

# MA 519: Homework 11

Max Jeter, Carlos Salinas

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## PROBLEM 11.1 (DASGUPTA 7.2 (A), (B), (C), (D), (E))

- (a) Suppose  $E|X_n - c|^\alpha \rightarrow 0$ , where  $0 < \alpha < 1$ . Does  $X_n$  necessarily converge in probability to  $c$ ?
- (b) Suppose  $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0, 1)$ . Under what condition on  $a_n$  can we conclude that  $X_n \xrightarrow{\mathcal{P}} \theta$ ?
- (c)  $o_p(1) + O_p(1) = ?$
- (d)  $o_p(1)O_p(1) = ?$
- (e)  $o_p(1) + o_p(1)O_p(1) = ?$

*SOLUTION.* For part (a) we show that indeed  $E(|X_n - c|^\alpha) \rightarrow 0$  implies  $X_n \xrightarrow{\mathcal{P}} c$ . Let  $\varepsilon > 0$  be given. By Markov's inequality, we have

$$P(|X_n - c| > \varepsilon) = P(|X_n - c|^\alpha > \varepsilon^\alpha) \leq \frac{E(|X_n - c|^\alpha)}{\varepsilon^\alpha}.$$

Since  $E(|X_n - c|^\alpha) \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that  $P(|X_n - c| > \varepsilon) \rightarrow 0$  as  $n \rightarrow \infty$ ; i.e.,  $X_n$  converges to  $c$  in probability.

For part (b), suppose  $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0, 1)$ ; i.e.,  $P(|a_n(X_n - \theta)| \leq x) \rightarrow \Phi(x)$  as  $n \rightarrow \infty$ . In words  $X_n \xrightarrow{\mathcal{P}} \theta$  means that for every  $\varepsilon > 0$  and every  $\eta > 0$  there exists a positive integer  $N$  depending on  $\varepsilon$  and  $\eta$  such that  $n \geq N$  implies

$$P(|X_n - \theta| \geq \varepsilon) < \eta.$$

First, let us find the PDF of the sequence  $a_n(X_n - \theta)$ . Let  $f_n$  denote the PDF of  $X_n$ , then the CDF of  $a_n(X_n - \theta)$  is

$$\begin{aligned} P(|a_n(X_n - \theta)| \leq x) &= P(-x \leq a_n(X_n - \theta) \leq x) \\ &= P\left(-\frac{x}{a_n} + \theta \leq X_n \leq \frac{x}{a_n} + \theta\right) \\ &= \int_{-x/a_n + \theta}^{x/a_n + \theta} f(y) dy \\ &= f(x/a_n + \theta) - f(-x/a_n + \theta), \end{aligned}$$

therefore its PDF is

$$\begin{aligned} \frac{dP}{dx}(|a_n(X_n - \theta)| \leq x) &= \frac{d}{dx} \left[ f\left(\frac{x}{a_n} + \theta\right) - f\left(-\frac{x}{a_n} + \theta\right) \right] \\ &= \frac{1}{a_n} (f(x/a_n + \theta) + f(-x/a_n + \theta)) \end{aligned}$$

For part (c), suppose  $\{a_n\}$  and  $\{b_n\}$  are sequences such that  $a_n = o_p(1)$  and  $b_n = O_p(1)$ , then for the sequence  $\{c_n := a_n + b_n\}$  the most we can expect is  $c_n = O_p(1)$ . Indeed, we know that if a sequence is  $o_p(1)$  then it is also  $O_p(1)$  therefore there exists  $K_1$  and  $K_2$  such that  $|a_n| \leq K_1$ ,  $|b_n| \leq K_2$  for all  $n \geq 1$ . Therefore,  $|c_n| \leq K_1 + K_2$  for all  $n \geq 1$ .

For part (d), suppose  $\{a_n\}$  and  $\{b_n\}$  are sequences such that  $a_n = o_p(1)$  and  $b_n = O_p(1)$ , then for the sequence  $\{c_n := a_n b_n\}$  the most we can expect is  $c_n = O_p(1)$ . Again, since  $\{a_n\}$  is  $o_p(1)$  it

is  $O_p(1)$  so there exists a constant  $K_1 \geq 0$  such that  $|a_n| \leq K_1$  for all  $n \geq 1$  and similarly for  $\{b_n\}$  there exists a constant  $K_2$  such that  $|b_n| \leq K_2$  for all  $n \geq 1$ . Therefore,  $|c_n| \leq K_1 K_2$  for all  $n \geq 1$  so  $c_n = O_p(1)$ .

For part (e), suppose  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  are sequences such that  $a_n, b_n = o_p(1)$  and  $c_n = O_p(1)$ , then for the sequence  $\{d_n := a_n + b_n c_n\}$  the most we can expect is  $d_n = O_p(1)$  since there exists constants  $K_1$ ,  $K_2$ , and  $K_3$  such that  $|a_n| \leq K_1$ ,  $|b_n| \leq K_2$ , and  $|c_n| \leq K_3$  for all  $n \geq 1$ . This implies that  $|d_n| \leq K_1 + K_2 K_3$  for all  $n \geq 1$ . Thus,  $d_n = O_p(1)$ . ■

## PROBLEM 11.2 (DASGUPTA 7.3 [MONTE CARLO])

Consider the purely mathematical problem of finding a definite integral  $\int_a^b f(x) dx$  for some (possibly complicated) function  $f(x)$ . Show that the SLLN provides a method for approximately finding the value of the integral by using appropriate averages  $\frac{1}{n} \sum_{k=1}^n f(X_k)$ .

Numerical analysts call this Monte Carlo integration.

*SOLUTION.* Let  $X_k$ , for  $1 \leq k \leq n$ , be independent and identically distributed  $U[a, b]$  random variables and let  $f: [a, b] \rightarrow \mathbb{R}$  be integrable on  $[a, b]$ . Moreover, let us denote the integral of  $f$  on  $[a, b]$  by

$$I := \int_a^b f(x) dx$$

and the average of  $n$  random sample points from  $[a, b]$  by

$$I_n := \frac{1}{n} \sum_{k=1}^n f(X_k).$$

By the strong law of large numbers, we immediately have

$$I_n \longrightarrow E(f(X_1)) = \int_{-\infty}^{\infty} f(x) \chi_{[a,b]}(x) dx = \int_a^b f(x) dx,$$

as desired. ■

## PROBLEM 11.3 (DASGUPTA 7.4 (A), (B))

Suppose  $X_1, \dots$ , are i.i.d. and that  $E(X_1) = \mu \neq 0$ ,  $\text{Var}(X_1) = \sigma^2 < \infty$ . Let  $S_{m,p} = \sum_{k=1}^m X_k^p$ ,  $m \geq 1$ ,  $p = 1, 2$ .

- (a) Identify with proof the almost sure limit of  $S_{m,1}/S_{n,1}$  for fixed  $m$ , and  $n \rightarrow \infty$ .
- (b) Identify with proof the almost sure limit of  $S_{n-m,1}/S_{n,1}$  for fixed  $m$ , and  $n \rightarrow \infty$ .

*SOLUTION.* For part (a), by the strong law of large numbers the average  $\bar{X}_n = S_{n,1}/n \xrightarrow{\text{a.s.}} \mu$  as  $n \rightarrow \infty$ , so  $S_{n,1} \xrightarrow{\text{a.s.}} \infty$  as  $n \rightarrow \infty$ . Therefore, since  $S_{m,1}$  is a fixed,  $S_{m,1}/S_{n,1} \xrightarrow{\text{a.s.}} 0$ .

For part (b), we have

$$\begin{aligned} \frac{S_{n-m,1}}{S_{n,1}} &= \frac{S_{n,1} - S_{m,1}}{S_{n,1}} \\ &= 1 - \frac{S_{m,1}}{S_{n,1}} \end{aligned}$$

which converges a.s. to 1 since  $S_{m,1}/S_{n,1} \xrightarrow{\text{a.s.}} 0$ . ■

## PROBLEM 11.4 (DASGUPTA 7.5 (A))

Let  $A_n$ ,  $n \geq 1$ ,  $A$  be events with respect to a common sample space  $\Omega$ .

- (a) Prove that  $I_{A_n} \xrightarrow{\mathcal{L}} I_A$  if and only if  $P(A_n) \rightarrow P(A)$ .

*SOLUTION.* One direction of this is obvious; namely, since  $I_{A_n}$  and  $I_A$  are indicator random variables  $E(I_{A_n}) = P(A_n)$  and  $E(I_A) = P(A)$  so  $E(I_{A_n}) = P(A_n) \rightarrow P(A) = E(I_A)$  implies  $I_{A_n} \xrightarrow{\mathcal{L}} I_A$ .

On the other hand, if  $I_{A_n} \xrightarrow{\mathcal{L}} I_A$ , then  $P(I_{A_n} \leq x) \rightarrow P(I_A \leq x)$  so letting  $x \rightarrow \infty$ ,  $P(A_n) = P(I_{A_n} \leq \infty) \rightarrow P(I_A \leq \infty) = P(A)$ . ■

## PROBLEM 11.5 (DASGUPTA 7.11 [SAMPLE MAXIMUM])

Let  $X_k$ ,  $k \geq 1$ , be an i.i.d. sequence, and  $X_{(n)}$  the maximum of  $X_1, \dots, X_n$ . Let  $\xi(F) = \sup\{x : F(x) < 1\}$ , where  $F$  is the common CDF of the  $X_k$ . Prove that  $X_{(n)} \xrightarrow{\text{a.s.}} \xi(F)$ .

*SOLUTION.* We point out that this is an immediate extension of Example 7.7 in DasGupta's book. Let  $\varepsilon > 0$ . Set  $\xi = \xi(F)$ . Then

$$\begin{aligned}
 P(|\forall n \geq m, \xi - X_{(n)}| \leq \varepsilon) &= P(\forall n \geq m, \xi - X_{(n)} \leq \varepsilon) \\
 &= P(\forall n \geq m, X_{(n)} \geq \xi - \varepsilon) \\
 &= P(X_{(m)} \geq \xi - \varepsilon) \\
 &= 1 - P(X_{(m)} < \xi - \varepsilon) \\
 &= 1 - P(X_i < \xi - \varepsilon)^m \\
 &\rightarrow 1
 \end{aligned}$$

with convergence above being as  $m \rightarrow \infty$ .

That is, by definition,  $X_{(n)} \rightarrow \xi$  almost surely. ■



## PROBLEM 11.6 (DASGUPTA 7.14 (A))

Suppose  $X_k$  are i.i.d. standard Cauchy. Show that

(a)  $P(|X_n| > n \text{ infinitely often}) = 1.$

*SOLUTION.* Recall that the sum of two independent Cauchy variables is again Cauchy. Therefore,  $\bar{X}_n = \sum_{k=1}^n X_k$  is Cauchy. Now by the weak law of large numbers, we have

$$P(\limsup |\bar{X}_n| = \infty) = 1.$$

This says that the average for any  $n$ , the average  $\bar{X}_n$  eventually exceeds  $n$  with probability 1. Therefore,  $X_n > n$  infinitely often. ■

## PROBLEM 11.7 (DASGUPTA 7.16 [COUPON COLLECTION])

Cereal boxes contain independently and with equal probability exactly one of  $n$  different celebrity pictures. Someone having the entire set of  $n$  pictures can cash them in for money. Let  $W_n$  be the minimum number of cereal boxes one would need to purchase to own a complete set of the pictures. Find a sequence  $a_n$  such that  $W_n/a_n \xrightarrow{\mathcal{P}} 1$ .  
(*Hint:* Approximate the mean of  $W_n$ .)

*SOLUTION.* We can model the scenario by  $X_k \sim \text{Geom}(\frac{n-k+1}{n})$  where initially ( $k = 1$ ) we have a 100% chance of finding obtaining new picture. Therefore, the minimum number of boxes required to obtain a complete set of pictures  $W_n$  is the sum  $\sum_{k=1}^n X_k$ . Define  $a_n := \sum_{k=1}^n n(n-k+1)^{-1} = n/(n^2 - nk + n)$ . Then by the weak law of large numbers ■

## PROBLEM 11.8 (DASGUPTA 7.17)

Let  $X_n \sim \text{Bin}(n, p)$ . Show that  $(X_n/n)^2$  and  $X_n(X_n - 1)/(n(n - 1))$  both converge in probability to  $p^2$ . Do they converge almost surely?

*SOLUTION.* First note that  $(X_n/n)^2 \sim X_n(X_n - 1)/(n(n - 1))$  so it suffices to show that  $(X_n/n)^2 \rightarrow p^2$ . We show this explicitly. Let  $\varepsilon$  be given then we show that

$$P\left(\left|\left(\frac{X_n}{n}\right)^2 - p^2\right| \geq \varepsilon\right) \rightarrow 0.$$

That is, given  $\eta > 0$  there exists  $N$  such that  $n \geq N$  implies

$$\begin{aligned} P\left(\left|\left(\frac{X_n}{n}\right)^2 - p^2\right| \geq \varepsilon\right) &= P\left(\left(\frac{X_n}{n}\right)^2 - p^2 \geq \varepsilon\right) + P\left(\left(\frac{X_n}{n}\right)^2 - p^2 \leq -\varepsilon\right) \\ &< \eta. \end{aligned}$$

From the calculations above, we have

$$\begin{aligned} P\left(\left(\frac{X_n}{n}\right)^2 - p^2 \geq \varepsilon\right) &= P\left(X_n \geq n\sqrt{\varepsilon + p^2}\right) \\ &= 1 - P\left(X_n < n\sqrt{\varepsilon + p^2}\right) \\ &\approx \frac{1}{\sqrt{2\pi np(1-p)}} \int_{-\infty}^{n\sqrt{\varepsilon + p^2}} e^{-(x-np)^2/(2np(1-p))} dx \\ &\sim C e^{-C''n^2} \int_{-\infty}^{C'n} e^{-C'''x^2} dx \end{aligned}$$

since both sequences in  $n$  above are convergent and the limit of the product of convergent sequences is the product of the limits, then the limit above equals 0. ■

## PROBLEM 11.9 (DASGUPTA 7.21)

Let  $X_1, X_2, \dots$ , be i.i.d.  $U[0, 1]$ . Let

$$G_n = (X_1 \cdots X_n)^{1/n}.$$

Find  $c$  such that  $G_n \xrightarrow{\mathcal{P}} c$ .

*SOLUTION.* Note that

$$\begin{aligned}\ln(G_n) &= \frac{1}{n} \ln(X_1 X_2 \cdots X_n) \\ &= \frac{1}{n} \sum_{i=1}^n \ln(X_i) \\ &\xrightarrow{\mathcal{P}} \int_0^1 \ln(x) dx \\ &= -1\end{aligned}$$

So that  $\ln(G_n) \xrightarrow{\mathcal{P}} -1$ ; that is,  $G_n \xrightarrow{\mathcal{P}} e^{-1}$ .

■

## PROBLEM 11.10 (DASGUPTA 7.30 [CONCEPTUAL])

Suppose  $X_n \xrightarrow{\mathcal{L}} X$ , and also  $Y_n \xrightarrow{\mathcal{L}} X$ . Does this mean that  $X_n - Y_n$  converge in distribution to (the point mass at) zero?

*SOLUTION.* No. Pick  $X_n = U(\{-1, 1\})$ , and  $Y_n = -X_n$ . Then  $X_n - Y_n = 2$  for all  $n \in \mathbb{N}$ , but  $X_n$  and  $Y_n$  are both uniformly distributed on  $\{-1, 1\}$ , so they both converge (in distribution) to  $U(\{-1, 1\})$ . ■

## PROBLEM 11.11 (DASGUPTA 7.31 (A))

- (a) Suppose  $a_n(X_n - \theta) \rightarrow N(0, \tau^2)$ ; what can be said about the limiting distribution of  $|X_n|$ , when  $\theta \neq 0$ ,  $\theta = 0$ ?

SOLUTION. ■