MA 523: Homework 8

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PROBLEM 8.1

Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x - 2k, t)$$

where

$$\Phi(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

is positive for $|x| < 1$, $t > 0$.

(*Hint:* Show that u satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{cases} u = 0 & \text{on } \{\, |x| = 1 \,\} \times \{\, t \geq 0 \,\}, \\ u = \delta_0 & \text{on } \{\, |x| = 1 \,\} \times \{\, t = 0 \,\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{\, |x| \leq 1 \,\} \times \{\, \varepsilon \leq t \leq T \,\}$ for small $\varepsilon > 0$ and large $T > 0$ pass to the limit as $\varepsilon \to 0^+$ and $T \to \infty$.)

SOLUTION. Taking the hint, let us verify that $u_t = u_{xx}$, for $t > 0$. By direct computation, we have

$$\Phi_x(x,t) = \frac{\partial}{\partial x}\left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \qquad\qquad \Phi_{xx}(x,t) = \frac{\partial}{\partial x}\left(-\frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}\right)$$

$$= -\frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}, \qquad\qquad = \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}$$

$$\qquad\qquad\qquad\qquad = \frac{(x^2 - 2t)e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}},$$

and

$$\Phi_t(x,t) = \frac{\partial}{\partial t}\left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right)$$

$$= \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}$$

$$= \frac{(x^2 - 2t)e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}}.$$

Since $\Phi_t = \Phi_{xx}$ it follows that $u_t = u_{xx}$ (for $t > 0$).

Next we show that $u = 0$ on $\{\,|x| = 1\,\} \times \{\,t \geq 0\,\}$ and $u = \delta_0$ on $\{\,|x| = 1\,\} \times \{\,t = 0\,\}$. To show $u = 0$ fix a $t \geq 0$ and after relabeling, we may assume that $x = 1$ which gives us

$$u(1, t) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-\frac{k^2}{4t}}}{\sqrt{4\pi t}}$$

$$= \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} (-1)^k \left(e^{-\frac{1}{4t}}\right)^{k^2}$$

$$=$$

$\blacksquare$

PROBLEM 8.2 (TIKHONOV'S EXAMPLE)

Let
$$g(t) := \begin{cases} e^{-t^2} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then $g \in C^\infty(\mathbf{R})$ and we define
$$u(x,t) := \sum_{k=0}^\infty \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that $u(x,t)$ solves the heat equation in $\mathbf{R} \times (0, \infty)$ with the initial condition $u(x,0) = 0$, $x \in \mathbf{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

SOLUTION.   ∎

PROBLEM 8.3

Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax) e^{-x^2} \, dx, \qquad (a > 0).$$

(*Hint:* Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.)

SOLUTION. ■