MA 519: Homework 9

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Problem 9.1 (Handout 13, # 7)

Let X have a *double exponential* density $f(x) = \frac{1}{2\sigma}\mathrm{e}^{-\frac{|x|}{\sigma}}$, $-\infty < x < \infty$, $\sigma > 0$.

(a) Show that all moments exist for this distribution.
(b) However, show that the MGF exists only for restricted values. Identify them and find a formula.

Solution. For part (a), we show that the moments $m_n := E(X^n)$ for all $n \in \mathbb{N}$. By direct calculation, we have

$$\begin{aligned} m_n &= \int_{-\infty}^{\infty} x^n f(x)\,dx \\ &= \int_{-\infty}^{\infty} \frac{x^n}{2\sigma}\mathrm{e}^{-\frac{|x|}{\sigma}}\,dx \\ &= \underbrace{\int_{-\infty}^{0} \frac{x^n}{2\sigma}\mathrm{e}^{\frac{x}{\sigma}}\,dx}_{L} + \int_{0}^{\infty} \frac{x^n}{2\sigma}\mathrm{e}^{-\frac{x}{\sigma}}\,dx, \end{aligned}$$

making the substitution $x \mapsto -y$ to L and relabeling y to x again, the above becomes

$$\begin{aligned} &= \int_{0}^{\infty} \frac{x^n + (-1)x^n}{2\sigma}\mathrm{e}^{-\frac{x}{\sigma}}\,dx \\ &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ I := \int_0^\infty \frac{x^n}{\sigma}\mathrm{e}^{-\frac{x}{\sigma}}\,dx & \text{if } n = 2k \text{ is even.} \end{cases} \end{aligned}$$

To evaluate I we apply integration by parts repeatedly to arrive at

$$\begin{aligned} I &= \int_{-\infty}^{0} \frac{x^n}{\sigma}\mathrm{e}^{-\frac{x}{\sigma}} \\ &= (-0+0) + \int_0^\infty n\sigma x^{n-1}\mathrm{e}^{-\frac{x}{\sigma}}\,dx \\ &= (-0+0) + (-0+0) + \int_0^\infty n(n-1)\sigma^2 x^{n-1}\mathrm{e}^{-\frac{x}{\sigma}}\,dx \qquad \vdots \\ &= (-0-0) + \cdots + (-0+0) + (-0+n!\sigma^n) \\ &= n!\sigma^n. \end{aligned}$$

Therefore, m_n exist and are finite for all $n \in \mathbb{N}$.

Fr part (b), the MGF associated to f is given by the series

$$m(t) = \sum_{n=0}^{\infty} \frac{t^n m_n}{n!} = \sum_{k=1}^{\infty} t^{2k}\sigma^{2k}. \tag{9.1}$$

This series is geometric and, as such, converges for all $-\frac{1}{\sigma} < t < \frac{1}{\sigma}$, in which case (9.1) becomes

$$m(t) = \frac{1}{1 - t^2\sigma^2}. \qquad \blacksquare$$

Problem 9.2 (Handout 13, # 10)

Suppose X has Cauchy distribution as in # 6. Which of the following functions have finite expectation

$$X; \quad -X; \quad |X|; \quad \frac{1}{X}; \quad \sin X; \quad \ln|X|; \quad e^X; \quad e^{-|X|}?$$

Solution. Suppose $X \sim \text{Cauchy}(0,1)$. Then the PDF of X is given by the expression

$$f(x) = \frac{1}{\pi(x^2+1)}.$$

Now we proceed to find the expectations of (i) X, (ii) $-X$, (iii) $\frac{1}{X}$, (iv) $\sin X$, (v) $\ln|X|$, (vi) e^X, (vii) $e^{-|X|}$.

For (i), the expectation does not even exist. We repeat the argument given in class: Consider

$$E(X) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx = \lim_{x_1,x_2} \frac{1}{\pi}\int_{-x_1}^{x_2} \frac{x}{x^2+1}\,dx.$$

Then, making the substitution $u = x^2+1$, $du = 2x\,dx$, the integral above evaluates to

$$E(X) = \lim_{x_1,x_2\to\infty} \frac{1}{2\pi}\ln\left(\frac{x_2^2+1}{x_1^2+1}\right).$$

However, the limit of this expression is undefined! Fix positive real number α and let $x_2 = \alpha x_1$. Then

$$E(X) = \lim_{x_1\to\infty} \frac{1}{2\pi}\ln\left(\frac{\alpha^2 x_1^2+1}{x_1^2+1}\right) = \frac{1}{2\pi}\ln\alpha^2.$$

This value is distinct for each α. Therefore, the limit is not unique and so is undefined.

For the rest of these ((ii) through (viii)), we use Theorem 1.33 to forgo computing the PDF.

For (ii), we have

$$E(-X) = \int_{-\infty}^{\infty} -xf(x)\,dx = -E(X).$$

Thus, $E(-X)$ is also undefined.

For (iii), we have

$$\begin{aligned} E(|X|) &= \int_{-\infty}^{\infty} |x| f(x)\,dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|x|}{x^2+1}\,dx \\ &= \frac{2}{\pi}\int_{0}^{\infty} \frac{x}{x^2+1}\,dx \\ &= \lim_{\xi\to\infty} \frac{2}{\pi}\ln(\xi^2+1) \\ &= \infty. \end{aligned}$$

For (iv), we have

$$
\begin{aligned}
E\left(\frac{1}{X}\right) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{x(x^2+1)}\,dx \\
&= \lim_{x_1,x_2\to\infty} \ln\left(\frac{x_2}{\sqrt{x_1^2+1}}\right)
\end{aligned}
$$

which is undefined. In fact, one can show that $\frac{1}{X} \sim \text{Cauchy}(0,1)$: First, let us find the CDF of $\frac{1}{X}$

$$
\begin{aligned}
F_{\frac{1}{X}}(x) &= P\left(\frac{1}{X} \leq x\right) \\
&= P\left(X \geq \frac{1}{x}\right) \\
&= 1 - P\left(X < \frac{1}{x}\right) \\
&= 1 - \frac{1}{\pi}\int_{-\infty}^{\frac{1}{x}} \frac{1}{y^2+1}\,dy \\
&= 1 - \tan^{-1}\left(\frac{1}{x}\right) - \frac{1}{2} \\
&= \frac{1}{2} - \tan^{-1}\left(\frac{1}{x}\right).
\end{aligned}
$$

Thus, the PDF of $\frac{1}{X}$ is

$$
\begin{aligned}
f_{\frac{1}{X}}(x) &= \frac{dF_{\frac{1}{X}}(x)}{dx} \\
&= -\left(-\frac{1}{x^2}\right)\left(\frac{1}{\left(\frac{1}{x}\right)^2+1}\right) \\
&= \frac{1}{x^2+1}.
\end{aligned}
$$

Thus, $\frac{1}{X} \sim \text{Cauchy}(0,1)$ giving us another argument for why $E(\frac{1}{X})$ is undefined.

For (v), we have

$$
\begin{aligned}
E(\sin X) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx \\
&\leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\sin x|}{x^2+1}\,dx
\end{aligned}
$$

since $|\sin x| \leq 1$, we further have

$$
\begin{aligned}
&\leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx \\
&= \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\
&= 1.
\end{aligned}
$$

Thus, $E(\sin X)$ exists and is finite.

For (vi), note that $\ln|X|$ by symmetry,

$$\begin{aligned} E(\ln|X|) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|x|}{x^2+1}\,dx \\ &= \frac{2}{\pi}\int_{0}^{\infty}\frac{\ln x}{x^2+1}\,dx \end{aligned}$$

making the substitution $\theta = \tan^{-1} x$, $(x^2+1)\,d\theta = dx$, we have

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \ln(\tan\theta)\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin\theta)\,d\theta - \int_0^{\frac{\pi}{2}} \ln(\cos\theta)\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin\theta)\,d\theta - \int_0^{\frac{\pi}{2}} \ln\Big(\sin\Big(\frac{\pi}{2}-\theta\Big)\Big)\,d\theta \end{aligned}$$

making the substitution $\varphi = \frac{\pi}{2} - \theta$, $-d\varphi = d\theta$,

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \ln(\sin\theta)\,d\theta + \int_{\frac{\pi}{2}}^{0} \ln(\sin\varphi)\,d\varphi \\ &= 0. \end{aligned}$$

For (vii), we have

$$\begin{aligned} E(\mathrm{e}^X) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^x}{x^2+1}\,dx \\ &= \frac{1}{\pi}\int_{0}^{\infty}\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{x^2+1}\,dx \\ &= \frac{2}{\pi}\int_{0}^{\infty}\frac{\cosh x}{x^2+1}\,dx \\ &= \end{aligned}$$

■

Problem 9.3 (Handout 13, # 16)

Give an example of each of the following phenomena:

(a) A continuous random variable taking values in $[0,1]$ with equal mean and median.
(b) A continuous random variable taking values in $[0,1]$ with mean equal to twice the median.
(c) A continuous random variable for which the mean does not exist.
(d) A continuous random variable for which the mean exists, but the variance does not exist.
(e) A continuous random variable with a PDF that is not differentiable at zero.
(f) a positive continuous random variable for which the mode is zero, but the mean does not exist.
(g) A continuous random variable for which all moments exist.
(h) A continuous random variable with median equal to zero, and 25^{th} and 75^{th} percentiles equal to 1.
(i) A continuous random variable X with mean equal to median equal to mode equal to zero, and $E(\sin X) = 0$.

Solution. First, note that $[0,1]$ is a probability space under the standard Lebesgue measure on $\mathbb{R}$. Therefore, it makes sense to consider $X\colon [0,1] \to \mathbb{R}$ random variables.

For part (a), consider the random variable $X\colon [0,1] \to \mathbb{R}$ defined by $x \mapsto x$ with $X \sim \text{Uniform}[0,1]$. Then the mean is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

and the median is

$$m = \inf\{\, x : F(x) = x \geq 0.5 \,\} = \frac{1}{2}.$$

For part (b), consider again the random variable $X(x) = x$ for $x \in [0,1]$, but this time let

$$f(x) = \begin{cases} & , \\ & . \end{cases}$$

be the PDF of X. Then the mean is ■

Problem 9.4 (Handout 13, # 17)

An exponential random variable with mean 4 is known to be larger than 6. What is the probability that it is larger than 8?

Solution. ∎

Problem 9.5 (Handout 13, # 18)

(Sum of Gammas). Suppose X, Y are independent random variables, and $X \sim \Gamma(\alpha, \lambda)$, $Y \sim \Gamma(\beta, \lambda)$. Find the distribution of $X + Y$ by using moment-generating functions.

Solution. ■

Problem 9.6 (Handout 13, # 19)

(Product of Chi Squares). Suppose $X_1, X_2, \ldots, X_n$ are independent chi square variables, with $X_i \sim \chi^2_{m_i}$. Find the mean and variance of $\prod_{i=1}^n X_i$.

Solution. ∎

Problem 9.7 (Handout 13, # 20)

Let $Z \sim \text{Normal}(0, 1)$. Find

$$P\left(0.5 < \left|Z - \frac{1}{2}\right| < 1.5\right); \quad P\left(\frac{\mathrm{e}^Z}{1 + \mathrm{e}^Z} > \frac{3}{4}\right); \quad P(\Phi(Z) < 0.5).$$

Solution. ■

Problem 9.8 (Handout 13, # 21)

Let $Z \sim \text{Normal}(0,1)$. Find the density of $\frac{1}{Z}$. Is the density bounded?

Solution. ■

Problem 9.9 (Handout 13, # 22)

The 25^{th} and the 75^{th} percentile of a normally distributed random variable are -1 and 1. What is the probability that the random variable is between -2 and 2?

Solution. ∎