Math 535 - General Topology
Fall 2012
Homework 14 Solutions

Problem 1. Let X be a topological space and (Y,d) a metric space. For each compact subset $K \subseteq X$, consider the pseudometric on $C(X,Y)$ defined by

$$d_K(f,g) = \sup_{x \in K} d\left(f(x), g(x)\right)$$

and its associated open balls $B_K(f,\epsilon) = \{g \in C(X,Y) \mid d_K(f,g) < \epsilon\}$.

Show that the collection of all open balls

$$\mathcal{B} = \{B_K(f,\epsilon) \mid K \subseteq X \text{ compact, } f \in C(X,Y), \epsilon > 0\}$$

forms a basis for a topology on $C(X,Y)$. More explicitly:

1. $\mathcal{B}$ covers $C(X,Y)$;
2. Finite intersections of members of $\mathcal{B}$ are unions of members of $\mathcal{B}$.

Solution. 1. For any $f \in C(X,Y)$, pick any compact $K \subseteq X$ (e.g. a singleton $\{x\}$) and any radius $\epsilon > 0$. Then f satisfies $d_K(f,f) = 0$ and therefore $f \in B_K(f,\epsilon)$.

2. It suffices to check the claim for an intersection of two members $B_1 = B_{K_1}(f_1,\epsilon_1)$ and $B_2 = B_{K_2}(f_2,\epsilon_2)$ of $\mathcal{B}$.

Let $g \in B_1 \cap B_2$. Consider the compact subset $K := K_1 \cup K_2 \subseteq X$ and the radius

$$\epsilon := \min\{\epsilon_1 - d_{K_1}(f_1,g), \epsilon_2 - d_{K_2}(f_2,g)\}.$$

By definition we have $g \in B_K(g,\epsilon)$, and moreover we claim:

$$g \in B_K(g,\epsilon) \subseteq B_1 \cap B_2$$

which proves the statement.

Proof of claim. Let $h \in B_K(g,\epsilon)$. Then we have:

$$\begin{aligned}
d_{K_1}(f_1,h) &\leq d_{K_1}(f_1,g) + d_{K_1}(g,h) \\
&\leq d_{K_1}(f_1,g) + d_K(g,h) \text{ since } K_1 \subseteq K \\
&< d_{K_1}(f_1,g) + \epsilon \text{ since } h \in B_K(g,\epsilon) \\
&\leq d_{K_1}(f_1,g) + \epsilon_1 - d_{K_1}(f_1,g) \\
&= \epsilon_1
\end{aligned}$$

and likewise $d_{K_2}(f_2,h) < \epsilon_2$. This proves $h \in B_1 \cap B_2$. $\square$

The following proposition will be relevant to Problem 2. **Do not** prove the proposition in your write-up.

Proposition. *1. Given a pseudometric d on a set X, there is a topologically equivalent pseudometric ρ on X which is bounded above by 1.*

For example, the formulas $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$ or $\rho(x,y) = \min\{d(x,y), 1\}$ work.

2. Given a countable family of pseudometrics $\{d_n\}_{n\in\mathbb{N}}$ on X which are bounded above by 1, the formula

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x,y) \tag{1}$$

defines a pseudometric d on X.

3. The topology $\mathcal{T}_d$ on X induced by d is the topology generated by $\bigcup_{n\in\mathbb{N}} \mathcal{T}_{d_n}$. More explicitly, this is the topology generated by the collection of all open balls

$$\{B_n(x,\epsilon) \mid n \in \mathbb{N}, x \in X, \epsilon > 0\}$$

where we used the notation $B_n(x,\epsilon) := \{y \in X \mid d_n(x,y) < \epsilon\}$.

Proof. Essentially Homework 6 Problem 4, more precisely:

1. Parts (a) and (b);
2. Part (c);
3. Slight generalization of part (d).

□

Problem 2. A family of pseudometrics $\{d_\alpha\}_{\alpha\in A}$ on a set X is **separating** if the following implication holds:

$$d_\alpha(x,y) = 0 \text{ for all } \alpha \in A \Rightarrow x = y.$$

In other words, for any distinct points $x \neq y$, there is an index $\alpha \in A$ satyisfying $d_\alpha(x,y) > 0$.

a. Let X be a set and $\{d_n\}_{n\in\mathbb{N}}$ a countable family of pseudometrics on X which are bounded above by 1. Let d be the pseudometric on X defined by the formula (1) as in the proposition above.

Show that d is a metric if and only if the family of pseudometrics $\{d_n\}_{n\in\mathbb{N}}$ is separating.

Solution. For any $x, y \in X$, consider the equivalent conditions:

$$\begin{aligned} d(x,y) = 0 \Leftrightarrow & \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x,y) = 0 \\ \Leftrightarrow & \frac{1}{2^n} d_n(x,y) = 0 \text{ for all } n \in \mathbb{N} \text{ since all terms are non-negative} \\ \Leftrightarrow & d_n(x,y) = 0 \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Therefore the implication $d(x,y) = 0 \Rightarrow x = y$, which says that d is a metric, is equivalent to the implication

$$d_n(x,y) = 0 \text{ for all } n \in \mathbb{N} \Rightarrow x = y$$

which says that the family $\{d_n\}_{n\in\mathbb{N}}$ is separating. $\square$

b. Let (Y, d) be a metric space and consider the mapping space $C(\mathbb{R}, Y)$. For all $n \in \mathbb{N}$, consider the compact interval $[-n, n] \subset \mathbb{R}$ and the associated pseudometric

$$d_n(f,g) = \sup_{x\in[-n,n]} d\left(f(x), g(x)\right).$$

Show that the family of pseudometrics $\{d_n\}_{n\in\mathbb{N}}$ on $C(\mathbb{R}, Y)$ is separating.

Solution. Note that the equality $d_n(f,g) = 0$ holds if and only if f and g agree on $[-n, n]$:

$$f|_{[-n,n]} = g|_{[-n,n]}.$$

Therefore the following conditions are equivalent:

$$\begin{aligned} & d_n(f,g) = 0 \ \text{ for all } n \in \mathbb{N} \\ \Leftrightarrow\ & f|_{[-n,n]} = g|_{[-n,n]} \ \text{ for all } n \in \mathbb{N} \\ \Leftrightarrow\ & f|_{\bigcup_{n\in\mathbb{N}}[-n,n]} = g|_{\bigcup_{n\in\mathbb{N}}[-n,n]} \\ \Leftrightarrow\ & f|_{\mathbb{R}} = g|_{\mathbb{R}} \\ \Leftrightarrow\ & f = g. \quad \square \end{aligned}$$

c. Show that the topology $\mathcal{T}$ on $C(\mathbb{R}, Y)$ generated by $\bigcup_{n\in\mathbb{N}} \mathcal{T}_{d_n}$ is the topology of compact convergence.

Solution. Denote by $\mathcal{T}_{comp}$ the topology of compact convergence on $C(\mathbb{R}, Y)$.

($\mathcal{T} \subseteq \mathcal{T}_{comp}$) Recall that $\mathcal{T}_{comp}$ is generated by

$$\bigcup_{\substack{K\subset\mathbb{R} \\ K \text{ compact}}} \mathcal{T}_{d_K}.$$

Since each closed interval $[-n, n] \subset \mathbb{R}$ is compact, we have the inclusion

$$\bigcup_{n\in\mathbb{N}} \mathcal{T}_{d_n} \subseteq \bigcup_{\substack{K\subset\mathbb{R} \\ K \text{ compact}}} \mathcal{T}_{d_K}$$

and therefore the inclusion $\mathcal{T} \subseteq \mathcal{T}_{comp}$ of topologies generated by these collections.

($\mathcal{T}_{comp} \subseteq \mathcal{T}$) Let $K \subset \mathbb{R}$ be compact and consider the open ball $B_K(f, \epsilon) \subseteq C(X, Y)$. By Problem 1, it suffices to find a $\mathcal{T}$-open U satisfying $f \in U \subseteq B_K(f, \epsilon)$.

Since K is bounded, pick $n \in \mathbb{N}$ large enough to satisfy $K \subseteq [-n, n]$. This inclusion implies the inequality

$$d_K(f, g) \leq d_n(f, g)$$

for all $f, g \in C(X, Y)$, and therefore the inclusion of open balls

$$B_n(f, \epsilon) \subseteq B_K(f, \epsilon).$$

Indeed, any element $g \in B_n(f, \epsilon)$ satisfies

$$d_K(f, g) \leq d_n(f, g) < \epsilon.$$

Therefore we have $f \in B_n(f, \epsilon) \subseteq B_K(f, \epsilon)$ where $B_n(f, \epsilon)$ is $\mathcal{T}$-open, as desired. $\square$

Problem 3. Let X be a topological space and kX its k-ification.

a. Show that the identity function $\mathrm{id}\colon kX \to X$ is a homeomorphism if and only if X is compactly generated.

Solution. We know that the identity function $\mathrm{id}\colon kX \to X$ is continuous, and it is a bijection. Therefore the following conditions are equivalent:

$$\begin{aligned}
&\mathrm{id}\colon kX \to X \text{ is a homeomorphism.}\\
\Leftrightarrow\ &\mathrm{id}\colon kX \to X \text{ is an open map.}\\
\Leftrightarrow\ &\text{Every open subset } A \subseteq kX \text{ is also open in } X.\\
\Leftrightarrow\ &\text{Every k-open subset } A \subseteq X \text{ is also open in } X.\\
\Leftrightarrow\ &X \text{ is compactly generated.} \quad \square
\end{aligned}$$

b. Show that kX is always compactly generated.

Solution.

Lemma. *Let $K \subseteq X$ be a compact subspace, and denote by $\widetilde{K} := \mathrm{id}^{-1}(K) \subseteq kX$ the same set viewed as a subspace of kX instead. Then the restriction*

$$\mathrm{id}|_{\widetilde{K}}\colon \widetilde{K} \to K$$

is a homeomorphism. In particular, $\widetilde{K}$ is compact, so that the map $\mathrm{id}\colon kX \to X$ is proper.

Proof. The restriction $\mathrm{id}|_{\widetilde{K}}$ is continuous (since id is), injective (since id is), and surjective (since $\mathrm{id}(\widetilde{K}) = K$). It remains to show that it is an open map.

Let $U \subseteq \widetilde{K}$ be an open subset. Since $\widetilde{K}$ is a subspace of kX, we have $U = O \cap \widetilde{K}$ for some open subset $O \subseteq kX$. Applying the identity function yields

$$\begin{aligned}
\mathrm{id}|_{\widetilde{K}}(U) &= \mathrm{id}|_{\widetilde{K}}(O \cap \widetilde{K})\\
&= \mathrm{id}(O) \cap \mathrm{id}(\widetilde{K})\\
&= \mathrm{id}(O) \cap K
\end{aligned}$$

which is open in K since $\mathrm{id}(O)$ is k-open in X (by definition of the topology on kX). $\square$

Let $A \subseteq kX$ be a k-open subset of kX. We want to show that A is open in kX, i.e. $\mathrm{id}(A) \subseteq X$ is k-open in X. Let $K \subseteq X$ be a compact subspace. Then $\widetilde{K} := \mathrm{id}^{-1}(K) \subseteq kX$ is compact, by the lemma. Since A is k-open in kX, $A \cap \widetilde{K}$ is open in $\widetilde{K}$. By the lemma, $\mathrm{id}(A) \cap K$ is open in K. Therefore $\mathrm{id}(A)$ is k-open in X. $\square$

c. Let W be a compactly generated space and $f\colon W \to X$ a continuous map. Show that there exists a unique continuous map $\widetilde{f}\colon W \to kX$ satisfying $f = \mathrm{id} \circ \widetilde{f}$, i.e. making the diagram

$$\begin{tikzcd} & kX \arrow[r, "\mathrm{id}"] & X \\ W \arrow[ur, dashed, "\widetilde{f}"] \arrow[urr, "f"'] & & \end{tikzcd}$$

commute.

Solution. Since the map $\mathrm{id}\colon kX \to X$ is bijective, there exists a unique function $\widetilde{f}\colon W \to kX$ satisfying $f = \mathrm{id} \circ \widetilde{f}$, namely the same function $\widetilde{f}(w) = f(w)$ for all $w \in W$. It remains to show that $\widetilde{f}\colon W \to kX$ is continuous. Since W is compactly generated, it suffices to show that the restriction $\widetilde{f}|_K\colon K \to kX$ to any compact subspace $K \subseteq W$ is continuous. Consider the commutative diagram:

$$\begin{tikzcd} & kX \arrow[r, "\mathrm{id}"] & X \\ K \arrow[ur, dashed, "\widetilde{f}|_K"] \arrow[urr, "f|_K"'] & & \end{tikzcd}$$

Let $O \subseteq kX$ be an open subset, i.e. $\mathrm{id}(O)$ is k-open in X. Then we have

$$\begin{aligned} \widetilde{f}|_K^{-1}(O) &= (\mathrm{id}^{-1} \circ f|_K)^{-1}(O) \\ &= f|_K^{-1}(\mathrm{id}(O)) \end{aligned}$$

which is open in K since $\mathrm{id}(O)$ is k-open in X and $f|_K\colon K \to X$ is a continuous map from a compact space. Therefore $\widetilde{f}|_K$ is continuous. $\square$

Problem 4. Show that any compactly generated space X is a quotient of a coproduct of compact spaces. In other words, there exists a collection $\{K_i\}_{i \in I}$ of compact spaces, indexed by some *set* I, and a quotient map $q \colon \coprod_{i \in I} K_i \twoheadrightarrow X$.

Solution. Consider the collection of all compact subspaces of X:

$$I := \{K \subseteq X \mid K \text{ is compact}\}.$$

Then I is a set, since it is a subcollection of the power set $\mathcal{P}(X)$ of X.

For each $K_i \in I$ (indexed tautologically), consider the inclusion map $\iota_i \colon K_i \hookrightarrow X$. Since each ι_i is continuous, they define together a continuous map

$$q \colon \coprod_{i \in I} K_i \to X$$

whose restriction to the summand K_i is ι_i.

q **is surjective.** Let $x \in X$. Then the singleton $\{x\} \subseteq X$ is a compact subspace, hence a summand of $\coprod_{i \in I} K_i$. Then the point

$$x \in \{x\} \subseteq \coprod_{i \in I} K_i$$

is mapped to $q(x) = \iota_{\{x\}}(x) = x \in X$.

q **is a quotient map.** It remains to show that any subset $U \subseteq X$ such that $q^{-1}(U)$ is open in $\coprod_{i \in I} K_i$ must be open in X. Consider the equivalent conditions:

$$\begin{aligned}
& q^{-1}(U) \text{ is open in } \coprod_{i \in I} K_i \\
\Leftrightarrow\; & q^{-1}(U) \cap K_i \text{ is open in } K_i \text{ for all } i \in I \\
\Leftrightarrow\; & \iota_i^{-1}(U) \text{ is open in } K_i \text{ for all } i \in I \\
\Leftrightarrow\; & U \cap K \text{ is open in } K \text{ for all compact subspace } K \subseteq X \\
\Leftrightarrow\; & U \text{ is k-open in } X.
\end{aligned}$$

Since X is compactly generated, being k-open in X is equivalent to being open in X. $\square$

Problem 5. Let X and Y be topological spaces, where Y is Hausdorff.

a. Consider the set Y^X of *all* functions from X to Y, endowed with the topology of pointwise convergence. Recall that via the correspondence $Y^X \cong \prod_{x \in X} Y$, this corresponds to the product topology.

Show that a collection of functions $\mathcal{F} \subseteq Y^X$ is compact if and only if the following two conditions hold:

1. $\mathcal{F}$ is closed in Y^X;
2. For all $x \in X$, the projection $p_x(\mathcal{F}) = \{f(x) \mid f \in \mathcal{F}\} \subseteq Y$ has compact closure in Y.

Solution. ($\Rightarrow$) The product topology on Y^X is Hausdorff since Y is Hausdorff. There any compact subset $\mathcal{F} \subseteq Y^X$ is closed in Y^X.

Since each projection $p_x \colon Y^X \twoheadrightarrow Y$ is continuous, the image $p_x(\mathcal{F}) \subseteq Y$ of the compact space $\mathcal{F}$ is compact, hence has compact closure in Y (since Y is Hausdorff).

($\Leftarrow$) Note the inclusion

$$\begin{aligned}
\mathcal{F} &\subseteq \bigcap_{x \in X} p_x^{-1}\left(p_x(\mathcal{F})\right) \\
&= \prod_{x \in X} p_x(\mathcal{F}) \\
&\subseteq \prod_{x \in X} \overline{p_x(\mathcal{F})}.
\end{aligned}$$

By assumption, each space $\overline{p_x(\mathcal{F})} \subseteq Y$ is compact, so that their product $\prod_{x \in X} \overline{p_x(\mathcal{F})}$ is compact, by Tychonoff's theorem. Since $\mathcal{F}$ is closed in Y^X and therefore in $\prod_{x \in X} \overline{p_x(\mathcal{F})}$, it follows that $\mathcal{F}$ is compact. $\square$

b. Let $C(X, Y)$ be endowed with the compact-open topology, and let $\mathcal{F} \subseteq C(X, Y)$ be a compact subspace. Show that $\mathcal{F}$ satisfies the conditions 1. and 2. listed in part (a), i.e.

1. $\mathcal{F}$ viewed as a subset of Y^X is closed **with respect to the topology of pointwise convergence**;
2. For all $x \in X$, the projection $p_x(\mathcal{F}) = \{f(x) \mid f \in \mathcal{F}\} \subseteq Y$ has compact closure in Y.

Solution. Denote by $\mathcal{T}_{co}$ the compact-open topology on Y^X or subspaces thereof (by abuse of notation). Denote by $\mathcal{T}_{prod}$ the topology of pointwise convergence on Y^X or subspaces thereof. The inclusion $\mathcal{T}_{prod} \subseteq \mathcal{T}_{co}$ of topologies implies that the composite

$$(C(X, Y), \mathcal{T}_{co}) \xrightarrow{\text{id}} (C(X, Y), \mathcal{T}_{prod}) \xrightarrow{\text{incl}} (Y^X, \mathcal{T}_{prod})$$

is continuous. If $\mathcal{F} \subseteq (C(X, Y), \mathcal{T}_{co})$ is compact, then $(\text{incl} \circ \text{id})(\mathcal{F}) \subseteq (Y^X, \mathcal{T}_{prod})$ is compact. By part (a), $(\text{incl} \circ \text{id})(\mathcal{F})$ satisfies conditions 1. and 2. But $(\text{incl} \circ \text{id})(\mathcal{F})$ is just $\mathcal{F}$ viewed as a subspace of Y^X. $\square$

Problem 6. (Munkres Exercise 47.1) For each of the following subsets $\mathcal{F} \subset C(\mathbb{R}, \mathbb{R})$, say if $\mathcal{F}$ is equicontinuous of not, and prove your answer.

a. $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ where $f_n(x) = x + \sin nx$.

Solution. $\mathcal{F}$ is not equicontinuous, since it is not equicontinuous at 0. Indeed, all functions f_n are differentiable, and their derivatives are

$$f_n'(x) = 1 + n \cos nx.$$

The derivatives $f_n'(0) = 1 + n$ are unbounded as n varies. □

Alternate solution. Take $\epsilon = 1$. Let $U \subseteq \mathbb{R}$ be any neighborhood of 0. Pick m large enough to satisfy $y := \frac{\pi}{2m} \in U$. Then we have

$$f_m(y) = y + \sin \frac{\pi}{2} = y + 1$$

while $f_m(0) = 0$, and in particular $|f_m(y) - f_m(0)| = y + 1 > 1$. This proves the non-inclusion

$$f_m(U) \nsubseteq B_1(f_m(0)). \quad \square$$

b. $\mathcal{F} = \{g_n \mid n \in \mathbb{N}\}$ where $g_n(x) = n + \sin x$.

Solution. $\mathcal{F}$ is equicontinuous, since the sine function is continuous, and translation $x \mapsto x+n$ is an isometry of $\mathbb{R}$. In other words, a neighborhood U of x which satisfies $\sin(U) \subseteq B_\epsilon(\sin x)$ will also satisfy $g_n(U) \subseteq B_\epsilon(g_n(x))$ for all $n \in \mathbb{N}$. □

c. $\mathcal{F} = \{h_n \mid n \in \mathbb{N}\}$ where $h_n(x) = |x|^{\frac{1}{n}}$.

Solution. $\mathcal{F}$ is not equicontinuous, since it is not equicontinuous at 0.

Take $\epsilon = \frac{1}{2}$. Let $U \subseteq \mathbb{R}$ be any neighborhood of 0, and let $x > 0$ be a point in U. The sequence of real numbers $h_n(x) = x^{\frac{1}{n}}$ converges to 1 as $n \to \infty$. Pick m large enough to satisfy $h_m(x) > \frac{1}{2}$. Noting $h_m(0) = 0$, we have

$$|h_m(x) - h_m(0)| = h_m(x) > \frac{1}{2}$$

which proves the non-inclusion

$$h_m(U) \nsubseteq B_{\frac{1}{2}}(h_m(0)). \quad \square$$

d. $\mathcal{F} = \{k_n \mid n \in \mathbb{N}\}$ where $k_n(x) = n \sin\left(\frac{x}{n}\right)$.

Solution. $\mathcal{F}$ is equicontinuous, in fact uniformly equicontinuous. All functions k_n are differentiable, and their derivatives are

$$k_n'(x) = n\,\frac{1}{n}\cos\left(\frac{x}{n}\right) = \cos\left(\frac{x}{n}\right)$$

whose magnitude is (uniformly) bounded by

$$|k_n'(x)| = |\cos\left(\frac{x}{n}\right)| \leq 1$$

for all $x \in \mathbb{R}$. $\square$