MA 544: Homework 9

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March 19, 2016

Problem 9.1 (Wheeden & Zygmund §6, Ex. 1)

(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{ y : (x, y) \in E \}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $\{ x : (x, y) \in E \}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every y. Show that for almost $y \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every x.

Proof. (a) That E has measure zero is a consequence of Fubini's theorem. Set $E_x \coloneqq \{ y : (x, y) \in E \}$ and $E_y \coloneqq \{ x : (x, y) \in E \}$ then, by Theorem 6.8, we have

$$|E| = \iint_{\mathbb{R}^2} \chi_E \, \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}} \left[\int_{E_x} 1 \, \mathrm{d}y \right] \mathrm{d}x = \int_{\mathbb{R}} \left[\int_{E_y} 1 \, \mathrm{d}x \right] \mathrm{d}y = 0. \tag{9.1}$$

Hence, E has measure zero. Moreover, we see that $\int_{\mathbb{R}} \left[\int_{E_y} 1 \, \mathrm{d}x \right] \mathrm{d}y = 0$ which means that for a.e. $y \in \mathbb{R}$, E_y has $\mathbb{R}^1$-measure zero.

(b) Let E be the set of all pairs $(x, y) \in \mathbb{R}^2$ such that $f(x, y)$ is not finite. By hypothesis, the set E_x has $\mathbb{R}^1$-measure zero for a.e. x. Therefore, by part (a) the set E_y has measure zero. Hence, for a.e. y, $f(x, y)$ is finite for a.e. x. ■

Problem 9.2 (Wheeden & Zygmund §6, Ex. 3)

Let f be measurable and finite a.e. on $[0,1]$. If $f(x)-f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$, show that $f \in L[0,1]$.

Proof. Put $I := [0,1]$. By Fubini's theorem, we have

$$\iint_{I\times I} f(x)-f(y)\,\mathrm{d}x\mathrm{d}y = \int_I \left[\int_I f(x)-f(y)\,\mathrm{d}y\right]\mathrm{d}x < \infty. \tag{9.2}$$

Here are some ideas. I suspect that in fact

$$\iint_{I\times I} f(x)-f(y)\,\mathrm{d}x\mathrm{d}y = 0$$

since we can do the following: Since $f(x)$ is a function of x, not of y, the integral

$$\int_I f(x)-f(y) = f(x)-M$$

where $M := \int_I f(y)\,\mathrm{d}y = \int_I f(x)\,\mathrm{d}x$. ■

Problem 9.3 (Wheeden & Zygmund §6, Ex. 4)

Let f be measurable and periodic with period 1: $f(t+1) = f(t)$. Suppose there is a finite c such that

$$\int_0^1 |f(a+t) - f(b+t)|\, dt \leq c$$

for all a and b. Show that $f \in L[0,1]$. (Set $a = x$, $b = -x$, integrate with respect to x, and make the change of variables $\chi = x + t$, $\eta = -x + t$.)

Proof. ■

Problem 9.4 (Wheeden & Zygmund §6, Ex. 6)

For $f \in L(\mathbb{R}^1)$, define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-ixt}\,dt$$

for $x \in \mathbb{R}^1$. (For complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i\int F_1$.) Show that if f and g belong to $L(\mathbb{R}^1)$, then

$$\widehat{(f * g)}(x) = 2\pi\hat{f}(x)\hat{g}(x).$$

Proof. ■

Problem 9.5 (Wheeden & Zygmund §6, Ex. 7)

Let F be a closd subset of $\mathbb{R}^1$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F, prove that the function

$$\int_{\mathbb{R}^1} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}}\, dt$$

is integrable over F and so is finite a.e. in F. (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

Proof. ∎

Problem 9.6 (Wheeden & Zygmund §6, Ex. 9)

(a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.

(b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subset \mathbb{R}^1$, and let M_α be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int M_\lambda \leq \lambda^{-1}|G|$, where $G = (a, b) - [c, d]$.

Proof. ■