MA571 Midterm 1: Practice Problems

Carlos Salinas

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Problem 1. Let $A \subset X$ and $B \subset Y$. Show that the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

Proof. Before we proceed, we need to prove the following nontrivial facts:

Claim 1 (Munkres §17, Ex. 3)**.** *If A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.*

Proof of claim. We will show that the complement of $A \times B$ is open in $X \times Y$. Let $(x, y) \in (X \times Y) \smallsetminus (A \times B)$. Then $x \notin A$ and $y \notin B$. Since A and B are closed in X and Y, respectively, there exist neighborhoods U and V of x and y, respectively, such that $U \subset X \smallsetminus A$ and $V \subset Y \smallsetminus B$. Then $U \times V \subset (X \times Y) \smallsetminus (A \times B)$ is a neighborhood of (x, y) so, by Lemma C, $(X \times Y) \smallsetminus (A \times B)$ is open. Thus, $A \times B$ is closed. ♣

Since $A \subset \overline{A}$ and $B \subset \overline{B}$ then $A \times B \subset \overline{A} \times \overline{B}$. Then by Lemma B $\overline{A \times B} \subset \overline{\overline{A} \times \overline{B}}$, but by Claim 1 $\overline{\overline{A} \times \overline{B}} = \overline{A} \times \overline{B}$ so $\overline{A \times B} \subset \overline{A} \times \overline{B}$. To see the reverse containment, take an element $(x, y) \in \overline{A} \times \overline{B}$ then for $x \in \overline{A}$ and $y \in \overline{B}$. Thus, by Theorem 17.5(a) for every neighborhood $U \ni x$ and $V \ni y$, we have $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. Thus, $U \times V \cap A \times B \neq \emptyset$ so by Theorem 17.5(b), since $U \times V$ is a basis element for the topology on $X \times Y$, $(x, y) \in \overline{A \times B}$. Thus, $\overline{A \times B} \supset \overline{A} \times \overline{B}$ and the equality $\overline{A \times B} = \overline{A} \times \overline{B}$ holds. ■

Problem 2. Let X be a topological space and let A be a dense subset of X. Let Y be a Hausdorff space and let $g, h \colon X \to Y$ be continuous functions which agree on A. Prove that $g = h$.

Proof. Suppose, towards a contradiction, that $g \neq h$. Then $g(x) \neq h(x)$ for some $x \in X \smallsetminus A$. Since Y is Hausdorff, there exists neighborhoods $U \ni g(x)$ and $V \ni h(x)$ with $U \cap V = \emptyset$. Since g and h are continuous, $g^{-1}(U)$ and $h^{-1}(V)$ are neighborhoods of x. In particular, $g^{-1}(U) \cap h^{-1}(U)$ is a nonempty neighborhood of x. Since $\overline{A} = X$, by Theorem 17.5(a), $\big(g^{-1}(U) \cap h^{-1}(V)\big) \cap A \neq \emptyset$. Let $x_0 \in \big(g^{-1}(U) \cap h^{-1}(V)\big) \cap A$. Then $g(x_0) = h(x_0) \in U \cap V$. This contradicts the fact that U and V were chosen to be disjoint. ■

Problem 3. Let X and Y be topological spaces and let $f \colon X \to Y$ be a continuous function. Let G_f (called the *graph* of f) be the subspace $\{\, x \times f(x) \mid x \in X \,\}$ of $X \times Y$. Prove that if Y is Hausdorff then G_f is closed.

Proof. We will show that the complement of G_f in $X \times Y$ is open. Let $(x, y) \in (X \times Y) \smallsetminus G_f$. Since Y is Hausdorff, choose neighborhoods U and V of y and $f(x)$ respectively, such that $f^{-1}(U) \cap V = \emptyset$. Then $f^{-1}(U) \times V \ni (x, y)$ is contained in the complement of G_f so, by Lemma C, G_f is open. ■

Problem 4. Let X be a topological space and let $f, g \colon X \to \mathbf{R}$ be continuous. Define $h \colon X \to \mathbf{R}$ by

$$h(x) = \min\{(f(x), g(x)\}.$$

Use the pasting lemma to prove that h is continuous. (You will not get full credit for any other method.)

Proof. Define the sets

$$A = \{\, x \in X \mid f(x) \leq g(x) \,\} \quad \text{and} \quad B = \{\, x \in X \mid f(x) \geq g(x) \,\}.$$

Note $X = A \cup B$ and $f(x) = g(x)$ for every $x \in A \cap B$. Moreover, we have that

$$h(x) = \min\{f(x), g(x)\} = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}.$$

Thus, by the pasting lemma, h is continuous if we can show that A and B are closed in X.

We will prove that the complement of A in X is open; the proof of B is similar. Let $x \in X \setminus A$. Then $f(x) > g(x)$. Thus we have the following result

Lemma 2. *Let $x, y \in X$ with the order topology. Then there exists a neighborhood $U \ni x$, $V \ni y$ with $U \cap V = \emptyset$ and $x' < y'$ for all $x' \in U$, $y' \in V$.*

Proof of lemma. We break the demonstration into the following cases:

Case 1: Suppose there exists $z \in X$ with $x < z < y$, i.e., $z \in (x, y)$. Let U be the ray $U = (-\infty, z)$ and V be the ray $V = (z, \infty)$. Then $U \cap V = \emptyset$ and for every $x' \in V$, $y' \in U$ $x' < c < y'$, in particular, $x' < y'$.

Case 2: Suppose that there does not exists $z \in X$ with $x < z < y$, i.e., $(x, y) = \emptyset$. Let U be the ray $U = (-\infty, x)$ and V be the ray $V = (y, \infty)$. Then $U \cap V = \emptyset$ and for every $x' \in U$, $y' \in V$ we have $x' < x < y < y'$, in particular, $x' < y'$. ♣

By Lemma 2, choose $U \ni g(x)$ and $V \ni f(x)$ as above. Then $g^{-1}(U) \cap f^{-1}(V)$ is a neighborhood of x with $g(x) < f(x)$ for all. Hence $g^{-1}(U) \cap f^{-1}(V) \subset X \setminus A$ and, by Lemma C, $X \setminus A$ is open. Thus, A is closed.

Having satisfied the conditions of the pasting lemma (Theorem 18.3), it follows that h is continuous. ■

Problem 5. Let X and Y be topological spaces and let $f \colon X \to Y$ be a function with the property that

$$f(\overline{A}) \subset \overline{f(A)}$$

for all subsets A of X. Prove that f is continuous.

Proof. Suppose that f has the property given above. Then we claim that:

Claim 3. *For every closed set B of Y, $f^{-1}(B)$, $f^{-1}(B)$ is closed in X.*

Proof of claim. Let B be closed in Y. We will show that $\overline{f^{-1}(B)} = f^{-1}(B)$. To that end, it suffices to show that $\overline{f^{-1}(B)} \subset f^{-1}(B)$ since the containment $\overline{f^{-1}(B)} \supset f^{-1}(B)$ is immediate (from the definition of the closure). By Munkres §2 Ex. 1(b), we have that $f(f^{-1}(B)) \subset B$ so if $x \in \overline{f^{-1}(B)}$ then $f(x) \in B$ since, by our assumption on f together with Lemma C, we have

$$f\big(\overline{f^{-1}(B)}\big) \subset \overline{f(f^{-1}(B))} \subset B.$$

Thus, $x \in f^{-1}(B)$ so $\overline{f^{-1}(B)} \subset f^{-1}(B)$ as desired. ♣

Let U be open in Y. Then $Y \setminus U$ is closed in Y. Then, by Claim 3, $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed in X so $X \setminus (X \setminus f^{-1}(U)) = f^{-1}(U)$ is open in X. Thus, f is continuous. ■

Problem 6. Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Prove that

$$f(\overline{A}) \subset \overline{f(A)}$$

for all subsets A of X.

Proof. Suppose f is continuous. Then, for every U open in Y, $f^{-1}(U)$ is open in X. Let $A \subset X$ and consider $\overline{f(A)}$. Then, $f^{-1}\big(Y \setminus \overline{f(A)}\big) = X \setminus f^{-1}\big(\overline{f(A)}\big)$ is open in X so its complement $f^{-1}\big(\overline{f(A)}\big)$ is closed in X. Moreover, by Munkres §2 Ex. 1(a), we we have $A \subset f^{-1}(f(A))$ and since, by Theorem 17.6, $\overline{f(A)} = f(A) \cup f(A)'$ we have that

$$A \subset f^{-1}\big(\overline{f(A)}\big) = f^{-1}(f(A) \cup f(A)') = f^{-1}(f(A)) \cup f^{-1}(f(A)').$$

In particular, by Lemma C, $\overline{A} \subset f^{-1}\big(\overline{f(A)}\big)$ so, by Munkres §2 Ex. 1(b), we have

$$f\big(\overline{A}\big) \subset f\big(f^{-1}\big(\overline{f(A)}\big)\big) \subset \overline{f(A)},$$

as desired. ■

Problem 7. Let X be any topological space and let Y be a Hausdorff space. Let $f, g\colon X \to Y$ be continuous functions. Prove that the set $\{\, x \in X \mid f(x) = g(x) \,\}$ is closed.

Proof. By Munkres §17 Ex. 13, Y is Hausdorff if and only if $\Delta_Y = \{\, (y, y) \mid y \in Y \,\}$ is closed in $Y \times Y$. By Theorem 18.4, the map $F = (f, g)\colon X \to Y \times Y$ is continuous since f and g are continuous. We claim that $F^{-1}(\Delta_Y) = \{\, x \in X \mid f(x) = g(x) \,\}$.

It is clear that if $f(x) = g(x) = y$ then $F(x) = (y, y) \in \Delta_Y$ so $F^{-1}(\Delta_Y) \supset \{\, x \in X \mid f(x) = g(x) \,\}$. Now suppose $x \in F^{-1}(\Delta_Y)$ then $F(x) = (f(x), g(x)) = (y, y) \in \Delta_Y$ so $f(x) = g(x) = y$ so $x \in \{\, x \in X \mid f(x) = g(x) \,\}$. Thus, $F^{-1}(\Delta_Y) = \{\, x \in X \mid f(x) = g(x) \,\}$ so, by Theorem 18.1(3), it follows that $\{\, x \in X \mid f(x) = g(x) \,\}$ is closed in X. ■

Problem 8. Let X be a topological space and A a subset of X. Suppose that

$$A \subset \overline{X \setminus \overline{A}}.$$

Prove that $\overline{A}$ does not contain any nonempty open set.

Proof. Suppose, seeking a contradiction, that $\operatorname{int} A \neq \emptyset$. Then there exists $x \in \operatorname{int} A \subset A$ and a neighborhood $U \ni x$ with $U \subset A$. Then $U \subset \overline{X \setminus \overline{A}}$. In particular, $x \in \overline{X \setminus \overline{A}}$ so $U \cap \overline{X \setminus \overline{A}} \neq \emptyset$. But $U \subset A \subset \overline{A}$ so $U \cap \big(X \setminus \overline{A}\big) = \emptyset$. This is a contradiction since $x \in \overline{X \setminus \overline{A}}$. Thus, $\operatorname{int} A = \emptyset$. ■

Problem 9. Let X be a topological space with a countable basis. Prove that every open cover of X has a countable subcover.

Proof. ■

Problem 10. Let X_α be an infinite family of topological spaces.

(a) Define the product topology on $\prod X_\alpha$.
(b) For each α, let A_α be a subspace of X_α. Prove that $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$.

Proof. (a)From Munkres §19, p. 114:

Definition. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \left\{ \pi_\beta^{-1}(U_\beta) \,\middle|\, U_\beta \text{ open in } X_\beta \right\},$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup \mathcal{S}_\beta.$$

The topology generated by the subbasis $\mathcal{S}$ is called the *product topology.*

Alternatively, we have the theorem:

Theorem (Munkres, Thm. 19.2)**.** *Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form*

$$\prod B_\alpha,$$

where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indicel α and $B_\alpha = X_\alpha$ for all the remaining indices is a basis for the product topology on $\prod X_\alpha$.

(b) (cf. Munkres §19, Theorem 19.5) Let $\mathbf{x} = (x_\alpha) \in \prod \overline{A}_\alpha$; we show that $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha \ni x$ be a basis element. Since $x_\alpha \in \overline{A}_\alpha$, there exists $y_\alpha \in U_\alpha \cap A_\alpha$ for each α. Then $\mathbf{y} = (y_\alpha)$ belongs to both U and $\prod A_\alpha$. Since U is arbitrary, it follows that $\mathbf{x} \in \overline{\prod A_\alpha}$.

Conversely, suppose that $\mathbf{x} = (x_\alpha) \in \overline{\prod A_\alpha}$; we show that $x_\beta \in \overline{A}_\beta$ for any index β. Let $V_\beta \ni x_\beta$ be an arbitrary neighborhood in X_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$, it contains a point $\mathbf{y} = (y_\alpha)$ of $\prod A_\alpha$. Then $y_\beta \in V_\beta \cap A_\beta$. It follows that $x_\beta \in \overline{A}_\beta$. ∎

Problem 11. Suppose that we are given an indexing set A, and for each $\alpha \in A$ a topological space X_α. Suppose also that for each $\alpha \in A$ we are given a point $b_\alpha \in X_\alpha$. Let $Y = \prod X_\alpha$ with the product topology. Let $\pi_\alpha \colon Y \to X_\alpha$ be the projection. Prove that the set

$$S = \{\, y \in Y \mid \pi_\alpha(y) = b_\alpha \text{ except for finitely many } \alpha \,\}$$

is dense in Y (that is, its closure is Y).

Proof. We want to show that $\overline{S} = X$ therefore, we will show that for every open subset U of X, $U \cap S \neq \emptyset$. By Theorem 17.5(b), it suffices to show this for basis elements. Let $\mathcal{B}_\alpha$ be a basis for X_α and $U = \prod U_\alpha$ be a basis element in the product topology on $\prod X_\alpha$. Then, by Theorem 19.2, $U_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $U_\alpha = X_\alpha$ for all the remaining indices. Hence, at least one $X_\alpha \ni b_\alpha$ so $U \cap S \neq \emptyset$. Since U was arbitrary, we conclude that $\overline{S} = X$. ∎

Problem 12. Let X be the Cartesian product $\mathbf{R}^\omega = \prod_{i=1}^\infty \mathbf{R}$ with the box topology (recall that a basis for this topology consists of all sets of the form $\prod_{i=1}^\infty U_i$, where each U_i is open in $\mathbf{R}$). Let $f \colon \mathbf{R} \to X$ be the function which takes t to $(t, t, t, ...)$. Prove that f is not continuous.

Proof. (cf. Example 2 in Munkres §19) It suffices to show that the preimage of a basis element U in the box topology is not open in $\mathbf{R}$. Let

$$U = \prod \left(-\frac{1}{n}, \frac{1}{n} \right).$$

Suppose that f is continuous. Then $f^{-1}(U)$ is open. Then by 18.1(4), for some $\delta > 0$, $(-\delta, \delta) \ni 0 \subset f^{-1}(U)$, $f((-\delta, \delta)) = \prod(-\delta, \delta) \subset B$. But, by the Archimedean principle, there exists $n \in \mathbf{Z}_+$ such that $1/n < \delta$ so $(-\delta, \delta) \not\subset (-1/N, 1/N)$ for any $N \geq n$. This is a contradiction. Therefore, f is not continuous on $\mathbf{R}^\omega$ with the box topology. ∎

Problem 13. Prove that the countable product $\mathbf{R}^\omega$ (with the product topology) has the following property: there is a countable family $\mathcal{F}$ of neighborhoods of the point $\mathbf{0} = (0, 0, 0, ...)$ such that for every neighborhood V of $\mathbf{0}$ there is a $U \in \mathcal{F}$ with $U \subset V$.

Note: the book proves that $\mathbf{R}^\omega$ is a metric space, but you may not use this in your proof. Use the definition of the product topology.

Proof. Define $\mathcal{F}$ to be the collection of all sets $U_{k,\ell} = \prod U_n$ where $U_n = (-1/k, 1/k)$ for $1 \leq n \leq \ell$ and $U_n = \mathbf{R}$ otherwise. Then we want to show that for every neighborhood V of $\mathbf{0}$, there exists $U \in \mathcal{F}$ with $U \subset V$. By Theorem 17.5(b) it suffices to prove this for basis elements containing $\mathbf{0}$. Hence, let $V = \prod V_n$ be a basis element containing $\mathbf{0}$. Then, by Theorem 19.2, V_n is a basis element for the standard topology on $\mathbf{R}$ containing 0, i.e, $V_n = (a_n, b_n)$ for $a_n < 0 < b_n$, for finitely many n and $V_n = \mathbf{R}$ otherwise. Without loss of generality, we may assume that $V = (a_1, b_1) \times \cdots (a_N, b_N) \times \mathbf{R} \times \cdots$. Let $\delta = \min\{|a_1|, b_1, ..., |a_N|, b_N\}$. Then by the Archimedean principle, there exists a positive integer m such that $1/m < \delta$. Thus, $U_{m,N} \subset V$. ∎

Problem 14. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X, thus an element of Y is a sequence of 0's and 1's. For each $n \geq 1$, let $y_0 \in Y$ be the element $(1, ..., 1, 0, ..)$, with n 1's at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1s. Prove that the set $\{y_n\}_{n \geq 1} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

Proof. Let $A = \{y_n\}_{n \geq 1} \cup \{y\}$. We will show that the complement of A in Y is open. By Lemma C, it suffices to find a basis element $U \ni \mathbf{x}$ with $U \cap A = \emptyset$. Let $\mathbf{x} \in Y \smallsetminus A$. Then $\mathbf{x}$ is a sequence of 0's and 1's where, say the first n terms, are not all 1. Let k, for $1 \leq k \leq n$, be the first zero to appear in the sequence $\mathbf{x}$ and ℓ, for $\ell > k$, be the first one to appear right after. Then the product $U = \prod U_n$ where

$$U_n = \begin{cases} \{0\} & \text{if } n = k, \\ \{1\} & \text{if } n = \ell, \\ X & \text{otherwise,} \end{cases}$$

is a basis element containing $\mathbf{x}$, but $U \cap A = \emptyset$ for otherwise there is a sequence $\mathbf{y} \in A$ with $y_k = 0$, but $y_\ell = 1$ which is impossible since $\ell > k$ and A consists of sequences $\mathbf{y}$ with the property that if $y_N = 1$ then $y_n = 1$ for all $n \leq N$. Thus, $Y \smallsetminus A$ is open so A is closed. ∎

Problem 15. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's. Let A be the subset of Y consisting of sequences with only a finite number of 1's. Is A closed? Prove or disprove.

Proof. A is not closed. Consider the point $\mathbf{1} = (1, 1, ...) \notin A$. But for every basis element $U = \prod U_n \ni \mathbf{1}$ where $U_n = X_n$ except for finitely many n's, $U \cap A \neq \emptyset$. ■

Problem 16. Let Y be a topological space.Let X be a set and let $f\colon X \to Y$ be a function. Give X the topology in which the open sets are the sets $f^{-1}(V)$ with V open in Y (you do not have to verify that this is a topology). Let $a \in X$ and let B be a closed set in X not containing a. Prove that $f(a)$ is not in the closure of $f(B)$.

Proof. Suppose B is closed in X and $a \in X \setminus B$. Then $X \setminus B$ is open in X so $X \setminus B = f^{-1}(V)$ for some V open in Y. Then $f(X \setminus B) \subset V \ni f(a)$ with $V \cap f(B) = \emptyset$ (otherwise $f(b) \in V$ for some $b \in B$, but the preimage of V lies in the complement of B). By Theorem 17.5(a), $f(a) \notin \overline{f(B)}$. ■

Problem 17. Let $f\colon X \to Y$ be a function that takes closed sets to closed sets. Let $y \in Y$ and let U be an open set containing $f^{-1}(y)$. Prove that there is an open set V containing y such that $f^{-1}(V)$ is contained in U.

Proof. Since U is open in X, $X \setminus U$ is closed in X. Since f is a closed mapping, $f(X \setminus U)$ is closed in Y so $Y \setminus f(X \setminus U)$ is open in Y. Moreover, $y \in Y \setminus f(X \setminus U)$ since $y \notin f(X \setminus U)$. Let $V \ni y$ open in Y. Then we claim that $f^{-1}(V) \subset U$. Otherwise, there exists $x \in f^{-1}(V) \cap (X \setminus U)$ so $f(x) \in V \cap f(X \setminus U)$, but this contradicts that $V \subset X \setminus f(X \setminus U)$. ■

Problem 18. Let X be a topological space with an equivalence relation $\sim$. Suppose that the quotient space $X/{\sim}$ is Hausdorff. Prove that the set $S = \{\, x \times y \in X \times X \mid x \sim y \,\}$ is a closed subset of $X \times X$.

Proof. Recall that a space Y is Hausdorff if and only if Δ_Y is closed in $Y \times Y$. Therefore, $X/{\sim}$ is Hausdorff implies $\Delta_{X/\sim}$ is closed in $X/{\sim} \times X/{\sim}$. Now consider the map $P = (p, p)\colon X \to X/{\sim} \times X/{\sim}$ where $p\colon X \to X/{\sim}$ is the quotient map on X. p is continuous by the definition of the quotient topology so by Theorem 18.4, the composite map P is continuous since it is continuous in each factor. Hence, we have that

$$\begin{aligned} P^{-1}(\Delta_{X/\sim}) &= \left\{\, (x, y) \in X \times X \;\middle|\; P(x, y) \in \Delta_{X/\sim} \,\right\} \\ &= \{\, (x, y) \in X \times X \mid p(x) = p(y) \,\} \\ &= \{\, (x, y) \in X \times X \mid x \sim y \,\} \\ &= S, \end{aligned}$$

so by Theorem 18.1(3), S is closed in X. ■

Problem 19. Let $p\colon X \to Y$ be a quotient map. Let us say that a subset S of X is *saturated* if it has the form $p^{-1}(T)$ for some subset T of Y. Suppose that for every $y \in Y$ and every open neighborhood U of $p^{-1}(y)$ there is a saturated open set V with $p^{-1}(y) \subset V \subset U$. Prove that p takes closed sets to closed sets.

Proof. Suppose that $W \neq X$ is closed so $X \setminus W$ is open. If $p(W) = Y$ we are done. Suppose $p(W) \neq Y$. Then there exists some $y \in Y \setminus p(W)$ so $p^{-1}(y) \subset X \setminus W$. Then, for some open $V \in Y$, $p^{-1}(y) \subset p^{-1}(V) \subset X \setminus W$. Thus, $y \in V \subset p(X \setminus W)$, but $p(X \setminus W) \subset Y \setminus p(W)$ since $y \in p(X \setminus W)$ if and only if $y = p(x)$ for $x \notin W$, but $y \in Y \setminus p(W)$ if and only if $y \neq p(x)$ for $x \in W$. Thus, $Y \setminus p(W)$ is open so $p(W)$ is closed. ■

Problem 20. Let X be a topological space, let D be a connected subset of X, and let $\{E_\alpha\}$ be a collection of connected subsets of X.

Prove that if $D \cap E_\alpha \neq \emptyset$ for all α, then $D \cup (\bigcup E_\alpha)$ is connected.

Proof. Consider the collection $\{D_\alpha\}$ where $D_\alpha = D \cup E_\alpha$. By Theorem 23.3, $D \cup E_\alpha$ is connected so every D_α is connected. Moreover $D_\alpha \cap D_\beta \supset D \neq \emptyset$ so by Theorem 23.3,

$$\bigcup D_\alpha = \bigcup D \cup E_\alpha = D \cup \left(\bigcup E_\alpha\right)$$

is connected. ∎

Problem 21. Let X and Y be connected. Prove that $X \times Y$ is connected.

Proof. Seeking a contradiction, suppose C, D is a separation of $X \times Y$. Fix an $y_0 \in Y$. Then the map $X \hookrightarrow X \times Y$ given by $x \mapsto (x, y_0)$ is continuous (by Theorem 18.4) so by Theorem 23.5 its image, $X \times y_0$, is connected. Similarly, the maps $y \mapsto (x, y)$ for fixed $x \in X$ are continuous and hence their images, $x \times Y$ are connected. Since $X \times y_0$ is connected, by Theorem 23.2, $X \times y_0 \subset C$ or D. Without loss of generality, suppose $X \times y_0 \subset C$. Then, since $x \times Y \cap X \times y_0 \ni (x, y_0) \neq \emptyset$ then $x \times Y \subset C$ for all x. Thus,

$$X \times y_0 \cup \left(\bigcup_{x \in X} x \times Y\right) = X \times Y \subset C$$

implies that $D = \emptyset$. This contradicts the assumption that C, D is a separation of $X \times Y$. ∎

Problem 22. For any space X, let us say that two points are "inseparable" if there is no separation $X = U \cup V$ into disjoint open sets such that $x \in U$ and $y \in V$.

Write $x \sim y$ if x and y are inseparable. Then $\sim$ is an equivalence relation (you don't have to prove this).

Now suppose that X is locally connected (this means that for every point x and every open neighborhood U of x, there is a connected open neighborhood V of x contained in U).

Prove that each equivalence class of the relation $\sim$ is connected.

Proof. Let x and C_x be the equivalence class of x. Then we claim that C_x is both closed and open.

To see that C_x is closed we will show that $C_x = \overline{C_x}$. Let $y \in \overline{C_x}$. Now, suppose $y \notin C_x$. Then, there exists disjoint neighborhoods $U \ni y$, $V \ni x$ with $X = U \cup V$. By Theorem 17.5(a), for every neighborhood $U \ni y$, $U \cap C_x \neq \emptyset$. Let $x' \in U \cap C_x$, then $U \ni x'$, $V \ni x$ is a separation of X so x' and x are separable. This contradicts that $x' \in C_x$. Thus, C_x is closed.

To see that C_x is open we will show that $X \setminus C_x$ is closed. Let $y \in \overline{X \setminus C_x}$ and suppose that $y \notin X \setminus X$. Then $y \in C_X$. Let $U \ni y$ be a neighborhood, then since X is locally connected, there exists a connected neighborhood $V \ni y$. But since $y \in \overline{X \setminus C_x}$, $V \cap X \setminus C_x \neq \emptyset$. Let $z \in V \cap X \setminus C_x$. Then there exist a separation $U_z \ni y$, $V_z \ni z$ of X, i.e., y and z are separable. Then $U_z \cap V$, $V_z \cap V$ is a separation of V. This is a contradiction. Thus, $X \setminus C_x$ is closed so C_x is open.

Now for the contradiction. Let C, D be a separation of C_x. Then by Theorem 16.2, C and D are open in X. Choose $x \in C$ and $y \in D$. Then $x \in C \cup (X \setminus C_x)$ and $y \in D$ is a separation of X so x and y are separable. This contradicts that $x, y \in C_x$, i.e., $x \sim y$. ∎

Problem 23. Let X be a topological space. Let $A \subset X$ be connected. Prove $\overline{A}$ is connected.

Proof. Seeking a contradiction, suppose C, D is a separation of $\overline{A}$. Then, by Theorem 23.2, $A \subset C$ or $A \subset D$. Suppose, without loss of generality, that $A \subset C$. Let B denote the closure of A in $\overline{A}$ with the subspace topology. Then by Theorem 17.4, $B = \overline{A} \cap \overline{A} = \overline{A}$. Let $x \in B \setminus C$. Then, for every neighborhood $U \ni x \subset D$, $U \cap A \neq \emptyset$. But $D \cap C = \emptyset$. This is a contradiction. Thus, $\overline{A}$ is connected. ∎

Problem 24. Let $X_1, X_2, \ldots$ be topological spaces. Suppose $\prod_{n=1}^{\infty} X_n$ is locally connected. Prove that all but finitely many X_n are connected.

Proof. Let $X = \prod X_n$ and U be a nonempty open set in X and pick an $x \in U$. Then, since X is locally connected, there exist a connected open neighborhood $V \ni x \subset U$. Now, since V is open let $W = \prod W_n$ be a basis element containing W. Then, by Theorem 19.2, $W_n = X_n$ for all but finitely many n. Then

$$\pi_n(V) \supset \pi_n(W) = W_n$$

is connected by Theorem 23.5. But $W_n = X_n$ for all but finitely many n so $\pi_n(V) = X_n$ for all but finitely many n. Thus, X_n is connected for all but finitely many n. ∎

Problem 25. Let X be a connected space and let $f : X \to Y$ be a function which is continuous and onto. Prove that Y is connected. (This is a theorem in Munkres—prove it from the definitions).

Proof. Seeking a contradiction, let C, D be a separation of Y. Then, since f is continuous and C and D are open, $f^{-1}(C)$ and $f^{-1}(D)$ are open. We claim that $f^{-1}(C), f^{-1}(D)$ is a separation of X. It is clear that $X = f^{-1}(C) \cup f^{-1}(D) = f^{-1}(C \cup D) = f^{-1}(Y)$. Moreover, $f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D) = \emptyset$. Thus, $f^{-1}(C), f^{-1}(D)$ is a separation. But X is connected. This is a contradiction. ∎

Problem 26. Given:

(i) $p : X \to Y$ is a quotient map.
(ii) Y is connected.
(iii) For every $y \in Y$, the set $p^{-1}(y)$ is connected.

Prove that X is connected.

Proof. ∎

Problem 27. Let A be a subset of $\mathbf{R}^2$ which is homeomorphic to the open unit interval $(0, 1)$. Prove that A does not contain a nonempty set which is open in $\mathbf{R}^2$.

Proof. Without loss of generality, assume that A is centered at the origin and contains an open ball D about 0 (by abuse of notation). Let $\varphi : A \to (0, 1)$ be a homeomorphism. Then $\varphi(D) \subset (0, 1)$ is connected. However, since φ is a homeomorphism, by Lemma A, $D \setminus \{0\} \approx \varphi(D) \setminus \{\varphi(0)\}$, but $\varphi(D) \setminus \{\varphi(0)\}$ is disconnected. This is a contradiction. ∎

Problem 28. Let X be a connected space. Let $\mathcal{U}$ be an open covering of X and let U be a nonempty set in $\mathcal{U}$. Say that a set V in $\mathcal{U}$ is *reachable from* U if there is a sequence $U = U_1, U_2, ..., U_n = V$ of sets in $\mathcal{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for each i from 1 to $n-1$. Prove that every nonempty V in $\mathcal{U}$ is reachable from U.

Proof. ∎

Problem 29. Suppose that X is connected and every point of X has a path-connected open neighborhood. Prove that X is path-connected.

Proof. ∎

Problem 30. Let X be a topological space and let $f, g \colon X \to [0,1]$ be continuous functions. Suppose that X is connected and f is onto. Prove that there must be a point $x \in X$ with $f(x) = g(x)$.

Proof. ∎