MA 523: Homework 1

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Problem 1.1 (Taylor's formula)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to \mathbf{0}$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. ▸ Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Proof of claim. We shall prove this by induction on m. The case $m = 1$ is straightforward by the chain rule

$$\begin{aligned}
\frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\
&= D^{(1,0,\ldots,0)} f(tx) x_1 + \cdots + D^{(0,\ldots,0,1)} f(tx) x_n \\
&= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^\alpha f(tx) x^\alpha.
\end{aligned}$$

Now, assume this for $m - 1$ and consider the m-th derivative of g we have

$$\begin{aligned}
\frac{d^m}{dt^m} g(t) &= \frac{d}{dt} \left[\frac{d^{m-1}}{dt^{m-1}} g(tx) \right] \\
&= \frac{d}{dt} \left[\sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} D^\alpha f(tx) x^\alpha \right] \\
&= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \frac{d}{dt} \left(\left[D^\alpha f(tx) x^\alpha \right] \right) \\
&= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \left[\sum_{|\beta|=1} \frac{1!}{\beta!} \left[D^\beta D^\alpha f(tx) x^{\alpha+\beta} + D^\alpha f(tx) D^\beta x^\alpha \right] \right] \\
&= \\
&= \sum_{|\alpha+\beta|=m} \frac{(m-1)!1!}{\alpha!\beta!} D^{\alpha+\beta} f(tx) x^{\alpha+\beta}
\end{aligned}$$

now, note that $(\alpha+\beta)! = \prod_{i=0}^{\beta_1}(\alpha_1+\beta_1-i)\cdots\prod_{i=0}^{\beta_n}(\alpha_n+\beta_n-i)\alpha! =$

$=$

$\square$

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Problem 1.2

Write down the characteristic equation for the p.d.e.

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ▸

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Problem 1.3

Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ▸

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Problem 1.4

For the equation
$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\left(u_{x_1}^2 + u_{x_2}^2\right)$$
find a solution with $u(x_1, 0) = \left(1 - x_1^2\right)/2$.

Solution. ►

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