

# MA571: Qual Preparation

Carlos Salinas

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# 1 Gepner's homework

## 1.1 Homework 1

**Exercise 1.1.** Let  $\{X_i : i \in I\}$  be an  $I$ -indexed family of topological spaces. Show that the Cartesian product

$$X = \prod_{i \in I} X_i,$$

equipped with the product topology, has the property that for each  $i \in I$  the projection  $p_i : X \rightarrow X_i$  is continuous, and moreover, that  $X$  has the following universal property: for any other topological space  $Y$ , the function

$$\mathrm{Hom}_{\mathbf{Top}}(Y, X) \longrightarrow \prod_{i \in I} \mathrm{Hom}_{\mathbf{Top}}(Y, X_i),$$

induced by the projections  $p_i : X \rightarrow X_i$ , is a bijection.

*SOLUTION.* ■

**Exercise 1.2.** Let  $X$  be the set equipped with a topology and let  $\{\mathcal{U}_i : i \in I\}$  a family of topologies on  $X$ . Show that

$$\mathcal{U} = \bigcap_{i \in I} \mathcal{U}_i$$

is a topology on  $X$ . Show that if  $\mathcal{B}$  is a basis for a topology on  $X$ , then the topology  $\mathcal{U}$  on  $X$  generated by  $\mathcal{B}$  is the intersection of all topologies on  $X$  which contain  $\mathcal{B}$ , and that this holds even if we only require that  $\mathcal{B}$  be a subbasis.

*SOLUTION.* ■

**Exercise 1.3.** A topological space  $X$  is said to be *Hausdorff* if, for every pair of points  $x_0, x_1 \in X$  with  $x_0 \neq x_1$ , there exists open subsets  $U_0, U_1$  of  $X$  such that  $x_0 \in U_0$ ,  $x_1 \in U_1$ , and  $U_0 \cap U_1 = \emptyset$ . Show that a topological space  $X$  is Hausdorff if and only if the diagonal inclusion  $X \rightarrow X \times X$  is closed.

*SOLUTION.* ■

**Exercise 1.4.** Let  $X$  be a topological space and let  $Y \subseteq X$  be a subset of  $X$ . Show that if  $Y$  is equipped with the subspace topology then the inclusion function  $i : Y \rightarrow X$  is continuous. Show that if there exists a continuous function  $q : X \rightarrow Y$  such that  $q \circ i = \mathrm{id}_Y$  then  $q$  is a quotient map (that is,  $Y$  is also a quotient topology). Give an example of such a situation.

*SOLUTION.* ■

**Exercise 1.5.** A *topological group* is a group  $G$  with a topology  $\mathcal{U}$  such that the multiplication  $\mu: G \times G \rightarrow G$  and inversion  $i: G \rightarrow G$  are continuous (it is standard to also assume that the topology  $\mathcal{U}$  on  $G$  is Hausdorff, which we shall do). Let  $H$  be a subgroup of  $G$ , and let  $G/H$  denote the quotient of  $G$  by the action of  $H$ , equipped with the quotient topology. Show that  $G/H$  is a homogeneous space and that the quotient map  $q: G \rightarrow G/H$  is open. If, moreover,  $H$  is a closed subset of  $G$ , show that  $G/H$  has the property that points are closed. Finally, show that if  $H$  is a normal subgroup of  $G$ , then  $G/H$  is a topological group. (Optional: is it Hausdorff?)

*SOLUTION.* ■

## 2 Homework 2

**Exercise 2.1.** A topological space  $X$  is said to be totally disconnected if a subspace  $Y \subseteq X$  is connected if and only if  $Y = \{x\}$  consists of only a single point  $x \in X$ . Show that if  $X$  is discrete (that is, all subsets of  $X$  are open) then  $X$  is totally disconnected. Find an example of a totally disconnected space which is not discrete.

*SOLUTION.* ■

**Exercise 2.2.** Let  $X$  be a simply ordered set equipped with the order topology. Show that if  $X$  is connected then  $X$  is a continuum.

*SOLUTION.* ■

**Exercise 2.3.** Show that a metric  $d: X \times X \rightarrow \mathbf{R}$  on a set  $X$  determines a coarsest topology  $\mathcal{U}$  on  $X$  for which the distance function  $d: X \times X \rightarrow \mathbf{R}$  is continuous, and give an explicit basis for this topology. Recall that a function  $f: X \rightarrow Y$  between metric spaces is said to be continuous at  $x$  if, for all  $\varepsilon > 0$  there exists a  $\delta > 0$  such that if  $d(x, x_0) < \delta$  then  $d(f(x), f(x_0)) < \varepsilon$ ; show that  $f$  is continuous (in the sense of topology) if and only if it is continuous at  $x$  for all  $x \in X$ . Finally, show that every compact subspace of a metric space is closed and bounded, and find an example of a metric space for which there exists a closed and bounded subspace which is not compact.

*SOLUTION.* ■

**Exercise 2.4.** Let  $X$  be a compact space,  $Y$  a Hausdorff space, and  $f: X \rightarrow Y$  a continuous function. Show that  $f$  is a closed map (that is,  $f$  sends closed sets to closed sets), and also that the projection  $p: X \times Y \rightarrow Y$  is a closed map.

*SOLUTION.* ■

**Exercise 2.5.** Let  $f: W \rightarrow X$  and  $g: W \rightarrow Y$  be continuous functions. The pushout  $X \amalg_W Y$  of  $f$  and  $g$  is the quotient of the disjoint union  $X \amalg Y$  by the equivalence relation generated by the relation  $x \sim y$  if there exists a  $w \in W$  such that  $x = f(w)$  and  $y = g(w)$ . Show that  $X \amalg_W Y$  comes equipped with continuous functions  $i: X \rightarrow X \amalg_W Y$  and  $j: Y \rightarrow X \amalg_W Y$  such that  $i \circ f = j \circ g$ , and is universal among topological spaces  $Z$  equipped with continuous functions  $i': X \rightarrow Z$  and  $j': Y \rightarrow Z$  such that  $i' \circ f = j' \circ g$  in the following sense: given any such space  $Z$ , there exists a unique continuous function  $k: X \amalg_W Y \rightarrow Z$  such that  $i' = k \circ i$  and  $j' = k \circ j$ .

*SOLUTION.* ■