# MA544: Qual Preparation

Carlos Salinas

July 18, 2016

## Contents

## 0.1 Danielli: Winter 2012

**Problem 1.** Let $f(x, y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each $x$, $f(x, y)$ is an integrable function of $y$, and $\partial f(x, y)/\partial x$ is a bounded function of $(x, y)$. Prove that $\partial f(x, y)/\partial x$ is a measurable function of $y$ for each $x$ and

$$\frac{d}{dx} \int_0^1 f(x, y) \, \mathrm{d}y = \int_0^1 \frac{\partial f(x, y)}{\partial x} \, \mathrm{d}y.$$

**Solution.** ► ◄

**Problem 2.** Let $f$ be a function of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with $g$ absolutely continuous and $h$ singular, show that

$$\int_a^b \varphi \, \mathrm{d}f = \int_a^b \varphi f' \, \mathrm{d}x + \int_a^b \varphi \, \mathrm{d}h.$$

*Hint*: A function $h$ is said to be singular if $h' = 0$.

**Solution.** ► ◄

**Problem 3.** Let $E \subset \mathbb{R}$ be a measurable set, and let $K$ be a measurable function on $E \times E$. Assume that there exists a positive constant $C$ such that

$$\int_E K(x, y) \, \mathrm{d}x \leq C \tag{1}$$

for a.e. $y \in E$, and

$$\int_E K(x, y) \, \mathrm{d}y \leq C \tag{2}$$

for a.e. $x \in E$.

Let $1 < p < \infty$, $f \in L^p(E)$, and define

$$T_f(x) = \int_E K(x, y) f(y) \, \mathrm{d}y.$$

(a) Prove that $T_f \in L^p(E)$ and

$$\|T_f\|_p \leq C\|f\|_p. \tag{3}$$

(b) Is (3) still valid if $p = 1$ or $\infty$? If so, are assumptions (1) and (2) needed?

**Solution.** ► ◄

**Problem 4.** Let $f$ be a nonnegative measurable function on $[0, 1]$ satisfying

$$|\{ x \in [0,1] : f(x) > \alpha \}| < \frac{1}{1+\alpha^2} \tag{4}$$

for $\alpha > 0$.

(a) Determine values of $p \in [1, \infty)$ for which $f \in L^p[0, 1]$.
(b) If $p_0$ is the minimum value of $p$ for which $p$ may fail to be in $L^p$, give an example of a function which satisfies (4), but which is not in $L^{p_0}[0, 1]$.

**Solution.** ► ◄

## 0.2 Danielli: Summer 2011

**Problem 1.** Let $f \in L^1(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x)\cos(tx)\,\mathrm{d}x$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

*Hint*: Start by proving the statement for $f = \chi_{[a,b]}$.

**Solution.** ►

◄

**Problem 2.** (a) Suppose that $f_k, f \in L^2(E)$, with $E$ a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E fg \tag{1}$$

as $k \to \infty$ for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \to \|f\|_2$ show that $f_k$ converges to $f$ in $L^2$, i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as $k \to \infty$.

(b) Provide an example of a sequence $f_k$ in $L^2$ and a function $f$ in $L^2$ satisfying (1), but such that $f_k$ does *not* converge to $f$ in $L^2$.

**Solution.** ► ◄

**Problem 3.** A bounded function $f$ is said to be of bounded variation on $\mathbb{R}$ if it is of bounded variation on any finite subinterval $[a,b]$, and moreover $A = \sup_{a,b} V[a,b;f] < \infty$. Here, $V[a,b;f]$ denotes the total variation of $f$ over the interval $[a,b]$. Show that:

(a) $\displaystyle\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x \le A|h|$ for all $h \in \mathbb{R}$.

*Hint*: For $h > 0$, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,\mathrm{d}x.$$

(b) $\left|\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x\right| \leq A$, where $\varphi$ is any function of class $C^1$, of bounded variation, compactly supported, with $\sup_{x\in\mathbb{R}} |\varphi(x)| \leq 1$.

**Solution.** ► ◄

**Problem 4.** (a) Prove the *generalized Hölder's inequality*: Assume $1 \leq p \leq \infty$, $j = 1, \ldots, n$, with $\sum_{j=1}^{\infty} 1/p_j = 1/r \leq 1$. If $E$ is a measurable set and $f_j \in L^{p_j}(E)$ for $j = 1, \ldots, n$, then $\prod_{j=1}^{n} f_j \in L^r(E)$ and

$$\|f_1 \cdots f_n\|_r \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if $1 \leq p, q, r \leq \infty$, with $1/p + 1/q = 1/r + 1$, $f \in L^p(\mathbb{R})$, and $g \in L^p(\mathbb{R})$, then

$$|(f * g)(x)| \leq \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \int |f(y)|^p |g(x-y)|^q \,\mathrm{d}y.$$

(Recall that $(f * g)(x) = \int f(y)g(x-y)\,\mathrm{d}y$.)

(c) Prove *Young's convolution theorem*: Assume that $p$, $q$, $r$, $f$, and $g$ are as in part (b). Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

**Solution.** ► ◄

# 1 Bañuelos

## 1.1 Bañuelos: Summer 2000

**Problem 1.** Let $(X, \mathscr{F}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions with the property that for all $n \geq 1$

$$\mu(\{\, x \in X : |f_n(x)| \geq \lambda \,\}) \leq C \exp(-\lambda^2/n)$$

for all $\lambda > 0$. (Here $C$ is a constant independent of $n$.) Let $n_k = 2^k$. Prove that

$$\limsup_{k\to\infty} \frac{|f_{n_k}|}{\sqrt{n_k \log(\log(n_k))}} \leq 1 \quad \text{a.e.}$$

**Solution.** ► Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions such that

$$\mu(\{\, x \in X : |f_n(x)| \geq \lambda \,\}) \leq C \exp(-\lambda^2/n) \tag{1}$$

for all $\lambda$. Now, consider the subsequence $\{f_{2^k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$. We aim to show that

$$\limsup_{k\to\infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} \leq 1$$

almost everywhere. To that end, it suffices to show that the set

$$E = \left\{ x \in X : \limsup_{k\to\infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} > 1 \right\}$$

has measure zero. Let $x \in E$ then

$$\limsup_{k\to\infty} \frac{|f_{2^k}(x)|}{\sqrt{2^k \log(\log(2^k))}} > 1.$$

This means that there exists some subsequence $\{k_m\}_{m=1}^{\infty} \subset \{k\}_{n=1}^{\infty}$ such that

$$\lim_{m\to\infty} \frac{|f_{2^{k_m}}(x)|}{\sqrt{2^{k_m} \log(\log(2^{k_m}))}} > 1.$$

This means that, for sufficiently large $N$

$$|f_{2^{k_n}}(x)| > \sqrt{2^{k_n} \log(\log(2^{k_n}))}$$

for all $n \geq N$. But by Equation (1) we have

$$\mu\left(\left\{ x \in X : \frac{|f_{2^{k_n}}(x)|}{\sqrt{2^{k_n}\log(\log(2^{k_n}))}} \geq 1 \right\}\right) \leq C\exp\left(-\left(\sqrt{2^{k_n}\log(\log(2^{k_n}))}\right)^2 \Big/ 2^{k_n}\right)$$
$$= C\exp\left(-2^{k_n}\log(\log(2^{k_n}))\big/2^{k_n}\right)$$
$$= C\exp\left(-\log(\log(2^{k_n}))\right)$$
$$= C\exp\left(\log(1/\log(2^{k_n}))\right)$$
$$= \frac{C}{\log(2^{k_n})}. \tag{2}$$

Letting $n \to \infty$, we see that the measure of the set on the left-hand side of Equation (2) must go to 0 so $\mu(E) = 0$. ◀

**Problem 2.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n$ be a sequence of measurable functions with $f_1 \in L^1(\mu)$ and with the property that

$$\mu(\{ x \in X : |f_n(x)| > \lambda \}) \leq \mu(\{ x \in X : |f_1(x)| > \lambda \})$$

for all $n$ and all $\lambda > 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_X \left[ \max_{1\leq j\leq n} |f_j| \right] \mathrm{d}\mu = 0.$$

[*Hint*: You may use the fact that $\|f\|_1 = \int_0^\infty \mu(\{ |f(x)| > \lambda \})\,\mathrm{d}\lambda$.]

**Solution.** ► Define $g_n, h_n : \mathcal{F} \to [0, \infty]$ for $n \in \mathbb{N}$ by

$$g_n(\lambda) = \mu(\{ x \in X : |f_n(x)| > \lambda \}), \quad h_n(\lambda) = \mu\left(\left\{ x \in X : \max_{1\leq i\leq n} |f_i(x)| > \lambda \right\}\right).$$

Now, note that, by the monotonicity of $\mu$, we have

$$h_n(\lambda) \leq \sum_{i=1}^{n} g_n(\lambda) \leq n g_1(\lambda).$$

Thus,

$$\frac{h_n(\lambda)}{n} \leq g_1(\lambda).$$

Since $\|f_1\|_1 = \int_0^\infty g_1(\lambda)\,\mathrm{d}\lambda$, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\int_X \left[\max_{1\le j\le n} |f_j|\right] \mathrm{d}\mu &= \lim_{n\to\infty}\int_X \frac{h_n(x)}{n}\,\mathrm{d}\mu \\
&= \int_X \lim_{n\to\infty} \frac{h_n(x)}{n}\,\mathrm{d}\mu \\
&\le \int_X \lim_{n\to\infty} \frac{\mu(X)}{n} \\
&= 0
\end{aligned}$$

as we wanted to show. ◄

**Problem 3.**

(i) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions. Prove that $f_n \to f$ is measurable if and only if every subsequence $\{f_{n_k}\}$ contains a further subsequence $\{f_{n_{k_j}}\}$ that converges a.e. to $f$.

(ii) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $F\colon \mathbb{R} \to \mathbb{R}$ be continuous and $f_n \to f$ in measure. Prove that $F(f_n) \to F(f)$ in measure. (You may assume, of course, that $f_n$, $F$, $F(f_n)$, and $F(f)$ are all measurable.)

**Solution.** ► Recall that a sequence of measurable functions $\{f_n\}$ converge in measure to a limit $f$ if for every $\varepsilon > 0$ the limit

$$\lim_{n\to\infty} \mu(\{\, x \in X : |f(x) - f_n(x)| \ge \varepsilon \,\}) = 0.$$

For part (i) $\implies$ suppose that $f_n \to f$ in measure. Then given $\varepsilon > 0$ and $\delta > 0$ there exists $N \in \mathbb{N}$ such that $n \ge N$ implies

$$\mu(\{\, x \in X : |f(x) - f_n(x)| \ge \varepsilon \,\}) < \delta.$$

In particular, given $\varepsilon = k^{-1}$ and $\delta = 2^{-k}$, consider the countable collection of measurable sets $\{E_k\}_{k=1}^\infty$ given by

$$E_k = \left\{\, x \in X : |f(x) - f_{n_k}(x)| \ge \frac{1}{k} \,\right\},$$

where $n_k \ge N(k)$ (which depends on our choice of $k$) such that

$$\mu(E_k) < \frac{1}{2^k}.$$

Now, by the Borel–Cantelli lemma, since

$$\sum_{k=1}^{\infty} \mu(E_k) < \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty,$$

for almost every $x \in X$, there exists $N_x \in \mathbb{N}$ such that $x \notin E_k$ for $k \geq N_x$. This means that for $k \geq N_x$, we have

$$|f(x) - f_{n_k}(x)| < \frac{1}{k}.$$

Let $\{f_{n_{k+1}}\}$ be the subsequence of $\{f_{n_k}\}$. Then

$$\lim_{k\to\infty} f_{n_{k+1}} = f$$

as desired.

$\Longleftarrow$ On the other hand, suppose that every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ contains a subsequence $\{f_{n_{k_j}}\}$ that converges to $f$. Seeking a contradiction, suppose that given $\varepsilon > 0$ there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$M = \mu\Big(\Big\{\, x \in X : |f(x) - f_{n_k}(x)| \geq \varepsilon \,\Big\}\Big) > 0.$$

But by assumption there exists a subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ that converges almost everywhere to $f$. We claim that this implies that $f_{n_{k_j}} \to f$ in measure.

*Proof of claim.* This is adapted from a proof in Royden, Proposition 3, Ch. 5.

First note that $f$ is measurable since it is the pointwise limit almost everywhere of a sequence of measurable functions. Let $\varepsilon, \delta > 0$ be given. Here is where the assumption that $\mu(X) < \infty$ is essential! By Egorov's theorem, there is a measurable subset $E \subset X$ with $\mu(X \setminus E) < \delta$ such that $f_n \to f$ uniformly on $E$. Thus, there is an index $N$ such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. Thus, for $n \geq N$,

$$\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\} \subset X \setminus E$$

so

$$\mu(\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\}) < \varepsilon.$$

Thus, we have

$$\lim_{n\to\infty} \mu(\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\}) = 0,$$

i.e., $f_n \to f$ in measure. ∎

Hence, since $f_{n_{k_j}} \to f$ in measure, but $M > 0$ we have a contradiction.

For (ii) since $F$ is continuous given $\varepsilon > 0$ there exist $\delta > 0$ such that $|x - x'| < \delta$ implies $|F(x) - F(x')| < \varepsilon$. By part (i), $f_n \to f$ in measure if and only if every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ contains a subsequence $\{f_{n_{k_j}}\}$ that converges to $f$ almost everywhere, i.e., given $\delta > 0$ there exists an index $N$ such that $n_{k_j} \geq N$ implies

$$|f(x) - f_{n_{k_j}}(x)| < \delta$$

for almost every $x \in X$. Thus,

$$\left|F(f(x)) - F(f_{n_{k_j}}(x))\right| < \varepsilon$$

and we see that for every subsequence $\{F \circ f_{n_k}\}$ of $\{F \circ f_n\}$ we can find a subsequence $\{F \circ f_{n_{k_j}}\}$ that converges almost everywhere to $F \circ f$. ◀

**Problem 4.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space and suppose $f \in L^1(\mu)$ is nonnegative. Suppose $1 < p < \infty$ and let $1 < q < \infty$ be its conjugate exponent, i.e., $1/p + 1/q = 1$. Suppose $f$ has the property that

$$\int_E f \, \mathrm{d}\mu \leq \mu(E)^{1/q}$$

for all measurable sets $E$. Prove that $f \in L^r(\mu)$ for any $1 \leq r < p$.

[*Hint*: Consider $\left\{ x \in X : 2^n \leq f(x) < 2^{n+1} \right\}$, if you like.]

**Solution.** ► By previous problems, we know that if $\mu(X) < \infty$ and $f \in L^p(X)$, then $f \in L^r(X)$ for $1 \leq r < p$, so it suffices to show that $\|f\|_p < \infty$.

Instead of following the hint, consider the set

$$E_t = \{ x \in X : f(x) \geq t \}$$

and let

$$\omega(t) = \mu(E_t),$$

i.e., the distribution function of $f$. Then, we have

$$\int_0^\infty \omega(t) \, \mathrm{d}t = \int_X f \, \mathrm{d}\mu.$$

In particular, if we make the substitution $\alpha = t^{1/p}$, $\mathrm{d}\alpha = t^{1/q}/p \, \mathrm{d}t = \alpha^{p/q}/p \, \mathrm{d}t$, we have

$$\int_X f^r \, \mathrm{d}\mu = \int_0^\infty p\alpha^{-p/q} \omega(\alpha) \, \mathrm{d}\alpha.$$

Now, by Chebyshev's inequality, we have

$$t\omega(t) \leq \int_{E_t} f \, \mathrm{d}\mu \leq \omega(t)^{1/q}$$

so

$$\omega(t) \leq t^{-p}.$$

Thus,

$$\int_X f^r \, \mathrm{d}\mu = \int_0^\infty p\alpha^{-p/q}\omega(\alpha) \, \mathrm{d}\alpha \leq \int_0^\infty p\alpha^{-p-p/q} \, \mathrm{d}\alpha.$$

Since $p + p/q > 1$, the integral above is finite. Thus, $f \in L^p(X)$ and we have $f \in L^r(X)$ for all $1 \leq r < p$. ◀

**Problem 5.** Let $f$ be a continuous function on $[-1, 1]$. Find

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|) \, \mathrm{d}x.$$

**Solution.** ▶ To find the limit of the integral

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|) \, \mathrm{d}x$$

we first make the following substitutions: Let $y = nx$, $\mathrm{d}y = n \, \mathrm{d}x$. Then

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|) \, \mathrm{d}x = \frac{1}{n} \int_{-1}^{1} f(y/n)(1 - |y|) \, \mathrm{d}y.$$

By the extreme value theorem, since $f$ is continuous and $[-1, 1]$ is compact $f$ is bounded on $[-1, 1]$ by, say $M$. Let $g(x) = M$. Then $g \in L^1(X)$ since $\|g\|_1 = 2M$. Thus, by the Lebesgue dominated convergence theorem, since

$$|f(y/n)(1 - |y|)| \leq M$$

on $[-1, 1]$ and $g \in L^1([-1, 1])$ it follows that

$$\begin{aligned}
\lim_{n\to\infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|) \, \mathrm{d}x &= \lim_{n\to\infty} \frac{1}{n} \int_{-1}^{1} f(y/n)(1 - |y|) \, \mathrm{d}y \\
&= \int_{-1}^{1} \lim_{n\to\infty} \left[ \frac{f(y/n)(1 - |y|)}{n} \right] \mathrm{d}y \\
&= \int_{-1}^{1} \lim_{n\to\infty} \left[ \frac{f(y/n)}{n} - \frac{|y|}{n} \right] \mathrm{d}y \\
&= 0.
\end{aligned}$$

◀

**Problem 6.** Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $f \in L^p(\mu)$, $1 \leq p < \infty$. Suppose $E_n$ is a sequence of measurable sets satisfying $\mu(E_n) = 1/n$ for all $n$. Prove that

$$\lim_{n\to\infty}\left[n^{(p-1)/p}\int_{E_n}|f|\,\mathrm{d}\mu\right] = 0.$$

**Solution.** ► The result follows immediately by Hölder's inequality. Let $C = \|f\|_p$. Since $f \in L^p(X)$, then $f \in L^p(E_n)$ for all $n \in \mathbb{N}$. Thus, by Hölder's inequality

$$\begin{aligned}
\|f\|_{L^1(E_n)} &\leq \|f\|_{L^p(E_n)}\mu(E)^{1/q} \\
&\leq C\mu(E)^{1/q} \\
&= C\mu(E)^{p/(p-1)} \\
&= Cn^{-p/(p-1)} \\
&= Cn^{p/(1-p)}.
\end{aligned}$$

Hence, the integral is bounded above by

$$\begin{aligned}
0 \leq n^{(p-1)/p}\int_{E_n}|f|\,\mathrm{d}\mu &\leq Cn^{(p-1)/p+p/(1-p)} \\
&= Cn^{(2p-1)/(p(1-p))}.
\end{aligned}$$

Since $p > 1$, $1 - p < 0$ and $2p - 1 > 0$ so the exponent $(2p-1)/(p(1-p)) < 0$. Thus, as $n \to \infty$

$$Cn^{(2p-1)/(p(1-p))} \longrightarrow 0.$$

It follows that

$$\lim_{n\to\infty}\left[n^{(p-1)/p}\int_{E_n}|f|\,\mathrm{d}\mu\right] = 0.$$

◄

**Problem 7.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{g_n\}$ be a sequence of nonnegative measurable functions with the property that $g_n \in L^1(\mu)$ for every $n$ and $g_n \to g$ in $L^1(\mu)$. Let $\{f_n\}$ be another sequence of nonnegative measurable functions on $(X, \mathcal{F}, \mu)$.

(i) If $f_n \leq g_n$ almost everywhere for every $n$, prove that

$$\limsup_{n\to\infty}\int_X f_n\,\mathrm{d}\mu \leq \int_X \limsup_{n\to\infty} f_n\,\mathrm{d}\mu.$$

[*Hint*: Start by considering a subsequence $\{f_{n_k}\}$ such that

$$\lim_{n_k\to\infty}\int_X f_{n_k}\,\mathrm{d}\mu = \limsup_{n\to\infty}\int_X f_n\,\mathrm{d}\mu$$

and let $\{g_{n_{k_j}}\}$ be a subsequence of $\{g_{n_k}\}$ such that $g_{n_{k_j}} \to g$ almost everywhere.]

(ii) If $f_n \to f$ almost everywhere and if $f_n \le g_n$ almost everywhere for all $n$, then $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

**Solution**. ► Part (i) is a generalization of what is colloquially known as the reverse Fatou's lemma. Consider the sequence of measurable functions $\{h_n\}$ where $h_n = g_n - f_n$. Note that $h_n \ge 0$ for all $x \in X$ since $g_n \ge f$ for all $x \in X$. Then

$$h_n \le \sup_{1\le k\le n} h_k$$

for all $x \in X$. ◄

**Problem 8**. Let $f \in L^1(\mathbb{R})$. Consider the function

$$F(x) = \int_{\mathbb{R}} \exp(\mathrm{i}xt) f(t)\,\mathrm{d}t.$$

(i) Show that $F \in L^\infty(\mathbb{R})$ and that $F$ is continuous at every $x \in \mathbb{R}$. Moreover, if $|t|^k f(t) \in L^\infty(\mathbb{R})$ for all $k \ge 1$, show that $F$ is infinitely differentiable, i.e., $F \in C^\infty(\mathbb{R})$.

(ii) Suppose $f$ is continuous as well as in $L^1(\mathbb{R})$. Show that $\lim_{|x|\to\infty} F(x) = 0$.

[*Hint*: Using $\exp(-\mathrm{i}\pi) = -1$, write $F(x) = \left(\int_{\mathbb{R}}(\exp(\mathrm{i}xt) - \exp(\mathrm{i}xt - \mathrm{i}\pi))\right)/2$.]

**Solution**. ► ◄

## 1.2 Bañuelos: Summer 2000

**Problem 1.** For any two subsets $A$ and $B$ of $\mathbb{R}$ define $A+B = \{ a + b : a \in A, b \in B \}$.

(i) Suppose $A$ is closed and $B$ is compact. Prove that $A + B$ is closed.
(ii) Give an example that shows that (i) may be false if we only assume that $A$ and $B$ are closed.

**Solution.** ► ◄

**Problem 2.** Suppose $f : [0, 1] \to \mathbb{R}$ is differentiable at every $x \in [0, 1]$ where by differentiability at 0 and 1 we mean right and left differentiability, respectively. Prove that $f'$ is continuous if and only if $f$ is uniformly differentiable. That is, if and only if for all $\varepsilon > 0$ there is an $h_0 > 0$ such that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

whenever $0 \le x, x + h \le 1$, $0 < |h| < h_0$.

**Solution.** ► ◄

**Problem 3.** Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$ and let $F_1, \ldots, F_{17}$ be seventeen measurable subsets of $X$ with $\mu(F_j) = 1/4$ for every $j$.

(i) Prove that five of these subsets must have an intersection of positive measure. That is, if $E_1, \ldots, E_k$ denotes the collection of all nonempty intersections of the $F_j$ taken five at a time ($k \le 6188$), show that at least one of these sets must have positive measure.
(ii) Is the conclusion in (i) true if we take sixteen sets instead of seventeen?

**Solution.** ► ◄

**Problem 4.** Let $f_n : X \to [0, \infty)$ be a sequence of measurable functions on the measure space $(X, \mathcal{F}, \mu)$. Suppose there is a positive constant $M$ such that the functions $g_n(x) = f(x)\chi_{\{ f_n \le M \}}(x)$ satisfy $\|g_n\|_1 \le A/n^{4/3}$ and for which $\mu(\{ x \in X : f_n(x) > M \}) \le B/n^{5/4}$, where $A$ and $B$ are positive constants independent of $n$. Prove that

$$\sum_{n=1}^{\infty} f_n < \infty$$

almost everywhere.

**Solution.** ► ◄

**Problem 5.** Let $\{g_n\}$ be a bounded sequence of functions on $[0, 1]$ which is uniformly Lipschitz. That isthere is a constant $M$ (independent of $n$) such that for all $n$, $|g_n(x) - g_n(y)| \leq M|x - y|$ for all $x, y \in [0, 1]$ and $|g_n(x)| \leq M$ for all $x \in [0, 1]$.

(i) Prove that for any $0 \leq a \leq b \leq 1$,

$$\lim_{n\to\infty} \int_a^b g_n(x) \sin(2n\pi x)\, \mathrm{d}x = 0.$$

(ii) Prove that for any $f \in L^1[0, 1]$,

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x) \sin(2n\pi x)\, \mathrm{d}x = 0.$$

**Solution.** ► ◄

**Problem 6.** Let $\{f_n\}$ be a sequence of nonnegative functions in $L^1[0, 1]$ with the property that $\int_0^1 f_n(t)\, \mathrm{d}t = 1$ and $\int_{1/n}^1 f_n(t)\, \mathrm{d}t \leq 1/n$ for all $n$. Define $h(x) = \sup_n f_n(x)$. Prove that $h \notin L^1[0, 1]$.

**Solution.** ► ◄

### 1.3 Bañuelos: Winter 2007

**Problem 1.** Let $f\colon [0,1] \to \mathbb{R}$.

(i) Define what it means for $f$ to be absolutely continuous.
(ii) Define what it means for $f$ to be of bounded variation.
(iii) Let $V(f;0,x)$ be the total variation of $f$ on $[0,x]$. Prove that if $f$ is absolutely continuous on $[0,1]$ so is $V(f;0,x)$.

**Solution.** ► ◄

**Problem 2.**

(i) Suppose that $f\colon [0,1] \to \mathbb{R}$ is nondecreasing with $f(0) = 0$ and $f(1) = 1$. For $a > 0$, let $A$ be set of all $x \in (0,1)$ for which

$$\limsup_{h\to 0} \frac{f(x+h) - f(x)}{h} > a.$$

Prove that $m^*(A) < 1/a$, where $m^*$ denotes the Lebesgue outer measure.
(ii) Prove that there is no Lebesgue measurable set $A$ in $[0,1]$ with the property that $m(A \cap I) = m(I)/4$ for every interval $I$.

[*Hint*: Consider the function $f(x) = \chi_A(x)$.]

**Solution.** ► ◄

**Problem 3.** Let $\{E_j\}_{j=1}^\infty$ be Lebesgue measurable sets in $[0,1]$ and let $E = \bigcup_{j=1}^\infty E_j$ and suppose there is an $\varepsilon > 0$ such that $\sum_{j=1}^\infty m(E_j) \le m(E) + \varepsilon$.

(i) Show that for all measurable sets $A \subset [0,1]$

$$\sum_{j=1}^\infty m(A \cap E_j) \le m(A \cap E) + \varepsilon.$$

(ii) Let $A$ be the set of all $x \in [0,1]$ which are in at least two of $E_j'$. Prove that $m(A) \le \varepsilon$.

**Solution.** ► ◄

**Problem 4.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n\colon X \to [0,\infty)$ be a sequence measurable functions and suppose that $\|f_n\|_p \le 1$, $1 < p < \infty$, and that $f_n \to f$ almost everywhere. Prove

(i) $f \in L^p(\mu)$.
(ii) $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

**Solution.** ► ◄

**Problem 5.**

**Solution.** ► ◄

**Problem 6.**

**Solution.** ► ◄

## 1.4 Bañuelos: Winter 2013

**Problem 1.**

(a)

(i) Define almost uniform convergence on the measure space $(X, \mathcal{F}, \mu)$.
(ii) Let $f_n$ be a sequence of nonnegative measurable functions converging almost uniformly to the nonnegative function $f$. Prove that $\sqrt{f_n}$ converges almost uniformly to $\sqrt{f}$.

(b)

(i) Suppose $f_n$ has the property that $\int_X |f_n| \, d\mu \to 0$.
(ii) Does it follow that $f_n \to 0$ almost everywhere? Justify your answer.
(iii) Does it follow that $f_n \to 0$ almost uniformly? Justify your answer.

**Solution.** ► ◄

**Problem 2.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1 \leq p \leq \infty$ and $q$ be its conjugate exponent. Suppose $f_n \to f$ in $L^p$ and $g_n \to g$ in $L^q$. Prove that $f_n g_n \to fg$ in $L^1$.

**Solution.** ► ◄

**Problem 3.** Let $\{a_k\}$ be a sequence of positive numbers converging to infinity. Prove that the following limit exists

$$\lim_{k\to\infty} \int_0^\infty \frac{\exp(-x)\cos x}{a_k x^2 + (1/a_k)} \, dx$$

and find it. Make sure to justify all steps.

**Solution.** ► ◄

**Problem 4.** Let $(X, \mathcal{F}, \mu)$ be $\sigma$-finite and $f$ be measurable such that for all $\lambda > 0$

$$\mu(\{ x \in X : |f(x)| > \lambda \}) \leq \frac{20}{\lambda^p}$$

where $1 < p < \infty$. Let $q$ be the conjugate exponent of $p$. Prove that there is a constant $C$ depending only on $p$ such that

$$\int_E |f(x)| \, d\mu \leq Cm(E)^{1/q},$$

for all measurable sets $E$ with $0 < \mu(E) < \infty$. (The inequality holds trivially when $\mu(E) = 0$ or $\mu(E) = \infty$.)

[*Hint*: Recall $\int_E |f(x)| \, d\mu = \int_0^\infty ? \, d\lambda$ and "break it" at the right place!]

**Solution.** ► ◄

**Problem 5.** Suppose $f\colon [0,1] \to \mathbb{R}$ is of bounded variation with $V(f; 0, 1) = \alpha$. For any $\beta > \alpha$, set

$$A = \left\{ x \in (0,1) : \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} > \beta \right\}.$$

Prove that for any $0 < p < 1$, $m(A) \leq (\alpha/\beta)^p$, where $m$ denotes the Lebesgue measure.

**Solution.** ► ◄

**Problem 6.** Let $f \in L^1(0,1)$ and for $x \in (0,1)$, define

$$h(x) = \int_x^1 \frac{f(t)}{t} \, dt.$$

(i) Prove that $h$ is continuous on $(0,1)$.
(ii) Show that

$$\int_0^1 h(t) \, dt = \int_0^1 f(t) \, dt.$$

**Solution.** ► ◄

## References

[1] Folland, G. *Real analysis: modern techniques and their applications.* Pure and applied mathematics. Wiley, 1984.

[2] Royden, H., and Fitzpatrick, P. *Real Analysis.* Featured Titles for Real Analysis Series. Prentice Hall, 2010.

[3] Rudin, W. *Principles of Mathematical Analysis.* International series in pure and applied mathematics. McGraw-Hill, 1976.

[4] Rudin, W. *Real and complex analysis.* Mathematics series. McGraw-Hill, 1987.

[5] Wheeden, R., and Zygmund, A. *Measure and Integral: An Introduction to Real Analysis.* Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1977.