# Math 535 - General Topology
# Fall 2012
# Homework 8 Solutions

**Problem 1.** (Willard Exercise 19B.1) Show that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the $n$-dimensional sphere $S^n$.

**Solution.** Note that $S^n$ is compact, and a punctured sphere $S^n \setminus \{p\}$ is dense in $S^n$, with complement consisting of a single point $\{p\}$. Therefore it suffices to show that $\mathbb{R}^n$ is homeomorphic to $S^n \setminus \{p\}$.

Consider $\mathbb{R}^n \cong \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ as the hyperplane $x_{n+1} = 0$ inside $\mathbb{R}^{n+1}$. Take the "North pole" $p = (0, \ldots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$. We will produce a homeomorphism $f \colon \mathbb{R}^n \to S^n \setminus \{p\}$ using "stereographic projection". Define the maps

$$f \colon \mathbb{R}^n \to S^n \setminus \{p\}$$

where $f(x)$ is the unique point of $S^n \setminus \{p\}$ which lies on the straight line through $x$ and $p$ and

$$g \colon S^n \setminus \{p\} \to \mathbb{R}^n$$

where $g(y)$ is the unique point of $\mathbb{R}^n$ which lies on the straight line through $y$ and $p$.

For all $x \in \mathbb{R}^n$, $g(f(x))$ is the unique point of $\mathbb{R}^n$ which lies on the straight line through $f(x)$ and $p$. By definition of $f$, $x$ is a point of $\mathbb{R}^n$ which lies on the straight line through $f(x)$ and $p$. This proves $g(f(x)) = x$.

For all $y \in S^n \setminus \{p\}$, $f(g(y))$ is the unique point of $S^n \setminus \{p\}$ which lies on the straight line through $g(y)$ and $p$. By definition of $g$, $y$ is a point of $S^n \setminus \{p\}$ which lies on the straight line through $g(y)$ and $p$. This proves $f(g(y)) = y$.

It remains to show that $f$ and $g$ are both continuous.

A straightforward calculation yields

$$f(x) = \frac{1}{\|x\|^2 + 1} \left[ 2x + (\|x\|^2 - 1)p \right]$$

which is a continuous function of $x$. Indeed, the norm $\|x\|$ is continuous in $x$, and the denominator satisfies $\|x\|^2 + 1 > 0$ for all $x \in \mathbb{R}^n$.

Another straightforward calculation yields

$$g(y) = \frac{1}{1 - y_{n+1}} \left[ y - y_{n+1} p \right]$$

which is a continuous function of $y$. Indeed, the projection $\mathbb{R}^{n+1} \twoheadrightarrow \mathbb{R}$ onto each coordinate is continuous (so that $y_{n+1}$ is a continuous function of $y$), and the denominator satisfies $1 - y_{n+1} > 0$ for all $y \in S^n \setminus \{p\}$. $\square$

**Problem 2.** (Munkres Exercise 38.4) Let $X$ be any topological space and $f\colon X \hookrightarrow Y$ a compactification of $X$, where moreover $Y$ is Hausdorff. Show that there is a unique continuous closed surjective map $g\colon \beta X \to Y$ that extends $f$, meaning $g \circ e = f$, where $e\colon X \to \beta X$ is the canonical evaluation map:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "f"'] & \beta X \arrow[d, dashed, "g"] \\ & Y. \end{tikzcd}$$

**Hint:** You do not need anything about a specific construction of $\beta X$, only its universal property.

**Solution.** Since $Y$ is compact Hausdorff, there is a unique continuous map $g\colon \beta X \to Y$ satisfying $g \circ e = f$ (by the universal property of the Stone-Čech compactification).

Since $\beta X$ is compact and $Y$ is Hausdorff, the continuous map $g\colon \beta X \to Y$ is closed.

Note that $g$ has dense image in $Y$:

$$\overline{g(\beta X)} \supseteq \overline{g(e(X))} = \overline{f(X)} = Y.$$

However, $g$ is closed and thus its image is closed:

$$g(\beta X) = \overline{g(\beta X)} = Y$$

which proves that $g$ is surjective. $\square$

*Remark.* If $X$ admits a Hausdorff compactification, we know that $X$ must be Tychonoff (a.k.a. $T_{3\frac{1}{2}}$), so that $e\colon X \to \beta X$ is in fact an embedding and $\beta X$ is therefore a compactification of $X$.

Problem 2 shows in particular that any Hausdorff compactification of $X$ is a quotient of the Stone-Čech compactification $\beta X$.

**Problem 3.** Let $f\colon X \to Y$ be a continuous map between topological spaces. Show that $f$ is closed if and only if for all $y \in Y$ and any open subset $U \subseteq X$ satisfying $f^{-1}(y) \subseteq U$, there is an open neighborhood $V$ of $y$ satisfying $f^{-1}(V) \subseteq U$.

**Solution.** Consider the equivalent statements:

$f$ is closed, i.e. for all $C \subseteq X$ closed, $f(C)$ is closed in $Y$.

$\Leftrightarrow$ For all $C \subseteq X$ closed and $y \notin f(C)$, there is an open neighborhood $V$ of $y$ satisfying $V \cap f(C) = \emptyset$.

$\Leftrightarrow$ For all $C \subseteq X$ closed and $y \in Y$ satisfying $f^{-1}(y) \cap C = \emptyset$, there is an open neighborhood $V$ of $y$ satisfying $f^{-1}(V) \cap C = \emptyset$.

$\Leftrightarrow$ (Taking $U := X \setminus C$) For all $U \subseteq X$ open and $y \in Y$ satisfying $f^{-1}(y) \subseteq U$, there is an open neighborhood $V$ of $y$ satisfying $f^{-1}(V) \subseteq U$. $\square$

**Problem 4.** Let $f\colon X \to Y$ be a continuous map between topological spaces.

**a.** Assume that $f\colon X \to Y$ is proper. Let $V \subseteq Y$ be an open subset. Show that the restriction $f|_{f^{-1}(V)}\colon f^{-1}(V) \to V$ is proper.

**Solution.** Let $K \subseteq V$ be a compact subspace. Then the preimage

$$(f|_{f^{-1}(V)})^{-1}(K) = f^{-1}(K) \subseteq f^{-1}(V) \subseteq X$$

is compact since $f\colon X \to Y$ is proper. $\square$

**b.** Assume that $Y$ is Hausdorff, and that for all $y \in Y$, there is an open neighborhood $V$ of $y$ such that the restriction $f|_{f^{-1}(V)} \colon f^{-1}(V) \to V$ is proper. Show that $f \colon X \to Y$ is proper.

**Solution.**

**Lemma.** *Let $K$ be a compact Hausdorff space and $\{V_\alpha\}_{\alpha \in A}$ an open cover of $K$. Then there are compact subspaces $K_1, \ldots, K_n \subseteq K$ satisfying $K = K_1 \cup \ldots \cup K_n$ and each $K_i$ is included in some $V_{\alpha_i}$.*

*Proof.* Since $K$ is compact, there is a finite subcover $\{V_{\alpha_1}, \ldots V_{\alpha_n}\}$. Thus it suffices to show the result for any finite open cover $\{V_1, \ldots V_n\}$ of $K$.

Writing $K$ as the union of two open subsets

$$K = V_1 \cup \ldots \cup V_n = V_1 \cup (V_2 \cup \ldots \cup V_n),$$

it suffices to show the result for an open cover consisting for two open subsets, and then apply the argument $n-1$ times.

Let $\{V_1, V_2\}$ be an open cover of $K$. Since $K$ is compact Hausdorff, it is normal. The equality $K = V_1 \cup V_2$ implies $V_2^c \subseteq V_1$, and note that $V_2^c$ is closed and $V_1$ is open. By normality, there is an open subset $U \subseteq X$ satisfying

$$V_2^c \subseteq U \subseteq \overline{U} \subseteq V_1.$$

Take $K_1 := \overline{U} \subseteq V_1$ and $K_2 := U^c \subseteq (V_2^c)^c = V_2$. Both $K_1$ and $K_2$ are closed in $K$, hence compact, and they cover $K$:

$$K_1 \cup K_2 = \overline{U} \cup U^c \supseteq U \cup U^c = K.$$

□

Now we solve the problem. Let $K \subseteq Y$ be compact. By assumption, there exists an open cover $\{V_\alpha\}_{\alpha \in A}$ of $Y$ such that the restrictions

$$f|_{f^{-1}(V_\alpha)} \colon f^{-1}(V_\alpha) \to V_\alpha$$

are proper.

By the lemma, $K$ can be written as a finite union

$$K = K_1 \cup \ldots \cup K_n$$

where each $K_i$ is compact and satisfies $K_i \subseteq V_{\alpha_i} \cap K \subseteq V_{\alpha_i}$. This implies that the preimage

$$f^{-1}(K_i) = \left(f|_{f^{-1}(V_{\alpha_i})}\right)^{-1}(K_i) \subseteq f^{-1}(V_{\alpha_i}) \subseteq X$$

is compact because $f|_{f^{-1}(V_{\alpha_i})}$ is proper. The preimage

$$\begin{aligned} f^{-1}(K) &= f^{-1}(K_1 \cup \ldots \cup K_n) \\ &= f^{-1}(K_1) \cup \ldots \cup f^{-1}(K_n) \end{aligned}$$

is a finite union of compact spaces, hence compact. □

**Alternate solution.** For each point $y \in K$, let $V_y$ be a neighborhood of $y$ such that the restriction $f|_{f^{-1}(V_y)}$ is proper. Note that $V_y \cap K$ is a $K$-neighborhood of $y$.

Since $K$ is compact Hausdorff (in particular locally compact Hausdorff), for each $y \in K$ there is a compact $K$-neighborhood $K_y$ of $y$ inside $V_y \cap K$.

The cover $\{K_y\}_{y \in K}$ of $K$ can be refined to an open cover of $K$. Hence, by compactness of $K$, there is a finite subcover

$$K = K_{y_1} \cup \ldots \cup K_{y_n}$$

where each of these satisfies $K_{y_i} \subseteq V_{y_i} \cap K \subseteq V_{y_i}$.

Conclude as above. $\square$

Recall that a **totally ordered set** is a partially ordered set $(X, \leq)$ where any two elements are comparable: for all $x, y \in X$, either $x \leq y$ or $y \leq x$.

**Problem 5.** Let $(X, \leq)$ be a totally ordered set. The **order topology** on $X$ is the topology generated by "open rays"

$$(a, \infty) := \{x \in X \mid x > a\}$$
$$(-\infty, a) := \{x \in X \mid x < a\}$$

for any $a \in X$.

**a.** Show that the order topology on a totally ordered set is always $T_1$.

**Solution.** Because $X$ is a totally ordered set, given any $a \in X$, every element $x \in X$ satisfies exactly one of the three conditions

$$\begin{cases} x < a \\ x = a \\ x > a \end{cases}$$

so that the "closed rays"

$$[a, \infty) := \{x \in X \mid x \geq a\} = X \setminus (-\infty, a)$$
$$(-\infty, a] := \{x \in X \mid x \leq a\} = X \setminus (a, \infty)$$

are indeed closed in $X$. Their intersection

$$(-\infty, a] \cap [a, \infty) = \{x \in X \mid x \leq a \text{ and } x \geq a\} = \{a\}$$

is therefore closed in $X$, so that all singletons of $X$ are closed. $\square$

**Alternate solution.** Let $x, y \in X$ be distinct points of $X$. Since $X$ is totally ordered, $x$ and $y$ are (strictly) comparable, WLOG $x < y$. Then the open ray $(-\infty, y)$ contains $x$ but not $y$, while the open ray $(x, \infty)$ contains $y$ but not $x$. $\square$

**b.** Show that the order topology on $\mathbb{R}$ with its usual order $\leq$ is the standard (metric) topology on $\mathbb{R}$.

**Solution.** ($\mathcal{T}_{\text{ord}} \leq \mathcal{T}_{\text{met}}$) For any $a \in \mathbb{R}$, the "open rays" $(a, \infty)$ and $(-\infty, a)$ are metrically open.

($\mathcal{T}_{\text{met}} \leq \mathcal{T}_{\text{ord}}$) The metric topology on $\mathbb{R}$ is generated by intervals $(a, b)$ for any $a < b$. But these are order-open since they are the finite intersection

$$(a, b) = (-\infty, b) \cap (a, \infty)$$

of "open rays". $\square$

**c.** An **interval** in a partially ordered set $(X, \leq)$ is a subset $I \subseteq X$ such that for all $x, y \in I$, the condition $x \leq z \leq y$ implies $z \in I$.

Let $(X, \leq)$ be a totally ordered set endowed with the order topology. Show that every connected subspace $A \subseteq X$ is an interval in $X$.

**Solution.** We will show the contrapositive: Any subset $A \subseteq X$ which is not an interval is not connected.

The fact that $A$ is not an interval means that there are elements $x, y \in A$ and an element $z \in X$ satisfying $x \leq z \leq y$ but $z \notin A$. This implies

$$\begin{aligned} A &\subseteq X \setminus \{z\} \\ &= (-\infty, z) \sqcup (z, \infty) \quad \text{since } X \text{ is totally ordered} \end{aligned}$$

so that $A$ can be written as the disjoint union

$$A = (A \cap (-\infty, z)) \sqcup (A \cap (z, \infty))$$

of subsets that are both open in $A$. Both subsets are non-empty, by the conditions $x \in A \cap (-\infty, z)$ and $y \in A \cap (z, \infty)$. Therefore $A$ is not connected. □

**d.** Find an example of totally ordered set $(X, \leq)$, endowed with the order topology, and an interval $A \subseteq X$ which is not a connected subspace.

**Solution.** Consider $X = \{0, 1\}$, and the subset $X \subseteq X$, which is trivially an interval in $X$.

Since $X$ is finite and $T_1$ (by part a), it is discrete. Since $X$ contains at least two elements, it is not connected. □

**Alternate solution.** Consider $\mathbb{N}$ with its usual order $\leq$. Then $\mathbb{N}$ is discrete, because every singleton

$$\{n\} = (-\infty, n+1) \cap (n-1, \infty)$$

is open. Since $\mathbb{N}$ is discrete and contains at least two elements, it is not connected. □

**Problem 6.** (Munkres Exercise 23.5) A space is **totally disconnected** if its only connected subspaces are singletons $\{x\}$.

**a.** Show that every discrete space is totally disconnected.

**Solution.** Every subspace $A \subseteq X$ of a discrete space $X$ is itself discrete. If $A$ contains at least two points, it is therefore not connected. $\square$

**b.** Find an example of totally disconnected space which is not discrete.

**Solution.** Take $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ viewed as a subspace of $\mathbb{R}$. Then $X$ is not discrete, since the singleton $\{0\}$ is not open in $X$. Indeed, any open ball around 0 contains infinitely many points of the form $\frac{1}{n}$.

However, $X$ is totally disconnected. If $A \subseteq X$ contains at least two points $x, y \in A$, then at least one of them (WLOG $y$) is of the form $y = \frac{1}{n} \neq 0$. We can write $A$ as the disjoint union

$$A = \{y\} \sqcup (A \setminus \{y\})$$

of two subsets that are open in $A$, and non-empty since $x \in A \setminus \{y\}$. Therefore $A$ is not connected. $\square$