MA 523: Homework 1

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August 31, 2016

Problem 1.1 (Taylor's formula)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to \mathbf{0}$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) \coloneqq f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha| = m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. ▸ Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) \coloneqq f(tx)$. We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha| = m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Proof of claim. We shall proceed by induction on m. The case $m = 1$ follows easily from the chain rule:

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\ &= D^{(1,0,\ldots,0)} f(tx) x_1 + \cdots + D^{(0,\ldots,0,1)} f(tx) x_n \\ &= \big(D^{(1,0,\ldots,0)} x_1 + \cdots + D^{(0,\ldots,0,1)} x_n\big) f(tx) \end{aligned}$$

which we can write compactly as

$$= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Now, assume the result for $n \leq m - 1$. Then

$$\begin{aligned} \frac{d^m}{dt^m} g(t) &= \frac{d}{dt}\left[\frac{d^{m-1}}{dt^{m-1}} g(t)\right] \\ &= \frac{d}{dt}\left[\sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} D^\alpha f(tx) x^\alpha\right] \end{aligned}$$

since the derivative is a linear operator, we have

$$\begin{aligned} &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \frac{d}{dt}\big[D^\alpha f(tx) x^\alpha\big] \\ &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \sum_{|\beta|=1} D^{\alpha+\beta} f(tx) x^{\alpha+\beta} \end{aligned}$$

but since f is smooth, the order in which we take derivatives does not matter and, hence the operators commute giving us

$$= \left[\sum_{|\beta|=1} (Dx)^\beta \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha \right] f(tx). \tag{1.1}$$

From here it suffices to do some combinatorics on the operators and reduce it to the desired expression. By the multinomial theorem, we have

$$\left(\sum\nolimits_{|\alpha'|=1} (Dx)^{\alpha'} \right)^{m-1} = \sum_{|\alpha|=m-1} \binom{|\alpha|}{\alpha} (Dx)^\alpha = \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha.$$

Thus (1.1) becomes

$$\begin{aligned} \left[\sum_{|\beta|=1} (Dx)^\beta \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha \right] f(tx) &= \left[\sum_{|\beta|=1} (Dx)^\beta \left(\sum\nolimits_{|\alpha'|=1} (Dx)^{\alpha'} \right)^{m-1} \right] f(tx) \\ &= \left[\left(\sum\nolimits_{|\beta|=1} (Dx)^\beta \right)^m \right] f(tx) \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} (Dx)^\alpha f(tx) \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha, \end{aligned}$$

as desired. $\square$

Now, applying Taylor's formula in 1 variable to $g(t)$ and evaluating at $t = 1$ we have

$$\begin{aligned} f(x) &= g(1) \\ &= \sum_{i=0}^{k} \frac{g^{(i)}(0)}{i!} 1^i + \mathrm{O}\big(|x|^{k+1}\big) \\ &= \sum_{i=0}^{k} \frac{1}{i!} \sum_{|\alpha|=i} \frac{i!}{\alpha!} D^\alpha f(tx) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big) \\ &= \sum_{i=0}^{k} \sum_{|\alpha|=i} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big) \\ &= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big) \end{aligned}$$

as desired. ◂

Problem 1.2

Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ▸ For reference, formula (5) in §2.1.2 of [E] is the solution to the nonhomogeneous problem

$$u(\mathbf{x}, t) = g(\mathbf{x} - t\mathbf{b}) + \int_0^1 f\big(\mathbf{x} + (s - t)\mathbf{b}, s\big)\, ds$$

where $\mathbf{x} \in \mathbb{R}^n$, $t > 0$.

To make the notation more bearable, we will use $\mathbf{b}$ and $\mathbf{x}$ to denote the original vectors in $(*)$. First, we write $(*)$ as the directional derivative along $(\mathbf{b}, 1)$, (note the abuse of notation)

$$\begin{aligned} f &= u_t + \mathbf{b} \cdot Du \\ &= (\mathbf{b}, 1) \cdot Du. \end{aligned}$$

Using the structure of characteristic ODE, we have

$$F(p, z, x) = (\mathbf{b}, 1) \cdot p.$$

This in turn gives us

$$\dot{x} = D_p F(p, z, x) = (\mathbf{b}, 1), \tag{1.2}$$

whose solution is easily seen to be the line $(\mathbf{b}t + \mathbf{x}, t)$ where $\mathbf{x}$ is some point in the hyperplane $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}$, and

$$\dot{z} = D_p F(p, z, x) = (\mathbf{b}, 1) \cdot p = f$$

with solution

$$z()$$

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Problem 1.3

Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ▸ For part (a), we have

$$\left(x^2\right) \cdot Du = u^2.$$

Using the method of characteristic gives us

$$F(p, z, x) = \left(x^2\right) \cdot p = z^2.$$

Thus, the characteristics are

$$\dot{x} = \left(x_1^2, x_2^2\right)$$

and

$$\dot{z} = \left(x_1^2, x_2^2\right) \cdot p = z^2$$

with initial conditions

$$x_1(0) = x^0 \qquad\qquad x_2(0) = 2x^0$$
$$z(0) = 1.$$

For part (b), we have

For part (c), we have ◂

Problem 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\left(u_{x_1}^2 + u_{x_2}^2\right)$$

find a solution with $u(x_1, 0) = \left(1 - x_1^2\right)/2$.

Solution. ►

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