Math 535 - General Topology
Fall 2012
Homework 10 Solutions

Problem 1. Let **Top** denote the category of topological spaces and continuous maps, and let **CHaus** denote the category of compact Hausdorff topological spaces and continuous maps. Show that the Stone-Čech construction

$$\beta\colon \mathbf{Top} \to \mathbf{CHaus}$$

is a functor.

Note: So far we know that β sends objects of **Top** to objects of **CHaus**. There remain three things to check.

Solution. **1) β sends morphisms to morphisms.** Let $f\colon X \to Y$ be a continuous map between topological spaces. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "f"'] & \beta X \arrow[d, dashed] \\ Y \arrow[r, "e_Y"] & \beta Y \end{tikzcd}$$

where the horizontal maps are the canonical evaluation maps. Note that βY is compact Hausdorff, and the composite $e_Y \circ f\colon X \to \beta Y$ is continuous. By the universival property of βX, there is a unique continuous map $h\colon \beta X \to \beta Y$ making the diagram commute, i.e. satisfying

$$h \circ e_X = e_Y \circ f.$$

Define βf to be this map h.

2) β preserves composition. Consider a composite $X \xrightarrow{f} Y \xrightarrow{g} Z$ and consider the diagram

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "f"] \arrow[dd, bend right=60, "g\circ f"'] & \beta X \arrow[d, "\beta f"'] \arrow[dd, bend left=60, "\beta g\circ\beta f"] \\ Y \arrow[r, "e_Y"] \arrow[d, "g"] & \beta Y \arrow[d, "\beta g"'] \\ Z \arrow[r, "e_Z"] & \beta Z. \end{tikzcd}$$

Note that the outer rectangle commutes because the two inner squares commute:

$$\begin{aligned} (\beta g \circ \beta f) \circ e_X &= \beta g \circ (\beta f \circ e_X) \\ &= \beta g \circ (e_Y \circ f) \\ &= (\beta g \circ e_Y) \circ f \\ &= (e_Z \circ g) \circ f \\ &= e_Z \circ (g \circ f). \end{aligned}$$

But $\beta(g \circ f)\colon \beta X \to \beta Z$ is the *unique* continuous map making the outer rectangle commute, which proves

$$\beta(g \circ f) = \beta g \circ \beta f.$$

3) β preserves identities. The diagram

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[d, "\mathrm{id}_X"'] & \beta X \arrow[d, "\mathrm{id}_{\beta X}"] \\
X \arrow[r, "e_X"] & \beta X
\end{tikzcd}$$

commutes:

$$\mathrm{id}_{\beta X} \circ e_X = e_X = e_X \circ \mathrm{id}_X.$$

But $\beta(\mathrm{id}_X)\colon \beta X \to \beta X$ is the *unique* continuous map making this diagram commute, which proves

$$\beta(\mathrm{id}_X) = \mathrm{id}_{\beta X}.$$

□

Problem 2. Let $\mathbf{Top}_*$ denote the category of pointed topological spaces and pointed continuous maps. To a space X, one can associate the pointed space

$$X_+ := X \amalg \{*\}$$

(with the coproduct topology) called "X with a disjoint basepoint", where $* \in X_+$ is the basepoint. To a continuous map $f\colon X \to Y$, one can assign the pointed continuous map

$$f_+\colon (X_+, *) \to (Y_+, *)$$

defined by

$$\begin{cases} f_+(x) = f(x) & \text{if } x \in X \\ f_+(*) = *. \end{cases}$$

One readily checks that this assignment makes the disjoint basepoint construction

$$(-)_+\colon \mathbf{Top} \to \mathbf{Top}_*$$

into a functor.

a. Show that for any space X and pointed space (Y, y_0), there is a bijection

$$\mathrm{Hom}_{\mathbf{Top}_*}\left((X_+, *), (Y, y_0)\right) \cong \mathrm{Hom}_{\mathbf{Top}}\left(X, Y\right).$$

Solution. Let $\varphi\colon \mathrm{Hom}_{\mathbf{Top}_*}\left((X_+, *), (Y, y_0)\right) \to \mathrm{Hom}_{\mathbf{Top}}\left(X, Y\right)$ be the restriction map defined by

$$\varphi(f) = f|_X.$$

In the other direction, consider the function $\psi\colon \mathrm{Hom}_{\mathbf{Top}}\left(X, Y\right) \to \mathrm{Hom}_{\mathbf{Top}_*}\left((X_+, *), (Y, y_0)\right)$ that sends a continuous map $g\colon X \to Y$ to the map $\psi(g)\colon X_+ \to Y$ defined by

$$\psi(g)(x) = \begin{cases} g(x) & \text{if } x \in X \\ y_0 & \text{if } x = *. \end{cases}$$

By construction, the map $\psi(g)\colon (X_+, *) \to (Y, y_0)$ is pointed.

To prove moreover that $\psi(g)$ is continuous, note that its restrictions $\psi(g)|_X = g$ and $\psi(g)|_{\{*\}}$ are both continuous. Thus $\psi(g)$ is continuous, since $X_+ = X \amalg \{*\}$ has the coproduct topology.

$\psi \circ \varphi = \mathrm{id}$**.** Let $f\colon (X_+, *) \to (Y, y_0)$ be a continuous pointed map. Then the map $\psi\varphi(f) = \psi(f|_X)\colon (X_+, *) \to (Y, y_0)$ is given by

$$\psi(f|_X)(x) = \begin{cases} f|_X(x) = f(x) & \text{if } x \in X \\ y_0 & \text{if } x = *. \end{cases}$$

Recall that f is pointed, i.e. it satisfies $f(*) = y_0$, so that $\psi(f|_X)$ agrees with f everywhere. This proves $\psi(f|_X) = f$.

$\varphi \circ \psi = \mathrm{id}$**.** Let $g\colon X \to Y$ be a continuous map. Then we have $\varphi\psi(f) = (\psi f)|_X = f$ by construction. $\square$

b. Show that the bijection in part (a) induces a bijection

$$[(X_+, *), (Y, y_0)]_* \cong [X, Y]$$

where $[(A, a_0), (B, b_0)]_* := \mathrm{Hom}_{h\mathbf{Top}_*}((A, a_0), (B, b_0))$ denotes the set of pointed homotopy classes of pointed continuous maps from (A, a_0) to (B, b_0).

As usual, $[X, Y] := \mathrm{Hom}_{h\mathbf{Top}}(X, Y)$ denotes the set of homotopy classes of continuous maps from X to Y.

Solution. Let $f, f' \colon (X_+, *) \to (Y, y_0)$ be pointed maps. The statement is that f and f' are pointed homotopic if and only if their restrictions $\varphi(f) = f|_X$ and $\varphi(f') = f'|_X$ are homotopic.

($\Rightarrow$) Let $F \colon X_+ \times [0, 1] \to Y$ be a pointed homotopy from f to f'. Then the restriction

$$F_{X\times[0,1]} \colon X \times [0, 1] \to Y$$

is a homotopy from $f|_X$ to $f'|_X$.

($\Leftarrow$) Let $G \colon X \times [0, 1] \to Y$ be a homotopy from $f|_X$ to $f'|_X$. Consider the map

$$\widetilde{G} \colon X_+ \times [0, 1] \cong (X \times [0, 1]) \amalg (\{*\} \times [0, 1]) \to Y$$

defined by

$$\begin{cases} \widetilde{G}|_{X\times[0,1]} = G & \\ \widetilde{G}(*, t) = y_0 & \text{for all } t \in [0, 1]. \end{cases}$$

In other words, $\widetilde{G}$ is obtained from G by applying ψ at each time: $\widetilde{g}_t = \psi(g_t)$.

Then $\widetilde{G}$ is continuous since its restriction to each summand $X \times [0, 1]$ and $\{*\} \times [0, 1]$ is continuous. By construction $\widetilde{G}$ is a pointed homotopy, i.e. it satisfies $\widetilde{G}(*, t) = y_0$ for all $t \in [0, 1]$. In fact, it is a pointed homotopy between the pointed maps

$$\widetilde{g}_0 = \psi(g_0) = \psi(f|_X) = f$$
$$\widetilde{g}_1 = \psi(g_1) = \psi(f'|_X) = f'$$

which are therefore pointed homotopic. $\square$

Problem 3. Let X be a topological space.

a. Let $w, x, y, z \in X$, $\alpha\colon [0,1] \to X$ a path from w to x, $\beta\colon [0,1] \to X$ a path from x to y, and $\gamma\colon [0,1] \to X$ a path from y to z. Show that concatenation of paths is associative up to homotopy, in the following sense:

$$(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma) \text{ rel } \{0,1\}.$$

Solution. Note that both sides are paths that go along α, β, and γ but at varying speeds. Given $0 < \sigma_1 < \sigma_2 < 1$, consider the path

$$\delta(\sigma_1, \sigma_2)\colon [0,1] \to X$$

from w to z defined by

$$\delta(\tau_1, \tau_2)(s) = \begin{cases} \alpha(\frac{s-0}{\sigma_1 - 0}) & \text{if } 0 \leq s \leq \sigma_1 \\ \beta(\frac{s-\sigma_1}{\sigma_2 - \sigma_1}) & \text{if } \sigma_1 \leq s \leq \sigma_2 \\ \gamma(\frac{s-\sigma_2}{1-\sigma_2}) & \text{if } \sigma_2 \leq s \leq 1. \end{cases}$$

In this notation, we have:

$$(\alpha * \beta) * \gamma = \delta(\frac{1}{4}, \frac{1}{2})$$
$$\alpha * (\beta * \gamma) = \delta(\frac{1}{2}, \frac{3}{4}).$$

Now if $\sigma_1(t)$ and $\sigma_2(t)$ are continuous functions of t satisfying $0 < \sigma_1(t) < \sigma_2(t) < 1$ for all $t \in [0,1]$, then the map $H\colon [0,1] \times [0,1] \to X$ defined by

$$H(-,t) = h_t = \delta(\sigma_1(t), \sigma_2(t))$$

is continuous and satisfies

$$H(0,t) = \delta(\sigma_1(t), \sigma_2(t))(0) = w$$
$$H(1,t) = \delta(\sigma_1(t), \sigma_2(t))(1) = z$$

for all $t \in [0,1]$, so that H is a path homotopy from h_0 to h_1.

In the case at hand, take $\sigma_1(t) = \frac{1}{4} + \frac{1}{4}t$ and $\sigma_2(t) = \frac{1}{2} + \frac{1}{4}t$ to obtain a path homotopy H between

$$h_0 = \delta(\sigma_1(0), \sigma_2(0)) = \delta(\frac{1}{4}, \frac{1}{2}) = (\alpha * \beta) * \gamma$$
$$h_1 = \delta(\sigma_1(1), \sigma_2(1)) = \delta(\frac{1}{2}, \frac{3}{4}) = \alpha * (\beta * \gamma).$$

$\square$

b. Let $\alpha\colon [0,1] \to X$ be a path in X from x to y. Denote by $\overline{\alpha}\colon [0,1] \to X$ the **reverse** path of α, defined by

$$\overline{\alpha}(s) = \alpha(1-s).$$

Show that $\overline{\alpha}$ is inverse to α up to homotopy, in the following sense:

$$\alpha * \overline{\alpha} \simeq 1_x \text{ rel } \{0,1\}$$

where $1_x\colon [0,1] \to X$ denotes the constant path at x.

Solution. The left-hand side is the path given by

$$(\alpha * \overline{\alpha})(s) = \begin{cases} \alpha(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \overline{\alpha}(2s-1) = \alpha(2-2s) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

which can be rewritten as $\alpha * \overline{\alpha} = \alpha \circ p$ where $p\colon [0,1] \to [0,1]$ is the "spike-shaped" function

$$p(s) = \begin{cases} 2s & \text{if } 0 \leq s \leq \frac{1}{2} \\ 2-2s & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

It suffices to show that p is homotopic rel $\{0,1\}$ to the constant function 0 in order to conclude

$$\begin{aligned} \alpha * \overline{\alpha} &= \alpha \circ p \\ &\simeq \alpha \circ 0 \ \text{ rel}\{0,1\} \\ &= \ \text{constant path at } \alpha(0) \\ &= 1_x \end{aligned}$$

as desired.

The map $H\colon [0,1] \times [0,1] \to [0,1]$ defined by

$$H(s,t) = tp(s)$$

is continuous and satisfies

$$\begin{aligned} H(0,t) &= tp(0) = 0 \\ H(1,t) &= tp(1) = 0 \end{aligned}$$

for all $t \in [0,1]$. Therefore H is a homotopy rel $\{0,1\}$ between $h_0 = H(-,0) \equiv 0$ and $h_1 = H(-,1) = p$ as desired. $\square$

Remark. No need to check the condition $\overline{\alpha} * \alpha \simeq 1_y$ rel $\{0,1\}$, which follows from part (b) applied to the path $\overline{\alpha}$ and observing $\overline{\overline{\alpha}} = \alpha$.

Remark. We have earned the right to adopt the notation $\overline{\alpha} = \alpha^{-1}$.

Definition. Let $A \subseteq X$ be a subspace of X, and denote by $i\colon A \to X$ the inclusion. Then A is called...

- a **retract** of X if there is a continuous map $r\colon X \to A$ satisfying $r \circ i = \mathrm{id}_A$, in other words $r(a) = a$ for all $a \in A$. Such a map r is called a **retraction** from X to A.
- a **deformation retract** of X if there is a retraction $r\colon X \to A$ which is moreover a homotopy equivalence, i.e. satisfying $i \circ r \simeq \mathrm{id}_X$.

  Explicitly: There is a homotopy $H\colon X \times [0,1] \to X$ satisfying $H(x,0) = x$ for all $x \in X$, $H(x,1) \in A$ for all $x \in X$, and $H(a,1) = a$ for all $a \in A$.
- a **strong deformation retract** of X if there is a retraction $r\colon X \to A$ which moreover satisfies
  $$i \circ r \simeq \mathrm{id}_X \text{ rel } A.$$
  Explicitly: There is a homotopy $H\colon X \times [0,1] \to X$ satisfying $H(x,0) = x$ for all $x \in X$, $H(x,1) \in A$ for all $x \in X$, and $H(a,t) = a$ for all $a \in A$ and all $t \in [0,1]$.

Problem 4. Consider the 2-simplex

$$\Delta^2 := \{(x,y) \in \mathbb{R}^2 \mid x+y \leq 1, x \geq 0, y \geq 0\}$$

and consider the subspace of Δ^2 consisting of points on the coordinate axes

$$A = \{(x,y) \in \Delta^2 \mid x = 0 \text{ or } y = 0\} = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})\,.$$

Show that A is a strong deformation retract of Δ^2.

Solution. Write Δ^2 as the union $\Delta^2 = C \cup D$ of two simplices with (ordered) vertices $(0,0), (0,1), (\frac{1}{2}, \frac{1}{2})$ and $(0,0), (1,0), (\frac{1}{2}, \frac{1}{2})$ respectively. In other words, C is the "top left" half of Δ^2 where $y \geq x$, while D is the "bottom right" half where $y \leq x$. Note that C and D are both closed (in $\mathbb{R}^2$ and therefore in Δ^2).

For $t \in [0,1]$, let C_t and D_t be the simplices with (ordered) vertices $(0,0), (0,1), t(\frac{1}{2}, \frac{1}{2})$ and $(0,0), (1,0), t(\frac{1}{2}, \frac{1}{2})$ respectively. Let $H \colon \Delta^2 \times [0,1] \to \Delta^2$ be the map defined as follows.

$H|_{C\times[0,1]}(-,t)$ is the unique affine transformation sending C to C_t.

$H|_{D\times[0,1]}(-,t)$ is the unique affine transformation sending D to D_t.

Note that both maps are continuous, since the vertices of C_t and D_t vary continuously as a function of t.

Also note that $C \cap D$ is the segment joining $(0,0)$ and $(\frac{1}{2}, \frac{1}{2})$, and those two vertices are sent to $(0,0)$ and $t(\frac{1}{2}, \frac{1}{2})$ by $H|_{C\times[0,1]}(-,t)$ and by $H|_{D\times[0,1]}(-,t)$. Therefore the two maps agree on the intersection

$$(C \times [0,1]) \cap (D \times [0,1]) = (C \cap D) \times [0,1]$$

and thus define a map H on $(C \times [0,1]) \cup (D \times [0,1]) = \Delta^2 \times [0,1]$. Moreover H is continuous since the restrictions $H|_{C\times[0,1]}$ and $H|_{D\times[0,1]}$ are continuous, and both subsets $C \times [0,1]$ and $D \times [0,1]$ are closed in $\Delta^2 \times [0,1]$.

For all $t \in [0,1]$, the map $H|_{C\times[0,1]}(-,t)$ sends $(0,0)$ to $(0,0)$ and $(0,1)$ to $(0,1)$ and therefore (since it is affine) leaves every point on the vertical segment between $(0,0)$ to $(0,1)$ fixed.

Likewise, the map $H|_{D\times[0,1]}(-,t)$ leaves every point on the horizontal segment between $(0,0)$ to $(1,0)$ fixed. This proves $H(a,t) = a$ for all $a \in A$ and $t \in [0,1]$.

The equalities $C_1 = C$ and $D_1 = D$ prove $H(-,1) = \mathrm{id}_{\Delta^2}$.

The equality $C_0 \cup D_0 = A$ proves $H(x,0) \in A$ for all $x \in \Delta^2$.

Therefore H is a homotopy rel A between id_{Δ^2} and a retraction $\Delta^2 \to A$. □

Remark. A straightforward calculation yields the explicit formula of H:

$$h_t(x,y) = \begin{cases} (tx, y - x + tx) & \text{if } y \geq x \\ (x - y + ty, ty) & \text{if } y \leq x. \end{cases}$$

Problem 5. Two objects X and Y of a category $\mathcal{C}$ are **connected by morphisms** if there is a zigzag of morphisms between them. More precisely, there is a finite sequence of objects

$$X = X_0, X_1, \ldots, X_{n-1}, X_n = Y$$

and for every $0 \leq i < n$, there is a morphism $f_i \colon X_i \to X_{i+1}$ or $f_i \colon X_{i+1} \to X_i$.

a. Show that two objects X and Y of a groupoid $\mathcal{G}$ are connected by morphisms if and only if there is a morphism $f \colon X \to Y$.

Solution. ($\Leftarrow$) The morphism $f \colon X \to Y$ exhibits X and Y as being connected by morphisms, i.e. $X_0 = X$, $X_1 = Y$, $f_0 = f$.

($\Rightarrow$) Assume there is a zigzag of morphisms f_i from $X = X_0$ to $Y = X_n$. Since $\mathcal{G}$ is a groupoid, every morphism has an inverse, and we can define morphisms $g_i \colon X_i \to X_{i+1}$ by

$$g_i = \begin{cases} f_i & \text{if } f_i \colon X_i \to X_{i+1} \\ f_i^{-1} & \text{if } f_i \colon X_{i+1} \to X_i. \end{cases}$$

Then the composite

$$\begin{tikzcd} X = X_0 \arrow[r, "g_0"] \arrow[rrrr, bend right, "f := g_{n-1} \circ \ldots \circ g_1 \circ g_0"'] & X_1 \arrow[r, "g_1"] & \ldots \arrow[r] & X_{n-1} \arrow[r, "g_{n-1}"] & X_n = Y \end{tikzcd}$$

is a morphism from X to Y. □

Remark. In particular, two points x and y in a space X are connected by morphisms in the fundamental groupoid $\Pi_1(X)$ if and only if they lie in the same path component of X.

b. Find an example of category $\mathcal{C}$ and objects X and Y of $\mathcal{C}$ that are connected by morphisms, but such that there are no morphisms from X to Y and no morphisms from Y to X:

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) = \emptyset \;\text{ and }\; \mathrm{Hom}_{\mathcal{C}}(Y,X) = \emptyset.$$

Solution. Let $\mathcal{C}$ be the category described by the graph

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \\ X. & \end{tikzcd}$$

More precisely, $\mathcal{C}$ has three objects $\mathrm{Ob}(\mathcal{C}) = \{X, Y, Z\}$ and only two non-identity morphisms $f\colon Z \to X$ and $g\colon Z \to Y$. This automatically forms a category, since no non-identity morphisms are composable.

The objects X and Y are connected by the zigzag of morphisms

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

but by definition, there are no morphisms $X \to Y$ or $Y \to X$. $\square$

c. Let X and Y be objects of a groupoid $\mathcal{G}$ that are connected by morphisms. Show that the vertex groups at X and Y are isomorphic (as groups):

$$\mathrm{Aut}_{\mathcal{G}}(X) \simeq \mathrm{Aut}_{\mathcal{G}}(Y).$$

Solution. By part (a), let $f\colon X \to Y$ be a morphism. Consider the map $\varphi^f\colon \mathrm{Aut}_{\mathcal{G}}(Y) \to \mathrm{Aut}_{\mathcal{G}}(X)$ defined by

$$\varphi^f(g) = f^{-1} \circ g \circ f.$$

φ^f is a group homomorphism.

$$\begin{aligned}\varphi^f(g_1 \circ g_2) &= f^{-1} \circ g_1 \circ g_2 \circ f \\ &= f^{-1} \circ g_1 \circ f \circ f^{-1} \circ g_2 \circ f \\ &= \varphi^f(g_1) \circ \varphi^f(g_2).\end{aligned}$$

φ^f is invertible. In fact its inverse is $\varphi^{f^{-1}}\colon \mathrm{Aut}_{\mathcal{G}}(X) \to \mathrm{Aut}_{\mathcal{G}}(Y)$.

For any $g \in \mathrm{Aut}_{\mathcal{G}}(Y)$ we have:

$$\begin{aligned}\varphi^{f^{-1}}\varphi^f(g) &= \varphi^{f^{-1}}\left(f^{-1} \circ g \circ f\right) \\ &= (f^{-1})^{-1} \circ f^{-1} \circ g \circ f \circ f^{-1} \\ &= f \circ f^{-1} \circ g \circ f \circ f^{-1} \\ &= \mathrm{id}_Y \circ g \circ \mathrm{id}_Y \\ &= g.\end{aligned}$$

Likewise, we have $\varphi^f\varphi^{f^{-1}}(g) = g$ for all $g \in \mathrm{Aut}_{\mathcal{G}}(X)$. □

Remark. This proves in particular that if x and y are two points in the same path component of a space X, then the fundamental groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic.

Problem 6. Let $f\colon X \xrightarrow{\simeq} Y$ be a homotopy equivalence between topological spaces. Show that for any choice of basepoint $x_0 \in X$, the induced group homomorphism

$$\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism.

Solution. Note: We omit "homotopy classes" of paths to ease the notation, i.e. write γ instead of $[\gamma]$.

Lemma. *Let $f, f'\colon X \to Y$ be two continuous maps, and let $H\colon X \times [0,1] \to Y$ be a homotopy (unpointed) from f to f'. Then for any basepoint $x_0 \in X$, the induced maps $\pi_1(f)$ and $\pi_1(f')$ differ by a "change of basepoint" isomorphism*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\pi_1(f)"] \arrow[dr, "\pi_1(f')"'] & \pi_1(Y, f(x_0)) \\
 & \pi_1(Y, f'(x_0)) \arrow[u, "\varphi^\alpha"', "\simeq"]
\end{tikzcd}$$

where $\varphi^\alpha\colon \pi_1(Y, f'(x_0)) \xrightarrow{\simeq} \pi_1(Y, f(x_0))$ is the group isomorphism (c.f. Problem 5c)

$$\varphi^\alpha(\gamma) = \alpha * \gamma * \alpha^{-1}$$

induced by the path α in Y from $f(x_0)$ to $f'(x_0)$ given by $\alpha(t) = H(x_0, t)$.

In particular, $\pi_1(f)$ is an isomorphism if and only if $\pi_1(f')$ is.

Proof. Let $\gamma\colon [0,1] \to X$ be a loop based at x_0. Then the two loops in Y being compared are

$$\pi_1(f)(\gamma) = f(\gamma) = h_0(\gamma)$$

and

$$\pi_1(f')(\gamma) = f'(\gamma) = h_1(\gamma)$$

and they are based at different points: $f(x_0)$ and $f'(x_0)$ respectively. In fact, for any $t \in [0,1]$, $h_t(\gamma)$ is a loop in Y based at $h_t(x_0) \in Y$.

Consider the loop in Y based at $f(x_0)$ that first runs along α up to $\alpha(t)$, then goes through the loop $h_t(\gamma)$, then comes back to $f(x_0)$ along α^{-1}:

$$a_t\colon [0, 1+2t] \to Y$$

$$a_t(s) = \begin{cases} \alpha(s) & \text{if } 0 \le s \le t \\ h_t(s-t) & \text{if } t \le s \le 1+t \\ \alpha(t-(s-1-t)) = \alpha(1+2t-s) & \text{if } 1+t \le s \le 1+2t. \end{cases}$$

Then a_t is continuous because α and H are. Reparametrizing to the interval $[0,1]$ yields the loop $b_t\colon [0,1] \to Y$ defined by

$$b_t(s) = a_t\left(s(1+2t)\right).$$

Now the map $B\colon [0,1]\times[0,1]\to Y$ defined by $B(s,t)=b_t(s)$ is continuous and satisfies the endpoint conditions

$$B(0,t)=b_t(0)=a_t(0)=f(x_0)$$
$$B(1,t)=b_t(1)=a_t(1+2t)=f(x_0)$$

for all $t\in[0,1]$. Therefore B is a pointed homotopy from the loop

$$b_0=a_0=h_0(\gamma)=f(\gamma)$$

to the loop

$$b_1\simeq \alpha * h_1(\gamma) * \alpha^{-1}=\varphi^\alpha(h_1(\gamma))=\varphi^\alpha(f'(\gamma))$$

which proves $\pi_1(f)=\varphi^\alpha\circ\pi_1(f')$. □

Let $g\colon Y\to X$ be a homotopy inverse of $f\colon X\to Y$. Consider the composite

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "\pi_1(f)"] \arrow[rr, bend right, "\simeq", "\pi_1(g\circ f)"'] & \pi_1(Y,f(x_0)) \arrow[r, "\pi_1(g)"] & \pi_1\left(X,g(f(x_0))\right) \end{tikzcd}$$

where $\pi_1(g)\circ\pi_1(f)=\pi_1(g\circ f)$ is an isomorphism by the lemma. Indeed, $g\circ f$ is (unpointed) homotopic to id_X, and $\pi_1(\mathrm{id}_X)$ is an isomorphism. Therefore $\pi_1(f)$ is injective and $\pi_1(g)$ is surjective.

Since the basepoint was arbitrary, the argument also applies to the homotopy equivalence $g\colon Y\xrightarrow{\simeq} X$ and basepoint $f(x_0)\in Y$, so that

$$\pi_1(g)\colon \pi_1\left(Y,f(x_0)\right)\to\pi_1\left(X,g(f(x_0))\right)$$

is also injective, and thus an isomorphism.

Therefore $\pi_1(f)=\pi_1(g)^{-1}\circ\pi_1(g\circ f)$ is an isomorphism. □

Alternate proof that $\pi_1(f)$ is surjective. Consider the diagram

$$\begin{tikzcd}
\pi_1(Y, f(x_0)) \arrow[r, "\pi_1(g)"] \arrow[rr, bend left, "\pi_1(f\circ g)", "\simeq"'] & \pi_1\left(X, g(f(x_0))\right) \arrow[r, "\pi_1(f)"] \arrow[d, "\varphi^\alpha"', "\simeq"] & \pi_1\left(Y, f(g(f(x_0)))\right) \arrow[d, "\varphi^{f(\alpha)}"', "\simeq"] \\
& \pi_1(X, x_0) \arrow[r, "\pi_1(f)"] & \pi_1(Y, f(x_0))
\end{tikzcd}$$

where the top composite is an isomorphism, again by the lemma. Here α denotes the path in X from x_0 to $g(f(x_0))$ given by $\alpha(t) = H(x_0, t)$ where H is a homotopy from id_X to $g \circ f$.

The square on the right commutes. For any $\gamma \in \pi_1\left(X, g(f(x_0))\right)$ we have

$$\begin{aligned}
\pi_1(f) \circ \varphi^\alpha(\gamma) &= \pi_1(f)\left(\alpha * \gamma * \alpha^{-1}\right) \\
&= f(\alpha * \gamma * \alpha^{-1}) \\
&= f(\alpha) * f(\gamma) * f(\alpha^{-1}) \\
&= f(\alpha) * f(\gamma) * f(\alpha)^{-1} \\
&= \varphi^{f(\alpha)}(f(\gamma)) \\
&= \varphi^{f(\alpha)} \circ \pi_1(f)(\gamma).
\end{aligned}$$

It follows that the composite

$$\varphi^{f(\alpha)} \circ \pi_1(f) \circ \pi_1(g) = \pi_1(f) \circ \varphi^\alpha \circ \pi_1(g)$$

is an isomorphism. Therefore the last step $\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective. $\square$