MA 544: Homework 10

Carlos Salinas

April 9, 2016

Problem 10.1 (Wheeden & Zygmund §7, Ex. 1)

Let f be measurable in $\mathbf{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(\mathbf{x}) \geq c\|\mathbf{x}\|^{-n}$ for $\|\mathbf{x}\| \geq 1$.

Proof. Let E be a measurable subset of $\mathbf{R}^n$ such that $f \neq 0$ on E. For now, let us assume that the measure of E is finite and that E contains a point $\mathbf{x}$ of magnitude $\|\mathbf{x}\| \geq 1$. By Vitali's lemma, given a finite collection of cubes $\{Q_j\}_{j=1}^N$ covering our set E, there exists a real number $\beta > 0$ such that the following inequality is observed

$$|E| < \frac{1}{\beta}\sum_{j=1}^{N}|Q_j|. \tag{1}$$

■

Problem 10.2 (Wheeden & Zygmund §7, Ex. 2)

Let $\varphi(\mathbf{x}), \mathbf{x} \in \mathbf{R}^n$, be a bounded measurable function such that $\varphi(\mathbf{x}) = 0$ for $\|\mathbf{x}\| \geq 1$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(\mathbf{x}) = \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon)$. ($\varphi_\varepsilon$ is called an *approximation to the identity.*) If $f \in L(\mathbf{R}^n)$, show that

$$\lim_{\varepsilon \to 0}(f * \varphi_\varepsilon)(x) = f(\mathbf{x})$$

in the Lebesgue set of f. (Note that $\int \varphi_\varepsilon = 1$, $\varepsilon > 0$, so that

$$(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x}) = \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{y})d\mathbf{y}.$$

Use Theorem 7.16.)

Proof. ∎

Problem 10.3 (Wheeden & Zygmund §7, Ex. 6)

Show that if $\alpha > 0$, then x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Proof. ■

Problem 10.4 (Wheeden & Zygmund §7, Ex. 8)

Prove the following converse of Theorem 7.31: If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Proof. ■

Problem 10.5 (Wheeden & Zygmund §7, Ex. 9)

If f is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then f is absolutely continuous on $[a, b]$. (For the second part, use Theorems 2.2(ii) and 7.24 to show that $V(x)$ is absolutely continuous and then use the result of Exercise 8).

Proof. ∎

Problem 10.6 (Wheeden & Zygmund §7, Ex. 12)

Use Jensen's inequality to prove that if $a, b \geq 0$, $p, q > 1$, $(1/p) + (1/q) = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}.$$

More generally, show that

$$a_1 \cdots a_N = \sum_{j=1}^{N} \frac{{a_j}^{p_j}}{p_j},$$

where $a_j \geq 0$, $p_j > 1$, $\sum_{j=1}^{N}(1/p_j) = 1$. (Write $a_j = e^{x_j/p_j}$ and use the convexity of e^x).

Proof. ∎

Problem 10.7 (Wheeden & Zygmund §7, Ex. 13)

Prove Theorem 7.36.

Proof. Recall the statement of Theorem 7.36

Theorem. (i) *If* φ_1 *and* φ_2 *are convex in* (a, b)*, then* $\varphi_1 + \varphi_2$ *is convex in* (a, b)*.*

(ii) *If* φ *is convex in* (a, b) *and* c *is a positive constant, then* $c\varphi$ *is convex in* (a, b)*.*

(iii) *If* φ_k*,* $k = 1, 2, \ldots$*, are convex in* (a, b) *and* $\varphi_k \to \varphi$ *in* (a, b)*, then* φ *is convex in* (a, b)*.*

■