MA571: Qual Problems

Carlos Salinas

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1 Covering Space Problems

Compiled from Prof. McClure's old quals.
Problem 1.1.

Proof. ∎

Problem 1.2.

Proof. ∎

Problem 1.3.

Proof. ∎

Problem 1.4.

Proof. ∎

Problem 1.5.

Proof. ∎

Problem 1.6.

Proof. ∎

Problem 1.7.

Proof. ∎

1.1 Kyle's Stuff

Problem 1.8 (No. 5)**.** Let X be a topological space and let $x_0 \in X$. Let U and V be open sets containing x_0, and suppose that the hypotheses of the seifert–van Kampen theorem are satisfied. Let $i_1 \colon U \cap V \to U$, $i_2 \colon U \cap V \to V$, $j_1 \colon U \to X$, and $j_2 \colon V \to X$ be the inclusion maps. Suppose that $(i_1)_* \colon \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is onto. Prove, using the Seifert–van Kampen theorem, that $(j_2)_* \colon \pi_1(V, x_0) \to \pi_1(X, x_0)$ is onto.

Proof. We use the classical Seifert–van Kampen theorem (Theorem 70.2). Suppose $(i_1)_* \colon \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is onto. Then for every element $\gamma \in \pi_1(U, x_0)$, $\gamma = (i_1)_*(\gamma')$ for some element $\gamma' \in \pi_1(U \cap V, x_0)$. Now, let $\gamma'' \in \pi_1(X, x_0)$. By the classical Seifert–van Kapmen theorem, the map

$$j \colon \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective and its kernel is the leans normal subgroup N of the free product that contanis all elements represented by words of the form $(i_1(g)^{-1}, i_2(g))$. ∎

Problem 1.9 (No. 6)**.** As in 5., but instead suppose that $(i_1)_* \colon \pi_1(U \cap V, x_0) \to \pi_1(X, x_0)$ is an isomorphism. Prove, using the Seifert–van Kampen theorem, that there is a homomorphism $\Phi \colon \pi_1(X, x_0) \to \pi_1(V, x_0)$ for which $\Phi \circ (j_2)_*$ is the identity map of $\pi_1(V, x_0)$.

Proof. ∎

2 August 2014

Problem 2.1. Let X be a topological space, let A be a subset of X, and let U be an open subset of X. Prove that $U \cap \bar{A} \subset \overline{U \cap A}$.

Proof. Let $x \in U \cap \bar{A}$. Then $x \in U$ and $x \in \bar{A}$. This means that, since U is open, by Lemma C there exist an open neighborhood V of x such that $V \subset U$. Moreover, since $x \in \bar{A}$, $V' \cap A \neq \emptyset$ for every open neighborhood V' of x. In particular, $V \cap A \neq \emptyset$. Thus, we have $V \cap U \neq \emptyset$ and $V \cap A \neq \emptyset$ so $V \cap (U \cap A) \neq \emptyset$. ■

Problem 2.2.

Proof. ■

Problem 2.3.

Proof. ■

Problem 2.4.

Proof. ■

Problem 2.5.

Proof. ■

Problem 2.6.

Proof. ■

Problem 2.7.

Proof. ■