MA 544: Homework 7

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February 28, 2016

Problem 7.1 (Wheeden & Zygmund §4, Ex. 9)

(a) Show that the limit of a decreasing (increasing) sequence of functions usc (lsc) at $\mathbf{x}_0$ is usc (lsc) at $\mathbf{x}_0$. In particular, the limit of a decreasing (increasing) sequence of functions continuous at $\mathbf{x}_0$ is usc (lsc) at $\mathbf{x}_0$.

(b) Let f be usc and less than ∞ on $[a,b]$. Show that there exists continuous f_k on $[a,b]$ such that $f_k \searrow f$.

Proof. (a) Without loss of generality, assume that the sequence of f_k's are define in all of $\mathbf{R}^n$. Suppose $\{f_k \colon \mathbf{R}^n \to \mathbf{R}\}_{k=1}^{\infty}$ is a sequence of decreasing functions which are usc at $\mathbf{x}_0$ and put $f \coloneqq \lim f_k$. Then, since the f_k's are decreasing, we have

$$f(\mathbf{x}) \leq f_k(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathbf{R}^n. \tag{7.1}$$

Moreover, f_k being usc at $\mathbf{x}_0$ means that for any sequence $\mathbf{x} \to \mathbf{x}_0$, $\overline{\lim}_{\mathbf{x}\to\mathbf{x}_0} f_k(\mathbf{x}) \leq f_k(\mathbf{x}_0)$. Thus, by (7.1) we have

$$\overline{\lim_{\mathbf{x}\to\mathbf{x}_0}} f(\mathbf{x}) \leq \overline{\lim_{\mathbf{x}\to\mathbf{x}_0}} f_k(\mathbf{x}) \leq f_k(\mathbf{x}_0). \tag{7.2}$$

Now, let $k \to \infty$ and we have $\overline{\lim}_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) \leq f(\mathbf{x}_0)$. Thus, f is usc at $\mathbf{x}_0$.

(b) Suppose f is usc and bounded on $[a,b]$. ■

Problem 7.2 (Wheeden & Zygmund §4, Ex. 11)

Let f be defined on $\mathbf{R}^n$ and let $B(\mathbf{x})$ denote the open ball $\{\, \mathbf{y} \mid |\mathbf{x}-\mathbf{y}| < r \,\}$ with center $\mathbf{x}$ and fixed radius r. Show that the function $g(\mathbf{x}) = \sup\{\, f(\mathbf{y}) \mid \mathbf{y} \in B(\mathbf{x}) \,\}$ is lsc and the function $h(\mathbf{x}) = \inf\{\, f(\mathbf{y}) \mid \mathbf{y} \in B(\mathbf{x}) \,\}$ is usc on $\mathbf{R}^n$. Is the same true for the closed ball $\{\, \mathbf{y} \mid |\mathbf{x}-\mathbf{y}| \le r \,\}$?

Proof. Fix $\mathbf{x}_0 \in \mathbf{R}^n$. Given any sequence $\mathbf{x} \to \mathbf{x}_0$, we must show that $\underline{\lim}_{x\to\mathbf{x}_0} g(\mathbf{x}) \ge g(\mathbf{x}_0)$. Take any such sequence $\mathbf{x}_k \to \mathbf{x}_0$ then convergence of such sequence means that for any $\varepsilon > 0$, the sequence $\{\mathbf{x}_k\}$ is eventually in $B_\delta(\mathbf{x}_0)$, i.e., for some index $N \in \mathbf{N}$ depending on ε, $\mathbf{x}_n \in B_\delta(\mathbf{x}_0)$ for all $n \ge N$. Hence, we have $g(\mathbf{x}_0) \le f(\mathbf{x}_n) \le g(\mathbf{x}_n)$. Passing to the lim sup, we have $g(\mathbf{x}_0) \le \overline{\lim}_{\mathbf{x}_k\to\mathbf{x}_0} g(\mathbf{x})$. Thus, g is lsc.

Fix $\mathbf{x}_0 \in \mathbf{R}^n$. Then, given any sequence $\mathbf{x}_k \to \mathbf{x}_0$ we must show that $\underline{\lim}_{\mathbf{x}_k\to\mathbf{x}_0} g(\mathbf{x}_k) \ge g(\mathbf{x}_0)$. Let $\mathbf{x}_k \to \mathbf{x}_0$ be a convergent sequence then convergence of said sequence means that for any $\varepsilon > 0$, $\mathbf{x}_k$ is eventually in $B_\varepsilon(\mathbf{x}_0)$, i.e., for some index $N \in \mathbf{N}$ depending on ε, $\mathbf{x}_n \in B_\varepsilon(\mathbf{x}_0)$ for $n \ge N$. By the definition of the supremum we have there exists $\mathbf{y} \in B(\mathbf{x}_0)$ such that $g(\mathbf{x}_0) < f(\mathbf{y}) + \varepsilon$. ∎

Problem 7.3 (Wheeden & Zygmund §4, Ex. 15)

Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. If $|f_k(M)| \leq M < \infty$ for all k for each $\mathbf{x} \in E$, show that given $\varepsilon > 0$, there is closed $F \subset E$ and finite M such that $|E \smallsetminus F| < \varepsilon$ and $|f_k(\mathbf{x})| \leq M$ for all $\mathbf{x} \in F$.

Proof. Define $f := \sup |f_k|$. Note that, since $|f_k| = f^+ + f^-$ and f^+ and f^- are measurable, $|f_k|$ is measurable hence, by 4.11, f is measurable. Now, given $\varepsilon > 0$ by Lusin's theorem f has the $\mathcal{C}$-property on E, i.e., there exists a closed subset F of E such that $|E \smallsetminus F| < \varepsilon/2$ and f is continuous when restricted to F. Take the $\delta > 0$ such that $\left|E \smallsetminus \overline{B_\delta(\mathbf{0})}\right| < \varepsilon/2$. Then $F \cap \overline{B_\delta(\mathbf{0})}$ is closed and compact and we have

$$\begin{aligned}\left|E \smallsetminus (F \cap \overline{B_\delta(\mathbf{0})})\right| &= \left|E \smallsetminus F \cup \left(E \smallsetminus \overline{B_\delta(\mathbf{0})}\right)\right| \\ &\leq |E \smallsetminus F| + \left|E \smallsetminus \overline{B_\delta(\mathbf{0})}\right| \\ &< \varepsilon.\end{aligned}$$

By Problem 6.2 (W&Z, 4.7) f achieves its maximum M on $F \cap \overline{B_\delta(\mathbf{0})}$. Thus, $|f_k| \leq M$ for all $\mathbf{x} \in F \cap \overline{B_\delta(\mathbf{0})}$. ∎

Problem 7.4 (Wheeden & Zygmund §4, Ex. 18)

If f is measurable on E, define $\omega_f(a) = |\{ f > a \}|$ for $-\infty < a < \infty$. If $f_k \nearrow f$, show that $\omega_{f_k} \nearrow \omega_f$. If $f_k \to f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f. [For the second part, show that if $f_k \to f$, then $\overline{\lim}_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$ and $\underline{\lim}_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$ for every $\varepsilon > 0$.]

Proof. ■

Problem 7.5 (Wheeden & Zygmund §5, Ex. 1)

If f is a simple measurable function (not necessarily positive) taking values a_j on E_j, $j = 1, ..., N$, show that $\int_E f = \sum_{j=1}^N a_j |E_j|$. [Use (5.24)].

Proof. Since f is a simple measurable function $E_k \cap E_\ell = \emptyset$ for $k \neq \ell$. Since $E \coloneqq \bigcup_{j=1}^N E_j$ is countable, by 5.24 we have

$$\int_E f = \sum_{j=1}^N \int_{E_j} f = \sum_{j=1}^N \int a_j \chi_{E_j} = \sum_{j=1}^N a_j |E_j|.$$

■

Problem 7.6 (Wheeden & Zygmund §5, Ex. 3)

Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \leq f$ a.e. on E, show that $\int_E f_k \to \int_E f$.

Proof. ∎