

# MA572 Hatcher Notes

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April 5, 2016

# 1 Homology

A summary of Hatcher's homology section from his *Algebraic Topology* book.

## 1.1 Simplicial and Singular Homology

Skip all this nonsense. I need to catch up.

## 1.2 Computations and Applications

### Degree

For a map  $f: S^n \rightarrow S^n$  with  $n > 0$ , the induced map  $f_*: H_n(S^n) \rightarrow H_n(S^n)$  is a homomorphism from an infinite cyclic group to itself and so must be of the form  $f_*(\alpha) = df(\alpha)$  for some integer  $d$  depending only on  $f$ . This integer is called the *degree* of  $f$  and is denoted by  $\deg f$ . Here are some basic properties of the degree

- (1)  $\deg \text{id}_{S^n} = 1$  since  $(\text{id}_{S^n})_* = \text{id}_{H_n(S^n)}$ .
- (2)  $\deg f = 0$  if  $f$  is not injective. For if we choose a point  $x_0 \in S^n \setminus f(S^n)$  then  $f$  can be factored as a composition  $S^n \rightarrow S^n \setminus \{x_0\} \hookrightarrow S^n$  and  $H_n(S^n \setminus \{x_0\}) = 0$  since  $S^n \setminus \{x_0\}$  is contractible.
- (3) If  $f \simeq g$  then  $\deg f = \deg g$  since  $f_* = g_*$ . The converse statement, that if  $\deg f = \deg g$ , is a fundamental theorem of Hopf from around 1925 which we prove in Corollary 4.25.
- (4)  $\deg fg = \deg f \deg g$ , since  $(f \circ g)_* = f_* \circ g_*$ . As a consequence,  $\deg f = \pm 1$  if  $f$  is a homotopy equivalence since  $f \circ g \simeq \text{id}_{S^n}$  implies that  $\deg f \deg g = \deg \text{id}_{S^n} = 1$ .
- (5)  $\deg f = -1$  if  $f$  is a reflection of  $S^n$ , fixing the points in some subsphere  $S^{n-1} \subset S^n$  and interchanging the two complementary hemispheres. For we can give  $S^n$  a  $\Delta$ -complex structure with these two hemispheres as its two  $n$ -simplices  $\Delta_1^n$  and  $\Delta_2^n$ , and the  $n$ -chain  $\Delta_1^n - \Delta_2^n$  represents a generator of  $H_n(S^n)$  as we saw in Example 2.23, so the reflection interchanging  $\Delta_1^n$  and  $\Delta_2^n$  sends this generator to its negative.
- (6) The antipodal map  $a: S^n \rightarrow S^n$ ,  $x \mapsto -x$ , has degree  $(-1)^{n+1}$  since it is the composition of  $n+1$  reflections, each changing the sign of one coordinate in  $\mathbf{R}^{n+1}$ .
- (7) If  $f: S^n \rightarrow S^n$  has no fixed points then  $\deg f = (-1)^{n+1}$ . For if  $f(x) \neq x$  for any  $x \in S^n$ , then the line segment from  $f(x)$  to  $-x$ , defined by  $t \mapsto (1-t)f(x) - tx$  for  $0 \leq t \leq 1$ , does not pass through the origin. Hence if  $f$  has no fixed points, the formula  $f_t(x) := [(1-t)f(x) - tx]/\|(1-t)f(x) - tx\|$  defines a homotopy from  $f$  to the antipodal map. Note that the antipodal map has no fixed points, so the fact that maps without fixed points are homotopic to the antipodal point is sort of a converse statement.