MA571 Homework 9

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Problem 9.1 (Munkres §46, Ex. 6)

Show that the compact-open topology, $\mathcal{C}(X,Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular. [*Hint:* If $\overline{U} \subset V$, then $\overline{S(C,U)} \subset S(C,V)$.]

Proof. Suppose that Y is regular. We shall proceed by the hint and Lemma 31.1(b). Consider the subbasis element $S(C,U)$. Since Y is regular, there exists a neighborhood $V \supset U$ such that $V \supset \overline{U}$. Let $f \in \overline{S(C,U)}$. Then, we claim that $f \in S(C,V)$. For suppose not, then there exists an element $x_0 \in C$ such that $f(x_0) \notin V$. Then, since $\overline{U} \subset V$, by hypothesis, $f(x_0) \notin \overline{U}$. Consider the subbasic neighborhood $S\big(\{x_0\}, Y - \overline{U}\big)$ of f. Then, $S\big(\{x_0\}, Y - \overline{U}\big) \cap S(C,U)$ is nonempty. Let g be in the aforementioned intersection. Then $g(x_0) \in g(C) \subset U$, but $g(x_0) \in Y - \overline{U}$. This is a contradiction. Thus, $\overline{S(C,U)} \subset S(C,V)$.

Now, let $f \in \mathcal{C}(X,Y)$ and let $V = \bigcap_{i=1}^n S(C_i, V_i)$ be a basic neighborhood of f. Then, since Y is regular, for every $y = f(x_i) \in f(C_i)$ there exists an open neighborhood U_{x_i} such that $\overline{U_{x_i}} \subset V_i$. These U_{x_i}'s form an open cover of $f(C_i)$ which is compact by Theorem 25.6 so there exists a finite collection of them, say $\{U_{x_i,j}\}_{j=1}^{n_i}$ that covers $f(C_i)$. Let $U_i = \bigcup_{j=1}^{n_i} U_{x_i,j}$. Then $\overline{U}_i = \bigcup_{j=1}^{n_i} \overline{U_{x_i,j}} \subset V_i$ by induction on Problem 2.2 (Munkres §17, Ex. 6(b)). Let $U = \bigcap_{i=1}^n S(C_i, U_i)$. We claim that U is the desired neighborhood of f that, by Theorem 31.1(b), shows that $\mathcal{C}(X,Y)$ is regular. Let us verify this. First, note that $f \in U$ since $f(C_i) \subset U_i$ for all i so U is indeed a neighborhood of U. Moreover, by the hint, we have that $\overline{S(C_i,U_i)} \subset S(C_i,V_i)$ since $\overline{U_i} \subset V_i$. Then $\overline{U} \subset \bigcap_{i=1}^n \overline{S(C_i,U_i)} \subset V$ by Lemma B. It follows, by Theorem 31.1(b), that $\mathcal{C}(X,Y)$ is regular. ■

Problem 9.2 (Munkres §46, Ex. 9(a,b,c))

Here is a (unexpected) application of Theorem 46.11 to quotient maps. (Compare Exercise 11 of §29.)

Theorem. *If* $p\colon A \to B$ *is a quotient map and* X *is locally compact Hausdorff, then* $(\mathrm{id}_X, p)\colon X \times A \to X \times B$ *is a quotient map.*

Proof. (a) Let Y be the quotient space induced by (id_X, p); let $q\colon X \times A \to Y$ be the quotient map. Show there is a bijective continuous map $f\colon Y \to X \times B$ such that $f \circ q = (\mathrm{id}_X, p)$.

(b) Let $g = f^{-1}$. Let $G\colon B \to \mathcal{C}(X, Y)$ and $Q\colon A \to \mathcal{C}(X, Y)$ be the maps induced by g and q, respectively. Show that $Q = G \circ p$.

(c) Show that Q is continuous; conclude that G is continuous, so that g is continuous.

Actual proof. (a) Note that, by Munkre's definition of the "quotient topology induced by (id_X, p)," i.e., the identification space $X \times A/\sim$ where two elements $(x_1, a_1) \sim (x_2, a_2)$ if and only if $(x_1, p(a_1)) = (x_2, p(a_2))$, it follows that the map (id_X, p) preserves the equivalence relation on $X \times A$ so that, by Theorem Q.3, the induced map $f\colon Y \to X \times B$ is continuous since (id_X, p) is. Lastly, it is clear by Theorem Q.2 that $f \circ q = (\mathrm{id}_X, p)$. This map is surjective since (id_X, p) is subjective. To see that f is injective, let $[x_1, a_1], [x_1, a_2] \in Y$ and suppose that $f([x_1, a_1]) = f([x_2, a_2])$. Then, taking a representative of each equivalence class, $(x_1, p(a_1)) = (x_2, p(a_2))$ implies $x_1 = x_2$ and $p(a_1) = p(a_2)$, i.e., $(x_1, a_1) \sim (x_2, a_2)$. Thus, f is injective.

(b) Recall from the definition given on Munkres §46, p. 287, that the induced map G (resp. Q) are defined by the equation $(G(b))(x) = f^{-1}(x, b)$ (resp. $(Q(a))(x) = q(x, a)$). Then we have that the composition

$$(G \circ p)(a)(x) = G(p(a))(x) = f^{-1}(x, p(a)) = (f^{-1} \circ (\mathrm{id}_X, p))(x, a) = Q(x, a)$$

as desired.

(c) By Theorem 46.11, since q is continuous with respect to the quotient topology on Y, it follows that the induced map Q is continuous. Additionally, since Q is equal to the composition $G \circ p$ by part (b) so by Theorem Q.2 G is continuous. Since X is locally compact Hausdorff, it follows by Theorem 46.11 that the map g is continuous. ■

Problem 9.3 (Munkres §51, Ex. 1)

Show that if $h, h' \colon X \to Y$ are homotopic and $k, k' \colon Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof. Let $H \colon X \times I \to Y$ and $K \colon Y \times I \to Z$ denote the homotopies from h to h' and k to k', respectively. Then, we claim that the map $L(x, t) = K(H(x, t), t)$ is a homotopy from $k \circ h$ to $k' \circ h'$. First, we check that L starts and ends where we want it to, i.e., $L(x, 0) = K(H(x, 0), 0) = k(h(x))$ and $L(x, 1) = K(H(x, 1), 1) = k'(h'(x))$. Lastly, we must assure ourselves that L is in fact continuous. But this last claim follows from the fact that L can be expressed as the composition $K \circ (h_t, t)$ where h_t denotes the continuous map $H(x, t)$ at time t. Since K is (by assumption) continuous and (h_t, t) are continuous by Theorem 18.4, it follows by Theorem 18.2(a) that L is continuous. Thus, $k \circ h \simeq k' \circ h'$ as desired. ∎

Problem 9.4 (Munkres §51, Ex. 2)

Given spaces X and Y, let $[X, Y]$ denote the homotopy classes of maps of X into Y

(a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.

(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

Proof. (a) Let $f, g \colon X \to I$ be arbitrary continuous maps. Then we claim that the straight line homotopy $H(x, t) = (1-t)f(x) + tg(x)$ gives a homotopy from f to g. Note that the image of $H(x, t)$ stays in the interval I since $(1-t)f(x) + tg(x) \leq (1-t) + t = 1$ for all x and for all t. Lastly, note that by Theorem 25.1 H is continuous since it is the sum of a product of continuous functions. Hence, $f \simeq g$. Since f and g were arbitrary, it follows that $[X, I]$ consists of a single equivalence class.

(b) Note that if $f, g \colon I \to Y$ are constant maps, say $f(x) = x_0$ and $g(x) = x_1$ for all $x \in I$, then the path $p \colon I \to Y$ where $p(0) = x_0$ and $p(1) = x_1$ defines a homotopy $H(x, t) = p(t)$. This map is clearly continuous since for any open neighborhood U of Y, since p is continuous, by Theorem 18.1(4) there exists a neighborhood $V \subset I$ such that $p(V) \subset U$ so $H(I \times V) = p(V) \subset U$ implies H is continuous by Theorem 18.1(4). Therefore, it suffices to show that given a continuous map $f \colon I \to Y$, f is nulhomotopic. Let $H(x, t)$ be the map $f((t-1)x)$. The map $(t-1)x$ is continuous by Theorem 25.1 so the composition $f \circ ((t-1)x)$ is continuous by Theorem 18.2(c). Then, observing that $H(x, 0) = f(x)$ and $H(x, 1) = f(0)$, $H(x, t)$ gives a homotopy from f to $f(0)$. It follows by Lemma 51.1 that given any $f, g \colon I \to Y$ continuous maps $f \simeq g$ by transitivity of homotopy. ■

Problem 9.5 (Munkres §51, Ex. 3(a,b,c,))

A space X is said to be *contractible* if the identity map $\mathrm{id}_X\colon X \to X$ is nullhomotopic.

(a) Show that I and $\mathbf{R}$ are contractible.

(b) Show that a contractible space is path connected.

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

Proof. Let us prove once an for all an exercise from Munkres that should have been assigned at the beginning of the semester:

Lemma 1 (Munkres, §18, Ex. 11)**.** *Let $F\colon X \times Y \to Z$. We say that F is* continuous in each variable separately *if for each y_0 in Y the map $h\colon X \to Z$ defined by $h(x) = F(x, y_0)$ is continuous, and for each x_0 in X the map $k\colon Y \to Z$ defined by $k(y) = F(x_0, y)$ is continuous. If F is continuous, then f is continuous in each variable.*

Proof. We will prove this only for the case of h; the proof of k is similar. Fix $y_0 \in Y$ and let W b an open neighborhood of $F(x, y_0)$. Then, by Theorem 18.1(4), there exists an open neighborhood W' of (x, y_0) such that $F(W') \subset W$. By Lemma C, we may restrict to a basic open neighborhood $U \times V$, U open in X and V open in Y, of (x, y_0) contained in W'. Then we have

$$h(U) = F(U \times y_0) \subset F(U \times V) \subset F(W') \subset W$$

so h is continuous. ♣

(a) It is clear that $\mathrm{id}_I\colon I \to I$ is nulhomotopic, say to the constant map 0, via the homotopy $H(x, t) = (1 - t)x$. Note that $H(x, 0) = x = \mathrm{id}_I(x)$ and $H(x, 1) = 0$ and $H(x, t)$ is continuous since $(1 - t)x$ is continuous by Theorem 25.1.[1] In the case of $\mathbf{R}$ the previous map $H(x, t)$ also works to show that $\mathrm{id}_{\mathbf{R}}$ is nulhomotopic since $H(x, 0) = x = \mathrm{id}_{\mathbf{R}}$ and $H(x, 1) = 0$ and $H(x, t)$ is continuous by Theorem 25.1.

(b) Suppose that X is contractible. Then there exists a homotopy $H(x, t)$ with $H(x, 0) = x$ and $H(x, 1) = x_0$ for some point $x_0 \in X$. Now, let $x_1, x_2 \in X$. Then the maps $p_1(t) = H(x_1, t)$ and $p_2(t) = H(x_2, t)$ are path homotopies from x_1 to x_0 and x_2 to x_0 by Lemma 1. It follows by the fact that $\simeq_p$ is an equivalence relation that $x_1 \simeq_p x_2$.

(c) Since Y is contractible there exist a homotopy $H(y, t)$ with $H(y, 0) = x$ and $H(y, 1) = y_0$ for some fixed $y_0 \in X$. Therefore, it suffices to show that an arbitrary continuous map $f\colon X \to Y$ is nulhomotopic. Consider the map $K(x, t) = H(f(x), t)$. This map is continuous since it is the composition $H \circ (f, \mathrm{id}_I)$. Moreover, $K(x, 0) = \mathrm{id}_Y(f(x)) = f(x)$ and $K(x, 1) = e_{y_0}(f(x)) = y_0$. Thus, f is nulhomotopic and it follows that $[X, Y]$ has a single element (all maps are null homotopic and Y is path connected by part (b)). ■

[1] More generally, we showed that products, sums and quotients (when they are defined) of maps from a metric space (X, d) to $\mathbf{R}$ (or a subspace of $\mathbf{R}$ by Theorem 18.2(d)) for that matter, are continuous.