MA 571: Homework 4 due Monday September 21.

Please read the "Note on Quotient Spaces" which is posted on my website. Then read from the beginning of Section 22 to the middle of page 141.

Please do:

p. 127 # 4(a) (Do the function h only; prove your answer.)
p. 127 # 4(b) (Do the sequences $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, but only for the box topology. Prove your answers.)
p. 127 # 6(b)

A) Prove Theorem Q.2 from the Note on Quotient Spaces.

B) Prove Proposition Q.5 from the Note on Quotient Spaces.

C) Prove Proposition Q.6 from the Note on Quotient Spaces.

D) (Do not use Problem E to do this problem). Let $\sim$ be the equivalence relation on the interval $[-1, 1]$ defined by $x \sim y$ if and only if $x = y$ or $x = -y$ with $y \in (-1, 1)$ (you do not have to prove that this is an equivalence relation). Prove that $[-1, 1]/\sim$ is not Hausdorff.

E) Let X be a topological space with an equivalence relation $\sim$. Suppose that the quotient space $X/\sim$ is Hausdorff.

Prove that the set

$$S = \{(x, y) \in X \times X \mid x \sim y\}$$

is a closed subset of $X \times X$.

For problem F you need the following definition: if Y is a topological space and S is a subset of Y, we write Y/S for the quotient space $Y/\sim$, where $\sim$ is defined by $x \sim y$ if and only if $x = y$ or $\{x, y\} \subset S$. (Intuitively, Y/S is obtained from Y by collapsing S to a point.)

F) Let X be a topological space. Let U be an open set in X, and let A be a subset of U. Give U the subspace topology. Let $i : U/A \to X/A$ be the map which takes $[x]$ to $[x]$ (you do not have to prove that this is well-defined).

(i) Prove that i is continuous.

(ii) Prove that i is an open map.

G) Let X be a topological space satisfying the first countability axiom (see the bottom of page 130 and the top of page 131). Let $A \subset X$ and let $x \in \overline{A}$. Prove that there is a sequence in A which converges to x (see the top of page 131 for a hint).