MA 519: Homework 13

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Problem 13.1 (Handout 17, # 16)

Suppose $X \sim \mathrm{Exp}(1)$, $Y \sim U[0,1]$, and X, Y are independent.

(a) Find the density of $X+Y$.
(b) Find the density of XY.

Solution. For part (a): Since X and Y are independent, the distribution of $X+Y$ is given by the convolution

$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_X(x-y) f_Y(y)\, dy,$$

where

$$f_X(x) = \begin{cases} \mathrm{e}^{-x} & \text{for } x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad f_Y(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, a straight forward calculation gives us

$$\begin{aligned} f_{X+Y}(x) &= \int_{-\infty}^{\infty} \chi_{[0,\infty)}(x-y)\mathrm{e}^{-(x-y)}\chi_{[0,1]}(y)\, dy \\ &= \mathrm{e}^{-x}\int_{-\infty}^{\infty} \mathrm{e}^{y}\chi_{[0,\infty)}(x-y)\chi_{[0,1]}(y)\, dy \\ &= \begin{cases} 0 & \text{for } x < 0, \\ 1-\mathrm{e}^{-x} & \text{for } 0 \leq x \leq 1, \\ (\mathrm{e}-1)\mathrm{e}^{-x} & \text{for } x > 1. \end{cases} \end{aligned}$$

Now let us run a sanity check by demonstrating that $\int_{-\infty}^{\infty} f_{X+Y}(x)\, dx = 1$,

$$\begin{aligned} \int_{-\infty}^{\infty} f_{X+Y}(x)\, dx &= \int_0^1 [1-\mathrm{e}^{-x}]\, dx + (\mathrm{e}-1)\int_1^{\infty} \mathrm{e}^{-x}\, dx \\ &= \big[1+\mathrm{e}^{-1}-1-0\big] + (\mathrm{e}-1)\big[0-(-\mathrm{e}^{-1})\big] \\ &= \mathrm{e}^{-1}+1-\mathrm{e}^{-1} \\ &= 1. \end{aligned}$$

For part (b): We can use the formula

$$f_{XY}(x) = \int_{-\infty}^{\infty} \Big[\tfrac{1}{|y|} f_X(y) f_Y(\tfrac{x}{y})\Big]\, dy$$

to arrive at the desired PDF. ■

Problem 13.2 (Handout 17, # 18)

Two points A, B are chosen at random from the unit circle. Find the probability that the circle centered at A with radius AB is fully contained within the original unit circle.

Solution. ∎

Problem 13.3 (Handout 17, # 19)

Let X, Y be i.i.d. $U[0,1]$ random variables. Find the correlation between $\max\{X, Y\}$ and $\min\{X, Y\}$.

Solution. ■