MA571 Problem Set 6

Carlos Salinas

October 5, 2015

Problem 6.1 (Munkres §25, Ex. 8)

Let $p\colon X \to Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected. [*Hint:* If C is a component of the open set U of Y, show that $p^{-1}(C)$ is a union of components of $p^{-1}(U)$.]

Proof. ■

Problem 6.2 (Munkres §25, Ex. 10(a,b))

Let X be a space. Let us define $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$.

(a) Show this relation is an equivalence relation. The equivalence classes are called *quasicomponents* of X.

(b) Show that each component of X lies in a quasicomponent of X, and that the components and quasicomponents of X are the same if X is locally connected.

Proof. ■

Problem 6.3 (Munkres §26, Ex. 4)

Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

Proof. ■

Problem 6.4 (Munkres §26, Ex. 5)

Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exists disjoint open sets U and V containing A and B, respectively.

Proof. ■

Problem 6.5 (Munkres §26, Ex. 7)

Show that if Y is compact, then the projection $\pi_X\colon X \times Y \to X$ is a closed map.

Proof. ∎

Problem 6.6 (A)

Let X be a compact space and let $\sim$ be an equivalence relation on X. Suppose that the set

$$S = \{ (x, y) \mid x \sim y \}$$

is a closed subset of $X \times X$. Prove that the quotient map $q \colon X \to X/\sim$ is a closed map.

Proof. ■

Problem 6.7 (B)

Let S^2 be the sphere

$$\{\, (x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1 \,\}.$$

Let S^2_+ be $S^2 \cap \{z \geq 0\}$ (the upper hemisphere), let $S^2 \cap \{z \leq 0\}$ (the lower hemisphere), and let E be $S^2 \cap \{z = 0\}$ (the equator). Recall the definition of Y/S from Homework #4. Prove that S^2/S^2_- is homeomorphic to S^2_+/E. [*Hint:* There are maps in both directions.]

Proof. ∎