MA 572: Homework 3

Carlos Salinas

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Problem 3.1 (Hatcher §2.1, Ex. 17)

(a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown. [What are X/A and X/B?]

Proof. (a) As a consequence of 2.16, we have a long exact sequence on relative homology

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots . \tag{1}$$

Now, specifying X to be S^2, we know, by 2.8, 2.6 and 2.14, that $H_n(A) \cong 0$ for all $n > 0$ and $H_0(A) \cong \bigoplus_{|A|} \mathbf{Z}$, and $H_n(S^2) \cong 0$ for $n \neq 2, 0$ and $H_n(S^2) \cong \mathbf{Z}$ otherwise. Hence, the long exact sequence (1) turns into

$$\begin{tikzcd} \cdots \arrow[r] & H_2(A) \arrow[r] & H_2(S^2) \arrow[r] & H_2(S^2, A) \arrow[dll, out=0, in=180] \\ & H_1(A) \arrow[r] & H_1(S^2) \arrow[r] & H_1(S^2, A) \arrow[dll, out=0, in=180] \\ & H_0(A) \arrow[r] & H_0(S^2) \arrow[r] & H_0(S^2, A) \arrow[r] & 0 \end{tikzcd} \tag{2}$$

which, filling in our computed values for $H_n(A)$ and $H_n(S^2)$, further becomes

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & \mathbf{Z} \arrow[r] & H_2(S^2, A) \arrow[dll, out=0, in=180] \\ & 0 \arrow[r] & 0 \arrow[r] & H_1(S^2, A) \arrow[dll, out=0, in=180] \\ & \bigoplus_{|A|} \mathbf{Z} \arrow[r] & \mathbf{Z} \arrow[r] & H_0(S^2, A) \arrow[r] & 0, \end{tikzcd} \tag{3}$$

with all other $n > 2$ homology groups for A and S^2 being 0. That last remark immediately tells us that, by exactness, $H_n(S^2, A) = 0$ for all $n > 2$. Starting from the bottom of (3), exactness at $H_2(S^2, A)$ tells us that $H_2(S^2, A) \cong \mathbf{Z}$ since the zero maps to the left and right

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{Z} \longrightarrow H_2(S^2, A) \longrightarrow 0 \longrightarrow \cdots$$

tells us that the map $\mathbf{Z} \to H_2(S^2, A)$ is an isomorphism. If we look at the reduced homology, the bottom row of (3) becomes

$$\cdots \longrightarrow 0 \longrightarrow \widetilde{H}_1(S^2, A) \longrightarrow \bigoplus_{|A|-1} \mathbf{Z} \longrightarrow 0 \longrightarrow \widetilde{H}_0(S^2, A) \longrightarrow 0.$$

By exactness at $\widetilde{H}_1(S^2, A)$, we have $H_1(S^2, A) \cong \widetilde{H}_1(S^2, A) \cong \bigoplus_{|A|-1} \mathbf{Z}$ and, last but not least, exactness at $\widetilde{H}_0(S^2, A) \cong 0$ gives us $H_0(S^2, A) \cong \mathbf{Z}$. In summary, we have

$$H_n(S^2, A) = \begin{cases} \mathbf{Z} & \text{if } n = 0, 2 \\ \bigoplus_{|A|-1} \mathbf{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

From 2.27 we know that $H_n^{\Delta}(S^1 \times S^1) \cong H_n(S^1 \times S^1)$ so from 2.3, we know that the homology of the torus $S^1 \times S^1$ is

$$H_n(S^1 \times S^1) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{if } n = 1 \\ \mathbf{Z} & \text{if } n = 2, 0 \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

Skipping directly, to our calculation, we have the long exact sequence

$$\begin{array}{c} \cdots \longrightarrow 0 \longrightarrow \mathbf{Z} \longrightarrow H_2(S^1 \times S^1, A) \longrightarrow \\ \longrightarrow 0 \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \longrightarrow H_1(S^1 \times S^1, A) \longrightarrow \\ \longrightarrow \bigoplus_{|A|} \mathbf{Z} \longrightarrow \mathbf{Z} \longrightarrow H_0(S^2, A) \longrightarrow 0. \end{array} \tag{6}$$

It is clear from exactness that $H_2(S^1 \times S^1, A) \cong \mathbf{Z}$ and $H_0(S^1 \times S^1, A) \cong \mathbf{Z}$. What is not clear is what $H_1(S^1 \times S^1, A)$ is. Exactness at $\mathbf{Z} \oplus \mathbf{Z}$ tells us that $\mathbf{Z} \oplus \mathbf{Z} \hookrightarrow H_1(S^1 \times S^1, A)$ and, looking at the reduced homology, exactness at $\bigoplus_{|A|-1} \mathbf{Z}$ tells us that $H_1(S^1 \times S^1, A) \twoheadrightarrow \bigoplus_{|A|-1} \mathbf{Z}$. Thus, we have $\bigoplus_{|A|-1} \mathbf{Z} \cong H_1(S^1 \times S^1, A)/\mathbf{Z} \oplus \mathbf{Z}$ from which we can deduce that $H_1(S^1 \times S^1, A) \cong \bigoplus_{|A|+1} \mathbf{Z}$.[1] In summary, the relative homology of $S^1 \times S^1$ with respect to A is

$$H_n(S^1 \times S^1, A) = \begin{cases} \mathbf{Z} & \text{if } n = 2, 0 \\ \bigoplus_{|A|+1} \mathbf{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

(b) ■

[1] I know this is not strictly correct, but the approach I took to solve the problem required me to construct an inverse map $H_1(S^1 \times S^1, A) \leftarrow \bigoplus_{|A|-1} \mathbf{Z}$, but this is difficult.

Problem 3.2 (Hatcher §2.2, Ex. 1)

Prove the Brouwer fixed point theorem for maps $f\colon D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f. [This was Brouwer's original proof.]

Proof. ■

Problem 3.3 (Hatcher §2.2, Ex. 6)

Show that every map $S^n \to S^n$ can be homotoped to have a fixed point if $n > 0$.

Proof. ■

Problem 3.4

Let $\mathcal{U}$ be an open cover of X. Prove that the inclusion of $C_*^{\mathcal{U}}(C)$ into $C_*(X)$ is a chain homotopy equivalence.

Proof. ■