MA557 Homework 12

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Problem 12.1

Let R be a Noetherian domain. Show that the following are equivalent:

(i) R is a unique factorization domain

(ii) every prime ideal of R of height one is principal

(iii) R is normal with $\mathrm{Cl}(R) = 0$.

Proof. (i) $\implies$ (ii) Suppose R is a Noetherian domain. Let $\mathfrak{p}$ be a height one prime. Then there exists at least one nonzero element $x \in \mathfrak{p}$. Let $x = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the factorization of x into irreducible (prime) elements of R. Set $p := p_i$ for any prime in the factorization of x. Then the ideal generated by p is a prime ideal contained in $\mathfrak{p}$, i.e., $\langle p \rangle \subset \mathfrak{p}$. But $\mathrm{ht}(\mathfrak{p}) = 1$. Thus, $\langle p \rangle = \mathfrak{p}$.

(ii) $\implies$ (ii) Suppose that every height one prime ideal in R is principal. To show that R is a UFD, it suffices to show that every irreducible element p is a prime element, that is, $\langle p \rangle$ is a prime ideal. Let $\mathfrak{p}$ be the minimal prime containing p. Since $\mathfrak{p}$ is principal, $\mathfrak{p} = \langle x \rangle$ for some $x \in \mathfrak{p}$. Thus, $p = xy$ for some $y \in R$. But p is prime hence, irreducible so either x or y is a unit. If x is a unit, then $\mathfrak{p} = R$, which is a contradiction. Thus, y must be a unit and we see that $\langle p \rangle = \langle xy \rangle = \mathfrak{p}$ is prime. ■

Problem 12.2

Let R be a ring with total ring of quotients K, M an R-module, and

$$\mathcal{T}(M) = \{\, x \in M \mid ax = 0 \text{ for some non zero-divisor } a \text{ of } RR \,\}.$$

The submodule $\mathcal{T}(M)$ is called the *torsion of* M, and M is called *torsion free* if $\mathcal{T}(M) = 0$. Show

(a) $\mathcal{T}(M) = \ker(M \to K \otimes_R M)$

(b) $M/\mathcal{T}(M)$ is torsion free.

Proof. ∎

Problem 12.3

Let R be a Dedekind domain and M a finitely generated R-module of rank r. Show that:

(a) If M is torsion free then M is projective (hint: induct on r).

(b) $M \cong \mathfrak{T}(M) \oplus P$ with P projective.

(c) If $M \neq 0$ is projective then $M \cong R^{r-1} \oplus I$ with $I \neq 0$ an ideal.

(d) If M is torsion (i.e., $M = \mathfrak{T}(M)$) then

$$M \cong R/I_1 \oplus \cdots \oplus R/I_n \qquad \text{with} \qquad I_1 \supset \cdots \supset I_n \neq 0$$

ideals (hint: for $p_1, ..., p_s$ the minimal primes of $\operatorname{ann}(M)$ and $S = R \smallsetminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s)$, show that $S^{-1}R$ is a PID).

Proof. ■