MA571 Problem Set 3

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Problem 3.1 (Munkres §18, p. 111, #7(a))

(a) Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is "continuous from the right," that is,

$$\lim_{x \to a+} f(x) = f(a).$$

for each $a \in \mathbf{R}$. Show that f is continuous when considered as a function from $\mathbf{R}_\ell$ to $\mathbf{R}$.

Proof. Recall the definition of "right-hand limit,":

Definition (Rudin §4, p. 94, Def. 4.25)**.** Let f be defined on (a, b). Consider any point x such that $a \leq x < b$. We write $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$.

■

Problem 3.2 (Munkres §18, p. 112, #13)

Let $A \subset X$; let $f\colon A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g\colon \overline{A} \to Y$, then g is uniquely determined by f.

Proof. ■

Problem 3.3 (Munkres §19, p. 118, #2)

Prove Theorem 19.3.

Proof. Recall the exact statement of Theorem 19.3 from Munkres §19, p. 116:

Theorem. *Let A_α be as subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

∎

Problem 3.4 (Munkres §19, p. 118, #3)

Prove Theorem 19.4.

Proof. Recall the exact statement of Theorem 19.4 from Munkres §19, p. 116:

Theorem. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

∎

Problem 3.5 (Munkres §19, p. 118, #6)

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Proof. ∎

Problem 3.6 (Munkres §19, p. 118, #7)

Let $\mathbf{R}^\infty$ be the subset of $\mathbf{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, ...)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbf{R}^\infty$ in $\mathbf{R}^\omega$ in the box and product topologies? Justify your answer.

Proof. ∎

Problem 3.7 (Munkres §20, p. 126, #3(b))

Let X be a metric space with metric d.

(b) Let X' denote a space having the same underlying set as X. show that if $d\colon X' \times X' \to \mathbf{R}$ is continuous, then the topology of X' is finer than the topology of X.

Proof. ∎

Problem 3.8 (Munkres §20, p. 127, #4(b))

Consider the product, uniform and box topologies on $\mathbf{R}^\omega$

(b) In which topologies do the following sequences converge?

$$\begin{aligned}
\mathbf{w}_1 &= (1,1,1,1,\ldots), & \mathbf{x}_1 &= (1,1,1,1,\ldots),\\
\mathbf{w}_2 &= (0,2,2,2,\ldots), & \mathbf{x}_2 &= \left(0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\ldots\right),\\
\mathbf{w}_3 &= (0,0,3,3,\ldots), & \mathbf{x}_3 &= \left(0,0,\tfrac{1}{3},\tfrac{1}{3},\ldots\right),\\
&\vdots & &\vdots\\
\mathbf{y}_1 &= (1,0,0,0,\ldots) & \mathbf{z}_1 &= (1,1,0,0,\ldots),\\
\mathbf{y}_2 &= \left(\tfrac{1}{2},\tfrac{1}{2},0,0,\ldots\right) & \mathbf{z}_2 &= \left(\tfrac{1}{2},\tfrac{1}{2},0,0,\ldots\right),\\
\mathbf{y}_3 &= \left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,\ldots\right) & \mathbf{z}_3 &= \left(\tfrac{1}{3},\tfrac{1}{3},0,0,\ldots\right),\\
&\vdots & &\vdots
\end{aligned}$$

Proof. ∎

Problem 3.9 (A)

Given: X a metric space, A a countable subset of X, and $\overline{A} = X$. To prove: the topology of X has a countable basis.

Proof. ■

Problem 3.10 (B)

Given: Y is an ordered set, (a, b) and (c, d) are disjoint open intervals, and there are elements $x \in (a, b)$ and $y \in (c, d)$ with $x < y$. To prove: every element of (a, b) less than every element of (c, d).

Proof. ■

Problem 3.11 (C)

(This problem will be used when we discuss quotient spaces). Let S and T be sets and let $f\colon S \to T$ be a function. Let $A \subset S$.

(i) Give an example to show that the equation

$$f^{-1}(f(A)) = A \tag{*}$$

isn't always valid.

(ii) Define an equivalence relation $\sim$ on S by $s \sim s'$ if and only if $f(s) = f(s')$. Using this equivalence relation, describe the subsets A of S for which (*) is true. Prove that your answer is correct.

Proof. ∎