

# MA 661: Homework 1

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**PROBLEM 1.1 (LEE, PROB. 3-1)**

Suppose  $(\widetilde{M}, \tilde{g})$  is a Riemannian  $m$ -manifold,  $M \subset \widetilde{M}$  is an embedded  $n$ -dimensional submanifold, and  $g$  is the induced Riemannian metric on  $M$ . For any point  $p$  show that there is a neighborhood  $\tilde{U}$  of  $p$  in  $\widetilde{M}$  and a smooth orthonormal frame  $(E_1, \dots, E_m)$  on  $\tilde{U}$  such that  $(E_1, \dots, E_m)$  form an orthonormal basis for  $T_q M$  at each  $q \in \tilde{U} \cap M$ . Any such frame is called an adapted orthonormal frame. [Hint: Apply the Gram-Schmidt algorithm to the coordinate frame  $\{\partial_i\}$  in slice coordinates.]

*Proof.*

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**PROBLEM 1.2 (LEE, PROB. 3-2)**

Suppose  $g$  is a pseudo-Riemannian metric on an  $n$ -manifold  $M$ . For any  $p \in M$ , show there is a smooth local frame  $(E_1, \dots, E_n)$  defined in a neighborhood of  $p$  such that  $g$  can be written locally in the form (3.4). Conclude that the index of  $g$  is constant on each component of  $M$ .

*Proof.*

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**PROBLEM 1.3 (LEE, PROB. 3-3)**

Let  $(M, g)$  be an oriented Riemannian manifold with volume element  $dV$ . The divergence operator  $\operatorname{div}: \mathcal{T}(M) \rightarrow C^\infty(M)$  is defined by

$$d(i_X dV) = (\operatorname{div} X) dV,$$

where  $i_x$  denotes interior multiplication by  $X$ : for any  $k$ -form  $\omega$ ,  $i_x \omega$  is the  $(k-1)$ -form defined by

$$i_X \omega(V_1, \dots, V_{k-1}) = \omega(X, V_1, \dots, V_{k-1}).$$

- (a) Suppose  $M$  is a compact, oriented Riemannian manifold with boundary. Prove the following divergence theorem for  $X \in \mathcal{T}(M)$ :

$$\int_M \operatorname{div} X dV = \int_{\partial M} \langle X, N \rangle d\tilde{V}.$$

where  $N$  is the outward unit normal to  $\partial M$  and  $d\tilde{V}$  is the Riemannian volume element of the induced metric on  $\partial M$ .

- (b) Show that the divergence operator satisfies the following product rule for a smooth function  $u \in C^\infty(M)$ :

$$\operatorname{div}(uX) = u \operatorname{div} X + \langle \operatorname{grad} u, X \rangle,$$

and deduce the following “integration by parts” formula:

$$\int_M \langle \operatorname{grad} u, X \rangle dV = - \int_M u \operatorname{div} X dV + \int_{\partial M} u \langle X, N \rangle d\tilde{V}.$$

*Proof.*

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**PROBLEM 1.4 (LEE, PROB. 3-4)**

Let  $(M, g)$  be a compact, connected, oriented Riemannian manifold with boundary. For  $u \in C^\infty M$ , the Laplacian of  $u$ , denoted  $\Delta u$ , is defined to be the function  $\Delta u := \operatorname{div}(\operatorname{grad} u)$ . A function  $u \in C^\infty(M)$  is said to be harmonic if  $\Delta u = 0$ .

(a) Prove Green's identities:

$$\begin{aligned} \int_M u \Delta v \, dV + \int_M \langle \operatorname{grad} u, \operatorname{grad} v \rangle \, dV &= \int_{\partial M} u N v \, d\tilde{V} \\ \int_M (u \Delta v - v \Delta u) \, dV &= \int_{\partial M} (u N v - v N u) \, d\tilde{V} \end{aligned}$$

(b) Show if  $\partial M \neq \emptyset$ , and  $u, v$  are harmonic functions on  $M$  whose restriction to  $\partial M$  agree, then  $u \equiv v$ .

(c) If  $\partial M = \emptyset$  show that the only harmonic functions on  $M$  are the constants.

*Proof.*

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**PROBLEM 1.5 (LEE, PROB. 3-5)**

Let  $M$  be a compact oriented Riemannian manifold (without boundary). A real number  $\lambda$  is called an eigenvalue of the Laplacian if there exists a smooth function  $u$  on  $M$ , not identically zero, such that  $\Delta u = \lambda u$ . In this case,  $u$  is called an eigenfunction corresponding to  $\lambda$ .

- (a) Prove that 0 is an eigenvalue of  $\Delta$ , and that all other eigenvalues are strictly negative.
- (b) If  $u$  and  $v$  are eigenfunctions corresponding to distinct eigenvalues, show that  $\int_M uv \, dV = 0$ .

*Proof.*

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