MA 523: Homework 5

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October 4, 2016

Problem 5.1

Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$, $x \in \mathbb{R}^n$, then $\Delta v = 0$.

Solution. Let

$$O = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be an orthogonal $n \times n$ matrix. We will show that $\Delta v = 0$, where $v(x) = u(Ox)$.

First, let us compute the gradient of v,

$$\begin{aligned} Dv(x) &= Du(Ox) \\ &= Du\big(a_{11}x_1 + \cdots + a_{1n}x_n, \ldots, a_{n1}x_1 + \cdots + a_{nn}x_n\big) \\ &= \Big(\textstyle\sum_{j=1}^n a_{j1}u_{x_j}, \ldots, \sum_{j=1}^n a_{jn}u_{x_j}\Big) \\ &= O^{\mathrm{T}} Du(x). \end{aligned}$$

Lastly, we compute the divergence of Dv,

$$\begin{aligned} \Delta v(x) &= \operatorname{div} Dv(x) \\ &= \operatorname{div}\Big(\textstyle\sum_{j=1}^n a_{j1}u_{x_j}, \ldots, \sum_{j=1}^n a_{jn}u_{x_j}\Big). \end{aligned}$$

Here the partial derivatives become unwieldy so we will first examine the partial $\frac{\partial}{\partial x_1}$ of the first term and proceed from there. In this case,

$$\begin{aligned} \frac{\partial}{\partial x_1}\sum_{j=1}^n a_{j1}u_{x_j} &= a_{11}\tfrac{\partial}{\partial x_1}u_{x_1} + a_{21}\tfrac{\partial}{\partial x_1}u_{x_2} + \cdots + a_{n1}\tfrac{\partial}{\partial x_1}u_{x_n} \\ &= a_{11}(a_{11}u_{x_1x_1} + a_{21}u_{x_1x_2} + \cdots + a_{n1}u_{x_1x_n}) \\ &\quad + \cdots + a_{n1}(a_{11}u_{x_1x_n} + a_{21}u_{x_2x_n} + \cdots + a_{n1}u_{x_nx_n}) \\ &= a_{11}^2u_{x_1x_1} + 2a_{11}a_{21}u_{x_1x_2} + 2a_{11}a_{31}u_{x_1x_3} + \cdots + a_{21}^2u_{x_2x_2} \\ &\quad + \cdots + a_{k1}^2u_{x_kx_k} + \cdots + a_{n1}^2u_{x_nx_n}. \end{aligned}$$

Similarly, taking the k^{th} partial of the k^{th} entry of Dv, we have

$$\begin{aligned} \frac{\partial}{\partial x_k}\sum_{j=1}^n a_{jk}u_{x_j} &= a_{1k}(a_{1k}u_{x_1x_1} + \cdots + a_{nk}u_{x_1x_n}) \\ &\quad + \cdots + a_{nk}(a_{1k}u_{x_1x_n} + \cdots + a_{nk}u_{x_nx_n}) \\ &= a_{1k}^2u_{x_1x_1} + a_{2k}^2u_{x_2x_2} + \cdots + a_{kk}^2u_{x_kx_k} \\ &\quad + \cdots + a_{nk}^2u_{x_nx_n} + \{\text{mixed terms}\}. \end{aligned} \tag{5.1}$$

Now, since O is orthogonal, we have

$$
\begin{aligned}
OO^{\mathrm{T}} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}^2 + \cdots + a_{1n}^2 & a_{11}a_{21} + \cdots + a_{1n}a_{2n} & \cdots & a_{11}a_{n1} + \cdots + a_{1n}a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} + \cdots + a_{nn}a_{1n} & a_{n1}a_{21} + \cdots + a_{nn}a_{2n} & \cdots & a_{n1}^2 + \cdots + a_{nn}^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}.
\end{aligned}
$$

We can sum up the results of our calculation as

$$
\begin{cases} \text{(a)} \ \sum_{j=1}^n a_{kj}a_{\ell j} = \sum_{j=1}^n a_{kj}^2 = 1 & \text{if } k = \ell, \\ \text{(b)} \ \sum_{j=1}^n a_{kj}a_{\ell j} = 0 & \text{if } k \neq \ell. \end{cases} \tag{5.2}
$$

for $1 \leq k, \ell \leq n$.

Now, going back to (5.1), we have

$$
\begin{aligned}
\operatorname{div} Dv &= \sum_{k=1}^n \left[\frac{\partial}{\partial x_k} \sum_{j=1}^n a_{jk} u_{x_j} \right] \\
&= (a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2) u_{x_1 x_1} + (a_{12}^2 + a_{22}^2 + \cdots + a_{2n}^2) u_{x_2 x_2} \\
&\quad + \cdots + (a_{1n}^2 + \cdots + a_{nn}^2) u_{x_n x_n} + \{\text{mixed terms}\} \\
&= u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} \\
&= 0,
\end{aligned}
$$

as desired.

All that is left to show as that the mixed terms in the expression above actually have coefficients of the form in (5.2) (b). A little routine calculation shows that indeed, the mixed terms have the form. Here is the first mixed term

$$
2(a_{11}a_{21} + a_{12}a_{22} + \cdots + a_{1n}a_{2n}) u_{x_1 x_2} = 0.
$$

■

Problem 5.2

Let $n = 2$ and U be the halfplane $\{ x_2 > 0 \}$. Prove that

$$\sup_U u = \sup_{\partial U} u$$

for $u \in C^2(U) \cap C(\bar{U})$ which are harmonic in U under the additional assumption that u is bounded from above in $\bar{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.)
[*Hint:* Take for $\varepsilon > 0$ the harmonic function

$$u(x_1, x_2) + \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region $\left\{ x_1^2 + (x_2 + 1)^2 < a_2, x_2 > 0 \right\}$ with large a. Let $\varepsilon \to 0$.]

Solution. Consider the harmonic function

$$u_\varepsilon(x_1, x_2) := u(x_1, x_2) + \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Set $U_a := \left\{ x_1^2 + (x_2 + 1)^2 < a_2, x_2 > 0 \right\}$ where $a = (a_1, a_2)$.

First, we note that $u_\varepsilon \downarrow u$ as $\varepsilon \to 0$ uniformly, i.e., given $\eta > 0$, for $0 < \varepsilon < 2\eta / \ln a_2$, we have

$$\begin{aligned}
|u_\varepsilon(x_1, x_2) - u(x_1, x_2)| &< \left| u(x_1, x_2) + \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} - u(x_1, x_2) \right| \\
&= \left| \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} \right| \\
&= \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} \\
&< \varepsilon \ln \sqrt{a_2} \\
&< \eta,
\end{aligned}$$

for any $(x_1, x_2) \in U_a$.

By the maximum principle,

$$\max_{\bar{U}_a} u_\varepsilon = \max_{\partial U_a} u_\varepsilon.$$

In particular,

$$\sup_{\bar{U}_a} u_\varepsilon = \sup_{\partial U_a} u_\varepsilon.$$

Now, since $u_\varepsilon \downarrow u$ uniformly,

$$\sup_{\bar{U}_a} u = \lim_{\varepsilon \to 0} \left[\sup_{\bar{U}_a} u_\varepsilon \right] = \lim_{\varepsilon \to 0} \left[\sup_{\partial U_a} u_\varepsilon \right] = \sup_{\partial U_a} u.$$

Thus, we have shown that for any a,

$$\sup_{\bar{U}_a} u = \sup_{\partial U_a} u. \tag{5.3}$$

We now extend this result to all of U.

By the mean value property

$$u(x_1, x_2) = \fint_{B_r(x_1,x_2)} u\, dy$$

for any open $B_r(x_1, x_2) \subset U$. Then given $(x_1, x_2) \in U$, for a sufficiently large so that $(x_1, x_2) \in U_a$,

$$\sup_{\partial U_a} u \geq u(x_1, x_2).$$

■

Problem 5.3

Let $U \subset \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \ldots, u^m$ are harmonic in U and

$$v := \varphi(u_1, \ldots, u_m).$$

Prove v is subharmonic.

[*Hint:* Convexity for a smooth function $\varphi(z)$ is equivalent to $\sum_{j,k=1}^m \varphi_{z_j,z_k}(z)\xi_j\xi_k \geq 0$ for any $\xi \in \mathbb{R}^m$.]

(b) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. (Assume that harmonic functions are C^∞.)

Solution. For part (a), by the chain rule, we have

$$v_{x_i} = \frac{\partial}{\partial x_i} v = \varphi_{u_1} u^1_{x_i} + \cdots + \varphi_{u_m} u^m_{x_i}.$$

Taking another partial, we have

$$\begin{aligned} v_{x_i x_i} &= \frac{\partial^2}{\partial x_i \partial x_i} v \\ &= \frac{\partial}{\partial x_i} v_{x_i} \\ &= \frac{\partial}{\partial x_i} \left(\varphi_{u_1} u^1_{x_i} + \cdots + \varphi_{u_m} u^m_{x_i}\right) \\ &= \varphi_{u_1} u^1_{x_i x_i} + \cdots + \varphi_{u_m} u^m_{x_i x_i} \\ &\quad + \left(\varphi_{u_1 u_1} u^1_{x_i} + \cdots + \varphi_{u_1 u_m} u^m_{x_i}\right) u^1_{x_i} \\ &\qquad + \cdots + \left(\varphi_{u_1 u_m} u^1_{x_i} + \cdots + \varphi_{u_m u_m} u^m_{x_i}\right) u^m_{x_i}. \end{aligned} \tag{5.4}$$

Now, taking the sum

$$\begin{aligned} \sum_{i=1}^n v_{x_i x_i} &= \sum_{i=1}^n \sum_{j=1}^m \varphi_{u_j} u^j_{x_i x_i} \\ &= \sum_{j=1}^m \sum_{i=1}^n \varphi_{u_j} u^j_{x_i x_i} \\ &= \sum_{j=1}^m \left(\varphi_{u_j} u^j_{x_1 x_1} + \cdots + \varphi_{u_j} u^j_{x_n x_n}\right) \\ &= \sum_{j=1}^m \varphi_{u_j} \left(u^j_{x_1 x_1} + \cdots + u^j_{x_n x_n}\right) \\ &= 0, \end{aligned}$$

since $\Delta u^j = 0$ for all j.

What about the remaining terms in (5.4)? These terms can be written in the form

$$\sum_{j,k=1}^{m} \varphi_{u_j u_k}(u)\xi_j \xi_k,$$

where $\xi_i = \big(u^1_{x_i}, \ldots, u^m_{x_i}\big)(x_1, \ldots, x_n) \in \mathbb{R}^m$ for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Since φ is convex, by assumption, this quantity is greater than or equal to 0.

Thus, $\Delta v \geq 0$ so v is subharmonic.

For part (b), we have

$$v = |Du|^2 = u^2_{x_1} + \cdots + u^2_{x_n}.$$

Taking the partial derivative with respect to x_i, we have

$$\begin{aligned} v_{x_i} &= \tfrac{\partial}{\partial x_i} v \\ &= \tfrac{\partial}{\partial x_i}\big(u^2_{x_1} + \cdots + u^2_{x_n}\big) \\ &= 2u_{x_1}u_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_n}, \end{aligned}$$

and again

$$\begin{aligned} v_{x_ix_i} &= \tfrac{\partial}{\partial x_i} v_{x_i} \\ &= \tfrac{\partial}{\partial x_i}\big(2u_{x_1}u_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_n}\big) \\ &= 2u_{x_1}u_{x_1x_ix_i} + 2u^2_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_ix_n} + 2u^2_{x_ix_n} \\ &= 2\sum_{j=1}^{n}\big(u_{x_j}u_{x_jx_ix_i} + u^2_{x_jx_i}\big). \end{aligned}$$

Then

$$\begin{aligned} \frac{\Delta v}{2} &= \sum_{i,j=1}^{n}\big(u_{x_j}u_{x_jx_ix_i} + u^2_{x_jx_i}\big) \\ &= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + \sum_{i,j=1}^{n} u^2_{x_jx_i}, \end{aligned}$$

splitting the second term into the sum

$$\begin{aligned} &= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + \sum_{1\leq j<i\leq n} u^2_{x_jx_i} \\ &\quad + \sum_{1\leq i<j\leq n} u^2_{x_jx_i} + \sum_{1\leq i=j\leq n} u^2_{x_ix_i}, \end{aligned}$$

where the last term is 0 since u is harmonic, giving us

$$
\begin{aligned}
&= \sum_{i,j=1}^{n} u_{x_j} u_{x_j x_i x_i} + \sum_{1\le j<i\le n} u_{x_j x_i}^2 + \sum_{1\le i<j\le n} u_{x_j x_i}^2 \\
&= \sum_{i,j=1}^{n} u_{x_j} u_{x_j x_i x_i} + 2 \sum_{1\le j<i\le n} u_{x_j x_i}^2,
\end{aligned}
$$

here $u_{x_i x_j x_j} = \Delta u_{x_i} = 0$ since the derivatives of harmonic functions are harmonic, so

$$
\begin{aligned}
&= \sum_{j=1}^{n} u_{x_j} (\Delta u_{x_j}) + 2 \sum_{1\le j<i\le n} u_{x_j x_i}^2 \\
&= 2 \sum_{1\le j<i\le n} u_{x_j x_i}^2 \\
&\ge 0,
\end{aligned}
$$

as desired. That is, $\Delta v \ge 0$ so v is subharmonic. ∎