

MA 519: Homework 5

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PROBLEM 5.1 (HANDOUT 7, # 6(D, F))

Find the variance of the following random variables

- (d) $X = \#$ of tosses of a fair coin necessary to obtain a head for the first time.
 - (f) $X = \#$ matches observed in random sitting of 4 husbands and their wives in opposite sides of a linear table.
- This is an example of the *matching problem*.

SOLUTION. Recall that the variance of a random variable can be computed as

$$\text{Var}(X) = E(X^2) - (E(X))^2.$$

For part (d), let X be as above. First, note that X takes every value on \mathbb{N} . Thus, its PMF is

$$p(n) = P(X = n) = \frac{1}{2^n}$$

and its expectation the value of the series

$$E(X) = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Using a little bit of analysis we can find the value of $E(X)$, e.g., by considering the function $f(x) := \sum_{n=1}^{\infty} nx^{n-1}$, taking its indefinite integral, and noting that it is a geometric series sans the first term. Concretely,

$$\int f(x) dx = \sum_{n=1}^{\infty} x^n = -1 + \sum_{n=0}^{\infty} x^n,$$

which, for $|x| < 1$, converges to the value $x/(1-x)$. Taking the derivative of this, we have $1/(1-x)^2$. Thus,

$$\begin{aligned} E(X) &= \sum_{n=1}^{\infty} \frac{n}{2^n} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \\ &= \frac{1/2}{(1 - (1/2))^2} \\ &= 2. \end{aligned}$$

This is the mean of X .

Next we must compute the mean of X^2 . We have already computed the PMF of X hence,

$$E(X^2) = \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

To find the limit of this series, we can use a similar method to the one in the last paragraph. That is, consider the function $g(x) := \sum_{n=1}^{\infty} n^2 x^{n-1}$. Taking its integral, we have

$$xG(x) = \int g(x) dx = \sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1}$$

and repeat this on G , giving us

$$\int G(x) dx = \sum_{n=1}^{\infty} x^n = -1 + \sum_{n=0}^{\infty} x^n = \frac{x}{1-x}.$$

Tracing back our steps,

$$\int g(x) = \frac{x}{(1-x)^2}$$

so

$$g(x) = \frac{1-x^2}{(1-x)^4}.$$

Thus,

$$\begin{aligned} E(X) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \\ &= \frac{(1/2)(1 - (1/2)^2)}{(1 - (1/2))^4} \\ &= 6. \end{aligned}$$

Putting all of this together, the variance is

$$\text{Var}(X) = 6 - (2)^2 = 2.$$

For part (f), again, we let X be as above. The PMF of X is given by

$$p(n) = P(X = n) =$$

■

PROBLEM 5.2 (HANDOUT 7, # 8)

(Nonexistence of variance).

- (a) Show that for a suitable positive constant c , the function $p(x) = c/x^3$, $x = 1, \dots$, is a valid probability mass function (PMF).
- (b) Show that in this case, the expectation of the underlying random variable exists, but the variance does not!

SOLUTION. ■

PROBLEM 5.3 (HANDOUT 7, # 9)

In a box, there are 2 black and 4 white balls. These are drawn out one by one at random (without replacement).

- (a) Let X be the draw at which the first black ball comes out. Find the mean and the variance of X .
- (b) Let X be the draw at which the second black ball comes out. Find the mean* the variance of X .

SOLUTION. For part (a), we must first find the PMF of X . This we do explicitly,

$$\begin{aligned} p(1) &= \frac{2}{6} = \frac{1}{3}, & p(2) &= \frac{2}{5} \cdot \frac{4}{6} = \frac{4}{15}, \\ p(3) &= \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} = \frac{1}{5}, & p(4) &= \frac{2}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} = \frac{2}{15}, \\ p(5) &= 1 \cdot \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} = \frac{1}{15}. \end{aligned}$$

Thus,

$$\begin{aligned} E(X) &= 1 \cdot \frac{1}{3} + 2 \cdot \frac{4}{15} + 3 \cdot \frac{1}{5} + 4 \cdot \frac{2}{15} + 5 \cdot \frac{1}{15} \\ &= \frac{7}{3} \\ &= 2.333. \end{aligned}$$

Similarly, we have

$$\begin{aligned} E(X^2) &= 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{4}{15} + 3^2 \cdot \frac{1}{5} + 4^2 \cdot \frac{2}{15} + 5^2 \cdot \frac{1}{15} \\ &= 7. \end{aligned}$$

Hence,

$$\text{Var}(X) = 7 - \left(\frac{7}{3}\right)^2 \approx 1.556.$$

For part (b) ■

*What is a meman? How do you pronounce meman? Is it mee-man or muh-man?

PROBLEM 5.4 (HANDOUT 7, # 10)

Suppose X has a *discrete uniform distribution* on the set $\{1, \dots, N\}$.

Find formulas for the mean and the variance of X .

SOLUTION. ■

PROBLEM 5.5 (HANDOUT 7, # 11)

(*Be Original*) Give an example of a random variable with mean 1 and variance 100.

SOLUTION.

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PROBLEM 5.6 (HANDOUT 7, # 13)

(*Be Original*). Suppose a random variable X has the property that its second and fourth moment are both 1.

What can you say about the nature of X ?

SOLUTION. ■

PROBLEM 5.7 (HANDOUT 7, # 14)

(Be Original). One of the following inequalities is true in general for all nonnegative random variables. Identify which one!

$$\begin{aligned} E(X)E(X^4) &\geq E(X^2)E(X^3); \\ E(X)E(X^4) &\leq E(X^2)E(X^2). \end{aligned}$$

SOLUTION. ■

PROBLEM 5.8 (HANDOUT 7, # 15)

Suppose X is the number of heads obtained in 4 tosses of a fair coin.

Find the expected value of the weird function

$$\log\left(2 + \sin\left(\frac{\pi}{4}x\right)\right).$$

SOLUTION.

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PROBLEM 5.9 (HANDOUT 7, # 16)

In a sequence of Bernoulli trials let X be the length of the run (of either successes or failures) started by the first trial.

- (a) Find the distribution of X , $E(X)$, $\text{Var}(X)$.

SOLUTION.

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PROBLEM 5.10 (HANDOUT 7, # 17)

A man with n keys wants to open his door and tries the keys independently and at random. Find the mean and variance of the number of trials

- (a) if unsuccessful keys are not eliminated from further selections;
- (b) if they are.

(Assume that only one key fits the door. The exact distributions are given in II, 7, but are not required for the present problem.)

SOLUTION.

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