Subject HW6

Sec 6.1 (p. 210)

#1 $\int_0^\infty e^{-st}(3t+12)\,dt$

$= 3\int_0^\infty e^{-st}t\,dt + 12\int_0^\infty e^{-st}\cdot 1\,dt$

$= 3\mathcal{L}(t) + 12\mathcal{L}(1)$

$= 3\cdot\frac{1}{s^2} + \frac{12}{s}$

#12 $\int_0^\infty (a-bt)^2 e^{-st}\,dt$

$= \int_0^\infty a^2e^{-st} - 2ab\,t\,e^{-st} + b^2t^2e^{-st}\,dt$

$= \frac{a^2}{s} - \frac{2ab}{s^2} + \frac{b^2\cdot 2!}{s^3}$

#5 $\int_0^\infty e^{-st}(e^{2t}\sinh t)\,dt$

$= \frac{1}{(s-2)^2-1}$

#13 $\int_0^\infty e^{-st}f(t)\,dt$

$= \int_0^1 e^{-st}\,dt + \int_1^2 (-1)e^{-st}\,dt$

$= \left[-\frac{1}{s}e^{-st}\right]_0^1 + \left[\frac{1}{s}e^{-st}\right]_1^2$

$= -\frac{1}{s}(e^{-s}-1) + \frac{1}{s}(e^{-2s}-e^{-s})$

$= \frac{1}{s}(1-2e^{-s}+e^{-2s})$

$= \frac{1}{s}(1-e^{-s})^2$

#14 $\int_0^\infty e^{-st} f(t)\,dt = \int_a^b e^{-st}\, k\, dt$

$= k\left[-\frac{1}{s} e^{-st}\right]_a^b$

$= k\left(-\frac{1}{s}\right)\left(e^{-bs} - e^{-as}\right)$

$= \frac{k}{s}\left(e^{-as} - e^{-bs}\right)$

#23. $\mathcal{L}(f(ct)) = \int_0^\infty e^{-st} f(ct)\,dt$

$\tau = ct$ $\quad = \int_0^\infty e^{-s\frac{\tau}{c}} f(\tau)\frac{1}{c}\,d\tau$

$d\tau = c \cdot dt$ $\quad = \int_0^\infty \frac{1}{c} e^{-\frac{s}{c}\tau} f(\tau)\,d\tau$

$\frac{1}{c}\,d\tau = dt$

$= \frac{1}{c}\int_0^\infty e^{-\frac{s}{c}\tau} f(\tau)\,d\tau$

$= \frac{1}{c} F\left(\frac{s}{c}\right)$

$\mathcal{L}(\cos \omega t) = \frac{1}{\omega} F\left(\frac{s}{\omega}\right)$

$F(s) = \mathcal{L}(\cos t) = \frac{s}{s^2+1}$

$\therefore \mathcal{L}(\cos \omega t) = \frac{1}{\omega} \cdot \frac{\left(\frac{s}{\omega}\right)}{\left(\frac{s}{\omega}\right)^2 + 1}$

$= \frac{s}{s^2 + \omega^2}$

#30 $\mathcal{L}^{-1}\left(\frac{4s+32}{s^2-16}\right) = 4\cosh(4t) + 8\cdot\sinh(4t)$

$$\left(\frac{4s+32}{s^2-16} = 4\cdot\frac{s}{s^2-4^2} + 8\cdot\frac{4}{s^2-4^2}\right)$$

$$= 4\cdot\frac{e^{4t}+e^{-4t}}{2} + 8\cdot\frac{e^{4t}-e^{-4t}}{2}$$

$$= 6e^{4t} - 2e^{-4t}$$

#32 $\mathcal{L}^{-1}\left(\frac{1}{(s+a)(s+b)}\right)$

$$\frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$$

$$1 = A(s+b) + B(s+a) = (A+B)s + Ab + Ba$$

$$\therefore \begin{cases} A+B=0 \\ Ab+Ba=1 \end{cases}$$

$$\therefore A(b-a) = 1 \Rightarrow A = \frac{1}{b-a} \quad B = \frac{1}{a-b}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{b-a}\cdot\frac{1}{s+a} + \frac{1}{a-b}\cdot\frac{1}{s+b}\right)$$

$$= \frac{1}{b-a}e^{-at} + \frac{1}{a-b}e^{-bt}$$

$$= \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)$$

Sec 6.2 (p. 216)

#4 $y'' + 9y = 10e^{-t}$ $\quad y(0) = 0 \quad y'(0) = 0$

$Y = \mathcal{L}(y)$

$s^2Y - sy(0) - y'(0) + 9Y = \dfrac{10}{s+1}$

$(s^2+9)Y = \dfrac{10}{s+1}$

$$Y = \frac{10}{(s+1)(s^2+9)} = \frac{a}{s+1} + \frac{bs+c}{s^2+9}$$

$10 = a(s^2+9) + (s+1)(bs+c)$

$\quad = (a+b)s^2 + (b+c)s + 9a + c$

$a+b=0 \Rightarrow b=-a$

$b+c=0 \Rightarrow c=-b=a$

$9a+c=10 \Rightarrow 9a+a=10$

$\therefore a=1 \quad b=-1 \quad c=1$

$$\therefore Y = \frac{1}{s+1} + \frac{-s+1}{s^2+9}$$

$$= \frac{1}{s+1} - \frac{s}{s^2+3^2} + \frac{1}{3}\cdot\frac{3}{s^2+3^2}$$

$$\therefore y = \mathcal{L}^{-1}\{Y\} = e^{-t} - \cos(3t) + \frac{1}{3}\sin(3t)$$

#5 $y'' - \frac{1}{4}y = 0 \quad y(0) = 12 \quad y'(0) = 0.$

$$s^2Y - sy(0) - y'(0) - \frac{1}{4}Y = 0$$

$$(s^2 - \frac{1}{4})Y = 12s$$

$$Y = \frac{12s}{s^2 - \frac{1}{4}} = 12 \cdot \frac{s}{s^2 - (\frac{1}{2})^2}$$

$$\therefore y = \mathcal{L}^{-1}(Y) = 12 \cdot \cosh(\frac{1}{2}t)$$

#17 $f = te^{-at}$

$f' = e^{-at} - ate^{-at}. \quad f(0) = 0. \quad f'(0) = 1$

$$\mathcal{L}(f') = sF(s) - f(0) = sF(s)$$

$$\mathcal{L}(f') = \mathcal{L}(e^{-at} - ate^{-at})$$

$$= \frac{1}{s+a} - a\underbrace{\mathcal{L}(te^{-at})}_{= F(s)}$$

$$\therefore sF(s) = \frac{1}{s+a} - aF(s)$$

$$\therefore (s+a)F(s) = \frac{1}{s+a}$$

$$\therefore F(s) = \frac{1}{(s+a)^2}$$

#19. $f(t) = \sin^2 \omega t$.

$$f' = 2\sin\omega t \cdot \cos\omega t \cdot \omega$$

$$= 2w \sin\omega t \cos\omega t$$

$$\mathcal{L}(f') = sF(s) - f(0) = sF(s)$$

$$\mathcal{L}(f') = \mathcal{L}(w \sin(2wt))$$

$$= w \cdot \frac{2w}{s^2+(2w)^2} = \frac{2w^2}{s^2+4w^2}$$

$$\therefore F(s) = \frac{2w^2}{s(s^2+4w^2)}$$

#26 $\mathcal{L}(F) = \frac{1}{s^4 - s^2} = \frac{1}{s^2} \frac{1}{(s^2-1)}$

$$\mathcal{L}^{-1}\left(\frac{1}{s} \cdot \frac{1}{(s^2-1)}\right) = \int_0^t \sinh(u)\, du$$

$$= \cosh(t) - 1 \quad := g(t)$$

$$\frac{1}{s^2}\frac{1}{(s^2-1)} = \frac{1}{s} \cdot \underbrace{\frac{1}{s}\frac{1}{(s^2-1)}}_{G(s)}$$

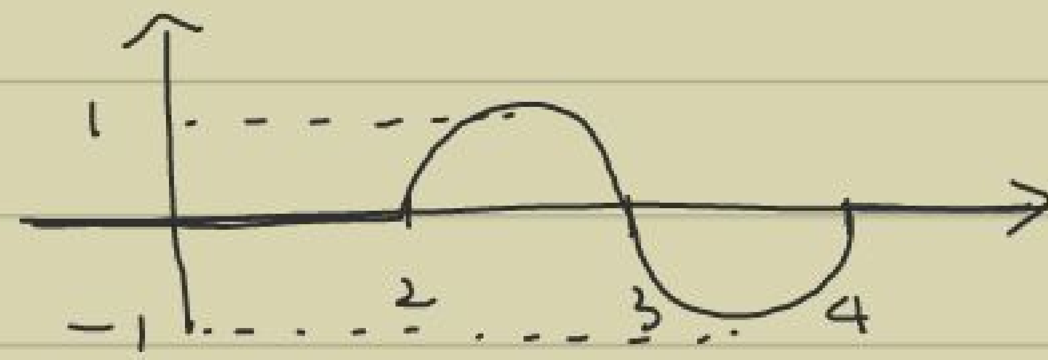
$$\mathcal{L}^{-1}\left(\frac{1}{s} \cdot \frac{1}{s}\frac{1}{s^2-1}\right) = \int_0^t g(u)\, du$$

$$= \int_0^t \cosh(u) - 1 \, du$$

$$= \sinh(t) - t$$

Sec 6.3 (p. 223)

#6 $f(t) = \sin \pi t \quad (2 < t < 4)$



$(u(t-2) - u(t-4)) \sin \pi t$

$\mathcal{L}\left((u(t-2) - u(t-4)) \sin \pi t \right)$

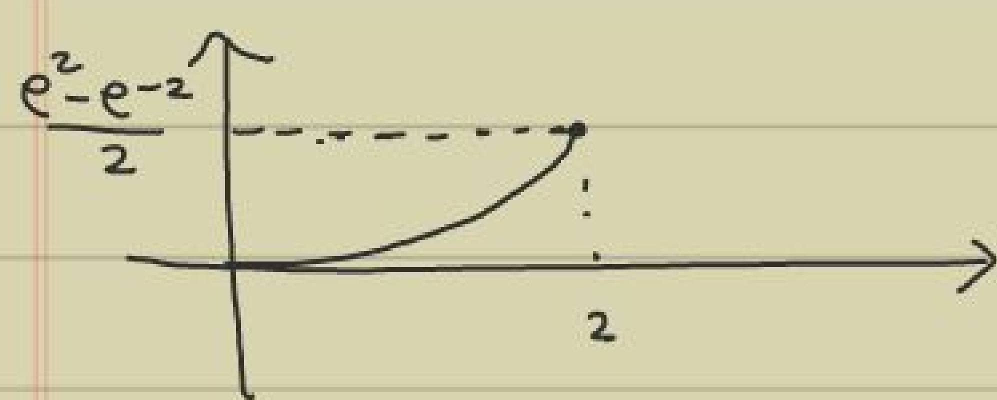
$= \mathcal{L}(u(t-2) \sin \pi t) - \mathcal{L}(u(t-4) \sin \pi t)$

$= \mathcal{L}(u(t-2) \sin \pi (t-2)) - \mathcal{L}(u(t-4) \sin \pi (t-4))$

$= e^{-2s} \dfrac{\pi}{s^2 + \pi^2} - e^{-4s} \dfrac{\pi}{s^2 + \pi^2}$

$= \dfrac{\pi}{s^2 + \pi^2} \left(e^{-2s} - e^{-4s} \right).$

#10 $f(t) = \sinh t \quad (0 < t < 2) \quad = \dfrac{e^t - e^{-t}}{2}$



$\sinh t \, (u(t-0) - u(t-2))$

$= \dfrac{e^t - e^{-t}}{2} (u(t) - u(t-2))$

$= \dfrac{e^t - e^{-t}}{2} u(t) - \dfrac{1}{2} \left(e^{(t-2)+2} - e^{-(t-2)-2} \right) \cdot u(t-2).$

$$= \frac{1}{2} e^{t} u(t) - \frac{1}{2} e^{-t} u(t) - \frac{1}{2}\Big(e^{2} \cdot e^{(t-2)} u(t-2)$$

$$- e^{-2} \cdot e^{-(t-2)} u(t-2) \Big)$$

$$\Rightarrow \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{2}\left(e^{2} \cdot \frac{e^{-2s}}{s-1} - e^{-2} \cdot \frac{e^{-2s}}{s+1} \right)$$

$$= \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{2}\left(\frac{e^{2}}{s-1} - \frac{e^{-2}}{s+1} \right) e^{-2s}$$

#13 $\mathcal{L}(f) = \dfrac{6(1 - e^{-\pi s})}{s^2+9} = \dfrac{6}{3} \dfrac{3}{s^2+9} - \dfrac{6}{3} \dfrac{3 \cdot e^{-\pi s}}{s^2+9}$

$$\therefore f(t) = 2 \cdot \sin 3t - 2\, u(t-\pi) \cdot \underbrace{\sin 3(t-\pi)}_{= -\sin 3t}$$

$$= 2\left(1 + u(t-\pi)\right) \sin 3t$$

#16 $\dfrac{2(e^{-s} - e^{-3s})}{s^2-4} = \dfrac{2e^{-s}}{s^2-4} - \dfrac{2e^{-3s}}{s^2-4}$

$$= u(t-1) \cdot \sinh 2(t-1) - u(t-3) \sinh 2(t-3)$$

#25 $y'' + y = \left\{ \begin{array}{ll} t & \text{if } 0<t<1 \\ 0 & \text{if } t>1. \end{array} \right\} := g(t)$

$y(0) = 0. \quad y'(0) = 0.$

$$g(t) = \big(u(t-0) - u(t-1)\big)t$$
$$= \big(u(t) - u(t-1)\big)t$$
$$= u(t)\cdot t - u(t-1)\big((t-1)+1\big)$$
$$= u(t)\cdot t - u(t-1)(t-1) - u(t-1)$$

$$\therefore G(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$\mathcal{L}[y''+y] = \big(s^2Y - sy(0) - y'(0)\big) + Y$$
$$= s^2Y + Y$$
$$= (s^2+1)Y$$

$$(s^2+1)Y = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$\therefore Y = \frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}$$

$$\frac{1}{s^2(s^2+1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{cs+d}{s^2+1}$$

$\Rightarrow \quad a=0 \quad b=1 \quad c=0 \quad d=-1$

$$\therefore \frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s+1)}\right) = t - \sin t$$

$$\frac{1}{s(s^2+1)} = \frac{a}{s} + \frac{bs+c}{s^2+1}$$

$$\Rightarrow a=1 \quad b=-1 \quad c=0$$

$$\therefore \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 - \cos t$$

$$\therefore y = \mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}\right)$$

$$= (t - \sin t) - u(t-1)\left[(t-1) - \sin(t-1)\right]$$

$$- u(t-1)\left[1 - \cos(t-1)\right]$$

$$= \begin{cases} t - \sin t & \text{if } 0<t<1 \\ \cos(t-1) + \sin(t-1) - \sin t & \text{if } t>1 \end{cases}$$

#39 $v(t) = \begin{cases} 1000 & \text{if } 0<t<2 \\ 0 & \text{if } t>2 \end{cases}$

$$i' + 2i + \frac{1}{0.5}\int_0^t i(\tau)\,d\tau = v(t)$$

$$= \left(u(t) - u(t-2)\right) 1000$$

$i(0)=0 \quad i'(0)=0$

$$\mathcal{L}(i) = I$$

$$\Rightarrow (sI - i(0)) + 2I + \frac{1}{0.5}\frac{I}{s} = \left(\frac{1}{s} - \frac{e^{-2s}}{s}\right)1000$$

$$\Rightarrow \left(s + 2 + \frac{2}{s}\right) I = \frac{1-e^{-2s}}{s} \cdot 1000$$

$$\Rightarrow \frac{s^2+2s+2}{s} I = \frac{1-e^{-2s}}{s} \cdot 1000$$

$$\Rightarrow I = \frac{1-e^{-2s}}{s^2+2s+2} \cdot 1000$$

$$= \frac{1-e^{-2s}}{(s+1)^2+1} \cdot 1000$$

$$= \left(\frac{1}{(s+1)^2+1} - \frac{e^{-2s}}{(s+1)^2+1}\right)1000$$

$$i = 1000\left(e^{-t}\sin t - u(t-2)\, e^{-(t-2)}\sin(t-2)\right)$$