MA 544: Homework 9

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Problem 9.1 (Wheeden & Zygmund §6, Ex. 1)

(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{ y : (x, y) \in E \}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $\{ x : (x, y) \in E \}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every y. Show that for almost $y \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every x.

Proof. (a) This follows from Fubini's theorem. Let $E_{\mathbf{x}} = \{ x : (x, y) \in E \}$ then by Theorem 6.8 we have

$$|E| = \iint_{\mathbb{R}^2} \chi_E \, dxdy = \int_{\mathbb{R}} \left[\int_{E_{\mathbf{x}}} dy \right] dx = 0. \tag{9.1}$$

Hence, $|E| = |E_{\mathbf{x}}| = 0$ for æ $y \in \mathbb{R}$.

(b) Suppose that $f(x, y) < \infty$ for a.e. $x \in \mathbb{R}$, for almost every $y \in \mathbb{R}$. ■

Problem 9.2 (Wheeden & Zygmund §6, Ex. 3)

Let f be measurable and finite a.e. on $[0,1]$. If $f(x)-f(y)$ is integrable over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, show that $f \in L[0,1]$.

Proof. Suppose that f is measurable and finite a.e. on $[0,1]$ and such that $f(x)-f(y) \in L([0,1]\times[0,1])$. Then, by Fubini's theorem we have

$$\iint\limits_{I\times I} f(x) - f(y)\,dxdy = \int_I \left[\int_I f(x) - f(y)\,dy\right] dx \tag{9.2}$$

■

Problem 9.3 (Wheeden & Zygmund §6, Ex. 4)

Let f be measurable and periodic with period 1: $f(t+1) = f(t)$. Suppose there is a finite c such that

$$\int_0^1 |f(a+t) - f(b+t)| dt \leq c$$

for all a and b. Show that $f \in L[0,1]$. (Set $a = x$, $b = -x$, integrate with respect to x, and make the change of variables $\chi = x + t$, $\eta = -x + t$.)

Proof. ■

Problem 9.4 (Wheeden & Zygmund §6, Ex. 6)

For $f \in L(\mathbb{R}^1)$, define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} \, dt$$

for $x \in \mathbb{R}^1$. (For complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i \int F_1$.) Show that if f and g belong to $L(\mathbb{R}^1)$, then

$$\widehat{(f * g)}(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

Proof. ■

PROBLEM 9.5 (WHEEDEN & ZYGMUND §6, EX. 7)

Let F be a closd subset of $\mathbb{R}^1$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F, prove that the function

$$\int_{\mathbb{R}^1} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}}\, dt$$

is integrable over F and so is finite a.e. in F. (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

Proof. ∎

Problem 9.6 (Wheeden & Zygmund §6, Ex. 9)

(a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.

(b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subset \mathbb{R}^1$, and let M_α be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int M_\lambda \leq \lambda^{-1}|G|$, where $G = (a, b) - [c, d]$.

Proof. ■