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MA 26500-215 Quiz 5

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1. Which of the following are *not* a basis for the vector space of all symmetric 2×2 matrices? Why? [Hint: Recall that a symmetric matrix must satisfy $A = A^{\mathrm{T}}$.]

   A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

   B. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$.

   C. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Solution: The correct choices are marked in **red**. Here is the rationale that accompanies it. We know that a symmetric matrix has the property that $A = A^{\mathrm{T}}$. That is, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\mathrm{T}} = A^{\mathrm{T}}.$$

This forces $b = c$. Then we can replace our original matrix A by one that looks like

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

since we know $b = c$ must be true for any 2×2 symmetric matrix. Thus, one basis for the set of all 2×2 symmetric matrices is the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \tag{$\star$}$$

The rest of the problem comes down to taking the set of vectors for options A, B and C and trying to reduce them to the set in ($\star$).

2. Which of the following are *not* a basis for $\mathbb{R}^3$? Why?

   A. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

B. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

C. $\left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \right\}$.

Solution: The correct choices are marked in **red**.

Since $\dim \mathbb{R}^3 = 3$ we know that B can't possible be a basis for $\mathbb{R}^3$ so we immediately disqualify it.

So that leaves A and B.

For A, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and find A_{rref} (which can be done very quickly). Then

$$A_{\text{rref}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is full rank. Thus, the set in A is a basis for $\mathbb{R}^3$.

For B we follow the same procedure. Let

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 5 & 0 \end{bmatrix}.$$

Then, doing some real quick row operations gets you to

$$A_{\text{rref}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is full rank. Thus, the set in B is also a basis for $\mathbb{R}^3$.