MA 544: Homework 9

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March 28, 2016

Problem 9.1 (Wheeden & Zygmund §6, Ex. 1)

(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}$, $\{ y : (x, y) \in E \}$ has $\mathbb{R}$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}$, $\{ x : (x, y) \in E \}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x, y)$ is finite for almost every y. Show that for almost $y \in \mathbb{R}$, $f(x, y)$ is finite for almost every x.

Proof. (a) That E has measure zero is a consequence of Fubini's theorem. Set $E_x := \{ y : (x, y) \in E \}$ and $E_y := \{ x : (x, y) \in E \}$ then, by Theorem 6.8, we have

$$|E| = \iint_{\mathbb{R}^2} \chi_E \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}} \left[\int_{E_x} 1 \, \mathrm{d}y \right] \mathrm{d}x = \int_{\mathbb{R}} \left[\int_{E_y} 1 \, \mathrm{d}x \right] \mathrm{d}y = 0. \tag{9.1}$$

Hence, E has measure zero. Moreover, we see that $\int_{\mathbb{R}} \left[\int_{E_y} 1 \, \mathrm{d}x \right] \mathrm{d}y = 0$ which means that for a.e. $y \in \mathbb{R}$, E_y has $\mathbb{R}$-measure zero.

(b) Let E be the set of all pairs $(x, y) \in \mathbb{R}^2$ such that $f(x, y)$ is not finite. By hypothesis, the set E_x has $\mathbb{R}$-measure zero for a.e. x. Therefore, by part (a) the set E_y has measure zero. Hence, for a.e. y, $f(x, y)$ is finite for a.e. x. ∎

Problem 9.2 (Wheeden & Zygmund §6, Ex. 3)

Let f be measurable and finite a.e. on $[0,1]$. If $f(x)-f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$, show that $f \in L[0,1]$.

Proof. Set $I \coloneqq [0,1]$. Suppose that $f(x)-f(y) \in L(I \times I)$. Then by Fubini's theorem we have

$$\iint_{I\times I} f(x)-f(y)\,\mathrm{d}x\,\mathrm{d}y = \int_I \left[\int_I f(x)-f(y)\,\mathrm{d}x\right]\mathrm{d}y = \int_I \left[\int_I f(x)-f(y)\,\mathrm{d}y\right]\mathrm{d}x < \infty. \tag{9.2}$$

Hence, for a.e. $y \in \mathbb{R}$, $f(x)-f(y)$ is integrable so $f(x)$ is integrable. ∎

Problem 9.3 (Wheeden & Zygmund §6, Ex. 4)

Let f be measurable and periodic with period 1: $f(t+1) = f(t)$. Suppose there is a finite c such that

$$\int_0^1 |f(a+t) - f(b+t)| \, \mathrm{d}t \leq c$$

for all a and b. Show that $f \in L[0,1]$. (Set $a = x$, $b = -x$, integrate with respect to x, and make the change of variables $\xi = x + t$, $\eta = -x + t$.)

Proof. Following the hint, write

$$c \geq \int_0^1 \int_0^1 |f(x+t) - f(-x+t)| \, \mathrm{d}x \, \mathrm{d}t$$

making the change of variables $\xi = x + t$, $\eta = -x + t$ and appropriate modification to the bounds of integration, i.e., $0 \leq \xi \leq 2$, $-1 \leq \eta \leq 1$ we have

$$= \int_{-1}^1 \int_0^2 |f(\xi) - f(\eta)| (\det \mathbf{J}(\xi, \eta)) \, \mathrm{d}\xi \, \mathrm{d}\eta$$

by Fubini's theorem

$$= \int_0^2 \int_{-1}^1 |f(\xi) - f(\eta)| (\det \mathbf{J}(\xi, \eta)) \, \mathrm{d}\eta \, \mathrm{d}\xi$$

where $\mathbf{J}(\xi, \eta) = \begin{bmatrix} \partial x/\partial \xi & \partial x/\partial \eta \\ \partial t/\partial \xi & \partial t/\partial \eta \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is the Jacobian of the linear transformation which sends the pair (ξ, η) to $(1/2(\xi - \eta), 1/2(\xi + \eta))$, hence we have

$$\begin{aligned} &= \frac{1}{2} \int_0^2 \int_{-1}^1 |f(\xi) - f(\eta)| \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &= \frac{1}{2} \int_0^2 \int_{-1}^0 |f(\xi) - f(\eta)| \, \mathrm{d}\xi \, \mathrm{d}\eta + + \frac{1}{2} \int_0^2 \int_0^1 |f(\xi) - f(\eta)| \, \mathrm{d}\xi \, \mathrm{d}\eta \end{aligned}$$

Here we use Theorem 3.35 to note that the translation $\eta \mapsto \eta + 1$ and the fact that f is periodic with period 1 gives us

$$= \int_0^2 \int_0^1 |f(\xi) - f(\eta)| \, \mathrm{d}\xi \, \mathrm{d}\eta$$

similarly, we have

$$= 2 \int_0^1 \int_0^1 |f(\xi) - f(\eta)| \, \mathrm{d}\xi \, \mathrm{d}\eta.$$

Hence, the inequality

$$\int_0^1 \int_0^1 |f(\xi) - f(\eta)| \, \mathrm{d}\xi \, \mathrm{d}\eta \leq \frac{c}{2} \tag{9.3}$$

holds so by Problem 9.2 (§6, Ex. 3), $|f| \in L[0,1]$ hence, $f \in L[0,1]$. ■

Problem 9.4 (Wheeden & Zygmund §6, Ex. 6)

For $f \in L(\mathbb{R})$, define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} \, \mathrm{d}t$$

for $x \in \mathbb{R}$. (For complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i \int F_1$.) Show that if f and g belong to $L(\mathbb{R})$, then

$$\widehat{(f * g)}(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

Proof. By direct computation we have

$$\widehat{(f * g)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(s-t) g(t) \, \mathrm{d}t \right] e^{-ixs} \, \mathrm{d}s$$

now do this

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s-t) g(t) e^{-ixs} \, \mathrm{d}t \, \mathrm{d}s$$

make the substitution $u = s - t$, then the above becomes

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) g(t) e^{-ix(u+t)} \, \mathrm{d}t \, \mathrm{d}u \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-ixu} g(t) e^{-ixt} \, \mathrm{d}t \, \mathrm{d}u \end{aligned}$$

by Fubini's theorem, this is just

$$\begin{aligned} &= 2\pi \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-ixu} \, \mathrm{d}u \right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) e^{-ixt} \, \mathrm{d}t \right) \\ &= 2\pi \hat{f}(x) \hat{g}(x) \end{aligned}$$

as desired. ∎

Problem 9.5 (Wheeden & Zygmund §6, Ex. 7)

Let F be a closed subset of $\mathbb{R}$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F, prove that the function

$$\int_{\mathbb{R}} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}} \, \mathrm{d}t$$

is integrable over F and so is finite a.e. in F. (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

Proof. Set $G \coloneqq \mathbb{R} \setminus F$. By assumption, we have

$$\int_G f(x) \, \mathrm{d}x < \infty. \tag{9.4}$$

By Tonelli's theorem, since $\delta(y) = 0$ for $y \in F$, we have

$$\begin{aligned} \int_F \left[\int_{\mathbb{R}} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}} \, \mathrm{d}y \right] \mathrm{d}x &= \int_F \left[\int_G \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}} \, \mathrm{d}y \right] \mathrm{d}x \\ &= \int_G \delta^\lambda(y) f(y) \left[\int_F \frac{\mathrm{d}x}{|x-y|^{1+\lambda}} \right] \mathrm{d}y. \end{aligned} \tag{9.5}$$

Now, by Marcinkiewwicz's theorem, we have

$$\int_F \frac{\mathrm{d}x}{|x-y|^{1+\lambda}} \le 2\lambda^{-1} \delta(y)^{-\lambda}. \tag{9.6}$$

Then, by (9.4), we have

$$\begin{aligned} \int_F \left[\int_{\mathbb{R}} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}} \, \mathrm{d}y \right] \mathrm{d}x &\le \int_G \delta^\lambda(y) f(y) \left[2\lambda^{-1} \delta(y)^{-\lambda} \right] \mathrm{d}y \\ &= 2\lambda^{-1} \int_G f(y) \, \mathrm{d}y \\ &< \infty \end{aligned} \tag{9.7}$$

as desired. ■

Problem 9.6 (Wheeden & Zygmund §6, Ex. 9)

(a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.

(b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subset \mathbb{R}$, and let M_α be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int M_\lambda \leq \lambda^{-1}|G|$, where $G = (a, b) - [c, d]$.

Proof. (a) Put $G \coloneqq (a, b) \setminus F$. Since $\delta(y) = 0$ for $y \in F$, by Tonelli's theorem we have

$$M_\lambda(x) = \int_G \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} \, \mathrm{d}y. \tag{9.8}$$

If $x \notin F$, then since G is open, there exists a sufficiently small $\varepsilon > 0$ such that $B_\varepsilon(x) \subset G$ and $m \coloneqq \inf_{y \in B_\varepsilon(x)} \delta(y) > 0$. Since $\delta^\lambda(y)/|x - y|^{1+\lambda}$ is nonnegative, we have

$$\begin{aligned}
\int_G \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} \, \mathrm{d}y &\geq \int_{B_\varepsilon(x)} \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} \, \mathrm{d}y \\
&\geq m^\lambda \int_{|x - y| < \varepsilon} \frac{1}{|x - y|^{1+\lambda}} \, \mathrm{d}y \\
&= 2m^\lambda \int_0^\varepsilon \frac{1}{u^{1+\lambda}} \, \mathrm{d}u \\
&= \left[2m^\lambda \lambda^{-1} u^{-\lambda}\right]_0^\varepsilon \\
&= \infty.
\end{aligned}$$

(b) ■