MA571 Problem Set 3

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Problem 3.1 (Munkres §18, p. 111, #7(a))

(a) Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is "continuous from the right," that is,

$$\lim_{x\to a+} f(x) = f(a).$$

for each $a \in \mathbf{R}$. Show that f is continuous when considered as a function from $\mathbf{R}_\ell$ to $\mathbf{R}$.

Proof. Recall the definition of "right-hand limit,":

Definition (Rudin §4, p. 94, Def. 4.25)**.** Let f be defined on (a,b). Consider any point x such that $a \le x < b$. We write $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x,b) such that $t_n \to x$.

This definition is not well suited for our purposes since it is defined in terms of limits of sequences, which Rudin validates in Theorem 4.6 (cf. Rudin, §4, p. 86) by proving that the limit-point formulation of continuity coincides with the ε-δ formulation. We shall, therefore, reformulate Rudin's definitions in terms of ε's and δ's as follows:

Definition. $f\colon \mathbf{R} \to \mathbf{R}$ is right-continuous at $x_0 \in \mathbf{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in [x_0, x_0+\delta)$ implies $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

We will prove that $f\colon \mathbf{R}_\ell \to \mathbf{R}$ is continuous in the sense: "for each open subset V of $\mathbf{R}$, the set $f^{-1}(V)$ is an open subset of $\mathbf{R}_\ell$" (cf. Munkres, §18, p. 102) and we shall do so in the spirit of Example 1 in Munkres §18, p. 103 and employ Theorem 18.1(4). Recall what basic open sets look like in the lower-limit topology (defined in Munkres §13, pp. 81-82), they are intervals of the form $[a,b) \subset \mathbf{R}$. Without loss of generality, consider the basic open set $V = (a,b)$ in $\mathbf{R}$. Let $x_0 \in f^{-1}(V)$. Then, since f is right-continuous, for $\varepsilon \le \min\{f(x_0) - a, b - f(x_0)\}$, there exists $\delta > 0$ such that $x \in [x_0, x_0+\delta)$ implies $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, i.e.,

$$f([x_0, x_0+\delta)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subset V.$$

By Theorem 18.1(4), f is continuous. ∎

Problem 3.2 (Munkres §18, p. 112, #13)

Let $A \subset X$; let $f\colon A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g\colon \overline{A} \to Y$, then g is uniquely determined by f.

Proof. We shall proceed by contradiction. Suppose that g_1 and g_2 are distinct continuous extensions of f to the closure of A, i.e., $g_1(x) \neq g_2(x)$ for some $x \in \overline{A} \setminus A$. Recall from Problem 2.7 (Munkres §13, p. 101, #13) that Y is Hausdorff if and only the diagonal $\Delta = \{\, y \times y \mid y \in Y \,\}$ is closed in $Y \times Y$. Now, consider the product map $G = g_1 \times g_2\colon X \to Y \times Y$. This map is continuous by Theorem 18.4 so, by Theorem 18.1(2), $G(\overline{A}) \subset \overline{G(A)}$. However, since $g_1 = g_2$ on A we have that $G(A) \subset \Delta$ so, by Lemma B (from Prof. McClure's lectures), we have that

$$G(\overline{A}) \subset \overline{G(A)} \subset \Delta.$$

But by assumption $g_1(x) \times g_2(x) \notin \Delta$. This is a contradiction. Therefore, $g_1 = g_2$ on $\overline{A}$, i.e., the extension of f to a continuous function g on $\overline{A}$ is unique. ∎

Problem 3.3 (Munkres §19, p. 118, #2)

Prove Theorem 19.3.

Proof. Recall the exact statement of Theorem 19.3 from Munkres §19, p. 116:

Theorem. *Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

Our goal is to show that the box (or product) topology on $\prod A_\alpha$ coincides with the subspace topology on $\prod A_\alpha$ as a subset of $\prod X_\alpha$ with the product (or box) topology. That is, we will show that U is open in the box (or product) topology on $\prod A_\alpha$ if and only if it is of the form $V \cap \prod A_\alpha$ where V is open in $\prod X_\alpha$ with the box (or product) topology (cf. Munkres §16 p. 88 and Munkres §19, p. 114 for the relevant definitions as we will avoid naming these topologies explicitly in this proof).

$\Longrightarrow$ By Theorem 13.1, it suffices to consider basic open sets in $\prod A_\alpha$ (equipped with the box or the product topology). Let U be a basic open set in $\prod A_\alpha$ with the box (or product) topology , then, by Theorem 19.2, $U = \prod U_\alpha$ where U_α is a basic open set in A_α (or all of A_α for all but finitely many α in the case of the product topology). Then, since A_α is a subspace of X_α, the set $U_\alpha = V_\alpha \cap A_\alpha$ for V_α open in X_α (or equal to X_α for all but finitely many α). Then, $V = \prod V_\alpha$ is a basic open set in $\prod X_\alpha$ (again, by Theorem 19.2) and we have that

$$U = \prod U_\alpha = \prod (V_\alpha \cap A_\alpha).$$

We must now prove that:

Lemma 6.
$$\prod (A_\alpha \cap B_\alpha) = \prod A_\alpha \cap \prod B_\alpha.$$

Proof of Lemma 6. The proof is just a matter of chasing definitions and so our demonstration will be compact, but slightly informal. Recall from the two lowermost definition in Munkres §19, p. 113 that $\mathbf{x} \in \prod_{\alpha\in J}(A_\alpha \cap B_\alpha)$ if, by definition, $\mathbf{x}\colon J \to \bigcup_{\alpha\in J}(A_\alpha \cap B_\alpha)$ such that $\mathbf{x}(\alpha) \in A_\alpha \cap B_\alpha$ for each $\alpha \in J$ if and only if $\mathbf{x}(\alpha) \in A_\alpha$, $\mathbf{x}(\alpha) \in B_\alpha$ if and only if $\mathbf{x}\colon J \to \bigcup_{\alpha\in J} A_\alpha$, $\mathbf{x}\colon J \to \bigcup_{\alpha\in J} B_\alpha$ if and only if $\mathbf{x} \in \prod_{\alpha\in J} A_\alpha$ and $\mathbf{x} \in \prod_{\alpha\in J} B_\alpha$ if and only if $\mathbf{x} \in \prod_{\alpha\in J} A_\alpha \cap \prod_{\alpha\in J} B_\alpha$. ♣

Thus, by Lemma 6

$$U = \prod (V_\alpha \cap A_\alpha) = \prod V_\alpha \cap \prod A_\alpha = V \cap \prod A_\alpha.$$

$\Longleftarrow$ Conversely, suppose U is of the form $V \cap \prod A_\alpha$, i.e., U is a basic open set in the subspace topology on $\prod A_\alpha$. Without loss of generality, assume that $V = \prod V_\alpha$ is a basic open set in $\prod X_\alpha$. Then, by Lemma 6,

$$U = V \cap \prod A_\alpha = \prod (V_\alpha \cap A_\alpha).$$

is a basic open set in the box (or product topology) on $\prod A_\alpha$ since $V\alpha \cap A_\alpha$ is a basic open set in A_α with the subspace topology. ■

Problem 3.4 (Munkres §19, p. 118, #3)

Prove Theorem 19.4.

Proof. Recall the exact statement of Theorem 19.4 from Munkres §19, p. 116:

Theorem. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

To show that $\prod X_\alpha$ equipped with the box (or the product) topology we will proceed in the following way: let $\mathbf{x}, \mathbf{y} \in \prod X_\alpha$, it is sufficient, although not necessary, to show that there exists basic open sets $U = \prod U_\alpha$ and $V = \prod V_\alpha$ neighborhoods of $\mathbf{x}$ and $\mathbf{y}$, respectively, such that $U \cap V = \emptyset$.

We will first demonstrate this for the product topology on $\prod X_\alpha$. Since X_α is Hausdorff for each α, there exists basic open sets U_α and V_α neighborhoods of x_α and y_α, respectively, such that $U_\alpha \cap V_\alpha = \emptyset$. For a finite collection of α's, say A, let $U'_\alpha = U_\alpha$ and $V'_\alpha = V_\alpha$ for all $\alpha \in A$ and $U'_\alpha = X_\alpha$, $V'_\alpha = X_\alpha$ otherwise. Then $U = \prod U'_\alpha$ and $V = \prod V'_\alpha$, by Theorem 19.2, are open in $\prod X_\alpha$ with the product topology and, by Lemma 6,

$$U \cap V = \prod U'_\alpha \cap \prod V'_\alpha = \prod_{\alpha \in A} (U_\alpha \cap V_\alpha) \times \prod_{\alpha \notin A} X_\alpha = \prod_{\alpha \in A} \emptyset \times \prod_{\alpha \notin A} X_\alpha = \emptyset.$$

Thus $\prod X_\alpha$ with the product topology is Hausdorff.

In the case of $\prod X_\alpha$ with the box topology, we take U and V to be the basis elements $U = \prod U_\alpha$ and $V = \prod V_\alpha$ such that $U_\alpha \cap V_\alpha = \emptyset$ for all α. Then, by Lemma 6, we have

$$U \cap V = \prod U_\alpha \cap \prod V_\alpha = \prod (U_\alpha \cap V_\alpha) = \prod \emptyset = \emptyset.$$

Hence $\prod X_\alpha$ with the box topology is Hausdorff. ■

Problem 3.5 (Munkres §19, p. 118, #6)

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Proof. $\Longrightarrow$ Suppose that the sequence $\{\mathbf{x}_n\}$ converges to $\mathbf{x} \in \prod X_\alpha$ then, from Munkres §17, p. 98, for every neighborhood U of $\mathbf{x}$, there is a positive integer N such that $\mathbf{x}_n \in U$ for $n \geq N$. Then, since $\pi_\beta \colon \prod X_\alpha \to X_\beta$ is continuous, $\pi_\beta(\mathbf{x}_n) = x_\beta^{(n)} \in \pi_\beta(U)$ a neighborhood of x_α for $n \geq N$ for all α. Note that this holds for $\prod X_\alpha$ with the box topology.

$\Longleftarrow$ Now, suppose that $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ for all α, i.e., the sequence $x_\alpha^{(n)} \to x_\alpha$ in X_α. Then, for every neighborhood U_α of x_α, there exists a positive integer N such that $x_\alpha^{(n)} \in U_\alpha$ for all $n \geq N$. Let A be a finite collection of α's ■

Problem 3.6 (Munkres §19, p. 118, #7)

Let $\mathbf{R}^\infty$ be the subset of $\mathbf{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, ...)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbf{R}^\infty$ in $\mathbf{R}^\omega$ in the box and product topologies? Justify your answer.

Proof. ■

Problem 3.7 (Munkres §20, p. 126, #3(b))

Let X be a metric space with metric d.

(b) Let X' denote a space having the same underlying set as X. Show that if $d\colon X' \times X' \to \mathbf{R}$ is continuous, then the topology of X' is finer than the topology of X.

Proof. It is enough to prove this for basic open sets $U = B_r(x) = \{ y \in X \; middle| \; d(x,y) < r \}$ for $r > 0$. Let f_x ■

Problem 3.8 (Munkres §20, p. 127, #4(b))

Consider the product, uniform and box topologies on $\mathbf{R}^\omega$

(b) In which topologies do the following sequences converge?

$$
\begin{aligned}
\mathbf{w}_1 &= (1,1,1,1,...), & \mathbf{x}_1 &= (1,1,1,1,...),\\
\mathbf{w}_2 &= (0,2,2,2,...), & \mathbf{x}_2 &= \left(0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},...\right),\\
\mathbf{w}_3 &= (0,0,3,3,...), & \mathbf{x}_3 &= \left(0,0,\tfrac{1}{3},\tfrac{1}{3},...\right),\\
&\vdots & &\vdots\\
\mathbf{y}_1 &= (1,0,0,0,...) & \mathbf{z}_1 &= (1,1,0,0,...),\\
\mathbf{y}_2 &= \left(\tfrac{1}{2},\tfrac{1}{2},0,0,...\right) & \mathbf{z}_2 &= \left(\tfrac{1}{2},\tfrac{1}{2},0,0,...\right),\\
\mathbf{y}_3 &= \left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,...\right) & \mathbf{z}_3 &= \left(\tfrac{1}{3},\tfrac{1}{3},0,0,...\right),\\
&\vdots & &\vdots
\end{aligned}
$$

Proof. ∎

Problem 3.9 (A)

Given: X a metric space, A a countable subset of X, and $\overline{A} = X$. To prove: the topology of X has a countable basis.

Proof. ■

Problem 3.10 (B)

Given: Y is an ordered set, (a, b) and (c, d) are disjoint open intervals, and there are elements $x \in (a, b)$ and $y \in (c, d)$ with $x < y$. To prove: every element of (a, b) less than every element of (c, d).

Proof. ■

Problem 3.11 (C)

(This problem will be used when we discuss quotient spaces). Let S and T be sets and let $f\colon S \to T$ be a function. Let $A \subset S$.

(i) Give an example to show that the equation
$$f^{-1}(f(A)) = A \tag{*}$$
isn't always valid.

(ii) Define an equivalence relation $\sim$ on S by $s \sim s'$ if and only if $f(s) = f(s')$. Using this equivalence relation, describe the subsets A of S for which (*) is true. Prove that your answer is correct.

Proof. ■