Central Limit Theorem (CLT)

Note Title 11/12/2015

(for Binomial r.v.'s) DeMoivre-Laplace Thm

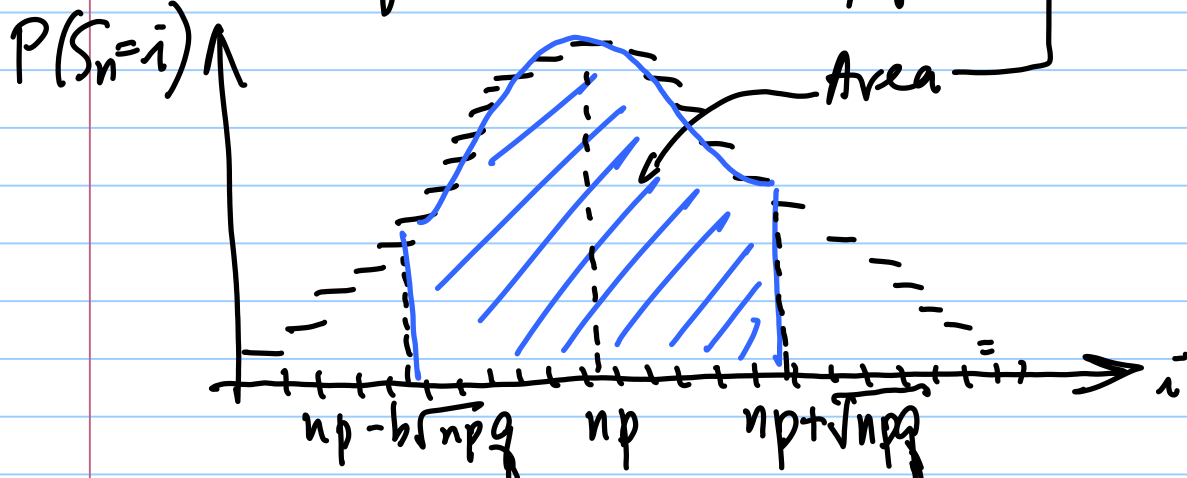
$$P(S_n = i) = \binom{n}{i} p^i q^{n-i}$$

$$\boxed{\begin{array}{l} \text{CLT: } \dfrac{S_n - np}{\sqrt{npq}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1) \\ \text{i.e. } \forall\ a<b \\ P\left(a < \dfrac{S_n - np}{\sqrt{npq}} < b\right) \xrightarrow{n\to\infty} \displaystyle\int_a^b \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz \end{array}}$$

In particular, for $a = -b$,

$$P\left(-b < \frac{S_n - np}{\sqrt{npq}} < b\right) \longrightarrow \int_{-b}^{b} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$

i.e. $P(np - b\sqrt{npq} < S_n < np + b\sqrt{npq})$



(1) Recall Stirling's Formula:

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \quad \text{for } n \gg 1$$

ie. $$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}} = 1$$

(2) Consider the max. value of $P(S_n = i)$ which occurs at $i = np$

$$P(S_n = np) = \binom{n}{i} p^i q^{n-i}, \quad i = np$$

$$= \frac{n!}{i!\,(n-i)!} p^i q^{n-i}$$

$$= \frac{n!}{(np)!\,(n-np)!} p^{np} q^{n-np}$$

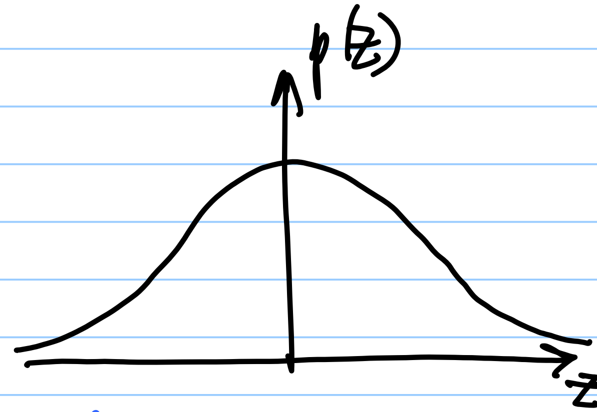
$$= \frac{n!}{(np)!\,(nq)!} p^{np} q^{nq}$$

$(p+q=1)$

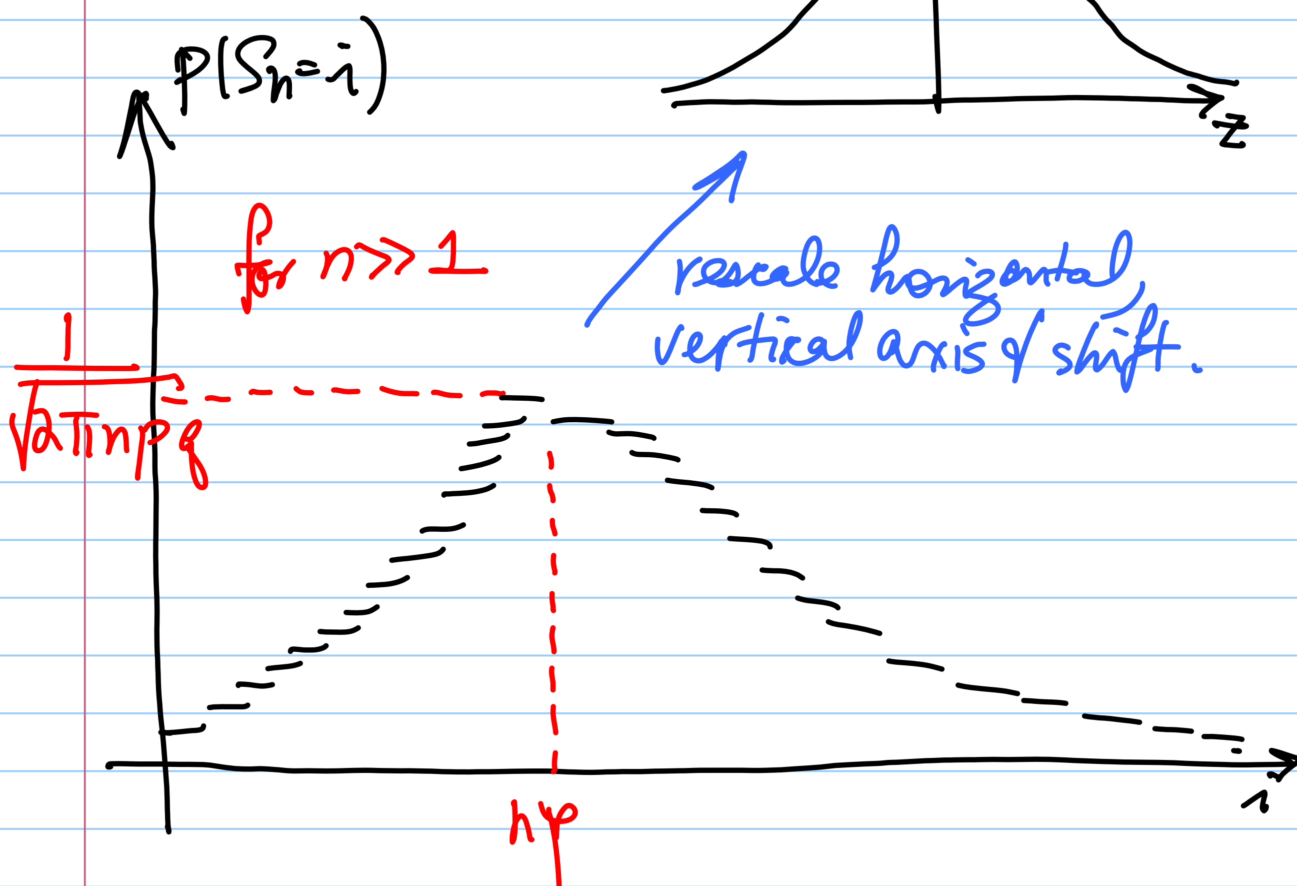
$$= \frac{\cancel{\sqrt{2\pi}}\; n^{n+\frac{1}{2}}\; \cancel{e^{-n}}\; p^{np}\, q^{nq}}{\cancel{\sqrt{2\pi}}\,(np)^{np+\frac{1}{2}}\,\cancel{e^{-np}}\,\sqrt{2\pi}\,(nq)^{nq+\frac{1}{2}}\,\cancel{e^{-nq}}}$$

$$= \frac{\cancel{n^{n}}\;\cancel{\sqrt{n}}\;\cancel{p^{np}}\;\cancel{q^{nq}}}{\sqrt{2\pi}\,\cancel{(np)^{np}}\,\cancel{(nq)^{nq}}\,\sqrt{\cancel{n}p}\,\sqrt{nq}}$$

$$= \frac{1}{\sqrt{2\pi npq}}$$



rescale horizontal, vertical axis & shift.



Local Limit Theorem

$$P(S_n = i) \xrightarrow{n \to \infty} ?$$

(n & i chosen appropriately)

$$= P\left(\frac{S_n - np}{\sqrt{npq}} = \frac{i - np}{\sqrt{npq}} \right)$$

Define $\quad z = \dfrac{i-np}{\sqrt{npq}}$

Choose $\quad n \to \infty, \; i \to \infty$ such that

z remains fixed.

Note: $\quad i = np + z\sqrt{npq}$

$n \gg \sqrt{n}$ for $n \gg 1$

Hence $\quad i = np + z\sqrt{npq} \longrightarrow \infty$

$$n - i = nq - z\sqrt{npq} \longrightarrow \infty$$

$$P(S_n = i)$$

$$= \binom{n}{i} p^i q^{n-i}$$

$$= \frac{n!}{i!\,(n-i)!} p^i q^{n-i}$$

$$= \frac{\cancel{\sqrt{2\pi}}\, n^{n+\cancel{\frac{1}{2}}}\, \cancel{e^{-n}} \quad p^i q^{n-i}}{\cancel{\sqrt{2\pi}}\; i^{i+\cancel{\frac{1}{2}}}\, \cancel{e^{-i}}\, \sqrt{2\pi}\,(n-i)^{n-i+\cancel{\frac{1}{2}}}\, \cancel{e^{-(n-i)}}}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{i(n-i)}} \left(\frac{np}{i}\right)^i \left(\frac{nq}{n-i}\right)^{n-i}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{\left(np + z\cancel{\sqrt{npq}}\right)\left(nq - z\cancel{\sqrt{npq}}\right)}} \left(\frac{np}{i}\right)^i \left(\frac{nq}{n-i}\right)^{n-i}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}} \underbrace{\left(\frac{np}{i}\right)^i \left(\frac{nq}{n-i}\right)^{n-i}}_{f_n(z)}$$

i.e. $P(S_n = i) \underset{\substack{n \gg 1 \\ z\text{-fixed}}}{\approx} \frac{1}{\sqrt{2\pi npq}} f_n(z)$

where $f_n(z) = \left(\frac{np}{i}\right)^i \left(\frac{nq}{n-i}\right)^{n-i}$

Compute asymptotics of $f_n(z)$:

$$\log f_n(z) = i \log\left(\frac{np}{i}\right) + (n-i)\log\left(\frac{nq}{n-i}\right)$$

$$= \left(np + z\sqrt{npq}\right) \log\left(\frac{np}{np + z\sqrt{npq}}\right)$$

$$+ \left(nq - z\sqrt{npq}\right) \log\left(\frac{nq}{nq - z\sqrt{npq}}\right)$$

$$= \left(np + z\sqrt{npq}\right) \log\left(1 + z\sqrt{\frac{q}{np}}\right)^{-1}$$

$$+ \left(nq - z\sqrt{npq}\right) \log\left(1 - z\sqrt{\frac{p}{nq}}\right)^{-1}$$

$$= -\left(np + z\sqrt{npq}\right)\log\left(1 + z\sqrt{\frac{q}{np}}\right)$$

$$- \left(nq - z\sqrt{npq}\right)\log\left(1 - z\sqrt{\frac{p}{nq}}\right)$$

[Taylor Expansion of $\log(1+x)$:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\cdots$$]

$$= -\left(np + z\sqrt{npq}\right)\left(z\sqrt{\frac{q}{np}} - \frac{1}{2}\frac{z^2 q}{np} + \cdots\cdots\right)$$

$$- \left(nq - z\sqrt{npq}\right)\left(-z\sqrt{\frac{p}{nq}} - \frac{1}{2}\frac{z^2 p}{nq} + \cdots\cdots\right)$$

$$= -z\sqrt{npq} + \frac{1}{2}z^2 q - z^2 q + O\left(\frac{1}{\sqrt{n}}\right)$$

$$z\sqrt{npq} + \frac{1}{2}z^2 p - z^2 p + O\left(\frac{1}{\sqrt{n}}\right)$$

these two cancel each other ! & $p+q=1$

$$= -\frac{z^2}{2} + O\left(\frac{1}{\sqrt{n}}\right) \xrightarrow{n\to\infty} -\frac{z^2}{2}$$

i.e. $\log f_n(z) \xrightarrow{n \to 1/2} -\frac{z^2}{2}$

Hence $f_n(z) \longrightarrow e^{-z^2/2}$

i.e. $$\boxed{P(S_n = i) \underset{n \gg 1}{\approx} \frac{1}{\sqrt{2\pi npq}} e^{-z^2/2}}$$

(Local CLT)

Putting the above together:

$$P\left(a \le \frac{S_n - np}{\sqrt{npq}} \le b\right)$$

$$= P\left(np + a\sqrt{npq} \le S_n \le np + b\sqrt{npq}\right)$$

$$= \sum_{np + a\sqrt{npq} \le i \le nq + b\sqrt{npq}} P(S_n = i)$$

$$= \sum_{np + a\sqrt{npq} \le i \le nq + b\sqrt{npq}} \frac{e^{-z^2/2}}{\sqrt{2\pi npq}}$$

$$= \sum_{a \le \frac{i-np}{\sqrt{npq}} \le b} \frac{e^{-z^2/2}}{\sqrt{2\pi npq}}$$

$$= \sum_{a \le z \le b} \frac{e^{-z^2/2}}{\sqrt{2\pi npq}}$$

[<u>Note</u> $\quad z = \dfrac{i-np}{\sqrt{npq}}$

i is integer valued, i.e. $\Delta i = 1$

Hence $\quad \Delta z = \dfrac{\Delta i}{\sqrt{npq}} = \dfrac{1}{\sqrt{npq}}$]

$$= \sum_{a \le z \le b} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}}$$

$$= \sum_{a \le z \le b} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \Delta z$$

Riemann Sum

$$\downarrow$$

$$\int_a^b \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz$$

integral