MA 523: Homework 8

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CARLOS SALINAS PROBLEM 8.1

Problem 8.1

Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t)$$

where

$$\Phi(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$$

is positive for |x| < 1, t > 0.

(Hint: Show that u satisfies $u_t = u_{xx}$ for t > 0,

$$\begin{cases} u = 0 & \text{on } \{ |x| = 1 \} \times \{ t \ge 0 \}, \\ u = \delta_0 & \text{on } \{ |x| = 1 \} \times \{ t = 0 \}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{|x| \le 1\} \times \{\varepsilon \le t \le T\}$ for small $\varepsilon > 0$ and large T > 0 pass to the limit as $\varepsilon \to 0^+$ and $T \to \infty$.)

SOLUTION.

PROBLEM 8.2 (TIKHONOV'S EXAMPLE)

Let

$$g(t) := \begin{cases} e^{-t^2} & t > 0, \\ 0 & t \le 0. \end{cases}$$

Then $g \in C^{\infty}(\mathbb{R})$ and we define

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that u(x,t) solves the heat equation in $\mathbb{R} \times (0,\infty)$ with the initial condition $u(x,0)=0, x\in\mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem.)

SOLUTION.

CARLOS SALINAS PROBLEM 8.3

Problem 8.3

Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax) e^{-x^2} dx, \qquad (a > 0).$$

 $(\mathit{Hint}:$ Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.)

Solution.