MA161Lesson Plan MicroTeachingSession

Carlos Salinas

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1 Indeterminate Forms and L'Hospital's Rule

Limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ is called an *indeterminate form of type* $\frac{0}{0}$.

Theorem 1 (L'Hospital's Rule)**.** *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that*

$$\lim_{x\to a} f(x) = 0 \qquad \textit{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \textit{and} \qquad \lim_{x\to a} g(x) = \pm\infty.$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

1.1 Indeterminate Products

Limit of the form

$$\lim_{x\to a} [f(x)g(x)]$$

where $f(x) \to 0$ and $g(x) \to \pm\infty$ as $x \to a$ is called an *indeterminate form of type* $0 \cdot \infty$. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}.$$

1.2 Indeterminate Differences

Limit of the form

$$\lim_{x\to a} [f(x) - g(x)]$$

where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$ is called an *indeterminate form of type* $\infty - \infty$. Try to convert the difference into a quotient (e.g., by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

1.3 Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x\to a} [f(x)]^{g(x)}.$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0.

2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0.
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞.

Each of these three cases can be treated by taking the natural logarithm: let $y = [f(x)]^{g(x)}$, then

$$\ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

1.4 Exercises

Exercise (§4.4, #11)**.**

$$\lim_{x\to(\pi/2)^+} \frac{\cos x}{1 - \sin x}.$$

Solution. Put $f(x) = \cos x$ and $g(x) = 1 - \sin x$ and note that

$$\lim_{x\to(\pi/2)^+} \cos x = 0 \quad \text{and} \quad \lim_{x\to(\pi/2)^+} 1 - \sin x = 0.$$

So we have

- Classify: type $\frac{0}{0}$.
- Check conditions for using l'Hospital's rules are satisfied: both f' and g' exist and they are

  $$f'(x) = -\sin x \qquad \text{and} \qquad g'(x) = -\cos x.$$

  Moreover $g'(x) \neq 0$ on $(0, \pi)$, in particular, $g'(0) = -1$ and $g'(\pi) = 1$.
- Use l'Hospital's rule:

  $$\lim_{x\to(\pi/2)^+} \frac{\cos x}{1 - \sin x} = \lim_{x\to(\pi/2)^+} \frac{-\sin x}{-\cos x} = \lim_{x\to(\pi/2)^+} \tan x = 0.$$

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Exercise (§4.4, #12)**.**

$$\lim_{x\to 0} \frac{\sin 4x}{\tan 5x}.$$

Solution. Put $f(x) = \sin 4x$ and $g(x) = \tan 5x$ and note that

$$\lim_{x\to 0} f(x) = 0 \quad \text{and} \quad \lim_{x\to 0} g(x) = 0.$$

So we have

- Classify: type $\frac{0}{0}$.

- Check conditions for using l'Hospital's rules are satisfied: both f' and g' exist and they are

$$f'(x) = 4\cos 4x \qquad \text{and} \qquad g'(x) = 5\sec^2 5x.$$

  Moreover $g'(x) \neq 0$ on $(-\pi/2, \pi/2)$.
- Use l'Hospital's rule:

$$\lim_{x\to 0} \frac{\sin 4x}{\tan 5x} = \lim_{x\to 0} \frac{4\cos 4x}{5\sec^2 5x} = \frac{4}{5} \lim_{x\to 0} \frac{\cos 4x}{\sec^2 5x} = \frac{4}{5}$$

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Exercise (§4.4, #25)**.**

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}.$$

Solution. Put $f(x) = e^x - 1 - x$ and $g(x) = x^2$ and note that

$$\lim_{x\to 0} f(x) = 0 \quad \text{and} \quad \lim_{x\to 0} g(x) = 0.$$

So we have

- Classify: type $\frac{0}{0}$.
- Check conditions for using l'Hospital's rules are satisfied: both f' and g' exist and they are

$$f'(x) = e^x - 1 \qquad \text{and} \qquad g'(x) = 2x.$$

  Moreover $g'(x) \neq 0$ on $(-\pi/2, \pi/2)$.
- $f'(x)/g'(x)$ is type $\frac{0}{0}$ so we apply L'Hospital's Rule again.
- Both f'' and g'' exist and they are

$$f'(x) = e^x \qquad \text{and} \qquad g''(x) = 2$$

- Use l'Hospital's rule:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

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Exercise (§4.4, #30)**.**

$$\lim_{x\to \infty} \frac{(\ln x)^2}{x}.$$

Solution. Put $f(x) = (\ln x)^2$ and $g(x) = x$ and note that

$$\lim_{x\to 0} f(x) = \infty \quad \text{and} \quad \lim_{x\to 0} g(x) = \infty.$$

So we have

- Classify: type $\frac{\infty}{\infty}$.
- Check conditions for using l'Hospital's rules are satisfied: both f' and g' exist and they are

$$f'(x) = \frac{2}{x}\ln x \qquad \text{and} \qquad g'(x) = 1.$$

- Quotient form, apply l'Hospital's Rule again:

$$F(x) = 2\ln x \qquad \text{and} \qquad G(x) = x,$$

then

$$F'(x) = 2/x \qquad \text{and} \qquad G'(x) = 1.$$

- Use l'Hospital's rule:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2/x}{1} = 2 \lim_{x\to\infty} \frac{1}{x} = 2 \cdot 0 = 0.$$

■

Exercise (§4.4, #33)**.**

$$\lim_{x\to 1} \frac{x+\sin x}{x+\cos x}.$$

Solution. ■

Exercise (§4.4, #43)**.**

$$\lim_{x\to 0}$$

Solution. ■

Exercise (§4.4., #50)**.**

$$\lim_{x\to 0} \csc x \sec 5x.$$

Solution. ■

Exercise (§4.4, #57)**.**

$$\lim_{x\to 0} (1-2x)^{1/x}.$$

Solution. ■

Exercise (§4.4, #61)**.**

$$\lim_{x\to\infty} x^{1/x}.$$

Solution. ■