MA571 Homework 14

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Problem 14.1 (Munkres §74, Ex. 6)

If $n > 1$, show that the fundamental group of the n-fold torus is not Abelian. [*Hint:* Let G be a free group on the set $\{\alpha_1, \beta_1, ..., \alpha_n, \beta_n\}$; let F be the free group on the set $\{\gamma, \delta\}$. Consider the homomorphism of G onto F that sends α_1 and β_1 to γ and all other α_i and β_i to δ.]

Proof. Let $\mathbf{T}^n$ denote the n-fold torus and let us fix a base-point $x_0 \in \mathbf{T}^n$. By Theorem 74.3, the fundamental group of $\mathbf{T}^n$, $\pi_1(\mathbf{T}^n, x_0)$, is isomorphic to the quotient of the free group G on the set $\{\alpha_1, \beta_1, ..., \alpha_n, \beta_n\}$, by the least normal subgroup N containing

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1}. \tag{1}$$

Now we proceed by the hint. Let F be the free group on the set $\{\gamma, \delta\}$. We define a homomorphism $\varphi\colon G \to F$ by the rule $\alpha_1 \mapsto \gamma$, $\beta_1 \mapsto \gamma$ and $\alpha_i \mapsto \delta$ and $\beta_i \mapsto \delta$ for all $i \neq 1$. By Lemma 69.1, φ determines a homomorphism $G \to F$. Moreover, note that φ is surjective so by the 1st isomorphism theorem, $G/\ker\varphi \cong F$. Now, our next goal is to use the universal mapping property of the group quotient which guarantees the existence and uniqueness of a map $\bar{\varphi}\colon G/N \to F$.[1] To that end, we need to show that $N < \ker\varphi$. But N is the intersection of all normal subgroups of G containing (1) hence, it suffices to show that $\ker\varphi$ contains (1). But this is immediate since

$$\begin{aligned}\varphi(\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1}) &= \varphi(\alpha_1)\varphi(\beta_1)\varphi(\alpha_1^{-1})\varphi(\beta_1)^{-1}\cdots\varphi(\alpha_n)\varphi(\beta_n)\varphi(\alpha_n^{-1})\varphi(\beta_n)^{-1}\\ &= \delta\delta\delta^{-1}\delta^{-1}\gamma\gamma\gamma^{-1}\gamma^{-1}\cdots\gamma\gamma\gamma^{-1}\gamma^{-1}\\ &= 1.\end{aligned}$$

Thus, there exists a map $\bar{\varphi}\colon G/N \to F$ such that $\varphi = \bar{\varphi} \circ \pi_N$ where π is the canonical (group) projection map $\pi\colon G \to G/N$ defined by the rule $g \mapsto g + N$. Since $\varphi(G) = F$, $\bar{\varphi}(G/N) = F$ which is non-Abelian so G/N.[2] ∎

[1] Munkres never explicitly calls it this in his short exposition of group theory or, indeed, the UMP of the free product, quotient topology, product topology. These concepts are very illuminating and makes the whole process of writing thinking about a particular algebraic/geometric object much easier. In my opinion of course. tl;dr I don't know where this is stated in Munkres; we all know some group theory—this is true; please don't take off points.

[2] This is by elementary group theory: Say G is Abelian and $\varphi\colon G \to F$ is an homomorphism. Then $\varphi(G) < F$ is Abelian since, by the properties of the homomorphism, for any g_1, g_2, $\varphi(g_1g_2) = \varphi(g_2g_1)$ so $\varphi(g_1)\varphi(g_2) = \varphi(g_2)\varphi(g_2)$.

Problem 14.2 (Munkres §75, Ex. 1)

Calculate $H_1(\mathbf{P}^2\#\mathbf{T})$. Assuming that the list of compact surfaces given in Theorem 75.5 is a complete list, to which of these surfaces is $\mathbf{P}^2\#\mathbf{T}$ homeomorphic?

Proof. We shall begin by setting up, but not actually computing the fundamental group, $\pi_1(\mathbf{P}^2\#\mathbf{T})$. On Munkres §74, p. 453, Munkres gives us the labeling scheme

$$aabcb^{-1}c^{-1} \tag{2}$$

for the connected sum of $\mathbf{P}^2$ and $\mathbf{T}$; we shall not prove that this, indeed, determines the quotient space $\mathbf{P}^2\#\mathbf{T}$ (unless we have time), but instead we will use the labeling scheme to do our computations. Now, given (2), by Theorem 74.3, the fundamental group of $\mathbf{P}^2\#\mathbf{T}$ is the quotient of the free group on the set $\{a, b, c\}$, G, by the least normal subgroup N containing (2). Now, by Corollary 75.2, the Abelianization of $\pi_1(\mathbf{P}^2\#bfT)$ is isomorphic to

$$H_1(\mathbf{P}^2\#\mathbf{T}) \cong \frac{\mathbf{Z}(p(a)) \times \mathbf{Z}(p(b)) \times \mathbf{Z}(p(c))}{p(\langle aabcb^{-1}c^{-1}\rangle)}$$

and simplifying the quotient above taking note that p is a homomorphism whose image is lies inside an Abelian group, we have

$$= \frac{\mathbf{Z}(p(a)) \times \mathbf{Z}(p(b)) \times \mathbf{Z}(p(c))}{\langle 2p(a)\rangle} \tag{3}$$

where we use the module notation $\mathbf{Z}(x)$ to denote the free Abelian group generated by x, because why not; these happen to be $\mathbf{Z}$-modules after all. By the 1st isomorphism theorem, (3) is isomorphic to

$$\mathbf{Z}/(2) \times \mathbf{Z} \times \mathbf{Z} \tag{4}$$

by the obvious homomorphism, i.e., the one sending $(p(a), 0, 0) \mapsto (1, 0, 0)$, $(0, p(b), 0) \mapsto (0, 1, 0)$, and $(0, 0, p(c)) \mapsto (0, 0, 1))$. This map is bijective since it has an inverse, the map sending $(1, 0, 0) \mapsto (p(a), 0, 0)$, $(0, 1, 0) \mapsto (0, p(b), 0)$, and $(0, 0, 1) \mapsto (0, 0, p(c))$. It follows from Lemma 67.7 that both of the maps described above are homomorphisms. From the list given in Theorem 75.5, $\mathbf{P}^2\#\mathbf{T}$ must be homeomorphic to $\mathbf{P}^2\#\mathbf{P}^2$. ∎

Problem 14.3 (Munkres §75, Ex. 2)

If $\mathbf{K}$ is the Klein bottle, calculate $H_1(\mathbf{K})$ directly.

Proof. For this problem, we use the labeling provided my Munkres on Ex. 3 of §74 (p. 454) and that is

$$aba^{-1}b. \tag{5}$$

We proceed exactly as in Problem 14.2: Given (5), by Theorem 74.3, the fundamental group of $\mathbf{K}$ is the quotient of the free group on the set $\{a, b\}$, G, by the least normal subgroup N containing (5). Now, by Corollary 75.2, the Abelianization of $\pi_1(\mathbf{K})$ is isomorphic to

$$H_1(\mathbf{K}) \cong \frac{\mathbf{Z}(p(a)) \times \mathbf{Z}(p(b))}{p(\langle aba^{-1}b \rangle)}$$

and simplifying the quotient above taking note that p is a homomorphism whose image is lies inside an Abelian group, we have

$$= \frac{\mathbf{Z}(p(a)) \times \mathbf{Z}(p(b))}{\langle 2p(b) \rangle}$$

the latter being homeomorphic to $\mathbf{Z} \times \mathbf{Z}/(2)$. ∎

Problem 14.4 (Munkres §75, Ex. 3(a,b,c))

Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $acadbcb^{-1}d$.

(a) Check that all vertices of P are mapped to the same point of the quotient space X by the pasting map.

(b) Calculate $H_1(X)$.

(c) Assuming X is homeomorphic to one of the surfaces given in Theorem 75.5 (which it is), which surface is it?

Proof. (a) Let $\{p_1, ..., p_8\}$ denote the 8 vertices of P and $p\colon P \to X$ denote the quotient map. Then, we want to show hat $p^{-1}([p_1]) = \{p_1, ..., p_8\}$.

(b) To compute $H_1(X)$ we will proceed exactly as in Problem 14.3: The labeling scheme is given by

$$acadbcb^{-1}d. \tag{6}$$

By Theorem 74.3, the fundamental group of X is the quotient of the free group on the set $\{a, b, c, d\}$, G, by the least normal subgroup N containing (6). Now, by Corollary 75.2, the Abelianization of $\pi_1(X)$ is isomorphic to

$$H_1(X) \cong \frac{\mathbf{Z}(p(a)) \times \mathbf{Z}(p(b)) \times \mathbf{Z}(p(c)) \times \mathbf{Z}(p(d))}{\langle 2(p(a) + p(c) + p(d))\rangle}. \tag{7}$$

Now we make a change of basis, like Munkres does in the proof of Theorem 74.4, i.e., we define an $\mathbf{Z}$-linear isomorphism which sends $a \mapsto a$, $b \mapsto b$, $c \mapsto c$, and $d \mapsto a + b + d$. Thus, (8) is isomorphic to

$$\frac{\mathbf{Z}(p(a)) \times \mathbf{Z}(p(b)) \times \mathbf{Z}(p(c)) \times \mathbf{Z}(p(d))}{\langle 2(p(a) + p(c) + p(d))\rangle} \cong \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}/(2).$$

(c) From the list given in Theorem 75.5, we see that $X \approx \mathbf{P}^2\#\mathbf{P}^2\#\mathbf{P}^2\#\mathbf{P}^2$. ∎

Problem 14.5 (A)

Define P^n to be the space $\mathbf{S}^n/{\sim}$ where $z \sim z'$ if and only if $z = z'$ or $z = -z'$. Use the Seifert–van Kampen Theorem to calculate $\pi_1(\mathbf{P}^n)$. (Hint: induction starting from the case $n = 2$ that was done in class.)

Proof. We shall prove, by induction on n, that $\pi_1(\mathbf{P}^n) \approx \mathbf{Z}/(2)$. The base cased was proven in class, although if the grader wishes to see, it is show on §73, p. 444 that P^2 is homeomorphic to the 2-dunce cap and a bit later Munkres shows that the fundamental group of the n-fold dunce cap is a cyclic group of order n (Theorem 73.4).

Suppose $\pi_1(\mathbf{P}^m) \cong \mathbf{Z}/(2)$ for all $m \leq n$. We generalize the $n = 2$ case to the general case. Let $q\colon \mathbf{S}^n \to \mathbf{P}^n$ denote the quotient map and define $U := q(\mathbf{B}^n \smallsetminus \mathbf{0})$ and $V := q(D^n)$. Then $U \cap V = q(D^n \smallsetminus \mathbf{0})$. ■

Problem 14.6 (B)

A topological space X is called *homogeneous* if for every pair of points $x, y \in X$ there is a homeomorphism $\varphi\colon X \to X$ with $\varphi(x) = y$. Prove that every connected 2-manifold is homogeneous. (Hint: use the optional problem from the previous assignment.)

Proof. Let $x \in X$ and let A be the set of all points $y \in A$ homogeneous with x and $B := X \setminus A$. We will show that if B is nonempty, the pair A, B constitutes a separation of X contradicting that X is a connected and hence, that it is a manifold. First, it is clear that $A \cap B = \emptyset$ and $A \cup B = X$ by definition. Next, we need to show that A and B are open. Let $z \in A$. Then, since X is a manifold, there exists an open neighborhood U of z homeomorphic to the unit disk $(\mathbf{B}^2)^\circ$ via the homeomorphism h. Let $y \in U$. Then $h(y), h(z) \in (\mathbf{B}^2)^\circ$. Since $\mathbf{R}^2$ is normal, there exists a closed disk, say D, containing $h(y)$ and $h(z)$ contained in $(\mathbf{B}^2)^\circ$. Then, by Lemma A, the restriction $\tilde{h} := h|_{h^{-1}(D)}$ is a homeomorphism. It is clear that $D \approx \mathbf{B}^2$ by an appropriate homeomorphism $k\colon D \to \mathbf{B}^2$, scaling and translation. Now, by the optional problem from Hw 13, there exists a homeomorphism $f\colon \mathbf{B}^2 \to \mathbf{B}^2$ taking $k(\tilde{h}(z))$ to $k(\tilde{h}(y))$ and fixing $\mathbf{S}$. Define the map

$$\psi := \tilde{h}^{-1} \circ k^{-1} \circ f \circ k \circ \tilde{h}. \tag{8}$$

Then we define a map on X by

$$\varphi(x) := \begin{cases} \mathrm{id}_X(x) & \text{if } x \in X \setminus (\tilde{h}^{-1}(D))^\circ \\ \psi(x) & \text{otherwise} \end{cases}. \tag{9}$$

This map is continuous by the pasting lemma since $X \setminus (\tilde{h}^{-1}(D))^\circ = \partial\tilde{h}^{-1}(D)$ so if $w \in \partial\tilde{h}^{-1}(D)$ then

$$\psi(w) = \tilde{h}^{-1}\Big(k^1\Big(f\Big(k\Big(\tilde{h}(w)\Big)\Big)\Big)\Big) = w = \mathrm{id}_X(w),$$

and each map, restricted to that particular subset, the identity by Theorem 18.2 and the latter by construction. Let $\tilde{\varphi}\colon X \to X$ be the homeomorphism which sends x to z. Thus, $\tilde{\varphi} \circ \varphi(x) = y$ so $y \in A$. This implies that $U \subset A$ so A is open by Lemma C. Now, we can also show that A is closed by the same method. If $z \in \overline{A}$ then for every neighborhood U of z, $U \cap A$ is nonempty. In particular, if U is a neighborhood of z homeomorphic to $\mathbf{B}^2 \setminus \mathbf{S}$ then we can apply the same argument to show that z is homogeneous to x, hence in A. Thus, A is an open and closed subset of X. Since X is connected, it follows that $A = X$. ∎