# MA 572: Homework 3

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February 17, 2016

## Problem 3.1 (Hatcher §2.1, Ex. 17)

(a) Compute the homology groups $H_n(X, A)$ when $X$ is $S^2$ or $S^1 \times S^1$ and $A$ is a finite set of points in $X$.

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for $X$ a closed orientable surface of genus two with $A$ and $B$ the circles shown. [What are $X/A$ and $X/B$?]

*Proof.* **(a)** As a consequence of 2.16, we have a long exact sequence on relative homology

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots . \tag{1}$$

Now, specifying $X$ to be $S^2$, we know, by 2.8, 2.6 and 2.14, that $H_n(A) \cong 0$ for all $n > 0$ and $H_0(A) \cong \bigoplus_{|A|} \mathbf{Z}$, and $H_n(S^2) \cong 0$ for $n \neq 2, 0$ and $H_n(S^2) \cong \mathbf{Z}$ otherwise. Hence, the long exact sequence (1) turns into

$$\begin{tikzcd} \cdots \arrow[r] & H_2(A) \arrow[r] & H_2(S^2) \arrow[r] & H_2(S^2, A) \arrow[dll, out=0, in=180] \\ & H_1(A) \arrow[r] & H_1(S^2) \arrow[r] & H_1(S^2, A) \arrow[dll, out=0, in=180] \\ & H_0(A) \arrow[r] & H_0(S^2) \arrow[r] & H_0(S^2, A) \arrow[r] & 0, \end{tikzcd} \tag{2}$$

which, filling in our computed values for $H_n(A)$ and $H_n(S^2)$, further becomes

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & \mathbf{Z} \arrow[r] & H_2(S^2, A) \arrow[dll, out=0, in=180] \\ & 0 \arrow[r] & 0 \arrow[r] & H_1(S^2, A) \arrow[dll, out=0, in=180] \\ & \bigoplus_{|A|} \mathbf{Z} \arrow[r] & \mathbf{Z} \arrow[r] & H_0(S^2, A) \arrow[r] & 0, \end{tikzcd} \tag{3}$$

with all other $n > 2$ homology groups for $A$ and $S^2$ being 0. That last remark immediately tells us that, by exactness, $H_n(S^2, A) = 0$ for all $n > 2$. Starting from the bottom of (3), exactness at $H_2(S^2, A)$ tells us that $H_2(S^2, A) \cong \mathbf{Z}$ since the zero maps to the left and right

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{Z} \longrightarrow H_2(S^2, A) \longrightarrow 0 \longrightarrow \cdots$$

tells us that the map $\mathbf{Z} \to H_2(S^2, A)$ is an isomorphism. If we look at the reduced homology, the bottom row of (3) becomes

$$\cdots \longrightarrow 0 \longrightarrow \widetilde{H}_1(S^2, A) \longrightarrow \bigoplus_{|A|-1} \mathbf{Z} \longrightarrow 0 \longrightarrow \widetilde{H}_0(S^2, A) \longrightarrow 0.$$

By exactness at $\widetilde{H}_1(S^2, A)$, we have $H_1(S^2, A) \cong \widetilde{H}_1(S^2, A) \cong \bigoplus_{|A|-1} \mathbf{Z}$ and, last but not least, exactness at $\widetilde{H}_0(S^2, A) \cong 0$ gives us $H_0(S^2, A) \cong \mathbf{Z}$. In summary, we have

$$H_n(S^2, A) = \begin{cases} \mathbf{Z} & \text{if } n = 0, 2 \\ \bigoplus_{|A|-1} \mathbf{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

**(b)** From 2.27 we know that $H_n^{\Delta}(S^1 \times S^1) \cong H_n(S^1 \times S^1)$ so from 2.3, we know that the homology of the torus $S^1 \times S^1$ is

$$H_n(S^1 \times S^1) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{if } n = 1 \\ \mathbf{Z} & \text{if } n = 2, 0 \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

Skipping directly, to our calculation, we have the long exact sequence

$$\begin{array}{l} \cdots \longrightarrow 0 \longrightarrow \mathbf{Z} \longrightarrow H_2(S^1 \times S^1, A) \longrightarrow \\ \longrightarrow 0 \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \longrightarrow H_1(S^1 \times S^1, A) \longrightarrow \\ \longrightarrow \bigoplus_{|A|} \mathbf{Z} \longrightarrow \mathbf{Z} \longrightarrow H_0(S^2, A) \longrightarrow 0. \end{array} \tag{6}$$

It is clear from exactness that $H_2(S^1 \times S^1, A) \cong \mathbf{Z}$ and $H_0(S^1 \times S^1, A) \cong \mathbf{Z}$. What is not clear is what $H_1(S^1 \times S^1, A)$ is. Exactness at $\mathbf{Z} \oplus \mathbf{Z}$ tells us that $\mathbf{Z} \oplus \mathbf{Z} \hookrightarrow H_1(S^1 \times S^1, A)$ and, looking at the reduced homology, exactness at $\bigoplus_{|A|-1} \mathbf{Z}$ tells us that $H_1(S^1 \times S^1, A) \twoheadrightarrow \bigoplus_{|A|-1} \mathbf{Z}$. Thus, we have $\bigoplus_{|A|-1} \mathbf{Z} \cong H_1(S^1 \times S^1, A)/\mathbf{Z} \oplus \mathbf{Z}$ from which we can deduce that $H_1(S^1 \times S^1, A) \cong \bigoplus_{|A|+1} \mathbf{Z}$.[1] In summary, the relative homology of $S^1 \times S^1$ with respect to $A$ is

$$H_n(S^1 \times S^1, A) = \begin{cases} \mathbf{Z} & \text{if } n = 2, 0 \\ \bigoplus_{|A|+1} \mathbf{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

■

[1] I know this is not strictly correct, but the approach I took to solve the problem required me to construct an inverse map $H_1(S^1 \times S^1, A) \leftarrow \bigoplus_{|A|-1} \mathbf{Z}$, but this is difficult.

## Problem 3.2 (Hatcher §2.2, Ex. 1)

Prove the Brouwer fixed point theorem for maps $f\colon D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of $S^n$ to the southern hemisphere via $f$. [This was Brouwer's original proof.]

*Proof.* Seeking a contradiction, suppose $f\colon D^n \to D^n$ has no fixed point. Let $N$ and $S$ denote, respectively, the northern and southern hemisphere meeting at the equator of $S^n$. Now, since the disk $D^n \approx S$, we may as well identify $D^n$ with $S$ and consider the map $f\colon D^n \to D^n$ as the map $S \to S$ by composing with the homeomorphism. Define a map $g\colon S^n \to S^n$ by

$$g \coloneqq \begin{cases} r & \text{on } N \\ \mathrm{id} & \text{on } S \end{cases}. \tag{8}$$

Note that the map $g$ is continuous by the pasting lemma since $g\restriction_S = \mathrm{id}$ and $g\restriction_N = r$ are continuous and $r$ fixes points at the equator $N \cap S$. Now, consider the map $F\colon S^n \to S^n$ given by the composition $\iota \circ f \circ g$ where $\iota\colon S \hookrightarrow S^n$ is the inclusion $S \subset S^n$. This map has no fixed points since $f$ has no fixed point hence, by property (g) of the degree, $\deg F = (-1)^{n+1}$. But $F$ is not onto, therefore $\deg F = 0$. This is a contradiction. ■

## Problem 3.3 (Hatcher §2.2, Ex. 6)

Show that every map $S^n \to S^n$ can be homotoped to have a fixed point if $n > 0$.

*Proof.* The result follows from 4.25 since a map $f\colon S^n \to S^n$ without any fixed points is homotopic to the antipodal map. Since the antipodal map has degree $-1$ or $1$ depending on $n$, it follows that the antipodal map is homotopic to either the identity map or a reflection map, both of which have fixed points. ∎

## Problem 3.4

Let $\mathcal{U}$ be an open cover of $X$. Prove that the inclusion of $C_*^{\mathcal{U}}(C)$ into $C_*(X)$ is a chain homotopy equivalence.

*Proof.* This is proposition 2.21 in the book. I will summarize the proof in four steps here.

**(1)** (Barycentric subdivision) Given a simplex $[v_0, ..., v_n]$ its barycenter is the point $b = \sum_i t_i v_i$ whose barycentric coordinates $t_i$ are all equal, i.e., $t_i = 1/(n+1)$ for all $i$. The barycentric subdivision of $[v_0, ..., v_n]$ is the decomposition of $[v_0, ..., v_n]$ into the $n$-simplices $[b, w_0, ..., w_{n-1}]$, where

**(2)**

**(3)**

**(4)**

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