MA 519: Homework 12

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Problem 12.1 (Handout 15, # 10)

Consider the experiment of picking one word at random from the sentence

All is well in the newell family

Let X be the length of the word selected and Y the number of Ls in it. Find in a tabular form the joint PMF of (X, Y), their marginal PMFs, means, and variances, and the correlation between X and Y.

Solution. ∎

Problem 12.2 (Handout 15, # 11)

Consider the joint PMF $p(x, y) = cxy$, $1 \leq x \leq 3$, $1 \leq y \leq 3$.

(a) Find the normalizing constant c.
(b) Are X and Y independent? Prove your claim.
(c) Find the expectations of X, Y, and XY.

Solution. ∎

Problem 12.3 (Handout 15, # 12)

A fair die is rolled twice. Let X be the maximum and Y the minimum of the two rolls. By using the joint PMF of X and Y worked out in the text, find the PMF of $\frac{X}{Y}$, and hence the mean of $\frac{X}{Y}$.

Solution. ■

Problem 12.4 (Handout 15, # 13)

Two random variables have the joint PMF $p(x, x+1) = \frac{1}{n+1}$, $x = 0, \ldots, n$. Answer the following question with as little calculation as possible.

(a) Are X and Y independent?
(b) What is the variance of $Y - X$?
(c) What is $\operatorname{Var}(Y \mid X = 1)$?

Solution. ∎

Problem 12.5 (Handout 15, # 14)

(Binomial Conditional Distribution). Suppose X and Y are independent random variables, and $X \sim \mathrm{Bin}(m, p)$, $Y \sim \mathrm{Bin}(n, p)$. Show that the conditional distribution of X given by $X + Y = t$ is a hypergeometric distribution; identify the parameters of this hypergeometric distribution.

Solution. ■

Problem 12.6 (Handout 15, # 15)

Suppose a fair die is rolled twice. Let X and Y be the two rolls. Find the following with as little calculation as possible.

(a) $E(X + Y \mid Y = y)$.
(b) $E(XY \mid Y = y)$.
(c) $\mathrm{Var}(X^2 Y \mid Y = y)$.
(d) $\rho_{X+Y,X-Y}$.

Solution. ■

Problem 12.7 (Handout 15, # 16)

(A Standard Deviation Inequality). Let X and Y be two random variables. Show that $\sigma_{X+Y} \leq \sigma_X + \sigma_Y$.

Solution. Suppose σ_X and σ_Y exist and are finite. We want to show

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y;$$

this is the same as showing that

$$\begin{aligned} \sigma_{X+Y}^2 &\leq \sigma_X + \sigma_Y^2 + 2\sigma_X\sigma_Y \\ \operatorname{Var}(X+Y) &\leq \operatorname{Var}(X) + \operatorname{Var}(Y) + 2[\operatorname{Var}(X)\operatorname{Var}(Y)]^{\frac{1}{2}}. \end{aligned}$$

First, let us expand $\operatorname{Var}(X+Y)$ using the definition of variance, we have

$$\begin{aligned} \operatorname{Var}(X+Y) &= E\big((X+Y)^2\big) - E(X+Y)^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 \\ &= \big(E(X^2) - E(X)^2\big) + \big(E(Y^2) - E(Y)^2\big) + 2[E(XY) - E(X)E(Y)] \\ &= \operatorname{Var}(X) + \operatorname{Var}(Y) + 2[E(XY) - E(X)E(Y)]. \end{aligned}$$

Therefore, it suffices to show that

$$E(XY) - E(X)E(Y) \leq [\operatorname{Var}(X)\operatorname{Var}(Y)]^{\frac{1}{2}}.$$

By the Cauchy–Schwartz inequality, we have

$$\begin{aligned} E(XY) - E(X)E(Y) &\leq [E(X^2)E(Y^2)]^{\frac{1}{2}} - [E(X)^2E(Y)^2]^{\frac{1}{2}} \\ &= \end{aligned}$$

■

Problem 12.8 (Handout 15, # 17)

Seven balls are distributed randomly in seven cells. Let X_k be the number of cells containing exactly k balls. Using the probabilities tabulated in II, 5, write down the joint distribution of X_2, X_3.

Solution. The tabled referenced in this problem is on p. 40 of Feller. Let us write down a table of our own for the joint distribution of (X_2, X_3):

$X_3 \backslash X_2$	0	1	2	3
0	0.048	0.156	0.321	0.107
1	0.109	0.214	0.027	0
2	0.018	0	0	0

Let us do a sanity check by summing over all of the entries in the table above

$$0.048 + 0.156 + 0.321 + 0.107 + 0.109 + 0.214 + 0.027 + 0 + 0.018 + 0 + 0 + 0 \approx 1. \qquad \blacksquare$$

Problem 12.9 (Handout 15, # 18)

Two ideal dice are thrown. Let X be the score on the first die and Y be the larger of two scores.

(a) Write down the joint distribution of X and Y.
(b) Find the means, the variances, and the covariance.

Solution. For part (a): The random variable X takes on integer values between zero and six and so does Y. Moreover, the dependence of Y on X tells us that $P(\{X = k\} \cap \{Y = \ell\}) = 0$ if $\ell < k$; this allows us to fill in a significant portion of the joint distribution table:

YX	0	1	2	3			
0							
1							
2							
3							
4							
5							
6							

■

Problem 12.10 (Handout 15, # 19)

Let X_1 and X_2 be independent and have the common geometric distribution $\{q^k p\}$ (as in problem 4). Show without calculations that the *conditional distribution of* X_1 *given* $X_1 + X_2$ *is uniform*, that is,

$$P(X_1 = k \mid X_1 + X_2 = n) = \frac{1}{n+1}, \quad k = 0, \dots, n. \tag{12.1}$$

Solution. ∎

Problem 12.11 (Handout 15, # 20)

If two random variables X and Y assume only two values each, and if $\mathrm{Cov}(X, Y) = 0$, then X and Y are independent.

Solution. ■