Math 527 - Homotopy Theory
Spring 2013
Homework 2 Solutions

Problem 1. Show that the reduced suspension $\Sigma X = X \wedge S^1$ of any pointed space X is a homotopy cogroup object in $\mathbf{Top}_*$, with structure maps coming from those of S^1 (c.f. Homework 1 Problem 3).

Solution. The functor $X \wedge -\colon \mathbf{Top}_* \to \mathbf{Top}_*$ preserves finite coproducts (by Homework 1 Problem 4), including the initial object (by the homeomorphism $* \xrightarrow{\cong} X \wedge *$). Therefore, applying $X \wedge -$ to the stucture maps of S^1 endows $X \wedge S^1$ with structure maps of the correct type. They satisfy the requisite equations up to pointed homotopy, because the functor $X \wedge -$ sends pointed-homotopic maps to pointed-homotopic maps (by Problem 3a.) $\square$

Problem 2. Show that a pointed homotopy between two pointed maps $X \to Y$ is the same as a pointed map

$$X \wedge (I_+) \to Y$$

where $(-)_+$ denotes the disjoint basepoint construction.

Solution. Given the homeomorphism

$$X \wedge (I_+) \cong X \times I / x_0 \times I$$

a (continuous) map $H\colon X \wedge (I_+) \to Y$ is the same as a (continuous) map $H\colon X \times I \to Y$ which is constant on the subset $x_0 \times I$. Thus, a *pointed* map $H\colon X \wedge (I_+) \to Y$ is the same as a map $H\colon X \times I \to Y$ with constant value y_0 on the subset $x_0 \times I$, i.e. a pointed homotopy between two pointed maps $X \to Y$. $\square$

Problem 3. Let X be a pointed space.

a. Show that the functor $X \wedge -\colon \mathbf{Top}_* \to \mathbf{Top}_*$ sends pointed-homotopic maps to pointed-homotopic maps.

Solution. Recall that two pointed maps $f, g\colon Y \to Z$ are pointed-homotopic if there exists a map H making the diagram

$$\begin{tikzcd}
Y \arrow[dr, "\iota_0"'] \arrow[drr, bend left, "f"] & & \\
& Y \wedge (I_+) \arrow[r, "H"] & Z \\
Y \arrow[ur, "\iota_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

commute, where $\iota_0\colon Y \cong Y \wedge S^0 \to Y \wedge (I_+)$ denotes the inclusion at $0 \in I$ and likewise for ι_1.

Note that ι_0 is the map obtained by applying the functor $Y \wedge -$ to the pointed map

$$S^0 \cong \{0\}_+ \hookrightarrow I_+.$$

By associativity of the smash product, applying the functor $X \wedge -$ yields

$$X \wedge Y \to X \wedge (Y \wedge I_+) \cong (X \wedge Y) \wedge I_+$$

which is still the inclusion at 0.

Therefore, applying $X \wedge -$ to the diagram above yields, up to natural isomorphism, the commutative diagram

$$\begin{tikzcd}
X \wedge Y \arrow[dr, "\iota_0"'] \arrow[drr, bend left, "X \wedge f"] & & \\
& X \wedge Y \wedge (I_+) \arrow[r, "X \wedge H"] & X \wedge Z \\
X \wedge Y \arrow[ur, "\iota_1"] \arrow[urr, bend right, "X \wedge g"'] & &
\end{tikzcd}$$

and thus a pointed homotopy from $X \wedge f$ to $X \wedge g$. $\square$

b. Show that the pointed map "inclusion at 0"

$$X \to X \wedge (I_+)$$
$$x \mapsto [x, 0]$$

is a pointed homotopy equivalence.

Solution. Note that the inclusion $\{0\} \hookrightarrow I$ is a homotopy equivalence. Since the disjoint base-point functor $(-)_+$ sends homotopic maps to pointed-homotopic maps, the inclusion $\{0\}_+ \hookrightarrow I_+$ is a pointed homotopy equivalence. By part (a), applying the functor $X \wedge -$ yields a pointed homotopy equivalence $X \to X \wedge (I_+)$. $\square$