MA 523: Homework 8

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Problem 8.1

Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t)$$

where

$$\Phi(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

is positive for $|x| < 1$, $t > 0$.
(*Hint:* Show that u satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{cases} u = 0 & \text{on } \{\,|x| = 1\,\} \times \{\,t \geq 0\,\}, \\ u = \delta_0 & \text{on } \{\,|x| = 1\,\} \times \{\,t = 0\,\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{\,|x| \leq 1\,\} \times \{\,\varepsilon \leq t \leq T\,\}$ for small $\varepsilon > 0$ and large $T > 0$ pass to the limit as $\varepsilon \to 0^+$ and $T \to \infty$.)

Solution. Taking the hint, let us verify that $u_t = u_{xx}$, for $t > 0$. By direct computation, we have

$$\begin{aligned} \Phi_x(x,t) &= \frac{\partial}{\partial x}\left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \\ &= -\frac{xe^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}, \end{aligned} \qquad \begin{aligned} \Phi_{xx}(x,t) &= \frac{\partial}{\partial x}\left(-\frac{xe^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}\right) \\ &= \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}}, \end{aligned}$$

and

$$\begin{aligned} \Phi_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \\ &= \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}}. \end{aligned}$$

Since $\Phi_t = \Phi_{xx}$ it follows that $u_t = u_{xx}$ (for $t > 0$).

Next we show that $u = 0$ on $\{ |x| = 1 \} \times \{ t \geq 0 \}$ and $u = \delta_0$ on $\{ |x| = 1 \} \times \{ t = 0 \}$. To show $u = 0$ fix a $t \geq 0$ and after relabeling, we may assume that $x = 1$ which gives us

$$\begin{aligned} u(1,t) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{-\frac{k^2}{4t}}}{\sqrt{4\pi t}} \\ &= \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} (-1)^k (\mathrm{e}^{-\frac{1}{4t}})^{k^2} \\ &= \end{aligned}$$

$\blacksquare$

Problem 8.2 (Tikhonov's example)

Let
$$g(t) := \begin{cases} \mathrm{e}^{-t^2} & t > 0, \\ 0 & t \leq 0. \end{cases}$$
Then $g \in C^\infty(\mathbf{R})$ and we define
$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$
Assuming that the series is convergent, show that $u(x,t)$ solves the heat equation in $\mathbf{R} \times (0, \infty)$ with the initial condition $u(x, 0) = 0$, $x \in \mathbf{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

Solution. Let u be as above. Then
$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}\right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}, \end{aligned}$$
and
$$\begin{aligned} u_x(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}\right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2k x^{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}, \end{aligned}$$
$$\begin{aligned} u_{xx}(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}\right) \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} (2k-1) x^{2k-2} + \tfrac{\partial}{\partial x} g^{(0)}(t) \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}. \end{aligned}$$
Thus, $u_t - \Delta u = 0$; i.e., u solves the heat equation. ■

Problem 8.3

Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax)\mathrm{e}^{-x^2}\,dx, \qquad (a > 0).$$

(*Hint:* Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.)

Solution. ■