

MA571 Homework 9

Carlos Salinas

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PROBLEM 9.1 (MUNKRES §46, EX. 6)

Show that the compact-open topology, $\mathcal{C}(X, Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular. [Hint: If $\overline{U} \subset V$, then $\overline{S(C, U)} \subset S(C, V)$.]

Proof. Suppose that Y is regular. We shall proceed by the hint and Lemma 31.1(b). Consider the subbasis element $S(C, U)$. Since Y is regular, there exists a neighborhood $V \supset U$ such that $V \supset \overline{U}$. Let $f \in \overline{S(C, U)}$. Then, we claim that $f \in S(C, V)$. For suppose not, then there exists an element $x_0 \in C$ such that $f(x_0) \notin V$. Then, since $\overline{U} \subset V$, by hypothesis, $f(x_0) \notin \overline{U}$. Consider the subbasic neighborhood $S(\{x_0\}, Y - \overline{U})$ of f . Then, $S(\{x_0\}, Y - \overline{U}) \cap S(C, U)$ is nonempty. Let g be in the aforementioned intersection. Then $g(x_0) \in g(C) \subset U$, but $g(x_0) \in Y - \overline{U}$. This is a contradiction. It follows by Lemma 31.1(b) that $\mathcal{C}(X, Y)$ is regular. ■

PROBLEM 9.2 (MUNKRES §46, EX. 9(A,B,C))

Here is a (unexpected) application of Theorem 46.11 to quotient maps. (Compare Exercise 11 of §29.)

Theorem. *If $p: A \rightarrow B$ is a quotient map and X is locally compact Hausdorff, then $(\text{id}_X, p): X \times A \rightarrow X \times B$ is a quotient map.*

Proof. (a) Let Y be the quotient space induced by (id_X, p) ; let $q: X \times A \rightarrow Y$ be the quotient map. Show there is a bijective continuous map $f: Y \rightarrow X \times B$ such that $f \circ q = (\text{id}_X, p)$.

(b) Let $g = f^{-1}$. Let $G: B \rightarrow \mathcal{C}(X, Y)$ and $Q: A \rightarrow \mathcal{C}(X, Y)$ be the maps induced by g and q , respectively. Show that $Q = G \circ p$.

(c) Show that Q is continuous; conclude that G is continuous, so that g is continuous.

Actual proof. (a) First, let us note that Y has the same underlying set as the Cartesian product $X \times B$ with the topology induced by the map (id_X, p) . Moreover, the quotient map q is (as a map between sets) equivalent to (id_X, p) . Thus, the natural bijective map to consider between Y and $X \times B$ is the identity map $f = \text{id}_{X \times B} = (\text{id}_X, \text{id}_B)$. It is clear that f is bijective. To see that f is continuous we note that since the composition

$$(f \circ q)(x, a) = (\text{id}_X, \text{id}_B) \circ (\text{id}_X, p)(x, a) = (\text{id}_X(a), \text{id}_B(p(a))) = (x, p(a)) = (\text{id}_X, p)(x, a)$$

and the map (id_X, p) is continuous by Theorem 18.4, it follows by Theorem Q.2 that f is continuous.

(b) Recall, from the definition given on Munkres §46, p. 287, that the induced map G (respectively Q) are defined by the equation $(G(b))(x) = (x, b)$ (respectively $(Q(a))(x) = (x, p(a))$). Then we have that the composition

$$(G \circ p)(a) = G(p(a)) = (G(p(a)))(x) = (x, p(a)) = (Q(a))(x) = Q(a)$$

as desired.

(c) By Theorem 46.11, since q is continuous with respect to the quotient topology on Y , it follows that the induced map Q is continuous. Additionally, since Q is equal to the composition $G \circ p$ by part (b) so by Theorem Q.2 G is continuous. Since X is locally compact Hausdorff, it follows by Theorem 46.11 that the map g is continuous. ■

PROBLEM 9.3 (MUNKRES §51, EX. 1)

Show that if $h, h': X \rightarrow Y$ are homotopic and $k, k': Y \rightarrow Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof. Let $H: X \times I \rightarrow Y$ and $K: Y \times I \rightarrow Z$ denote the homotopies from h to h' and k to k' , respectively. Then, we claim that the map $L(x, t) = K(H(x, t), t)$ is a homotopy from $k \circ h$ to $k' \circ h'$. First, we check that L starts and ends where we want it to, i.e., $L(x, 0) = K(H(x, 0), 0) = k(h(x))$ and $L(x, 1) = K(H(x, 1), 1) = k'(h'(x))$. Lastly, we must assure ourselves that L is in fact continuous. But this last claim follows from the fact that L can be expressed as the composition $K \circ (h_t, t)$ where h_t denotes the continuous map $H(x, t)$ at time t . Since K is (by assumption) continuous and (h_t, t) are continuous by Theorem 18.4, it follows by Theorem 18.2(a) that L is continuous. Thus, $k \circ h \simeq k' \circ h'$ as desired. ■

PROBLEM 9.4 (MUNKRES §51, EX. 2)

Given spaces X and Y , let $[X, Y]$ denote the homotopy classes of maps of X into Y

- (a) Let $I = [0, 1]$. Show that for any X , the set $[X, I]$ has a single element.
- (b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

Proof. (a) Let $f, g: X \rightarrow I$ be arbitrary continuous maps. Then we claim that the straight line homotopy $H(x, t) = (1 - t)f(x) + tg(x)$ gives a homotopy from f to g . Note that the image of $H(x, t)$ stays in the interval I since $(1 - t)f(x) + tg(x) \leq (1 - t) + t = 1$ for all x and for all t . Lastly, note that by Theorem 25.1 H is continuous since it is the sum of a product of continuous functions. Hence, $f \simeq g$. Since f and g were arbitrary, it follows that $[X, I]$ consists of a single equivalence class.

(b) ■

PROBLEM 9.5 (MUNKRES §51, EX. 3(A,B,C,))

A space X is said to be *contractible* if the identity map $\text{id}_X: X \rightarrow X$ is nullhomotopic.

- (a) Show that I and \mathbf{R} are contractible.
- (b) Show that a contractible space is path connected.
- (c) Show that if Y is contractible, then for any X , the set $[X, Y]$ has a single element.

Proof.

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