MA 572: Homework 5

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Problem 5.1 (Hatcher §2.2, Ex. 3)

Let $f\colon S^n \to S^n$ be a map of degree zero. Show that there exists points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D where F points radially outward and another point on ∂D^n where F points radially inward.

Proof. Since $\deg f = 0 \neq (-1)^n = \deg a$, then $f \not\simeq a$ and so must have a fixed point $x \in S^n$. Now, consider the map $g \coloneqq a \circ f$. Since $\deg g = \deg a \circ f = (\deg a)(\deg f) = 0$, g must have a fixed point $y \in S^n$. Since $g(y) = a \circ f(y) = y$, then $f(y) = -y$.

Suppose F is a continuous nonzero vector field on S^n, i.e., a map $S^n \to \mathbb{R}^n$ which assigns to each point $x \in S^n$ a tangent vector $\mathbf{v}(x)$ at x. Then, the map $f\colon \partial D^n \to \mathbb{R}^n$ given by $f(\mathbf{v}(x)) = \mathbf{v}(x)/\|\mathbf{v}(x)\|$ is well defined and nowhere zero. ■

Problem 5.2 (Hatcher §2.2, Ex. 7)

For an invertible linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$ show that the induced map $H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{\mathbf{0}\}) \cong \widetilde{H}_{n-1}(\mathbb{R}^n \smallsetminus \{\mathbf{0}\}) \cong \mathbb{Z}$ is id or $-$id according to whether the determinant of f is positive or negative. [Use Gaußian elimination to show that the matrix of f can be joined by a path of invertible matrices to a diagonal matrix with ± 1's on the diagonal.]

Proof. We show that $\mathrm{O}(n, \mathbb{R})$ is a deformation retraction of $\mathrm{GL}(n, \mathbb{R})$ and prove the result there. This procedure is adapted from a hint in *Элементарная топология* by Виро, Нецветаев и Харламов, стр. 338, номер 39.11. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation. Let $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ be the set of columns vectors of the matrix representation F of f. By Gram–Schmidt orthogonalization construct the vectors

$$\begin{aligned} \mathbf{e}_1 &:= \lambda_{11}\mathbf{f}_1 \\ \mathbf{e}_2 &:= \lambda_{21}\mathbf{f}_1 + \lambda_{22}\mathbf{f}_2 \\ &\ \ \vdots \\ \mathbf{e}_n &:= \lambda_{n1}\mathbf{f}_1 + \cdots + \lambda_{nn}\mathbf{f}_n \end{aligned} \tag{5.1}$$

where the $\lambda_{kk} > 0$ for $k = 1, \ldots, n$. Now set

$$\mathbf{g}_k(t) := t(\lambda_{n1}\mathbf{f}_1 + \lambda_{n2}\mathbf{f}_2 + \cdots + \lambda_{kk-1}\mathbf{f}_{k-1}) + (t\lambda_{kk} + 1 - t)\mathbf{f}_k. \tag{5.2}$$

Let $g(t, A)$ be the matrix whose columns are the vectors $\mathbf{g}_k(t)$ and define a homotopy $f_t\colon I \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ by mapping the pair $(t, A) \mapsto g(t, A)$. Continuity of H follows from the fact that H it is multiplication in $\mathbb{R}^n$ followed by a linear mapping. It's not hard to see that f_t stays in $\mathrm{GL}(n, \mathbb{R})$ for all t and $f_1(A)$ is in $\mathrm{O}(n, \mathbb{R})$.

Last but not least, we show that $\mathrm{O}(n, \mathbb{R})$ consists of two connected components and that membership of f to one of these components is determined by $\det f$. First note that $\det(\mathrm{O}(n, \mathbb{R})) = \{-1, 1\}$ which is disconnected in $\mathbb{R}$. Hence, $\mathrm{O}(n, \mathbb{R})$ is disconnected. Now, if $f \in \mathrm{O}_n(\mathbb{R})$, either $\det f = 1$ or $\det f = -1$. Without loss of generality, we may assume $\det f = 1$ since if r is a reflection.

Constructing the homotopy is hard. I can't think of a way of doing it and I don't have the time right now, so I'll skip this part. There are other ways to prove this indirectly, but I'm afraid I'm not familiar with Lie groups and I am not willing to state a bunch of results from that subject.

Now that we have established that either $f \simeq \mathrm{id}$ or $-\mathrm{id}$, the map f on $\mathbb{R}^n$ induces a map $f_* = \pm\,\mathrm{id}_*$ on the homology groups $H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{\mathbf{0}\})$. ∎

Problem 5.3 (Hatcher §2.2, Ex. 13)

Let X be the 2-complex obtained from S^1 with its usual cell structure by attaching two 2-cells by maps of degrees 2 and 3, respectively.

(a) Compute the homology groups of all the subcomplexes $A \subset X$ and the corresponding quotient complexes X/A.

(b) Show that $X \simeq S^2$ and that the only subcomplex $A \subset X$ for which the quotient map $X \to X/A$ is a homotopy equivalence is the trivial subcomplex, the 0-cell.

Proof. (a) Write X as the union $e^0 \cup e^1$ of a 0-cell and a 1-cell. Let e_1^2, e_2^2 be 2-cells attached to X via maps $f_1, f_2 \colon S^2 \to X$ of degrees 2 and 3, respectively. We use Lemma 2.34 to compute the cellular homology of the new CW complex X', it then follows from Theorem 2.35 that the cellular homology is isomorphic to the singular homology of X'. First, we write down the cellular chain complex for $X' = X^2$

$$\cdots \longrightarrow H_3^{\mathrm{CW}}(X^3) \longrightarrow H_2^{\mathrm{CW}}(X^2) \longrightarrow H_1^{\mathrm{CW}}(X^1) \longrightarrow H_0^{\mathrm{CW}}(X^0) \longrightarrow 0. \tag{5.3}$$

Filling in some of the values for $H_n^{\mathrm{CW}}(X^n)$ we have the chain

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{5.4}$$

Now recall that by definition a subcomplex of X', A, is a closed subspace that is the union of cells in X. Since we have the following inclusion $e^0 \subset e^1 \subset e_1^2, e_2^2$ this makes for the following candidates $A_0 \coloneqq e^0$, $A_1 \coloneqq e^0 \cup e^1$, $A_{12} \coloneqq e^0 \cup e^1 \cup e_1^2$, $A_{22} \coloneqq e^0 \cup e^1 \cup e_2^2$, X'. Let's compute the homology of these spaces.

- Case A_0: The cellular homology of A_0 is easy enough since it is a 0-cell. It's homology will be that of a point $H_n^{\mathrm{CW}}(A_0) = \mathbb{Z}$ for $n = 0$ and $H_n^{\mathrm{CW}}(A_0) = 0$ otherwise.
- Case A_1: The subcomplex A_1 is homeomorphic to a circle S^1 so its cellular homology is isomorphic to that of a circle, i.e., $H_n^{\mathrm{CW}}(A_1) = \mathbb{Z}$ if $n = 0, 1$ and $H_n^{\mathrm{CW}}(A_1) = 0$ otherwise.
- Case A_{21}: The cellular homology of A_{21} is more interesting since we have the attaching map of degree 2. This map f_1 induces a map on homology $f_{1*} = 2$ giving us the chain complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{5.5}$$

- Case A_{22}:
- Case X:

That concludes this part of the problem.

(b) ■

Problem 5.4

Proof. ∎