MA 523: Homework 4

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Problem 4.1 (Legendre transform)

Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u\big(\mathbf{x}(p)\big),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29)).

Solution. ■

PROBLEM 4.2

Find the solution $u(x,t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$\begin{cases} u(x,0) = f(x), \qquad u_t(x,0) = g(x), & \text{for } x > 0 \\ u_t(0,t) = \alpha u_x(0,t), & \text{for } t > 0, \end{cases}$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$. (*Hint:* Use a representation $u(x,t) = F(x-t) + G(x+t)$ for the solution.)

SOLUTION. ■

PROBLEM 4.3

(a) Let u be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0,t) = u(\pi,t) = 0$. Show that the *energy*

$$E(t) = \frac{1}{2}\int_0^\pi \left(u_t^2 + c^2 u_x^2\right) dx, \quad t > 0$$

is independent of t; i.e., $\frac{d}{dt}E = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x,t) = \sum_{n=1}^\infty \big(a_n \cos(nct) + b_n \sin(nct)\big) \sin(nx)$$

in terms of coefficients a_n, b_n.

SOLUTION. For part (a), suppose that u is, as above, a solution to the wave equation which is C^2 up to the boundary. We show that its energy is independent of t, i.e., that $dE/dt = 0$. Assuming the energy is bounded, the dominated convergence theorem allows us to permute the order of integration and differentiation like so

$$\begin{aligned}\frac{dE(t)}{dt} &= \frac{d}{dt}\left(\frac{1}{2}\int_0^\pi \left(u_t^2 + c^2 u_x^2\right) dx\right)\\ &= \frac{1}{2}\int_0^\pi \tfrac{\partial}{\partial t}\left(u_t^2 + c^2 u_x^2\right) dx\\ &= \frac{1}{2}\int_0^\pi 2u_t u_{tt} + 2c^2 u_x u_{xt}\, dx\end{aligned}$$

which, after using the relation $u_{tt} = c^2 u_{xx}$, becomes

$$\begin{aligned}&= c^2 \int_0^\pi u_t u_{xx} + u_x u_{xt}\, dx\\ &= c^2 \int_0^\pi \tfrac{\partial}{\partial x}(u_x u_t)\, dx\\ &= c^2\big(u_x(\pi,t)u_t(\pi,t) - u_x(0,t)u_t(0,t)\big)\\ &= 0\end{aligned}$$

since the boundary conditions, i.e., $u = 0$, implies $u_x = u_t = 0$ at the boundary.

For part (b), suppose u is a Fourier series solution to the wave equation, i.e.,

$$u(x,t) = \sum_{n=1}^\infty \big(a_n \cos(nct) + b_n \sin(nct)\big) \sin(nx).$$

First we compute u_t and u_x. They are

$$\begin{aligned} u_t(x,t) &= \tfrac{\partial}{\partial t} u(x,t) \\ &= \sum_{n=1}^{\infty} cn\big(b_n \cos(nct) - a_n \sin(nct)\big)\sin(nx) \end{aligned}$$

and

$$\begin{aligned} u_x(x,t) &= \tfrac{\partial}{\partial x} u(x,t) \\ &= \sum_{n=1}^{\infty} n\big(a_n \cos(nct) + b_n \sin(nct)\big)\cos(nx). \end{aligned}$$

Thus,

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^{\pi}\Bigg[\left(\sum_{n=1}^{\infty} cn\big(b_n \cos(nct) - a_n \sin(nct)\big)\sin(nx)\right)^2 \\ &\qquad + c^2\left(\sum_{n=1}^{\infty} n\big(a_n \cos(nct) + b_n \sin(nct)\big)\cos(nx)\right)^2\Bigg] \\ &= \end{aligned}$$

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