Fall 2016 Notes

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Contents

Chapter 1

Probability

We will devote this chapter to the material that is covered in MA 51900 (discrete probability) as it was covered in DasGupta's class. We will, for the most part, reference Feller's *An introduction to probability theory and its applications, Volume 1* [5] (especially for the discrete noncalculus portion of the class) and DasGupta's own book *Fundamentals of Probability: A First Course* [3].

1.1 Discrete Probability

The material in this section is pulled almost entirely from [5] with minor detours to [3]. We will not reference any particular pages in either book (unless we feel particularly lazy).

Background

Given a discrete sample space Ω with sample points $\omega_1, \omega_2, \ldots$, we shall assume that with each point ω_j there is associated a number, called the probability of ω_j and denoted by $P(\omega_i)$. It is nonnegative and such that

$$\sum_{i \in \mathbf{N}} P(\omega_i) = 1. \tag{1.1}$$

Definition 1.1. The probability $P(A)$ of an event A is the sum of the probabilities of all sample points in it.

Since the probability of Ω is 1 by (1.1), it follows that for any event A

$$0 \leq P(A) \leq 1. \tag{1.2}$$

Let A_1 and A_2 be arbitrary events. To compute the probability $P(A_1 \cup A_2)$ that either A_1 or A_2 or both occur, we have to add the probabilities of the sample points contained either in A_1 or in A_2, but each point is to be counted only once. Therefore, we have

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2). \tag{1.3}$$

Now, if ω is any point contained in both A_1 and A_2 the probability of ω, $P(\omega)$, appears on the right-hand side of (**??**) twice but only once in the left-hand side. This analysis leads us to conclude that the probability $P(A_1 \cap A_2)$ occurs twice on right-hand side of (1.3), and we have the important result

Theorem 1.2. *For any two events A_1 and A_2 the probability that either A_1 or A_2 or both occur is given by*

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2). \tag{1.4}$$

If $A_1 \cap A_2 = \emptyset$, that is, if A_1 and A_2 are mutually exclusive, then (1.4) *reduces to*

$$P(A_1 \cup A_2) = P(A_1) + P(A_2).$$

We may similarly continue to consider the probability of (countably) arbitrarily many events $A_1, A_2, \ldots$,

$$P\Big(\bigcup\nolimits_{i \in \mathbf{N}} A_i\Big) \leq \sum_{i \in \mathbf{N}} P(A_i). \tag{1.5}$$

This equation is referred to as *Boole's inequality*. In the special case where the events $A_1, A_2, \ldots$ are mutually exclusive, we have

$$P\Big(\bigcup\nolimits_{i \in \mathbf{N}} A_i\Big) = \sum_{i \in \mathbf{N}} P(A_i).$$

Chapter 2

Introduction to Partial Differential Equations

Here we summarize some important points about PDEs. The material is mostly taken from Evans's *Partial Differential Equations* [4] with occasional detours to Strauss's *Partial Differential Equations: An Introduction* [7].

Chapter 3

Algebraic Geometry

A summary to a course on an introduction to sheaf cohomology. We will mostly reference Donu's notes available here `https://www.math.purdue.edu/~dvb/classroom.html`, but also cite Ravi Vakil's *Fundamentals of Algebraic Geometry* [8] available here `https://math216.wordpress.com/`.

3.1 The statement of de Rham's theorem

These are almost verbatim Arapura's notes on the de Rham Complex and cohomology.

Before doing anything fancy, let's start at the beginning. Let $U \subseteq \mathbf{R}^3$ be an open set. In calculus class, we learn about operations

$$\{\,\text{functions}\,\} \xrightarrow{\nabla} \{\,\text{vector fields}\,\} \xrightarrow{\nabla\times} \{\,\text{vector fields}\,\} \xrightarrow{\nabla\cdot} \{\,\text{functions}\,\}$$

such that $(\nabla\times)(\nabla) = 0$ and $(\nabla\cdot)(\nabla\times) = 0$. This is a prototype for a *complex.* An obvious question: does $\nabla \times v = 0$ imply that v is a gradient? Answer: sometimes yes (e.g. if $U = \mathbf{R}^3$) and sometimes no (e.g. if $U = \mathbf{R}^3$ minus a line).

Chapter 4

Algebraic Topology

From my meetings with Mark. We reference Hatcher's *Algebraic Topology* [6] freely available here `https://www.math.cornell.edu/~hatcher/#ATI`.

4.1 Cohomology

Bibliography

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[8] R. Vakil. Math 216: Foundations of algebraic geometry, 2016.