MA571 Homework 10

Carlos Salinas

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Problem 10.1 (Munkres §52, Ex. 2)

Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. By Theorem 52.1, the paths α and β induce a group homomorphism $\hat{\alpha}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $\hat{\beta}\colon \pi_1(X, x_1) \to \pi_1(X, x_2)$, respectively. We want to show therefore that the induced homomorphism $\hat{\gamma} = \widehat{\alpha * \beta}$ is in fact equivalent to the composition $\hat{\beta} \circ \hat{\alpha}$. Let $[f]$ be a loop based at x_0 then

$$\begin{aligned} \hat{\gamma}([f]) &= \widehat{\alpha * \beta}([f]) \\ &= \left[\, \overline{\alpha * \beta}\, \right] * [f] * [\alpha * \beta] \\ &= \left[\bar{\beta} * \bar{\alpha}\right] * [f] * [\alpha] * [\beta] \end{aligned}$$

by the well-definedness of the path product operation, we have

$$= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta]$$

by associativity of the path product,

$$\begin{aligned} &= [\bar{\beta}] * ([\bar{\alpha}] * [f] * [\alpha]) * [\beta] \\ &= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta] \end{aligned}$$

where $\alpha([f])$ is a loop based at x_1 so

$$\begin{aligned} &= \hat{\beta}(\hat{\alpha}([f])) \\ &= \left(\hat{\beta} \circ \hat{\alpha}\right)([f]). \end{aligned}$$

Thus, the following diagram commutes

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\hat{\alpha}"] \arrow[dr, "\hat{\gamma} = \widehat{\alpha * \beta}"'] & \pi_1(X, x_1) \arrow[d, "\hat{\beta}"] \\ & \pi_1(X, x_2). \end{tikzcd}$$

■

Problem 10.2 (Munkres §52, Ex. 3)

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is Abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. $\Longrightarrow$ Suppose that $\pi_1(X, x_0)$ is Abelian. Then for any class of loops about x_0, say $[f]$ and $[g]$, the product $[f] * [g] = [g] * [f]$. Let α and β be paths from x_0 to x_1. Then the induced map on fundamental groups $\hat{\alpha}$ and $\hat{\beta}$ yield isomorphism by Theorem 52.1 so that the map $\hat{\bar{\beta}} \circ \hat{\alpha}$ is an automorphism of $\pi_1(X, x_0)$. Moreover, we have

$$\begin{aligned} \hat{\bar{\beta}} \circ \hat{\alpha}([f]) &= \hat{\bar{\beta}}\big(\hat{\alpha}([f])\big) \\ &= \hat{\bar{\beta}}\big([\bar{\alpha}] * [f] * [\alpha]\big) \\ &= [\beta] * \big([\bar{\alpha}] * [f] * [\alpha]\big) * [\bar{\beta}] \end{aligned}$$

by associativity of the path product, we may rewrite the above expression as

$$= ([\beta] * [\bar{\alpha}]) * [f] * ([\alpha] * [\bar{\beta}])$$

noting that $[\beta] * [\bar{\alpha}]$ and $[\alpha] * [\bar{\beta}]$ are loops based at x_0, since $\pi_1(X, x_0)$ is Abelian, we have

$$\begin{aligned} &= ([\beta] * [\bar{\alpha}]) * ([\alpha] * [\bar{\beta}]) * [f] \\ &= [e_{x_0}] * [f] \\ &= [f]. \end{aligned}$$

Thus, $\hat{\bar{\beta}} \circ \hat{\alpha} = \mathrm{id}_{\pi_1(X, x_0)}$, i.e., $\hat{\alpha} = \hat{\beta}$.

$\Longleftarrow$ Let f and g be loops about x_0. Then, since X is path connected, we claim that f and g are homotopic to the path product $\alpha_1 * \bar{\beta}_1$ and $\alpha_2 * \bar{\beta}_2$ where α_i, β_i are paths from x_0 to x_1. More precisely, split f into the paths $f_1 = f(t/2)$ and $f_2 = f((t+1)/2)$; it is clear that $f = f_1 * f_2$. Let $x_2 \coloneqq f_1(1)$ then there exists a path α from x_2 to x_1 since X is path connected. Now we claim that the following

$$H(x, t) \coloneqq f_1(x) * \alpha(tx) * \bar{\alpha}(tx) * f_2(x)$$

is a homotopy from $f = f_1 * f_2$ to the extended loop $\tilde{f} = f_1 * \alpha * \bar{\alpha} * f_2$. It is clear that H is continuous since it is a path products and multiplication on the interval I, tx, is continuous. Lastly, $H(x, 0) = f_1(x) * \alpha(0) * \bar{\alpha}(0) * f_2()$ ■

Problem 10.3 (Munkres §52, Ex. 4)

Let $A \subset X$; suppose $r \colon X \to A$ is continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A.) If $a_0 \in A$, show that

$$r_* \colon \pi_1(X, x_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

Proof. ∎

Problem 10.4 (Munkres §53, Ex. 6)

Show that if X is path connected, the homomorphism induced by a continuous map is independent of the base point, up to isomorphisms of the groups involved. More precisely, let $h\colon X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1}) \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps "commutes"

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ {\scriptstyle\hat{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle\hat{\beta}} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1). \end{array}$$

Proof. ■

Problem 10.5 (Munkres §55, Ex. 1)

Show that if A is a retract of B^2, then every continuous map $f\colon A \to A$ has a fixed point.

Proof. ∎

Problem 10.6 (Munkres §55, Ex. 2)

Show that if $h\colon S^1 \to S^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof. ∎

Problem 10.7 ((A))

Prove that every m-manifold is locally path-connected.

Proof. ■

Problem 10.8 ((B))

Prove that every m-manifold is regular.

Proof. ■

Problem 10.9 ((C))

Prove that there is no 1-1 continuous function $\iota\colon S^1 \to \mathbf{R}$. You may assume any fact about trigonometric functions. (Note: this shows in particular that there is no $\iota\colon S^1 \to \mathbf{R}$ with $p \circ \iota$ equal to the identity map, where p is the map in the note on the Fundamental Group of the Circle.)

Proof. ■

Problem 10.10 ((D))

Prove Proposition C from the note on the Fundamental Group of the Circle.

Proof. ■