MA 519: Homework 8

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PROBLEM 8.1 (HANDOUT 12, # 2)

Let X be Uniform$[a, b]$. Find the PDF, CDF, mean, and variance of X.

SOLUTION. By definition, the PDF of X is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Next we compute the CDF:

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} f(y)\, dy \\ &= \begin{cases} \int_{-\infty}^{x} 0\, dy & \text{if } x < a, \\ \int_{-\infty}^{a} 0\, dy + \int_{a}^{x} \frac{1}{b-a}\, dy & \text{if } a \leq x < b, \\ \int_{-\infty}^{a} 0\, dy + \int_{a}^{b} \frac{1}{b-a}\, dy + \int_{b}^{\infty} 0\, dy & \text{if } x \geq b \end{cases} \\ &= \begin{cases} 0 & \text{if } x < a, \\ \frac{x-a}{b-a} & \text{if } a \leq x < b, \\ 1 & \text{if } x \geq b. \end{cases} \end{aligned}$$

The mean of X is given by

$$\int_{a}^{b} \frac{x}{b-a}\, dx = \frac{b-a}{2}$$

The second moment of X is given by

$$\int_{a}^{b} \frac{x^2}{b-a}\, dx = \frac{b^3 - a^3}{3(b-a)}$$

so that the variance of X is given by

$$E(X^2) - E(X)^2 = \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b-a}{2}\right)^2$$

∎

PROBLEM 8.2 (HANDOUT 12, # 8)

The diameter of a circular disk cut out by a machine has the following PDF

$$f(x) = \begin{cases} \frac{4x - x^2}{9} & \text{if } 1 \le x \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Find the average diameter of disks coming from this machine (in inches).

SOLUTION. The expected value of the diameter of the disks coming from this machine is

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_{1}^{4} x \frac{4x - x^2}{9}\, dx \\ &= \int_{1}^{4} \frac{4x^2 - x^3}{9}\, dx \\ &= \int_{1}^{4} \frac{4x^2 - x^3}{9}\, dx \\ &= \frac{9}{4} \\ &= 2.25 \end{aligned}$$

∎

PROBLEM 8.3 (HANDOUT 12, # 9)

Suppose X is Uniform$[0, 2\pi]$. Find $P(-0.5 \leq \sin X \leq 0.5)$.

SOLUTION. Note to the grader: the sheet says "$[0, 2]$". After talking with Dr. Dasgupta, he told me that he meant "$[0, 2\pi]$."

The probability that $sin(X) \in [-0.5, 0.5]$ is the same as the probability that $X \in [0, \pi/6] \cup [5\pi/6, 7\pi/6] \cup [11\pi/6, 2\pi]$.

Because X is uniform, this is

$$\frac{1}{2\pi} \left(\frac{\pi}{6} + \frac{2\pi}{6} + \frac{\pi}{6} \right) = \frac{1}{3} \approx 0.33333$$

∎

PROBLEM 8.4 (HANDOUT 12, # 13)

X has a piecewise uniform distribution on $[0, 1]$, $[1, 3]$, $[3, 6]$, and $[6, 10]$. Write its density function.

SOLUTION. Let f be the PDF of X. Then

$$f(x) = \begin{cases} \frac{12}{25} & x \in [0, 1) \\ \frac{6}{25} & x \in [1, 3) \\ \frac{4}{25} & x \in [3, 6) \\ \frac{3}{25} & x \in [6, 10) \end{cases}$$

∎

PROBLEM 8.5 (HANDOUT 12, # 16)

Show that for every p, $0 \leq p \leq 1$, the function $f(x) = p \sin x + (1 - p) \cos x$, $0 \leq x \leq \pi/2$ (and $f(x) = 0$ otherwise), is a density function. Find its CDF and use it to find all the medians.

SOLUTION. Let $p \in [0, 1]$. Let $f_p(x) = p \sin x + (1 - p) \cos x$. Then

$$\int_0^{\pi/2} f_p(x)\, dx = \int_0^{\pi/2} p \sin(x) + (1 - p) \cos(x)\, dx$$
$$= p + 1 - p$$
$$= 1$$

and because $\sin(x)$ and $\cos(x)$ are both positive on $[0, \pi/2]$ and p and $1 - p$ are both positive, then f_p is positive on $[0, \pi/2]$. That is, f_p is a PDF.

Now, the CDF of f_p is

$$F_p(x) = \int_0^x f_p(y)\, dy$$
$$= p(1 - \cos(x)) + (1 - p) \sin(x)$$

Last, our median occurs when $F_p(x) = \frac{1}{2}$.   ∎

PROBLEM 8.6 (HANDOUT 12, # 17)

Give an example of a density function on $[0, 1]$ by giving a formula such that the density is finite at zero, unbounded at one, has a unique minimum in the open interval $(0, 1)$ and such that the median is 0.5.

SOLUTION. The density function given by

$$f(x) = \begin{cases} 2(1/2 - x) & x \in [0, 1/2), \\ 8(x - 1/2) & x \in [1/2, 3/4), \\ -\frac{1}{1+\ln(4)} \ln(-(x-1)) & x \in [3/4, 1), \\ 0 & x \notin [0, 1) \end{cases}$$

satisfies these requirements. First, by integrating, we see that $\int_0^1 f(x)\,dx = 1$;

$$\int_0^1 f(x)\,dx = \int_0^{1/2} 2(1/2 - x)\,dx + \int_{1/2}^{3/4} 8(x - 1/2)\,dx + \int_{3/4}^1 -\frac{1}{1+\ln(4)}\ln(-(x-1))\,dx$$
$$= 1/2 + 1/4 + 1/4 = 1.$$

It is also somewhat clear that f is nonnegative. Also, f is finite at 0 (indeed, $f(0) = 1$), that the median of f is taken at $1/2$, that f is unbounded at 1, and that the unique minimum of f occurs at $1/2$.   ■

Problem 8.7 (Handout 12, # 18)

(*A Mixed Distribution*). Suppose the damage claims on a particular type of insurance policy are uniformly distributed on $[0, 5]$ (in thousands of dollars), but the maximum by the insurance company is 2500 dollars. Find the CDF and the expected value of the payout, and plot the CDF. What is unusual about this CDF?

Solution. The CDF of the payout is given by

$$F(x) = \begin{cases} 1.25x & x \in [0, 2.5), \\ 1 & x \in [2.5, \infty) \end{cases}$$

The expected value of the payout is

$$\begin{aligned} \int_0^5 f(x)\, dx &= \int_0^{2.5} x f(x) + \frac{2.5}{2} \\ &= \int_0^{2.5} x \frac{1/5}{+} \frac{2.5}{2} \\ &= 0.625 + 1.25 \\ &= 1.875 \end{aligned}$$

(with this figure being in thousands of dollars).

The CDF is discontinuous at 2.5, which is somewhat interesting.   ■

PROBLEM 8.8 (HANDOUT 12, # 19)

(*Random Distribution*). Jen's dog broke her six-inch long pencil off at a random point on the pencil. Find the density function and the expected value of the ratio of the lengths of the shorter piece and the longer piece of the pencil.

SOLUTION. Let $I = [0, 6]$ denote Jen's six-inch pencil and let $X \colon \Omega \to I \smallsetminus \{0, 6\}$ denote the point in I (excluding the end points) at which Jen's dog broke her pencil. Assuming $X \sim \text{Uniform}[0, 6]$, the PDF of X is given by

$$f_X(x) = \begin{cases} \frac{1}{6} & \text{if } 0 \leq x \leq 6, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we wish to find the PDF and mean of Y the ratio of the lengths of the shorter piece and the longer piece of the pencil. First, let us write Y in terms of X:

$$Y = \begin{cases} \frac{X}{6-X} & \text{if } 0 < X < 3, \\ \frac{6-X}{X} & \text{if } 3 \leq X < 6, \\ 0 & \text{otherwise.} \end{cases}$$

To find the PDF of Y we first compute its CDF and differentiate:

$$F_Y(x) = P(Y \leq x)$$

$$= \begin{cases} P(0 \leq 0) & \text{if } x \leq 0 \\ P(0 \leq 0) + P\left(\frac{X}{6-X} \leq x\right) & \text{if } 0 < x < 3, \\ P(0 \leq 0) + P\left(\frac{X}{6-X} \leq 3\right) + P\left(\frac{6-X}{X} \leq x\right) & \text{if } 3 \leq x \end{cases}$$

$$=$$

∎

PROBLEM 8.9 (HANDOUT 12, # 20)

(*Square of a PDF Need Not Be a PDF*). Give an example of a density function $f(x)$ on $[0, 1]$ such that $cf^2(x)$ cannot be a density function for any c.

SOLUTION. Set $C = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)}$. Consider $f : [0, 1] \to$ where $f(x) = \frac{1}{C}\sqrt{n}$ if $x \in [\frac{1}{n+1}, \frac{1}{n}]$. Then

$$\int_0^1 f(x)\,dx = \frac{1}{C}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} = 1,$$

but

$$\int_0^1 f(x)^2\,dx = \frac{1}{C^2}\sum_{n=1}^{\infty} \frac{1}{n+1}$$

which diverges. That is, f is a density function, but f^2 cannot be rescaled to be a density function. ∎

PROBLEM 8.10 (HANDOUT 12, # 21)

(*Percentiles of the Standard Cauchy*). Find the pth percentile of the standard Cauchy density for a general p, and compute it for $p = 0.75$.

SOLUTION.   ∎

Problem 8.11 (Handout 12, # 22)

(*Integer Part*). Suppose X has a uniform distribution on $[0, 10.5]$. Find the expected value of the integer part of X.

Solution.   ∎

PROBLEM 8.12 (HANDOUT 12, # 23)

X is uniformly distributed on some interval $[a, b]$. If its mean is 2, and variance is 3, what are the values of a, b?

SOLUTION. From the first problem in this homework, this means that

$$2 = \frac{1}{2}(b + a)$$

$$3 = \frac{b^3 - a^3}{3(b - a)} - \left(\frac{b - a}{2}\right)^2$$

This system of equations is satisfied by

$$a = 2 - \sqrt{\frac{3}{2}}$$

$$b = 2 + \sqrt{\frac{3}{2}}$$

∎