MA 544: Homework 1

Carlos Salinas

January 17, 2016

Problem 1.1 (Wheeden & Zygmund §2, Ex. 1)

Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0, 1]$, but that $V[f; 0, 1] = +\infty$.

Proof. Moreover, f is continuous on $(0, 1]$ since it is the product of continuous functions on $(0, 1]$. To see that f is continuous at 0 is suffices to show that $f(0+) = f(0) = 0$. To that end, let $\{x_n\} \subset [0, 1]$ be a sequence such that $x_n \to 0$ and consider $\lim_{n\to\infty} f(x_n)$. Since $x_n \to 0$, for every $\varepsilon > 0$, there exists a natural number N such that $n \geq N$ implies $|0 - x_n| < \varepsilon$. Thus, for $n \geq N$ we have

$$|0 - f(x_n)| = |f(x_n)| = |x_n||\sin(1/x_n)| \leq \varepsilon|\sin(1/\varepsilon)| \leq \varepsilon.$$

Thus, $f(x_n) \to 0$ and we see that $f(0+) = 0$. Hence, f is continuous on $[0, 1]$.

It is easy to see that f is bounded since $|\sin(1/x)| \leq 1$ for all $x \in (0, 1]$. More explicitly, we have that

$$|f(x)| \leq |x\sin(1/x)| = |x| \cdot |\sin(1/x)| \leq 1 \cdot 1.$$

Thus, $|f(x)| \leq 1$ and we see that f is bounded.

Moreover, f is continuous on $(0, 1]$ since it is the product of continuous functions on $(0, 1]$. To see that f is continuous at 0, it suffices to show that $f(0+) = 0$. To that end, we shall use the following limiting argument: Let $\varepsilon > 0$ and consider the limit (from the right) of $f(\varepsilon)$ as $\varepsilon \to 0$. This is

$$\lim_{\varepsilon\to 0} f(\varepsilon) \lim_{\varepsilon\to 0} \varepsilon \sin(1/\varepsilon) \leq \lim_{\varepsilon\to 0} |\varepsilon||\sin(1/\varepsilon)| \leq \lim_{\varepsilon\to 0} |\varepsilon| \cdot 1 = 0.$$

Thus, $f(0+) = 0$ and we see that f is continuous on $[0, 1]$.

Last but not least, we show that f is BV. Define the family of partitions $\{\Gamma_n\}_{n=1}^{\infty}$ by $x_i :=$ ■

Problem 1.2 (Wheeden & Zygmund §2, Ex. 2)

Prove theorem (2.1).

Proof. Recall the statement of theorem (2.1):

Theorem (Wheeden & Zygmund, 2.1)**.** *(a) If f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$.*

(b) Let f and g be of bounded variation on $[a, b]$. Then cf (for any real constant c), $f + g$, and fg are of bounded variation on $[a, b]$. Moreover, f/g is of bounded variation on $[a, b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a, b]$.

(a) We shall proceed by contradiction. Suppose that f is not bounded, i.e., for every positive real number $M > 0$, there exists $x \in [a, b]$ such that $|f(x)| > M$. In particular, if V is the variation of f, then $|f(x_0)| > V + (f(a) + f(b))/2$ for some $x_0 \in [a, b]$. Then, putting $\Gamma = \{a, x_0, b\} \subset [a, b]$, we have

$$\begin{aligned}
S_\Gamma &= |f(b) - f(x_0)| + |f(x_0) - f(a)| \\
&= |f(x_0) - f(b)| + |f(x_0) - f(a)| \\
&\geq |2f(x_0) - f(a) - f(b)| \\
&= |2(V + (f(a) + f(b))/2) - f(a) - f(b)| \\
&= |2V + f(a) + f(b) - f(a) - f(b)| \\
&= 2V \\
&> V.
\end{aligned}$$

This is a contradiction since V is the supremum over all such sums.

(b) Let f and g be BV on $[a, b]$ and c a real number. Then, for every partition Γ of $[a, b]$, we have $V[f; a, b] \geq S_\Gamma[f; a, b]$ ■

Problem 1.3 (Wheeden & Zygmund §2, Ex. 3)

If $[a', b']$ is a subinterval of $[a, b]$ show that $P[a', b'] \leq P[a, b]$ and $N[a', b'] \leq N[a, b]$.

Proof. ∎

Problem 1.4 (Wheeden & Zygmund §2, Ex. 11)

Show that $\int_a^b f \, \mathrm{d}\phi$ exists if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|, |\Gamma'| < \delta$.

Proof. ∎

Problem 1.5 (Wheeden & Zygmund §2, Ex. 13)

Prove theorem (2.16).

Proof.

Theorem (Wheeden & Zygmund, 2.16). *(i) If $\int_a^b f\,\mathrm{d}\phi$ exists, then so do $\int_a^b cf\,\mathrm{d}\phi$ and $\int_a^b f\,\mathrm{d}(c\phi)$ for any constant c, and*

$$\int_a^b cf\,\mathrm{d}\phi = \int_a^b f\,\mathrm{d}(c\phi) = c\int_a^b f\,\mathrm{d}\phi.$$

(ii) If $\int_a^b f_1\,\mathrm{d}\phi$ and $\int_a^b f_2\,\mathrm{d}\phi$ both exist, so does $\int_a^b (f_1+f_2)\,\mathrm{d}\phi$, and

$$\int_a^b (f_1+f_2)\,\mathrm{d}\phi = \int_a^b f_1\,\mathrm{d}\phi + \int_a^b f_2\,\mathrm{d}\phi.$$

(iii) If $\int_a^b f\,\mathrm{d}\phi_1$ and $\int_a^b f\,\mathrm{d}\phi_2$ both exist, so does $\int_a^b f\,\mathrm{d}(\phi_1+\phi_2)$, and

$$\int_a^b f\,\mathrm{d}(\phi_1+\phi_2) = \int_a^b f\,\mathrm{d}\phi_1 + \int_a^b f\,\mathrm{d}\phi_2.$$

■