# MA557 Homework 12

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## Problem 12.1

Let $R$ be a Noetherian domain. Show that the following are equivalent:

(i) $R$ is a unique factorization domain

(ii) every prime ideal of $R$ of height one is principal

(iii) $R$ is normal with $\mathrm{Cl}(R) = 0$.

*Proof.* (i) $\implies$ (ii) Suppose $R$ is a Noetherian domain. Let $\mathfrak{p}$ be a height one prime. Then there exists at least one nonzero element $x \in \mathfrak{p}$. Let $x = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the factorization of $x$ into irreducible (prime) elements of $R$. Set $p \coloneqq p_i$ for any prime in the factorization of $x$. Then the ideal generated by $p$ is a prime ideal contained in $\mathfrak{p}$, i.e., $\langle p \rangle \subset \mathfrak{p}$. But $\mathrm{ht}(\mathfrak{p}) = 1$. Thus, $\langle p \rangle = \mathfrak{p}$.

(ii) $\implies$ (ii) Suppose that every height one prime ideal in $R$ is principal. To show that $R$ is a UFD, it suffices to show that every irreducible element $p$ is a prime element, that is, $\langle p \rangle$ is a prime ideal. Let $\mathfrak{p}$ be the minimal prime containing $p$. Since $\mathfrak{p}$ is principal, $\mathfrak{p} = \langle x \rangle$ for some $x \in \mathfrak{p}$. Thus, $p = xy$ for some $y \in R$. But $p$ is prime hence, irreducible so either $x$ or $y$ is a unit. If $x$ is a unit, then $\mathfrak{p} = R$, which is a contradiction. Thus, $y$ must be a unit and we see that $\langle p \rangle = \langle xy \rangle = \mathfrak{p}$ is prime.

Now, for the following implications we need to know a couple of denfinitions and a theorem: Let $D(R)$ denote the set of divisional fractional $R$-ideals and $F(R)$ denote the set of all principal fractional ideals. Then the *divisor class group of* $R$ is the quotient $\mathrm{Cl}(R) \coloneqq D(R)/F(R)$.

**Theorem** Krull's Principal Ideal Theorem**.** *In a Noetherian ring, every minimal prime ideal of a principal ideal has height at most* 1.

■

## Problem 12.2

Let $R$ be a ring with total ring of quotients $K$, $M$ an $R$-module, and

$$\operatorname{Tor}(M) = \{\, x \in M \mid ax = 0 \text{ for some non zero-divisor } a \text{ of } R \,\}.$$

The submodule $\operatorname{Tor}(M)$ is called the *torsion of* $M$, and $M$ is called *torsion free* if $\operatorname{Tor}(M) = 0$. Show

(a) $\operatorname{Tor}(M) = \ker(M \to K \otimes_R M)$

(b) $M/\operatorname{Tor}(M)$ is torsion free.

*Proof.* (a) Let $S$ denote the set of all regular elements of $R$ and let $\varphi\colon R \to K$, where $K \coloneqq S^{-1}R$, be the canonical localization map $a \mapsto a/1$. We show, by way of double inclusion, that $\operatorname{Tor}(M) = \ker\Phi$, where $\Phi\colon M \to K \otimes_R M$ is the canonical map $x \mapsto 1 \otimes x$. Note that this map, $\Phi$, is well defined by the UMP of the tensor product (HW 2). Now let us show the containment $\operatorname{Tor}(M) \subset \ker\Phi$: Let $x \in \operatorname{Tor}(M)$, then $x$ is a non-zero divisor of $R$ such that $ax = 0$. Since $a$ is a non-zero divisior, $a \in S$ so $a/1 = 0/1$ in $K$. Thus, we have

$$\Phi(xm) = 1 \otimes x = a/1 \otimes x = 0 \otimes x = 0,$$

so $x \in \ker\Phi$. Conversely, suppose that $x \in \ker(\Phi)$. By some theorem from the localization section[1] we have $K \otimes_R M \cong S^{-1}M$. Thus $1 \otimes x = 0$ implies that $x = 0$ in the localization $S^{-1}M$. This is true if and only if $ax = 0$ for some non-zero divisor $a$ of $R$. Thus, $x \in \ker\Phi$ and equality holds.

(b) We prove the statement elementwise. Let $x \coloneqq x' + \operatorname{Tor}(M)$ be in $M/\operatorname{Tor}(M)$. Then $ax = 0$ for some non zero-divisor $a \in R$. This implies that $ax' + \operatorname{Tor}(M) = 0 + \operatorname{Tor}(M)$ or $ax' \in \operatorname{Tor}(M)$. Then $b(ax') = 0$ for some non zero-divisor $b \in R$. Since both $a$ and $b$ are non-zero divisors, and $(ba)x' = 0$ then $x' \in \operatorname{Tor}(M)$. Thus, $\operatorname{Tor}(M) = 0$. ■

[1] Sorry! I misplaced my notebook and I've been taking notes on sheets of computer paper so I hate going through the mess.

## Problem 12.3

Let $R$ be a Dedekind domain and $M$ a finitely generated $R$-module of rank $r$. Show that:

(a) If $M$ is torsion free then $M$ is projective (hint: induct on $r$).

(b) $M \cong \operatorname{Tor}(M) \oplus P$ with $P$ projective.

(c) If $M \neq 0$ is projective then $M \cong R^{r-1} \oplus I$ with $I \neq 0$ an ideal.

(d) If $M$ is torsion (i.e., $M = \operatorname{Tor}(M)$) then

$$M \cong R/I_1 \oplus \cdots \oplus R/I_n \qquad \text{with} \qquad I_1 \supset \cdots \supset I_n \neq 0$$

ideals (hint: for $p_1, ..., p_s$ the minimal primes of $\operatorname{ann}(M)$ and $S = R \smallsetminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s)$, show that $S^{-1}R$ is a PID).

*Proof.* (a) We prove the statement for $r = 1$ and then induct on $r$. Let $M$ be generated by $x$. Then $M \coloneqq \langle x \rangle$ is an $R$-module. Hello ■