Tropicalization of Character Varieties

Tropical Geometry Group

June 10, 2016

Visualizing Tropical Character Varieties

Part I

Summary of work: Our group worked on visualization of tropicalized character varieties into $\mathrm{SL}_2\,\mathbb{C}$ of $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} * \mathbb{Z}$ as well as several knot groups whose A-polynomials we obtained from a paper by Eric Chesebro *Formulas for Character Varieties of* 2*-Bridge Knots* which can be found on his website `http://hs.umt.edu/math/research/technical-reports/documents/2012/KnotFormulas.pdf`.
With the help of `Mathematica`, we made the following subdivision of the Newton polytopes of these character varieties.

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Part II

Some pictures

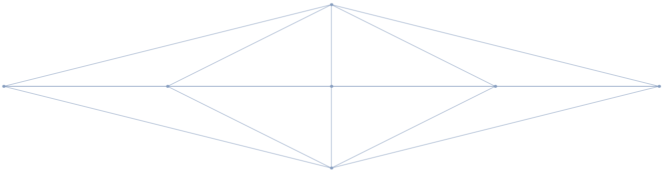

Figure: Subdivided Newton polytope for the A-polynomial of the figure-8 knot.

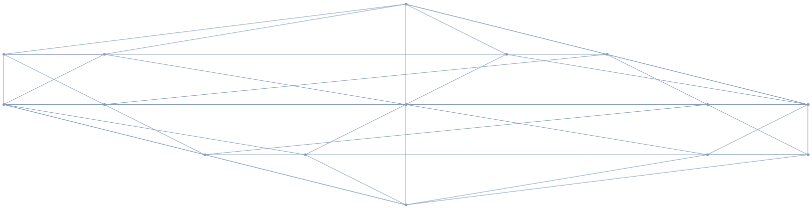

Figure: Subdivided Newton polytope for the A-polynomial the 4-twisted knot.

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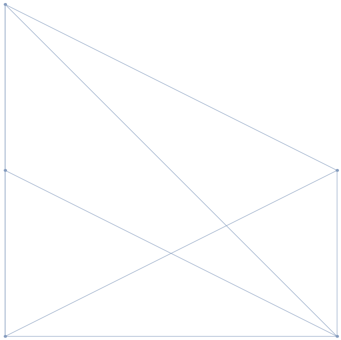
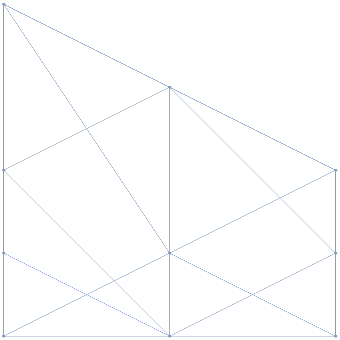
Part III

Some more pictures for Newton polytopes of two 2-bridge knots corresponding to $\varphi(1)$ and $\varphi(2)$ of Chesebro's equation.





$S_{2,g}$ - a recursive approach

Definition

$S_{2,g}$ is the number of words over an alphabet of size g such that no letter appears (considering absolute multiplicities) more than twice.

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$$S_{2,g} = \sum_{d=0}^{g} \sum_{\ell=0}^{2g} \sum_{u=0}^{g} S_{2,g}^{d,\ell,u},$$

and

$$S_{2,0}^{0,0,0} = 1.$$

The recursion

$$
\begin{aligned}
S_{2,g}^{d,\ell,u} &= S_{2,g-1}^{d,\ell,u} \\
&+ \sum_{j=0}^{u-1} \binom{g-1}{j} j! 2^j (\ell - 2j) \left(S_{2,g-1-j}^{d,\ell-1-2j,u-j-1} + S_{2,g-1-j}^{d-1,\ell-1-2j,u-j-1} \right) 2 \\
&+ \sum_{j=0}^{u-1} \sum_{k=0}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \left(\frac{(\ell - 2(j+k) - 1)(\ell - 2(j+k) - 2)}{2} \right) S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&+ \sum_{j=0}^{0} \sum_{k=1}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \left(\frac{(\ell - 2(j+k) - 1)2}{2} \right) S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&+ \sum_{j=1}^{u-1} \sum_{k=0}^{0} \binom{g-1}{j+k} (j+k)! 2^{j+k} \left(\frac{(\ell - 2(j+k) - 1)2}{2} \right) S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&- \sum_{j=1}^{u-1} \sum_{k=1}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \left(\frac{(\ell - 2(j+k) - 1)2}{2} \right) S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&+ \sum_{j=1}^{u-1} \sum_{k=0}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \left(\frac{(\ell - 2(j+k) - 1)2(j-1)}{2} \right) S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&+ \sum_{j=1}^{u-1} \sum_{k=1}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \left(\frac{(\ell - 2(j+k) - 1)2)}{2} \right) S_{2,g-1-j-k}^{d,-2(j+k)-2,u-1-j-k} 4
\end{aligned}
$$

Some calculations

This has been implemented, with the following results

$$
\begin{aligned}
S_{2,0} &= 1\\
S_{2,1} &= 5\\
S_{2,2} &= 105\\
S_{2,3} &= 6061\\
S_{2,4} &= 668753\\
&\vdots
\end{aligned}
$$

Conclusion: it is impractical to consider naïve generators when examining representations F_n to $\mathrm{SL}_2\,\mathbb{C}$.

Tropicalization of $\mathfrak{X}(F_3, \mathrm{SL}_2\,\mathbb{C})$

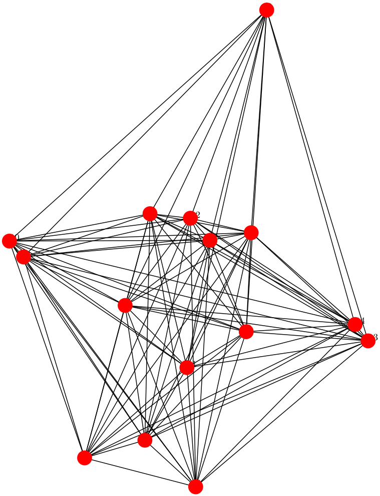
The character variety $\mathfrak{X}(F_3, \mathrm{SL}_2\,\mathbb{C})$ of F_3 into $\mathrm{SL}_2\,\mathbb{C}$ is cut out by the polynomial

$$\begin{aligned} f = abcg - def + d^2 + e^2 + f^2 + a^2 + b^2 + c^2 \\ + g^2 + afg + beg + cdg + abd + ace + bcf - 4. \end{aligned}$$

Its tropicalization $\mathrm{Trop}(\mathfrak{X}(F_3, \mathrm{SL}_2\,\mathbb{C}))$ is the codimension 1 cones of the dual fan of the newton polytope $\mathrm{N}(f)$.

Small Graph

The edge graph of the Newton polytope of $\mathfrak{X}(F_3, \mathrm{SL}_2\,\mathbb{C})$ is shown here:



Tropicalization of $\mathfrak{X}(F_2, \mathrm{SL}_3\,\mathbb{C}))$

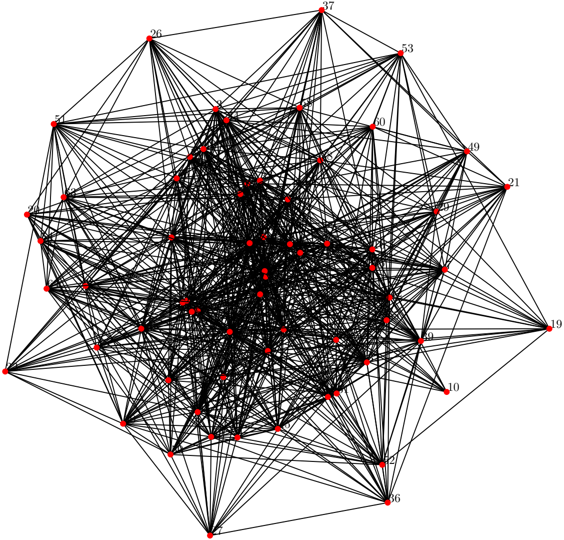
The character variety $\mathfrak{X}(F_2, \mathrm{SL}_3\,\mathbb{C}))$ is cut out by the polynomial

$$
\begin{aligned}
f = 9 &+ 3i - 6ab - 6cd - 6ef - 6gh + i^2 - abi - cdi - efi - ghi \\
&+ a^3 + c^3 + e^3 + g^3 + b^3 + d^3 + f^3 + h^3 - 3bhf - 3aeg - 3ceh \\
&- 3dfg + 3adh + 3bcg + 3acf + 3bde - abcdi + acfi + bdei + adhi \\
&+ bcgi + abcd + cdef + abgh + cdgh + abef + efgh + dh^2f + ceg^2 \\
&+ b^2dh + a^2cg + ad^2f + bc^2e + a^2hf + b^2eg + chf^2 + de^2g \\
&+ b^2cf + a^2ed + ac^2h + bd^2g + d^2eh + c^2fg + aeh^2 + bfg^2 \\
&+ be^2h + af^2g - 2bdf^2 - 2ace^2 - 2bch^2 - 2adg^2 + b^2d^2f \\
&+ a^2c^2e + b^2c^2h + a^2d^2g - acd^2h - bc^2dg - a^2bcf - ab^2de \\
&- ac^2df - bcd^2e - a^2bdh - ab^2cg - abd^3 - abc^3 - b^3cd - a^3cd \\
&- bcdhf - acdeg - abceh - abdfg + a^2b^2cd + abc^2d^2
\end{aligned}
$$

Its tropicalization $\mathrm{Trop}(\mathfrak{X}(F_2, \mathrm{SL}_3\,\mathbb{C}))$ is the codimension 1 cones of the dual fan of the newton polytope $\mathrm{N}(f)$.

Big Graph

The edge graph of the Newton polytope is shown here $\mathfrak{X}(F_2, \mathrm{SL}_3\,\mathbb{C})$:



Generators

Let $F_3 = \langle A, B, C \rangle$. $\mathbb{C}[\mathfrak{X}(F_3, \mathrm{PSL}_2\,\mathbb{C})] = \mathbb{C}[\mathfrak{X}(F_3, \mathrm{SL}_2\,\mathbb{C})]^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ is generated by:

Type χ

$$\begin{aligned} \chi_A &:= (\operatorname{tr} A)^2 & \chi_B &:= (\operatorname{tr} B)^2 & \chi_C &:= (\operatorname{tr} C)^2 \\ \chi_{AB} &:= (\operatorname{tr} AB)^2 & \chi_{AC} &:= (\operatorname{tr} AC)^2 & & \\ \chi_{BC} &:= (\operatorname{tr} BC)^2 & \chi_{ABC} &:= (\operatorname{tr} ABC)^2 & & \end{aligned}$$

Generators cont.

Type τ

$$\tau_{AB} := \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AB \quad \tau_{AC} := \operatorname{tr} A \operatorname{tr} C \operatorname{tr} AC \quad \tau_{BC} := \operatorname{tr} B \operatorname{tr} C \operatorname{tr} BC$$

Generators cont.

Type τ

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Type Λ

$$\Lambda_A := \operatorname{tr} B \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} AC \qquad \Lambda_B := \operatorname{tr} A \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} BC$$
$$\Lambda_C := \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AC \operatorname{tr} BC$$

Generators cont.

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$$\Lambda_C := \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AC \operatorname{tr} BC$$

Lonely Δ

$$\Delta := \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C \operatorname{tr} ABC$$

Generators cont.

Equally lonely Σ

$$\Sigma := \operatorname{tr} AB \operatorname{tr} AC \operatorname{tr} BC$$

Generators cont.

Equally lonely Σ

$$\Sigma := \operatorname{tr} AB \operatorname{tr} AC \operatorname{tr} BC$$

Type Θ

$$\Theta_A := \operatorname{tr} A \operatorname{tr} BC \operatorname{tr} ABC \qquad \Theta_B := \operatorname{tr} B \operatorname{tr} AC \operatorname{tr} ABC$$
$$\Theta_C := \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} ABC$$

Relations

Explicit example:

$$\Sigma^2 = (\operatorname{tr} AB \operatorname{tr} AC \operatorname{tr} BC)^2 = \chi_{AB}\chi_{AC}\chi_{BC}$$

(Binomial) Relations

$$\tau_{AB}^2 = \chi_A \chi_B \chi_{AB} \qquad \tau_{AC}^2 = \chi_A \chi_C \chi_{AC}$$
$$\tau_{BC}^2 = \chi_B \chi_C \chi_{BC}$$

$$\Lambda_A^2 = \chi_B \chi_C \chi_{AB} \chi_{AC} \qquad \Lambda_B^2 = \chi_A \chi_C \chi_{AB} \chi_{BC}$$
$$\Lambda_C^2 = \chi_A \chi_B \chi_{AC} \chi_{BC}$$

$$\Theta_A^2 = \chi_A \chi_{BC} \chi_{ABC} \qquad \Theta_B^2 = \chi_B \chi_{AC} \chi_{ABC}$$
$$\Theta_C^2 = \chi_C \chi_{AB} \chi_{ABC}$$

$$\Sigma^2 = \chi_{AB} \chi_{AC} \chi_{BC} \qquad \Delta^2 = \chi_A \chi_B \chi_C \chi_{ABC}.$$

…and finally the relation coming from $\mathfrak{X}(F_3, \mathrm{SL}_2\,\mathbb{C})$ can be written as

$$\begin{matrix} \chi_A + \chi_B + \chi_C + \chi_{AB} + \chi_{AC} \\ + \chi_{BC} + \chi_{ABC} + \Sigma + \Delta \end{matrix} = \begin{matrix} \tau_{AB} + \tau_{AC} + \tau_{BC} \\ + \Theta_A + \Theta_B + \Theta_C + 4. \end{matrix}$$