# MA 519: Homework 13

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December 1, 2016

## Problem 13.1 (Handout 17, # 16)

Suppose $X \sim \mathrm{Exp}(1)$, $Y \sim U[0,1]$, and $X$, $Y$ are independent.

(a) Find the density of $X + Y$.
(b) Find the density of $XY$.

*Solution.* For part (a): Since $X$ and $Y$ are independent, the distribution of $X+Y$ is given by the convolution

$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_X(x-y) f_Y(y)\, dy,$$

where

$$f_X(x) = \begin{cases} \mathrm{e}^{-x} & \text{for } x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad f_Y(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, a straight forward calculation gives us

$$\begin{aligned} f_{X+Y}(x) &= \int_{-\infty}^{\infty} \chi_{[0,\infty)}(x-y)\mathrm{e}^{-(x-y)}\chi_{[0,1]}(y)\, dy \\ &= \mathrm{e}^{-x}\int_{-\infty}^{\infty} \mathrm{e}^{y}\chi_{[0,\infty)}(x-y)\chi_{[0,1]}(y)\, dy \\ &= \begin{cases} 0 & \text{for } x < 0, \\ 1-\mathrm{e}^{-x} & \text{for } 0 \leq x \leq 1, \\ (\mathrm{e}-1)\mathrm{e}^{-x} & \text{for } x > 1. \end{cases} \end{aligned}$$

Now let us run a sanity check by demonstrating that $\int_{-\infty}^{\infty} f_{X+Y}(x)\, dx = 1$,

$$\begin{aligned} \int_{-\infty}^{\infty} f_{X+Y}(x)\, dx &= \int_0^1 [1-\mathrm{e}^{-x}]\, dx + (\mathrm{e}-1)\int_1^{\infty} \mathrm{e}^{-x}\, dx \\ &= \big[1+\mathrm{e}^{-1}-1-0\big] + (\mathrm{e}-1)\big[0-(-\mathrm{e}^{-1})\big] \\ &= \mathrm{e}^{-1}+1-\mathrm{e}^{-1} \\ &= 1. \end{aligned}$$

For part (b): Since $X$ and $Y$ are independent, we have

$$F_{XY}(z) = \iint_{\{\,(x,y):xy\leq z\,\}} f_X(x)f_Y(y)\, dx\, dy.$$

Let us find the CDF of $XY$. By a direct computation

$$
\begin{aligned}
F_{XY}(z) &= \iint\limits_{\{(x,y):xy\leq z\}} f_X(x) f_Y(y)\,dx\,dy \\
&= \iint\limits_{\{(x,y):xy\leq z\}} \mathrm{e}^{-x}\chi_{[0,\infty)}(x)\chi_{[0,1]}(y)\,dx\,dy \\
&=
\end{aligned}
$$

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## Problem 13.2 (Handout 17, # 18)

Two points $A$, $B$ are chosen at random from the unit circle. Find the probability that the circle centered at $A$ with radius $AB$ is fully contained within the original unit circle.

*Solution.* The probability that a circle centered at $A$ with radius $AB$ is contained in the original circle is zero. What the professor means is "two points $A$, $B$ are chosen at random from *inside* the unit circle". We can think of choosing $A$ as choosing a random variable $0 < R < 1$ representing the distance of $A$ from the origin and we ask what is the probability that the point $B$ lands inside the circle of radius $1 - R$ centered at $A$.

First, let us find the distribution for the radius $R$. We can find the CDF of $R$ as the ratio of the area of the circle centered at the origin with radius $x$ and the unit circle; i.e.,

$$P(R \le x) = \frac{\pi x^2}{\pi \cdot 1^2} = x^2 \qquad \text{for } 0 < x < 1.$$

Thus the PDF of $R$ is

$$f_R(x) = 2x \qquad \text{for } 0 < x < 1.$$

Then $A = R\Theta$ where $\Theta \sim U(0, 2\pi)$.

Now, the probability we are after is

$$P\big(B \in \{\, x : |x - A| < 1 - R \,\}\big).$$

To find this probability we use a bit of calculus

$$\begin{aligned} P\big(B \in \{\, x : |x - A| < 1 - R \,\} \mid A = x'\big) &= \frac{1}{\pi} \int_{\{\, x : |x - x'| < 1 - R \,\}} \chi_{\{\, |x| = 1 \,\}}(y)\, dy \\ &= \end{aligned}$$

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## Problem 13.3 (Handout 17, # 19)

Let $X, Y$ be i.i.d. $U[0,1]$ random variables. Find the correlation between $\max\{X, Y\}$ and $\min\{X, Y\}$.

*Solution.* First, let us find the CDF of $W := \max\{X, Y\}$ and $Z := \min\{X, Y\}$. These are

$$\begin{aligned} P(W \leq x) &= P(\max\{X,Y\} \leq x) \\ &= P(X \leq x \text{ and } Y \leq x) \\ &= P(X \leq x)P(Y \leq x) \\ &= \begin{cases} x^2 & \text{for } 0 \leq x \leq 1, \\ 1 & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

and

$$\begin{aligned} P(Z \leq x) &= 1 - P(Z \geq x) \\ &= 1 - P(\min\{X,Y\} \geq x) \\ &= 1 - P(X \geq x \text{ and } Y \geq x) \\ &= 1 - P(X \geq x)P(Y \geq x) \\ &= \begin{cases} 1-(1-x)^2 & \text{for } 0 \leq x \leq 1, \\ 1 & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, their corresponding PDFs are

$$f_W(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases} \qquad f_Z(x) = \begin{cases} 2(1-x) & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

With this data we can now compute the mean of $W$ and $Z$ and find the correlation between $W$ and $Z$. First, let us find the first and second moments of $W$, $Z$.

The first moment of $W$ and $Z$ are

$$\begin{aligned} E(W) &= \int_0^1 2x^2\,dx \\ &= \frac{2}{3}, \end{aligned}$$

and

$$\begin{aligned} E(Z) &= \int_0^1 2x(1-x)\,dx \\ &= \frac{1}{3}. \end{aligned}$$

The second moment of $W$ and $Z$ are

$$\begin{aligned} E(W) &= \int_0^1 2x^3\,dx \\ &= \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} E(Z) &= \int_0^1 2x^2(1-x)\,dx \\ &= \frac{2}{3} - \frac{1}{2} \\ &= \frac{1}{6}. \end{aligned}$$

Thus, the variances are

$$\begin{aligned} \mathrm{Var}(W) &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{1}{18} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(Z) &= \frac{1}{6} - \frac{1}{9} \\ &= \frac{1}{18}. \end{aligned}$$

Thus,

$$\sigma_W = \sigma_Z = \sqrt{1/18}.$$

Lastly, we must find the covariance of $W$ and $Z$. That is,

$$\begin{aligned} \mathrm{Cov}(W,Z) &= E(WZ) - E(W)E(Z) \\ &= E(\max\{X,Y\}\min\{X,Y\}) - E(W)E(Z) \\ &= E(XY) - E(W)E(Z) \end{aligned}$$

since $X$ and $Y$ are independent

$$\begin{aligned} &= E(X)E(Y) - E(W)E(Z) \\ &= \frac{1}{4} - \frac{2}{9} \\ &= \frac{1}{36}. \end{aligned}$$

At last we come to the correlation of $W$ and $Z$,

$$\rho_{WZ} = \frac{\mathrm{Cov}(W,Z)}{\sigma_W\sigma_Z} = \frac{1/36}{1/18} = \frac{1}{2}. \qquad \blacksquare$$