

# MA544: Qual Problems

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# 1 Notes

Notes based off of Wheeden and Zygmund's *Measure and Integral* book.

## 1.1 Exam 1 Review

This is all of the material we covered before exam 1.

Introductory material I should have known from 504.

If  $\mathcal{F}$  is a countable (i.e., finite or countably infinite), it will be called a *sequence of sets* and denoted  $\mathcal{F} = \{E_k : k = 1, 2, \dots\}$ . The corresponding union and intersection will be written  $\bigcup_k E_k$  and  $\bigcap_k E_k$ . A sequence  $\{E_k\}$  of sets is said to *increase* to  $\bigcup_k E_k$  if  $E_k \subset E_{k+1}$  for all  $k$  and to *decrease* to  $\bigcap_k E_k$  if  $E_k \supset E_{k+1}$  for all  $k$ ; we use the notation  $E_k \nearrow \bigcap_k E_k$  and  $E_k \searrow \bigcup_k E_k$  to denote these two possibilities. If  $\{E_k\}_{k=1}^\infty$  is a sequence of sets, we define

$$\limsup E_k = \bigcap_{j=1}^\infty \bigcup_{k=j}^\infty E_k, \quad \liminf E_k = \bigcup_{j=1}^\infty \bigcap_{k=j}^\infty E_k, \quad (1)$$

noting that the sets  $U_j = \bigcup_{k=j}^\infty E_k$  and  $V_j = \bigcap_{k=j}^\infty E_k$  satisfy  $U_j \searrow \limsup E_k$  and  $V_j \nearrow \liminf E_k$ . Then  $\limsup E_k$  consists of those points of  $\mathbb{R}^n$  that belong to infinitely many  $E_k$  and  $\liminf E_k$  to those that belong to all  $E_k$  for  $k \geq k_0$  (where  $k_0$  may vary from point to point). Thus  $\liminf E_k \subset \limsup E_k$ .

If  $E_1$  and  $E_2$  are two sets, we define  $E_1 \setminus E_2$  by  $E_1 \setminus E_2 = E_1 \cap \mathbb{C}E_2$  and call it the *difference* of  $E_1$  and  $E_2$  or the *relative complement* of  $E_2$  in  $E_1$ . We will often have occasion to use *de Morgan laws*, which govern relations between complements, unions, and intersections; these state that

$$\mathbb{C}\left(\bigcup_{E \in \mathcal{F}} E\right) = \bigcap_{E \in \mathcal{F}} \mathbb{C}E, \quad \mathbb{C}\left(\bigcap_{E \in \mathcal{F}} E\right) = \bigcup_{E \in \mathcal{F}} \mathbb{C}E, \quad (2)$$

and are easily verified.

If  $\mathbf{x} \in \mathbb{R}^n$ , we say that a sequence  $\{\mathbf{x}_k\}$  *converges* to  $\mathbf{x}$ , or that  $\mathbf{x}$  is the *limit point* of  $\{\mathbf{x}_k\}$ , if  $\|\mathbf{x} - \mathbf{x}_k\| \rightarrow 0$  as  $k \rightarrow \infty$ . We denote this by writing either  $\mathbf{x} = \lim_{k \rightarrow \infty} \mathbf{x}_k$  or  $\mathbf{x}_k \rightarrow \mathbf{x}$  as  $k \rightarrow \infty$ . A point  $\mathbf{x} \in \mathbb{R}^n$  is called a *limit point of a set*  $E$  if it is the limit point of a sequence of distinct points of  $E$ . A point  $\mathbf{x} \in E$  is called a *isolated point* of  $E$  if it is not the limit point of any sequence in  $E$  (excluding the trivial sequence  $\{\mathbf{x}_k\}$  where  $\mathbf{x}_k = \mathbf{x}$  for all  $k \in \mathbb{N}$ ). It follows that  $\mathbf{x}$  is isolated if and only if there is a  $\delta > 0$  such that  $\|\mathbf{x} - \mathbf{y}\| > \delta$  for every  $\mathbf{y} \in E$ ,  $\mathbf{y} \neq \mathbf{x}$ .

For sequences  $\{x_k\}$  in  $\mathbb{R}$ , we will write  $\lim_{k \rightarrow \infty} x_k = \infty$ , or  $x_k \rightarrow \infty$  as  $k \rightarrow \infty$ , if given  $M > 0$  there is an integer  $N$  such that  $x_k \geq M$  whenever  $k \geq N$ .

A sequence  $\{\mathbf{x}_k\}$  in  $\mathbb{R}^n$  is called a *Cauchy sequence* if given  $\varepsilon > 0$  there exists an integer  $N$  such that  $\|\mathbf{x}_k - \mathbf{x}_\ell\| < \varepsilon$  for all  $k, \ell \geq N$ .  $\mathbb{R}^n$  is a complete metric space, i.e., every Cauchy sequence in  $\mathbb{R}^n$  converges to a point of  $\mathbb{R}^n$ .

A set  $E \subset E_1$  is said to be *dense* in  $E_1$  if for every  $\mathbf{x}_1 \in E_1$  and  $\varepsilon > 0$  there is a point  $\mathbf{x} \in E$  such that  $0 < \|\mathbf{x} - \mathbf{x}_1\| < \varepsilon$ . Thus,  $E$  is dense in  $E_1$  if every point of  $E_1$  is a limit point of  $E$ . If  $E = E_1$ , we say  $E$  is *dense in itself*. As an example, the set of limit points of  $\mathbb{R}^n$  each of whose coordinates is a rational number is dense in  $\mathbb{R}^n$ . Since this set is also countable, it follows that  $\mathbb{R}^n$  is *separable*, by which we mean that  $\mathbb{R}^n$  has a countable dense subset.

For nonempty subsets  $E$  of  $\mathbb{R}$ , we use the standard notation  $\sup E$  and  $\inf E$  for the *supremum* (least upper bound) and *infimum* (greatest lower bound) of  $E$ . In case  $\sup E$  belong to  $E$ , it will be called  $\max E$ ; similarly,  $\inf E$  will be called  $\min E$  if it belongs to  $E$ .

If  $\{a_k\}_{k=1}^{\infty}$  is a sequence of points in  $\mathbb{R}$ , let  $b_j = \sup_{k \geq j} a_k$  and  $c_j = \inf_{k \geq j} a_k$ ,  $j = 1, 2, \dots$ . Then  $-\infty \leq c_j \leq b_j \leq \infty$  and  $\{b_j\}$  and  $\{c_j\}$  are monotone decreasing and increasing, respectively; that is,  $b_j \geq b_{j+1}$  and  $c_j \leq c_{j+1}$ . Define  $\limsup_{k \rightarrow \infty} a_k$  and  $\liminf_{k \rightarrow \infty} a_k$  by

$$\begin{aligned}\limsup_{k \rightarrow \infty} a_k &= \lim_{j \rightarrow \infty} b_j = \lim_{j \rightarrow \infty} \left\{ \lim_{k \geq j} a_k \right\}, \\ \liminf_{k \rightarrow \infty} a_k &= \lim_{j \rightarrow \infty} c_j = \lim_{j \rightarrow \infty} \left\{ \lim_{k \geq j} a_k \right\}.\end{aligned}\tag{3}$$

**Theorem 1** (1.4). (a)  $L = \limsup_{k \rightarrow \infty} a_k$  if and only if (i) there is a subsequence  $\{a_{k_j}\}$  of  $\{a_k\}$  that converges to  $L$  and (ii) if  $L' > L$ , there is an integer  $N$  such that  $a_k < L'$  for  $k \geq N$ .

(b)  $\ell = \liminf_{k \rightarrow \infty} a_k$  if and only if (i) there is a subsequence  $\{a_{k_j}\}$  of  $\{a_k\}$  that converges to  $\ell$  and (ii) if  $\ell' < \ell$ , there is an integer  $N$  such that  $a_k > \ell'$  for  $k \geq N$ .

Thus, when they are finite,  $\limsup a_k$  and  $\liminf a_k$  are the largest and smallest limit points of  $\{a_k\}$ , respectively.

We can also use the metric on  $\mathbb{R}$  to define the *diameter* of a set  $E$  by letting

$$\text{diam } E = \{ \|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in E \}.\tag{4}$$

If the diameter of  $E$  is finite,  $E$  is said to be *bounded*. Equivalently,  $E$  is bounded if there is a finite constant  $M$  such that  $\|\mathbf{x}\| \leq M$  for all  $\mathbf{x} \in E$ . If  $E_1$  and  $E_2$  are two sets, the *distance between*  $E_1$  and  $E_2$  is defined by

$$d(E_1, E_2) = \inf \{ \|\mathbf{x} - \mathbf{y}\| : \mathbf{x} \in E_1, \mathbf{y} \in E_2 \}.\tag{5}$$

For  $\mathbf{x} \in \mathbb{R}^n$  and  $\delta > 0$ , the set

$$B(\mathbf{x}, \delta) = \{ \mathbf{y} : \|\mathbf{y} - \mathbf{x}\| < \delta \}\tag{6}$$

is called the *open ball with center*  $\mathbf{x}$  and *radius*  $\delta$ . A point  $\mathbf{x}$  of a set  $E$  is called an *interior point* of  $E$  if there exists  $\delta > 0$  such that  $B(\mathbf{x}, \delta) \subset E$ . The collection of all interior points of  $E$  is called the *interior* of  $E$  and denoted  $E^\circ$ . A set  $E$  is said to be *open* if  $E^\circ = E$ ; that is,  $E$  is open if for each  $\mathbf{x} \in E$  there exists  $\delta > 0$  such that  $B(\mathbf{x}, \delta) \subset E$ . The empty set  $\emptyset$  is open by convention. The whole space  $\mathbb{R}^n$  is clearly open and  $B(\mathbf{x}, \delta)$  is evidently open. We will generally denote open sets by the letter  $G$ .

A set  $E$  is called *closed* if  $\mathbb{C}E$  is open. Note that  $\emptyset$  and  $\mathbb{R}^n$  are closed. Closed sets will generally be denoted by the letter  $F$ . The union of a set  $E$  and all its limit points is called the *closure* of  $E$  and written  $\bar{E}$ . By the *boundary* of  $E$ , we mean  $\partial E = \bar{E} \setminus E^\circ$ .

**Theorem 2** (1.5). (i)  $B(\bar{\mathbf{x}}, \delta) = \{ \mathbf{y} : \|\mathbf{x} - \mathbf{y}\| \leq \delta \}$

(ii)  $E$  is closed if and only if  $E = \bar{E}$ ; that is,  $E$  is closed if and only if it contains all of its limit points.

(iii)  $\bar{E}$  is closed, and  $\bar{E}$  is the smallest closed set containing  $E$ ; that is,  $F$  is closed and  $E \subset F$ , then  $\bar{E} \subset F$ .

(The rest of this is a bunch of theorems that can be expressed in more generality from a more topological perspective. At any rate, they are very basic.)

Consider a collection  $\{A\}$  of sets  $A$ . A set is said to be of *type*  $A_\delta$  if it can be written as a countable intersection of sets  $A$  and of *type*  $A_\sigma$  if it can be written as a countable union of sets  $A$ . The most common uses of this notation are  $G_\delta$  and  $F_\sigma$ , where  $\{G\}$  denotes open sets in  $\mathbb{R}^n$  and  $\{F\}$  closed sets. Hence,  $H$  is of *type*  $G_\delta$  if

$$H = \bigcap_k G_k, \quad G_k \text{ open}, \quad (7)$$

and is of *type*  $F_\sigma$  if

$$H = \bigcup_k F_k, \quad F_k \text{ closed}. \quad (8)$$

The complement of a  $G_\delta$  set is an  $F_\sigma$  and vice-a-versa.

Another type of set that we have the occasion to use is the *perfect set*, by which we mean a closed set  $C$  each of whose points is a limit point of  $C$ . Thus, a perfect set is a closed set that is dense in itself.

**Theorem 3 (1.9).** *A perfect set is uncountable.*

Other special sets that will be important are  $n$ -dimensional intervals. When  $n = 1$  and  $a < b$ , we will use the usual notations  $[a, b] = \{x : a \leq x \leq b\}$ ,  $(a, b) = \{x : a < x < b\}$ , etc. Whenever we use just the word interval, we generally mean closed interval. An  $n$ -dimensional interval  $I$  is a subset of  $\mathbb{R}^n$  of the form  $I = \{\mathbf{x} = (x_1, \dots, x_n) : a_k \leq x_k \leq b_k, k = 1, \dots, n\}$ , where  $a_k < b_k, k = 1, \dots, n$ . An interval is thus closed, and we say it has edges parallel to the coordinate axes. If the edge lengths  $b_k - a_k$  are all equal,  $I$  will be called an  $n$ -dimensional cube with edges parallel to the coordinate axes. Cubes will usually be denoted by the letter  $Q$ . Two intervals are said to be *nonoverlapping* if their interiors are disjoint, that is, if the most they have in common is some part of their boundaries. A set equal to an interval minus will be called a *partly-open interval*. By definition, the *volume*  $\text{vol}(I)$  of the interval  $I = \{(x_1, \dots, x_n) : a_k \leq x_k \leq b_k, k = 1, \dots, n\}$  is

$$\text{vol}(I) = \prod_{k=1}^n (b_k - a_k). \quad (9)$$

More generally, if  $\{\mathbf{e}_k\}_{k=1}^n$  is any given set of  $n$  vectors emanating from a point in  $\mathbb{R}^n$ , we will consider the closed *parallelepiped*

$$P = \left\{ \mathbf{x} : \mathbf{x} = \sum_{k=1}^n t_k \mathbf{e}_k, 0 \leq t_k \leq 1 \right\}. \quad (10)$$

Note that the edges of  $P$  are parallel translates of  $\mathbf{e}_k$ . Thus,  $P$  is an interval if the  $\mathbf{e}_k$  are parallel to the coordinate axes. The *volume*  $\text{vol } P$  of  $P$  is *by definition* the absolute value of the  $n \times n$  determinant having  $\mathbf{e}_1, \dots, \mathbf{e}_n$  as rows. In case  $P$  is an interval, this definition agrees with the one given earlier. A linear transformation  $T$  of  $\mathbb{R}^n$  transforms a parallelepiped  $P$  into a parallelepiped  $P'$  with volume  $\text{vol } P' = |\det T| \text{vol } P$ . In particular, rotation of the axes in  $\mathbb{R}^n$  does not change the volume of a parallelepiped.

**Theorem 4** (1.10). *Every open set in  $\mathbb{R}$  can be written as a countable union of disjoint open intervals.*

*Proof.* Let  $G$  be an open subset of  $\mathbb{R}$ . For each  $x$  in  $G$ , by Zorn's lemma, we may choose a maximal interval  $I_x \subset G$ . Now, if  $x, x' \in G$  are distinct points, then, by maximality, either  $I_x = I_{x'}$  or  $I_x \cap I_{x'} = \emptyset$ . Clearly,  $G = \bigcup_{x \in G} I_x$ . Since each  $I_x$  contains a rational number, the number of distinct  $I_x$  must be countable, and the theorem follows. ■

**Theorem 5** (1.11). *Every open set in  $\mathbb{R}^n$ ,  $n \geq 1$ , can be written as a countable union of nonoverlapping (closed) cubes. It can also be written as a countable union of disjoint partly open cubes.*

*Proof.* The proof is analogous to that of Theorem 1.10, but more general. Consider a lattice of points of  $\mathbb{R}^n$  with integral coordinates and the corresponding net  $K_0$  of cubes with edge length 1 and vertices. Bisecting each edge of a cube in  $K_0$ , we obtain from it  $2^n$  subcubes of edge length  $1/2$ . The total collection of these subcubes for every cube in  $K_0$  forms a net  $K_1$  of cubes. If we continue bisecting, we obtain finer and finer nets  $K_j$  of cubes such that each cube in  $K_j$  has edge length  $2^{-j}$  and is the union of  $2^n$  nonoverlapping cubes in  $K_{j+1}$ .

Now let  $G$  be any open set in  $\mathbb{R}^n$ . Let  $S_0$  be the collection of all cubes  $K_0$  that lie entirely in  $G$ . Let  $S_1$  be those cubes in  $K_1$  that lie in  $G$  but are not subcubes of any cube in  $S_0$ . More generally, for  $j \geq 1$ , let  $S_j$  be the cubes in  $K_j$  that lie in  $G$  but that are not subcubes of any cube in  $S_0, \dots, S_{j-1}$ . If  $S$  denotes the total collection of cubes from all the  $S_j$ , then  $S$  is countable since each  $K_j$  is countable, and the cubes in  $S$  are nonoverlapping by construction. Hence,  $G = \bigcup_{Q \in S} Q$ , which proves the first statement.

The second part of the statement is left as an exercise to me, but I'm not interested in solving it; there is nothing to be gained from attempting a solution to it. ■

The collection  $\{Q : Q \in K_j, j = 1, 2, \dots\}$  constructed above is called a family of dyadic cubes. In general, by *dyadic cubes*, we mean the family of cubes obtained from repeated bisection of any initial net of cubes in  $\mathbb{R}^n$ .

It follows from Theorem 1.10 that any closed set in  $\mathbb{R}$  can be constructed by deleting a countable number of open disjoint intervals from  $\mathbb{R}$ .

By cover of a set  $E$ , we mean a family  $\mathcal{F}$  of sets  $A$  such that  $E \subset \bigcup_{A \in \mathcal{F}} A$ . A *subcover*  $\mathcal{F}'$  of a cover  $\mathcal{F}$  is a cover with the property that  $A' \in \mathcal{F}$  whenever  $A' \in \mathcal{F}'$ . A cover  $\mathcal{F}$  is called an *open cover* if each set in  $\mathcal{F}$  is open.

**Theorem 6** (1.12). (a) *(The Heine–Borel theorem) A set  $E \subset \mathbb{R}^n$  is compact if and only if it is closed and bounded.*

(b) *A set  $E \subset \mathbb{R}^n$  is compact if and only if every sequence of points in  $E$  has a subsequence that converges to a point of  $E$ .*

By a function  $f$  defined for  $\mathbf{x}$  in a set  $E \subset \mathbb{R}^n$ , we will always mean a *real-valued* function, unless explicitly stated otherwise. By *real-valued*, we generally mean *extended real-valued*, i.e.,  $f$  may take the values  $\pm\infty$ ; if  $|f(\mathbf{x})| < \infty$  for all  $\mathbf{x} \in E$ , we say  $f$  is *finite* (or *finite-valued*) on  $E$ . A finite function  $f$  is said to be *bounded* on  $E$  if there is a finite number  $M$  such that  $|f(\mathbf{x})| \leq M$  for  $\mathbf{x} \in E$ ; that is,  $f$  is bounded on  $E$  if  $\sup |f(\mathbf{x})|$ , where  $\mathbf{x} \in E$ , is finite. A sequence  $\{f_k\}$  of functions is said to be *uniformly bounded* on  $E$  if there is a finite  $M$  such that  $|f_k(\mathbf{x})| \leq M$  for  $\mathbf{x} \in E$  and all  $k$ .

By the *support* of  $f$ , we mean the closure of the set where  $f$  is not zero. Thus, the support of a function is always closed. It follows that a function defined in  $\mathbb{R}^n$  has *compact support* if and only if it vanishes outside some bounded set.

A function  $f$  defined on an interval  $I$  in  $\mathbb{R}$  is called *monotone increasing (decreasing)* if  $f(x) \leq f(y)$  [ $f(x) \geq f(y)$ ] whenever  $x < y$  and  $x, y \in I$ . By *strictly monotone increasing (decreasing)*, we mean that  $f(x) < f(y)$  [ $f(x) > f(y)$ ] if  $x < y$  and  $x, y \in I$ .

Let  $f$  be defined on  $E \subset \mathbb{R}^n$  and let  $\mathbf{x}_0$  be a limit point of  $E$ . Let  $B'(\mathbf{x}_0, \delta) = B(\mathbf{x}_0, \delta) \setminus \{\mathbf{x}_0\}$  denote the punctured ball with center  $\mathbf{x}_0$  and radius  $\delta$ , and let

$$M(\mathbf{x}_0, \delta) = \sup_{\mathbf{x} \in B'(\mathbf{x}_0, \delta) \cap E} f(\mathbf{x}), \quad m(\mathbf{x}_0, \delta) = \inf_{\mathbf{x} \in B'(\mathbf{x}_0, \delta) \cap E} f(\mathbf{x}). \quad (11)$$

As  $\delta \searrow 0$ ,  $M(\mathbf{x}_0, \delta)$  decreases and  $m(\mathbf{x}_0, \delta)$  increases, and we define

$$\begin{aligned} \limsup_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \\ \mathbf{x} \in E}} f(\mathbf{x}) &= \lim_{\delta \rightarrow 0} M(\mathbf{x}_0, \delta) \\ \liminf_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \\ \mathbf{x} \in E}} f(\mathbf{x}) &= \lim_{\delta \rightarrow 0} m(\mathbf{x}_0, \delta). \end{aligned} \quad (12)$$

**Theorem 7 (1.14).** (a)  $M = \limsup_{\mathbf{x} \rightarrow \mathbf{x}_0; \mathbf{x} \in E} f(\mathbf{x})$  if and only if (i) there exist  $\mathbf{x}_k$  in  $E \setminus \{\mathbf{x}_0\}$  such that  $\mathbf{x}_k \rightarrow \mathbf{x}_0$  and  $f(\mathbf{x}_k) \rightarrow M$  and (ii) if  $M' > M$ , there exists  $\delta > 0$  such that  $f(\mathbf{x}) < M'$  for  $\mathbf{x} \in B'(\mathbf{x}_0, \delta) \cap E$ .

(b)  $m = \liminf_{\mathbf{x} \rightarrow \mathbf{x}_0; \mathbf{x} \in E} f(\mathbf{x})$  if and only if (i) there exist  $\mathbf{x}_k$  in  $E \setminus \{\mathbf{x}_0\}$  such that  $\mathbf{x}_k \rightarrow \mathbf{x}_0$  and  $f(\mathbf{x}_k) \rightarrow m$  and (ii) if  $m' < m$ , there exists  $\delta > 0$  such that  $f(\mathbf{x}) > m'$  for  $\mathbf{x} \in B'(\mathbf{x}_0, \delta) \cap E$ .

A function  $f$  defined on a neighborhood of  $\mathbf{x}_0$  is said to be *continuous* at  $\mathbf{x}_0$  if  $f(\mathbf{x}_0)$  is finite and  $\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$ . If  $f$  is not continuous at  $\mathbf{x}_0$ , it follows that unless  $f(\mathbf{x}_0)$  is infinite, either  $\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x})$  does not exist or is different from  $f(\mathbf{x}_0)$ .

For functions on  $\mathbb{R}$ , we will use the notation

$$f(x_0+) = \lim_{\substack{x \rightarrow x_0 \\ x > x_0}} f(x) \quad f(x_0-) = \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} f(x). \quad (13)$$

for the *right-* and *left-hand limits* of  $f$  at  $x_0$ , when they exist. If  $f(x_0+)$ ,  $f(x_0-)$ , and  $f(x_0)$  exist and are finite, but  $f$  is not continuous at  $x_0$ , then either  $f(x_0+) \neq f(x_0-)$  or  $f(x_0+) = f(x_0-) \neq f(x_0)$ . In the first case,  $x_0$  is called a *jump discontinuity* of  $f$  and in the second, a *removable discontinuity* of  $f$  (since by changing the value of  $f$  at  $x_0$ , we can make it continuous there). Such discontinuities are said to be of the *first kind*, as distinguished from those of the *second kind*, for which either  $f(x_0+)$  or  $f(x_0-)$  does not exist or for which  $f(x_0+)$ ,  $f(x_0-)$  or  $f(x_0)$  are infinite.

If  $f$  is defined only in a set  $E$  containing  $\mathbf{x}_0$ ,  $E \subset \mathbb{R}^n$ , then  $f$  is said to be *continuous at  $\mathbf{x}_0$  relative to  $E$*  if  $f(\mathbf{x}_0)$  is finite and either  $\mathbf{x}_0$  is an isolated point of  $E$  or  $\mathbf{x}_0$  is a limit point of  $E$  and  $\lim_{\mathbf{x} \rightarrow \mathbf{x}_0; \mathbf{x} \in E} f(\mathbf{x}) = f(\mathbf{x}_0)$ . If  $E' \subset E$ , a function is said to be *continuous in  $E'$  relative to  $E$*  if it is continuous relative to  $E$  at every point of  $E'$ .

**Theorem 8 (1.15).** Let  $E$  be a compact set in  $\mathbb{R}^n$  and  $f$  be continuous in  $E$  relative to  $E$ . Then the following are true:

(i)  $f$  is bounded on  $E$ ,  $\sup_{\mathbf{x} \in E} |f(\mathbf{x})| < \infty$ .

- (ii)  $f$  attains its supremum and infimum on  $E$ ; i.e., there exists  $\mathbf{x}_1, \mathbf{x}_2 \in E$  such that  $f(\mathbf{x}_1) = \sup_{\mathbf{x} \in E} f(\mathbf{x})$ ,  $f(\mathbf{x}_2) = \inf_{\mathbf{x} \in E} f(\mathbf{x})$ .
- (iii)  $f$  is uniformly continuous on  $E$  relative to  $E$ ; i.e., given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$  if  $\|\mathbf{x} - \mathbf{y}\| < \delta$  and  $\mathbf{x}, \mathbf{y} \in E$ .

**Theorem 9** (1.16). *Let  $\{f_k\}$  be a sequence of functions defined on  $E$  that are continuous in  $E$  relative to  $E$  and that converge uniformly on  $E$  to a finite  $f$ . Then  $f$  is continuous in  $E$  relative to  $E$ .*

A transformation  $T$  of a set  $E \subset \mathbb{R}^n$  into  $\mathbb{R}^n$  is a mapping  $\mathbf{y} = T\mathbf{x}$  that carries points  $\mathbf{x} \in E$  into points  $\mathbf{y} \in \mathbb{R}^n$ . If  $\mathbf{y} = (y_1, \dots, y_n)$ , then  $T$  can be identified with the collection of coordinate functions  $y_k = f_k(\mathbf{x})$ ,  $k = 1, \dots, n$ , which are induced by  $T$ . The image of  $E$  under  $T$  is the set  $\{\mathbf{y} : \mathbf{y} = T\mathbf{x} \text{ for some } \mathbf{x} \in E, \}$ .  $T$  is continuous at  $\mathbf{x}_0 \in E$  relative to  $E$ .

**Theorem 10** (1.17). *Let  $\mathbf{y} = T\mathbf{x}$  be a transformation of  $\mathbb{R}^n$  that is continuous in  $E$  relative to  $E$ . If  $E$  is compact, then so is the image  $TE$ .*

If  $f$  is defined and bounded on an interval  $I = \{\mathbf{x} : \mathbf{x} = (x_1, \dots, x_n), a_k \leq x_k \leq b_k, k = 1, \dots, n\}$  in  $\mathbb{R}^n$ , its Riemann integral will be denoted

$$(R) \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n) dx_1 \cdots dx_n \quad \text{or} \quad (R) \int_I f(\mathbf{x}) d\mathbf{x} \quad (14)$$

and is defined as follows. Partition  $I$  into a finite collection  $\Gamma$  of nonoverlapping intervals,  $\Gamma = \{I_k\}_{k=1}^N$ , and define the norm  $\|\Gamma\|$  of  $\Gamma$  to be  $\|\Gamma\| = \max_k \text{diam } I_k$ . Select a point  $\vec{\xi}_k$  in  $I_k$  for  $k \geq 1$ , and let

$$\begin{aligned} R_\Gamma(\vec{\xi}_1, \dots, \vec{\xi}_n) &= \sum_{k=1}^N f(\vec{\xi}_k) \text{vol}(I_k) \\ U_\Gamma(\vec{\xi}_1, \dots, \vec{\xi}_n) &= \sum_{k=1}^N \left[ \sup_{\mathbf{x} \in I_k} f(\mathbf{x}) \right] \text{vol}(I_k) \\ L_\Gamma(\vec{\xi}_1, \dots, \vec{\xi}_n) &= \sum_{k=1}^N \left[ \inf_{\mathbf{x} \in I_k} f(\mathbf{x}) \right] \text{vol}(I_k). \end{aligned} \quad (15)$$

We define the Riemann integral by saying that  $A = (R) \int_I f(\mathbf{x}) d\mathbf{x}$  if  $\lim_{\|\Gamma\| \rightarrow 0} R_\Gamma$  exists and equals  $A$ ; that is, if given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|A - R_\Gamma| < \varepsilon$  for any  $\Gamma$  and any chosen  $\{\vec{\xi}_k\}$ , provided only that  $\|\Gamma\| < \delta$ . This definition is actually equivalent to the statement that

$$\inf_\Gamma U_\Gamma = \sup_\Gamma L_\Gamma = A. \quad (16)$$

The integral of course exists if  $f$  is continuous on  $I$ .

Let  $f$  be a real-valued function that is defined and finite for all  $x$  in a closed bounded interval  $a \leq x \leq b$ . Let  $\Gamma = \{x_0, \dots, x_m\}$  be a partition of  $[a, b]$ ; that is,  $\Gamma$  is a collection of points  $x_i$ ,  $i = 0, 1, \dots, m$ , satisfying  $x_0 = a$ ,  $x_m = b$ , and  $x_{i-1} < x_i$  for  $i = 1, \dots, m$ . With each partition  $\Gamma$  we associate the sum

$$S_\Gamma = S_\Gamma[f; a, b] = \sum_{i=1}^m |f(x_i) - f(x_{i-1})|. \quad (17)$$

The *variation of  $f$  over  $[a, b]$*  is defined as

$$V = V[f; a, b] = \sup_{\Gamma} S_{\Gamma}, \quad (18)$$

where the supremum is taken over all partitions  $\Gamma$  of  $[a, b]$ . The variation of  $V[f; a, b]$  will sometimes also be denoted by  $V[a, b]$  or  $V(f)$ . Since  $0 \leq S_{\Gamma} < \infty$ , we have  $0 \leq V \leq \infty$ . If  $V < \infty$ ,  $f$  is said to be of *bounded variation on  $[a, b]$* ; if  $V = \infty$ ,  $f$  is of *unbounded variation on  $[a, b]$* .

Here are several examples

**Examples 1.** Suppose  $f$  is monotone in  $[a, b]$ . Then, clearly, each  $S_{\Gamma}$  equals  $|f(b) - f(a)|$ , and therefore  $V = |f(b) - f(a)|$ .

**Examples 2.** Suppose the graph of  $f$  can be split into a finite number of monotone arcs; that is, suppose  $[a, b] = \bigcup_{i=1}^k [a_i, a_{i+1}]$  and  $f$  monotone in each  $[a_i, a_{i+1}]$ . Then  $V = \sum_{i=1}^k |f(a_{i+1}) - f(a_i)|$ . To see this, we use the result of the previous example and the fact, to be proved, in Theorem 2.2, that  $V = \sum_{i=1}^k V[a_i, a_{i+1}]$ .

**Examples 3.** Let  $f$  be defined by  $f(x) = 0$  when  $x \neq 0$  and  $f(0) = 1$ , and let  $[a, b]$  be any interval containing 0 in its interior. Then  $S_{\Gamma}$  is either 2 or 0 depending on whether  $x = 0$  is a partition point or not. Thus,  $V[a, b] = 2$ .

If  $\Gamma = \{x_0, x_1, \dots, x_m\}$  is a partition of  $[a, b]$ , let  $\|\Gamma\|$ , called the *norm of  $\Gamma$* , be defined as the longest subinterval of  $\Gamma$ :

$$\|\Gamma\| = \max_{i=1, \dots, m} x_i - x_{i-1}. \quad (19)$$

If  $f$  is continuous on  $[a, b]$  and  $\{\Gamma_j\}$  is a sequence of partitions  $[a, b]$  with  $\|\Gamma_j\| \rightarrow 0$ , we shall see in Theorem 2.9 that  $V = \lim_{j \rightarrow \infty} S_{\Gamma_j}$ . The example above shows that this may fail for functions that are discontinuous even at a single point: if we take  $f$  and  $[a, b]$  is in the example above and choose  $\Gamma_j$  such that  $x = 0$  is never a partition in the point, then  $\lim S_{\Gamma_j} = 0$ , while if we choose the  $\Gamma_j$  such that  $x = 0$  alternatively is and is not a point, then  $\lim S_{\Gamma_j}$  does not exist.

**Examples 4.** Let  $f$  be the *Dirichlet function*, defined by  $f(x) = 1$  for  $x$  rational and  $f(x) = 0$  for  $x$  irrational. Then, clearly,  $V[a, b] = \infty$  for any interval  $[a, b]$ .

**Examples 5.** A function that is continuous on an interval is not necessarily of bounded variation on the interval. To see this, let  $\{a_j\}$  and  $\{d_j\}$ ,  $j = 1, 2, \dots$ , be two monotone decreasing sequences in  $(0, 1]$  with  $a_1 = 1$ ,  $\lim_{j \rightarrow \infty} a_j = \lim_{j \rightarrow \infty} d_j = 0$  and  $\sum d_j = \infty$ . Construct a continuous function  $f$  as follows. On each subinterval  $[a_{j+1}, a_j]$ , the graph of  $f$  consists of sides of the isosceles with base  $[a_{j+1}, a_j]$  and height  $d_j$ . Thus,  $f(a_j) = 0$ , and if  $m_j$  denotes the midpoint of  $[a_{j+1}, a_j]$ , then  $f(m_j) = d_j$ . If we further define  $f(0) = 0$ , then  $f$  is continuous on  $[0, 1]$ . Taking  $\Gamma_k$  to be the partition defined by the points  $0, \{a_j\}_{j=1}^{k+1}$ , and  $\{m_j\}_{j=1}^k$ , we see that  $S_{\Gamma} = 2 \sum_{j=1}^k d_j$ . Hence,  $V[f; 0, 1] = \infty$ .

**Examples 6.** A function  $f$  defined on  $[a, b]$  is said to satisfy the *Lipschitz condition* on  $[a, b]$ , or be a *Lipschitz function* on  $[a, b]$ , if there is a constant  $C$  such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all  $x, y \in [a, b]$ . Such a function is clearly of bounded variation, with  $V[f; a, b] \leq C(b - a)$ . For example, if  $f$  has a continuous derivative on  $[a, b]$ , or even just a bounded derivative, then (by the mean-value theorem)  $f$  satisfies the Lipschitz condition on  $[a, b]$ .



**Theorem 11 (2.1).** (i) *If  $f$  is of bounded variation on  $[a, b]$ , then  $f$  is bounded on  $[a, b]$ .*

(ii) *Let  $f$  and  $g$  be of bounded variation on  $[a, b]$ . Then  $cf$  (for any real constant  $c$ ),  $f + g$ , and  $fg$  are of bounded variation on  $[a, b]$ . Moreover,  $f/g$  is of bounded variation if there exist some  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for  $x \in [a, b]$ .*

*Proof by Carlos.* And the proof of these two is rather clear.

For (i), we proceed by contradiction. Suppose that  $f$  is of bounded variation on the interval  $[a, b]$ . Then the variation  $V$  of  $f$  over  $[a, b]$  is finite. However, if  $f$  is unbounded on  $[a, b]$ , for every positive real number  $M$ , there exists some  $x \in [a, b]$  such that  $|f(x)| > M$ . In particular, for any  $x' \in [a, b]$  we have

$$|f(x) - f(x')| > M.$$

In turn, this tells us that  $V > |f(x) - f(x')| > M$  for any partition  $\Gamma$  containing  $x$ , so  $V = \infty$ . This yields a contradiction.

For (ii), the proofs are simple. Suppose  $f$  is of bounded variation on  $[a, b]$  with variation  $V_f$ . Let  $c$  be a real constant, then

$$\begin{aligned} V[cf; a, b] &= \sup_{\Gamma} \sum_{i=1}^n |cf(x_i) - cf(x_{i-1})| \\ &= |c| \sup_{\Gamma} \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \\ &= |c| V_f \\ &< \infty. \end{aligned}$$

Hence,  $cf$  is of bounded variation on  $[a, b]$ . Suppose  $f$  and  $g$  are of bounded variation on  $[a, b]$  with variation  $V_f$  and  $V_g$ , respectively. Then

$$\begin{aligned} V[f + g; a, b] &= \sup_{\Gamma} \sum_{i=1}^n |(f + g)(x_i) - (f + g)(x_{i-1})| \\ &= \sup_{\Gamma} \sum_{i=1}^n |(f(x_i) - f(x_{i-1})) + (g(x_i) - g(x_{i-1}))| \\ &\leq \sup_{\Gamma} \sum_{i=1}^n [|f(x_i) - f(x_{i-1})| + |g(x_i) - g(x_{i-1})|] \\ &= \sup_{\Gamma} \sum_{i=1}^n |f(x_i) - f(x_{i-1})| + \sup_{\Gamma} \sum_{i=1}^n |g(x_i) - g(x_{i-1})| \\ &= V_f + V_g \\ &< \infty. \end{aligned}$$

Hence,  $f + g$  is of bounded variation. Suppose  $f$  and  $g$  are of bounded variation on  $[a, b]$  with variation  $V_f$  and  $V_g$ , respectively. Then

$$\begin{aligned}
V[fg; a, b] &= \sup_{\Gamma} \sum_{i=1}^n |(fg)(x_i) - (fg)(x_{i-1})| \\
&= \sup_{\Gamma} \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sup_{\Gamma} \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_i) + f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sup_{\Gamma} \sum_{i=1}^n |(f(x_i)g(x_i) - f(x_{i-1})g(x_i)) - (f(x_{i-1})g(x_{i-1}) - f(x_{i-1})g(x_i))| \\
&= \sup_{\Gamma} \sum_{i=1}^n |g(x_i)||f(x_i) - f(x_{i-1})| + \sup_{\Gamma} \sum_{i=1}^n |f(x_{i-1})||g(x_i) - g(x_{i-1})|
\end{aligned}$$

by part (i), since  $f$  and  $g$  are b.v. on  $[a, b]$ , they are bounded so there exists  $M$  and  $N$  such that  $|f(x)| < M$ ,  $|g(x)| < M$  for all  $x \in [a, b]$

$$\begin{aligned}
&= MV_f + NV_g \\
&< \infty.
\end{aligned}$$

Hence,  $fg$  is of bounded variation on  $[a, b]$ . Suppose that  $f$  and  $g$  are of bounded variation on  $[a, b]$  with variation  $V_f$  and  $V_g$ , respectively. Suppose, additionally that there exists  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for all  $x \in [a, b]$ . Then, we have

$$\begin{aligned}
V[f/g; a, b] &= \sup_{\Gamma} \sum_{i=1}^n |(f/g)(x_i) - (f/g)(x_{i-1})| \\
&= \sup_{\Gamma} \sum_{i=1}^n \left| \frac{f(x_i)}{g(x_i)} - \frac{f(x_{i-1})}{g(x_{i-1})} \right| \\
&= \sup_{\Gamma} \sum_{i=1}^n \left| \frac{g(x_{i-1})f(x_i) - g(x_i)f(x_{i-1})}{g(x_i)g(x_{i-1})} \right|
\end{aligned}$$

and since we have  $g(x) > \varepsilon$  for any  $x \in [a, b]$ ,  $1/g(x) < 1/\varepsilon$  for any  $x \in [a, b]$ , so

$$\begin{aligned}
&\leq \sup_{\Gamma} \left[ \frac{1}{\varepsilon^2} \sum_{i=1}^n |g(x_{i-1})f(x_i) - g(x_i)f(x_{i-1})| \right] \\
&\leq \sup_{\Gamma} \left[ \frac{1}{\varepsilon^2} \sum_{i=1}^n |g(x_{i-1})f(x_i) - g(x_{i-1})f(x_{i-1}) \right. \\
&\quad \left. - (g(x_i)f(x_{i-1}) - g(x_{i-1})f(x_{i-1}))| \right] \\
&=
\end{aligned}$$

etc. etc. etc. ■

**Theorem 12 (2.2).** (i) If  $[a', b']$  is a subinterval of  $[a, b]$ , then  $V[a', b'] \leq V[a, b]$ ; that is, variation increases with interval.

(ii) If  $a < c < b$ , then  $V[a, b] = V[a, c] + V[c, b]$ ; that is, variation is additive on adjacent intervals.

*Carlos's proof.* (i) follows from (ii). By recursively applying part (ii), we have

$$V[a, b] = V[a, a'] + V[a', b'] + V[b', b].$$

Hence,

$$\begin{aligned} V[a', b'] &= V[a, b] - V[a, a'] - V[b', b] \\ &\leq V[a, b] \end{aligned}$$

as desired.

To see part (ii) let  $f$  be a real-valued function defined on  $[a, b]$ . If  $V[f; a, b] = \infty$ , there is nothing to show so suppose  $V[f; a, b] < \infty$ . Let  $c$  be a point in  $[a, b]$  not equal to either endpoint  $a$  or  $b$ . ■

*Proof of (ii).* Let  $I = [a, b]$ ,  $I_1 = [a, c]$ ,  $I_2 = [c, b]$ ,  $V = V[a, b]$ ,  $V_1 = V[a, c]$ , and  $V_2 = V[c, b]$ . If  $\Gamma_1$  and  $\Gamma_2$  are any partitions of  $I_1$  and  $I_2$ , respectively, then  $\Gamma = \Gamma_1 \cup \Gamma_2$  is one of  $I$ , and  $S_\Gamma[I] = S_{\Gamma_1}[I_1] + S_{\Gamma_2}[I_2]$ . Thus,  $S_{\Gamma_1}[I_1] + S_{\Gamma_2}[I_2] \leq V$ . Therefore, taking the supremum over  $\Gamma_1$  and  $\Gamma_2$  separately, we obtain  $V_1 + V_2 \leq V$ .

To show the opposite inequality, let  $\Gamma$  be any partition  $I$ , and let  $\bar{\Gamma}$  be  $\Gamma$  with  $c$  adjoined. Then  $S_\Gamma[I] \leq S_{\bar{\Gamma}}[I]$ , and  $\bar{\Gamma}$  splits into partitions  $\Gamma_1$  of  $I_1$  and  $\Gamma_2$  of  $I_2$ . Thus, we have

$$S_\Gamma[I] \leq S_{\bar{\Gamma}}[I] = S_{\Gamma_1}[I_1] + S_{\Gamma_2}[I_2] \leq V_1 + V_2.$$

Therefore,  $V \leq V_1 + V_2$ , which completes the proof of (ii). ■

For any real number  $x$ , define

$$x^+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \quad x^- = \begin{cases} 0 & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}.$$

These are called the *positive* and *negative parts* of  $x$ , respectively, and satisfy the relations

$$x^+, x^- \geq 0; \quad |x| = x^+ + x^-; \quad x = x^+ - x^-.$$

Given a finite function  $f$  on  $[a, b]$  and a partition  $\Gamma = \{x_i\}_{i=0}^\infty$  of  $[a, b]$ , define

$$\begin{aligned} P_\Gamma &= P_\Gamma[f; a, b] = \sum_{i=1}^m [f(x_i) - f(x_{i-1})]^+ \\ N_\Gamma &= N_\Gamma[f; a, b] = \sum_{i=1}^m [f(x_i) - f(x_{i-1})]^- \end{aligned}$$

Thus,  $P_\Gamma$  is the sum of the positive terms of  $S_\Gamma$ , and  $-N_\Gamma$  is the sum of the negative terms of  $S_\Gamma$ . In particular, we have  $P_\Gamma \geq 0$ ,  $N_\Gamma \geq 0$ ,

$$\begin{aligned} P_\Gamma + N_\Gamma &= S_\Gamma, \\ P_\Gamma - N_\Gamma &= f(b) - f(a). \end{aligned} \tag{20}$$

## 1.2 Exam 2 Review

This is all of the material we covered before exam 2.

Let  $f$  be defined on  $E$ , and let  $\mathbf{x}_0$  be a limit point of  $E$  in  $E$ . Then  $f$  is said to be *upper semicontinuous at  $\mathbf{x}_0$*  if

$$\limsup_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \\ \mathbf{x} \in E}} f(\mathbf{x}) \leq f(\mathbf{x}_0). \quad (21)$$

Note that if  $f(\mathbf{x}_0) = \infty$ , then  $f$  is usc at  $\mathbf{x}_0$  automatically; otherwise, the statement that  $f$  is usc at  $\mathbf{x}_0$  means that given any  $M > f(\mathbf{x}_0)$ , there exists  $\delta > 0$  such that  $f(\mathbf{x}) < M$  for all  $\mathbf{x} \in E$  that lie in the ball  $B_\delta(\mathbf{x}_0)$ .

Similarly,  $f$  is said to be *lower semicontinuous at  $\mathbf{x}_0$*  if  $-f$  is usc at  $\mathbf{x}_0$ .

**Theorem (4.14).** *A function  $f$  is usc relative to  $E$  if and only if  $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$  is relatively closed (equivalently, if  $\{\mathbf{x} \in E : f(\mathbf{x}) < a\}$  is relatively open) for all finite  $a$*

*Proof of theorem 4.14.* Suppose that  $f$  is usc relative to  $E$ . Given  $a$ , let  $\mathbf{x}_0$  be a limit point of  $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$  in  $E$ . Then there exists  $\mathbf{x}_k \in E$  such that  $\mathbf{x}_k \rightarrow \mathbf{x}_0$  and  $f(\mathbf{x}_k) > a$ . Since  $f$  is usc at  $\mathbf{x}_0$ , we have  $f(\mathbf{x}_0) \geq \limsup_{k \rightarrow \infty} f(\mathbf{x}_k)$ . Therefore,  $f(\mathbf{x}_0) > a$ , so  $\mathbf{x}_0 \in \{\mathbf{x} \in E : f(\mathbf{x}) > a\}$ . Hence,  $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$  is relatively closed.

Conversely, let  $\mathbf{x}_0$  be a limit point of  $E$  that is in  $E$ . If  $f$  is not usc at  $\mathbf{x}_0$ , then  $f(\mathbf{x}_0) < \infty$ , and there exists  $M$  and  $\{\mathbf{x}_k\}$  such that  $f(\mathbf{x}_0) < M$ ,  $\mathbf{x}_k \in E$ ,  $\mathbf{x}_k \rightarrow \mathbf{x}_0$ , and  $f(\mathbf{x}_k) \geq M$ . Hence,  $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$  is not relatively closed since it does not contain all its limit points in  $E$ . ■

**Theorem (4.17, Egorov's theorem).** *Suppose that  $\{f_k\}$  is a sequence of measurable functions that converge a.e. in a set  $E$  of finite measure to a finite limit  $f$ . Then given  $\varepsilon > 0$  there exists a closed subset  $F$  of  $E$  such that  $|E \setminus F| < \varepsilon$  and  $f_k \rightarrow f$  uniformly on  $F$ .*

A function  $f$  defined on a measurable set  $E$  has *property C* on  $E$  if given  $\varepsilon > 0$ , there is a closed set  $F \subset E$  such that

(i)  $|E \setminus F| < \varepsilon$

(ii)  $f$  is continuous relative to  $F$ .

**Theorem (4.20, Lusin's theorem).** *Let  $f$  be defined and finite on a measurable set  $E$ . Then  $f$  is measurable if and only if it has property C on  $E$ .*

We start with a nonnegative function  $f$  defined on a measurable subset  $E$  of  $\mathbb{R}^n$ . Let's

$$\begin{aligned} \Gamma(f, E) &= \{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{R}^{n+1} : \mathbf{x} \in E, f(\mathbf{x}) < \infty\}, \\ R(f, E) &= \{(\mathbf{x}, y) \in \mathbb{R}^{n+1} : \mathbf{x} \in E, 0 \leq y \leq f(\mathbf{x}) \text{ if } f(\mathbf{x}) < \infty \text{ and } 0 \leq y < \infty \text{ if } f(\mathbf{x}) = \infty\}. \end{aligned} \quad (22)$$

$\Gamma(f, E)$  is called the *graph of  $f$  over  $E$*  and  $R(f, E)$  the *region under  $f$  over  $E$* .

If  $R(f, E)$  is measurable (as a subset of  $\mathbb{R}^{n+1}$ ), its measure  $|R(f, E)|_{\mathbb{R}^{n+1}}$  is called the *Lebesgue integral over  $E$* , and we write

$$\int_E f(\mathbf{x}) d\mathbf{x} = |R(f, E)|_{\mathbb{R}^{n+1}}. \quad (23)$$

This is sometimes written as

$$\int_E f$$

or at times the lengthy notation

$$\int \cdots \int_E f(x_1, \dots, x_n) dx_1 \cdots dx_n$$

is convenient.

**Theorem (5.1).** *Let  $f$  be a nonnegative function defined on a measurable set  $E$ . Then  $\int_E f$  exists if and only if  $f$  is measurable.*

**Lemma (5.3).** *If  $f$  is a nonnegative measurable function on  $E$ ,  $0 \leq |E| \leq \infty$ , then  $|\Gamma(f, E)| = 0$ .*

**Theorem (5.5).** (i) *If  $f$  and  $g$  are measurable and if  $0 \leq g \leq f$  on  $E$ ,  $\int_E g \leq \int_E f$ . In particular,  $\int_E \inf f \leq \int_E f$ .*

(ii) *If  $f$  is nonnegative and measurable on  $E$  and if  $\int_E f$  is finite, then  $f < \infty$  a.e. in  $E$ .*

(iii) *Let  $E_1$  and  $E_2$  be measurable and  $E_1 \subset E_2$ . If  $f$  is nonnegative and measurable on  $E_2$ , then  $\int_{E_1} f \leq \int_{E_2} f$ .*

**Theorem (5.6, the monotone convergence theorem for nonnegative functions).** *If  $\{f_k\}$  is a sequence of nonnegative functions such that  $f_k \nearrow f$  on  $E$ , then*

$$\int_E f \rightarrow \int_E f.$$

*Proof.* By Theorem 4.12,  $f$  is measurable since it is the limit of a sequence of measurable functions. Since  $R(f_k, E) \cup \Gamma(f, E) \nearrow R(f, E)$  and  $|\Gamma(f, E)| = 0$ , the result follows by Theorem 3.26 on the measure of a monotone convergent sequences of measurable sets.  $\blacksquare$

**Theorem (5.9).** *Let  $f$  be nonnegative on  $E$ . If  $|E| = 0$ , then  $\int_E f = 0$ .*

**Theorem (5.10).** *If  $f$  and  $g$  are nonnegative and measurable on  $E$  and if  $g \leq f$  a.e. in  $E$ , then  $\int_E g \leq \int_E f$ .*

*In particular, if  $f = g$  a.e. in  $E$ , then  $\int_E f = \int_E g$ .*

**Theorem (5.11).** *Let  $f$  be nonnegative and measurable on  $E$ . Then  $\int_E f = 0$  if and only if  $f = 0$  a.e. in  $E$ .*

**Corollary (5.12, Chebyshev's inequality).** *Let  $f$  be nonnegative and measurable on  $E$ . If  $a > 0$ , then*

$$\frac{1}{a} \int_E f \geq |\{\mathbf{x} \in E : f(\mathbf{x}) > a\}|.$$

**Theorem (5.13).** *If  $f$  is nonnegative and measurable, and if  $c$  is any nonnegative constant, then*

$$\int_E cf = c \int_E f.$$

**Theorem (5.14).** *If  $f$  and  $g$  are nonnegative and measurable, then*

$$\int_E (f + g) = \int_E f + \int_E g.$$

**Corollary.** *Suppose that  $f$  and  $\varphi$  are measurable on  $E$ ,  $0 \leq f \leq \varphi$ , and  $\int_E \varphi$  is finite. Then*

$$\int_E (\varphi - f) = \int_E \varphi - \int_E f.$$

**Theorem (5.16).** *If  $f_k$ ,  $k = 1, 2, \dots$ , are nonnegative and measurable, then*

$$\int_E \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_E f_k.$$

**Theorem (5.17, Fatou's lemma).** *If  $\{f_k\}$  is a sequence of nonnegative measurable functions on  $E$ , then*

$$\int_E \liminf_{k \rightarrow \infty} f_k \leq \liminf_{k \rightarrow \infty} \int_E f_k.$$

*Proof of Fatou's lemma.* ■

**Theorem (5.19, Lebesgue's dominated convergence theorem for nonnegative functions).** *Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $E$  such that  $f_k \rightarrow f$  a.e. in  $E$ . If there exists a measurable function  $\varphi$  such that  $f_k \leq \varphi$  a.e. for all  $k$  and if  $\int_E \varphi$  is finite, then*

$$\int_E f_k \longrightarrow \int_E f.$$

**Theorem (5.21).** *Let  $f$  be measurable in  $E$ . Then  $f$  is integrable over  $E$  if and only if  $|f|$  is.*

**Theorem (5.22).** *If  $f \in L(E)$ , then  $f$  is finite a.e. in  $E$ .*

**Theorem (5.24).** *If  $\int_E f$  exists and  $E = \bigcup_{k \in \mathbb{N}} E_k$  is the countable union of disjoint measurable sets  $E_k$ , then*

$$\int_E f = \sum_{k \in \mathbb{N}} \int_{E_k} f.$$

**Theorem (5.25).** *If  $|E| = 0$  or if  $f = 0$  a.e. in  $E$ , then  $\int_E f = 0$ .*

**Theorem (5.32, monotone convergence theorem).** *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$ :*

- (i) *If  $f_k \nearrow f$  a.e. on  $E$  and there exists  $\varphi \in L(E)$  such that  $f_k \leq \varphi$  a.e. on  $E$  for all  $k$ , then  $\int_E f_k \rightarrow \int_E f$ .*
- (ii) *If  $f_k \searrow f$  a.e. on  $E$  and there exists  $\varphi \in L(E)$  such that  $f_k \leq \varphi$  a.e. on  $E$  for all  $k$ , then  $\int_E f_k \rightarrow \int_E f$ .*

**Theorem (5.33, uniform convergence theorem).** *Let  $f_k \in L(E)$  for  $k \in \mathbb{N}$  and let  $\{f_k\}$  converge uniformly to  $f$  on  $E$ ,  $|E| < \infty$ . Then  $f \in L(E)$  and  $\int_E f_k \rightarrow \int_E f$ .*

**Theorem** (5.34, Fatou's lemma). *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$ . If there exists  $\varphi \in L(E)$  such that  $f_k \geq \varphi$  a.e. on  $E$  for all  $k$ , then*

$$\int_E \liminf_{k \rightarrow \infty} f_k \leq \liminf_{k \rightarrow \infty} \int_E f_k.$$

**Corollary** (5.35, reverse Fatou's lemma). *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$ . If there exists  $\varphi \in L(E)$  such that  $f_k \leq \varphi$  a.e. on  $E$  for all  $k$ , then*

$$\int_E \limsup_{k \rightarrow \infty} f_k \geq \limsup_{k \rightarrow \infty} \int_E f_k.$$

**Theorem** (5.36, Lebesgue's dominated convergence theorem). *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$  such that  $f_k \rightarrow f$  a.e. in  $E$ . If there exists  $\varphi \in L(E)$  such that  $|f_k| \leq \varphi$  a.e. in  $E$  for all  $k \in \mathbb{N}$ , then  $\int_E f_k \rightarrow \int_E f$ .*

**Corollary** (5.37, bounded convergence theorem). *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$  such that  $f_k \rightarrow f$  a.e. in  $E$ . If  $|E| < \infty$  there is a finite constant  $M$  such that  $|f_k| \leq M$  a.e. in  $E$ , then  $\int_E f_k \rightarrow \int_E f$ .*

**Theorem** (6.1 Fubini's theorem). *Let  $f(\mathbf{x}, \mathbf{y}) \in L(I)$ ,  $I = I_1 \times I_2$ . Then*

- (i) *For almost every  $\mathbf{x} \in I_1$ ,  $f(\mathbf{x}, \mathbf{y})$  is measurable and integrable on  $I_2$  as a function of  $\mathbf{y}$ ;*
- (ii) *As a function of  $\mathbf{x}$ ,  $\int_{I_2} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$  is measurable and integrable on  $I_1$ , and*

$$\iint_I f(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = \int_{I_1} \left[ \int_{I_2} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right] d\mathbf{x}.$$

**Theorem** (6.8). *Let  $f(\mathbf{x}, \mathbf{y})$  be a measurable function defined on a measurable subset  $E$  of  $\mathbb{R}^{n+m}$ , and let  $E_{\mathbf{x}} = \{\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in E\}$ .*

- (i) *For almost every  $\mathbf{x} \in \mathbb{R}^n$ ,  $f(\mathbf{x}, \mathbf{y})$  is a measurable function of  $\mathbf{y}$  on  $E_{\mathbf{x}}$ .*
- (ii) *If  $f(\mathbf{x}, \mathbf{y}) \in L(E)$ , then for almost every  $\mathbf{x} \in \mathbb{R}^n$ ,  $f(\mathbf{x}, \mathbf{y})$  is an integrable function on  $E_{\mathbf{x}}$  with respect to  $\mathbf{y}$ ; moreover  $\int_{E_{\mathbf{x}}} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$  is an integrable function of  $\mathbf{x}$  and*

$$\iint_E f(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = \int_{\mathbb{R}^n} \left[ \int_{E_{\mathbf{x}}} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right] d\mathbf{x}.$$

**Theorem** (6.10, Tonelli's theorem). *Let  $f(\mathbf{x}, \mathbf{y})$  be nonnegative and measurable on an interval  $I = I_1 \times I_2$  of  $\mathbb{R}^{n+m}$ . Then, for almost every  $\mathbf{x} \in I_1$ ,  $f(\mathbf{x}, \mathbf{y})$  is a measurable function of  $\mathbf{y}$  on  $I_2$ . Moreover, as a function of  $\mathbf{x}$ ,  $\int_{I_2} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$  is measurable on  $I_1$ , and*

$$\iint_I f(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = \int_{I_1} \left[ \int_{I_2} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right] d\mathbf{x}$$

If  $f$  and  $g$  are measurable in  $\mathbb{R}^n$ , their *convolution*  $(f * g)(\mathbf{x})$  is defined by

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y}) d\mathbf{y},$$

provided the integral exists.

**Theorem (6.14).** *If  $f \in L(\mathbb{R}^n)$  and  $g \in L(\mathbb{R}^n)$ , then  $(f * g)(\mathbf{x})$  exists for almost every  $\mathbf{x} \in \mathbb{R}^n$  and is measurable. Moreover,  $f * g \in L(\mathbb{R}^n)$  and*

$$\begin{aligned} \int_{\mathbb{R}^n} |f * g| d\mathbf{x} &\leq \left( \int_{\mathbb{R}^n} |f| d\mathbf{x} \right) \left( \int_{\mathbb{R}^n} |g| d\mathbf{x} \right) \\ \int_{\mathbb{R}^n} (f * g)(\mathbf{x}) d\mathbf{x} &= \left( \int_{\mathbb{R}^n} f d\mathbf{x} \right) \left( \int_{\mathbb{R}^n} g d\mathbf{x} \right). \end{aligned}$$

**Corollary (6.16).** *If  $f$  and  $g$  are nonnegative and measurable on  $\mathbb{R}^n$ , then  $f * g$  is measurable on  $\mathbb{R}^n$  and*

$$\int_{\mathbb{R}^n} (f * g) d\mathbf{x} = \left( \int_{\mathbb{R}^n} f d\mathbf{x} \right) \left( \int_{\mathbb{R}^n} g d\mathbf{x} \right).$$

**Theorem (6.17, Marcinkiewicz).** *Let  $F$  be a closed subset of a bounded open interval  $(a, b)$ , and let  $\delta(x) = \delta(x, F)$  be the corresponding distance function. Then, given  $\lambda > 0$ , the integral*

$$M_\lambda(x) = \int_a^b \frac{\delta(y)^\lambda}{|x - y|^{1+\lambda}} dy$$

*is finite a.e. in  $F$ . Moreover,  $M_\lambda \in L(F)$  and*

$$\int_F M_\lambda dx \leq 2\lambda^{-1} |G|,$$

*where  $G = (a, b) \setminus F$ .*



### 1.3 Final Exam Review

Material covered since exam 2.

If  $f$  is a Riemann integrable function on an interval  $[a, b]$  in  $\mathbb{R}$ , then the familiar definition of its indefinite integral is

$$F(x) = \int_a^x f(y)dy, \quad a \leq x \leq b.$$

The fundamental theorem of calculus asserts that  $F' = f$  if  $f$  is continuous. We will study an analogue of this result for Lebesgue integrable  $f$  and higher dimensions.

We must first find an appropriate definition of the indefinite integral. In two dimensions, for example, we might choose

$$F(x_1, x_2) = \int_{a_1}^{x_1} \int_{a_2}^{x_2} f(y_1, y_2)dy_1dy_2.$$

It turns out, however, to be better to abandon the notion that the indefinite integral be a function of point and adopt the idea that it be a function of set. Thus, given  $f \in L(A)$ , where  $A$  is a measurable subset of  $\mathbb{R}^n$ , we define the *indefinite integral of  $f$*  to be the function

$$F(E) = \int_E f,$$

where  $E$  is any measurable subset of  $A$ .

$F$  is an example of a *set function*, by which we mean any real-valued function  $F$  defined on a  $\sigma$ -algebra  $\Sigma$  of measurable sets such that

- (i)  $F(E)$  is finite for every  $E \in \Sigma$ .
- (ii)  $F$  is *countably additive*; that is, if  $E = \bigcup_k E_k$  is a union of disjoint  $E_k \in \Sigma$ , then

$$F(E) = \sum_k F(E_k).$$

By Theorem 5.5 and 5.24, the indefinite integral of  $f \in L(A)$  satisfies (i) and (ii) for the  $\sigma$ -algebra of measurable subsets of  $A$ .

Recall that the diameter of a set  $E$  is the value

$$\sup\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in E\}.$$

A set function  $F(E)$  is called *continuous* if  $F(E)$  tends to zero as the diameter of  $E$  tends to zero; i.e.,  $F(E)$  is continuous if, given  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $|F(E)| < \varepsilon$  whenever the diameter of  $E$  is less than  $\delta$ . An example of a function that is *not* continuous can be obtained by setting  $F(E) = 1$  for any measurable set that contains the origin, and  $F(E) = 0$  otherwise.<sup>1</sup>

A set function  $F$  is called *absolutely continuous with respect to the Lebesgue measure*, or simply *absolutely continuous* if  $F(E)$  tends to zero as the measure of  $E$  tends to zero. Thus,  $F$  is absolutely

---

<sup>1</sup>Why is this function not continuous. Consider the following argument: Let  $\varepsilon = 1/2$  and let  $B_k = B(\mathbf{0}, 1/k)$ . Then as the diameter of  $B_k$  goes to zero,  $F(B_k) = 1$  for all  $k$  so  $F(B_k) \rightarrow 1 > 1/2$ .

continuous if given a  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|F(E)| < \varepsilon$  whenever the measure of  $E$  is less than  $\delta$ .

A set function that is absolutely continuous is clearly continuous<sup>2</sup>, however, the converse is false, as shown in the following example. Let  $A$  be the unit square in  $\mathbb{R}^2$ , let  $D$  be the diagonal of  $A$ , and consider the  $\sigma$ -algebra of measurable subsets  $E$  of  $A$  for which  $E \cap D$  is linearly measurable. For such  $E$ , let  $F(E)$  be the linear measure of  $E \cap D$ . Then  $F$  is a continuous set function. However, it is not absolutely continuous since the sets  $E$  containing a fixed segment of  $D$  whose  $\mathbb{R}^2$ -measures are arbitrarily small.

**Theorem 13** (7.1). *If  $f \in L(A)$ , its definite integral is absolutely continuous.*

*Proof.* We may assume that  $f \geq 0$  by considering  $f^+$  and  $f^-$ . Fix  $k$  and write  $f = g + h$ , where  $g = f$  whenever  $f \leq k$  and  $g = k$  otherwise. Given  $\varepsilon > 0$ , we may choose  $k$  so large that  $0 \leq \int_A h < \varepsilon/2$  and, *a fortiori*,  $0 \leq \int_E f < \varepsilon/2$ . Since

$$\int |f - C| \leq \int |f - f_{k_0}| + \int |f_{k_0} - C| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

we see that  $f$  has property  $\mathcal{A}$ .

To prove the lemma, let  $f \in L(\mathbb{R})$ . Writing  $f = f^+ - f^-$ , we may assume that  $f \geq 0$ . Then

$$\int |\chi_G - \chi_E| = |G \setminus E| < \varepsilon.$$

so we may assume that  $f = \chi_G$  for open set  $G$  of finite measure. Using Theorem 1.11, write  $G$  as the union of (partly open) disjoint intervals  $G = \bigcup I_k$ . If we let  $f_N$  be the characteristic function of  $\bigcup_{k=1}^N I_k$ , we obtain

$$\int |f - f_N| = \sum_{k=N+1}^{\infty} |I_k| \rightarrow 0$$

since  $\sum_k |I_k| = |G| < \infty$ , i.e., the series converges. By (2), it is enough to show that each  $f_N$  has property  $\mathcal{A}$ . But  $f_N$  is the sum  $\chi_{I_k}$ ,  $k = 1, \dots, N$ , so it suffices by (1) to show that the characteristic function of any partly open interval  $I$  has property  $\mathcal{A}$ . This is practically self-evident: if  $I^*$  denotes an interval that contains the closure of  $I$  in its interior and that satisfies  $|I^* \setminus I| < \varepsilon$ , then for any continuous  $C$ ,  $0 \leq C \leq 1$ , which is 1 in  $I$  and 0 outside  $I^*$ , we have

$$\int |\chi_I - C| \leq |I^* - I| < \varepsilon.$$

■

**Theorem 14** (Simple Vitali lemma). *Let  $E$  be a subset of  $\mathbb{R}^n$  with  $|E|_e < \infty$ , and let  $K$  be a collection of cubes  $Q$  covering  $E$ , then there exists a positive constant  $\beta$ , depending only on  $n$ , and a finite number of disjoint cubes,  $Q_1, \dots, Q_N$  in  $K$  such that*

$$\sum_{j=1}^N |Q_j| \geq \beta |E|_e$$

---

<sup>2</sup>Suppose  $F$  is absolutely continuous. Then, given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|F(E)| < \varepsilon$  whenever  $|E| < \delta$ .

As an application of Vitali's covering lemma, we will prove some basic result concerning the differentiability of monotone functions on  $\mathbb{R}$ . If  $f(x)$  is a real-valued function defined and finite in a neighborhood of  $x_0$ , consider the four *Dini numbers* (or *derivatives*),

$$\begin{aligned} D_1 f(x_0) &= \limsup_{h \rightarrow 0+} \frac{f(x_0 + h) - f(x_0)}{h}, \\ D_2 f(x_0) &= \limsup_{h \rightarrow 0+} \frac{f(x_0 + h) - f(x_0)}{h}, \\ D_3 f(x_0) &= \limsup_{h \rightarrow 0-} \frac{f(x_0 + h) - f(x_0)}{h}, \\ D_4 f(x_0) &= \limsup_{h \rightarrow 0-} \frac{f(x_0 + h) - f(x_0)}{h}. \end{aligned}$$

Clearly,  $D_2 f \leq D_1 f$  and  $D_4 f \leq D_3 f$ . If all four Dini numbers are equal, that is, if  $\lim_{h \rightarrow 0} [f(x_0 + h) - f(x_0)]/h$  exist, finite or infinite, we say that  $f$  has a *derivative at  $x_0$*  and call the value the *derivative  $f'(x_0)$*  at  $x_0$ . Thus,  $-\infty \leq f'(x_0) \leq \infty$  if  $f'(x_0)$  exists.

**Theorem 15** (7.21). *Let  $f$  be monotone increasing function and finite on an open interval  $(a, b) \subset \mathbb{R}$ . Then  $f$  has a measurable, nonnegative, finite derivative  $f'$  a.e. in  $(a, b)$ . Moreover,*

$$0 \leq \int_a^b f' \leq f(b-) - f(a+).$$

*Proof.* We may assume that  $(a, b)$  is finite; the general case follows from this by passage to the limit: We will show that the set  $\{x \in (a, b) : D_1 f(x) > D_4 f(x)\}$  has measure zero. A similar argument will apply to any two Dini numbers of  $f$ . It is enough to show that each set

$$A_{r,s} := \{x \in (a, b) : D_1 f(x) > r > s > D_4 f(x)\}$$

has measure zero since the original set is the union of these over rational  $r$  and  $s$ . ■

夜明け生まれくるものよ 決して見失わないで ` You are you 少女よ今こそ 振りほどき立ち  
上がれ!! 胸の傷をしたまま 風を切り進む君を ただ見守る術など何もない ° その瞳はずっと先の  
自分を探しているの でも聞こえるよ深い悲しみの鼓動が叫んでる ° I'm just 夜明け前の暗闇が  
一生を運命づける 大丈夫少女よ今こそ 悔やみ恥じることはない 十字を背負い死ぬでほのをかけ  
ろ!!

## 2 MA 544 Spring 2016

### 2.1 Exam 1 Prep

**Problem 2.1.** Let  $E \subset \mathbb{R}^n$  be a measurable set,  $r \in \mathbb{R}$  and define the set  $rE = \{r\mathbf{x} : \mathbf{x} \in E\}$ . Prove that  $rE$  is measurable, and that  $|rE| = |r|^n|E|$ .

*Proof.* Define a linear map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $\mathbf{x} \mapsto r\mathbf{x}$ . Using the standard basis for  $\mathbb{R}^n$ , this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \quad (24)$$

which has determinant  $\det T = r^n$ . By 3.35, we have  $|E| = |T(E)| = r^n|E| = |rE|$ . ■

**Problem 2.2.** Let  $\{E_k\}$ ,  $k \in \mathbb{N}$  be a collection of measurable sets. Define the set

$$\liminf_{k \rightarrow \infty} E_k = \bigcup_{k=1}^{\infty} \left( \bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|.$$

*Proof.* If the  $\liminf_{k \rightarrow \infty} |E_k| = \infty$  the inequality holds trivially. Hence, we may, without loss of generality, assume that  $\liminf_{k \rightarrow \infty} |E_k| < \infty$ . By 3.20, the set  $\liminf_{k \rightarrow \infty} E_k$  is measurable and we have

$$\left| \liminf_{k \rightarrow \infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \quad (25)$$

where  $F_k = \bigcap_{n=k}^{\infty} E_n$ . Now, note that the collection of sets  $F'_k = \bigcup_{\ell=1}^k F_\ell$  forms an increasing sequence of measurable sets  $F'_k \nearrow F'$ , where  $F' = \bigcup_{k=1}^{\infty} F_k = \liminf_{k \rightarrow \infty} E_k$ . Then, by 3.26 (i), we have

$$\lim_{k \rightarrow \infty} |F'_k| = |F'| = \left| \liminf_{k \rightarrow \infty} E_k \right|. \quad (26)$$

Hence, it suffices to show that  $|F'_k| \leq |E_k|$  for all  $k$ , but this follows by monotonicity of the outer measure, 3.3, since  $F'_k \subset E_k$ . Thus, we have the desired inequality

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|. \quad (27)$$

■

**Problem 2.3.** Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here  $B(\mathbf{0}, r) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r\}$ . Prove that  $F$  is monotonic increasing and continuous.

*Proof.* That  $F$  is increasing is immediate from the monotonicity of the outer measure since for  $x < x'$  we have  $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$  so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \leq |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that  $F$  is continuous, we will prove the following lemma

**Lemma 16.** *For any  $x > 0$ ,  $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$ .*

*Proof of lemma.* If  $\mathbf{y} \in xB(\mathbf{0}, 1)$  then  $\mathbf{y} = x\mathbf{y}'$  for  $\mathbf{y}' \in B(\mathbf{0}, 1)$ . Thus,  $|\mathbf{y}'| = |\mathbf{y}|/x < 1$  so  $|\mathbf{y}| < x$  implies that  $\mathbf{y} \in B(\mathbf{0}, x)$ . Hence, we have the containment  $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$ .

On the other hand, if  $\mathbf{y} \in B(\mathbf{0}, x)$  then  $|\mathbf{y}| < x$  so  $|\mathbf{y}|/x < 1$ . Hence,  $\mathbf{y}/x \in B(\mathbf{0}, 1)$  so  $x(\mathbf{y}/x) = \mathbf{y} \in xB(\mathbf{0}, 1)$ . Thus,  $B(\mathbf{0}, x) \subset xB(\mathbf{0}, 1)$  and equality holds. ♣

In light of Lemma 16 and 3.35, for  $x > 0$ , we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n |B(\mathbf{0}, 1)|. \quad (28)$$

It is clear that  $F$  is continuous on the interval  $[0, \infty)$  since  $F$  is a polynomial in  $x$ . ■

**Problem 2.4.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $C$  be the set of all points at which  $f$  is continuous. Show that  $C$  is a set of type  $G_\delta$ .

*Proof.* From the topological definition of continuity,  $f$  is continuous at  $x \in C$  if and only if for every neighborhood  $U$  of  $f(x)$ , the preimage  $f^{-1}(U)$  is a neighborhood of  $x$ . Now, ■

Let  $x \in C$ . Then, by the definition of continuity, for every natural number  $n > 0$  there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies

$$|f(x) - f(x')| < \frac{1}{2n}. \quad (29)$$

Let  $x'', x' \in B(x, \delta)$ . Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x'')| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \quad (30)$$

In view of these estimates, define the set

$$A_n = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \quad (31)$$

Good Lord, that was a long definition! We claim that  $C = \bigcap_{n=1}^{\infty} A_n$  and that  $A_n$  is open for all  $n$ .

First, let us show that  $C = \bigcap_{n=1}^{\infty} A_n$ . Let  $x \in C$ . Then for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . Thus,  $x \in A_n$  for all  $n$  so  $x \in \bigcap A_n$ . On the other hand, if  $x \in \bigcap A_n$  for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ .

Fix  $\varepsilon > 0$ . By the Archimedean principle, there exists  $N > 0$  such that  $\varepsilon > 1/N$ . Then, since  $x \in A_N$  it follows that for some  $\delta' > 0$ ,  $|x - x'| < \delta'$  implies  $|f(x) - f(x')| < 1/N < \varepsilon$ . Thus,  $x \in C$  and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$ .

Lastly, we show that  $A_n$  is open. Let  $x \in A_n$ . Then there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . In particular, this means that  $B(x, \delta) \subset A_n$  for any  $x' \in B(x, \delta)$  satisfies  $|f(x) - f(x')| < 1/n$ . Thus,  $A_n$  is open and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$  is a  $G_\delta$  set.

**Problem 2.5.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Is it true that if the sets  $\{f = r\}$  are measurable for all  $r \in \mathbb{R}$ , then  $f$  is measurable?

*Proof.* No. Recall that, by definition, or 4.1,  $f$  is measurable if and only if  $\{f > a\}$  for all  $a \in \mathbb{R}$ . ■

**Problem 2.6.** Let  $\{f_k\}_{k=1}^{\infty}$  be a sequence of measurable functions on  $\mathbb{R}$ . Prove that the set  $\{x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists}\}$  is measurable.

*Proof.* The idea here should be to rewrite

$$E = \left\{ x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists} \right\} \quad (32)$$

as a countable union/intersection of measurable sets. Let  $x \in E$ . By the Cauchy criterion, for every  $N > 0$  there exists a positive integer  $M$  such that  $m, n \geq M$  implies  $|f_n(x) - f_m(x)| < 1/N$ . With this in mind, define

$$E_N = \left\{ x : \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \quad (33)$$

Then, like for Problem 1.4, it is not too hard to see that the  $E_n$ 's are open and that  $E = \bigcap_{n=1}^{\infty} E_n$ . Thus,  $E$  is a  $G_\delta$  set and therefore measurable. ■

**Problem 2.7.** A real valued function  $f$  on an interval  $[a, b]$  is said to be *absolutely continuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Show that an absolutely continuous function on  $[a, b]$  is of bounded variation on  $[a, b]$ .

*Proof.* Suppose  $f: [a, b] \rightarrow \mathbb{R}$  is absolutely continuous. Then for fixed  $\varepsilon = 1$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , we have  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Let  $\Gamma = \{x_k\}_{k=1}^N$  be a partition of  $[a, b]$  into closed intervals such that  $x_{k+1} - x_k < \delta$ , then by absolute continuity we have

$$\begin{aligned} V[f; \Gamma] &= \sum_{k=1}^N |f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \quad (34)$$

Thus,  $f$  is b.v. on  $[a, b]$ . ■

**Problem 2.8.** Let  $f$  be a continuous function from  $[a, b]$  into  $\mathbb{R}$ . Let  $\chi_{\{c\}}$  be the characteristic function of a singleton  $\{c\}$ , i.e.,  $\chi_{\{c\}}(x) = 0$  if  $x \neq c$  and  $\chi_{\{c\}}(c) = 1$ . Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}.$$

*Proof.*

■

### 3 Exam 1



### 3.1 Exam 2 Prep

**Problem 3.1.** Define for  $\mathbf{x} \in \mathbb{R}^n$ ,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that  $f$  is integrable outside any ball  $B_\varepsilon(\mathbf{0})$ , and that there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x}) d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

*Proof.* Recall that a real-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is (Lebesgue) integrable over a subset  $E$  of  $\mathbb{R}^n$  (or, alternatively,  $f$  belongs to  $L(E)$ ) if

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty.$$

Put  $E = \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$ . Then, to show that  $f$  belongs to  $L(E)$  it suffices to prove the inequality

$$\int_E f(\mathbf{x}) d\mathbf{x} < \frac{C}{\varepsilon} \tag{35}$$

for some appropriate constant  $C$ . We proceed by directly computing the Lebesgue integral of  $f$  and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x}) d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{(x_1^2 + \cdots + x_n^2)^{(n+1)/2}} \end{aligned}$$

let  $E_i$  denote the projection of  $E$  onto its  $i$ -th coordinate and make the trigonometric substitution  $x_1 = \sqrt{x_2^2 + \cdots + x_n^2} \tan \theta$ ,  $dx_1 = \sqrt{x_2^2 + \cdots + x_n^2} \sec^2 \theta d\theta$  with  $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$  giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[ \frac{\cos^{n-1} \theta}{(x_2^2 + \cdots + x_n^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{(x_2^2 + \cdots + x_n^2)^{n/2}} \left[ \int_{E_\theta} \cos^{n-1} \theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1} \theta d\theta < \infty. \tag{36}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned}
\int \cdots \int_E f(\mathbf{x}) d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{x_n^2} \\
&= 2C' \int_\varepsilon^\infty \frac{dx_n}{x_n^2} \\
&= \frac{C}{\varepsilon}
\end{aligned} \tag{37}$$

as desired. ■

**Problem 3.2.** Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $\mathbb{R}^n$ , and assume that  $f_k$  converges pointwise almost everywhere to a function  $f$ . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets  $E$  of  $\mathbb{R}^n$ . Moreover, show that this is not necessarily true if  $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$ .

*Proof.* This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit  $f$  of  $\{f_k\}$  must be nonnegative a.e. in  $\mathbb{R}^n$ . For assume otherwise. Then there exists a collection of points  $\mathbf{x}$  in  $\mathbb{R}^n$  of nonzero  $\mathbb{R}^n$ -Lebesgue measure such that  $f(\mathbf{x}) < 0$ . But  $f_k(\mathbf{x}) \geq 0$  for all  $k \in \mathbb{N}$ . Set  $0 < \varepsilon < |f(\mathbf{x})|$  then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{38}$$

for all  $k$  which contradicts our assumption that  $f_k \rightarrow f$  a.e. on  $\mathbb{R}^n$ . Therefore, the set of points  $\mathbf{x} \in \mathbb{R}^n$  where  $f(\mathbf{x}) < 0$  must have measure zero.

Now, based on pointwise convergence a.e. to  $f$ , given  $\varepsilon > 0$  for a.e.  $\mathbf{x} \in \mathbb{R}^n$  we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{39}$$

for sufficiently large  $k$ ; say  $k$  greater than or equal to some index  $N \in \mathbb{N}$ . Moreover, we are given convergence in  $L(\mathbb{R}^n)$  of  $f_k$  to  $f$

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f < \infty. \tag{40}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{41}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{42}$$

for all  $k \in \mathbb{N}$ . By Theorem 5.5(ii),  $f$  and the  $f_k$ 's are finite a.e. in  $\mathbb{R}^n$  so for some sufficiently large real number  $M$ ,  $|f|, |f_k| \leq M$  for a.e.  $\mathbf{x} \in \mathbb{R}^n$ . In particular, for any measurable subset  $E$  of  $\mathbb{R}^n$ ,  $|f|, |f_k| \leq M$  for a.e.  $\mathbf{x} \in E$  so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \rightarrow \int_E f < \infty. \quad (43)$$

However, if  $f$  does not belong to  $L(\mathbb{R}^n)$ , i.e., its integral over  $\mathbb{R}^n$  is infinity, there is no guarantee that  $f$  will be finite a.e. in  $\mathbb{R}^n$ . This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset  $E$  of  $\mathbb{R}^n$ . Let us demonstrate this with an example. Consider the sequence of functions ■

**Problem 3.3.** Assume that  $E$  is a measurable set of  $\mathbb{R}^n$ , with  $|E| < \infty$ . Prove that a nonnegative function  $f$  defined on  $E$  is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}| < \infty.$$

*Proof.* If  $f$  is integrable over a measurable subset  $E$  of  $\mathbb{R}^n$ , then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \quad (44)$$

Set  $E_k = \{\mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k\}$  and  $F_k = \{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}$ . Note the following properties about the sets we have just defined: first, the  $E_k$ 's are pairwise disjoint and the  $F_k$ 's are nested in the following way  $F_{k+1} \subset F_k$ ; second,  $E = \bigcup_{k=1}^{\infty} E_k$  and  $E_k = F_k \setminus F_{k+1}$ . By Theorem 3.23, since the  $E_k$ 's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \quad (45)$$

Now, since  $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$  on  $E_k$ , we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \quad (46)$$

Then we have the following upper and lower estimates on the integral of  $f$  over  $E$

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \quad (47)$$

But note that  $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$  by Corollary 3.25 since the measures of  $E_k$ ,  $F_k$ , and  $F_{k+1}$  are all finite. Hence, (47) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \quad (48)$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned} \tag{50}$$

Thus, from (49) and (50)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \tag{51}$$

so the integral  $\int_E f$  converges if and only if the sum  $\sum_{k=0}^{\infty} |F_k|$  converges. ■

**Problem 3.4.** Suppose that  $E$  is a measurable subset of  $\mathbb{R}^n$ , with  $|E| < \infty$ . If  $f$  and  $g$  are measurable functions on  $E$ , define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$  if and only if  $f_k$  converges to  $f$  as  $k \rightarrow \infty$ .

*Proof.*  $\implies$  : First note that  $\rho$  is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$ . Then, given  $\varepsilon > 0$  there exist an

sufficiently large index  $N$  such that for every  $k \geq N$  we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \quad (52)$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in  $E$  which happens if  $|f_k - f| = 0$  a.e. in  $E$ .

$\Leftarrow$  : Suppose that  $f_k \rightarrow f$  as  $k \rightarrow \infty$ .

I don't know how to solve this. This is the intended solution:

$\Rightarrow$  : Given  $\varepsilon > 0$ ,  $\rho(f_k, f) \rightarrow 0$  implies that

$$\int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \rightarrow 0.$$

Observe that the function  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}$  given by  $\Phi(x) = x/(1+x)$  is increasing on  $\mathbb{R}^+$  and  $0 < \Phi(x) < 1$ , hence

$$\begin{aligned} \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1 + \varepsilon} |\{x \in E : |f_k(x) - f(x)| > \varepsilon\}|. \end{aligned}$$

Therefore,

$$|\{x \in E : |f_k(x) - f(x)| > \varepsilon\}| \leq \frac{1 + \varepsilon}{\varepsilon} \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \rightarrow 0$$

as  $k \rightarrow \infty$ .

$\Leftarrow$  : Conversely, given  $\delta > 0$ , we have

$$\begin{aligned} \rho(f_k, f) &= \int_{\{x \in E : |f_k(x) - f(x)| > \delta\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\quad + \int_{\{x \in E : |f_k(x) - f(x)| \leq \delta\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\leq |\{x \in E : |f_k(x) - f(x)| > \delta\}| + \frac{\delta}{1 + \delta} |E|. \end{aligned}$$

Since  $|E| < \infty$  and  $\delta/(1+\delta) \searrow 0$ , then for any  $\varepsilon > 0$ , there exists  $\delta' > 0$  such that

$$\frac{\delta'}{1 + \delta'} |E| < \frac{\varepsilon}{2}.$$

If  $f_k \rightarrow f$  as  $k \rightarrow \infty$  in measure, then for the above  $\delta'$  there is an index  $N > 0$  such that  $k \geq N$  implies

$$|\{x \in E : |f_k(x) - f(x)| > \delta'\}| < \frac{\varepsilon}{2}.$$

Therefore,  $f_k \rightarrow f$  in measure implies  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$ . ■

**Problem 3.5.** Define the *gamma function*  $\Gamma: \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function*  $\beta: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function  $u \mapsto e^{-u} u^{y-1}$  is in  $L(\mathbb{R}^+)$  for all  $y \in \mathbb{R}^+$ .

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

*Proof.* (a) Fix  $y \in \mathbb{R}^+$ . Then we must show that  $\Gamma(y) < \infty$ . First, since  $(0, 1)$  and  $[1, \infty)$  are disjoint measurable subsets of  $\mathbb{R}$ , by Theorem 5.7 we can split the integral  $\Gamma(y)$  into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u} u^{y-1} du}_{I_1} + \underbrace{\int_1^\infty e^{-u} u^{y-1} du}_{I_2}. \quad (53)$$

We will show, separately, that  $I_1$  and  $I_2$  are finite.

To see that  $I_1$  is finite, note that

$$\begin{aligned} e^{-u} u^{y-1} &= e^{-u} e^{(y-1) \log u} \\ &= e^{-u+(y-1) \log u} \\ &\leq e^{(y-1) \log u} \\ &= u^{y-1} \end{aligned} \quad (54)$$

since  $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u} u^{y-1} du \\ &\leq \int_0^1 u^{y-1} du \\ &= \left[ \frac{u^y}{y} \right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \quad (55)$$

To see that  $I_2$  is finite, note that

$$e \quad (56)$$

**Intended solution:**

(b)

■

**Problem 3.6.** Let  $f \in L(\mathbb{R}^n)$  and for  $\mathbf{h} \in \mathbb{R}^n$  define  $f_{\mathbf{h}}: \mathbb{R}^n \rightarrow \mathbb{R}$  be  $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$ . Prove that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

*Proof.* Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{57}$$

■

**Problem 3.7.** (a) If  $f_k, g_k, f, g \in L(\mathbb{R}^n)$ ,  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ ,  $|f_k| \leq g_k$  and

$$\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if  $f_k, f \in L(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e. in  $\mathbb{R}^n$ , then

$$\int_{\mathbb{R}^n} |f_k - f| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \rightarrow \int_{\mathbb{R}^n} |f| \quad \text{as} \quad k \rightarrow \infty.$$

*Proof.* (a) Since  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. and  $|f_k| \leq g_k$ , then by Fatou's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} (g - f) &= \int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} g_k - f_k \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_k - f_k, \\ \int_{\mathbb{R}^n} g + f &= \int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} g_k + f_k \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_k + f_k. \end{aligned}$$

Since  $f_k, g_k, f, g \in L(\mathbb{R}^n)$  and  $\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g$ , then using the similar argument as problem 2, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f &\geq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k, \\ \int_{\mathbb{R}^n} f &\leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k. \end{aligned}$$

Therefore,  $\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f$ .

(b)  $\Rightarrow$  : This direction is obvious by the inequality

$$\left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \leq \int_{\mathbb{R}^n} ||f_k| - |f|| \leq \int_{\mathbb{R}^n} |f_k - f|.$$

$\Leftarrow$  : Let  $g_k = |f_k| + |f|$  and  $g = 2|f|$ . Since  $f_k, f \in L(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e., then  $g_k, g \in L(\mathbb{R}^n)$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ . By the assumption,  $\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g$ .

Let  $\tilde{f}_k = |f_k - f|$ . Then  $\tilde{f}_k \rightarrow 0$  a.e. in  $\mathbb{R}^n$  and  $\tilde{f}_k \leq g_k$ . Applying part (a) to  $\tilde{f}_k$  we have

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

■



### 3.2 MA 544 - Midterm 2

**Problem 3.8.** Assume that  $f \in L(\mathbb{R}^n)$ . Show that for every  $\varepsilon > 0$  there exists a ball  $B$ , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

*Proof.* Recall that  $f \in L(\mathbb{R}^n)$  if and only if  $|f| \in L(\mathbb{R}^n)$ . Let  $B_k = B(\mathbf{0}, k)$  for  $k \in \mathbb{N}$  and  $\chi_{B_k}$  be the indicator function associated with  $B_k$ . Then, the sequence of maps  $\{|f_k|\}$  defined  $f_k = f\chi_{B_k}$  converge pointwise to  $|f|$ . Since  $|f| \in L(\mathbb{R}^n)$ , by the monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbb{R}^n} |f|. \quad (58)$$

But this means, exactly, that for every  $\varepsilon > 0$  there exists sufficiently large  $N \in \mathbb{N}$  such that

$$\begin{aligned} \varepsilon &> \left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \\ &= - \int_{\mathbb{R}^n} |f_k| + \int_{\mathbb{R}^n} |f| \\ &= - \int_{\mathbb{R}^n} |f| + \int_{\mathbb{R}^n} |f| \\ &= - \int_{B_k} |f| + \int_{\mathbb{R}^n} |f| \\ &= \int_{\mathbb{R}^n \setminus B_k} |f| \end{aligned} \quad (59)$$

as desired. ■

**Problem 3.9.** Let  $f \in L(E)$ , and let  $\{E_j\}$  be a countable collection of pairwise disjoint measurable subsets of  $E$ , such that  $E = \bigcup_{j=1}^{\infty} E_j$ . Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

*Proof.* First, since the  $E_j$ 's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \quad (60)$$

Let  $\chi_{E_j}$  be the characteristic function of the subset  $E_j$  of  $E$  and define  $f_j = f\chi_{E_j}$  for  $j \in \mathbb{N}$ . Note that, since both  $f$  and  $\chi_{E_j}$  are measurable on  $E$ ,  $f_j$  is measurable on  $E$  and  $\sum_{j=1}^{\infty} f_j = f$ . Moreover, since  $E_j \subset E$ , by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \setminus E_j} f = \int_E f_j + \int_{E \setminus E_j} f. \quad (61)$$

Hence, because the  $E_j$ 's are disjoint  $(E \setminus E_k) \setminus E_\ell = (E \setminus E_\ell) \setminus E_k$  so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \quad (62)$$

as desired. ■

**Problem 3.10.** Let  $\{f_k\}$  be a family in  $L(E)$  satisfying the following property: For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|A| < \delta$  implies

$$\int_A |f_k| < \varepsilon$$

for all  $k \in \mathbb{N}$ . Assume  $|E| < \infty$ , and  $f_k(x) \rightarrow f(x)$  as  $k \rightarrow \infty$  for a.e.  $x \in E$ . Show that

$$\lim_{k \rightarrow \infty} \int_E f_k = \int_E f.$$

(Hint: Use Egorov's theorem.)

*Proof.* Let  $\varepsilon > 0$  be given. Then, by the hypothesis, there exists  $\delta > 0$  such that  $|A| < \delta$  implies

$$\int_A |f_k| < \varepsilon \quad (63)$$

for all  $k \in \mathbb{N}$ . By Egorov's theorem, there exists a closed subset  $F$  of  $E$  such that  $|E \setminus F| < \delta$  and  $f_k \rightarrow f$  uniformly on  $F$ . Then, by the uniform convergence theorem,

$$\int_F f_k \rightarrow \int_F f \quad (64)$$

as  $k \rightarrow \infty$ . But by hypothesis, we have

$$\int_{E \setminus F} |f_k| < \varepsilon. \quad (65)$$

Letting  $\varepsilon \rightarrow 0$ , we achieved the desired convergence. ■

**Problem 3.11.** Let  $I = [0, 1]$ ,  $f \in L(I)$ , and define  $g(x) = \int_x^1 t^{-1} f(t) dt$  for  $x \in I$ . Prove that  $g \in L(I)$  and

$$\int_I g = \int_I f.$$

*Proof.* By Lusin's theorem, there exists a closed subset  $F$  of  $I$  with  $|I \setminus F| < \varepsilon$  such that the restriction of  $f$  to  $F = I \setminus E$  is continuous. Now, since  $F$  is closed in  $I$  and  $I$  is compact, it follows that  $F$  is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials  $\{p_k\}$  such that  $p_k \rightarrow f$  uniformly on  $F$ . Then, by the uniform convergence theorem, we have

$$\int_F p_k \rightarrow \int_F f \quad (66)$$

so

$$\begin{aligned}
\int_F \left[ \int_x^1 t^{-1} p_k(t) dt \right] dx &= \int_F \left[ \int_x^1 a t^{-1} + q_k(t) dt \right] dx \\
&= \int_F q'_k(x) - a \log(x) dx \\
&< \infty
\end{aligned} \tag{67}$$

for all  $k$  and converges uniformly to  $g$  so  $g \in L(I)$ . I don't know how to show that in fact  $\int_I g = \int_I f$ . Perhaps you show that the places where they differ is a set of measure zero. ■