MA 544: Homework 10

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Problem 10.1 (Wheeden & Zygmund §7, Ex. 1)

Let f be measurable in $\mathbf{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(\mathbf{x}) \geq c\|\mathbf{x}\|^{-n}$ for $\|\mathbf{x}\| \geq 1$.

Proof. Let E be a measurable subset of $\mathbf{R}^n$ with positive measure such that $f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$. Moreover, for now, assume that $|E| < \infty$. By Vitali's lemma, for $\{Q_k\}_{k=1}^N$ a finite cover of E by cubes, there exists a real number $\beta > 0$ such that

$$|E| < \frac{1}{\beta} \sum_{k=1}^{N} |Q_k|. \tag{1}$$

■

Problem 10.2 (Wheeden & Zygmund §7, Ex. 2)

Let $\varphi(\mathbf{x}), \mathbf{x} \in \mathbf{R}^n$, be a bounded measurable function such that $\varphi(\mathbf{x}) = 0$ for $\|\mathbf{x}\| \geq 1$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(\mathbf{x}) \coloneqq \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon)$. ($\varphi_\varepsilon$ is called an *approximation to the identity.*) If $f \in L^1(\mathbf{R}^n)$, show that

$$\lim_{\varepsilon \to 0}(f * \varphi_\varepsilon)(x) = f(\mathbf{x})$$

in the Lebesgue set of f. (Note that $\int \varphi_\varepsilon = 1$, $\varepsilon > 0$, so that

$$(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x}) = \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{y})d\mathbf{y}.$$

Use Theorem 7.16.)

Proof. ■

Problem 10.3 (Wheeden & Zygmund §7, Ex. 6)

Show that if $\alpha > 0$, then x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Proof. ■

Problem 10.4 (Wheeden & Zygmund §7, Ex. 8)

Prove the following converse of Theorem 7.31: If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Proof. ■

Problem 10.5 (Wheeden & Zygmund §7, Ex. 9)

If f is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then f is absolutely continuous on $[a, b]$. (For the second part, use Theorems 2.2(ii) and 7.24 to show that $V(x)$ is absolutely continuous and then use the result of Exercise 8).

Proof. ■

Problem 10.6 (Wheeden & Zygmund §7, Ex. 11)

Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a \coloneqq g(\alpha)$, $b \coloneqq g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt.$$

(Consider the case when f is the characteristic function of an interval, an open set, etc.)

Proof. ∎