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MA 26500-215 Quiz 7

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1. For the following problems write **T** for true, **F** for false. You do not need to justify your answers.

   (a) (3 points) For all $m \times n$ matrices A and B, $\text{nullity}(A + B) = \text{nullity}\, A + \text{nullity}\, B$.

   (b) (3 points) For all $n \times n$ matrices A and B, $\text{nullity}(AB) = (\text{nullity}\, A)(\text{nullity}\, B)$.

   (c) (3 points) For all $n \times n$ matrices A and B, where A is an elementary matrix, $\text{nullity}(AB) = \text{nullity}\, B$.

   (d) (3 points) If $\mathbf{x}_p$ is a solution to the system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{y} + \mathbf{x}_p$ is also a solution to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{y} \in \text{Nullspace}\, A$.

Solution: The answers for part (a), (b), (c) and (d) are **F**, **F**, **T** and **T** respectively. For part (c) and (d) you should refer to Kolman and Hill (particularly Ch. 4.7 on *Homogeneous Systems*).

To see that (a) is false consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The nullity of A and B is both 0, but

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which has nullity 2 and $0 + 0$ is by no means equal to 2.

To see that (b) is false, consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then nullity of A is 1 whereas the nullity of B is 0, but

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which has nullity 1. Again, $0 \cdot 1$ is not equal to 1.

2. (8 points) Prove that if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are in $\mathbb{R}^3$ and $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to every vector in $\text{span}\{\mathbf{v}, \mathbf{w}\}$.

   [*Hint*: What does it mean for a vector $\mathbf{x}$ to be in $\text{span}\{\mathbf{v}, \mathbf{w}\}$ and what does it mean for two vectors to be orthogonal?]

Solution: Starting from the top. We know that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \cdot \mathbf{w} = 0$. Now, what does it mean for $\mathbf{x}$ to be in $\text{span}\{\mathbf{v}, \mathbf{w}\}$? It means that there exists scalars $a_1, a_2 \in \mathbb{R}$ (both can possibly be 0) such that $\mathbf{x} = a_1\mathbf{v} + a_2\mathbf{w}$. Thus

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{x} &= \mathbf{u} \cdot (a_1\mathbf{v} + a_2\mathbf{w}) \\
&= \mathbf{u} \cdot (a_1\mathbf{v}) + \mathbf{u} \cdot (a_2\mathbf{w}) \\
&= a_1(\mathbf{u} \cdot \mathbf{v}) + a_2(\mathbf{u} \cdot \mathbf{w}) \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

so $\mathbf{u}$ is orthogonal to $\mathbf{x}$. Since the choice of $\mathbf{x}$ was arbitrary, we conclude that $\mathbf{u}$ is orthogonal to every vector in $\text{span}\{\mathbf{v}, \mathbf{w}\}$.