Math 535 - General Topology
Fall 2012
Homework 2 Solutions

Problem 1. (Brown Exercise 2.4.5) Consider $X = [0, 2] \setminus \{1\}$ as a subspace of the real line $\mathbb{R}$. Show that the subset $[0, 1) \subset X$ is both open and closed in X.

Solution. $[0, 1)$ is open in X because we can write

$$[0, 1) = (-8, 1) \cap X$$

and $(-8, 1)$ is open in $\mathbb{R}$.

On the other hand, $[0, 1)$ is closed in X because we can write

$$[0, 1) = [0, 1] \cap X$$

and $[0, 1]$ is closed in $\mathbb{R}$. $\square$

Problem 2. (Bredon Exercise I.3.8) Let X be a topological space that can be written as a union $X = A \cup B$ where A and B are *closed* subsets of X. Let $f\colon X \to Y$ be a function, where Y is any topological space. Assume that the restrictions of f to A and to B are both continuous. Show that f is continuous.

Solution.

Lemma. *Let $A \subseteq X$ be a closed subset. If $C \subseteq A$ is closed in A, then C is also closed in X.*

Proof. Since C is closed in A, it can be written as $C = \widetilde{C} \cap A$ for some closed subset $\widetilde{C} \subseteq X$. Therefore C is an intersection of closed subsets of X, and thus is closed in X. □

Let $C \subset Y$ be a closed subset. Its preimage under f is the union

$$\begin{aligned} f^{-1}(C) &= \big(f^{-1}(C) \cap A\big) \cup \big(f^{-1}(C) \cap B\big) \\ &= (f|_A)^{-1}(C) \cup (f|_B)^{-1}(C). \end{aligned}$$

Since the restriction $f|_A\colon A \to Y$ is continuous, $(f|_A)^{-1}(C)$ is closed in A, and thus closed in X by the lemma. Likewise, $(f|_B)^{-1}(C)$ is closed in X. Therefore their union

$$f^{-1}(C) = (f|_A)^{-1}(C) \cup (f|_B)^{-1}(C).$$

is closed in X, so that f in continuous. □

Remark. The same proof shows that the statement still holds if A and B are both *open* in X.

Problem 3. A map between topological spaces $f\colon X \to Y$ is called an **open** map if for every open subset $U \subseteq X$, its image $f(U) \subseteq Y$ is open in Y.

a. (Munkres Exercise 2.16.4) Let X and Y be topological spaces. Show that the projection maps $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ are open maps.

Solution.

Lemma. *A map $f\colon X \to Y$ is open if and only if $f(B) \subseteq Y$ is open in Y for every $B \in \mathcal{B}$ belonging to some basis $\mathcal{B}$ of the topology on X.*

Proof. ($\Rightarrow$) Each member $B \in \mathcal{B}$ is open in X.

($\Leftarrow$) Let $U \subseteq X$ be open in X. Then U is a union $U = \bigcup_\alpha B_\alpha$ of basic open subsets $B_\alpha \in \mathcal{B}$. Its image under f is

$$\begin{aligned} f(U) &= f\left(\bigcup_\alpha B_\alpha\right) \\ &= \bigcup_\alpha f(B_\alpha) \end{aligned}$$

where each $f(B_\alpha)$ is open in Y by assumption. Thus $f(U)$ is a union of open subsets and hence open. $\square$

Take an "open box" $U \times V \subseteq X \times Y$, where $U \subseteq X$ is open and $V \subseteq Y$ is open. Its projection onto the first factor is

$$p_X(U \times V) = U \subseteq X$$

which is open in X. Since open boxes form a basis of the topology on $X \times Y$, the lemma guarantees that p_X is an open map, and likewise for p_Y. $\square$

b. Find an example of *metric* spaces X and Y, and a closed subset $C \subseteq X \times Y$ such that the projection $p_X(C) \subseteq X$ is *not* closed in X.

In other words, the projection maps are (usually) not closed maps.

Solution. Take $X = Y = \mathbb{R}$ and consider the hyperbola in $\mathbb{R} \times \mathbb{R}$

$$C = \{(x, \frac{1}{x}) \mid x \neq 0\} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid xy = 1\}.$$

Its projection onto the first factor is

$$p_X(C) = \mathbb{R} \setminus \{0\}$$

which is *not* closed in $\mathbb{R}$.

To show that C is closed in $\mathbb{R} \times \mathbb{R}$, note that the function $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = xy$ is continuous, and C is the preimage $C = f^{-1}(\{1\})$. Since the singleton $\{1\}$ is closed in $\mathbb{R}$, C is closed in $\mathbb{R} \times \mathbb{R}$. $\square$

Problem 4. (Munkres Exercise 2.19.7) Consider the set of sequences of real numbers

$$\mathbb{R}^{\mathbb{N}} = \{(x_1, x_2, \ldots) \mid x_n \in \mathbb{R} \text{ for all } n \in \mathbb{N}\} \cong \prod_{n\in\mathbb{N}} \mathbb{R}$$

and consider the subset of sequences that are "eventually zero"

$$\mathbb{R}^\infty := \{x \in \mathbb{R}^{\mathbb{N}} \mid x_n \neq 0 \text{ for at most finitely many } n\}.$$

a. In the box topology on $\mathbb{R}^{\mathbb{N}}$, is $\mathbb{R}^\infty$ a closed subset?

Solution. Yes, $\mathbb{R}^\infty$ is closed in the box topology.

Let $x \in \mathbb{R}^{\mathbb{N}} \setminus \mathbb{R}^\infty$, which means that the sequence x has infinitely many non-zero entries $x_n \neq 0$. For all those indices n, pick an open neighborhood U_n of $x_n \in \mathbb{R}$ which does not contain 0. For other values of n, take $U_n = \mathbb{R}$. Then the open box $\prod_n U_n$ is an open neighborhood of x which does not intersect $\mathbb{R}^\infty$.

Indeed, for any $y \in \prod_n U_n$ and every index n such that $x_n \neq 0$, we have $y_n \in U_n$ so that $y_n \neq 0$ by construction. Because there are infinitely many such indices, we conclude $y \notin \mathbb{R}^\infty$. □

b. In the product topology on $\mathbb{R}^{\mathbb{N}}$, is $\mathbb{R}^\infty$ a closed subset?

Solution. No, $\mathbb{R}^\infty$ is not closed in the product topology.

Let $x \in \mathbb{R}^{\mathbb{N}} \setminus \mathbb{R}^\infty$ and consider any open neighborhood $U = \prod_n U_n$ of x which is a "large box", i.e. $U_n \subseteq \mathbb{R}$ is open for all n and $U_n = \mathbb{R}$ except for finitely many n. In particular, there is a number N such that $U_n = \mathbb{R}$ for all $n \geq N$. Consider a sequence y with $y_n = 0$ for all $n \geq N$ and $y_n \in U_n$ for $1 \leq n < N$. Then we have $y \in U \cap \mathbb{R}^\infty$.

Because "large boxes" form a basis of the product topology, every open neighborhood of x intersects $\mathbb{R}^\infty$. Therefore $\mathbb{R}^\infty$ is not closed. □

Remark. In fact, the argument shows that x is not an interior point of $\mathbb{R}^{\mathbb{N}} \setminus \mathbb{R}^\infty$, so that the interior of $\mathbb{R}^{\mathbb{N}} \setminus \mathbb{R}^\infty$ is empty. Equivalently, the closure of $\mathbb{R}^\infty$ is all of $\mathbb{R}^{\mathbb{N}}$, i.e. $\mathbb{R}^\infty$ is dense in $\mathbb{R}^{\mathbb{N}}$.

Problem 5. Let X be a topological space, S a set, and $f\colon X \to S$ a function. Consider the collection of subsets of S

$$\mathcal{T} := \{U \subseteq S \mid f^{-1}(U) \text{ is open in } X\}.$$

a. Show that $\mathcal{T}$ is a topology on S.

Solution.

1. The preimage $f^{-1}(S) = X$ is open in X, so that the entire set S is in $\mathcal{T}$. Likewise, $f^{-1}(\emptyset) = \emptyset$ is open in X, so that the empty set $\emptyset$ is in $\mathcal{T}$.
2. Let U_α be a family of members of $\mathcal{T}$. Then we have
$$f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha)$$
where each $f^{-1}(U_\alpha)$ is open in X by assumption. Thus $f^{-1}\left(\bigcup_\alpha U_\alpha\right)$ is also open in X, so that the union $\bigcup_\alpha U_\alpha$ is in $\mathcal{T}$.
3. Let U and U' be members of $\mathcal{T}$. Then we have
$$f^{-1}\left(U \cap U'\right) = f^{-1}(U) \cap f^{-1}(U')$$
where $f^{-1}(U)$ and $f^{-1}(U')$ are open in X by assumption. Thus $f^{-1}\left(U \cap U'\right)$ is also open in X, so that the finite intersection $U \cap U'$ is in $\mathcal{T}$. $\square$

b. Show that $\mathcal{T}$ is the largest topology on S making f continuous.

Solution. Note that $\mathcal{T}$ makes f continuous by construction: for all $U \in \mathcal{T}$, the preimage $f^{-1}(U) \subseteq X$ is open in X.

Let $\mathcal{T}'$ be a topology on S making f continuous. Then for every $U \in \mathcal{T}'$, the preimage $f^{-1}(U)$ is open in X, which means $U \in \mathcal{T}$. This proves $\mathcal{T}' \leq \mathcal{T}$. $\square$

c. Let Y be a topological space. Show that a map $g\colon S \to Y$ is continuous if and only if the composite $g \circ f\colon X \to Y$ is continuous.

Solution. $(\Rightarrow)$ The maps f and g are continuous, hence so is their composite $g \circ f$.

$(\Leftarrow)$ Assume $g \circ f$ is continuous; we want to show that g is continuous. Let $U \subseteq Y$ be open and take its preimage $g^{-1}(U) \subseteq S$. To check that this subset is open, consider its preimage

$$f^{-1}\left(g^{-1}(U)\right) = (g \circ f)^{-1}(U) \subseteq X$$

which is open in X since $g \circ f$ is continuous. By definition of $\mathcal{T}$, $g^{-1}(U)$ is indeed open in S. $\square$

d. Show that $\mathcal{T}$ is the smallest topology on S with the property that a map $g\colon S \to Y$ is continuous whenever $g \circ f$ is continuous.

Solution. Let $\mathcal{T}'$ be a topology on S with said property. We know that $f\colon X \to (S, \mathcal{T})$ is continuous, but it can be written as the composite

$$X \xrightarrow{f} (S, \mathcal{T}') \xrightarrow{\mathrm{id}} (S, \mathcal{T}).$$

By the property of $\mathcal{T}'$, the composite $\mathrm{id} \circ f$ being continuous guarantees that the identity $\mathrm{id}\colon (S, \mathcal{T}') \to (S, \mathcal{T})$ is continuous, i.e. $\mathcal{T} \leq \mathcal{T}'$. $\square$

Problem 6. Consider the subset $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ viewed as a subspace of the real line $\mathbb{R}$. As a *set*, X is the disjoint union of the singletons $\{0\}$ and $\{\frac{1}{n}\}$ for all $n \in \mathbb{N}$. However, show that X does *not* have the coproduct topology on $\{0\} \amalg \coprod_{n \in \mathbb{N}} \{\frac{1}{n}\}$.

Solution. In the coproduct topology on $\{0\} \amalg \coprod_{n \in \mathbb{N}} \{\frac{1}{n}\}$ (which happens to be the discrete topology), the summand $\{0\}$ is open.

However, in the subspace topology on X, the singleton $\{0\}$ is *not* open. Indeed, any open ball $B_r(0)$ around 0 will contain other points $\frac{1}{n} \in B_r(0)$, for all n such that $\frac{1}{n} < r$. $\square$