MA572 Hatcher Notes

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1 Homology

A summary of Hatcher's homology section from his *Algebraic Topology* book.

1.1 Simplicial and Signular Homology

Skip all this nonsense. I need to catch up.

1.2 Computations and Applications

Degree

For a map $f\colon S^n \to S^n$ with $n > 0$, the induced map $f_*\colon H_n(S^n) \to H_n(S^n)$ is a homomorphism from an infinite cyclic group to itself and so must be of the form $f_*(\alpha) = df(\alpha)$ for some integer d depending only on f. This integer is called the *degree* of f and is denoted by $\deg f$. Here are some basic properties of the degree

(1) $\deg \mathrm{id}_{S^n} = 1$ since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{H_n(S^n)}$.

(2) $\deg f = 0$ if f is not injective. For if we choose a point $x_0 \in S^n \smallsetminus f(S^n)$ then f can be factored as a composition $S^n \to S^n \smallsetminus \{x_0\} \hookrightarrow S^n$ and $H_n(S^n \smallsetminus \{x_0\}) = 0$ since $S^n \smallsetminus \{x_0\}$ is contractible.

(3) If $f \simeq g$ then $\deg f = \deg g$ since $f_* = g_*$. The converse statement, that if $\deg f = \deg g$, is a fundamental theorem of Hopf from around 1925 which we prove in Corollary 4.25.

(4) $\deg fg = \deg f \deg g$, since $(f \circ g)_* = f_* \circ g_*$. As a consequence, $\deg f = \pm 1$ if f is a homotopy equivalence since $f \circ g \simeq \mathrm{id}_{S^n}$ implies that $\deg f \deg g = \deg \mathrm{id}_{S^n} = 1$.

(5) $\deg f = -1$ if f is a reflection of S^n, fixing the points in some subsphere $S^{n-1} \subset S^n$ and interchanging the two complementary hemispheres. For we can give S^n a Δ-complex structure with these two hemispheres as its two n-simplices Δ_1^n and Δ_2^n, and the n-chain $\Delta_1^n - \Delta_2^n$ represents a generator of $H_n(S^n)$ as we saw in Example 2.23, so the reflection interchanging Δ_1^n and Δ_2^n sends this generator to its negative.

(6) The antipodal map $a\colon S^n \to S^n$, $x \mapsto -x$, has degree $(-1)^{n+1}$ since it is the composition of $n+1$ reflections, each changing the sign of one coordinate in $\mathbf{R}^{n+1}$.

(7) If $f\colon S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$. For if $f(x) \neq x$ for any $x \in S^n$, then the line segment from $f(x)$ to $-x$, defined by $t \mapsto (1-t)f(x) - tx$ for $0 \le t \le 1$, does not pass through the origin. Hence if f has no fixed points, the formula $f_t(x) \coloneqq [(1-t)f(x) - tx]/\|(1-t)f(x) - tx\|$ defines a homotopy from f to the antipodal map. Note that the antipodal map has no fixed points, so the fact that maps without fixed points are homotopic to the antipodal point is sort of a converse statement.

Theorem 1 (2.8)**.** *S^n has a continuous field of nonzero tangent vectors if and only if n is odd.*

Proposition (2.29)**.** *$\mathbf{Z}/2\mathbf{Z}$ is the only nontrivial group that can act freely on S^n if n is even.*

Recall that the action of a group G on a space X is a homomorphism from G to the group $\mathrm{Homeo}(X)$ of homeomorphisms $X \to X$, and the action is free if the homeomorphism corresponding to each nontrivial element of G has no fixed points. In the case of S^n, the antipodal map $x \mapsto -x$ generates a free action of $\mathbf{Z}/2\mathbf{Z}$.

Next we describe a technique for computing degrees which can be applied to most maps that arise in practice. Suppose $f\colon S^n \to S^n$, $n > 0$, has the property that for some point $y \in S^n$, the preimage $f^{-1}(y)$ consists of only finitely many points, say $x_1, \ldots, x_m$. Let $U_1, \ldots, U_m$ be disjoint neighborhoods of these points, mapped by f into a neighborhood V of y. Then $f(U_i \smallsetminus \{x_i\}) \subset V \smallsetminus \{y_i\}$ for each i, and we have a commutative diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i \smallsetminus \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V \smallsetminus \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \smallsetminus \{x_i\}) & H_n(S^n, S^n \smallsetminus \{f^{-1}(y)\}) \arrow[l, "p_i"] \arrow[r, "f_*"] & H_n(S^n, S^n \smallsetminus \{y\}) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\cong"] \arrow[u, "j"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

where all the maps are the obvious ones, and in particular k_i and p_i are induced by inclusions, so the triangles and squares commute. The two isomorphisms in the upper half of the diagram come from excision, while the lower two isomorphisms come from exact sequences of pairs. Via these four isomorphisms, the top two groups in the diagram can be identified with $H_n(S^n) \cong \mathbf{Z}$, and the top homomorphism f_* becomes multiplication by an integer called the *local degree* of f at x_i, written $\deg f|_{x_i}$.

For example, if f is a homeomorphism, then y can be any point and there is only one corresponding to x_i, so all the maps in the diagram are isomorphisms and $\deg f|_{x_i} = \deg f = \pm 1$. The situation occurs quite often in applications, and it is usually not hard to determine the correct signs.

Here is the formula that reduces degree calculations to computing local degrees:

Proposition 2 (2.30)**.** $\deg f = \sum_i \deg f|_{x_i}$.

Proof. By excision the central term $H_n(S^n, S^n \smallsetminus \{f^{-1}(y)\})$ in the preceding lemma is the direct sum of the groups $H_n(U_i, U_i \smallsetminus \{x_i\}) \cong \mathbf{Z}$, with k_i the inclusion of the i-th summand and p_i the projection onto the ith summand. Identifying the outer groups in the diagram with $\mathbf{Z}$ as before, commutativity of the lower triangle says that that $p_i \circ j(1) = 1$, hence $j(1) = (1, \ldots, 1) = \sum_i k_i(1)$. Commutativity of the upper square says that the middle f_* takes $k_i(1)$ to $\deg f|_{x_i}$, hence the sum $\sum_i k_i(1)$ is taken to $\sum_i \deg f|_{x_i}$. Commutativity of the lower square then gives the formula $\deg f = \sum_i \deg f|_{x_i}$. ■