MA 571: Note on quotient spaces.

You may use anything in this note on all future homework.

DEFINITION OF THE QUOTIENT TOPOLOGY.

Given a set S with an equivalence relation $\sim$, we define the *quotient set* $S/\!\sim$ to be the set of equivalence classes. The function $q : S \to S/\!\sim$ that takes a point of S to its equivalence class is called the *quotient map*.

Now suppose X is a topological space with an equivalence relation $\sim$ (we don't assume any relationship between the topology and $\sim$). What topology should we put on the set $X/\!\sim$?

As a hint, we recall that the subspace topology is designed to make the inclusion map continuous: it has exactly the open sets needed to make this happen. Similarly, the product topology has exactly the open sets needed to make the projection maps continuous.

For $X/\!\sim$ we want the quotient map $q : X \to X/\!\sim$ to be continuous.

So we define the *quotient topology* to consist of the sets U in $X/\!\sim$ whose inverse image $q^{-1}(U)$ is open in X. You can check that this is really a topology.

For later use we record an elementary fact:

Lemma Q.1. The quotient map $q : X \to X/\!\sim$ is continuous.

THE MOST IMPORTANT PROPERTY OF THE QUOTIENT TOPOLOGY.

Theorem Q.2. A function $f : X/\!\sim \to Y$ is continuous if and only if the composite $X \xrightarrow{q} X/\!\sim \xrightarrow{f} Y$ is continuous.

The proof is a homework problem.

It may be helpful to think about the relationship between Theorem Q.2 and Theorem 18.4. Theorem 18.4 says that a map *into* a product is continuous if and only if its composite with each of the projections is continuous. Theorem Q.2 says that a map *out of* a quotient space is continuous if and only if its composite with the quotient map is continuous.

FACTORING A MAP THROUGH A QUOTIENT SPACE.

Suppose that we are given a space X with an equivalence relation $\sim$ and a continuous function $g : X \to Y$ that *preserves* the equivalence relation; that is, if $x \sim x'$ then $g(x) = g(x')$. Then we get a well-defined map

$$\bar{g} : X/\!\sim\, \to Y$$

taking $[x]$ to $g(x)$ (where $[x]$ denotes the equivalence class).

Theorem Q.3. $\bar{g}$ is continuous.

Proof. This is an immediate consequence of Theorem Q.2, because $\bar{g} \circ q = g$. QED

MUNKRES QUOTIENT MAPS.

Next observe that, for any function $p : X \to Y$, we can define an equivalence relation $\sim_p$ on X by $x \sim_p x' \Leftrightarrow p(x) = p(x')$. This is obviously reflexive, symmetric and transitive.

Clearly, p preserves the equivalence relation $\sim_p$, so by Theorem Q.3 we get a continuous function $\bar{p} : X/\!\sim_p\, \to Y$.

Definition Q.4. Let X and Y be topological spaces. A map $p : X \to Y$ is a *Munkres quotient map* if

$$\bar{p} : X/\!\sim_p\, \to Y$$

is a homeomorphism.

Proposition Q.5. A map $p : X \to Y$ satisfies Definition Q.4 if and only if it satisfies the definition at the top of page 137 in Munkres.

The proof is a homework problem.

Next we observe that Theorem Q.2 generalizes to Munkres quotient maps.

Proposition Q.6. Let $p : X \to Y$ be a Munkres quotient map. A function $f : Y \to Z$ is continuous if and only if the composite $X \xrightarrow{p} Y \xrightarrow{f} Z$ is continuous.

The proof is a homework problem

EQUIVALENCE RELATIONS AND PARTITIONS.

You know that an equivalence relation on a set S gives a *partition* of S into disjoint sets (the equivalence classes).

In the other direction, if we are given a partition of S into disjoint sets, we can define an equivalence relation $\sim$ by letting $x \sim y$ if and only if x and y are in the same set of the partition. Munkres does this in Examples 4 and 5 on page 139.

Also, Munkres uses the notation X^* instead of $X/\!\sim$ (see the top of page 139). You may use either notation on the homework.