MA52300 Fall 2016

Homework Assignment 5

Due Wed, Oct 5, 2016

1. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define
$$v(x) := u(Ox), \quad x \in \mathbb{R}^n,$$
then $\Delta v = 0$.

2. Let $n = 2$ and U be the halfplane $\{x_2 > 0\}$. Prove that
$$\sup_U u = \sup_{\partial U} u$$
for $u \in C^2(U) \cap C(\overline{U})$ which are harmonic in U, under the additional assumption that u is bounded from above in $\overline{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.) [*Hint:* Take for $\epsilon > 0$ the harmonic function
$$u(x_1, x_2) - \epsilon \log \sqrt{x_1^2 + (x_2 + 1)^2}.$$
Apply the maximum principle to a region $\{x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0\}$ with large a. Let $\epsilon \to 0$.]

3. Let $U \subset \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if
$$-\Delta v \leq 0 \quad \text{in } U$$
   (a) Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \ldots, u^m$ are harmonic in U and
$$v := \phi(u^1, \ldots, u^m).$$
   Prove v is subharmonic. [*Hint:* Convexity for a smooth function $\phi(z)$ is equivalent to $\sum_{j,k=1}^m \phi_{z_j z_k}(z)\xi_j \xi_k \geq 0$ for any $\xi \in \mathbb{R}^m$.]

   (b) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. (Assume that harmonic functions are C^∞.)