MA 166: HW 18 Problem 4 Solution

TA: Carlos Salinas

February 29, 2016

Problem 1 (HW #18, #4)**.** Find the centroid of the region bounded by the given curves.

$$y = 6\sin 4x, \qquad y = 6\cos 4x, \qquad x = 0, \qquad x = \pi/16.$$

Solution. So I made a mistake when I calculated M_x on the board, but this should be correct now. Hope that you can adapt it to your specific problem.

First we find the area

$$\begin{aligned}
A &= \int_0^{\pi/16} 6\cos 4x - 6\sin 4x \, dx \\
&= 6\int_0^{\pi/16} \cos 4x - \sin 4x \, dx \\
&= 6\left[\frac{\sin 4x + \cos 4x}{4}\right]_0^{\pi/16} \\
&= \frac{6}{4}\left[\frac{\sin 4x + \cos 4x}{4}\right]_0^{\pi/16} \\
&= \frac{3}{2}(\sin(4\pi/16) + \cos(4\pi/16) - (\sin 4\cdot 0 + \cos 4 \cdot 0)) \\
&= \frac{3}{2}(\sin \pi/4 + \cos \pi/4 - \sin 0 - \cos 0) \\
&= \frac{3}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1\right) \\
&= \boxed{\frac{3}{2}\left(\sqrt{2} - 1\right).}
\end{aligned}$$

Now we find $\bar{x}$ and $\bar{y}$

$$\begin{aligned}
\bar{x} &= \frac{1}{A}\int_0^{\pi/16} x(6\cos 4x - 6\sin 4x) \, dx \\
&= \frac{6}{A}\int_0^{\pi/16} x\cos + x - x\sin 4x \, dx
\end{aligned}$$

Now use integration by parts or tabular integration to get that $\int x \cos x = (x \sin 4x)/4 + (\cos 4x)/16$ and $\int -x \sin 4x = (x \cos 4x)/4 - (\sin 4x)/16$

$$
\begin{aligned}
&= \frac{6}{A}\left[\frac{x\sin 4x}{4} + \frac{\cos 4x}{16} + \frac{x\cos 4x}{4} - \frac{\sin 4x}{16}\right]_0^{\pi/16} \\
&= \frac{6}{A}\left[\frac{1}{4}\left(x\sin 4x + \frac{\cos 4x}{4} + x\cos 4x - \frac{\sin 4x}{4}\right)\right]_0^{\pi/16} \\
&= \frac{3}{2A}\left[x\sin 4x + \frac{\cos 4x}{4} + x\cos 4x - \frac{\sin 4x}{4}\right]_0^{\pi/16} \\
&= \frac{3}{2A}\left[x(\sin 4x + \cos 4x) - \frac{1}{4}(\cos 4x - \sin 4x)\right]_0^{\pi/16} \\
&= \frac{3}{2\big(3/2(\sqrt{2}-1)\big)}\left[x(\sin 4x + \cos 4x) - \frac{1}{4}(\cos 4x - \sin 4x)\right]_0^{\pi/16} \\
&= \frac{1}{\sqrt{2}-1}\left(\frac{\pi\sqrt{2}}{16} - \frac{1}{4}\right) \\
&= \frac{1}{\sqrt{2}-1}\left(\frac{\pi\sqrt{2}}{16} - \frac{4}{16}\right) \\
&= \frac{1}{\sqrt{2}-1}\,\frac{\pi\sqrt{2}-4}{16} \\
&= \boxed{\frac{\pi\sqrt{2}-4}{16\big(\sqrt{2}-1\big)}}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{y} &= \frac{1}{A}\int_0^{\pi/16} \frac{(6\cos 4x)^2 - (6\sin 4x)^2}{2}\,dx \\
&= \frac{1}{A}\int_0^{\pi/16} \frac{36\cos^2 4x - 36\sin^2 4x}{2}\,dx \\
&= \frac{1}{A}\int_0^{\pi/16} \frac{36}{2}\big(\cos^2 4x - \sin^2 4x\big)\,dx \\
&= \frac{18}{A}\int_0^{\pi/16} \cos^2 4x - \sin 4x^2\,dx
\end{aligned}
$$

use the identity $\cos^2\theta - \sin^2\theta = \cos 2\theta$

$$
\begin{aligned}
&= \frac{18}{A}\int_0^{\pi/16} \cos 8x \, dx \\
&= \frac{18}{A}\left[\frac{\sin 8x}{8}\right]_0^{\pi/16} \\
&= \frac{18}{(3/2)\left(\sqrt{2}-1\right)}\left(\frac{\sin(8(\pi/16))}{8} - \frac{\sin(8\cdot 0)}{8}\right) \\
&= \frac{18}{(3/2)\left(\sqrt{2}-1\right)}\left(\frac{\sin(\pi/2)}{8} - \frac{\sin 0}{8}\right) \\
&= \frac{18}{(3/2)\left(\sqrt{2}-1\right)}\left(\frac{1}{8} - 0\right) \\
&= \frac{2\cdot 3\cdot 3}{(3/2)2\cdot 2\cdot 2\left(\sqrt{2}-1\right)} \\
&= \boxed{\frac{3}{2\left(\sqrt{2}-1\right)}.}
\end{aligned}
$$

☺