# MA 544: Homework 11

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## Problem 11.1 (Wheeden & Zygmund §7, Ex. 11)

Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let $f$ be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt.$$

(Consider the case when $f$ is the characteristic function of an interval, an open set, etc.)

*Proof.* Recall that, by Theorem 5.21, $f$ is integrable (or in $L^1$) on $[\alpha, \beta]$ if and only if $|f|$ is integrable on $[\alpha, \beta]$. Therefore, it suffices to prove the result for the case $f \geq 0$. We split the proof of the result into a series of claims and then proceed to show the more general result.

**Claim 1.** *Let $g$ be as above and $G$ be an open subset of $[\alpha, \beta]$. Then*

$$|g(G)| = \int_G g'(t)dt.$$

*Proof of claim 1.* Let $G$ be an open subset of $(a, b)$ then, by Theorem 1.10, $G$ can be written as the countable union of disjoint open intervals $\{I_k\}$. By Theorem 5.7, since $g'$ is nonnegative and measurable and $\int_G g'$ is finite (in particular, it is bounded above by $\int_a^b g'$), we have

$$\int_G g'(t)dt = \sum_k \int_{I_k} g'(t)dt. \tag{11.1}$$

But by Theorem 7.27, since $g$ is absolutely continuous on $[\alpha, \beta]$, $g$ is b.v. on $[\alpha, \beta]$ so by Theorem 7.30

$$|g(I_k)| = g(\beta_k) - g(\alpha_k) = V[g; \alpha_k, \beta_k] = \int_{\alpha_k}^{\beta_k} g'(t)dt$$

where $\alpha_k$ is the left-most endpoint of $I_k$ and $\beta_k$ the right-most. By Equation (11.1), on the right-hand side, we have

$$\int_{I_k} g'(t)dt = |g(I_k)|$$

so, by Theorem 3.23, we have

$$\int_G g'(t)dt = \sum_k |g(I_k)| = |g(\textstyle\bigcup_k I_k)| = |g(G)| \tag{11.2}$$

as desired. ♣

**Claim 2.** *Let $g$ be as above and $E$ be a $G_\delta$-subset of $[\alpha, \beta]$. Then*

$$|g(E)| = \int_E g'(t)dt.$$

*Proof of claim 2.* Suppose $E$ is a $G_\delta$-set, then $E$ is the countable intersection of open subsets $\{G_k\}$ of $[\alpha, \beta]$. We may choose $G_k$'s such that $G_k \searrow E$ (for example, taking our original collection of open subsets $\{G_k\}$ and taking the finite intersection $\bigcap_{j=1}^k G_j$). Hence, we have $\chi_{G_k} \searrow \chi_E$ and consequently $\chi_{G_k} g' \searrow \chi_E g'$. Thus, we have

$$\lim_{k\to\infty} \int_E \chi_{G_k} g'(t)dt = \lim_{k\to\infty} |g(G_k)| = |g(E)| \tag{11.3}$$

by Claim 1 and Theorem 3.10. Thus, by the monotone convergence theorem together with Equation (11.3), we have

$$|g(E)| = \lim_{k\to\infty} \int_E \chi_{G_k} g'(t)dt = \int_E \chi_{G_k} g'(t)dt \tag{11.4}$$

as desired. ♣

**Claim 3.** *Let $g$ be as above and $E$ be a $G_\delta$-subset of $[\alpha, \beta]$. Then*

$$|g(E)| = \int_E g'(t)dt.$$

**Claim 4.** *Let $g$ be as above and $E$ be a $G_\delta$-subset of $[\alpha, \beta]$ and $f$ be a simple function. Then*

$$\int_E f(x)dx = \int_E f(g(t))g'(t)dt.$$

The general idea is like this, you prove a sequence of less general claims and make a limiting argument on simple functions. For now, assume Claim 4. Suppose $f \geq 0$ is bounded and measurable. Then, by Theorem 4.12, we may take a sequence $\{f_k\}$ of bounded and measurable simple functions such that $f_k \nearrow f$ a.e. on $[a, b]$. Then, the $f_k \circ g$'s are measurable and $f_k \circ g \nearrow f \circ g$. Thus, $(f_k \circ g)g'$ is measurable so by the monotone convergence theorem

$$\int_\alpha^\beta f_k(g(t))g'(t)dt \longrightarrow \int_\alpha^\beta f(g(t))g'(t)dt$$

as $k \to \infty$. But, assuming the result holds for simple functions, we have

$$\int_\alpha^\beta f_k(g(t))g'(t)dt = \int_a^b f_k(x)dx$$

so taking the limit as $k \to \infty$, by the monotone convergence theorem, we have

$$\int_\alpha^\beta f(g(t))g'(t)dt = \int_a^b f(x)dx$$

as desired. ■

## Problem 11.2 (Wheeden & Zygmund §7, Ex. 15)

Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\varphi(x) = \int_a^x f(t)dt + \varphi(a)$ in $(a, b)$ and $f$ is monotone increasing, then $\varphi$ is convex in $(a, b)$. (Use Exercise 14.)

*Proof.* We will assume the result in Exercise 14. First we check that $\varphi$ is continuous. Since $f$ is monotone increasing, $f$ is b.v. on $[a, b]$ so $f$ is bounded a.e. on $(a, b)$ by a previous exercise. Thus, $f \in L(a, b)$ so by Theorem 7.1, $\varphi$ is absolutely continuous and hence, continuous.

Now, let $x_1, x_2 \in (a, b)$ and, without loss of generality, assume $x_1 < x_2$. Then, we have

$$\begin{aligned}\varphi\left(\frac{x_1+x_2}{2}\right) &= \int_a^{(x_1+x_2)/2} f(t)dt + \varphi(a)\\ &= \int_a^{x_1} f(t)dt + \int_{x_1}^{(x_1+x_2)/2} f(t)dt + \varphi(a)\end{aligned}$$

since $f$ is monotone increasing, we have $\int_{x_1}^{(x_1+x_2)/2} f(t)dt \leq \int_{(x_1+x_2)/2}^{x_2} f(t)dt$ so

$$\begin{aligned}&= \int_a^{x_1} f(t)dt + \frac{1}{2}\left[2\int_{x_1}^{(x_1+x_2)/2} f(t)dt\right] + \varphi(a)\\ &\leq \int_a^{x_1} f(t)dt + \frac{1}{2}\left[\int_{x_1}^{(x_1+x_2)/2} f(t)dt + \int_{(x_1+x_2)/2}^{x_2} f(t)dt\right] + \varphi(a)\\ &= \frac{1}{2}\left[\int_a^{x_1} f(t)dt + \varphi(a)\right] + \frac{1}{2}\left[\int_a^{x_1} f(t)dt + \int_{x_1}^{(x_1+x_2)/2} f(t)dt + \int_{(x_1+x_2)/2}^{x_2} f(t)dt + \varphi(a)\right]\\ &= \frac{1}{2}\left[\int_a^{x_1} f(t)dt + \varphi(a)\right] + \frac{1}{2}\left[\int_a^{x_2} f(t)dt + \varphi(a)\right]\\ &= \frac{\varphi(x_1)+\varphi(x_2)}{2}.\end{aligned}$$

Thus, by Exercise 14, $\varphi$ is convex. ■

## Problem 11.3 (Wheeden & Zygmund §5, Ex. 8)

Prove (5.49).

*Proof.* Recall the content of equation 5.49: For $f$ measurable, we have

$$\omega(\alpha) \leq \frac{1}{\alpha^p} \int\limits_{\{ f>\alpha \}} f^p, \quad \alpha > 0. \tag{11.5}$$

Consider the $L^p$-norm of $f$ raised to the $p$-th power

$$\|f\|_p{}^p = \int |f(x)|^p dx$$

since $f$ is measurable, $f$ is measurable so $\{ f > \alpha \}$ is measurable hence, by the monotonicity of the Lebesgue integral, we have

$$\begin{aligned} &\geq \int_{\{ f>\alpha \}} f^p dx \\ &\geq \int_{\{ f>\alpha \}} \alpha^p dx \\ &= \alpha^p |\{ f > \alpha \}| \\ &= \alpha^p \omega(\alpha). \end{aligned}$$

Thus, we have

$$\omega(\alpha) \leq \frac{1}{\alpha^p} \int_{\{ f>\alpha \}} f^p$$

as desired. ∎

## Problem 11.4 (Wheeden & Zygmund §5, Ex. 11)

For which $p$ does $1/x \in L^p(0,1)$? $L^p(1,\infty)$? $L^p(0,\infty)$?

*Proof.* For the first case, we know that $1/x \in L^p(0,1)$ if and only if

$$\int_0^1 \frac{dx}{x^p} < \infty. \tag{11.6}$$

This happens if and only if $p < 1$.

In the second, case $1/x \in L^p(1,\infty)$ if and only if

$$\int_1^\infty \frac{dx}{x^p} < \infty \tag{11.7}$$

if and only if $p > 1$

Lastly, $1/x \in L^p(0,\infty)$ if and only if

$$\int_0^\infty \frac{dx}{x^p} = \int_0^1 \frac{dx}{x^p} + \int_1^\infty \frac{dx}{x^p} < \infty$$

if and only if $1/x$ satisfies both (11.6) and (11.6). This is impossible. Thus, $1/x \notin L^p(0,\infty)$ for any $p \geq 0$ ■

## PROBLEM 11.5 (WHEEDEN & ZYGMUND §5, EX. 12)

Give an example of a bounded continuous $f$ on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0,\infty)$ for any $p > 0$.

*Proof.* An example, given in class, is the following: Set

$$f(x) \coloneqq \begin{cases} 1 & x \leq e \\ 1/\ln x & x \geq e. \end{cases} \tag{11.8}$$

This function is bounded (above by 1), continuous ($\lim_{x+\to e} f(x) = 1 = \lim_{x-\to e} f(x)$) and $\lim_{x\to\infty} f(x) = 0$ Now, because $\ln(x) \prec x^{1/p}$ (i.e., ln grows slower than any monomial $x^k$), for sufficiently large values of $x$, say $x \geq K$, depending on $p$, we have

$$\int_K^\infty \frac{dx}{(\ln x)^p} \geq \int_K^\infty \frac{dx}{(x^{1/p})^p} \int_K^\infty \frac{dx}{x} = \infty.$$

so $f$ cannot be in $L^p(0,\infty)$ for any $p > 0$. ■

## Problem 11.6 (Wheeden & Zygmund §5, Ex. 17)

If $f \geq 0$ and $\omega(\alpha) \leq c(1+\alpha)^p$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.

*Proof.* If we assume the results of Exercise 16, it suffices to show that

$$\frac{1}{r}\int f^r = \int_0^\infty \alpha^{r-1}\omega(\alpha)d\alpha \leq c\int_0^\infty \frac{\alpha^{r-1}}{(1+\alpha)^p}d\alpha < \infty \tag{11.9}$$

for all $0 < r < p$. Make a change of variables $\beta \coloneqq \alpha + 1$, $d\beta = d\alpha$ and (on the right-hand side of (11.9)) and consider the improper integral

$$c\int_1^x \left[\sum_{k=0}^r a_k \beta^k\right] = c\sum_{k=0}^r \left[\frac{a_k}{k+1}\beta^{k+1}\right]_0^x$$

By Problem 11.4, we know that the sum on the left converges as $x \to \infty$ since, we have $k \geq p - (r-1) > 1$. Hence, (11.9) holds and we have $f \in L^r$. ■

## Problem 11.7 (Wheeden & Zygmund §8, Thm. 8.3)

If $f, g \in L^p(E)$, $p > 0$, then $f + g \in L^p(E)$ and $cf \in L^p(E)$ for any constant $c$.

*Proof.* Suppose $f, gi \in L^p(E)$ and $c$ is any constant, then, by Minkowski's inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p < \infty$$

and

$$\|cf\|_p = \left(\int_E |cf|^p\right)^{1/p} = \left(\int_E |c|^p|f|^p\right)^{1/p} = |c|\left(\int_E |c|^p|f|^p\right)^{1/p} < \infty.$$

Thus, $f + g, cf \in L^p(E)$ ■