# MA 523: Homework 1

Carlos Salinas

August 30, 2016

## Problem 1.1 (Taylor's formula)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to 0$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

*Hint*: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on $m$.

**Solution.** ► Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

*Proof of claim.* We shall proceed by induction on $m$. The case $m = 1$ follows easily from the chain rule:

$$
\begin{aligned}
\frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\
&= D^{(1,0,\ldots,0)} f(tx) x_1 + \cdots + D^{(0,\ldots,0,1)} f(tx) x_n \\
&= \big( D^{(1,0,\ldots,0)} x_1 + \cdots + D^{(0,\ldots,0,1)} x_n \big) f(tx)
\end{aligned}
$$

which we can write compactly as

$$= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Now, assume the result for $n \leq m - 1$. Then

$$
\begin{aligned}
\frac{d^m}{dt^m} g(t) &= \frac{d}{dt} \left[ \frac{d^{m-1}}{dt^{m-1}} g(t) \right] \\
&= \frac{d}{dt} \left[ \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} D^\alpha f(tx) x^\alpha \right]
\end{aligned}
$$

since the derivative is a linear operator, we have

$$
\begin{aligned}
&= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \frac{d}{dt} \big[ D^\alpha f(tx) x^\alpha \big] \\
&= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \sum_{|\beta|=1} D^{\alpha+\beta} f(tx) x^{\alpha+\beta}
\end{aligned}
$$

but since $f$ is smooth, the order in which we take derivatives does not matter and, hence the operators commute giving us

$$= \left[ \sum_{|\beta|=1} (Dx)^\beta \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha \right] f(tx). \qquad (1.1)$$

From here it suffices to do some combinatorics on the operators and reduce it to the desired expression. By the multinomial theorem, we have

$$\left( \sum_{|\alpha'|=1} (Dx)^{\alpha'} \right)^{m-1} = \sum_{|\alpha|=m-1} \binom{|\alpha|}{\alpha} (Dx)^\alpha = \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha.$$

Thus (1.1) becomes

$$\left[ \sum_{|\beta|=1} (Dx)^\beta \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha \right] f(tx) = \left[ \sum_{|\beta|=1} (Dx)^\beta \left( \sum_{|\alpha'|=1} (Dx)^{\alpha'} \right)^{m-1} \right] f(tx)$$

$$= \left[ \left( \sum_{|\beta|=1} (Dx)^\beta \right)^m \right] f(tx)$$

$$= \sum_{|\alpha|=m} \frac{m!}{\alpha!} (Dx)^\alpha f(tx)$$

$$= \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

as desired.                                                                                       ■

Now, applying Taylor's formula in 1 variable to $g(t)$ and evaluating at $t = 1$ we have

$$f(x) = g(1)$$

$$= \sum_{i=0}^{k} \frac{g^{(i)}(0)}{i!} 1^i + O(|x|^{k+1})$$

$$= \sum_{i=0}^{k} \frac{1}{i!} \sum_{|\alpha|=i} \frac{i!}{\alpha!} D^\alpha f(tx) x^\alpha + O(|x|^{k+1})$$

$$= \sum_{i=0}^{k} \sum_{|\alpha|=i} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$$

$$= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$$

as desired.                                                                                       ◀

## Problem 1.2

Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

**Solution.** ▸ For reference, formula (5) in §2.1.2 of [E] is the solution to the nonhomogeneous problem

$$u(x, t) = g(x - tb) + \int_0^1 f\big(x + (s - t)b, s\big)\, ds$$

where $x \in \mathbb{R}^n$, $t > 0$.

First, we write $(*)$ as the directional derivative along $(b, 1)$

$$\begin{aligned}
f &= u_t + b \cdot Du \\
&= (b, 1) \cdot Du \\
&= D_{(b,1)}u.
\end{aligned}$$

Using the structure of characteristic ODE, we have

$$\begin{aligned}
\dot{p}(s) &= \\
\dot{z}(s) &= \\
\dot{x}(s) &=
\end{aligned}$$

with initial conditions

$$\begin{aligned}
\dot{p}(s) &= \\
\dot{z}(s) &= \\
\dot{x}(s) &=.
\end{aligned}$$

◂

## PROBLEM 1.3

Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $u u_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

**Solution.** ▶

◀

## Problem 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\left(u_{x_1}^2 + u_{x_2}^2\right)$$

find a solution with $u(x_1, 0) = \left(1 - x_1^2\right)/2$.

**Solution.** ▶

◀