Representation Theory

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1 What is Representation Theory?

Groups arise in nature as "sets of symmetries (of an object), which are closed under composition and under taking inverses". For example, the *symmetric group* S_n is the group of all permutations (symmetries) of $\{1, \ldots, n\}$; the *alternating group* A_n is the set of all symmetries preserving the parity of the number of ordered pairs; the *dihedral group* D_{2n} is the group of symmetries of the regular n-gon in the plane. The *orthogonal group* $\mathrm{O}(3)$ is the group of distance-preserving transformations of Euclidean space which fix the origin. There is also the group of *all* distance preserving transformations, which includes the translations along with $\mathrm{O}(3)$.*

The official definition is of course more abstract, a group is a set G with a binary operation $*$ which is associative, has a unit element e and for which inverses exist. Associativity allows a convenient abuse of notation, where we write gh for $g * h$; we have $ghk = (gh)k = g(hk)$ and parentheses are unnecessary. I will often write 1 for e, but this is dangerous on rare occasions, such that when studying the group $\mathbb{Z}$ under addition; in that case, $e = 0$.

The abstract definition notwithstanding, the interesting situation involves a group "acting" on a set. Formally, an action of a group G on a set X is an "action map" $a\colon G \times X \to X$ which is *compatible with the group law*, in the sense that

$$\begin{aligned} a\big(h, a(g, x)\big) &= a(hg, x) \\ a(e, x) &= x. \end{aligned}$$

This justifies the abuse of notation $a(g, x) = gx$, for we have $h(gx) = (hg)x$.

From this point of view, geometry asks, "Given a geometric object X, what is its group of symetries?" Representation theory reverses the quostion to "Given a group G, what objects X does it act on?" and attempts to answer this question by classifying such X up to isomorphism.

Before restricting to the linear case, our main concern, let us remember another way to describe an action of G on X. Every $g \in G$ defines a map $a(g)\colon X \to X$ by $x \mapsto gx$. This map is a bijection, with inverse map $a(g^{-1})$: indeed, $\big(a(g^{-1}) \circ a(g)\big)(x) = g^{-1}gx = ex = x$ from the properties of the action. Hence $a(g)$ belongs to the set $\operatorname{Sym} X$ of bijective self-maps of X. This set forms a group under composition, and the properties of an action imply that

Proposition 1.1. *An action of G on X "is the same as" a group homomorphism $\alpha\colon G \to \operatorname{Sym} X$.*

The formulation of Prop. 1.1 leads to the following observation. For any action a of H on X and group homomorphism $\varphi\colon G \to H$, there is defined a *restricted* or *pulled-back* action $\varphi^* a$ of G on X, as $\varphi^* a = a \circ \varphi$. In the original definition, the action sends (g, x) to $\varphi(g)(x)$.

Example 1.1 (Tautological action of $\operatorname{Sym} X$ on X)**.** This is the obvious action, call it T, sending f, x to $f(x)$, where $f\colon X \to X$ is a bijection and $x \in X$. In this language, the action a of G on X is $\alpha^* T$ with the homomorphism α of the proposition – the pull-back under α of the tautological action.

Example 1.2 (Linearity)**.** The question of classifying all possible X with action of G is hopeless in such generality, but one should recall that, in first approximation, mathematics is linear. So we shall take our X to be a*vector space* over some ground *field*, and ask that the action of G be linear, as well, in other words, that it should preserve the vector space structure. Our interest is mostly confined to the case when the field of scalars is $\mathbb{C}$, although we shall occasionally mention how the picture changes when other fields are studied.

*This group is isomorphic to the *semi-direct product* $\mathrm{O}(3) \ltimes \mathbb{R}^3$.

Definition 1.1. A linear representation ρ of G on a complex vector space V is a set-theoretic action on V which preserves the linear structure, i.e.,

$$\begin{aligned} \rho(g)(\mathbf{v}_1 + \mathbf{v}_2) &= \rho(g)\mathbf{v}_1 + \rho(g)\mathbf{v}_2, && \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V, \\ \rho(g)(k\mathbf{v}) &= k\rho(g)\mathbf{v} && \text{for all } k \in \mathbb{C}, \mathbf{v} \in V. \end{aligned}$$

Unless otherwise mentioned, a *representation* will mean a *finite-dimensional complex representation*.

Example 1.3 (The general linear group)**.** Let V be a complex vector space of dimension $n < \infty$. After choosing a basis, we can identify it with $\mathbb{C}^n$, although we shal lavoid doing so without good reason. Recall that the *endomorphism algebra* $\mathrm{End}(V)$ is the set of all linear maps (or *operators*) $L\colon V \to V$, with the natural addition of linear maps and the composition as multiplication. If V has been identified with $\mathbb{C}^n$, a linear map is uniquely representable by a matrix, and the addition of linear maps becomes the entrywise addition, while the composition becomes the matrix multiplication.

Inside $\mathrm{End}(V)$, there is contained the group $\mathrm{GL}(V)$ of invertible linear operators; the group operation, of course, is composition.

Proposition 1.2. *V is naturally a representation of* $\mathrm{GL}(V)$.

It is called the *standard* representation of $\mathrm{GL}(V)$. The following corresponds to Prop. 1.1, involvinge the same abuse of language.

Proposition 1.3. *A representation of G on V "is the same as" a group homomorphism from G to* $\mathrm{GL}(V)$.

Proof. Observe that, to give a linear action of G on V, we must assign to each $g \in G$ a linear self-map $\rho(g) \in \mathrm{End}(V)$. Compatibility of the action with the group law requires

$$\rho(h)\big(\rho(g)(\mathbf{v})\big) = \rho(hg)(\mathbf{v}), \qquad \rho(1)(\mathbf{v}) = \mathbf{v},$$

for all $\mathbf{v} \in V$, whence we conclude that $\rho(1) = \mathrm{id}$, $\rho(hg) = \rho(h) \circ \rho(g)$. Taking $h = g^{-1}$ shows that $\rho(g)$ is invertible, hence lands in $\mathrm{GL}(V)$. The first relation then says that we are dealing with a group homomorphism. ∎

Definition 1.2. An *isomorphism* φ between two representation (ρ_1, V_1) and (ρ_2, V_2) of G is a linear isomorphism $\varphi\colon V_1 \to V_2$ which intertwines with the action of G, that is, satisfies

$$\varphi\big(\rho_1(h)(\mathbf{v})\big) = \rho_2(g)\big(\varphi(\mathbf{v})\big).$$

Note that the equality makes sense even if φ is not invertible, in which case it is just called an *intertwining operator* or *G-linear map*. However, if φ is invertible, we can write instead

$$\rho_2 = \varphi \circ \rho_1 \circ \varphi^{-1}, \tag{1}$$

meaning that we have an equality of linear maps after intserting any group element g. Observe that this relation determines ρ_2 if ρ_1 and φ are known. We can finally formulate the basic problem of representation theory: Classify all representation of a given group G, up to isomorphism.

For arbitrary G, this is very hard! We shall concentrate on finite groups, where a very good general theory exists. Later on, we shall study some examples of topological compact groups, such as $\mathrm{U}(1)$ and $\mathrm{SU}(2)$. The general theory for compact groups is also completely understood, but requires more difficult methods.

I close with a simple observation, tying in with Def. 1.2. Given any representation ρ of G no a space V of dimension n, a choice of basis in V identifies this linearly with $\mathbb{C}^n$. Call the isomorphism φ. Then, by formula (1), we can define a new representation ρ_2 of G on $\mathbb{C}^n$, which is isomorphic to (ρ, V). So any n-dimensional representation of G is isomorphic to a representation on $\mathbb{C}^n$. The use of an abstract vector space does not lead to 'new' representation, but it does free us from the presence of a distinguished basis.

2 Lecture 2

Hello