MA 519: Homework 1

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Problem 1.1 (Handout 1, # 5 [Feller Vol. 1])

A closet contains five pairs of shoes. If four shoes are selected at random, what is the probability that there is at least one complete pair among the four?

Solution. ► Let Ω denote the sample space and A denote the event that at least 1 complete pair of shoes is among the 4. We can reduce the problem of finding $P(A)$ into finding the probabilities of the mutually exclusive events

$$A_1 := \left\{ \text{exactly 1 pair is among the 4} \right\}$$

and

$$A_1 := \left\{ \text{exactly 2 pairs are among the 4} \right\}.$$

since $A = A_1 \cup A_2$, and using the additivity of P,

$$P(A) = P(A_1) + P(A_2).$$

(To keep the problem short, we will not show that $A_1 \cap A_2 = \emptyset$ and $A = A_1 \cup A_2$.)

First, let us count the number of sample points in Ω: since the closet contains 5 pairs of shoes it contains a total of 10 choose out of which we are selecting 4. Hence, the number of sample points is

$$\#\Omega = \binom{10}{4} = \frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{4 \cdot 3 \cdot 2 \cdot 6!} = 10 \cdot 3 \cdot 7 = 210. \tag{1.1}$$

Now we count the sample points in A_1 and A_2: counting the points in A_2 is immediate since we are not taking into consideration the order in which we select the pair

$$\#A_2 = \binom{5}{2} = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3!}{2 \cdot 3!} = 5 \cdot 2 = 10. \tag{1.2}$$

Counting the points in A_1 is not much harder: first, we observe that there are 5 pairs to choose from and for the remaining two shoes we must choose one shoe (either a right or a left) from the remaining 4 pairs which leaves $7 - 1 = 6$ other shoes to choose from; *i.e.* the number of sample points in A_1 is

$$5 \cdot 4 \cdot 6 = 120. \tag{1.3}$$

Taking the results of (**??**), (**??**) and (**??**), the probability that there is at least one complete pair among the four is

$$P(A) = P(A_1) + P(A_2) = \frac{120}{210} + \frac{10}{210} = \frac{130}{210} \approx 0.6190.$$

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Problem 1.2 (Handout 1, # 7 [Feller Vol. 1])

A gene consists of 10 subunits, each of which is normal or mutant. For a particular cell, there are 3 mutant and 7 normal subunits. Before the cell divides into 2 daughter cells, the gene duplicates. The corresponding gene of cell 1 consists of 10 subunits chosen from the 6 mutant and 14 normal units. Cell 2 gets the rest. What is the probability that one of the cells consists of all normal subunits.

Solution. ► We shall employ the sames strategy as that of Problem 1.1. Let A denote the event that one of the cells contains all normal units. Then, like Problem 1.1, we can reduce the problem of finding the probability of A to finding the probability of

$$A_1 := \left\{ \text{cell 1 consists of all normal subunits} \right\}$$

and

$$A_2 := \left\{ \text{cell 1 contains 6 mutant cells} \right\}$$

and taking their sum.

Now, let us count the number of points in our sample space Ω. Assuming the configuration of the subunits in a gene does not matter, we have

$$\#\Omega = \binom{20}{10} = 184756 \tag{1.4}$$

sample points.

Now we count the number of points in A_1 and A_2 these are: for A_1 we choose 10 subunits from among the 14 normal subunits giving us

$$\#A_1 = \binom{14}{10} = 1001 \tag{1.5}$$

sample points. For A_2, we must choose all 6 mutant subunits leaving 4 choices from among the 14 normal subunits giving us

$$\#A_1 = \binom{14}{4} = 1001. \tag{1.6}$$

Thus, we have

$$P(A) = P(A_1) + P(A_2) = \frac{1001}{184756} + \frac{1001}{184756} \approx 0.01083.$$

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Problem 1.3 (Handout 1, # 9 [Feller Vol. 1])

From a sample of size n, r elements are sampled at random. Find the probability that none of the N prespecified elements are included in the sample, if sampling is

(a) with replacement;

(b) without replacement.

Compute it for $r = N = 10$, $n = 100$.

Solution. ► For part (a), with replacement, the number of points in the sample space Ω_a is given by the expression

$$\#\Omega_a = n^r.$$

Let A_a be the event that none of the N prespecified elements appear (with $N \leq r$). Now to find $P(A_a)$, we count the sample points in A_a these are: there are N elements to avoid so $n - N$ elements to choose from with replacement. This gives us

$$\#A_a = (n - N)^r$$

Thus, the probability of A_a happening is

$$P(A_a) = \frac{(n-N)^r}{n^r} = \left(\frac{n-N}{n}\right)^r. \tag{1.7}$$

For part (b), without replacement, the number of points in the sample space Ω_b is given by the expression

$$\#\Omega_b = \binom{n}{r}.$$

Let A_b be the event that none of the N prespecified elements appear (with $N \leq r$). Again, to find $P(A_b)$ we need only count the sample points in A_b: there are N elements to avoid so $n - N$ elements to choose from without replacement. Hence,

$$\#A_b = \binom{n-N}{r}.$$

Thus, the probability of A_b happening is

$$P(A_b) = \binom{n-N}{r} \Big/ \binom{n}{r} = \frac{(n-1)\cdots(n-N)}{(n+r-1)\cdots(n+r-N)}. \tag{1.8}$$

Lastly, we compute, using Eqs. (**??**) and (**??**), we compute the probabilities in (a) and (b) with $r = N = 10$ and $n = 100$. These are:

$$P(A_a) = \left(\frac{90}{100}\right)^{100} \approx 0.3487,$$

and

$$P(A_b) = \frac{90\cdots 81}{100\cdots 91} \approx 0.3305.$$

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Problem 1.4 (Handout 1, # 11 [Text 1.3])

A telephone number consists of ten digits, of which the first digit is one of $1, 2, \ldots, 9$ and the others can be $0, 1, 2, \ldots, 9$. What is the probability that 0 appears at most once in a telephone number, if all the digits are chosen completely at random?

Solution. ► Let Ω be the sample space and let A be the event that at 0 appears at most once in a telephone number if all the digits are chosen completely at random. First, let us count the number of elements in the sample space, this is

$$\#\Omega = 9 \cdot 10^9$$

where the first digit is taken from among $1, 2, \ldots, 9$ and the remaining 9 out of $0, 1, 2 \ldots, 9$. Assuming randomness (*i.e.* that every sample point is equally likely), it suffices to count the sample points in the event. We do this by decomposing A into the union of mutually exclusive events

$$A_i = \left\{ \text{telephone numbers with exactly one 0 in the } i\text{-th position} \right\}.$$

The number of sample points in A_i is

$$\#A_i = 9 \cdot 9^8$$

since we must choose 8 digits of the number from among $1, \ldots, 9$ digits (with repetition). Thus,

$$P(A) = P(A_1) + \cdots P(A_9) = \frac{9 \cdot 9 \cdot 9^8}{9 \cdot 10^9} = \left(\frac{9}{8}\right)^9 \approx 0.3874.$$

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Problem 1.5 (Handout 1, # 12 [Text 1.6])

Events A, B and C are defined in a sample space Ω. Find expressions for the following probabilities in terms of $P(A)$, $P(B)$, $P(C)$, $P(AB)$, $P(AC)$, $P(BC)$ and $P(ABC)$; here AB means $A \cap B$, *etc.*:

(a) the probability that exactly two of A, B, C occur;

(b) the probability that exactly one of these events occur;

(c) the probability that none of these events occur.

Solution. ► These are all easy consequences of the inclusion-exclusion formula. For part (a) we have is $AB + AC + BC - ABC$

$$P(AB + AC + BC) = P(AB) + P(AC) + P(BC) - 2P(ABC).$$

For part (b) we have

$$P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

Lastly, for part (c) we have

$$P(\Omega) - P(A) - P(B) - P(C) + P(AB) + P(AC) + P(BC) + 2P(ABC).$$

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Problem 1.6 (Handout 1, # 13 [Text 1.8])

Mrs. Jones predicts that if it rains tomorrow it is bound to rain the day after tomorrow. She also thinks that the chance of rain tomorrow is $1/2$ and that the chance of rain the day after tomorrow is $1/3$. Are these subjective probabilities consistent with the axioms and theorems of probability?

Solution. ► No. Let $\Omega = \{(0,0),(0,1),(1,0),(1,1)\}$ be the sample space, where $(0,0)$ corresponds to the event that it rains on neither day, $(0,1)$ corresponds to the event that it rains tomorrow only, $(1,0)$ corresponds to the event that it rains the day after tomorrow only, and $(1,1)$ corresponds to the event that it rains on both days. Mrs. Jones has predicted that $P\big((0,1)\big) = 0$, $P\big((0,1)\big) + P\big((1,1)\big) = 1/2$, and $P\big((1,0)\big) + P\big((1,1)\big) = 1/3$. We can solve this system of equations: We get that

$$P\big((0,1)\big) = 0, \qquad P\big((1,1)\big) = \frac{1}{2}, \qquad P\big((1,0)\big) = -\frac{1}{6}$$

which fails our axioms of probability: we have an event that has negative probability. ◄

Problem 1.7 (Handout 1, # 16)

Consider a particular player, say North, in a Bridge game. Let X be the number of aces in his hand. Find the distribution of X.

Solution. ► First, we recall that any one player in a Bridge game has 13 cards out of the 52 in the deck, and that there are four (distinct) aces in the 52 card deck. Let Ω be the sample space of hands that North could have drawn. Let A_i be the event that North has drawn i aces. It is clear that $P(A_i) = 0$ for all $i \neq 0, 1, 2, 3, 4$.

First,

$$\#\Omega = \binom{52}{13}$$

and

$$\#A_i = \binom{4}{i}\binom{48}{13-i}$$

so that

$$P(A_i) = \binom{4}{i}\binom{48}{13-i}\Big/\binom{52}{13}$$

(note that this holds even when $i \neq 0, 1, 2, 3, 4$, as the $\binom{4}{i}$ term is zero in those cases.) ◄

Problem 1.8 (Handout 1, # 20)

If 100 balls are distributed completely at random into 100 cells, find the expected value of the number of empty cells.

Replace 100 by n and derive the general expression. Now approximate it as n tends to ∞.

Solution. ► First, we do the general case since that is what was tackled first. In the general case, we can reduce the problem to counting the solutions to

$$\sum_{j=1}^{n-i} X_j = n$$

where $1 < X_j \leq n$. Letting $Y_j = X_j - 1$ we can forget about the condition that $X_j > 1$ and we have

$$\sum_{j=1}^{n-i} Y_j = n - (n-i) = i$$

for $0 \leq Y_j \leq n-1$. It is shown in Feller that the number of distinguishable distributions are

$$\binom{n+i-1}{n}.$$

Thus, if we let A_i be the event that i bins are left empty,

$$\#A_i = \binom{n+i-1}{n} = \binom{n+i-1}{i-1}.$$

Lastly, we must count the number of points in our sample space Ω. This is given by

$$\#\Omega = \binom{2n-1}{n}.$$

This gives us an expected value of

$$\begin{aligned}
\mathbf{E}[\Omega] &= \frac{1}{\binom{2n-1}{n}} \sum_{i=1}^{n} i \binom{n+i-1}{n} \\
&= \frac{1}{\binom{2n-1}{n}} \sum_{i=1}^{n} i \binom{n+i-1}{i-1} \\
&= \frac{1}{\binom{2n-1}{n}} \sum_{i=0}^{n} (i-1) \binom{n+i}{i} \\
&= \frac{1}{\binom{2n-1}{n}} \left[\sum_{i=0}^{n} i \binom{n+i}{i} - \sum_{i=0}^{n} \binom{n+i}{i} \right]
\end{aligned}$$

Now, for the last part, consider the estimates we must ◄