

## MA 523: Homework 1

Carlos Salinas

August 29, 2016



**PROBLEM 1.1 (TAYLOR'S FORMULA)**

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be smooth,  $n \geq 2$ . Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$$

as  $x \rightarrow 0$  for each  $k = 1, 2, \dots$ , assuming that you know this formula for  $n = 1$ .

*Hint:* Fix  $x \in \mathbb{R}^n$  and consider the function of one variable  $g(t) := f(tx)$ . Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on  $m$ .

**Solution.** ▶ Taking the hint, fix  $x \in \mathbb{R}^n$  and consider the function of one variable  $g(t) := f(tx)$ . We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

*Proof of claim.* We shall prove this by induction on  $m$ . The case  $m = 1$  is straightforward by the chain rule

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\ &= D^{(1,0,\dots,0)} f(tx) x_1 + \dots + D^{(0,\dots,0,1)} f(tx) x_n \\ &= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^\alpha f(tx) x^\alpha. \end{aligned}$$

Now, assume this for  $m - 1$  and consider the  $m$ -th derivative of  $g$  we have

$$\begin{aligned} \frac{d^m}{dt^m} g(t) &= \frac{d}{dt} \left[ \frac{d^{m-1}}{dt^{m-1}} g(tx) \right] \\ &= \frac{d}{dt} \left[ \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} D^\alpha f(tx) x^\alpha \right] \\ &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \frac{d}{dt} (D^\alpha f(tx) x^\alpha) \\ &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \left[ \sum_{|\beta|=1} \frac{1!}{\beta!} D^\beta D^\alpha (f(tx) x^\alpha) x^\beta \right] \\ &= \sum_{|\alpha+\beta|=m} \frac{(m-1)!1!}{\alpha!\beta!} D^{\alpha+\beta} f(tx) x^{\alpha+\beta} \end{aligned}$$

now, note that  $(\alpha + \beta)! = \prod_{i=0}^{\beta_1} (\alpha_1 + \beta_1 - i) \cdots \prod_{i=0}^{\beta_n} (\alpha_n + \beta_n - i) \alpha! =$   
=

□

◀

**PROBLEM 1.2**

Write down the characteristic equation for the p.d.e.

$$u_t + b \cdot Du = f \tag{*}$$

on  $\mathbb{R}^n \times (0, \infty)$ , where  $b \in \mathbb{R}^n$ . Using the characteristic equation, solve (\*) subject to the initial condition

$$u = g$$

on  $\mathbb{R}^n \times \{t = 0\}$ . Make sure the answer agrees with formula (5) in §2.1.2 of [E].

**Solution.** ►

◀

**PROBLEM 1.3**

Solve using the characteristics:

- (a)  $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$ ,  $u = 1$  on the line  $x_2 = 2x_1$ .
- (b)  $uu_{x_1} + u_{x_2} = 1$ ,  $u(x_1, x_2) = x_1/2$ .
- (c)  $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$ ,  $u(x_1, x_2, 0) = g(x_1, x_2)$ .

**Solution.** ►

◀

**PROBLEM 1.4**

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with  $u(x_1, 0) = (1 - x_1^2)/2$ .

**Solution.** ►

◀