MA 519: Homework 9

Max Jeter,   Carlos Salinas

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PROBLEM 9.1 (HANDOUT 13, # 7)

Let X have a *double exponential* density $f(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}$, $-\infty < x < \infty$, $\sigma > 0$.

(a) Show that all moments exist for this distribution.
(b) However, show that the MGF exists only for restricted values. Identify them and find a formula.

SOLUTION. For part (a), we show that $E(X^n) < \infty$ for all $n \in \mathbb{N}$. By direct calculation, we have

$$
\begin{aligned}
E(X^n) &= \int_{-\infty}^{\infty} x^n f(x)\, dx \\
&= \int_{-\infty}^{\infty} \frac{x^n}{2\sigma} e^{-\frac{|x|}{\sigma}}\, dx \\
&= \underbrace{\int_{-\infty}^{0} \frac{x^n}{2\sigma} e^{\frac{x}{\sigma}}\, dx}_{L} + \int_{0}^{\infty} \frac{x^n}{2\sigma} e^{-\frac{x}{\sigma}}\, dx,
\end{aligned}
$$

making the substitution $x \mapsto -y$ to L and relabeling y to x again, we see that

$$
\begin{aligned}
&= \int_{-\infty}^{0} \frac{x^n + (-1)x^n}{2\sigma} e^{-\frac{x}{\sigma}}\, dx \\
&= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ I := \int_{0}^{\infty} \frac{x^n}{\sigma} e^{-\frac{x}{\sigma}}\, dx & \text{if } n = 2k \text{ is even.} \end{cases}
\end{aligned}
$$

To evaluate I we use integration by parts recursively to arrive at

$$
\begin{aligned}
I &= \int_{-\infty}^{0} \frac{x^n}{\sigma} e^{-\frac{x}{\sigma}} \\
&= (-0 + 0) + \int_{0}^{\infty} n\sigma x^{n-1} e^{-\frac{x}{\sigma}}\, dx \\
&= (-0 + 0) + (-0 + 0) + \int_{0}^{\infty} n(n-1)\sigma^2 x^{n-1} e^{-\frac{x}{\sigma}}\, dx \\
&\;\;\vdots \\
&= (-0 - 0) + \cdots + (-0 + 0) + (-0 + n!\sigma^n) \\
&= n!\sigma^n.
\end{aligned}
$$

Therefore, $E(X^n) < \infty$ for all $n \in \mathbb{N}$, i.e., all moments of this distribution exist.

For part (b), the MGF associated to f is

$$
m(t) = \sum_{n=0}^{\infty} \frac{t^n E(X^n)}{n!} = \sum_{n=0}^{\infty} t^n \sigma^n. \tag{9.1}
$$

This is a geometric series and so converges for all $-\frac{1}{\sigma} < t < \frac{1}{\sigma}$, in which case (9.1) becomes

$$m(t) = \frac{1}{1 - t\sigma}.$$ ∎

PROBLEM 9.2 (HANDOUT 13, # 10)

Suppose X has Cauchy distribution as in # 6. Which of the following functions have finite expectation

$$X; \quad -X; \quad |x|; \quad \frac{1}{X}; \quad \sin X; \quad \ln|X|; \quad \mathrm{e}^X; \quad \mathrm{e}^{-|X|}?$$

SOLUTION. Recall from Handout 13, # 6, that X is the horizontal distance along the wall when a flashlight is aimed at random with angle $\Theta \sim \mathrm{Uniform}[-\pi, \pi]$. First, let us find the PDF of X. Let ℓ be the vertical distance from the light source to the wall. Then, by elementary trigonometry, we have the following relation between X and Θ:

$$X = \ell \tan \Theta. \tag{9.2}$$

Using (9.2), we can figure out the CDF of X and consequently its PDF:

$$\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(\ell \tan \Theta \leq x) \\
&= P\big(\Theta \leq \tan^{-1}\big(\tfrac{x}{\ell}\big)\big) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\tan^{-1}\left(\frac{x}{\ell}\right)} 1 \, dy \\
&= \frac{1}{2\pi} \Big[\tan^{-1}\Big(\frac{x}{\ell}\Big) + \pi\Big] \\
&= \frac{1}{2\pi} \tan^{-1}\Big(\frac{x}{\ell}\Big) + \frac{1}{2}.
\end{aligned}$$

∎

PROBLEM 9.3 (HANDOUT 13, # 16)

Give an example of each of the following phenomena:

(a) A continuous random variable taking values in $[0, 1]$ with equal mean and median.
(b) A continuous random variable taking values in $[0, 1]$ with mean equal to twice the median.
(c) A continuous random variable for which the mean does not exist.
(d) A continuous random variable for which the mean exists, but the variance does not exist.
(e) A continuous random variable with a PDF that is not differentiable at zero.
(f) a positive continuous random variable for which the mode is zero, but the mean does not exist.
(g) A continuous random variable for which all moments exist.
(h) A continuous random variable with median equal to zero, and 25th and 75th percentiles equal to 1.
(i) A continuous random variable X with mean equal to median equal to mode equal to zero, and $E(\sin X) = 0$.

SOLUTION. First, note that $[0, 1]$ is a probability space under the standard Lebesgue measure on $\mathbb{R}$. Therefore, it makes sense to consider $X \colon [0, 1] \to \mathbb{R}$ random variables.

For part (a), consider the random variable $X \colon [0, 1] \to \mathbb{R}$ defined by $x \mapsto x$ with $X \sim \text{Uniform}[0, 1]$. Then the mean is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

and the median is

$$m = \inf\{\, x : F(x) = x \geq 0.5 \,\} = \frac{1}{2}.$$

For part (b), consider again the random variable $X(x) = x$ for $x \in [0, 1]$, but this time let

$$f(x) = \left\{ \begin{array}{l} \text{,} \\ \text{.} \end{array} \right.$$

be the PDF of X. Then the mean is   ∎

PROBLEM 9.4 (HANDOUT 13, # 17)

An exponential random variable with mean 4 is known to be larger than 6. What is the probability that it is larger than 8?

SOLUTION.   ■

Problem 9.5 (Handout 13, # 18)

(Sum of Gammas). Suppose X, Y are independent random variables, and $X \sim \Gamma(\alpha, \lambda)$, $Y \sim \Gamma(\beta, \lambda)$. Find the distribution of $X + Y$ by using moment-generating functions.

Solution.   ∎

Problem 9.6 (Handout 13, # 19)

(Product of Chi Squares). Suppose $X_1, X_2, \ldots, X_n$ are independent chi square variables, with $X_i \sim \chi^2_{m_i}$. Find the mean and variance of $\prod_{i=1}^{n} X_i$.

Solution.   ∎

PROBLEM 9.7 (HANDOUT 13, # 20)

Let $Z \sim \text{Normal}(0,1)$. Find

$$P\big(0.5 < |Z - \tfrac{1}{2}| < 1.5\big); \quad P\Big(\tfrac{e^Z}{1 + e^Z} > \tfrac{3}{4}\Big); \quad P(\Phi(Z) < 0.5).$$

SOLUTION.   ∎

PROBLEM 9.8 (HANDOUT 13, # 21)

Let $Z \sim \text{Normal}(0,1)$. Find the density of $\frac{1}{Z}$. Is the density bounded?

SOLUTION.   ∎

Problem 9.9 (Handout 13, # 22)

The 25th and the 75th percentile of a normally distributed random variable are -1 and 1. What is the probability that the random variable is between -2 and 2?

Solution.   ■