MA 572: Homework 2

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Problem 2.1 (Hatcher §2.1, Ex. 16)

(a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.

(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof. (a) $\implies$ Suppose that the relative 0th homology of X with respect to A, $H_0(X, A)$, is trivial. Let $\{X_\alpha\}$ be the set of path-components of X. We aim to show that $A \cap X_\alpha \neq \emptyset$ for all α. Let $i\colon A \hookrightarrow X$ denote the canonical inclusion map $A \subset X$. Now, the map i can be extended to a chain map between chain complexes which, by proposition 2.9, induces a homomorphism $i_*\colon H_n(A) \to H_n(X)$ between the homology groups of A and X. Similarly, the map $j\colon C_n(X) \to C_n(X, A)$ induces a map $j_*\colon H_n(X) \to H_n(X, A)$ so, by theorem 2.16, we have a long exact sequence

$$\cdots \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0. \tag{1}$$

In particular, the short exact sequence

$$0 \xrightarrow{0} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0. \tag{2}$$

But $H_0(X, A) = 0$ so the map $j_* = 0$. By short exactness of (2) we have $\operatorname{im} i_* = \ker j_* = H_0(X)$, so i_* is surjective.

(b) ∎

Problem 2.2 (Hatcher §2.1, Ex. 18)

Show that for the subspace $\mathbf{Q} \subset \mathbf{R}$, the relative homology group $H_1(\mathbf{R}, \mathbf{Q})$ is free abelian and find a basis.

Proof. ∎

Problem 2.3

Homotopy invariance of homology.

Proof. ■