MA166: Exam 3 Solutions

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1 Exam III (Spring 2016)

Here are the solutions to Exam III.
Version 01: CCEA ECCD BADC
Version 02: DECA ECBC DABE

Problem 1 (# 1, # 11)**.** Test the following series for convergence:

(I) $\sum_{n=1}^{\infty} e^{-n}$;

(II) $\sum_{n=1}^{\infty} 2^n/n$;

(III) $\sum_{n=1}^{\infty} 3/n^2$.

Solution. (I) By the integral test, we have

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-n} &\sim \int_1^{\infty} e^{-x} \\ &= \left[-e^{-x}\right]_1^{\infty} \\ &= -0-(-e^{-1}) \\ &= e^{-1} \\ &< \infty.\end{aligned}$$

Hence, $\sum_{n=1}^{\infty} e^{-n}$ converges.

(II) This series does not converge by the test for divergence since the limit of $2^n/n$ as $n \to \infty$ is no 0. In particular, you can use l'Hôpital's rule to get $\lim_{x\to\infty} 2^x/x = \lim_{x\to\infty} 2^x \ln 2/1 = \infty \neq 0$.

(III) Write $\sum_{n=1}^{\infty} 3/n^2$ as the product $3\sum_{n=1}^{\infty} n^2$. Then, since $\sum_{n=1}^{\infty} 1/n^2$ is a p-series with $p = 2 > 1$, it converges. Hence $3\sum_{n=1}^{\infty} 1/n^2 = \sum_{n=1}^{\infty} 3/n^2$ converges.

Answer: C, B. ■

Problem 2 (# 2, # 4)**.** Determine which of the following statements are true and which are false.

(I) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(II) If $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} b_n$ is convergent, and $b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(III) If $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} b_n$ is convergent, and $\lim_{n\to\infty} a_n/b_n = 5$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Solution. (I) This is false. Consider the series $\sum_{n=1}^{\infty}(-1)^n/n$. This series is convergent by the alternating series test, but $\sum_{n=1}^{\infty} 1/n$ is divergent since it is the harmonic series.

(II) This is false. Consider the series $\sum_{n=1}^{\infty} 1/n^2$ and $\sum_{n=1}^{\infty} n$; $1/n^2 < n$, but $\sum_{n=1}^{\infty} n$ diverges, whereas $\sum_{n=1}^{\infty} 1/n^2$ converges since it is a p-series with $p = 2 > 1$.

(III) This is true by the limit comparison test.

Answer: C, A. ■

Problem 3 (# 3, # 6)**.** The series $\sum_{n=1}^{\infty}(-1)^n/n^2$ is convergent by the alternating series test. According to the alternating series estimation theorem, what is the smallest number of terms needed to find he sum of the series with error less than $1/15$

Solution. Compute the first few terms of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \approx -1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\cdots.$$

Since the term following $-1/9$ is less that $1/15$, we only need 3, by the alternating series estimation theorem, we only need 3 terms.

Answer: E, C ■

Problem 4 (# 4, # 1)**.** Test the following series for convergence:

(I) $\sum_{n=1}^{\infty} e^{2n}/n!$;

(II) $\sum_{n=1}^{\infty} n/(2n+4)$;

(III) $\sum_{n=1}^{\infty}(-1)^n n/(2n+4)$.

Solution. (I) By the ratio test, we have

$$\begin{aligned}\left(\frac{e^{2(n+1)}}{(n+1)!}\right)\left(\frac{n!}{e^{(2n+1)}}\right) &= \frac{e^{2n+2}n!}{(n+1)n!e^{2n}} \\ &= \frac{e^2}{n+1}\end{aligned}$$

which goes to 0 as $n\to\infty$. Thus, the series in (I) converges.

(II) By the ratio test, we have

$$\begin{aligned}\left(\frac{n+1}{2(n+1)+4}\right)\left(\frac{2n+4}{n}\right) &= \frac{(n+1)(2n+4)}{(2n+6)n} \\ &= \frac{2n^2+6n+4}{2n^2+6n}\end{aligned}$$

(III)

Answer: , ■

Problem 5 (# 5, # 3)**.** What is the interval of convergence of the series $\sum_{n=1}^{\infty} n(x-1)^n/(n^2+1)$?

Solution. Answer: , ■

Problem 6 (# 6, # 7)**.** Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

(I) $\sum_{n=1}^{\infty} \cos(n)/(n^2+1)$;

(II) $\sum_{n=1}^{\infty} (-1)^n/(2n)$.

Solution. Answer: , ■

Problem 7 (# 7, # 9)**.** Determine which of the following statements are true and which are false.

(I) $\sum_{n=1}^{\infty} 1/(n^2+1)$ is convergent by the ratio test.

(II) $\sum_{n=1}^{\infty} 1/(n^2+1)$ is convergent by the limit comparison test with $\sum_{n=1}^{\infty} 1/n^2$.

(III) $\sum_{n=1}^{\infty} 1/(n^2+1)$ is convergent by the direct comparison test with $\sum_{n=1}^{\infty} 1/n^2$.

Solution. Answer: , ■

Problem 8 (# 8, # 10)**.** The series $\sum_{n=1}^{\infty} (-1)^n n x^n$ converges for $|x| < 1$ to ?

Solution. Answer: , ■

Problem 9 (# 9, # 12)**.** Let $\sin x = \sum_{n=0}^{\infty} a_n (x - \pi/6)^n$ be the Taylor series for $\sin x$ at $a = \pi/6$. Then $a_2 =$?

Solution. Answer: , ■

Problem 10 (# 10, # 5)**.** What is the coefficient of x^8 in the power series representation for $\int x \cos(x^3)\, dx$? You may assume $\cos(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n!)$.

Solution. Answer: , ■

Problem 11 (# 11, # 8)**.** A particle moves along the curve $x = \tan t$, $y = \sec t$, $0 \leq t \leq \pi/4$. The trajectory of the particle is a segment of

Solution. Answer: , ■

Problem 12 (# 12, # 2)**.** The length of the curve $x = \cos\theta + \theta \sin\theta$, $y = \sin\theta - \theta\cos\theta$, $0 \leq \theta \leq \pi$, is

Solution. Answer: , ■