Math 527 - Homotopy Theory
Spring 2013
Homework 7 Solutions

Problem 1. (May § 10.7 Problem 2)

a. Let $f\colon X \xrightarrow{\sim} Y$ be a weak homotopy equivalence. Assuming X is a CW-complex and Y has the *homotopy type* of a CW-complex, show that f is a homotopy equivalence.

Solution. Let $g\colon Y \xrightarrow{\simeq} K$ be a homotopy equivalence to a CW complex K. Then the composite $gf\colon X \to K$ is a weak homotopy equivalence, hence a homotopy equivalence by Whitehead. Therefore f is a homotopy equivalence, since g and gf are (2-out-of-3 property). $\square$

b. Show that the space $A := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ does *not* have the homotopy type of a CW-complex.

Solution. By part (a), it suffices to produce a CW-complex W and a weak homotopy equivalence $f\colon W \xrightarrow{\sim} A$ which is not a homotopy equivalence.

Consider the discrete space $W = \coprod_{n\in\mathbb{N}\cup\{0\}}\{x_n\}$ and the obvious bijection $f\colon W \to A$, which is continuous since W is discrete. Note that W is a (0-dimensional) CW-complex.

***f* is a weak homotopy equivalence.** Since both W and A are totally disconnected, f induces a bijection $f_*\colon \pi_0(W) \xrightarrow{\simeq} \pi_0(A)$ and all higher homotopy groups $\pi_n(W)$ and $\pi_n(A)$ are trivial at any basepoint.

***f* is not a homotopy equivalence.** Let $g\colon A \to W$ be any continuous map and consider $g(0) \in W$. Since the singleton $\{g(0)\}$ is open in W, there is a neighborhood $U \subseteq A$ of 0 satisfying $g(U) \subseteq \{g(0)\}$. Since U contains more than one point (in fact infinitely many), g is not bijective and therefore does not induce an isomorphism on $\pi_0(A) \to \pi_0(W)$. $\square$

Problem 2. Consider the "equatorial" embeddings

$$S^0 \subset S^1 \subset S^2 \subset \dots$$

of spheres, and define the infinite-dimensional sphere $S^\infty := \operatorname{colim}_n S^n$. Show that S^∞ is contractible.

Solution. The embeddings $S^0 \subset S^1 \subset \dots$ define the standard CW-structure on S^∞, with two cells in each dimension $0, 1, \dots$, where the n-skeleton is S^n. Note that S^∞ is path-connected (since its 1-skeleton S^1 is) and its homotopy groups are trivial:

$$\pi_k(S^\infty) = \operatorname*{colim}_n \pi_k(S^n) = 0$$

as $\pi_k(S^n) = 0$ for $n > k$. Therefore the inclusion $* \hookrightarrow S^\infty$ is a weak homotopy equivalence, and thus a homotopy equivalence, by Whitehead. $\square$

Remark. More is true: The inclusion $* \hookrightarrow S^\infty$ of one of the two points in S^0 is the inclusion of a subcomplex of S^∞, and thus a strong deformation retract.

Problem 3. (Hatcher § 4.1 Exercise 14 and more)

a. Let X and Y be homotopy equivalent spaces. Assuming that X and Y admit CW-structures without $(n+1)$-cells (for some $n \geq 0$), show that the n-skeleta X_n and Y_n are homotopy equivalent.

Solution. Let $f\colon X \xrightarrow{\simeq} Y$ be a homotopy equivalence with homotopy inverse $g\colon Y \xrightarrow{\simeq} X$. By cellular approximation, WLOG f and g are cellular maps. Consider their restrictions to the n-skeleta $f|_{X_n}\colon X_n \to Y_n$ and $g|_{Y_n}\colon Y_n \to X_n$.

Let $H\colon X \times I \to X$ be a homotopy from gf to id_X. By cellular approximation, H is homotopic rel $X \times \partial I$ to a cellular map $H'\colon X \times I \to X$, which in particular restricts to

$$H'|_{X_n \times I}\colon X_n \times I \to X_{n+1} = X_n$$

using the fact that X has no $(n+1)$-cells. Thus the composite $g|_{Y_n} \circ f|_{X_n}\colon X_n \to X_n$ is homotopic to the identity. Likewise, since Y has no $(n+1)$-cells, the composite $f|_{X_n} \circ g|_{Y_n}\colon Y_n \to Y_n$ is homotopic to the identity. $\square$

b. Find an example of homotopy equivalent spaces X and Y, and CW-structures on X and Y such that for all $n \geq 0$, the n-skeleta X_n and Y_n are *not* homotopy equivalent.

Solution. Take the space $*$ with a single 0-cell, and the infinite-dimensional sphere S^∞. By problem 2, the inclusion $* \hookrightarrow S^\infty$ is a homotopy equivalence. However, their respective n-skeleta $(*)_n = *$ and $(S^\infty)_n = S^n$ are not homotopy equivalent, for any $n \geq 0$. $\square$

Problem 4. (Hatcher § 4.1 Exercise 16 and more)

a. Let (X, x_0) be a pointed space. Show that the summand inclusion $\iota \colon X \hookrightarrow X \vee S^n$ induces isomorphisms on homotopy groups π_i (based at any point) for all $i < n$.

Solution. Since $\iota \colon X \hookrightarrow X \vee S^n$ is obtained by attaching an n-cell (via the constant attaching map $S^{n-1} \to X$ to the basepoint), it is an $(n-1)$-connected map. Moreover, ι admits a retraction $X \vee S^n \twoheadrightarrow X$ sending the second summand S^n to the basepoint. Therefore the induced map $\iota_* \colon \pi_i(X) \to \pi_i(X \vee S^n)$ is injective for all i, in particular for $i = n - 1$. $\square$

b. Let X and Y be connected CW-complexes. Show that any map $f\colon X \to Y$ factors as a composite $X \xrightarrow{g} Z \xrightarrow{h} Y$ where $g\colon X \to Z$ induces isomorphisms on π_i for $i \leq n$ and $h\colon Z \to Y$ induces isomorphisms on π_i for $i \geq n+1$.

Solution. Start with the factorization $X \xrightarrow{\mathrm{id}} X \xrightarrow{f} Y$. Clearly the first map $\mathrm{id}\colon X \to X$ induces isomorphisms on homotopy groups π_i for $i \leq n$. The second map $f\colon X \to Y$ need not be surjective on π_{n+1}. Attach $(n+1)$-cells to X as follows:

$$X \underset{=: g_{(1)}}{\overset{\iota}{\hookrightarrow}} X \vee \bigvee_{\alpha \in \pi_{n+1}(Y)} S^{n+1} \underset{=: h_{(1)}}{\xrightarrow{(f,(\theta_\alpha))}} Y$$

where $\theta_\alpha\colon S^{n+1} \to Y$ is a chosen representative of the class $\alpha \in \pi_{n+1}(Y)$. Rename the middle term $Z_{(1)}$. By part (a), the map $g_{(1)}\colon X \to Z_{(1)}$ still induces isomorphisms on π_i for $i \leq n$. By construction, the map $h_{(1)}\colon Z_{(1)} \to Y$ induces a surjection on π_{n+1}.

Inductive step. For $k \geq 1$, assume we have a factorization of f

$$X \xrightarrow{g_{(k)}} Z_{(k)} \xrightarrow{h_{(k)}} Y$$

such that $g_{(k)}$ induces isomorphisms on π_i for $i \leq n$ and $h_{(k)}\colon Z_{(k)} \to Y$ induces isomorphisms on π_i for $n < i < n+k$ and a surjection on π_{n+k}.

Injectivity on π_{n+k}. For each $\alpha \in \ker\big(h_{(k)*}\colon \pi_{n+k}(Z_{(k)}) \to \pi_{n+k}(Y)\big)$, pick a representative $\theta_\alpha\colon S^{n+k} \to Z_{(k)}$ and a null-homotopy $H_\alpha\colon D^{n+k+1} \to Y$ of $h_{(k)}\theta_\alpha$. Attach a $(n+k+1)$-cell to $Z_{(k)}$ via the attaching map θ_α and map this new cell to Y via H_α.

Surjectivity on π_{n+k+1}. For each $\alpha \in \pi_{n+k+1}(Y)$, pick a representative $\theta_\alpha\colon S^{n+k+1} \to Y$ and attach a $(n+k+1)$-cell to $Z_{(k)}$ via the constant attaching map, and map this new cell to Y via θ_α.

After all those cell attachments, we obtain a new factorization of f

$$X \xrightarrow{g_{(k+1)}} Z_{(k+1)} \xrightarrow{h_{(k+1)}} Y$$

where $g_{(k+1)}$ still induces isomorphisms on π_i for $i \leq n$, since attaching $(n+k+1)$-cells does not affect homotopy groups below dimension $n+k$. Moreover, $h_{(k+1)}$ still induces isomorphisms on π_i for $n < i < n+k$ and now induces an isomorphism on π_{n+k} and a surjection on π_{n+k+1}.

Repeating this process inductively, we obtain a factorization of f

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

with the desired properties. $\square$