MA 661: Homework 1

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Problem 1.1 (Lee, Prob. 3-1)

Suppose $(\widetilde{M}, \tilde{g})$ is a Riemannian m-manifold, $M \subset \widetilde{M}$ is an embedded n-dimensional submanifold, and g is the induced Riemannian metric on M. For any point p show that there is a neighborhood $\widetilde{U}$ of p in $\widetilde{M}$ and a smooth orthonormal frame $(E_1, ..., E_m)$ on $\widetilde{U}$ such that $(E_1, ..., E_m)$ form an orthonormal basis for T_qM at each $q \in \widetilde{U} \cap M$. Any such frame is called an adapted orthonormal frame. [Hint: Apply the Gram–Schmidt algorithm to the coordinate frame $\{\partial_i\}$ in slice coordinates.]

Proof. ■

Problem 1.2 (Lee, Prob. 3-2)

Suppose g is a pseudo-Riemannian metric on an n-manifold M. For any $p \in M$, show there is a smooth local frame $(E_1, ..., E_n)$ defined in a neighborhood of p such that g can be written locally in the form (3.4). Conclude that the index of g is constant on each component of M.

Proof. ■

Problem 1.3 (Lee, Prob. 3-3)

Let (M, g) be an oriented Riemannian manifold with volume element $\mathrm{d}V$. The divergence operator $\operatorname{div} \colon \mathcal{T}(M) \to C^\infty(M)$ is defined by

$$d(i_X \ \mathrm{d}V) = (\operatorname{div} X) \ \mathrm{d}V,$$

where i_x denotes interior multiplication by X: for any k-form ω, $i_x\omega$ is the $(k-1)$-form defined by

$$i_X\omega(V_1, ..., V_{k_1}) = \omega(X, V_1, ..., V_{k-1}).$$

(a) Suppose M is a compact, oriented Riemannian manifold with boundary. Prove the following divergence theorem for $X \in \mathcal{T}(M)$:

$$\int_M \operatorname{div} X \ \mathrm{d}V = \int_{\partial M} \langle X, N \rangle \ \mathrm{d}\widetilde{V}.$$

where N is the outward unit normal to ∂M and $\mathrm{d}\widetilde{V}$ is the Riemannian volume element of the induced metric on ∂M.

(b) Show that the divergence operator satisfies the following product rule for a smooth function $u \in C^\infty(M)$:

$$\operatorname{div}(uX) = u \operatorname{div} X + \langle \operatorname{grad} u, X \rangle,$$

and deduce the following "integration by parts" formula:

$$\int_M \langle \operatorname{grad} u, X \rangle \ \mathrm{d}V = -\int_M u \operatorname{div} X \ \mathrm{d}V + \int \partial M u \langle X, N \rangle \, d\widetilde{V}.$$

Proof. ∎

Problem 1.4 (Lee, Prob. 3-4)

Let (M, g) be a compact, connected, oriented Riemannian manifold with boundary. For $u \in C^\infty M$, the Laplacian of u, denoted Δu, is defined to be the function $\Delta u := \operatorname{div}(\operatorname{grad} u)$. A function $u \in C^\infty(m)$ is said to be harmonic if $\Delta u = 0$.

(a) Prove Green's identities:

$$\int_M u \, \Delta \, v \; \mathrm{d}V + \int_M \langle \operatorname{grad} u, \operatorname{grad} v \rangle = \int_{\partial M} uNv \; \mathrm{d}\widetilde{V}$$

$$\int_M (u \, \Delta \, v - v \, \Delta \, u) \; \mathrm{d}V = \int_{\partial M} (uNv - vNu) \; \mathrm{d}\widetilde{V}$$

(b) Show if $\partial M \neq \emptyset$, and u, v are harmonic functions on M whose restriction to ∂M agree, then $u \equiv v$.

(c) If $\partial M = \emptyset$ show that the only harmonic functions on M are the constants.

Proof. ■

Problem 1.5 (Lee, Prob. 3-5)

Let M be a compact oriented Riemannian manifold (without boundary). A real number λ is called an eigenvalue of the Laplacian if there exists a smooth function u on M, not identically zero, such that $\Delta\, u = \lambda u$. In this case, u is called an eigenfunction corresponding to λ.

(a) Prove that 0 is an eigenvalue of Δ, and that all other eigenvalues are strictly negative.

(b) If u and v are eigenfunctions corresponding to distinct eigenvalues, show that $\int_M uv \, \mathrm{d}V = 0$.

Proof. ∎