

# MA544: Qual Preparation

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July 18, 2016

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**o.1 Danielli: Winter 2012**

**Problem 1.** Let  $f(x, y)$ ,  $0 \leq x, y \leq 1$ , satisfy the following conditions: for each  $x$ ,  $f(x, y)$  is an integrable function of  $y$ , and  $\partial f(x, y)/\partial x$  is a bounded function of  $(x, y)$ . Prove that  $\partial f(x, y)/\partial x$  is a measurable function of  $y$  for each  $x$  and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial f(x, y)}{\partial x} dy.$$

**Solution.** ►

◀

**Problem 2.** Let  $f$  be a function of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular, show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh.$$

*Hint:* A function  $h$  is said to be singular if  $h' = 0$ .

**Solution.** ►

◀

**Problem 3.** Let  $E \subset \mathbb{R}$  be a measurable set, and let  $K$  be a measurable function on  $E \times E$ . Assume that there exists a positive constant  $C$  such that

$$\int_E K(x, y) dx \leq C \tag{1}$$

for a.e.  $y \in E$ , and

$$\int_E K(x, y) dy \leq C \tag{2}$$

for a.e.  $x \in E$ .

Let  $1 < p < \infty$ ,  $f \in L^p(E)$ , and define

$$T_f(x) = \int_E K(x, y) f(y) dy.$$

(a) Prove that  $T_f \in L^p(E)$  and

$$\|T_f\|_p \leq C \|f\|_p. \tag{3}$$

(b) Is (3) still valid if  $p = 1$  or  $\infty$ ? If so, are assumptions (1) and (2) needed?

**Solution.** ►

◀

**Problem 4.** Let  $f$  be a nonnegative measurable function on  $[0, 1]$  satisfying

$$|\{x \in [0, 1] : f(x) > \alpha\}| < \frac{1}{1 + \alpha^2} \quad (4)$$

for  $\alpha > 0$ .

- (a) Determine values of  $p \in [1, \infty)$  for which  $f \in L^p[0, 1]$ .
- (b) If  $p_0$  is the minimum value of  $p$  for which  $p$  may fail to be in  $L^p$ , give an example of a function which satisfies (4), but which is not in  $L^{p_0}[0, 1]$ .

**Solution.** ►

◀

## o.2 Danielli: Summer 2011

**Problem 1.** Let  $f \in L^1(\mathbb{R})$ , and let  $F(t) = \int_{\mathbb{R}} f(x) \cos(tx) \, dx$ .

- (a) Prove that  $F(t)$  is continuous for  $t \in \mathbb{R}$ .
- (b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

*Hint:* Start by proving the statement for  $f = \chi_{[a,b]}$ .

**Solution.** ►

◀

**Problem 2.** (a) Suppose that  $f_k, f \in L^2(E)$ , with  $E$  a measurable set, and that

$$\int_E f_k g \rightarrow \int_E f g \quad (1)$$

as  $k \rightarrow \infty$  for all  $g \in L^2(E)$ . If, in addition,  $\|f_k\|_2 \rightarrow \|f\|_2$  show that  $f_k$  converges to  $f$  in  $L^2$ , i.e., that

$$\int_E |f - f_k|^2 \rightarrow 0$$

as  $k \rightarrow \infty$ .

- (b) Provide an example of a sequence  $f_k$  in  $L^2$  and a function  $f$  in  $L^2$  satisfying (1), but such that  $f_k$  does *not* converge to  $f$  in  $L^2$ .

**Solution.** ►

◀

**Problem 3.** A bounded function  $f$  is said to be of bounded variation on  $\mathbb{R}$  if it is of bounded variation on any finite subinterval  $[a, b]$ , and moreover  $A = \sup_{a,b} V[a, b; f] < \infty$ . Here,  $V[a, b; f]$  denotes the total variation of  $f$  over the interval  $[a, b]$ . Show that:

- (a)  $\int_{\mathbb{R}} |f(x+h) - f(x)| \, dx \leq A|h|$  for all  $h \in \mathbb{R}$ .

*Hint:* For  $h > 0$ , write

$$\int_{\mathbb{R}} |f(x+h) - f(x)| \, dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)| \, dx.$$

- (b)  $\left| \int_{\mathbb{R}} f(x) \varphi'(x) dx \right| \leq A$ , where  $\varphi$  is any function of class  $C^1$ , of bounded variation, compactly supported, with  $\sup_{x \in \mathbb{R}} |\varphi(x)| \leq 1$ .

**Solution.** ►

◀

**Problem 4.** (a) Prove the *generalized Hölder's inequality*: Assume  $1 \leq p \leq \infty$ ,  $j = 1, \dots, n$ , with  $\sum_{j=1}^{\infty} 1/p_j = 1/r \leq 1$ . If  $E$  is a measurable set and  $f_j \in L^{p_j}(E)$  for  $j = 1, \dots, n$ , then  $\prod_{j=1}^n f_j \in L^r(E)$  and

$$\|f_1 \cdots f_n\|_r \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

- (b) Use part (a) to show that that if  $1 \leq p, q, r \leq \infty$ , with  $1/p + 1/q = 1/r + 1$ ,  $f \in L^p(\mathbb{R})$ , and  $g \in L^q(\mathbb{R})$ , then

$$|(f * g)(x)| \leq \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q dy.$$

(Recall that  $(f * g)(x) = \int f(y)g(x-y) dy$ .)

- (c) Prove *Young's convolution theorem*: Assume that  $p, q, r, f$ , and  $g$  are as in part (b). Then  $f * g \in L^r(\mathbb{R})$  and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

**Solution.** ►

◀

# 1 Bañuelos

## 1.1 Bañuelos: Summer 2000

**Problem 1.** Let  $(X, \mathcal{F}, \mu)$  be a measure space and suppose  $\{f_n\}$  is a sequence of measurable functions with the property that for all  $n \geq 1$

$$\mu(\{x \in X : |f_n(x)| \geq \lambda\}) \leq C \exp(-\lambda^2/n)$$

for all  $\lambda > 0$ . (Here  $C$  is a constant independent of  $n$ .) Let  $n_k = 2^k$ . Prove that

$$\limsup_{k \rightarrow \infty} \frac{|f_{n_k}|}{\sqrt{n_k \log(\log(n_k))}} \leq 1 \quad \text{a.e.}$$

**Solution.** ► Suppose  $\{f_n\}_{n=1}^\infty$  is a sequence of measurable functions such that

$$\mu(\{x \in X : |f_n(x)| \geq \lambda\}) \leq C \exp(-\lambda^2/n) \quad (1)$$

for all  $\lambda$ . Now, consider the subsequence  $\{f_{2^k}\}_{k=1}^\infty$  of  $\{f_n\}_{n=1}^\infty$ . We aim to show that

$$\limsup_{k \rightarrow \infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} \leq 1$$

almost everywhere. To that end, it suffices to show that the set

$$E = \left\{ x \in X : \limsup_{k \rightarrow \infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} > 1 \right\}$$

has measure zero. Let  $x \in E$  then

$$\limsup_{k \rightarrow \infty} \frac{|f_{2^k}(x)|}{\sqrt{2^k \log(\log(2^k))}} > 1.$$

This means that there exists some subsequence  $\{k_m\}_{m=1}^\infty \subset \{k\}_{n=1}^\infty$  such that

$$\lim_{m \rightarrow \infty} \frac{|f_{2^{k_m}}(x)|}{\sqrt{2^{k_m} \log(\log(2^{k_m}))}} > 1.$$

This means that, for sufficiently large  $N$

$$|f_{2^{k_n}}(x)| > \sqrt{2^{k_n} \log(\log(2^{k_n}))}$$

for all  $n \geq N$ . But by Equation (1) we have

$$\begin{aligned}
\mu\left(\left\{x \in X : \frac{|f_{2^{k_n}}(x)|}{\sqrt{2^{k_n} \log(\log(2^{k_n}))}} \geq 1\right\}\right) &\leq C \exp\left(-\left(\sqrt{2^{k_n} \log(\log(2^{k_n}))}\right)^2 / 2^{k_n}\right) \\
&= C \exp\left(-2^{k_n} \log(\log(2^{k_n})) / 2^{k_n}\right) \\
&= C \exp\left(-\log(\log(2^{k_n}))\right) \\
&= C \exp\left(\log(1/\log(2^{k_n}))\right) \\
&= \frac{C}{\log(2^{k_n})}.
\end{aligned} \tag{2}$$

Letting  $n \rightarrow \infty$ , we see that the measure of the set on the left-hand side of Equation (2) must go to 0 so  $\mu(E) = 0$ .  $\blacktriangleleft$

**Problem 2.** Let  $(X, \mathcal{F}, \mu)$  be a finite measure space. Let  $f_n$  be a sequence of measurable functions with  $f_1 \in L^1(\mu)$  and with the property that

$$\mu(\{x \in X : |f_n(x)| > \lambda\}) \leq \mu(\{x \in X : |f_1(x)| > \lambda\})$$

for all  $n$  and all  $\lambda > 0$ . Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_X \left[ \max_{1 \leq j \leq n} |f_j| \right] d\mu = 0.$$

[Hint: You may use the fact that  $\|f\|_1 = \int_0^\infty \mu(\{|f(x)| > \lambda\}) d\lambda$ .]

**Solution.**  $\blacktriangleright$  Define  $g_n, h_n : \mathcal{F} \rightarrow [0, \infty]$  for  $n \in \mathbb{N}$  by

$$g_n(\lambda) = \mu(\{x \in X : |f_n(x)| > \lambda\}), \quad h_n(\lambda) = \mu\left(\left\{x \in X : \max_{1 \leq i \leq n} |f_i(x)| > \lambda\right\}\right).$$

Now, note that, by the monotonicity of  $\mu$ , we have

$$h_n(\lambda) \leq \sum_{i=1}^n g_n(\lambda) \leq n g_1(\lambda).$$

Thus,

$$\frac{h_n(\lambda)}{n} \leq g_1(\lambda).$$

Since  $\|f_1\|_1 = \int_0^\infty g_1(\lambda) d\lambda$ , by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \int_X \left[ \max_{1 \leq j \leq n} |f_j| \right] d\mu &= \lim_{n \rightarrow \infty} \int_X \frac{h_n(x)}{n} d\mu \\ &= \int_X \lim_{n \rightarrow \infty} \frac{h_n(x)}{n} d\mu \\ &\leq \int_X \lim_{n \rightarrow \infty} \frac{\mu(X)}{n} \\ &= 0 \end{aligned}$$

as we wanted to show. ◀

### Problem 3.

- (i) Let  $(X, \mathcal{F}, \mu)$  be a finite measure space. Let  $\{f_n\}$  be a sequence of measurable functions. Prove that  $f_n \rightarrow f$  is measurable if and only if every subsequence  $\{f_{n_k}\}$  contains a further subsequence  $\{f_{n_{k_j}}\}$  that converges a.e. to  $f$ .
- (ii) Let  $(X, \mathcal{F}, \mu)$  be a finite measure space. Let  $F: \mathbb{R} \rightarrow \mathbb{R}$  be continuous and  $f_n \rightarrow f$  in measure. Prove that  $F(f_n) \rightarrow F(f)$  in measure. (You may assume, of course, that  $f_n, F, F(f_n)$ , and  $F(f)$  are all measurable.)

**Solution.** ▶ Recall that a sequence of measurable functions  $\{f_n\}$  converge in measure to a limit  $f$  if for every  $\varepsilon > 0$  the limit

$$\lim_{n \rightarrow \infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) = 0.$$

For part (i)  $\implies$  suppose that  $f_n \rightarrow f$  in measure. Then given  $\varepsilon > 0$  and  $\delta > 0$  there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies

$$\mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) < \delta.$$

In particular, given  $\varepsilon = k^{-1}$  and  $\delta = 2^{-k}$ , consider the countable collection of measurable sets  $\{E_k\}_{k=1}^\infty$  given by

$$E_k = \left\{ x \in X : |f(x) - f_{n_k}(x)| \geq \frac{1}{k} \right\},$$

where  $n_k \geq N(k)$  (which depends on our choice of  $k$ ) such that

$$\mu(E_k) < \frac{1}{2^k}.$$



Now, by the Borel–Cantelli lemma, since

$$\sum_{k=1}^{\infty} \mu(E_k) < \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty,$$

for almost every  $x \in X$ , there exists  $N_x \in \mathbb{N}$  such that  $x \notin E_k$  for  $k \geq N_x$ . This means that for  $k \geq N_x$ , we have

$$|f(x) - f_{n_k}(x)| < \frac{1}{k}.$$

Let  $\{f_{n_{k+1}}\}$  be the subsequence of  $\{f_{n_k}\}$ . Then

$$\lim_{k \rightarrow \infty} f_{n_{k+1}} = f$$

as desired.

$\Leftarrow$  On the other hand, suppose that every subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  contains a subsequence  $\{f_{n_{k_j}}\}$  that converges to  $f$ . Seeking a contradiction, suppose that given  $\varepsilon > 0$  there exists a subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  such that

$$M = \mu(\{x \in X : |f(x) - f_{n_k}(x)| \geq \varepsilon\}) > 0.$$

But by assumption there exists a subsequence  $\{f_{n_{k_j}}\}$  of  $\{f_{n_k}\}$  that converges almost everywhere to  $f$ . We claim that this implies that  $f_{n_{k_j}} \rightarrow f$  in measure.

*Proof of claim.* This is adapted from a proof in Royden, Proposition 3, Ch. 5.

First note that  $f$  is measurable since it is the pointwise limit almost everywhere of a sequence of measurable functions. Let  $\varepsilon, \delta > 0$  be given. **Here is where the assumption that  $\mu(X) < \infty$  is essential!** By Egorov's theorem, there is a measurable subset  $E \subset X$  with  $\mu(X \setminus E) < \delta$  such that  $f_n \rightarrow f$  uniformly on  $E$ . Thus, there is an index  $N$  such that  $n \geq N$  implies

$$|f_n(x) - f(x)| < \varepsilon$$

for all  $x \in E$ . Thus, for  $n \geq N$ ,

$$\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\} \subset X \setminus E$$

so

$$\mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) < \varepsilon.$$

Thus, we have

$$\lim_{n \rightarrow \infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}) = 0,$$

i.e.,  $f_n \rightarrow f$  in measure. ■

Hence, since  $f_{n_{k_j}} \rightarrow f$  in measure, but  $M > 0$  we have a contradiction.

For (ii) since  $F$  is continuous given  $\varepsilon > 0$  there exist  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|F(x) - F(x')| < \varepsilon$ . By part (i),  $f_n \rightarrow f$  in measure if and only if every subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  contains a subsequence  $\{f_{n_{k_j}}\}$  that converges to  $f$  almost everywhere, i.e., given  $\delta > 0$  there exists an index  $N$  such that  $n_{k_j} \geq N$  implies

$$|f(x) - f_{n_{k_j}}(x)| < \delta$$

for almost every  $x \in X$ . Thus,

$$|F(f(x)) - F(f_{n_{k_j}}(x))| < \varepsilon$$

and we see that for every subsequence  $\{F \circ f_{n_k}\}$  of  $\{F \circ f_n\}$  we can find a subsequence  $\{F \circ f_{n_{k_j}}\}$  that converges almost everywhere to  $F \circ f$ .  $\blacktriangleleft$

**Problem 4.** Let  $(X, \mathcal{F}, \mu)$  be a finite measure space and suppose  $f \in L^1(\mu)$  is nonnegative. Suppose  $1 < p < \infty$  and let  $1 < q < \infty$  be its conjugate exponent, i.e.,  $1/p + 1/q = 1$ . Suppose  $f$  has the property that

$$\int_E f \, d\mu \leq \mu(E)^{1/q}$$

for all measurable sets  $E$ . Prove that  $f \in L^r(\mu)$  for any  $1 \leq r < p$ .

[Hint: Consider  $\{x \in X : 2^n \leq f(x) < 2^{n+1}\}$ , if you like.]

**Solution.**  $\blacktriangleright$  By previous problems, we know that if  $\mu(X) < \infty$  and  $f \in L^p(X)$ , then  $f \in L^r(X)$  for  $1 \leq r < p$ , so it suffices to show that  $\|f\|_p < \infty$ .

Instead of following the hint, consider the set

$$E_t = \{x \in X : f(x) \geq t\}$$

and let

$$\omega(t) = \mu(E_t),$$

i.e., the distribution function of  $f$ . Then, we have

$$\int_0^\infty \omega(t) \, dt = \int_X f \, d\mu.$$

In particular, if we make the substitution  $\alpha = t^{1/p}$ ,  $d\alpha = t^{1/q}/p \, dt = \alpha^{p/q}/p \, dt$ , we have

$$\int_X f^p \, d\mu = \int_0^\infty p\alpha^{-p/q} \omega(\alpha) \, d\alpha.$$

Now, by Chebyshev's inequality, we have

$$t\omega(t) \leq \int_{E_t} f \, d\mu \leq \omega(t)^{1/q}$$

so

$$\omega(t) \leq t^{-p}.$$

Thus,

$$\int_X f^r \, d\mu = \int_0^\infty p\alpha^{-p/q} \omega(\alpha) \, d\alpha \leq \int_0^\infty p\alpha^{-p-p/q} \, d\alpha.$$

Since  $p + p/q > 1$ , the integral above is finite. Thus,  $f \in L^p(X)$  and we have  $f \in L^r(X)$  for all  $1 \leq r < p$ . ◀

**Problem 5.** Let  $f$  be a continuous function on  $[-1, 1]$ . Find

$$\lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|) \, dx.$$

**Solution.** ▶ To find the limit of the integral

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|) \, dx$$

we first make the following substitutions: Let  $y = nx$ ,  $dy = n \, dx$ . Then

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|) \, dx = \frac{1}{n} \int_{-1}^1 f(y/n)(1 - |y|) \, dy.$$

By the extreme value theorem, since  $f$  is continuous and  $[-1, 1]$  is compact  $f$  is bounded on  $[-1, 1]$  by, say  $M$ . Let  $g(x) = M$ . Then  $g \in L^1(X)$  since  $\|g\|_1 = 2M$ . Thus, by the Lebesgue dominated convergence theorem, since

$$|f(y/n)(1 - |y|)| \leq M$$

on  $[-1, 1]$  and  $g \in L^1([-1, 1])$  it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|) \, dx &= \lim_{n \rightarrow \infty} \frac{1}{n} \int_{-1}^1 f(y/n)(1 - |y|) \, dy \\ &= \int_{-1}^1 \lim_{n \rightarrow \infty} \left[ \frac{f(y/n)(1 - |y|)}{n} \right] dy \\ &= \int_{-1}^1 \lim_{n \rightarrow \infty} \left[ \frac{f(y/n)}{n} - \frac{|y|}{n} \right] dy \\ &= 0. \end{aligned}$$

◀

**Problem 6.** Let  $(X, \mathcal{F}, \mu)$  be a measure space and suppose  $f \in L^p(\mu)$ ,  $1 \leq p < \infty$ . Suppose  $E_n$  is a sequence of measurable sets satisfying  $\mu(E_n) = 1/n$  for all  $n$ . Prove that

$$\lim_{n \rightarrow \infty} \left[ n^{(p-1)/p} \int_{E_n} |f| \, d\mu \right] = 0.$$

**Solution.** ► The result follows immediately by Hölder's inequality. Let  $C = \|f\|_p$ . Since  $f \in L^p(X)$ , then  $f \in L^p(E_n)$  for all  $n \in \mathbb{N}$ . Thus, by Hölder's inequality

$$\begin{aligned} \|f\|_{L^1(E_n)} &\leq \|f\|_{L^p(E_n)} \mu(E_n)^{1/q} \\ &\leq C \mu(E_n)^{1/q} \\ &= C \mu(E)^{p/(p-1)} \\ &= C n^{-p/(p-1)} \\ &= C n^{p/(1-p)}. \end{aligned}$$

Hence, the integral is bounded above by

$$\begin{aligned} 0 \leq n^{(p-1)/p} \int_{E_n} |f| \, d\mu &\leq C n^{(p-1)/p + p/(1-p)} \\ &= C n^{(2p-1)/(p(1-p))}. \end{aligned}$$

Since  $p > 1$ ,  $1 - p < 0$  and  $2p - 1 > 0$  so the exponent  $(2p - 1)/(p(1 - p)) < 0$ . Thus, as  $n \rightarrow \infty$

$$C n^{(2p-1)/(p(1-p))} \longrightarrow 0.$$

It follows that

$$\lim_{n \rightarrow \infty} \left[ n^{(p-1)/p} \int_{E_n} |f| \, d\mu \right] = 0.$$

◀

**Problem 7.** Let  $(X, \mathcal{M}, \mu)$  be a measure space and let  $\{g_n\}$  be a sequence of non-negative measurable functions with the property that  $g_n \in L^1(\mu)$  for every  $n$  and  $g_n \rightarrow g$  in  $L^1(\mu)$ . Let  $\{f_n\}$  be another sequence of nonnegative measurable functions on  $(X, \mathcal{F}, \mu)$ .

(i) If  $f_n \leq g_n$  almost everywhere for every  $n$ , prove that

$$\limsup_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \int_X \limsup_{n \rightarrow \infty} f_n \, d\mu.$$

[Hint: Start by considering a subsequence  $\{f_{n_k}\}$  such that

$$\lim_{n_k \rightarrow \infty} \int_X f_{n_k} d\mu = \limsup_{n \rightarrow \infty} \int_X f_n d\mu$$

and let  $\{g_{n_{k_j}}\}$  be a subsequence of  $\{g_{n_k}\}$  such that  $g_{n_{k_j}} \rightarrow g$  almost everywhere.]

- (ii) If  $f_n \rightarrow f$  almost everywhere and if  $f_n \leq g_n$  almost everywhere for all  $n$ , then  $\|f_n - f\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ .

**Solution.** ► Part (i) is a generalization of what is colloquially known as the reverse Fatou's lemma. Consider the sequence of measurable functions  $\{h_n\}$  where  $h_n = g_n - f_n$ . Note that  $h_n \geq 0$  for all  $x \in X$  since  $g_n \geq f$  for all  $x \in X$ . Then

$$h_n \leq \sup_{k \geq n} h_k$$

for all  $x \in X$ . Then  $\sup_{k \geq n} h_n \nearrow \limsup_{n \rightarrow \infty} h_n$  so by the monotone convergence theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_X \sup_{k \geq n} h_n d\mu &= \int_X \limsup_{n \rightarrow \infty} h_n d\mu \\ &= \int_X \limsup_{n \rightarrow \infty} (g_n - f_n) d\mu \\ &\leq \int_X \limsup_{n \rightarrow \infty} g_n d\mu + \int_X \limsup_{n \rightarrow \infty} (-f) d\mu \\ &\leq \int_X g d\mu - \int_X \liminf_{n \rightarrow \infty} f d\mu. \end{aligned}$$

Then, we have ◀

**Problem 8.** Let  $f \in L^1(\mathbb{R})$ . Consider the function

$$F(x) = \int_{\mathbb{R}} \exp(ixt) f(t) dt.$$

- (i) Show that  $F \in L^\infty(\mathbb{R})$  and that  $F$  is continuous at every  $x \in \mathbb{R}$ . Moreover, if  $|t|^k f(t) \in L^\infty(\mathbb{R})$  for all  $k \geq 1$ , show that  $F$  is infinitely differentiable, i.e.,  $F \in C^\infty(\mathbb{R})$ .
- (ii) Suppose  $f$  is continuous as well as in  $L^1(\mathbb{R})$ . Show that  $\lim_{|x| \rightarrow \infty} F(x) = 0$ .

[Hint: Using  $\exp(-i\pi) = -1$ , write  $F(x) = \left( \int_{\mathbb{R}} (\exp(ixt) - \exp(ixt - i\pi)) \right) / 2$ .]

**Solution.** ► ◀

## 1.2 Bañuelos: Summer 2000

**Problem 1.** For any two subsets  $A$  and  $B$  of  $\mathbb{R}$  define  $A+B = \{a+b : a \in A, b \in B\}$ .

- (i) Suppose  $A$  is closed and  $B$  is compact. Prove that  $A+B$  is closed.
- (ii) Give an example that shows that (i) may be false if we only assume that  $A$  and  $B$  are closed.

**Solution.** ►

◀

**Problem 2.** Suppose  $f: [0, 1] \rightarrow \mathbb{R}$  is differentiable at every  $x \in [0, 1]$  where by differentiability at 0 and 1 we mean right and left differentiability, respectively. Prove that  $f'$  is continuous if and only if  $f$  is uniformly differentiable. That is, if and only if for all  $\varepsilon > 0$  there is an  $h_0 > 0$  such that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

whenever  $0 \leq x, x+h \leq 1, 0 < |h| < h_0$ .

**Solution.** ►

◀

**Problem 3.** Let  $(X, \mathcal{F}, \mu)$  be a measure space with  $\mu(X) = 1$  and let  $F_1, \dots, F_{17}$  be seventeen measurable subsets of  $X$  with  $\mu(F_j) = 1/4$  for every  $j$ .

- (i) Prove that five of these subsets must have an intersection of positive measure. That is, if  $E_1, \dots, E_k$  denotes the collection of all nonempty intersections of the  $F_j$  taken five at a time ( $k \leq 6188$ ), show that at least one of these sets must have positive measure.
- (ii) Is the conclusion in (i) true if we take sixteen sets instead of seventeen?

**Solution.** ►

◀

**Problem 4.** Let  $f_n: X \rightarrow [0, \infty)$  be a sequence of measurable functions on the measure space  $(X, \mathcal{F}, \mu)$ . Suppose there is a positive constant  $M$  such that the functions  $g_n(x) = f(x)\chi_{\{f_n \leq M\}}(x)$  satisfy  $\|g_n\|_1 \leq A/n^{4/3}$  and for which  $\mu(\{x \in X : f_n(x) > M\}) \leq B/n^{5/4}$ , where  $A$  and  $B$  are positive constants independent of  $n$ . Prove that

$$\sum_{n=1}^{\infty} f_n < \infty$$

almost everywhere.

**Solution.** ►

◀

**Problem 5.** Let  $\{g_n\}$  be a bounded sequence of functions on  $[0, 1]$  which is uniformly Lipschitz. That is there is a constant  $M$  (independent of  $n$ ) such that for all  $n$ ,  $|g_n(x) - g_n(y)| \leq M|x - y|$  for all  $x, y \in [0, 1]$  and  $|g_n(x)| \leq M$  for all  $x \in [0, 1]$ .

(i) Prove that for any  $0 \leq a \leq b \leq 1$ ,

$$\lim_{n \rightarrow \infty} \int_a^b g_n(x) \sin(2n\pi x) \, dx = 0.$$

(ii) Prove that for any  $f \in L^1[0, 1]$ ,

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) g_n(x) \sin(2n\pi x) \, dx = 0.$$

**Solution.** ►

◀

**Problem 6.** Let  $\{f_n\}$  be a sequence of nonnegative functions in  $L^1[0, 1]$  with the property that  $\int_0^1 f_n(t) \, dt = 1$  and  $\int_{1/n}^1 f_n(t) \, dt \leq 1/n$  for all  $n$ . Define  $h(x) = \sup_n f_n(x)$ . Prove that  $h \notin L^1[0, 1]$ .

**Solution.** ►

◀

### 1.3 Bañuelos: Winter 2007

**Problem 1.** Let  $f: [0, 1] \rightarrow \mathbb{R}$ .

- (i) Define what it means for  $f$  to be absolutely continuous.
- (ii) Define what it means for  $f$  to be of bounded variation.
- (iii) Let  $V(f; 0, x)$  be the total variation of  $f$  on  $[0, x]$ . Prove that if  $f$  is absolutely continuous on  $[0, 1]$  so is  $V(f; 0, x)$ .

**Solution.** ►

◀

**Problem 2.**

- (i) Suppose that  $f: [0, 1] \rightarrow \mathbb{R}$  is nondecreasing with  $f(0) = 0$  and  $f(1) = 1$ . For  $a > 0$ , let  $A$  be set of all  $x \in (0, 1)$  for which

$$\limsup_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} > a.$$

Prove that  $m^*(A) < 1/a$ , where  $m^*$  denotes the Lebesgue outer measure.

- (ii) Prove that there is no Lebesgue measurable set  $A$  in  $[0, 1]$  with the property that  $m(A \cap I) = m(I)/4$  for every interval  $I$ .

[Hint: Consider the function  $f(x) = \chi_A(x)$ .]

**Solution.** ►

◀

**Problem 3.** Let  $\{E_j\}_{j=1}^\infty$  be Lebesgue measurable sets in  $[0, 1]$  and let  $E = \bigcup_{j=1}^\infty E_j$  and suppose there is an  $\varepsilon > 0$  such that  $\sum_{j=1}^\infty m(E_j) \leq m(E) + \varepsilon$ .

- (i) Show that for all measurable sets  $A \subset [0, 1]$

$$\sum_{j=1}^\infty m(A \cap E_j) \leq m(A \cap E) + \varepsilon.$$

- (ii) Let  $A$  be the set of all  $x \in [0, 1]$  which are in at least two of  $E_j$ . Prove that  $m(A) \leq \varepsilon$ .

**Solution.** ►

◀

**Problem 4.** Let  $(X, \mathcal{F}, \mu)$  be a finite measure space. Let  $f_n: X \rightarrow [0, \infty)$  be a sequence measurable functions and suppose that  $\|f_n\|_p \leq 1$ ,  $1 < p < \infty$ , and that  $f_n \rightarrow f$  almost everywhere. Prove



- (i)  $f \in L^p(\mu)$ .
- (ii)  $\|f_n - f\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ .

**Solution.** ►



**Problem 5.**

**Solution.** ►



**Problem 6.**

**Solution.** ►



## 1.4 Bañuelos: Winter 2013

### Problem 1.

(a)

- (i) Define almost uniform convergence on the measure space  $(X, \mathcal{F}, \mu)$ .
- (ii) Let  $f_n$  be a sequence of nonnegative measurable functions converging almost uniformly to the nonnegative function  $f$ . Prove that  $\sqrt{f_n}$  converges almost uniformly to  $\sqrt{f}$ .

(b)

- (i) Suppose  $f_n$  has the property that  $\int_X |f_n| d\mu \rightarrow 0$ .
- (ii) Does it follow that  $f_n \rightarrow 0$  almost everywhere? Justify your answer.
- (iii) Does it follow that  $f_n \rightarrow 0$  almost uniformly? Justify your answer.

Solution. ►

◀

**Problem 2.** Let  $(X, \mathcal{F}, \mu)$  be a measure space and let  $1 \leq p \leq \infty$  and  $q$  be its conjugate exponent. Suppose  $f_n \rightarrow f$  in  $L^p$  and  $g_n \rightarrow g$  in  $L^q$ . Prove that  $f_n g_n \rightarrow fg$  in  $L^1$ .

Solution. ►

◀

**Problem 3.** Let  $\{a_k\}$  be a sequence of positive numbers converging to infinity. Prove that the following limit exists

$$\lim_{k \rightarrow \infty} \int_0^\infty \frac{\exp(-x) \cos x}{a_k x^2 + (1/a_k)} dx$$

and find it. Make sure to justify all steps.

Solution. ►

◀

**Problem 4.** Let  $(X, \mathcal{F}, \mu)$  be  $\sigma$ -finite and  $f$  be measurable such that for all  $\lambda > 0$

$$\mu(\{x \in X : |f(x)| > \lambda\}) \leq \frac{20}{\lambda^p}$$

where  $1 < p < \infty$ . Let  $q$  be the conjugate exponent of  $p$ . Prove that there is a constant  $C$  depending only on  $p$  such that

$$\int_E |f(x)| \, d\mu \leq C m(E)^{1/q},$$

for all measurable sets  $E$  with  $0 < \mu(E) < \infty$ . (The inequality holds trivially when  $\mu(E) = 0$  or  $\mu(E) = \infty$ .)

[Hint: Recall  $\int_E |f(x)| \, d\mu = \int_0^\infty \lambda \, d\lambda$  and “break it” at the right place!]

**Solution.** ►

◀

**Problem 5.** Suppose  $f: [0, 1] \rightarrow \mathbb{R}$  is of bounded variation with  $V(f; 0, 1) = \alpha$ . For any  $\beta > \alpha$ , set

$$A = \left\{ x \in (0, 1) : \limsup_{h \rightarrow 0} \frac{|f(x+h) - f(x)|}{|h|} > \beta \right\}.$$

Prove that for any  $0 < p < 1$ ,  $m(A) \leq (\alpha/\beta)^p$ , where  $m$  denotes the Lebesgue measure.

**Solution.** ►

◀

**Problem 6.** Let  $f \in L^1(0, 1)$  and for  $x \in (0, 1)$ , define

$$h(x) = \int_x^1 \frac{f(t)}{t} \, dt.$$

- (i) Prove that  $h$  is continuous on  $(0, 1)$ .
- (ii) Show that

$$\int_0^1 h(t) \, dt = \int_0^1 f(t) \, dt.$$

**Solution.** ►

◀

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