

## MA 523: Homework 2

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September 9, 2016



**Problem 2.1**

Verify assertion (36) in [E, §3.2.3], that when  $\Gamma$  is not flat near  $x^0$  the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here  $\nu(x^0)$  denotes the normal to the hypersurface  $\Gamma$  at  $x^0$ ).

**Solution.** ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the standard noncharacteristic boundary condition if  $\Gamma$  is flat near  $x^0$  because in such case we have  $\nu(x^0) = (0, \dots, 0, 1)$  so

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall verify the noncharacteristic condition (2.1) by first flattening the boundary near  $x^0$  and then applying the noncharacteristic boundary conditions to the flattened region. Assuming some degree of regularity near  $x^0$ , e.g., that the boundary of  $U$  be smooth, we may express  $\Gamma$  near  $x^0$  as the graph of a smooth function  $f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ , i.e.,  $x = (x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}))$  on  $\Gamma$  and  $x_n \geq f(y)$  after reorienting the coordinate axes. Then we flatten out  $\Gamma$  via the map  $\Phi(x): \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by

$$\left\{ \begin{array}{l} y_1 = x_1 = \Phi^1(x), \\ \vdots \\ y_{n-1} = x_{n-1} = \Phi^{n-1}(x), \\ y_n = x_n - f(x_1, \dots, x_{n-1}) = \Phi^n(x) \end{array} \right.$$

and write  $y = \Phi(x)$ . Let  $\Psi = \Phi^{-1}$  and rewrite our PDE  $F$  in terms of  $y$  as follows,

$$0 = F(Du(\Psi(y)), u(\Psi(y)), \Psi(y)). \tag{2.2}$$

Since  $\Delta = \Phi(\Gamma)$  is flat near  $y^0 = \Phi(x^0) = (y_1^0, \dots, y_{n-1}^0, 0)$ , we may apply the standard noncharacteristic condition on (2.2) and get

$$0 \neq F_{u_{y_n}}(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)).$$

Before we move on to finding an expression for this derivative, let us consider the gradient  $Du(\Psi(y))$ . By the chain rule, we have

$$\begin{aligned} u_{y_i}(\Psi(y)) &= \sum_{j=1}^n u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_i} \\ &= u_{x_i}(\Psi(y)) + u_{x_n}(\Psi(y)) f_{y_i}(y_1, \dots, y_{n-1}), \\ u_{y_n}(\Psi(y)) &= \sum_{j=1}^n u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_n} \\ &= u_{x_n}(\Psi(y)), \end{aligned}$$

Then, substituting  $u_{y_n}$  for  $u_{x_n}$ , we have

$$u_{y_i}(\Psi(y)) = u_{x_i}(\Psi(y)) + u_{y_n}(\Psi(y))f_{y_i}(y_1, \dots, y_{n-1}),$$

Now, by the chain rule on (2.2), we have

$$\begin{aligned} 0 &\neq F_{u_{y_n}}(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)) \\ &= F_{u_{y_n}}(u_{x_1} + u_{y_n}f_{y_1}, \dots, u_{x_{n-1}} + u_{y_n}f_{y_{n-1}}, u_{y_n}, z^0, x^0) \\ &= F_{u_{x_1}}f_{y_1} + \dots + F_{u_{x_{n-1}}}f_{y_{n-1}} + F_{u_{x_n}} \\ &= D_p F(p^0, z^0, x^0) \cdot (Df(x^0), 1) \\ &= D_p F(p^0, z^0, x^0) \cdot \nu(x^0), \end{aligned}$$

as we set out to show. ◀

**Problem 2.2**

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions  $u(x, 0) = g(x)$  is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some  $t > 0$ , unless  $a(g(x))$  is a nondecreasing function of  $x$ .

**Solution.** ► The characteristic ODEs of this PDE are

$$\dot{t} = 1, \quad \dot{x} = a(z), \quad \dot{z} = 0. \quad (2.3)$$

with initial conditions  $t_0 = 0$ ,  $x_0 = x(0)$  and  $z(x_0, 0) = g(x_0)$  with  $(x_0, 0) \in \mathbb{R} \times (0, \infty)$ . Hence, we have

$$t(s) = s, \quad x(s) = a(g(x_0))s + x_0, \quad z(s) = g(x_0).$$

Thus, solving for  $x_0$  and  $s$  in terms of  $t$ ,  $x$  and  $z$ , we have

$$\begin{aligned} x &= a(g(x_0))s + x_0 \\ &= a(z)t + x_0, \end{aligned}$$

so, moving  $x_0$  to the left-hand side

$$x_0 = x - a(z)t$$

hence,

$$z = g(x - a(z)t),$$

i.e.,

$$u = g(x - a(u)t),$$

as desired.

For the latter half of the problem, write

$$u(x + a(g(x))t, t) = g(x).$$

Suppose that  $a(g(x))$  is not a nondecreasing function of  $x$ . Then, there exists  $0 < x_1 < x_2$  such that  $a(g(x_1)) > a(g(x_2))$ . Define

$$y = -\frac{x_1 - x_2}{a(g(x_1)) - a(g(x_2))} > 0. \quad (2.4)$$

Then, we have

$$t_0 = x_1 + a(g(x_1))y = x_2 + a(g(x_2))y.$$

Thus,

$$\begin{aligned}u(x, t_0) &= g(x_1) \\&= u(x_1 + a(g(x_1))t_0, t_0) \\&= g(x_2) \\&= u(x_2 + a(g(x_2))t_0, t_0).\end{aligned}$$

However,  $g(x_1) \neq g(x_2)$  since  $a(g(x_1)) > a(g(x_2))$ . ◀

**Problem 2.3**

Show that the function  $u(x, t)$  defined for  $t \geq 0$  by

$$u(x, t) = \begin{cases} -\frac{2}{3} \left( t + \sqrt{3x + t^2} \right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law  $u_t + (u^2/2)_x = 0$  (*inviscid Burgers' equation*).

**Solution.** ► The shock occurs along the curve  $C$  given by  $s(t) = -t^2/4$ . First, we verify that the equation given by  $u$  above is in fact a solution to the inviscid Burgers' equation to the right and to the left of  $C$ : to the left of  $C$ ,  $4x + t^2 < 0$ , the equation is trivially satisfied whereas to the right,  $4x + t^2 > 0$ , we have,

$$-\frac{2}{3} \left( 1 + \frac{t}{\sqrt{3x + t^2}} \right) + \frac{2}{9} \left( 3 + \frac{3t}{\sqrt{3x + t^2}} \right) = 0.$$

So  $u$  is indeed a solution to the inviscid Burgers' equation.

Now we examine the behavior of  $u$  along the curve  $C$ . First, we have

$$\begin{aligned} \sigma = \dot{s}(t), & \quad \llbracket u \rrbracket = u_\ell - u_r, & \quad \llbracket F \rrbracket = F(u_\ell) - F(u_r) \\ = -\frac{t}{2} & \quad = 0 + \frac{2}{3} \left( t + \sqrt{-\frac{3}{4}t^2 + t^2} \right) & \quad = 0 - \frac{\llbracket u_r \rrbracket^2}{2} \\ & \quad = 0 + \frac{2}{3} \left( \frac{3}{2}t \right) & \quad = 0 - \frac{t^2}{2} \\ & \quad = t & \quad = -\frac{t^2}{2}. \end{aligned}$$

Thus,

$$\llbracket F \rrbracket = -\frac{t^2}{2} = \left( -\frac{t}{2} \right) t = \sigma \llbracket u \rrbracket$$

so the PDE satisfies the Rankine–Hugoniot condition and hence, is an integral solution.

Lastly, we verify that  $u$  satisfies the entropy condition, i.e., the one-sided jump estimate, Eq. (36) from [E, §3.4.3]. We break this test into two cases, (a)  $u$  is to the left of  $C$  and (b)  $u$  is to the right of  $C$ .

For (a), we have  $x < -t^2/4$ ,  $t > 0$ , and  $z > 0$ . Then,

$$u(x + z, t) - u(x, t) \leq \sup_{x < -t^2/4}$$

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