MA 523: Homework, Midterms and Practice Problems Solutions

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Contents

1 Homework Solutions

These are my (corrected) solutions to Petrosyan's Math 523 homework for the fall semester of 2016.

1.1 Homework 1

PROBLEM 1.1.1 (Taylor's formula). Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O\big(|x|^{m+1}\big)$$

as $x \to \mathbf{0}$ for each $m = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. Taking the hint, let us consider the function in one variable $g(t) := f(tx)$ for $x \in \mathbb{R}^n$ fixed. We show by induction on m that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha. \tag{1.1.1}$$

Once we have shown (1.1.1) holds, evaluating g at $t = 1$ gives us the desired equality; i.e,

$$f(x) = g(1)$$

which, by Taylor's formula in one variable, is

$$= \sum_{j=0}^{m} \frac{g^{(j)}(0)}{j!} 1^j + O\big(|x|^{m+1}\big)$$

applying (1.1.1) here gives us

$$\begin{aligned}
&= \sum_{k=0}^{m} \frac{1}{k!} \left[\sum\nolimits_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha f(tx) x^\alpha\right] + O\big(|x|^{m+1}\big) \\
&= \sum_{k=0}^{m} \left[\sum\nolimits_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha\right] + O\big(|x|^{m+1}\big) \\
&= \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O\big(|x|^{m+1}\big)
\end{aligned}$$

as desired.

Let us now show that (1.1.1) holds. To prove this we consider the algebra on the differential operator d/dt. By the chain rule, we have

$$\frac{d}{dt}(\,\cdot\,) = \sum_{k=1}^{m} x_k \frac{\partial}{\partial x_k}(\,\cdot\,).$$

Since f is smooth by Schwartz's theorem the differential operators $\partial/\partial x_k$ and $\partial/\partial x_l$ commute for all $1 \leq k, l \leq n$. Therefore, by the multinomial theorem,

$$\frac{d^m}{dt^m}(\,\cdot\,) = \left(\sum\nolimits_{k=1}^{m} x_k \frac{\partial}{\partial x_k}(\,\cdot\,)\right)^k = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha D^\alpha(\,\cdot\,). \qquad \blacksquare$$

Problem 1.1.2. Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition $u = g$ on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. Write

$$F(p, z, x, t) := (b, 1) \cdot p - f = 0.$$

Then the characteristic equations to the problem $(*)$ with the initial value $u(\,\cdot\,, 0) = g(\,\cdot\,)$ are given by

$$\begin{cases} \dot{p} = -D_{x,t}F - D_z F p = 0, \\ \dot{z} = D_p F \cdot p = (b, 1) \cdot p = f, \\ (\dot{x}, \dot{t}) = D_p F = (b, 1). \end{cases}$$

Now let us solve the characteristic equations above subject to the initial values $(x(0), t(0)) = (x^0, 0) \in \mathbb{R}^n \times (0, \infty)$; these are

$$\begin{cases} x(s) = x^0 + bs, \quad t(s) = s, \\ z(s) = z(0) + \displaystyle\int_0^s f(x(\tau), t(\tau))\, d\tau \\ \qquad = g(x^0) + \displaystyle\int_0^s f(x^0 + b\tau, \tau)\, d\tau. \end{cases}$$

Solving back, we have $t = s$, $x^0 = x - bs = x - bt$, and therefore

$$u(x, t) = z(s) = g(x - bt, t) + \int_0^t f(x - b\tau, \tau)\, d\tau$$

solves the transport equation $(*)$ with initial value $u(\,\cdot\,, 0) = g(\,\cdot\,)$.

This verifies formula [5] from [E, §2.1.2]. $\blacksquare$

Problem 1.1.3. Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_1) = \frac{x_1}{2}$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. For part (a): Write $F := (x_1^2, x_2^2)\cdot p - z^2 = 0$. We have the characteristic equations

$$\begin{cases} \dot{p} = -D_x F - D_z F p = 2\big((x_1 - z)p_1, (x_2 - z)p_2\big), \\ \dot{z} = D_p \cdot p = z^2, \\ \dot{x} = (x_1^2, x_2^2). \end{cases}$$

We can solve the characteristic equations with respect to the initial conditions $x(0) = (x^0, 2x^0)$, $z(0) = 1$ on the line $x_1 = 2x_2$; these are

$$\begin{cases} x_1(s) = \dfrac{x^0}{1 + x^0 s}, \quad x_2(s) = \dfrac{2x^0}{1 + 2x^0 s}, \\ z(s) = \dfrac{1}{1 - s}. \end{cases}$$

Now we solve these in terms of the coordinates (x_1, x_2). Assuming $x^0 \neq 0$, we have

$$s = \frac{1}{x^0} - \frac{1}{x_1} \quad \text{and} \quad s = \frac{1}{2x^0} - \frac{1}{x_2}.$$

Therefore,

$$\begin{aligned} s &= 2\left(\frac{1}{2x^0} - \frac{1}{x_2}\right) - \left(\frac{1}{x^0} - \frac{1}{x_1}\right) \\ &= \frac{1}{x_1} - \frac{2}{x_2}. \end{aligned}$$

Thus,

$$u(x_1, x_2) = \frac{1}{1 - \left(\frac{1}{x_1} - \frac{2}{x_2}\right)} = \frac{x_1 x_2}{x_1 x_2 - x_2 - 2x_1}$$

solves the PDE F for (x_1, x_2) on the line $x_1 = 2x_2$ away from the origin.

For part (b): Write $F = (z, 1)\cdot p - 1 = 0$. Then we have the characteristic equations

$$\begin{cases} \dot{p} = -D_x F - D_z p = -(p_1, 0) \\ \dot{z} = D_p \cdot p = 1 \\ \dot{x} = D_p F = (z, 1) \end{cases}$$

Next we solve the characteristic equations subject to the initial conditions $x(0) = (x^0, x^0)$, $z(0) = \frac{x^0}{2}$ on the line $x_1 = x_2$; these are

$$\begin{cases} z(s) = \frac{1}{2}x^0 + s, \\ x_1(s) = x^0 + \frac{1}{2}\big(x^0 s + s^2\big), \quad x_2(s) = x^0 + s. \end{cases}$$

Then, solving in terms of the coordinates (x_1, x_2), we have

$$x^0 = 2(x_2 - z) \quad \text{and} \quad s = 2z - x_2.$$

Therefore,

$$\begin{aligned} x_1 &= 2(x_2 - z) + (x_2 - z)(2z - x_2) + \tfrac{1}{2}(2z - x_2)^2 \\ &= -\tfrac{1}{2}x_2(x_2 - 4) + (x_2 - 2)z. \end{aligned}$$

Hence,

$$u(x_1, x_2) = \frac{2x_1 + x_2^2 - 4x_2}{2(x_2 - 2)}$$

solves the PDE F subject to the condition $u(x_1, x_1) = \frac{x_1}{2}$ provided $x_2 \neq 2$.

For part (c): Write $F := (x_1, 2x_2, 1) \cdot p - 3z = 0$. Then the characteristic equations are

$$\begin{cases} \dot{p} = -D_x F - D_z p = (2p_1, p_2, 3p_3) \\ \dot{z} = D_p F \cdot p = 3z \\ \dot{x} = D_p F = (x_1, 2x_2, 1) \end{cases}$$

Next we sole the characteristic equations subject to the initial conditions $x(0) = (x_1^0, x_2^0, 0)$, $z(s) = g(x_1^0, x_2^0)$; these are

$$\begin{cases} x_1(s) = x_1^0 \mathrm{e}^{s}, \quad x_2(s) = x_2^0 \mathrm{e}^{2s}, \quad x_3(s) = s, \\ \ z(s) = g(x_1^0, x_2^0)\mathrm{e}^{3s}. \end{cases}$$

Then, solving for u in terms of the coordinates (x_1, x_2, x_3), we have

$$s = x_3, \quad x_1^0 = x_1 \mathrm{e}^{-s}, \quad \text{and} \quad x_2^0 = x_2 \mathrm{e}^{-2s}.$$

Thus,

$$u(x_1, x_2, x_3) = g(x_1 \mathrm{e}^{-x_3}, x_2 \mathrm{e}^{-2x_3})\mathrm{e}^{3x_3}$$

solves the PDE F subject to the condition $u(x_1, x_2, 0) = g(x_1, x_2)$. ■

Problem 1.1.4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \tfrac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with $u(x_1, 0) = \frac{1 - x_1^2}{2}$.

Solution. The equation is nonlinear and therefore, we do not expect the method of characteristics to yield a unique solution to the PDE

$$F := x_1 p_1 + x_2 p_2 + \tfrac{1}{2}(p_1^2 + p_2^2) - z.$$

Let us find the characteristic equations for F; these are

$$\begin{cases} \dot{p} = -D_x F - D_z F p = -(p_1, p_2) - (-1)(p_1, p_2) = 0, \\ \dot{z} = D_p F \cdot p = (x_1 + p_1, x_2 + p_2) \cdot (p_1, p_2) = (x_1 + p_1)p_1 + (x_2 + p_2)p_2, \\ \dot{x} = D_p F = (x_1 + p_1, x_2 + p_2), \end{cases}$$

Next we solve the characteristic equations subject to the initial values $x(0) = (x^0, 0)$, $z(0) = \frac{1}{2}(1 - (x^0)^2)$ and, after revisiting the equation F, $p_1(0) = -x^0$ and

$$p_2(0)^2 = 2\Big(-(x^0)^2 + \tfrac{1}{2}(x^0)^2 + \tfrac{1}{2}(1 - (x^0)^2)\Big) = 1$$

so $p_2(0) = \pm 1$. Therefore, the solution to the characteristic equations subject to these initial values is

$$\begin{cases} p_1(s) = -x^0, \quad p_2(s) = \pm 1, \\ x_1(s) = x^0, \quad x_2(s) = \pm 1(\mathrm{e}^s - 1), \\ z(s) = \frac{1}{2}\big(1 - (x^0)^2\big) + (\mathrm{e}^s - 1). \end{cases}$$

Thus, solving for s and x^0 in terms of the coordinates (x_1, x_2), we have

$$u(x_1, x_2) = \tfrac{1}{2}(1 - x_1^2) \pm x_2. \qquad \blacksquare$$

1.2 Homework 2

PROBLEM 1.2.1. Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

SOLUTION. Throughout this, let (p^0, z^0, x^0) denote an admissible triple to the PDE F at some point x^0 in its domain. First, note that the condition

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 since in that case the normal to the hypersurface at x^0 will be $(0, \dots, 0, 1)$; i.e.,

$$\begin{aligned} 0 &\neq D_pF(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall therefore proceed to flatten the hypersurface Γ near x^0 and apply the standard noncharacteristic boundary condition.

Assuming Γ is reasonably tame, by the implicit function theorem, it can be written as the graph $\{ x_n = \varphi(x_1, \dots, x_{n-1}) \}$ on some neighborhood U of x^0 for φ smooth. Now consider the smooth mapping $\Phi \colon U \to V$ given by

$$\begin{cases} y_j := \Phi^j(x) := x_j, & 1 \leq j \leq n-1, \\ y_n := \Phi^n(x) := x_n - \varphi(x_1, \dots, x_{n-1}), \end{cases}$$

where we use y to denote new coordinates on the image of Φ. Note that $\nu(x^0)$ is parallel to the gradient $D_x\Phi^n = (-\varphi_{x_1}, \dots, -\varphi_{x_{n-1}}, 1)$ so the inner product of the latter with $F_{p_n}(p^0, z^0, x^0)$ is nonzero if and only if the inner product of $\nu(x^0)$ with $F_{p_n}(p^0, z^0, x^0)$ is nonzero.

Set $\Delta := \Phi(\Gamma)$ and define $v(y) := u(\Phi^{-1}(y))$. Then $u(x) = v(\Phi(x))$. Moreover, by the chain rule we have

$$D_{x_i}u = \sum_{j=1}^{n} D_{y_j}vD_{x_j}\Phi^j, \quad 1 \leq j \leq n;$$

i.e., $D_xu = D_yvD_x\Phi$. Thus, v satisfies the PDE

$$G(D_yv, v, y) := F(D_yvD_x\Phi, v, \Phi^{-1}(y)) = 0$$

in Δ and, since Δ has been flattened near $y^0 := \Phi(x^0)$, applying the noncharacteristic condition, we have

$$\begin{aligned} D_{p_n}G &= (D_{p_1}F)(D_{x_1}\Phi^n) + \cdots + (D_{p_n}F)(D_{x_n}\Phi) \\ &= D_pF \cdot D_x\Phi^n. \end{aligned}$$

Therefore, if (p^0, z^0, x^0) is a compatible triple for F and $(q^0, z^0, y^0) = (p^0D_x\Phi(x^0), z^0, \Phi(x^0))$ is the corresponding for G, then

$$D_{p_n}G(q^0, z^0, y^0) = D_pF(p^0, z^0, x^0) \cdot D_x\Phi^n(x^0);$$

i.e.,

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$$

where $\nu(x^0)$ is the normal vector to Γ at x^0. ■

PROBLEM 1.2.2. Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. Write $F(p, z, x, t) := (a(z), 1) \cdot p = 0$. Using the method of characteristics, we have the following characteristic ODEs to solve

$$\begin{cases} \dot{p} = -D_{x,t}F - D_zFp = -a'(z)p_1(p_1, p_2), \\ \dot{z} = D_pF \cdot p = (a(z), 1) \cdot p = 0, \\ \dot{x} = D_{p_1}F = a(z), \quad \dot{t} = D_{p_2}F = 1. \end{cases}$$

Solving these subject to the initial conditions $x(0) = x^0$, $t(0) = 0$, and $z(0) = g(x^0)$, we have

$$\begin{cases} x(s) = x^0 + a(g(x^0))s, \quad t(s) = s, \\ z(s) = g(x^0). \end{cases}$$

Thus, we see that u is constant on the projected characteristics

$$x = x^0 + a(g(x^0))t; \tag{1.2.1}$$

i.e., $u = g(x^0)$.

Solving for u in terms of x and t, we have

$$x^0 = x - a(u)t$$

so

$$u = g(x - a(u)t).$$

Now, choose another starting point $y^0 < x^0$. Then we must have $u = g(y^0)$ on the curve

$$x = y^0 + a(g(y^0))t. \tag{1.2.2}$$

Thus, if $g(y^0) > g(x^0)$ the two characteristics (1.2.1) and (1.2.2) will cross at some $t^0 > 0$ and we cannot have a continuous solution up to that point. If $g(y^0) < g(x^0)$ the characteristics (1.2.1) and (1.2.2) will not cross and therefore, u solves the PDE on the upper halfplane $\{ t > 0 \}$. ■

PROBLEM 1.2.3. Show that the function $u(x,t)$ defined for $t \geq 0$ by

$$u(x,t) = \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{for } 4x + t^2 > 0, \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burgers' equation*).

SOLUTION. First we show that u is in fact a weak solution of the inviscid Burgers' equation. That is, write u_l and u_r for u restricted to the domains $\{\, 4x + t^2 < 0 \,\}$ and $\{\, 4x + t^2 > 0 \,\}$, respectively. Then a trivial calculation shows that u_l and u_r are solutions to the inviscid Burgers' equation: It is clear that u_l is a solution of the Burgers' equation since it is the trivial solution; for u_r we must work a little harder;

$$\begin{aligned} (u_r)_t &= -\frac{2}{3}\left(1 + \frac{t}{\sqrt{3x + t^2}}\right), \\ (u_r)_x &= -\left(\frac{1}{\sqrt{3x + t^2}}\right) \end{aligned}$$

hence,

$$\begin{aligned} u_t + (u^2/2)_x &= u_t + u_x u \\ &= -\frac{2}{3}\left(1 + \frac{t}{\sqrt{3x + t^2}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{3x + t^2}}\right)\left(t + \sqrt{3x + t^2}\right) \\ &= 0. \end{aligned}$$

Lastly, we need to verify the Rankine–Hugoniot condition on the curve $\Gamma := \{\, 4x + t^2 = 0 \,\}$. Write $s(t) = x = -t^2/4$ (the natural parametrization of the curve Γ), then

$$\begin{aligned} \sigma &= \dot{s}(t), & [\![u]\!] &= u_l - u_r, & [\![F]\!] &= F(u_l) - F(u_r) \\ &= -\frac{t}{2} & &= 0 + \frac{2}{3}\left(t + \sqrt{-\frac{3}{4}t^2 + t^2}\right) & &= 0 - \frac{[\![u_r]\!]^2}{2} \\ & & &= 0 + \frac{2}{3}\left(\frac{3}{2}t\right) & &= 0 - \frac{t^2}{2} \\ & & &= t & &= -\frac{t^2}{2}. \end{aligned}$$

Thus,

$$[\![F]\!] = -\frac{t^2}{2} = \left(-\frac{t}{2}\right)t = \sigma[\![u]\!]$$

so the PDE satisfies the Rankine–Hugoniot condition and hence, is an integral solution.

Finally, to check that u is an entropy solution we note that $u^2/2$ is strictly convex and therefore, we must show that $u_l > u_r$. But this is obvious since $u_l - u_r = t$ which is strictly greater than zero. ∎

1.3 Homework 3

PROBLEM 1.3.1. Consider the initial value problem

$$\begin{cases} u_t = \sin u_x, \\ u(x,0) = \frac{\pi}{4}x. \end{cases}$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

SOLUTION. The equation $F(p,z,x,t) := \sin p_1 - p_2 = 0$ is a fully nonlinear first-order PDE. We first verify that the curve Γ is characteristic near the origin; i.e., we must show that $F_p \cdot \nu \neq 0$ where ν is the normal vector to Γ at the origin. In this case, $\nu = (0,1)$ and $F_p = (\cos p_1, -1)$; hence,

$$F_p \cdot \nu = (\cos p_1, -1) \cdot (0,1) = -1 \neq 0.$$

Moreover, the curve Γ is analytic (since it is cut out by the equation $\frac{\pi}{4}x$) and the initial conditions are analytic. Therefore, the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and we can obtain an analytic solution

$$u(x,t) = \sum_{m,n} \frac{a_{m,n}}{m!n!} x^m t^n$$

about the origin.

First, we must compute the coefficients $a_{m,n}$. To this end, we must find the partial derivatives $u_{m,n}$ and potentially, relations among them which will help us to find these coefficients. Naïvely listing the partials with respect to t and x, we have

$$\begin{aligned} u(0,0) &= 0 & u_x(0,0) &= \tfrac{\pi}{4} \\ u_t(0,0) &= \sin u_x(0,0) = \tfrac{\sqrt{2}}{2} & u_{xx}(0,0) &= 0 \\ u_{tx}(0,0) &= 0 & u_{tt}(0,0) &= -\cos\big(u_x(0,0)\big)u_{xt}(0,0) = 0 \\ u_{xxx}(0,0) &= 0 & u_{ttx}(0,0) &= 0 \\ &\vdots & &\vdots \end{aligned}$$

It is not difficult to see that higher derivatives of u will be zero. Thus,

$$u(x,t) = \tfrac{\pi}{4}x + \tfrac{\sqrt{2}}{2}t.$$

Plugging this equation into F we see that

$$u_t - \sin u_x = \tfrac{\sqrt{2}}{2} - \sin(\tfrac{\pi}{4}) = 0;$$

i.e., $u(x,t)$, as defined above, is an analytic solution to the PDE F. ■

PROBLEM 1.3.2. Consider the Cauchy problem for $u(x,y)$

$$\begin{cases} \quad u_y = a(x,y,u)u_x + b(x,y,u), \\ u(x,0) = 0 \end{cases}$$

let a and b be analytic functions of their arguments. Assume that $D^\alpha a(0,0,0) \geq 0$ and $D^\alpha b(0,0,0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

(a) Show that $D^\beta u(0,0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $D^\beta u(0,0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$.
Hint: Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in β_2.

SOLUTION. For part (a): We compute all partial $D^\beta u$ at $(0,0)$ for $|\beta| \leq 2$ explicitly; these are

$$\begin{aligned} u(0,0) &= u_x(0,0) = u_{xx}(0,0) = 0, \\ u_y(0,0) &= a(0,0,0)u_x(0,0) + b(0,0,0) = b \geq 0, \\ u_{xy}(0,0) &= (a_x(0,0,0) + a_u(0,0,0)u_x(0,0)) + a(0,0,0)u_{xx}(0,0) + b_x(0,0,0) \\ &\quad + b_z(0,0,0)u_x(0,0) \geq 0, \\ u_{yy}(0,0) &= (a_y(0,0,0) + a_u(0,0,0)u_y(0,0))u_x(0,0) \\ &\quad + b_y(0,0,0) + b_u(0,0,0)u_y(0,0) \geq 0. \end{aligned}$$

For part (b): Following the proof of the Cauchy–Kovalevskaya theorem, we use induction on β_2. The case $\beta_2 = 0$ is clear as $D^{(\beta_1,0)}u = \frac{\partial^{\beta_1} u}{\partial x^{\beta_1}} = 0$ by our previous work. Now suppose the proposition is true for all $\beta_2 \leq n-1$, we show the proposition holds for $\beta_2 = n$. From our previous work above, we have

$$\begin{aligned} D^\beta u &= D^{\beta_1, n-1} u_y \\ &= D^{\beta_1, n-1}[a(x,y,u)u_x + b(x,y,u)] \\ &= P_\beta(D^\gamma u, D^\delta a, D^\varepsilon b), \end{aligned}$$

where P_β is some polynomial with nonnegative coefficients depending only on $D^\gamma u$ with $|\gamma| \leq |\beta|$ and $|\gamma_2| \leq n-1$. Since all partial derivatives in $D^\beta u$ are nonnegative at the origin and P_β is a polynomial with positive coefficients, it follows that $D^\beta u(0,0) \geq 0$ for all β. ■

PROBLEM 1.3.3. (Kovalevskaya's example) show that the line $\{\, t = 0 \,\}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution u of the heat equation in $\mathbb{R} \times \mathbb{R}$, with $u = \frac{1}{1+x^2}$ on $\{t = 0\}$.

Hint: Assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of $(0,0)$.

SOLUTION. First we show that the line $\Gamma := \{\, t = 0 \,\}$ is characteristic for the heat equation. With $\nu = (1,0)$ the normal to the line Γ, the noncharacteristic condition reads

$$\sum_{|\alpha|=2} a_\alpha \nu^\alpha \neq 0.$$

However,

$$\sum_{|\alpha|=2} a_\alpha \nu^\alpha = 1 \cdot 1 + a_{0,2} \cdot 0 = 1 \neq 0.$$

Thus, Γ is characteristic for $u_t = u_{xx}$.

Now, suppose u is an analytic solution to the heat equation $u_t - u_{xx} = 0$ given by

$$u(x,t) = \sum_{m,n} \frac{a_{m,n}}{m!n!} x^m t^n.$$

Let us compute the coefficients $a_{m,n}$ near $(0,0)$. From the PDE, we have the relation

$$\begin{aligned} a_{m,n} &= D^{(m,n)} u(0,0) \\ &= D^{(m,n-1)} u_t(0,0) \\ &= D^{(m,n-1)} u_{xx}(0,0) \\ &= D^{(m+2,n-1)} u(0,0) \\ &= a_{m+2,n-1}. \end{aligned} \tag{1.3.1}$$

Form the initial condition, we have

$$u(x,0) = \sum_{k=1}^{\infty} (-1)^k x^{2k} \tag{1.3.2}$$

for a sufficiently small neighborhood about $(0,0)$, where the right-hand side is given taylor series of $\frac{1}{1+x^2}$. Taking the m^{th} x-partial derivative at $(0,0)$, with the help of Eq. (1.3.2) we find the coefficients

$$a_{m,0} = \begin{cases} 0 & \text{if } m = 2k+1 \text{ is odd} \\ (-1)^k (2k)! & \text{if } m = 2k \text{ is even.} \end{cases} \tag{1.3.3}$$

Putting all of this information together, we deduce that

$$a_{2m+1,n} = 0$$

for all m, n and, recursively,

$$a_{2m,n} = a_{2m+2,n-1} = \cdots = a_{2(m+n),0} = (-1)^{m+n} \big(2(m+n)\big)!.$$

Thus, for small $t > 0$ we have

$$u(0,t) = \sum_n a_{0,n} t^n. \tag{1.3.4}$$

However, by the ratio test, we see that the coefficients of the form $a_{0,n}$ grow very quickly; i.e.,

$$\begin{aligned} \frac{|a_{0,n+1}|}{|a_{0,n}|} &= \frac{(2n+2)!/2n!}{(n+1)!/n!} \\ &= \frac{(2n+2)(2n+1)}{n+1} \\ &= 2(2n+1) \end{aligned}$$

which approaches ∞ as $n \to \infty$. Therefore, the radius of convergence for (1.3.4) is zero. This contradicts the assumption that u is analytic. $\frac{1}{R} = \infty$ so ■

1.4 Homework 4

Problem 1.4.1 (Legendre transform). Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u\big(\mathbf{x}(p)\big),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

Hint: See [E, §4.4.3b], prove the identities (29).

Solution. Assuming the regularity on v prescribed above, we compute $v_{p_1p_1}$, $v_{p_1p_2}$ and $v_{p_2p_2}$.

First, we compute $v_{p_1p_2}$ since in the case of $v_{p_1p_1}$ and $v_{p_2p_2}$, there is some symmetry we can exploit. Taking the first partial with respect to p^1, we have

$$\begin{aligned} v_{p_1} &= \tfrac{\partial}{\partial p_1}\big(x^1(p)p^1 + x^2(p)p^2 - u\big(\mathbf{x}(p)\big)\big) \\ &= x^1(p) + x^1_{p_1}(p)p^1 + x^2_{p_1}(p)p^2 - u_{x_1}\big(\mathbf{x}(p)\big)x^1_{p_1}(p) - u_{x_2}\big(\mathbf{x}(p)\big)x^2_{p_1}(p) \\ &= x^1 + x^1_{p_1}p^1 + x^2_{p_1}p^2 - p^1x^1_{p_1} - p^2x^2_{p_1} \\ &= x^1, \end{aligned} \tag{1.4.1}$$

since $u_{x_1} = p^1$ and $u_{x_2} = p^2$.

Similarly, for v_{p_2}, we have

$$\begin{aligned} v_{p_2} &= \tfrac{\partial}{\partial p_2}\big(x^1(p)p^1 + x^2(p)p^2 - u\big(\mathbf{x}(p)\big)\big) \\ &= x^1_{p_2}(p)x^1(p) + x^2(p) + x^2_{p_2}(p)p^2 - u_{x_1}\big(\mathbf{x}(p)\big)x^1_{p_2}(p) - u_{x_2}\big(\mathbf{x}(p)\big)x^2_{p_2}(p) \\ &= x^1_{p_2}x^1 + x^2 + x^2_{p_2}p^2 - p^1x^1_{p_2} - p^2x^2_{p_2} \\ &= x^2. \end{aligned} \tag{1.4.2}$$

Now, taking the partial of (1.4.1) with respect to p_1 and then p_2, we have

$$v_{p_1p_1} = x^1_{p_1} = x^1_{u_{x_1}}, \qquad v_{p_1p_2} = x^1_{p_2} = x^1_{u_{x_2}},$$

and similarly for (1.4.2),

$$v_{p_1p_2} = x^2_{p_1} = x^2_{u_{x_1}}, \qquad v_{p_2p_2} = x^2_{p_2} = x^2_{u_{x_2}}.$$

By the inverse function theorem, we have

$$\begin{aligned}
\begin{bmatrix} v_{p_1p_1} & v_{p_1p_2} \\ v_{p_1p_2} & v_{p_2p_2} \end{bmatrix} &= \begin{bmatrix} x^1_{u_{x_1}} & x^1_{u_{x_2}} \\ x^2_{u_{x_1}} & x^2_{u_{x_2}} \end{bmatrix} \\
&= \begin{bmatrix} u_{x_1x_1} & u_{x_1x_2} \\ u_{x_1x_2} & u_{x_2x_2} \end{bmatrix}^{-1} \\
&= \frac{1}{J} \begin{bmatrix} u_{x_2x_2} & -u_{x_1x_2} \\ -u_{x_1x_2} & u_{x_1x_1} \end{bmatrix}.
\end{aligned}$$

Hence,

$$\begin{cases} u_{x_1x_1} = Jv_{p_2p_2} \\ u_{x_1x_2} = -Jv_{p_1p_2} \\ u_{x_2x_2} = Jv_{p_1p_1}, \end{cases} \tag{1.4.3}$$

which verifies Equation (29) from [E, §4.4.3b]. Substituting (1.4.3) into the original equation,

$$\begin{aligned}
0 &= Ja^{11}(p)v_{p_2p_2} - Ja^{12}(p)v_{p_1p_2} + Ja^{22}(p)v_{p_1p_1} \\
&= a^{22}(p)v_{p_1p_1} - a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_2p_2},
\end{aligned}$$

as was to be shown. ∎

PROBLEM 1.4.2. Find the solution $u(x,t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$\begin{cases} u = f, \quad u_t = g, & \text{for } (0,\infty) \times \{\, t = 0 \,\}, \\ u_t = \alpha u_x, & \text{for } \{\, x = 0 \,\} \times (0,\infty), \end{cases}$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$.

Hint: Use a representation $u(x,t) = F(x-t) + G(x+t)$ for the solution.

SOLUTION. Suppose $u(x,t) = F(x-t)+G(x+t)$ is a classical solution to the one-dimensional wave equation with the prescribed initial conditions. Then, we want to extend the data to all of x so that we can exploit d'Alembert's formula. Suppose we gave done this by, e.g., taking the odd reflection of $\tilde{f}$, $\tilde{g}$, and $\tilde{h}(x,t) := \int_{x-t}^{x+t} \tilde{g}(s)ds$. All we need to do is use the initial data to find the relation between $\tilde{f}$, $\tilde{g}$, and $\tilde{g}$ at $x = 0$.

Using d'Alembert's formula,

$$\tilde{u}(x,t) = \frac{\tilde{f}(x+t) + \tilde{f}(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \tilde{g}(s)\, ds,$$

we compute $\tilde{u}_t(0,t)$ and $\alpha\tilde{u}_x(0,t)$ to match our initial data.

Hence,

$$\begin{aligned} u_t(0,t) &= \tfrac{1}{2}\big(-\tilde{f}(-t) + \tilde{f}(t) + \tilde{g}(t) - \tilde{g}(-t)\big), \\ u_x(0,t) &= \tfrac{1}{2}\big(\tilde{f}(-t) + \tilde{f}(t) + \tilde{g}(t) - \tilde{g}(-t)\big), \end{aligned}$$

so

$$0 = \tfrac{1}{2}\big(-(1+\alpha)\tilde{f}(-t) + (1-\alpha)\tilde{f}(t) + (1-\alpha)(\tilde{g}(t) - \tilde{g}(t))\big). \qquad \blacksquare$$

PROBLEM 1.4.3. (a) Let u be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0,t) = u(\pi,t) = 0$. Show that the *energy*

$$E(t) = \frac{1}{2}\int_0^\pi \left(u_t^2 + c^2 u_x^2\right) dx, \quad t > 0$$

is independent of t; i.e., $\frac{d}{dt}E = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x,t) = \sum_{n=1}^{\infty}\big(a_n\cos(nct) + b_n\sin(nct)\big)\sin(nx)$$

in terms of coefficients a_n, b_n.

SOLUTION. For part (a): Suppose that u is, as above, a solution to the wave equation which is C^2 up to the boundary. We show that its energy is independent of t, i.e., that $\frac{d}{dt}E = 0$. Assuming the energy is bounded, the dominated convergence theorem allows us to permute the order of integration and differentiation like so

$$\begin{aligned} \tfrac{d}{dt}E(t) &= \frac{d}{dt}\left(\frac{1}{2}\int_0^\pi \left(u_t^2 + c^2u_x^2\right) dx\right) \\ &= \frac{1}{2}\int_0^\pi \tfrac{\partial}{\partial t}\left(u_t^2 + c^2u_x^2\right) dx \\ &= \frac{1}{2}\int_0^\pi 2u_tu_{tt} + 2c^2u_xu_{xt}\, dx \end{aligned}$$

which, after using the relation $u_{tt} = c^2u_{xx}$, becomes

$$\begin{aligned} &= c^2\int_0^\pi u_tu_{xx} + u_xu_{xt}\, dx \\ &= c^2\int_0^\pi \tfrac{\partial}{\partial x}(u_xu_t)\, dx \\ &= c^2\big(u_x(\pi,t)u_t(\pi,t) - u_x(0,t)u_t(0,t)\big) \\ &= 0 \end{aligned}$$

since the boundary conditions, i.e., $u = 0$, implies $u_x = u_t = 0$ at the boundary.

For part (b): Suppose u is a Fourier series solution to the wave equation, i.e.,

$$u(x,t) = \sum_{k=1}^{\infty}\big(a_k\cos(kct) + b_k\sin(kct)\big)\sin(kx).$$

First, we compute u_x and u_t, they are

$$u_x(x,t) = \sum_{k=1}^{\infty} k\big(a_k \cos(kct) + b_k \sin(kct)\big)\cos(kx),$$
$$u_t(x,t) = \sum_{k=1}^{\infty} kc\big(-a_k \sin(kct) + b_k \cos(kct)\big)\sin(kx).$$

Let $u_x^n(x,t)$ and $u_t^n(x,t)$ be the partial sums of the two equations above. Then

$$E_n(t) = \frac{1}{2}\int_0^{\pi} (u_t^n)^2 + c^2(u_x^n)^2 \, dt$$

taking into account orthogonality relations of cos and sin, we have

$$\begin{aligned}
&= \frac{1}{2}\sum_{k=1}^{n} k^2c^2\Bigg[\bigg(\int_0^{\pi}\big(a_k^2 \sin(kct) + b_k^2 \cos(kct)\big)\bigg)\sin^2(kx) \\
&\qquad\qquad + \bigg(\int_0^{\pi}\big(a_k^2 \cos(kct) + a_k^2 \sin(kct)\big)\bigg)\cos^2(kx)\Bigg] \\
&= \frac{\pi}{2}\sum_{k=1}^{n} k^2c^2(a_k^2 + b_k^2).
\end{aligned}$$

Thus, since the limit is uniform

$$E(t) = \lim_{n\to\infty} E_n(t) = \frac{\pi}{4}\sum_{n=1}^{\infty} n^2c^2(a_n^2 + b_n^2). \qquad \blacksquare$$

1.5 Homework 5

PROBLEM 1.5.1. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$, $x \in \mathbb{R}^n$, then $\Delta v = 0$.

SOLUTION. Let

$$O = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be an orthogonal $n \times n$ matrix. We will show that $\Delta v = 0$, where $v(x) = u(Ox)$.

First, let us compute the gradient of v,

$$\begin{aligned} Dv(x) &= Du(Ox) \\ &= Du\big(a_{11}x_1 + \cdots + a_{1n}x_n, \dots, a_{n1}x_1 + \cdots + a_{nn}x_n\big) \\ &= \Big(\textstyle\sum_{j=1}^n a_{j1}u_{x_j}, \dots, \sum_{j=1}^n a_{jn}u_{x_j}\Big) \\ &= O^{\mathrm{T}} Du(x). \end{aligned}$$

Lastly, we compute the divergence of Dv,

$$\begin{aligned} \Delta v(x) &= \operatorname{div} Dv(x) \\ &= \operatorname{div}\Big(\textstyle\sum_{j=1}^n a_{j1}u_{x_j}, \dots, \sum_{j=1}^n a_{jn}u_{x_j}\Big). \end{aligned}$$

Here the partial derivatives become unwieldy so we will first examine the partial $\frac{\partial}{\partial x_1}$ of the first term and proceed from there. In this case,

$$\begin{aligned} \frac{\partial}{\partial x_1}\sum_{j=1}^n a_{j1}u_{x_j} &= a_{11}(u_{x_1})_{x_1} + a_{21}(u_{x_2})_{x_1} + \cdots + a_{n1}(u_{x_n})_{x_1} \\ &= a_{11}(a_{11}u_{x_1x_1} + a_{21}u_{x_1x_2} + \cdots + a_{n1}u_{x_1x_n}) \\ &\quad + \cdots + a_{n1}(a_{11}u_{x_1x_n} + a_{21}u_{x_2x_n} + \cdots + a_{n1}u_{x_nx_n}) \\ &= a_{11}^2u_{x_1x_1} + 2a_{11}a_{21}u_{x_1x_2} + 2a_{11}a_{31}u_{x_1x_3} + \cdots + a_{21}^2u_{x_2x_2} \\ &\quad + \cdots + a_{k1}^2u_{x_kx_k} + \cdots + a_{n1}^2u_{x_nx_n}. \end{aligned}$$

Similarly, taking the k^{th} partial of the k^{th} entry of Dv, we have

$$\begin{aligned} \frac{\partial}{\partial x_k}\sum_{j=1}^n a_{jk}u_{x_j} &= a_{1k}(a_{1k}u_{x_1x_1} + \cdots + a_{nk}u_{x_1x_n}) \\ &\quad + \cdots + a_{nk}(a_{1k}u_{x_1x_n} + \cdots + a_{nk}u_{x_nx_n}) \\ &= a_{1k}^2u_{x_1x_1} + a_{2k}^2u_{x_2x_2} + \cdots + a_{kk}^2u_{x_kx_k} \\ &\quad + \cdots + a_{nk}^2u_{x_nx_n} + \{\text{mixed terms}\}. \end{aligned} \tag{1.5.1}$$

Now, since O is orthogonal, we have

$$\begin{aligned} OO^{\mathrm{T}} &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}^2 + \cdots + a_{1n}^2 & a_{11}a_{21} + \cdots + a_{1n}a_{2n} & \cdots & a_{11}a_{n1} + \cdots + a_{1n}a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} + \cdots + a_{nn}a_{1n} & a_{n1}a_{21} + \cdots + a_{nn}a_{2n} & \cdots & a_{n1}^2 + \cdots + a_{nn}^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \end{aligned}$$

We can sum up the results of our calculation as

$$\begin{cases} \text{(a)} \ \sum_{j=1}^n a_{kj}a_{lj} = \sum_{j=1}^n a_{kj}^2 = 1 & \text{if } k = l, \\ \text{(b)} \ \sum_{j=1}^n a_{kj}a_{lj} = 0 & \text{if } k \neq l. \end{cases} \tag{1.5.2}$$

for $1 \leq k, l \leq n$.

Now, going back to (1.5.1), we have

$$\begin{aligned} \operatorname{div} Dv &= \sum_{k=1}^n \left[\frac{\partial}{\partial x_k} \sum_{j=1}^n a_{jk} u_{x_j} \right] \\ &= (a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2) u_{x_1x_1} + (a_{12}^2 + a_{22}^2 + \cdots + a_{2n}^2) u_{x_2x_2} \\ &\quad + \cdots + (a_{1n}^2 + \cdots + a_{nn}^2) u_{x_nx_n} + \{\text{mixed terms}\} \\ &= u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} \\ &= 0, \end{aligned} \tag{1.5.3}$$

as desired.

All that is left to show as that the mixed terms in the expression above actually have coefficients of the form in (1.5.2) (b). And in fact one can see, expanding (1.5.3), that the mixed terms have the form

$$\sum_{j=1}^n a_{kj}a_{lj} = 0.$$

For example, the first member in the mixed terms sequence is

$$2(a_{11}a_{21} + a_{12}a_{22} + \cdots + a_{1n}a_{2n}) u_{x_1x_2} = 0.$$

(Time permits, I will provide a better argument than simply expanding (1.5.3); but a little routine calculation shows that these terms in fact have the form we have described.) ■

PROBLEM 1.5.2. Let $n = 2$ and U be the halfplane $\{ x_2 > 0 \}$. Prove that

$$\sup_U u = \sup_{\partial U} u$$

for $u \in C^2(U) \cap C(\bar{U})$ which are harmonic in U under the additional assumption that u is bounded from above in $\bar{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.)

Hint: Take for $\varepsilon > 0$ the harmonic function

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2+1)^2}.$$

Apply the maximum principle to a region $\left\{ x_1^2 + (x_2+1)^2 < a^2, x_2 > 0 \right\}$ with large a. Let $\varepsilon \to 0$.

Solution. Consider the harmonic function

$$u_\varepsilon(x_1, x_2) := u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2+1)^2}.$$

Set $U_a := \left\{ x_1^2 + (x_2+1)^2 < a^2, x_2 > 0 \right\}$.

First, we note that $u_\varepsilon \uparrow u$ as $\varepsilon \to 0$ pointwise, i.e., given $\eta > 0$, for

$$0 < \varepsilon(x_1, x_2) < \eta / \ln \sqrt{x_1^2 + (x_2+1)^2},$$

we have

$$\begin{aligned}
|u_\varepsilon(x_1, x_2) - u(x_1, x_2)| &= \left| u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2+1)^2} - u(x_1, x_2) \right| \\
&= \left| \varepsilon \ln \sqrt{x_1^2 + (x_2+1)^2} \right| \\
&= \varepsilon \ln \sqrt{x_1^2 + (x_2+1)^2} \\
&< \eta,
\end{aligned}$$

for any $(x_1, x_2) \in U_a$.

Moreover, a simple calculation shows that u_ε is in fact harmonic. By the linearity the Laplacian, it suffices to show that the Laplacian of

$$v_\varepsilon(x) := \varepsilon \ln \sqrt{x_1^2 + (x_2+1)^2}$$

is 0. First, we calculate he partial derivatives $\frac{\partial^2}{\partial x_1 \partial x_1}$ and $\frac{\partial^2}{\partial x_2 \partial x_2}$

$$\begin{aligned}
\frac{(v_\varepsilon)_{x_1}}{\varepsilon} &= -\frac{x_1}{x_1^2 + (x_2+1)^2} & \frac{(v_\varepsilon)_{x_2}}{\varepsilon} &= -\frac{(x_2+1)}{x_1^2 + (x_2+1)^2} \\
\frac{(v_\varepsilon)_{x_1 x_1}}{\varepsilon} &= -\frac{-x_1^2 + (x_2+1)^2}{\left(x_1^2 + (x_2+1)^2\right)^2} & \frac{(v_\varepsilon)_{x_2 x_2}}{\varepsilon} &= -\frac{x_1^2 - (x_2+1)^2}{\left(x_1^2 + (x_2+1)^2\right)^2}.
\end{aligned}$$

Thus,

$$\Delta u_\varepsilon = \Delta u + \Delta v_\varepsilon = \Delta v_\varepsilon = \varepsilon \left(-\frac{-x_1^2 + (x_2+1)^2}{\left(x_1^2 + (x_2+1)^2\right)^2} - \frac{x_1^2 - (x_2+1)^2}{\left(x_1^2 + (x_2+1)^2\right)^2} \right) = 0.$$

Now, observe that, for any a, by the maximum principle, we have

$$\max_{\bar{U}_a} u_\varepsilon = \max_{\partial U_a} u_\varepsilon$$

for any a. Choose a large enough so

$$\sup_{\partial U_a} u_\varepsilon \leq \sup_{\partial U} u.$$

Then,

$$\sup_{\bar{U}_a} u_\varepsilon \leq \sup_{\partial U} u$$

so, taking $a \to \infty$, we have

$$\sup_{\bar{U}} u_\varepsilon \leq \sup_{\partial U} u.$$

Thus, for every $x_1, x_2 \in U$,

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} < \sup_{\partial U} u.$$

Letting $\varepsilon \to 0$, we achieve the desired inequality, i.e.,

$$\sup_{\bar{U}} u \leq \sup_{\partial U} u.$$

The last inequality is obvious and stems from the fact that $\partial U \subseteq \bar{U}$, i.e., the inequality

$$\sup_{\partial U} u \leq \sup_{\bar{U}} u.$$

We conclude that

$$\sup_{\partial U} u = \sup_{\bar{U}} u,$$

as was to be shown. ■

PROBLEM 1.5.3. Let $U \subseteq \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \ldots, u^m$ are harmonic in U and

$$v := \varphi(u_1, \ldots, u_m).$$

Prove v is subharmonic.

Hint: Convexity for a smooth function $\varphi(z)$ is equivalent to $\sum_{j,k=1}^{m} \varphi_{z_j, z_k}(z)\xi_j\xi_k \geq 0$ for any $\xi \in \mathbb{R}^m$.

(b) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. (Assume that harmonic functions are C^∞.)

Solution. For part (a): By the chain rule, we have

$$v_{x_i} = \varphi_{u_1} u^1_{x_i} + \cdots + \varphi_{u_m} u^m_{x_i}.$$

Taking another partial, we have

$$\begin{aligned} v_{x_i x_i} &= (v_{x_i})_{x_i} \\ &= \frac{\partial}{\partial x_i}\big(\varphi_{u_1} u^1_{x_i} + \cdots + \varphi_{u_m} u^m_{x_i}\big) \\ &= \varphi_{u_1} u^1_{x_i x_i} + \cdots + \varphi_{u_m} u^m_{x_i x_i} \\ &\quad + \big(\varphi_{u_1 u_1} u^1_{x_i} + \cdots + \varphi_{u_1 u_m} u^m_{x_i}\big) u^1_{x_i} \\ &\qquad + \cdots + \big(\varphi_{u_1 u_m} u^1_{x_i} + \cdots + \varphi_{u_m u_m} u^m_{x_i}\big) u^m_{x_i}. \end{aligned} \tag{1.5.4}$$

Now, taking the sum

$$\begin{aligned} \sum_{i=1}^n v_{x_i x_i} &= \sum_{i=1}^n \sum_{j=1}^m \varphi_{u_j} u^j_{x_i x_i} \\ &= \sum_{j=1}^m \sum_{i=1}^n \varphi_{u_j} u^j_{x_i x_i} \\ &= \sum_{j=1}^m \big(\varphi_{u_j} u^j_{x_1 x_1} + \cdots + \varphi_{u_j} u^j_{x_n x_n}\big) \\ &= \sum_{j=1}^m \varphi_{u_j} \big(u^j_{x_1 x_1} + \cdots + u^j_{x_n x_n}\big) \\ &= 0, \end{aligned}$$

since $\Delta u^j = 0$ for all j.

What about the remaining terms in (1.5.4)? These terms can be written in the form

$$\sum_{j,k=1}^m \varphi_{u_j u_k}(u) \xi_j \xi_k,$$

where $\xi_i = \big(u^1_{x_i}, \ldots, u^m_{x_i}\big)(x_1, \ldots, x_n) \in \mathbb{R}^m$ for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Since φ is convex, by assumption, this quantity is greater than or equal to 0.

Thus, $\Delta v \geq 0$ so v is subharmonic.

For part (b): We have

$$v = |Du|^2 = u^2_{x_1} + \cdots + u^2_{x_n}.$$

Taking the partial derivative with respect to x_i, we have

$$\begin{aligned} v_{x_i} &= \frac{\partial}{\partial x_i}\big(u^2_{x_1} + \cdots + u^2_{x_n}\big) \\ &= 2u_{x_1} u_{x_1 x_i} + \cdots + 2u_{x_n} u_{x_i x_n}, \end{aligned}$$

and again

$$\begin{aligned}
v_{x_i x_i} &= (v_{x_i})_{x_i} \\
&= \frac{\partial}{\partial x_i}\big(2u_{x_1}u_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_n}\big) \\
&= 2u_{x_1}u_{x_1x_ix_i} + 2u_{x_1x_i}^2 + \cdots + 2u_{x_n}u_{x_ix_ix_n} + 2u_{x_ix_n}^2 \\
&= 2\sum_{j=1}^{n}\big(u_{x_j}u_{x_jx_ix_i} + u_{x_jx_i}^2\big).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\Delta v}{2} &= \sum_{i,j=1}^{n}\big(u_{x_j}u_{x_jx_ix_i} + u_{x_jx_i}^2\big) \\
&= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + \sum_{i,j=1}^{n} u_{x_jx_i}^2,
\end{aligned}$$

splitting the second term into the sum

$$\begin{aligned}
&= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + \sum_{1\le j<i\le n} u_{x_jx_i}^2 \\
&\quad + \sum_{1\le i<j\le n} u_{x_jx_i}^2 + \sum_{1\le i=j\le n} u_{x_ix_i}^2,
\end{aligned}$$

where the last term is 0 since u is harmonic, giving us

$$\begin{aligned}
&= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + \sum_{1\le j<i\le n} u_{x_jx_i}^2 + \sum_{1\le i<j\le n} u_{x_jx_i}^2 \\
&= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + 2\sum_{1\le j<i\le n} u_{x_jx_i}^2,
\end{aligned}$$

here $\sum_{j=1}^{n} u_{x_ix_jx_j} = \Delta u_{x_i} = 0$ since the derivatives of harmonic functions are harmonic, so

$$\begin{aligned}
&= \sum_{j=1}^{n} u_{x_j}(\Delta u_{x_j}) + 2\sum_{1\le j<i\le n} u_{x_jx_i}^2 \\
&= 2\sum_{1\le j<i\le n} u_{x_jx_i}^2 \\
&\ge 0,
\end{aligned}$$

as desired. That is, $\Delta v \ge 0$ so v is subharmonic. ∎

1.6 Homework 6

PROBLEM 1.6.1. For $n = 2$ find Green's function for the quadrant $U := \{ x_1, x_2 > 0 \}$ by repeated reflection.

SOLUTION. Taking the hit, set $x' := (x_1, -x_2)$, $x'' := (-x_1, x_2)$, $x''' := (-x_1, -x_2)$, and define

$$\varphi^x(y) := \Phi(y - x') + \Phi(y - x'') - \Phi(y - x'''). \tag{1.6.1}$$

We claim that φ^x, as defined above, solves

$$\begin{cases} \Delta\varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(y - x) & \text{on } \partial U. \end{cases}$$

It is clear that $\Delta\varphi^x = 0$ since it is built up from the fundamental solutions on $\mathbb{R}^n$ (this follows from the linearity of the Laplace operator). To see that $\varphi^x(y) = \Phi(x - y)$ on ∂U, we do a case by case analysis.

Note that on $\{ x_1 = 0 \} \subseteq \partial U$, we have

$$\varphi^x(y_1, 0) = \Phi(-x_1, y_2 + x_2) + \Phi(-x_1, y_2 - x_2) - \Phi(x_1, y_2 + x_2),$$

where, since the fundamental solution is radial, we have $\Phi(-x_1, y_2 + x_2) = \Phi(x_1, y_2 + x_2)$, and hence the above equals

$$\begin{aligned} &= \Phi(-x_1, y_2 - x_2) \\ &= \Phi(y - x) \end{aligned}$$

and on $\{ x_2 = 0 \} \subseteq \partial U$, we have

$$\varphi^x(0, y_2) = \Phi(y_1 - x_1, x_2) + \Phi(y_1 + x_1, -x_2) - \Phi(y_1 + x_1, x_2)$$

where, again because Φ is radial, $\Phi(y_1 + x_1, -x_2) = \Phi(y_1 + x_1, x_2)$, thus the above equals

$$\begin{aligned} &= \Phi(y_1 - x_1, x_2) \\ &= \Phi(y - x). \end{aligned}$$

Thus, $\varphi^x(y) = \Phi(y - x)$ on ∂U.

Therefore, Green's function on U is

$$G(x, y) = \Phi(y - x) - \varphi^x(y) = \Phi(y - x) - \Phi(y - x') - \Phi(y - x'') + \Phi(y - x'''). \qquad \blacksquare$$

PROBLEM 1.6.2. (Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B(0, r) = \{ x \in \mathbb{R}^n : |x| < r \}$.

Solution. Recall Poisson's formula for the ball

$$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y), \tag{1.6.2}$$

where $x \in B(0,r)$ and u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,r), \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

For fixed $x \in B(0,r)$, write

$$u(x) = r^{n-2}(r+|x|)(r-|x|)\left[\frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y)\right].$$

Now, since $r+|x| \geq |x-y| \geq r-|x|$ for all $y \in \partial B(0,r)$, we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} \fint_{\partial B(0,r)} g(y)\, dS(y) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} \fint_{\partial B(0,r)} g(y)\, dS(y). \tag{1.6.3}$$

Since $u = g$ on the boundary $\partial B(0,r)$, by applying the mean-value property on (1.6.3) we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} u(0),$$

as desired. ∎

Problem 1.6.3. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$\begin{cases} P_k(\lambda x) = \lambda^k P_k(x), & P_m(\lambda x) = \lambda^m P_m(x) & \text{for every } x \in \mathbb{R}^n,\ \lambda > 0, \\ \Delta P_k = 0, & \Delta P_m = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

(a) Show that

$$\begin{cases} \dfrac{\partial P_k}{\partial \nu} = kP_k(x), \quad \dfrac{\partial P_m}{\partial \nu} = mP_m(x) \quad \text{on } \partial B(0,1), \end{cases}$$

where $B(0,1) = \{\, x \in \mathbb{R}^n : |x| < 1 \,\}$ and ν is the outward normal on $\partial B(0,1)$.

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x)P_m(x)\, d\sigma = 0, \qquad \text{if } k \neq m.$$

Solution. For part (a), let

$$P_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha.$$

Then, since $\nu = (x_1, \dots, x_n)$, the derivative along ν is given by

$$
\begin{aligned}
\frac{\partial P_k(x)}{\partial \nu} &= \sum_{j=1}^{n} (P_k)_{x_j} x_j \\
&= \sum_{j=1}^{n} \left(\sum\nolimits_{|\alpha|=k} a_\alpha x^\alpha \right)_{x_j} x_j \\
&= \sum_{j=1}^{n} \left(\sum\nolimits_{l=1}^{m} a_\alpha x_1^{\alpha_1^l} \cdots x^{\alpha_j^l} \cdots x^{\alpha_n^l} \right)_{x_j} x_j
\end{aligned}
$$

where $\sum_{j=1}^{n} \alpha_j^l = k$ and $1 \le j \le \binom{n+k-1}{n} =: m$ (by the stars and bars theorem)

$$
\begin{aligned}
&= \sum_{j=1}^{n} \sum_{l=1}^{m} \left(\alpha_j^l a_\alpha x_1^{\alpha_1^l} \cdots x^{\alpha_j^l - 1} \cdots x^{\alpha_n^l} \right) x_j \\
&= \sum_{j=1}^{n} \sum_{l=1}^{m} \alpha_j^l a_\alpha x_1^{\alpha_1^l} \cdots x^{\alpha_j^l} \cdots x^{\alpha_n^l} \\
&= \sum_{j=1}^{n} \sum_{l=1}^{m} \alpha_j^l a_\alpha x^\alpha
\end{aligned}
$$

switching the order of summation, we have

$$
\begin{aligned}
&= \sum_{l=1}^{m} \sum_{j=1}^{n} \alpha_j^l a_\alpha x^\alpha \\
&= \sum_{l=1}^{m} k a_\alpha x^\alpha \\
&= k \sum_{l=1}^{m} a_\alpha x^\alpha \\
&= k P_k(x).
\end{aligned}
$$

This argument, of course, applies to every $k \in \mathbb{N}$.

For part (b), by Green's theorem, we have

$$
\begin{aligned}
\int_{B(0,r)} P_k(x) \Delta P_m(x) - (\Delta P_k(x)) P_m(x) \, dx &= \int_{\partial B(0,r)} P_k(x) \frac{\partial}{\partial \nu} P_m(x) - \frac{\partial}{\partial \nu} P_k(x) P_m(x) \, dS(x) \\
&= \int_{\partial B(0,r)} (m-k) P_k(x) P_m(x) \, dS(x),
\end{aligned}
$$

where the left-hand side is equal to zero since both ΔP_k and ΔP_m are zero. Since $m \neq k$, it must be the case that

$$
\int_{\partial B(0,r)} P_k(x) P_m(x) \, dS(x) = 0. \qquad \blacksquare
$$

1.7 Homework 7

Problem 1.7.1. Solve the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ u = x_1 & \text{on } |x| = 1, \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

Hint: Use Laurent series.

Solution. First, let us make the change of variables $(x_1, x_2) \mapsto r\mathrm{e}^{\mathrm{i}\theta}$ to the Dirichlet problem in question:

$$\begin{cases} \Delta u = 0 & \text{in } 1 < r < 2, \\ u = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) & \text{on } r = 1, \\ u = 1 + \frac{1}{\mathrm{i}}(\mathrm{e}^{\mathrm{i}2\theta} - \mathrm{e}^{-\mathrm{i}2\theta}) & \text{on } r = 2. \end{cases} \tag{1.7.1}$$

Now, suppose u is a solution, of the form

$$u(r\mathrm{e}^{\mathrm{i}\theta}) = b \ln r + \sum_{n=-\infty}^{\infty} (a_n r^n + \overline{a_{-n}} r^{-n}) \mathrm{e}^{\mathrm{i}n\theta},$$

to the problem (1.7.1). It is easy to see that u is in fact harmonic:

$$\begin{aligned} \Delta u &= u_{rr} + \tfrac{1}{r} u_r + \tfrac{1}{r^2} u_{\theta\theta} \\ &= -br^{-2} + br^{-2} \\ &\quad + \sum_{n=-\infty}^{\infty} \big[\big(n(n-1) + n - n^2\big) a_n r^n + \big(n(n-1) + n - n^2\big) \overline{a_{-n}} r^{-n}\big] \mathrm{e}^{\mathrm{i}n\theta} \\ &= 0. \end{aligned}$$

Next we use the boundary data to compute the coefficients a_n, $n \in \mathbb{Z}$. Using the data (1.7.1), on $r = 1$ we have

$$\tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) = \sum_{n=-\infty}^{\infty} (a_n + \overline{a_{-n}}) \mathrm{e}^{\mathrm{i}n\theta},$$

and on $r = 2$

$$1 + \tfrac{1}{\mathrm{i}}(\mathrm{e}^{\mathrm{i}2\theta} - \mathrm{e}^{-\mathrm{i}2\theta}) = b \ln 2 + \sum_{n=-\infty}^{\infty} (2^n a_n + 2^{-n} a_{-n}) \mathrm{e}^{\mathrm{i}n\theta}.$$

These equations immediately tell us that $b = \frac{1}{\ln 2}$. Moreover, the following relations on the coefficients hold

$$\begin{cases} \frac{1}{2} = a_1 + \overline{a_{-1}}, \quad \frac{1}{2} = a_{-1} + \overline{a_1}, \\ \frac{1}{\mathrm{i}} = 2^2 a_2 + 2^{-2} \overline{a_{-2}}, \quad -\frac{1}{\mathrm{i}} = 2^2 a_{-2} + 2^{-2} \overline{a_2}, \\ 0 = a_n + \overline{a_{-n}} & \text{for } n \neq \pm 1, \\ 0 = 2^n a_n + 2^{-n} \overline{a_{-n}} & \text{for } n \neq \pm 2. \end{cases}$$

A little calculation shows that

$$\begin{cases} a_1 = -\frac{1}{6}, \quad a_{-1} = \frac{2}{3}, \\ a_2 = -\frac{4\mathrm{i}}{15}, \quad a_{-2} = -\frac{4\mathrm{i}}{15}, \\ a_n = 0 & \text{for } n \neq \pm 1, \pm 2. \end{cases}$$

Thus,

$$\begin{aligned} u(r\mathrm{e}^{\mathrm{i}\theta}) &= \tfrac{1}{\ln 2}\ln r + \big(-\tfrac{4\mathrm{i}}{15}r^{-2} + \tfrac{4\mathrm{i}}{15}r^2\big)\mathrm{e}^{-\mathrm{i}2\theta} + \big(\tfrac{2}{3}r^{-1} - \tfrac{1}{6}r\big)\mathrm{e}^{-\mathrm{i}\theta} \\ &\quad + \big(-\tfrac{1}{6}r + \tfrac{2}{3}r^{-1}\big)\mathrm{e}^{\mathrm{i}\theta} + \big(-\tfrac{4\mathrm{i}}{15}r^2 + \tfrac{4\mathrm{i}}{15}r^{-2}\big)\mathrm{e}^{\mathrm{i}2\theta} \\ &= \tfrac{1}{\ln 2}\ln r - \tfrac{8}{15}r^{-4}\left(\frac{r^2\mathrm{e}^{\mathrm{i}2\theta} - r^2\mathrm{e}^{-\mathrm{i}2\theta}}{2\mathrm{i}}\right) + \tfrac{8}{15}\left(\frac{r^2\mathrm{e}^{\mathrm{i}2\theta} - r^2\mathrm{e}^{-\mathrm{i}2\theta}}{2\mathrm{i}}\right) \\ &\quad + \tfrac{4}{3}r^{-2}\left(\frac{r\mathrm{e}^{\mathrm{i}\theta} + r\mathrm{e}^{-\mathrm{i}\theta}}{2}\right) - \tfrac{1}{3}\left(\frac{r\mathrm{e}^{\mathrm{i}\theta} + r\mathrm{e}^{-\mathrm{i}\theta}}{2}\right). \end{aligned}$$

Substituting back, we have

$$u(x_1, x_2) = \tfrac{1}{\ln 2}\ln(x_1^2 + x_2^2) - \frac{16x_1x_2}{15(x_1^2 + x_2^2)^2} + \frac{16x_1x_2}{15} + \frac{4x_1}{3(x_1^2 + x_2^2)} - \frac{x_1}{3}$$

which clearly satisfies the boundary data at $|x| = 1$ and $|x| = 2$. ■

Problem 1.7.2. Let Ω be a bounded domain with a C^1 boundary, $g \in C^2(\partial\Omega)$ and $f \in C(\bar{\Omega})$. Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where ν is the outer normal on $\partial\Omega$. Show that the solution of $(*)$ in $C^2(\Omega) \cap C^1(\bar{\Omega})$ is unique up to a constant; i.e., if u_1 and u_2 are both solutions of $(*)$, then $u_2 = u_1 + \text{const.}$ in Ω.

Hint: Look at the proof of the uniqueness for the Dirichlet problem by energy methods [E, 2.2.5a].

Solution. Suppose u_1 and u_2 are solutions to the Neumann problem $(*)$. Define $v := u_1 - u_2$. Then v is harmonic in Ω and $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$. Consider the energy functional

$$E[v] = \frac{1}{2}\int_\Omega |Dv|^2\,dx.$$

By Green's formula,

$$\begin{aligned} E[v] &= \frac{1}{2}\int_\Omega |Dv|^2\,dx \\ &= -\frac{1}{2}\int_\Omega v\Delta v\,dx + \int_{\partial U}\tfrac{\partial v}{\partial \nu}v\,dS(x) \\ &= 0. \end{aligned}$$

This implies that $|Dv|^2 = Dv \cdot Dv = 0$ which, since the standard inner product on $\mathbb{R}^n$ is positive-definite, implies that $Dw \equiv 0$. It follows that $u_1 = u_2 + \text{const}$, i.e., the solution u to $(*)$ is unique up to a constant. ■

Problem 1.7.3. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta_x u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\}, \end{cases}$$

where $c \in \mathbb{R}$.

Hint: Rewrite the problem in terms of $v(x,t) := \mathrm{e}^{ct} u(x,t)$.

Solution. Taking the hint, let us rewrite the problem in terms of $v(x,t) = \mathrm{e}^{ct} u(x,t)$:

$$\begin{cases} v_t - \Delta_x v = \mathrm{e}^{ct} f & \text{in } \mathbb{R}^n \times (0,\infty), \\ v = g & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\}. \end{cases} \tag{1.7.2}$$

By Duhamel's principle, the problem (1.7.2) is solved by

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) \mathrm{e}^{cs} f(y,s)\, dy ds,$$

where Φ is the fundamental solution to the heat equation. Thus, the formula

$$u(x,t) = \mathrm{e}^{-ct} v(x,t) = \mathrm{e}^{-ct} \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy + \mathrm{e}^{-ct} \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) \mathrm{e}^{cs} f(y,s)\, dy\, ds$$

solves the original problem. ■

1.8 Homework 8

PROBLEM 1.8.1. Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t)$$

where

$$\Phi(x,t) = \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

is positive for $|x| < 1$, $t > 0$.

Hint: Show that u satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{cases} u = 0 & \text{on } \{\, |x| = 1 \,\} \times \{\, t \geq 0 \,\}, \\ u = \delta_0 & \text{on } \{\, |x| \leq 1 \,\} \times \{\, t = 0 \,\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{\, |x| \leq 1 \,\} \times \{\, \varepsilon \leq t \leq T \,\}$ for small $\varepsilon > 0$ and large $T > 0$ pass to the limit as $\varepsilon \to 0+$ and $T \to \infty$.

SOLUTION. Taking the hint, let us verify that $u_t = u_{xx}$, for $t > 0$. By direct computation, we have

$$\Phi_x(x,t) = \frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) = -\frac{x\mathrm{e}^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}},$$

$$\begin{aligned} \Phi_{xx}(x,t) &= \frac{\partial}{\partial x}\left(-\frac{x\mathrm{e}^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}\right) \\ &= \frac{x^2\mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)\mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}}, \end{aligned}$$

and

$$\begin{aligned} \Phi_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \\ &= \frac{x^2\mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)\mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}}. \end{aligned}$$

Since $\Phi_t = \Phi_{xx}$ it follows that $u_t = u_{xx}$ (assuming uniform convergence of u).

Next we show that $u = 0$ on $\{\, |x| = 1 \,\} \times \{\, t \geq 0 \,\}$ and $u = \delta_0$ on $\{\, |x| = 1 \,\} \times \{\, t = 0 \,\}$.

To show $u = 0$ fix a $t \geq 0$ and, after relabeling if necessary, assume that $x = 1$ which gives us

$$\begin{aligned} u(1,t) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{-\frac{(1-2k)^2}{4t}}}{\sqrt{4\pi t}} \\ &= \frac{1}{\sqrt{4\pi t}} \left(\cdots - \mathrm{e}^{-\frac{9}{4t}} + \mathrm{e}^{-\frac{1}{4t}} - \mathrm{e}^{-\frac{1}{4t}} + \mathrm{e}^{-\frac{9}{4t}} + \cdots \right) \\ &= 0. \end{aligned}$$

Similarly for $u(-1,t) = 0$.

For $u(|x| \leq 1, 0)$, we have a

$$\begin{aligned} u(|x| \leq 1, 0) &= \sum_{k=-\infty}^{\infty} (-1)^k \lim_{t \to 0+} \left[\mathrm{e}^{-\frac{(x-2k)^2}{4t}} / \sqrt{4\pi t} \right] \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \delta_0(x - 2k) \\ &= \delta_0(x) \end{aligned}$$

since $|x| \leq 1$ and values δ_0 is zero for values $x - 2k$ outside of the interval $[-1, 1]$.

At last we show that u is positive for $|x| < 1$, $t > 0$. Seeking a contradiction, suppose u is negative on some point (x_0, t_0) in $\{\, |x| < 1 \,\} \times \{\, \varepsilon \leq t \leq T \,\}$. Then by the minimum principle, u achieves its minimum somewhere on the bottom boundary $\{\, |x| = 1 \,\} \times \{\, t = \varepsilon \,\}$. Therefore, there exists a sequence $(x_n, t_n+) \to (x, 0)$, where $|x_n|, |x| < 1$, such that $u(x, 0) < 0$. However, we have shown above that $u(x, 0) = \delta_0(x)$ for $|x| < 1$; i.e., either $u(x, 0) = 0$ or $u(x, 0) = +\infty$. This is a contradiction. Therefore, it must be the case that $u \geq 0$ for $|x| < 1$, $t > 0$. ■

PROBLEM 1.8.2 (Tikhonov's example). Let

$$g(t) := \begin{cases} \mathrm{e}^{-t^{-2}} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then $g \in C^\infty(\mathbb{R})$ and we define

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that $u(x,t)$ solves the heat equation in $\mathbb{R} \times (0, \infty)$ with the initial condition $u(x, 0) = 0$, $x \in \mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

Solution. Let u be as above. Then

$$\begin{aligned}
u_t(x,t) &= \frac{\partial}{\partial t}\left(\sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k)!}x^{2k}\right)\\
&= \sum_{k=0}^{\infty}\frac{g^{(k+1)}(t)}{(2k)!}x^{2k}\\
&= \sum_{k=2}^{\infty}\frac{g^{(k)}(t)}{(2k-2)!}x^{2k-2},
\end{aligned}$$

and

$$\begin{aligned}
u_x(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k)!}x^{2k}\right) & u_{xx}(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k-1)!}x^{2k-1}\right)\\
&= \sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k)!}2kx^{2k-1} & &= \sum_{k=1}^{\infty}\frac{g^{(k)}(t)}{(2k-1)!}(2k-1)x^{2k-2}+\frac{\partial}{\partial x}g^{(0)}(t)\\
&= \sum_{k=1}^{\infty}\frac{g^{(k)}(t)}{(2k-1)!}x^{2k-1}, & &= \sum_{k=2}^{\infty}\frac{g^{(k)}(t)}{(2k-2)!}x^{2k-2}.
\end{aligned}$$

Thus, $u_t - \Delta u = 0$; i.e., u solves the heat equation. As this example shows, unless some assumptions on u such as subexponential (cf. [E §2.3], Theorem 7) growth is assumed. ■

Problem 1.8.3. Evaluate the integral

$$\int_{-\infty}^{\infty}\cos(ax)\mathrm{e}^{-x^2}\,dx, \qquad (a>0).$$

Hint: Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.

Solution. By separation of variables,

$$u(x,t)=\cos(ax)\mathrm{e}^{-a^2t}$$

is a solution to the Cauchy problem

$$\begin{cases} u_t-\Delta u=0 & \text{in } \mathbb{R}\times(0,\infty),\\ u=\cos(ax) & \text{on } \mathbb{R}\times\{\,t=0\,\}. \end{cases}$$

However, the convolution

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\cos(ay)\mathrm{e}^{-\frac{|x-y|^2}{4t}}\,dy$$

is also a solution to the Cauchy problem. Now note that

$$\begin{aligned}\int_{-\infty}^{\infty} \cos(ay)\mathrm{e}^{-y^2}\, dy &= \sqrt{\pi} \cdot u(0, \tfrac{1}{4}) \\ &= \sqrt{\pi}\mathrm{e}^{-\frac{a^2}{4}}.\end{aligned}$$ ■

1.9 Homework 9

PROBLEM 1.9.1. (a) Show that for $n = 3$ the general solution to the wave equation $u_{tt} - \Delta u = 0$ with spherical symmetry about the origin has the form

$$u = \frac{1}{r}F(r+t) + \frac{1}{r}G(r-t), \quad r = |x|,$$

with suitable F and G.

(b) Show that the solution with initial data of the form

$$u(r,0) = 0, \quad u_t(r,0) = h(r)$$

(h is an even function of r) is given by

$$u = \frac{1}{2r}\int_{r-t}^{r+t} \rho h(\rho)\, d\rho.$$

SOLUTION. ∎

PROBLEM 1.9.2. Show that the solution $w(x_1, t)$ of the initial-value problem for the *Klein–Gordon equation*

$$\begin{cases} w_{tt} = w_{x_1 x_1} - \lambda^2 w, & \\ w(x_1, 0) = 0, & w_t(x_1, 0) = h(x_1) \end{cases} \tag{1.9.1}$$

is given by

$$w(x_1, t) = \frac{1}{2}\int_{x_1-t}^{x_1+t} J_0(\lambda s) h(y_1)\, dy_1.$$

Here $s^2 = t^2 - (x_1 - y_1)^2$, while J_0 denotes the Bessel function defined by

$$J_0(z) := \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(z \sin\theta)\, d\theta.$$

Hint: Descend to (1.9.1) from the two-dimensional wave equation satisfied by

$$u(x_1, x_2, t) = \cos(\lambda x_2) w(x_1, t).$$

SOLUTION. ∎

PROBLEM 1.9.3. Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u = g, \quad u_t = h & \text{on } \mathbb{R}^3 \times \{\, t = 0 \,\} \end{cases}$$

where g and h are smooth and have compact support. Show there exists a constant C such that

$$|u(x,t)| \le C t^{-1} \quad (x \in \mathbb{R}^3,\ t > 0).$$

SOLUTION. ∎

2 Exams

2.1 Midterm Practice Problems

Problem 2.1.1. Solve $u_{x_1}^2 + x_2 u_{x_2} = u$ with initial conditions $u(x_1, 1) = x_1^2/4 + 1$.

Solution. By inspection, we may suspect that $v(x_1, x_2) = x_1^2/4 + x_2$ is a solution to the PDE. It certainly satisfies the boundary condition. A routine calculation shows that v is in fact a solution to the PDE. Lucky guess!

More formally, let us solve this problem using the method of characteristics. First, write

$$F(p, z, x) = \left(p^1(s)\right)^2 + x^2(s)p^2(s) - z(s) = 0.$$

Then, the characteristic ODEs are

$$\begin{cases} \left(\dot{p}^1(s), \dot{p}^2(s)\right) = -(0, p^2(s)) + (p^1(s), p^2(s)) \\ \qquad\qquad\qquad = (p^1(s), 0), \\ \dot{z}(s) = (2p^1(s), x^2(s)) \cdot (p^1(s), p^2(s)) \\ \qquad\qquad\qquad = 2p^1(s)^2 + x^2(s)p^2(s), \\ \left(\dot{x}^1(s), \dot{x}^2(s)\right) = (2p^1(s), x^2(s)). \end{cases}$$

Now, for $(x^1(0), x^2(0)) = (x^0, 1)$, integrating the characteristics, we get

$$\begin{cases} \left(p^1(s), p^2(s)\right) = \left(p_0^1 \mathrm{e}^s, p_0^2\right), \\ \left(x^1(s), x^2(s)\right) = \left(2p_0^1 \mathrm{e}^s + x_0^1, x_0^2 \mathrm{e}^s\right), \\ z(s) = \dfrac{\left(x^0\right)^2}{4} \mathrm{e}^{2s} + p_0^2 \mathrm{e}^s + z^0 \end{cases}$$

Using the initial condition and the PDE, we find that

$$\begin{aligned} &p_0^1 = \frac{x^0}{2}, \quad p_0^2 = \frac{\left(x^0\right)^2}{4} + 1 - \frac{\left(x^0\right)^2}{4} = 1, \\ &x_0^1 = 0, \qquad x_0^2 = 1 \\ &z^0 = 0, \end{aligned}$$

and consequently

$$\begin{cases} \left(x^1(s), x^2(s)\right) = \left(x^0 \mathrm{e}^s, \mathrm{e}^s\right), \\ z(s) = \dfrac{\left(x^0\right)^2}{4} \mathrm{e}^{2s} + \mathrm{e}^s \end{cases}$$

so, rewriting z in terms of (x^1, x^2), we have

$$\begin{aligned} z(s) &= \frac{\left(x^0\right)^2}{4} \mathrm{e}^{2s} + \mathrm{e}^s \\ &= \frac{\left(x^1(s)\right)^2}{4} + x^2(s), \end{aligned}$$

so the solution in terms of (x_1, x_2), is

$$u(x_1, x_2) = \frac{x_1^2}{4} + x_2,$$

just as we suspected. ∎

PROBLEM 2.1.2. Find the maximal $t_0 > 0$ for which the (classical) solution of the Cauchy problem

$$\begin{cases} uu_x + u_t = 0, \\ \quad u(x,0) = \mathrm{e}^{-\frac{x^2}{2}}, \end{cases}$$

exists in $\mathbb{R} \times [0,t)$; i.e., the first time $t = t_0$ when the shock develops.

SOLUTION. First, let us find a solution to the PDE using the method of characteristics. Write

$$F(p, z, x) = z(s)p^1(s) + p^2(s).$$

Then, the characteristic ODEs are

$$\begin{cases} \big(\dot{p}^1(s), \dot{p}^2(s)\big) = -(0,0) - p^1(p^1(s), p^2(s)) \\ \qquad\qquad\qquad = \big(-p^1(s)^2, -p^1(s)p^2(s)\big), \\ \qquad\qquad \dot{z}(s) = (z(s), 1) \cdot (p^1(s), p^2(s)) \\ \qquad\qquad\qquad = z(s)p^1(s) + p^2(s) \\ \qquad\qquad\qquad = 0, \\ \quad \big(\dot{x}(s), \dot{t}(s)\big) = (z(s), 1). \end{cases}$$

Thus, integrating the characteristic ODEs from $(x^0, 0)$, we have

$$\begin{cases} \qquad \dot{z}(s) = z^0, \\ \big(x(s), t(s)\big) = (z^0 s + x^0, s); \end{cases}$$

since the PDE is quasilinear, we disregard (p^1, p^2).

Applying the boundary conditions, we see that

$$z^0 = u(x^0, 0) = \mathrm{e}^{-(x^0)^2/2}.$$

Here's where it gets tricky. After a little struggling, we see that there is really no way to solve for z in terms of $(x(s), t(s))$. However, we can solve for the projected characteristics:

$$(x(t,y), t) = (\mathrm{e}^{-y^2/2}t + y, t);$$

and this is really all that matters for us to find the time t_0 when the shock develops, i.e., the time when the projected characteristic fails to be injective.

A little calculation shows that this happens precisely when $t = \mathrm{e}^{-1/2}$. ∎

Problem 2.1.3. If ρ_0 denotes the maximum density of cars on a highway (i.e., under bumpet-to-bumper conditions), then a reasonable model for traffic density ρ is given by

$$\begin{cases} \rho_t + (F(\rho))_x = 0, \\ F(\rho) = c\rho\left(1 - \dfrac{\rho}{\rho_0}\right), \end{cases}$$

where $c > 0$ is a constant (free speed of highway). Suppose the initial density is

$$\rho(x,0) = \begin{cases} \frac{1}{2}\rho_0 & \text{if } x < 0, \\ \rho_0 & \text{if } x > 0. \end{cases}$$

Find the shock curve and describe the weak solution. (Interpret your result for the traffic flow.)

Solution. First, note that

$$\begin{aligned} (F(\rho))_x &= F'(\rho)\rho_x \\ &= \left[-c\frac{\rho}{\rho_0} + c\left(1 - \frac{\rho}{\rho_0}\right)\right]\rho_x \\ &= \left(c - \frac{2c\rho}{\rho_0}\right)\rho_x. \end{aligned}$$

Let us find a solution to the PDE using the method of characteristics. Write

$$G(p,z,x) = p^2(s) + F'(z(s))p^1(s).$$

Then, the characteristic ODEs are

$$\begin{cases} \big(\dot{p}^1(s), \dot{p}^2(s)\big) = \big(-F''(z(s))p^1(s), -F''(z(s))p^2(s)\big), \\ \dot{z}(s) = F'(z(s))p^1(s) + p^2(s) \\ \phantom{\dot{z}(s)} = 0, \\ \big(\dot{x}^1(s), \dot{x}^2(s)\big) = \big(F'(z(s)), 1\big). \end{cases}$$

Now, integrating the characteristics, we have

$$\begin{cases} z(s) = z^0, \\ \big(x^1(s), x^2(s)\big) = \big(F'(z^0)s + x^0, s\big). \end{cases}$$

We have two cases to consider, $x^0 < 0$ or $x^0 > 0$. For $x^0 < 0$, $z^0 = \frac{\rho_0}{2}$ and the projected characteristics look like

$$\begin{aligned} \left(F'(\frac{\rho_0}{2})t + x^0, t\right) &= \left(\left[c - \frac{2c(\frac{\rho_0}{2})}{\rho_0}\right]t + x^0, t\right) \\ &= (0 \cdot t + x^0, t) \\ &= (x^0, t) \end{aligned}$$

(where we have replaced s with the more appropriate t). Whereas for $x^0 > 0$, we have

$$\begin{aligned}\big(F'(\rho_0)t + x^0, t\big) &= \left(\left[c - \frac{2c\rho_0}{\rho_0}\right]t + x^0, t\right)\\ &= (-ct + x^0, t).\end{aligned}$$

These characteristics intersect precisely when

$$t = \frac{x_1^0 - x_2^0}{c},$$

where $x_1^0 > 0$, $x_2^0 < 0$. ■

PROBLEM 2.1.4. Find the characteristics of the second order equation

$$u_{xx} - (2\cos x)u_{xy} - (3 + \sin^2 x)u_{yy} - yu_y = 0,$$

and transform it to the canonical form.

SOLUTION. First, writing the PDE in the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + 2Du_x + 2Eu_y + Fu = 0,$$

we see that $A = 1$, $B = -\cos x$, $C = -3\sin^2 x$, and $E = -\frac{y}{2}$. We solve for the characteristic curve by find a solution to the ODEs

$$\begin{aligned}\frac{dy}{dx} &= \frac{B \pm \sqrt{B^2 - AC}}{A}\\ &= -\cos x \pm \sqrt{\cos^2 x + 3 + \sin^2 x}\\ &= -\cos x \pm 2.\end{aligned}$$

The solutions give us the following ODEs

$$\begin{cases} y = -\sin x + 2x + \xi(x, y),\\ y = -\sin x - 2x + \eta(x, y).\end{cases}$$

Integrating these equations, we have

$$\begin{cases} \xi(x, y) = y + \sin x - 2x,\\ \eta(x, y) = y + \sin x + 2x.\end{cases}$$

These are the characteristic strips for the PDE.

To put this PDE in canonical form, we first compute the following partial derivatives

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x,\\ u_y &= u_\xi \xi_y + u_\eta \eta_y,\\ u_{xx} &= u_\xi \xi_{xx} + u_\eta \eta_{xx} + (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + (u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x)\eta_x\\ &= u_{\xi\xi}(\xi_x)^2 + u_{\eta\eta}(\eta_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx},\end{aligned}$$

exploiting symmetry, we can find u_{yy} by replacing x with y above

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + u_{\eta\eta}(\eta_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_\xi\xi_{yy} + u_\eta\eta_{yy},$$

the last thing we need to figure out is the mixed partial

$$\begin{aligned} u_{xy} &= u_\xi\xi_{xy} + u_\eta\eta_{xy} + (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + (u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y)\eta_x \\ &= u_{\xi\xi}\xi_x\xi_y + u_{\eta\eta}\eta_x\eta_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_\xi\xi_{xy} + u_\eta\eta_{xy}. \end{aligned}$$

Now find the partials $\xi_x, \eta_x, \xi_y, \eta_y, \xi_{xy}, \ldots$, etc.

$$\begin{aligned} \xi_x &= \cos x - 2, & \eta_x &= \cos x + 2, \\ \xi_{xx} &= -\sin x, & \eta_{xx} &= -\sin x, \\ \xi_{xy} &= 0, & \eta_{xy} &= 0, \\ \xi_y &= 1, & \eta_y &= 1, \\ \xi_{yy} &= 0, & \eta_{yy} &= 0. \end{aligned}$$

Thus,

$$\begin{cases} u_x = (\cos x - 2)u_\xi + (\cos x + 2)u_\eta, \\ u_y = u_\xi + u_\eta, \\ u_{xx} = (\cos x - 2)^2 u_{\xi\xi} + (\cos x + 2)^2 u_{\eta\eta} \\ \qquad + 2(\cos x + 2)(\cos x - 2)u_{\xi\eta} - (\sin x)u_\xi - (\sin x)u_\eta \\ \quad = (\cos^2 x - 4\cos x + 4)u_{\xi\xi} + (\cos^2 x + 4\cos x + 4)u_{\eta\eta} \\ \qquad + 2(\cos^2 x - 4)u_{\xi\eta} - (\sin x)u_\xi - (\sin x)u_\eta \\ u_{yy} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}, \\ u_{xy} = (\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}, \end{cases}$$

so the canonical form is

$$\begin{aligned} 0 &= u_{xx} - (2\cos x)u_{xy} - (3\sin^2 x)u_{yy} - yu_y \\ &= \xi^2 u_{\xi\xi} + \eta^2 u_{\eta\eta} \\ &\quad + 2\xi\eta u_{\xi\eta} - (\sin x)u_\xi - (\sin x)u_\eta \\ &\quad - (2\cos x)\big((\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}\big) \\ &\quad - (3\sin^2 x)(u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}) \\ &\quad - y(u_\xi + u_\eta) \end{aligned}$$

Who cares. ■

PROBLEM 2.1.5. Let $Lu := u_{xx} - 4u_{yy} + \sin(y + 2x)u_x = 0$.

(a) Consider the level curve $\Gamma = \{\, (x, y) : \varphi(x, y) = C \,\}$ where $|D\varphi| \neq 0$ on Γ. Define what it means for Γ to be characteristic with respect to L at a point $(x_0, y_0) \in \Gamma$.
(b) Find the points at which the curve $x^2 + y^2 = 5$ is characteristic.

(c) Is it true that every smooth simple closed curve Γ in $\mathbb{R}^2$ has at least one point at which it is characteristic with respect to L?

Solution. ■

Problem 2.1.6. Consider the second order equation

$$Lu := u_{xx} - 2xu_{xy} + x^2 u_{yy} - 2u_y = 0.$$

(a) Find the characteristic curves of $Lu = 0$. What is the type of this equation?
(b) Find the points on the line $\Gamma := \{ (x, y) \in \mathbb{R}^2 : x + y = 1 \}$ at which Γ is characteristic with respect to $Lu = 0$.

Solution. ■

Problem 2.1.7. Solve the initial boundary value problem for the equation $u_{tt} = u_{xx}$ in $\{ x > 0, t > 0 \}$ satisfying

$$\begin{cases} u(x,0) = \sin^2 x, \quad u_t(x,0) = \sin x, \\ u(0,t) = 0. \end{cases}$$

Solution. ■

Problem 2.1.8. Consider the initial/boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } 0 < x < \pi,\ t > 0, \\ u(x,0) = x, \quad u_t(x,0) = 0 & \text{for } 0 < x < \pi, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0 & \text{for } t > 0. \end{cases}$$

(a) Find a weak solution of the problem.
(b) Is the solution unique? Continuous? C^1?

Solution. ■

Problem 2.1.9. Let B_1^+ denote the open half-ball $\{ x \in \mathbb{R}^n : |x| < 1, x_n > 0 \}$. Assume $u \in C(\bar{B}_1^+)$ is harmonic in B_1^+ with $u = 0$ on $\partial B_1^+ \cap \{ x_n = 0 \}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0, \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0, \end{cases}$$

for $x \in B_1$. Prove v is harmonic in B_1.

Hint: It will be enough to prove that $\int_B \nabla v \nabla \eta \, dx = 0$ for any test function $\eta \in C_0^\infty(B_1)$. Split $\int_{B_1} = \int_{B_1^+} + \int_{B_1^-}$ and apply the integration by parts formula to each of $\int_{B_1^\pm}$.

Solution. ∎

Problem 2.1.10. Let u and v be harmonic functions in the unit ball $B_1 \subseteq \mathbb{R}^n$. What can you conclude about u and v if

(a) $D^\alpha u(0) = D^\alpha v(0)$ for every multiindex α?
(b) $u(x) \leq v(x)$ for every $x \in B_1$ and $u(0) = v(0)$?

Justify your answer in each case.

Solution. ∎

Problem 2.1.11. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

is a solution to the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial, i.e., $f(y) = f(|y|)$, and supported in $B_R := \{\, |x| < R \,\}$, then

$$u(x) = c\Phi(x)$$

for any $x \in \mathbb{R}^n \setminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y)\, dy.$$

Hint: Use polar (spherical) coordinates and apply the mean value property for harmonic functions.

Solution. ∎

2.2 Final Practice Problems

PROBLEM 2.2.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Show that the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u + \alpha \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

has at most one solution in $C^2(\Omega) \cap C(\bar{\Omega})$ if $\alpha > 0$. Here ν is the outward normal on $\partial\Omega$ and f, g are assumed to be smooth.

SOLUTION. Let us assume that Ω is also a connected subset of $\mathbb{R}^n$. We will use energy methods to show that there is only one solution to the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u + \alpha \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases} \tag{2.2.1}$$

Suppose u_1 and u_2 are two distinct solutions to the problem (2.2.1). Define $v := u_1 - u_2$. Then v solves the problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ v + \alpha \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the energy

$$E[v] = \frac{1}{2} \int_\Omega |Dv|^2 \, dx.$$

By Green's formula, we may recast the expression above as the sum

$$\begin{aligned} E[v] &= -\frac{1}{2} \left[\int_\Omega v \Delta v \, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} v \, dS(x) \right] \\ &= -\frac{\alpha}{2} \int_{\partial\Omega} v^2 \, dS(x) \\ &\geq 0. \end{aligned}$$

However, since $\alpha > 0$ and v^2 is strictly positive, it must be the case that $v \equiv 0$ on $\partial\Omega$. The maximum principle then implies that $v \equiv 0$ in Ω. It follows that $u_1 = u_2$; i.e., the solution is unique. ■

PROBLEM 2.2.2. Let g be a continuous function with compact support in $\mathbb{R}^n$. Write the formula for the bounded solution of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{ t = 0 \}. \end{cases}$$

Prove that $\lim_{t\to\infty} u(x,t) = 0$, where the convergence is uniform in $x \in \mathbb{R}^n$.

Solution. From previous work on the heat equation, we know that the convolution

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

the initial-value problem above. A crude estimate on the magnitude of u gives us

$$\begin{aligned} |u(x,t)| &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left| \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|x-y|^2}{4t}} g(y)\, dy \right| \\ &\leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left| \mathrm{e}^{-\frac{|x-y|^2}{4t}} g(y) \right| dy \\ &< M t^{-\frac{n}{2}}; \end{aligned} \tag{2.2.2}$$

where $M < \infty$ is chosen such that

$$\frac{1}{(4\pi)^{\frac{n}{2}}} \int_{\operatorname{Supp} g} \left| \mathrm{e}^{-\frac{|x-y|^2}{4t}} g(y) \right| dy < M.$$

Thus, using the estimate (2.2.2) we see that $\lim_{t\to\infty} u(x,t) = 0$ uniformly. ∎

Problem 2.2.3. Find an explicit solution to the problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty), \\ u = \mathrm{e}^{3x} & \text{in } \mathbb{R} \times \{\, t = 0 \,\}. \end{cases}$$

Solution. By separation of variables, suppose we can write $u(x,t)$ as the product $X(x)T(t)$. Then

$$\begin{cases} X(x)T'(t) - X''(x)T(t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ X(x)T(0) = \mathrm{e}^{3x} & \text{on } \mathbb{R} \times \{\, t = 0 \,\}. \end{cases} \tag{2.2.3}$$

After some algebraic maneuvers, we see $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = 9$ so it suffices to solve the system of ODEs

$$\begin{cases} X''(x) - 9X(x) = 0, \\ \quad T'(t) - 9T(t) = 0. \end{cases}$$

The solution to these are

$$\begin{cases} X(x) = C_1 \mathrm{e}^{3x}, \\ \ T(t) = C_2 \mathrm{e}^{9t}. \end{cases}$$

Thus,

$$u(x,t) = X(x)T(t) = C\mathrm{e}^{3x+9t}$$

solves (2.2.3). Analyzing the initial conditions, we conclude that $C = 1$. In conclusion,

$$u(x,t) = \mathrm{e}^{3x+9t}$$

solves the original problem.

Another way to solve this problem is by computing the convolution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \mathrm{e}^{3x}\, dy.$$

Putting this through WolframAlpha gives

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \left[\sqrt{4\pi t} \mathrm{e}^{9t+3x} \right] = \mathrm{e}^{9t+3x}$$

which agrees with our 'separation of variables' solution. ■

PROBLEM 2.2.4. Find a formula for the solution of

$$\begin{cases} u_{tt} - u_{xx} + u = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = f, \quad u_t = g & \text{on } \mathbb{R} \times \{ t = 0 \} \end{cases}$$

where $f, g \in C_0^\infty(\mathbb{R})$.

Hint: Method I: Use Hadamard's method of descent. Namely, find $h(y)$ such that $v(x,y,t) := h(y)u(x,t)$ solves

$$v_{tt} - (v_{xx} + v_{yy}) = 0.$$

Method II: Use the Fourier transform.

SOLUTION. By Method I: Set $h(y) := \cos y$ and $v(x,y,t) := h(y)u(x,t)$. Then v solves the initial-value problem

$$\begin{cases} v_{tt} - (v_{xx} + v_{yy}) = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ v = \tilde{f}, \quad v_t = \tilde{g} & \text{on } \mathbb{R}^2 \times \{ t = 0 \} \end{cases}$$

where $\tilde{f} := hf$ and $\tilde{g} := hg$. The solution to this problem is given by the average integral

$$v(x,y,t) = \frac{1}{2\pi t^2} \iint_{B(x,y,t)} \left[\frac{(tf(\xi) + t^2 g(\xi))\cos\eta + tD((\cos\eta)f(\xi)) \cdot (\xi - x, \eta - y)}{(t^2 - (\xi - x)^2 - (\eta - y)^2)^{\frac{1}{2}}} \right] d\xi\, d\eta$$

Therefore, the equation

$$v(x,0,t) = \frac{1}{2\pi t^2} \iint_{B(x,0,t)} \left[\frac{(tf(\xi) + t^2 g(\xi))\cos\eta + t[f'(\xi)(\xi - x) - \eta\sin(\eta)]}{(t^2 - (\xi - x)^2 - \eta^2)^{\frac{1}{2}}} \right] d\xi\, d\eta$$

solves the original problem.

To simplify this, let us first compute the following integrals since the former integral is too large to work with directly,

$$I_1 = \frac{1}{2\pi t^2} \iint_{B(x,0,t)} \left[\frac{(tf(\xi) + t^2 g(\xi)) \cos\eta}{(t^2 - (\xi - x)^2 - \eta^2)^{\frac{1}{2}}} \right] d\xi \, d\eta,$$

$$I_2 = \frac{1}{2\pi t^2} \iint_{B(x,0,t)} \left[\frac{-t\eta \sin\eta}{(t^2 - (\xi - x)^2 - \eta^2)^{\frac{1}{2}}} \right] d\xi \, d\eta,$$

$$I_3 = \frac{1}{2\pi t^2} \iint_{B(x,0,t)} \left[\frac{tf'(\xi)(\xi - x)}{(t^2 - (\xi - x)^2 - \eta^2)^{\frac{1}{2}}} \right] d\xi \, d\eta.$$

Throughout the following analysis, set $s := \sqrt{t^2 - (\xi - x)^2}$. For I_1, we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi t^2} \int_{x-t}^{x+t} \left[\int_{-s}^{s} \frac{\cos\eta}{\sqrt{s^2 - \eta^2}} \, d\eta \right] (tf(\xi) + t^2 g(\xi)) \, d\xi \\ &= \frac{1}{2t} \int_{x-t}^{x+t} J_0(s)[f(\xi) + tg(\xi)] \, d\xi; \end{aligned} \tag{2.2.4}$$

for I_2, we have

$$\begin{aligned} I_2 &= \frac{1}{2\pi t^2} \int_{x-t}^{x+t} \left[\int_{-s}^{s} \frac{\eta \sin\eta}{\sqrt{s - \eta^2}} \, d\eta \right] (-t) \, d\xi \\ &= -\frac{1}{2t} \int_{x-t}^{x+t} s J_1(s) \, d\xi; \end{aligned} \tag{2.2.5}$$

and for I_3, we have

$$\begin{aligned} I_3 &= \frac{1}{2\pi t^2} \int_{x-t}^{x+t} \left[\int_{-s}^{s} \frac{1}{\sqrt{s^2 - \eta^2}} \, d\eta \right] tf'(\xi)(\xi - x) \, d\xi \\ &= \frac{1}{2t} \int_{x-t}^{x+t} f'(\xi)(\xi - x) \, d\xi. \end{aligned} \tag{2.2.6}$$

Putting (2.2.4), (2.2.5), and (2.2.6) together, we have

$$u(x,t) = \frac{1}{2t} \int_{x-t}^{x+t} [J_0(s)(f(\xi) + tg(\xi)) + f'(\xi)(\xi - x) - sJ_1(s)] \, d\xi$$

which solves the initial-value problem in question. ■

Problem 2.2.5. Let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ satisfy

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{in } \mathbb{R}^n \times \{\, t = 0 \,\}. \end{cases}$$

Show that if both g and h are radial, then so is $u(\,\cdot\,,t)$ for any $t > 0$. (Recall that the function f is called radial if $f(x) = f(|x|)$.)

Solution. Let $O \in \mathrm{SO}(n)$ be a rotation matrix. Set $v(x) := u(Ox)$. Then v solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ v(x,0) = g(Ox) = g(x), \quad v_t(x,0) = h(Ox) = h(x) & \text{in } \mathbb{R}^n \times \{\, t = 0 \,\}. \end{cases}$$

By the uniqueness for the wave equation there exist at most one C^2 solution to the initial-value problem above. Thus, it must be the case that $v = u$; i.e., $u(x) = u(|x|)$ since $O \in \mathrm{SO}(3)$ was arbitrary. ■

Problem 2.2.6. Find the value of the solution u of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{for } x \in \mathbb{R}^3,\ t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = \varphi(x), \end{cases}$$

where

$$\varphi(x) := \begin{cases} 1 & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a \end{cases}$$

at a point (x,t) such that $|x| + t < a$.

Solution. By Kirchhoff's formula, to solution to this initial-value problem is given by

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t\varphi(y)\, dS(y).$$

Then, since $\varphi(y) \equiv 1$ for $|y| = |x| + t < a$, the integral above becomes

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t\, dS(y) = t. \qquad ■$$

Problem 2.2.7. Let u be a nonzero harmonic function in $B(0,R) := \{\, x \in \mathbb{R}^n : |x| < R \,\}$. Define

$$E(r) := \fint_{\partial B(0,r)} u^2(y)\, dS(y).$$

Show that $\ln E(r)$ is a convex function of $\ln r$; i.e.,

$$E\big(\sqrt{ab}\big)^2 \leq E(a)E(b), \quad \text{for } a, b > 0,$$

for any $0 < a \leq c < R$.

Hint: Use the representation of u as a uniformly convergent series

$$u(x) = \sum_{k=0}^{\infty} p_k(x), \qquad |x| < R,$$

where $p_k(x)$ is a homogeneous harmonic polynomial of order k.

Solution. Write
$$u(x) = \lim_{n\to\infty} \sum_{k=0}^{n} p_k(x), \qquad |x| < R,$$
where $p_k(x)$ is a homogeneous harmonic polynomial of order k; the limit converges uniformly. Then
$$E(r) = ⨍_{\partial B(0,r)} \Big[\lim_{n\to\infty} \sum\nolimits_{k=0}^{n} p_k(x)\Big]^2 \, dS(y)$$
expand the sum by the multinomial theorem
$$= ⨍_{\partial B(0,r)} \lim_{n\to\infty} \left[\sum_{k_0+\cdots+k_n=2} \binom{2}{k_0,\ldots,k_n} p_0(x)^{k_0} \cdots p_n(x)^{k_n} \right] dS(y)$$
since the limit is uniform, we may interchange the limit with the integral
$$= \lim_{n\to\infty} \left[⨍_{\partial B(0,r)} \sum_{k_0+\cdots+k_n=2} \binom{2}{k_0,\ldots,k_n} p_0(x)^{k_0} \cdots p_n(x)^{k_n} \, dS(y) \right]$$
$$= \lim_{n\to\infty} ⨍_{\partial B(0,r)} p_0(x)^2 + \cdots + p_n(x)^2 \, dS(y)$$
since harmonic polynomials of distinct degrees are orthogonal
$$= ⨍_{\partial B(0,r)} \Big[\sum\nolimits_{k=0}^{\infty} p_k(x)^2 \Big] \, dS(y);$$
i.e.,
$$E(r) = ⨍_{\partial B(0,r)} \sum_{k=0}^{\infty} p_k(x)^2 \, dS(y) \tag{2.2.7}$$
for $r < R$.

Now, applying the Cauchy–Schwartz to (2.2.7) with $r = \sqrt{ab}$ we achieve the desired inequality. ∎

Problem 2.2.8. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution to the following Cauchy problem,
$$\begin{cases} u_{tt} - \Delta u = \mathrm{e}^{-t} f(x) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u = u_t = 0, & \text{on } \mathbb{R}^3 \times \{\, t = 0 \,\}. \end{cases}$$
Verify that the integral representation reduces to the obvious solution $u = \mathrm{e}^{-t} + t - 1$ when $f(x) = 1$.

Solution. Proceeding by Duhamel's principle, define $v := u(x,t;s)$. Then v is a solution of
$$\begin{cases} v_{tt} - \Delta v = \mathrm{e}^{-t} f(x) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ v = 0, \quad v_t(\,\cdot\,; s) = \mathrm{e}^{-s} f(\,\cdot\,), & \text{on } \mathbb{R}^3 \times \{\, t = 0 \,\}. \end{cases} \tag{2.2.8}$$

By Kirchhoff's formula,

$$
\begin{aligned}
v(x,t;s) &= \fint_{\partial B(x,t)} t\mathrm{e}^{-s} f(y)\, dS(y) \\
&= \frac{\mathrm{e}^{-s}}{4\pi t} \int_{\partial B(x,t)} f(y)\, dS(y) \\
&= \frac{\mathrm{e}^{-s}}{4\pi(t-s)} \int_{\partial B(x,t-s)} f(y)\, dS(y)
\end{aligned}
$$

solves (2.2.8).

Then,

$$
\begin{aligned}
u(x,t) &= \int_0^t \left[\int_{\partial B(x,t-s)} f(y)\, dS(y) \right] \left(\frac{\mathrm{e}^{-s}}{4\pi(t-s)} \right) ds \\
&= \int_0^t \left[\int_{\partial B(x,t-s)} \left(\frac{f(y)}{t-s} \right) dS(y) \right] \left(\frac{\mathrm{e}^{-s}}{4\pi} \right) ds \qquad (2.2.9) \\
&= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,r)} \frac{\mathrm{e}^{t-r} f(y)}{r}\, dS(y)\, dr
\end{aligned}
$$

solves the original problem.

In the case $f(x) = 1$, (2.2.9) becomes

$$
\begin{aligned}
u(x,t) &= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,r)} \frac{\mathrm{e}^{t-r}}{r}\, dS(y)\, dr \\
&= \frac{1}{4\pi} \int_0^t \left[\int_{\partial B(x,r)} dS(y) \right] \frac{\mathrm{e}^{-r}}{r}\, dr \\
&= \frac{1}{4\pi} \int_0^t 4\pi r^2 \left(\frac{\mathrm{e}^{-r}}{r} \right) dr \\
&= \int_0^t r\mathrm{e}^{-r}\, dr \\
&= \mathrm{e}^{-t} + t - 1.
\end{aligned}
$$

■

PROBLEM 2.2.9. Let $f(x) = \mathrm{e}^{-|x|^2}$, $x \in \mathbb{R}^n$. Find $f * f$.

Hint: Use either the heat equation or the Fourier transform.

SOLUTION. First we proceed by the heat equation. Suppose u is a solution to the initial-value problem

$$
\begin{cases}
u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\
u = f & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\}
\end{cases} \qquad (2.2.10)
$$

where $f(x) := \mathrm{e}^{-|x|^2}$. Then

$$
u(x,t) = f(x) * \Phi(x,t),
$$

where Φ is the fundamental solution to the heat equation, solves (2.2.10). But

$$\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|x|^2}{4t}} = \frac{1}{(4\pi t)^{\frac{n}{2}}} [f(x)]^{\frac{1}{4t}}.$$

Therefore, the convolution we are after is precisely

$$(f * f)(x) = \pi^{\frac{n}{2}} u(x, \tfrac{1}{4}).$$

Solving for $u(x, \frac{1}{4})$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-|x-y|^2} f(y)\, dy \\ &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-|x-y|^2} \mathrm{e}^{-|y|^2}\, dy \\ &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-|x-y|^2 - |y|^2}\, dy \end{aligned}$$

which, by Fubini's theorem, becomes the product of integrals in one coordinate

$$\begin{aligned} &= \frac{1}{\pi^{\frac{n}{2}}} \prod_{k=1}^{n} \left[\int_{\mathbb{R}} \mathrm{e}^{-|x_k - y_k|^2 - |y_k|^2}\, dy_k \right] \\ &= \frac{1}{\pi^{\frac{n}{2}}} \prod_{k=1}^{n} \left[\int_{\mathbb{R}} \mathrm{e}^{-(x_k^2 - 2x_k y_k + y_k^2) - y_k^2}\, dy_k \right] \\ &= \frac{1}{\pi^{\frac{n}{2}}} \mathrm{e}^{-|x|^2} \prod_{k=1}^{n} \underbrace{\left[\int_{\mathbb{R}} \mathrm{e}^{2x_k y_k - 2y_k^2}\, dy_k \right]}_{I_k}. \end{aligned}$$

Let us find I_k and complete the solution of u above. This is,

$$\begin{aligned} I_k &= \int_{\mathbb{R}} \mathrm{e}^{-2\left(y_k - \frac{1}{2}x_k\right)^2 + \frac{1}{2}x_k^2}\, dy_k \\ &= \frac{\mathrm{e}^{\frac{1}{2}x_k^2}}{2} \int_{\mathbb{R}} \mathrm{e}^{-z^2}\, dz \\ &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}x_k^2}. \end{aligned}$$

Thus, $u(x, \frac{1}{4})$ is

$$u(x, \tfrac{1}{4}) = \frac{1}{2^{\frac{n}{2}}} \mathrm{e}^{-\frac{|x|^2}{2}}$$

so

$$f * f = \left(\frac{\pi}{2}\right)^{\frac{n}{2}} \mathrm{e}^{-\frac{|x|^2}{2}}. \qquad \blacksquare$$

Remarks. We still had to compute the convolution $f * f$ barehanded. Realizing it as the solution to the heat equation was of no help. Perhaps finding a solution through separation of variables is supposed to make this problem easier.

PROBLEM 2.2.10. Recall that a solution to the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dt,$$

where, for $t > 0$,

$$\Phi(z,t) = (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x|^2}{4t}}.$$

Assume that g is continuous and compactly supported. Show that there exists a $C > 0$ such that

$$|Du(x,t)| \leq Ct^{-\frac{1}{2}} \|g\|_{L^\infty}.$$

SOLUTION. Let us immediately jump to the partial derivative D_{x_k} of u,

$$D_{x_j} u = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(y) \left[-\frac{(x_j - y_j)}{2t} \right] \mathrm{e}^{-\frac{-|x-y|^2}{4t}}\,dy.$$

Hence,

$$\begin{aligned} |D_x u(x,t)| &\leq \frac{\|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi)^{\frac{n}{2}}} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|x-y|}{2t} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\,dy \\ &= \frac{\|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi)^{\frac{n}{2}}} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|z|}{2t} \mathrm{e}^{-\frac{|z|^2}{4t}}\,dz \end{aligned}$$

setting $w = |z|/\sqrt{t}$, we have

$$\begin{aligned} &= \frac{\|g\|_{L^\infty(\mathbb{R}^n)}}{2(4\pi)^{\frac{n}{2}}} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|w|}{t^{\frac{1}{2}}} \mathrm{e}^{-\frac{w^2}{4}} t^{\frac{n}{2}}\,dw \\ &= \frac{\|g\|_{L^\infty(\mathbb{R}^n)}}{2(4\pi)^{\frac{n}{2}}} \frac{1}{\sqrt{t}} \int_{\mathbb{R}^n} |w| \mathrm{e}^{-\frac{w^2}{4}}\,dw \\ &= \frac{\|g\|_{L^\infty(\mathbb{R}^n)} C_n}{\sqrt{t}}, \end{aligned}$$

where

$$C_n := \frac{1}{2(4\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |w| \mathrm{e}^{-\frac{w^2}{4}}\,dw > 0. \qquad \blacksquare$$

3 Qualifying Exams

3.1 Qualifying Exam, August '04

PROBLEM 3.1.1. Consider the initial value problem

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = -u, \\ \qquad\qquad\qquad u = f \qquad \text{on } S^1 = \{\, x^2+y^2=1 \,\}, \end{cases}$$

where a and b satisfy

$$a(x,y) + b(x,y)y > 0$$

for any $x, y \in \mathbb{R}^n \setminus \{(0,0)\}$.

(a) Show that the initial value problem has a unique solution in a neighborhood of S^1. Assume that a, b, and f are smooth.
(b) Show that the solution of the initial value problem actually exists in $\mathbb{R}^2 \setminus \{(0,0)\}$.

SOLUTION. ■

PROBLEM 3.1.2. Let $u \in C^2\big(\mathbb{R} \times [0,\infty)\big)$ be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty), \\ \qquad u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where f and g have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx,$$

and the potential energy by

$$P(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx.$$

Show that

(a) $K(t) + P(t)$ is constant in t,
(b) $K(t) = P(t)$ for all large enough times t.

SOLUTION. ■

PROBLEM 3.1.3. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = \mathrm{e}^{-t} g(x) & \text{for } x \in \mathbb{R}^3,\ t > 0, \\ \quad u(x,0) = u_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution $u = \mathrm{e}^{-t} + t - 1$ when $g(x) = 1$.

Solution. ■

Problem 3.1.4. Let Ω be a bounded open set in $\mathbb{R}^n$ and $g \in C_0^\infty(\Omega)$. Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega,\ t > 0, \\ u(x,0) = g(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } xi \in \partial\Omega,\ t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n,\ t > 0, \\ v(x,0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put $g = 0$ outside Ω.

(a) Show that

$$-v(x,t) \leq u(x,t) \leq v(x,t)$$

for any $x \in \Omega$, $t > 0$.

(b) Use (a) to conclude that

$$\lim_{t\to\infty} u(x,t) = 0,$$

for any $x \in \Omega$.

Solution. ■

Problem 3.1.5. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \qquad P_m(\lambda x) = \lambda^m P_m(x),$$

for any $x \in \mathbb{R}^n$, $\lambda > 0$,

$$\Delta P_k = 0, \qquad \Delta P_m = 0$$

in $\mathbb{R}^n$.

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = kP_k(x), \qquad \frac{\partial P_m(x)}{\partial \nu} = mP_m(x)$$

on ∂B_1, where $B_1 = \{\, |x| < 1 \,\}$ and ν is the outward normal on ∂B_1.

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x)\, dS = 0,$$

if $k \neq m$.

Solution. ■

3.2 Qualifying Exam, August ‘05

PROBLEM 3.2.1.

(a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ \qquad\quad u = 1 \qquad \text{on } S^1 = \{\, x^2 + y^2 = 1 \,\}. \end{cases}$$

(b) Is the solution unique in a neighborhood of the point $(1,0)$? Justify your answer.

SOLUTION. The solution to teh first part is

$$u(x,y) = \frac{x^2 + y^2}{4} + \frac{3}{4}.$$

■

PROBLEM 3.2.2. Consider the second order PDE in $\{\, x > 0, y > 0 \,\} \subseteq \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

(a) Classify the equation and reduce it to the canonical form.
(b) Show that the general solution of the equation is given by the formula

$$u(x,y) = F(x,y) + \sqrt{xy}\, G(x/y).$$

SOLUTION. ■

PROBLEM 3.2.3. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^3$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial (i.e., $f(y) = f(|y|)$) and supported in $B_R = \{\, |x| < R \,\}$, then

$$u(x) = c\Phi(x),$$

for any $x \in \mathbb{R}^n \smallsetminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y)\, dy.$$

[*Hint:* Use spherical (polar) coordinates and the mean value property.]

SOLUTION. ■

Problem 3.2.4. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \qquad u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where $g, h \in C_0^\infty(\mathbb{R}^2)$ and $\alpha \geq 0$ is a constant.

(a) Show that for an appropriate choice of constant λ and μ the function

$$v(x_1, x_2, x_3, t) := \mathrm{e}^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation $v_{tt} - \Delta v = 0$.

(b) Use (a) to prove the following domain of dependence result: for any point $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$ the value $u(x_0, t_0)$ is uniquely determined by values of g and h in $\overline{B_{t_0}(x_0)} := \{\, |x - x_0| \leq t_0 \,\}$. (You may use the corresponding result for the wave equation without proof.)

Solution. ∎

Problem 3.2.5. Let $u(x,t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0, \infty)$ with the initial condition

$$u(x,0) = u_0(x)$$

for $x \in \mathbb{R}$, where $u_0 \in C^\infty$ is 2π-periodic, i.e., $u_0(x + 2\pi) = u_0(x)$. Show that

$$\lim_{t \to \infty} u(x,t) = a_0,$$

uniformly in $x \in \mathbb{R}$, where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x)\, dx.$$

Solution. ∎

3.3 Qualifying Exam, January ‘14

PROBLEM 3.3.1. Consider the first order equation in $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

(a) Find the characteristic curves of the equation.
(b) Consider the Cauchy problem for this equation prescribed on the line $x_1 = 1$:

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on f so that the proble is solvable in a neighborhood of the point $(1, 0)$.

SOLUTION. ■

PROBLEM 3.3.2. Let u be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{ x_n > 0 \}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where g is a continuous function with compact support in $\mathbb{R}^{n-1}$. Here $n \geq 2$. Prove that

$$u(x) \longrightarrow 0, \qquad \text{as } |x| \longrightarrow \infty,$$

for $x \in \{ x_n > 0 \}$.

SOLUTION. ■

PROBLEM 3.3.3. Let u be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \qquad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions $u(x, 0) = g(x)$, where g is a bounded continuous function on $\mathbb{R}$ satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^\alpha, \qquad x, y \in \mathbb{R}$$

with a constant $\alpha \in (0, 1]$. Show that

$$\begin{aligned} |u(x,t) - u(y,t)| &\leq M|x - y|^\alpha, & x, y \in \mathbb{R}, t > 0, \\ |u(x,t) - u(x,s)| &\leq C_\alpha M|t - s|^{\alpha/2}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of $u(x,t)$ as a convolution with the heat kernel $\Phi(y,t)$, make a change of variables $z = y/\sqrt{t}$ and use that $\left|\sqrt{t} - \sqrt{s}\right| \leq \sqrt{|t - s|}$.]

SOLUTION. ■

PROBLEM 3.3.4. Let u be a positive harmonic function in the unit ball B_1 in $\mathbb{R}^n$. Show that

$$\bigl|D(\ln u)\bigr| \le M \qquad \text{in } B_{1/2}$$

for a constant M depending only on the dimension n.
[*Hint:* Use the interior derivative estimate $\bigl|Du(x)\bigr| \le (C_n/r)\sup_{B_r(x)}|u|$ for $B_r(x) \subseteq B_1$ as well as the Harnack inequality for harmonic functions.]

SOLUTION. ■

PROBLEM 3.3.5. Let u be a C^2 solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0,\infty), \\ \qquad u = 0, \qquad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some $k \ge 0$. Prove that there exists a function $\varphi(r)$ such that

$$u(x,t) = t^{k+2}\varphi\bigl(|x|/t\bigr).$$

[*Hint:* As one of the steps show that u is $(k+2)$-homogeneous in (x,t) variables, i.e., $u(\lambda x, \lambda t) = \lambda^{k+2}u(x,t)$ for any $\lambda > 0$.]

SOLUTION. ■