MA544: Qual Problems

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1 MA 544 Spring 2016

1.1 Exam 1 Prep

Problem 1.1. Let $E \subset \mathbf{R}^n$ be a measurable set, $r \in \mathbf{R}$ and define the set $rE = \{\, r\mathbf{x} : \mathbf{x} \in E \,\}$. Prove that rE is measurable, and that $|rE| = |r|^n|E|$.

Proof. Define a linear map $T \colon \mathbf{R}^n \to \mathbf{R}^n$ by $\mathbf{x} \mapsto r\mathbf{x}$. Using the standard basis for $\mathbf{R}^n$, this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \tag{1}$$

which has determinant $\det T = r^n$. By 3.35, we have $|E| = |T(E)| = r^n|E| = |rE|$. ∎

Problem 1.2. Let $\{E_k\}$, $k \in \mathbf{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty} E_n\right).$$

Show that

$$\left|\liminf_{k\to\infty} E_k\right| \le \liminf_{k\to\infty}|E_k|.$$

Proof. If the $\liminf|E_k| = \infty$ the inequality holds trivially. Hence, we may, without loss of generality, assume that $\liminf|E_k| < \infty$. By 3.20, the set $\liminf E_k$ is measurable and we have

$$\left|\liminf_{k\to\infty} E_k\right| = \left|\bigcup_{k=1}^{\infty} F_k\right|, \tag{2}$$

where $F_k \coloneqq \bigcap_{n=k}^{\infty} E_n$. Now, note that the collection of sets $F_k' \coloneqq \bigcup_{\ell=1}^{k} F_\ell$ forms an increasing sequence of measurable sets $F_k' \nearrow F'$, where $F' = \bigcup_{k=1}^{\infty} F_k = \liminf E_k$. Then, by 3.26 (i), we have

$$\lim_{k\to\infty}|F_k'| = |F'| = \left|\liminf_{k\to\infty} E_k\right|. \tag{3}$$

Hence, it suffices to show that $|F_k'| \le |E_k|$ for all k, but this follows by monotonicity of the outer measure, 3.3, since $F_k' \subset E_k$. Thus, we have the desired inequality

$$\left|\liminf_{k\to\infty} E_k\right| \le \liminf_{k\to\infty}|E_k|. \tag{4}$$

∎

Problem 1.3. Consider the function

$$F(x) \coloneqq \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) \coloneqq \{\, \mathbf{y} \in \mathbf{R}^n : |\mathbf{y}| < r \,\}$. Prove that F is monotonic increasing and continuous.

Proof. That F is increasing is immediate from the monotonicity of the outer measure since for $x < x'$ we have $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$ so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \leq |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that F is continuous, we will prove the following lemma

Lemma 1. *For any $x > 0$, $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$.*

Proof of lemma. If $\mathbf{y} \in xB(\mathbf{0}, 1)$ then $\mathbf{y} = x\mathbf{y}'$ for $\mathbf{y}' \in B(\mathbf{0}, 1)$. Thus, $|\mathbf{y}'| = |\mathbf{y}|/x < 1$ so $|\mathbf{y}| < x$ implies that $\mathbf{y} \in B(\mathbf{0}, x)$. Hence, we have the containment $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$.

On the other hand, if $\mathbf{y} \in B(\mathbf{0}, x)$ then $|\mathbf{y}| < x$ so $|\mathbf{y}/x| < 1$. Hence, $\mathbf{y}/x \in B(\mathbf{0}, 1)$ so $x(\mathbf{y}/x) = \mathbf{y} \in B(\mathbf{0}, x)$. Thus, $B(\mathbf{0}, x) \subset xB(\mathbf{0},)$ and equality holds. ♣

In light of Lemma 1 and 3.35, for $x > 0$, we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n |B(\mathbf{0}, 1)|. \tag{5}$$

It is clear that F is continuous on the interval $[0, \infty)$ since F is a polynomial in x. ■

Problem 1.4. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. From the topological definition of continuity, f is continuous at $x \in C$ if and only if for every neighborhood U of $f(x)$, the preimage $f^{-1}(U)$ is a neighborhood of x. Now, ■

Let $x \in C$. Then, by the definition of continuity, for every natural number $n > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f(x) - f(x')| < \frac{1}{2n}. \tag{6}$$

Let $x'', x' \in B(x, \delta)$. Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x)''| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \tag{7}$$

In view of these estimates, define the set

$$A_n \coloneqq \left\{ x \in \mathbf{R} : \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \tag{8}$$

Good Lord, that was a long definition! We claim that $C = \bigcap_{n=1}^{\infty} A_n$ and that A_n is open for all n.

First, let us show that $C = \bigcap_{n=1}^{\infty} A_n$. Let $x \in C$. Then for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Thus, $x \in A_n$ for all n so $x \in \bigcap A_n$. On the other hand, if $x \in \bigcap A_n$ for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$.

Fix $\varepsilon > 0$. By the Archimedean principle, there exists $N > 0$ such that $\varepsilon > 1/N$. Then, since $x \in A_N$ it follows that for some $\delta' > 0$, $|x - x'| < \delta'$ implies $|f(x) - f(x')| < 1/N < \varepsilon$. Thus, $x \in C$ and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$.

Lastly, we show that A_n is open. Let $x \in A_n$. Then there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. In particular, this means that $B(x, \delta) \subset A_n$ for any $x' \in B(x, \delta)$ satisfies $|f(x) - f(x')| < 1/n$. Thus, A_n is open and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$ is a G_δ set.

Problem 1.5. Let $f \colon \mathbf{R} \to \mathbf{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbf{R}$, then f is measurable?

Proof. No. Recall that, by definition, or 4.1, f is measurable if and only if $\{ f > a \}$ for all $a \in \mathbf{R}$. ∎

Problem 1.6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbf{R}$. Prove that the set $\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \}$ is measurable.

Proof. The idea here should be to rewrite

$$E \coloneqq \left\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \right\} \tag{9}$$

as a countable union/intersection of measurable sets. Let $x \in E$. By the Cauchy criterion, for every $N > 0$ there exists a positive integer M such that $m, n \geq M$ implies $|f_n(x) - f_m(x)| < 1/N$. With this in mind, define

$$E_N \coloneqq \left\{ x : \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \tag{10}$$

Then, like for Problem 1.4, it is not too hard to see that the E_n's are open and that $E = \bigcap_{n=1}^{\infty} E_n$. Thus, E is a G_δ set and therefore measurable. ∎

Problem 1.7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, one has $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. Suppose $f \colon [a, b] \to \mathbf{R}$ is absolutely continuous. Then for fixed $\varepsilon = 1$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, we have $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Let $\Gamma \coloneqq \{x_k\}_{k=1}^{N}$ be a partition of $[a, b]$ into closed intervals such that $x_{k+1} - x_k < \delta$, then by absolute continuity we have

$$\begin{aligned} V[f; \Gamma] &= \sum_{k=1}^{N} |f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \tag{11}$$

Thus, $f \in \mathrm{BV}[a, b]$. ∎

Problem 1.8. Let f be a continuous function from $[a, b]$ into $\mathbf{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, i.e., $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f \,\mathrm{d}\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(a) & \text{if } c = b \end{cases} .$$

Proof. ■

2 Exam 1

2.1 Exam 2 Prep

Problem 2.1. Define for $\mathbf{x} \in \mathbf{R}^n$,

$$f(\mathbf{x}) \coloneqq \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B_\varepsilon(\mathbf{0})$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbf{R}^n \smallsetminus B_\varepsilon(\mathbf{0})} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f\colon \mathbf{R}^n \to \mathbf{R}$ is (Lebesgue) integrable over a subset E of $\mathbf{R}^n$ (or, alternatively, f belongs to $L^1(E)$) if

$$\int_E f(\mathbf{x})\, \mathrm{d}\mathbf{x} < \infty.$$

Put $E \coloneqq \mathbf{R}^n \smallsetminus B_\varepsilon(\mathbf{0})$. Then, to show that f belongs to $L^1(E)$ it suffices to prove the inequality

$$\int_E f(\mathbf{x})\, \mathrm{d}\mathbf{x} < \frac{C}{\varepsilon} \tag{12}$$

for some appropriate constant C. We proceed by directly computing the Lebesgue integral of f and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x})\, \mathrm{d}\mathbf{x} &= \int_E \frac{\mathrm{d}\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{({x_1}^2 + \cdots + {x_n}^2)^{(n+1)/2}} \end{aligned}$$

let E_i denote the projection of E onto its i-th coordinate and make the trigonometric substitution $x_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \tan\theta$, $\mathrm{d}x_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \sec^2\theta\, \mathrm{d}\theta$ with $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$ giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[\frac{\cos^{n-1}\theta}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}}\, \mathrm{d}\theta \right] \mathrm{d}x_2 \cdots \mathrm{d}x_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{\mathrm{d}x_2 \cdots \mathrm{d}x_n}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1}\theta\, \mathrm{d}\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1}\theta\, \mathrm{d}\theta < \infty. \tag{13}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned}\int \cdots \int_E f(\mathbf{x})\,\mathrm{d}\mathbf{x} &< C' \int_{E_n} \frac{\mathrm{d}x_n}{{x_n}^2} \\ &= 2C' \int_{\varepsilon}^{\infty} \frac{\mathrm{d}x_n}{{x_n}^2} \\ &= \frac{C}{\varepsilon}\end{aligned} \tag{14}$$

as desired. ■

Problem 2.2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbf{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbf{R}^n} f = \lim_{k\to\infty} \int_{\mathbf{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbf{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbf{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit f of $\{f_k\}$ must be nonnegative a.e. in $\mathbf{R}^n$. For assume otherwise. Then there exists a collection of points $\mathbf{x}$ in $\mathbf{R}^n$ of nonzero $\mathbf{R}^n$-Lebesgue measure such that $f(\mathbf{x}) < 0$. But $f_k(\mathbf{x}) \geq 0$ for all $k \in \mathbf{N}$. Set $0 < \varepsilon < |f(\mathbf{x})|$ then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{15}$$

for all k which contradicts our assumption that $f_k \to f$ a.e. on $\mathbf{R}^n$. Therefore, the set of points $\mathbf{x} \in \mathbf{R}^n$ where $f(\mathbf{x}) < 0$ must have measure zero.

Now, based on pointwise convergence a.e. to f, given $\varepsilon > 0$ for a.e. $\mathbf{x} \in \mathbf{R}^n$ we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{16}$$

for sufficiently large k; say k greater than or equal to some index $N \in \mathbf{N}$. Moreover, we are given convergence in $L^1(\mathbf{R}^n)$ of f_k to f

$$\int_{\mathbf{R}^n} f_k \to \int_{\mathbf{R}^n} f < \infty. \tag{17}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbf{R}^n} f < \infty \tag{18}$$

and

$$\int_E f_k \leq \int_{\mathbf{R}^n} f_k < \infty \tag{19}$$

for all $k \in \mathbf{N}$. By Theorem 5.5(ii), f and the f_k's are finite a.e. in $\mathbf{R}^n$ so for some sufficiently large real number M, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in \mathbf{R}^n$. In particular, for any measurable subset E of $\mathbf{R}^n$, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in E$ so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \to \int_E f < \infty. \tag{20}$$

However, if f does not belong to $L^1(\mathbf{R}^n)$, i.e., its integral over $\mathbf{R}^n$ is infinity, there is no guarantee that f will be finite a.e. in $\mathbf{R}^n$. This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset E of $\mathbf{R}^n$. Let us demonstrate this with an example. Consider the sequence of functions ■

Problem 2.3. Assume that E is a measurable set of $\mathbf{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of $\mathbf{R}^n$, then

$$\int_E f(\mathbf{x})\, \mathrm{d}\mathbf{x} < \infty. \tag{21}$$

Set $E_k \coloneqq \{\, \mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k \,\}$ and $F_k \coloneqq \{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{22}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \leq (k+1)|E_k|. \tag{23}$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x})\, \mathrm{d}\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \tag{24}$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (24) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x})\, \mathrm{d}\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \tag{25}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (26) \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \qquad (27) \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned}$$

Thus, from (26) and (27)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \qquad (28)$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ■

Problem 2.4. Suppose that E is a measurable subset of $\mathbf{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f, g) := \int_E \frac{|f-g|}{1+|f-g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. $\implies$: First note that ρ is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that $\rho(f_k, f) \to 0$ as $k \to \infty$. Then, given $\varepsilon > 0$ there exist an sufficiently large index N such that for every $k \geq N$ we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \tag{29}$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in E which happens if $|f_k - f| = 0$ a.e. in E.

$\Longleftarrow$: Suppose that $f_k \to f$ as $k \to \infty$.

I don't know how to solve this. This is the intended solution:

$\Longrightarrow$: Given $\varepsilon > 0$, $\rho(f_k, f) \to 0$ implies that

$$\int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f} \, \mathrm{d}x \longrightarrow 0.$$

Observe that the function $\Phi \colon \mathbf{R}^+ \to \mathbf{R}$ given by $\Phi(x) \coloneqq x/(1+x)$ is increasing on $\mathbf{R}^+$ and $0 < \Psi(x) < 1$, hence

$$\begin{aligned}\int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f|} \, \mathrm{d}x &\geq \int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{\varepsilon}{1 + \varepsilon} \, \mathrm{d}x \\ &= \frac{\varepsilon}{1+\varepsilon} |\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}|.\end{aligned}$$

Therefore,

$$|\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}| \leq \frac{1+\varepsilon}{\varepsilon} \int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f|} \, \mathrm{d}x \longrightarrow 0$$

as $k \to \infty$.

$\Longleftarrow$: Conversely, given $\delta > 0$, we have

$$\begin{aligned}\rho(f_k, f) = &\int_{\{ x \in E : |f_k(x) - f(x)| > \delta \}} \frac{|f_k - f|}{1 + |f_k - f|} \, \mathrm{d}x \\ &+ \int_{\{ x \in E : |f_k(x) - f(x)| \leq \delta \}} \frac{|f_k - f|}{1 + |f_k - f|} \, \mathrm{d}x \\ &\leq |\{ x \in E : |f_k(x) - f(x)| > \delta \}| + \frac{\delta}{1+\delta} |E|.\end{aligned}$$

Since $|E| < \infty$ and $\delta/(1+\delta) \searrow 0$, then for any $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\frac{\delta'}{1+\delta'} |E| < \frac{\varepsilon}{2}.$$

If $f_k \to f$ as $k \to \infty$ in measure, then for the above δ' there is an index $N > 0$ such that $k \geq N$ implies

$$|\{ x \in E : |f_k(x) - f(x)| > \delta' \}| < \frac{\varepsilon}{2}.$$

Therefore, $f_k \to f$ in measure implies $\rho(f_k, f) \to 0$ as $k \to \infty$. ■

Problem 2.5. Define the *gamma function* $\Gamma\colon \mathbf{R}^+ \to \mathbf{R}$ by

$$\Gamma(y) := \int_0^\infty e^{-u} u^{y-1}\,\mathrm{d}u,$$

and the *beta function* $\beta\colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$ by

$$\beta(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbf{R}^+)$ for all $y \in \mathbf{R}^+$.

(b) Show that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. (a) Fix $y \in \mathbf{R}^+$. Then we must show that $\Gamma(y) < \infty$. First, since $(0,1)$ and $[1,\infty)$ are disjoint measurable subsets of $\mathbf{R}$, by Theorem 5.7 we can split the integral $\Gamma(y)$ into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u}u^{y-1}\,\mathrm{d}u}_{I_1} + \underbrace{\int_1^\infty e^{-u}u^{y-1}\,\mathrm{d}u}_{I_2}. \tag{30}$$

We will show, separately, that I_1 and I_2 are finite.

To see that I_1 is finite, note that

$$\begin{aligned} e^{-u}u^{y-1} &= e^{-u}e^{(y-1)\log u} \\ &= e^{-u+(y-1)\log u} \\ &\le e^{(y-1)\log u} \\ &= u^{y-1} \end{aligned} \tag{31}$$

since $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u}u^{y-1}\,\mathrm{d}u \\ &\le \int_0^1 u^{y-1}\,\mathrm{d}u \\ &= \left[\frac{u^y}{y}\right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \tag{32}$$

To see that I_2 is finite, note that

$$e \tag{33}$$

Intended solution:

(b) ■

Problem 2.6. Let $f \in L^1(\mathbf{R}^n)$ and for $\mathbf{h} \in \mathbf{R}^n$ define $f_{\mathbf{h}} \colon \mathbf{R}^n \to \mathbf{R}$ be $f_{\mathbf{h}}(\mathbf{x}) \coloneqq f(\mathbf{x}-\mathbf{h})$. Prove that
$$\lim_{\mathbf{h}\to\mathbf{0}} \int_{\mathbf{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. Note that by the triangle inequality, we have the following estimate on the integral
$$\int_{\mathbf{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{x} \leq \tag{34}$$
■

Problem 2.7. (a) If $f_k, g_k, f, g \in L^1(\mathbf{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbf{R}^n$, $|f_k| \leq g_k$ and
$$\int_{\mathbf{R}^n} g_k \to \int_{\mathbf{R}^n} g,$$
prove that
$$\int_{\mathbf{R}^n} f_k \to \int_{\mathbf{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L^1(\mathbf{R}^n)$ and $f_k \to f$ a.e. in $\mathbf{R}^n$, then
$$\int_{\mathbf{R}^n} |f_k - f| \to 0 \qquad \text{as} \qquad k \to \infty$$
if and only if
$$\int_{\mathbf{R}^n} |f_k| \to \int_{\mathbf{R}^n} |f| \qquad \text{as} \qquad k \to \infty.$$

Proof. (a) Since $f_k \to f$ and $g_k \to g$ a.e. and $|f_k| \leq g_k$, then by Fatou's theorem,
$$\begin{aligned} \int_{\mathbf{R}^n} (g-f) = \int_{\mathbf{R}^n} \liminf_{k\to\infty} g_k - f_k \leq \liminf_{k\to\infty} \int_{\mathbf{R}^n} g_k - f_k, \\ \int_{\mathbf{R}^n} g + f \int_{\mathbf{R}^n} \liminf_{k\to\infty} g_k + f_k \leq \liminf_{k\to\infty} \int_{\mathbf{R}^n} g_k + f_k. \end{aligned}$$
Since $f_k, g_k, f, g \in L^1(\mathbf{R}^n)$ and $\int_{\mathbf{R}^n} g_k \to \int_{\mathbf{R}^n} g$, then using the similar argument as problem 2, we have
$$\begin{aligned} \int_{\mathbf{R}^n} f \geq \limsup_{k\to\infty} \int_{\mathbf{R}^n} f_k, \\ \int_{\mathbf{R}^n} f \leq \liminf_{k\to\infty} \int_{\mathbf{R}^n} f_k. \end{aligned}$$
Therefore, $\int_{\mathbf{R}^n} f_k \to \int_{\mathbf{R}^n} f$.

(b) $\implies$: This direction is obvious by the inequality
$$\left| \int_{\mathbf{R}^n} |f_k| - |f| \right| \leq \int_{\mathbf{R}^n} ||f_k| - |f|| \leq \int_{\mathbf{R}^n} |f_k - f|.$$

$\Longleftarrow$: Let $g_k := |f_k| + |f|$ and $g := 2|f|$. Since $f_k, f \in L^1(\mathbf{R}^n)$ and $f_k \to f$ a.e., then $g_k, g \in L^1(\mathbf{R}^n)$ and $g_k \to g$ a.e. in $\mathbf{R}^n$. By the assumption, $\int_{\mathbf{R}^n} g_k \to \int_{\mathbf{R}^n} g$.

Let $\tilde{f}_k := |f_k - f|$. Then $\tilde{f}_k \to 0$ a.e. in $\mathbf{R}^n$ and $\tilde{f}_k \le g_k$. Applying part (a) to $\tilde{f}_k$ we have

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} \tilde{f}_k = \lim_{k\to\infty} \int_{\mathbf{R}^n} |f_k - f| = 0.$$

■

Review of concepts

To conclude this review sheet, here are some important lemmas, theorems, and corollaries from the book:

Let f be defined on E, and let $\mathbf{x}_0$ be a limit point of E in E. Then f is said to be *upper semicontinuous at* $\mathbf{x}_0$ if

$$\limsup_{\substack{\mathbf{x}\to\mathbf{x}_0 \\ \mathbf{x}\in E}} f(\mathbf{x}) \le f(\mathbf{x}_0). \tag{35}$$

Note that if $f(\mathbf{x}_0) = \infty$, then f is usc at $\mathbf{x}_0$ automatically; otherwise, the statement that f is usc at $\mathbf{x}_0$ means that given any $M > f(\mathbf{x}_0)$, there exists $\delta > 0$ such that $f(\mathbf{x}) < M$ for all $\mathbf{x} \in E$ that lie in the ball $B_\delta(\mathbf{x}_0)$.

Similarly, f is said to be *lower semicontinuous at* $\mathbf{x}_0$ if $-f$ is usc at $\mathbf{x}_0$.

Theorem (4.14)**.** *A function f is usc relative to E if and only if $\{\, \mathbf{x} \in E : f(\mathbf{x}) > a \,\}$ is relatively closed (equivalently, if $\{\, \mathbf{x} \in E : f(\mathbf{x}) < a \,\}$ is relatively open) for all finite a*

Proof of theorem 4.14. Suppose that f is usc relative to E. Given a, let $\mathbf{x}_0$ be a limit point of $\{\, \mathbf{x} \in E : f(\mathbf{x}) > a \,\}$ in E. Then there exists $\mathbf{x}_k \in E$ such that $\mathbf{x}_k \to \mathbf{x}_0$ and $f(\mathbf{x}_k) \ge a$. Since f is usc at $\mathbf{x}_0$, we have $f(\mathbf{x}_0) \ge \limsup_{k\to\infty} f(\mathbf{x}_k)$. Therefore, $f(\mathbf{x}_0) \ge a$, so $\mathbf{x}_0 \in \{\, \mathbf{x} \in E : f(\mathbf{x}) > a \,\}$. Hence, $\{\, \mathbf{x} \in E : f(\mathbf{x}) > a \,\}$ is relatively closed.

Conversely, let $\mathbf{x}_0$ be a limit point of E that is in E. If f is not usc at $\mathbf{x}_0$, then $f(\mathbf{x}_0) < \infty$, and there exists M and $\{\mathbf{x}_k\}$ such that $f(\mathbf{x}_0) < M$, $\mathbf{x}_k \in E$, $\mathbf{x}_k \to \mathbf{x}_0$, and $f(\mathbf{x}_k) \ge M$. Hence, $\{\, \mathbf{x} \in E : f(\mathbf{x}) > a \,\}$ is not relatively closed since it does not contain all its limit points in E¿ ■

Theorem (4.17, Egorov's theorem)**.** *Suppose that $\{f_k\}$ is a sequence of measurable functions that converge a.e. in a set E of finite measure to a finite limit f. Then given $\varepsilon > 0$ there exits a closed subset F of E such that $|E \setminus F| < \varepsilon$ and $f_k \to f$ uniformly on F.*

A function f defined on a measurable set E has *property* $\mathcal{C}$ on E if given $\varepsilon > 0$, there is a closed set $F \subset E$ such that

(i) $|E \setminus F| < \varepsilon$

(ii) f is continuous relative to F.

Theorem (4.20, Lusin's theorem)**.** *Let f be defined and finite on a measurable set E. Then f is measurable if and only if it has property $\mathcal{C}$ on E.*

We start with a nonnegative function f defined on a measurable subset E of $\mathbf{R}^n$. Let's

$$\begin{aligned}&\Gamma(f,E) \coloneqq \left\{\, (\mathbf{x}, f(\mathbf{x})) \in \mathbf{R}^{n+1} : \mathbf{x} \in E,\ f(\mathbf{x}) < \infty \,\right\},\\ &R(f,E) \coloneqq \left\{\, (\mathbf{x}, y) \in \mathbf{R}^{n+1} : \mathbf{x} \in E,\ 0 \le y \le f(\mathbf{x}) \text{ if } f(\mathbf{x}) < \infty \text{and } 0 \le y < \infty \text{ if } f(\mathbf{x}) = \infty \,\right\}.\end{aligned} \tag{36}$$

$\Gamma(f,E)$ is called the *graph of f over E* and $R(f,E)$ the *region under f over E*.

If $R(f,E)$ is measurable (as a subset of $\mathbf{R}^{n+1}$), its measure $|R(f,E)|_{\mathbf{R}^{n+1}}$ is called the *Lebesgue integral over E*, and we write

$$\int_E f(\mathbf{x})\,\mathrm{d}\mathbf{x} \coloneqq |R(f,E)|_{\mathbf{R}^{n+1}}. \tag{37}$$

This is sometimes written as

$$\int_E f$$

or at times the lengthy notation

$$\int \cdots \int_E f(x_1, \ldots, x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

is convenient.

Theorem (5.1)**.** *Let f be a nonnegative function defined on a measurable set E. Then $\int_E f$ exists if and only if f is measurable.*

Lemma (5.3)**.** *If f is a nonnegative measurable function on E, $0 \le |E| \le \infty$, then $|\Gamma(f,E)| = 0$.*

Theorem (5.5)**.**

(i) *If f and g are measurable and if $0 \le g \le f$ on E, $\int_E g \le \int_E f$. In particular, $\int_E \inf f \le \int_E f$.*

(ii) *If f is nonnegative and measurable on E and if $\int_E f$ is finite, then $f < \infty$ a.e. in E.*

(iii) *Let E_1 and E_2 be measurable and $E_1 \subset E_2$. If f is nonnegative and measurable on E_2, then $\int_{E_1} f \le \int_{E_2} f$.*

Theorem (5.6, the monotone convergence theorem for nonnegative functions)**.** *If $\{f_k\}$ is a sequence of nonnegative functions such that $f_k \nearrow f$ on E, then*

$$\int_E f \to \int_E f.$$

Proof. By Theorem 4.12, f is measurable since it is the limit of a sequence of measurable functions. Since $R(f_k, E) \cup \Gamma(f,E) \nearrow R(f,E)$ and $|\Gamma(f,E)| = 0$, the result follows by Theorem 3.26 on the measure of a monotone convergent sequences of measurable sets. ∎

Theorem (5.9)**.** *Let f be nonnegative on E. If $|E| = 0$, then $\int_E f = 0$.*

Theorem (5.10)**.** *If f and g are nonnegative and measurable on E and if $g \le f$ a.e. in E, then $\int_E g \le \int_E f$.*

In particular, if $f = g$ a.e. in E, then $\int_E f = \int_E g$.

Theorem (5.11)**.** *Let f be nonnegative and measurable on E. Then $\int_E f = 0$ if and only if $f = 0$ a.e. in E.*

Corollary (5.12, Chebyshev's inequality)**.** *Let f be nonnegative and measurable on E. If $a > 0$, then*

$$\frac{1}{a}\int_E f \geq |\{\, \mathbf{x} \in E : f(\mathbf{x}) > a \,\}|.$$

Theorem (5.13)**.** *If f is nonnegative and measurable, and if c is any nonnegative constant, then*

$$\int_E cf = c\int_E f.$$

Theorem (5.14)**.** *If f and g are nonnegative and measurable, then*

$$\int_E (f+g) = \int_E f + \int_E g.$$

Corollary. *Suppose that f and φ are measurable on E, $0 \leq f \leq \varphi$, and $\int_E f$ is finite. Then*

$$\int_E (\varphi - f) = \int_E \varphi - \int_E f.$$

Theorem (5.16)**.** *If f_k, $k = 1, 2, \ldots,$ are nonnegative and measurable, then*

$$\int_E \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_E f_k.$$

Theorem (5.17, Fatou's lemma)**.** *If $\{f_k\}$ is a sequence of nonnegative measurable functions on E, then*

$$\int_E \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int_E f_k.$$

Proof of Fatou's lemma. ■

Theorem (5.19, Lebesgue's dominated convergence theorem for nonnegative functions)**.** *Let $\{f_k\}$ be a sequence of nonnegative measurable functions on E such that $f_k \to f$ a.e. in E. If there exists a measurable function φ such that $f_k \leq \varphi$ a.e. for all k and if $\int_E \varphi$ is finite, then*

$$\int_E f_k \longrightarrow \int_E f.$$

Theorem (5.21)**.** *Let f be measurable in E. Then f is integrable over E if and only if $|f|$ is.*

Theorem (5.22)**.** *If $f \in L^1(E)$, then f is finite a.e. in E.*

Theorem (5.24)**.** *If $\int_E f$ exists and $E = \bigcup_{k\in\mathbf{N}} E_k$ is the countable union of disjoint measurable sets E_k, then*

$$\int_E f = \sum_{k\in\mathbf{N}} \int_{E_k} f.$$

Theorem (5.25)**.** *If $|E| = 0$ or if $f = 0$ a.e. in E, then $\int_E f = 0$.*

Theorem (5.32, monotone convergence theorem)**.** *Let $\{f_k\}$ be a sequence of measurable functions on E:*

(i) *If $f_k \nearrow f$ a.e. on E and there exists $\varphi \in L^1(E)$ such that $f_k \geq \varphi$ a.e. on E for all k, then $\int_E f_k \to \int_E f$.*

(ii) *If $f_k \searrow f$ a.e. on E and there exists $\varphi \in L^1(E)$ such that $f_k \leq \varphi$ a.e. on E for all k, then $\int_E f_k \to \int_E f$.*

Theorem (5.33, uniform convergence theorem)**.** *Let $f_k \in L^1(E)$ for $k \in \mathbf{N}$ and let $\{f_k\}$ converge uniformly to f on E, $|E| < \infty$. Then $f \in L^1(E)$ and $\int_E f_k \to \int_E f$.*

Theorem (5.34, Fatou's lemma)**.** *Let $\{f_k\}$ be a sequence of measurable functions on E. If there exists $\varphi \in L^1(E)$ such that $f_k \geq \varphi$ a.e. on E for all k, then*

$$\int_E \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int_E f_k.$$

Corollary (5.35, reverse Fatou's lemma)**.** *Let $\{f_k\}$ be a sequence of measurable functions on E. If there exits $\varphi \in L^1(E)$ such that $f_k \leq \varphi$ a.e. on E for all k, then*

$$\int_E \limsup_{k\to\infty} f_k \geq \limsup_{k\to\infty} \int_E f_k.$$

Theorem (5.36, Lebesgue's dominated convergenge theorem)**.** *Let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If there exists $\varphi \in L^1(E)$ such that $|f_k| \leq \varphi$ such that $|f_k| \leq \varphi$ a.e. in E for all $k \in \mathbf{N}$, then $\int_E f_k \to \int_E f$.*

Corollary (5.37, bounded convergence theeorem)**.** *Let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If $|E| < \infty$ there is a finite constant M such that $|f_k| \leq M$ a.e. in E, then $\int_E f_k \to \int_E f$.*

Theorem (6.1 Fubini's theorem)**.** *Let $f(\mathbf{x}, \mathbf{y}) \in L^1(I)$, $I := I_1 \times I_2$. Then*

(i) *For almost every $\mathbf{x} \in I_1$, $f(\mathbf{x}, \mathbf{y})$ is measurable and integrable on I_2 as a function of $\mathbf{y}$;*

(ii) *As a function of $\mathbf{x}$, $\int_{I_2} f(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{y}$ is measurable and integrable on I_1, and*

$$\iint_I f(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = \int_{I_1} \left[\int_{I_2} f(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{y} \right] \mathrm{d}\mathbf{x}.$$

Theorem (6.8)**.** *Let $f(\mathbf{x}, \mathbf{y})$ be a measurable function defined on a measurable subset E of $\mathbf{R}^{n+m}$, and let $E_{\mathbf{x}} := \{\, \mathbf{y} : (\mathbf{x}, \mathbf{y}) \in E \,\}$.*

(i) *For almost every $\mathbf{x} \in \mathbf{R}^n$, $f(\mathbf{x}, \mathbf{y})$ is a measurable function of $\mathbf{y}$ on $E_{\mathbf{x}}$.*

(ii) *If $f(\mathbf{x}, \mathbf{y}) \in L^1(E)$, then for almost every $\mathbf{x} \in \mathbf{R}^n$, $f(\mathbf{x}, \mathbf{y})$ is an integrable on $E_{\mathbf{x}}$ with respect to $\mathbf{y}$; moreover $\int_{E_{\mathbf{x}}} f(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{y}$ is an integrable function of $\mathbf{x}$ and*

$$\iint_E f(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = \int_{\mathbf{R}^n} \left[\int_{E_{\mathbf{x}}} f(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{y} \right] \mathrm{d}\mathbf{x}.$$

Theorem (6.10, Tonelli's theorem)**.** *Let $f(\mathbf{x},\mathbf{y})$ be nonnegative and measurable on an interval $I = I_1 \times I_2$ of $\mathbf{R}^{n+m}$. Then, for almost every $\mathbf{x} \in I_1$, $f(\mathbf{x},\mathbf{y})$ is a measurable function of $\mathbf{y}$ on I_2. Moreover, as a function of $\mathbf{x}$, $\int_{I_2} f(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}$ is measurable on I_1, and*

$$\iint_I f(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = \int_{I_1}\left[\int_{I_2} f(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\right]\mathrm{d}\mathbf{x}$$

If f and g are measurable in $\mathbf{R}^n$, their *convolution* $(f*g)(\mathbf{x})$ is defined by

$$(f*g)(\mathbf{x}) := \int_{\mathbf{R}^n} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

provided the integral exists.

Theorem (6.14)**.** *If $f \in L^1(\mathbf{R}^n)$ and $g \in L^1(\mathbf{R}^n)$, then $(f*g)(\mathbf{x})$ exists for almost every $\mathbf{x} \in \mathbf{R}^n$ and is measurable. Moreover, $f* \in L^1(\mathbf{R}^n)$ and*

$$\int_{\mathbf{R}^n} |f*g|\,\mathrm{d}\mathbf{x} \le \left(\int_{\mathbf{R}^n} |f|\,\mathrm{d}\mathbf{x}\right)\left(\int_{\mathbf{R}^n} |g|\,\mathrm{d}\mathbf{x}\right)$$
$$\int_{\mathbf{R}^n} (f*g)(\mathbf{x})\,\mathrm{d}\mathbf{x} = \left(\int_{\mathbf{R}^n} f\,\mathrm{d}\mathbf{x}\right)\left(\int_{\mathbf{R}^n} g\,\mathrm{d}\mathbf{x}\right).$$

Corollary (6.16)**.** *If f and g are nonnegative and measurable on $\mathbf{R}^n$, then $f*g$ is measurable on $\mathbf{R}^n$ and*

$$\int_{\mathbf{R}^n} (f*g)\,\mathrm{d}\mathbf{x} = \left(\int_{\mathbf{R}^n} f\,\mathrm{d}\mathbf{x}\right)\left(\int_{\mathbf{R}^n} g\,\mathrm{d}\mathbf{x}\right).$$

Theorem (6.17, Marcinkiewicz)**.** *Let F be a closed subset of a bounded open interval (a,b), and let $\delta(x) := \delta(x,F)$ be the corresponding distance function. Then, given $\lambda > 0$, the integral*

$$M_\lambda(x) := \int_a^b \frac{\delta(y)^\lambda}{|x-y|^{1+\lambda}}\,\mathrm{d}y$$

is finite a.e. in F. Moreover, $M_\lambda \in L^1(F)$ and

$$\int_F M_\lambda\,\mathrm{d}x \le 2\lambda^{-1}|G|,$$

where $G := (a,b) \setminus F$.

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Problem 2.8. Assume that $f \in L^1(\mathbf{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbf{R}^n \setminus B} |f| < \varepsilon.$$

Proof. Recall that $f \in L^1(\mathbf{R}^n)$ if and only if $|f| \in L^1(\mathbf{R}^n)$. Let $B_k \coloneqq B(\mathbf{0}, k)$ for $k \in \mathbf{N}$ and χ_{B_k} be the indicator function associated with B_k. Then, the sequence of maps $\{|f_k|\}$ defined $f_k \coloneqq f\chi_{B_k}$ converge pointwise to $|f_k|$. Since $|f| \in L^1(\mathbf{R}^n)$, by the monotone convergence theorem, we have

$$\int_{\mathbf{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbf{R}^n} |f|. \tag{38}$$

But this means, exactly, that for every $\varepsilon > 0$ there exists sufficiently large $N \in \mathbf{N}$ such that

$$\begin{aligned} \varepsilon &> \left| \int_{\mathbf{R}^n} |f_k| - \int_{\mathbf{R}^n} |f| \right| \\ &= - \int_{\mathbf{R}^n} |f_k| + \int_{\mathbf{R}^n} |f| \\ &= - \int_{\mathbf{R}^n} |f| + \int_{\mathbf{R}^n} |f| \\ &= - \int_{B_k} |f| + \int_{\mathbf{R}^n} |f| \\ &= \int_{\mathbf{R}^n \setminus B_k} |f| \end{aligned} \tag{39}$$

as desired. ∎

Problem 2.9. Let $f \in L^1(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. First, since the E_j's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \tag{40}$$

Let χ_{E_j} be the characteristic function of the subset E_j of E and define $f_j \coloneqq f\chi_{E_j}$ for $j \in \mathbf{N}$. Note that, since both f and χ_{E_j} are measurable on E, f_j is measurable on E and $\sum_{j=1}^{\infty} f_j = f$. Moreover, since $E_j \subset E$, by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \setminus E_j} f = \int_E f_j + \int_{E \setminus E_j} f. \tag{41}$$

Hence, because the E_j's are disjoint $(E \smallsetminus E_k) \smallsetminus E_\ell = (E \smallsetminus E_\ell) \smallsetminus E_k$ so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \tag{42}$$

as desired. ■

Problem 2.10. Let $\{f_k\}$ be a family in $L^1(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbf{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Proof. Let $\varepsilon > 0$ be given. Then, by the hypothesis, there exists $\delta > 0$ such that such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon \tag{43}$$

for all $k \in \mathbf{N}$. By Egorov's theorem, there exists a closed subset F of E such that $|E \smallsetminus F| < \delta$ and $f_k \to f$ uniformly on F. Then, by the uniform convergence theorem,

$$\int_F f_k \longrightarrow \int_F f \tag{44}$$

as $k \to \infty$. But by hypothesis, we have

$$\int_{E \smallsetminus F} |f_k| < \varepsilon. \tag{45}$$

Letting $\varepsilon \to 0$, we achieved the desired convergence. ■

Problem 2.11. Let $I := [0, 1]$, $f \in L^1(I)$, and define $g(x) := \int_x^1 t^{-1} f(t)\,\mathrm{d}t$ for $x \in I$. Prove that $g \in L^1(I)$ and

$$\int_I g = \int_I f.$$

Proof. By Lusin's theorem, there exists a closed subset F of I with $|I \smallsetminus F| < \varepsilon$ such that the restriction of f to $F := I \smallsetminus E$ is continuous. Now, since F is closed in I and I is compact, it follows that I is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials $\{p_k\}$ such that $p_k \to f$ uniformly on F. Then, by the uniform convergence theorem, we have

$$\int_F p_k \longrightarrow \int_F f \tag{46}$$

so

$$
\begin{aligned}
\int_F \left[\int_x^1 t^{-1} p_k(t)\,\mathrm{d}t \right] \mathrm{d}x &= \int_F \left[\int_x^1 a t^{-1} + q_k(t)\,\mathrm{d}t \right] \mathrm{d}x \\
&= \int_F q_k'(x) - a\log(x)\,\mathrm{d}x \\
&< \infty
\end{aligned} \tag{47}
$$

for all k and converges uniformly to g so $g \in L^1(I)$. I don't know how to show that in fact $\int_I g = \int_I f$. Perhaps you show that the places where they differ is a set of measure zero. ■