Exercises in Basic Mathematics

Carlos Salinas

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Definition 1. Given any ideal $\mathfrak{a} \subset k[X_1, \dots, X_q]$, define $\mathcal{V}_k(\mathfrak{a})$ by

$$\mathcal{V}_k(\mathfrak{a}) := \{\, \mathbf{x} \in \mathbb{A}^q : \text{for every } f \in \mathfrak{a},\ f(\mathbf{x}) = 0 \,\}.$$

We call $\mathcal{V}_k(\mathfrak{a})$ the *set of Ω-points of the affine k-variety determined by* $\mathfrak{a}$. With a slight abuse of language, we call $\mathcal{V}_k(\mathfrak{a})$ the *affine k-variety determined by* $\mathfrak{a}$. Similarly, given by any ideal $\mathfrak{a} \subset \bar{k}[X_1, \dots, X_q]$, defined by $\mathcal{V}_{\bar{k}}(\mathfrak{a})$ by

$$\mathcal{V}_{\bar{k}}(\mathfrak{a}) := \{\, \mathbf{x} \in \mathbb{A}^q : \text{for every } f \in \mathfrak{a},\ f(\mathbf{x}) = 0 \,\}.$$

We call $\mathcal{V}_{\bar{k}}(\mathfrak{a})$ the *set of Ω-points of the (geometric) affine $\bar{k}$-variety determined by* $\mathfrak{a}$, or for short, the *(geometric) affine variety determined by* $\mathfrak{a}$.

To ease the notation, we usually drop the subscript k or $\bar{k}$ and simply write $\mathcal{V}$.

If A is a (commutative) ring (with unit 1), recall that the *radical*, $\sqrt{\mathfrak{b}}$, of an ideal, $\mathfrak{a} \subset A$, is defined by

$$\sqrt{\mathfrak{a}} := \{\, a \in A : \text{there exists } n \geq 1,\ a^n \in \mathfrak{a} \,\}.$$

A *radical ideal* is an ideal, $\mathfrak{a}$, such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

The following properties are easily verified. We state them for $\mathcal{V}_k$, but they also hold for $\mathcal{V}_{\bar{k}}$:

$$\begin{aligned}
&\mathcal{V}(0) = \mathbb{A}^n,\ \mathcal{V}(A) = \emptyset \\
&\mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{V}(\mathfrak{a}\mathfrak{b}) = \mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) \\
&\mathfrak{a} \subset \mathfrak{b} \text{ implies that } \mathcal{V}(\mathfrak{b}) \subset \mathcal{V}(\mathfrak{b}) \\
&\mathcal{V}\Big(\textstyle\sum_\alpha \mathfrak{a}_\alpha\Big) = \textstyle\bigcap_\alpha \mathcal{V}(\mathfrak{a}_\alpha) \\
&\mathcal{V}(\sqrt{\mathfrak{a}}) = \mathcal{V}(\mathfrak{a})
\end{aligned}$$

From the relations above, it follows that the sets $\mathcal{V}(\mathfrak{a})$ can be taken as closed subsets of $\mathbb{A}^q$, and we obtain a topology on $\mathbb{A}^q$. This is the *k-topology on $\mathbb{A}^q$.* If we consider ideals in $\bar{k}[X_1, \ldots, X_q]$ (i.e., sets of the form $\mathcal{V}_{\bar{k}}(\mathfrak{a})$), we obtain the *Zariski topology on $\mathbb{A}^q$.*

The Zariski topology is not necessarily Hausdorff (except when $\mathcal{V}(\mathfrak{a})$ consits of a finite set of points.)

Let us see that $\mathbb{A}^q$ is not Hausdorff in the Zariski topology. Let $P, Q \in \mathbb{A}^q$, with $P \neq Q$. The line $\overleftrightarrow{PQ}$ is isomorphic to $\mathbb{A}^1$. Thus, it is enough to show that $\mathbb{A}^1$ is not Hausdorff. Consider any ideal $\mathfrak{a} \subset \bar{k}[X]$. Then, $\mathfrak{a}$ is a principal ideal, and thus

$$\mathfrak{a} = (f)$$

for some polynomial f, which shows that $\mathcal{V}(\mathfrak{a}) = \mathcal{V}(f)$ is a finite set. As a consequence, the closed sets of $\mathbb{A}^1$ (other than $\mathbb{A}^1$) are finite. Then, the union of two closed sets (distinct from $\mathbb{A}^1$) is also finite, and thus distinct from $\mathbb{A}^1$.

The topology on $\mathbb{A}^q$ is not the product topology on $\prod_{i=1}^{q} \mathbb{A}^1$.

For example, when $n = 2$, the closed set in $\mathbb{A}^1 \times \mathbb{A}^1$ are those sets consisting of finitely many horizontal and vertical lines, and intersections of such sets. However

$$X^2 + Y^2 - 1 = 0$$

defines a closed set in $\mathbb{A}^2$ not of the previous form.

To go backwards from subsets of $\mathbb{A}^q$ to ideals, we make the following definition.

Definition 2. Given any subset $S \subset \mathbb{A}^q$, define $\mathcal{I}_k(S)$ and $\mathcal{I}_{\bar{k}}(S)$ by

$$\mathcal{I}_k(S) := \left\{ f \in k[X_1, \ldots, X_q] : \text{for every } s \in S,\ f(s) = 0 \right\}$$

and

$$\mathcal{I}_{\bar{k}}(S) := \left\{ f \in \bar{k}[X_1, \ldots, X_q] : \text{for every } s \in S,\ f(s) = 0 \right\}$$

The following properties are easily shown (following our conventions, they are stated for $\mathcal{I}_k$, but they are easily shown for $\mathcal{I}_{\bar{k}}$).

CHAPTER 4

Differential Geometry Exercises

4.1 The Matrix Exponential; Some Matrix Lie Groups

The Exponential Map