MA 519: Homework 3

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Problem 3.1 (Handout 3, # 3)

n sticks are broken into one short and one long part. The $2n$ parts are then randomly paired up to form n new sticks. Find the probability that

(a) the parts are joined in their original order, i.e., the new sticks are the same as the old sticks;
(b) each long part is paired up with a short part.

Solution. For part (a): let A_i denote the event that at the ith we choose a pair of sticks we get pair that joins up to one of the original sticks. We are after the probability of A the event that all the parts are joined in their original order, i.e., $A = \bigcap_{i=1}^{n} A_i$. First, we find $P(A_i)$. There are $2n$ ways of choosing a part of a stick and, once we have made that choice, only one way of choosing the original complement to it. The probability of making this choice on our first try is

$$P(A_1) = \frac{2n}{2n(2n-1)} = \frac{1}{2n-1}.$$

Now, assuming that the event of choosing a part of a stick and its original complement is independent from our other choices, we have $2n-2$ choices for our next stick and only one way to chose its complement. Therefore, the probability of making this choice is

$$P(A_2) = \frac{2n-2}{(2n-2)(2n-3)} = \frac{1}{2n-3}.$$

Proceeding in this way we see that at the ith step, the probability of choosing a two broken sticks that make up an original stick is

$$P(A_i) = \frac{1}{2(n-i+1)-1}.$$

Thus, by the hierarchical multiplicative formula the probability that the sticks are paired in their original order is

$$P(A) = \left(\frac{1}{2n-1}\right)\left(\frac{1}{2n-3}\right)\cdots\left(\frac{1}{3}\right)\left(\frac{1}{1}\right).$$

For part (b): let A_i denote the event that at the ith step we pair a long stick with a short stick. Then, to find the probability of $A = \bigcap_{i=1}^{n} A_i$, we first find the probabilities of the A_i. There are $2n$ ways to choose the first stick and n ways to chose either a long or a short part. Thus, the probability of choosing a long and a short part on the first try is

$$P(A_1) = \frac{2n \cdot n}{2n(2n-1)} = \frac{n}{2n-1}.$$

As in part (a), at the ith step, the probability of choosing a long and a short stick together is

$$P(A_i) = \frac{n-i+1}{n} 2(n-i+1)-1.$$

Thus, the probability of event A, that each long and short stick is paired together, is

$$P(A) = \left(\frac{n}{2n-1}\right)\left(\frac{n-1}{2n-3}\right)\cdots\left(\frac{2}{3}\right)\left(\frac{1}{1}\right).$$

■

Problem 3.2 (Handout 3, # 5)

In a town, there are 3 plumbers. On a certain day, 4 residents need a plumber and they each call one plumber at random.

Solution. ∎

Problem 3.3 (Handout 4, # 7)

(*Polygraphs*). Polygraphs are routinely administered to job applicants for sensitive government positions. Suppose someone actually lying fails the polygraph 90% of the time. But someone telling the truth also fails the polygraph 15% of the time. If a polygraph indicates that an applicant is lying, what is the probability that he is in fact telling the truth? Assume a general prior probability p that the person is telling the truth.

Solution. By the total probability formula, we have

$$P(C) = P(C|A)P(A) + P(A)$$

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Problem 3.4 (Handout 4, # 8)

In a bolt factory machines A, B, C manufacture, respectively, 25, 35, and 40 per cent of the total. Of their output 5, 4, and 2 per cent are defective bolts. A bolt is drawn at random from the produce and is found defective. What are the probabilities that it was manufactured by machines A, B, C?

Solution. ■

Problem 3.5 (Handout 4, # 9)

Suppose that 5 men out of 100 and 25 women out of 10 000 are colorblind. A colorblind person is chosen at random. What is the probability of his being male? (Assume males and females to be in equal numbers.)

Solution. ∎

Problem 3.6 (Handout 4, # 10)

(*Bridge*). In a bridge party West has no ace. What probability should he attribute to the event of his partner having

(a) no ace,
(b) two or more aces?

Verify the result by a direct argument.

Solution. ■

Problem 3.7 (Handout 4, # 12)

A true-false question will be posed to a couple on a game show. The husband and the wife each has a probability p of picking the correct answer. Should they decide to let one of the answer the question, or decide that they will give the common answer if they agree and toss a coin to pick the answer if they disagree?

Solution. ■

Problem 3.8 (Handout 4, # 13)

An urn containing 5 balls has been filled up by taking 5 balls at random from a second urn which originally had 5 black and 5 white balls. A ball is chosen at random from the first urn and is found to be black. What is the probability of drawing a white ball if a second ball is chosen from among the remaining 4 balls in the first urn?

Solution. We shall use the total probability formula to figure out the probabilities in question. Let B denote the event that that on our second drawing we draw a white ball. First, we must find a suitable partition $A_1, \dots, A_n$ of Ω such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, for $1 \leq i, j \leq n$, i.e., the events A_i are mutually exclusive. Consider the following partition of Ω,

$$A_i = \left\{ \text{exactly } i \text{ white balls are put into the second urn} \right\}.$$

The events A_i, $1 \leq i \leq 4$, are clearly mutually exclusive (if we have exactly i white balls in the urn, we cannot simultaneously have j white balls in the urn for $i \neq j$). Therefore, to find $P(A)$, we need only find the probabilities $P(B|A_i)$ and $P(A_i)$.

The probabilities of A_i are easy to calculate: there are $\binom{10}{5} = 252$ ways to chose 5 balls from the urn containing the 5 white and 5 black balls, and the number of ways of choosing exactly i black balls are $\binom{5}{5-i}$. Thus,

$$P(A_i) = \frac{\binom{5}{i}}{252} = \binom{5}{i}^2 \Big/ 252.$$

The probabilities of B given A_i are also easy to calculate

$$P(B|A_i) = \frac{i}{4},$$

since we have removed one ball and it was not white and there are i white balls remaining.

Thus, by the total probability formula,

$$\begin{aligned} P(B) &= \left(\frac{1}{4}\right)\left(\frac{25}{252}\right) + \left(\frac{2}{4}\right)\left(\frac{100}{252}\right) \\ &\quad + \left(\frac{3}{4}\right)\left(\frac{100}{252}\right) + \left(\frac{4}{4}\right)\left(\frac{25}{252}\right) \\ &\approx 0.620\,039\,682\,539\,682\,4. \end{aligned}$$

■

Problem 3.9 (Handout 4, # 15)

Events A, B, C have probabilities p_1, p_2, p_3. Given that exactly two of the three events occured, the probability that C occured is greater than 1/2 if and only if ... (write down the necessary and sufficient condition).

Solution. For convenience, let D denote the event that two of A, B, C occured. By Bayes' theorem, we have

$$\begin{aligned}
P(C|D) &= \frac{P(D|C)P(C)}{P(D)} \\
&= \frac{P(D|C)P(C)}{P(D|C)P(C) + P(D|\neg C)P(\neg C)} \\
&= \frac{P\big((A \cup B) \setminus (A \cap B)\big)p_3}{P\big((A \cup B) \setminus (A \cap B)\big)p_3 + P\big((A \cup B) \setminus C\big)(1 - p_3)},
\end{aligned}$$

but by the inclusion-exclusion principle, $P\big((A \cup B) \setminus (A \cap B)\big) = P(A) + P(B) - 2P(A \cap B)$ so, letting $x = P(A \cap B)$, the above becomes

$$\begin{aligned}
&= \frac{(p_1 + p_2 - 2x)p_3}{(p_1 + p_2 - 2x)p_3 + x(1 - p_3)} \\
&= \frac{p_1p_3 + p_2p_3 - 2xp_3}{p_1p_3 + p_2p_3 + x - 3xp_3} \\
&\geq \frac{1}{2}
\end{aligned}$$

if and only if

$$2(p_1p_3 + p_2p_3 - 2xp_3) \geq p_1p_3 + p_2p_3 + x - 3xp_3,$$

that is,

$$p_1p_3 + p_2p_3 \geq x + xp_3.$$

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Problem 3.10 (Handout 5, # 1)

There are five coins on a desk: 2 are double-headed, 2 are double-tailed, and 1 is a normal coin.
One of the coins is selected at random and tossed. It shows heads.
What is the probability that the other side of this coin is a tail?

Solution. Let A denote the event that the other side of a coin is tail and let B denote the event that after picking a coin at random and tossing it, it comes up heads. By Bayes' theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)};$$

where $P(B|A) = 1/3$ since there are 3 coins whose backside is tail, but only one of which can come up heads; $P(A) = 3/5$ since 3 out of the 5 coins have a tail; and $P(B) = 1/2$ since $1 + 4 = 5$ of the 10 faces are heads. Thus,

$$P(A|B) = \frac{(1/3)(3/5)}{1/2} = \frac{2}{5} = 0.4.$$

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Problem 3.11 (Handout 5, # 2)

(*Genetic testing*). There is a 50-50 chance that the Queen carries the gene for hemophilia. If she does, then each Prince has a 50-50 chance of carrying it. Three Princesses were recently tested and found to be non-carriers. Find the following probabilities:

(a) that the Queen is a carrier;
(b) that the fourth Princess is a carrier.

Solution. For part (a): let A denote the event that the Queen has hæmophilia and let B be the event that three princesses were tested and found to be non-carriers. Then, by Bayes' theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

By assumption, $P(A) = 1/2$. We cannot calculate $P(B)$ directly but, by the total probability formula,

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|\neg A)P(\neg A) \\ &= \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) \\ &= \frac{9}{16}. \end{aligned}$$

Thus,

$$P(A|B) = \frac{(1/8)(1/2)}{9/16} = \frac{1}{9} \approx 0.1111.$$

For part (b): let A denote the event that the fourth princess is a carrier and B remain as above, i.e., the event that three princesses were found to be non-carriers. By Bayes' theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

If the fourth princess is a carrier then the Queen is a carrier. Thus, $P(B) = 9/16$ and $P(B|A) = 1/8$ as above. Now, $P(A) = P(A|C)P(C) + P(A|\neg C)P(\neg C)$ where C denotes the event that the Queen is a carrier. Thus,

$$P(A) = \left(\frac{1}{2}\right)\frac{1}{2} = \frac{1}{4}.$$

Thus,

$$P(A|B) = \frac{(1/8)(1/4)}{9/16} = \frac{1}{18} \approx 0.0556.$$

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Problem 3.12 (Handout 5, # 4)

(*Is Johnny in Jail*). Johnny and you are roommates. You are a terrific student and spend Friday evenings drowned in books. Johnny always goes out on Friday evenings. 40% of the times, he goes out with his girlfriend, and 60% of the times he goes to a bar. If he goes out with his girlfriend, 30% of the times he is just too lazy to come back and spends the night at hers. If he goes to a bar, 40% of the times he gets mad at the person sitting on his right, beats him up, and goes to jail. On one Saturday morning, you wake up to see Johnny is missing. Where is Johnny?

Solution. Let A denote the event that Johnny is in jail and B denote the event that Johnny is missing. Then, by Bayes' theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Now, let C denote the event that Johnny went to the bar. Then, by the total probability formula,

$$P(A) = P(A|C)P(C) + P(A|\neg C)P(\neg C) = (0.4)(0.4) + 0 = 0.16.$$

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