MA544: Qual Preparation

Carlos Salinas

August 5, 2016

Contents

1 MA 544 Spring 2016

This is material from the course MA 544 as it was taught in the spring of 2016.

1.1 Homework

These exercises were assigned from Wheeden and Zygmund's *Measure and Integral*, therefore, most of the theorems I reference will be from [4]. Other resources include [1] and [2]. For more elementary results, I cite [3]. Unless otherwise stated, whenever we quote a result, e.g., Theorem 1.1, it is understood to come from Wheeden and Zygmund's *Measure and Integral*.

Throughout these notes

$\mathbb{R}$	is the set of real numbers
$\mathbb{R}^+$	is the set of positive real numbers, that is, $x \in \mathbb{R}$ with $x \geq 0$
$\mathbb{C}$	is the set of complex numbers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{Z}$	is the set of the integers
$\mathbb{Z}^+$	is the set of positive integers, that is, $x \in \mathbb{Z}$ with $x \geq 0$
$\mathbb{N}$	is the set of the natural numbers $1, 2, \ldots$
$A \setminus B$	is the set difference of A and B, that is, the complement of $A \cap B$ in A
$m^*(E)$	the outer measure of E
$m_*(E)$	the inner measure of E
$m(E)$	the Lebesgue measure of E
$\|-\|$	the standard Euclidean norm on $\mathbb{R}^n$
$f \asymp g$	means f is asymptotically equivalent to g, that is, $\lim_{x\to\infty} g(x)/f(x) = 1$

1.1.1 Homework 1

Problem 1 (Wheeden & Zygmund Ch. 2, Ex. 1)**.** Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0,1]$, but that $V[f;0,1] = \infty$.

Solution. ▶ Let f equal $x\sin(1/x)$. We will show that f is bounded and continuous on $[0,1]$, but that it is not of bounded variation on $[0,1]$.

First we will show that f is bounded. Note that both $|x|$ and $|\sin(1/x)|$ are bounded by 1 on the interval $[0,1]$. Since $|f| = |x||\sin(1/x)|$, it follows that $|f| \leq 1$ on $[0,1]$. Thus, f is bounded on $[0,1]$.

Next we show that f is continuous. It is easy to show that f is continuous on the subinterval $(0,1]$ since both $|x|$ and $\sin(1/x$ are continuous on that interval and we know that the product of continuous functions is continuous. To see that f is continuous at 0 we must show that $f(x^+) = f(0)$; that is, the limit of f as x approaches 0 from the right is $f(0)$ which by definition is 0. To this end, it suffices to take a (monotonically decreasing) sequence $x_n \downarrow 0$ and show that the limit of the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is 0. Let $\varepsilon > 0$ be given then, since x_n converges to 0 there exists an index N such that $|0 - x_n| < \varepsilon$ whenever $n \geq N$. Since $|f(x_n)| \leq |x_n|$ on $[0,1]$, the following inequality holds

$$\begin{aligned} |0 - f(x_n)| &= |0 - x_n \sin(1/x_n)| \\ &\leq |x_n| \\ &< \varepsilon. \end{aligned}$$

Thus, f is continuous at 0 and it converges to 0.

Despite the nice properties that f seemingly possesses, f is not b.v. on $[0,1]$. To show that f is not b.v. on $[0,1]$ we must show that for any positive real number M there exists some partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of $[0,1]$ such that the sum associated to Γ

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| > M.$$

Let N be the smallest integer greater than M and let n be the smallest integer greater than or equal to $N/2$. Then the partition $\Gamma = \{\, x_0 = 1 < x_1 < \cdots < x_{n+1} = 1 \,\}$ where $x_i = 2/((3 + (n-i))\pi$ for $1 \leq i \leq N$. Then we have the inequality

$$\begin{aligned} S_\Gamma &= \sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})| + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &= N + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &> M. \end{aligned}$$

Thus, f is not b.v. on $[0,1]$. ◀

Problem 2 (Wheeden & Zygmund Ch. 2, Ex. 2)**.** Prove theorem (2.1).

Solution. ▶ Recall the statement of Theorem 2.1:

(a) If f is of bounded variation on $[a,b]$, then f is bounded on $[a,b]$.

(b) Let f and g be of bounded variation on $[a,b]$. Then cf (for any real constant c), $f+g$, and fg are of bounded variation on $[a,b]$. Moreover, f/g is of bounded variation on $[a,b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a,b]$.

We shall prove these in alphabetical order:

For part (a) we shall proceed by contradiction. First, without loss of generality, we may assume that $f(a) = 0$ since the function the variation of $g(x) = f(x) - f(a)$ is equal to the variation of f and $g(a) = 0$. Suppose that f is b.v. on $[a,b]$ with variation $V = V[f; a, b]$, but that f is unbounded on $[a,b]$; that is, given a positive real number M there exists a point x in $[a,b]$ such that $|f(x)| > M$. In particular, there exists $x \in [a,b]$ such that $|f(x)| > V$. Hence, for any $x \in [a,b]$ by the triangle inequality we have

$$\begin{aligned} V &< |f(x)| \\ &= |f(x) - f(a) + f(a)| \\ &\leq |f(x) - f(a)| + |f(a)| \\ &\leq V. \end{aligned}$$

This is a contradiction. Therefore, it must be the case that if f is b.v. on $[a,b]$ then f is bounded on $[a,b]$.

We break part (b) into three sections. Suppose f and g are b.v. on $[a,b]$ with variation V and V', respectively. We will show that (i) cf; (ii) $f+g$; and (iii) fg are b.v. on $[a,b]$. Moreover, we show that (iv) f/g is b.v. on $[a,b]$ if there exists $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for all $x \in [a,b]$.

For part (i) above let c be a real number. Given a partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of $[a,b]$, we have

$$\begin{aligned} S_\Gamma &= \sum_{i=1}^{n} |cf(x_i) - cf(x_{i-1})| \\ &= \sum_{i=1}^{n} |c||f(x_i) - cf(x_{i-1})| \\ &= |c| \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &\leq |c|V \end{aligned}$$

since V is the supremum of the sums of the form $\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|$ over all partitions of $[a,b]$. Thus, $V[cf; a, b] \leq |c|V$ so cf is b.v. on $[a,b]$.

For part (ii) given a partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of the interval $[a,b]$, by the triangle inequality we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^{n} |(f(x_i) + g(x_i)) - (f(x_{i-1}) + g(x_{i-1}))| \\
&= \sum_{i=1}^{n} |(f(x_i) - f(x_{i-1})) + (g(x_i) - g(x_{i-1}))| \\
&\le \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \\
&\le V + V'.
\end{aligned}$$

Thus, $f + g$ is b.v. on $[a, b]$

For part (iii) since f and g are b.v. on $[a, b]$ by part (a) f and g are bounded on $[a, b]$ by, say, M and N, respectively. Now, given a partition $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ of $[a, b]$, by the triangle inequality we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1}) \\
&\qquad\qquad + f(x_i)g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^{n} |(f(x_i)g(x_i) - f(x_i)g(x_{i-1})) \\
&\qquad\qquad - (f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1}))| \\
&\le \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_i)g(x_{i-1})| \\
&\qquad\qquad + \sum_{i=1}^{n} |f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^{n} |f(x_i)||g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} |g(x_{i-1})||f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^{n} M|g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} N|f(x_i) - f(x_{i-1})| \\
&\le MV' + NV.
\end{aligned}$$

Thus, fg is b.v. on $[a, b]$.

Finally, for part (iv) suppose there exists $\varepsilon > 0$ such that $|g(x)| \ge \varepsilon$ for all $x \in [a, b]$. Then, given a partition $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ of $[a,b]$, largely by the triangle inequality, we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^{n} |f(x_i)/g(x_i) - f(x_{i-1})/g(x_{i-1})| \\
&= \sum_{i=1}^{n} \left| \frac{f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)}{g(x_i)g(x_{i-1})} \right| \\
&\le \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1}) \\
&\qquad\qquad - (f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1}))| \\
&\le \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |g(x_{i-1})||f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_{i-1})||g(x_i) - g(x_{i-1})| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^{n} M_g |f(x_i) - f(x_i)| + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} M_f |g(x_i) - g(x_i)| \\
&= \frac{1}{\varepsilon^2} M_g \sum_{i=1}^{n} |f(x_i) - f(x_i)| + \frac{1}{\varepsilon^2} M_f \sum_{i=1}^{n} |g(x_i) - g(x_i)| \\
&\le \frac{1}{\varepsilon^2}(NV + MV')
\end{aligned}$$

where, as above, f is bounded by M and g is bounded by N. Thus, f/g is b.v. on $[a,b]$.

This concludes the proof of Theorem 2.1. ◂

Problem 3 (Wheeden & Zygmund Ch. 2, Ex. 3)**.** If $[a', b']$ is a subinterval of $[a,b]$ show that $P[a', b'] \le P[a,b]$ and $N[a', b'] \le N[a,b]$.

Solution. ▸ We will prove this by digging in to the definition of N and P. Recall that given a partition $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ of the interval $[a,b]$, P and N are defined to be the supremum over the sum of the positive and, respectively, the sum negative terms of S_Γ; that is, P and N are the supremum over every partition Γ of $[a,b]$ of

$$P_\Gamma = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+ \quad \text{and} \quad N_\Gamma = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^-.$$

Let $f\colon [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$ and let $[a', b']$ be a subinterval of $[a,b]$. Without loss of generality, we may assume that $[a', b']$ is strictly contained in $[a,b]$; that is, $a' \ne a$ and $b' \ne b$. We aim to show that $P[a', b'] \le P[a,b]$ and $N[a', b'] \le N[a,b]$. Since the argument for N is similar to that of P, we will omit it here for the sake of brevity. Now, consider the closure of the complement of $[a', b']$ in $[a,b]$, $\overline{[a,b] \setminus [a', b']} = [a, a'] \cup [b', b]$. Since $[a, a']$, $[a', b']$

and $[b', b]$ are close intervals we may take partitions

$$\Gamma_a = \{ x_0 < x_1 \cdots < x_\ell \},$$
$$\Gamma_{ab} = \{ x_\ell < x_{\ell+1} < \cdots < x_m \}$$

and

$$\Gamma_b = \{ x_m < x_{m+1} < \cdots < x_n \}$$

of $[a, a']$, $[a', b']$ and $[b', b]$, respectively and extend this to a partition

$$\Gamma = \{ x_0 < x_1 < \cdots < x_\ell < x_{\ell+1} \cdots < x_m < x_{m+1} \cdots < x_n \}$$

of $[a, b]$. Then, by the definition of N we have the string of inequalities

$$\begin{aligned} P_\Gamma &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+ \\ &= \sum_{i=1}^{\ell} [f(x_i) - f(x_{i-1})]^+ \\ &\qquad + \sum_{i=\ell+1}^{m} [f(x_i) - f(x_{i-1})]^+ \\ &\qquad + \sum_{i=m+1}^{n} [f(x_i) - f(x_{i-1})]^+ \\ &= P_{\Gamma_{ab}} + P_{\Gamma_a} + P_{\Gamma_b} \\ &\leq P[a, b]. \end{aligned}$$

Taking the supremum on the left, we have

$$P[a, a'] + P[a', b'] + P[b', a'] \leq P[a, b].$$

Since P is strictly positive, it must be the case that $P[a', b'] \leq P[a, b]$. ◀

Problem 4 (Wheeden & Zygmund Ch. 2, Ex. 11)**.** Show that $\int_a^b f \,\mathrm{d}\varphi$ exists if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|, |\Gamma'| < \delta$.

Solution. ▶ One direction is straightforward. Namely $\Longleftarrow$: suppose that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ whenever $|\Gamma|$ and $|\Gamma'|$ are less than δ. Let $\{\Gamma_n\}_{n=1}^\infty$ be a decreasing sequence of partitions (by which we mean $\Gamma_n \subseteq \Gamma_{n+1}$ of $[a, b]$ such that $|\Gamma_n| \to 0$. Then, by convergence, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|\Gamma_n| < \delta$. Then, for $n, m \geq N$, we have

$$|R_{\Gamma_n} - R_{\Gamma_m}| < \varepsilon.$$

Thus, by the Cauchy criterion for convergence, the sequence $\{R_{\Gamma_n}\}_{n=0}^\infty$ converges and its limit is by definition the Riemann–Stieltjes integral $\int_a^b f \,\mathrm{d}\varphi$.

On the other hand $\Longrightarrow$: suppose that $I = \int_a^b f \, \mathrm{d}\varphi$ exists. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that $|I - R_\Gamma| < \varepsilon/2$ whenever $|\Gamma| < \delta$. Let Γ and Γ' be two partitions of $[a, b]$ with norm $|\Gamma|, |\Gamma'| < \delta$. Then we have

$$\begin{aligned} |R_\Gamma - R_{\Gamma'}| &= |R_\Gamma - I - (R_{\Gamma'} - I)| \\ &\leq |R_\Gamma - I| + |R_{\Gamma'} - I| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, I satisfies the Cauchy condition. ◂

Problem 5 (Wheeden & Zygmund Ch. 2, Ex. 13)**.** Prove theorem (2.16).

Solution. ▸ Recall the statement of Theorem 2.16:

(i) If $\int_a^b f \, \mathrm{d}\varphi$ exists, then so do $\int_a^b cf \, \mathrm{d}\varphi$ and $\int_a^b f \, \mathrm{d}(c\varphi)$ for any constant c, and

$$\int_a^b cf \, \mathrm{d}\varphi = \int_a^b f \, \mathrm{d}(c\varphi) = c \int_a^b f \, \mathrm{d}\varphi.$$

(ii) If $\int_a^b f_1 \, \mathrm{d}\varphi$ and $\int_a^b f_2 \, \mathrm{d}\varphi$ both exist, so does $\int_a^b (f_1 + f_2) \, \mathrm{d}\varphi$, and

$$\int_a^b (f_1 + f_2) \, \mathrm{d}\varphi = \int_a^b f_1 \, \mathrm{d}\varphi + \int_a^b f_2 \, \mathrm{d}\varphi.$$

(iii) If $\int_a^b f \, \mathrm{d}\varphi_1$ and $\int_a^b f \, \mathrm{d}\varphi_2$ both exist, so does $\int_a^b f \, \mathrm{d}(\varphi_1 + \varphi_2)$, and

$$\int_a^b f \, \mathrm{d}(\varphi_1 + \varphi_2) = \int_a^b f \, \mathrm{d}\varphi_1 + \int_a^b f \, \mathrm{d}\varphi_2.$$

We prove this in (Roman) numerical order.

For (i) suppose that $I = \int_a^b f \, \mathrm{d}\varphi$ exists. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|I - R_\Gamma| < \varepsilon/|c|$ whenever Γ is a partition of $[a, b]$ with $|\Gamma| < \delta$. We claim that $\int_a^b cf \, \mathrm{d}\varphi = |c|I$. Let $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ be a partition $[a, b]$ with $|\Gamma| < \delta$. Then the Riemann–Stieltjes sums R'_Γ of the pair (cf, φ) associated to Γ give us the chain of inequalities

$$\begin{aligned} ||c|I - R'_\Gamma| &= \left| |c|I - \sum_{i=1}^n cf(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c| \left| \sum_{i=1}^n cf(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c||I - R_\Gamma| \\ &< |c| \frac{\varepsilon}{|c|} \\ &= \varepsilon. \end{aligned}$$

Thus, $\int_a^b cf\,\mathrm{d}\varphi$ is Riemann–Stieltjes integrable and its integral is equal to $|c|I$. A similar argument shows that $\int_a^b f\,\mathrm{d}(c\varphi)$ is Riemann–Stieltjes integrable with integral $|c|I$.

For (ii) let $I_1 = \int_a^b f_1\,\mathrm{d}\varphi$ and $I_2 = \int_a^b f_2\,\mathrm{d}\varphi$. Then, we claim that $I = \int_a^b (f_1 + f_2)\,\mathrm{d}\varphi$ exists and that $I = I_1 + I_2$. Since both I_1 and I_2 exist, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|I_1 - R_\Gamma^1| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |I_2 - R_\Gamma^2| < \frac{\varepsilon}{2}$$

whenever $|\Gamma| < \delta$. Let $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ be a partition of $[a, b]$ with $|\Gamma| < \delta$. Then the Riemann–Stieltjes sums R_Γ of the pair $(f_1 + f_2, \varphi)$ associated to Γ give is the following chain of inequalities

$$\begin{aligned}
|(I_1 + I_2) - R_\Gamma| &= \left|(I_1 + I_2) - \sum_{i=1}^n (f_1(\xi_i) + f_2(\xi_i))[\varphi(x_i) - \varphi(x_{i-1})]\right| \\
&= \left| I_1 - \sum_{i=1}^n f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1}) \right. \\
&\qquad \left. + I_2 - \sum_{i=1}^n f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\
&\leq \left| I_1 - \sum_{i=1}^n f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1}) \right| \\
&\qquad + \left| I_2 - \sum_{i=1}^n f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\
&= |I_1 - R_\Gamma^1| + |I_2 - R_\Gamma^2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus, I exists and it is equal to the sum $I_1 + I_2$.

Part (iii) is similar to part (ii) in the above equation except that instead of splitting the sum at $f_1 + f_2$ part, we split it at $\varphi_1 + \varphi_2$ part. ◀

1.1.2 Homework 2

Problem 1. Show that the boundary of any interval has outer measure zero.

Solution. ▸ Let $I = \prod_{i=1}^{n} I_i$ be a closed interval in $\mathbb{R}^n$ and let J be the boundary of I. We must show that given $\varepsilon > 0$ there exists a countable collection of intervals $\{I_n\}_{n\in J}$ covering J such that

$$\sum_{n\in J} \mathrm{vol}(I_n) < \varepsilon.$$

First, note that we can write J as the union $\bigcup_{i=1}^{n} J_i$ where

$$J_i = [a_1, b_1] \times \cdots \times \{a_i\} \times \cdots \times [a_n, b_n] \cup [a_1, b_1] \times \cdots \times \{b_i\} \times \cdots \times [a_n, b_n].$$

Since the countable union of null sets has measure zero, it suffices to show that the set

$$[a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$$

has measure zero. Consider the collection $\{I_\varepsilon\}$ consisting of the single interval

$$I_\varepsilon = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \left[a_n - \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)}, a_n + \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)}\right].$$

It is clear that $I_\varepsilon \supseteq J$. Now, computing the volume of this interval, we have

$$\begin{aligned}
\mathrm{vol}(I_\varepsilon) &= \prod_{i=1}^{n-1}(b_i - a_i)\left[a_n + \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)} - \left(a_n - \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)}\right)\right] \\
&= \left[\prod_{i=1}^{n-1}(b_i - a_i)\right] \frac{\varepsilon}{\prod_{i=1}^{n-1}(b_i - a_i)} \\
&= \varepsilon.
\end{aligned}$$

Thus, J has measure zero. ◂

Problem 2. Show that a set consisting of a single point has outer measure zero.

Solution. ▸ Let $\{a\}$ be the set consisting of a single point $a \in \mathbb{R}$. Then we must show that given $\varepsilon > 0$ there exists a countable collection of intervals $\{I_n\}$ such that

$$\sum_{n\in J} m(I_n) < \varepsilon.$$

Consider the collection $\{I_\varepsilon\}$ consisting of the single interval

$$I_\varepsilon = \left[a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2}\right].$$

It is clear that $\{a\} \subseteq I_\varepsilon$. Moreover,

$$\begin{aligned}
\mathrm{vol}(I_\varepsilon) &= a + \frac{\varepsilon}{2} - \left(a - \frac{1}{\varepsilon}\right) \\
&= \varepsilon.
\end{aligned}$$

Thus, $\{a\}$ has measure zero. ◂

1.1.3 Homework 3

Problem 1 (Wheeden & Zygmund Ch. 3, Ex. 5)**.** Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage each interval removed has length $\delta 3^{-k}$, $0<\delta<1$. Show that the resulting set is perfect, has measure $1-\delta$, and contains no interval.

Solution. ▸ We construct the prescribed subset as follows: take the open interval $(1/2-\delta/6, 1/2+\delta/6)$ and remove it from the closed interval $[0,1]$ the result is a union of two disjoint closed intervals

$$E_{1,1}=\left[0,\frac{1}{2}-\frac{1}{6}\delta\right],\quad E_{1,2}=\left[\frac{1}{2}+\frac{1}{6}\delta,1\right],$$

whose union we call E_1; this marks the first step in the construction of this Cantor-like set. Next, we remove the set

$$\left(\frac{1}{4}-\frac{5}{36}\delta,\frac{1}{4}+\frac{1}{36}\delta\right)\cup\left(\frac{3}{4}+\frac{\delta}{36},\frac{3}{4}+\frac{5}{36}\delta\right)$$

from the set E_1 which yields E_2 the union of the four closed intervals

$$\begin{aligned} E_{2,1}&=\left[0,\frac{1}{4}-\frac{5}{36}\delta\right], & E_{2,2}&=\left[\frac{1}{4}+\frac{1}{36}\delta,\frac{1}{2}-\frac{1}{6}\delta\right],\\ E_{2,3}&=\left[\frac{1}{2}+\frac{1}{6}\delta,\frac{3}{4}+\frac{\delta}{36}\right], & E_{2,4}&=\left[\frac{3}{4}+\frac{5}{36}\delta,1\right]. \end{aligned}$$

In the nth step of the construction, we remove an open interval of length $3^{-n}\delta$ from the center of each interval $E_{n-1,i}$ yielding E_n which is the union of 2^n intervals $E_{n,i}$ of length $2^{-n}-\delta 2^{-n}\sum_{i=1}^{n}2^{i-1}3^{-i}$. Let E be the intersection $\bigcap_{i=1}^{\infty}E_i$. This concludes our construction.

Next we show that E is perfect, has measure $1-\delta$ and contains no interval.

To see that E is perfect, we must show that E is closed and that and dense in itself. The set E is closed because it is the (arbitrary) intersection of closed intervals. To see that E is dense in itself, we must show that for every $\varepsilon>0$, for every $x\in E$, the intersection $(B(x,\varepsilon)\cap E)\setminus\{x\}$ is nonempty. Let $\varepsilon>0$ and $x\in E$ be given. Then, since $x\in E$, $x\in E_n$ for every n. Thus, x is in some closed interval $E_{n,i}\subseteq E_n$. Let N be the smallest integer such that the length of $E_{N,i}=[a,b]$ is less that ε. Then, $a,b\in E$ and $a,b\in B(x,\varepsilon)$ and x is cannot be equal to both a and b. Thus, $(E\cap B(x,\varepsilon))\setminus\{x\}\neq\emptyset$. It follows that E is a perfect set.

To see that the measure of E is $1-\delta$ by Theorem 3.26 (ii) since $m(E_1)=1-\delta/3<\infty$ and

$E_n \searrow E$ we have

$$
\begin{aligned}
m(E) &= m\left(\bigcap_{i=1}^{\infty} E_i\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} m(E_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{2^n} \left[\frac{1}{2^n} - \frac{\delta}{2^n} \sum_{i=1}^{n} \frac{2^{i-1}}{3^i}\right] \\
&= \lim_{n\to\infty} \left[1 - \delta \sum_{i=1}^{n} \frac{2^{i-1}}{3^i}\right] \\
&= \lim_{n\to\infty} \left[1 - \frac{\delta}{3} \sum_{i=1}^{n} \left(\frac{2}{3}\right)^{i-1}\right]
\end{aligned}
$$

letting $j = i - 1$, we can rewrite the series above as the geometric series

$$
\begin{aligned}
&= 1 - \frac{\delta}{3} \lim_{n\to\infty} \sum_{j=0}^{n} \left(\frac{2}{3}\right)^{j} \\
&= 1 - \delta,
\end{aligned}
$$

as desired.

Lastly, we must show that E contains no interval. Seeking a contradiction, suppose that E contains an interval $I = [a, b]$ of length $b - a$. Then, since $I \subseteq E$, $I \subseteq E_n$ for all n so, since I is connected, it must be contained in one of the $E_{n,i}$ for all n. Let N be the smallest integer such that $m(E_{N,i}) < b - a$ and $E_{N,i} = [c, d]$ contains I. Then, since $I \subseteq E_{N,i}$, both a and b are points in I, $|b - a| \leq |d - c| = m(E_{N,i})$. This is a contradiction. Thus, it must be the case that E contains no interval. ◀

Problem 2 (Wheeden & Zygmund Ch. 3, Ex. 7)**.** Prove (3.15).

Solution. ▸ Here is the statement of the lemma:

> *If $\{I_k\}_{k=1}^N$ is a finite collection of nonoverlapping intervals, then $\bigcup_{k=1}^N I_k$ is measurable and $m\left(\bigcup_{k=1}^N I_k\right) = \sum_{k=1}^N m(I_k)$.*

By Theorem 3.12, the union $\bigcup_{n=1}^N I_n$ is measurable. Hence, it remains to show that $m\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n)$.

We take the approach of extending the argument provided in Theorem 3.2. As in Theorem 3.2, we note that, since $\{I_n\}_{n=1}^N$ covers the union $\bigcup_{n=1}^N I_n$, then

$$m\left(\bigcup_{n=1}^{N} I_n\right) \leq \sigma\left(\bigcup_{n=1}^{N} I_n\right) = \sum_{n=1}^{N} m(I_n).$$

On the other hand, note that I_n is the union $I_n^\circ \cup \partial I_n$ of its interior and its boundary. In the previous homework, we showed that the boundary of an interval has measure zero. Hence, we have

$$m(I_n^\circ) \leq m(I_n) \leq m(I_n^\circ) + m(\partial I_n) = m(I_n^\circ)$$

so $m(I_n) = m(I_n^\circ)$. Now, note that

$$m\left(\bigcup_{n=1}^{N} I_n^\circ\right) = \sum_{n=1}^{N} m(I_n^\circ) = \sum_{n=1}^{N} m(I_n).$$

Hence, we have

$$\begin{aligned}\sum_{n=1}^{N} m(I_n) &= m\left(\bigcup_{n=1}^{N} I_n^\circ\right) \\ &\leq m\left(\bigcup_{n=1}^{N} I_n\right) \\ &\leq \sum_{n=1}^{N} m(I_n).\end{aligned}$$

Thus, equality $m\left(\bigcup_{n=1}^{N} I_n\right) = \sum_{n=1}^{N} m(I_n)$ holds. ◂

Problem 3 (Wheeden & Zygmund Ch. 3, Ex. 8)**.** Show that the Borel algebra $\mathcal{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.

Solution. ▸ Since $\mathcal{B}$ is the smallest σ-algebra containing all of the open sets of $\mathbb{R}^n$, it contains all of the closed sets of $\mathbb{R}^n$. Now, suppose that $\mathcal{B}'$ is another σ-algebra containing the closed sets in $\mathbb{R}^n$. Then, $\mathcal{B}' \subseteq \mathcal{B}$ since $\mathcal{B}$ contains all of the closed sets in $\mathbb{R}^n$. However, since $\mathcal{B}'$ is a σ-algebra, it contains all of the open sets in $\mathbb{R}^n$, so $\mathcal{B}' \subseteq \mathcal{B}$ since $\mathcal{B}$ is the smallest σ-algebra containing the open sets in $\mathbb{R}^n$. Thus, $\mathcal{B}' = \mathcal{B}$. ◂

Problem 4 (Wheeden & Zygmund Ch. 3, Ex. 9)**.** If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum m^*(E_k) < \infty$, show that $\limsup E_k$ (and also $\liminf E_k$ has measure zero.

Solution. ▸ First, since $\{E_n\}_{n=1}^{\infty}$ is a sequence of sets with

$$\sum_{i=1}^{\infty} m^*(E_i) < \infty$$

for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\sum_{i=n}^{\infty} m^*(E_i) < \varepsilon.$$

Let's put this aside for now.

Define $E = \limsup_{n\to\infty} E_n$ and $E'_n = \bigcup_{i=n}^{\infty} E_n$. It is easy to see that $\{E'_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets whose intersection $\bigcap_{n=1}^{\infty} E_n$ is the limit supremum E. By the monotonicity of the outer measure, we have

$$m^*(E) \leq m^*(E'_n)$$

for all $n \in \mathbb{N}$. On the other hand,

$$m^*(E'_n) \leq \sum_{i=n}^{\infty} m^*(E_i) < \varepsilon$$

for every ε. Letting ε go to 0 we have $m^*(E) = 0$.

Lastly, we note that $E' = \liminf_{n\to\infty} E_n$ is a subset of $\limsup_{n\to\infty} E_n$, so that $m^*(E') = 0$. ◂

Problem 5 (Wheeden & Zygmund Ch. 3, Ex. 10)**.** If E_1 and E_2 are measurable, show that $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

Solution. ▸ We may, without loss of generality, assume that $m(E_1), m(E_2) < \infty$ for otherwise there is nothing to show as equality holds trivially.

Now, by Carathéodory's theorem we have the following characterization of measurability: a set E is measurable if and only if for every set A we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E).$$

Therefore, the following equalities hold

$$\begin{aligned} m(E_1) &= m(E_1 \cap E_2) + m(E_1 \setminus E_2) \\ m(E_2) &= m(E_1 \cap E_2) + m(E_2 \setminus E_1). \end{aligned}$$

Moreover, from elementary set theory we have

$$(E_1 \cup E_2) \setminus E_2 = E_1 \setminus (E_1 \cap E_2),$$

$E_1 \subseteq E_1 \cup E_2$ and $E_1 \cap E_2 \subseteq E_1$ so

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

as desired. ◂

1.1.4 Homework 4

Problem 1 (Wheeden & Zygmund Ch. 3, Ex. 12)**.** If E_1 and E_2 are measurable sets in $\mathbb{R}^1$, show $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $m(E_1 \times E_2) = m(E_1)m(E_2)$. (Interpret $0 \cdot \infty$ as 0.) [*Hint:* Use a characterization of measurability.]

Solution. ▸ The proof of this result is rather long and we shall omit it for now as I gain nothing from retracing my steps on this one. ◂

Problem 2 (Wheeden & Zygmund Ch. 3, Ex. 13)**.** Motivated by (3.7), define the *inner measure* of E by $m_*(E) = \sup m(F)$, where the supremum is taken over all closed subsets F of E. Show that

(i) $m_*(E) \leq m^*(E)$, and

(ii) if $m^*(E) < \infty$, then E is measurable if and only if $m_*(E) = m^*(E)$.

[Use (3.22).]

Solution. ▸ First we show part (i). If $m^*(E) = \infty$, the inequality holds trivially. Suppose that $m^*(E) < \infty$. Then, since F is closed, it is measurable and $m(F) = m^*(F)$. Moreover, $F \subseteq E$ so by the monotonicity of the outer measure,

$$m(F) = m^*(F) < m^*(E).$$

Taking the supremum over all F on the left, we have

$$m_*(E) = \sup_{F \subseteq E} m(F) < m^*(E)$$

as we set out to show.

Next we show part (ii). Let $E \subseteq \mathbb{R}^n$ with $m^*(E) < \infty$. $\Longrightarrow$ Suppose that E is measurable. Then, by Lemma 3.22, there exists a closed set $F \subseteq E$ such that $m^*(E \setminus F) < \varepsilon$. Since closed sets are measurable, by Corollary 3.31, we have

$$m^*(E \setminus F) = m(E) - m(F) < \varepsilon$$

so

$$m(E) < m(F) + \varepsilon.$$

Letting ε go to 0, we have

$$m(E) \leq m(F);$$

and taking the supremum on the right

$$m(E) \leq m_*(E).$$

But, by part (i), $m_*(E) \leq m^*(E) = m(E)$. Thus, $m_*(E) = m^*(E)$ as was to be shown.

$\Longleftarrow$ On the other hand, suppose that $m_*(E) = m^*(E)$. Then, given $\varepsilon > 0$ there exists an open set G containing E and a closed set F contained in E such that

$$m(G) - m^*(E) < \frac{\varepsilon}{2}$$
$$m_*(E) - m(F) < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}
m^*(E \setminus F) &< m^*(G \setminus F) \\
&= m^*(G) - m^*(G \cap F) \\
&= m^*(G) - m^*(F) \\
&< \frac{\varepsilon}{2} + m^*(E) - \Big(m^*(E) - \frac{\varepsilon}{2}\Big) \\
&= \varepsilon.
\end{aligned}$$

Thus, by Lemma 3.22, E is measurable. ◀

Problem 3 (Wheeden & Zygmund Ch. 3, Ex. 15)**.** If E is measurable and A is any subset of E, show that $m(E) = m_*(A) + m^*(E \setminus A)$. (See Exercise 13 for the definition of $m_*(A)$.)

Solution. ▶ Suppose $A \subseteq E$. If A is measurable, by Problem 2, the outer and inner measure of A agree; symbolically, we have $m(A) = m^*(A) = m_*(A)$. Thus, we have

$$m^*(E \setminus A) = m^*(E) - m^*(A) = m^*(E) - m_*(A).$$

If A is not measurable and $m(E) < \infty$, then we must have $m^*(A), m^*(E \setminus A) < \infty$ by the monotonicity of the outer measure; since both A an $E \setminus A$ are subsets of E. Hence, we may, without any ambiguity, subtract the quantity $m^*(E \setminus A)$ from $m(E)$ and we have

$$\begin{aligned}
m(E) - m^*(E \setminus A) &= m(E) - \inf\{\, m(G) : E \setminus A \subseteq G \text{ and } G \text{ is open}\,\} \\
&= m(E) - \inf\{\, m(G) : E \setminus A \subseteq G \subseteq E \text{ and } G \text{ is open}\,\} \\
&=
\end{aligned}$$

◀

1.1.5 Homework 5

Problem 1 (Wheeden & Zygmund Ch. 3, Ex. 14)**.** Show that the conclusion of part (ii) of Exercise 13 is false if $m^*(E) = \infty$.

Solution. ▸ Part (ii) of Exercise 13 is part (ii) of Problem 2 from the last section (Homework 4). In that problem we showed that if the outer measure of E is finite, then E is measurable if and only if its outer and inner measure agree. Here we construct a counter example to this when the outer measure of E is ∞; that is, we show that there exists a set E with $m^*(E) = \infty$ such that $m^*(E) \neq m_*(E)$. So, which set shall it be? Since we are unoriginal, we will pull an example from Wheeden and Zygmund itself.

Let $V \subseteq [0, 1]$ be Vitali's unmeasurable (Theorem 3.38) and consider the union $E = V \cup (2, \infty)$. It is clear that the inner and outer measure of E are both ∞. However, E itself must be unmeasurable for otherwise $E \cap [0, 1] = V$ is measurable. ◂

Problem 2 (Wheeden & Zygmund Ch. 3, Ex. 16)**.** Prove (3.34).

Solution. ▸ We must prove Equation 3.34; that is, if P is a parallellepiped

$$m(P) = \mathrm{vol}(P).$$

We may, without loss of generality, assume that one of the vertices of P is $\mathbf{0}$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a set of vectors such that

$$P = \left\{ x \in \mathbb{R}^n : x = \sum_{k=1}^n t_k \mathbf{e}_k,\ 0 \leq t_k \leq 1 \right\}.$$

By definition, the measure of P is

$$m(P) = \inf_{\mathcal{S}} \left[\sum_{I_n \in \mathcal{S}} \mathrm{vol}(I_n) \right]$$

where $\mathcal{S}$ is a cover of P by intervals. Take the set of

> *Remarks.* Literally nobody cares about this problem. I don't remember how to do it, but it must have been painful if I can't figure it out now, even.

◂

Problem 3 (Wheeden & Zygmund Ch. 3, Ex. 18)**.** Prove that outer measure is *translation invariant*; that is, if $E_h = \{ x + h : x \in E \}$ is the translate of E by h, $h \in \mathbb{R}^n$, show that $m^*(E_h) = m^*(E)$. If E is measurable, show that E_h is also measurable. [This fact was used in proving (3.37).]

Solution. ▸ Let $E \subseteq \mathbb{R}^n$ and $h \in \mathbb{R}^n$ and define the set E_h to be the set $E_h = \{ x + h : x \in E \}$. We will show that the outer measure of E is preserved under such translations. But first, let us point out that E_h is nothing more than the image of E under the linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$

given by $x \mapsto x + h$. By Theorem 3.35, such a map preserves measurability of sets and for any measurable set $E' \subseteq \mathbb{R}^n$, $m(T(E')) = (\det T)m(E') = m(E')$ (since $\det T = 1$. Now, by Theorem 3.6, for every $\varepsilon > 0$, there exist an open set $G \supseteq E$ such that $m^*(G) \leq m^*(E) + \varepsilon$. Consider the image of G under T, $T(G)$ is an open set containing E_h so $m^*(G) \geq m^*(E)$ and

$$m^*(T(G)) = m^*(G) < m^*(E) + \varepsilon.$$

Letting $\varepsilon \to 0$, we achieve the inequality

$$m^*(E_h) \leq m^*(E).$$

To get the other inequality, take the map $T^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ which takes $x \mapsto x - h$; this sends E_h to E and the same argument shows that

$$m^*(E) \leq m^*(E).$$

Thus, we have $m^*(E) = m^*(E_h)$, as was to be shown. ◂

Problem 4 (Wheeden & Zygmund Ch. 4, Ex. 1)**.** Prove corollary (4.2) and theorem (4.8)

Solution. ▸ The corollary and theorem in question are:

> *If f is measurable, then $\{ f > -\infty \}$, $\{ f < +\infty \}$, $\{ f = +\infty \}$, $\{ a \leq f \leq b \}$, $\{ f = a \}$, etc., are all measurable. Moreover f is measurable if and only if $\{ a < f < +\infty \}$ is measurable for every finite a.*

and

> *If f is measurable and λ is any real number, then $f + \lambda$ and λf are measurable.*

Their proofs are quite simple. For the corollary: Suppose $f \colon E \to \mathbb{R}$ is a measurable function. By Theorem 4.1, f is measurable if and only if for every finite $\alpha \in \mathbb{R}$, the sets

$$\begin{gathered} \{ x \in E : f(x) \geq \alpha \} \\ \{ x \in E : f(x) < \alpha \} \\ \{ x \in E : f(x) \leq \alpha \} \end{gathered}$$

are measurable. Since measurable sets form a σ-algebra on $\mathbb{R}^n$, we know that the countable union and intersection of measurable sets is measurable. Thus,

$$\begin{aligned} \{ x \in E : f(x) > -\infty \} &= \bigcup_{\alpha \in \mathbb{Z}} \{ x \in E : f(x) > \alpha \} \\ \{ x \in E : f(x) = \infty \} &= \bigcap_{n=1}^{\infty} \{ x \in E : f(x) > n \} \\ \{ x \in E : f(x) < \infty \} &= \bigcup_{\alpha \in \mathbb{Z}} \{ x \in E : f(x) < \alpha \} \end{aligned}$$

are easily seen to be measurable.

Showing that $\{ x \in E : f(x) = \alpha \}$ and $\{ x \in E : \alpha < f(x) < \beta \}$ are measurable requires some clever (but not too clever) intersection/union of the sets we get from Theorem 4.1.

For the theorem: Suppose f is measurable and λ is a constant. By Theorem 4.1, for any finite $\alpha \in \mathbb{R}$ we have

$$\{ x \in E : f(x) > \alpha - \lambda \}$$

so

$$\{ x \in E : f(x) + \lambda > \alpha \}$$

is measurable. Thus, $f + \lambda$ is measurable. Similarly, for $\lambda \neq 0$, taking the set

$$\{ x \in E : f(x) > \alpha/\lambda \} = \{ x \in E : \lambda f(x) > \alpha \}$$

shows that λf is measurable; otherwise, if $\lambda = 0$, $\lambda f = 0$ is constant and hence is continuous which in turn implies that it is measurable. ◀

Problem 5 (Wheeden & Zygmund Ch. 4, Ex. 2)**.** Let f be a simple function, taking its distinct values on disjoint sets $E_1, \ldots, E_N$. Show that f is measurable if and only if $E_1, \ldots, E_N$ are measurable.

Solution. ▶ $\Longrightarrow$ Suppose that f is measurable. Then, by Corollary 4.2, the sets of the form $\{ f = \alpha_n \} = E_n$ are measurable. So the sets E_n are measurable.

$\Longleftarrow$ On the other hand, suppose that the sets E_n are measurable. Then, χ_{E_n} is measurable so by Theorem 4.8, f is measurable since it is the sum

$$f = \sum_{n=1}^{N} \alpha_{E_n}.$$

◀

1.1.6 Homework 6

Problem 1 (Wheeden & Zygmund Ch. 4, Ex. 4)**.** Let f be defined and measurable in $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(T(x))$ is measurable. [If $E_1 = \{ x : f(x) > a \}$ and $E_2 = \{ x : f(T(x)) > a \}$, show $E_2 = T^{-1}(E_1)$.]

Solution. ▶ Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then, we show that the composition $f \circ T$ is measurable. Fix a finite $\alpha \in \mathbb{R}$ and let

$$\begin{aligned} E_1 &= \{ x : f(x) > \alpha \} \\ E_2 &= \{ x : f(T(x)) > \alpha \}. \end{aligned}$$

Then, by Theorem 3.35, it suffices to show that $E_2 = T^{-1}(E_1)$ since T^{-1} is a nonsingular linear transformation so it sends measurable sets to measurable sets. But this equality is obvious: Suppose $x \in E_2$; then $f(T(x)) > \alpha$ so, because T is nonsingular and therefore bijective, clearly $x \in T^{-1}(E_1)$ so $E_2 \subseteq T^{-1}(E_1)$. One the other hand, if $x \in T^{-1}(E_1)$ then x is a point in E such that $f(T(x)) > \alpha$ so $x \in E_2$. Thus, $E_2 = T^{-1}(E_1)$ and consequently, $f \circ T$ is a measurable function. ◀

Problem 2 (Wheeden & Zygmund Ch. 4, Ex. 7)**.** Let f be usc and less that ∞ on a compact set E. Show that f is bounded above on E. Show also that f assumes its maximum on E, i.e., that there exists $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.

Solution. ▶ First we show that f is bounded. Suppose that f is u.s.c. on E. Then, by Theorem 4.14 (i), sets of the form $\{ x \in E : f(x) < \alpha \}$ are relatively open. Let $\mathcal{G} = \{G_\alpha\}_{\alpha \in \mathbb{Z}}$ where $G_a = \{ x \in E : f(x) < \alpha \}$. Then $\mathcal{G}$ forms an open cover of E and since E is compact there exists a finite subset $\{G_{\alpha_n}\}_{n=1}^N$ for some finite subset $\{ \alpha_1, \dots, \alpha_N \}$ of $\mathbb{Z}$. Let $\alpha = \max\{ \alpha_1, \dots, \alpha_N \}$. Then, $f(x) < \alpha$ for all $x \in E$ so f is bounded above by α.

Next, we show that f in fact assumes its maximum (locally) on E by using only topological properties of f. Since sets of the form $\{ x \in E : f(x) \geq \alpha \}$ are relatively closed, by Theorem 4.14 (i), for fixed $x \in E$ the sets $F_x = \{ y \in E : f(y) \geq f(x) \}$ are relatively closed. Consider the collection $\{F_x\}_{x \in E}$ of closed subsets of E. First, note that each of these sets is nonempty since $f(x) \geq f(x)$ so $x \in F_x$ for every $x \in E$. Now, let $\{x_n\}_{n=1}^N \subseteq E$ and consider the collection $\{F_{x_n}\}_{n=1}^N$. Then $\bigcap_{n=1}^N F_{x_n} \neq \emptyset$ since for x the point in $\{ x_1, \dots, x_N \}$ such that $f(x) = \min\{ f(x_1), \dots, f(x_N) \}$, $x \in F_{x_n}$ for all $1 \leq n \leq N$. Thus, by the finite intersection property, the intersection $F = \bigcap_{x \in E} F_x$ is nonempty. Let $y \in \bigcap_{x \in E} F_x$, then $f(y) \geq f(x)$ for all $x \in E$ so f achieves its maximum (locally) on E. ◀

Problem 3 (Wheeden & Zygmund Ch. 4, Ex. 8)**.**

(a) Let f and g be two functions which are u.s.c. at x_0. Show that $f + g$ is u.s.c. at x_0. Is $f - g$ u.s.c. at x_0? When is fg u.s.c. at x_0?

(b) If $\{f_k\}$ is a sequence of functions are u.s.c. at x_0, show that $\inf f_k(x)$ is u.s.c. at x_0.

(c) If $\{f_k\}$ is a sequence of functions which are u.s.c. at x_0 and which converge uniformly near x_0, show that $\lim f_k$ is u.s.c. at x_0.

Solution. ▸ We prove these in alphabetical order (a) → (b) → (c).

For (a), suppose that f and g are u.s.c. at x_0. Then given $M > f(x_0), g(x_0)$ there exists $\delta_1, \delta_2 > 0$ such that $f(x), g(x) < M/2$ for all $|x_1 - x_0| < \delta_1, |x_2 - x_0| < \delta_2$, respectively. Let δ be the minimum of $\{\delta_1, \delta_2\}$. Then for any x such that $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) + g(x) - (f(x_0) + g(x_0))| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\ &\leq |(f(x) - f(x_0))| + |(g(x) - g(x_0))| \\ &< \frac{M}{2} + \frac{M}{2} \\ &= M. \end{aligned}$$

Thus, $f + g$ is u.s.c.

For that second little part of (a), the one that asks "Is $f - g$ u.s.c. at x_0?" we provide a counter example. In fact, the following is enough of a counterexample: Take $f = 0$ (which is continuous everywhere) and g any function that is u.s.c., but not continuous, at x_0 then $f - g = -g$ is l.s.c. at x_0. Another counterexample is provided by the equations u_1 and u_2 from Ch. 4 of Wheeden and Zygmund: Fix an $x_0 \in \mathbb{R}$ and define

$$u_1(x) = \begin{cases} 0 & \text{if } x < x_0, \\ 1 & \text{if } x \geq x_0, \end{cases} \qquad u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

Then

$$u_1(x) - u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

is not u.s.c. at x_0 since being u.s.c. at x_0 implies that for $1/2 > f(x_0) = 0$ there exists $\delta > 0$ such that $f(x) < 1/2$ for all $x \in (x_0 - \delta, x_0 + \delta)$. But for any $x' > x_0$ in $(x_0 - \delta, x + \delta)$, $u(x') = 1 > 1/2$ which contradicts the assumption that u is u.s.c. at x_0.

For (b), suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions that are u.s.c. at x_0. Then

$$\limsup_{\substack{x \to x_0 \\ x \in E}} f_n(x) \leq f_n(x_0)$$

for all $n \in \mathbb{N}$. We must show that

$$\limsup_{\substack{x \to x_0 \\ x \in E}} \big[\inf f_n(x)\big] \leq \inf f_n(x_0).$$

◂

1.1.7 Homework 7

Problem 1 (Wheeden & Zygmund Ch. 4, Ex. 9)**.**

(a) Show that the limit of a decreasing (increasing) sequence of functions u.s.c. (l.s.c.) at x_0 is u.s.c. (l.s.c.) at x_0. In particular, the limit of a decreasing (increasing) sequence of functions continuous at x_0 is u.s.c. (l.s.c.) at x_0.

(b) Let f be u.s.c. and less than ∞ on $[a, b]$. Show that there exists continuous f_k on $[a, b]$ such that $f_k \downarrow f$.

Solution. ▸ For part (a) we may as well assume that $f \geq 0$ for all x. Let $\{f_n\}$, $n \in \mathbb{N}$, be a sequence of decreasing functions with limit f which are u.s.c. at x_0. Then, for every $n \in \mathbb{N}$, for every sequence $x \to x_0$,

$$\limsup_{x \to x_0} f_n(x) \leq f_n(x_0).$$

Now, we claim that $f(x) \leq f_n(x)$ for every x and every $n \in \mathbb{N}$.

Proof of claim. Suppose $f(x) > f_{N_1}(x)$ for some x, $N_1 \in \mathbb{N}$. Then there exists a real number $\varepsilon > 0$ such that $0 < \varepsilon < |f(x) - f_n(x)|$ (we may, for example, take ε to be in $\mathbb{Q}$ which is dense in $\mathbb{R}$. Then, since $f_n \downarrow f$, there exists an index $N_1 \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \varepsilon.$$

However, since the sequence f_n decreases to f, for $n \geq \max\{N_1, N_2\}$, $f_n(x) \leq f_{N_1}(x)$ so

$$|f(x) - f_n(x)| > |f(x) - f_{N_1}(x)| > \varepsilon.$$

This is a contradiction. ■

Having established this, for every sequence $x \to x_0$, we have

$$\limsup_{x \to x_0} f(x) \leq \limsup_{x \to x_0} f_n(x) \leq f_n(x_0).$$

Letting $n \to \infty$,

$$\limsup_{x \to x_0} f(x) \leq \lim_{n \to \infty} f_n(x_0) = f(x_0).$$

For part (b) suppose $f\colon [a, b] \to \mathbb{R}$ is u.s.c. on $[a, b]$ and $f(x) < \infty$ for all $x \in [a, b]$. For a fixed $x \in [a, b]$, f is u.s.c. at x if for every $\varepsilon > 0$, there exists a neighborhood $B(x, \delta)$ such that $f(y) < f(x) + \varepsilon$. Now, let $\varepsilon = 1/n$. Then, for each $x \in [a, b]$, there exists a neighborhood $B(x, \delta_x)$ such that $f(y) < f(x) + \varepsilon$ for $y \in B(x, \delta_x)$.

The following post on the Mathematics StackExchange contains a solution to part (b) of this problem.

First, we claim that $f(x) \neq \infty$ for any $x \in [a, b]$, it must be bounded.

Proof of claim. By Theorem 4.14 (a), sets of the form $\{\, x \in [a, b] : f(x) < a \,\}$ is relatively open for all finite a. Define

$$E_n = \{\, x \in [a, b] : f(x) < n \,\}.$$

Then, the collection $\mathcal{E} = \{E_n\}$, $n \in \mathbb{N}$, is an open cover of $[a, b]$. Since $[a, b]$ is compact, there exists a finite subcover $\{E_{n_1}, \ldots, E_{n_m}\}$ of $\mathcal{E}$. Letting $M = \max\{n_1, \ldots, n_m\}$, we have $f < M$ for all $x \in [a, b]$. Thus, f is bounded on $[a, b]$. ■

Now that we have established that f is bounded on $[a, b]$ by, say, M then $\sup_{x\in[a,b]} f \leq M$. Define

$$f_n(x) = \sup_{y\in[a,b]} \big[f(y) - n|x - y|\big].$$

We claim that this family of functions $\{f_n\}$, $n \in \mathbb{N}$, is continuous and that $f_n \to f$. To see that f is continuous, we observe that this family of functions is in fact Lipschitz continuous

$$\begin{aligned}
|f_n(x) - f_n(y)| &= \left| \sup_{z\in[a,b]} \big[f(z) - n|x - z|\big] - \sup_{z\in[a,b]} \big[f(z) - n|y - z|\big] \right| \\
&\leq \left| \sup_{z\in[a,b]} \big[f(z) - n|x - z| - f(z) - n|y - z|\big] \right| \\
&= \left| \sup_{z\in[a,b]} \big[-n|x - z| - n|y - z|\big] \right| \\
&= \left| \sup_{z\in[a,b]} \big[-n|x - y + (y - z)| - n|y - z|\big] \right| \\
&\leq \left| \sup_{z\in[a,b]} \big[-n|x - y| - 2n|y - z|\big] \right| \\
&= n|x - y|.
\end{aligned}$$

Thus, f_n is Lipschitz and in particular, it is continuous.

To see that $f_n \to f$ pointwise, let $\varepsilon > 0$ be given then we must show that there exists some index N such that $n \geq N$ implies

$$|f(x) - f_n(x)| < \varepsilon.$$

Expanding the equation above, we see that

$$|f(x) - f_n(x)| = \left| f(x) - \sup_{y\in[a,b]} \big[f(y) - n|x - y|\big]. \right|$$

◀

Problem 2 (Wheeden & Zygmund Ch. 4, Ex. 11)**.** Let f be defined on $\mathbb{R}^n$ and let $B(x)$ denote the open ball $\{\, y : |x - y| < r \,\}$ with center x and fixed radius r. Show that the function $g(x) = \sup\{\, f(y) : y \in B(x) \,\}$ is l.s.c. and the function $h(x) = \inf\{\, f(y) : y \in B(x) \,\}$ is u.s.c. on $\mathbb{R}^n$. Is the same true for the closed ball $\{\, y : |x - y| \leq r \,\}$?

Solution. ▶ Note that, by properties of the infimum/supremum for any set of real numbers $S \subset \mathbb{R}$,

$$\sup S = -\inf(-S)$$

where $-S = \{ -s : s \in S \}$. Thus,

$$\begin{aligned} g(x) &= -\inf\{ -f(y) : y \in B(x,r) \} \\ &= \sup\{ f(y) : y \in B(x,r) \}. \end{aligned}$$

Letting $f' = -f$, it suffices to show that $g'(x) = \inf\{ f'(y) : y \in B(x,r) \}$ is u.s.c. since for any u.s.c. function f, $-f$ is l.s.c. Therefore, we show that h is u.s.c.

To see that h is u.s.c., let $M > h(x_0)$. Then we must show that there exists a neighborhood $B(x_0,\delta)$ such that $M > h(x)$ for every $x \in B(x_0,\delta)$. Since $h(x_0)$ is the infimum of $f(x)$ over all $x \in B(x_0,r)$, given $\varepsilon > 0$ there exists $x \in B(x_0,r)$ such that $f(x) < h(x_0) + \varepsilon < M$. Define $\delta = (r - |x-y|)/2$. Then we claim that for any $x \in B(x_0,\delta)$,

$$g(x) < M.$$

Proof of claim. Let $x \in B(x_0,\delta)$. Then $y \in B(x_0,\delta)$ since

$$\begin{aligned} |x-y| &= |x - x_0 - (y - x_0)| \\ &\leq |x - x_0| + |y - x_0| \\ &= (r - |y - x_0|)/2 + |y - x_0| \\ &= r/2 + |y - x_0|/2 \\ &< r. \end{aligned}$$

Thus,

$$g(x) \leq f(y) < g(x_0) + \varepsilon < M.$$

■

It follows that g is u.s.c. ◀

Problem 3 (Wheeden & Zygmund Ch. 4, Ex. 15)**.** Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$. If $|f_k(x)| \leq M_x < \infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there is closed $F \subseteq E$ and finite M such that $m(E \setminus F) < \varepsilon$ and $|f_k(x)| \leq M$ for all $x \in F$.

Solution. ▶ Set $f = \sup_{n\in\mathbb{N}} |f_n|$; then, f is measurable since it is the supremum of measurable functions $|f_n|$. By Lusin's theorem f satisfies the $\mathcal{C}$-property, i.e., there exists a closed subset F' of E with $m(E \setminus F') < \varepsilon/2$ and a continuous function $\bar{f} \colon E \to \mathbb{R}$ such that $f\restriction_{F'} = \bar{f}\restriction_{F'}$. Now, let B be the closed ball centered at $\mathbf{0}$ such that $|E \setminus B| < \varepsilon/2$ (remember, this is all taking place in $\mathbb{R}^n$, so we can do this). Thus, $F' \cap B$ is compact since it is a closed subset of B the latter being a compact set. Let $F = F' \cap B$ then,

$$\begin{aligned} |E \setminus F| &= |E \setminus (F' \cap B)| \\ &= |(E \setminus F') \cup (E \setminus B)| \\ &\leq |E \setminus F'| + |E \setminus B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

so F has the desired measure. Lastly, by the mean value theorem, f achieves its maximum, call it M, on F since F is compact. It follows that $f_n\restriction_F \leq M$ for all $n \in \mathbb{N}$. ◀

Problem 4 (Wheeden & Zygmund Ch. 4, Ex. 18)**.** If f is measurable on E, define $\omega_f(a) = m\{\, f > a\,\}$ for $-\infty < a < \infty$. If $f_k \uparrow f$, show that $\omega_{f_k} \uparrow \omega_f$. If $f_k \to f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f. [For the second part, show that if $f_k \to f$, then $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a-\varepsilon)$ and $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a+\varepsilon)$ for every $\varepsilon > 0$.]

Solution. ▶ For the first part of this problem we will show that the sequence of distribution functions $\{\omega_{f_n}\}$, $n \in \mathbb{N}$, is increasing and that its limit is ω_f. It is easy to verify that this sequence is in fact increasing: if $x \in \{\, f_{n-1} \geq M\,\}$ then $x \in \{\, f_n \geq M\,\}$ since $f_n \geq f_{n-1}$ for all $x \in E$. Thus, $\omega_{f_n} \geq \omega_{f_{n-1}}$. Now we need to show that the limit of this sequence is in fact ω_f: fix an $x \in E$ and let $\varepsilon > 0$ be given. Then there exists an index N' such that $n \geq N'$ implies $|f(x) - f_n(x)| < \varepsilon$. Now, we want to use this ε and index N' (with some possible alterations), for some fixed M, we want to show that the difference

$$|\omega_f(M) - \omega_{f_n}(M)| < \varepsilon.$$

First, by properties of the Lebesgue measure

$$m\{\, f > M\,\} - m\{\, f_n > M\,\} \leq m(\{\, f > M\,\} \setminus \{\, f_n > M\,\}).$$

In turn, it is easy to see that the latter set is in fact

$$\begin{aligned} E_{M,n} &= \{\, x \in E : f(x) > M \text{ and } f_n(x) \leq M\,\} \\ &= \{\, x \in E : f(x) > M \text{ and } f(x) - f_n(x) > 0\,\}. \end{aligned}$$

Then, $E_{M,n} \subseteq \{\, x \in E : f(x) - f_n(x) > M\,\} = E_{0,n}$ and the the measure of the latter set converges to 0 since $f_n \to f$ and this implies that f_n converges to f in measure (a weaker form of pointwise convergence). Let N'' be the index such that $n \geq N'$ implies $m(E_{0,n}) < \varepsilon$. Then for $n \geq N$ with $N = \max\{N', N''\}$, the difference

$$|\omega_f(M) - \omega_{f_n}(M)| < \varepsilon.$$

Thus, we have shown that $\omega_{f_n} \uparrow \omega_f$. ◀

Problem 5 (Wheeden & Zygmund Ch. 5, Ex. 1)**.** If f is a simple measurable function (not necessarily positive) taking values a_j on E_j, $j = 1, \dots, N$, show that $\int_E f = \sum_{j=1}^N a_j m(E_j)$. [Use (5.24)].

Solution. ▶ It is enough to consider simple positive measurable functions f since we can split f into the difference of two positive simple measurable functions, namely, $f = f^+ - f^-$. Now, since f is a simple function, $f = \sum_{n=1}^N a_n \chi_{E_n}$ for measurable subsets $E_n \subseteq E$. Now, by Theorem 5.24, we have

$$\begin{aligned}\int_E f \,\mathrm{d}x &= \int_E \left[\sum_{n=1}^{N} a_n \chi_{E_n}\right] \mathrm{d}x \\ &= \sum_{n=1}^{N} \int_{E_n} a_n \,\mathrm{d}x \\ &= \sum_{n=1}^{N} a_n m(E_n),\end{aligned}$$

as we set out to show. ◂

Problem 6 (Wheeden & Zygmund Ch. 5, Ex. 3)**.** Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \le f$ a.e. on E, show that $\int_E f_k \to \int_E f$.

Solution. ▸ The result follows from a simple application of Fatou's lemma. Consider the sequence of integrals $\left\{\int_E f_n\right\}$, $n \in \mathbb{N}$. By Fatou's lemma

$$\begin{aligned}\int_E \liminf_{n\to\infty} f_n \,\mathrm{d}x &= \int_E f \,\mathrm{d}x \\ &\le \liminf_{n\to\infty} \int_E f_n \,\mathrm{d}x.\end{aligned}$$

By Theorem 5.10, since $f_n \le f$, we have

$$\limsup_{n\to\infty} \int_E f_n \,\mathrm{d}x \le \int_E f \,\mathrm{d}x.$$

Thus, we have

$$\limsup_{n\to\infty} \int_E f_n \,\mathrm{d}x \le \liminf_{n\to\infty} \int_E f_n \,\mathrm{d}x,$$

which implies that

$$\limsup_{n\to\infty} \int_E f_n \,\mathrm{d}x = \liminf_{n\to\infty} \int_E f_n \,\mathrm{d}x$$

so

$$\lim_{n\to\infty} \int_E f_n \,\mathrm{d}x = \int_E f \,\mathrm{d}x$$

as we set out to show. ◂

1.1.8 Homework 8

Problem 1 (Wheeden & Zygmund Ch. 5, Ex. 2)**.** Show that the conclusion of (5.32) are not true without the assumption that $\varphi \in L(E)$. [In part (ii), for example, take $f_k = \chi_{(k,\infty)}$.]

Solution. ▸ ◂

Problem 2 (Wheeden & Zygmund Ch. 5, Ex. 4)**.** If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and $\int_0^1 x^k f(x)\,\mathrm{d}x \to 0$.

Solution. ▸ ◂

Problem 3 (Wheeden & Zygmund Ch. 5, Ex. 6)**.** Let $f(x,y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $\partial f(x,y)/\partial x$ is a bounded function of (x,y). Show that $\partial f(x,y)/\partial x$ is a measurable function of y for each x and

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^1 f(x,y)\,\mathrm{d}y = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,\mathrm{d}y.$$

Solution. ▸ ◂

Problem 4 (Wheeden & Zygmund Ch. 5, Ex. 7)**.** Give an example of an f that is not integrable, but whose improper Riemann integral exists and is finite.

Solution. ▸ ◂

Problem 5 (Wheeden & Zygmund Ch. 5, Ex. 21)**.** If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

Solution. ▸ ◂

Problem 6 (Wheeden & Zygmund Ch. 6, Ex. 10)**.** Let V_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theorem that

$$V_n = 2V_{n-1} \int_0^1 \left(1 - t^2\right)^{(n-1)/2} \mathrm{d}t.$$

(We also observe that by setting $w = t^2$, the integral is a multiple of a classical β-function and so can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,\mathrm{d}t$, $s > 0$.)

Solution. ▸ ◂

Problem 7 (Wheeden & Zygmund Ch. 6, Ex. 11)**.** Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,\mathrm{d}x = \pi^{n/2}.$$

(For $n = 1$, write $\left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x dy$ and use polar. For $n > 1$, use the formula $e^{-|x|^2} = e^{-x_1{}^2}\cdots e^{-x_n{}^2}$ and Fubini's theorem to reduce the case $n = 1$.)

Solution. ▶ ◀

1.1.9 Homework 9

Problem 1 (Wheeden & Zygmund Ch. 6, Ex. 1)**.**

(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}$, $\{ y : (x, y) \in E \}$ has $\mathbb{R}$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}$, $\{ x : (x, y) \in E \}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x, y)$ is finite for almost every y. Show that for almost $y \in \mathbb{R}$, $f(x, y)$ is finite for almost every x.

Solution. ▸ ◂

Problem 2 (Wheeden & Zygmund Ch. 6, Ex. 3)**.** Let f be measurable and finite a.e. on $[0, 1]$. If $f(x) - f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$, show that $f \in L[0, 1]$.

Solution. ▸ ◂

Problem 3 (Wheeden & Zygmund Ch. 6, Ex. 4)**.** Let f be measurable and periodic with period 1: $f(t + 1) = f(t)$. Suppose there is a finite c such that

$$\int_0^1 |f(a + t) - f(b + t)| \,\mathrm{d}t \le c$$

for all a and b. Show that $f \in L[0, 1]$. (Set $a = x$, $b = -x$, integrate with respect to x, and make the change of variables $\xi = x + t$, $\eta = -x + t$.)

Solution. ▸ ◂

Problem 4 (Wheeden & Zygmund Ch. 6, Ex. 6)**.** For $f \in L(\mathbb{R})$, define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} \,\mathrm{d}t$$

for $x \in \mathbb{R}$. (For complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i \int F_1$.) Show that if f and g belong to $L(\mathbb{R})$, then

$$\widehat{(f * g)}(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

Solution. ▸ ◂

Problem 5 (Wheeden & Zygmund Ch. 6, Ex. 7)**.** Let F be a closed subset of $\mathbb{R}$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F, prove that the function

$$\int_{\mathbb{R}} \frac{\delta^\lambda(y) f(y)}{|x - y|^{1+\lambda}} \,\mathrm{d}t$$

is integrable over F and so is finite a.e. in F. (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

Solution. ▶ ◀

Problem 6 (Wheeden & Zygmund Ch. 6, Ex. 9)**.**

(a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.

(b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subseteq \mathbb{R}$, and let M_α be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int M_\lambda \leq \lambda^{-1}|G|$, where $G = (a, b) - [c, d]$.

Solution. ▶ ◀

1.1.10 Homework 10

Problem 1 (Wheeden & Zygmund Ch. 7, Ex. 1)**.** Let f be measurable in $\mathbb{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(x) \geq c\|x\|^{-n}$ for $\|x\| \geq 1$.

Solution. ▶ ◀

Problem 2 (Wheeden & Zygmund Ch. 7, Ex. 2)**.** Let $\varphi(x), x \in \mathbb{R}^n$, be a bounded measurable function such that $\varphi(x) = 0$ for $\|x\| \geq 1$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. ($\varphi_\varepsilon$ is called an *approximation to the identity.*) If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon \to 0}(f * \varphi_\varepsilon)(x) = f(x)$$

in the Lebesgue set of f. (Note that $\int \varphi_\varepsilon = 1$, $\varepsilon > 0$, so that

$$(f * \varphi_\varepsilon)(x) - f(x) = \int [f(x-y) - f(x)]\varphi_\varepsilon(y)\,\mathrm{d}y.$$

Use Theorem 7.16.)

Solution. ▶ ◀

Problem 3 (Wheeden & Zygmund Ch. 7, Ex. 6)**.** Show that if $\alpha > 0$, then x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Solution. ▶ ◀

Problem 4 (Wheeden & Zygmund Ch. 7, Ex. 8)**.** Prove the following converse of Theorem 7.31: If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Solution. ▶ ◀

Problem 5 (Wheeden & Zygmund Ch. 7, Ex. 9)**.** If f is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then f is absolutely continuous on $[a, b]$. (For the second part, use Theorems 2.2(ii) and 7.24 to show that $V(x)$ is absolutely continuous and then use the result of Exercise 8).

Solution. ▶ ◀

Problem 6 (Wheeden & Zygmund Ch. 7, Ex. 12)**.** Use Jensen's inequality to prove that if $a, b \geq 0$, $p, q > 1$, $(1/p) + (1/q) = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

More generally, show that

$$a_1 \cdots a_N = \sum_{j=1}^{N} \frac{{a_j}^{p_j}}{p_j},$$

where $a_j \geq 0$, $p_j > 1$, $\sum_{j=1}^{N}(1/p_j) = 1$. (Write $a_j = e^{x_j/p_j}$ and use the convexity of e^x.

Solution. ▶ ◀

Problem 7 (Wheeden & Zygmund Ch. 7, Ex. 13)**.** Prove Theorem 7.36.

Solution. ▶ Recall the statement of Theorem 7.36

(i) *If φ_1 and φ_2 are convex in (a, b), then $\varphi_1 + \varphi_2$ is convex in (a, b).*

(ii) *If φ is convex in (a, b) and c is a positive constant, then $c\varphi$ is convex in (a, b).*

(iii) *If φ_k, $k = 1, 2, \ldots$, are convex in (a, b) and $\varphi_k \to \varphi$ in (a, b), then φ is convex in (a, b).*

◀

1.1.11 Homework 11

Problem 1 (Wheeden & Zygmund Ch. 7, Ex. 11)**.** Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t.$$

(Consider the case when f is the characteristic function of an interval, an open set, etc.)

Solution. ▸ ◂

Problem 2 (Wheeden & Zygmund Ch. 7, Ex. 15)**.** Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\varphi(x) = \int_a^x f(t)\,\mathrm{d}t + \varphi(a)$ in (a, b) and f is monotone increasing, then φ is convex in (a, b). (Use Exercise 14.)

Solution. ▸ ◂

Problem 3 (Wheeden & Zygmund Ch. 5, Ex. 8)**.** Prove (5.49).

Solution. ▸ ◂

Problem 4 (Wheeden & Zygmund Ch. 5, Ex. 11)**.** For which p does $1/x \in L^p(0, 1)$? $L^p(1, \infty)$? $L^p(0, \infty)$?

Solution. ▸ ◂

Problem 5 (Wheeden & Zygmund Ch. 5, Ex. 12)**.** Give an example of a bounded continuous f on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

Solution. ▸ ◂

Problem 6 (Wheeden & Zygmund Ch. 5, Ex. 17)**.** If $f \geq 0$ and $\omega(\alpha) \leq c(1 + \alpha)^p$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.

Solution. ▸ ◂

Problem 7 (Wheeden & Zygmund Ch. 8, Thm. 8.3)**.** If $f, g \in L^p(E)$, $p > 0$, then $f + g \in L^p(E)$ and $cf \in L^p(E)$ for any constant c.

Solution. ▸ ◂

1.1.12 Homework 12

Problem 1 (Wheeden & Zygmund Ch. 8, Ex. 2)**.** Prove the converse of Hölder's inequality for $p = 1$ and ∞. Show also that for $1 \le p \le \infty$, a real-valued measurable f belongs to $L^p(E)$ if $fg \in L^1(E)$ for every $g \in L^{p'}(E)$, $1/p + 1/p' = 1$. The negation is also of interest: if $f \in L^p(E)$ then there exists $g \in L^{p'}(E)$ such that $fg \notin L^1(E)$. (To verify the negation, construct g of the form $\sum a_k g_k$ satisfying $\int_E fg_k \to \infty$.)

Solution. ▸ ◂

Problem 2 (Wheeden & Zygmund Ch. 8, Ex. 3)**.** Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when $p < 1$.

Solution. ▸ ◂

Problem 3 (Wheeden & Zygmund Ch. 8, Ex. 4)**.** Let f and g be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let $1 < p < \infty$. Prove that equality holds in the inequality $|\int fg| \le \|f\|_p \|g\|_{p'}$ if and only if fg has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e.

If $\|f + g\|_p = \|f\|_p + \|g\|_p$ and $g \neq 0$ in Minkowski's inequality, show that f is a multiple of g.

Find analogues of these results for the spaces ℓ^p.

Solution. ▸ ◂

Problem 4 (Wheeden & Zygmund Ch. 8, Ex. 5)**.** For $0 < p \le \infty$ and $0 < |E| < \infty$, define

$$N_p[f] = \left(\frac{1}{E} \int_E |f|^p \right)^{1/p},$$

where $N_\infty[f]$ means $\|f\|_\infty$. Prove that if $p_1 < p_2$, then $N_{p_1}[f] \le N_{p_2}[f]$. Prove also that if $1 \le p \le \infty$, then $N_p[f+g] \le N_p[f] + N_p[g]$, $(1/|E|) \int_E |fg| \le N_p[f] N_{p'}[g]$, $1/p + 1/p' = 1$, and $\lim_{p\to\infty} N_p[f] = \|f\|_\infty$. Thus, N_p behaves like $\|\cdot\|_p$ but has the advantage of being monotone in p. Recall Exercise 28 of Chapter 5.

Solution. ▸ ◂

Problem 5 (Wheeden & Zygmund Ch. 8, Ex. 6)**.**

(a) Let $1 \le p_i,\ r \le \infty$ and $\sum_{i=1}^k 1/p_i = 1/r$. Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

(b) Let $1 \le p < r < q \le \infty$ and define $\theta \in (0,1)$ by $1/r = \theta/p + (1-\theta)/q$. Prove the interpolation estimate

$$\|f\|_r \le \|f\|_p{}^{\theta} \|f\|_q{}^{1-\theta}.$$

In particular, if $A = \max\{\|f\|_p, \|f\|_q\}$, then $\|f\|_r \le A$.

Solution. ▶ ◀

Problem 6 (Wheeden & Zygmund Ch. 8, Ex. 9)**.** If f is real-valued and measurable on E, $|E| > 0$, define its essential infimum on E by

$$\operatorname{ess\,inf} f = \sup\{\, \alpha : |\{\, x \in E : f(x) < \alpha \,\}| = 0 \,\}.$$

If $f \ge 0$, show that $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup} 1/f)^{-1}$.

Solution. ▶ ◀

Problem 7 (Wheeden & Zygmund Ch. 8, Ex. 11)**.** If $f_k \to f$ in L^p, $1 \le p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty < M$ for all k, prove that $f_k g_k \to fg$ in L^p.

Solution. ▶ ◀

2 Danielli

2.1 Danielli: Practice Exams Spring 2016

2.1.1 Exam 1 Practice

Problem 1. Let $E \subseteq \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{ rx : x \in E \}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Solution. ▸ Define a map a linear map $T \colon \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = rx$. Since a the image of a measurable set E under linear map is measurable and $m(T(E)) = |\det T| m(E) = |r|^n m(E)$, it suffices to show that $T(E) = rE$.

Let $y \in T(E)$ then $y = rx$ for some $x \in E$. Thus, $y \in rE$. Let $y \in rE$. Then, $y = rx = T(x)$ for some $x \in E$. Thus, $y \in T(E)$. It follows that $m(rE) = |r|^n m(E)$. ◂

Problem 2. Let $\{E_n\}$, $n \in \mathbb{N}$ be a collection of measurable sets. Define the set

$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} E_k \right).$$

Show that

$$m\Big(\liminf_{n\to\infty} E_n\Big) \le \liminf_{n\to\infty} m(E_n).$$

Solution. ▸ Here's a quick and dirty way of proving this: let $\mathbf{1}_{E_n}$ be the characteristic function of E_n. Then, by Fatou's lemma,

$$\int \liminf_{n\to\infty} \mathbf{1}_{E_n}(x)\,\mathrm{d}x \le \liminf_{n\to\infty} \int \mathbf{1}_{E_n}(x)\,\mathrm{d}x. \tag{1}$$

By definition of the characteristic function, it is easy to see that the right hand-side of the Equation (1) is

$$\liminf_{k\to\infty} m(E_k).$$

But what about the left-hand side of (1)? We claim that

$$\liminf_{n\to\infty} \mathbf{1}_{E_n} = \mathbf{1}_E$$

where $E = \liminf_{n\to\infty} E_n$.

Proof of claim. Suppose $x \in E$. We must show that $\liminf_{n\to\infty} \mathbf{1}_{E_n}(x) = 1$. By definition

$$\liminf_{n\to\infty} \mathbf{1}_{E_n} = \lim_{n\to\infty} \left[\inf_{k\ge n} \mathbf{1}_{E_k} \right].$$

Now $x \in E$ if and only if $x \in \bigcap_{k=N}^{\infty} E_k$ for some $N \in \mathbb{N}$. Then for $k \ge N$

$$\inf_{k\ge n} \mathbf{1}_{E_k}(x) = 1$$

so $\liminf_{n\to\infty} \mathbf{1}_{E_n}(x) = 1$.

On the other hand, if $x \notin E$ then $x \notin \bigcap_{k=n}^{\infty} E_k$ for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$,

$$\inf_{k\geq n} \mathbf{1}_{E_k}(x) = 0$$

so $\liminf_{n\to\infty} \mathbf{1}_{E_k} = 0$. ■

Having established this equivalence, we have

$$m\Big(\liminf_{n\to\infty} E_n\Big) = \int \liminf_{n\to\infty} \mathbf{1}_{E_n}(x)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int \mathbf{1}_{E_n}(x)\,\mathrm{d}x = \liminf_{n\to\infty} m(E_n).$$

◄

Problem 3. Consider the function

$$F(x) = \begin{cases} m\big(B(\mathbf{0}, x)\big) & x > 0, \\ 0 & x = 0. \end{cases}$$

Here $B(\mathbf{0}, r) = \{\, y \in \mathbb{R}^n : |y| < r \,\}$. Prove that F is monotonic increasing and continuous.

Solution. ► Let $T\colon \mathbb{R}^n \times [0, x) \to \mathbb{R}^n$ be the linear map given by $T(x, r) = rx$. By Problem 1, we know that $T\big(B(\mathbf{0}, 1), r\big) = B(\mathbf{0}, r)$ and consequently, $m\big(B(\mathbf{0}, 1)\big) = |r|^n m\big(B(\mathbf{0}, 1)\big)$. Interpreting $B(\mathbf{0}, 0) = \emptyset$, we have $F(x) = |r|^n m\big(B(\mathbf{0}, 1)\big)$ and it is easy to see that F is both monotonically increasing and continuous since it is a polynomial in r. ◄

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Solution. ► Let C be the subset of $\mathbb{R}$ where f is continuous, i.e., the set

$$C = \Big\{\, x \in \mathbb{R} : \text{given } \varepsilon > 0 \text{ there exist } \delta > 0 \text{ such that } |f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \,\Big\}.$$

In light of the latter equality, for each $n \in \mathbb{N}$ define the following family of subsets of C,

$$G_n = \Big\{\, x \in \mathbb{R} : \text{there exists } \delta_n > 0 \text{ such that } |f(x) - f(y)| < \frac{1}{n} \text{ whenever } |x - y| < \delta_n \,\Big\}.$$

We claim that (i) the G_n are open and (ii) $C = \bigcap_{n\in\mathbb{N}} G_n$.

The proof of (i) is easy: let $x \in G_n$ then there exists $\delta_n > 0$ such that

$$|f(x) - f(y)| < \frac{1}{n}.$$

Then $B(x, \delta_n) \subseteq G_n$ since $x' \in B(x, \delta_n)$ implies that $|x - x'| < \delta$ so

$$|f(x) - f(x')| < \frac{1}{n}.$$

The proof of (ii) is also straight forward: let $x \in C$ then given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $|x - y| < \delta$. In particular, if $\varepsilon = 1/n$ then there exists δ_n such that $|x - y| < \delta_n$ implies

$$|f(x) - f(y)| < \frac{1}{n}$$

for ever $n \in \mathbb{N}$. Thus, $x \in \bigcap_{n \in \mathbb{N}} G_n$. On the other hand, if $x \in \bigcap_{ni \in \mathbb{N}} G_n$, then $x \in G_n$ for all $n \in \mathbb{N}$. Thus, given $\varepsilon > 0$, by the Archimedean property of the real numbers, there exists a positive integer N such that $1/N < \varepsilon$ and hence for $\delta = \delta_N > 0$ we have

$$|f(x) - f(y)| < \frac{1}{N}$$

whenever $|x - y| < \delta_N$. Thus, $x \in C$.

It follows that $C = \bigcap_{n \in \mathbb{N}} G_n$ and hence is a G_δ set. ◂

Problem 5. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Solution. ▸ The statement is false and, of course, the counterexample involves existence of non-measurable sets. Let $V \subseteq [0, 1]$ be a Vitali set and consider the function $f \colon \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} x & \text{if } x \in V, \\ -x & \text{if } x \in \mathbb{R} \smallsetminus V. \end{cases}$$

Then, $\{ f = r \}$ is measurable for all $r \in \mathbb{R}$ since the set either consists of a single point or is the empty set. However, $\{ f \geq 0 \} = V$ is not measurable. ◂

Problem 6. Let $\{f_k\}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set

$$\left\{ x : \lim_{k \to \infty} f_k(x) \text{ exists} \right\}$$

is measurable.

Solution. ▸ Suppose $\{f_n\}$, $n \in \mathbb{N}$, is a sequence of measurable functions and let

$$E = \left\{ x : \lim_{n \to \infty} f_n(x) \text{ exists} \right\}.$$

Then, by general properties of the limit supremum and the limit infimum, we know that $\lim_{n \to \infty} f_n(x)$ exists if and only if

$$\limsup_{n \to \infty} f_n(x) = \liminf_{n \to \infty} f_n(x).$$

Both of these functions are measurable so the set

$$E = \left\{ x : \limsup_{n \to \infty} f_n(x) = \liminf_{n \to \infty} f_n(x) \right\}.$$

is measurable. ◂

Problem 7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Solution. ▸ Let $\varepsilon = 1$ then, since $f\colon [a, b] \to \mathbb{R}$ is absolutely continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $y - x < \delta$ (assuming $x < y$). Partition the closed interval $[a, b]$ into subintervals $\{[a_n, b_n] : 1 \le n \le N\}$ of length less than or equal to δ. Then

$$\operatorname{var}(f; [a_n, b_n]) \le 1.$$

Thus,

$$\operatorname{var}(f; [a, b]) \le N$$

for every partition Γ of $[a, b]$. ◂

Problem 8. Let f be a continuous function from $[a, b]$ into $\mathbb{R}$. Let $\mathbf{1}_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, that is, $\mathbf{1}_{\{c\}}(x) = 0$ if $x \ne c$ and $\mathbf{1}_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\mathbf{1}_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b), \\ -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

Solution. ▸ There are three cases to consider (1) $c \in (a, b)$, (2) $c = a$ and (3) $c = b$. Cases (2) and (3) can be handled easily: if $c = a$ then the Rieman–Stieltjes integral of f with respect to $\mathbf{1}_{\{c\}}$ is the supremum over all sums

$$\sum_{n=1}^N f(\xi_n)\big[\mathbf{1}_{\{c\}}(x_n) - \mathbf{1}_{\{c\}}(x_{n-1})\big]$$

where $x_0 = a$ and $x_N = b$ for all partitions $\Gamma = \{x_0, \dots, x_N\}$ of $[a, b]$. Thus, the sum

$$\sum_{n=1}^N f(\xi_n)\big[\mathbf{1}_{\{c\}}(x_n) - \mathbf{1}_{\{c\}}(x_{n-1})\big] = \begin{cases} -f(\xi_0) & \text{if } c = a, \\ f(\xi_N) & \text{if } c = b. \end{cases}$$

Letting $\Delta(\Gamma) \to 0$, $\xi_0 \to a$ and $\xi_N \to b$ giving us

$$\int_a^b f \, \mathrm{d}\mathbf{1}_{\{c\}} = \begin{cases} -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

It remains to show that

$$\int_a^b f \, \mathrm{d}\mathbf{1}_{\{c\}} = 0$$

if $c \in (a, b)$. To that end, note that if Γ_c is a partition containing the point c, say, $x_m = c$ for some $1 \le m \le N$, the partial sums yield

$$\sum_{n=1}^N f(\xi_n)\big[\mathbf{1}_{\{c\}}(x_n) - \mathbf{1}_{\{c\}}(x_{n-1})\big] = f(\xi_{m+1}) - f(\xi_m).$$

Letting $\Delta(\Gamma_c) \to 0$, since f is continuous, $f(\xi_{m+1}) \to f(\xi_m)$. Thus,

$$\int_a^b f \, \mathrm{d}\mathbf{1}_{\{c\}} = 0.$$

◀

2.1.2 Exam 1

I lost this exam. These are the questions I could recall explicitly. For the first problem, we were asked to show that the Dichlet function $\mathbf{1}_{\mathbb{Q}}(x)$ is not Riemann integrable and prove something about $\mathbb{Q}$. For the second question, we were asked to show that the measure of countable union of disjoint measurable sets $\{E_n : n \in \mathbb{N}\}$, is equal to the sum of their individual measures (or something to that effect).

Problem 1.

Problem 2.

Problem 3.

(i) Show that if $B_r = \{ x \in \mathbb{R}^n : |x| < r \}$, then there exists a constant C such that $|B_r| = Cr^n$. (*Hint*: Think of B_r as $\{ rx : x \in B_1 \}$.)

(ii) Let $E \subseteq \mathbb{R}^n$ be a measurable set and let $\varphi_E \colon \mathbb{R}^n \to \mathbb{R}$ be defined $\varphi_E(x) = m(E \cap B_{|x|})$. Use part (i) to prove that φ_E is continuous.

Solution. ▸ For part (i), as in the practice problems, define the linear map $T \colon \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = rx$. Note that this map is Lipschitz so the image of a measurable set E under T is measurable and $m(T(E)) = |\det T| m(E) = |r|^n m(E)$. It is not too difficult to see that

$$T(B_1) = B_r$$

as sets, so $m(B_r) = |r|^n m(B_1)$. Now, let $C = m(B_1)$.

For part (ii), note that we can write the map φ_E as ◂

Problem 4. Assume that $f \colon [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$. Prove that f is measurable.

Solution. ▸ ◂

2.1.3 Exam 2 Practice Problems

Problem 1. Define for $x \in \mathbb{R}^n$,

$$f(x) = \begin{cases} |x|^{-(n+1)} & \text{if } x \neq \mathbf{0}, \\ 0 & \text{if } x = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B(\mathbf{0}, \varepsilon)$, and that there exists a constant $C > 0$ such that

$$\int\limits_{\mathbb{R}^n \setminus B(\mathbf{0},\varepsilon)} f(x)\,\mathrm{d}x \leq \frac{C}{\varepsilon}.$$

Solution. ▶ Suppose that the following is true. ◀

Problem 2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Solution. ▶ ◀

Problem 3. Assume that E is a measurable set of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, x \in E : f(x) \geq k \,\}| < \infty.$$

Solution. ▶ ◀

Problem 4. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Solution. ▶ ◀

Problem 5. Define the *gamma function* $\Gamma\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1}\,\mathrm{d}u,$$

and the *beta function* $\beta\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Solution. ▶ ◀

Problem 6. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}}\colon \mathbb{R}^n \to \mathbb{R}$ be $f_{\mathbf{h}}(x) = f(x - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h}\to\mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Solution. ▶ ◀

Problem 7. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \le g_k$ and

$$\int_{\mathbb{R}^n} g_k \longrightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \qquad \text{as } k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \longrightarrow \int_{\mathbb{R}^n} |f| \qquad \text{as } k \to \infty.$$

Solution. ▶ ◀

2.1.4 Exam 2 (2010)

Problem 1. Suppose $f \in L^1(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \smallsetminus B} |f| < \varepsilon.$$

Hint: Use the monotone convergence theorem.

Solution. ▶ ◀

Problem 2.

(a) Prove the following generalization of *Chebyshev's inequality*: Let $0 < p < \infty$ and $E \subseteq \mathbb{R}^n$ be measurable. assume that $|f|^p \in L^1(E)$. Then

$$|\{\, x \in E : f(x) > \alpha \,\}| \leq \frac{1}{\alpha^p} \int_{\{\, f > \alpha \,\}} f^p,$$

for $\alpha > 0$.

(b) Let p, E, and f be as in part (a). In addition, assume that $\{f_k\}$ is a sequence such that $\int_E |f_k - f|^p \to 0$ as $k \to \infty$. Show that $f_k \to f$ in measure on E.

Recall that $f_k \to f$ in measure on E if and only if for every $\varepsilon > 0$

$$\lim_{k\to\infty} |\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}| = 0.$$

Solution. ▶ ◀

Problem 3. Let $f \in L^1(\mathbb{R})$, and define

$$F(\xi) = \int_{\mathbb{R}} f(x) \cos(2\pi x \xi)\, \mathrm{d}x.$$

Prove that F is continuous and bounded on $\mathbb{R}$.

Solution. ▶ ◀

Problem 4. Use repeated integration techniques to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, \mathrm{d}x = \pi^{n/2}.$$

Hint: Start from the case $n = 1$ by using the polar coordinates in

$$\left[\int_{\mathbb{R}} e^{-x^2}\, \mathrm{d}x\right]^2 = \left[\int_{\mathbb{R}} e^{-x^2}\, \mathrm{d}x\right]\left[\int_{\mathbb{R}} e^{-x^2}\, \mathrm{d}y\right]$$

Solution. ▶ ◀

Problem 5.

Solution. ▶ ◀

2.1.5 Exam 2

Problem 1. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Solution. ▶ ◀

Problem 2. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Solution. ▶ ◀

Problem 3. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbb{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Solution. ▶ ◀

Problem 4. Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)\,\mathrm{d}t$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Solution. ▶ ◀

2.1.6 Final Exam Practice Problems

Problem 1. Suppose $f \in L^1(\mathbb{R}^n)$ and that x is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) = \frac{1}{|r|^n} \int_{B(0,r)} |f(x-y) - f(x)| \, \mathrm{d}y.$$

Show that:

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$;

(b) there exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

Solution. ▸ (a) Without loss of generality, we may assume $r < s$. Then, we want to show that as $r \to s$, the quantity

$$|A(s) - A(r)| \longrightarrow 0.$$

Set $F(y) = |f(x-y) - f(x)|$ and consider said quantity

$$\begin{aligned}
|A(s) - A(r)| &= \left| \frac{1}{|s|^n} \int_{B_s} F(y) \, \mathrm{d}y - \frac{1}{|r|^n} \int_{B_r} F(y) \, \mathrm{d}y \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y) \, \mathrm{d}y + \frac{1}{|s|^n} \int_{B_r} F(y) \, \mathrm{d}y - \frac{1}{|r|^n} \int_{B_r} F(y) \, \mathrm{d}y \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y) \, \mathrm{d}y + \left(\frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(y) \, \mathrm{d}y \right| \\
&\leq \underbrace{\frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y) \, \mathrm{d}y}_{I_1} + \underbrace{\left(\frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(y) \, \mathrm{d}y}_{I_2}.
\end{aligned}$$

Hence, we must show that the quantities $I_1, I_2 \to 0$ as $r \to s$.

To see that $A(r) \to 0$ as $r \to 0$, note that x is a point of the Lebesgue set of f and that

$$0 = \lim_{B_r \searrow x} \frac{1}{|B_1||r|^n} \int_{B_r} |f(y) - f(x)| \, \mathrm{d}y = \frac{1}{|B_1|} \lim_{B_r \searrow x} \frac{1}{|r|^n} \int_{B_r} |f(t) - f(x)| \, \mathrm{d}t = \lim_{r \to 0} A(r).$$

by making the change of variables $t = x - y$.

(b) ◂

Problem 2. Let $E \subseteq \mathbb{R}^n$ be a measurable set, $1 \leq n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0$$

if and only if

$$\|f_k\|_p \longrightarrow \|f\|_p$$

as $k \to \infty$.

Solution. ▸ ◂

Problem 3. Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.

(b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

Solution. ▸ ◂

Problem 4. Let $f \in L(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x)\cos(tx)dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.

(b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Solution. ▸ ◂

Problem 5. Let f be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular. Show that

$$\int_a^b \varphi\, \mathrm{d}f = \int_a^b \varphi f' dx + \int_a^b \varphi\, \mathrm{d}h$$

for all functions φ continuous on $[a, b]$.

Solution. ▸ ◂

2.1.7 Final Exam 2010

Problem 1. Suppose that $f \in L^1(\mathbb{R}^n)$, and that x is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) = \frac{1}{r^n} \int_{B_r} |f(x-y) - f(x)| \, \mathrm{d}y,$$

where $B_r = B(\mathbf{0}, r)$.

Show that

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$.

(b) There exists a constant $M > 0$ such that $A(r) \le M$ for all $r > 0$.

Solution. ▶ (a)

(b) ◀

Problem 2. Let $E \subseteq \mathbb{R}^n$ be a measurable set, $1 \le p < \infty$. assume that $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0 \iff \|f_k\|_p \longrightarrow \|f\|_p$$

Hint: To prove one of the implications, you can use the following fact without proving it:

$$\left| \frac{a-b}{2} \right| \le \frac{|a|^p + |b|^p}{2}$$

for all $a, b \in \mathbb{R}$.

Solution. ▶ ◀

Problem 3. Let $0 < p < q < r \le \infty$, $E \subseteq \mathbb{R}^n$ be a measurable set. Show that each $f \in L^q(E)$ is the sum of a function $g \in L^p(E)$ and a function $h \in L^r(E)$.

Solution. ▶ ◀

Problem 4. Prove that $f\colon [a,b] \to \mathbb{R}$ is Lipschitz continuous if and only if f is absolutely continuous and there exists a constant $M > 0$ such that $|f'| < M$ a.e. on $[a,b]$.

Solution. ▶ ◀

Problem 5. Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.

(b) Does this conclusion continue to be valid when $p = 1$ or $p = \infty$?.

Solution. ▶ ◀

2.1.8 Final Exam

Never went to get it.

Problem 1.

Problem 2.

Problem 3.

Problem 4.

2.2 Danielli: Summer 2011

Problem 1. Let $f \in L^1(\mathbb{R})$, and let $\hat{f}(x) = \int_{\mathbb{R}} f(t)\cos(xt)\,\mathrm{d}t$.

(a) Prove that $\hat{f}(x)$ is continuous for $x \in \mathbb{R}$.

(b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{x\to\infty} \hat{f}(x) = 0.$$

Hint: Start by proving the statement for $f = \mathbf{1}_{[a,b]}$.

Solution. ▸ For part (a): let $\varepsilon > 0$ be given. Then, since $\cos(xt)$ is continuous there exists $\delta' > 0$ such that $|x - y| < \delta$ implies

$$|\cos(xt) - \cos(yt)| < \frac{\varepsilon}{\|f\|_1}.$$

Now, let $\delta = \delta'$. Then we have

$$\begin{aligned}
|\hat{f}(x) - \hat{f}(y)| &= \left| \int_{\mathbb{R}} f(t)\cos(xt)\,\mathrm{d}t - \int_{\mathbb{R}} f(t)\cos(yt)\,\mathrm{d}t \right| \\
&\le \int_{\mathbb{R}} |f(t)||\cos(xt) - \cos(yt)|\,\mathrm{d}t \\
&< \frac{\varepsilon}{\|f\|_1}\int_{\mathbb{R}} |f(t)|\,\mathrm{d}t \\
&= \frac{\varepsilon}{\|f\|_1}\|f\|_1 \\
&= \varepsilon.
\end{aligned}$$

Since this can be done for any $x \in \mathbb{R}$, $\hat{f}$ is continuous on $\mathbb{R}$.

For part (b): since simple functions are dense in $L^1(\mathbb{R})$, f there exists a sequence of simple functions $\{s_n\}$, $n \in \mathbb{N}$, such that $\int_{\mathbb{R}} s_n \to \|f\|_1$. Therefore, it suffices to prove the result for characteristic functions. Let $f = \mathbf{1}_{[a,b]}$ and consider the limit

$$\lim_{x\to\infty} \hat{f}(x) = \lim_{x\to\infty} \int_{\mathbb{R}} f(t)\cos(xt)\,\mathrm{d}t.$$

Since $f = \mathbf{1}_{[a,b]}$, we have

$$\begin{aligned}
\lim_{x\to\infty} \int_{\mathbb{R}} f(t)\cos(xt)\,\mathrm{d}t &= \lim_{x\to\infty} \int_a^b \cos(xt)\,\mathrm{d}t \\
&= \lim_{x\to\infty} \left[\frac{1}{x}(\sin(xa) - \sin(xb)) \right] \\
&= \lim_{x\to\infty} \left[\frac{\sin(xa)}{x} - \frac{\sin(xb)}{x} \right] \\
&= \left[\lim_{x\to\infty} \frac{\sin(xa)}{x} \right] - \left[\lim_{x\to\infty} \frac{\sin(xb)}{x} \right] \\
&= 1 - 1 \\
&= 0,
\end{aligned}$$

as we set out to show. ◂

Problem 2.

(a) Suppose that $f_k, f \in L^2(E)$, with E a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E fg \qquad (\bigstar)$$

as $k \to \infty$ for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \to \|f\|_2$ show that f_k converges to f in L^2, i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as $k \to \infty$.

(b) Provide an example of a sequence f_k in L^2 and a function f in L^2 satisfying ($\bigstar$), but such that f_k does *not* converge to f in L^2.

Solution. ▸ For part (a): expand the limit

$$\begin{aligned}\lim_{n\to\infty}\int_E |f - f_n|^2 \,\mathrm{d}x &= \lim_{n\to\infty}\left[\int_E \left(|f|^2 - 2|ff_n| + |f|_n^2\right)\mathrm{d}x\right] \\ &= \lim_{n\to\infty}\left[\|f_n\|_2 + \|f\|_2 - 2\int_E f_n f \,\mathrm{d}x\right] \\ &= \lim_{n\to\infty}\|f_n\|_2 + \lim_{n\to\infty}\|f\|_2 - 2\lim_{n\to\infty}\int_E f_n f \,\mathrm{d}x.\end{aligned} \tag{1}$$

Since

$$\int_E f_n g \,\mathrm{d}x \longrightarrow \int_E fg \,\mathrm{d}x$$

for every $g \in L^p(E)$,

$$\int_E f_n f \,\mathrm{d}x \longrightarrow \int_E f^2 \,\mathrm{d}x = \|f\|_2{}^2.$$

Moreover, $\|f_n\|_2 \to \|f\|_2$ so the limit in (1) converges to

$$\lim_{n\to\infty}\|f_n\|_2 + \lim_{n\to\infty}\|f\|_2 - 2\lim_{n\to\infty}\int_E f_n f \,\mathrm{d}x = \|f\|_2 + \|f\|_2 - 2\|f\|_2 = 0$$

as $n \to \infty$.

For part (b), consider the sequence $\{f_n\}$, $n \in \mathbb{N}$, where $f_n(x) = \log(n)\exp(-nx)$. Then, we claim that $f_n \xrightarrow{L^2[0,1]} 0$, but that $f_n \nrightarrow 0$ pointwise. To see the former, first note that

$$\begin{aligned}\lim_{n\to\infty}\left[\int_0^1 f_n(x)\,\mathrm{d}x\right] &= \lim_{n\to\infty}\left[\int_0^1 \log(n)\exp(-nx)\,\mathrm{d}x\right]\\ &= \lim_{n\to\infty}\left[\log(n)\exp(-nx)\big|_0^1\right]\\ &= \lim_{n\to\infty}\left[\frac{1}{n}\log(n) - \frac{1}{n}\log(n)\exp(-n)\right]\\ &= \lim_{n\to\infty}\left[\left(\frac{1-\exp(-n)}{n}\right)\log(n)\right]\\ &= 0.\end{aligned}$$

However, f_n does not converge to 0 a.e. since, for $x = 0$ there exist no $N \in \mathbb{N}$ such that

$$|\log(n)| < 1.$$

for all $n \geq N$. ◀

Problem 3. A bounded function f is said to be of bounded variation on $\mathbb{R}$ if it is of bounded variation on any finite subinterval $[a, b]$, and moreover $A := \sup_{a,b} V[a, b; f] < \infty$. Here, $V[a, b; f]$ denotes the total variation of f over the interval $[a, b]$. Show that:

(a) $\displaystyle\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x \leq A|h|$ for all $h \in \mathbb{R}$.

Hint: For $h > 0$, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x = \sum_{n=-\infty}^{\infty}\int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,\mathrm{d}x.$$

(b) $\displaystyle\left|\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x\right| \leq A$, where φ is any function of class C^1, of bounded variation, compactly supported, with $\sup_{x\in\mathbb{R}} |\varphi(x)| \leq 1$.

Solution. ▶ For part (a), it suffices to consider only positive h as, making the change of variables $u = x + h$ yields

$$\int_{\mathbb{R}} |f(u) - f(u-h)|\,\mathrm{d}u = \int_{\mathbb{R}} |f(u + (-h)) - f(u)|\,\mathrm{d}u$$

where $-h$ is positive (and letting $h = 0$, we have a trivial inequality). Now, taking the hint, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x = \sum_{n=-\infty}^{\infty}\int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,\mathrm{d}x.$$

Now, since $|f((n+1)h) - f(nh)|$ is a sum in the total variation of f on the interval $[nh, (n+1)h]$, $|f(x+h) - f(x)|$ is bounded by $V[nh, (n+2)h; f]$. Thus, we have

$$\begin{aligned}
\int_{\mathbb{R}} |f(x+h) - f(x)| \, \mathrm{d}x &= \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)| \, \mathrm{d}x \\
&\le \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} V[nh, (n+2)h; f] \, \mathrm{d}x \\
&= \sum_{n=-\infty}^{\infty} V[nh, (n+2)h; f] \int_{nh}^{(n+1)h} \mathrm{d}x \\
&= \sum_{n=-\infty}^{\infty} V[nh, (n+2)h; f] |h| \\
&= 2A|h|.
\end{aligned}$$

I suspect there is an error here as the most obvious bound we can get is $2A|h|$ and not the stricter $A|h|$.

For part (b), f is absolutely continuous since it is of bounded variation and φ is absolutely continuous since it is Lipschitz (φ is differentiable on a compact set, thus, by the mean value theorem $|\varphi(x) - \varphi(y)| \le \varphi'(\xi)|x-y|$ for some $\xi \in \operatorname{Supp}\varphi$). Assuming $\operatorname{Supp}\varphi$ has nonempty interior, $\operatorname{Supp}\varphi$ contains a closed interval $I = [a,b]$ (in fact, $\operatorname{Supp}\varphi$ is of the form $[a,b] \smallsetminus \bigcup_{n\in A} I_n$, $A \subseteq \mathbb{N}$, where $I_n = (a_n, b_n)$ with $a_n, b_n \in \operatorname{Supp}\varphi$) and thus, by integration by parts, we have

$$\begin{aligned}
\int_a^b f\varphi' \, \mathrm{d}x &= f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b f'\varphi \, \mathrm{d}x \\
&\le f(b) - f(a) - \int_a^b f' \, \mathrm{d}x \\
&= 2(f(b) - f(a)) \\
&\le 2V[a,b;f]
\end{aligned}$$

Thus, summing over every

$$\sum_{n=0}^{\infty} \int_{a_n}^{b_n} f\varphi' \, \mathrm{d}x \le 2|A|.$$

◀

Problem 4.

(a) Prove the *generalized Hölder's inequality*: Assume $1 \le p_j \le \infty$, $j = 1, \ldots, n$, with $\sum_{j=1}^n 1/p_j = 1/r \le 1$. If E is a measurable set and $f_j \in L^{p_j}(E)$ for $j = 1, \ldots, n$, then $\prod_{j=1}^n f_j \in L^r(E)$ and

$$\|f_1 \cdots f_n\|_r \le \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if $1 \le p, q, r \le \infty$, with $1/p + 1/q = 1/r + 1$, $f \in L^p(\mathbb{R})$, and $g \in L^q(\mathbb{R})$, then

$$|(f*g)(x)|^r \le \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q \, \mathrm{d}y.$$

(Recall that $(f * g)(x) = \int f(y)g(x-y)\,\mathrm{d}y$.)

(c) Prove *Young's convolution theorem*: Assume that p, q, r, f, and g are as in part (b). Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Solution. ▸ For (a) we shall proceed by induction on n the number of measurable functions $f_j \in L^{p_j}(E)$, $1 \leq j \leq n$. The case $n = 2$ holds by using Hölder's inequality on the exponents $r/p + r/q = 1$,

$$\begin{aligned}\left[\int_E |f_1 f_2|^r\right]^{1/r} \mathrm{d}x &= \|{f_1}^r {f_2}^r\|_1 \\ &\leq \|{f_1}^r\|_{p/r} \|{f_2}^r\|_{q/r} \\ &= \|f_1\|_p \|f_2\|_q.\end{aligned}$$

Now, suppose this holds for $n-1$ measurable functions $f_j \in L^{p_j}(E)$, $1 \leq j \leq n-1$. Then for $f_j \in L^{p_j}(E)$ with $\sum_{j=1}^{n} 1/p_j = 1/r$, we have $r' = \sum_{j=1}^{n-1} 1/p_j = 1/r - 1/p_n$ so by the inductive step

$$\|f_1 \cdots f_{n-1}\|_{r'} \leq \|f_1\|_{p_1} \cdots \|f_{n-1}\|_{p_{n-1}}$$

hence, $f_1 \cdots f_{n-1} \in L^{r'}(E)$. Thus,

$$\begin{aligned}\|f_1 \cdots f_{n-1} f_n\|_r &\leq \|f_1 \cdots f_{n-1}\|_{r'} \|f_n\|_{p_n} \\ &\leq \|f_1\|_{p_1} \cdots \|f_{n-1}\|_{p_{n-1}} \|f_n\|_{p_n},\end{aligned}$$

as we set out to show.

For part (b), applying the generalized Hölder's inequality we proved in part (a),

$$\begin{aligned}|f * g| &= \left|\int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y\right| \\ &\leq \int_{\mathbb{R}} \big|f(y)g(x-y)\big|\,\mathrm{d}y \\ &= \int_{\mathbb{R}} |f(y)|^{1+p/r-p/r} |g(x-y)|^{1+q/r-q/r}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} |f(y)|^{p/r} |g(x)|^{q/r} |f(y)|^{1-p/r} |g(x-y)|^{1-q/r}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} |f(y)|^{p/r} |g(x)|^{q/r} |f(y)|^{(r-p)/r} |g(x-y)|^{(r-q)/r}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \big(|f(y)|^p |g(x)|^q\big)^{1/r} |f(y)|^{(r-p)/r} |g(x-y)|^{(r-q)/r}\,\mathrm{d}y \\ &\leq \big\|\big(|f(y)|^p |g(x)|^q\big)^{1/r}\big\|_r \big\||f(y)|^{(r-p)/r}\big\|_{pr/(r-p)} \big\||g(x-y)|^{(r-q)/r}\big\|_{qr/(r-q)} \\ &= \|f\|_p{}^{(r-p)/r} \|g\|_q{}^{(r-q)/r} \left[\int_{\mathbb{R}} |f(y)|^p |g(x-y)|^q\,\mathrm{d}y\right]^{1/r}.\end{aligned}$$

Raising both sides to the power r, we have

$$|(f*g)(x)|^r \le \|f\|_p^{r-p}\|g\|_q^{r-q}\int |f(y)|^p|g(x-y)|^q\,\mathrm{d}y,$$

as desired.

For part (c), using the estimate we worked out in part (b) together with Tonelli's theorem, we have

$$\begin{aligned}
\|f*g\|_r^r &= \int_{\mathbb{R}} |f*g(x)|^r\,\mathrm{d}x \\
&\le \int_{\mathbb{R}}\left[\|f\|_p^{r-p}\|g\|_q^{r-q}\int_{\mathbb{R}}|f(y)|^p|g(x-y)|^q\,\mathrm{d}y\right] \\
&= \|f\|_p^{r-p}\|g\|_q^{r-q}\iint\limits_{\mathbb{R}\times\mathbb{R}}|f(y)|^p|g(x-y)|^q\,\mathrm{d}y \\
&= \|f\|_p^{r-p}\|g\|_q^{r-q}\int_{\mathbb{R}}|f(y)|^p\left[\int_{\mathbb{R}}|g(x-y)|^q\,\mathrm{d}y\right]\mathrm{d}x \\
&\le \|f\|_p^{r-p}\|g\|_q^{r-q}\|f\|_p^p\|g_q\|^q \\
&= \|f\|_p^r\|g\|_q^r.
\end{aligned}$$

Taking the rth root on each side, we achieve the desired estimate

$$\|f*g\|_r \le \|f\|_p\|g\|_q.$$

◀

2.3 Danielli: Winter 2012

Problem 1. Let $f(x, y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $\partial f(x, y)/\partial x$ is a bounded function of (x, y). Prove that $\partial f(x, y)/\partial x$ is a measurable function of y for each x and

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^1 f(x, y)\,\mathrm{d}y = \int_0^1 \frac{\partial f(x, y)}{\partial x}\,\mathrm{d}y.$$

Solution. ▸ The end points can be dealt with separately. Fix a point $x_0 \in (0, 1)$ and consider the sequence of measurable functions $\{f_n'\}$ where

$$f_n'(y) = \frac{f(x_0 + h_n, y) - f(x_0, y)}{h_n}$$

where $\{h_n\}$ is a sequence of numbers converging to 0. Since f is differentiable as a function of x, the sequence $\{f_n'(x_0, y)\}$ converges to $\partial f/\partial x(x_0, y)$. Now, since $|\partial f/\partial x(x, y)| \le M$ for some $M \in \mathbb{R}^+$ for all $(x, y) \in [0, 1] \times [0, 1]$, by the bounded convergence theorem

$$\begin{aligned} \lim_{n\to\infty} \int_0^1 f_n'(y)\,\mathrm{d}y &= \int_0^1 \lim_{n\to\infty} f_n'(y)\,\mathrm{d}y \\ &= \int_0^1 \frac{\partial f(x_0, y)}{\partial x}\,\mathrm{d}y. \end{aligned} \tag{1}$$

It remains to show that the left side of (1) is the derivative of the integral of $f(x_0, y)$ as a function of y. But this is exactly

$$\begin{aligned} \lim_{n\to\infty} \int_0^1 f_n'(y)\,\mathrm{d}y &= \lim_{n\to\infty} \int_0^1 \frac{f(x_0 + h_n, y) - f(x_0, y)}{h_n} \mathrm{d}x \\ &= \lim_{n\to\infty} \frac{\int_0^1 f(x_0 + h_h, y) - f(x_0, y)}{h_n} \mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \int_0^1 f(x_0, y)\,\mathrm{d}y. \end{aligned}$$

It follows that for any $x \in [0, 1]$,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^1 f(x, y)\,\mathrm{d}y = \int_0^1 \frac{\partial f(x, y)}{\partial x}\,\mathrm{d}y$$

so $\partial f/\partial x(x, y)$ is a measurable function of y. ◂

Problem 2. Let f be a function of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular, show that

$$\int_a^b \varphi\,\mathrm{d}f = \int_a^b \varphi f'\,\mathrm{d}x + \int_a^b \varphi\,\mathrm{d}h.$$

Hint: A function h is said to be singular if $h' = 0$.

Solution. ▶ Let ◀

Problem 3. Let $E \subseteq \mathbb{R}$ be a measurable set, and let K be a measurable function on $E \times E$. Assume that there exists a positive constant C such that

$$\int_E K(x,y)\,\mathrm{d}x \le C \tag{$\bigstar$}$$

for a.e. $y \in E$, and

$$\int_E K(x,y)\,\mathrm{d}y \le C \tag{$\clubsuit$}$$

for a.e. $x \in E$.

Let $1 < p < \infty$, $f \in L^p(E)$, and define

$$T_f(x) = \int_E K(x,y) f(y)\,\mathrm{d}y.$$

(a) Prove that $T_f \in L^p(E)$ and

$$\|T_f\|_p \le C\|f\|_p. \tag{$\spadesuit$}$$

(b) Is ($\spadesuit$) still valid if $p = 1$ or ∞? If so, are assumptions ($\bigstar$) and ($\clubsuit$) needed?

Solution. ▶ ◀

Problem 4. Let f be a nonnegative measurable function on $[0,1]$ satisfying

$$m\{\, x \in [0,1] : f(x) > \alpha \,\} < \frac{1}{1+\alpha^2} \tag{$\blacklozenge$}$$

for $\alpha > 0$.

(a) Determine values of $p \in [1, \infty)$ for which $f \in L^p[0,1]$.

(b) If p_0 is the minimum value of p for which p may fail to be in L^p, give an example of a function which satisfies ($\blacklozenge$), but which is not in $L^{p_0}[0,1]$.

Solution. ▶ ◀

References

[1] Folland, G. *Real analysis: modern techniques and their applications.* Pure and applied mathematics. Wiley, 1984.

[2] Royden, H., and Fitzpatrick, P. *Real Analysis.* Featured Titles for Real Analysis Series. Prentice Hall, 2010.

[3] Rudin, W. *Principles of Mathematical Analysis.* International series in pure and applied mathematics. McGraw-Hill, 1976.

[4] Wheeden, R., and Zygmund, A. *Measure and Integral: An Introduction to Real Analysis.* Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1977.