MA571 Problem Set 5

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September 24, 2015

Problem 5.1 (Munkres §23, Ex. 3)

Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \emptyset$ for all α, then $A \cup (\bigcup A_\alpha)$ is connected.

Proof. We shall aim to prove this result by using Theorem 23.3 from Munkres. Define the collection $\{B_\alpha\}$ by setting $B_\alpha = A \cup A_\alpha$. Note that by Theorem 23.3, B_α is connected for all α, since $A \cap A_\alpha \neq \emptyset$ and both A and A_α are connected. Next observe that the intersection $B_\alpha \cap B_\beta \neq \emptyset$ for all α and β, in particular, the subspace A is contained in the intersection since $A \subset B_\alpha$ and $A \subset B_\beta$ for all α and β. Therefore, $\{B_\alpha\}$ is a collection of connected subspaces of X that have a point in common. Applying Theorem 23.3 one last time, we see that the union

$$\bigcup B_\alpha = \bigcup (A \cup A_\alpha) = A \cup \left(\bigcup A_\alpha\right)$$

is connected. ∎

Problem 5.2 (Munkres §23, Ex. 6)

Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X \setminus A$, then C intersects ∂A.

Proof. We shall proceed by contradiction. Suppose that $C \cap \partial A = \emptyset$, then we shall show that the pair $C \cap A$ and $C \cap (X \setminus A)$ forms a separation of C. Recall that by definition (see Munkres §17, p. 102) the boundary $\partial A = \overline{A} \cap \overline{X \setminus A}$. Then we claim that $\overline{A} = \partial A \cup \operatorname{int} A$:

Lemma 13. *Let X be a topological space and $A \subset X$. Then ∂A and $\operatorname{int} A$ are disjoint and $\overline{A} = \partial A \cup \operatorname{int} A$.*

Proof of lemma. The point $x \in \partial A$ if and only if $x \in \overline{A}$ and $x \in \overline{X \setminus A}$. Thus, for every neighborhood U of x, the intersection $U \cap X \setminus A \neq \emptyset$, in particular $U \not\subset A$ so x is not an interior point of A. Hence, we see that $\partial A \cap \operatorname{int} A = \emptyset$. To prove the last statement note that $\partial A \subset \overline{A}$ and $\operatorname{int} A \subset A \subset \overline{A}$ (cf. Munkres §17, p. 95), so that $\partial A \cup \operatorname{int} A \subset \overline{A}$ hence, it suffices to show the reverse inclusion, namely, $\overline{A} \subset \partial A \cup \operatorname{int} A$. Let $x \in \overline{A}$. If $x \in \operatorname{int} A$, then clearly $x \in \partial A \cup \operatorname{int} A$. Suppose $x \notin \operatorname{int} A$. Then, by Theorem 17.5(a), for every neighborhood U of x, the intersection $U \cap A \neq \emptyset$ and $U \not\subset A$. Thus, $U \cap (X \setminus A) \neq \emptyset$ so $x \in \overline{X \setminus A}$. It follows that $x \in \overline{A} \cap \overline{X \setminus A} = \partial A$. ♣

Lemma 14. *Let X be a topological space and $A \subset X$. Then $\partial A = \partial(X \setminus A)$.*

Proof of lemma. Replace A by $X \setminus A$ in the definition of the boundary of A. Then we have:

$$\begin{aligned}
\partial(X \setminus A) &= \overline{X \setminus A} \cap \overline{X \setminus (X \setminus A)} \\
&= \overline{X \setminus A} \cap \overline{A} \\
&= \overline{A} \cap \overline{X \setminus A} \\
&= \partial A.
\end{aligned}$$

♣

Now, by Theorem 17.4, we have that $\overline{C \cap A} = C \cap \overline{A}$ and $\overline{C \cap (X \setminus A)} = C \cap \overline{X \setminus A}$. But by Lemma 13 and Lemma 14, the latter sets are equivalent to $\overline{C \cap A} = C \cap (\partial A \cup \operatorname{int} A)$ and $\overline{C \cap (X \setminus A)} = C \cap (\partial A \cup \operatorname{int}(X \setminus A))$. But since $C \cap \partial A = \emptyset$ by assumption, we have

$$\begin{aligned}
\overline{C \cap A} \cap (C \cap (X \setminus A)) &= (C \cap (\partial A \cup \operatorname{int} A)) \cap (C \cap (X \setminus A)) \\
&= ((C \cap \partial A) \cup (C \cap \operatorname{int} A)) \cap (C \cap (X \setminus A)) \\
&= (C \cap \operatorname{int} A) \cap (C \cap (X \setminus A)) \\
&= \emptyset
\end{aligned}$$

since $C \cap \operatorname{int} A \subset A$ and $C \cap (X \setminus A) \subset X \setminus A$. Similarly, we have that the intersection $\overline{C \cap (X \setminus A)} \cap (C \cap A) = \emptyset$. So by Lemma 23.1, $C \cap A$ and $C \cap (X \setminus A)$ form a separation of C. This contradicts the assumption that C is connected. Therefore, we conclude that $C \cap \partial A \neq \emptyset$. ■

Problem 5.3 (Munkres §23, Ex. 7)

Is the space $\mathbf{R}_\ell$ connected? Justify your answer.

Proof. No. The space $\mathbf{R}_\ell$ is not connected and we may exhibit an explicit separation. Namely, consider the basis elements $(-\infty, 0)$ and $[0, \infty)$. Then $\mathbf{R} = (-\infty, 0) \cup [0, \infty)$, hence $(-\infty, 0)$ and $[0, \infty)$ form a separation of $\mathbf{R}$ with the lower limit topology.

Alternatively, one may note that $\mathbf{R} \smallsetminus (-\infty, 0) = [0, \infty)$ is open in $\mathbf{R}_\ell$ so $(-\infty, 0)$ is both open and closed. Hence, by Munkres's alternative formulation of connectedness (cf. Munkres §23, p. 148 the italicized paragraph), $\mathbf{R}_\ell$ is disconnected. ■

Problem 5.4 (Munkres §23, Ex. 9)

Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that

$$(X \times Y) \smallsetminus (A \times B)$$

is connected.

Proof. Consider the family of embeddings $\{i_\alpha\}$ where $i_\alpha \colon X \hookrightarrow X \times Y$ maps $x \mapsto x \times y_\alpha$ for $y_\alpha \notin B$, for all α. By Theorem 23.5, $i_\alpha(X) = X \times y_\alpha$ is connected subspace of $X \times Y$. Moreover $X \times y_\alpha \subset (X \times Y) \smallsetminus (A \times B)$ so $X \times y_0$, in particular, we have that is a connected subspace of $(X \times Y) \smallsetminus (A \times B)$. Similarly, consider the family of embeddigs $\{j_\alpha\}$ where $j_\alpha \colon Y \hookrightarrow X \times Y$ maps $y \mapsto x_\alpha \times y$ for $x_\alpha \notin A$. We similarly have that $j_\alpha(Y) = x_\alpha \times Y$ is a connected subspace of $(X \times Y) \smallsetminus (A \times B)$. Then we claim that

$$(X \times Y) \smallsetminus (A \times B) = \bigcup (X \times y_\alpha) \cup (x_\beta \times Y).$$

It is clear that the union on the right is a subset of $(X \times Y) \smallsetminus (A \times B)$ since each $X \times y_\alpha$ and $x_\beta \times Y$ is a subset of $(X \times Y) \smallsetminus (A \times B)$. To see the reverse containment, take $x \times y$ in the union $\bigcup (X \times y_\alpha) \cup (x_\beta \times Y)$. Then $x \times y$ is in some $(X \times y_\alpha) \cup (x_\beta \times Y)$ so $x \times y \in X \times y_\alpha$ or $x \times y \in x_\beta \times Y$. If $x \times y \in \bigcup X \times y_\alpha$, then $y_\alpha \notin B$ so $x \times y \notin A \times B$, hence $x \times y \in (X \times Y) \smallsetminus (A \times B)$. If $x \times y \in \bigcup x_\beta \times Y$ then $x \notin A$, hence $x \times y \notin A \times B$ so $x \times y \in (X \times Y) \smallsetminus (A \times B)$. Thus, we have that $(X \times Y) \smallsetminus (A \times B) = \bigcup (X \times y_\alpha) \cup (x_\beta \times Y)$. Then, note that by Theorem 23.3, since $X \cap y_\alpha \cap x_\beta \cap Y \neq \emptyset$, in particular, $x_\beta \times y_\alpha$ is in the intersection, $(X \times y_\alpha) \cup (x_\beta \times Y)$ is connected for all α and all β. Thus, the subspace $(X \times Y) \smallsetminus (A \times B)$ is connected. ∎

Problem 5.5 (Munkres §24, Ex. 1(ac))

(a) Show that no two of the spaces $(0, 1)$, $(0, 1]$ and $[0, 1]$ are homeomorphic. [*Hint:* What happens if you remove a point from each of these spaces?]
(c) Show $\mathbf{R}^n$ and $\mathbf{R}$ are not homeomorphic if $n > 1$.

Proof. (a) Suppose $\varphi\colon (0, 1] \to (0, 1)$ is a homeomorphism. We claim that the restriction of φ to $(0, 1) \subset (0, 1]$ gives a homeomorphism to $(0, 1) \setminus \{\varphi(1)\}$:

Lemma 15. *Suppose $\varphi\colon X \to Y$ is a homeomorphism. Then the restricted map $\varphi_0\colon X \setminus x_0 \to Y \setminus \{\varphi(x_0)\}$ of φ is a homeomorphism.*

Proof of lemma. at ♣

Now remove 1 from $(0, 1]$. Then, since $\varphi(1)$ is bijective, there exists $y \in (0, 1)$ such that $\varphi(1) = y$. Then $(0, 1) \setminus \{y\} = (0, y) \cup (y, 1)$ is disconnected, but $(0, 1) \setminus \{1\} = (0, 1)$ is connected. This contradicts Theorem 23.5 that the image of.

(b) ■

Problem 5.6 (Munkres §24, Ex. 2)

Let $f\colon S^1 \to \mathbf{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

Proof. ■

Problem 5.7 (Munkres §25, Ex. 2(b))

(b) Consider $\mathbf{R}^\omega$ in the uniform topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same component of $\mathbf{R}^\omega$ if and only if the sequence

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, ...)$$

is bounded. [*Hint:* It suffices to consider the case where $\mathbf{y} = \mathbf{0}$.]

Proof. ∎

Problem 5.8 (Munkres §25, Ex. 4)

Let X be locally path connected. Show that every connected open set in X is path connected.

Proof. ■

Problem 5.9 (Munkres §25, Ex. 6)

A space X is said to be *weakly locally path connected at* x if for every neighborhood U of x, there is a connected subspace of X contained in U that contains a neighborhood of x. Show that if X is weakly locally connected at each of its points, then X is locally connected. [*Hint:* H]

Proof. ∎

Problem 5.10 (A)

Let X be a topological space. The quotient space $(X \times [0,1])/(X \times 0)$ is called the *cone* of X and denoted CX.

Prove that if X is homeomorphic to Y then CX is homeomorphic to CY (*Hint:* There are maps in both directions).

Proof. ■

Problem 5.11 (Extra problem)

Notation: for positive integers i, n, I, N, let us write $(i, n) \gg (I, N)$ if $i > I$ and $n > N$.

Theorem 16. *A sequence $\{\mathbf{x}_n\}$ in $\mathbf{R}^\omega$ converges to $\mathbf{0}$ in the box topology if and only if two conditions hold:*

(i) for each k, $\lim_{n\to\infty} x_n^{(k)} = 0$, and
(ii) there is a pair (I, N) with $x_n^{(k)} = 0$ whenever $(i, n) \gg (I, N)$.

Proof. ■