MA166: Recitation 5 Prep

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1 Recitation 5 Script

Here are the recitation notes for the 11th of February, 2016.

Section 1.1: Homework for the week

Homework 11

Problem 1.1 (WebAssign, HW 11, #1)**.** Evaluate the integral using the indicated trigonometric substitution. (Use C for the constant of integration.)

$$\int \frac{x^3}{\sqrt{x^2+49}}\,\mathrm{d}\,x \quad x = 7\tan\theta.$$

Proof. We solve this problem by using the suggested trigonometric substitution $x = 7\tan\theta$. Using this substitution, we have $\mathrm{d}\,x = 7\sec^2\theta\,\mathrm{d}\,\theta$ and

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^2+49}}\,\mathrm{d}\,x &= \int \frac{343\tan^3\theta}{\sqrt{49\tan^2\theta+49}} 7\sec^2\theta\,\mathrm{d}\,\theta \\
&= \int \frac{2401\tan^3\theta\sec^2\theta}{\sqrt{49\tan^2\theta+49}}\,\mathrm{d}\,\theta \\
&= \int \frac{2401\tan^3\theta\sec^2\theta}{\sqrt{49(\tan^2\theta+1)}}\,\mathrm{d}\,\theta \\
&= \int \frac{2401\tan^3\theta\sec^2\theta}{7\sqrt{\tan^2\theta+1}}\,\mathrm{d}\,\theta \\
&= \int \frac{343\tan^3\theta\sec^2\theta}{\sqrt{\tan^2\theta+1}}\,\mathrm{d}\,\theta
\end{aligned}$$

now, recall the trigonometric identity for tangent $\tan^2\theta + 1 = \sec^2\theta$, so the integral above turns into

$$\begin{aligned}
&= \int \frac{343\tan^3\theta\sec^2\theta}{\sqrt{\sec^2\theta}}\,\mathrm{d}\,\theta \\
&= \int \frac{343\tan^3\theta\sec^2\theta}{\sec\theta}\,\mathrm{d}\,\theta \\
&= 343\int \tan^3\theta\sec\theta\,\mathrm{d}\,\theta
\end{aligned}$$

Now, what? Use the tangent identity again, and we get

$$\begin{aligned}
&= 343\int \tan^2\theta\tan\theta\sec\theta\,\mathrm{d}\,\theta \\
&= 343\int (\sec^2\theta - 1)\tan\theta\sec\theta\,\mathrm{d}\,\theta
\end{aligned}$$

So far so god, but now what? Make the substitution $u = \sec\theta$ since $\mathrm{d}\,u = \sec\theta\tan\theta\,\mathrm{d}\,\theta$ which turns our integral into

$$= 343\int (u^2-1)\tan\theta\sec\theta\frac{\mathrm{d}\,u}{\sec\theta\tan\theta}$$

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Problem 1.2 (WebAssign, HW 11, #2)**.** Evaluate the integral

$$2\int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}\,x.$$

Proof. Make the substitution $x=\sin\theta$ so that $\mathrm{d}\,x=\cos\theta\,\mathrm{d}\,\theta$. Replacing our original bounds for x in terms of $\sin^1(0)=0$ and $\sin^{-1}(1)=\pi/2$ we have the following integral

$$\begin{aligned}
I &= 2\int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}\,x \\
&= 2\int_0^{\pi/2} \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\,\theta
\end{aligned}$$

which, by the Pythagorean identity $\cos^2\theta+\sin^2\theta=1$, turns into

$$\begin{aligned}
&= 2\int_0^{\pi/2} \sin^3\theta\cos\theta\cos\theta\,\mathrm{d}\,\theta \\
&= 2\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,\mathrm{d}\,\theta \\
&= 2\int_0^{\pi/2} \sin^2\theta\sin\theta\cos^2\theta\,\mathrm{d}\,\theta \\
&= 2\int_0^{\pi/2} (1-\cos^2\theta)\sin\theta\cos^2\theta\,\mathrm{d}\,\theta
\end{aligned}$$

Now we make the substitution $u=\cos\theta$ so $\mathrm{d}\,u=-\cos\theta \iff \theta$ and we have

$$\begin{aligned}
&= 2\int_1^0 (1-u^2)\sin\theta u^2\theta\frac{\mathrm{d}\,u}{-\sin\theta} \\
&= 2\int_1^0 -(1-u^2)u^2\,\mathrm{d}\,u \\
&= 2\int_0^1 (1-u^2)u^2\,\mathrm{d}\,u \\
&= 2\int_0^t 1u^2-u^4\,\mathrm{d}\,u \\
&= 2\big[\tfrac{1}{3}u^3-\tfrac{1}{5}u^5\big]_0^1 \\
&= 2\big(\tfrac{1}{3}-\tfrac{1}{5}-(0-0)\big) \\
&= \boxed{\frac{4}{15}.}
\end{aligned}$$

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Problem 1.3 (WebAssign, HW 11, #3)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int\frac{\mathrm{d}\,x}{\sqrt{x^2+49}}.$$

Proof. This is essentially the same as problem 1 with no pesky x^3 in the numerator. Make the substitution $x = 7\tan\theta$ so $\mathrm{d}\,x = 7\sec^2\theta\,\mathrm{d}\,\theta$ and we have

$$\begin{aligned}
\int \frac{\mathrm{d}\,x}{\sqrt{x^2+49}} &= \int \frac{7\sec^2\theta\,\mathrm{d}\,\theta}{\sqrt{49\tan^2\theta+49}} \\
&= \int \frac{7\sec^2\theta\,\mathrm{d}\,\theta}{\sqrt{49(\tan^2\theta+1)}} \\
&= \int \frac{7\sec^2\theta\,\mathrm{d}\,\theta}{7\sqrt{\tan^2\theta+1}} \\
&= \int \frac{\sec^2\theta\,\mathrm{d}\,\theta}{\sqrt{\tan^2\theta+1}} \\
&= \int \frac{\sec^2\theta\,\mathrm{d}\,\theta}{\sqrt{\sec^2\theta}} \\
&= \int \frac{\sec^2\theta\,\mathrm{d}\,\theta}{\sec\theta} \\
&= \int \frac{\sec^2\theta\,\mathrm{d}\,\theta}{\sec\theta} \\
&= \int \sec\theta\,\mathrm{d}\,\theta \\
&= \ln|\sec\theta+\tan\theta| + C'
\end{aligned}$$

Now, substituting back into our original integral, we have

$$\begin{aligned}
&= \ln\left|\frac{\sqrt{x^2+49}}{7} + \frac{x}{7}\right| + C' \\
&= \ln\left|\sqrt{x^2+49}+x\right| - \ln 7 + C'
\end{aligned}$$

and we can group C' and $\ln 7$ into a constant C tho get our answer

$$= \boxed{\ln\left|\sqrt{x^2+49}+x\right| + C.}$$ ☺

Problem 1.4 (WebAssign, HW 11, #4)**.** Evaluate the integral.

$$\int_0^a 3x^2\sqrt{a^2-x^2}\,\mathrm{d}\,x.$$

Proof. Alright! Make the substitution $x = a\sin\theta$. Then $\mathrm{d}\,x = a\cos\theta\,\mathrm{d}\,\theta$ and the integral above turns into

$$\begin{aligned}
\int_0^a 3x^2\sqrt{a^2-x^2}\,\mathrm{d}\,x &= \int_0^{\pi/2} 3a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}a\cos\theta\,\mathrm{d}\,\theta \\
&= \int_0^{\pi/2} 3a^3\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}a\cos\theta\,\mathrm{d}\,\theta
\end{aligned}$$

$$= 3a^3 \int_0^{\pi/2} \sin^2\theta \sqrt{a^2 - a^2\sin^2\theta}\cos\theta \,\mathrm{d}\,\theta$$
$$= 3a^3 \int_0^{\pi/2} \sin^2\theta \sqrt{a^2(1-\sin^2\theta)}\cos\theta \,\mathrm{d}\,\theta$$
$$= 3a^3 \int_0^{\pi/2} \sin^2\theta \left(a\sqrt{1-\sin^2\theta}\right)\cos\theta \,\mathrm{d}\,\theta$$
$$= 3a^4 \int_0^{\pi/2} \sin^2\theta \sqrt{1-\sin^2\theta}\cos\theta \,\mathrm{d}\,\theta$$

using he Pythagorean identity, we have

$$= \int_0^{\pi/2} \sin^2\theta \sqrt{\cos^2\theta}\cos\theta \,\mathrm{d}\,\theta$$
$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta \,\mathrm{d}\,\theta$$
$$= \int_0^{\pi/2} (\sin\theta\cos\theta)^2 \,\mathrm{d}\,\theta$$

use the identity $2\sin\theta\cos\theta = \sin^2 2\theta$

$$= 3a^4 \int_0^{\pi/2} \tfrac{1}{4}(2\sin\theta\cos\theta)^2 \,\mathrm{d}\,\theta$$
$$= 3a^4 \int_0^{\pi/2} \tfrac{1}{4}(\sin 2\theta)^2 \,\mathrm{d}\,\theta$$
$$= 3a^4 \int_0^{\pi/2} \tfrac{1}{4}\sin^2 2\theta \,\mathrm{d}\,\theta$$
$$= \frac{3a^4}{4} \int_0^{\pi/2} \sin^2 2\theta \,\mathrm{d}\,\theta$$

use another trig substitution $\sin^2\theta = 1 - \cos 2\theta$ so

$$= \frac{3a^4}{4} \int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 2(2\theta)) \,\mathrm{d}\,\theta$$
$$= \frac{3a^4}{8} \int_0^{\pi/2} 1 - \cos 4\theta \,\mathrm{d}\,\theta$$
$$= \frac{3a^4}{8} \left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$
$$= \frac{3a^4}{8} \left[\frac{\pi}{2} - \sin 2\pi - (0-0)\right]$$
$$= \frac{3a^4}{8} \left[\frac{\pi}{2} - 0 - (0-0)\right]$$
$$= \boxed{\frac{3a^4\pi}{16}.}$$

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Problem 1.5 (WebAssign, HW 11, #5)**.** Evaluate the integral.

$$\int_0^7 \sqrt{x^2+49}\,\mathrm{d}\,x.$$

Proof. Like in problem 1, let $x = 7\tan\theta$. Then $\mathrm{d}\,x = 7\sec^2\theta\,\mathrm{d}\,\theta$ and

$$\begin{aligned}
\int_0^7 \sqrt{x^2+49}\,\mathrm{d}\,x &= \int_0^{\pi/4} \sqrt{49\tan^2\theta+49}7\sec^2\theta\,\mathrm{d}\,\theta \\
&= \int_0^{\pi/4} \sqrt{49(\tan^2\theta+1)}7\sec^2\theta\,\mathrm{d}\,\theta \\
&= \int_0^{\pi/4} 7\sqrt{\tan^2\theta+1}7\sec^2\theta\,\mathrm{d}\,\theta \\
&= \int_0^{\pi/4} 49\sqrt{\tan^2\theta+1}\sec^2\theta\,\mathrm{d}\,\theta \\
&= 49\int_0^{\pi/4} \sqrt{\tan^2\theta+1}\sec^2\theta\,\mathrm{d}\,\theta \\
&= 49\int_0^{\pi/4} \sqrt{\sec^2\theta}\sec^2\theta\,\mathrm{d}\,\theta \\
&= 49\int_0^{\pi/4} \sec\theta\sec^2\theta\,\mathrm{d}\,\theta \\
&= 49\int_0^{\pi/4} \sec^3\theta\,\mathrm{d}\,\theta \\
&= 49\int_0^{\pi/4} \sec\theta\sec^2\theta\,\mathrm{d}\,\theta
\end{aligned}$$

here we use integration by parts with $u = \sec\theta$ and $\mathrm{d}\,v = \sec^2\theta$ so we have $\mathrm{d}\,u = \sec\theta\tan\theta\,\mathrm{d}\,\theta$ and $v = \tan\theta$

$$\begin{aligned}
&= 49\left([\sec\theta\tan\theta]_0^{\pi/4} - \int_0^{\pi/4} \sec\theta\tan^2\theta\right) \\
&= 49\left([\sec\theta\tan\theta]_0^{\pi/4} - \int_0^{\pi/4} \sec\theta(\sec^2\theta-1)\theta\right) \\
&= 49\left([\sec\theta\tan\theta]_0^{\pi/4} - \int_0^{\pi/4} \sec^3 - \sec\theta\,\mathrm{d}\,\theta\right)
\end{aligned}$$

now, move the $\int_0^{\pi/4}\sec^3\theta\,\mathrm{d}\,\theta$ to the left and divide by 2 and we get

$$= \frac{49}{2}\left([\sec\theta\tan\theta]_0^{\pi/4} + \int_0^{\pi/4} \sec\theta\,\mathrm{d}\,\theta\right)$$

$$
\begin{aligned}
&= \frac{49}{2}[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|]_0^{\pi/4} \\
&= \frac{49}{2}\left(\sqrt{2}\cdot 1 + \ln\left(1+\sqrt{2}\right) - (0+\ln(1+0))\right) \\
&= \frac{49}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right) \\
&= \boxed{\frac{49}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right).}
\end{aligned}
$$

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Section 1.2: Solutions to Exam 1

Problem 1.6 (#1, #11)**.** If a and b are the values of k for which the angle between $\langle 1,2,2\rangle$ and $\langle 1,0,k\rangle$ equals $\pi/4$, then $a+b=$?

Solution. All you needed to do for this problem was remember the law of cosines for vectors which tells us that

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta. \tag{1}$$

Applying the equation above to our vectors, we get

$$
\begin{aligned}
\langle 1,2,2\rangle\cdot\langle 1,0,k\rangle &= \left(\sqrt{1^2+2^2+2^2}\right)\left(\sqrt{1^2+0^2+k^2}\right)\cos(\pi/4) \\
1\cdot 1 + 2\cdot 0 + 2\cdot k &= \left(\sqrt{1+4+4}\right)\left(\sqrt{1+0+k^2}\right)1/\sqrt{2} \\
1+2k &= \left(\sqrt{1+4+4}\right)\left(\sqrt{1+0+k^2}\right) \\
\sqrt{2}(1+2k) &= \sqrt{9}\left(\sqrt{1+k^2}\right) \\
&= 3\sqrt{1+k^2},
\end{aligned}
$$

now, squaring both sides, we get

$$
\begin{aligned}
2(1+2k)^2 &= 9\left(1+k^2\right) \\
2\left(1+4k+4k^2\right) &= 9+9k^2 \\
2+8k+8k^2 &= 9+9k^2
\end{aligned}
$$

now move everything on the right to the left and reorder by the highest exponent of k

$$0 = k^2 - 8k + 7. \tag{2}$$

Can you see what $a+b$ is already? No? Well consider the following quadratic polynomial $(x-a)(x-b)$. What are the roots of $(x-a)(x-b)$? Well, they are a and b of course. Now, expand $(x-a)(x-b)$ like so

$$(x-a)(x-b) = x^2 - ax - bx + ab = x^2 - (a+b)x + ab$$

so $a+b$ is the same as the negative of the coefficient in front of x in our quadratic polynomial. In this case, it would be $\boxed{a+b=-(-8)=8}$.

Is that not satisfying? Well, we can go ahead and compute the roots of equation (2) by using the quadratic formula. From doing that, we get the roots

$$\begin{aligned} x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 7}}{2} \\ &= 4 \pm \sqrt{\frac{8^2 - 4 \cdot 7}{4}} \\ &= 4 \pm \sqrt{\frac{8^2 - 4 \cdot 7}{4}} \\ &= 4 \pm \sqrt{\frac{4 \cdot 2 \cdot 8 - 4 \cdot 7}{4}} \\ &= 4 \pm \sqrt{2 \cdot 8 - 7} \\ &= 4 \pm \sqrt{16 - 7} \\ &= 4 \pm \sqrt{9} \\ &= 4 \pm 3 \end{aligned}$$

so $a = 7$ and $b = 1$. Hence, $\boxed{a + b = 8}$ like we said before. ☺

Problem 1.7 (#2, #1)**.** Let $\langle a, b, c\rangle$ be the vector projection of $\vec{u} = \langle 2, -1, 9\rangle$ onto $\vec{v} = \langle 1, 2, 2\rangle$. Compute $a + b + c$.

Solution. Recall the formula for the projection of $\vec{u}$ onto $\vec{v}$:

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}. \tag{3}$$

Plugging in our values of $\vec{u}$ and $\vec{v}$ into this equation, we have

$$\begin{aligned} \operatorname{proj}_{\vec{v}} \vec{u} &= \frac{\langle 2, -1, 9\rangle \cdot \langle 1, 2, 2\rangle}{|\langle 1, 2, 2\rangle|^2} \langle 1, 2, 2\rangle \\ &= \frac{2 \cdot 1 - 1 \cdot 2 + 9 \cdot 2}{\left(\sqrt{1^2 + 2^2 + 2^2}\right)^2} \langle 1, 2, 2\rangle \\ &= \frac{2 - 2 + 18}{\left(\sqrt{9}\right)^2} \langle 1, 2, 2\rangle \\ &= \frac{18}{9} \langle 1, 2, 2\rangle \\ &= 2\langle 1, 2, 2\rangle \text{ or } \langle 2, 4, 4\rangle. \end{aligned}$$

So $\boxed{a + b + c = 2 + 4 + 4 = 10}$. ☺

Problem 1.8 (#3, #3)**.** Let $\vec{u} = \langle 0, 1, 2\rangle$, $\vec{v} = \langle 3, 1, 0\rangle$, and $\vec{w} = \langle a, b, c\rangle$. Suppose $\vec{w}$ is a unit vector with $c > 0$ that is perpendicular to both $\vec{u}$ and $\vec{v}$. Compute $a + b + c$.

Solution. Now, you all remember that to find a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$ all we need to do is find their cross product $\vec{u} \times \vec{v}$, right? Let's start by finding this

$$\begin{aligned}
\vec{u} \times \vec{v} &= \langle 0,1,2\rangle \times \langle 3,1,0\rangle \\
&= \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\hat{\imath} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\hat{\jmath} + \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}\hat{k} \\
&= (1\cdot 0 - 2\cdot 1)\hat{\imath} + (2\cdot 3 - 0\cdot 0)\hat{\jmath} + (0\cdot 1 - 1\cdot 3)\hat{k} \\
&= -2\hat{\imath} + 6\hat{\jmath} - 3\hat{k} \\
&= \langle -2, 6, -3\rangle.
\end{aligned}$$

We are not quite done yet since we want a unit vector. All we need to do is divide by the magnitude of $\vec{u} \times \vec{v}$ and we are one step closer to the solution

$$\begin{aligned}
\frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} &= \frac{\langle -2,6,-3\rangle}{\sqrt{(-2)^2 + 6^2 + (-3)^2}} \\
&= \frac{\langle -2,6,-3\rangle}{\sqrt{4+36+9}} \\
&= \frac{\langle -2,6,-3\rangle}{\sqrt{49}} \\
&= \frac{\langle -2,6,-3\rangle}{7} \\
&= \left\langle -\frac{2}{7}, \frac{6}{7}, -\frac{3}{7}\right\rangle.
\end{aligned}$$

Notice that the third entry $-3/7$ on the vector above is negative so we need to take the negative of the vector above and we call this $\vec{w}$

$$\vec{w} = \langle a,b,c\rangle = -\left\langle -\frac{2}{7}, \frac{6}{7}, -\frac{3}{7}\right\rangle = \left\langle -\left(-\frac{2}{7}\right), -\frac{6}{7}, -\left(-\frac{3}{7}\right)\right\rangle = \left\langle \frac{2}{7}, -\frac{6}{7}, \frac{3}{7}\right\rangle.$$

Now, all we need to do is take the sum of the entries of the vector $\vec{w}$ above and we are done!

$$\boxed{a+b+c\frac{2}{7} - \frac{6}{7} + \frac{3}{7} = \frac{2-6+3}{7} = -\frac{1}{7}.}$$

☺

Problem 1.9 (#4, #5)**.** Find the area of the region by the curves $y = x^2 - 2$ and $y = |x|$.

Solution. Alright! Remember the definition of the absolute value of a function anyone? Here it is: If $f(x)$ is a function of x, i.e., $f(x) = \cos x$ or $f(x) = e^x + \cos \pi x - x^\pi$ or what have you, then

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}. \tag{4}$$

All this is saying is that if we plug in a value into our function $f(x)$ and it returns a negative value we make it positive and if the value is positive we leave it positive. What does this mean for $y = |x|$? It means that

$$|x| = \begin{cases} x & \text{if } f(x) \geq 0 \\ -x & \text{if } f(x) < 0 \end{cases},$$

i.e., $|x|$ is $-x$ from $-\infty$ to 0 and x from 0 to $+\infty$, if this notation makes sense to you. This means that we must consider two cases when solving for the intersection of $|x|$ with $x^2 - 2$: We must consider the possibility that $x^2 - 2$ intersects $|x|$ for some value $x < 0$ and $x^2 - 2$ intersects $|x|$ for some value $x \geq 0$. So we must solve both equations

$$\begin{aligned} x &= x^2 - 2 \\ -x &= x^2 - 2. \end{aligned}$$

By some simple algebra, we can rearrange the equations above into

$$0 = x^2 - x - 2 \tag{5}$$

$$0 = x^2 + x - 2 \tag{6}$$

and solve for x. By the quadratic equation on equation (5) we have

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-2)}}{2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = -1 \text{ or } 2.$$

Since we are only looking at positive values of x, -1 makes no sense so we throw it out and 2 remains behind.

We do the same thing for equation (6)

$$x = \frac{-1 \pm \sqrt{1^2 - 4(-2)}}{2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2 \text{ or } 1.$$

Since we are only only looking at negative values of, 1 makes no sense so we throw it out and keep -2.

Now all we need to do is observe that for $0 \leq x \leq 2$ the equation $x > x^2 - 2$ and for $-2 \leq x \leq 0$ the equation $-x > x^2 - 2$ so the area enclosed by $y = |x|$ and $y = x^2 - 2$ is given by the integral

$$\begin{aligned} \int_{-2}^{2} \left| |x| - \left(x^2 - 2\right) \right| \mathrm{d}\,x &= \int_{-2}^{0} \left| |x| - \left(x^2 - 2\right) \right| \mathrm{d}\,x + \int_{0}^{2} \left| |x| - \left(x^2 - 2\right) \right| \mathrm{d}\,x \\ &= \underbrace{\int_{-2}^{0} -x - \left(x^2 - 2\right) \mathrm{d}\,x}_{\text{Int. 1}} + \underbrace{\int_{0}^{2} x - \left(x^2 - 2\right) \mathrm{d}\,x}_{\text{Int. 2}}. \end{aligned}$$

Let's compute Int. 1 and Int. 2 separately

$$\begin{aligned} \text{Int. } 1 &= \int_{-2}^{0} -x - \left(x^2 - 2\right) \mathrm{d}\,x \\ &= \int_{-2}^{0} -x - x^2 + 2 \,\mathrm{d}\,x \end{aligned}$$

$$
\begin{aligned}
&= \int_{-2}^{0} -x^2 - x + 2 \,\mathrm{d}\, x \\
&= -\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + 2x\Big|_{-2}^{0} \\
&= -\tfrac{1}{3}\cdot 0^3 - \tfrac{1}{2}\cdot 0^2 + 2\cdot 0 \\
&\quad - \left(-\tfrac{1}{3}(-2)^3 - \tfrac{1}{2}(-2)^2 + 2(-2)\right) \\
&= 6 - \tfrac{8}{3} \\
&= \frac{6\cdot 3 - 8}{3} \\
&= \frac{18-8}{3} \\
&= \frac{10}{3}.
\end{aligned}
$$

Now, you can either compute Int. 2 and add that quantity to Int. 1 to get the area of your bounded region, or you can plot the curves and notice that the areas are symmetric and double Int. 2 to get our total area $\boxed{2(10/3) = 20/3}$.

Let's compute Int. 1 just to make sure

$$
\begin{aligned}
\text{Int. } 2 &= \int_{0}^{2} x - \left(x^2 - 2\right) \mathrm{d}\, x \\
&= \int_{0}^{2} x - x^2 + 2 \,\mathrm{d}\, x \\
&= \int_{0}^{2} -x^2 + x + 2 \,\mathrm{d}\, x \\
&= -\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 2x\Big|_{0}^{2} \\
&= -\tfrac{1}{3}2^3 + \tfrac{1}{2}2^2 + 2\cdot 2 \\
&\quad - \left(-\tfrac{1}{3}\cdot 0^3 + \tfrac{1}{2}\cdot 0^2 + 2\cdot 0\right) \\
&= -\frac{8}{3} + 2 + 4 \\
&= -\frac{8}{3} + 6 \\
&= \frac{-8 + 2\cdot 6}{3} \\
&= \frac{-8+18}{3} \\
&= \frac{10}{3}.
\end{aligned}
$$

Then

$$
\int_{-2}^{2} \left||x| - \left(x^2 - 2\right)\right| \mathrm{d}\, x = \text{Int. } 1 + \text{Int. } 2 = \frac{10}{3} + \frac{10}{3} = \boxed{\frac{20}{3}.} \qquad ☺
$$

Problem 1.10 (#5, #4)**.** What is the radius of the sphere $x^2 + y^2 + z^2 + 8x - 2y - 4z = 15$?

Solution. Remember the standard equation for the sphere of radius r with center $C = (a, b, c)$? Here it is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \tag{7}$$

So all we need to do is to manipulate our equation $x^2 + y^2 + z^2 + 8x - 2y - 4z = 15$ until it looks like the equation (7)

$$\begin{aligned} x^2 + y^2 + z^2 + 8x - 2y - 4z &= 15 \\ \left(x^2 + 8x\right) + \left(y^2 - 2y\right) + \left(z^2 - 4z\right) &= 15 \end{aligned}$$

now we complete the square, not forgetting to balance the equation on the right-hand side

$$\begin{aligned} \left(x^2 + 8x + 16\right) + \left(y^2 - 2y + 1\right) + \left(z^2 - 4z + 4\right) &= 15 + 16 + 1 + 4 \\ (x+4)^2 + (y-1)^2 + (z-2)^2 &= 36. \end{aligned}$$

We didn't need to factor the left-hand side, but why not do it anyway? Now, looking at our equation (7) we see that our radius $r^2 = 36$ so $\boxed{r = \sqrt{36} = 6}$. ☺

Problem 1.11 (#6, #6)**.** Consider the region enclosed by the graph of the function $y = x^4$ and the x-axis between $x = 0$ and $x = 1$. Find the volume of the solid obtained by rotating the region about the x-axis using the disks/washers method.

Solution. This one is easy enough. All we need to do is find an equation for the area of the perpendicular cross section A which (as a rule of thumb, you want to express in terms of the axis which is perpendicular to your cross section) will be in terms of x

$$A(x) = \pi y^2 = \pi\left(x^4\right)^2 = \pi x^8.$$

Now, compute

$$\begin{aligned} \int_0^1 \pi A(x)\,\mathrm{d}\,x &= \int_0^1 \pi x^8\,\mathrm{d}\,x \\ &= \pi \int_0^1 x^8\,\mathrm{d}\,x \\ &= \pi\, \tfrac{1}{9}x^9\Big|_0^1 \\ &= \pi\left(\tfrac{1}{9}1^9 - \left(\tfrac{1}{9}0^9\right)\right) \\ &= \boxed{\frac{\pi}{9}.} \end{aligned}$$

☺

Problem 1.12 (#7, #9)**.** Consider the region enclosed by the graph of the function $y = x - x^4$ and the x-axis. Find the volume of the solid obtained by rotating the region about the y-axis using the cylindrical shells method.

Solution. This one is also easy. All we need to do is find the cylindrical area. Since we are revolving about the y-axis, the length of our cylinder will be x and point along the x-axis so we probably want to express our cross sectional area A in terms of x like so

$$A(x) = 2\pi x(x - x^4).$$

Now, we need to find when $x - x^4$ intersects the line $y = 0$. This happens when $x = 0$ since $0 - 0^4 = 0$ and, factoring, $x(1 - x^3)$ when $x = 1$. Using the formula for our cross section, we integrate $A(x)$ from 0 to 1 to find our volume

$$\begin{aligned}
V &= \int_0^1 2\pi x\big(x - x^4\big)\,\mathrm{d}\,x \\
&= 2\pi \int_0^1 x^2 - x^5\,\mathrm{d}\,x \\
&= 2\pi\Big(\tfrac{1}{3}x^3 - \tfrac{1}{6}x^6\big|_0^1\Big) \\
&= 2\pi\big(\tfrac{1}{3}1^3 - \tfrac{1}{6}1^6 - \big(\tfrac{1}{3}0^3 - \tfrac{1}{6}0^6\big)\big) \\
&= 2\pi\left(\frac{2-1}{6}\right) \\
&= \frac{2\pi}{6} \\
&= \boxed{\frac{\pi}{3}.}
\end{aligned}$$

☺

Problem 1.13 (#8, #8)**.** Let $\langle a, b, c\rangle$ be the unit vector of length 6 in the opposite direction to $\langle -2, 1, -2\rangle$. Compute $a + b + c$.

Solution. The wording of this question is very wonky. What was meant, I believe, was "Let $\langle a, b, c\rangle$ be the vector which is 6 times as long as the unit vector pointing in the opposite direction to $\langle -2, 1, -2\rangle$".

First, make turn the vector $\langle -2, 1, -2\rangle$ into a unit vector like so

$$\frac{\langle -2,1,-2\rangle}{|\langle -2,1,-2\rangle|} = \frac{\langle -2,1,-2\rangle}{\sqrt{(-2)^2 + 1^2 + (-2)^2}} = \frac{\langle -2,1,-2\rangle}{\sqrt{4+1+4}} = \frac{\langle -2,1,-2\rangle}{\sqrt{9}} = \left\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle.$$

To get a unit vector pointing in the opposite way, we just multiply by -1.[1] Thus, we have

$$\left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle.$$

We are told that our vector must be 6 times as long as the unit vector so

$$\langle a, b, c\rangle = 6\left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle = \langle 4, -2, 4\rangle,$$

and $\boxed{a + b + c = 4 - 2 + 4 = 6}$. ☺

Problem 1.14 (#9, #10)**.** A force of 8 lb is required to hold a spring stretched 4 in beyond its natural length. How much work is done in stretching the same spring from its natural length to 6 in?

[1] Why? Well, recall the law of cosines, equation (1), which says that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$. If two vectors are pointing in opposite directions, that means that the angle between them is π or $180°$ so $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$. Now, divide on both sides and we get $(\vec{a}/|\vec{a}|)\cdot(\vec{b}/|\vec{b}|) = -1$. It can be shown that, in fact, $\vec{a}/|\vec{a}| = -\vec{b}/|\vec{b}|$, but you have to solve a system of equations and that requires a bit more math that I am willing to write on this footnote.

Solution. Recall the definition for the force required to move a spring a distance x from its natural length

$$F(x) = kx. \tag{8}$$

This is called Hooke's law and, without a doubt, you will see it in physics and, should you decide to become a mechanical engineer, you will see it again[2] Now, to find the work needed to move the spring from x_1 to x_2 is given by taking the integral

$$W(x_1, x_2) = \int_{x_1}^{x_2} kx \,\mathrm{d}\, x = \tfrac{1}{2}kx^2\Big|_{x_1}^{x_2} = \tfrac{1}{2}k\big({x_2}^2 - {x_1}^2\big). \tag{9}$$

Since they want the work in terms of lb-ft, it would be best to convert from in to ft now. Let's do that: So initially the spring is stretched to 4 in which is $4/12 = 1/3$ ft and we want to know how much work is required to stretch it from its natural length 0 ft to $6/12 = 1/2$ ft. To proceed, we need to find out what the value of k is

$$k = \frac{8}{1/3} = 3 \cdot 8 = 24 \text{ lb/ft}.$$

Now, plug in our values into equation (9) and we have

$$\begin{aligned} W(0, 1/2) &= \tfrac{1}{2} \cdot 24\big((1/2)^2 - 0^2\big) \\ &= 12 \cdot \frac{1}{4} \\ &= \boxed{3 \text{ lb-ft.}} \end{aligned}$$

☺

Problem 1.15 (#10, #12)**.** Evaluate $\int_1^e x \ln x \,\mathrm{d}\, x$ using integration by parts.

Solution. Since it's hard to take the integral of $\ln x$ and easy to take the integral of x take $\mathrm{d}\, v = x$ and $u = \ln x$, then $\mathrm{d}\, u = x^{-1} \,\mathrm{d}\, x$ and $v = \frac{1}{2}x^2$ so

$$\begin{aligned} \int_1^e u \,\mathrm{d}\, v &= \tfrac{1}{2}x^2 \ln x\Big|_1^e - \int \tfrac{1}{2}x^{-1}x^2 \,\mathrm{d}\, x \\ &= \tfrac{1}{2}x^2 \ln x\Big|_1^e - \int \tfrac{1}{2}x \,\mathrm{d}\, x \\ &= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2\Big|_1^e \\ &= \tfrac{1}{4}x^2(2\ln x - 1)\Big|_1^e \\ &= \tfrac{1}{4}e^2(2-1) - \tfrac{1}{4}1(2\ln 1 - 1) \\ &= \tfrac{1}{4}e^2 + \tfrac{1}{4} \\ &= \boxed{\frac{e^2+1}{4}.} \end{aligned}$$

☺

Problem 1.16 (#11, #2)**.** Evaluate $\int_0^\pi \sin^3 x \,\mathrm{d}\, x$.

[2]The analogue of the spring in electrical engineering is the inductor. By analogue, I mean that, mathematically, the inductor and the spring behave the same way in their appropriate contexts.

Solution. Using the Pythagorean identity

$$\cos^2 x + \sin^2 x = 1, \tag{10}$$

we get $\sin^2 x = 1 - \cos^2 x$ so

$$\begin{aligned}\int_0^\pi \sin^3 x \,\mathrm{d}\, x &= \int_0^\pi \sin^2 x \sin x \,\mathrm{d}\, x \\ &= \int_0^\pi \left(1 - \cos^2 x\right) \sin x \,\mathrm{d}\, x.\end{aligned}$$

Now, what is a good substitution to use at this point? We want to get rid of the $\sin x$ so let's do $u = \cos x$. Then $\mathrm{d}\, u = -\sin x \,\mathrm{d}\, x$ and the integral above turns into

$$\int_1^{-1} \left(1 - u^2\right) \sin x \frac{\mathrm{d}\, u}{-\sin x} = -\int_{-1}^{1} 1 - u^2 \,\mathrm{d}\, u.$$

Now, remember the identity about the integral that says that $\int_a^b f(x) \,\mathrm{d}\, x = -\int_b^a f(x) \,\mathrm{d}\, x$, so the above turns into

$$\begin{aligned}-\int_1^{-1} 1 - u^2 \,\mathrm{d}\, u &= \int_1^{-1} 1 - u^2 \,\mathrm{d}\, u \\ &= u - \tfrac{1}{3}u^3\Big|_{-1}^{1} \\ &= 1 - \tfrac{1}{3} - \left(-1 - \tfrac{1}{3}(-1)^3\right) \\ &= \tfrac{2}{3} - \left(-1 + \tfrac{1}{3}\right) \\ &= \frac{2}{3} - \left(-\tfrac{2}{3}\right) \\ &= \frac{2}{3} + \frac{2}{3} \\ &= \boxed{\frac{4}{3}.}\end{aligned}$$

☺

Why did the limits of integration change from $0 \le x \le \pi$ to $1 \ge u \ge -1$, well u is the new variable we are integration with respect to and the relation ship between u and x is that $u = \cos x$ so the limits of u will be from $\cos 0 = 1$ to $\cos \pi = -1$. Makes sense, right?

Problem 1.17 (#12, #7)**.** Evaluate $\int_0^{\pi/4} \cos^2 x \,\mathrm{d}\, x$. Hint: $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

Solution. All we need to do is modify the hint to express $\cos^2 x$ in terms of $\cos 2x$ like so

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1 \\ \cos 2x + 1 &= 2\cos^2 x \\ \frac{\cos 2x + 1}{2} &= \cos^2 x\end{aligned}$$

so our integral turns into

$$\begin{aligned}
\int_0^{\pi/4} \cos^2 x \,\mathrm{d}\, x &= \int_0^{\pi/4} \frac{\cos 2x + 1}{2} \,\mathrm{d}\, x \\
&= \frac{1}{2} \int_0^{\pi/4} \cos 2x + 1 \,\mathrm{d}\, x \\
&= \frac{1}{2} \Big(\tfrac{1}{2} \sin 2x + x \Big|_0^{\pi/4} \Big) \\
&= \frac{1}{2} \big(\tfrac{1}{2} \sin 2(\pi/4) + \tfrac{pi}{4} - \big(\tfrac{1}{2} \sin 2 \cdot 0 - 0 \big) \big) \\
&= \frac{1}{2} \big(\tfrac{1}{2} \cdot 1 + \tfrac{\pi}{4} - (0 - 0) \big) \\
&= \tfrac{1}{2} \left(\frac{2}{4} + \frac{\pi}{4} \right) \\
&= \tfrac{1}{2} \left(\frac{2 + \pi}{4} \right) \\
&= \frac{2 + \pi}{8} \\
&= \boxed{\frac{1}{4} + \frac{\pi}{8} \text{ or } \frac{\pi}{8} + \frac{1}{4}.}
\end{aligned}$$

☺