MA166: Solutions to Homework 1-10

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1 Homework 8 Solutions

Problem 1.1 (WebAssign, HW8, 1)**.** A variable force of $2x^{-2}$ pounds moves an object along a straight line when it is x feet from the origin. Calculate the work done in moving the object from $x = 1$ ft to $x = 15$ ft. (Round your answer to two decimal places.)

Solution. Recall the formula for the work done by a fore as a function of distance? It is given by

$$W = \int_1^1 5F(x)\,\mathrm{d}x = 2\int_1^{15} x^{-2}\,\mathrm{d}x = -2\int_1^{15} \frac{1}{x} = -\frac{2}{x}\bigg|_1^{15} = -\frac{2}{15} + \frac{2}{1} = \frac{28}{15}.$$

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Problem 1.2 (WebAssign, HW8, 2)**.** Shown is the graph of a force function (in Newtons) that increases to its maximum value and then remains constant. How much work W is done by the force in moving an object a distance of 24 m?

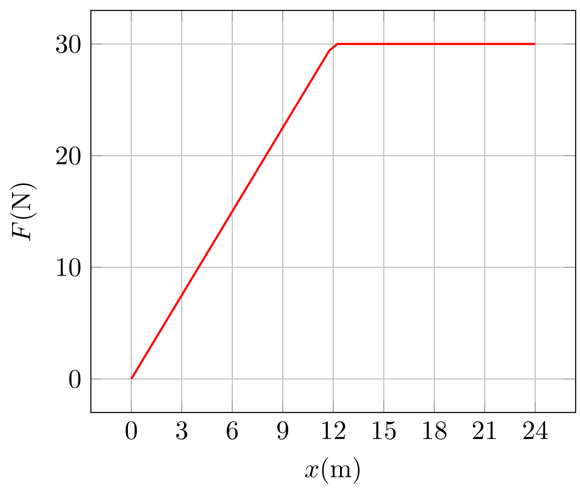

Figure 1.1: Graph of the force F with respect to distance.

Solution. Remember that the work done by a force $F(x)$ over a distance $a \leq x \leq b$ is given by the formula

$$W = \int_a^b F(x)\,\mathrm{d}x. \tag{1}$$

Since the graph in Figure (1.1) is piecewise, all we need to do is break up our interval $0 \leq x \leq 24$ into the part where y is a diagonal line and where y is a horizontal line, compute the work done on each interval, W_1 and W_2, and add them up to get the total work $W = W_1 + W_2$. First, note that over $0 \leq x \leq 12$, $F(x) = \frac{30}{12}x = \frac{5}{2}x$ so the work done from $0 \leq x \leq 12$ is

$$W_1 = \int_0^{12} \frac{5}{2}x\,\mathrm{d}x = \frac{5}{4}x^2\bigg|_0^{12} = \frac{5}{4}(12)^2 = 180\,\mathrm{J}.$$

Next, we see that $F(x) = 30$ is constant for all $12 \leq x \leq 24$ so we have

$$W_2 = \int_{12}^{24} F(x)\,\mathrm{d}x = \int_{12}^{24} 30\,\mathrm{d}x = 30x|_{12}^{24} = 30 \cdot 24 - 30 \cdot 12 = 30 \cdot 12 = 360\,\mathrm{J}.$$

Hence, the total work done by $F(x)$ over $0 \leq x \leq 24$ is

$$\boxed{W = W_1 + W_2 = 180\,\mathrm{J} + 360\,\mathrm{J} = 540\,\mathrm{J}.}$$

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Problem 1.3 (WebAssign, HW8, 3)**.** A force of 6 lb is required to hold a spring stretched 8 in beyond its natural length. How much work W is done in stretching it from its natural length to 14 in beyond its natural length?

Solution. Recall Hooke's law for the force required to stretch a spring a distance x beyond its natural length

$$F(x) = kx. \tag{2}$$

Now, we are given a force of 6 lb and a distance of 8 in. Using this information, we can figure out what the coefficient k must be:

$$k = F/x = 6/8\text{lb/in}.$$

Using the Equation (1) for work, we have that the work done on a spring by stretching it from x_1 to x_2 is

$$W(x') = \int_{x_1}^{x_2} F(x)\,\mathrm{d}x = \int_{0}^{x'} kx\,\mathrm{d}x = \frac{1}{2}kx^2\Big|_0^{x'} = \frac{1}{2}k\big({x_1}^2 - {x_2}^2\big) \tag{3}$$

so, plugging in 14, into our equation $W(x')$ above we get

$$W(14) = \frac{1}{2} \cdot \frac{6}{8} \cdot 14^2 = 7 \cdot \frac{6}{8} = \boxed{\frac{49}{8}\ \text{ft-lb.}}$$

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Problem 1.4 (WebAssign, HW8, 4)**.** If the work required to stretch a spring 3 ft beyond its natural length is 9 ft-lb, how much work is needed to stretch it 9 in beyond its natural length?

Solution. We employ the same idea as the one we used to calculate the work on the last problem. First, we find the coefficient k, wit the help of Equation (3) this value will be

$$k = \frac{2W}{x^2} = \frac{2 \cdot 9}{9} = 2\ \text{lb/ft}.$$

Now, convert our 9 in to ft we get 3/4 ft. Lastly, applying Equation (3) on 3/4 ft we get

$$W(3/4) = \frac{1}{2} \cdot 2 \cdot (3/4)^2 = \boxed{\frac{9}{16}\ \text{in-lb.}}$$

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Problem 1.5 (WebAssign, HW8, 5)**.** An aquarium 6 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use 9.8 m/s^2 for g and the fact that the density of water is 1000 kg/m^3.)

- Show how to approximate the required work by a Riemann sum. (Let x be the height in meters below the top of the tank. Enter x_i^* as x_i.)
- Express the work as an integral.

Solution. Let's place the origin at the top of the tank. Then, the expression for the work required to pump out half the water from our tank will be given approximately by

$$W(x) \approx \lim_{n\to\infty} \sum_{i=1}^{n} 9.8 \cdot 1000 \cdot 6 \cdot x_i \Delta x = \boxed{\sum_{i=1}^{n} 58800 x_i^* \Delta x.}$$

Now we compute the integral from $0 \le x \le 1/2$. We have

$$W = 58800 \int_0^{1/2} x \, \mathrm{d}x = 5800 \left(\frac{1}{2} x^2 \Big|_0^{1/2} \right) = 29400\big((1/2)^2 - 0\big) = \boxed{7350\,\text{J.}}$$

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Problem 1.6 (WebAssign, HW8, 6)**.** A tank is full of water (see Figure 1.2). Find the work W required to pump the water out of the spout. (Use 9.8 m/s^2 for g. Use 1000 kg/m^3 as the weight density of water. Assume that $a = 4$ m, $b = 4$ m, $c = 12$ m, and $d = 2$ m.)

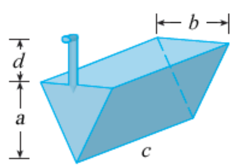

Figure 1.2: A sketch of the tank.

Solution. The work required to pump the water out of the spout of a tank with the dimensions above is given by the

$$W = \lim_{n\to\infty} \sum_{i=0}^{n} F(x_i)\Delta x$$

where $0\,\text{m} \le x \le 8\,\text{m}$. Now, the first thing we need to do is to find the way the force changes as we move up the tank. Let x be the vertical distance from the bottom of the tank and y be the horizontal distance from. Then, the work needed to lift the slice slice of volume ΔV that is a distance $(4+2) - x$ from the spout is

$$W(x) \approx g\rho(6 - x)\Delta V, \tag{4}$$

where ρ is the density of the water and g is gravity. Now, how does our little slice of volume change as we move up the tank? If y is the side of the right-triangle made by going up a distance x up the tank, then

$$\frac{x}{y} = \frac{4}{4} = 1$$

since the triangles are similar. Hence, $y = 2x$ so the volume change will be

$$V(x) \approx 12 \cdot \frac{1}{2}(2x)\Delta x = 12x\Delta x(6 - x),$$

this is just the area of a triangle $x\Delta x$ (base times height) times the depth 12 m. Hence, the, plugging in the last equation into our equation for the force (Equation (4)) will be

$$W(x) \approx g\rho(6 - x)(12x\Delta x) = 117600(6 - x)x\Delta x$$

Last but not least, we compute the limit as $n \to \infty$,

$$W = \lim_{n\to\infty} \sum_{i=0}^{n} W(x_i)\Delta x$$

this is just the integral

$$\begin{aligned}
&= 117600 \int_0^4 (6 - x)x \, \mathrm{d}x \\
&= 117600 \int_0^4 6x - x^2 \, \mathrm{d}x \\
&= 117600 \left(3x^2 - \frac{1}{3}x^3 \Big|_0^4 \right) \\
&= 117600 \left(3 \cdot 4^2 - \frac{4^3}{3} - (0 - 0) \right) \\
&= 117600 \left(\frac{3 \cdot 3 \cdot 4^2 - 4^3}{3} \right) \\
&= 117600 \left(4^2 \cdot \frac{9 - 4}{3} \right) \\
&= 117600 \left(4^2 \cdot \frac{5}{3} \right) \\
&= \boxed{3\,136\,000\,\mathrm{J}.}
\end{aligned}$$

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Problem 1.7 (WebAssign, HW8, 7)**.** Consider the given function and the given interval

$$f(x) = 10 \sin x - 5 \sin 2x, \quad [0, \pi]$$

(a) Find the average value f_{ave} of f on the given interval.

(b) Find c such that $f_{\text{ave}} = f(c)$.

(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

Solution. (a) Recall the definition of the average of a function over an interval $[a, b]$, it is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f \, \mathrm{d}x. \tag{5}$$

Now, all we need to do is calculate the integral of our f and divide that value by $\pi - 0$, i.e.,

$$\begin{aligned}
f_{\text{ave}} &= \frac{1}{\pi}\int_0^\pi f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\pi 10\sin x - 5\sin 2x \,\mathrm{d}x \\
&= \frac{1}{\pi}\left(-10\cos x + \frac{5}{2}\cos 2x\Big|_0^\pi\right) \\
&= \frac{1}{\pi}\left(-10\cos\pi + \frac{5}{2}\cos 2\pi - \left(-10\cos 0 + \frac{5}{2}\cos 0\right)\right) \\
&= \frac{1}{\pi}\left(10 + \frac{5}{2} + 10 - \frac{5}{2}\right) \\
&= \frac{1}{\pi}\cdot 20 \\
&= \boxed{\frac{20}{\pi}.}
\end{aligned}$$

(b) For this part all we need to do is set the equation $f(x) = 10\sin x - 5\sin 2x$ equal to $20/\pi$ and solve for x:

$$10\sin x - 5\sin 2x = 5(2\sin x - \sin 2x) = \frac{20}{\pi}$$

so

$$2\sin x - \sin 2x = \frac{4}{\pi}.$$

I can't think of any way to solve this than plugging it into your calculator and having your calculator approximate the solution (your calculator probably uses Newton's method, which you will learn about later in your career). The values are about $\underline{c \approx 1.238 \text{ and } 2.808.}$

(c) Sketching the graph is easy and the box having the same area as will have height equal to $20/\pi$ and length π. Multiply these together and we get the area under the curve, which we computed to be 20. ☺

Problem 1.8 (WebAssign, HW8, 8)**.**

Solution. ☺

2 Homework 9 Solutions

Problem 2.1 (Stewart §7.1, Exercise 1).

Solution. ☺

Problem 2.2 (Stewart §7.1, Exercise 3).

Solution. ☺

Problem 2.3 (Stewart §7.1, Exercise 10).

Solution. ☺

Problem 2.4 (Stewart §7.1, Exercise 17).

Solution. ☺

Problem 2.5 (Stewart §7.1, Exercise 27).

Solution. ☺

Problem 2.6 (Stewart §7.1, Exercise 37).

Solution. ☺

Problem 2.7 (Stewart §7.1, Exercise 62).

Solution. ☺

3 Homework 10 Solutions

Problem 3.1 (Stewart §7.2, Exercise 1)**.**

Solution. ☺

Problem 3.2 (Stewart §7.2, Exercise 7)**.**

Solution. ☺

Problem 3.3 (Stewart §7.2, Exercise 11)**.**

Solution. ☺

Problem 3.4 (Stewart §7.2, Exercise 17)**.**

Solution. ☺

Problem 3.5 (Stewart §7.2, Exercise 23)**.**

Solution. ☺

Problem 3.6 (Stewart §7.2, Exercise 24)**.**

Solution. ☺

Problem 3.7 (Stewart §7.2, Exercise 35)**.**

Solution. ☺

Problem 3.8 (Stewart §7.2, Exercise 61)**.**

Solution. ☺