MRC 2016 Report: Asymptotics, Free Groups, Counting Polynomials

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1 Overview

The $\mathrm{SL}(2,\mathbb{C})$-character variety of a finitely presented group Γ, denoted $\mathfrak{X}_\Gamma$, is defined over $\mathbb{Z}[1/2]$. During the introductory talks by Lawton an algorithm was presented to determine its generators and relations. This working group broke into 3 subgroups. The first was tasked with writing a computer program in *Mathematica* to implement the aforementioned algorithm. That is, given a presentation for Γ as an input the program outputs a finite presentation for the coordinate ring $\mathbb{C}[\mathfrak{X}_\Gamma]$ over $\mathbb{Z}[1/2]$. The second group, using these ring presentations, were tasked with determining whether certain examples of $\mathfrak{X}_\Gamma$ are polynomial count. That is, are the functions that input a finite field $\mathbb{F}_q$ and output the cardinality of the $\mathbb{F}_q$-points of $\mathfrak{X}_\Gamma$ polynomials. The third group, also using the explicit ring theoretic presentations of the first group were tasked with determining the growth rates of the maximal orbits of the outer automorphisms on the $\mathbb{F}_q$-points of $\mathfrak{X}_\Gamma$ as a function of q. As a whole we were working to use computational and experimental methods to explore arithmetic structure of rank 1 character varieties.

2 Algebraic Structure

In Sean Lawton's MRC lectures, an algorithm was given for generating presentations of $\mathrm{SL}(2,\mathbb{C})$ character varieties with a sufficient number of generators. This algorithm works for any finitely presented group Γ.

One of this group's goals was to implement this algorithm in *Mathematica*, run some tests and compute some examples. The program has now been completed and includes three main parts : trace simplification, generators, and relations.

For trace simplification, we added some *Mathematica* replacement rules that automatically reduce any expression in terms of traces of words to an expression having only words of length 3 or less. The generators are just traces of all reduced words of length 3 or less, and the relations of type 1 and 2 were taken from Lawton's slides.

All three parts of the program are packaged into one function called *Presentation* which essentially takes in as input a presentation for a finitely presented group Γ and outputs a presentation for $\mathfrak{X}_\Gamma$. This program should be useful to many of the other groups that started to work together at MRC, including the Tropical Geometry group and the other subgroups of this group since they need explicit presentations in order to run experiments.

There is currently a paper being written in collaboration with Sean Lawton and Christopher Manon on the topic of minimal character variety presentations, and this program will be published together with the paper.

3 Character Varieties over Finite Fields

Understanding the geometry of character varieties is a challenging problem which have recently been explored in [2], [3], [4]. Inspired by Weil conjectures and recent work on E-polynomials of character varieties such as [6], [8], [9], [11], we took the approach of counting the number of points on the character varieties over the finite fields $\mathbb{F}_q$ for q a prime power in hopes that this information will shed light on the geometric and arithmetic structures of the character varieties themselves.

We chose to investigate this problem for the character varieties of torus knot groups. Specifically, let $\Gamma_{n,m}$ be the fundamental group of the complement of the (n,m)-torus knot in the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$. This group has a simple presentation $\Gamma_{n,m} = \{x, y \mid x^n = y^m\}$. Let $\mathfrak{X}_{n,m}(\mathbb{K}) := \mathfrak{X}(\Gamma_{n,m}, \mathrm{SL}_2(\mathbb{K}))$ denote the $\mathrm{SL}_2(\mathbb{K})$-character variety of the torus knot group $\Gamma_{n,m}$, where $\mathbb{K}$ is a field. Given coprime integers n and m, we study the cardinality $\#\mathfrak{X}_{n,m}(\mathbb{F}_q)$ as a function of q.

For dimensional reasons, we expect $\#\mathfrak{X}_{n,m}(\mathbb{F}_q)$ to be asymptotically linear. Nonetheless, it is still desirable to obtain an exact formula for $\#\mathfrak{X}_{n,m}(\mathbb{F}_q)$. Guided by V. Munoz's work [12] on $\mathfrak{X}_{n,m}(\mathbb{C})$, we make a distinction between the absolutely reducible and the absolutely irreducible characters, where 'absolutely' indicates that we consider the characters over the algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$.

As in the case of complex characters, the absolutely reducible part has one component while the absolutely irreducible one has many nonintersecting components. Moreover, each component are simple and can easily be parameterized. However, unlike the complex case, the number of absolutely irreducible components varies with q and some of these components might not intersect the absolutely reducible one.

We were able to figure out the combinatorial arrangement of all components and obtain a precise formula of $\#\mathfrak{X}_{n,m}(\mathbb{F}_q)$ in the case that q is odd and n, m, q are pairwise coprime; see [5]. Define

$$R_q := \{(\lambda, \mu) \in \overline{\mathbb{F}_q} \mid \lambda^n = \mu^m = \pm 1;\ \lambda, \mu \neq \pm 1;\ \lambda + \lambda^{-1}, \mu + \mu^{-1} \in \mathbb{F}_q\}.$$

For each pair (λ, μ) in R_q, a simple lemma yields a unique $t \in \overline{\mathbb{F}_q}$ such that $\lambda = t^m$ and $\mu = t^n$. Let r_1 be the number of pairs such that $t + t^{-1} \in \mathbb{F}_q$ and $r_2 := \#R_q - r_1$ the size of the rest of R_q. Then, we obtain

$$\#\mathfrak{X}_{n,m}(\mathbb{F}_q) = \frac{r_1}{4}(q-2) + \frac{r_2}{4}q + q.$$

Moving forward, we hope to extend our formula to the case where q may not be coprime with either n or m. In light of recent results on (complex) character varieties of torus knot groups for SU_2 and SL_3 ([13] and [14]), we plan to expand our investigation to these algebraic groups and others, such as GL_n. Hopefully, these steps will allow us to build a general framework for studying character varieties over finite fields.

4 Dynamics over finite fields

Let F_r be a free group of rank r with generators $\{\gamma_1, ..., \gamma_r\}$. Then by [7] the group of outer automorphisms of F_r defined as $Aut(F_r)/Inn(F_r)$ is given by the presentation:

$$Out(F_r) = \langle \{\rho_{ij}, \epsilon_i | i, j = 1, ..., n, i \neq j\} \rangle \text{ where}$$

$$\epsilon_i : \begin{array}{rcl} \gamma_i \mapsto & \gamma_i^{-1} & \\ \gamma_k \mapsto & \gamma_k & i \neq j \end{array}$$

$$\rho_{ij} : \begin{array}{rcl} \gamma_i \mapsto & \gamma_i\gamma_j & \\ \gamma_k \mapsto & \gamma_k & k \neq i \end{array}$$

For $N = \frac{r(r^2+5)}{6}$, the minimal (trace) embedding of $\mathfrak{X}_r$ into $\mathbb{C}^N$ is given by

$$[\rho] \mapsto \begin{array}{l} \big(tr(\rho(\gamma_1)), ..., tr(\rho(\gamma_1)), tr(\rho(\gamma_1\gamma_2)), ..., tr(\rho(\gamma_{r-1}\gamma_r)), \\ \quad tr(\rho(\gamma_1\gamma_2\gamma_3)), ..., tr(\rho(\gamma_{r-2}\gamma_{r-1}\gamma_r))\big) \end{array}$$

Let $tr(\gamma_i) = x_i$, $tr(\gamma_i\gamma_j) = x_{ij}$ and $tr(\gamma_i\gamma_j\gamma_k) = x_{ijk}$. Then the image of the generators under the minimal trace embedding can be computed as follows.
For $\epsilon_i \in Out(F_r)$:

$$\begin{array}{rrcl}
1. & tr(\epsilon_i(\gamma_i)) & = & tr((\gamma_i^{-1}) = x_i \\
2. & tr(\epsilon_i(\gamma_i\gamma_j)) & = & x_i x_j - x_{ij} \\
3. & tr(\epsilon_i(\gamma_h\gamma_i)) & = & x_h x_i - x_{hi} \\
4. & tr(\epsilon_i(\gamma_i\gamma_j\gamma_k)) & = & x_i x_{jk} - x_{ijk} \\
5. & tr(\epsilon_i(\gamma_h\gamma_i\gamma_j)) & = & x_i x_{hk} - x_{ijk} \\
6. & tr(\epsilon_i(\gamma_g\gamma_h\gamma_i)) & = & x_i x_{gh} - x_{ijk}.
\end{array}$$

Similarly for $\rho_{ij} \in Out(F_r)$,

$$\begin{array}{rrcc}
1. & tr(\rho_{ij}(\gamma_i)) & = & tr(\rho_i(\gamma_i\gamma_j)) = x_{ij} \\
2. & tr(\rho_{ij}(\gamma_i\gamma_j)) & = & x_j x_{ij} - x_i \\
3. & tr(\rho_{ij}(\gamma_j\gamma_i)) & = & x_j x_{ij} - x_i \\
4. & tr(\rho_{ij}(\gamma_h\gamma_i)) & = & x_{hij} \\
5. & tr(\rho_{ij}(\gamma_i\gamma_h)) & = & x_{ijh} \\
6. & tr(\rho_{ij}(\gamma_i\gamma_j\gamma_k)) & = & x_j x_{ijk} - x_{ik} \\
7. & tr(\rho_{ij}(\gamma_i\gamma_h\gamma_j)) & = & \frac{1}{2}(x_i x_j^2 x_h - 2x_j x_h x_{i,j} + x_h(-x_i + x_j x_{i,j}) - (-2 + x_j^2)x_{i,h} - 2x_i x_j x_{j,h} \\
 & & & +2x_{i,j}x_{j,h} + x_i(-x_h + x_j x_{j,h}) + x_j(-x_i x_j x_h + x_h x_{i,j} + x_j x_{i,h} \\
 & & & +x_i x_{j,h} - x_{i,j,h}) + x_j x_{i,j,h}) \\
8. & tr(\rho_{ij}(\gamma_i\gamma_h\gamma_k)) & = & \frac{1}{2}(-x_h x_k x_{i,j} - x_i x_k x_{j,h} + x_k x_{i,j,h} - x_j x_h x_{i,k} + x_{i,k}x_{j,h} - x_{i,h}x_{j,k} \\
 & & & -x_i x_j x_{h,k} + x_{i,j}x_{h,k} + x_h x_{i,j,k} + x_j x_{i,h,k} + x_i x_{j,h,k} + x_i x_j x_h x_k) \\
9. & tr(\rho_{ij}(\gamma_j\gamma_i\gamma_k)) & = & \frac{1}{2}(x_i x_j^2 x_k - 2x_j x_k x_{i,j} + x_k(-x_i + x_j x_{i,j}) - (-2 + x_j^2)x_{i,k} - 2x_i x_j x_{j,k}+ \\
 & & & 2x_{i,j}x_{j,k} + x_i(-x_k + x_j x_{j,k}) + x_j(-x_i x_j x_k + x_k x_{i,j} + x_j x_{i,k}+ \\
 & & & x_i x_{j,k} - x_{i,j,k}) + x_j x_{i,j,k}) \\
10. & tr(\rho_{ij}(\gamma_h\gamma_i\gamma_j)) & = & x_j x_{hij} - x_{hi} \\
11. & tr(\rho_{ij}(\gamma_h\gamma_i\gamma_k)) & = & \frac{1}{2}(-x_h x_k x_{i,j} - x_i x_k x_{j,h} + x_k x_{i,j,h} - x_j x_h x_{i,k} + x_{i,k}x_{j,h} - x_{i,h}x_{j,k} \\
 & & & -x_i x_j x_{h,k} + x_{i,j}x_{h,k} + x_h x_{i,j,k} + x_j x_{i,h,k} + x_i x_{j,h,k} + x_i x_j x_h x_k) \\
12. & tr(\rho_{ij}(\gamma_j\gamma_h\gamma_i)) & = & \frac{1}{2}(-x_i x_k x_{j,h} - x_j x_k x_{h,i} + x_k x_{j,h,i} - x_h x_i x_{j,k} + x_{j,k}x_{h,i} - x_{j,i}x_{h,k} \\
 & & & -x_j x_h x_{i,k} + x_{j,h}x_{i,k} + x_i x_{j,h,k} + x_h x_{j,i,k} + x_j x_{h,i,k} + x_j x_h x_i x_k) \\
13. & tr(\rho_{ij}(\gamma_h\gamma_j\gamma_i)) & = & \frac{1}{2}(x_h x_j^2 x_i - 2x_j x_i x_{h,j} + x_i(-x_h + x_j x_{h,j}) - (-2 + x_j^2)x_{h,i} - 2x_h x_j x_{j,i}+ \\
 & & & 2x_{h,j}x_{j,i} + x_h(-x_i + x_j x_{j,i}) + x_j(-x_h x_j x_i + x_i x_{h,j} + x_j x_{h,i}+ \\
 & & & x_h x_{j,i} - x_{h,j,i}) + x_j x_{h,j,i}) \\
14. & tr(\rho_{ij}(\gamma_h\gamma_k\gamma_i)) & = & \frac{h}{k}(-x_i x_j x_{h,k} - x_h x_j x_{k,i} + x_j x_{h,k,i} - x_k x_i x_{h,j} + x_{h,j}x_{k,i} - x_{h,i}x_{k,j} \\
 & & & -x_h x_k x_{i,j} + x_{h,k}x_{i,j} + x_i x_{h,k,j} + x_k x_{h,i,j} + x_h x_{k,i,j} + x_h x_k x_i x_j)
\end{array}$$

These transformations defined on generators can be used to find the image of any outer automorphism under the trace embedding. Using the above algorithm along with the program by Jean-Philippe Burelle we were able to write Matematica code which outputs the transformation in terms of trace coordinates when such an element of $Out(F_r)$ is given as input.

Let $\Gamma = F_2$ and consider $\mathfrak{X} = \mathfrak{X}(\Gamma, \mathrm{SL}_2(\mathbb{F}_q))$. There is natural action of $Out(\Gamma)$ on $\mathfrak{X}$ by precomposition. This action has been extensively studied in the case when $\mathfrak{X}$ is the full $\mathrm{SL}_2(\mathbb{C})$ character variety, as an algebraic analogue of the mapping class group acting on Teichmüller space. There are however interesting questions that arise around the finite field variant of the character variety. One question we hope to address is: what is the largest orbit for a fixed element of $Out(\Gamma)$? In [10], a quadratic bound in q , relies heavily on the fact that every element of $Out(\Gamma)$ is geometrically realizable as a mapping class of the once punctured torus; see [1]. What we hope to address: is there an element which asymptotically realizes this bound, and if so what type of element is it (psuedo-Anosov, reducible, periodic)? It is clear that there are psuedo-Anosov elements which act periodically on $\mathfrak{X} = \mathbb{A}^3$, hence a characterization of mapping classes cannot happen. However, it would be useful to develop a theoretic check for determining if a mapping class was psuedo-anosov given the growth of its "arithmetic orbits".

We develop an algorithm to compute the following function over the primes. Fix $\alpha \in Out(\Gamma)$ and the function, $\mathbf{O}_\alpha(p)$, is defined by:

$$p \mapsto \max_{x \in \mathbb{A}^3} |Orb_\alpha(x)| \tag{1}$$

We were able to compute the first 60 prime for a properly chosen psuedo-Anosov, α, which is the composition of an involution with a Dehn-twist about the meridian. The results are as follows. The best fit quadratic regression has leading coefficient approaching zero, suggesting less than quadratic growth. However, the best fit linear regression has slope tending to infinity, suggesting greater than linear growth. It would be interesting if $\mathbf{O}_\alpha(p) \sim p \log(p)$.

Future work:

1. Compute $\mathbf{O}_\alpha(p)$ for more primes and other α 's.

2. Generalize algorithm for $q = p^d$.

3. Develop an algorithm for $\Gamma = \mathbb{F}_3$, this might prove fruitful, in more than one way. Come up with a similar theoretic bound using a mapping class group model; finding traces which are fixed by the orbits.

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