

# MA553: Qual Problems

Carlos Salinas

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## 1 Ulrich's MA 553 Problems for Spring '16

### 1.1 Homework 1

**Problem 1.1.** Let  $G$  be a group,  $a \in G$  an element of finite order  $m$ , and  $n$  a positive integer. Prove that

$$|a^n| = \frac{m}{\gcd(m, n)}.$$

*Proof.* ■

**Problem 1.2.** Let  $G$  be a group, and let  $a, b$  be elements of finite order  $m, n$  respectively. Show that if  $ba = ab$  and  $\langle a \rangle \cap \langle b \rangle = \{e\}$ , then  $|ab| = \text{lcm}(m, n)$ .

*Proof.* ■

**Problem 1.3.** Let  $G$  be a group and  $H, K$  normal subgroups with  $H \cap K = \{e\}$ . Show that

- (a)  $hk = kh$  for every  $h \in H, k \in K$ .
- (b)  $HK$  is a subgroup of  $G$  with  $HK \cong H \times K$ .

*Proof.* ■

**Problem 1.4.** Show that  $A_4$  has no subgroup of order 6 (although  $6 \mid 12 = |A_4|$ ).

*Proof.* ■

## 1.2 Homework 2

**Problem 1.5.** Let  $G$  be the group of order  $2^3 \cdot 3$ ,  $n \geq 2$ . Show that  $G$  has a normal 2-subgroup  $\neq \{e\}$ .

*Proof.* ■

**Problem 1.6.** Let  $G$  be a group of order  $p^2q$ ,  $p$  and  $q$  primes. Show that the Sylow  $p$ -Sylow subgroup or the  $q$ -Sylow subgroup of  $G$  is normal in  $G$ .

*Proof.* ■

**Problem 1.7.** Let  $G$  be a subgroup of order  $pqr$ ,  $p < q < r$  primes. Show that the  $r$ -Sylow subgroup of  $G$  is normal in  $G$ .

*Proof.* ■

**Problem 1.8.** Let  $G$  be a group of order  $n$  and let  $\varphi: G \rightarrow S_n$  be given by the action of  $G$  on  $G$  via translation.

- (a) For  $a \in G$  determine the number and the lengths of the disjoint cycles of the permutation  $\phi(a)$ .
- (b) Show that  $\varphi(G) \not\subset A_n$  if and only if  $n$  is even and  $G$  has a cyclic 2-Sylow subgroup.
- (c) If  $n = 2m$ ,  $m$  odd, show that  $G$  has a subgroup of index 2.

*Proof.* ■

**Problem 1.9.** Show that the only simple groups  $\neq \{e\}$  of order  $< 60$  are the groups of prime order.

*Proof.* ■

## 1.3 Homework 3

**Problem 1.10.** Let  $G$  be a finite group,  $p$  a prime number,  $N$  the intersection of all  $p$ -Sylow subgroups of  $G$ . Show that  $N$  is a normal  $p$ -subgroup of  $G$  and that every normal  $p$ -subgroup of  $G$  is contained in  $N$ .

*Proof.* ■

**Problem 1.11.** Let  $G$  be a group of order 231 and let  $H$  be an 11-Sylow subgroup of  $G$ . Show that  $H \subset Z(G)$ .

*Proof.* ■

**Problem 1.12.** Let  $G = \{e, a_1, a_2, a_3\}$  be a non-cyclic group of order 4 and define  $\varphi: S_3 \rightarrow (G)$  by  $\varphi(\sigma)(e) = e$  and  $f(\sigma)(a_1) = a_{\sigma(i)}$ . Show that  $\varphi$  is well-defined and an isomorphism of groups.

*Proof.* ■

**Problem 1.13.** Determine all groups of order 18.

*Proof.*

