

## MA571 Homework 14

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**PROBLEM 14.1 (MUNKRES §74, EX. 6)**

If  $n > 1$ , show that the fundamental group of the  $n$ -fold torus is not Abelian. [Hint: Let  $G$  be a free group on the set  $\{\alpha_1, \beta_1, \dots, \alpha_n, \beta_n\}$ ; let  $F$  be the free group on the set  $\{\gamma, \delta\}$ . Consider the homomorphism of  $G$  onto  $F$  that sends  $\alpha_1$  and  $\beta_1$  to  $\gamma$  and all other  $\alpha_i$  and  $\beta_i$  to  $\delta$ .]

*Proof.* Let  $\mathbf{T}^n$  denote the  $n$ -fold torus and let  $x_0 \in \mathbf{T}^n$ . By Theorem 74.3, the  $\pi_1(\mathbf{T}^n, x_0)$  is isomorphic to the quotient of the free group on  $2n$  letters, say  $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ , by the least normal subgroup,  $N$ , containing  $[\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_n, \beta_n]$  where  $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ , i.e., the commutator of  $\alpha$  and  $\beta$ . ■

**PROBLEM 14.2 (MUNKRES §76, EX. 1)**

Calculate  $H_1(P^2 \# T)$ . Assuming that the list of compact surfaces given in Theorem 75.5 is a complete list, to which of these surfaces is  $P^2 \# T$  homeomorphic?

*Proof.*

■

**PROBLEM 14.3 (MUNKRES §76, EX. 2)**

If  $K$  is the Klein bottle, calculate  $H_1(K)$  directly.

*Proof.*

■

**PROBLEM 14.4 (MUNKRES §76, EX. 3(A,B,C))**

Let  $X$  be the quotient space obtained from an 8-sided polygonal region  $P$  by pasting its edges together according to the labelling scheme  $acadbc b^{-1}d$ .

- (a) Check that all vertices of  $P$  are mapped to the same point of the quotient space  $X$  by the pasting map.
- (b) Calculate  $H_1(X)$ .
- (c) Assuming  $X$  is homeomorphic to one of the surfaces given in Theorem 75.5 (which it is), which surface is it ?

*Proof.*

■

**PROBLEM 14.5 (A)**

Define  $P^n$  to be the space  $S^n/\sim$  where  $z \sim z'$  if and only if  $z = z'$  or  $z = -z'$ . Use the Seifert–van Kampen Theorem to calculate  $\pi_1(P^n)$ . (Hint: induction starting from the case  $n = 2$  that was done in class.)

*Proof.*

■

**PROBLEM 14.6 (B)**

A topological space  $X$  is called *homogeneous* if for every pair of points  $x, y \in X$  there is a homeomorphism  $\varphi: X \rightarrow X$  with  $\varphi(x) = y$ . Prove that every connected 2-manifold is homogeneous. (Hint: use the optional problem from the previous assignment.)

*Proof.*

■



**PROBLEM 14.7 (OPTIONAL PROBLEM)**

(i) Let  $x \subset \mathbb{R}^3$  be the cylinder

$$\left\{ (x, y, z) \mid |z| \leq \frac{1}{\sqrt{2}} \text{ and } x^2 + y^2 = \frac{1}{\sqrt{2}} \right\}$$

and let  $f : X \rightarrow \mathbb{R}^3$  be the map

$$f(x, y, z) = \left( 2^{1/4} x \sqrt{1 - z^2}, 2^{1/4} y \sqrt{1 - z^2}, z \right).$$

Prove that  $f$  is a homeomorphism from  $X$  to the subspace

$$Y = S^2 \cap \left\{ (x, y, z) \mid |z| \leq \frac{1}{\sqrt{2}} \right\}.$$

(ii) Prove that the Möbius band is homeomorphic to  $P^2$  with an open disk removed (think of  $P^2$  as  $S^2/\sim$  and use part (i)).

*Proof.*

■