MA 523: Homework 8

Carlos Salinas

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Problem 8.1

Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t)$$

where

$$\Phi(x,t) = \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

is positive for $|x| < 1$, $t > 0$.
(*Hint:* Show that u satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{cases} u = 0 & \text{on } \{\, |x| = 1 \,\} \times \{\, t \geq 0 \,\}, \\ u = \delta_0 & \text{on } \{\, |x| = 1 \,\} \times \{\, t = 0 \,\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{\, |x| \leq 1 \,\} \times \{\, \varepsilon \leq t \leq T \,\}$ for small $\varepsilon > 0$ and large $T > 0$ pass to the limit as $\varepsilon \to 0^+$ and $T \to \infty$.)

Solution. Taking the hint, let us first show that $u_t = u_{xx}$. By direct verification, we have

$$\begin{aligned}
\Phi_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \\
&= -\frac{x^2 \mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi} t^{\frac{5}{2}}} - \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi} t^{\frac{3}{2}}} \\
&= -\frac{(x^2 + 4t)\mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi} t^{\frac{5}{2}}}, \\
\Phi_x(x,t) &= \frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \\
&= -\frac{x^2 \mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi} t^{\frac{3}{2}}}, \\
\Phi_{xx}(x,t) &= \frac{\partial}{\partial x}\left(-\frac{x^2 \mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi} t^{\frac{3}{2}}}\right) \\
&= \frac{x^3 \mathrm{e}^{-\frac{x^2}{4t}}}{8\sqrt{4\pi} t^{\frac{5}{2}}} - \frac{x \mathrm{e}^{-\frac{x^2}{4t}}}{2\sqrt{4\pi} t^{\frac{3}{2}}} \\
&= \frac{(x^3 - 4xt)\mathrm{e}^{-\frac{x^2}{4t}}}{8\sqrt{4\pi} t^{\frac{5}{2}}}
\end{aligned}$$

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Problem 8.2 (Tikhonov's example)

Let

$$g(t) := \begin{cases} \mathrm{e}^{-t^2} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then $g \in C^\infty(\mathbb{R})$ and we define

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that $u(x,t)$ solves the heat equation in $\mathbb{R} \times (0, \infty)$ with the initial condition $u(x, 0) = 0$, $x \in \mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem.)

Solution. ■

Problem 8.3

Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax)\mathrm{e}^{-x^2}\,dx, \qquad (a > 0).$$

(*Hint:* Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.)

Solution. ■