MA 519: Homework 9

Max Jeter, Carlos Salinas

October 27, 2016

Problem 9.1 (Handout 13, # 7)

Let X have a *double exponential* density $f(x) = \frac{1}{2\sigma}\mathrm{e}^{-\frac{|x|}{\sigma}}$, $-\infty < x < \infty$, $\sigma > 0$.

(a) Show that all moments exist for this distribution.
(b) However, show that the MGF exists only for restricted values. Identify them and find a formula.

Solution. For part (a), we show that the moments $m_n := E(X^n)$ for all $n \in \mathbb{N}$. By direct calculation, we have

$$\begin{aligned} m_n &= \int_{-\infty}^{\infty} x^n f(x)\, dx \\ &= \int_{-\infty}^{\infty} \frac{x^n}{2\sigma}\mathrm{e}^{-\frac{|x|}{\sigma}}\, dx \\ &= \underbrace{\int_{-\infty}^{0} \frac{x^n}{2\sigma}\mathrm{e}^{\frac{x}{\sigma}}\, dx}_{L} + \int_{0}^{\infty} \frac{x^n}{2\sigma}\mathrm{e}^{-\frac{x}{\sigma}}\, dx, \end{aligned}$$

making the substitution $x \mapsto -y$ to L and relabeling y to x again, the above becomes

$$\begin{aligned} &= \int_0^{\infty} \frac{x^n + (-1)x^n}{2\sigma}\mathrm{e}^{-\frac{x}{\sigma}}\, dx \\ &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ I := \int_0^{\infty} \frac{x^n}{\sigma}\mathrm{e}^{-\frac{x}{\sigma}}\, dx & \text{if } n = 2k \text{ is even.} \end{cases} \end{aligned}$$

To evaluate I we apply integration by parts repeatedly to arrive at

$$\begin{aligned} I &= \int_{-\infty}^{0} \frac{x^n}{\sigma}\mathrm{e}^{-\frac{x}{\sigma}} \\ &= (-0+0) + \int_0^{\infty} n\sigma x^{n-1}\mathrm{e}^{-\frac{x}{\sigma}}\, dx \\ &= (-0+0) + (-0+0) + \int_0^{\infty} n(n-1)\sigma^2 x^{n-1}\mathrm{e}^{-\frac{x}{\sigma}}\, dx \qquad \vdots \\ &= (-0-0) + \cdots + (-0+0) + (-0+n!\sigma^n) \\ &= n!\sigma^n. \end{aligned}$$

Therefore, m_n exist and are finite for all $n \in \mathbb{N}$.

Fr part (b), the MGF associated to f is given by the series

$$m(t) = \sum_{n=0}^{\infty} \frac{t^n m_n}{n!} = \sum_{k=1}^{\infty} t^{2k}\sigma^{2k}. \tag{9.1}$$

This series is geometric and, as such, converges for all $-\frac{1}{\sigma} < t < \frac{1}{\sigma}$, in which case (9.1) becomes

$$m(t) = \frac{1}{1 - t^2\sigma^2}. \qquad \blacksquare$$

Problem 9.2 (Handout 13, # 10)

Suppose X has Cauchy distribution as in # 6. Which of the following functions have finite expectation

$$X; \quad -X; \quad |X|; \quad \frac{1}{X}; \quad \sin X; \quad \ln|X|; \quad e^X; \quad e^{-|X|}?$$

Solution. Suppose $X \sim \text{Cauchy}(0,1)$. Then the PDF of X is given by the expression

$$f(x) = \frac{1}{\pi(x^2+1)}.$$

Now we proceed to find the expectations of (i) X, (ii) $-X$, (iii) $\frac{1}{X}$, (iv) $\sin X$, (v) $\ln|X|$, (vi) e^X, (vii) $e^{-|X|}$.

For (i), the expectation does not even exist. We repeat the argument given in class: Consider

$$E(X) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx = \lim_{x_1,x_2} \frac{1}{\pi}\int_{-x_1}^{x_2} \frac{x}{x^2+1}\,dx.$$

Then, making the substitution $u = x^2+1$, $du = 2x\,dx$, the integral above evaluates to

$$E(X) = \lim_{x_1,x_2\to\infty} \frac{1}{2\pi}\ln\left(\frac{x_2^2+1}{x_1^2+1}\right).$$

However, the limit of this expression is undefined! Fix positive real number α and let $x_2 = \alpha x_1$. Then

$$E(X) = \lim_{x_1\to\infty} \frac{1}{2\pi}\ln\left(\frac{\alpha^2 x_1^2+1}{x_1^2+1}\right) = \frac{1}{2\pi}\ln\alpha^2.$$

This value is distinct for each α. Therefore, the limit is not unique and so is undefined.

For (ii), the CDF of $-X$ is given by

$$\begin{aligned} F_{-X}(x) &= P(-X \le x) \\ &= P(X \ge -x) \\ &= 1 - P(X < -x) \\ &= 1 - \int_{-\infty}^{-x} \frac{1}{\pi(y^2+1)}\,dy \\ &= \frac{1}{2} - \frac{1}{\pi}\tan^{-1}(-x). \end{aligned}$$

Thus, the PDF of $-X$ is

$$f_{-X}(x) = \frac{dF_{-X}(x)}{dx} = \frac{1}{\pi(x^2+1)}.$$

Thus, $-X \sim \text{Cauchy}(0,1)$ and as we have previously shown, $E(X) = E(-X)$ is undefined.

For (iii), we have

$$X = \begin{cases} X & \text{if } X > 0, \\ -X & \text{if } X \le 0. \end{cases}$$

Thus, the CDF of $|X|$ is

$$\begin{aligned}
F_{|X|}(x) &= P(|X| \leq x) \\
&= P(-x \leq X \leq x) \\
&= \frac{1}{\pi}\int_{-x}^{x} \frac{1}{y^2+1}\,dy \\
&= \frac{1}{\pi}\big(\tan^{-1}(x) - \tan^{-1}(-x)\big) \\
&= \frac{2}{\pi}\tan^{-1}(x),
\end{aligned}$$

and hence its PDF is

$$f_{|X|}(x) = \frac{dF_{|X|}(x)}{dx} = \begin{cases} \frac{2}{\pi(x^2+1)} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned}
E(X) &= \int_0^\infty \frac{2x}{x^2+1}\,dx \\
&= \lim_{x\to\infty} \ln(x^2+1) \\
&= \infty.
\end{aligned}$$

For (iv), as discussed in class $\frac{1}{X} \sim \text{Cauchy}(0,1)$. Let us show this. First, we find the CDF of X:

$$\begin{aligned}
F_{\frac{1}{X}}(x) &= P\left(\frac{1}{X} \leq x\right) \\
&= P\left(X \geq \frac{1}{x}\right) \\
&= 1 - P\left(X < \frac{1}{x}\right) \\
&= 1 - \frac{1}{\pi}\int_{-\infty}^{\frac{1}{x}} \frac{1}{y^2+1}\,dy \\
&= 1 - \tan^{-1}\left(\frac{1}{x}\right) - \frac{1}{2} \\
&= \frac{1}{2} - \tan^{-1}\left(\frac{1}{x}\right).
\end{aligned}$$

Thus, the PDF of $\frac{1}{X}$ is

$$\begin{aligned}
f_{\frac{1}{X}}(x) &= \frac{dF_{\frac{1}{X}}(x)}{dx} \\
&= -\left(-\frac{1}{x^2}\right)\left(\frac{1}{\left(\frac{1}{x}\right)^2+1}\right) \\
&= \frac{1}{x^2+1}.
\end{aligned}$$

Thus, $\frac{1}{X} \sim \text{Cauchy}(0,1)$ so its expectation is not defined.

For (v), the CDF of $\sin X$ is given by

$$F_{\sin X}(x) = P(\sin X \le x),$$

for $-1 \le x \le 1$, we have

$$\begin{aligned}
&= \sum_{-\infty<n<\infty} P\big((2n+1)\pi - \sin^{-1} x \le X \le (2n+2)\pi + \sin^{-1} x\big) \\
&= \frac{1}{\pi} \sum_{-\infty<n<\infty} \int_{(2n+1)\pi-\sin^{-1} x}^{(2n+2)\pi+\sin^{-1} x} \frac{1}{y^2+1}\, dy \\
&= \frac{1}{\pi} \sum_{-\infty<n<\infty} \tan^{-1}\big((2n+2)\pi + \sin^{-1} x\big) + \tan^{-1}\big((2n+1)\pi - \sin^{-1} x\big).
\end{aligned}$$

Thus, the PDF of $\sin X$ is

For (vi), the CDF of $\ln |X|$ is given by

$$\begin{aligned}
F_{\ln|X|}(x) &= P(\ln |X| \le x) \\
&= P(|X| \le \mathrm{e}^x) \\
&= P(-\mathrm{e}^x \le X \le \mathrm{e}^x) \\
&= \frac{1}{\pi} \int_{-\mathrm{e}^x}^{\mathrm{e}^x} \frac{1}{y^2+1}\, dy \\
&= \frac{2}{\pi} \tan^{-1}(\mathrm{e}^x).
\end{aligned}$$

Thus, the PDF of $\ln |X|$ is

$$f_{\ln|X|}(x) = \frac{dF_{\ln|X|}(x)}{dx} = \frac{2\mathrm{e}^x}{\pi(\mathrm{e}^{2x}+1)}.$$

Thus, the expectation of $\ln |X|$ is

$$\begin{aligned}
E(\ln |X|) &= \int_{-\infty}^{\infty} \frac{2\mathrm{e}^x}{\pi(\mathrm{e}^{2x}+1)}\, dx \\
&=
\end{aligned}$$

■

Problem 9.3 (Handout 13, # 16)

Give an example of each of the following phenomena:

(a) A continuous random variable taking values in $[0, 1]$ with equal mean and median.
(b) A continuous random variable taking values in $[0, 1]$ with mean equal to twice the median.
(c) A continuous random variable for which the mean does not exist.
(d) A continuous random variable for which the mean exists, but the variance does not exist.
(e) A continuous random variable with a PDF that is not differentiable at zero.
(f) a positive continuous random variable for which the mode is zero, but the mean does not exist.
(g) A continuous random variable for which all moments exist.
(h) A continuous random variable with median equal to zero, and 25^{th} and 75^{th} percentiles equal to 1.
(i) A continuous random variable X with mean equal to median equal to mode equal to zero, and $E(\sin X) = 0$.

Solution. First, note that $[0, 1]$ is a probability space under the standard Lebesgue measure on $\mathbb{R}$. Therefore, it makes sense to consider $X\colon [0, 1] \to \mathbb{R}$ random variables.

For part (a), consider the random variable $X\colon [0, 1] \to \mathbb{R}$ defined by $x \mapsto x$ with $X \sim \text{Uniform}[0, 1]$. Then the mean is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

and the median is

$$m = \inf\{\, x : F(x) = x \geq 0.5 \,\} = \frac{1}{2}.$$

For part (b), consider again the random variable $X(x) = x$ for $x \in [0, 1]$, but this time let

$$f(x) = \begin{cases} & , \\ & . \end{cases}$$

be the PDF of X. Then the mean is ■

Problem 9.4 (Handout 13, # 17)

An exponential random variable with mean 4 is known to be larger than 6. What is the probability that it is larger than 8?

Solution. ■

Problem 9.5 (Handout 13, # 18)

(Sum of Gammas). Suppose X, Y are independent random variables, and $X \sim \Gamma(\alpha, \lambda)$, $Y \sim \Gamma(\beta, \lambda)$. Find the distribution of $X + Y$ by using moment-generating functions.

Solution. ■

Problem 9.6 (Handout 13, # 19)

(Product of Chi Squares). Suppose $X_1, X_2, \dots, X_n$ are independent chi square variables, with $X_i \sim \chi^2_{m_i}$. Find the mean and variance of $\prod_{i=1}^n X_i$.

Solution. ∎

Problem 9.7 (Handout 13, # 20)

Let $Z \sim \text{Normal}(0,1)$. Find

$$P\left(0.5 < \left|Z - \frac{1}{2}\right| < 1.5\right); \quad P\left(\frac{\mathrm{e}^Z}{1+\mathrm{e}^Z} > \frac{3}{4}\right); \quad P(\Phi(Z) < 0.5).$$

Solution. ■

Problem 9.8 (Handout 13, # 21)

Let $Z \sim \text{Normal}(0, 1)$. Find the density of $\frac{1}{Z}$. Is the density bounded?

Solution. $\blacksquare$

Problem 9.9 (Handout 13, # 22)

The 25^{th} and the 75^{th} percentile of a normally distributed random variable are -1 and 1. What is the probability that the random variable is between -2 and 2?

Solution. ∎