MA544: Qual Preparation

Carlos Salinas

July 8, 2016

Contents

1 MA 544 Spring 2016

This is material from the course MA 544 as it was taught in the spring of 2016.

1.1 Homework

These exercises were assigned from Wheeden and Zygmund's *Measure and Integral*, therefore, most of the theorems I reference will be from [5]. Other resources include [1] and [2]. For more elementary results, I cite [3]. Unless otherwise stated, whenever we quote a result, e.g., Theorem 1.1, it is understood to come from Wheeden and Zygmund's *Measure and Integral*.

Throughout these notes

$\mathbb{R}$	is the set of real numbers
$\mathbb{R}^+$	is the set of positive real numbers, that is, $x \in \mathbb{R}$ with $x \geq 0$
$\mathbb{C}$	is the set of complex numbers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{Z}$	is the set of the integers
$\mathbb{Z}^+$	is the set of positive integers, that is, $x \in \mathbb{Z}$ with $x \geq 0$
$\mathbb{N}$	is the set of the natural numbers $1, 2, \ldots$
$A \smallsetminus B$	is the set difference of A and B, that is, the complement of $A \cap B$ in A
$m^*(E)$	the outer measure of E
$m_*(E)$	the inner measure of E
$m(E)$	the Lebesgue measure of E
$\|-\|$	the standard Euclidean norm on $\mathbb{R}^n$
$f \asymp g$	means f is asymptotically equivalent to g, that is, $\lim_{x\to\infty} g(x)/f(x) = 1$

1.1.1 Homework 1

Problem 1 (Wheeden & Zygmund Ch. 2, Ex. 1)**.** Let $f(x) = x\sin(1/x)$ for $0 < x \le 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0,1]$, but that $V[f; 0, 1] = \infty$.

Solution. ► Let f equal $x\sin(1/x)$. We will show that f is bounded and continuous on $[0,1]$, but that it is not of bounded variation on $[0,1]$.

First we will show that f is bounded. Note that both $|x|$ and $|\sin(1/x)|$ are bounded by 1 on the interval $[0,1]$. Since $|f| = |x||\sin(1/x)|$, it follows that $|f| \le 1$ on $[0,1]$. Thus, f is bounded on $[0,1]$.

Next we show that f is continuous. It is easy to show that f is continuous on the subinterval $(0,1]$ since both $|x|$ and $\sin(1/x)$ are continuous on that interval and we know that the product of continuous functions is continuous. To see that f is continuous at 0 we must show that $f(x^+) = f(0)$; that is, the limit of f as x approaches 0 from the right is $f(0)$ which by definition is 0. To this end, it suffices to take a (monotonically decreasing) sequence $x_n \searrow 0$ and show that the limit of the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is 0. Let $\varepsilon > 0$ be given then, since x_n converges to 0 there exists an index N such that $|0 - x_n| < \varepsilon$ whenever $n \ge N$. Since $|f(x_n)| \le |x_n|$ on $[0,1]$, the following inequality holds

$$\begin{aligned} |0 - f(x_n)| &= |0 - x_n \sin(1/x_n)| \\ &\le |x_n| \\ &< \varepsilon. \end{aligned}$$

Thus, f is continuous at 0 and it converges to 0.

Despite the nice properties that f seemingly possesses, f is not b.v. on $[0,1]$. To show that f is not b.v. on $[0,1]$ we must show that for any positive real number M there exists some partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of $[0,1]$ such that the sum associated to Γ

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| > M.$$

Let N be the smallest integer greater than M and let n be the smallest integer greater than or equal to $N/2$. Then the partition $\Gamma = \{\, x_0 = 1 < x_1 < \cdots < x_{n+1} = 1 \,\}$ where $x_i = 2/((3 + (n-i))\pi)$ for $1 \le i \le N$. Then we have the inequality

I took care to choose x_i suc
$x_i < x_{i+1}$ in the partition.

$$\begin{aligned} S_\Gamma &= \sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})| + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &= N + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &> M. \end{aligned}$$

Thus, f is not b.v. on $[0,1]$. ◄

Problem 2 (Wheeden & Zygmund Ch. 2, Ex. 2)**.** Prove theorem (2.1).

Solution. ► Recall the statement of Theorem 2.1:

(a) If f is of bounded variation on $[a,b]$, then f is bounded on $[a,b]$.
(b) Let f and g be of bounded variation on $[a,b]$. Then cf (for any real constant c), $f+g$, and fg are of bounded variation on $[a,b]$. Moreover, f/g is of bounded variation on $[a,b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a,b]$.

We shall prove these in alphabetical order:

For part (a) we shall proceed by contradiction. First, without loss of generality, we may assume that $f(a) = 0$ since the function the variation of $g(x) = f(x) - f(a)$ is equal to the variation of f and $g(a) = 0$. Suppose that f is b.v. on $[a,b]$ with variation $V = V[f;a,b]$, but that f is unbounded on $[a,b]$; that is, given a positive real number M there exists a point x in $[a,b]$ such that $|f(x)| > M$. In particular, there exists $x \in [a,b]$ such that $|f(x)| > V$. Hence, for any $x \in [a,b]$ by the triangle inequality we have

$$\begin{aligned} V &< |f(x)| \\ &= |f(x) - f(a) + f(a)| \\ &\leq |f(x) - f(a)| + |f(a)| \\ &\leq V. \end{aligned}$$

This is a contradiction. Therefore, it must be the case that if f is b.v. on $[a,b]$ then f is bounded on $[a,b]$.

We break part (b) into three sections. Suppose f and g are b.v. on $[a,b]$ with variation V and V', respectively. We will show that (i) cf; (ii) $f+g$; and (iii) fg are b.v. on $[a,b]$. Moreover, we show that (iv) f/g is b.v. on $[a,b]$ if there exists $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for all $x \in [a,b]$.

For part (i) above let c be a real number. Given a partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of $[a,b]$, we have

$$\begin{aligned} S_\Gamma &= \sum_{i=1}^{n} |cf(x_i) - cf(x_{i-1})| \\ &= \sum_{i=1}^{n} |c||f(x_i) - cf(x_{i-1})| \\ &= |c| \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &\leq |c|V \end{aligned}$$

since V is the supremum of the sums of the form $\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|$ over all partitions of $[a,b]$. Thus, $V[cf;a,b] \leq |c|V$ so cf is b.v. on $[a,b]$.

For part (ii) given a partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of the interval $[a,b]$, by the triangle inequality we have

$$
\begin{aligned}
S_\Gamma &= \sum_{i=1}^{n} |(f(x_i)+g(x_i)) - (f(x_{i-1})+g(x_{i-1}))| \\
&= \sum_{i=1}^{n} |(f(x_i)-f(x_{i-1})) + (g(x_i)-g(x_{i-1}))| \\
&\le \sum_{i=1}^{n} |f(x_i)-f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i)-g(x_{i-1})| \\
&\le V + V'.
\end{aligned}
$$

Thus, $f+g$ is b.v. on $[a,b]$

For part (iii) since f and g are b.v. on $[a,b]$ by part (a) f and g are bounded on $[a,b]$ by, say, M and N, respectively. Now, given a partition $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ of $[a,b]$, by the triangle inequality we have

$$
\begin{aligned}
S_\Gamma &= \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^{n} |f(x_i)g(x_i) \quad - f(x_{i-1})g(x_{i-1}) \\
&\qquad\qquad\qquad + f(x_i)g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^{n} |(f(x_i)g(x_i) \quad - f(x_i)g(x_{i-1})) \\
&\qquad\qquad\qquad - (f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1}))| \\
&\le \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_i)g(x_{i-1})| \\
&\qquad\qquad + \sum_{i=1}^{n} |f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^{n} |f(x_i)||g(x_i)-g(x_{i-1})| + \sum_{i=1}^{n} |g(x_{i-1})||f(x_i)-f(x_{i-1})| \\
&= \sum_{i=1}^{n} M|g(x_i)-g(x_{i-1})| + \sum_{i=1}^{n} N|f(x_i)-f(x_{i-1})| \\
&\le MV' + NV.
\end{aligned}
$$

Thus, fg is b.v. on $[a,b]$.

Finally, for part (iv) suppose there exists $\varepsilon > 0$ such that $|g(x)| \ge \varepsilon$ for all $x \in [a,b]$. Then, given a partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of $[a, b]$, largely by the triangle inequality, we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^{n} |f(x_i)/g(x_i) - f(x_{i-1})/g(x_{i-1})| \\
&= \sum_{i=1}^{n} \left| \frac{f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)}{g(x_i)g(x_{i-1})} \right| \\
&\le \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1}) \\
&\qquad\qquad\qquad - (f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1}))| \\
&\le \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |g(x_{i-1})||f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_{i-1})||g(x_i) - g(x_{i-1})| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^{n} M_g |f(x_i) - f(x_i)| + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} M_f |g(x_i) - g(x_i)| \\
&= \frac{1}{\varepsilon^2} M_g \sum_{i=1}^{n} |f(x_i) - f(x_i)| + \frac{1}{\varepsilon^2} M_f \sum_{i=1}^{n} |g(x_i) - g(x_i)| \\
&\le \frac{1}{\varepsilon^2} (NV + MV')
\end{aligned}$$

where, as above, f is bounded by M and g is bounded by N. Thus, f/g is b.v. on $[a, b]$.

This concludes the proof of Theorem 2.1. ◀

Problem 3 (Wheeden & Zygmund Ch. 2, Ex. 3)**.** If $[a', b']$ is a subinterval of $[a, b]$ show that $P[a', b'] \le P[a, b]$ and $N[a', b'] \le N[a, b]$.

Solution. ▶ We will prove this by digging in to the definition of N and P. Recall that given a partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of the interval $[a, b]$, P and N are defined to be the supremum over the sum of the positive and, respectively, the sum negative terms of S_Γ; that is, P and N are the supremum over every partition Γ of $[a, b]$ of

$$P_\Gamma = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+ \quad \text{and} \quad N_\Gamma = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^- .$$

Let $f\colon [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and let $[a', b']$ be a subinterval of $[a, b]$. Without loss of generality, we may assume that $[a', b']$ is strictly contained in $[a, b]$; that is, $a' \ne a$ and $b' \ne b$. We aim to show that $P[a', b'] \le P[a, b]$ and $N[a', b'] \le N[a, b]$. Since the argument for N is similar to that of P, we will omit it here for the sake of brevity. Now, consider the closure of the complement of $[a', b']$ in $[a, b]$, $\overline{[a, b] \setminus [a', b']} = [a, a'] \cup [b', b]$. Since $[a, a']$, $[a', b']$ and $[b', b]$ are close intervals we may take partitions

cond thought, this is not the
ct way to go, but the argu-
is almost correct. You need
rt with a partition Γ of $[a, b]$
efined it so that some sub-
of Γ is a partition of $[a', b']$
rgue that because $P[a, b]$ is
remum and for any partition
is in the sum P_Γ, it follows
$P[a', b'] \le P[a, b]$.

$$\Gamma_a = \{ x_0 < x_1 \cdots < x_\ell \},$$
$$\Gamma_{ab} = \{ x_\ell < x_{\ell+1} < \cdots < x_m \}$$

and

$$\Gamma_b = \{ x_m < x_{m+1} < \cdots < x_n \}$$

of $[a, a']$, $[a', b']$ and $[b', b]$, respectively and extend this to a partition

$$\Gamma = \{ x_0 < x_1 < \cdots < x_\ell < x_{\ell+1} \cdots < x_m < x_{m+1} \cdots < x_n \}$$

of $[a, b]$. Then, by the definition of N we have the string of inequalities

$$\begin{aligned}
P_\Gamma &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+ \\
&= \sum_{i=1}^{\ell} [f(x_i) - f(x_{i-1})]^+ \\
&\qquad + \sum_{i=\ell+1}^{m} [f(x_i) - f(x_{i-1})]^+ \\
&\qquad + \sum_{i=m+1}^{n} [f(x_i) - f(x_{i-1})]^+ \\
&= P_{\Gamma_{ab}} + P_{\Gamma_a} + P_{\Gamma_b} \\
&\leq P[a, b].
\end{aligned}$$

Taking the supremum on the left, we have

$$P[a, a'] + P[a', b'] + P[b', a'] \leq P[a, b].$$

Since P is strictly positive, it must be the case that $P[a', b'] \leq P[a, b]$. ◀

Problem 4 (Wheeden & Zygmund Ch. 2, Ex. 11)**.** Show that $\int_a^b f \, \mathrm{d}\varphi$ exists if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|, |\Gamma'| < \delta$.

Solution. ▶ One direction is straightforward. Namely $\Longleftarrow$: suppose that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ whenever $|\Gamma|$ and $|\Gamma'|$ are less than δ. Let $\{\Gamma_n\}_{n=1}^\infty$ be a decreasing sequence of partitions (by which we mean $\Gamma_n \subset \Gamma_{n+1}$) of $[a, b]$ such that $|\Gamma_n| \to 0$. Then, by convergence, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|\Gamma_n| < \delta$. Then, for $n, m \geq N$, we have

$$|R_{\Gamma_n} - R_{\Gamma_m}| < \varepsilon.$$

Thus, by the Cauchy criterion for convergence, the sequence $\{R_{\Gamma_n}\}_{n=0}^\infty$ converges and its limit is by definition the Riemann–Stieltjes integral $\int_a^b f \, \mathrm{d}\varphi$.

On the other hand $\Longrightarrow$: suppose that $I = \int_a^b f \, \mathrm{d}\varphi$ exists. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that $|I - R_\Gamma| < \varepsilon/2$ whenever $|\Gamma| < \delta$. Let Γ and Γ' be two partitions of $[a, b]$ with norm

$|\Gamma|, |\Gamma'| < \delta$. Then we have

$$\begin{aligned}|R_\Gamma - R_{\Gamma'}| &= |R_\Gamma - I - (R_{\Gamma'} - I)| \\ &\leq |R_\Gamma - I| + |R_{\Gamma'} - I| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Thus, I satisfies the Cauchy condition. ◀

Problem 5 (Wheeden & Zygmund Ch. 2, Ex. 13)**.** Prove theorem (2.16).

Solution. ▶ Recall the statement of Theorem 2.16:

(i) If $\int_a^b f \, \mathrm{d}\varphi$ exists, then so do $\int_a^b cf \, \mathrm{d}\varphi$ and $\int_a^b f \, \mathrm{d}(c\varphi)$ for any constant c, and

$$\int_a^b cf \, \mathrm{d}\varphi = \int_a^b f \, \mathrm{d}(c\varphi) = c \int_a^b f \, \mathrm{d}\varphi.$$

(ii) If $\int_a^b f_1 \, \mathrm{d}\varphi$ and $\int_a^b f_2 \, \mathrm{d}\varphi$ both exist, so does $\int_a^b (f_1 + f_2) \, \mathrm{d}\varphi$, and

$$\int_a^b (f_1 + f_2) \, \mathrm{d}\varphi = \int_a^b f_1 \, \mathrm{d}\varphi + \int_a^b f_2 \, \mathrm{d}\varphi.$$

(iii) If $\int_a^b f \, \mathrm{d}\varphi_1$ and $\int_a^b f \, \mathrm{d}\varphi_2$ both exist, so does $\int_a^b f \, \mathrm{d}(\varphi_1 + \varphi_2)$, and

$$\int_a^b f \, \mathrm{d}(\varphi_1 + \varphi_2) = \int_a^b f \, \mathrm{d}\varphi_1 + \int_a^b f \, \mathrm{d}\varphi_2.$$

We prove this in (Roman) numerical order.

For (i) suppose that $I = \int_a^b f \, \mathrm{d}\varphi$ exists. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|I - R_\Gamma| < \varepsilon/|c|$ whenever Γ is a partition of $[a, b]$ with $|\Gamma| < \delta$. We claim that $\int_a^b cf \, \mathrm{d}\varphi = |c|I$. Let $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ be a partition $[a, b]$ with $|\Gamma| < \delta$. Then the Riemann–Stieltjes sums R'_Γ of the pair (cf, φ) associated to Γ give us the chain of inequalities

$$\begin{aligned}||c|I - R'_\Gamma| &= \left| |c|I - \sum_{i=1}^n cf(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c| \left| \sum_{i=1}^n cf(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c||I - R_\Gamma| \\ &< |c| \frac{\varepsilon}{|c|} \\ &= \varepsilon.\end{aligned}$$

Thus, $\int_a^b cf \, \mathrm{d}\varphi$ is Riemann–Stieltjes integrable and its integral is equal to $|c|I$. A similar argument shows that $\int_a^b f \, \mathrm{d}(c\varphi)$ is Riemann–Stieltjes integrable with integral $|c|I$.

For (ii) let $I_1 = \int_a^b f_1 \, \mathrm{d}\varphi$ and $I_2 = \int_a^b f_2 \, \mathrm{d}\varphi$. Then, we claim that $I = \int_a^b (f_1 + f_2) \, \mathrm{d}\varphi$ exists and that $I = I_1 + I_2$. Since both I_1 and I_2 exist, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|I_1 - R_\Gamma^1| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |I_2 - R_\Gamma^2| < \frac{\varepsilon}{2}$$

whenever $|\Gamma| < \delta$. Let $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ be a partition of $[a, b]$ with $|\Gamma| < \delta$. Then the Riemann–Stieltjes sums R_Γ of the pair $(f_1 + f_2, \varphi)$ associated to Γ give is the following chain of inequalities

$$\begin{aligned}
|(I_1 + I_2) - R_\Gamma| &= \left| (I_1 + I_2) - \sum_{i=1}^{n} (f_1(\xi_i) + f_2(\xi_i))[\varphi(x_i) - \varphi(x_{i-1})] \right| \\
&= \Bigg| I_1 - \sum_{i=1}^{n} f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1}) \\
&\qquad + I_2 - \sum_{i=1}^{n} f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \Bigg| \\
&\le \left| I_1 - \sum_{i=1}^{n} f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1}) \right| \\
&\qquad + \left| I_2 - \sum_{i=1}^{n} f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\
&= |I_1 - R_\Gamma^1| + |I_2 - R_\Gamma^2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus, I exists and it is equal to the sum $I_1 + I_2$.

Part (iii) is similar to part (ii) in the above equation except that instead of splitting the sum at $f_1 + f_2$ part, we split it at $\varphi_1 + \varphi_2$ part. ◀

1.1.2 Homework 2

Problem 1. Show that the boundary of any interval has outer measure zero.

Solution. ▶ Let $I :- \prod_{i=1}^{n} I_i$ be a closed interval in $\mathbb{R}^n$ and let J be the boundary of I. We must show that given $\varepsilon > 0$ there exists a countable collection of intervals $\{I_n\}_{n\in J}$ covering J such that

$$\sum_{n\in J} \operatorname{vol}(I_n) < \varepsilon.$$

First, note that we can write J as the union $\bigcup_{i=1}^{n} J_i$ where

$$J_i :- [a_1, b_1] \times \cdots \times \{a_i\} \times \cdots \times [a_n, b_n] \cup [a_1, b_1] \times \cdots \times \{b_i\} \times \cdots \times [a_n, b_n].$$

Since the countable union of null sets has measure zero, it suffices to show that the set

$$[a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$$

has measure zero. Consider the collection $\{I_\varepsilon\}$ consisting of the single interval

$$I_\varepsilon :- [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \left[a_n - \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)}, a_n + \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)}\right].$$

It is clear that $I_\varepsilon \supset J$. Now, computing the volume of this interval, we have

$$\begin{aligned}
\operatorname{vol}(I_\varepsilon) &= \prod_{i=1}^{n-1}(b_i - a_i)\left[a_n + \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)} - \left(a_n - \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)}\right)\right] \\
&= \left[\prod_{i=1}^{n-1}(b_i - a_i)\right]\frac{\varepsilon}{\prod_{i=1}^{n-1}(b_i - a_i)} \\
&= \varepsilon.
\end{aligned}$$

Thus, J has measure zero. ◀

Problem 2. Show that a set consisting of a single point has outer measure zero.

Solution. ▶ Let $\{a\}$ be the set consisting of a single point $a \in \mathbb{R}$. Then we must show that given $\varepsilon > 0$ there exists a countable collection of intervals $\{I_n\}$ such that

$$\sum_{n\in J} m(I_n) < \varepsilon.$$

Consider the collection $\{I_\varepsilon\}$ consisting of the single interval

$$I_\varepsilon :- \left[a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2}\right].$$

It is clear that $\{a\} \subset I_\varepsilon$. Moreover,

$$\begin{aligned}
\operatorname{vol}(I_\varepsilon) &= a + \frac{\varepsilon}{2} - \left(a - \frac{1}{\varepsilon}\right) \\
&= \varepsilon.
\end{aligned}$$

Thus, $\{a\}$ has measure zero. ◀

1.1.3 Homework 3

Problem 1 (Wheeden & Zygmund Ch. 3, Ex. 5)**.** Construct a subset of $[0, 1]$ in the same manner as the Cantor set, except that at the kth stage each interval removed has length $\delta 3^{-k}$, $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no interval.

Solution. ▶ We construct the prescribed subset as follows: take the open interval $(1/2 - \delta/6, 1/2 + \delta/6)$ and remove it from the closed interval $[0, 1]$ the result is a union of two disjoint closed intervals

$$E_{1,1} = \left[0, \frac{1}{2} - \frac{1}{6}\delta\right], \quad E_{1,2} = \left[\frac{1}{2} + \frac{1}{6}\delta, 1\right],$$

whose union we call E_1; this marks the first step in the construction of this Cantor-like set. Next, we remove the set

$$\left(\frac{1}{4} - \frac{5}{36}\delta, \frac{1}{4} + \frac{1}{36}\delta\right) \cup \left(\frac{3}{4} + \frac{\delta}{36}, \frac{3}{4} + \frac{5}{36}\delta\right)$$

from the set E_1 which yields E_2 the union of the four closed intervals

$$\begin{aligned} E_{2,1} &= \left[0, \frac{1}{4} - \frac{5}{36}\delta\right], & E_{2,2} &= \left[\frac{1}{4} + \frac{1}{36}\delta, \frac{1}{2} - \frac{1}{6}\delta\right], \\ E_{2,3} &= \left[\frac{1}{2} + \frac{1}{6}\delta, \frac{3}{4} + \frac{\delta}{36}\right], & E_{2,4} &= \left[\frac{3}{4} + \frac{5}{36}\delta, 1\right]. \end{aligned}$$

In the nth step of the construction, we remove an open interval of length $3^{-n}\delta$ from the center of each interval $E_{n-1,i}$ yielding E_n which is the union of 2^n intervals $E_{n,i}$ of length $2^{-n} - \delta 2^{-n} \sum_{i=1}^{n} 2^{i-1}3^{-i}$. Let E be the intersection $\bigcap_{i=1}^{\infty} E_i$. This concludes our construction.

Next we show that E is perfect, has measure $1 - \delta$ and contains no interval.

To see that E is perfect, we must show that E is closed and that and dense in itself. The set E is closed because it is the (arbitrary) intersection of closed intervals. To see that E is dense in itself, we must show that for every $\varepsilon > 0$, for every $x \in E$, the intersection $(B(x, \varepsilon) \cap E) \setminus \{x\}$ is nonempty. Let $\varepsilon > 0$ and $x \in E$ be given. Then, since $x \in E$, $x \in E_n$ for every n. Thus, x is in some closed interval $E_{n,i} \subset E_n$. Let N be the smallest integer such that the length of $E_{N,i} = [a, b]$ is less that ε. Then, $a, b \in E$ and $a, b \in B(x, \varepsilon)$ and x is cannot be equal to both a and b. Thus, $(E \cap B(x, \varepsilon)) \setminus \{x\} \neq \emptyset$. It follows that E is a perfect set.

To see that the measure of E is $1 - \delta$ by Theorem 3.26 (ii) since $m(E_1) = 1 - \delta/3 < \infty$ and

$E_n \searrow E$ we have

$$\begin{aligned}
m(E) &= m\left(\bigcap_{i=1}^{\infty} E_i\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} m(E_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{2^n} \left[\frac{1}{2^n} - \frac{\delta}{2^n} \sum_{i=1}^{n} \frac{2^{i-1}}{3^i}\right] \\
&= \lim_{n\to\infty} \left[1 - \delta \sum_{i=1}^{n} \frac{2^{i-1}}{3^i}\right] \\
&= \lim_{n\to\infty} \left[1 - \frac{\delta}{3} \sum_{i=1}^{n} \left(\frac{2}{3}\right)^{i-1}\right]
\end{aligned}$$

letting $j = i - 1$, we can rewrite the series above as the geometric series

$$\begin{aligned}
&= 1 - \frac{\delta}{3} \lim_{n\to\infty} \sum_{j=0}^{n} \left(\frac{2}{3}\right)^{j} \\
&= 1 - \delta,
\end{aligned}$$

as desired.

Lastly, we must show that E contains no interval. Seeking a contradiction, suppose that E contains an interval $I = [a, b]$ of length $b - a$. Then, since $I \subset E$, $I \subset E_n$ for all n so, since I is connected, it must be contained in one of the $E_{n,i}$ for all n. Let N be the smallest integer such that $m(E_{N,i}) < b - a$ and $E_{N,i} = [c, d]$ contains I. Then, since $I \subset E_{N,i}$, both a and b are points in I, $|b - a| \le |d - c| = m(E_{N,i})$. This is a contradiction. Thus, it must be the case that E contains no interval. ◀

Problem 2 (Wheeden & Zygmund Ch. 3, Ex. 7)**.** Prove (3.15).

Solution. ▶ Here is the statement of the lemma:

> *If $\{I_k\}_{k=1}^N$ is a finite collection of nonoverlapping intervals, then $\bigcup_{k=1}^N I_k$ is measurable and $m\left(\bigcup_{k=1}^N I_k\right) = \sum_{k=1}^N m(I_k)$.*

By Theorem 3.12, the union $\bigcup_{n=1}^N I_n$ is measurable. Hence, it remains to show that $m\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n)$.

We take the approach of extending the argument provided in Theorem 3.2. As in Theorem 3.2, we note that, since $\{I_n\}_{n=1}^N$ covers the union $\bigcup_{n=1}^N I_n$, then

$$m\left(\bigcup_{n=1}^{N} I_n\right) \le \sigma\left(\bigcup_{n=1}^{N} I_n\right) = \sum_{n=1}^{N} m(I_n).$$

On the other hand, note that I_n is the union $I_n^\circ \cup \partial I_n$ of its interior and its boundary. In the previous homework, we showed that the boundary of an interval has measure zero. Hence, we have

$$m(I_n^\circ) \le m(I_n) \le m(I_n^\circ) + m(\partial I_n) = m(I_n^\circ)$$

so $m(I_n) = m(I_n^\circ)$. Now, note that

Let us substantiate this
Suppose $\{I'_n\}_{n=1}^M$ a
tion of intervals which
$\bigcup_{n=1}^N I_n$ with $\sigma\left(\{I'_n\}_{n=1}^M\right.$
$\sum_{n=1}^M m(I_n) < \sum_{n=1}^N$
Then, the collection of
vals $\{I_n \cap I'_m\}_{1\le m\le M}^{1\le n\le N}$
$\bigcup_{n=1}^N I_n$ and has me
$\sigma(\{I_n \cap I'_m\}) < \sum_{n=1}^M m($
$\sum_{n=1}^N m(I_n)$. This implies
one of the intervals, say, I_k
completely covered by int
of the form $I_k \cap I'_\ell$, $1 \le \ell$
This is a contradiction.

$$m\left(\bigcup_{n=1}^N I_n^\circ\right) = \sum_{n=1}^N m(I_n^\circ) = \sum_{n=1}^N m(I_n).$$

Hence, we have

$$\begin{aligned}
\sum_{n=1}^N m(I_n) &= m\left(\bigcup_{n=1}^N I_n^\circ\right) \\
&\le m\left(\bigcup_{n=1}^N I_n\right) \\
&\le \sum_{n=1}^N m(I_n).
\end{aligned}$$

Thus, equality $m\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n)$ holds. ◀

Problem 3 (Wheeden & Zygmund Ch. 3, Ex. 8)**.** Show that the Borel algebra $\mathcal{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.

Solution. ▶ Since $\mathcal{B}$ is the smallest σ-algebra containing all of the open sets of $\mathbb{R}^n$, it contains all of the closed sets of $\mathbb{R}^n$. Now, suppose that $\mathcal{B}'$ is another σ-algebra containing the closed sets in $\mathbb{R}^n$. Then, $\mathcal{B}' \subset \mathcal{B}$ since $\mathcal{B}$ contains all of the closed sets in $\mathbb{R}^n$. However, since $\mathcal{B}'$ is a σ-algebra, it contains all of the open sets in $\mathbb{R}^n$, so $\mathcal{B}' \subset \mathcal{B}$ since $\mathcal{B}$ is the smallest σ-algebra containing the open sets in $\mathbb{R}^n$. Thus, $\mathcal{B}' = \mathcal{B}$. ◀

Problem 4 (Wheeden & Zygmund Ch. 3, Ex. 9)**.** If $\{E_k\}_{k=1}^\infty$ is a sequence of sets with $\sum m^*(E_k) < \infty$, show that $\limsup E_k$ (and also $\liminf E_k$) has measure zero.

Solution. ▶ First, since $\{E_n\}_{n=1}^\infty$ is a sequence of sets with

$$\sum_{i=1}^\infty m^*(E_i) < \infty$$

for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \ge N$ implies

$$\sum_{i=n}^\infty m^*(E_i) < \varepsilon.$$

Let's put this aside for now.

Define $E = \limsup_{n\to\infty} E_n$ and $E'_n = \bigcup_{i=n}^{\infty} E_n$. It is easy to see that $\{E'_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets whose intersection $\bigcap_{n=1}^{\infty} E_n$ is the limit supremum E. By the monotonicity of the outer measure, we have

This is because E $E_{n+1} \cup E'_n$.

$$m^*(E) \le m^*(E'_n)$$

for all $n \in \mathbb{N}$. On the other hand,

$$m^*(E'_n) \le \sum_{i=n}^{\infty} m^*(E_i) < \varepsilon$$

for every ε. Letting ε go to 0 we have $m^*(E) = 0$.

Lastly, we note that $E' = \liminf_{n\to\infty} E_n$ is a subset of $\limsup_{n\to\infty} E_n$, so that $m^*(E') = 0$. ◀

Problem 5 (Wheeden & Zygmund Ch. 3, Ex. 10)**.** If E_1 and E_2 are measurable, show that $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

Solution. ▶ We may, without loss of generality, assume that $m(E_1), m(E_2) < \infty$ for otherwise there is nothing to show as equality holds trivially.

Now, by Carathéodory's theorem we have the following characterization of measurability: a set E is measurable if and only if for every set A we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E).$$

that we remove the $*$ from ce for measurable sets the measure and measure agree.

Therefore, the following equalities hold

$$\begin{aligned} m(E_1) &= m(E_1 \cap E_2) + m(E_1 \setminus E_2) \\ m(E_2) &= m(E_1 \cap E_2) + m(E_2 \setminus E_1). \end{aligned}$$

Moreover, from elementary set theory we have

$$(E_1 \cup E_2) \setminus E_2 = E_1 \setminus (E_1 \cap E_2),$$

orollary to Carathéodory's em would have been enough this result; that is, Corol- .31.

$E_1 \subset E_1 \cup E_2$ and $E_1 \cap E_2 \subset E_1$ so

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

as desired. ◀

1.1.4 Homework 4

Problem 1 (Wheeden & Zygmund Ch. 3, Ex. 12)**.** If E_1 and E_2 are measurable sets in $\mathbb{R}^1$, show $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $m(E_1 \times E_2) = m(E_1)m(E_2)$. (Interpret $0 \cdot \infty$ as 0.) [*Hint:* Use a characterization of measurability.]

Solution. ▶ The proof of this result is rather long and we shall omit it for now as I gain nothing from retracing my steps on this one. ◀

Problem 2 (Wheeden & Zygmund Ch. 3, Ex. 13)**.** Motivated by (3.7), define the *inner measure* of E by $m_*(E) = \sup m(F)$, where the supremum is taken over all closed subsets F of E. Show that

(i) $m_*(E) \leq m^*(E)$, and
(ii) if $m^*(E) < \infty$, then E is measurable if and only if $m_*(E) = m^*(E)$.

[Use (3.22).]

Solution. ▶ First we show part (i). If $m^*(E) = \infty$, the inequality holds trivially. Suppose that $m^*(E) < \infty$. Then, since F is closed, it is measurable and $m(F) = m^*(F)$. Moreover, $F \subset E$ so by the monotonicity of the outer measure,

$$m(F) = m^*(F) < m^*(E).$$

Taking the supremum over all F on the left, we have

$$m_*(E) = \sup_{F \subset E} m(F) < m^*(E)$$

as we set out to show.

Next we show part (ii). Let $E \subset \mathbb{R}^n$ with $m^*(E) < \infty$. $\implies$ Suppose that E is measurable. Then, by Lemma 3.22, there exists a closed set $F \subset E$ such that $m^*(E \setminus F) < \varepsilon$. Since closed sets are measurable, by Corollary 3.31, we have

$$m^*(E \setminus F) = m(E) - m(F) < \varepsilon$$

so

$$m(E) < m(F) + \varepsilon.$$

Letting ε go to 0, we have

$$m(E) \leq m(F);$$

and taking the supremum on the right

$$m(E) \leq m_*(E).$$

But, by part (i), $m_*(E) \leq m^*(E) = m(E)$. Thus, $m_*(E) = m^*(E)$ as was to be shown.

$\impliedby$ On the other hand, suppose that $m_*(E) = m^*(E)$. Then, given $\varepsilon > 0$ there exists an open set G containing E and a closed set F contained in E such that These are the definitions of

$$m(G) - m^*(E) < \frac{\varepsilon}{2}$$
$$m_*(E) - m(F) < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned} m^*(E \setminus F) &< m^*(G \setminus F) \\ &= m^*(G) - m^*(G \cap F) \\ &= m^*(G) - m^*(F) \\ &< \frac{\varepsilon}{2} + m^*(E) - \left(m^*(E) - \frac{\varepsilon}{2}\right) \\ &= \varepsilon. \end{aligned}$$

Thus, by Lemma 3.22, E is measurable. ◀

Problem 3 (Wheeden & Zygmund Ch. 3, Ex. 15)**.** If E is measurable and A is any subset of E, show that $m(E) = m_*(A) + m^*(E \setminus A)$. (See Exercise 13 for the definition of $m_*(A)$.)

Solution. ► Suppose $A \subset E$. If A is measurable, by Problem 2, the outer and inner measure of A agree; symbolically, we have $m(A) = m^*(A) = m_*(A)$. Thus, we have

$$m^*(E \setminus A) = m^*(E) - m^*(A) = m^*(E) - m_*(A).$$

though A is not measur- we can still find its inner uter measure.

If A is not measurable and $m(E) < \infty$, then we must have $m^*(A), m^*(E \setminus A) < \infty$ by the monotonicity of the outer measure; since both A an $E \setminus A$ are subsets of E. Hence, we may, without any ambiguity, subtract the quantity $m^*(E \setminus A)$ from $m(E)$ and we have

$$\begin{aligned} m(E) - m^*(E \setminus A) &= m(E) - \inf \{ m(G) : E \setminus A \subset G \text{ and } G \text{ is open} \} \\ &= m(E) - \inf \{ m(G) : E \setminus A \subset G \subset E \text{ and } G \text{ is open} \} \\ &= \end{aligned}$$

Etcetera. The rest of this is uninteresting. ◀

1.1.5 Homework 5

Problem 1 (Wheeden & Zygmund Ch. 3, Ex. 14)**.** Show that the conclusion of part (ii) of Exercise 13 is false if $m^*(E) = \infty$.

Solution. ▶ Part (ii) of Exercise 13 is part (ii) of Problem 2 from the last section (Homework 4). In that problem we showed that if the outer measure of E is finite, then E is measurable if and only if its outer and inner measure agree. Here we construct a counter example to this when the outer measure of E is ∞; that is, we show that there exists a set E with $m^*(E) = \infty$ such that $m^*(E) \neq m_*(E)$. So, which set shall it be? Since we are unoriginal, we will pull an example from Wheeden and Zygmund itself.

Let $V \subset [0,1]$ be Vitali's unmeasurable (Theorem 3.38) set as constructed in Wheeden and Zygmund and consider the union $E = V \cup (2, \infty)$. It is clear that the inner and outer measure of E are both ∞. However, E itself must be unmeasurable for otherwise $E \cap [0,1] = V$ is measurable. ◀

Problem 2 (Wheeden & Zygmund Ch. 3, Ex. 16)**.** Prove (3.34).

Solution. ▶ We must prove Equation 3.34; that is, if P is a parallellepiped

$$m(P) = \mathrm{vol}(P).$$

This is an uninteresting and slightly technical problem. We gain nothing from retracing its proof. ◀

Problem 3 (Wheeden & Zygmund Ch. 3, Ex. 18)**.** Prove that outer measure is *translation invariant*; that is, if $E_h = \{\, x + h : x \in E \,\}$ is the translate of E by h, $h \in \mathbb{R}^n$, show that $m^*(E_h) = m^*(E)$. If E is measurable, show that E_h is also measurable. [This fact was used in proving (3.37).]

To use **boldface** or not t
boldface with points in $\mathbb{R}^n$
is the question. Whether 't
bler...
For now, let us just use
italic for points in a set and
face math for vectors. Yes,
this convention.

Solution. ▶ Let $E \subset \mathbb{R}^n$ and $h \in \mathbb{R}^n$ and define the set E_h to be the set $E_h = \{\, x + h : x \in E \,\}$. We will show that the outer measure of E is preserved under such translations. But first, let us point out that E_h is nothing more than the image of E under the linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $x \mapsto x + h$. By Theorem 3.35, such a map preserves measurability of sets and for any measurable set $E' \subset \mathbb{R}^n$, $m(T(E')) = (\det T) m(E') = m(E')$ (since $\det T = 1$). Now, by Theorem 3.6, for every $\varepsilon > 0$, there exist an open set $G \supset E$ such that $m^*(G) \leq m^*(E) + \varepsilon$. Consider the image of G under T, $T(G)$ is an open set containing E_h so $m^*(G) \geq m^*(E)$ and

$$m^*(T(G)) = m^*(G) < m^*(E) + \varepsilon.$$

Letting $\varepsilon \to 0$, we achieve the inequality

$$m^*(E_h) \leq m^*(E).$$

To get the other inequality, take the map $T^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ which takes $x \mapsto x - h$; this sends E_h to E and the same argument shows that

$$m^*(E) \leq m^*(E).$$

Thus, we have $m^*(E) = m^*(E_h)$, as was to be shown. ◀

Problem 4 (Wheeden & Zygmund Ch. 4, Ex. 1)**.** Prove corollary (4.2) and theorem (4.8)

Solution. ▶ The corollary and theorem in question are:

> *If f is measurable, then $\{ f > -\infty \}$, $\{ f < +\infty \}$, $\{ f = +\infty \}$, $\{ a \le f \le b \}$, $\{ f = a \}$, etc., are all measurable. Moreover f is measurable if and only if $\{ a < f < +\infty \}$ is measurable for every finite a.*

and

> *If f is measurable and λ is any real number, then $f + \lambda$ and λf are measurable.*

Their proofs are quite simple. For the corollary: Suppose $f\colon E \to \mathbb{R}$ is a measurable function. By Theorem 4.1, f is measurable if and only if for every finite $\alpha \in \mathbb{R}$, the sets

$$\begin{gathered} \{ x \in E : f(x) \ge \alpha \} \\ \{ x \in E : f(x) < \alpha \} \\ \{ x \in E : f(x) \le \alpha \} \end{gathered}$$

are measurable. Since measurable sets form a σ-algebra on $\mathbb{R}^n$, we know that the countable union and intersection of measurable sets is measurable. Thus,

$$\begin{aligned} \{ x \in E : f(x) > -\infty \} &= \bigcup_{\alpha \in \mathbb{Z}} \{ x \in E : f(x) > \alpha \} \\ \{ x \in E : f(x) = \infty \} &= \bigcap_{n=1}^{\infty} \{ x \in E : f(x) > n \} \\ \{ x \in E : f(x) < \infty \} &= \bigcup_{\alpha \in \mathbb{Z}} \{ x \in E : f(x) < \alpha \} \end{aligned}$$

are easily seen to be measurable.

Showing that $\{ x \in E : f(x) = \alpha \}$ and $\{ x \in E : \alpha < f(x) < \beta \}$ are measurable requires some clever (but not too clever) intersection/union of the sets we get from Theorem 4.1.

For the theorem: Suppose f is measurable and λ is a constant. By Theorem 4.1, for any finite $\alpha \in \mathbb{R}$ we have

$$\{ x \in E : f(x) > \alpha - \lambda \}$$

so

$$\{ x \in E : f(x) + \lambda > \alpha \}$$

is measurable. Thus, $f + \lambda$ is measurable. Similarly, for $\lambda \ne 0$, taking the set

$$\{ x \in E : f(x) > \alpha/\lambda \} = \{ x \in E : \lambda f(x) > \alpha \}$$

shows that λf is measurable; otherwise, if $\lambda = 0$, $\lambda f = 0$ is constant and hence is continuous which in turn implies that it is measurable. ◀

Problem 5 (Wheeden & Zygmund Ch. 4, Ex. 2)**.** Let f be a simple function, taking its distinct values on disjoint sets $E_1, \ldots, E_N$. Show that f is measurable if and only if $E_1, \ldots, E_N$ are measurable.

Solution. ► $\Longrightarrow$ Suppose that f is measurable. Then, by Corollary 4.2, the sets of the form $\{ f = \alpha_n \} = E_n$ are measurable. So the sets E_n are measurable.

$\Longleftarrow$ On the other hand, suppose that the sets E_n are measurable. Then, χ_{E_n} is measurable so by Theorem 4.8, f is measurable since it is the sum

$$f = \sum_{n=1}^{N} \alpha_{E_n}.$$ ◄

1.1.6 Homework 6

Problem 1 (Wheeden & Zygmund Ch. 4, Ex. 4)**.** Let f be defined and measurable in $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(T(x))$ is measurable. [If $E_1 = \{ x : f(x) > a \}$ and $E_2 = \{ x : f(T(x)) > a \}$, show $E_2 = T^{-1}(E_1)$.]

Solution. ▶ Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function and $T \colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then, we show that the composition $f \circ T$ is measurable. Fix a finite $\alpha \in \mathbb{R}$ and let

$$\begin{aligned} E_1 &= \{ x : f(x) > \alpha \} \\ E_2 &= \{ x : f(T(x)) > \alpha \} . \end{aligned}$$

Then, by Theorem 3.35, it suffices to show that $E_2 = T^{-1}(E_1)$ since T^{-1} is a nonsingular linear transformation so it sends measurable sets to measurable sets. But this equality is obvious: Suppose $x \in E_2$; then $f(T(x)) > \alpha$ so, because T is nonsingular and therefore bijective, clearly $x \in T^{-1}(E_1)$ so $E_2 \subset T^{-1}(E_1)$. One the other hand, if $x \in T^{-1}(E_1)$ then x is a point in E such that $f(T(x)) > \alpha$ so $x \in E_2$. Thus, $E_2 = T^{-1}(E_1)$ and consequently, $f \circ T$ is a measurable function. ◀

Problem 2 (Wheeden & Zygmund Ch. 4, Ex. 7)**.** Let f be usc and less that ∞ on a compact set E. Show that f is bounded above on E. Show also that f assumes its maximum on E, i.e., that there exists $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.

Solution. ▶ First we show that f is bounded. Suppose that f is u.s.c. on E. Then, by Theorem 4.14 (i), sets of the form $\{ x \in E : f(x) < \alpha \}$ are relatively open. Let $\mathcal{G} = \{G_\alpha\}_{\alpha \in \mathbb{Z}}$ where $G_a = \{ x \in E : f(x) < \alpha \}$. Then $\mathcal{G}$ forms an open cover of E and since E is compact there exists a finite subset $\{G_{\alpha_n}\}_{n=1}^N$ for some finite subset $\{ \alpha_1, \ldots, \alpha_N \}$ of $\mathbb{Z}$. Let $\alpha = \max\{ \alpha_1, \ldots, \alpha_N \}$. Then, $f(x) < \alpha$ for all $x \in E$ so f is bounded above by α.

Next, we show that f in fact assumes its maximum (locally) on E by using only topological properties of f. Since sets of the form $\{ x \in E : f(x) \geq \alpha \}$ are relatively closed, by Theorem 4.14 (i), for fixed $x \in E$ the sets $F_x = \{ y \in E : f(y) \geq f(x) \}$ are relatively closed. Consider the collection $\{F_x\}_{x \in E}$ of closed subsets of E. First, note that each of these sets is nonempty since $f(x) \geq f(x)$ so $x \in F_x$ for every $x \in E$. Now, let $\{x_n\}_{n=1}^N \subset E$ and consider the collection $\{F_{x_n}\}_{n=1}^N$. Then $\bigcap_{n=1}^N F_{x_n} \neq \emptyset$ since for x the point in $\{ x_1, \ldots, x_N \}$ such that $f(x) = \min\{ f(x_1), \ldots, f(x_N) \}$, $x \in F_{x_n}$ for all $1 \leq n \leq N$. Thus, by the finite intersection property, the intersection $F = \bigcap_{x \in E} F_x$ is nonempty. Let $y \in \bigcap_{x \in E} F_x$, then $f(y) \geq f(x)$ for all $x \in E$ so f achieves its maximum (locally) on E. ◀

Problem 3 (Wheeden & Zygmund Ch. 4, Ex. 8)**.**

(a) Let f and g be two functions which are u.s.c. at x_0. Show that $f + g$ is u.s.c. at x_0. Is $f - g$ u.s.c. at x_0? When is fg u.s.c. at x_0?

(b) If $\{f_k\}$ is a sequence of functions are u.s.c. at x_0, show that $\inf f_k(x)$ is u.s.c. at x_0.

(c) If $\{f_k\}$ is a sequence of functions which are u.s.c. at x_0 and which converge uniformly near x_0, show that $\lim f_k$ is u.s.c. at x_0.

Solution. ► We prove these in alphabetical order (a) → (b) → (c).

For (a), suppose that f and g are u.s.c. at x_0. Then given $M > f(x_0), g(x_0)$ there exists $\delta_1, \delta_2 > 0$ such that $f(x), g(x) < M/2$ for all $|x_1 - x_0| < \delta_1, |x_2 - x_0| < \delta_2$, respectively. Let δ be the minimum of $\{ \delta_1, \delta_2 \}$. Then for any x such that $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) + g(x) - (f(x_0) + g(x_0))| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\ &\leq |(f(x) - f(x_0))| + |(g(x) - g(x_0))| \\ &< \frac{M}{2} + \frac{M}{2} \\ &= M. \end{aligned}$$

Thus, $f + g$ is u.s.c.

For that second little part of (a), the one that asks "Is $f - g$ u.s.c. at x_0?" we provide a counter example. In fact, the following is enough of a counterexample: Take $f = 0$ (which is continuous everywhere) and g any function that is u.s.c., but not continuous, at x_0 then $f - g = -g$ is l.s.c. at x_0. Another counterexample is provided by the equations u_1 and u_2 from Ch. 4 of Wheeden and Zygmund: Fix an $x_0 \in \mathbb{R}$ and define

In fact, we could have pre[...] this, since if g is u.s.c. at x[...] is l.s.c. at x_0.

$$u_1(x) = \begin{cases} 0 & \text{if } x < x_0, \\ 1 & \text{if } x \geq x_0, \end{cases} \qquad u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

Then

$$u_1(x) - u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

is not u.s.c. at x_0 since being u.s.c. at x_0 implies that for $1/2 > f(x_0) = 0$ there exists $\delta > 0$ such that $f(x) < 1/2$ for all $x \in (x_0 - \delta, x_0 + \delta)$. But for any $x' > x_0$ in $(x_0 - \delta, x + \delta)$, $u(x') = 1 > 1/2$ which contradicts the assumption that u is u.s.c. at x_0.

For (b), suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions that are u.s.c. at x_0. Then

$$\limsup_{\substack{x \to x_0 \\ x \in E}} f_n(x) \leq f_n(x_0)$$

for all $n \in \mathbb{N}$. We must show that

$$\limsup_{\substack{x \to x_0 \\ x \in E}} \big[\inf f_n(x)\big] \leq \inf f_n(x_0).$$

◄

1.1.7 Homework 7

Problem 1 (Wheeden & Zygmund Ch. 4, Ex. 9)**.**

(a) Show that the limit of a decreasing (increasing) sequence of functions usc (lsc) at x_0 is usc (lsc) at x_0. In particular, the limit of a decreasing (increasing) sequence of functions continuous at x_0 is usc (lsc) at x_0.

(b) Let f be usc and less than ∞ on $[a, b]$. Show that there exists continuous f_k on $[a, b]$ such that $f_k \searrow f$.

Solution. ▶ ◀

Problem 2 (Wheeden & Zygmund Ch. 4, Ex. 11)**.** Let f be defined on $\mathbb{R}^n$ and let $B(x)$ denote the open ball $\{ y : |x - y| < r \}$ with center x and fixed radius r. Show that the function $g(x) = \sup \{ f(y) : y \in B(x) \}$ is lsc and the function $h(x) = \inf \{ f(y) : y \in B(x) \}$ is usc on $\mathbb{R}^n$. Is the same true for the closed ball $\{ y : |x - y| \leq r \}$?

Solution. ▶ ◀

Problem 3 (Wheeden & Zygmund Ch. 4, Ex. 15)**.** Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$. If $|f_k(M)| \leq M < \infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there is closed $F \subset E$ and finite M such that $m(E \setminus F) < \varepsilon$ and $|f_k(x)| \leq M$ for all $x \in F$.

Solution. ▶ ◀

Problem 4 (Wheeden & Zygmund Ch. 4, Ex. 18)**.** If f is measurable on E, define $\omega_f(a) = |\{ f > a \}|$ for $-\infty < a < \infty$. If $f_k \nearrow f$, show that $\omega_{f_k} \nearrow \omega_f$. If $f_k \to f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f. [For the second part, show that if $f_k \to f$, then $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$ and $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$ for every $\varepsilon > 0$.]

Solution. ▶ ◀

Problem 5 (Wheeden & Zygmund Ch. 5, Ex. 1)**.** If f is a simple measurable function (not necessarily positive) taking values a_j on E_j, $j = 1, ..., N$, show that $\int_E f = \sum_{j=1}^N a_j |E_j|$. [Use (5.24)].

Solution. ▶ ◀

Problem 6 (Wheeden & Zygmund Ch. 5, Ex. 3)**.** Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \leq f$ a.e. on E, show that $\int_E f_k \to \int_E f$.

Solution. ▶ ◀

1.1.8 Homework 8

Problem 1 (Wheeden & Zygmund Ch. 5, Ex. 2)**.** Show that the conclusion of (5.32) are not true without the assumption that $\varphi \in L(E)$. [In part (ii), for example, take $f_k = \chi_{(k,\infty)}$.]

Solution. ► ◄

Problem 2 (Wheeden & Zygmund Ch. 5, Ex. 4)**.** If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, ...,$ and $\int_0^1 x^k f(x)\,\mathrm{d}x \to 0$.

Solution. ► ◄

Problem 3 (Wheeden & Zygmund Ch. 5, Ex. 6)**.** Let $f(x,y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $\partial f(x,y)/\partial x$ is a bounded function of (x,y). Show that $\partial f(x,y)/\partial x$ is a measurable function of y for each x and

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 f(x,y)\,\mathrm{d}y = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,\mathrm{d}y.$$

Solution. ► ◄

Problem 4 (Wheeden & Zygmund Ch. 5, Ex. 7)**.** Give an example of an f that is not integrable, but whose improper Riemann integral exists and is finite.

Solution. ► ◄

Problem 5 (Wheeden & Zygmund Ch. 5, Ex. 21)**.** If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

Solution. ► ◄

Problem 6 (Wheeden & Zygmund Ch. 6, Ex. 10)**.** Let V_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theorem that

$$V_n = 2V_{n-1}\int_0^1 \left(1 - t^2\right)^{(n-1)/2}\,\mathrm{d}t.$$

(We also observe that by setting $w = t^2$, the integral is a multiple of a classical β-function and so can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,\mathrm{d}t$, $s > 0$.)

Solution. ► ◄

Problem 7 (Wheeden & Zygmund Ch. 6, Ex. 11)**.** Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x} = \pi^{n/2}.$$

(For $n = 1$, write $\left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x dy$ and use polar. For $n > 1$, use the formula $e^{-|\mathbf{x}|^2} = e^{-x_1{}^2}\cdots e^{-x_n{}^2}$ and Fubini's theorem to reduce the case $n = 1$.)

Solution. ▶ ◀

1.1.9 Homework 9

Problem 1 (Wheeden & Zygmund Ch. 6, Ex. 1)**.**

(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}$, $\{ y : (x, y) \in E \}$ has $\mathbb{R}$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}$, $\{ x : (x, y) \in E \}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x, y)$ is finite for almost every y. Show that for almost $y \in \mathbb{R}$, $f(x, y)$ is finite for almost every x.

Solution. ► ◄

Problem 2 (Wheeden & Zygmund Ch. 6, Ex. 3)**.** Let f be measurable and finite a.e. on $[0, 1]$. If $f(x) - f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$, show that $f \in L[0, 1]$.

Solution. ► ◄

Problem 3 (Wheeden & Zygmund Ch. 6, Ex. 4)**.** Let f be measurable and periodic with period 1: $f(t + 1) = f(t)$. Suppose there is a finite c such that

$$\int_0^1 |f(a + t) - f(b + t)| \,\mathrm{d}t \le c$$

for all a and b. Show that $f \in L[0, 1]$. (Set $a = x$, $b = -x$, integrate with respect to x, and make the change of variables $\xi = x + t$, $\eta = -x + t$.)

Solution. ► ◄

Problem 4 (Wheeden & Zygmund Ch. 6, Ex. 6)**.** For $f \in L(\mathbb{R})$, define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} \,\mathrm{d}t$$

for $x \in \mathbb{R}$. (For complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i \int F_1$.) Show that if f and g belong to $L(\mathbb{R})$, then

$$\widehat{(f * g)}(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

Solution. ► ◄

Problem 5 (Wheeden & Zygmund Ch. 6, Ex. 7)**.** Let F be a closed subset of $\mathbb{R}$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F, prove that the function

$$\int_{\mathbb{R}} \frac{\delta^\lambda(y) f(y)}{|x - y|^{1+\lambda}} \,\mathrm{d}t$$

is integrable over F and so is finite a.e. in F. (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

Solution. ▶ ◀

Problem 6 (Wheeden & Zygmund Ch. 6, Ex. 9)**.**

(a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.

(b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subset \mathbb{R}$, and let M_α be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int M_\lambda \leq \lambda^{-1}|G|$, where $G = (a, b) - [c, d]$.

Solution. ▶ ◀

1.1.10 Homework 10

Problem 1 (Wheeden & Zygmund Ch. 7, Ex. 1)**.** Let f be measurable in $\mathbb{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(\mathbf{x}) \geq c\|\mathbf{x}\|^{-n}$ for $\|\mathbf{x}\| \geq 1$.

Solution. ► ◄

Problem 2 (Wheeden & Zygmund Ch. 7, Ex. 2)**.** Let $\varphi(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$, be a bounded measurable function such that $\varphi(\mathbf{x}) = 0$ for $\|\mathbf{x}\| \geq 1$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(\mathbf{x}) = \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon)$. ($\varphi_\varepsilon$ is called an *approximation to the identity*.) If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon \to 0}(f * \varphi_\varepsilon)(\mathbf{x}) = f(\mathbf{x})$$

in the Lebesgue set of f. (Note that $\int \varphi_\varepsilon = 1$, $\varepsilon > 0$, so that

$$(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x}) = \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\, \varphi_\varepsilon(\mathbf{y})\, \mathrm{d}\mathbf{y}.$$

Use Theorem 7.16.)

Solution. ► ◄

Problem 3 (Wheeden & Zygmund Ch. 7, Ex. 6)**.** Show that if $\alpha > 0$, then x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Solution. ► ◄

Problem 4 (Wheeden & Zygmund Ch. 7, Ex. 8)**.** Prove the following converse of Theorem 7.31: If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Solution. ► ◄

Problem 5 (Wheeden & Zygmund Ch. 7, Ex. 9)**.** If f is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then f is absolutely continuous on $[a, b]$. (For the second part, use Theorems 2.2(ii) and 7.24 to show that $V(x)$ is absolutely continuous and then use the result of Exercise 8).

Solution. ► ◄

Problem 6 (Wheeden & Zygmund Ch. 7, Ex. 12)**.** Use Jensen's inequality to prove that if $a, b \geq 0$, $p, q > 1$, $(1/p) + (1/q) = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

More generally, show that

$$a_1 \cdots a_N = \sum_{j=1}^{N} \frac{{a_j}^{p_j}}{p_j},$$

where $a_j \geq 0$, $p_j > 1$, $\sum_{j=1}^{N}(1/p_j) = 1$. (Write $a_j = e^{x_j/p_j}$ and use the convexity of e^x).

Solution. ▶ ◀

Problem 7 (Wheeden & Zygmund Ch. 7, Ex. 13)**.** Prove Theorem 7.36.

Solution. ▶ Recall the statement of Theorem 7.36

(i) *If φ_1 and φ_2 are convex in (a, b), then $\varphi_1 + \varphi_2$ is convex in (a, b).*
(ii) *If φ is convex in (a, b) and c is a positive constant, then $c\varphi$ is convex in (a, b).*
(iii) *If φ_k, $k = 1, 2, \ldots,$ are convex in (a, b) and $\varphi_k \to \varphi$ in (a, b), then φ is convex in (a, b).*

◀

1.1.11 Homework 11

Problem 1 (Wheeden & Zygmund Ch. 7, Ex. 11)**.** Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)\, \mathrm{d}t.$$

(Consider the case when f is the characteristic function of an interval, an open set, etc.)

Solution. ▶ ◀

Problem 2 (Wheeden & Zygmund Ch. 7, Ex. 15)**.** Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\varphi(x) = \int_a^x f(t)\, \mathrm{d}t + \varphi(a)$ in (a, b) and f is monotone increasing, then φ is convex in (a, b). (Use Exercise 14.)

Solution. ▶ ◀

Problem 3 (Wheeden & Zygmund Ch. 5, Ex. 8)**.** Prove (5.49).

Solution. ▶ ◀

Problem 4 (Wheeden & Zygmund Ch. 5, Ex. 11)**.** For which p does $1/x \in L^p(0, 1)$? $L^p(1, \infty)$? $L^p(0, \infty)$?

Solution. ▶ ◀

Problem 5 (Wheeden & Zygmund Ch. 5, Ex. 12)**.** Give an example of a bounded continuous f on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

Solution. ▶ ◀

Problem 6 (Wheeden & Zygmund Ch. 5, Ex. 17)**.** If $f \geq 0$ and $\omega(\alpha) \leq c(1 + \alpha)^p$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.

Solution. ▶ ◀

Problem 7 (Wheeden & Zygmund Ch. 8, Thm. 8.3)**.** If $f, g \in L^p(E)$, $p > 0$, then $f + g \in L^p(E)$ and $cf \in L^p(E)$ for any constant c.

Solution. ▶ ◀

1.1.12 Homework 12

Problem 1 (Wheeden & Zygmund Ch. 8, Ex. 2)**.** Prove the converse of Hölder's inequality for $p = 1$ and ∞. Show also that for $1 \leq p \leq \infty$, a real-valued measurable f belongs to $L^p(E)$ if $fg \in L^1(E)$ for every $g \in L^{p'}(E)$, $1/p + 1/p' = 1$. The negation is also of interest: if $f \in L^p(E)$ then there exists $g \in L^{p'}(E)$ such that $fg \notin L^1(E)$. (To verify the negation, construct g of the form $\sum a_k g_k$ satisfying $\int_E f g_k \to \infty$.)

Solution. ► ◄

Problem 2 (Wheeden & Zygmund Ch. 8, Ex. 3)**.** Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when $p < 1$.

Solution. ► ◄

Problem 3 (Wheeden & Zygmund Ch. 8, Ex. 4)**.** Let f and g be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let $1 < p < \infty$. Prove that equality holds in the inequality $|\int fg| \leq \|f\|_p \|g\|_{p'}$ if and only if fg has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e.

If $\|f + g\|_p = \|f\|_p + \|g\|_p$ and $g \neq 0$ in Minkowski's inequality, show that f is a multiple of g.

Find analogues of these results for the spaces ℓ^p.

Solution. ► ◄

Problem 4 (Wheeden & Zygmund Ch. 8, Ex. 5)**.** For $0 < p \leq \infty$ and $0 < |E| < \infty$, define

$$N_p[f] = \left(\frac{1}{E} \int_E |f|^p \right)^{1/p},$$

where $N_\infty[f]$ means $\|f\|_\infty$. Prove that if $p_1 < p_2$, then $N_{p_1}[f] \leq N_{p_2}[f]$. Prove also that if $1 \leq p \leq \infty$, then $N_p[f + g] \leq N_p[f] + N_p[g]$, $(1/|E|) \int_E |fg| \leq N_p[f] N_{p'}[g]$, $1/p + 1/p' = 1$, and $\lim_{p\to\infty} N_p[f] = \|f\|_\infty$. Thus, N_p behaves like $\|\cdot\|_p$ but has the advantage of being monotone in p. Recall Exercise 28 of Chapter 5.

Solution. ► ◄

Problem 5 (Wheeden & Zygmund Ch. 8, Ex. 6)**.**

(a) Let $1 \leq p_i,\ r \leq \infty$ and $\sum_{i=1}^k 1/p_i = 1/r$. Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

(b) Let $1 \le p < r < q \le \infty$ and define $\theta \in (0,1)$ by $1/r = \theta/p + (1-\theta)/q$. Prove the interpolation estimate

$$\|f\|_r \le \|f\|_p{}^{\theta} \|f\|_q{}^{1-\theta}.$$

In particular, if $A = \max\{\|f\|_p, \|f\|_q\}$, then $\|f\|_r \le A$.

Solution. ► ◄

Problem 6 (Wheeden & Zygmund Ch. 8, Ex. 9)**.** If f is real-valued and measurable on E, $|E| > 0$, define its essential infimum on E by

$$\operatorname{ess\,inf} f = \sup\{\,\alpha : |\{\,x \in E : f(x) < \alpha\,\}| = 0\,\}\,.$$

If $f \ge 0$, show that $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup} 1/f)^{-1}$.

Solution. ► ◄

Problem 7 (Wheeden & Zygmund Ch. 8, Ex. 11)**.** If $f_k \to f$ in L^p, $1 \le p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty < M$ for all k, prove that $f_k g_k \to fg$ in L^p.

Solution. ► ◄

1.2 Exam Preparation

1.2.1 Exam 1 Practice

Problem 1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{ r\mathbf{x} : \mathbf{x} \in E \}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Define a map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by $T\mathbf{x} = r\mathbf{x}$. Note that T is *Lipschitz continuous* since for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the equality

$$|T\mathbf{x} - T\mathbf{y}| = |r\mathbf{x} - r\mathbf{y}| = |r||\mathbf{x} - \mathbf{y}| \tag{1}$$

is satisfied. By Theorem 3.33 from [5, Ch. 3, p.55], the image of E under T is measurable. Moreover, by Theorem 3.35 [5, Ch. 3, p. 56], since T is linear, it follows that $|T(E)| = |\det T||E|$ where $\det T = |r|^n$. Lastly, we note that the image of E under T is precisely the set rE so that $|T(E)| = |rE| = |r|^n|E|$, as was to be shown. ■

Problem 2. Let $\{E_k\}$, $k \in \mathbf{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k\to\infty} E_k \right| \le \liminf_{k\to\infty} |E_k| .$$

Proof. Following the style of [5, Ch. 1, p. 2], particularly, the sets defined after the introduction of equation (1.1), set

$$V_k = \bigcap_{\ell=k}^{\infty} E_\ell. \tag{2}$$

Note that the collection of sets $\{V_k\}$ forms an increasing sequence, that is, if $\mathbf{x} \in V_k$ then, by (2), $\mathbf{x}$ is in the intersection $E_k \cap \left(\bigcap_{\ell=k+1} E_\ell\right)$, but, by (2), $\bigcap_{\ell=k+1} E_\ell = V_{k+1}$ thus, $\mathbf{x}$ is in V_{k+1} so $V_{k+1} \supset V_k$. Hence, we have $V_k \nearrow \liminf E_k$.

Now, consider the sequence $\{|V_k|\}$ formed by the Lebesgue measure of the V_k. By Theorem 3.26 from [5, Ch. 3, p. 51], since $V_k \nearrow \liminf E_k$,

$$\lim_{k\to\infty} |V_k| = \lim_{k\to\infty} \left| \bigcap_{\ell=k}^{\infty} E_\ell \right| = \left| \liminf_{k\to\infty} E_k \right| . \tag{3}$$

But note that, by the monotonicity of the Lebesgue measure, we have

$$\left| \bigcap_{\ell=k}^{\infty} E_\ell \right| \le |E_k|, \tag{4}$$

so, by properties of the lim inf, in particular, by Theorem 19(v) from [2, Ch. 1, p. 23], we have

$$\limsup_{k\to\infty} |V_k| \le \liminf_{k\to\infty} |E_k|. \tag{5}$$

Hence, by (3) and Proposition 19 (iv), since the sequence $\{|V_k|\}$ converges and is equal to the measure of $\liminf E_k$, by (5), we have

$$\left|\liminf_{k\to\infty} E_k\right| \le \liminf_{k\to\infty} |E_k| \tag{6}$$

as was to be shown. ■

Problem 3. Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) = \{\, \mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r \,\}$. Prove that F is monotonic increasing and continuous.

Proof. Define the linear map $T\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ by $T(r)\mathbf{x} = r\mathbf{x}$. We claim that $B(\mathbf{0}, r) = T(r, B(\mathbf{0}, 1))$. To reduce notation, set $B_1 = B(\mathbf{0}, 1)$ and $B_r = B(\mathbf{0}, r)$.

Proof of claim. $\subset$: Let $\mathbf{x} \in B_r$. Then $|\mathbf{x}| < r$ so $|\mathbf{x}|/r < 1$. Thus, $|\mathbf{x}|/r \in B_1$ so it is in the image of B_1 under the map $T(r, \cdot)$.

$\supset$: On the other hand, suppose $\mathbf{x} \in T(r, B_1)$. Then $\mathbf{x} = r\mathbf{y}$ for some $\mathbf{y} \in B_1$. Then, since $|\mathbf{y}| < 1$, $|\mathbf{x}| = r|\mathbf{y}| < r$ so $\mathbf{x} \in B_r$. ♣

From the claim, we see that $F(x) = |T(x, B(\mathbf{0}, 1))|$ which, by Problem 1, is nothing more that the polynomial $|B_1|x^n$. It is clear, from this equivalence, that F is monotonically increasing: Take $x, y \in [0, \infty)$ such that $x < y$, then $x^n < y^n$ so

$$F(x) = |B_1|x^n < |B_1|y^n = F(y). \tag{7}$$

Thus, F is monotonically increasing.

In the argument above, since $F(x) = |B_1|x^n$ is a polynomial in $[0, \infty)$ (and polynomials are continuous on $\mathbb{R}$) F is continuous on $[0, \infty)$. ■

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. (Without much motivation) let us consider the collection of sets $\{E_k\}$ defined by

$$E_k = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } y, z \in B(x, \delta) \text{ implies } |f(y) - f(z)| < \frac{1}{k} \right\}. \tag{8}$$

We claim that $C = \bigcap_{k=1}^{\infty} E_k$ and that each E_k is open.

Proof of claim. First, we demonstrate equality. $\subset$: Suppose $x \in C$. Then, by the definition of continuity, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $y \in B(x, \delta)$ implies $|f(x) - f(y)| < \delta$. In particular, for every k, there exists $\delta > 0$ such that for $y \in B(x, \delta)$ the inequality $|f(x) - f(y)| < 1/k$ holds. Thus, x is in $\bigcap_{k=1}^{\infty} E_k$.

$\supset$: On the other hand, suppose that $x \in \bigcap_{k=1}^{\infty} E_k$. Then, given $\varepsilon > 0$, by the Archimedean property, there exists a positive integer N such that $1/N < \varepsilon$. Then, since $x \in \bigcap_{k=1}^{\infty} E_k$, $x \in E_N$ so

$$|f(x) - f(y)| < \frac{1}{N} < \varepsilon. \tag{9}$$

Thus, x is in C and $C = \bigcap_{k=1}^{\infty} E_k$.

All that remains to be shown is that the E_k are open. But this is clear by the way we defined E_k in (8): Let $x \in E_k$, then there exists $\delta > 0$ such that for any $y, z \in B(x, \delta)$, $|f(y) - f(z)| < 1/k$; Let $x' \in B(x, \delta)$ and set $\delta' = \min\{|(x+\delta) - x'|, |(x-\delta) - x|\}$. Then, since $B(x', \delta') \subset B(x, \delta)$, for every $y, z \in B(x', \delta')$, we have $|f(y) - f(z)| < 1/k$. Hence, $x' \in E_k$ for any $x' \in B(x, \delta)$ so $B(x, \delta) \subset E_k$. ♣

Since C can be expressed as the countable intersection of open sets E_k, it follows that C is a G_δ set. ■

Problem 5. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Proof. If $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, it is not necessarily the case that f is measurable. Consider the following construction: Let $E \subset (0,1)$ be an unmeasurable set.* Define a map $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{R} \setminus ((0,1) \setminus E), \\ x+1 & \text{if } x \in (0,1) \setminus E. \end{cases} \tag{10}$$

By the definition, it is clear that $\{ f = r \}$ is measurable and $|\{ f = r \}| = 0$ since $\{ f = r \}$ contains at most two elements. However, the set $\{ 0 < f < 1 \} = E$ is not measurable. Thus, f is not measurable. ■

Problem 6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set $\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \}$ is measurable.

Proof. By Theorem 4.12 from [5, Ch. 4, p. 67], $\liminf_{k\to\infty} f_k$ and $\limsup_{k\to\infty} f_k$ are measurable. By Theorem 4.7 from [5, Ch. 4, p. 66]

$$\left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\} \tag{11}$$

is measurable. Since

$$\left\{ \lim_{k\to\infty} f_k \text{ exists} \right\} = \left\{ \limsup_{k\to\infty} f_k = \liminf_{k\to\infty} f_k \right\} = \mathbb{R} \setminus \left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\}, \tag{12}$$

by Theorem 3.17 from [5, Ch. 3, p. 48], the set $\{ \lim_{k\to\infty} f_k \text{ exists} \}$ is measurable. ■

*It's construction does not concern us. The interested reader such direct their refer to Theorem 3.38 from [5, Ch. 3, p. 57-58] or Theorem 17 from [2, Ch. 2§7, p. 48].

Problem 7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. Suppose f is absolutely continuous on $[a, b]$. Let $\varepsilon = 1$. Then, there exists $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < 1$. Let $N = \lceil (b-a)/\delta \rceil$, that is, N is the smallest integer greater than $(b - a)/\delta$, and consider the partition $\Gamma = \{x_k\}$ where $x_k = a + k(b - a)/N$, for $k = 0, \dots, N$. Then $x_k - x_{k-1} < (b - a)/N < \delta$ so, by Theorem 2.2(i) from [5, Ch. 2, p. 19], we have $V[f; x_{k-1}, x_k] < 1$ for $k = 0, \dots, N$. In follows by Theorem 2.2(ii) that

$$V[f; a, b] = \sum_{k=1}^{N} V[f; x_{k-1}, x_k] < N. \tag{13}$$

Thus, f is b.v. on $[a, b]$. ∎

Problem 8. Let f be a continuous function from $[a, b]$ into $\mathbb{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, that is, $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b), \\ -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

Proof. The result follows quite easily from more sophisticated measure theoretic arguments. At this point, however, such language has not been discussed so we shall prove this using nothing but the definition of the Riemann–Stieltjes integral and properties thereof.

Let us consider each case $c \in (a, b)$, $c = a$, and $c = b$ separately.

Recall that the given a partition $\Gamma = \{x_0, \dots, x_m\}$ of $[a, b]$, the Riemann–Stieltjes sum of f with respect to φ is

$$R_\Gamma = \sum_{k=1}^{m} f(\xi_k)[\varphi(x_k) - \varphi(x_{k-1})]. \tag{14}$$

The Riemann–Stieltjes integral is defined as the limit

$$\int_a^b f \,\mathrm{d}\varphi = \lim_{|\Gamma| \to 0} R_\Gamma \tag{15}$$

if it exists.

Suppose $c \in (a, b)$. Then, for any partition Γ of $[a, b]$, either $c \in \Gamma$ or $c \notin \Gamma$. In the latter case, $R_\Gamma = 0$. In the former case c is one of the x_k, say $c = x_\ell$ for $0 < \ell < m$. Then

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^{m} f(\xi_k)[\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= 0 + \dots + 0 + f(\xi_{\ell-1}) - f(\xi_\ell) + 0 + \dots + 0 \\ &= f(\xi_{\ell-1}) - f(\xi_\ell). \end{aligned} \tag{16}$$

Since f is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi_\ell - \xi_{\ell-1}| < \delta$ implies $|f(\xi_\ell) - f(\xi_{\ell-1})| < \varepsilon$. It follows that the quantity in (16) approaches 0 as $|\Gamma|$ approaches 0. Therefore, $\int_a^b f \, \mathrm{d}\chi_{\{c\}} = 0$.

Suppose $c = a$. Then, since any partition Γ of $[a, b]$ must contain the point a, we have

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^{m} f(\chi_k)[\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= f(\xi_1)[\chi_{\{c\}}(x_1) - \chi_{\{c\}}(x_0)] + f(\xi_2)[\chi_{\{c\}}(x_2) - \chi_{\{c\}}(x_1)] \\ &\qquad\qquad + \cdots + f(\xi_m)[\chi_{\{c\}}(x_m) - \chi_{\{c\}}(x_{m-1})] \\ &= -f(\xi_1) + 0 + \cdots + 0 \\ &= -f(\xi_1) \end{aligned} \tag{17}$$

Taking the limit as $|\Gamma| \to 0$, $\xi_1 \to a$ so, by continuity of f, $f(\xi_1) \to f(a)$. Thus, $\int_a^b f \, \mathrm{d}\chi_{\{c\}} = -f(a)$.

A similar argument to the one above shows that, if $c = b$, the Riemann–Stieltjes integral $\int_a^b f \, \mathrm{d}\chi_{\{c\}} = f(b)$. ■

1.2.2 Exam 1

Problem 1.

Proof. ■

Problem 2.

Proof. ■

Problem 3.

(i) Show that if $B_r = \{ \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < r \}$, then there exists a constant C such that $|B_r| = Cr^n$.

(*Hint*: Think of B_r as $\{ r\mathbf{x} : \mathbf{x} \in B_1 \}$.)

(ii) Let $E \subset \mathbb{R}^n$ be a measurable set and let $\varphi_E \colon \mathbb{R}^n \to \mathbb{R}$ be defined $\varphi_E(\mathbf{x}) = \left|E \cap B_{|\mathbf{x}|}\right|$. Use part (i) to prove that φ_E is continuous.

Proof. (i) To prove this result, we use the map constructed in Problem 1 of the review sheet for Exam 1, the map $T \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Set $T_r \colon \mathbb{R}^n \to \mathbb{R}^n$ to be $T_r = T(r)$. Then, we claim $B_r = T_r(B_1)$ and $|B_r| = |T_r(B_1)|$, which, as we saw in Problem 1 of the review sheet, has measure $|B_1||r|^n$. Setting $C = |B_1|$, we have $|B_r| = C|r|^n$ as desired.

(ii) To prove that φ_E is continuous, we provide an (ε, δ)-argument. Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that $\mathbf{y} \in B(\mathbf{x}, \delta)$ implies

$$|\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| < \varepsilon. \tag{18}$$

First, note that since $\mathbf{x} \mapsto |\mathbf{x}|$ is continuous and polynomials $p \colon \mathbb{R}^n \to \mathbb{R}^n$ are continuous, then the composition $\mathbf{x} \mapsto |\mathbf{x}|^n$ is continuous. Therefore, there exists $\delta > 0$ such that $\mathbf{y} \in B(\mathbf{x}, \delta)$ implies

$$||\mathbf{x}|^n - |\mathbf{y}|^n| < \frac{\varepsilon}{C}, \tag{19}$$

where $C = |B_1|$.

Now, let $x \in \mathbb{R}^n$ and $\mathbf{y} \in B(\mathbf{x}, \delta)$ as above. Then, by (19) we have

$$\begin{aligned} |\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| &= \left|\left|E \cap B_{|\mathbf{x}|}\right| - \left|E \cap B_{|\mathbf{y}|}\right|\right| \\ &\leq \left||B_{|\mathbf{x}|}| - |B_{|\mathbf{y}|}|\right| \\ &= C|\,|\mathbf{x}|^n - |\mathbf{y}|^n| \\ &\leq C\left[\frac{\varepsilon}{C}\right] \\ &= \varepsilon. \end{aligned} \tag{20}$$

It follows that φ_E is continuous. ■

Problem 4. Assume that $f\colon [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$. Prove that f is measurable.

Proof. By Jordan's theorem (Corollary 2.7 from [5, Ch. 2, p. 21]), the function f is of bounded variation on $[a,b]$ if and only if it can be written as the difference $f_1 - f_2$ of two bounded functions f_1 and f_2 that are monotone increasing on $[a,b]$. Then, f_1 and f_2 are continuous a.e. on $[a,b]$ and hence, are measurable. ■

1.2.3 Exam 2 Practice Problems

Problem 1. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B(\mathbf{0}, \varepsilon)$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \smallsetminus B(\mathbf{0},\varepsilon)} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f \colon \mathbb{R}^n \to \mathbb{R}$ is Lebesgueintegrable on a subset E of $\mathbb{R}^n$ if

$$\int_E f(\mathbf{x})\,\mathrm{d}\mathbf{x} < \infty. \tag{21}$$

Let f be as given in the statement of the problem and set $B_\varepsilon = B(\mathbf{0}, \varepsilon)$. Consider the change of variables to *hyperspherical coordinates* $(x_1, \ldots, x_n) \mapsto (r, \Theta)$ where $\Theta = (\theta_1, \ldots, \theta_{n-1})$.† By Theorem 7.26(iii) from [4, Ch. 7, p. 123], we have

$$\begin{aligned} \int_{\mathbb{R}^n \smallsetminus B_\varepsilon} f(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^n \smallsetminus B_\varepsilon} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n \smallsetminus B_\varepsilon} \frac{1}{|\mathbf{x}|^{n+1}}\,\mathrm{d}\mathbf{x}. \\ &= \int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}}\,\mathrm{d}r dV, \end{aligned} \tag{22}$$

where S_r^{n-1} is the $(n-1)$-sphere centered at $\mathbf{0}$ with radius r, that is, the subset $\{\, \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1 \,\}$ of $\mathbb{R}^n$ and dV is the *volume element* of S_r^{n-1}. Since $1/|r|^{n+1}$ is nonnegative, by Tonelli's theorem the the iterated integrals in (22) may be exchange, that is,

$$\int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}}\,\mathrm{d}r dV = \int_\varepsilon^\infty \left(\int_{S_r^{n-1}} 1\,\mathrm{d}V \right) \frac{1}{|r|^{n+1}}\,\mathrm{d}r. \tag{23}$$

Now, note that from Problem 1 of the review sheet for Exam 1, we have

$$\int_{S_r^{n-1}} 1\,\mathrm{d}V = |S_r^{n-1}|_{\mathbb{R}^{n-1}} = |S^{n-1}|_{\mathbb{R}^{n-1}} |r|^{n-1}. \tag{24}$$

Set $C = |S^{n-1}|_{\mathbb{R}^{n-1}}$. Putting equations (22), (23), and (24) together, we have

$$\begin{aligned} \int_{\mathbb{R}^n \smallsetminus B_\varepsilon} f(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_\varepsilon^\infty C|r|^{n-1} \frac{1}{|r|^{n+1}}\,\mathrm{d}r \\ &= \int_\varepsilon^\infty \frac{C}{|r|^2}\,\mathrm{d}r \\ &= \lim_{x \to \infty} \left[-\frac{C}{x} - \left(-\frac{C}{\varepsilon} \right) \right] \\ &= \frac{C}{\varepsilon}, \end{aligned} \tag{25}$$

†The explicit construction of the map $(x_1, \ldots, x_n) \mapsto (r, \Theta)$ is of no concern to us for now. What is important is that it exists.

as was to be shown. $\blacksquare$

Problem 2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. Let $E \subset \mathbb{R}^n$ be a measurable subset of $\mathbb{R}^n$. Then, since $f_k \to f$ pointwise a.e. on $\mathbb{R}^n$, then $f_k \to f$ pointwise a.e. on E and $\mathbb{R}^n \setminus E$. To prove that the limit of the sequence of integrals $\left\{\int_E f_k\right\}$ exist and is equal to $\int_E f$, it suffices to prove that

$$\int_E f \le \liminf_{k\to\infty} \int_E f_k \le \limsup_{k\to\infty} \int_E f_k \le \int_E f. \tag{26}$$

The lower bound in (26) follows from an application of Fatou's lemma:

$$\int_E f = \int_E \liminf_{k\to\infty} f \le \liminf_{k\to\infty} \int_E f_k. \tag{27}$$

Also by Fatou's lemma, we have

$$\int_{\mathbb{R}^n\setminus E} f = \int_{\mathbb{R}^n\setminus E} \liminf_{k\to\infty} f \le \liminf_{k\to\infty} \int_{\mathbb{R}^n\setminus E} f_k. \tag{28}$$

Now, since $f \in L^1(\mathbb{R}^n)$, by equation (28) and properties of the $\liminf$ and $\limsup$[‡] we have

$$\begin{aligned}
\int_E f = \int_{\mathbb{R}^n} f - \int_{\mathbb{R}^n\setminus E} f &\ge \limsup_{k\to\infty} \int_{\mathbb{R}^n} f - \liminf_{k\to\infty} \int_{\mathbb{R}^n\setminus E} f_k \\
&\ge \limsup_{k\to\infty} \int_{\mathbb{R}^n} f_k - \limsup_{k\to\infty} \int_{\mathbb{R}^n\setminus E} f_k \\
&= \limsup_{k\to\infty} \left[\int_{\mathbb{R}^n} f_k - \int_{\mathbb{R}^n\setminus E} f_k\right] \\
&= \limsup_{k\to\infty} \int_E f_k.
\end{aligned} \tag{29}$$

By equations (27) and (29) it follows that $\lim_{k\to\infty} \int_E f_k$ exists and is equal to $\int_E f$.

[‡]Namely, for any sequence of positive real numbers $\{a_k\}$ the inequality $\liminf a_k \le \limsup a_k$ holds

To see that the result need not be true if $\int_E f = \infty$, consider the following example: Let $f_k\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} k^2/2 & \text{if } x \in (-1/k, 1/k), \\ 1 & \text{otherwise} \end{cases} \tag{30}$$

and $f = 1$.

It is easy to see that $f_k \to f$ a.e. in $\mathbb{R}$ and that both $\int_{\mathbb{R}} f = \infty$ and $\lim_{k\to\infty} f_k = \infty$. However, if $E = (-1,1)$ then $\int_E f = 1$, but $\lim_{k\to\infty} \int_E f = \infty$. ■

Problem 3. Assume that E is a measurable set of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of $\mathbb{R}^n$, then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \tag{31}$$

Set $E_k = \{\, \mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k \,\}$ and $F_k = \{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \smallsetminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{32}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \tag{33}$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \tag{34}$$

But note that $|E_k| = |F_k \smallsetminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (34) becomes

$$\sum_{k=0}^{\infty} k\left(|F_k| - |F_{k+1}|\right) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)\left(|F_k| - |F_{k+1}|\right). \tag{35}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k\left(|F_k| - |F_{k+1}|\right) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty}(k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (36) \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty}(k+1)\left(|F_k| - |F_{k+1}|\right) &= \sum_{k=0}^{\infty}(k+1)|F_k| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty}(k+1)|F_k| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty}(k+2)|F_{k+1}| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \qquad (37) \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned}$$

Thus, from (36) and (37)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x})d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \qquad (38)$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ■

Problem 4. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f,g) = \int_E \frac{|f-g|}{1+|f-g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. ■

Problem 5. Define the *gamma function* $\Gamma\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1}\, \mathrm{d}u,$$

and the *beta function* $\beta\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. ■

Problem 6. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}}\colon \mathbb{R}^n \to \mathbb{R}$ be $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \to \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. ■

Problem 7. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \le g_k$ and

$$\int_{\mathbb{R}^n} g_k \longrightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \qquad \text{as } k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \longrightarrow \int_{\mathbb{R}^n} |f| \qquad \text{as } k \to \infty.$$

Proof. (a) $\implies$ (b): Assume part (a) then $\implies$ if

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \tag{39}$$

as $k \to \infty$, we have

(b): ■

1.2.4 Exam 2 (2010)

Problem 1. Suppose $f \in L^1(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Hint: Use the monotone convergence theorem.

Proof. ∎

Problem 2. (a) Prove the following generalization of *Chebyshev's inequality*: Let $0 < p < \infty$ and $E \subset \mathbb{R}^n$ be measurable. assume that $|f|^p \in L^1(E)$. Then

$$|\{ x \in E : f(\mathbf{x}) > \alpha \}| \leq \frac{1}{\alpha^p} \int_{\{ f > \alpha \}} f^p,$$

for $\alpha > 0$.

(b) Let p, E, and f be as in part (a). In addition, assume that $\{f_k\}$ is a sequence such that $\int_E |f_k - f|^p \to 0$ as $k \to \infty$. Show that $f_k \to f$ in measure on E.

Recall that $f_k \to f$ in measure on E if and only if for every $\varepsilon > 0$

$$\lim_{k \to \infty} |\{ \mathbf{x} \in E : |f_k(\mathbf{x}) - f(\mathbf{x})| > \varepsilon \}| = 0.$$

Proof. ∎

Problem 3. Let $f \in L^1(\mathbb{R})$, and define

$$F(\xi) = \int_{\mathbb{R}} f(x) \cos(2\pi x \xi) \, \mathrm{d}x.$$

Prove that F is continuous and bounded on $\mathbb{R}$.

Proof. ∎

Problem 4. Use repeated integration techniques to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} \, \mathrm{d}\mathbf{x} = \pi^{n/2}.$$

Hint: Start from the case $n = 1$ by using the polar coordinates in

$$\left[\int_{\mathbb{R}} e^{-x^2} \, \mathrm{d}x \right]^2 = \left[\int_{\mathbb{R}} e^{-x^2} \, \mathrm{d}x \right] \left[\int_{\mathbb{R}} e^{-x^2} \, \mathrm{d}y \right]$$

Proof. ■

Problem 5.

Proof. ■

1.2.5 Exam 2

Problem 1. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Proof. ■

Problem 2. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. ■

Problem 3. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbf{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k \to \infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Proof. ■

Problem 4. Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)\, \mathrm{d}t$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Proof. ■

1.2.6 Final Exam Practice Problems

Problem 1. Suppose $f \in L^1(\mathbb{R}^n)$ and that x is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) = \frac{1}{|r|^n} \int_{B(0,r)} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| \,\mathrm{d}\mathbf{y}.$$

Show that:

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$;
(b) there exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

Proof. (a) Without loss of generality, we may assume $r < s$. Then, we want to show that as $r \to s$, the quantity

$$|A(s) - A(r)| \longrightarrow 0.$$

Set $F(\mathbf{y}) = |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})|$ and consider said quantity

$$\begin{aligned}
|A(s) - A(r)| &= \left| \frac{1}{|s|^n} \int_{B_s} F(\mathbf{y}) \,\mathrm{d}\mathbf{y} - \frac{1}{|r|^n} \int_{B_r} F(\mathbf{y}) \,\mathrm{d}\mathbf{y} \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) \,\mathrm{d}\mathbf{y} + \frac{1}{|s|^n} \int_{B_r} F(\mathbf{y}) \,\mathrm{d}\mathbf{y} - \frac{1}{|r|^n} \int_{B_r} F(\mathbf{y}) \,\mathrm{d}\mathbf{y} \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) \,\mathrm{d}\mathbf{y} + \left(\frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(\mathbf{y}) \,\mathrm{d}\mathbf{y} \right| \\
&\leq \underbrace{\frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) \,\mathrm{d}\mathbf{y}}_{I_1} + \underbrace{\left(\frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(\mathbf{y}) \,\mathrm{d}\mathbf{y}}_{I_2}.
\end{aligned}$$

Hence, we must show that the quantities $I_1, I_2 \to 0$ as $r \to s$.

To see that $A(r) \to 0$ as $r \to 0$, note that x is a point of the Lebesgue set of f and that

$$0 = \lim_{B_r \searrow \mathbf{x}} \frac{1}{|B_1||r|^n} \int_{B_r} |f(\mathbf{y}) - f(\mathbf{x})| \,\mathrm{d}\mathbf{y} = \frac{1}{|B_1|} \lim_{B_r \searrow \mathbf{x}} \frac{1}{|r|^n} \int_{B_r} |f(\mathbf{t}) - f(\mathbf{x})| \,\mathrm{d}\mathbf{t} = \lim_{r \to 0} A(r).$$

by making the change of variables $\mathbf{t} = \mathbf{x} - \mathbf{y}$.

(b) ∎

Problem 2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \leq n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0$$

if and only if

$$\|f_k\|_p \longrightarrow \|f\|_p$$

as $k \to \infty$.

Proof. ■

Problem 3. Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

Proof. ■

Problem 4. Let $f \in L(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x)\cos(tx)dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Proof. ■

Problem 5. Let f be of bounded variation on $[a,b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular. Show that

$$\int_a^b \varphi \, \mathrm{d}f = \int_a^b \varphi f' dx + \int_a^b \varphi \, \mathrm{d}h$$

for all functions φ continuous on $[a,b]$.

Proof. ■

1.2.7 Final Exam 2010

Problem 1. Suppose that $f \in L^1(\mathbb{R}^n)$, and that $\mathbf{x}$ is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) = \frac{1}{r^n} \int_{B_r} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{y},$$

where $B_r = B(\mathbf{0}, r)$.

Show that

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$.
(b) There exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

Proof. (a)

(b) ■

Problem 2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \leq p < \infty$. assume that $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0 \iff \|f_k\|_p \longrightarrow \|f\|_p$$

Hint: To prove one of the implications, you can use the following fact without proving it:

$$\left|\frac{a-b}{2}\right| \leq \frac{|a|^p + |b|^p}{2}$$

for all $a, b \in \mathbb{R}$.

Proof. ■

Problem 3. Let $0 < p < q < r \leq \infty$, $E \subset \mathbb{R}^n$ be a measurable set. Show that each $f \in L^q(E)$ is the sum of a function $g \in L^p(E)$ and a function $h \in L^r(E)$.

Proof. ■

Problem 4. Prove that $f \colon [a, b] \to \mathbb{R}$ is Lipschitz continuous if and only if f is absolutely continuous and there exists a constant $M > 0$ such that $|f'| < M$ a.e. on $[a, b]$.

Proof. ■

Problem 5. Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ or $p = \infty$?.

Proof. ■

1.2.8 Final Exam

2 MA 544 Past Quals

2.1 Danielli: Winter 2012

Problem 1. Let $f(x, y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $\partial f(x, y)/\partial x$ is a bounded function of (x, y). Prove that $\partial f(x, y)/\partial x$ is a measurable function of y for each x and

$$\frac{d}{dx} \int_0^1 f(x, y)\, \mathrm{d}y = \int_0^1 \frac{\partial f(x, y)}{\partial x}\, \mathrm{d}y.$$

Proof. ■

Problem 2. Let f be a function of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular, show that

$$\int_a^b \varphi\, \mathrm{d}f = \int_a^b \varphi f'\, \mathrm{d}x + \int_a^b \varphi\, \mathrm{d}h.$$

Hint: A function h is said to be singular if $h' = 0$.

Proof. ■

Problem 3. Let $E \subset \mathbb{R}$ be a measurable set, and let K be a measurable function on $E \times E$. Assume that there exists a positive constant C such that

$$\int_E K(x, y)\, \mathrm{d}x \leq C \tag{40}$$

for a.e. $y \in E$, and

$$\int_E K(x, y)\, \mathrm{d}y \leq C \tag{41}$$

for a.e. $x \in E$.

Let $1 < p < \infty$, $f \in L^p(E)$, and define

$$T_f(x) = \int_E K(x, y) f(y)\, \mathrm{d}y.$$

(a) Prove that $T_f \in L^p(E)$ and

$$\|T_f\|_p \leq C\|f\|_p. \tag{42}$$

(b) Is (42) still valid if $p = 1$ or ∞? If so, are assumptions (40) and (41) needed?

Proof. ■

Problem 4. Let f be a nonnegative measurable function on $[0,1]$ satisfying

$$|\{\, x \in [0,1] : f(x) > \alpha \,\}| < \frac{1}{1+\alpha^2} \tag{43}$$

for $\alpha > 0$.

(a) Determine values of $p \in [1, \infty)$ for which $f \in L^p[0,1]$.
(b) If p_0 is the minimum value of p for which p may fail to be in L^p, give an example of a function which satisfies (43), but which is not in $L^{p_0}[0,1]$.

Proof. ■

2.2 Danielli: Summer 2011

Problem 1. Let $f \in L^1(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x)\cos(tx)\,\mathrm{d}x$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Hint: Start by proving the statement for $f = \chi_{[a,b]}$.

Proof. ∎

Problem 2. (a) Suppose that $f_k, f \in L^2(E)$, with E a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E fg \tag{44}$$

as $k \to \infty$ for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \to \|f\|_2$ show that f_k converges to f in L^2, i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as $k \to \infty$.

(b) Provide an example of a sequence f_k in L^2 and a function f in L^2 satisfying (44), but such that f_k does *not* converge to f in L^2.

Proof. ∎

Problem 3. A bounded function f is said to be of bounded variation on $\mathbb{R}$ if it is of bounded variation on any finite subinterval $[a, b]$, and moreover $A = \sup_{a,b} V[a, b; f] < \infty$. Here, $V[a, b; f]$ denotes the total variation of f over the interval $[a, b]$. Show that:

(a) $\displaystyle\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x \le A|h|$ for all $h \in \mathbb{R}$.

Hint: For $h > 0$, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,\mathrm{d}x.$$

(b) $\displaystyle\left|\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x\right| \le A$, where φ is any function of class C^1, of bounded variation, compactly supported, with $\sup_{x\in\mathbb{R}} |\varphi(x)| \le 1$.

Proof. ∎

Problem 4. (a) Prove the *generalized Hölder's inequality*: Assume $1 \le p \le \infty$, $j = 1, \dots, n$, with $\sum_{j=1}^{\infty} 1/p_j = 1/r \le 1$. If E is a measurable set and $f_j \in L^{p_j}(E)$ for $j = 1, \dots, n$, then $\prod_{j=1}^{n} f_j \in L^r(E)$ and

$$\|f_1 \cdots f_n\|_r \le \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if $1 \le p, q, r \le \infty$, with $1/p + 1/q = 1/r + 1$, $f \in L^p(\mathbb{R})$, and $g \in L^p(\mathbb{R})$, then

$$|(f * g)(x)| \le \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \int |f(y)|^p |g(x-y)|^q \, \mathrm{d}y.$$

(Recall that $(f * g)(x) = \int f(y) g(x-y) \, \mathrm{d}y$.)

(c) Prove *Young's convolution theorem*: Assume that p, q, r, f, and g are as in part (b). Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Proof. ■

3 Bañuelos: Summer 2000

Problem 1. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions with the property that for all $n \geq 1$

$$\mu\left(\{\, x \in X : |f_n(x)| \geq \lambda \,\}\right) \leq C \exp(-\lambda^2/n)$$

for all $\lambda > 0$. (Here C is a constant independent of n.) Let $n_k = 2^k$. Prove that

$$\limsup_{k\to\infty} \frac{|f_{n_k}|}{\sqrt{n_k \log(\log(n_k))}} \leq 1 \quad \text{a.e.}$$

Solution. ▶ Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions such that

$$\mu\left(\{\, x \in X : |f_n(x)| \geq \lambda \,\}\right) \leq C \exp(-\lambda^2/n) \tag{1}$$

for all λ. Now, consider the subsequence $\{f_{2^k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$. We aim to show that

$$\limsup_{k\to\infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} \leq 1$$

almost everywhere. To that end, it suffices to show that the set

$$E = \left\{ x \in X : \limsup_{k\to\infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} > 1 \right\}$$

has measure zero. Let $x \in E$ then

$$\limsup_{k\to\infty} \frac{|f_{2^k}(x)|}{\sqrt{2^k \log(\log(2^k))}} > 1.$$

This means that there exists some subsequence $\{k_m\}_{m=1}^{\infty} \subset \{k\}_{n=1}^{\infty}$ such that

$$\lim_{m\to\infty} \frac{|f_{2^{k_m}}(x)|}{\sqrt{2^{k_m} \log(\log(2^{k_m}))}} > 1.$$

This means that, for sufficiently large N

$$|f_{2^{k_n}}(x)| > \sqrt{2^{k_n} \log(\log(2^{k_n}))}$$

for all $n \geq N$. But by Equation (1) we have

$$\begin{aligned}
\mu\left(\left\{ x \in X : \frac{|f_{2^{k_n}}(x)|}{\sqrt{2^{k_n} \log(\log(2^{k_n}))}} \geq 1 \right\}\right) &\leq C \exp\left(- \left(\sqrt{2^{k_n} \log(\log(2^{k_n}))} \right)^2 \Big/ \, 2^{k_n} \right) \\
&= C \exp\left(- \, 2^{k_n} \log(\log(2^{k_n}))/\, 2^{k_n}\right) \\
&= C \exp\left(- \log(\log(2^{k_n}))\right) \\
&= C \exp\left(\log(1/\log(2^{k_n}))\right) \\
&= \frac{C}{\log(2^{k_n})}.
\end{aligned} \tag{2}$$

Letting $n \to \infty$, we see that the measure of the set on the left-hand side of Equation (2) must go to 0 so $\mu(E) = 0$. ◀

Problem 2. Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let f_n be a sequence of measurable functions with $f_1 \in L^1(\mu)$ and with the property that

$$\mu(\{ x \in X : |f_n(x)| > \lambda \}) \leq \mu(\{ x \in X : |f_1(x)| > \lambda \})$$

for all n and all $\lambda > 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_X \left[\max_{1\leq j\leq n} |f_j| \right] \mathrm{d}\mu = 0.$$

[*Hint*: You may use the fact that $\|f\|_1 = \int_0^\infty \mu(\{ |f(x)| > \lambda \})\, \mathrm{d}\lambda$.]

Solution. ▶ Define $g_n, h_n \colon \mathcal{F} \to [0, \infty]$ for $n \in \mathbb{N}$ by

$$g_n(\lambda) = \mu(\{ x \in X : |f_n(x)| > \lambda \}), \qquad h_n(\lambda) = \mu\left(\left\{ x \in X : \max_{1\leq i\leq n} |f_i(x)| > \lambda \right\} \right).$$

Now, note that, by the monotonicity of μ, we have

$$h_n(\lambda) \leq \sum_{i=1}^{n} g_n(\lambda) \leq n g_1(\lambda).$$

Thus,

$$\frac{h_n(\lambda)}{n} \leq g_1(\lambda).$$

Since $\|f_1\|_1 = \int_0^\infty g_1(\lambda)\, \mathrm{d}\lambda$, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \int_X \left[\max_{1\leq j\leq n} |f_j| \right] \mathrm{d}\mu &= \lim_{n\to\infty} \int_X \frac{h_n(x)}{n} \,\mathrm{d}\mu \\
&= \int_X \lim_{n\to\infty} \frac{h_n(x)}{n} \,\mathrm{d}\mu \\
&\leq \int_X \lim_{n\to\infty} \frac{\mu(X)}{n} \\
&= 0
\end{aligned}$$

as we wanted to show. ◀

Problem 3.

(i) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions. Prove that $f_n \to f$ is measurable if and only if every subsequence $\{f_{n_k}\}$ contains a further subsequence $\{f_{n_{k_j}}\}$ that converges a.e. to f.

(ii) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $F \colon \mathbb{R} \to \mathbb{R}$ be continuous and $f_n \to f$ in measure. Prove that $F(f_n) \to F(f)$ in measure. (You may assume, of course, that f_n, F, $F(f_n)$, and $F(f)$ are all measurable.)

Solution. ► Recall that a sequence of measurable functions $\{f_n\}$ converge in measure to a limit f if for every $\varepsilon > 0$ the limit

$$\lim_{n\to\infty} \mu(\{ x \in X : |f(x) - f_n(x)| \geq \varepsilon \}) = 0.$$

For part (i) $\Longleftarrow$ suppose that every subsequence $\{f_{n_k}\}$ contains a subsequence $\{f_{n_{k_\ell}}\}$ that converges almost everywhere to f. ◄

Problem 4. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and suppose $f \in L^1(\mu)$ is nonnegative. Suppose $1 < p < \infty$ and let $1 < q < \infty$ be its conjugate exponent, i.e., $1/p + 1/q = 1$. Suppose f has the property that

$$\int_E f \, \mathrm{d}\mu \leq \mu(E)^{1/q}$$

for all measurable sets E. Prove that $f \in L^r(\mu)$ for any $1 \leq r < p$.

[*Hint*: Consider $\left\{ x \in X : 2^n \leq f(x) < 2^{n+1} \right\}$.]

Solution. ► ◄

Problem 5. Let f be a continuous function on $[-1, 1]$. Find

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|) \, \mathrm{d}x.$$

Solution. ► ◄

Problem 6. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $f \in L^p(\mu)$, $1 \leq p < \infty$. Suppose E_n is a sequence of measurable sets satisfying $\mu(E_n) = 1/n$ for all n. Prove that

$$\lim_{n\to\infty} \left[n^{p-1} p \int_{E_n} |f| \, \mathrm{d}\mu \right] = 0.$$

References

[1] G.B. Folland. *Real analysis: modern techniques and their applications*. Pure and applied mathematics. Wiley, 1984.

[2] H.L. Royden and P. Fitzpatrick. *Real Analysis*. Featured Titles for Real Analysis Series. Prentice Hall, 2010.

[3] W. Rudin. *Principles of Mathematical Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1976.

[4] W. Rudin. *Real and complex analysis*. Mathematics series. McGraw-Hill, 1987.

[5] R. Wheeden and A. Zygmund. *Measure and Integral: An Introduction to Real Analysis*. Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1977.