MA571: Qual Preparation

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1 Gepner

1.1 Gepner's homework

Homework 1

Exercise 1.1. Let $\{X_i : i \in I\}$ be an I-indexed family of topologictal spaces. Show that the Cartesian product

$$X = \prod_{i \in I} X_i,$$

equipped with the product topology, has the property that for each $i \in I$ the projection $p_i \colon X \to X_i$ is continuous, and moreover, that X has the following universal property: for any other topological space Y, the function

$$\mathrm{Hom}_{\mathbf{Top}}(Y, X) \longrightarrow \prod_{i \in I} \mathrm{Hom}_{\mathbf{Top}}(Y, X_i),$$

induced by the projections $p_i \colon X \to X_i$, is a bijection.

Solution. ∎

Exercise 1.2. Let X be the set equipped with a topology and let $\{\mathcal{U}_i : i \in I\}$ a family of topologies on X. Show that

$$\mathcal{U} = \bigcap_{i \in I} \mathcal{U}_i$$

is a topology on X. Show that if $\mathcal{B}$ is a basis for a topology on X, then the topology $\mathcal{U}$ on X generated by $\mathcal{B}$ is the intersection of all topologies on X which contain $\mathcal{B}$, and that this holds even if we only require that $\mathcal{B}$ be a subbasis.

Solution. ∎

Exercise 1.3. A topological space X is said to be *Hausdorff* if, for every pair of points $x_0, x_1 \in X$ with $x_0 \neq x_1$, there exists open subsets U_0, U_1 of X such that $x_0 \ni U_0$, $x_1 \in U_1$, and $U_0 \cap U_1 = \varnothing$. Show that a topological space X is Hausdorff if and only if the diagonal inclusion $X \to X \times X$ is closed.

Solution. ∎

Exercise 1.4. Let X be a topological space and let $Y \subset X$ be a subset of X. Show that if Y is equipped with the subspace topology then the inclusion function $i \colon Y \to X$ is continuous. Show that if there exists a continuous function $q \colon X \to Y$ such that $q \circ i = \mathrm{Id}_Y$ then q is a quotient map (that is, Y is also a quotient topology). Give an example of such a situation.

Solution. ∎

Exercise 1.5. A *topological group* is a group G with a topology $\mathcal{U}$ such that the multiplication $\mu \colon G \times G \to G$ and inversion $i \colon G \to G$ are continuous (it is standard to also assume that the topology $\mathcal{U}$ on G is Hausdorff, which we shall do). Let H be a subgroup of G, and let G/H denote the quotient of G by the action of H, equipped with the quotient topology. Show that G/H is a homogeneous space and that the quotient map $q \colon G \to G/H$ is open. If, moreover, H is a closed subset of G, show that G/H has the property that points are closed. Finally, show that if H is a normal subgroup of G, then G/H is a topological group. (Optional: is it Hausdorff?)

Solution. ∎

1.2 Homework 2

Exercise 1.6. A topological space X is said to be totally disconnected if a subspace $Y \subset X$ is connected if and only if $Y = \{x\}$ consists of only a single point $x \in X$. Show that if X is discrete (that is, all subsets of X are open) then X is totally disconnected. Find an example of a totally disconnected space which is not discrete.

Solution. ■

Exercise 1.7. Let X be a simply ordered set equipped with the order topology. Show that if X is connected then X is a continuum.

Solution. ■

Exercise 1.8. Show that a metric $d\colon X \times X \to \mathbb{R}$ on a set X determines a coarsest topology $\mathcal{U}$ on X for which the distance function $d\colon X \times X \to \mathbb{R}$ is continuous, and give an explicit basis for this topology. Recall that a function $f\colon X \to Y$ between metric spaces is said to be continuous at x if, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d(x, x_0) < \delta$ then $d\big(f(x), f(x_0)\big) < \varepsilon$; show that f is continuous (in the sense of topology) if and only if it is continuous at x for all $x \in X$. Finally, show that every compact subspace of a metric space is closed and bounded, and find an example of a metric space for which there exists a closed and bounded subspace which is not compact.

Solution. ■

Exercise 1.9. Let X be a compact space, Y a Hausdorff space, and $f\colon X \to Y$ a continuous function. Show that f is a closed map (that is, f sends closed sets to closed sets), and also that the projection $p\colon X \times Y \to Y$ is a closed map.

Solution. ■

Exercise 1.10. Let $f\colon W \to X$ and $g\colon W \to Y$ be continuous functions. The pushout $X \amalg_W Y$ of f and g is the quotient of the disjoint union $X \amalg Y$ by the equivalence relation generated by the relation $x \sim y$ if there exists a $w \in W$ such that $x = f(w)$ and $y = g(w)$. Show that $X \amalg_W Y$ comes equipped with continuous functions $i\colon X \to X \amalg_W Y$ and $j\colon Y \to X \amalg_W Y$ such that $i \circ f = j \circ g$, and is universal among topological spaces Z equipped with continuous functions $i'\colon X \to Z$ and $j'\colon Y \to Z$ such that $i' \circ f = j' \circ g$ in the following sense: given any such space Z, there exists a unique continuous function $k\colon X \amalg_W Y \to Z$ such that $i' = k \circ i$ and $j' = k \circ j$.

Solution. ■