MA 598 PG: Homework 1

Carlos Salinas

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1 Notes

Let's just turn this file into notes.

1.1 Preliminaries

Topological spaces

By a topological space we mean a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a set of subsets of X satisfying:

(i) $\emptyset, X \in \mathcal{T}$.

(ii) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$.

(iii) For any subset $\mathcal{S} \subset \mathcal{T}$, $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.

The sets $U \in \mathcal{T}$ are referred to as $\mathcal{T}$-*open sets* or, simply, *open sets* when the topology $\mathcal{T}$ is understood. A subset $C \subset X$ is $\mathcal{T}$-*closed* or, simply, *closed* if $X \setminus C \in \mathcal{T}$. Given a subset $Y \subset X$, we define the *closure* of Y in X to be the intersection of all closed sets $C \subset X$ such that $Y \subset C$. We denote the closure of Y by $\overline{Y}$. We say $Y \subset X$ is *dense* if $\overline{Y} = X$. For each $x \in X$, we say $U \in \mathcal{T}$ is an *open neighborhood* of x or, simply, a *neighborhood* of x if $x \in U$. A *base* for a topology $\mathcal{T}$ is any subset $\mathcal{B}$ of $\mathcal{T}$ such that every $U \in \mathcal{T}$ can be expressed as a union of open sets in $\mathcal{B}$. A *neighborhood base* at x is any collection $\mathcal{B}_x$ of neighborhoods of x such that every every neighborhood of x can be expressed as a union of sets in $\mathcal{B}_x$.

Examples 1 (Discrete topology)**.** If $\mathcal{T}$ is the power set $\mathcal{P}(x)$ of X then $\mathcal{T}$ is a topology. This topology is called the *discrete topology.* Every set of X is both open and closed.

Given a topological space $(X, \mathcal{T})$ and a subset $Y \subset X$, we define the *subspace topology* $\mathcal{T}_{X,Y}$ on Y by

$$\mathcal{T}_{X,Y} \coloneqq \{\, U \cap Y : U \in \mathcal{T} \,\}. \tag{1}$$

We will refer to $(Y, \mathcal{T}_{X,Y})$ as a subspace of $(X, \mathcal{T})$. We say that $(X, \mathcal{T})$ is *compact* if given any subset $\mathcal{S} \subset \mathcal{T}$ such that $X = \bigcup_{U \in \mathcal{S}} U$, there exists a finite subset $\mathcal{S}_0 \subset \mathcal{S}$ such that $X = \bigcup_{U \in \mathcal{S}_0} U$. We will say that a subset $Y \subset (X, \mathcal{T})$ is compact if $(Y, \mathcal{T}_{X,Y})$ is a compact space. The following lemma is immediate from the definition of closed and compact.

Lemma 1. *Let $(X, \mathcal{T})$ be a compact space. If $\mathcal{S}$ is a collection of closed sets of X such that for any finite subset $\mathcal{S}_0 \subset \mathcal{S}$, we have $\bigcap_{C \in \mathcal{S}_0} C \neq \emptyset$, then $\bigcap_{C \in \mathcal{S}} C \neq \emptyset$.*

We say a space $(X, \mathcal{T})$ is *Hausdorff* if given distinct $x_1, x_2 \in X$, there exist disjoint open sets $U_1, U_2 \in \mathcal{T}$ such that $x_i \in U_i$ for $i = 1, 2$. If X is a Hausdorff space, then $\{x\}$ is closed for all $x \in X$. We say a space $(X, \mathcal{T})$ is *connected* if X cannot be expressed as the union of two disjoint closed sets. We say a space $(X, \mathcal{T})$ is totally disconnected if every connected subspace has at most one element.

Lemma 2. *Let X be a compact Hausdorff space.*

(a) If C_1, C_2 are disjoint closed subsets of X, then there exists disjoint open subsets U_1, U_2 of X such that $C_i \subset U_i$ for $i = 1, 2$.

(b) If $x \in X$ and A_x is the intersection of all sets U containing x such that are both open and closed, then A_x is connected.

(c) If X is also totally disconnected, then every open set is a union of sets that are both open and closed.

Proof. We start with (a). First, we assert that for each $x \in C_1$ there exists disjoint open sets U_x and V_x such that $x \in U_x$ and $C_2 \subset V_x$. For each $y \in C_2$, there exists disjoint open sets $U_{x,y}$ and $V_{x,y}$ such that $x \in U_{x,y}$, and $y \in V_{x,y}$. The set of open sets $\mathcal{C}_x = \{X \setminus C_2\} \cup \{V_{x,y}\}_{y \in C_2}$ is an open cover of X. Since X is compact, there exists a finite subset $\{y_1, ..., y_n\}$ of C_2 such that X is a union of $X - C_2$ and the sets V_{x,y_i}. Taking

$$U_x := \bigcap_{i=1}^{n} U_{x,y_i} \quad , \text{ and} \quad V_x := \bigcup_{i=1}^{n} V_{x,y_i},$$

verifies our first assertion. ■