MA571 Problem Set 2

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Problem 2.1 (Munkres §17, p. 100, 2)

Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. Let C denote the closure of A in X then, by Theorem 17.4, $A = \bar{A} = C \cap Y$ is the closure of A in Y. Thus, A is closed in X since it is the intersection of two closed subsets of X. ∎

Problem 2.2 (Munkres §17, p. 100, 3)

Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Proof. Before proceeding we will prove the following set theoretic result (which was adapted from Exercises 2(n) and 2(o) from §1, p. 14 of Munkres):

Lemma 5. *For sets A, B, C and D we the following equalities hold:*

(a) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(b) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.
(c) $(A \setminus C) \times B = (A \times B) \setminus (C \times B)$.

that is, the Cartesian product distributes over taking complements.

Proof of Lemma 4. (a) The equality follows (rather straightforwardly) from the definition of the Cartesian product and the complement of a set for $x \times y \in (A \times B) \cap (C \times D)$ if and only if $x \times y \in A \times B$ and $x \times y \in C \times D$ if and only if $x \in A$ and $x \in C$ and $y \in B$ and $y \in D$ if and only if $x \in A \cap C$ and $y \in B \cap D$ if and only if $x \times y \in (A \cap C) \times (B \cap D)$.

(b) The point $x \times y \in A \times (B \setminus C)$ if and only if $x \in A$ and $y \in B \setminus C$ if and only if $x \in A$ and $y \in B$ and $y \notin C$ if and only if $x \times y \in A \times B$ and $x \times y \notin A \times C$ if and only if $x \times y \in (A \times B) \setminus (A \times C)$.

(c) The very same argument as part (b) can be used, taking B to be a subset of A and replacing (where appropriate) A by $A \setminus B$ and $B \setminus C$ by C, to prove that

$$(A \setminus B) \times C = (A \times C) \setminus (B \times C). \qquad \blacklozenge$$

Now, since A is closed in X and B is closed in Y, their complements, $X \setminus A$ and $Y \setminus B$ are, by definition, open in X and Y, respectively. Then, the sets

$$(X \setminus A) \times Y \quad \text{and} \quad X \times (Y \setminus B)$$

are open since they are basic open sets in the product topology on $X \times Y$. So, applying Lemma 4(b) and (c), their complements

$$(X \times Y) \setminus (X \setminus A) \times Y = A \times Y \quad \text{and} \quad (X \times Y) \setminus X \times (Y \setminus B) = X \times B$$

are closed in $X \times Y$. At last, we have that

$$(A \times Y) \cap (X \times B)$$

is the intersection of closed sets, hence, by Theorem 17.1(b), is closed. By Lemma 4(a),

$$(A \times Y) \cap (X \times B) = (A \cap X) \times (Y \cap B) = A \times B$$

so $A \times B$ is closed in $X \times Y$. ∎

Problem 2.3 (Munkres §17, p. 101, 6(b))

Let A, B and A_α denote subsets of a space X. Prove the following:

(b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof. By definition, the closure of a set is the intersection of all closed sets which contain it therefore, $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ since $\bar{A} \cup \bar{B}$ is a closed set, by Theorem 17.1(a), which contains $A \cup B$. To see the reverse containment note that $\bar{A} \subset \overline{A \cup B}$ since $\overline{A \cup B}$ is a closed set which contains A. Similarly $\bar{B} \subset \overline{A \cup B}$ so $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. Therefore, $\overline{A \cup B} = \bar{A} \cup \bar{B}$ holds.

Naturally this results extends, by induction, to the case of finite unions of sets. ∎

Problem 2.4 (Munkres §17, p. 101, 6(c))

Let A, B and A_α denote subsets of a space X. Prove the following:

(b) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$.

Proof. Let C denote the set $\overline{\bigcup A_\alpha}$. It is clear, by the definition of the closure of a set, that $\bar{A}_\alpha \subset C$ for every α since C is a closed set which contains A_α, so $\bigcup_\alpha \bar{A}_\alpha \subset C$.

The reverse is not true in general; in fact, as Theorem 17.1(3) suggests, an arbitrary union of closed sets is not even necessarily closed. For a concrete example consider the family $A_r = \{r\}$ for $r \in \mathbf{Q}$. The closure of a point r in $\mathbf{R}$ is itself since its complement, $\mathbf{R} \setminus \{r\}$, is the union of the open intervals $(-\infty, r)$ and (r, ∞); in particular, $\{r\}$ is the "smallest" closed set containing $\{r\}$. Hence, we see that the union

$$\bigcup_{r \in \mathbf{Q}} \bar{A}_r = \mathbf{Q},$$

but, by Example 6, $\bar{\mathbf{Q}} = \mathbf{R}$. ■

Problem 2.5 (Munkres §17, p. 101, 7)

Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \bar{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so x must belong to the closure of some A_α. Therefore, $x \in \bigcup \bar{A}_\alpha$.

Critique. The main argument, that "x must belong to the closure of some A_α", is what is wrong here. The point x may belong to the closure of multiple A_α's, in fact uncountably many of them, so that one would have to prove that if x belongs the closure of some family A_β of set, then x must belong to the union of their closures. This takes us right back to what we are trying to prove. ■

Problem 2.6 (Munkres §17, p. 101, 9)

Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}.$$

Proof. By Problem 2.2, $\bar{A} \times \bar{B}$ is a closed set which contains $A \times B$ so it must contain the closure of $A \times B$, i.e., $\overline{A \times B} \subset \bar{A} \times \bar{B}$. To see the reverse containment, take a point $x \times y \in \bar{A} \times \bar{B}$. Then, by Theorem 17.5(a), for every neighborhood U of x and every neighborhood V of y, the intersections $U \cap A$ and $V \cap B$ are nonempty. Thus, by Lemma 5(a), the set

$$(V \times U) \cap (A \times B) = (V \cap A) \times (U \cap B)$$

is nonempty. Then, since $U \times V$ is an arbitrary basis element containing $x \times y$, by Theorem 17.5(b) $x \times y \in \overline{A \times B}$. Thus, $\overline{A \times B} = \bar{A} \times \bar{B}$. ∎

Problem 2.7 (Munkres §17, p. 101, 10)

Show that every order topology is Hausdorff.

Proof. Let $(X, <)$ denote a nonempty set equipped with a simple order relation. Then by the definition on Munkres §14, p. 84, a basis for the order topology on X are sets of the following types:

(1) All open intervals (a, b) in X.
(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

Let a and b be two distinct points in X; we may assume, without loss of generality, that $a < b$. Then, we must show that there exists neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$.

If X set with finite cardinality the order topology on X will coincide with the discrete topology so that we may take $\{a\}$ and $\{b\}$ to be neighborhoods of a and b. Then, $\{a\} \cap \{b\} = \emptyset$ so X is Hausdorff.

Now, suppose X is not of finite cardinality. Define the sets

$$C = (a, b), \quad A = \{\, x \in X \mid x < a\} \quad \text{and} \quad B = \{\, x \in X \mid x > b\}.$$

Then at least one of A, B or C is nonempty and has infinite cardinality.

Suppose A is nonempty, but B and C are empty. Take any element $x \in A$, then (x, b) is a neighborhood of a and b must be a largest element so $(a, b_0] = C \cup \{b\} = \{b\}$ is a neighborhood of b satisfying $(x, b) \cap \{b\} = \emptyset$. Similarly, if B is nonempty, but A and C are empty, $\{a\}$ and (a, x) for some $x \in B$ are neighborhoods of a and b, respectively, with $\{a\} \cap (a, x) = \emptyset$.

If C is nonempty but A and B are empty, a must be a smallest element and b must be a largest element. Then, since X is not finite, there exist at least two distinct elements x and y in C with $x < y$ so $[a, x)$ and $(y, b]$ are neighborhoods of a and b, respectively, with $[a, x) \cap (y, b] = \emptyset$.

Now, suppose at least two of A, B and C are nonempty. If C is empty, but A and B are nonempty. Then the intervals $(x, b) = (x, a]$ and $(a, y) = [b, y)$ are neighborhoods of a and b respectively with $(x, b) \cap (a, y) = \emptyset$. If A is empty, but B and C are nonempty, then a is a smallest element. Then there exists at least two distinct elements x and y with $x < y$ in C so that $[a, x)$ and (y, b) are neighborhoods of a and b, respectively, with $[a, x) \cap (y, b) = \emptyset$. Similarly, if B is empty, but A and C are nonempty, for any $x < y$ in C, (a, x) and $(y, b]$ are neighborhoods of a and b, respectively, with $(a, x) \cap (y, b]$.

Lastly, if A, B and C are nonempty we win! Then, for any $x \in A$, $y \in B$ and $z, w \in C$ with $z < w$ the intervals (x, z) and (w, y) are neighborhoods of a and b, respectively, with $(x, z) \cap (w, y) = \emptyset$.

In every case, X satisfies the Hausdorff property. ■

***Remarks**.* Perhaps there is a better way to approach this problem. The demonstration is thorough and covers every case, but we still desire a more elegant proof.

Problem 2.8 (Munkres §17, p. 101, 13)

Show that X is Hausdorff if and only if the *diagonal* $\Delta = \{\, x \times x \mid x \in X \,\}$ is closed in $X \times X$.

Proof. $\Longrightarrow$ Suppose X is Hausdorff. The diagonal Δ is closed, by definition, if and only if its complement, $(X \times X) \setminus \Delta$, is open in $X \times X$. Let $x \times y \in (X \times X) \setminus \Delta$. Since X is Hausdorff, there exists neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$. Thus, $U \times V$ is a neighborhood of $x \times y$ contained in $(X \times X) \setminus \Delta$. By the definition on Munkres §13 p. 78, since for every point $x \times y \in (X \times X) \setminus \Delta$ we may find a basis element $U \times V \subset (X \times X) \setminus \Delta$ containing $x \times y$, it follows that $(X \times X) \setminus \Delta$ is open. Thus, Δ is closed.

$\Longleftarrow$ Suppose Δ is closed. Then the complement of Δ is open in $X \times X$. Thus, for every $x \times y$ in the complement of Δ, we may find a basis element $U \times V \subset (X \times X) \setminus \Delta$ containing $x \times y$. Thus, U and V are neighborhoods of x and y, respectively, such that $U \cap V = \emptyset$ (for otherwise $z \times z \in U \times V$ but $U \times V$ is in the complement of Δ). Thus, X is Hausdorff. ■

Problem 2.9 (Munkres §18, p. 111, 4)

Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f\colon X \to X \times Y$ and $g\colon Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

Proof. We will prove the result for f only as the proof for g is analogous. Let $Z = \operatorname{im} f$. To show that $f\colon X \to X \times Y$ is an imbedding, we must demonstrate that the map $f'\colon X \to Z$, obtained by restricting the range of f, is a continuous injection with a continuous inverse. To see that f' is indeed injective suppose $f'(x) = f(x) = f(x)' = f'(x')$ for $x, x' \in X$. Then $x \times y_0 = x' \times y_0$ so $x = x'$. Thus f' is injective. Now, f is continuous since, by Theorem 18.1(4), for each $x \in X$ and each neighborhood V of $f(x) = x \times y_0$, there is an open set U, namely $\pi_X(V)$ which is open since $\pi_X\colon Z \to X$ is an open map by Problem 1.7 (Munkres §16, Exercise 4), with $f(U) = \{\, x \times y_0 \mid x \in \pi_X(V) \,\} \subset V$ ∎

Problem 2.10 (Munkres §18, p. 111-112, 8(a,b))

Let Y be an ordered set in the order topology. Let $f, g \colon X \to Y$ be continuous.

(a) Show that the set $\{ x \mid f(x) \leq g(x) \}$ is closed in X.
(b) Let $h \colon X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [*Hint:* Use the pasting lemma.]

Proof. ■

Problem 2.11

Given: X is a topological space with open sets $U_1, ..., U_n$ such that $\bar{U}_i = X$ for all i. Prove that the closure of $U_1 \cap \cdots \cap U_n$ is X.

Proof. ∎