MA52300 Fall 2016

Homework Assignment 1

Due Wed, Aug 31, 2016

1. (Taylor's formula) Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}) \quad \text{as } x \to 0$$

   for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

   Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha| = m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

   by induction in m.

2. Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{$*$}$$

   Where $b \in \mathbb{R}^n$, $f = f(x, t)$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}.$$

   Make sure your answer agrees with formula (5) in §2.1.2 of [E].

3. Solve using characteristics:
   (a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2, \quad u = 1$ on the line $x_2 = 2x_1$.
   (b) $u u_{x_1} + u_{x_2} = 1, \quad u(x_1, x_1) = \frac{1}{2} x_1$.
   (c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \quad u(x_1, x_2, 0) = g(x_1, x_2)$

4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

   find a solution with $u(x_1, 0) = \frac{1}{2}(1 - x_1^2)$.