T_n x = nx_n = 0$$

and therefore

$$\sup_{n\in\mathbb{N}}|T_n x| = \max_{1\le n\le N}|T_n x| < \infty.$$

Not uniformly bounded. Consider the standard basis vectors $e^k \in X$ whose coordinates are

$$e_i^k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

and note that these are unit vectors: $\|e^k\|_\infty = 1$.

The equality $T_n(e^n) = n(e_n^n) = n(1) = n$ implies

$$\begin{aligned} \|T_n\| &= \sup_{x\in X\setminus\{0\}} \frac{\|T_n x\|}{\|x\|} \\ &\ge \frac{\|T_n e^n\|}{\|e^n\|} \\ &= \frac{|n|}{1} \\ &= n. \end{aligned}$$

It follows that the collection $\{T_n\}_{n\in\mathbb{N}}$ is not uniformly bounded:

$$\sup_{n\in\mathbb{N}}\|T_n\| = \infty. \quad \square$$

b. Part (a) implies that X cannot be complete. Show explicitly that X is not complete by exhibiting a Cauchy sequence in X that does not converge in X.

Solution. Let us denote the sequence index as a superscript. Consider the sequence $(x^{(n)})_{n\in\mathbb{N}}$ in X consisting of the following vectors:

$$x_i^{(n)} = \begin{cases} \frac{1}{i} & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

Note that each vector $x^{(n)}$ is eventually zero, hence a legitimate element of X.

The sequence is Cauchy. For any $N \in \mathbb{N}$ and $m, n \geq N$ (with $m \leq n$), the distance

$$\begin{aligned} d(x^{(m)}, x^{(n)}) &= \|x^{(m)} - x^{(n)}\| \\ &= \max\{\frac{1}{m+1}, \frac{1}{m+2}, \dots, \frac{1}{n}\} \\ &= \frac{1}{m+1} \\ &< \frac{1}{N} \end{aligned}$$

converges to 0 as $N \to \infty$.

The sequence does not converge in X. Let $x \in X$ and let $N \in \mathbb{N}$ be large enough so that $x_i = 0$ for all $i > N$. Then for all $n > N$, the distance

$$\begin{aligned} d(x^{(n)}, x) &= \|x^{(n)} - x\| \\ &= \sup_{i\in\mathbb{N}} |x_i^{(n)} - x_i| \\ &\geq \sup_{i>N} |x_i^{(n)} - x_i| \\ &= \sup_{i>N} |x_i^{(n)}| \\ &= |x_{N+1}^{(n)}| \\ &= \frac{1}{N+1} \end{aligned}$$

is bounded away from 0. Therefore the sequence $(x^{(n)})_{n\in\mathbb{N}}$ does not converge to $x \in X$. $\square$