MA 572: Homework 3

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PROBLEM 3.1 (HATCHER §2.1, Ex. 17)

- (a) Compute the homology groups $H_n(X,A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.
- (b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown. [What are X/A and X/B?]

Proof. (a) As a consequence of 2.16, we have a long exact sequence on relative homology

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$
 (1)

Now, specifying X to be S^2 , we know, by 2.8, 2.6 and 2.14, that $H_n(A) \cong 0$ for all n > 0 and $H_0(A) \cong \bigoplus_{|A|} \mathbb{Z}$, and $H_n(S^2) \cong 0$ for $n \neq 2, 0$ and $H_n(S^2) \cong \mathbb{Z}$ otherwise. Hence, the long exact sequence (1) turns into

$$\cdots \longrightarrow H_2(A) \longrightarrow H_2(S^2) \longrightarrow H_2(S^2, A) \longrightarrow$$

$$\longrightarrow H_1(A) \longrightarrow H_1(S^2) \longrightarrow H_1(S^2, A) \longrightarrow$$

$$\longrightarrow H_0(A) \longrightarrow H_0(S^2) \longrightarrow H_0(S^2, A) \longrightarrow 0$$
(2)

which, filling in our computed values for $H_n(A)$ and $H_n(S^2)$, further becomes

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_2(S^2, A) \longrightarrow 0 \longrightarrow H_1(S^2, A) \longrightarrow 0 \longrightarrow H_1(S^2, A) \longrightarrow 0,$$

$$(3)$$

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with all other n > 2 homology groups for A and S^2 being 0. That last remark immediately tells us that, by exactness, $H_n(S^2, A) = 0$ for all n > 2. Starting from the bottom of (3), exactness at $H_2(S^2, A)$ tells us that $H_2(S^2, A) \cong \mathbb{Z}$ since the zero maps to the left and right

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_2(S^2, A) \longrightarrow 0 \longrightarrow \cdots$$

tells us that the map $\mathbb{Z} \to H_2(S^2, A)$ is an isomorphism. If we look at the reduced homology, the bottom row of (3) becomes

$$\cdots \longrightarrow 0 \longrightarrow \widetilde{H}_1(S^2, A) \longrightarrow \bigoplus_{|A|-1} \mathbb{Z} \longrightarrow 0 \longrightarrow \widetilde{H}_0(S^2, A) \longrightarrow 0.$$

By exactness at $\widetilde{H}_1(S^2, A)$, we have $H_1(S^2, A) \cong \widetilde{H}_1(S^2, A) \cong \bigoplus_{|A|=1} \mathbb{Z}$ and, last but not least, exactness at $\widetilde{H}_0(S^2, A) \cong 0$ gives us $H_0(S^2, A) \cong \mathbb{Z}$. In summary, we have

$$H_n(S^2, A) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2\\ \bigoplus_{|A| - 1} \mathbb{Z} & \text{if } n = 1\\ 0 & \text{otherwise} \end{cases}$$
 (4)

From 2.27 we know that $H_n^{\Delta}(S^1 \times S^1) \cong H_n(S^1 \times S^1)$ so from 2.3, we know that the homology of the torus $S^1 \times S^1$ is

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1\\ \mathbb{Z} & \text{if } n = 2, 0.\\ 0 & \text{otherwise} \end{cases}$$
 (5)

Skipping directly, to our calculation, we have the long exact sequence

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_2(S^1 \times S^1, A) \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(S^1 \times S^1, A) \longrightarrow 0 \longrightarrow \bigoplus_{|A|} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow H_0(S^2, A) \longrightarrow 0.$$

$$(6)$$

It is clear from exactness that $H_2(S^1 \times S^1, A) \cong \mathbb{Z}$ and $H_0(S^1 \times S^1, A) \cong \mathbb{Z}$. What is not clear is what $H_1(S^1 \times S^1, A)$ is. Exactness at $\mathbb{Z} \oplus \mathbb{Z}$ tells us that $\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow H_1(S^1 \times S^1, A)$ and, looking at the reduced homology, exactness at $\bigoplus_{|A|-1} \mathbb{Z}$ tells us that $H_1(S^1 \times S^1, A) \twoheadrightarrow \bigoplus_{|A|-1} \mathbb{Z}$. Thus, we have $\bigoplus_{|A|-1} \mathbb{Z} \cong H_1(S^1 \times S^1, A)/\mathbb{Z} \oplus \mathbb{Z}$ from which we can deduce that $H_1(S^1 \times S^1, A) \cong \bigoplus_{|A|+1} \mathbb{Z}$. In summary, the relative homology of $S^1 \times S^1$ with respect to A is

$$H_n(S^1 \times S^1, A) = \begin{cases} \mathbb{Z} & \text{if } n = 2, 0\\ \bigoplus_{|A|+1} \mathbb{Z} & \text{if } n = 1\\ 0 & \text{otherwise} \end{cases}$$
 (7)

(b) For this problem, we'll freely assume

 $^{^1}$ I know this is not strictly correct, but the approach I took to solve the problem required me to construct an inverse map $H_1(S^1 \times S^1, A) \leftarrow \bigoplus_{|A|=1} \mathbb{Z}$, but this is difficult.

PROBLEM 3.2 (HATCHER §2.2, Ex. 1)

Prove the Brouwer fixed point theorem for maps $f: D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f. [This was Brouwer's original proof.]

Proof. Seeking a contradiction, suppose $f: D^n \to D^n$ has no fixed point. Let N and S denote, respectively, the northern and southern hemisphere meeting at the equator of S^n . Now, since the disk $D^n \approx S$, we may as well identify D^n with S and consider the map $f: D^n \to D^n$ as the map $S \to S$ by composing with the homeomorphism. Define a map $g: S^n \to S^n$ by

$$g := \begin{cases} r & \text{on } N \\ \text{id} & \text{on } S \end{cases}$$
 (8)

Note that the map g is continuous by the pasting lemma since $g \upharpoonright_S = \operatorname{id}$ and $g \upharpoonright_N = r$ are continuous and r fixes points at the equator $N \cap S$. Now, consider the map $F \colon S^n \to S^n$ given by the composition $\iota \circ f \circ g$ where $\iota \colon S \hookrightarrow S^n$ is the inclusion $S \subset S^n$. This map has no fixed points since f has no fixed point hence, by property (g) of the degree, $\deg F = (-1)^{n+1}$. But F is not onto, therefore $\deg F = 0$. This is a contradiction.

PROBLEM 3.3 (HATCHER §2.2, Ex. 6)

Show that every map $S^n \to S^n$ can be homotoped to have a fixed point if n > 0.

Proof. The result follows from 4.25 since a map $f \colon S^n \to S^n$ without any fixed points is homotopic to the antipodal map. Since the antipodal map has degree -1 or 1 depending on n, it follows that the antipodal map is homotopic to either the identity map or a reflection map, both of which have fixed points.

CARLOS SALINAS PROBLEM 3.4

Problem 3.4

Let \mathcal{U} be an open cover of X. Prove that the inclusion of $C_*^{\mathcal{U}}(C)$ into $C_*(X)$ is a chain homotopy equivalence.

Proof. This is proposition 2.21 in the book. I will summarize the proof in four steps here.

- (1) (Barycentric subdivision) Given a simplex $[v_0,...,v_n]$ its barycenter is the point $b = \sum_i t_i v_i$ whose barycentric coordinates t_i are all equal, i.e., $t_i = 1/(n+1)$ for all i. The barycentric subdivision of $[v_0,...,v_n]$ is the decomposition of $[v_0,...,v_n]$ into the n-simplices $[b,w_0,...,w_{n-1}]$, where
- **(2)**
- (3)
- **(4)**