

# MA553: Qual Preparation

Carlos Salinas

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# Chapter 1

## MA 553 Spring 2016

This is material from the course MA 533 as it was taught in the spring of 2016.

### 1.1 Homework

Most of the homework is Ulrich original (or as original as elementary exercises in abstract algebra can be). However, an excellent resource and one that I will often quote on these solutions is [3]. Other resources include [1] and (to a lesser extent) [2]. I may also cite Milne's *Group Theory*, *Field Theory*, and *Commutative Algebra: A Primer* notes, respectively, [4], [5], and (no reference for the last).

Throughout these notes

$\mathbb{R}$	is the set of real numbers
$\mathbb{C}$	is the set of complex numbers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{F}_q$	is the finite field of order $q = p^n$ for some prime $p$
$\mathbb{Z}$	is the set of the integers
$\mathbb{N}$	is the set of the natural numbers $1, 2, \dots$
$k$	is used to denote the base field with characteristic $\text{char } k$
$K, E, L$	is used to denote field extensions over the base field $k$
$C_n$	is the cyclic group of order $n$ not necessarily equal (but isomorphic) to $\mathbb{Z}/p\mathbb{Z}$
$S_n$	is the symmetric group on $\{1, \dots, n\}$
$A_n$	is the alternating group on $\{1, \dots, n\}$
$D_n$	is the dihedral group of order $n$
$A \setminus B$	is the set difference of $A$ and $B$ , that is, the complement of $A \cap B$ in $A$
$X \simeq Y$	means $X$ and $Y$ are isomorphic as groups, rings, $R$ -modules, or fields

**1.1.1 Homework 1**

**Problem 1.** Let  $G$  be a group,  $a \in G$  an element of finite order  $m$ , and  $n$  a positive integer. Prove that

$$|a^n| = \frac{m}{\gcd(m, n)}.$$

*Proof.* Without loss of generality, we may assume  $n < m$ ; otherwise, by the fundamental theorem of arithmetic, there exist  $q$  and  $r$  with  $r < m$  such that  $n = qm + r$  so  $a^n = a^{qm+r} = a^{qm}a^r = a^r$ . ■

**Problem 2.** Let  $G$  be a group, and let  $a, b$  be elements of finite order  $m, n$  respectively. Show that if  $ba = ab$  and  $\langle a \rangle \cap \langle b \rangle = \{e\}$ , then  $|ab| = \text{lcm}(m, n)$ .

*Proof.* ■

**Problem 3.** Let  $G$  be a group and  $H, K$  normal subgroups with  $H \cap K = \{e\}$ . Show that

- (a)  $hk = kh$  for every  $h \in H, k \in K$ .
- (b)  $HK$  is a subgroup of  $G$  with  $HK \simeq H \times K$ .

*Proof.* ■

**Problem 4.** Show that  $A_4$  has no subgroup of order 6 (although  $6 \mid 12 = |A_4|$ ).

*Proof.* ■

**1.1.2 Homework 2**

**Problem 1.** Let  $G$  be the group of order  $2^3 \cdot 3$ ,  $n \geq 2$ . Show that  $G$  has a normal 2-subgroup  $\neq \{e\}$ .

*Proof.* ■

**Problem 2.** Let  $G$  be a group of order  $p^2q$ ,  $p$  and  $q$  primes. Show that the Sylow  $p$ -Sylow subgroup or the  $q$ -Sylow subgroup of  $G$  is normal in  $G$ .

*Proof.* ■

**Problem 3.** Let  $G$  be a subgroup of order  $pqr$ ,  $p < q < r$  primes. Show that the  $r$ -Sylow subgroup of  $G$  is normal in  $G$ .

*Proof.* ■

**Problem 4.** Let  $G$  be a group of order  $n$  and let  $\varphi: G \rightarrow S_n$  be given by the action of  $G$  on  $G$  via translation.

- (a) For  $a \in G$  determine the number and the lengths of the disjoint cycles of the permutation  $\varphi(a)$ .
- (b) Show that  $\varphi(G) \not\subset A_n$  if and only if  $n$  is even and  $G$  has a cyclic 2-Sylow subgroup.
- (c) If  $n = 2m$ ,  $m$  odd, show that  $G$  has a subgroup of index 2.

*Proof.* ■

**Problem 5.** Show that the only simple groups  $\neq \{e\}$  of order  $< 60$  are the groups of prime order.

*Proof.* ■

**1.1.3 Homework 3**

**Problem 1.** Let  $G$  be a finite group,  $p$  a prime number,  $N$  the intersection of all  $p$ -Sylow subgroups of  $G$ . Show that  $N$  is a normal  $p$ -subgroup of  $G$  and that every normal  $p$ -subgroup of  $G$  is contained in  $N$ .

*Proof.* ■

**Problem 2.** Let  $G$  be a group of order 231 and let  $H$  be an 11-Sylow subgroup of  $G$ . Show that  $H \subset Z(G)$ .

*Proof.* ■

**Problem 3.** Let  $G = \{e, a_1, a_2, a_3\}$  be a non-cyclic group of order 4 and define  $\varphi: S_3 \rightarrow \text{Aut}(G)$  by  $\varphi(\sigma)(e) = e$  and  $\varphi(\sigma)(a_i) = a_{\sigma(i)}$ . Show that  $\varphi$  is well-defined and an isomorphism of groups.

*Proof.* ■

**Problem 4.** Determine all groups of order 18.

*Proof.* ■

**1.1.4 Homework 4**

**Problem 1.** Let  $p$  be a prime and let  $G$  be a nonAbelian group of order  $p^3$ . Show that  $G' = Z(G)$ .

*Proof.* ■

**Problem 2.** Let  $p$  be an odd prime and let  $G$  be a nonAbelian group of order  $p^3$  having an element of order  $p^2$ . Show that there exists an element  $b \notin \langle a \rangle$  of order  $p$ .

*Proof.* ■

**Problem 3.** Let  $p$  be an odd prime. Determine all groups of order  $p^3$ .

*Proof.* ■

**Problem 4.** Show that  $(S_n)' = A_n$ .

*Proof.* ■

**Problem 5.** Show that every group of order  $< 60$  is solvable.

*Proof.* ■

**Problem 6.** Show that every group of order 60 that is simple (or not solvable) is isomorphic to  $A_5$ .

*Proof.* ■

**1.1.5 Homework 5**

**Problem 1.** Find all composition series and the composition factors of  $D_6$ .

*Proof.* ■

**Problem 2.** Let  $T$  be the subgroup of  $\text{GL}(n, \mathbb{R})$  consisting of all upper triangular invertible matrices. Show that  $T$  is solvable.

*Proof.* ■

**Problem 3.** Let  $p \in \mathbb{Z}$  be a prime number. Show:

- (a)  $(p-1)! \equiv -1 \pmod{p}$ .
- (b) If  $p \equiv 1 \pmod{4}$  then  $x^2 \equiv -1 \pmod{p}$  for some  $x \in \mathbb{Z}$ .

*Proof.* ■

**Problem 4.** (a) Show that the following are equivalent for an odd prime number  $p \in \mathbb{Z}$ :

- (i)  $p \equiv 1 \pmod{4}$ .
- (ii)  $p = a^2 + b^2$  for some  $a, b$  in  $\mathbb{Z}$ .
- (iii)  $p$  is not prime in  $\mathbb{Z}[i]$ .

- (b) Determine all prime ideals of  $\mathbb{Z}[i]$ .

*Proof.* ■



**1.1.6 Homework 6**

**Problem 1.** Let  $R$  be a domain. Show that  $R$  is a UFD if and only if every nonzero nonunit in  $R$  is a product of irreducible elements and the intersection of any two principal ideals is again principal.

*Proof.* ■

**Problem 2.** Let  $R$  be a PID and  $\mathfrak{p}$  a prime ideal of  $R[X]$ . Show that  $\mathfrak{p}$  is principal or  $\mathfrak{p} = (a, f)$  for some  $a \in R$  and some monic polynomial  $f \in R[X]$ .

*Proof.* ■

**Problem 3.** Let  $k$  be a field and  $n \geq 1$ . Show that  $Z^n + Y^3 + X^2 \in k(X, Y)[Z]$  is irreducible.

*Proof.* ■

**Problem 4.** Let  $k$  be a field of characteristic zero and  $n \geq 1, m \geq 2$ . Show that  $X_1^n + \cdots + X_m^n - 1 \in k[X_1, \dots, X_m]$  is irreducible.

*Proof.* ■

**Problem 5.** Show that  $X^{3^n} + 2 \in \mathbb{Q}(i)[X]$  is irreducible.

*Proof.* ■

## 1.1.7 Homework 7

**Problem 1.** Let  $k \subset K$  and  $k \subset L$  be finite field extensions contained in some field. Show that:

- (a)  $[KL : L] \leq [K : k]$ .
- (b)  $[KL : k] \leq [K : k][L : k]$ .
- (c)  $K \cap L = k$  if equality holds in (b).

*Proof.* ■

**Problem 2.** Let  $k$  be a field of characteristic  $\neq 2$  and  $a, b$  elements of  $k$  so that  $a, b, ab$  are not squares in  $k$ . Show that  $[k(\sqrt{a}, \sqrt{b}) : k] = 4$ .

*Proof.* ■

**Problem 3.** Let  $R$  be a UFD, but not a field, and write  $K := \text{Quot}(R)$ . Show that  $[\bar{K} : k] = \infty$ .

*Proof.* ■

**Problem 4.** Let  $k \in K$  be an algebraic field extension. Show that every  $k$ -homomorphism  $\delta : K \rightarrow K$  is an isomorphism.

*Proof.* ■

**Problem 5.** Let  $K$  be the splitting field of  $X^6 - 4$  over  $\mathbb{Q}$ . Determine  $K$  and  $[K : \mathbb{Q}]$ .

*Proof.* ■

## 1.1.8 Homework 8

**Problem 1.** Let  $k$  be a field,  $f \in k[X]$  is a polynomial of degree  $n \geq 1$ , and  $K$  the splitting field of  $f$  over  $k$ . Show that  $[K : k] \mid n!$ .

*Proof.* ■

**Problem 2.** Let  $k$  be a field and  $n \geq 0$ . Define a map  $\Delta_n : k[X] \rightarrow k[X]$  by  $\Delta_n(\sum a_i X^i) := \sum a_i \binom{i}{n} X^{i-n}$ . Show:

- (a)  $\Delta_n$  is  $k$ -linear, and for  $f, g$  in  $k[X]$ ,  $\Delta_n(fg) = \sum_{j=0}^n \Delta_j(f) \Delta_{n-j}(g)$ ;
- (b)  $f^{(n)} = n! \Delta_n(f)$ ;
- (c)  $f(X+a) = \sum \Delta_n(f)(a) X^n$ , where  $a \in k$ ;
- (d)  $a \in k$  is a root of  $f$  of multiplicity  $n$  if and only if  $\Delta_i(f)(a) = 0$  for  $0 \leq i \leq n-1$  and  $\Delta_n(f)(a) \neq 0$ .

*Proof.* ■

**Problem 3.** Let  $k \subset K$  be a finite field extension. Show that  $k$  is perfect if and only if  $K$  is perfect.

*Proof.* ■

**Problem 4.** Let  $K$  be the splitting field of  $X^p - X - 1$  over  $k := \mathbb{Z}/p\mathbb{Z}$ . Show that  $k \subset K$  is normal, separable, of degree  $p$ .

*Proof.* ■

**Problem 5.** Let  $k$  be a field of characteristic  $p > 0$ , and  $k(X, Y)$  the field of rational functions in two variables.

- (a) Show that  $[k(X, Y) : k(X^p, Y^p)] = p^2$ .
- (b) Show that the extension  $k(X^p, Y^p) \subset k(X, Y)$  is not simple.
- (c) Find infinitely many distinct fields  $L$  with  $k(X^p, Y^p) \subset L \subset k(X, Y)$ .

*Proof.* ■

### 1.1.9 Homework 9

**Problem 1.** Let  $k \subset K$  be a finite extension of fields of characteristic  $p > 0$ . Show that if  $p \nmid [K : k]$ , then  $k \subset K$  is separable.

*Proof.* ■

**Problem 2.** Let  $k \subset K$  be an algebraic extension of fields of characteristic  $p > 0$ , let  $L$  be an algebraically closed field containing  $K$ , and let  $\delta: k \rightarrow L$  be an embedding. Show that  $k \subset K$  is purely inseparable if and only if there exists exactly one embedding  $\tau: K \rightarrow L$  extending  $\delta$ .

*Proof.* ■

**Problem 3.** Let  $k \subset K = k(\alpha, \beta)$  be an algebraic extension of fields of characteristic  $p > 0$ , where  $\alpha$  is separable over  $k$  and  $\beta$  is purely inseparable over  $k$ . Show that  $K = k(\alpha + \beta)$ .

*Proof.* ■

**Problem 4.** Let  $f(X) \in \mathbb{F}_q[X]$  be irreducible. Show that  $f(X) \mid X^{q^n} - X$  if and only if  $\deg f(X) \mid n$ .

*Proof.* ■

**Problem 5.** Show that  $\text{Aut}_{\mathbb{F}_q}(\bar{\mathbb{F}}_q)$  is an infinite Abelian group which is torsionfree (i.e.,  $\delta^n = \text{id}$  implies  $\delta = \text{id}$  or  $n = 0$ ).

*Proof.* ■

**Problem 6.** Show that in a finite field, every element can be written as a sum of two perfect squares.

*Proof.* ■

## 1.1.10 Homework 10

**Problem 1.** Let  $k \subset K := k(\alpha)$  be a simple field extension, let  $G := \{\delta_1, \dots, \delta_n\}$  be a finite subgroup of  $\text{Aut}_k(K)$ , and write  $f(X) := \prod_{i=1}^n (X - \delta_i(\alpha)) = \sum_{i=0}^n a_i X^i$ . Show that  $f(X)$  is the minimal polynomial of  $\alpha$  over  $K^G$  and that  $K^G = k(a_0, \dots, a_{n-1})$ .

*Proof.* ■

**Problem 2.** Let  $k$  be a field,  $k(X)$  the field of rational functions, and  $u \in k(X) \setminus k$ . Write  $u := f/g$  with  $f$  and  $g$  relatively prime in  $k[X]$ . Show that  $[k(X) : k(u)] = \max\{\deg f, \deg g\}$ .

*Proof.* ■

**Problem 3.** Let  $k$  be a field and  $K := k(X)$  the field of rational functions. Show that for every  $\delta \in \text{Aut}_k(K)$ ,  $\delta(X) := (aX + b)/(cX + d)$  for some  $a, b, c, d$  in  $k$  with  $ad - bc \neq 0$ , and that conversely, every such rational functions uniquely determines an automorphism  $\delta \in \text{Aut}_k(K)$ .

*Proof.* ■

**Problem 4.** With the notion of the previous problem let  $\delta \in \text{Aut}_k(K)$  and  $G := \langle \delta \rangle$ .

- (a) Assume  $\delta(X) = 1/(1 - X)$ . Show that  $|G| = 3$  and determine  $K^G$ .
- (b) Assume  $\text{char } k = 0$  and  $\delta(X) = X + 1$ . Show that  $G$  is infinite and determine  $K^G$ .

*Proof.* ■

**Problem 5.** Let  $k \subset K$  be a finite Galois extension with  $G := \text{Gal}(K/k)$ , let  $L$  be a subfield of  $K$  containing  $k$  with  $H := \text{Gal}(K/L)$ , and let  $L'$  be the compositum in  $K$  of the fields  $\delta(L)$ ,  $\delta \in G$ . Show that:

- (a)  $L'$  is the unique smallest subfield of  $K$  that contains  $L$  and is Galois over  $k$ .
- (b)  $\text{Gal}(K/L') = \bigcap_{\delta \in G} \delta H \delta^{-1}$ .

*Proof.* ■

## 1.1.11 Homework 11

**Problem 1.** Show that every algebraic extension of a finite field is Galois and Abelian.

*Proof.* ■

**Problem 2.** Let  $k$  be a field of characteristic  $\neq 2$  and  $f(X) \in k[X]$  a cubic whose discriminant is a square. Show that  $f$  is either irreducible or a product of linear polynomials in  $k[X]$ .

*Proof.* ■

**Problem 3.** Let  $k$  be a field of characteristic  $\neq 2$ , and let  $f(X) := X^4 + aX^2 + b \in k[X]$  be irreducible with Galois group  $G$ . Show:

- (i) If  $b$  is a square in  $k$ , then  $G = H$ .
- (ii) If  $b$  is not a square in  $k$ , but  $b(a^2 - 4b)$  is, then  $G \simeq C_4$ .
- (iii) If neither  $b$  nor  $b(a^2 - 4b)$  is a square in  $k$ , then  $G \simeq D_4$ .

*Proof.* ■

**Problem 4.** Determine the Galois group of:

- (a)  $X^4 - 5$  over  $\mathbb{Q}$ , over  $\mathbb{Q}(\sqrt{5})$ , over  $\mathbb{Q}(\sqrt{-5})$ ;
- (b)  $X^3 - 10$  over  $\mathbb{Q}$ ;
- (c)  $X^4 - 4X^2 + 5$  over  $\mathbb{Q}$ ;
- (d)  $X^4 + 3X^3 + 3X - 2$  over  $\mathbb{Q}$ ;
- (e)  $X^4 + 2X^2 + X + 3$  over  $\mathbb{Q}$ .

*Proof.* ■

**Problem 5.** Let  $K$  be the splitting field of  $X^4 - X^2 - 1$  over  $\mathbb{Q}$ . Determine all intermediate fields  $L$ ,  $\mathbb{Q} \subset L \subset K$ . Which of these are Galois over  $\mathbb{Q}$ ?

*Proof.* ■

**1.1.12 Homework 12**

**Problem 1.** Prove that the resolvent cubic  $X^4 + aX^2 + bX + c$  is given by  $X^3 - aX^2 - 4cX + 4ac - b^2$ .

*Proof.* ■

**Problem 2.** Show that the general polynomial  $g(Y) := Y^n + u_1Y^{n-1} + \cdots + u_n$  is irreducible in  $k(u_1, \dots, u_n)[Y]$ .

*Proof.* ■

**Problem 3.** Let  $k$  be a field.

- (a) compute the discriminant  $Y^3 - Y \in k[Y]$  and  $Y^3 - 1 \in k[Y]$ .
- (b) Show that the discriminant of the polynomial  $(Y - X_1)(Y - X_2)(Y - X_3)$  over  $k(X_1, X_2, X_3)$  is of the form

$$\lambda_1 s_1^4 + \lambda_2 s_1^4 s_2 + \lambda_3 s_1^3 s_3 + \lambda_4 s_1^2 s_2^2 + \lambda_5 s_1 s_2 s_3 + \lambda_6 s_2^3 + \lambda_7 s_3^2$$

with  $\lambda_i \in k$ .

- (c) From (b) and (a) conclude that the discriminant  $Y^3 + aY + b \in k[Y]$  is  $-4a^3 - 27b^2$ .

*Proof.* ■

**Problem 4.** Let  $\Phi_n(X)$  be the  $n$ th cyclotomic polynomial over  $\mathbb{Q}$ .

- (a) Let  $n = p_1^{r_1} \cdots p_s^{r_s}$  with  $p_i$  distinct prime numbers and  $r_i > 0$ . Show that  $\Phi(X) = \Phi_{p_1 \cdots p_s}(X^{p_1^{r_1-1} \cdots p_s^{r_s-1}})$ .
- (b) For a prime number  $p$  with  $p \nmid n$  show that  $\Phi_{pn}(X) = \Phi_n(X^p)/\Phi_n(X)$ .

*Proof.* ■

**1.1.13 Homework 13**

**Problem 1.** Let  $n \geq 3$  and  $\rho$  a primitive  $n$ th root of unity over  $\mathbb{Q}$ . Show that  $[\mathbb{Q}(\rho + \rho^{-1}) : \mathbb{Q}] = \varphi(n)/2$ .

*Proof.* ■

**Problem 2.** Let  $\rho$  be a primitive  $n$ th root of unity over  $\mathbb{Q}$ . Determine all  $n$  so that  $\mathbb{Q} \subset \mathbb{Q}(\rho)$  is cyclic.

*Proof.* ■

**Problem 3.** Let  $k \subset K$  be an extension of finite fields. Show that  $N_k^K$  and  $\text{Tr}_k^K$  are surjective maps from  $K$  to  $k$ .

*Proof.* ■

**Problem 4.** Let  $f(X) \in k[X]$  be a separable polynomial of degree  $n \geq 3$  with Galois group isomorphic to  $S_n$ , and let  $\alpha \in \bar{k}$  be a root of  $f(X)$ .

- (a) Show that  $f(X)$  is irreducible.
- (b) Show that  $\text{Aut}_k(k(\alpha)) = \{\text{id}\}$ .
- (c) Show that  $\alpha^n \notin k$  if  $n \geq 4$ .

*Proof.* ■

**Problem 5.** Let  $k \subset K$  be a Galois extension.

- (a) For  $k \subset L \subset K$  show that  $\text{Gal}(K/L)$  is solvable if  $\text{Gal}(K/k)$  is solvable.
- (b) For  $k \subset L \subset K$  with  $k \subset L$  normal show that  $\text{Gal}(L/k)$  and  $\text{Gal}(K/L)$  are solvable if and only if  $\text{Gal}(K/k)$  is solvable.
- (c) For  $k \subset L$  with  $K$  and  $L$  in a common field show that  $\text{Gal}(KL/L)$  is solvable if  $\text{Gal}(K/k)$  is solvable.

*Proof.* ■



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