

# MA571: Qual Problems

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## 1 Covering Space Problems

Compiled from Prof. McClure's old quals.

**Problem 1.1.**

*Proof.* ■

**Problem 1.2.**

*Proof.* ■

**Problem 1.3.**

*Proof.* ■

**Problem 1.4.**

*Proof.* ■

**Problem 1.5.**

*Proof.* ■

**Problem 1.6.**

*Proof.* ■

**Problem 1.7.**

*Proof.* ■

### 1.1 Kyle's Stuff

**Problem 1.8** (No. 5). Let  $X$  be a topological space and let  $x_0 \in X$ . Let  $U$  and  $V$  be open sets containing  $x_0$ , and suppose that the hypotheses of the Seifert–van Kampen theorem are satisfied. Let  $i_1: U \cap V \rightarrow U$ ,  $i_2: U \cap V \rightarrow V$ ,  $j_1: U \rightarrow X$ , and  $j_2: V \rightarrow X$  be the inclusion maps. Suppose that  $(i_1)_*: \pi_1(U \cap V, x_0) \rightarrow \pi_1(U, x_0)$  is onto. Prove, using the Seifert–van Kampen theorem, that  $(j_2)_*: \pi_1(V, x_0) \rightarrow \pi_1(X, x_0)$  is onto.

*Proof.* We use the classical Seifert–van Kampen theorem (Theorem 70.2). Suppose  $(i_1)_*: \pi_1(U \cap V, x_0) \rightarrow \pi_1(U, x_0)$  is onto. Then for every element  $\gamma \in \pi_1(U, x_0)$ ,  $\gamma = (i_1)_*(\gamma')$  for some element  $\gamma' \in \pi_1(U \cap V, x_0)$ . Now, let  $\gamma'' \in \pi_1(X, x_0)$ . By the classical Seifert–van Kampen theorem, the map

$$j: \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective and its kernel is the least normal subgroup  $N$  of the free product that contains all elements represented by words of the form  $(i_1(g))^{-1}, i_2(g)$ . ■

**Problem 1.9** (No. 6). As in 5., but instead suppose that  $(i_1)_*: \pi_1(U \cap V, x_0) \rightarrow \pi_1(X, x_0)$  is an isomorphism. Prove, using the Seifert–van Kampen theorem, that there is a homomorphism  $\Phi: \pi_1(X, x_0) \rightarrow \pi_1(V, x_0)$  for which  $\Phi \circ (j_2)_*$  is the identity map of  $\pi_1(V, x_0)$ .

*Proof.* ■

## 2 August 2014

**Problem 2.1.** Let  $X$  be a topological space, let  $A$  be a subset of  $X$ , and let  $U$  be an open subset of  $X$ . Prove that  $U \cap \bar{A} \subset \overline{U \cap A}$ .

*Proof.* Let  $x \in U \cap \bar{A}$ . Then  $x \in U$  and  $x \in \bar{A}$ . This means that, since  $U$  is open, by Lemma C there exist an open neighborhood  $V$  of  $x$  such that  $V \subset U$ . Moreover, since  $x \in \bar{A}$ ,  $V' \cap A \neq \emptyset$  for every open neighborhood  $V'$  of  $x$ . In particular,  $V \cap A \neq \emptyset$ . Thus, we have  $V \cap U \neq \emptyset$  and  $V \cap A \neq \emptyset$  so  $V \cap (U \cap A) \neq \emptyset$ . ■

**Problem 2.2.** Let  $X$  be the following subspace of  $\mathbb{R}^2$ :

$$((0, 1] \times [0, 1]) \cup ([2, 3] \times [0, 1]).$$

Let  $\sim$  be the equivalence relation on  $X$  with  $(1, t) \sim (2, t)$  (that is  $(s, t) \sim (s', t') \iff (s, t) = (s', t')$  or  $t = t'$  and  $\{s, s'\} = \{1, 2\}$ ; you do *not* have to prove that this is an equivalence relation). Prove that  $X/\sim$  is homeomorphic to  $(0, 2) \times [0, 1]$ . (*Hint*: construct maps in both directions).

*Proof.* We shall proceed by the hint. Let  $q: X \rightarrow X/\sim$  denote the quotient map. Then, for  $(x, y) \in X$ , we define the map

We shall proceed by the hint. Let  $q: X \rightarrow X/\sim$  denote the quotient map. Then, for  $x \in X$ , we define the map

$$h(s, t) := \begin{cases} (s, t) & \text{if } (s, t) \in (0, 1] \times [0, 1] \\ (s - 1, t) & \text{if } (s, t) \in [2, 3] \times [0, 1] \end{cases}$$

from  $X \rightarrow (0, 2) \times [0, 1]$ .

By the UMP of the quotient space (Theorem Q.3), if we can show that  $h$  is continuous and preserves the equivalence relation, the induced map on the quotient space,  $h': X/\sim \rightarrow (0, 2) \times [0, 1]$  will be continuous. To that end, we will use the pasting lemma. First, note that  $(0, 1] \times [0, 1]$  and  $[2, 3] \times [0, 1]$  are closed subsets of  $X$  since  $(0, 1] \times [0, 1]$  is the complement of  $((1, \infty) \times (-2, 2)) \cap X$  which is open in  $X$  (since  $X$  inherits its topology from  $\mathbb{R}^2$ ), similarly,  $[2, 3] \times [0, 1]$  is closed in  $X$  since it is the complement of  $((-\infty, 2) \times (-2, 2)) \cap X$  which is open in  $X$  for the same reasons. It is clear that the maps  $x \mapsto x$  and  $x \mapsto x - 1$  are continuous onto their image, since the latter is nothing more than the inclusion map and the former is nothing more than subtraction, which is continuous by Theorem 21.5. Thus, by the pasting lemma,  $h$  is continuous.

Now we show that  $h$  does in fact preserve the equivalence relation. Suppose  $(s, t) \sim (s', t')$ . Then either  $(s, t) = (s', t')$  or  $t = t'$  and  $s, s' \in \{1, 2\}$ . In the former case, we have  $h(s, t) = h(s', t')$  (whether  $(s, t), (s', t') \in (0, 1] \times [0, 1]$  or its complement). In the latter case, we may, without loss of generality, assume that  $(s, t) = (1, t)$  and  $(s', t') = (2, t)$ . Then  $h(s, t) = (1, t) = (2 - 1, t) = h(s', t')$ . Thus, by Theorem Q.3, the induced map  $h': X/\sim \rightarrow (0, 2) \times [0, 1]$  is continuous. Moreover, the map is bijective with inverse

$$(h')^{-1} := \begin{cases} [s, t] & \text{if } x \in (0, 1] \\ [s + 1, t] & \text{if } x \in [1, 2). \end{cases}$$

This is clearly an inverse as

$$h' \circ (h')^{-1} = \text{id}_{X/\sim}$$

and

$$(h')^{-1} \circ h' = \text{id}_{(0, 2) \times [0, 1]}.$$

Thus, by Theorem 26.6,  $h'$  is a homeomorphism. ■

**Problem 2.3.** Prove that there is an equivalence relation  $\sim$  on the interval  $[0, 1]$  such that  $[0, 1]/\sim$  is homeomorphic to  $[0, 1] \times [0, 1]$ . As part of your proof *explain* how you are using one or more properties of the quotient topology.

*Proof.* First, it suffices to find a continuous surjective map  $f: [0, 1] \rightarrow [0, 1] \times [0, 1]$  and quotient out by the preimage of every point  $x \in [0, 1] \times [0, 1]$ . These maps are hard to describe in general, but they exist (take for example a space-filling curve). Next, note that if  $C$  is a closed subset of  $[0, 1]$  then it is compact so  $f(C)$  is compact. But since  $[0, 1] \times [0, 1]$  is compact Hausdorff, then  $f(C) \subset [0, 1] \times [0, 1]$  will be closed. It follows by that  $f$  will be a Munkres quotient map, so by Theorem Q.4,  $f': [0, 1]/\sim \rightarrow [0, 1] \times [0, 1]$  is a homeomorphism for some equivalence relation  $\sim$  on  $[0, 1]$ . ■

**Problem 2.4.** Let  $D$  be the closed unit disk in  $\mathbb{R}^2$ , that is, the set

$$\{(x, y) \mid x^2 + y^2 \leq 1\}.$$

Let  $E$  be the open unit disk

$$\{(x, y) \mid x^2 + y^2 < 1\}.$$

Let  $X$  be the one-point compactification of  $E$ , and let  $f: D \rightarrow X$  be the map defined by

$$f(x, y) = \begin{cases} (x, y) & \text{if } x^2 + y^2 < 1 \\ \infty & \text{if } x^2 + y^2 = 1. \end{cases}$$

Prove that  $f$  is continuous.

*Proof.* By the section on the one-point-compactification, it suffices to check two cases of open sets (1) all sets  $U$  open in  $E$ , and (2) all sets of the form  $U = X - C$  containing the point at infinity,  $\infty$ , where  $C$  is compact. In the first case, it is clear that  $f$  is continuous since it is just the inclusion map and is in fact bijective on  $E$ . For the second case, suppose that  $U$  is a neighborhood of  $\infty$ . Then  $Y - U$  is a compact subset of  $E$ , hence closed since  $X$  is a compact Hausdorff space. But since  $f$  is bijective, continuous on  $E$ , then  $f^{-1}(X - U)$  is a closed subset of  $E$ . Thus, by theorem 18.2,  $f$  is continuous. ■

**Problem 2.5.** Let  $X$  and  $Y$  be homotopy-equivalent topological spaces. Suppose that  $X$  is path-connected. Prove that  $Y$  is path-connected.

*Proof.* First we prove the following important result:

**Lemma 1.** *Path-connectedness is a topological property, i.e., if  $X$  is path-connected and  $f: X \rightarrow Y$  is a continuous map then,  $f(X)$  is path connected.*

*Proof.* Since  $X$  is path-connected, for any pair of points  $x, x' \in X$  there exists a continuous map  $p: [0, 1] \rightarrow X$  such that  $p(0) = x$  and  $p(1) = x'$ . Since composition of continuous maps is continuous,  $f \circ p: [0, 1] \rightarrow Y$  is a path from  $f(x)$  to  $f(x')$ . Since this property holds for any  $y \in f(X)$ , it follows that  $f(X)$  is path-connected. ♣

Now, suppose that  $X$  is homotopy-equivalent to  $Y$ . Then there exists continuous maps  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  such that  $g \circ f \simeq \text{id}_X$  and  $f \circ g \simeq \text{id}_Y$ . Now, since  $X$  is path-connected, by Lemma (1) we have  $f(X)$  is path-connected. Thus, it suffices to show that for every point  $y \in Y$  there exists a path  $p: [0, 1] \rightarrow Y$  from  $y$  to some point  $y' \in f(X)$ . Now, since  $f \circ g \simeq \text{id}_Y$ , there exists a homotopy, say  $H: Y \times [0, 1] \rightarrow Y$  such that  $H(s, 0) = f \circ g(s)$  and  $H(s, 1) = s$ . Consider the evaluation  $H_y := H(y, t) \circ H(y, t)$  where the map  $(y, t): [0, 1] \rightarrow Y \times [0, 1]$  is the imbedding of  $[0, 1]$  at  $y$  (which is continuous by Theorem 18.4) thus,  $H_y$  is continuous. Moreover,  $H_y(0) = f \circ g(y) \in f(Y)$  and  $H_y(1) = \text{id}_Y(y) = y$  so  $H_y$  is a path from  $y$  to a point  $f \circ g(y)$  in  $f(X)$ . Since we can do this for any point  $y \in Y$ , it follows, since path-connectedness is an equivalence relation, that  $Y$  is path-connected. ■

**Problem 2.6.** Let  $a$  and  $b$  denote the points  $(-1, 0)$  and  $(1, 0)$  in  $\mathbb{R}^2$ . Let  $x_0$  denote the origin  $(0, 0)$ . Use the Seifert–van Kampen theorem to calculate  $\pi_1(\mathbb{R}^2 - \{a, b\}, x_0)$ . You may not use any other method.

*Proof.* We'll use Theorem 70.2's version of the Seifert–van Kampen theorem. Define

$$U := (-\infty, \frac{1}{2}) \times \mathbb{R} \quad \text{and} \quad V := (-\frac{1}{2}, \infty) \times \mathbb{R}.$$

Then  $U \cap V = (-1/2, 1/2) \times \mathbb{R}$  is clearly path-connected since it is a convex set. Moreover, note that  $U \simeq \mathbb{R}^2 \setminus \{x_0\}$  and  $V \simeq \mathbb{R}^2 \setminus \{x_0\}$  (in the case of  $U$ , first consider the homeomorphism  $(x, y) \mapsto (x + 1, y)$  which sends  $a$  to  $(0, 0)$  and then the homotopy  $(x, y) \mapsto \frac{1}{t}(x, ys)$ ).

Once we have established the above, since the fundamental group of a space is invariant under homotopy-equivalence,  $\pi_1(U, x_0) \cong \pi_1(\mathbb{R}^2 \setminus \{x_0\}, y_0) \cong \mathbb{Z}$  for some arbitrary  $y_0 \neq x_0$  and similarly  $\pi_1(V, x_0) \cong \mathbb{Z}$ . ■

**Problem 2.7.** Let  $p: E \rightarrow B$  be a covering map with  $B$  locally connected, and let  $x \in B$ . Prove that  $x$  has a neighborhood  $W$  with the following property: for every connected component  $C$  of  $p^{-1}(W)$ , the map  $p: C \rightarrow W$  is a homeomorphism.

*Proof.* Let  $U$  be an evenly covered neighborhood of  $x$ . Then  $p^{-1}(U) = \bigcup_{\alpha} V_{\alpha}$  where the  $V_{\alpha}$  are open in  $E$  and  $V_{\alpha} \cap V_{\beta} = \emptyset$  whenever  $\alpha \neq \beta$ . For any  $\alpha$ , let  $C$  be a connected component of  $p^{-1}(U)$  containing  $p^{-1}(x) \cap V_{\alpha}$  (the latter is a one point set since  $p|_{V_{\alpha}}$  is a bijection). Then  $C \subset V_{\alpha}$  for at most one such  $\alpha$  for otherwise  $C \cap V_{\beta} \neq \emptyset$  for some  $\beta \neq \alpha$ , so  $C \cap V_{\beta}$  and  $C \cap V_{\alpha}$  form a separation of (note that  $C \setminus (C \cap V_{\beta}) = C \cap V_{\alpha}$  and vice-versa thus,  $C \cap V_{\beta}$  and  $C \cap V_{\alpha}$  are open and closed in the subspace topology on  $C$ , conversely) by Lemma 23.1.

Thus,  $p(C) \subset U$  is connected by Theorem 23.5. Moreover, since  $V_{\alpha} \supset C$  is homeomorphic to  $U$  by the restriction  $p|_{V_{\alpha}}$ ,  $p(C)$  is a connected component of  $U$  as the following lemma shows

**Lemma 2.** Suppose  $C$  is a connected component of  $X$  and  $h: X \rightarrow Y$  is a homeomorphism. Then  $h(C)$  is a connected component of  $Y$ .

*Proof of lemma.* Let  $C$  be a connected component of  $X$ . By theorem 23.5,  $h(C)$  is a connected subset of  $Y$ , moreover, is open. By Theorem 25.1,  $h(C)$  is contained in a connected component of  $Y$ , say  $D$ . Hence, we must show that  $D \subset h(C)$ . Now, since  $h$  is a homeomorphism,  $h^{-1}(D)$  is a connected subset of  $X$ , by Theorem 23.5, so is contained in only one component of  $X$ . But  $h^{-1}(D) \cap C \neq \emptyset$  so  $h^{-1}(D) \subset C$ . Thus, since  $h$  is a set-bijection,  $D \subset h(C)$ . ♣

so by Theorem 25.3,  $p(C)$  is open in  $B$  since  $B$  is locally connected. Thus, the restriction  $p|_C$  is a homeomorphism onto its image  $W := p(C)$ , by Lemma A, which is a neighborhood of  $x$ . ■