MA557 Problem Set 3

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Problem 3.1

Let R be a domain and Γ the set of all principal ideals in R. Show that R is a unique factorization domain if and only if Γ satisfies the ascending chain condition and every irreducible element of R is prime.

Proof. ■

Problem 3.2

Let M be an Artinian R-module. Show that every injective R-lnear map $\varphi\colon M \to M$ is an isomorphism.

Proof. ∎

Problem 3.3

Let M be a finitely generated Artinian module. Show that M is Noetherian.

Proof. ■

Problem 3.4

Let R be a ring that is Artinian or Noetherian, and $x \in R$. Show that for some $n > 0$, the image of x in $R/(0 : x)^n$ is a nonzero-divisor on that ring.

Proof. ■

Problem 3.5

Let R be an Artinian ring. Show that $R \cong R_1 \times \cdots \times R_n$ with R_i Artinian local rings.

Proof. ■

Problem 3.6

Let R be an Artinian ring all of whose maximal ideals are principal. Show that every ideal in R is principal.

Proof. ■

Problem 3.7

Prove 2.12.

Proof. Recall the statement of Theorem 2.12:

Theorem. *Let R be a ring, M, M' and M'' be R-modules. Then*

(a) *The following are equivalent:*

   (1) $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M''$ *is exact*

   (2) $0 \to \hom(N, M') \xrightarrow{\hom(N,\varphi)} \hom(N, M) \xrightarrow{\hom(N,\psi)} \hom(N, M'')$ *is exact for all modules N.*

(b) *The following are equivalent:*

   (1) $M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ *is exact.*

   (2) $0 \to \hom(M'', N) \xrightarrow{\hom(\psi,N)} \hom(M, N) \xrightarrow{\hom(\psi,N)} \hom(M', N)$ *is exact for all modules N.*

   (3) $M' \otimes N \xrightarrow{\varphi\otimes N} M \otimes N \xrightarrow{\psi\otimes N} M'' \otimes N \to 0$ *is exact for all modules N.*

■