MA557 Problem Set 3

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Problem 3.1

Find an example of a finitely generated ring extension $R \subset S$ where S is a Noetherian ring, but R is not.

Proof. ■

Problem 3.2

Consider the homomorphism of rings

$$\begin{tikzcd} & S \arrow[d, "\psi"] \\ R \arrow[r, "\varphi"] & T. \end{tikzcd}$$

The *fiber product* of R and S over T is the subring $R \times_T S = \{ (r, s) \mid \varphi(t) = \psi(s) \}$ of $R \times S$. Assume φ and ψ are surjective. Show that if R and S are Noetherian rings then so is $R \times_T S$.

Proof. Suppose that R and S are Noetherian rings with surjective ring maps $\varphi \colon R \to T$ and $\psi \colon S \to T$. Then, by (3.5), the product $R \times S$ is Noetherian. Define the ring map $\Phi \colon R \times S \to T \times T$ by $\Phi = (\varphi, \psi)$. Then the diagonal, $\Delta_T = \{ (t, t) \mid t \in T \}$, of $T \times T$ is exactly the image of the fiber product of R and S under the ring map Φ. And this is not terribly difficult to see: It is clear, by the definition of the fiber product, that $\Phi(R \times_T S) \subset \Delta_T$. To show the reverse containment, take an element $(t, t) \in \Delta_T$. Then, since φ and ψ are surjective, there are corresponding elements r and s of the rings R and S, respectively, such that $\varphi(r) = t$ and $\psi(s) = t$. Hence, (t, t) are in the image $R \times_T S$ under Φ.

Now, it is clear that $R \times S$ and $T \times T$ have an $R \times S$-module structure ($R \times S$ by the usual ring multiplication and $T \times T$ by $(r, s)(t, t') = (\varphi(r)t, \psi(s)t'))$ so they have an $R \times_T S$-module structure by restriction to the subring $R \times_T S$ of $R \times S$. Consider the quotient module $T \times T/\Delta_T$. $T \times T/\Delta_T$ also inherits an $R \times_T S$-module structure from $T \times T$. Note that the map $\Phi \colon R \times S \to T \times T$ is an $R \times_T S$-linear map: It is clear that Φ is linear with respect to "+", what is not so obvious is that multiplication by scalars is preserved so take $(r', s') \in R \times_T S$ and $(r, s) \in R \times S$, then

$$\begin{aligned} \Phi((r', s')(r, s) &= \Phi(r'r, s's) \\ &= (\varphi(r'r), \psi(s's')) \\ &= (\varphi(r')\varphi(r), \psi(s')\psi(s)) \\ &= (\varphi(r'), \psi(s'))(\varphi(r), \psi(s)) \\ &= \Phi(r', s')\Phi(s, r) \end{aligned}$$

as desired. Therefore, Φ induces an $R \times_T S$-linear map $\Phi^* \colon R \times S \to T \times T/\Delta_T$ via composition with the quotient map, i.e., $\Phi^* = \pi \circ \Phi$ and we have the following exact sequence of $R \times_T S$-modules

$$0 \longrightarrow R \times_T S \xhookrightarrow{\iota} R \times S \xrightarrow{\Phi^*} \frac{T \times T}{\Delta_T} \longrightarrow 0.$$

By (3.4), $R \times_T S$ are Noetherian. ■

Problem 3.3

Let M be an R-module. Show that M is a flat R-module if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.

Proof. $\Longrightarrow$ ■

Problem 3.4

Let M be an R-module and $\mathfrak{a}$ an R-ideal.

(a) Show that if $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, then $M = \mathfrak{a}M$.
(b) Show that the converse holds in case M is finite.

Proof. (a) Suppose that $M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. ■

Problem 3.5

Prove that every power of a maximal ideal is primary.

Proof. ■

PROBLEM 3.6

(a) Show that the radical of a primary ideal is prime.
(b) Find an example of a power of a prime ideal that is not primary.
(c) Let $\mathfrak{p}$ be a prime ideal of a ring R and $n \in \mathbf{N}$. The R-ideal $\mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$ s called the *nth symbolic power of* $\mathfrak{p}$. Show that $\mathfrak{p}^{(n)}$ is primary.

Proof. ■