

# MA 523: Homework 7

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## PROBLEM 7.1

Solve the Dirichlet problem for the Laplace equation in  $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ u = x_1 & \text{on } |x| = 1, \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

(*Hint:* Use Laurent series.)

*SOLUTION.* First, let us convert the Dirichlet problem above to polar coordinates by the transformation  $(x, y) \mapsto (r \cos \theta, r \sin \theta)$ :

$$\begin{cases} \Delta u = 0 & \text{in } 1 < r < 2, \\ u = r \cos \theta & \text{on } r = 1, \\ u = 1 + \frac{1}{2} r^2 \sin(2\theta) & \text{on } r = 2. \end{cases} \quad (7.1)$$

Now, suppose  $u(r, \theta)$  is a Laurent series solution to (7.1)

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n r^n (\cos(n\theta) + \sin(n\theta)).$$

■

## PROBLEM 7.2

Let  $\Omega$  be a bounded domain with a  $C^1$  boundary,  $g \in C^2(\partial\Omega)$  and  $f \in C(\bar{\Omega})$ . Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \quad (*)$$

where  $\nu$  is the outer normal on  $\partial\Omega$ . Show that the solution of  $(*)$  in  $C^2(\Omega) \cap C^1(\bar{\Omega})$  is unique up to a constant; i.e., if  $u_1$  and  $u_2$  are both solutions of  $(*)$ , then  $u_2 = u_1 + \text{const.}$  in  $\Omega$ .

(Hint: Look at the proof of the uniqueness for the Dirichlet problem by energy methods [E, 2.2.5a].)

*SOLUTION.* Suppose  $u_1$  and  $u_2$  are solutions to the Neumann problem  $(*)$ . Define  $v := u_1 - u_2$ . Then  $v$  is harmonic in  $\Omega$  and  $\frac{\partial v}{\partial \nu} = 0$  on  $\partial\Omega$ . Consider the energy functional

$$E[v] = \frac{1}{2} \int_{\Omega} |Dv|^2 dx.$$

By Green's formula version (ii),

$$\begin{aligned} E[v] &= \frac{1}{2} \int_{\Omega} |Dv|^2 dx \\ &= -\frac{1}{2} \int_{\Omega} v \Delta v dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} v dS(x) \\ &= 0. \end{aligned}$$

This implies that  $|Dv|^2 = Dv \cdot Dv = 0$  which, since the standard inner product on  $\mathbb{R}^n$  is positive-definite, implies that  $Dv \equiv 0$ . It follows that  $u_1 = u_2 + \text{const.}$ , i.e., the solution  $u$  to  $(*)$  is unique up to a constant. ■

## PROBLEM 7.3

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where  $c \in \mathbb{R}$ .

(*Hint:* Rewrite the problem in terms of  $v(x, t) := e^{ct}u(x, t)$ .)

*SOLUTION.*

■