MA 523: Homework 1

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Problem 1.1 (Taylor's formula)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to \mathbf{0}$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. ▸ Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. We claim that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

The proof of this follows from Leibniz's formula (which I found easier to prove)

Lemma (Leibniz's rule)**.** *Let $u, v\colon \mathbb{R}^n \to \mathbb{R}$ be smooth. Then*

$$D^\alpha uv = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v.$$

Proof of lemma. We proceed by induction on $m = |\alpha|$. For $m = 1$, we have

$$\begin{aligned} D^\alpha uv &= \frac{\partial}{\partial x_i} uv \\ &= u_{x_i} v + u v_{x_i} \\ &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v. \end{aligned}$$

Now assume the result for all $n \leq m-1$. Then for $|\alpha| = m$ we have

$$D^\alpha uv = \frac{\partial}{\partial x_i} D^{\alpha'} uv$$

where $\alpha' = \alpha - (0, \ldots, 1, \ldots, 0)$ in the i-th position

$$\begin{aligned} &= \frac{\partial}{\partial x_i} \left[\sum_{\beta\leq\alpha'} \binom{\alpha'}{\beta} D^\beta u D^{\alpha'-\beta} v \right] \\ &= \end{aligned}$$

□

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Problem 1.2

Write down the characteristic equation for the p.d.e.

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ▸

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Problem 1.3

Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ▸

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Problem 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\left(u_{x_1}^2 + u_{x_2}^2\right)$$

find a solution with $u(x_1, 0) = \left(1 - x_1^2\right)/2$.

Solution. ►

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