MA 523: Homework 1

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Problem 1.1 (Taylor's formula)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to \mathbf{0}$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) \coloneqq f(tx)$. Prove that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t) = \sum_{|\alpha| = m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. ▸ Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) \coloneqq f(tx)$. We claim that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t) = \sum_{|\alpha| = m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Proof of claim. We shall proceed by induction on m. The case $m = 1$ follows easily from the chain rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} g(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f(tx) \\
&= D^{(1,0,\ldots,0)} f(tx) x_1 + \cdots + D^{(0,\ldots,0,1)} f(tx) x_n \\
&= \big(D^{(1,0,\ldots,0)} x_1 + \cdots + D^{(0,\ldots,0,1)} x_n\big) f(tx)
\end{aligned}$$

More generally, applying the equation above recursively, we have

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} g(t) = \big(D^{(1,0,\ldots,0)} x_1 + \cdots + D^{(0,\ldots,0,1)} x_n\big)^m f(tx)$$

by the multinomial theorem

$$\begin{aligned}
&= \sum_{|\alpha| = m} \binom{|\alpha|}{\alpha} D^\alpha x^\alpha f(tx) \\
&= \sum_{|\alpha| = m} \binom{|\alpha|}{\alpha} D^\alpha f(tx) x^\alpha \\
&= \sum_{|\alpha| = m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha
\end{aligned}$$

as desired. □

Now, applying Taylor's formula in 1 variable to $g(t)$

$$\begin{aligned}
g(t) &= \sum_{i=0}^{k} \frac{g^{(i)}(0)}{i!} t^i + \mathrm{R}_k(g) \\
&= \sum_{i=0}^{k} \frac{1}{i!} \sum_{|\alpha|=i} \frac{i!}{\alpha!} D^\alpha f(tx) x^\alpha + \mathrm{R}_k(g) \\
&= \sum_{i=0}^{k} \sum_{|\alpha|=i} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha t^i + \mathrm{R}_k(g) \\
&= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha t^i + \mathrm{R}_k(g)
\end{aligned}$$

and evaluating at $t = 1$ we have

$$\begin{aligned}
g(1) &= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha t^i \\
&= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{R}_k(g) \\
&= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{R}_k(g)
\end{aligned}$$

where the remainder $\mathrm{R}_k(g) \asymp |x|^{k+1}$

$$= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{R}_k\left(|x|^{k+1}\right).$$

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Problem 1.2

Write down the characteristic equation for the p.d.e.

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ▸

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Problem 1.3

Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ▸

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Problem 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\left(u_{x_1}^2 + u_{x_2}^2\right)$$

find a solution with $u(x_1, 0) = \left(1 - x_1^2\right)/2$.

Solution. ►