

## MA571 Homework 8

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**PROBLEM 8.1 (MUNKRES §46, EX. 6)**

Show that the compact-open topology,  $\mathcal{C}(X, Y)$  is Hausdorff if  $Y$  is Hausdorff, and regular if  $Y$  is regular. [Hint: If  $\overline{U} \subset V$ , then  $\overline{S(C, U)} \subset S(C, V)$ .]

*Proof.* Suppose that  $Y$  is Hausdorff. Let  $f$  and  $g$  be distinct continuous functions from  $X$  to  $Y$ . Then there exists a point  $x_0 \in X$  such that  $f(x_0) \neq g(x_0)$ . Since  $Y$  is Hausdorff there exists disjoint neighborhoods  $U$  and  $V$  of  $f(x_0)$  and  $g(x_0)$ , respectively. Now, we claim that

**Claim.** *If  $C \subset X$  is finite,  $C$  is compact.*

*Proof.* Write  $C = \{x_1, \dots, x_n\}$ . Let  $\mathcal{A}$  be an open cover of  $C$ . Then since  $C \subset \bigcup_{U_\alpha \in \mathcal{A}} U_\alpha$  we can choose  $A_i$  containing  $x_i$  for every  $1 \leq i \leq n$ . Thus, the subcollection  $\{U_i\}_{i=1}^n$  covers  $C$ . ♣

Let  $U' = S(\{x_0\}, U)$  and  $V' = S(\{x_0\}, V)$ . Note that  $U'$  and  $V'$  are nonempty since  $f \in U'$  and  $g \in V'$ . Moreover, their intersection is empty for suppose  $h \in U' \cap V'$ , then  $h(x_0) \in U \cap V$ , but  $U \cap V = \emptyset$ . Then, since  $U'$  and  $V'$  are subbasis elements for the compact-open topology on  $\mathcal{C}(X, Y)$  and they “separate”  $f$  and  $g$ , it follows that  $\mathcal{C}(X, Y)$  is Hausdorff.

Now, suppose that  $Y$  is regular. We shall proceed by the hint and Lemma 31.1(b). Consider the subbasis element  $S(C, U)$ . Since  $Y$  is regular, there exists a neighborhood  $V \supset U$  such that  $V \supset \overline{U}$ . Let  $f \in \overline{S(C, U)}$ . Then, we claim that  $f \in S(C, V)$ . For suppose not, then there exists an element  $x_0 \in C$  such that  $f(x_0) \notin V$ . Then, since  $\overline{U} \subset V$ , by hypothesis,  $f(x_0) \notin \overline{U}$ . Consider the subbasic neighborhood  $S(\{x_0\}, Y - \overline{U})$  of  $f$ . Then,  $S(\{x_0\}, Y - \overline{U}) \cap S(C, U)$  is nonempty. Let  $g$  be in the aforementioned intersection. Then  $g(x_0) \in g(C) \subset U$ , but  $g(x_0) \in Y - \overline{U}$ . This is a contradiction. It follows by Lemma 31.1(b) that  $\mathcal{C}(X, Y)$  is regular. ■

**PROBLEM 8.2 (MUNKRES §46, EX. 7)**

Show that if  $Y$  is locally compact Hausdorff, then composition of maps

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$$

is continuous, provided the compact-open topology is used throughout. [Hint: If  $g \circ f \in S(C, U)$ , find  $V$  such that  $f(C) \subset V$  and  $g(\overline{V}) \subset U$ .]

*Proof.* Let  $F: \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$  given by  $(f, g) \mapsto g \circ f$ . Suppose  $g \circ f \in S(C, U)$ . Then  $g(f(C)) \subset U$  and since  $g$  is continuous, we have that  $g^{-1}(U)$  is an open set containing  $f(C)$ . Thus, by theorem 29.2, for every  $x \in f(C)$  there exists an open neighborhood  $V_x$  of  $x$  such that  $\overline{V_x} \subset g^{-1}(U)$  is compact. Then the collection of all such open neighborhoods,  $\{V_x\}_{x \in f(C)}$ , forms an open cover of  $f(C)$ . Since  $f(C)$  is compact, by Theorem 26.5 since  $C$  is compact and  $f$  is continuous, then by Lemma 26.1 there exists a finite subcollection, say  $\{V_i\}_{i=1}^n$ , that covers  $C$ . Let  $V = \bigcup_{i=1}^n V_i$ . We claim that  $\overline{V} \subset U$  and is compact. More generally, we have

**Lemma 16** (Munkres §26, Ex. 3). *A finite union of compact subspaces of  $X$  is compact.*

*Proof of lemma.* Suppose  $C_1, \dots, C_n \subset X$  are compact and write  $C = \bigcup_{i=1}^n C_i$ . Let  $\mathcal{A} = \{U_\alpha\}$  be an open cover of  $C$ . Then  $C_i \subset \bigcup U_\alpha$  so, since  $C_i$  is compact, there exists a finite subcollection  $\mathcal{A}_i = \{U_j^i\}_{j=1}^{n_i}$  that covers  $C_i$ . Then  $\mathcal{B} = \bigcup_{i=1}^n \mathcal{A}_i$  is a finite subcollection of  $\mathcal{A}$  that covers  $C$ , i.e.,  $C$  is compact. ♣

By Lemma 16,  $\overline{V}$  is compact since, by induction on Problem 2.2 (Munkres §17, Ex. 6(b)), it is the union of finitely many compact sets  $\overline{V} = \bigcup_{i=1}^n \overline{V_i}$ . Moreover, by Lemma 5 (from HW # 2<sup>1</sup>) we have that  $f(C) \subset V \subset \overline{V} \subset g^{-1}(U)$ . At last, tying these results together, we have

$$F(S(C, V) \times (\overline{V}, U)) \subset S(C, U),$$

since  $f' \in S(C, V)$  if  $f'(C) \subset V$  and  $g' \in S(\overline{V}, U)$  if  $g'(\overline{V}) \subset U$  so  $g'(f'(C)) \subset g'(\overline{V}) \subset U$  so  $g' \circ f' \in S(C, U)$ . It follows, by Theorem 18.1(4), that  $F$  is continuous. ■

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<sup>1</sup>This states that if  $A_\alpha \subset C$  then  $\bigcup A_\alpha \subset C$ .

**PROBLEM 8.3 (MUNKRES §46, EX. 8)**

Let  $\mathcal{C}'(X, Y)$  denote the set  $\mathcal{C}(X, Y)$  in some topology  $\mathcal{T}$ . Show that if the evaluation map

$$e: X \times \mathcal{C}'(X, Y) \longrightarrow Y$$

is continuous, then  $\mathcal{T}$  contains the compact-open topology. [*Hint:* The induced map  $E: \mathcal{C}'(X, Y) \rightarrow \mathcal{C}(X, Y)$  is continuous.]

*Proof.* Suppose that the evaluation map  $e: X \times \mathcal{C}'(X, Y) \longrightarrow Y$  is continuous. Then, by Theorem 46.11 the induced map  $E: \mathcal{C}'(X, Y) \rightarrow \mathcal{C}(X, Y)$  in

$$X \times \mathcal{C}'(X, Y) \xrightarrow{(\text{id}_X, E)} X \times \mathcal{C}(X, Y) \xrightarrow{e'} Y$$

is continuous. In fact, it is easy to see that the induced map  $E$  is the identity map on  $\mathcal{C}(X, Y)$  for  $e(x, f) = f(x) = f'(x) = e'(f', x) = e'(E(f), x)$  for all  $x$  so  $f = f'$ . Now, let  $S(C, U)$  be a subbasic open set in  $\mathcal{C}(X, U)$ . Then  $E^{-1}(S(C, U)) = S(C, U)$  is open in  $\mathcal{C}'(X, Y)$ . Thus  $\mathcal{T}$  is finer than the compact-open topology. ■

**PROBLEM 8.4 ((A))**

**Definition 1.** Definition. If  $X$  is a locally compact Hausdorff space then the space  $Y$  given by Theorem 29.1 is called the *one-point compactification* of  $X$ .

Let  $X$  be a compact Hausdorff space and let  $W$  be an open subset of  $X$  (so  $W$  is locally compact by Corollary 29.3) with  $W \neq X$ . Prove that the one-point compactification of  $W$  is homeomorphic to the quotient space  $X/(X - W)$ .

*Proof.* Let  $W_\infty$  denote the one-point compactification of  $W$  and define the map  $p: X \rightarrow W_\infty$  by

$$p(x) = \begin{cases} x, & x \in W \\ \infty, & x \in X - W. \end{cases}$$

We claim that  $p$  is continuous. It suffices to show that the preimage of a basic open set in  $W_\infty$  is open in  $X$ . Suppose  $U$  is a type 1 open subset of  $W_\infty$ , that is,  $U$  does not contain the point at infinity. Then  $U \subset W$  so is open in  $X$  by Theorem 16.2. Suppose that  $U$  is a type 2 open subset of  $W_\infty$ . Then  $C = W_\infty - U$  is a compact subset of  $W_\infty$ . Moreover  $C \subset W$  so  $C$  is a compact subset of  $X$ , that is to say, if  $\{U_\alpha\}$  is an open cover of  $C$  in  $X$ , then  $\{U_\alpha \cap W\}$  is an open cover of  $C$  in  $W$  and since  $C$  is compact in  $W$ , there exists a finite subcollection  $\{U_i \cap W\}_{i=1}^n$  in  $Y$  that covers  $C$  hence, the collection  $\{U_i\}_{i=1}^n$  is a finite subcollection in  $X$  that covers  $C$ . It follows by Theorem 26.3 that  $C$  is closed so  $p^{-1}(U) = X - C$  is open in  $X$ . Thus,  $p$  is continuous. By Theorem Q.3, it follows that the induced map  $\bar{p}: X/(X - W) \rightarrow W_\infty$  is continuous. Moreover,  $p$  preserves the equivalence relation: Suppose  $x \sim y$  then either  $x = y \in W$  or  $x, y \in X - W$ ; in the former we have  $p(x) = x = y = p(y)$ ; in the latter we have  $p(x) = \infty = p(y)$ .

By Theorem 26.6, since the quotient  $X/(X - W)$  is compact and  $W_\infty$  is Hausdorff, it suffices to show that  $\bar{p}$  is bijective. It is clear that  $\bar{p}$  is surjective since  $p$  is surjective ( $p(X) = p(W \cup (X - W)) = p(W) \cup p(X - W) = W \cup \{\infty\} = W_\infty$ ). To see that  $\bar{p}$  is injective suppose  $p([x]) = p([y])$ . Then  $p([x]) = \infty$  or  $p([x]) \neq \infty$ . If  $p([x]) \neq \infty$ , then  $p([x]) = x = y = p([y])$  or  $p([x]) = \infty = p([y])$ . In either case,  $x \sim y$  so  $[x] = [y]$ . Thus,  $\bar{p}$  is bijective. It follows that  $\bar{p}$  is a homeomorphism. ■

**PROBLEM 8.5 ((B))**

Let  $X$  be a compact Hausdorff space, let  $Y$  be a topological space, and let  $p: X \rightarrow Y$  be a closed surjective continuous map. Prove that  $Y$  is Hausdorff. [*Hint*: one ingredient in the proof is p. 171 # 5.]

Note: combining this with HW 4 Problem E and HW 6 Problem A gives a necessary and sufficient condition for a quotient of a compact Hausdorff space to be Hausdorff.

*Proof.* Let  $x$  and  $y$  be distinct points in  $Y$ . Since  $p$  is surjective, there exist  $x_0$  and  $y_0$  in  $X$  such that  $p(x_0) = x$  and  $p(y_0) = y$ . Then, since  $X$  is Hausdorff, by Theorem 17.8,  $x_0$  and  $y_0$  are closed in  $X$  so  $x$  and  $y$  are closed in  $Y$ . Then  $p^{-1}(x)$  and  $p^{-1}(y)$  are closed since

$$X - p^{-1}(x) = p^{-1}(Y - x)$$

which is open in  $X$  since  $Y - x$  is open in  $Y$  and  $p$  is continuous. Moreover,  $p^{-1}(x)$  and  $p^{-1}(y)$  are clearly disjoint for otherwise  $p(z) = x = y$ , but  $x \neq y$ . Now, by Theorem 32.3,  $X$  is normal since it is a compact Hausdorff space (alternatively we may appeal to Theorem 26.3 and Munkres §26, Ex.5 as suggested in the hint) so there exist disjoint open sets  $U$  and  $V$  containing  $p^{-1}(x)$  and  $p^{-1}(y)$ , respectively. Then  $X - U$  and  $X - V$  are closed so  $p(X - U)$  and  $p(X - V)$  are closed in  $Y$ . Then, we claim  $U' = Y - p(X - U)$  and  $V' = Y - p(X - V)$  are disjoint neighborhoods of  $x$  and  $y$ , respectively. It is clear that  $U'$  and  $V'$  are open, since their complements are closed. Moreover,  $U' \ni x$  and  $V' \ni y$  since  $Y - U' = p(X - U)$  does not contain  $x$  and  $Y - V' = p(X - V)$  does not contain  $y$ . Lastly,  $U' \cap V' = \emptyset$  for otherwise there is  $z \in U' \cap V'$  so  $z \notin p(X - U)$  and  $z \notin p(X - V)$  so  $z \in Y - (p(X - U) \cup p(X - V))$ , but  $p(X - U) \cup p(X - V) \supset p((X - U) \cup p(X - V)) = p(X)$  so  $z \in \emptyset$ , this is a contradiction. Thus,  $Y$  is Hausdorff. ■

**PROBLEM 8.6 ((C))**

Let  $S^2 \subset \mathbf{R}^3$  be the subspace

$$\{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}.$$

Prove that  $S^2$  is a 2-manifold. (The definition of  $m$ -manifold, where  $m$  is a positive whole number, is given at the top of page 225.)

*Proof.* Note that, by Problem 3.9 (A) from HW # 3, since  $\mathbf{Q}^3$  is a dense countable subset of  $\mathbf{R}^3$ ,  $\mathbf{R}^3$  has a countable basis so  $S^2$  has a countable basis. Next, we define the following system of charts in  $S^2$  with inverses

$$\begin{aligned} \varphi_z(x, y, z) &= (x, y) & \varphi_z^{-1}(x, y) &= \left( x, y, \sqrt{1 - x^2 - y^2} \right) & U_z &= \{ (x, y, z) \in S^2 \mid z > 0 \} \\ \varphi_{-z}(x, y, z) &= (x, y) & \varphi_{-z}^{-1}(x, y) &= \left( x, y, -\sqrt{1 - x^2 - y^2} \right) & U_{-z} &= \{ (x, y, z) \in S^2 \mid z < 0 \} \\ \varphi_y(x, y, z) &= (x, z) & \varphi_y^{-1}(x, z) &= \left( x, \sqrt{1 - x^2 - z^2}, z \right) & U_y &= \{ (x, y, z) \in S^2 \mid y > 0 \} \\ \varphi_{-y}(x, y, z) &= (x, z) & \varphi_{-y}^{-1}(x, z) &= \left( x, -\sqrt{1 - x^2 - z^2}, z \right) & U_{-y} &= \{ (x, y, z) \in S^2 \mid y < 0 \} \\ \varphi_x(x, y, z) &= (y, z) & \varphi_x^{-1}(y, z) &= \left( \sqrt{1 - y^2 - z^2}, y, z \right) & U_x &= \{ (x, y, z) \in S^2 \mid x > 0 \} \\ \varphi_{-x}(x, y, z) &= (y, z) & \varphi_{-x}^{-1}(y, z) &= \left( -\sqrt{1 - y^2 - z^2}, y, z \right) & U_{-x} &= \{ (x, y, z) \in S^2 \mid x < 0 \} \end{aligned}$$

These maps are obviously continuous by Theorem 18.4 and 21.4, invertible, and they are in fact homeomorphisms by Theorem 26.6. Now, note that the open sets  $\{U_x, U_{-x}, U_y, U_{-y}, U_z, U_{-z}\}$  cover  $S^2$  so every point  $(x, y, z) \in S^2$  is contained in one. Then, by the definition of  $m$ -manifold in Munkres §36, p. 225, it follows that  $S^2$  is a 2-manifold, i.e., a surface. ■



**PROBLEM 8.7 ((D))**

Prove that the union of the  $x$  and  $y$ -axes in  $\mathbf{R}^2$  is not a 1-manifold.

*Proof.* Let  $X$  denote the union of the  $x$  and  $y$ -axes, that is,

$$X = \{ (x, 0) \} \cup \{ (0, x) \}$$

for  $x \in \mathbf{R}$ . Suppose  $X$  is a 1-manifold. Then around every open subset  $U$  of  $X$ , there exists a homeomorphism  $\varphi: U \rightarrow V$  for some  $V$  open in  $\mathbf{R}$ . Without loss of generality, we may assume  $V = (a, b)$  for some real numbers  $a < b$ . Now, consider the open neighborhood  $U = B((0, 0), \varepsilon) \cap X = ((-\varepsilon, \varepsilon) \times 0) \cup (0 \times (-\varepsilon, \varepsilon))$ . Since  $X$  is a 1-manifold, there exists a homeomorphism  $\varphi: U \rightarrow (a, b)$ . Then, by Lemma A,  $\varphi(U - 0 \times 0) \approx (a, b) - \varphi(0 \times 0)$ . However,

$$U - 0 \times 0 = ((-\varepsilon, 0) \times 0) \cup ((0, \varepsilon) \times 0) \cup (0 \times (-\varepsilon, 0)) \cup (0 \times (0, \varepsilon))$$

is a union of four disjoint open subsets of  $X$ , therefore,  $U$  consists of four connected components. However,  $(a, b) - \varphi(0 \times 0) = (a, \varphi(0 \times 0)) \cup (\varphi(0 \times 0), b)$  consists of only two connected components. This is a contradiction. It follows that  $X$  is not a 1-manifold. ■