MA 519: Homework 10

Max Jeter, Carlos Salinas

November 3, 2016

Problem 10.1 (Handout 14, # 5)

Approximately find the probability of getting a total exceeding 3600 in 1000 rolls of a fair die.

Solution. Let X_k, $1 \leq k \leq 1000$, denote the roll of a fair die. Then, as we have surely shown before, the mean and variance of the X_k are $\mu = 3.5$ and $\sigma^2 = 2.917$, respectively. By the central limit theorem, we can approximate $P(\sum_{k=1}^{100} X_k \geq 3600)$ by

$$P\left(\sum_{k=1}^{1000} X_k \geq 3600\right) \approx \int_{3600}^{\infty} \frac{\mathrm{e}^{-(x-3500)^2/5833.333}}{\sqrt{2\pi} \cdot 54.006}\, dx$$
$$\approx 0.514.$$

■

Problem 10.2 (Handout 14, # 6)

A basketball player has a history of converting 80% of his free throws. Find a normal approximation with a continuity correction of the probability that he will make between 18 and 22 throws out of 25 free throws.

Solution. Let X denote number of free shots (out of 25) the player has made. Since the outcome of the player's free shots is binary (the player can either score or not score the throw) $X \sim \mathrm{Bin}(25, 0.8)$. Therefore, by the de Moivre–Laplace central limit theorem with continuity correction, we have

$$\begin{aligned} P(18 \leq X \leq 22) &\approx \Phi\left(\frac{22.5 - 20}{\sqrt{25 \cdot 0.8 \cdot 0.2}}\right) - \Phi\left(\frac{17.5 - 20}{\sqrt{25 \cdot 0.8 \cdot 0.2}}\right) \\ &= \Phi(1.25) - \Phi(-1.25) \\ &\approx 0.789. \end{aligned}$$

■

Problem 10.3 (Handout 14, # 7)

Suppose $X_1, \ldots, X_n$ are independent $N(0,1)$ variables. Find an approximation to the probability that $\sum_{k=1}^n X_k$ is larger than $\sum_{k=1}^n X_k^2$, when $n = 10, 20, 30$.

Solution. Suppose $X_1, \ldots, X_n$ are independent $N(0,1)$ variables. We know that the sum $S_2 := \sum_{k=1}^n X_k^2$ is a chi-square distribution with mean n, variance $2n$, and density

$$f_{S_2}(x) = \frac{\mathrm{e}^{-x/2} x^{n/2-1}}{2^{n/2}\Gamma(m/2)}.$$

Moreover, by Theorem 1.46, the distribution of the sum $S_1 := \sum_{k=1}^n X_k$ is exactly a $N(0,n)$ distribution. Therefore,

$$P(S_1 > S_2) =$$

■

Problem 10.4 (Handout 14, # 8)

(A Product Problem). Suppose $X_1, \ldots, X_{30}$ are 30 independent variables, each distributed as $U[0,1]$. Find an approximation to the probability that their *geometric mean* exceeds 0.4; exceeds 0.5.

Solution. Write $Y_k := \ln X_k$, $1 \leq k \leq 30$. Then we can write the geometric mean of $X_1, \ldots, X_{30}$ as

$$\sqrt[30]{\prod\nolimits_{k=1}^{30} X_k} = \exp\left(\frac{1}{30}\sum\nolimits_{k=1}^{30} Y_k\right).$$

First, let us find the mean and the variance of Y_k, $1 \leq k \leq 30$. ■

Problem 10.5 (Handout 14, # 9)

(Comparing a Poisson Approximation and a Normal Approximation). Suppose 1.5% of residents of a town never read a newspaper. Compute the exact value, a Poisson approximation, and a normal approximation of the probability that at least one resident in a sample of 50 residents never reads a newspaper.

Solution. ■

Problem 10.6 (Handout 14, # 10)

(Test Your Intuition). Suppose a fair coin is tossed 100 times. Which is more likely: you will get exactly 50 heads, or you will get more than 60 heads?

Solution. ■

Problem 10.7 (Handout 14, # 11)

Find the probability that among 10 000 random digits the digit 7 appears not more than 968 times.

Solution. ■

Problem 10.8 (Handout 14, # 12)

Find a number k such that the probability is about 0.5 that the number of heads obtained in 1000 tossings of a coin will be between 490 and k.

Solution. ∎

Problem 10.9 (Handout 14, # 13)

In 10 000 tossings, a coin fell heads 5400 times. Is it reasonable to assume that the coin is skew?

Solution. ■

Problem 10.10 (Handout 14, # 14)

Interpret in plain words the statement the problem: *(Normal approximation to the Poisson distribution).* Using Stirling's formula, show that, if $\lambda \to \infty$, then for fixed $\alpha < \beta$

$$\sum_{\lambda+\alpha\sqrt{\lambda}<k<\lambda+\beta\sqrt{\lambda}} p(k;\lambda) \longrightarrow \Phi(\beta) - \Phi(\alpha).$$

Solution. Recall that $p(k;\lambda)$ is the discrete Poisson distribution

$$p(k;\lambda) = \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}.$$

$\blacksquare$

Problem 10.11 (Handout 14, # 15)

Give a proof that as $x \to \infty$,

$$1 - \Phi(x) \asymp \frac{\varphi(x)}{x}.$$

Remark: This gives the exact rate at which the standard normal right tail area goes to zero. It is even faster than the rate at which the standard normal density goes to zero, because of the extra x in the denominator.

Solution. We show that the limit of the ratios

$$\frac{1 - \Phi(x)}{\varphi(x)/x} = \frac{x - x\Phi(x)}{\varphi(x)}$$

tends to 1 as x tends to ∞. By l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{1 - \Phi(x)}{\varphi(x)/x} &= \lim_{x\to\infty} \frac{-\varphi(x)}{\varphi'(x)x^{-1} - \varphi(x)x^{-2}} \\
&= \lim_{x\to\infty} \frac{\varphi(x)}{\varphi(x)x^{-2} - \varphi'(x)x^{-1}} \\
&= \lim_{x\to\infty} \frac{\mathrm{e}^{-x^2/2}}{x^{-2}\mathrm{e}^{-x^2/2} + \mathrm{e}^{-x^2/2}} \\
&= \lim_{x\to\infty} \frac{1}{x^{-2} + 1} \\
&= \lim_{x\to\infty} \frac{x^2}{x^2 + 1}
\end{aligned}$$

applying l'Hôpital's rule again gives us

$$\begin{aligned}
&= \lim_{x\to\infty} \frac{2x}{2x} \\
&= 1
\end{aligned}$$

as was to be shown. ∎