MA571 Homework 8

Carlos Salinas

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Problem 8.1 (Munkres §46, Ex. 6)

Show that the compact-open topology, $\mathcal{C}(X, Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular. [*Hint:* If $\overline{U} \subset V$, then $\overline{S(C, U)} \subset S(C, V)$.]

Proof. We will first prove the following fact:

Lemma. *If $C \subset X$ is finite, it is compact.*

Proof. Let $C \subset X$ be finite. Put $C = \{x_1, ..., x_n\}$ and let $\{U_\alpha\}$ be an open cover of C. Suppose that there is no finite subcollection of $\{U_\alpha\}$ which covers C. Then, for every U_α there is a distinct point $x \in C \cap U_\alpha$. This contradicts the fact that C is finite. ♣

Now, suppose Y is Hausdorff. Let $f, g \in \mathcal{C}(X, Y)$ with $f \neq g$, i.e., there exists a point $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Since Y is Hausdorff, there exists disjoint neighborhoods U and V of $f(x_0)$ and $g(x_0)$, respectively. Let $U' = S(\{x_0\}, U)$ and $V' = S(\{x_0\}, V)$; note that $\{x_0\}$ is compact by the lemma and U' and V' are subbasis elements of the compact-open topology by the definition on Munkres §46, p. 285. Then $U' \cap V' = \emptyset$ for otherwise, there is a function $h \in U' \cap V'$ such that $h(x_0) \in U \cap V$, but this contradicts $U \cap V = \emptyset$. Thus, $\mathcal{C}(X, Y)$ is Hausdorff.

Now, suppose Y is regular. We will proceed by the hint: Suppose $f \in \overline{S(C, U)}$. ■

Problem 8.2 (Munkres §46, Ex. 7)

Show that if Y is locally compact Hausdorff, then composition of maps

$$\mathcal{C}(X,Y)\times\mathcal{C}(Y,Z)\longrightarrow\mathcal{C}(X,Z)$$

is continuous, provided the compact-open topology is used throughout. [*Hint:* If $g\circ f\in S(C,U)$, find V such that $f(C)\subset V$ and $g(\overline{V})\subset U$.]

Proof. ∎

PROBLEM 8.3 (MUNKRES §46, EX. 8)

Let $\mathcal{C}'(X, Y)$ denote the set $\mathcal{C}(X, Y)$ in some topology $\mathcal{T}$. Show that if the evaluation map

$$e\colon X \times \mathcal{C}'(X, Y) \longrightarrow Y$$

is continuous, then $\mathcal{T}$ contains the compact-open topology. [*Hint:* The induced map $E\colon \mathcal{C}'(X, Y) \to \mathcal{C}(X, Y)$ is continuous.]

Proof. ∎

Problem 8.4 ((A))

Definition 1. Definition. If X is a locally compact Hausdorff space then the space Y given by Theorem 29.1 is called the *one-point compactification* of X.

Let X be a compact Hausdorff space and let W be an open subset of X (so W is locally compact by Corollary 29.3) with $W \neq X$. Prove that the one-point compactification of W is homeomorphic to the quotient space $X/(X - W)$.

Proof. ■

Problem 8.5 ((B))

Let X be a compact Hausdorff space, let Y be a topological space, and let $p\colon X \to Y$ be a closed surjective continuous map. Prove that Y is Hausdorff. [*Hint:* one ingredient in the proof is p. 171 # 5.]

Note: combining this with HW 4 Problem E and HW 6 Problem A gives a necessary and sufficient condition for a quotient of a compact Hausdorff space to be Hausdorff.

Proof. ■

Problem 8.6 ((C))

Let $S^2 \subset \mathbf{R}^3$ be the subspace

$$\{\, (x, y, z) \mid x^2 + y^2 + z^2 = 1 \,\}.$$

Prove that S^2 is a 2-manifold. (The definition of m-manifold, where m is a positive whole number, is given at the top of page 225.)

Proof. ■

Problem 8.7 ((D))

Prove that the union of the x and y-axes in $\mathbf{R}^2$ is not a 1-manifold.

Proof. ∎