

MA557 Homework 6

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PROBLEM 6.1

Let R be a Noetherian ring and I, J R -ideals. Write $I^{(J)} = \bigcup_{n \geq 1} (I : J^n)$, which is called the *saturation of I with respect to J* . Show:

- (a) If $I = \bigcap_{i=1}^m \mathfrak{q}_i$ with \mathfrak{q}_i \mathfrak{p}_i -primary, then $I^{(J)} = \bigcap_{J \not\subset \mathfrak{p}_i} \mathfrak{q}_i$.
- (b) $I^{(J)}$ is the unique largest R -ideal that coincides with I locally on the open set $\text{Spec}(R) \setminus V(J)$.

Proof. (a) We shall demonstrate double inclusion: Let $\bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal decomposition of I into primary ideals where \mathfrak{q}_i is \mathfrak{p}_i -primary. \implies Suppose $x \in I^{(J)}$ then $xJ^n \subset I$ for some $n \geq 1$. Given i such that $\mathfrak{p}_i \not\supset J^*$ take $y \in J \setminus \mathfrak{p}_i$. Then $xy^n \in \mathfrak{q}_i$ so $x \in \mathfrak{q}_i$ since \mathfrak{q}_i is primary and $y \notin \mathfrak{p}_i$. Hence, $I^{(J)} \subset \bigcap_{J \not\subset \mathfrak{p}_i} \mathfrak{q}_i$. \Leftarrow Conversely, suppose that $x \in \bigcap_{J \not\subset \mathfrak{p}_i} \mathfrak{q}_i$ then $x \in \mathfrak{q}_i$ for all $\mathfrak{q}_i \not\supset J$. Take any \mathfrak{p}_j containing J . Then $\mathfrak{p}_j = \text{nil}(R/\mathfrak{q}_j)^c$ (this is easily seen from the fact that $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, i.e., \mathfrak{q}_i is \mathfrak{p}_i -primary and the correspondence theorem for ideals) so there exists n_j with $xJ^{n_j} \subset \mathfrak{q}_j$ (since, in the quotient, \bar{J} is nilpotent). Let n be the maximum of all such n_j then $xJ^n \subset \mathfrak{q}_i$ for all i , i.e., $x \in (I : J^n) = \bigcap_i (\mathfrak{q}_i : J^n)$. Thus, $x \in I^{(J)}$.

(b) We will prove that $I^{(J)}$ is precisely the set of all $x \in R$ such that $x/1$ vanishes in $R_{\mathfrak{p}}$ for all $\mathfrak{p} \not\supset J$. \implies Given $x \in I^{(J)}$, $xJ^n \subset I$ for some $n \geq 1$. Let \mathfrak{p} be a prime ideal not containing J and let $y \in J \setminus \mathfrak{p}$. Then $xy^n \in I$ and $y^n \notin \mathfrak{p}$ so $x/1 = 0$ in $R_{\mathfrak{p}}$. \Leftarrow Conversely, suppose that $x/1$ vanishes in $R_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset R$. Then $xy = 0$ for some $y \in R \setminus \mathfrak{p}$. Since $\mathfrak{p} = \sqrt{\mathfrak{q}_i}$ for some i , $y^n \in \mathfrak{q}_i$ for some $n \geq 1$. Let $\bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal decomposition of 0 (one exists since R is Noetherian) where \mathfrak{q}_i is \mathfrak{p}_i -primary. By part (a), it suffices to show that \blacksquare

*Why does such an ideal exist? Well, suppose that $\mathfrak{p}_i \supset J$ for all $1 \leq i \leq m$. Then $J \subset \bigcap_{i=1}^m \mathfrak{p}_i = \bigcap_{i=1}^m \sqrt{\mathfrak{q}_i} = \sqrt{\bigcap_{i=1}^m \mathfrak{q}_i} = \sqrt{I}$. What next?

PROBLEM 6.2

Let R be a Noetherian ring. Show that R is reduced if and only if $\text{Quot}(R)$ is a finite direct product of fields.

Proof. \implies Suppose that R is reduced. Then $\sqrt{0} = 0$. Since R is Noetherian, by 5.13 every ideal has a primary decomposition. Let $\bigcap_{i=1}^n \mathfrak{q}_i$, where \mathfrak{q}_i is \mathfrak{p}_i -primary, be such a decomposition for 0. Then we have that $0 = \sqrt{0} = \sqrt{\bigcap_{i=1}^n \mathfrak{q}_i} = \bigcap_{i=1}^n \sqrt{\mathfrak{q}_i} = \bigcap_{i=1}^n \mathfrak{p}_i$ so that $\mathfrak{q}_i = \mathfrak{p}_i$. In particular, \mathfrak{p}_i are minimal over 0 so by 5.15 the decomposition is unique. Thus, by the Chinese remainder theorem we have an exact sequence

$$0 \longrightarrow R \xrightarrow{\varphi} \prod_{i=1}^n R/\mathfrak{p}_i \xrightarrow{\pi} \text{coker } \varphi \longrightarrow 0.$$

Now, consider $\text{Quot}(R) = S^{-1}R$ where $S = R \setminus \bigcup_{i=1}^n \mathfrak{p}_i$. By 4.6, since $S^{-1}R$ is a flat R -module, we have an exact sequence

$$0 \longrightarrow S^{-1}R \xrightarrow{\varphi} S^{-1}\left(\prod_{i=1}^n R/\mathfrak{p}_i\right) \cong \prod_{i=1}^n S^{-1}R/S^{-1}\mathfrak{p}_i \xrightarrow{\pi} S^{-1}\text{coker } \varphi \longrightarrow 0.$$

In fact, φ is an R -linear isomorphism since $\text{coker } \varphi$ is annihilated by an element of S .[†] It follows that $\text{Quot}(R) \cong \prod_{i=1}^n k_i$ where $k_i = \text{Quot}(R/\mathfrak{p}_i)$.

\Leftarrow Conversely, suppose that $\text{Quot}(R) \cong \prod_{i=1}^n k_i$ where k_i is a field. ■

[†]Why? We need to show that the set $\text{ann}_R(\text{coker } \varphi)$ is nonempty.

PROBLEM 6.3

Let R be a Noetherian ring and $x \in R$ an R -regular element. Show that $\text{Ass}_R(R/(x^n)) = \text{Ass}_R(R/(x))$ for every $n \geq 1$.

Proof. Suppose that x is an R -regular element then $x \notin \text{nil}(R)$, i.e., x is not a nilpotent element. Recall that the associated of (x^n) are precisely $\text{Ass}_R(R/(x^n))$. We will show double inclusion: One direction is easy namely \implies suppose that $\mathfrak{p} \in \text{Ass}_R(R/(x^n))$. Then $\mathfrak{p} = \text{ann}(\bar{y}) = ((x^n) : y)$ for some $y \in R$. Then $\sqrt{\mathfrak{p}} = \sqrt{((x^n) : y)} = (\sqrt{(x^n)} : y) = ((x) : y)$ so $\mathfrak{p} \in \text{Ass}_R(R/(x))$. \longleftarrow Conversely, suppose that \mathfrak{p} an associated prime of (x) . ■

PROBLEM 6.4

Let $\varphi: R \rightarrow T$ be a homomorphism of rings where T is Noetherian, let ${}^a\varphi$ be the induced map on the spectra, and let N be a T -module. Show:

- (a) $\text{Ass}_R(N) = {}^a\varphi(\text{Ass}_T(N))$.
- (b) If N is finitely generated as a T -module then $\text{Ass}_R(N)$ is finite.

Proof.

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PROBLEM 6.5

Let K be a field that is a finitely generated \mathbf{Z} -algebra. Show that K is a finite field.

Proof.

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PROBLEM 6.6

Let k be a Noetherian ring, R a finitely generated k -algebra, and $\text{Aut}_k(R)$ the group of k -algebra automorphisms of R . For a subgroup G of $\text{Aut}_k(R)$ write $R^G = \{x \in R \mid \sigma(x) = x \text{ for every } \sigma \in G\}$, which is called the *ring of invariants* of G . Show that if G is finite then R^G is a finitely generated k -algebra (and hence a Noetherian ring).

Proof.

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