MA571 Midterm 1: Practice Problems

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October 2, 2015

Problem 1. Let $A \subset X$ and $B \subset Y$. Show that the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

Proof. Before we proceed, we need to prove the following nontrivial facts:

Claim 1 (Munkres §17, Ex. 3)**.** *If A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.*

Proof of claim. We will show that the complement of $A \times B$ is open in $X \times Y$. Let $(x, y) \in (X \times Y) \setminus (A \times B)$. Then $x \notin A$ and $y \notin B$. Since A and B are closed in X and Y, respectively, there exist neighborhoods U and V of x and y, respectively, such that $U \subset X \setminus A$ and $V \subset Y \setminus B$. Then $U \times V \subset (X \times Y) \setminus (A \times B)$ is a neighborhood of (x, y) so, by Lemma C, $(X \times Y) \setminus (A \times B)$ is open. Thus, $A \times B$ is closed. ♣

Since $A \subset \overline{A}$ and $B \subset \overline{B}$ then $A \times B \subset \overline{A} \times \overline{B}$. Then by Lemma B $\overline{A \times B} \subset \overline{\overline{A} \times \overline{B}}$, but by Claim 1 $\overline{\overline{A} \times \overline{B}} = \overline{A} \times \overline{B}$ so $\overline{A \times B} \subset \overline{A} \times \overline{B}$. To see the reverse containment, take an element $(x, y) \in \overline{A} \times \overline{B}$ then for $x \in \overline{A}$ and $y \in \overline{B}$. Thus, by Theorem 17.5(a) for every neighborhood $U \ni x$ and $V \ni y$, we have $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. Thus, $U \times V \cap A \times B \neq \emptyset$ so by Theorem 17.5(b), since $U \times V$ is a basis element for the topology on $X \times Y$, $(x, y) \in \overline{A \times B}$. Thus, $\overline{A \times B} \supset \overline{A} \times \overline{B}$ and the equality $\overline{A \times B} = \overline{A} \times \overline{B}$ holds. ■

Problem 2. Let X be a topological space and let A be a dense subset of X. Let Y be a Hausdorff space and let $g, h \colon X \to Y$ be continuous functions which agree on A. Prove that $g = h$.

Proof. Suppose, towards a contradiction, that $g \neq h$. Then $g(x) \neq h(x)$ for some $x \in X \setminus A$. Since Y is Hausdorff, there exists neighborhoods $U \ni g(x)$ and $V \ni h(x)$ with $U \cap V = \emptyset$. Since g and h are continuous, $g^{-1}(U)$ and $h^{-1}(V)$ are neighborhoods of x. In particular, $g^{-1}(U) \cap h^{-1}(U)$ is a nonempty neighborhood of x. Since $\overline{A} = X$, by Theorem 17.5(a), $\big(g^{-1}(U) \cap h^{-1}(V)\big) \cap A \neq \emptyset$. Let $x_0 \in \big(g^{-1}(U) \cap h^{-1}(V)\big) \cap A$. Then $g(x_0) = h(x_0) \in U \cap V$. This contradicts the fact that U and V were chosen to be disjoint. ■

Problem 3. Let X and Y be topological spaces and let $f \colon X \to Y$ be a continuous function. Let G_f (called the *graph* of f) be the subspace $\{\, x \times f(x) \mid x \in X \,\}$ of $X \times Y$. Prove that if Y is Hausdorff then G_f is closed.

Proof. We will show that the complement of G_f in $X \times Y$ is open. Let $(x, y) \in (X \times Y) \setminus G_f$. Since Y is Hausdorff, choose neighborhoods U and V of y and $f(x)$ respectively, such that $f^{-1}(U) \cap V = \emptyset$. Then $f^{-1}(U) \times V \ni (x, y)$ is contained in the complement of G_f so, by Lemma C, G_f is open. ■

Problem 4. Let X be a topological space and let $f, g \colon X \to \mathbf{R}$ be continuous. Define $h \colon X \to \mathbf{R}$ by

$$h(x) = \min\{(f(x), g(x)\}.$$

Use the pasting lemma to prove that h is continuous. (You will not get full credit for any other method.)

Proof. Define the sets

$$A = \{ x \in X \mid f(x) \le g(x) \} \quad \text{and} \quad B = \{ x \in X \mid f(x) \ge g(x) \}.$$

Note $X = A \cup B$ and $f(x) = g(x)$ for every $x \in A \cap B$. Moreover, we have that

$$h(x) = \min\{f(x), g(x)\} = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}.$$

Thus, by the pasting lemma, h is continuous if we can show that A and B are closed in X.

We will prove that the complement of A in X is open; the proof of B is similar. Let $x \in X \setminus A$. Then $f(x) > g(x)$. Thus we have the following result

Lemma 2. *Let $x, y \in X$ with the order topology. Then there exists a neighborhood $U \ni x$, $V \ni y$ with $U \cap V = \emptyset$ and $x' < y'$ for all $x' \in U$, $y' \in V$.*

Proof of lemma. We break the demonstration into the following cases:

Case 1: Suppose there exists $z \in X$ with $x < z < y$, i.e., $z \in (x, y)$. Let U be the ray $U = (-\infty, z)$ and V be the ray $V = (z, \infty)$. Then $U \cap V = \emptyset$ and for every $x' \in V$, $y' \in U$ $x' < c < y'$, in particular, $x' < y'$.

Case 2: Suppose that there does not exists $z \in X$ with $x < z < y$, i.e., $(x, y) = \emptyset$. Let U be the ray $U = (-\infty, x)$ and V be the ray $V = (y, \infty)$. Then $U \cap V = \emptyset$ and for every $x' \in U$, $y' \in V$ we have $x' < x < y < y'$, in particular, $x' < y'$. ♣

By Lemma 2, choose $U \ni g(x)$ and $V \ni f(x)$ as above. Then $g^{-1}(U) \cap f^{-1}(V)$ is a neighborhood of x with $g(x) < f(x)$ for all. Hence $g^{-1}(U) \cap f^{-1}(V) \subset X \setminus A$ and, by Lemma C, $X \setminus A$ is open. Thus, A is closed.

Having satisfied the conditions of the pasting lemma, it follows that h is continuous. ■

Problem 5. Let X and Y be topological spaces and let $f : X \to Y$ be a function with the property that

$$f(\overline{A}) \subset \overline{f(A)}$$

for all subsets A of X. Prove that f is continuous.

Proof. Suppose that f has the property given above then, we claim that

Claim 3 (Munkres §18, Theorem 18.1(3))**.** *For every closed set B of Y, $f^{-1}(B)$ is closed in X.*

Proof of claim. Let B be closed in Y. Then we have $f(f^{-1}(B)) \subset B$ so if $x \in \overline{f^{-1}(B)}$ then

$$f(x) \in f(f^{-1}(B)) \subset \overline{f(f^{-1}(B))} \subset \overline{B} = B,$$

so that $x \in f^{-1}(B)$. Thus $\overline{f}^{-1}(B) \subset f^{-1}(B)$ and $\overline{f^{-1}(B)} = f^{-1}(B)$ as desired. ♣

■

Problem 6. Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Prove that

$$f(\overline{A}) \subset \overline{f(A)}$$

for all subsets A of X.

Proof. ■

Problem 7. Let X be any topological space and let Y be a Hausdorff space. Let $f, g\colon X \to Y$ be continuous functions. Prove that the set $\{\, x \in X \mid f(x) = g(x) \,\}$ is closed.

Proof. ■

Problem 8. Let X be a topological space and A a subset of X. Suppose that

$$A \subset \overline{X \setminus \overline{A}}.$$

Prove that $\overline{A}$ does not contain any nonempty open set.

Proof. ■

Problem 9. Let X be a topological space with a countable basis. Prove that every open cover of X has a countable subcover.

Proof. ■

Problem 10. Let X_α be an infinite family of topological spaces.

(a) Define the product topology on $\prod X_\alpha$.
(b) For each α, let A_α be a subspace of X_α. Prove that $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$.

Proof. ■

Problem 11. Suppose that we are given an indexing set A, and for each $\alpha \in A$ a topological space X_α. Suppose also that for each $\alpha \in A$ we are given a point $b_\alpha \in X_\alpha$. Let $Y = \prod X_\alpha$ with the product topology. Let $\pi_\alpha\colon Y \to X_\alpha$ be the projection. Prove that the set

$$S = \{\, y \in Y \mid \pi_\alpha(y) = b_\alpha \text{ except for finitely many } \alpha \,\}$$

is dense in Y (that is, its closure is Y).

Proof. ■

Problem 12. Let X be the Cartesian product $\mathbf{R}^\omega = \prod_{i=1}^\infty \mathbf{R}$ with the box topology (recall that a basis for this topology consists of all sets of the form $\prod_{i=1}^\infty U_i$, where each U_i is open in $\mathbf{R}$). Let $f\colon \mathbf{R} \to X$ be the function which takes t to $(t, t, t, ...)$. Prove that f is not continuous.

Proof. ■

Problem 13. Prove that the countable product $\mathbf{R}^\omega$ (with the product topology) has the following property: there is a countable family $\mathcal{F}$ of neighborhoods of the point $\mathbf{0} = (0, 0, 0, ...)$ such that for every neighborhood V of $\mathbf{0}$ there is a $U \in \mathcal{F}$ with $U \subset V$.

Note: the book proves that $\mathbf{R}^\omega$ is a metric space, but you may not use this in your proof. Use the definition of the product topology.

Proof. ■

Problem 14. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X, thus an element of Y is a sequence of 0's and 1's. For each $n \geq 1$, let $y_0 \in Y$ be the element $(1, 1, 1, ..., 1, 0, 0, 0, ..)$, with n 1's at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1s. Prove that the set $\{y_n\}_{n\geq 1} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

Proof. ■

Problem 15. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's. Let A be the subset of Y consisting of sequences with only a finite number of 1's. Is A closed? Prove or disprove.

Proof. ■

Problem 16. Let Y be a topological space.Let X be a set and let $f : X \to Y$ be a function. Give X the topology in which the open sets are the sets $f^{-1}(V)$ with V open in Y (you do not have to verify that this is a topology). Let $a \in X$ and let B be a closed set in X not containing a. Prove that $f(a)$ is not in the closure of $f(B)$.

Proof. ■

Problem 17. Let $f : X \to Y$ be a function that takes closed sets to closed sets. Let $y \in Y$ and let U be an open set containing $f^{-1}(y)$. Prove that there is an open set V containing y such that $f^{-1}(V)$ is contained in U.

Proof. ■

Problem 18. Let X be a topological space with an equivalence relation $\sim$. Suppose that the quotient space $X/\sim$ is Hausdorff. Prove that the set $S = \{ x \times y \in X \times X \mid x \sim y \}$ is a closed subset of $X \times X$.

Proof. ■

Problem 19. Let $p : X \to Y$ be a quotient map. Let us say that a subset S of X is *saturated* if it has the form $p^{-1}(T)$ for some subset T of Y. Suppose that for every $y \in Y$ and every open neighborhood U of $p^{-1}(y)$ there is a saturated open set V with $p^{-1}(y) \subset V \subset U$. Prove that p takes closed sets to closed sets.

Proof. ■

Problem 20. Let X be a topological space, let D be a connected subset of X, and let $\{E_\alpha\}$ be a collection of connected subsets of X.

Proof. ■

Problem 21. Let X and Y be connected. Prove that $X \times Y$ is connected.

Proof. ■

Problem 22. For any space X, let us say that two points are "inseparable" if there is no separation $X = U \cup V$ into disjoint open sets such that $x \in U$ and $y \in V$. Write $x \sim y$ if x and y are inseparable. Then $\sim$ is an equivalence relation (you don't have to prove this). Now suppose that X is locally connected (this means that for every point x and every open neighborhood U of x, there is a connected open neighborhood V of x contained in U). Prove that ecah equivalence class of the relation $\sim$ is connected.

Proof. ■

Problem 23. Let X be a topological space. Let $A \subset X$ be connected. Prove $\overline{A}$ is connected.

Proof. ■

Problem 24. Let $X_1, X_2, ...$ be topological spaces. Suppose $\prod_{n=1}^{\infty} X_n$ is locally connected. Prove that al lbut finitely many X_n are connected.

Proof. ■

Problem 25. LEt X be a connected space and let $f\colon X \to Y$ be a function which is continuous and onto. Prove that Y is connected. (This is a theorem in Munkres—prove it from the definitions).

Proof. ■

Problem 26. Given:

(i) $p\colon X \to Y$ is a quotient map.
(ii) Y is connected.
(iii) For every $y \in Y$, the set $p^{-1}(y)$ is connected.

Prove that X is connected.

Proof. ■

Problem 27. Let A be a subset of $\mathbf{R}^2$ which is homeomorphic to the open unit interval $(0, 1)$. Prove that A does not contain a nonempty set which is open in $\mathbf{R}^2$.

Proof. ■

Problem 28. Let X be a connected space. Let $\mathcal{U}$ be an open covering of X and let U be a nonempty set in $\mathcal{U}$. Say that a set V in $\mathcal{U}$ is *reachable from* U if there is a sequence $U = U_1, U_2, ..., U_n = V$ of sets in $\mathcal{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for each i from 1 to $n - 1$. Prove that every nonempty V in $\mathcal{U}$ is reachable from U.

Proof. ■

Problem 29. Suppose that X is connected and every point of X has a path-connected open neighborhood. Prove that X is path-connected.

Proof. ■

Problem 30. Let X be a topological space and let $f, g \colon X \to [0, 1]$ be continuous functions. Suppose that X is connected and f is onto. Prove that there must be a point $x \in X$ with $f(x) = g(x)$.

Proof. ■