QUALIFYING EXAMINATION
AUGUST 2004
MATH 523 - Prof. Petrosyan

1. Consider the initial value problem

$$\begin{aligned} &a(x,y)\,u_x + b(x,y)\,u_y = -u \\ &u = f \quad \text{on } S = \{(x,y) : x^2 + y^2 = 1\}, \end{aligned}$$

where a and b satisfy

$$a(x,y)\,x + b(x,y)\,y > 0 \quad \text{for any } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

a. Show that the initial value problem has a unique solution in a neighborhood of S. Assume that a, b and f are smooth.

b. Show that the solution of the initial value problem actually exists in $\mathbb{R}^2 \setminus \{(0,0)\}$.

2. Let $u \in C^2(\mathbb{R} \times [0,\infty))$ be a solution of the initial value problem for the one dimensional wave equation

$$\begin{array}{ll} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = f,\ u_t = g & \text{on } \mathbb{R} \times \{0\}, \end{array}$$

where f and g have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$$

and the potential energy by

$$P(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx.$$

Show that

a. $K(t) + P(t)$ is constant in t,

b. $K(t) = P(t)$ for all large enough times t.

3. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{array}{ll} u_{tt} - \Delta u = e^{-t} g(x) & \text{for } x \in \mathbb{R}^3,\ t > 0 \\ u(x,0) = u_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{array}$$

Verify that the integral representation reduces to the obvious solution $u = e^{-t} + t - 1$ when $g(x) = 1$.

4. Let Ω be a bounded open set in $\mathbb{R}^n$ and $g \in C_0^\infty(\Omega)$. Consider the solutions of the initial boundary value problem

$$\begin{array}{ll} \Delta u - u_t = 0 & \text{for } x \in \Omega,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \Omega \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t \geq 0 \end{array}$$

and the Cauchy problem

$$\begin{array}{ll} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n,\ t > 0 \\ v(x,0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{array}$$

where we put $g = 0$ outside Ω.

a. Show that

$$-v(x,t) \leq u(x,t) \leq v(x,t), \quad \text{for any } x \in \Omega,\ t > 0.$$

b. Use **a** to conclude that

$$\lim_{t \to \infty} u(x,t) = 0, \quad \text{for any } x \in \Omega.$$

5. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \quad P_m(\lambda x) = \lambda^m P_m(x), \quad \text{for any } x \in \mathbb{R}^n,\ \lambda > 0,$$

$$\Delta P_k = 0, \quad \Delta P_m = 0 \quad \text{in } \mathbb{R}^n.$$

a. Show that

$$\frac{\partial P_k(x)}{\partial \nu} = k P_k(x), \quad \frac{\partial P_m(x)}{\partial \nu} = m P_m(x) \quad \text{on } \partial B_1,$$

where $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ and ν is the outward normal on ∂B_1.

b. Use **a** and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x)\, dS = 0, \quad \text{if } k \neq m.$$