3.4: 1, 2, 3
3.5: 1, 2, 3

3.4.1 Verify Theorem 3.11 for the matrix $A=\begin{bmatrix}-2&3&0\\4&1&-3\\2&0&1\end{bmatrix}$ by computing $a_{11}A_{12}+a_{21}A_{22}+a_{31}A_{32}$.

This is Thm (3.11) for $j=1$ and $k=2$.

$a_{11}A_{12}+a_{21}A_{22}+a_{31}A_{32} = (-2)(-1)^{1+1}\begin{vmatrix}4&-3\\2&1\end{vmatrix} + (4)(-1)^{2+1}\begin{vmatrix}-2&0\\2&1\end{vmatrix} + (2)(-1)^{3+1}\begin{vmatrix}-2&0\\4&-3\end{vmatrix}$

$=(-2)(4+6)-4(-2)+2(6) = -20+8+12=0.$

3.4.2 Let $A=\begin{bmatrix}2&1&3\\-1&2&0\\3&-2&1\end{bmatrix}$. (a) Find adj(A). (b). Compute det(A). (c) Verify Theorem 3.12; that is, $A(\text{adj}(A)) = (\text{adj}(A))A = \det(A)I_3$.

(a) $A_{11}=(-1)^{1+1}\begin{vmatrix}2&0\\-2&1\end{vmatrix}=2$, $A_{12}=(-1)^{1+2}\begin{vmatrix}-1&0\\3&1\end{vmatrix}=+1$, $A_{13}=(-1)^{1+3}\begin{vmatrix}-1&2\\3&-2\end{vmatrix}=2-6=-4$,

$A_{21}=(-1)^{2+1}\begin{vmatrix}1&3\\-2&1\end{vmatrix}=-(1+6)=-7$, $A_{22}=(-1)^{2+2}\begin{vmatrix}2&3\\3&1\end{vmatrix}=2-9=-7$, $A_{23}=(-1)^{2+3}\begin{vmatrix}2&1\\3&-2\end{vmatrix}=-(-4-3)=7$,

$A_{31}=(-1)^{3+1}\begin{vmatrix}1&3\\2&0\end{vmatrix}=-6$, $A_{32}=(-1)^{3+2}\begin{vmatrix}2&3\\-1&0\end{vmatrix}=-3$, $A_{33}=(-1)^{3+3}\begin{vmatrix}2&1\\-1&2\end{vmatrix}=4+1=5$.

Then $\text{adj}(A)=\begin{bmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{bmatrix}=\begin{bmatrix}2&-7&-6\\1&-7&-3\\-4&7&5\end{bmatrix}$

(b) $\det(A)=a_{13}A_{13}+a_{23}A_{23}+a_{33}A_{33}=(3)(-4)+(0)(7)+(1)(5)=-12+5=-7.$

(c) $A(\text{adj}(A)) = \begin{bmatrix}2&1&3\\-1&2&0\\3&-2&1\end{bmatrix}\begin{bmatrix}2&-7&-6\\1&-7&-3\\-4&7&5\end{bmatrix}=\begin{bmatrix}-7&0&0\\0&-7&0\\0&0&-7\end{bmatrix}=\det(A)I_3.$

$\parallel$

$(\text{adj}(A))A = \begin{bmatrix}2&-7&-6\\1&-7&-3\\-4&7&5\end{bmatrix}\begin{bmatrix}2&1&3\\-1&2&0\\3&-2&1\end{bmatrix}=\begin{bmatrix}-7&0&0\\0&-7&0\\0&0&-7\end{bmatrix}=\det(A)I_3$

3.4.3 | Let $A = \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix}$. Follow the directions of Exercise 2.

(a) $A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 1 \\ -4 & 5 \end{vmatrix} = 20+4 = 24$, $A_{12} = (-1)^{1+2}\begin{vmatrix} -3 & 1 \\ 4 & 5 \end{vmatrix} = -(-15-4) = 19$, $A_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 4 \\ 4 & -4 \end{vmatrix} = 12-16 = -4$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 8 \\ -4 & 5 \end{vmatrix} = -(10+32) = -42$, $A_{22} = (-1)^{2+2}\begin{vmatrix} 6 & 8 \\ 4 & 5 \end{vmatrix} = 30-32 = -2$, $A_{23} = (-1)^{2+3}\begin{vmatrix} 6 & 2 \\ 4 & -4 \end{vmatrix} = -(-24-8) = 32$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 8 \\ 4 & 1 \end{vmatrix} = 2-32 = -30$, $A_{32} = (-1)^{3+2}\begin{vmatrix} 6 & 8 \\ -3 & 1 \end{vmatrix} = -(6+24) = -30$, $A_{33} = (-1)^{3+3}\begin{vmatrix} 6 & 2 \\ -3 & 4 \end{vmatrix} = 24+6 = +30$

Then

$$\operatorname{adj}(A) = \begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & +30 \end{bmatrix}$$

(b) $\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (6)(24) + (2)(19) + (8)(-4) = 150$

(c) $$A(\operatorname{adj}(A)) = \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix} = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 150 \end{bmatrix} = \det(A) I_3$$

$$\shortparallel$$

$$(\operatorname{adj}(A))A = \begin{bmatrix} 24 & -42 & -30 \\ 19 & -2 & -30 \\ -4 & 32 & 30 \end{bmatrix}\begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 150 \end{bmatrix} = \det(A) I_3.$$

3.5.1 | If possible, solve the following linear system by Cramer's rule:
$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 2 \\ x_1 \quad\;\; + 2x_3 &= 0 \\ 2x_1 + 3x_2 - x_3 &= -5. \end{aligned}$$

$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 0 & 2 \\ 2 & 3 & -1 \end{bmatrix}$ $\det(A) = -4\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 2 & 6 \\ 1 & 2 \end{vmatrix} = -4(-1-4) - 3(4-6) = 26.$

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 2 & 4 & 6 \\ 0 & 0 & 2 \\ -5 & 3 & -1 \end{vmatrix}}{26} = \frac{1}{26}\left[2\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 4 & 6 \\ 3 & -1 \end{vmatrix} - 5\begin{vmatrix} 4 & 6 \\ 0 & 2 \end{vmatrix}\right] = \frac{1}{26}\left[(2)(-6) - 5(8)\right] = \frac{-52}{26} = -2$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{1}{26}\begin{vmatrix} 2 & 2 & 6 \\ 1 & 0 & 2 \\ 2 & -5 & -1 \end{vmatrix} = \frac{1}{26}\left[-2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ 2 & -1 \end{vmatrix} + 5\begin{vmatrix} 2 & 6 \\ 1 & 2 \end{vmatrix}\right] = \frac{1}{26}\left[(-2)(-5) + (5)(-2)\right] = \frac{1}{26}(10 + -10) = 0$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{1}{26}\begin{vmatrix} 2 & 4 & 2 \\ 1 & 0 & 0 \\ 2 & 3 & -5 \end{vmatrix} = \frac{1}{26}\left[2\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} - 5\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix}\right] = \frac{1}{26}\left[(2)(3) - (5)(-4)\right] = \frac{26}{26} = 1.$$

3.5.21 Repeat Exercise 1 for the linear system $\begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & -1 & 2 \\ 1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 5 \\ 4 \end{bmatrix}$.

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & -2 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & -1 & 2 \\ 1 & -1 & 0 & 1 \end{vmatrix}_{\substack{-2r_1+r_3 \\ -r_1+r_4}} = \begin{vmatrix} 1 & 1 & 1 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & -1 & -3 & 6 \\ 0 & -2 & -1 & 3 \end{vmatrix}_{r_3+r_2} = \begin{vmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & -2 & 9 \\ 0 & -1 & -3 & 6 \\ 0 & -2 & -1 & 3 \end{vmatrix}_{\substack{r_2+r_3 \\ 2r_2+r_4}} = \begin{vmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & -5 & 15 \\ 0 & 0 & -5 & 21 \end{vmatrix}_{-r_3+r_4} = \begin{vmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & -5 & 15 \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

$$= -30.$$

$$|A_1| = \begin{vmatrix} -4 & 1 & 1 & -2 \\ 4 & 2 & 1 & 3 \\ 5 & 1 & -1 & 2 \\ 4 & -1 & 0 & 1 \end{vmatrix}_{r_3+r_1} = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 4 & 2 & 1 & 3 \\ 5 & 1 & -1 & 2 \\ 4 & -1 & 0 & 1 \end{vmatrix}_{\substack{-4r_1+r_2 \\ -5r_1+r_3 \\ -4r_1+r_4}} = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & -6 & 1 & 3 \\ 0 & -9 & -1 & 2 \\ 0 & -9 & 0 & 1 \end{vmatrix}_{-r_3+r_4} = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & -6 & 1 & 3 \\ 0 & -9 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -6 & 1 & 3 \\ -9 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix} = -6\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} + 9\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -6(1-2) + 9(-1-3) = 6 - 36 = -30$$

$$x_1 = \frac{-30}{-30} = 1.$$

$$|A_2| = \begin{vmatrix} 1 & -4 & 1 & -2 \\ 0 & 4 & 1 & 3 \\ 2 & 5 & -1 & 2 \\ 1 & 4 & 0 & 1 \end{vmatrix}_{\substack{-2r_1+r_3 \\ -r_1+r_4}} = \begin{vmatrix} 1 & -4 & 1 & -2 \\ 0 & 4 & 1 & 3 \\ 0 & 13 & -3 & 6 \\ 0 & 8 & -1 & 3 \end{vmatrix}_{\substack{-3r_2+r_3 \\ -2r_2+r_4}} = \begin{vmatrix} 1 & -4 & 1 & -2 \\ 0 & 4 & 1 & 3 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & -3 & -3 \end{vmatrix}_{r_2 \leftrightarrow r_3} = -\begin{vmatrix} 1 & -4 & 1 & -2 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 1 & 3 \\ 0 & 0 & -3 & -3 \end{vmatrix}_{-4r_2+r_3} = -\begin{vmatrix} 1 & -4 & 1 & -2 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 25 & 15 \\ 0 & 0 & -3 & -3 \end{vmatrix}$$

$$= -(1)\begin{vmatrix} 1 & -6 & -3 \\ 0 & 25 & 15 \\ 0 & -3 & -3 \end{vmatrix} = -(1)(1)\begin{vmatrix} 25 & 15 \\ -3 & -3 \end{vmatrix} = -(-75 + 45) = 30$$

$$x_2 = \frac{30}{-30} = -1.$$

$$|A_3| = \begin{vmatrix} 1 & 1 & -4 & -2 \\ 0 & 2 & 4 & 3 \\ 2 & 1 & 5 & 2 \\ 1 & -1 & 4 & 1 \end{vmatrix}_{\substack{-2r_1+r_3 \\ -r_1+r_4}} = \begin{vmatrix} 1 & 1 & -4 & -2 \\ 0 & 2 & 4 & 3 \\ 0 & -1 & 13 & 6 \\ 0 & -2 & 8 & 3 \end{vmatrix}_{r_3+r_2} = \begin{vmatrix} 1 & 1 & -4 & -2 \\ 0 & 1 & 17 & 9 \\ 0 & -1 & 13 & 6 \\ 0 & -2 & 8 & 3 \end{vmatrix}_{\substack{r_2+r_3 \\ 2r_2+r_3}} = \begin{vmatrix} 1 & 1 & -4 & -2 \\ 0 & 1 & 17 & 9 \\ 0 & 0 & 30 & 15 \\ 0 & 0 & 42 & 21 \end{vmatrix}_{r_4-r_3} = \begin{vmatrix} 1 & 1 & -4 & -2 \\ 0 & 1 & 17 & 9 \\ 0 & 0 & 30 & 15 \\ 0 & 0 & 12 & 6 \end{vmatrix}$$

$$= (1)\begin{vmatrix} 1 & 17 & 9 \\ 0 & 30 & 15 \\ 0 & 12 & 6 \end{vmatrix} = (1)(1)\begin{vmatrix} 30 & 15 \\ 12 & 6 \end{vmatrix} = 6 \cdot 30 - 12 \cdot 15 = 0$$

$$x_3 = \frac{0}{-30} = 0$$

$$|A_4| = \begin{vmatrix} 1 & 1 & 1 & -4 \\ 0 & 2 & 1 & 4 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 0 & 4 \end{vmatrix}_{\substack{-2r_1+r_3 \\ -r_1+r_4}} = \begin{vmatrix} 1 & 1 & 1 & -4 \\ 0 & 2 & 1 & 4 \\ 0 & -1 & -3 & 13 \\ 0 & -2 & -1 & 8 \end{vmatrix}_{r_3+r_2} = \begin{vmatrix} 1 & 1 & 1 & -4 \\ 0 & 1 & -2 & 17 \\ 0 & -1 & -3 & 13 \\ 0 & -2 & -1 & 8 \end{vmatrix}_{\substack{r_2+r_3 \\ 2r_2+r_4}} = \begin{vmatrix} 1 & 1 & 1 & -4 \\ 0 & 1 & -2 & 17 \\ 0 & 0 & -5 & 30 \\ 0 & 0 & -5 & 42 \end{vmatrix}_{-r_3+r_4} = \begin{vmatrix} 1 & 1 & 1 & -4 \\ 0 & 1 & -2 & 17 \\ 0 & 0 & -5 & 30 \\ 0 & 0 & 0 & 12 \end{vmatrix} = -5 \cdot 12 = -60$$

$$x_4 = \frac{-60}{-30} = 2.$$

3.5.3 | Solve the following linear system for x_3, by Cramer's rule:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 6 \\ 3x_1 + 2x_2 - 2x_3 &= -2 \\ x_1 + x_2 + 2x_3 &= -4. \end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & -2 \\ 1 & 1 & 2 \end{bmatrix}. \quad \det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & -2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} -r_3+r_1 \\ -3r_3+r_2 \end{matrix} \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & -8 \\ 1 & 1 & 2 \end{vmatrix} \underset{r_2+r_3}{=} \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & -8 \\ 1 & 0 & -6 \end{vmatrix} \underset{-r_1+r_3}{=} \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & -8 \\ 0 & 0 & -5 \end{vmatrix} = 5$$

$$|A_3| = \begin{vmatrix} 2 & 1 & 6 \\ 3 & 2 & -2 \\ 1 & 1 & -4 \end{vmatrix} \begin{matrix} -r_3+r_1 \\ -3r_3+r_2 \end{matrix} = \begin{vmatrix} 1 & 0 & 10 \\ 0 & -1 & +10 \\ 1 & 1 & -4 \end{vmatrix} \underset{r_2+r_3}{=} \begin{vmatrix} 1 & 0 & 10 \\ 0 & -1 & +10 \\ 1 & 0 & 6 \end{vmatrix} \underset{-r_1+r_3}{=} \begin{vmatrix} 1 & 0 & 10 \\ 0 & -1 & -10 \\ 0 & 0 & -4 \end{vmatrix} = 4$$

$$x_3 = \frac{4}{5}$$