MA 519: Homework, Midterms and Practice Problems Solutions

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1 Midterms, Exams, and Qualifying Exams

1.1 Qualifying Exams, August '99

Exercise 1.1. The number of fish that Anirban catches on any given day has a Poisson distribution with mean 20. Due to the legendary softness of his heart, he sets free, on average, 3 out of the 4 fish he catches. Find the mean and the variance of the number of fish Anirban takes home on a given day.

Solution. ■

Exercise 1.2. A fair die is rolled and at the same time a fair coin is tossed. This is done repeatedly. Find the probability that head occurs (strictly) before six occurs.

Solution. ■

Exercise 1.3. X, Y are independent random variables with a common density $f(x) = e^{-|x|}/2$, $x \in (-\infty, \infty)$. Find the density function of $X + Y$.

Solution. ■

Exercise 1.4. Let X_n denote the distance between two points chosen independently at random from the unit cube in $\mathbb{R}^n$. Evaluate

$$\lim_{n\to\infty} \frac{E(X_n)}{\sqrt{n}}.$$

Solution. ■

Exercise 1.5. Let X be distributed as Uniform$[0, 1]$. What is the probability that the digit 5 does not occur in the decimal expansion of X?

Solution. ■

1.2 Qualifying Exam, January ‘06

Exercise 1.6. The birthdays of 5 people are known to fall in exactly 3 calendar months. What is the probability that exactly two of the 5 were born in January?

Solution. ■

Exercise 1.7. Coupons are drawn, independently, with replacement, one at a time, from a set of 10 coupons. Find, explicitly, the expected number of draws

(a) until the first draw coupon is drawn again;
(b) until a duplicate occurs.

Solution. ■

Exercise 1.8. Let N be a positive integer. Choose an integer at random from $\{1, \dots, N\}$. Let E be the event that your chosen random number is divisible by 3, and divisible by at least one of 4 and 6, but not divisible by 5. Find, explicitly, $\lim_{N\to\infty} P(E)$.

Solution. ■

Exercise 1.9. Anirban is driving his Dodge on a highway with 4 lanes each way. He is wired to change lanes every minute on the minute. He changes with equal probability to either adjacent lane if there are two adjacent lanes, and the successive changes are mutually independent. Find, explicitly, the probability that after 4 minutes, Anirban is back to the lane he started from

(a) if he started at an outside lane;
(b) if he started at an inside lane.

Solution. ■

Exercise 1.10. Burgess is going to Moose Pass, Alaska. He is driving his Dodge. He puts his car on cruise control at 70 mph. Gas stations are located every 30 miles, starting from his home. His car runs out of gas at a time distributed as an exponential with mean 4 hours. When that happens, he gets out, takes his bik out of his trunk, and bikes to the next gas station say M, at 10 mph. Let the time elapsed between when Burgess starts his trip and when he arrives at the gas station M be T. Find $E(T)$.

Solution. ■

Exercise 1.11. A fair coin is tossed n times. Suppose X heads are obtained. Given $X = x$, let Y be generated according to the Poisson distribution with mean x. Find the unconditional variance of Y, and then find the limit of the probability $P\big(|Y - n/2| > n^{3/4}\big)$, as $n \to \infty$.

Solution. ∎

Exercise 1.12. Anirban plays a game repeatedly. On each play he wins an amount uniformly distributed in $(0, 1)$ dollars, and then he tips the lady in charge of the game the square of the amount he has won. Then he plays again, tips again, and so on. Approximately calculate the probability that if he plays and tips six hundred times, his total winnings minus his total tips will exceed \$105.

Solution. ∎

Exercise 1.13. Anirban's dog got mad at him and broke his walking cane, first uniformly into two peices, and then the long piece again uniformly into two pieces. Find the probability that Anirban can make a triangle out of the three pieces of his cane.

Solution. ∎

Exercise 1.14. Suppose X, Y, Z are identically independently distributed Exp(1) random variables. Find the joint density of (X, XY, XYZ).

Solution. ∎

Exercise 1.15. Let X be the number of Kings and Y the number of Hearts in a Bridge hand. Find the correlation between X and Y.

Solution. ∎

1.3 Qualifying Exam, August '14

Exercise 1.16.

(a) 3 balls are distributed one by one and at random in 3 boxes. What is the probability that exactly one box remains empty?
(b) n balls are distributed one by one and at random in n boxes. Find the probability that exactly one box remains empty.
(c) n balls are distributed one by one and at random in n boxes Find the probability that exactly two boxes remain empty.

Solution. ■

Exercise 1.17. n players each roll a fair die. For any pair of players i, j, $i < j$, who roll the same number, the group is awarded one point.

(a) Find the mean of the total points of the group.
(b) Find the variance of the total points of the group.

Solution. ■

Exercise 1.18. Suppose $X_1, X_2, \ldots,$ is an infinite sequence of independently identically distributed Uniform$[0,1]$ random variables. Find the limit

$$\lim_{n\to\infty} P\left[\frac{\left(\prod_{i=1}^n X_i\right)^{1/n}}{\left(\sum_{i=1}^n X_i\right)/n} > \frac{3}{4}\right].$$

Solution. ■

Exercise 1.19. Suppose X is an exponential random variable with density $e^{-x/\sigma_1}/\sigma_1$ and Y is another exponential random variable with density $e^{-y/\sigma_2}/\sigma_2$, and that X, Y are independent.

(a) Find the CDF of $X/(X+Y)$.
(b) In the case $\sigma_1 = 2$, $\sigma_2 = 1$, find the mean of $X/(X+Y)$.

Solution. ■

Exercise 1.20. Ten independently picked Uniform$[0,100]$ numbers are each rounded to the nearest integer. Use the central limit theorem to approximate the probability that the sum of the ten rounded numbers equals the rounded value of the sum of the ten original numbers.

Solution. ■

Exercise 1.21. Suppose for some given $m \geq 2$, we choose m independently identically distributed Uniform$[0, 1]$ random variables $X_1, \dots, X_m$. Let $X_{\min}$ denote their minimum and $X_{\max}$ denote their maximum. Now continue sampling $X_{m+1}, \dots,$ from the Uniform$[0, 1]$ density. Let N be the first index k such that X_{m+k} falls outside the interval $[X_{\min}, X_{\max}]$.

(a) Find a formula for $P(N > n)$ for a general n.
(b) Hence, explicitly find $E(N)$.

Solution. ■

Exercise 1.22. A $G_{n,p}$ graph on n vartices is obtained by adding each of the $\binom{n}{2}$ possible edges into the graph mutually independently with probability p. If vertex subsets A, B both have k vertices, and each vertex A shares an edge with each vertex in B, but there are no edges among the vertices within A or within B, then A, B generate a complete bipartate subgraph of order k denoted as $K_{k,k}$.

(a) For a given n and p, find an expression for the expected number of complete bipartate subgraphs $K_{3,3}$ of order $k = 3$ in a $G_5 n, p$ graph.
(b) Let p_n denote the value of p for which the expected value in part (a) equals one. Identify constants α, β such that $\lim_{n\to\infty} n^\alpha p_n = \beta$.

Solution. ■