MA557 Problem Set 3

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Problem 3.1

Find an example of a finitely generated ring extension $R \subset S$ where S is a Noetherian ring, but R is not.

Proof. Let k be a field and consider its polynomial ring $k[X, Y]$ in two variables. Then we claim that the subring $k[XY, XY^2, ...]$ is non-Noetherian but its extension to $k[X, Y]$ (by adjoining the indeterminants X and Y) is Noetherian by Hilbert's basis theorem. Consider the increasing chain of ideals

$$(XY) \subsetneq (XY, XY^2) \subsetneq (XY, XY^2, XY^3) \subsetneq \cdots .$$

This chain does not stabilize for suppose that it did, then for some positive integer n, we have $(XY, XY^2, ..., XY^n) = (XY, XY^2, ..., XY^n, XY^{n+1})$ so $XY^{n+1} \in (XY, XY^2, ..., XY^n)$. But this implies that $XY^{n+1} = p(XY, XY^2, ...)q(XY, ..., XY^n)$ for some polynomials $q \in k[XY, XY^2, ...]$, $q \in (XY, ..., XY^n)$. Thus, we have that

$$\begin{aligned} \deg_Y(XY^{n+1}) &= n + 1 \\ &= \deg_Y p + \deg_Y q \end{aligned} \qquad \begin{aligned} \deg_X(XY^{n+1}) &= 1 \\ &= \deg_X p + \deg_X q. \end{aligned}$$

Since $\deg_Y q \leq n$, $\deg_Y p \geq 1$. Thus, $\deg_X p = 1$ so $q \in k$, i.e., q is a unit. This is a contradiction since $(XY, ..., XY^n)$ is a proper ideal. ■

Problem 3.2

Consider the homomorphism of rings

$$\begin{tikzcd} & S \arrow[d, "\psi"] \\ R \arrow[r, "\varphi"] & T. \end{tikzcd}$$

The *fiber product* of R and S over T is the subring $R \times_T S = \{ (r,s) \mid \varphi(t) = \psi(s) \}$ of $R \times S$. Assume φ and ψ are surjective. Show that if R and S are Noetherian rings then so is $R \times_T S$.

Proof. We first prove the following result:

Lemma 1 (Matsumura, Ex. 3.1)**.** *Let $\mathfrak{a}_1, ..., \mathfrak{a}_n$ be ideals of a ring A such that $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = 0$. If each $A/\mathfrak{a}_i$ is a Noetherian ring then so is R.*

Proof of lemma. If the $\mathfrak{a}_1, .., \mathfrak{a}_n$ are coprime, by the Chinese remainder theorem, we have $A \cong A_1/\mathfrak{a}_1 \times \cdots A_n/\mathfrak{a}_n$ so A is Noetherian. Otherwise, we have a canonical injection $\varphi\colon A \hookrightarrow A_1/\mathfrak{a}_1 \times \cdots \times A_n/\mathfrak{a}_n$ which gives rise to the exact sequence of A-modules

$$0 \longrightarrow A \overset{\varphi}{\hookrightarrow} \frac{A}{\mathfrak{a}_1} \times \cdots \times \frac{A}{\mathfrak{a}_n} \longrightarrow \operatorname{coker} \varphi \longrightarrow 0.$$

By 3.4(a), R is Noetherian. ♣

Now consider the canonical projections $\pi_R\colon R \times_T S \to R$ and $\pi_S\colon R \times_T S \to S$. Then, by the isomorphism theorem we have $R \cong R \times_T S/\ker \pi_R$ and $S \cong R \times_T S/\ker \pi_S$. Then we have the following

$$\begin{tikzcd} 0 \arrow[r] & \ker \pi_R \arrow[dr, hook] & & R \arrow[r] & 0 \\ & & R \times_T S \arrow[ur, "\pi_R"] \arrow[dr, "\pi_S"'] & & \\ 0 \arrow[r] & \ker \pi_S \arrow[ur, hook] & & S \arrow[r] & 0 \end{tikzcd}$$

which is short exact along the top and bottom, left to right. Since

$$R \times S \cong \frac{R \times_T S}{\ker \pi_R} \times \frac{R \times_T S}{\ker \pi_S}$$

by Lemma 1, we need only show that $\ker \pi_R \cap \ker \pi_S = 0$. But this is straightforward for suppose $(x,y) \in \ker \pi_R \cap \ker \pi_S$ then $(x,y) \in \ker \pi_R$ implies that $(x,y) = (0,y)$ for all $y \in \ker \psi$ and $(x,y) \in \ker \pi_S$ implies that $(x,y) = (x,0)$ where $x \in \ker \varphi$ so $(x,y) = (0,0)$. Applying Lemma 1, it follows that $R \times_T S$ is Noetherian. ∎

Problem 3.3

Let M be an R-module. Show that M is a flat R-module if and only if $M_\mathfrak{m}$ is a flat $R_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ of R.

Proof. We prove the following result:

Lemma 2 (Atiyah & MacDonald, Ex.2.8(i))**.** *If M and N are flat A-modules, then so is $M \otimes_A N$.*

Proof of lemma. Let
$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$
be a short exact sequence of A-modules. Since M is a flat A-module, by , the following is a short exact sequence
$$0 \longrightarrow M \otimes N' \longrightarrow M \otimes N \longrightarrow M \otimes N'' \longrightarrow 0.$$
Moreover, since N is also a flat A-module, the following is also short exact
$$0 \longrightarrow N \otimes (M \otimes N') \longrightarrow N \otimes (M \otimes N) \longrightarrow N \otimes (M \otimes N'') \longrightarrow 0$$
so, by 2.7, we have that
$$0 \longrightarrow (N \otimes M) \otimes N' \longrightarrow (N \otimes M) \otimes N \longrightarrow (N \otimes M) \otimes N'' \longrightarrow 0$$
is short exact. Hence, $N \otimes M$ is a flat A-module. ♣

Let $\mathfrak{m} \subset R$ be a maximal ideal. Then $R_\mathfrak{m}$ is R-algebra via the canonical inclusion map $\iota\colon R \hookrightarrow R_\mathfrak{m}$. Thus, M admits an $(R, R_\mathfrak{p})$-bimodule structure, by 2.8, so that given an R-module N and an $R_\mathfrak{p}$-module P there is a canonical isomorphism
$$N \otimes_R (M \otimes_{R_\mathfrak{m}} P) \cong (N \otimes_R M) \otimes_{R_\mathfrak{m}} P.$$

Now, suppose that M is a flat R-module. Then, given an injective R-linear map $\varphi\colon N \hookrightarrow P$ the induced R-linear map
$$M \otimes_R N \longhookrightarrow M \otimes_R P$$
is injective (in this case, the mapping is $(m, n) \mapsto (m, \varphi(n))$. But by 4.5 $(M \otimes_R N)_\mathfrak{m} \cong (R_\mathfrak{m} \otimes_R M) \otimes_{R_\mathfrak{m}} (R_\mathfrak{m} \otimes_R N)$ so by 4.6 and Lemma 2, the induced $R_\mathfrak{m}$-linear map
$$M_\mathfrak{m} \otimes_{R_\mathfrak{m}} N_\mathfrak{m} \longhookrightarrow M_\mathfrak{m} \otimes_{R_\mathfrak{m}} N_\mathfrak{m}$$
is injective (in this case, the mapping factors through a number of isomorphism, but element-wise rule is not important). Hence, $M_\mathfrak{m}$ is flat.

Conversely, suppose that $M_\mathfrak{m}$ is flat for every maximal ideal $\mathfrak{m} \subset R$. Then, given an injective $R_\mathfrak{m}$-linear map $\varphi\colon N \hookrightarrow P$ the induced map
$$M_\mathfrak{m} \otimes_{R_\mathfrak{m}} N \longhookrightarrow M_\mathfrak{m} \otimes_{R_\mathfrak{m}} P$$
is injective. By restriction of scalars, we make N and P into R-modules via $N' = R_\mathfrak{m} \otimes_R N$ and $M' = R_\mathfrak{m} \otimes_R P$ so $(M \otimes_R N')_\mathfrak{m} \cong M_\mathfrak{m} \otimes_{R_\mathfrak{m}} N$ and $(M \otimes_R P')_\mathfrak{m} \cong M_\mathfrak{m} \otimes_{R_\mathfrak{m}} P$ so we have that the map
$$(M \otimes_R N')_\mathfrak{m} \longhookrightarrow (M \otimes_R P')_\mathfrak{m}$$

is injective for all $\mathfrak{m}$. We claim that this implies that the mapping

$$\Phi\colon M \otimes_R N' \longrightarrow M \otimes_R P'$$

is injective. Consider the short exact sequence

$$0 \longrightarrow \ker \Phi \longrightarrow M \otimes_R N' \longrightarrow M \otimes_R P' \longrightarrow 0.$$

By assumption, when we localize we have an injection $(M \otimes_R N')_{\mathfrak{m}} \hookrightarrow (M \otimes_R P')_{\mathfrak{m}}$ so

$$0 \longrightarrow (\ker \Phi)_{\mathfrak{m}} \longrightarrow (M \otimes_R N')_{\mathfrak{m}} \hookrightarrow (M \otimes_R P')_{\mathfrak{m}} \longrightarrow 0$$

implies that $(\ker \Phi)_{\mathfrak{m}} = 0$ for all $\mathfrak{m}$. By 4.9, this implies that $\ker \Phi = 0$ so the map Φ is indeed an injection. Thus, M is a flat R-module. ■

Problem 3.4

Let M be an R-module and $\mathfrak{a}$ an R-ideal.

(a) Show that if $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, then $M = \mathfrak{a}M$.
(b) Show that the converse holds in case M is finite.

Proof. (a) Consider the quotient ring $R/\mathfrak{a}$. The pair $(R/\mathfrak{a}, \pi)$, where $\pi\colon R \twoheadrightarrow R/\mathfrak{a}$ is the canonical projection, is an R-algebra so it makes sense to talk about restriction of scalars. Now, by 2.13, we have that $R/\mathfrak{a} \otimes_R M \cong M/\mathfrak{a}M$ is an $R/\mathfrak{a}$-module. By 1.2, there is a one-one correspondence between maximal ideals $\overline{\mathfrak{m}}$ of $R/\mathfrak{a}$ and maximal ideals $\mathfrak{m} \supset \mathfrak{a}$. Thus, $(M/\mathfrak{a}M)_{\overline{\mathfrak{m}}} = 0$ for every maximal ideal $\overline{\mathfrak{m}}$ implies, by 4.9, that $(M/\mathfrak{a}M) = 0$. Thus, $M = \mathfrak{a}M$.

(b) Now suppose M is finitely generated and $M = \mathfrak{a}M$. Then, by Nakayama's lemma, there exists $x \in 1 + \mathfrak{a}$ such that $xM = 0$. In particular, we have that $x \in \operatorname{ann} M$ so by 4.8, $x/1 \in \operatorname{ann}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$. However, $x/1$ is a unit in $M_{\mathfrak{m}}$ (since $x \notin \mathfrak{m}$) so $(x/1)M_{\mathfrak{m}} = 0$, but $(x/1)M_{\mathfrak{m}} = M_{\mathfrak{m}}$ so $M_{\mathfrak{m}} = 0$. ∎

PROBLEM 3.5

Prove that every power of a maximal ideal is primary.

Proof. Let R be a ring, $\mathfrak{m} \subset R$ be a maximal ideal and k a positive integer. Consider the quotient $R/\mathfrak{m}^k$. We must show that every zero-divisor of $R/\mathfrak{m}^k$ is nilpotent. Note that $R/\mathfrak{m}^k$ is a local ring with maximal ideal $\bar{\mathfrak{m}}$ the projection of $\mathfrak{m}$ (for suppose $\bar{\mathfrak{n}}$ is another maximal ideal of $R/\mathfrak{m}^k$, then, by 1.2, there is a corresponding maximal ideal $\mathfrak{n} \supset \mathfrak{m}^k$ of R, but $\sqrt{\mathfrak{m}^k} = \mathfrak{m} \subset \mathfrak{n}$ implies $\mathfrak{m} = \mathfrak{n}$ by maximality). Suppose $\bar{x}\bar{y} = \bar{0}$ where $\bar{x} \neq \bar{0}$ and $\bar{y} \neq \bar{0}$ are in $R/\mathfrak{m}^k$. Then, if either $\bar{x}$ or $\bar{y}$ is a unit we are done. Suppose $\bar{x}$ and $\bar{y}$ are non-units. Then $\bar{x}, \bar{y} \in \bar{\mathfrak{m}}$ so $\bar{x}^k = \bar{y}^k = \bar{0}$ are nilpotent since $x^k, y^k \in \mathfrak{m}^k$. It follows that $\mathfrak{m}^k$ is primary. ■

Problem 3.6

(a) Show that the radical of a primary ideal is prime.
(b) Find an example of a power of a prime ideal that is not primary.
(c) Let $\mathfrak{p}$ be a prime ideal of a ring R and $n \in \mathbf{N}$. The R-ideal $\mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$ is called the *nth symbolic power of* $\mathfrak{p}$. Show that $\mathfrak{p}^{(n)}$ is primary.

Proof. (a) Let $\mathfrak{a} \subset R$ be primary. Suppose $xy \in \sqrt{\mathfrak{a}}$. Then $x^k y^k \in \mathfrak{a}$. Since $\mathfrak{a}$ is primary, either $x^k \in \mathfrak{a}$ (in which case, $x \in \mathfrak{a}$) or $y \in \sqrt{\mathfrak{a}}$. In either case, $x \in \sqrt{\mathfrak{a}}$ or $y \in \sqrt{\mathfrak{a}}$ hence, $\sqrt{\mathfrak{a}}$ is prime.

(b) Consider the following example (taken from Atiyah & MacDonald §4, Example 2): Consider the quotient of the polynomial ring in three variables $A = k[X, Y, Z]/(XY - Z^2)$ and let $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ be the images of X, Y and Z, respectively, in the quotient. Then $\mathfrak{p} = (\overline{X}, \overline{Z})$ is prime (since $A/\mathfrak{p} \cong k[Y]$ via $\overline{Y} + \mathfrak{p} \mapsto Y$ is a domain) however $\mathfrak{p}^2$ is not primary since

$$\overline{X}\overline{Y} = XY + (XY - Z^2) = Z^2 + (XY - Z^2) = \overline{Z}^2 \in \mathfrak{p}^2,$$

but $\overline{X} \notin \mathfrak{p}^2$ and $\overline{Y} \notin \sqrt{\mathfrak{p}^2} = \mathfrak{p}$. Hence, $\mathfrak{p}^2$ is not primary.

(c) Note that that since $\mathfrak{p}^n \subset \mathfrak{p}$ then $(\mathfrak{p}^n)^e \subset \mathfrak{p}^e$ so by 4.13(c) $(\mathfrak{p}^n)^{ec} \subset \mathfrak{p}^{ec} = \mathfrak{p}$ so by 4.13(e) it suffices to show that $(\mathfrak{p}^n)^e$ is primary. But this follows from Problem 3.5 since $(\mathfrak{p}^n)^e$ is a power of the unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$. ■