

## MA 26500-215 Quiz 5

July 22, 2016

1. Which of the following are *not* a basis for the vector space of all symmetric  $2 \times 2$  matrices? Why? [HINT: Recall that a symmetric matrix must satisfy  $A = A^T$ .]

A.  $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$

B.  $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$

C.  $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$

**Solution:** The correct choices are marked in **red**. Here is the rationale that accompanies it. We know that a symmetric matrix has the property that  $A = A^T$ . That is, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = A^T.$$

This forces  $b = c$ . Then we can replace our original matrix  $A$  by one that looks like

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

since we know  $b = c$  must be true for any  $2 \times 2$  symmetric matrix. Thus, one basis for the set of all  $2 \times 2$  symmetric matrices is the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \quad (\star)$$

The rest of the problem comes down to taking the set of vectors for options A, B and C and trying to reduce them to the set in  $(\star)$ .

2. Which of the following are *not* a basis for  $\mathbb{R}^3$ ? Why?

A.  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

$$\text{B. } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

$$\text{C. } \left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \right\}.$$

**Solution:** The correct choices are marked in **red**.

Since  $\dim \mathbb{R}^3 = 3$  we know that B can't possibly be a basis for  $\mathbb{R}^3$  so we immediately disqualify it.

So that leaves A and B.

For A, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and find  $A_{\text{rref}}$  (which can be done very quickly). Then

$$A_{\text{rref}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is full rank. Thus, the set in A is a basis for  $\mathbb{R}^3$ .

For B we follow the same procedure. Let

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 5 & 0 \end{bmatrix}.$$

Then, doing some real quick row operations gets you to

$$A_{\text{rref}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is full rank. Thus, the set in B is also a basis for  $\mathbb{R}^3$ .