MA 544: Homework 8

Carlos Salinas

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Problem 8.1 (Wheeden & Zygmund §5, Ex. 2)

Show that the conclusion of (5.32) are not true without the assumption that $\varphi \in L(E)$. [In part (ii), for example, take $f_k = \chi_{(k,\infty)}$.]

Proof. ■

Problem 8.2 (Wheeden & Zygmund §5, Ex. 4)

If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, ...,$ and $\int_0^1 x^k f(x)\, dx \to 0$.

Proof. ■

Problem 8.3 (Wheeden & Zygmund §5, Ex. 6)

Let $f(x, y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $(\partial(x, y)/\partial x)$ is a bounded function of (x, y). Show that $(\partial(x, y)\partial x)$ is a measurable function of y for each x and

$$\frac{d}{dx} \int_0^1 f(x, y)\, dy = \int_0^1 \frac{\partial}{\partial x} f(x, y)\, dy.$$

Proof. ∎

Problem 8.4 (Wheeden & Zygmund §5, Ex. 7)

Give an example of an f that is not integrable, but whose improper Riemann integral exists and is finite.

Proof. ■

PROBLEM 8.5 (WHEEDEN & ZYGMUND §5, EX. 21)

If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

Proof. ∎

Problem 8.6 (Wheeden & Zygmund §6, Ex. 10)

Let V_n be the volume of the unit ball in $\mathbf{R}^n$. Show by using Fubini's theorem that

$$V_n = 2V_{n-1}\int_0^1 \left(1-t^2\right)^{(n-1)/2}\,dt.$$

(We also observe that by setting $w = t^2$, the integral is a multiple of a classical β-function and so can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$, $s > 0$.)

Proof. ■

Problem 8.7 (Wheeden & Zygmund §6, Ex. 11)

Use Fubini's theorem to prove that

$$\int_{\mathbf{R}^n} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = \pi^{n/2}.$$

(For $n = 1$, write $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dxdy$ and use polar. For $n > 1$, use the formula $e^{-|\mathbf{x}|^2} = e^{-{x_1}^2} \cdots e^{-{x_n}^2}$ and Fubini's theorem to reduce the case $n = 1$.)

Proof. ■