3.2: 23, 25, 26
3.3: 4, 6, 7

3.2.23 If $\det(A)=2$, find $\det(A^5)$.

$$\det(A^5) \underset{\text{Thm}(3.9)}{=} (\det(A))^5 = 2^5 = 32.$$

3.2.25 Use Theorem (3.8) to determine which of the following matrices are nonsingular:

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 1 & -7 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 2 & 0 \\ 0 & 1 & 2 & -7 \end{bmatrix}$

Theorem (3.8) says A is nonsingular iff $\det(A) \neq 0$.

(a) $$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 1 & -7 & 2 \end{vmatrix} \underset{\substack{-2r_1+r_2 \\ -r_1+r_3}}{=} \begin{vmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -10 & 0 \end{vmatrix} \underset{-2r_2+r_3}{=} \begin{vmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$ so the matrix is singular.

(b) $$\begin{vmatrix} 1 & 2 & 0 & 5 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 2 & 0 \\ 0 & 1 & 2 & -7 \end{vmatrix} \underset{\substack{-3r_1+r_2 \\ 2r_1+r_3}}{=} \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & -2 & 1 & -8 \\ 0 & 9 & 2 & 10 \\ 0 & 1 & 2 & -7 \end{vmatrix} \underset{r_2 \leftrightarrow r_4}{=} - \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & -7 \\ 0 & 9 & 2 & 10 \\ 0 & -2 & 1 & -8 \end{vmatrix} \underset{\substack{-9r_2+r_3 \\ 2r_2+r_4}}{=} - \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & -16 & 73 \\ 0 & 0 & 5 & -22 \end{vmatrix} \underset{3r_4+r_3}{=} - \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & 5 & -22 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 5 & -22 \end{vmatrix} \underset{-5r_3+r_4}{=} \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 13 \end{vmatrix} = 13 \neq 0$$ so the matrix is nonsingular.

3.2.26 Use Theorem (3.8) to determine all values of t so that the following matrices are nonsingular:

(a) $\begin{bmatrix} t & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} t & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & t \end{bmatrix}$ (c) $\begin{bmatrix} t & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 1 & 0 & 0 & t \end{bmatrix}$.

(a) $$\begin{vmatrix} t & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} \underset{c_1 \leftrightarrow c_2}{=} - \begin{vmatrix} 1 & t & 2 \\ 4 & 3 & 5 \\ 7 & 6 & 8 \end{vmatrix} \underset{\substack{-4r_1+r_2 \\ -7r_1+r_3}}{=} \begin{vmatrix} 1 & t & 2 \\ 0 & 3-4t & -3 \\ 0 & 6-7t & -6 \end{vmatrix} \underset{-2r_2+r_3}{=} \begin{vmatrix} 1 & t & 2 \\ 0 & 3-4t & -3 \\ 0 & t & 0 \end{vmatrix} \underset{4r_3+r_2}{=} \begin{vmatrix} 1 & t & 2 \\ 0 & 3 & -3 \\ 0 & t & 0 \end{vmatrix} \underset{r_2 \leftrightarrow r_3}{=} - \begin{vmatrix} 1 & t & 2 \\ 0 & t & 0 \\ 0 & 3 & -3 \end{vmatrix} \tfrac{1}{t} r_2$$

$$= -t \begin{vmatrix} 1 & t & 2 \\ 0 & 1 & 0 \\ 0 & 3 & -3 \end{vmatrix} \underset{-3r_2+r_3}{=} -t \begin{vmatrix} 1 & t & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = 3t.$$ Thus $\begin{vmatrix} t & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} \neq 0$ iff $t \neq 0$.

(b) $$\begin{vmatrix} t & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & t \end{vmatrix} \underset{r_1 \leftrightarrow r_3}{=} - \begin{vmatrix} 1 & 0 & t \\ 0 & 1 & 1 \\ t & 1 & 2 \end{vmatrix} \underset{-tr_1+r_3}{=} - \begin{vmatrix} 1 & 0 & t \\ 0 & 1 & 1 \\ 0 & 1 & 2-t^2 \end{vmatrix} \underset{-r_2+r_3}{=} - \begin{vmatrix} 1 & 0 & t \\ 0 & 1 & 1 \\ 0 & 0 & 1-t^2 \end{vmatrix} = -1+t^2 \neq 0$$ iff $t \neq 1, -1$.

(c) $$\begin{vmatrix} t & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 1 & 0 & 0 & t \end{vmatrix} \underset{r_1 \leftrightarrow r_4}{=} - \begin{vmatrix} 1 & 0 & 0 & t \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ t & 0 & 0 & 1 \end{vmatrix} \underset{-tr_1+r_4}{=} - \begin{vmatrix} 1 & 0 & 0 & t \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1-t \end{vmatrix} = t^2(t-1) \neq 0$$ iff $t \neq 0$ or $t \neq 1$.

3.3.4 Let $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & -4 & -1 \\ 3 & 2 & 4 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix}$. Find the following cofactors: (a) A_{12} (b) A_{23} (c) A_{33} (d) A_{41}

$A_{ij} = (-1)^{i+j} \det(M_{ij})$.

(a) $A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -4 & -1 \\ 3 & 4 & 0 \\ 0 & -1 & 0 \end{vmatrix} \underset{r_2 - r_1}{=} - \begin{vmatrix} 2 & -1 & -1 \\ 1 & 8 & 1 \\ 0 & -1 & 0 \end{vmatrix} \underset{-2r_2+r_1}{=} - \begin{vmatrix} 0 & -17 & -3 \\ 1 & 8 & 1 \\ 0 & -1 & 0 \end{vmatrix} \underset{17r_3+r_1}{=} - \begin{vmatrix} 0 & 0 & -3 \\ 1 & 8 & 1 \\ 0 & -1 & 0 \end{vmatrix} \underset{r_1 \leftrightarrow r_2}{=} \begin{vmatrix} 1 & 8 & 1 \\ 0 & 0 & -3 \\ 0 & -1 & 0 \end{vmatrix}$

$= - \begin{vmatrix} 1 & 8 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -1(1)(-1)(-3) = -3$

(b) $A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 3 & 0 \end{vmatrix} = 0$ as a column is 0.

(c) $A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 3 & 0 \end{vmatrix} \underset{-2r_1+r_2}{=} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & 0 \end{vmatrix} \underset{-3r_2+r_3}{=} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{vmatrix} = 3$

(d) $A_{41} = (-1)^{4+1} \begin{vmatrix} 0 & 3 & 0 \\ 1 & -4 & -1 \\ 2 & 4 & 0 \end{vmatrix} \underset{r_1 \leftrightarrow r_2}{=} \begin{vmatrix} 1 & -4 & -1 \\ 0 & 3 & 0 \\ 2 & 4 & 0 \end{vmatrix} \underset{-2r_1+r_3}{=} \begin{vmatrix} 1 & -4 & -1 \\ 0 & 3 & 0 \\ 0 & 12 & 2 \end{vmatrix} \underset{-4r_2+r_3}{=} \begin{vmatrix} 1 & -4 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6$

3.3.6 Use Thm (3.10) to evaluate the determinants in Exercis 1(a), (c), and (f) of Section 3.2.

1a) $\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = 3(1) - (0)(2) = 3$ expansion of first row

1c) $\begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 4\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 4 \cdot 2 \cdot 3 = 24$ expansion of first row

1f) $\begin{vmatrix} 4 & 2 & 3 & -4 \\ 3 & -2 & 1 & 5 \\ -2 & 0 & 1 & -3 \\ 8 & -2 & 6 & 4 \end{vmatrix} = (-2)\begin{vmatrix} 2 & 3 & -4 \\ -2 & 1 & 5 \\ -2 & 6 & 4 \end{vmatrix} - 0\begin{vmatrix} 4 & 3 & -4 \\ 3 & 1 & 5 \\ 8 & 6 & 4 \end{vmatrix} + (1)\begin{vmatrix} 4 & 2 & -4 \\ 3 & -2 & 5 \\ 8 & -2 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 4 & 2 & 3 \\ 3 & -2 & 1 \\ 8 & -2 & 6 \end{vmatrix}$ expansion of third row

$= (-2)\left[2\begin{vmatrix} 1 & 5 \\ 6 & 4 \end{vmatrix} - 3\begin{vmatrix} -2 & 5 \\ -2 & 4 \end{vmatrix} + (-4)\begin{vmatrix} -2 & 1 \\ -2 & 6 \end{vmatrix}\right] + \left[4\begin{vmatrix} -2 & 5 \\ -2 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & 5 \\ 8 & 4 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -2 \\ 8 & -2 \end{vmatrix}\right] + 3\left[4\begin{vmatrix} -2 & 1 \\ -2 & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 8 & 6 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ 8 & -2 \end{vmatrix}\right]$

$= -2\left[2(4-30) - 3(-8+10) - 4(-12+2)\right] + \left[4(-8+10) - 2(12-40) - 4(-6+16)\right] + 3\left[4(-12+2) - 2(18-8) + 3(-6+16)\right]$

$= -2\left[-52 - 6 + 40\right] + \left[8 + 56 - 40\right] + 3\left[-40 - 20 + 30\right] = 36 + 24 - 90 = -30$

3.3.7 Use Thm (3.10) to evaluate the determinants in Exercise 2(a), (c), and (f) of Section 3.2.

2a) $\begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix} = 2(-1) - (-2)(3) = -2 + 6 = 4$ expanding the first row.

2c) $\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 6 \end{vmatrix} = 3\begin{vmatrix} 4 & 2 \\ 5 & 0 \end{vmatrix} - 0\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} + 0\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 3(4\cdot 0 - 5\cdot 2) = -30$ expanding third row.

2f) $\begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & -4 & -2 \\ 3 & -1 & 0 \\ 8 & -4 & 6 \end{vmatrix} \pm 0\begin{vmatrix} 3 & -4 & -2 \\ 2 & -1 & 0 \\ 11 & -4 & 6 \end{vmatrix} + (1)\begin{vmatrix} 3 & 2 & -2 \\ 2 & 3 & 0 \\ 11 & 8 & 6 \end{vmatrix} - 4\begin{vmatrix} 3 & 2 & -4 \\ 2 & 3 & -1 \\ 11 & 8 & -4 \end{vmatrix}$ expanding first row

$= 2\left[-3\begin{vmatrix} -4 & -2 \\ -4 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -2 \\ 8 & 6 \end{vmatrix} - 0\begin{vmatrix} 2 & -4 \\ 8 & -4 \end{vmatrix}\right] + \left[-2\begin{vmatrix} 2 & -2 \\ 8 & 6 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ 11 & 6 \end{vmatrix} - 0\begin{vmatrix} 3 & 2 \\ 11 & 8 \end{vmatrix}\right] - 4\left[3\begin{vmatrix} 3 & -1 \\ 8 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 11 & -4 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 11 & 8 \end{vmatrix}\right]$

$= 2\left[-3(-24-8) - (12+16)\right] + \left[-2(12+16) + 3(18+22)\right] - 4\left[3(-12+8) - 2(-8+11) - 4(16-33)\right]$

$= 2\left[96 - 28\right] + \left[-56 + 120\right] - 4\left[-12 - 6 + 68\right] = 136 + 64 - 200 = 0$