MA571 Homework 13

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Problem 13.1 (Munkres §58, Ex.9(a,b,c))

We define the *degree* of a continuous map $h\colon S^1 \to S^1$ as follows:

Let b_0 be the point $(0,1)$ of S^1; choose a generator γ for the infinite cyclic group $\pi_1(S^1, b_0)$. If x_0 is any point of S^1, choose a path α in S^1 from b_0 to x_0 and define $\gamma(x_0) \coloneqq \hat{\alpha}(\gamma)$. Then $\gamma(x_0)$ generates $\pi_1(S^1, x_0)$. The element $\gamma(x_0)$ is independent of the choice of the path α, since the fundamental group of S^1 is Abelian.

Now given $h\colon S^1 \to S^1$, choose $x_0 \in S^1$ and let $h(x_0) = x_1$. Consider the homomorphism

$$h_*\colon \pi_1(S^1, x_0) \longrightarrow \pi_1(S^1, x_1).$$

Since both groups are infinite cyclic, we have

$$h_*(\gamma(x_0)) = d \cdot \gamma(x_1) \tag{*}$$

for some integer d, if the group is written additively. The integer d is called the *degree* of h and is denoted by $\deg h$.

The degree of h is independent of the choice of the generator γ; choosing the other generator woul merely change the sign of both sides of (*).

(e) Show that if $h, k\colon S^1 \to S^1$ have the same degree, they are homotopic.

Proof. ■

Problem 13.2 (Munkres §69, Ex. 1)

Check the details of Example 1.

Proof. The following is the statement of Example 1 as found in the book:

Examples 1. Consider the group P of bijections of the set $\{0, 1, 2\}$ with itself. For $i = 1, 2$, define an element π_1 of P by setting $\pi_i(i) = i - 1$ and $\pi_i(i - 1) = i$ and $\pi_i(j) = j$ otherwise. Then π_i generates a subgroup G_i of P of order 2. The group $G_1$and G_2 generate P, as you can check. But P is not their free product. The reduced words (π_1, π_2, π_1) and (π_2, π_1, π_2), for instance, represent the same element of P.

■

Problem 13.3 (Munkres §69, Ex. 2(a,b,c))

Let $G = G_1 * G_2$, where G_1 and G_2 are nontrivial groups.

(a) Show G is not Abelian.

(b) If $x \in G$, define the *length* of x to be the length of the unique reduced word in the elements of G_1 and G_2 that represents x. Show that if x has even length (at least 2), then x does not have finite order. Show that if x has odd length (at least 3), then x is conjugate to an element of shorter length.

(c) Show that the only elements of G that have finite order are the elements of G_1 and G_2 that have finite order, and their conjugates.

Proof. ■

Problem 13.4 (Munkres §69, Ex. 3)

Let $G = G_1 * G_2$. Given $c \in G$, let cG_1c^{-1} denrote the set of all elements of the form cxc^{-1}, for $x \in G_1$. It is a subgroup of G; show that the intersection with G_2 is the identity alone.

Proof. ■

Problem 13.5

Proof. ■

Problem 13.6

Proof. ∎

Problem 13.7

Proof. ■

Problem 13.8

Proof. ∎