

MA 544: Homework 7

Carlos Salinas

February 29, 2016

PROBLEM 7.1 (WHEEDEN & ZYGMUND §4, EX. 9)

- (a) Show that the limit of a decreasing (increasing) sequence of functions usc (lsc) at \mathbf{x}_0 is usc (lsc) at \mathbf{x}_0 . In particular, the limit of a decreasing (increasing) sequence of functions continuous at \mathbf{x}_0 is usc (lsc) at \mathbf{x}_0 .
- (b) Let f be usc and less than ∞ on $[a, b]$. Show that there exists continuous f_k on $[a, b]$ such that $f_k \searrow f$.

Proof. (a) Without loss of generality, assume that the sequence of f_k 's are define in all of \mathbf{R}^n . Suppose $\{f_k : \mathbf{R}^n \rightarrow \mathbf{R}\}_{k=1}^{\infty}$ is a sequence of decreasing functions which are usc at \mathbf{x}_0 and put $f := \lim f_k$. Then, since the f_k 's are decreasing, we have

$$f(\mathbf{x}) \leq f_k(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbf{R}^n. \quad (7.1)$$

Moreover, f_k being usc at \mathbf{x}_0 means that for any sequence $\mathbf{x}_k \rightarrow \mathbf{x}_0$, $\overline{\lim}_{\mathbf{x} \rightarrow \mathbf{x}_0} f_k(\mathbf{x}) \leq f_k(\mathbf{x}_0)$. Thus, by (7.1) we have

$$\overline{\lim}_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) \leq \overline{\lim}_{\mathbf{x} \rightarrow \mathbf{x}_0} f_k(\mathbf{x}) \leq f_k(\mathbf{x}_0). \quad (7.2)$$

Now, let $k \rightarrow \infty$ and we have $\overline{\lim}_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) \leq f(\mathbf{x}_0)$. Thus, f is usc at \mathbf{x}_0 .

(b) Mimicking the construction of the function associated to the cantor set divide the interval into $2n$ closed intervals $I_1 := [y_0, y_1], \dots, I_{2n} := [y_{2n-1}, y_{2n}]$ and define

$$f_n(y_k) := \max_{I_k \cup I_{k+1}} f(x) \quad (7.3)$$

and $f_n(x)$ is linear on I_n . Then the function f_n is clearly continuous on $[a, b]$. Moreover, we claim that $f_n \rightarrow f$. ■

PROBLEM 7.2 (WHEEDEN & ZYGMUND §4, EX. 11)

Let f be defined on \mathbf{R}^n and let $B(\mathbf{x})$ denote the open ball $\{\mathbf{y} \mid |\mathbf{x} - \mathbf{y}| < r\}$ with center \mathbf{x} and fixed radius r . Show that the function $g(\mathbf{x}) = \sup\{f(\mathbf{y}) \mid \mathbf{y} \in B(\mathbf{x})\}$ is lsc and the function $h(\mathbf{x}) = \inf\{f(\mathbf{y}) \mid \mathbf{y} \in B(\mathbf{x})\}$ is usc on \mathbf{R}^n . Is the same true for the closed ball $\{\mathbf{y} \mid |\mathbf{x} - \mathbf{y}| \leq r\}$?

Proof. Fix $\mathbf{x}_0 \in \mathbf{R}^n$. By 4.14(ii), given a real number m such that $f(\mathbf{x}_0) > m$, it suffices to show that there exists $\delta > 0$ such that for every $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ we have $g(\mathbf{x}) > m$. Since $g(\mathbf{x}_0)$ is the supremum of $f(\mathbf{x})$ for all $\mathbf{x} \in B(\mathbf{x}_0)$, given $\varepsilon > 0$ such that $g(\mathbf{x}_0) - \varepsilon > m$ there exists $\mathbf{y} \in B(\mathbf{x}_0)$ such that $f(\mathbf{y}) > g(\mathbf{x}_0) - \varepsilon$. Let $\delta := \frac{1}{2}(r - |\mathbf{y} - \mathbf{x}_0|)$ (δ is positive since $\mathbf{y} \in B(\mathbf{x}_0)$ hence $|\mathbf{y} - \mathbf{x}_0| < r$). Then for any $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ we have

$$\begin{aligned} |\mathbf{x} - \mathbf{y}| &= |\mathbf{x} - \mathbf{x}_0 - (\mathbf{y} - \mathbf{x}_0)| \\ &\leq |\mathbf{x} - \mathbf{x}_0| + |\mathbf{y} - \mathbf{x}_0| \\ &= \frac{1}{2}(r - |\mathbf{y} - \mathbf{x}_0|) + |\mathbf{y} - \mathbf{x}_0| \\ &= \frac{1}{2}r + \frac{1}{2}|\mathbf{y} - \mathbf{x}_0| \\ &< r \end{aligned} \tag{7.4}$$

so $\mathbf{y} \in B(\mathbf{x})$. Hence, $g(\mathbf{x}) \geq f(\mathbf{y}) > g(\mathbf{x}_0) - \varepsilon > m$, i.e., $B_\delta(\mathbf{x}_0) \subset \{\mathbf{x} \mid f(\mathbf{x}) > m\}$. Thus, g is lsc on \mathbf{R}^n .

The proof for h is similar to g . In fact, note that for any set $E \subset \mathbf{R}$ we have $\inf E = -\sup(-E)$ so that if we set $f' := -f$ and define $g'(\mathbf{x}) := \sup\{f'(\mathbf{y}) \mid \mathbf{y} \in B(\mathbf{x})\}$. By the above, g' is lsc in \mathbf{R}^n so $h = -g'$ is usc in \mathbf{R}^n . ■

PROBLEM 7.3 (WHEEDEN & ZYGMUND §4, EX. 15)

Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. If $|f_k(M)| \leq M < \infty$ for all k for each $\mathbf{x} \in E$, show that given $\varepsilon > 0$, there is closed $F \subset E$ and finite M such that $|E \setminus F| < \varepsilon$ and $|f_k(\mathbf{x})| \leq M$ for all $\mathbf{x} \in F$.

Proof. Define $f := \sup |f_k|$. Note that, since $|f_k| = f^+ + f^-$ and f^+ and f^- are measurable, $|f_k|$ is measurable hence, by 4.11, f is measurable. Now, given $\varepsilon > 0$ by Lusin's theorem f has the \mathcal{C} -property on E , i.e., there exists a closed subset F of E such that $|E \setminus F| < \varepsilon/2$ and f is continuous when restricted to F . Take the $\delta > 0$ such that $|E \setminus \overline{B_\delta(\mathbf{0})}| < \varepsilon/2$. Then $F \cap \overline{B_\delta(\mathbf{0})}$ is closed and compact and we have

$$\begin{aligned} |E \setminus (F \cap \overline{B_\delta(\mathbf{0})})| &= |E \setminus F \cup (E \setminus \overline{B_\delta(\mathbf{0})})| \\ &\leq |E \setminus F| + |E \setminus \overline{B_\delta(\mathbf{0})}| \\ &< \varepsilon. \end{aligned}$$

By Problem 6.2 (W&Z, 4.7) f achieves its maximum M on $F \cap \overline{B_\delta(\mathbf{0})}$. Thus, $|f_k| \leq M$ for all $\mathbf{x} \in F \cap \overline{B_\delta(\mathbf{0})}$. ■

PROBLEM 7.4 (WHEEDEN & ZYGMUND §4, EX. 18)

If f is measurable on E , define $\omega_f(a) = |\{f > a\}|$ for $-\infty < a < \infty$. If $f_k \nearrow f$, show that $\omega_{f_k} \nearrow \omega_f$. If $f_k \rightarrow f$, show that $\omega_{f_k} \rightarrow \omega_f$ at each point of continuity of ω_f . [For the second part, show that if $f_k \rightarrow f$, then $\overline{\lim}_{k \rightarrow \infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$ and $\underline{\lim}_{k \rightarrow \infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$ for every $\varepsilon > 0$.]

Proof. For fixed a define $E_k := \{f_k > a\}$. Then we have $E_1 \subset E_2 \subset \cdots$ so $E_k \nearrow \bigcup E_k$. Now, it is clear that $\{f > a\} \supset \bigcup E_k$ since $\mathbf{x} \in \bigcup E_k$ implies that $\mathbf{x} \in E_k$ for some k so $f_k(\mathbf{x}) > a$ for all $K \geq k$. In particular, $f(\mathbf{x}) > a$. Thus, $\mathbf{x} \in \{f > a\}$. On the other hand, if $\mathbf{x} \in \{f > a\}$ then $f(\mathbf{x}) > a$ so $f_k \nearrow f$ implies that for sufficiently large N we have $f_N(\mathbf{x}) > a$. Thus, $\mathbf{x} \in E_N$ so $\mathbf{x} \in \bigcup E_k$ and we have

$$\{f > a\} = \bigcup E_k. \quad (7.5)$$

It follows by 3.26(i) that $\omega_{f_k} \nearrow \omega_f$ point-wise. ■

PROBLEM 7.5 (WHEEDEN & ZYGMUND §5, EX. 1)

If f is a simple measurable function (not necessarily positive) taking values a_j on E_j , $j = 1, \dots, N$, show that $\int_E f = \sum_{j=1}^N a_j |E_j|$. [Use (5.24)].

Proof. Since f is a simple measurable function $E_k \cap E_\ell = \emptyset$ for $k \neq \ell$. Since $E := \bigcup_{j=1}^N E_j$ is countable, by 5.24 we have

$$\int_E f = \sum_{j=1}^N \int_{E_j} f = \sum_{j=1}^N \int a_j \chi_{E_j} = \sum_{j=1}^N a_j |E_j|.$$

■

PROBLEM 7.6 (WHEEDEN & ZYGMUND §5, EX. 3)

Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E . If $f_k \rightarrow f$ and $f_k \leq f$ a.e. on E , show that $\int_E f_k \rightarrow \int_E f$.

Proof. By Fatou's lemma we have

$$\int_E f = \int_E \liminf_{k \rightarrow \infty} f_k \leq \liminf_{k \rightarrow \infty} \int_E f_k. \quad (7.6)$$

Now, by 5.10 since $f_k \leq f$ a.e. on E we have $\int_E f_k \leq \int_E f$ so

$$\overline{\lim}_{k \rightarrow \infty} \int_E f_k \leq \int_E f. \quad (7.7)$$

But $\liminf \int_E f_k \leq \overline{\lim} \int_E f_k$ so we must have $\lim \int_E f_k \rightarrow \int_E f$ as $k \rightarrow \infty$. ■