MA 572: Homework 3

Carlos Salinas

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Problem 3.1 (Hatcher §2.1, Ex. 17)

(a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown. [What are X/A and X/B?]

Proof. (a) Since A is a finite collection of points in S^2, let us enumerate the set A via $\{a_1, ..., a_n\}$ and denote by A_k the subset $\{a_1, ..., a_k\}$ of A, where $k \leq n$. Now, by the generalization of theorem 2.16 to triples, we have the long exact sequence

$$\cdots \longrightarrow H_m(A_n, A_{n-1}) \longrightarrow H_m(S^2, A_{n-1}) \longrightarrow H_m(S^2, A_n) \longrightarrow H_{m-1}(A_n, A_{n-1}) \longrightarrow \cdots . \quad (1)$$

Exactness of (1) tells us that for $m \geq 2$ we have $H(S^2, A_{n-1}) \cong H(S^2, A_n)$ since

$$H_m(A_n, A_{n-1}) = 0 \longrightarrow H_m(S^2, A_{n-1}) \longrightarrow H_m(S^2, A_n) \longrightarrow 0 = H_{m-1}(A_n, A_{n-1})$$

is exact. Evidently, $H_m(A_n, A_{n-1}) = 0$ for $m > 1$.footnoteI will prove this if time permits.

(b) ∎

Problem 3.2 (Hatcher §2.2, Ex. 1)

Prove the Brouwer fixed point theorem for maps $f\colon D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f. [This was Brouwer's original proof.]

Proof. ∎

Problem 3.3 (Hatcher §2.2, Ex. 6)

Show that every map $S^n \to S^n$ can be homotoped to have a fixed point if $n > 0$.

Proof. ■

Problem 3.4

Let $\mathcal{U}$ be an open cover of X. Prove that the inclusion of $C_*^{\mathcal{U}}(C)$ into $C_*(X)$ is a chain homotopy equivalence.

Proof. ∎