HwK12 p.1

4.4: 3, 4, 5, 6, 7, 10, 12, 13, 15

4.4.3 In each part, determine whether the given vector $p(t)$ in P_2 belongs to Span $\{p_1(t), p_2(t), p_3(t)\}$, where $p_1(t) = t^2+2t+1$, $p_2(t) = t^2+3$, and $p_3(t) = t-1$

(a) $p(t) = t^2+t+2$ (b) $p(t) = 2t^2+2t+3$ (c) $p(t) = -t^2+t-4$ (d) $p(t) = -2t^2+3t+1$.

Need c_1, c_2, c_3 so that $c_1p_1(t) + c_2p_2(t) + c_3p_3(t) = p(t)$.

This is $c_1(t^2+2t+1) + c_2(t^2+3) + c_3(t-1) = (c_1+c_2)t^2 + (2c_1+c_3)t + (c_1+3c_2-c_3)$.

So its the system $c_1+c_2 = a$, $2c_1+c_3 = b$, $c_1+3c_2-c_3 = c$ for at^2+bt+c.

(a) $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & -1 & 2 \end{array}\right] \xrightarrow[-r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{array}\right] \xrightarrow{r_2+r_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-r_2+r_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & +\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So yes with $c_1 = +\frac{1}{2}$ and $c_2 = \frac{1}{2}$.

(b) $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \\ 1 & 3 & -1 & 3 \end{array}\right] \xrightarrow[-r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -2 \\ 0 & 2 & -1 & 1 \end{array}\right] \xrightarrow{r_2+r_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$ So No.

(c) $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & -1 & -4 \end{array}\right] \xrightarrow[-r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -2 & 1 & 3 \\ 0 & 2 & -1 & -3 \end{array}\right] \xrightarrow{r_2+r_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-r_2+r_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so yes with $c_1 = \frac{1}{2}$, $c_2 = -\frac{3}{2}$

(d) $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 2 & 0 & 1 & 3 \\ 1 & 3 & -1 & 1 \end{array}\right] \xrightarrow[-r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & -2 & 1 & 7 \\ 0 & 2 & -1 & 5 \end{array}\right] \xrightarrow{r_2+r_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & -2 & 1 & 7 \\ 0 & 0 & 0 & 12 \end{array}\right]$$ So No.

4.4.4 In each part, determine whether the given vector A in M_{22} belongs to Span $\{A_1, A_2, A_3\}$, where $A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, note $c_1A_1 + c_2A_2 + c_3A_3 = A$ gives eqns

$$\begin{aligned} c_1 + c_2 + 2c_3 &= a \\ -c_1 + c_2 + 2c_3 &= b \\ -c_3 &= c \\ 3c_1 + 2c_2 + c_3 &= d \end{aligned}$$

~~S~~ Going to find R s.t. $RS = S$ in RREF.

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} r_1+r_2 \\ \\ \\ -3r_1+r_4 \end{array} \left[\begin{array}{ccc|cccc} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -5 & -3 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} +r_4+r_2 \\ -r_3 \end{array} \left[\begin{array}{ccc|cccc} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -5 & -3 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} -2r_3+r_1 \\ +r_3+r_2 \\ \\ 5r_3+r_4 \end{array} \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -3 & 0 & -5 & 1 \end{array}\right]$$

$$\begin{array}{c} -r_2+r_1 \\ \\ \\ r_2+r_4 \end{array} \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 3 & -1 & 3 & -1 \\ 0 & 1 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -5 & 1 & -6 & 2 \end{array}\right] \quad \text{so } R = \begin{bmatrix} 3 & -1 & 3 & -1 \\ -2 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ -5 & 1 & -6 & 2 \end{bmatrix}$$

Thus for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we just need to see if we can solve $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

(a) $A = \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$. $\begin{bmatrix} 3 & -1 & 3 & -1 \\ -2 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ -5 & 1 & -6 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ -1 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ so $c_1 = 2$, $c_2 = 1$, $c_3 = 1$

(b) $A = \begin{bmatrix} -3 & -1 \\ 3 & 2 \end{bmatrix}$ $\begin{bmatrix} 3 & -1 & 3 & -1 \\ -2 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ -5 & 1 & -6 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -3 \\ 0 \end{bmatrix}$ so $c_1 = -1$, $c_2 = 4$, $c_3 = -3$

(c) $A = \begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}$ $\begin{bmatrix} 3 & -1 & 3 & -1 \\ -2 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ -5 & 1 & -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ -9 \\ -3 \\ -31 \end{bmatrix}$ so not possible.

(d) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 3 & -1 & 3 & -1 \\ -2 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ -5 & 1 & -6 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -2 \\ -15 \end{bmatrix}$ so not possible.

4.4.5 Which of the following vectors span R_2?

(a) $[1\ 2], [-1\ 1]$ (b) $[0\ 0], [1, 1], [-2\ -2]$ (c) $[1\ 3], [2\ -3], [0\ 2]$

(d) $[2\ 4], [-1\ 2]$.

Let $V = [a\ b]$ in R_2. The vectors span R_2 if $c_1V_1 + c_2V_2\ (+ c_3V_3) = V$.

(a) $$\left[\begin{array}{cc|c} 1 & -1 & a \\ 2 & 1 & b \end{array}\right] \xrightarrow{-2r_1+r_2} \left[\begin{array}{cc|c} 1 & -1 & a \\ 0 & 3 & b-2a \end{array}\right] \xrightarrow{\frac{1}{3}r_2} \left[\begin{array}{cc|c} 1 & -1 & a \\ 0 & 1 & \frac{1}{3}(b-2a) \end{array}\right] \xrightarrow{r_2+r_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3}b - \frac{1}{3}a \\ 0 & 1 & \frac{1}{3}b - \frac{2}{3}a \end{array}\right]$$

So yes.

(b) $$\left[\begin{array}{ccc|c} 0 & 1 & -2 & a \\ 0 & 1 & -2 & b \end{array}\right] \xrightarrow{-r_1+r_2} \left[\begin{array}{ccc|c} 0 & 1 & -2 & a \\ 0 & 0 & 0 & b-a \end{array}\right]$$ So no, particularly for $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not in the span

(c) $$\left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 3 & -3 & 2 & b \end{array}\right] \xrightarrow{-3r_1+r_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 0 & -9 & 2 & b-3a \end{array}\right]$$ this does span R_2

(d) $$\left[\begin{array}{cc|c} 2 & -1 & a \\ 4 & 2 & b \end{array}\right] \xrightarrow{-2r_1+r_2} \left[\begin{array}{cc|c} 2 & -1 & a \\ 0 & 4 & b-2a \end{array}\right]$$ this does span R_2.

4.4.6 Which of the following sets of vectors span R^4?

(a) $\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix} 3 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -2 \\ -1 \end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}\right\}$

(a) No. $$\left[\begin{array}{cc|c} 1 & 0 & a \\ -1 & 1 & b \\ 2 & 1 & c \\ 0 & 1 & d \end{array}\right] \xrightarrow[-2r_1+r_3]{r_1+r_2} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & a+b \\ 0 & 1 & -2a+c \\ 0 & 1 & d \end{array}\right] \xrightarrow[-r_4+r_3]{-r_4+r_2} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 0 & a+b+d \\ 0 & 0 & -2a+c+d \\ 0 & 1 & d \end{array}\right]$$ forces $a+b+d=0$, $-2a+c+d=0$.

(b) No. $$\left[\begin{array}{ccc|c} 3 & 1 & 0 & a \\ 2 & 2 & 0 & b \\ 1 & -1 & 0 & c \\ 0 & 0 & 1 & d \end{array}\right] \xrightarrow{r_3 \leftrightarrow r_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & c \\ 2 & 2 & 0 & b \\ 3 & 1 & 0 & a \\ 0 & 0 & 1 & d \end{array}\right] \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & c \\ 0 & 4 & 0 & b-2c \\ 0 & 4 & 0 & a-3c \\ 0 & 0 & 1 & d \end{array}\right] \xrightarrow{-r_2+r_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & c \\ 0 & 4 & 0 & b-2c \\ 0 & 0 & 0 & a-b-c \\ 0 & 0 & 1 & d \end{array}\right]$$ forces $a-b-c=0$.

(c) $$\left[\begin{array}{ccccc|c} 3 & 4 & 3 & 5 & 0 & a \\ 2 & 0 & 2 & 6 & 4 & b \\ -1 & 0 & -1 & -3 & -2 & c \\ 2 & 2 & 2 & 2 & -1 & d \end{array}\right] \xrightarrow[\substack{+2r_3+r_2 \\ 2r_3+r_4}]{-r_2+r_1} \left[\begin{array}{ccccc|c} 1 & 4 & 1 & -1 & -4 & a-b \\ 0 & 0 & 0 & 0 & 0 & b+2c \\ -1 & 0 & -1 & -3 & -2 & c \\ 0 & 2 & 0 & -4 & -5 & 2c+d \end{array}\right]$$ So no as needs $b+2c=0$.

(d) $$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a \\ 1 & 2 & 0 & 1 & b \\ 0 & -1 & 1 & 2 & c \\ 0 & 1 & -1 & -1 & d \end{array}\right] \xrightarrow{-r_1+r_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a \\ 0 & 1 & 0 & -1 & -a+b \\ 0 & -1 & 1 & 2 & c \\ 0 & 1 & -1 & -1 & d \end{array}\right] \xrightarrow[\substack{r_2+r_3 \\ -r_2+r_4}]{-r_2+r_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2a-b \\ 0 & 1 & 0 & -1 & -a+b \\ 0 & 0 & 1 & 1 & -a+b+c \\ 0 & 0 & -1 & 0 & a-b+d \end{array}\right] \xrightarrow{r_3+r_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2a-b \\ 0 & 1 & 0 & -1 & -a+b \\ 0 & 0 & 1 & 1 & -a+b+c \\ 0 & 0 & 0 & 1 & c+d \end{array}\right]$$

So yes.

4.4.7 Which of the following sets of vectors span R_4?

(a) [1 0 0 1], [0 1 0 0], [1 1 1 1], [1 1 1 0] (b) [1 2 1 0], [1 1 -1 0], [0 0 0 1]

(c) [6 4 -2 4], [2 0 0 1], [3 2 -1 2], [5 6 -3 2], [0 4 -2 -1]

(d) [1 1 0 0], [1 2 -1 1], [0 0 1 1], [2 1 2 1].

Repeat 4.4.6. $V = [a\ b\ c\ d]$ in R_4

(a)
$$\left[\begin{array}{cccc|c} 1&0&1&1&a\\ 0&1&1&1&b\\ 0&0&1&1&c\\ 1&0&1&0&d \end{array}\right] \xrightarrow{-r_1+r_4} \left[\begin{array}{cccc|c} 1&0&1&1&a\\ 0&1&1&1&b\\ 0&0&1&1&c\\ 0&0&1&0&-a+d \end{array}\right] \xrightarrow{r_4 \leftrightarrow r_3} \left[\begin{array}{cccc|c} 1&0&1&1&a\\ 0&1&1&1&b\\ 0&0&1&0&-a+d\\ 0&0&1&1&c \end{array}\right] \xrightarrow{-r_3+r_4} \left[\begin{array}{cccc|c} 1&0&1&1&a\\ 0&1&1&1&b\\ 0&0&1&0&-a+d\\ 0&0&0&1&a+c-d \end{array}\right]$$
so yes.

(b) No.
$$\left[\begin{array}{ccc|c} 1&1&0&a\\ 2&1&0&b\\ 1&-1&0&c\\ 0&0&1&d \end{array}\right] \xrightarrow[-r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1&1&0&a\\ 0&-1&0&b-2a\\ 0&-2&0&c-a\\ 0&0&1&d \end{array}\right] \xrightarrow{-2r_2+r_3} \left[\begin{array}{ccc|c} 1&1&0&a\\ 0&-1&0&b-2a\\ 0&0&0&3a-2b+c\\ 0&0&1&d \end{array}\right]$$
needs $3a-2b=-c$.

(c)
$$\left[\begin{array}{ccccc|c} 6&2&3&5&0&a\\ 4&0&2&6&4&b\\ -2&0&-1&-3&-2&c\\ 4&1&2&2&1&d \end{array}\right] \xrightarrow{2r_3+r_4} \left[\begin{array}{ccccc|c} 6&2&3&5&0&a\\ 0&0&0&0&0&b+2c\\ -2&0&-1&-3&-2&c\\ 4&1&2&2&1&d \end{array}\right]$$
No. needs $b+2c=0$.

(d)
$$\left[\begin{array}{cccc|c} 1&1&0&2&a\\ 1&2&0&1&b\\ 0&-1&1&2&c\\ 0&1&1&1&d \end{array}\right] \xrightarrow[r_3+r_4]{-r_1+r_2} \left[\begin{array}{cccc|c} 1&1&0&2&a\\ 0&1&0&-1&-a+b\\ 0&-1&1&2&c\\ 0&0&2&3&c+d \end{array}\right] \xrightarrow{r_2+r_3} \left[\begin{array}{cccc|c} 1&1&0&2&a\\ 0&1&0&-1&-a+b\\ 0&0&1&1&-a+b+c\\ 0&0&2&3&c+d \end{array}\right] \xrightarrow{-2r_3+r_4} \left[\begin{array}{cccc|c} 1&1&0&2&a\\ 0&1&0&-1&a+b\\ 0&0&1&1&-a+b+c\\ 0&0&0&1&2a-2b-c+d \end{array}\right]$$
So yes.

4.4.10 Does the set $S = \left\{\begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}\right\}$ span M_{22}?

$$c_1\begin{bmatrix}1&1\\0&0\end{bmatrix} + c_2\begin{bmatrix}0&0\\1&1\end{bmatrix} + c_3\begin{bmatrix}1&0\\0&1\end{bmatrix} + c_4\begin{bmatrix}0&1\\1&1\end{bmatrix} = \begin{bmatrix}a&b\\c&d\end{bmatrix}$$ is

$$\left[\begin{array}{cccc|c} 1&0&1&0&a\\ 1&0&0&1&b\\ 0&1&0&1&c\\ 0&1&1&1&d \end{array}\right] \xrightarrow[-r_3+r_4]{-r_1+r_2} \left[\begin{array}{cccc|c} 1&0&1&0&a\\ 0&0&-1&1&-a+b\\ 0&1&0&1&c\\ 0&0&1&0&-c+d \end{array}\right] \xrightarrow{r_4+r_2} \left[\begin{array}{cccc|c} 1&0&1&0&a\\ 0&0&0&1&-a+b-c+d\\ 0&1&0&1&c\\ 0&0&1&0&-c+d \end{array}\right]$$
does span M_{22}.

4.4.12 Find a set of vectors spanning the null space of $A = \begin{bmatrix}1&1&2&-1\\2&3&6&-2\\-2&1&2&2\\0&-2&-4&0\end{bmatrix}$.

Null space is $\vec{x}$ with $A\vec{x}=\vec{0}$. Thus we need A in rref to determine solutions.

$$\begin{bmatrix}1&1&2&-1\\2&3&6&-2\\-2&1&2&2\\0&-2&-4&0\end{bmatrix} \xrightarrow[2r_1+r_3]{-2r_1+r_2} \begin{bmatrix}1&1&2&-1\\0&1&2&0\\0&3&6&0\\0&-2&-4&0\end{bmatrix} \xrightarrow[-3r_2+r_3,\ 2r_2+r_4]{-r_2+r_1} \begin{bmatrix}1&0&0&-1\\0&1&2&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$
has solutions $x_1 - x_4 = 0$, $x_2 + 2x_3 = 0$

So $x_1 = x_4 = s$, $x_2 = -2x_3 = -2r$, $x_3 = r$ so

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}s\\-2r\\r\\s\end{bmatrix} = s\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + r\begin{bmatrix}0\\-2\\1\\0\end{bmatrix}$$, so $\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\-2\\1\\0\end{bmatrix}$ span the null space.

4.4.13 | The set W of all 2x2 matrices A with trace equal to zero is a subspace of M_{22}.

Let $S = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$. Show Span (S) = W.

Recall $tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d$, so $tr(A) = 0$ forces $a = -d$.

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = c_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ gives } c_1 = b,\ c_2 = c,\ c_3 = a$$

So Span (S) = W.

This is harder for 3x3 and higher.

4.4.15 | The set W of all 3x3 matrices of the form $\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$ is a subspace of M_{33}.

Determine a subset S of W so that Span (S) = W.

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So let $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$.