MA598: Lie Groups

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CHAPTER 1

Prologue

This summer, we will be making our way through Knapp's *Lie Groups Beyond an Introduction* [2] although, I (the writer of these notes) will occasionally refer to [1] for examples.

1.1 Representation of Finite Groups

Definitions

A *representation* of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$; we say that such a map ρ *gives* V *the structure of a* G*-module.* When there is little ambiguity about the map ρ we will call V itself as a representation of G; in this vein, we supress the symbol ρ and write gv for $\rho(g)(v)$. The dimension of V is sometimes called the *degree* of ρ.

A map φ between two representations V and W of G is a vector space map $\varphi\colon V \to W$ such that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

commutes for every $g \in G$. (We will call this a G*-linear map* when we want to distinguish it from an arbitrary linear map between the vector spaces V and W). We can then define $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$, and $\operatorname{Coker}\varphi$, which are also G-modules.

A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under G. A representation V is called *irreducible* if there is no proper nonzero invariant subspace W of V.

If V and W are representations, so are $V \oplus W$ and $V \otimes W$ with $g(v \otimes w) := gv \otimes gw$. Moreover, the nth tensor power $\bigotimes^n V$, the exterior power $\bigwedge^n V$ and symmetric powers $\mathrm{Sym}^n V$ are subrepresentations of it. The dual $V^* = \mathrm{Hom}(V, \mathbb{C})$ of V is also a representation, though not in the most obvious way: We want the two representations of G with respect to the natural pairing between V and V^*, so that if $\rho\colon G \to \mathrm{GL}(V)$ is a representation and $\rho^*\colon G \to \mathrm{GL}(V)$ is its dual, then we have

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle \tag{1}$$

for all $g \in G$, $v \in V$, and $v^* \in V^*$. This in turn forces us to define the dual representation by

$$\rho^*(g) := {}^{\mathrm{t}}\rho(g^{-1})\colon V^* \longrightarrow V^*$$

for all $g \in G$.

Exercise 1. Let us verify that (1) is satisfied by the definition of ρ^*.

Proof. With ρ^* as defined above, choose $g \in G$, $v \in V$ and $v^* \in V$. Then, we have

$$\begin{aligned} \langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle &= \langle \rangle \\ &= \end{aligned}$$

■

Having defined the dual representation of the tensor product of two representations, it is likewise the case that if V and W are representations, then $\mathrm{Hom}(V, W)$ is also a representation, via the identification $\mathrm{Hom}(V, W) = V^* \otimes W$. Unraveling this, if we view an element of $\mathrm{Hom}(V, W)$ as a linear map φ from V to W, we have

$$(g\varphi)(v) = g\varphi(g^{-1}v)$$

for all $v \in V$. In other words, the definition is such that the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "g\varphi"'] & W \end{tikzcd}$$

commutes. Note that the dual representation is, in turn, a special case of this: When $W = \mathbb{C}$ is the trivial representation, i.e., $gw = w$ for all $w \in \mathbb{C}$, this makes V^* into a G-module, with $g\varphi(v) = \varphi(g^{-1}v)$, i.e., $g\varphi = {}^{\mathrm{t}}(g^{-1})$.

Exercise 2. We verify that in general the vector space of G-linear maps between two representations V and W of G is just the subspace $\mathrm{Hom}(V, W)^G$ of elements of $\mathrm{Hom}(V, G)$ fixed under the action of G. We will often denote this space by $\mathrm{Hom}_G(V, W)$.

Proof. ■

We have taken the identification $\mathrm{Hom}(V, W) = V^* \otimes W$ as the definition of the representation $\mathrm{Hom}(V, W)$. More generally, the usual identities for vector spaces are also true for representations, e.g.,

$$
\begin{aligned}
V \otimes (U \oplus W) &= (V \otimes U) \oplus (V \otimes W) \\
\bigwedge^{k}(V \oplus W) &= \bigoplus_{a+b=k} \bigwedge\nolimits^{a} V \otimes \bigwedge\nolimits^{b} W \\
\bigwedge\nolimits^{k} V^* &= \left(\bigwedge\nolimits^{k} V\right)^*
\end{aligned}
$$

Bibliography

[1] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Graduate Texts in Mathematics. Springer, 2003.

[2] A.W. Knapp. *Lie Groups Beyond an Introduction.* Progress in Mathematics. Birkhäuser Boston, 2002.