# MA553: Qual Problems

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## 1 Ulrich's MA 553 Exercises for Spring '16

### 1.1 Homework 1

**Exercise 1.1.** Let $G$ be a group, $a \in G$ an element of finite order $m$, and $n$ a positive integer. Prove that

$$|a^n| = \frac{m}{\gcd(m,n)}.$$

*Proof.* ■

**Exercise 1.2.** Let $G$ be a group, and let $a$, $b$ be elements of finite order $m$, $n$ respectively. Show that if $ba = ab$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$, then $|ab| = \operatorname{lcm}(m,n)$.

*Proof.* ■

**Exercise 1.3.** Let $G$ be a group and $H$, $K$ normal subgroups with $H \cap K = \{e\}$. Show that

(a) $hk = kh$ for every $h \in H$, $k \in K$.

(b) $HK$ is a subgroup of $G$ with $HK \cong H \times K$.

*Proof.* ■

**Exercise 1.4.** Show that $A_4$ has no subgroup of order 6 (although $6 \mid 12 = |A_4|$).

*Proof.* ■

## 1.2 Homework 2

**Exercise 1.5.** Let $G$ be the group of order $2^3 \cdot 3$, $n \geq 2$. Show that $G$ has a normal 2-subgroup $\neq \{e\}$.

*Proof.* ∎

**Exercise 1.6.** Let $G$ be a group of order $p^2q$, $p$ and $q$ primes. Show that the Sylow $p$-Sylow subgroup or the $q$-Sylow subgroup of $G$ is normal in $G$.

*Proof.* ∎

**Exercise 1.7.** Let $G$ be a subgroup of order $pqr$, $p < q < r$ primes. Show that the $r$-Sylow subgroup of $G$ is normal in $G$.

*Proof.* ∎

**Exercise 1.8.** Let $G$ be a group of order $n$ and let $\varphi\colon G \to S_n$ be given by the action of $G$ on $G$ via translation.

(a) For $a \in G$ determine the number and the lengths of the disjoint cycles of the permutation $\phi(a)$.

(b) Show that $\varphi(G) \not\subset A_n$ if and only if $n$ is even and $G$ has a cyclic 2-Sylow subgroup.

(c) If $n = 2m$, $m$ odd, show that $G$ has a subgroup of index 2.

*Proof.* ∎

**Exercise 1.9.** Show that the only simple groups $\neq \{e\}$ of order $< 60$ are the groups of prime order.

*Proof.* ∎

## 1.3 Homework 3