Proof of Central Limit Theorem for general random variables using moment generating functions. (MGF).

Def Given X, $\boxed{M_X(t) = E(e^{tX})}$, $t \in \mathbb{R}$.

(Laplace transform of X)

assumed such a function exists, ie. the expectation of e^{tX} is finite for all t.

[**Note**: a companion function, characteristic function,

$$\boxed{\varphi_X(t) = E(e^{itX})}, \quad t \in \mathbb{R}$$

(Fourier transform of X)

always exists as $|e^{itX}|$ is a bounded function for any t and X:

$$e^{itX} = (\cos tX) + i(\sin tX)$$

bounded function.]

Some basic properties of $M_X(t)$

(1) $M(t) = E(e^{tX})$

$$M(0) = E(e^{tX})\Big|_{t=0} = 1$$

$$M'(t) = \frac{d}{dt} E[e^{tX}]$$

$$= E\left[\frac{d}{dt} e^{tX}\right]$$

$$= E[X e^{tX}]$$

$$M'(0) = E[X e^{tX}]\Big|_{t=0} = E[X]$$

$$M''(t) = \frac{d^2}{dt^2} E[e^{tX}]$$

$$= E[X^2 e^{tX}]$$

$$M''(0) = E[X^2 e^{tX}]\Big|_{t=0} = E(X^2)$$

Hence $E(X) = M'(0)$

$$\mathrm{Var}(X) = E(X^2) - (EX)^2 = M''(0) - (M'(0))^2$$

(>0)

(2) If X and Y are independent, then

$$M_{X+Y}(t) = E[e^{t(X+Y)}]$$
$$= E[e^{tX}e^{tY}]$$
$$= E(e^{tX})\,E(e^{tY})$$
$$= M_X(t)\,M_Y(t)$$

[In fact, X & Y are independent

$$\Updownarrow$$

$$M_{X+Y}(t) = M_X(t)\,M_Y(t).$$]

(3) The distribution function are in one to one correspondence to $M(t)$:

If $M_X(t) = M_Y(t)$ for all t,

then $X \overset{d}{=} Y$, i.e. for all a, $P(X \le a) = P(Y \le a)$

equal in the sense of distribution

(4) Convergence in distribution is equivalent to pointwise convergence of MGF.

$$\boxed{X_i \xrightarrow{\mathcal{D}} X}$$

("for all a", $P(X_i \le a) \xrightarrow{i\to\infty} P(X \le a)$.)

$$\Updownarrow$$

$$\boxed{M_{X_i}(t) \xrightarrow{i\to\infty} M_X(t), \text{ for all } t}$$

point-wise convergence of MGF.

Moment Generating Function for $X \sim N(\mu, \sigma^2)$

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} \frac{e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dx$$

$$z = \frac{x-\mu}{\sigma}$$

$$x = \sigma z + \mu$$

$$dx = \sigma \, dz$$

$$= \int_{-\infty}^{\infty} \frac{e^{t[\sigma z + \mu]} e^{-\frac{z^2}{2}}}{\sqrt{2\pi\cancel{\sigma^2}}} \cancel{\sigma} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2} + t\sigma z} dz$$

completing square

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2 - 2t\sigma z + t^2\sigma^2 - t^2\sigma^2}{2}} dz$$

$$= e^{t\mu + \frac{t^2\sigma^2}{2}} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t\sigma)^2}{2}} \, dz}_{=1}$$

$$\boxed{M_{(\mu,\sigma^2)}(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}}$$

<u>Remarks:</u>

(1) $M_{(0,1)}(t) = e^{t^2/2}$

$\qquad M_{(0,1)} \leftrightarrow N(0,1)$

(2) $M_{(\mu_1,\sigma_1^2)}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}}$

$M_{(\mu_2,\sigma_2^2)}(t) = e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}$

$$M_{(\mu_1,\sigma_1^2)}(t) \times M_{(\mu_2,\sigma_2^2)}(t) = M_{((\mu_1+\mu_2),\, \sigma_1^2+\sigma_2^2)}(t)$$

$$\Updownarrow$$

$$\boxed{N(\mu_1,\sigma_1^2) + N(\mu_2,\sigma_2^2) = N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)}$$

Pf of CLT $\left(\dfrac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1) \right)$

$$\left[\text{Need to show:} \quad M_{\frac{S_n - n\mu}{\sigma\sqrt{n}}}(t) \xrightarrow{n\to\infty} e^{t^2/2} \right]$$

$$M_{\frac{S_n - n\mu}{\sigma\sqrt{n}}}(t) = E\left[e^{t\left(\frac{S_n - n\mu}{\sigma\sqrt{n}}\right)}\right]$$

$$= E\left[e^{\left(\frac{t}{\sigma\sqrt{n}}\right)\left[(X_1-\mu) + \cdots (X_n-\mu)\right]}\right]$$

iid

$$= E\left[e^{\frac{t}{\sigma\sqrt{n}}(X_1-\mu)}\right] \cdots\cdots E\left[e^{\frac{t}{\sigma\sqrt{n}}(X_n-\mu)}\right]$$

the same function.

$$= \left(E\left[e^{\frac{t}{\sigma\sqrt{n}}(X_1-\mu)}\right]\right)^n$$

$$= \left(e^{-\frac{t\mu}{\sigma\sqrt{n}}} \, E \, e^{\frac{t}{\sigma\sqrt{n}}X_1} \right)^n$$

$$= e^{-\frac{t\mu\sqrt{n}}{\sigma}} \left[E\left(e^{\frac{t}{\sigma\sqrt{n}}X} \right) \right]^n$$

$$= e^{-\frac{t\mu\sqrt{n}}{\sigma}} \left[M_X\left(\frac{t}{\sigma\sqrt{n}} \right) \right]^n$$

(t - fixed)

$n \to \infty$

$$0 \quad \times \quad [M(0)]^{\infty}$$

$$= 0 \times 1^{\infty} = ?$$

Let $f(t) = e^{-\frac{t\mu\sqrt{n}}{\sigma}} M_X^n\left(\frac{t}{\sigma\sqrt{n}}\right)$

$$\log f(t) = -\frac{t\mu\sqrt{n}}{\sigma} + n \log M_X\left(\frac{t}{\sigma\sqrt{n}}\right)$$

$$-\infty + \infty \log 1$$

$$= -\infty + \infty \times 0 = ?$$

Use Taylor expansion

(1) $M(t) = M(0) + M'(0)t + \frac{1}{2}M''(0)t^2 + \cdots$

$= 1 + (EX)t + \frac{1}{2}E(X^2)t^2 + \cdots$

$= 1 + \mu t + \frac{1}{2}(\sigma^2+\mu^2)t^2 + \cdots$

$\left(t \to \frac{t}{\sigma\sqrt{n}}\right)$

$$\boxed{M\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{\mu t}{\sigma\sqrt{n}} + \frac{1}{2}\left(\frac{\sigma^2+\mu^2}{\sigma^2}\right)\frac{t^2}{n} + \cdots}$$

(2) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for $(|x|<1)$

(3) $$\log f(t) = -\frac{t\mu\sqrt{n}}{\sigma} + n\log\Big[\overbrace{1 + \underbrace{\frac{\mu t}{\sigma\sqrt{n}} + \frac{1}{2}\left(\frac{\sigma^2+\mu^2}{\sigma^2}\right)\frac{t^2}{n} + \cdots}_{x}}^{M\left(\frac{t}{\sigma\sqrt{n}}\right)}\Big]$$

$$\log f(t) = -\frac{t\mu\sqrt{n}}{\sigma}$$

$$+ n\log\Big[1 + \overbrace{\Big(\frac{\mu t}{\sigma\sqrt{n}} + \frac{1}{2}\Big(\frac{\sigma^2+\mu^2}{\sigma^2}\Big)\frac{t^2}{n}\Big)}^{x} + \cdots\Big]$$

$$= -\frac{t\mu\sqrt{n}}{\sigma}$$

$$+ n\Big[\Big(\frac{\mu t}{\sigma\sqrt{n}} + \Big(\frac{\sigma^2+\mu^2}{2\sigma^2}\Big)\frac{t^2}{n}\Big)$$

$$- \frac{1}{2}\Big(\frac{\mu t}{\sigma\sqrt{n}} + \Big(\frac{\sigma^2+\mu^2}{2\sigma^2}\Big)\frac{t^2}{n}\Big)^2$$

$$+ \cdots\Big]$$

cancel exactly

$$= -\cancel{\frac{t\mu\sqrt{n}}{\sigma}} + \cancel{\frac{\mu t}{\sigma\sqrt{n}}n} + \frac{\sigma^2+\mu^2}{2\sigma^2}t^2$$

$$- \frac{\cancel{n}}{2}\Big[\frac{\mu^2 t^2}{\sigma^2 \cancel{n}} + O\Big(\frac{1}{n^{3/2}}\Big) + O\Big(\frac{1}{n^2}\Big)\Big] + \cdots$$

$$= \frac{\sigma^2 + \cancel{\mu^2}}{2\sigma^2} t^2 - \frac{\cancel{\mu^2} t^2}{2\sigma^2} + O\left(\frac{1}{\sqrt{n}}\right) + \cdots$$

$$= \frac{t^2}{2} + \cdots$$

Hence $\log f(t) \xrightarrow{n \to \infty} \frac{t^2}{2}$

$$\boxed{f(t) \longrightarrow e^{t^2/2}}$$

i.e. $\boxed{M_{\frac{S_n - n\mu}{\sigma\sqrt{n}}}(t) \xrightarrow{n \to \infty} e^{t^2/2}} = M_{N(0,1)}(t)$

$$\Updownarrow$$

$$\boxed{\frac{S_n - n\mu}{\sigma\sqrt{n}} \xRightarrow{\mathcal{D}} N(0,1)}$$