# MA544: Qual Preparation

Carlos Salinas

August 7, 2016

## Contents

# 1 MA 544 Spring 2016

This is material from the course MA 544 as it was taught in the spring of 2016.

## 1.1 Homework

These exercises were assigned from Wheeden and Zygmund's *Measure and Integral*, therefore, most of the theorems I reference will be from [4]. Other resources include [1] and [2]. For more elementary results, I cite [3]. Unless otherwise stated, whenever we quote a result, e.g., Theorem 1.1, it is understood to come from Wheeden and Zygmund's *Measure and Integral*.

Throughout these notes

| | |
|---|---|
| $\mathbb{R}$ | is the set of real numbers |
| $\mathbb{R}^+$ | is the set of positive real numbers, that is, $x \in \mathbb{R}$ with $x \geq 0$ |
| $\mathbb{C}$ | is the set of complex numbers |
| $\mathbb{Q}$ | is the set of rational numbers |
| $\mathbb{Z}$ | is the set of the integers |
| $\mathbb{Z}^+$ | is the set of positive integers, that is, $x \in \mathbb{Z}$ with $x \geq 0$ |
| $\mathbb{N}$ | is the set of the natural numbers $1, 2, \ldots$ |
| $A \setminus B$ | is the set difference of $A$ and $B$, that is, the complement of $A \cap B$ in $A$ |
| $m^*(E)$ | the outer measure of $E$ |
| $m_*(E)$ | the inner measure of $E$ |
| $m(E)$ | the Lebesgue measure of $E$ |
| $\lVert - \rVert$ | the standard Euclidean norm on $\mathbb{R}^n$ |
| $f \asymp g$ | means $f$ is asymptotically equivalent to $g$, that is, $\lim_{x\to\infty} g(x)/f(x) = 1$ |

### 1.1.1 Homework 1

**Exercise 1** (Wheeden & Zygmund Ch. 2, Ex. 1)**.** Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that $f$ is bounded and continuous on $[0, 1]$, but that $V[f; 0, 1] = \infty$.

**Solution**. ► Let $f$ equal $x\sin(1/x)$. We will show that $f$ is bounded and continuous on $[0, 1]$, but that it is not of bounded variation on $[0, 1]$.

First we will show that $f$ is bounded. Note that both $|x|$ and $|\sin(1/x)|$ are bounded by 1 on the interval $[0, 1]$. Since $|f| = |x||\sin(1/x)|$, it follows that $|f| \leq 1$ on $[0, 1]$. Thus, f is bounded on $[0, 1]$.

Next we show that $f$ is continuous. It is easy to show that $f$ is continuous on the subinterval $(0, 1]$ since both $|x|$ and $\sin(1/x$ are continuous on that interval and we know that the product of continuous functions is continuous. To see that $f$ is continuous at 0 we must show that $f(x^+) = f(0)$; that is, the limit of $f$ as x approaches 0 from the right is $f(0)$ which by definition is 0. To this end, it suffices to take a (monotonically decreasing) sequence $x_n \downarrow 0$ and show that the limit of the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is 0. Let $\varepsilon > 0$ be given then, since $x_n$ converges to 0 there exists an index $N$ such that $|0 - x_n| < \varepsilon$ whenever $n \geq N$. Since $|f(x_n)| \leq |x_n|$ on $[0, 1]$, the following inequality holds

$$\begin{aligned} |0 - f(x_n)| &= |0 - x_n \sin(1/x_n)| \\ &\leq |x_n| \\ &< \varepsilon. \end{aligned}$$

Thus, $f$ is continuous at 0 and it converges to 0.

Despite the nice properties that $f$ seemingly possesses, $f$ is not b.v. on $[0, 1]$. To show that $f$ is not b.v. on $[0, 1]$ we must show that for any positive real number $M$ there exists some partition $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ of $[0, 1]$ such that the sum associated to $\Gamma$

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| > M.$$

Let $N$ be the smallest integer greater than $M$ and let $n$ be the smallest integer greater than or equal to $N/2$. Then the partition $\Gamma = \{ x_0 = 1 < x_1 < \cdots < x_{n+1} = 1 \}$ where $x_i = 2/((3 + (n - i))\pi$ for $1 \leq i \leq N$. Then we have the inequality

$$\begin{aligned} S_\Gamma &= \sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})| + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &= N + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &> M. \end{aligned}$$

Thus, $f$ is not b.v. on $[0, 1]$. ◄

**Exercise 2** (Wheeden & Zygmund Ch. 2, Ex. 2)**.** Prove theorem (2.1).

**Solution**. ► Recall the statement of Theorem 2.1:

(a) If $f$ is of bounded variation on $[a, b]$, then $f$ is bounded on $[a, b]$.

(b) Let $f$ and $g$ be of bounded variation on $[a, b]$. Then $cf$ (for any real constant $c$), $f+g$, and $fg$ are of bounded variation on $[a, b]$. Moreover, $f/g$ is of bounded variation on $[a, b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a, b]$.

We shall prove these in alphabetical order:

For part (a) we shall proceed by contradiction. First, without loss of generality, we may assume that $f(a) = 0$ since the function the variation of $g(x) = f(x) - f(a)$ is equal to the variation of $f$ and $g(a) = 0$. Suppose that $f$ is b.v. on $[a, b]$ with variation $V = V[f; a, b]$, but that $f$ is unbounded on $[a, b]$; that is, given a positive real number $M$ there exists a point $x$ in $[a, b]$ such that $|f(x)| > M$. In particular, there exists $x \in [a, b]$ such that $|f(x)| > V$. Hence, for any $x \in [a, b]$ by the triangle inequality we have

$$\begin{aligned} V &< |f(x)| \\ &= |f(x) - f(a) + f(a)| \\ &\leq |f(x) - f(a)| + |f(a)| \\ &\leq V. \end{aligned}$$

This is a contradiction. Therefore, it must be the case that if $f$ is b.v. on $[a, b]$ then $f$ is bounded on $[a, b]$.

We break part (b) into three sections. Suppose $f$ and $g$ are b.v. on $[a, b]$ with variation $V$ and $V'$, respectively. We will show that (i) $cf$; (ii) $f+g$; and (iii) $fg$ are b.v. on $[a, b]$. Moreover, we show that (iv) $f/g$ is b.v. on $[a, b]$ if there exists $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for all $x \in [a, b]$.

For part (i) above let $c$ be a real number. Given a partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of $[a, b]$, we have

$$\begin{aligned} S_\Gamma &= \sum_{i=1}^{n} |cf(x_i) - cf(x_{i-1})| \\ &= \sum_{i=1}^{n} |c||f(x_i) - cf(x_{i-1})| \\ &= |c| \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &\leq |c|V \end{aligned}$$

since $V$ is the supremum of the sums of the form $\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|$ over all partitions of $[a, b]$. Thus, $V[cf; a, b] \leq |c|V$ so $cf$ is b.v. on $[a, b]$.

For part (ii) given a partition $\Gamma = \{\, x_0 < x_1 < \cdots < x_n \,\}$ of the interval $[a, b]$, by the triangle inequality we have

$$\begin{aligned} S_\Gamma &= \sum_{i=1}^{n} |(f(x_i) + g(x_i)) - (f(x_{i-1}) + g(x_{i-1}))| \\ &= \sum_{i=1}^{n} |(f(x_i) - f(x_{i-1})) + (g(x_i) - g(x_{i-1}))| \\ &\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \\ &\leq V + V'. \end{aligned}$$

Thus, $f + g$ is b.v. on $[a, b]$

For part (iii) since $f$ and $g$ are b.v. on $[a, b]$ by part (a) $f$ and $g$ are bounded on $[a, b]$ by, say, $M$ and $N$, respectively. Now, given a partition $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ of $[a, b]$, by the triangle inequality we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^{n} |f(x_i)g(x_i) \quad - f(x_{i-1})g(x_{i-1}) \\
&\qquad\qquad + f(x_i)g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^{n} |(f(x_i)g(x_i) \quad - f(x_i)g(x_{i-1})) \\
&\qquad\qquad - (f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1}))| \\
&\leq \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_i)g(x_{i-1})| \\
&\qquad\qquad + \sum_{i=1}^{n} |f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^{n} |f(x_i)||g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} |g(x_{i-1})||f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^{n} M|g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} N|f(x_i) - f(x_{i-1})| \\
&\leq MV' + NV.
\end{aligned}$$

Thus, $fg$ is b.v. on $[a, b]$.

Finally, for part (iv) suppose there exists $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for all $x \in [a, b]$. Then, given a partition $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ of $[a, b]$, largely by the triangle inequality, we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^{n} |f(x_i)/g(x_i) - f(x_{i-1})/g(x_{i-1})| \\
&= \sum_{i=1}^{n} \left| \frac{f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)}{g(x_i)g(x_{i-1})} \right| \\
&\le \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1}) \\
&\qquad\qquad - (f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1}))| \\
&\le \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |g(x_{i-1})||f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_{i-1})||g(x_i) - g(x_{i-1})| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^{n} M_g |f(x_i) - f(x_i)| + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} M_f |g(x_i) - g(x_i)| \\
&= \frac{1}{\varepsilon^2} M_g \sum_{i=1}^{n} |f(x_i) - f(x_i)| + \frac{1}{\varepsilon^2} M_f \sum_{i=1}^{n} |g(x_i) - g(x_i)| \\
&\le \frac{1}{\varepsilon^2} (NV + MV')
\end{aligned}$$

where, as above, $f$ is bounded by $M$ and $g$ is bounded by $N$. Thus, $f/g$ is b.v. on $[a, b]$.

This concludes the proof of Theorem 2.1. ◂

**Exercise 3** (Wheeden & Zygmund Ch. 2, Ex. 3)**.** If $[a', b']$ is a subinterval of $[a, b]$ show that $P[a', b'] \le P[a, b]$ and $N[a', b'] \le N[a, b]$.

**Solution**. ▸ We will prove this by digging in to the definition of $N$ and $P$. Recall that given a partition $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ of the interval $[a, b]$, $P$ and $N$ are defined to be the supremum over the sum of the positive and, respectively, the sum negative terms of $S_\Gamma$; that is, $P$ and $N$ are the supremum over every partition $\Gamma$ of $[a, b]$ of

$$P_\Gamma = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+ \quad \text{and} \quad N_\Gamma = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^- .$$

Let $f\colon [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$ and let $[a', b']$ be a subinterval of $[a, b]$. Without loss of generality, we may assume that $[a', b']$ is strictly contained in $[a, b]$; that is, $a' \ne a$ and $b' \ne b$. We aim to show that $P[a', b'] \le P[a, b]$ and $N[a', b'] \le N[a, b]$. Since the argument for $N$ is similar to that of $P$, we will omit it here for the sake of brevity. Now, consider the closure of the complement of $[a', b']$ in $[a, b]$,

$\overline{[a,b] \setminus [a',b']} = [a,a'] \cup [b',b]$. Since $[a,a']$, $[a',b']$ and $[b',b]$ are close intervals we may take partitions

$$\begin{aligned} \Gamma_a &= \{ x_0 < x_1 \cdots < x_\ell \}, \\ \Gamma_{ab} &= \{ x_\ell < x_{\ell+1} < \cdots < x_m \} \end{aligned}$$

and

$$\Gamma_b = \{ x_m < x_{m+1} < \cdots < x_n \}$$

of $[a,a']$, $[a',b']$ and $[b',b]$, respectively and extend this to a partition

$$\Gamma = \{ x_0 < x_1 < \cdots < x_\ell < x_{\ell+1} \cdots < x_m < x_{m+1} \cdots < x_n \}$$

of $[a,b]$. Then, by the definition of $N$ we have the string of inequalities

$$\begin{aligned} P_\Gamma &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+ \\ &= \sum_{i=1}^{\ell} [f(x_i) - f(x_{i-1})]^+ \\ &\qquad + \sum_{i=\ell+1}^{m} [f(x_i) - f(x_{i-1})]^+ \\ &\qquad + \sum_{i=m+1}^{n} [f(x_i) - f(x_{i-1})]^+ \\ &= P_{\Gamma_{ab}} + P_{\Gamma_a} + P_{\Gamma_b} \\ &\le P[a,b]. \end{aligned}$$

Taking the supremum on the left, we have

$$P[a,a'] + P[a',b'] + P[b',a'] \le P[a,b].$$

Since $P$ is strictly positive, it must be the case that $P[a',b'] \le P[a,b]$. ◂

**Exercise 4** (Wheeden & Zygmund Ch. 2, Ex. 11)**.** Show that $\int_a^b f \, \mathrm{d}\varphi$ exists if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|, |\Gamma'| < \delta$.

**Solution**. ▸ One direction is straightforward. Namely $\Longleftarrow$ : suppose that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ whenever $|\Gamma|$ and $|\Gamma'|$ are less than $\delta$. Let $\{\Gamma_n\}_{n=1}^\infty$ be a decreasing sequence of partitions (by which we mean $\Gamma_n \subseteq \Gamma_{n+1}$ of $[a,b]$ such that $|\Gamma_n| \to 0$. Then, by convergence, there exists $N \in \mathbb{N}$ such that $n \ge N$ implies $|\Gamma_n| < \delta$. Then, for $n, m \ge N$, we have

$$|R_{\Gamma_n} - R_{\Gamma_m}| < \varepsilon.$$

Thus, by the Cauchy criterion for convergence, the sequence $\{R_{\Gamma_n}\}_{n=0}^\infty$ converges and its limit is by definition the Riemann–Stieltjes integral $\int_a^b f \, \mathrm{d}\varphi$.

On the other hand $\Longrightarrow$ : suppose that $I = \int_a^b f \,\mathrm{d}\varphi$ exists. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that $|I - R_\Gamma| < \varepsilon/2$ whenever $|\Gamma| < \delta$. Let $\Gamma$ and $\Gamma'$ be two partitions of $[a, b]$ with norm $|\Gamma|, |\Gamma'| < \delta$. Then we have

$$\begin{aligned} |R_\Gamma - R_{\Gamma'}| &= |R_\Gamma - I - (R_{\Gamma'} - I)| \\ &\le |R_\Gamma - I| + |R_{\Gamma'} - I| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, $I$ satisfies the Cauchy condition. ◂

**Exercise 5** (Wheeden & Zygmund Ch. 2, Ex. 13)**.** Prove theorem (2.16).

**Solution**. ▸ Recall the statement of Theorem 2.16:

(i) If $\int_a^b f \,\mathrm{d}\varphi$ exists, then so do $\int_a^b cf \,\mathrm{d}\varphi$ and $\int_a^b f \,\mathrm{d}(c\varphi)$ for any constant $c$, and

$$\int_a^b cf \,\mathrm{d}\varphi = \int_a^b f \,\mathrm{d}(c\varphi) = c \int_a^b f \,\mathrm{d}\varphi.$$

(ii) If $\int_a^b f_1 \,\mathrm{d}\varphi$ and $\int_a^b f_2 \,\mathrm{d}\varphi$ both exist, so does $\int_a^b (f_1 + f_2) \,\mathrm{d}\varphi$, and

$$\int_a^b (f_1 + f_2) \,\mathrm{d}\varphi = \int_a^b f_1 \,\mathrm{d}\varphi + \int_a^b f_2 \,\mathrm{d}\varphi.$$

(iii) If $\int_a^b f \,\mathrm{d}\varphi_1$ and $\int_a^b f \,\mathrm{d}\varphi_2$ both exist, so does $\int_a^b f \,\mathrm{d}(\varphi_1 + \varphi_2)$, and

$$\int_a^b f \,\mathrm{d}(\varphi_1 + \varphi_2) = \int_a^b f \,\mathrm{d}\varphi_1 + \int_a^b f \,\mathrm{d}\varphi_2.$$

We prove this in (Roman) numerical order.

For (i) suppose that $I = \int_a^b f \,\mathrm{d}\varphi$ exists. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|I - R_\Gamma| < \varepsilon/|c|$ whenever $\Gamma$ is a partition of $[a, b]$ with $|\Gamma| < \delta$. We claim that $\int_a^b cf \,\mathrm{d}\varphi = |c|I$. Let $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ be a partition $[a, b]$ with $|\Gamma| < \delta$. Then the Riemann–Stieltjes sums $R'_\Gamma$ of the pair $(cf, \varphi)$ associated to $\Gamma$ give us the chain of inequalities

$$\begin{aligned} ||c|I - R'_\Gamma| &= \left| |c|I - \sum_{i=1}^n cf(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c| \left| \sum_{i=1}^n cf(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c||I - R_\Gamma| \\ &< |c| \frac{\varepsilon}{|c|} \\ &= \varepsilon. \end{aligned}$$

Thus, $\int_a^b cf\,\mathrm{d}\varphi$ is Riemann–Stieltjes integrable and its integral is equal to $|c|I$. A similar argument shows that $\int_a^b f\,\mathrm{d}(c\varphi)$ is Riemann–Stieltjes integrable with integral $|c|I$.

For (ii) let $I_1 = \int_a^b f_1\,\mathrm{d}\varphi$ and $I_2 = \int_a^b f_2\,\mathrm{d}\varphi$. Then, we claim that $I = \int_a^b (f_1 + f_2)\,\mathrm{d}\varphi$ exists and that $I = I_1 + I_2$. Since both $I_1$ and $I_2$ exist, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|I_1 - R_\Gamma^1| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |I_2 - R_\Gamma^2| < \frac{\varepsilon}{2}$$

whenever $|\Gamma| < \delta$. Let $\Gamma = \{ x_0 < x_1 < \cdots < x_n \}$ be a partition of $[a, b]$ with $|\Gamma| < \delta$. Then the Riemann–Stieltjes sums $R_\Gamma$ of the pair $(f_1 + f_2, \varphi)$ associated to $\Gamma$ give is the following chain of inequalities

$$\begin{aligned}
|(I_1 + I_2) - R_\Gamma| &= \left|(I_1 + I_2) - \sum_{i=1}^{n}(f_1(\xi_i) + f_2(\xi_i))[\varphi(x_i) - \varphi(x_{i-1})]\right| \\
&= \left|I_1 - \sum_{i=1}^{n} f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})\right. \\
&\qquad \left. + I_2 - \sum_{i=1}^{n} f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})]\right| \\
&\leq \left|I_1 - \sum_{i=1}^{n} f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})\right| \\
&\qquad + \left|I_2 - \sum_{i=1}^{n} f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})]\right| \\
&= |I_1 - R_\Gamma^1| + |I_2 - R_\Gamma^2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus, $I$ exists and it is equal to the sum $I_1 + I_2$.

Part (iii) is similar to part (ii) in the above equation except that instead of splitting the sum at $f_1 + f_2$ part, we split it at $\varphi_1 + \varphi_2$ part. ◂

### 1.1.2 Homework 2

**Exercise 1.** Show that the boundary of any interval has outer measure zero.

**Solution.** ▸ Let $I = \prod_{i=1}^{n} I_i$ be a closed interval in $\mathbb{R}^n$ and let $J$ be the boundary of $I$. We must show that given $\varepsilon > 0$ there exists a countable collection of intervals $\{I_n\}_{n\in J}$ covering $J$ such that

$$\sum_{n\in J} \operatorname{vol}(I_n) < \varepsilon.$$

First, note that we can write $J$ as the union $\bigcup_{i=1}^{n} J_i$ where

$$J_i = [a_1, b_1] \times \cdots \times \{a_i\} \times \cdots \times [a_n, b_n] \cup [a_1, b_1] \times \cdots \times \{b_i\} \times \cdots \times [a_n, b_n].$$

Since the countable union of null sets has measure zero, it suffices to show that the set

$$[a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$$

has measure zero. Consider the collection $\{I_\varepsilon\}$ consisting of the single interval

$$I_\varepsilon = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \left[a_n - \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)}, a_n + \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)}\right].$$

It is clear that $I_\varepsilon \supseteq J$. Now, computing the volume of this interval, we have

$$\begin{aligned}
\operatorname{vol}(I_\varepsilon) &= \prod_{i=1}^{n-1}(b_i - a_i)\left[a_n + \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)} - \left(a_n - \frac{\varepsilon}{2\prod_{i=1}^{n-1}(b_i - a_i)}\right)\right] \\
&= \left[\prod_{i=1}^{n-1}(b_i - a_i)\right] \frac{\varepsilon}{\prod_{i=1}^{n-1}(b_i - a_i)} \\
&= \varepsilon.
\end{aligned}$$

Thus, $J$ has measure zero. ◂

**Exercise 2.** Show that a set consisting of a single point has outer measure zero.

**Solution.** ▸ Let $\{a\}$ be the set consisting of a single point $a \in \mathbb{R}$. Then we must show that given $\varepsilon > 0$ there exists a countable collection of intervals $\{I_n\}$ such that

$$\sum_{n\in J} m(I_n) < \varepsilon.$$

Consider the collection $\{I_\varepsilon\}$ consisting of the single interval

$$I_\varepsilon = \left[a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2}\right].$$

It is clear that $\{a\} \subseteq I_\varepsilon$. Moreover,

$$\begin{aligned}
\operatorname{vol}(I_\varepsilon) &= a + \frac{\varepsilon}{2} - \left(a - \frac{1}{\varepsilon}\right) \\
&= \varepsilon.
\end{aligned}$$

Thus, $\{a\}$ has measure zero. ◂

### 1.1.3 Homework 3

**Exercise 1** (Wheeden & Zygmund Ch. 3, Ex. 5)**.** Construct a subset of $[0, 1]$ in the same manner as the Cantor set, except that at the $k$th stage each interval removed has length $\delta 3^{-k}$, $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no interval.

**Solution**. ▸ We construct the prescribed subset as follows: take the open interval $(1/2 - \delta/6, 1/2 + \delta/6)$ and remove it from the closed interval $[0, 1]$ the result is a union of two disjoint closed intervals

$$E_{1,1} = \left[0, \frac{1}{2} - \frac{1}{6}\delta\right], \quad E_{1,2} = \left[\frac{1}{2} + \frac{1}{6}\delta, 1\right],$$

whose union we call $E_1$; this marks the first step in the construction of this Cantor-like set. Next, we remove the set

$$\left(\frac{1}{4} - \frac{5}{36}\delta, \frac{1}{4} + \frac{1}{36}\delta\right) \cup \left(\frac{3}{4} + \frac{\delta}{36}, \frac{3}{4} + \frac{5}{36}\delta\right)$$

from the set $E_1$ which yields $E_2$ the union of the four closed intervals

$$E_{2,1} = \left[0, \frac{1}{4} - \frac{5}{36}\delta\right], \qquad E_{2,2} = \left[\frac{1}{4} + \frac{1}{36}\delta, \frac{1}{2} - \frac{1}{6}\delta\right],$$
$$E_{2,3} = \left[\frac{1}{2} + \frac{1}{6}\delta, \frac{3}{4} + \frac{\delta}{36}\right], \quad E_{2,4} = \left[\frac{3}{4} + \frac{5}{36}\delta, 1\right].$$

In the $n$th step of the construction, we remove an open interval of length $3^{-n}\delta$ from the center of each interval $E_{n-1,i}$ yielding $E_n$ which is the union of $2^n$ intervals $E_{n,i}$ of length $2^{-n} - \delta 2^{-n} \sum_{i=1}^{n} 2^{i-1}3^{-i}$. Let $E$ be the intersection $\bigcap_{i=1}^{\infty} E_i$. This concludes our construction.

Next we show that $E$ is perfect, has measure $1 - \delta$ and contains no interval.

To see that $E$ is perfect, we must show that $E$ is closed and that and dense in itself. The set $E$ is closed because it is the (arbitrary) intersection of closed intervals. To see that $E$ is dense in itself, we must show that for every $\varepsilon > 0$, for every $x \in E$, the intersection $(B(x, \varepsilon) \cap E) \setminus \{x\}$ is nonempty. Let $\varepsilon > 0$ and $x \in E$ be given. Then, since $x \in E$, $x \in E_n$ for every $n$. Thus, $x$ is in some closed interval $E_{n,i} \subseteq E_n$. Let $N$ be the smallest integer such that the length of $E_{N,i} = [a, b]$ is less that $\varepsilon$. Then, $a, b \in E$ and $a, b \in B(x, \varepsilon)$ and $x$ is cannot be equal to both $a$ and $b$. Thus, $(E \cap B(x, \varepsilon)) \setminus \{x\} \neq \emptyset$. It follows that $E$ is a perfect set.

To see that the measure of $E$ is $1 - \delta$ by Theorem 3.26 (ii) since $m(E_1) = 1 - \delta/3 < \infty$ and $E_n \searrow E$ we have

$$\begin{aligned}
m(E) &= m\left(\bigcap_{i=1}^{\infty} E_i\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} m(E_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{2^n} \left[\frac{1}{2^n} - \frac{\delta}{2^n} \sum_{i=1}^{n} \frac{2^{i-1}}{3^i}\right] \\
&= \lim_{n\to\infty} \left[1 - \delta \sum_{i=1}^{n} \frac{2^{i-1}}{3^i}\right] \\
&= \lim_{n\to\infty} \left[1 - \frac{\delta}{3} \sum_{i=1}^{n} \left(\frac{2}{3}\right)^{i-1}\right]
\end{aligned}$$

letting $j = i - 1$, we can rewrite the series above as the geometric series

$$= 1 - \frac{\delta}{3} \lim_{n\to\infty} \sum_{j=0}^{n} \left(\frac{2}{3}\right)^j$$
$$= 1 - \delta,$$

as desired.

Lastly, we must show that $E$ contains no interval. Seeking a contradiction, suppose that $E$ contains an interval $I = [a, b]$ of length $b - a$. Then, since $I \subseteq E$, $I \subseteq E_n$ for all $n$ so, since $I$ is connected, it must be contained in one of the $E_{n,i}$ for all $n$. Let $N$ be the smallest integer such that $m(E_{N,i}) < b - a$ and $E_{N,i} = [c, d]$ contains $I$. Then, since $I \subseteq E_{N,i}$, both $a$ and $b$ are points in $I$, $|b - a| \le |d - c| = m(E_{N,i})$. This is a contradiction. Thus, it must be the case that $E$ contains no interval. ◂

**Exercise 2** (Wheeden & Zygmund Ch. 3, Ex. 7)**.** Prove (3.15).

**Solution.** ▸ Here is the statement of the lemma:

*If $\{I_k\}_{k=1}^N$ is a finite collection of nonoverlapping intervals, then $\bigcup_{k=1}^N I_k$ is measurable and $m\left(\bigcup_{k=1}^N I_k\right) = \sum_{k=1}^N m(I_k)$.*

By Theorem 3.12, the union $\bigcup_{n=1}^N I_n$ is measurable. Hence, it remains to show that $m\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n)$.

We take the approach of extending the argument provided in Theorem 3.2. As in Theorem 3.2, we note that, since $\{I_n\}_{n=1}^N$ covers the union $\bigcup_{n=1}^N I_n$, then

$$m\left(\bigcup_{n=1}^N I_n\right) \le \sigma\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n).$$

On the other hand, note that $I_n$ is the union $I_n^\circ \cup \partial I_n$ of its interior and its boundary. In the previous homework, we showed that the boundary of an interval has measure zero. Hence, we have

$$m(I_n^\circ) \le m(I_n) \le m(I_n^\circ) + m(\partial I_n) = m(I_n^\circ)$$

so $m(I_n) = m(I_n^\circ)$. Now, note that

$$m\left(\bigcup_{n=1}^N I_n^\circ\right) = \sum_{n=1}^N m(I_n^\circ) = \sum_{n=1}^N m(I_n).$$

Hence, we have

$$\begin{aligned}\sum_{n=1}^N m(I_n) &= m\left(\bigcup_{n=1}^N I_n^\circ\right)\\ &\le m\left(\bigcup_{n=1}^N I_n\right)\\ &\le \sum_{n=1}^N m(I_n).\end{aligned}$$

Thus, equality $m\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n)$ holds. ◂

**Exercise 3** (Wheeden & Zygmund Ch. 3, Ex. 8)**.** Show that the Borel algebra $\mathcal{B}$ in $\mathbb{R}^n$ is the smallest $\sigma$-algebra containing the closed sets in $\mathbb{R}^n$.

**Solution**. ▸ Since $\mathcal{B}$ is the smallest $\sigma$-algebra containing all of the open sets of $\mathbb{R}^n$, it contains all of the closed sets of $\mathbb{R}^n$. Now, suppose that $\mathcal{B}'$ is another $\sigma$-algebra containing the closed sets in $\mathbb{R}^n$. Then, $\mathcal{B}' \subseteq \mathcal{B}$ since $\mathcal{B}$ contains all of the closed sets in $\mathbb{R}^n$. However, since $\mathcal{B}'$ is a $\sigma$-algebra, it contains all of the open sets in $\mathbb{R}^n$, so $\mathcal{B}' \subseteq \mathcal{B}$ since $\mathcal{B}$ is the smallest $\sigma$-algebra containing the open sets in $\mathbb{R}^n$. Thus, $\mathcal{B}' = \mathcal{B}$. ◂

**Exercise 4** (Wheeden & Zygmund Ch. 3, Ex. 9)**.** If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum m^*(E_k) < \infty$, show that $\limsup E_k$ (and also $\liminf E_k$ has measure zero.

**Solution**. ▸ First, since $\{E_n\}_{n=1}^{\infty}$ is a sequence of sets with

$$\sum_{i=1}^{\infty} m^*(E_i) < \infty$$

for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\sum_{i=n}^{\infty} m^*(E_i) < \varepsilon.$$

Let's put this aside for now.

Define $E = \limsup_{n\to\infty} E_n$ and $E_n' = \bigcup_{i=n}^{\infty} E_n$. It is easy to see that $\{E_n'\}_{n=1}^{\infty}$ is a decreasing sequence of sets whose intersection $\bigcap_{n=1}^{\infty} E_n$ is the limit supremum $E$. By the monotonicity of the outer measure, we have

$$m^*(E) \leq m^*(E_n')$$

for all $n \in \mathbb{N}$. On the other hand,

$$m^*(E_n') \leq \sum_{i=n}^{\infty} m^*(E_i) < \varepsilon$$

for every $\varepsilon$. Letting $\varepsilon$ go to 0 we have $m^*(E) = 0$.

Lastly, we note that $E' = \liminf_{n\to\infty} E_n$ is a subset of $\limsup_{n\to\infty} E_n$, so that $m^*(E') = 0$. ◂

**Exercise 5** (Wheeden & Zygmund Ch. 3, Ex. 10)**.** If $E_1$ and $E_2$ are measurable, show that $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

**Solution**. ▸ We may, without loss of generality, assume that $m(E_1), m(E_2) < \infty$ for otherwise there is nothing to show as equality holds trivially.

Now, by Carathéodory's theorem we have the following characterization of measurability: a set $E$ is measurable if and only if for every set $A$ we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E).$$

Therefore, the following equalities hold

$$m(E_1) = m(E_1 \cap E_2) + m(E_1 \setminus E_2)$$
$$m(E_2) = m(E_1 \cap E_2) + m(E_2 \setminus E_1).$$

Moreover, from elementary set theory we have

$$(E_1 \cup E_2) \setminus E_2 = E_1 \setminus (E_1 \cap E_2),$$

$E_1 \subseteq E_1 \cup E_2$ and $E_1 \cap E_2 \subseteq E_1$ so

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

as desired. ◂

### 1.1.4 Homework 4

**Exercise 1** (Wheeden & Zygmund Ch. 3, Ex. 12)**.** If $E_1$ and $E_2$ are measurable sets in $\mathbb{R}^1$, show $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $m(E_1 \times E_2) = m(E_1)m(E_2)$. (Interpret $0 \cdot \infty$ as 0.) [*Hint:* Use a characterization of measurability.]

**Solution**. ▸ The proof of this result is rather long and we shall omit it for now as I gain nothing from retracing my steps on this one. ◂

**Exercise 2** (Wheeden & Zygmund Ch. 3, Ex. 13)**.** Motivated by (3.7), define the *inner measure* of $E$ by $m_*(E) = \sup m(F)$, where the supremum is taken over all closed subsets $F$ of $E$. Show that

(i) $m_*(E) \leq m^*(E)$, and

(ii) if $m^*(E) < \infty$, then $E$ is measurable if and only if $m_*(E) = m^*(E)$.

[Use (3.22).]

**Solution**. ▸ First we show part (i). If $m^*(E) = \infty$, the inequality holds trivially. Suppose that $m^*(E) < \infty$. Then, since $F$ is closed, it is measurable and $m(F) = m^*(F)$. Moreover, $F \subseteq E$ so by the monotonicity of the outer measure,

$$m(F) = m^*(F) < m^*(E).$$

Taking the supremum over all $F$ on the left, we have

$$m_*(E) = \sup_{F \subseteq E} m(F) < m^*(E)$$

as we set out to show.

Next we show part (ii). Let $E \subseteq \mathbb{R}^n$ with $m^*(E) < \infty$. $\implies$ Suppose that $E$ is measurable. Then, by Lemma 3.22, there exists a closed set $F \subseteq E$ such that $m^*(E \setminus F) < \varepsilon$. Since closed sets are measurable, by Corollary 3.31, we have

$$m^*(E \setminus F) = m(E) - m(F) < \varepsilon$$

so

$$m(E) < m(F) + \varepsilon.$$

Letting $\varepsilon$ go to 0, we have

$$m(E) \leq m(F);$$

and taking the supremum on the right

$$m(E) \leq m_*(E).$$

But, by part (i), $m_*(E) \leq m^*(E) = m(E)$. Thus, $m_*(E) = m^*(E)$ as was to be shown.

$\impliedby$ On the other hand, suppose that $m_*(E) = m^*(E)$. Then, given $\varepsilon > 0$ there exists an open set $G$ containing $E$ and a closed set $F$ contained in $E$ such that

$$m(G) - m^*(E) < \frac{\varepsilon}{2}$$
$$m_*(E) - m(F) < \frac{\varepsilon}{2}.$$

Then

$$
\begin{aligned}
m^*(E \setminus F) &< m^*(G \setminus F) \\
&= m^*(G) - m^*(G \cap F) \\
&= m^*(G) - m^*(F) \\
&< \frac{\varepsilon}{2} + m^*(E) - \left(m^*(E) - \frac{\varepsilon}{2}\right) \\
&= \varepsilon.
\end{aligned}
$$

Thus, by Lemma 3.22, $E$ is measurable. ◂

**Exercise 3** (Wheeden & Zygmund Ch. 3, Ex. 15)**.** If $E$ is measurable and $A$ is any subset of $E$, show that $m(E) = m_*(A) + m^*(E \setminus A)$. (See Exercise 13 for the definition of $m_*(A)$.)

**Solution**. ▸ Suppose $A \subseteq E$. If $A$ is measurable, by Problem 2, the outer and inner measure of $A$ agree; symbolically, we have $m(A) = m^*(A) = m_*(A)$. Thus, we have

$$
m^*(E \setminus A) = m^*(E) - m^*(A) = m^*(E) - m_*(A).
$$

If $A$ is not measurable and $m(E) < \infty$, then we must have $m^*(A), m^*(E \setminus A) < \infty$ by the monotonicity of the outer measure; since both $A$ an $E \setminus A$ are subsets of $E$. Hence, we may, without any ambiguity, subtract the quantity $m^*(E \setminus A)$ from $m(E)$ and we have

$$
\begin{aligned}
m(E) - m^*(E \setminus A) &= m(E) - \inf \{ m(G) : E \setminus A \subseteq G \text{ and } G \text{ is open} \} \\
&= m(E) - \inf \{ m(G) : E \setminus A \subseteq G \subseteq E \text{ and } G \text{ is open} \} \\
&=
\end{aligned}
$$

◂

### 1.1.5 Homework 5

**Exercise 1** (Wheeden & Zygmund Ch. 3, Ex. 14)**.** Show that the conclusion of part (ii) of Exercise 13 is false if $m^*(E) = \infty$.

**Solution**. ▸ Part (ii) of Exercise 13 is part (ii) of Problem 2 from the last section (Homework 4). In that problem we showed that if the outer measure of $E$ is finite, then $E$ is measurable if and only if its outer and inner measure agree. Here we construct a counter example to this when the outer measure of $E$ is $\infty$; that is, we show that there exists a set $E$ with $m^*(E) = \infty$ such that $m^*(E) \neq m_*(E)$. So, which set shall it be? Since we are unoriginal, we will pull an example from Wheeden and Zygmund itself.

Let $V \subseteq [0, 1]$ be Vitali's unmeasurable (Theorem 3.38) and consider the union $E = V \cup (2, \infty)$. It is clear that the inner and outer measure of $E$ are both $\infty$. However, $E$ itself must be unmeasurable for otherwise $E \cap [0, 1] = V$ is measurable. ◂

**Exercise 2** (Wheeden & Zygmund Ch. 3, Ex. 16)**.** Prove (3.34).

**Solution**. ▸ We must prove Equation 3.34; that is, if $P$ is a parallellepiped

$$m(P) = \mathrm{vol}(P).$$

We may, without loss of generality, assume that one of the vertices of $P$ is $\mathbf{0}$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a set of vectors such that

$$P = \left\{ x \in \mathbb{R}^n : x = \sum\nolimits_{k=1}^{n} t_k \mathbf{e}_k, 0 \leq t_k \leq 1 \right\}.$$

By definition, the measure of $P$ is

$$m(P) = \inf_{\mathcal{S}} \left[ \sum_{I_n \in \mathcal{S}} \mathrm{vol}(I_n) \right]$$

where $\mathcal{S}$ is a cover of $P$ by intervals. Take the set of

> *Remarks.* Literally nobody cares about this problem. I don't remember how to do it, but it must have been painful if I can't figure it out now, even.

◂

**Exercise 3** (Wheeden & Zygmund Ch. 3, Ex. 18)**.** Prove that outer measure is *translation invariant*; that is, if $E_h = \{ x + h : x \in E \}$ is the translate of $E$ by $h$, $h \in \mathbb{R}^n$, show that $m^*(E_h) = m^*(E)$. If $E$ is measurable, show that $E_h$ is also measurable. [This fact was used in proving (3.37).]

**Solution**. ▸ Let $E \subseteq \mathbb{R}^n$ and $h \in \mathbb{R}^n$ and define the set $E_h$ to be the set $E_h = \{ x + h : x \in E \}$. We will show that the outer measure of $E$ is preserved under such translations. But first, let us point out that $E_h$ is nothing more than the image of $E$ under the linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $x \mapsto x + h$. By Theorem 3.35, such a map preserves measurability of sets and for any measurable set $E' \subseteq \mathbb{R}^n$, $m(T(E')) = (\det T) m(E') = m(E')$ (since $\det T = 1$. Now, by Theorem 3.6, for every $\varepsilon > 0$, there exist an open set $G \supseteq E$ such that $m^*(G) \leq m^*(E) + \varepsilon$. Consider the image of $G$ under $T$, $T(G)$ is an open set containing $E_h$ so $m^*(G) \geq m^*(E)$ and

$$m^*(T(G)) = m^*(G) < m^*(E) + \varepsilon.$$

Letting $\varepsilon \to 0$, we achieve the inequality

$$m^*(E_h) \leq m^*(E).$$

To get the other inequality, take the map $T^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ which takes $x \mapsto x - h$; this sends $E_h$ to $E$ and the same argument shows that

$$m^*(E) \leq m^*(E).$$

Thus, we have $m^*(E) = m^*(E_h)$, as was to be shown. ◂

**Exercise 4** (Wheeden & Zygmund Ch. 4, Ex. 1)**.** Prove corollary (4.2) and theorem (4.8)

**Solution**. ▸ The corollary and theorem in question are:

> *If $f$ is measurable, then $\{ f > -\infty \}$, $\{ f < +\infty \}$, $\{ f = +\infty \}$, $\{ a \leq f \leq b \}$, $\{ f = a \}$, etc., are all measurable. Moreover $f$ is measurable if and only if $\{ a < f < +\infty \}$ is measurable for every finite $a$.*

and

> *If $f$ is measurable and $\lambda$ is any real number, then $f + \lambda$ and $\lambda f$ are measurable.*

Their proofs are quite simple. For the corollary: Suppose $f \colon E \to \mathbb{R}$ is a measurable function. By Theorem 4.1, $f$ is measurable if and only if for every finite $\alpha \in \mathbb{R}$, the sets

$$\{ x \in E : f(x) \geq \alpha \}$$
$$\{ x \in E : f(x) < \alpha \}$$
$$\{ x \in E : f(x) \leq \alpha \}$$

are measurable. Since measurable sets form a $\sigma$-algebra on $\mathbb{R}^n$, we know that the countable union and intersection of measurable sets is measurable. Thus,

$$\{ x \in E : f(x) > -\infty \} = \bigcup_{\alpha \in \mathbb{Z}} \{ x \in E : f(x) > \alpha \}$$
$$\{ x \in E : f(x) = \infty \} = \bigcap_{n=1}^{\infty} \{ x \in E : f(x) > n \}$$
$$\{ x \in E : f(x) < \infty \} = \bigcup_{\alpha \in \mathbb{Z}} \{ x \in E : f(x) < \alpha \}$$

are easily seen to be measurable.

Showing that $\{ x \in E : f(x) = \alpha \}$ and $\{ x \in E : \alpha < f(x) < \beta \}$ are measurable requires some clever (but not too clever) intersection/union of the sets we get from Theorem 4.1.

For the theorem: Suppose $f$ is measurable and $\lambda$ is a constant. By Theorem 4.1, for any finite $\alpha \in \mathbb{R}$ we have

$$\{ x \in E : f(x) > \alpha - \lambda \}$$

so

$$\{ x \in E : f(x) + \lambda > \alpha \}$$

is measurable. Thus, $f + \lambda$ is measurable. Similarly, for $\lambda \neq 0$, taking the set

$$\{ x \in E : f(x) > \alpha/\lambda \} = \{ x \in E : \lambda f(x) > \alpha \}$$

shows that $\lambda f$ is measurable; otherwise, if $\lambda = 0$, $\lambda f = 0$ is constant and hence is continuous which in turn implies that it is measurable. ◂

**Exercise 5** (Wheeden & Zygmund Ch. 4, Ex. 2)**.** Let $f$ be a simple function, taking its distinct values on disjoint sets $E_1, \dots, E_N$. Show that $f$ is measurable if and only if $E_1, \dots, E_N$ are measurable.

**Solution**. ▸ $\Longrightarrow$ Suppose that $f$ is measurable. Then, by Corollary 4.2, the sets of the form $\{ f = \alpha_n \} = E_n$ are measurable. So the sets $E_n$ are measurable.

$\Longleftarrow$ On the other hand, suppose that the sets $E_n$ are measurable. Then, $\chi_{E_n}$ is measurable so by Theorem 4.8, $f$ is measurable since it is the sum

$$f = \sum_{n=1}^{N} \alpha_{E_n}.$$

◂

### 1.1.6 Homework 6

**Exercise 1** (Wheeden & Zygmund Ch. 4, Ex. 4)**.** Let $f$ be defined and measurable in $\mathbb{R}^n$. If $T$ is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(T(x))$ is measurable. [If $E_1 = \{ x : f(x) > a \}$ and $E_2 = \{ x : f(T(x)) > a \}$, show $E_2 = T^{-1}(E_1)$.]

**Solution**. ▸ Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function and $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then, we show that the composition $f \circ T$ is measurable. Fix a finite $\alpha \in \mathbb{R}$ and let

$$\begin{aligned} E_1 &= \{ x : f(x) > \alpha \} \\ E_2 &= \{ x : f(T(x)) > \alpha \} . \end{aligned}$$

Then, by Theorem 3.35, it suffices to show that $E_2 = T^{-1}(E_1)$ since $T^{-1}$ is a nonsingular linear transformation so it sends measurable sets to measurable sets. But this equality is obvious: Suppose $x \in E_2$; then $f(T(x)) > \alpha$ so, because $T$ is nonsingular and therefore bijective, clearly $x \in T^{-1}(E_1)$ so $E_2 \subseteq T^{-1}(E_1)$. One the other hand, if $x \in T^{-1}(E_1)$ then $x$ is a point in $E$ such that $f(T(x)) > \alpha$ so $x \in E_2$. Thus, $E_2 = T^{-1}(E_1)$ and consequently, $f \circ T$ is a measurable function. ◂

**Exercise 2** (Wheeden & Zygmund Ch. 4, Ex. 7)**.** Let $f$ be usc and less that $\infty$ on a compact set $E$. Show that $f$ is bounded above on $E$. Show also that $f$ assumes its maximum on $E$, i.e., that there exists $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.

**Solution**. ▸ First we show that $f$ is bounded. Suppose that $f$ is u.s.c. on $E$. Then, by Theorem 4.14 (i), sets of the form $\{ x \in E : f(x) < \alpha \}$ are relatively open. Let $\mathcal{G} = \{G_\alpha\}_{\alpha \in \mathbb{Z}}$ where $G_a = \{ x \in E : f(x) < \alpha \}$. Then $\mathcal{G}$ forms an open cover of $E$ and since $E$ is compact there exists a finite subset $\{G_{\alpha_n}\}_{n=1}^N$ for some finite subset $\{ \alpha_1, \ldots, \alpha_N \}$ of $\mathbb{Z}$. Let $\alpha = \max \{ \alpha_1, \ldots, \alpha_N \}$. Then, $f(x) < \alpha$ for all $x \in E$ so $f$ is bounded above by $\alpha$.

Next, we show that $f$ in fact assumes its maximum (locally) on $E$ by using only topological properties of $f$. Since sets of the form $\{ x \in E : f(x) \geq \alpha \}$ are relatively closed, by Theorem 4.14 (i), for fixed $x \in E$ the sets $F_x = \{ y \in E : f(y) \geq f(x) \}$ are relatively closed. Consider the collection $\{F_x\}_{x \in E}$ of closed subsets of $E$. First, note that each of these sets is nonempty since $f(x) \geq f(x)$ so $x \in F_x$ for every $x \in E$. Now, let $\{x_n\}_{n=1}^N \subseteq E$ and consider the collection $\{F_{x_n}\}_{n=1}^N$. Then $\bigcap_{n=1}^N F_{x_n} \neq \emptyset$ since for $x$ the point in $\{ x_1, \ldots, x_N \}$ such that $f(x) = \min \{ f(x_1), \ldots, f(x_N) \}$, $x \in F_{x_n}$ for all $1 \leq n \leq N$. Thus, by the finite intersection property, the intersection $F = \bigcap_{x \in E} F_x$ is nonempty. Let $y \in \bigcap_{x \in E} F_x$, then $f(y) \geq f(x)$ for all $x \in E$ so $f$ achieves its maximum (locally) on $E$. ◂

**Exercise 3** (Wheeden & Zygmund Ch. 4, Ex. 8)**.**

(a) Let $f$ and $g$ be two functions which are u.s.c. at $x_0$. Show that $f + g$ is u.s.c. at $x_0$. Is $f - g$ u.s.c. at $x_0$? When is $fg$ u.s.c. at $x_0$?

(b) If $\{ f_k \}$ is a sequence of functions are u.s.c. at $x_0$, show that $\inf f_k(x)$ is u.s.c. at $x_0$.

(c) If $\{ f_k \}$ is a sequence of functions which are u.s.c. at $x_0$ and which converge uniformly near $x_0$, show that $\lim f_k$ is u.s.c. at $x_0$.

**Solution**. ▸ We prove these in alphabetical order (a) → (b) → (c).

For (a), suppose that $f$ and $g$ are u.s.c. at $x_0$. Then given $M > f(x_0), g(x_0)$ there exists $\delta_1, \delta_2 > 0$ such that $f(x), g(x) < M/2$ for all $|x_1 - x_0| < \delta_1, |x_2 - x_0| < \delta_2$, respectively. Let $\delta$ be the minimum of $\{ \delta_1, \delta_2 \}$. Then for any $x$ such that $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) + g(x) - (f(x_0) + g(x_0))| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\ &\leq |(f(x) - f(x_0))| + |(g(x) - g(x_0))| \\ &< \frac{M}{2} + \frac{M}{2} \\ &= M. \end{aligned}$$

Thus, $f + g$ is u.s.c.

For that second little part of (a), the one that asks "Is $f - g$ u.s.c. at $x_0$?" we provide a counter example. In fact, the following is enough of a counterexample: Take $f = 0$ (which is continuous everywhere) and $g$ any function that is u.s.c., but not continuous, at $x_0$ then $f - g = -g$ is l.s.c. at $x_0$. Another counterexample is provided by the equations $u_1$ and $u_2$ from Ch. 4 of Wheeden and Zygmund: Fix an $x_0 \in \mathbb{R}$ and define

$$u_1(x) = \begin{cases} 0 & \text{if } x < x_0, \\ 1 & \text{if } x \geq x_0, \end{cases} \qquad u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

Then

$$u_1(x) - u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

is not u.s.c. at $x_0$ since being u.s.c. at $x_0$ implies that for $1/2 > f(x_0) = 0$ there exists $\delta > 0$ such that $f(x) < 1/2$ for all $x \in (x_0 - \delta, x_0 + \delta)$. But for any $x' > x_0$ in $(x_0 - \delta, x + \delta)$, $u(x') = 1 > 1/2$ which contradicts the assumption that $u$ is u.s.c. at $x_0$.

For (b), suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions that are u.s.c. at $x_0$. Then

$$\limsup_{\substack{x \to x_0 \\ x \in E}} f_n(x) \leq f_n(x_0)$$

for all $n \in \mathbb{N}$. We must show that

$$\limsup_{\substack{x \to x_0 \\ x \in E}} \bigl[\inf f_n(x)\bigr] \leq \inf f_n(x_0).$$

◂

#### 1.1.7 Homework 7

**Exercise 1** (Wheeden & Zygmund Ch. 4, Ex. 9)**.**

(a) Show that the limit of a decreasing (increasing) sequence of functions u.s.c. (l.s.c.) at $x_0$ is u.s.c. (l.s.c.) at $x_0$. In particular, the limit of a decreasing (increasing) sequence of functions continuous at $x_0$ is u.s.c. (l.s.c.) at $x_0$.

(b) Let $f$ be u.s.c. and less than $\infty$ on $[a, b]$. Show that there exists continuous $f_k$ on $[a, b]$ such that $f_k \downarrow f$.

**Solution**. ▸ For part (a) we may as well assume that $f \geq 0$ for all $x$. Let $\{f_n\}$, $n \in \mathbb{N}$, be a sequence of decreasing functions with limit $f$ which are u.s.c. at $x_0$. Then, for every $n \in \mathbb{N}$, for every sequence $x \to x_0$,

$$\limsup_{x \to x_0} f_n(x) \leq f_n(x_0).$$

Now, we claim that $f(x) \leq f_n(x)$ for every $x$ and every $n \in \mathbb{N}$.

*Proof of claim.* Suppose $f(x) > f_{N_1}(x)$ for some $x$, $N_1 \in \mathbb{N}$. Then there exists a real number $\varepsilon > 0$ such that $0 < \varepsilon < |f(x) - f_n(x)|$ (we may, for example, take $\varepsilon$ to be in $\mathbb{Q}$ which is dense in $\mathbb{R}$. Then, since $f_n \downarrow f$, there exists an index $N_1 \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \varepsilon.$$

However, since the sequence $f_n$ decreases to $f$, for $n \geq \max\{N_1, N_2\}$, $f_n(x) \leq f_{N_1}(x)$ so

$$|f(x) - f_n(x)| > |f(x) - f_{N_1}(x)| > \varepsilon.$$

This is a contradiction. ■

Having established this, for every sequence $x \to x_0$, we have

$$\limsup_{x \to x_0} f(x) \leq \limsup_{x \to x_0} f_n(x) \leq f_n(x_0).$$

Letting $n \to \infty$,

$$\limsup_{x \to x_0} f(x) \leq \lim_{n \to \infty} f_n(x_0) = f(x_0).$$

For part (b) suppose $f : [a, b] \to \mathbb{R}$ is u.s.c. on $[a, b]$ and $f(x) < \infty$ for all $x \in [a, b]$. For a fixed $x \in [a, b]$, $f$ is u.s.c. at $x$ if for every $\varepsilon > 0$, there exists a neighborhood $B(x, \delta)$ such that $f(y) < f(x) + \varepsilon$. Now, let $\varepsilon = 1/n$. Then, for each $x \in [a, b]$, there exists a neighborhood $B(x, \delta_x)$ such that $f(y) < f(x) + \varepsilon$ for $y \in B(x, \delta_x)$.

The following post on the Mathematics StackExchange contains a solution to part (b) of this problem.

First, we claim that $f(x) \neq \infty$ for any $x \in [a, b]$, it must be bounded.

*Proof of claim.* By Theorem 4.14 (a), sets of the form $\{ x \in [a, b] : f(x) < a \}$ is relatively open for all finite $a$. Define

$$E_n = \{ x \in [a, b] : f(x) < n \}.$$

Then, the collection $\mathcal{E} = \{E_n\}$, $n \in \mathbb{N}$, is an open cover of $[a, b]$. Since $[a, b]$ is compact, there exists a finite subcover $\{E_{n_1}, \ldots, E_{n_m}\}$ of $\mathcal{E}$. Letting $M = \max\{n_1, \ldots, n_m\}$, we have $f < M$ for all $x \in [a, b]$. Thus, $f$ is bounded on $[a, b]$. ■

Now that we have established that $f$ is bounded on $[a, b]$ by, say, $M$ then $\sup_{x\in[a,b]} f \leq M$. Define

$$f_n(x) = \sup_{y\in[a,b]} \big[f(y) - n|x-y|\big].$$

We claim that this family of functions $\{f_n\}$, $n \in \mathbb{N}$, is continuous and that $f_n \to f$. To see that $f$ is continuous, we observe that this family of functions is in fact Lipschitz continuous

$$\begin{aligned}
|f_n(x) - f_n(y)| &= \left|\sup_{z\in[a,b]} \big[f(z) - n|x-z|\big] - \sup_{z\in[a,b]} \big[f(z) - n|y-z|\big]\right| \\
&\leq \left|\sup_{z\in[a,b]} \big[f(z) - n|x-z| - f(z) - n|y-z|\big]\right| \\
&= \left|\sup_{z\in[a,b]} \big[-n|x-z| - n|y-z|\big]\right| \\
&= \left|\sup_{z\in[a,b]} \big[-n|x-y+(y-z)| - n|y-z|\big]\right| \\
&\leq \left|\sup_{z\in[a,b]} \big[-n|x-y| - 2n|y-z|\big]\right| \\
&= n|x-y|.
\end{aligned}$$

Thus, $f_n$ is Lipschitz and in particular, it is continuous.

To see that $f_n \to f$ pointwise, let $\varepsilon > 0$ be given then we must show that there exists some index $N$ such that $n \geq N$ implies

$$|f(x) - f_n(x)| < \varepsilon.$$

Expanding the equation above, we see that

$$|f(x) - f_n(x)| = \left|f(x) - \sup_{y\in[a,b]} \big[f(y) - n|x-y|\big].\right|$$

◂

**Exercise 2** (Wheeden & Zygmund Ch. 4, Ex. 11)**.** Let $f$ be defined on $\mathbb{R}^n$ and let $B(x)$ denote the open ball $\{\, y : |x-y| < r \,\}$ with center $x$ and fixed radius $r$. Show that the function $g(x) = \sup\{\, f(y) : y \in B(x) \,\}$ is l.s.c. and the function $h(x) = \inf\{\, f(y) : y \in B(x) \,\}$ is u.s.c. on $\mathbb{R}^n$. Is the same true for the closed ball $\{\, y : |x-y| \leq r \,\}$?

**Solution**. ▸ Note that, by properties of the infimum/supremum for any set of real numbers $S \subset \mathbb{R}$,

$$\sup S = -\inf(-S)$$

where $-S = \{\, -s : s \in S \,\}$. Thus,

$$\begin{aligned}
g(x) &= -\inf\{\, -f(y) : y \in B(x,r) \,\} \\
&= \sup\{\, f(y) : y \in B(x,r) \,\}.
\end{aligned}$$

Letting $f' = -f$, it suffices to show that $g'(x) = \inf\{ f'(y) : y \in B(x, r) \}$ is u.s.c. since for any u.s.c. function $f$, $-f$ is l.s.c. Therefore, we show that $h$ is u.s.c.

To see that $h$ is u.s.c., let $M > h(x_0)$. Then we must show that there exists a neighborhood $B(x_0, \delta)$ such that $M > h(x)$ for every $x \in B(x_0, \delta)$. Since $h(x_0)$ is the infimum of $f(x)$ over all $x \in B(x_0, r)$, given $\varepsilon > 0$ there exists $x \in B(x_0, r)$ such that $f(x) < h(x_0) + \varepsilon < M$. Define $\delta = (r - |x - y|)/2$. Then we claim that for any $x \in B(x_0, \delta)$,

$$g(x) < M.$$

*Proof of claim.* Let $x \in B(x_0, \delta)$. Then $y \in B(x_0, \delta)$ since

$$\begin{aligned}
|x - y| &= |x - x_0 - (y - x_0)| \\
&\leq |x - x_0| + |y - x_0| \\
&= (r - |y - x_0|)/2 + |y - x_0| \\
&= r/2 + |y - x_0|/2 \\
&< r.
\end{aligned}$$

Thus,

$$g(x) \leq f(y) < g(x_0) + \varepsilon < M.$$

■

It follows that $g$ is u.s.c. ◂

**Exercise 3** (Wheeden & Zygmund Ch. 4, Ex. 15)**.** Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set $E$ with $m(E) < \infty$. If $|f_k(x)| \leq M_x < \infty$ for all $k$ for each $x \in E$, show that given $\varepsilon > 0$, there is closed $F \subseteq E$ and finite $M$ such that $m(E \setminus F) < \varepsilon$ and $|f_k(x)| \leq M$ for all $x \in F$.

**Solution**. ▸ Set $f = \sup_{n\in\mathbb{N}} |f_n|$; then, $f$ is measurable since it is the supremum of measurable functions $|f_n|$. By Lusin's theorem $f$ satisfies the $C$-property, i.e., there exists a closed subset $F'$ of $E$ with $m(E \setminus F') < \varepsilon/2$ and a continuous function $\bar{f} \colon E \to \mathbb{R}$ such that $f\restriction_{F'} = \bar{f}\restriction_{F'}$. Now, let $B$ be the closed ball centered at $\mathbf{0}$ such that $|E \setminus B| < \varepsilon/2$ (remember, this is all taking place in $\mathbb{R}^n$, so we can do this). Thus, $F' \cap B$ is compact since it is a closed subset of $B$ the latter being a compact set. Let $F = F' \cap B$ then,

$$\begin{aligned}
|E \setminus F| &= |E \setminus (F' \cap B)| \\
&= |(E \setminus F') \cup (E \setminus B)| \\
&\leq |E \setminus F'| + |E \setminus B| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

so $F$ has the desired measure. Lastly, by the mean value theorem, $f$ achieves its maximum, call it $M$, on $F$ since $F$ is compact. It follows that $f_n\restriction_F \leq M$ for all $n \in \mathbb{N}$. ◂

**Exercise 4** (Wheeden & Zygmund Ch. 4, Ex. 18)**.** If $f$ is measurable on $E$, define $\omega_f(a) = m\{ f > a \}$ for $-\infty < a < \infty$. If $f_k \uparrow f$, show that $\omega_{f_k} \uparrow \omega_f$. If $f_k \to f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of $\omega_f$. [For the second part, show that if $f_k \to f$, then $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$ and $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$ for every $\varepsilon > 0$.]

**Solution**. ▸ For the first part of this problem we will show that the sequence of distribution functions $\{\omega_{f_n}\}$, $n \in \mathbb{N}$, is increasing and that its limit is $\omega_f$. It is easy to verify that this sequence is in fact increasing: if $x \in \{ f_{n-1} \geq M \}$ then $x \in \{ f_n \geq M \}$ since $f_n \geq f_{n-1}$ for all $x \in E$. Thus, $\omega_{f_n} \geq \omega_{f_{n-1}}$. Now we need to show that the limit of this sequence is in fact $\omega_f$: fix an $x \in E$ and let $\varepsilon > 0$ be given. Then there exists an index $N'$ such that $n \geq N'$ implies $|f(x) - f_n(x)| < \varepsilon$. Now, we want to use this $\varepsilon$ and index $N'$ (with some possible alterations), for some fixed $M$, we want to show that the difference

$$\left|\omega_f(M) - \omega_{f_n}(M)\right| < \varepsilon.$$

First, by properties of the Lebesgue measure

$$m\{ f > M \} - m\{ f_n > M \} \leq m(\{ f > M \} \setminus \{ f_n > M \}).$$

In turn, it is easy to see that the latter set is in fact

$$\begin{aligned} E_{M,n} &= \{ x \in E : f(x) > M \text{ and } f_n(x) \leq M \} \\ &= \{ x \in E : f(x) > M \text{ and } f(x) - f_n(x) > 0 \}. \end{aligned}$$

Then, $E_{M,n} \subseteq \{ x \in E : f(x) - f_n(x) > M \} = E_{0,n}$ and the the measure of the latter set converges to 0 since $f_n \to f$ and this implies that $f_n$ converges to $f$ in measure (a weaker form of pointwise convergence). Let $N''$ be the index such that $n \geq N'$ implies $m(E_{0,n}) < \varepsilon$. Then for $n \geq N$ with $N = \max\{N', N''\}$, the difference

$$\left|\omega_f(M) - \omega_{f_n}(M)\right| < \varepsilon.$$

Thus, we have shown that $\omega_{f_n} \uparrow \omega_f$. ◂

**Exercise 5** (Wheeden & Zygmund Ch. 5, Ex. 1)**.** If $f$ is a simple measurable function (not necessarily positive) taking values $a_j$ on $E_j$, $j = 1, \ldots, N$, show that $\int_E f = \sum_{j=1}^N a_j m(E_j)$. [Use (5.24)].

**Solution**. ▸ It is enough to consider simple positive measurable functions $f$ since we can split $f$ into the difference of two positive simple measurable functions, namely, $f = f^+ - f^-$. Now, since $f$ is a simple function, $f = \sum_{n=1}^N a_n \chi_{E_n}$ for measurable subsets $E_n \subseteq E$. Now, by Theorem 5.24, we have

$$\begin{aligned} \int_E f \,\mathrm{d}x &= \int_E \Big[\sum_{n=1}^N a_n \chi_{E_n}\Big] \mathrm{d}x \\ &= \sum_{n=1}^N \int_{E_n} a_n \,\mathrm{d}x \\ &= \sum_{n=1}^N a_n m(E_n), \end{aligned}$$

as we set out to show. ◂

**Exercise 6** (Wheeden & Zygmund Ch. 5, Ex. 3)**.** Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on $E$. If $f_k \to f$ and $f_k \leq f$ a.e. on $E$, show that $\int_E f_k \to \int_E f$.

**Solution**. ▸ The result follows from a simple application of Fatou's lemma. Consider the sequence of integrals $\left\{\int_E f_n\right\}$, $n \in \mathbb{N}$. By Fatou's lemma

$$\begin{aligned}\int_E \liminf_{n\to\infty} f_n \,\mathrm{d}x &= \int_E f \,\mathrm{d}x \\ &\leq \liminf_{n\to\infty} \int_E f_n \,\mathrm{d}x.\end{aligned}$$

By Theorem 5.10, since $f_n \leq f$, we have

$$\limsup_{n\to\infty} \int_E f_n \,\mathrm{d}x \leq \int_E f \,\mathrm{d}x.$$

Thus, we have

$$\limsup_{n\to\infty} \int_E f_n \,\mathrm{d}x \leq \liminf_{n\to\infty} \int_E f_n \,\mathrm{d}x,$$

which implies that

$$\limsup_{n\to\infty} \int_E f_n \,\mathrm{d}x = \liminf_{n\to\infty} \int_E f_n \,\mathrm{d}x$$

so

$$\lim_{n\to\infty} \int_E f_n \,\mathrm{d}x = \int_E f \,\mathrm{d}x$$

as we set out to show. ◂

### 1.1.8 Homework 8

**Exercise 1** (Wheeden & Zygmund Ch. 5, Ex. 2)**.** Show that the conclusion of (5.32) are not true without the assumption that $\varphi \in L(E)$. [In part (ii), for example, take $f_k = \chi_{(k,\infty)}$.]

**Solution**. ► ◄

**Exercise 2** (Wheeden & Zygmund Ch. 5, Ex. 4)**.** If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and $\int_0^1 x^k f(x)\,\mathrm{d}x \to 0$.

**Solution**. ► ◄

**Exercise 3** (Wheeden & Zygmund Ch. 5, Ex. 6)**.** Let $f(x, y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each $x$, $f(x, y)$ is an integrable function of $y$, and $\partial f(x, y)/\partial x$ is a bounded function of $(x, y)$. Show that $\partial f(x, y)/\partial x$ is a measurable function of $y$ for each $x$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 f(x, y)\,\mathrm{d}y = \int_0^1 \frac{\partial}{\partial x} f(x, y)\,\mathrm{d}y.$$

**Solution**. ► ◄

**Exercise 4** (Wheeden & Zygmund Ch. 5, Ex. 7)**.** Give an example of an $f$ that is not integrable, but whose improper Riemann integral exists and is finite.

**Solution**. ► ◄

**Exercise 5** (Wheeden & Zygmund Ch. 5, Ex. 21)**.** If $\int_A f = 0$ for every measurable subset A of a measurable set $E$, show that $f = 0$ a.e. in $E$.

**Solution**. ► ◄

**Exercise 6** (Wheeden & Zygmund Ch. 6, Ex. 10)**.** Let $V_n$ be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theorem that

$$V_n = 2V_{n-1}\int_0^1 \left(1 - t^2\right)^{(n-1)/2}\,\mathrm{d}t.$$

(We also observe that by setting $w = t^2$, the integral is a multiple of a classical $\beta$-function and so can be expressed in terms of the $\Gamma$-function: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,\mathrm{d}t$, $s > 0$.)

**Solution**. ► ◄

**Exercise 7** (Wheeden & Zygmund Ch. 6, Ex. 11)**.** Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,\mathrm{d}x = \pi^{n/2}.$$

(For $n = 1$, write $\left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x dy$ and use polar. For $n > 1$, use the formula $e^{-|x|^2} = e^{-x_1{}^2} \cdots e^{-x_n{}^2}$ and Fubini's theorem to reduce the case $n = 1$.)

**Solution.** ► ◄

### 1.1.9 Homework 9

**Exercise 1** (Wheeden & Zygmund Ch. 6, Ex. 1)**.**

(a) Let $E$ be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}$, $\{ y : (x, y) \in E \}$ has $\mathbb{R}$-measure zero. Show that $E$ has measure zero and that for almost every $y \in \mathbb{R}$, $\{ x : (x, y) \in E \}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x, y)$ is finite for almost every $y$. Show that for almost $y \in \mathbb{R}$, $f(x, y)$ is finite for almost every $x$.

**Solution.** ▸ ◂

**Exercise 2** (Wheeden & Zygmund Ch. 6, Ex. 3)**.** Let $f$ be measurable and finite a.e. on $[0, 1]$. If $f(x) - f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$, show that $f \in L[0, 1]$.

**Solution.** ▸ ◂

**Exercise 3** (Wheeden & Zygmund Ch. 6, Ex. 4)**.** Let $f$ be measurable and periodic with period 1: $f(t+1) = f(t)$. Suppose there is a finite $c$ such that

$$\int_0^1 |f(a + t) - f(b + t)| \, \mathrm{d}t \le c$$

for all $a$ and $b$. Show that $f \in L[0, 1]$. (Set $a = x$, $b = -x$, integrate with respect to $x$, and make the change of variables $\xi = x + t$, $\eta = -x + t$.)

**Solution.** ▸ ◂

**Exercise 4** (Wheeden & Zygmund Ch. 6, Ex. 6)**.** For $f \in L(\mathbb{R})$, define the *Fourier transform* $\hat{f}$ of $f$ by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} \, \mathrm{d}t$$

for $x \in \mathbb{R}$. (For complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts $F_0$ and $F_1$ are integrable, we define $\int F = \int F_0 + i \int F_1$.) Show that if $f$ and $g$ belong to $L(\mathbb{R})$, then

$$\widehat{(f * g)}(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

**Solution.** ▸ ◂

**Exercise 5** (Wheeden & Zygmund Ch. 6, Ex. 7)**.** Let $F$ be a closed subset of $\mathbb{R}$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and $f$ is nonnegative and integrable over the complement of $F$, prove that the function

$$\int_{\mathbb{R}} \frac{\delta^\lambda(y) f(y)}{|x - y|^{1+\lambda}} \, \mathrm{d}t$$

is integrable over $F$ and so is finite a.e. in $F$. (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

**Solution.** ► ◄

**Exercise 6** (Wheeden & Zygmund Ch. 6, Ex. 9)**.**

(a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.

(b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subseteq \mathbb{R}$, and let $M_\alpha$ be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that $M_\lambda$ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int M_\lambda \leq \lambda^{-1}|G|$, where $G = (a, b) - [c, d]$.

**Solution.** ► ◄

### 1.1.10 Homework 10

**Exercise 1** (Wheeden & Zygmund Ch. 7, Ex. 1)**.** Let $f$ be measurable in $\mathbb{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant $c$ such that $f^*(x) \geq c\|x\|^{-n}$ for $\|x\| \geq 1$.

**Solution**. ▸ ◂

**Exercise 2** (Wheeden & Zygmund Ch. 7, Ex. 2)**.** Let $\varphi(x)$, $x \in \mathbb{R}^n$, be a bounded measurable function such that $\varphi(x) = 0$ for $\|x\| \geq 1$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. ($\varphi_\varepsilon$ is called an *approximation to the identity*.) If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon \to 0}(f * \varphi_\varepsilon)(x) = f(x)$$

in the Lebesgue set of $f$. (Note that $\int \varphi_\varepsilon = 1$, $\varepsilon > 0$, so that

$$(f * \varphi_\varepsilon)(x) - f(x) = \int [f(x-y) - f(x)]\, \varphi_\varepsilon(y)\, \mathrm{d}y.$$

Use Theorem 7.16.)

**Solution**. ▸ ◂

**Exercise 3** (Wheeden & Zygmund Ch. 7, Ex. 6)**.** Show that if $\alpha > 0$, then $x^\alpha$ is absolutely continuous on every bounded subinterval of $[0, \infty)$.

**Solution**. ▸ ◂

**Exercise 4** (Wheeden & Zygmund Ch. 7, Ex. 8)**.** Prove the following converse of Theorem 7.31: If $f$ is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then $f$ is absolutely continuous on $[a, b]$.

**Solution**. ▸ ◂

**Exercise 5** (Wheeden & Zygmund Ch. 7, Ex. 9)**.** If $f$ is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then $f$ is absolutely continuous on $[a, b]$. (For the second part, use Theorems 2.2(ii) and 7.24 to show that $V(x)$ is absolutely continuous and then use the result of Exercise 8).

**Solution**. ▸ ◂

**Exercise 6** (Wheeden & Zygmund Ch. 7, Ex. 12)**.** Use Jensen's inequality to prove that if $a, b \geq 0$, $p, q > 1$, $(1/p) + (1/q) = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

More generally, show that

$$a_1 \cdots a_N = \sum_{j=1}^{N} \frac{a_j{}^{p_j}}{p_j},$$

where $a_j \geq 0$, $p_j > 1$, $\sum_{j=1}^{N}(1/p_j) = 1$. (Write $a_j = e^{x_j/p_j}$ and use the convexity of $e^x$.

**Solution**. ▸ ◂

**Exercise 7** (Wheeden & Zygmund Ch. 7, Ex. 13)**.** Prove Theorem 7.36.

**Solution**. ▸ Recall the statement of Theorem 7.36

(i) *If $\varphi_1$ and $\varphi_2$ are convex in $(a, b)$, then $\varphi_1 + \varphi_2$ is convex in $(a, b)$.*

(ii) *If $\varphi$ is convex in $(a, b)$ and $c$ is a positive constant, then $c\varphi$ is convex in $(a, b)$.*

(iii) *If $\varphi_k$, $k = 1, 2, \ldots$, are convex in $(a, b)$ and $\varphi_k \to \varphi$ in $(a, b)$, then $\varphi$ is convex in $(a, b)$.*

◂

#### 1.1.11 Homework 11

**Exercise 1** (Wheeden & Zygmund Ch. 7, Ex. 11)**.** Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let $f$ be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t.$$

(Consider the case when $f$ is the characteristic function of an interval, an open set, etc.)

**Solution.** ► ◄

**Exercise 2** (Wheeden & Zygmund Ch. 7, Ex. 15)**.** Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\varphi(x) = \int_a^x f(t)\,\mathrm{d}t + \varphi(a)$ in $(a, b)$ and $f$ is monotone increasing, then $\varphi$ is convex in $(a, b)$. (Use Exercise 14.)

**Solution.** ► ◄

**Exercise 3** (Wheeden & Zygmund Ch. 5, Ex. 8)**.** Prove (5.49).

**Solution.** ► ◄

**Exercise 4** (Wheeden & Zygmund Ch. 5, Ex. 11)**.** For which $p$ does $1/x \in L^p(0, 1)$? $L^p(1, \infty)$? $L^p(0, \infty)$?

**Solution.** ► ◄

**Exercise 5** (Wheeden & Zygmund Ch. 5, Ex. 12)**.** Give an example of a bounded continuous $f$ on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

**Solution.** ► ◄

**Exercise 6** (Wheeden & Zygmund Ch. 5, Ex. 17)**.** If $f \geq 0$ and $\omega(\alpha) \leq c(1 + \alpha)^p$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.

**Solution.** ► ◄

**Exercise 7** (Wheeden & Zygmund Ch. 8, Thm. 8.3)**.** If $f, g \in L^p(E)$, $p > 0$, then $f + g \in L^p(E)$ and $cf \in L^p(E)$ for any constant $c$.

**Solution.** ► ◄

### 1.1.12 Homework 12

**Exercise 1** (Wheeden & Zygmund Ch. 8, Ex. 2)**.** Prove the converse of Hölder's inequality for $p = 1$ and $\infty$. Show also that for $1 \le p \le \infty$, a real-valued measurable $f$ belongs to $L^p(E)$ if $fg \in L^1(E)$ for every $g \in L^{p'}(E)$, $1/p + 1/p' = 1$. The negation is also of interest: if $f \in L^p(E)$ then there exists $g \in L^{p'}(E)$ such that $fg \notin L^1(E)$. (To verify the negation, construct $g$ of the form $\sum a_k g_k$ satisfying $\int_E f g_k \to \infty$.)

**Solution.** ▸ ◂

**Exercise 2** (Wheeden & Zygmund Ch. 8, Ex. 3)**.** Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when $p < 1$.

**Solution.** ▸ ◂

**Exercise 3** (Wheeden & Zygmund Ch. 8, Ex. 4)**.** Let $f$ and $g$ be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let $1 < p < \infty$. Prove that equality holds in the inequality $|\int fg| \le \|f\|_p \|g\|_{p'}$ if and only if $fg$ has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e.

If $\|f + g\|_p = \|f\|_p + \|g\|_p$ and $g \neq 0$ in Minkowski's inequality, show that $f$ is a multiple of $g$.

Find analogues of these results for the spaces $\ell^p$.

**Solution.** ▸ ◂

**Exercise 4** (Wheeden & Zygmund Ch. 8, Ex. 5)**.** For $0 < p \le \infty$ and $0 < |E| < \infty$, define

$$N_p[f] = \left(\frac{1}{E}\int_E |f|^p\right)^{1/p},$$

where $N_\infty[f]$ means $\|f\|_\infty$. Prove that if $p_1 < p_2$, then $N_{p_1}[f] \le N_{p_2}[f]$. Prove also that if $1 \le p \le \infty$, then $N_p[f+g] \le N_p[f] + N_p[g]$, $(1/|E|)\int_E |fg| \le N_p[f]N_{p'}[g]$, $1/p + 1/p' = 1$, and $\lim_{p\to\infty} N_p[f] = \|f\|_\infty$. Thus, $N_p$ behaves like $\|\cdot\|_p$ but has the advantage of being monotone in $p$. Recall Exercise 28 of Chapter 5.

**Solution.** ▸ ◂

**Exercise 5** (Wheeden & Zygmund Ch. 8, Ex. 6)**.**

(a) Let $1 \le p_i, r \le \infty$ and $\sum_{i=1}^k 1/p_i = 1/r$. Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

(b) Let $1 \le p < r < q \le \infty$ and define $\theta \in (0, 1)$ by $1/r = \theta/p + (1-\theta)/q$. Prove the interpolation estimate

$$\|f\|_r \le \|f\|_p{}^\theta \|f\|_q{}^{1-\theta}.$$

In particular, if $A = \max\{\|f\|_p, \|f\|_q\}$, then $\|f\|_r \le A$.

**Solution**. ► ◄

**Exercise 6** (Wheeden & Zygmund Ch. 8, Ex. 9)**.** If $f$ is real-valued and measurable on $E$, $|E| > 0$, define its essential infimum on $E$ by

$$\operatorname{ess\,inf} f = \sup \{ \alpha : |\{ x \in E : f(x) < \alpha \}| = 0 \} .$$

If $f \geq 0$, show that $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup} 1/f)^{-1}$.

**Solution**. ► ◄

**Exercise 7** (Wheeden & Zygmund Ch. 8, Ex. 11)**.** If $f_k \to f$ in $L^p$, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty < M$ for all $k$, prove that $f_k g_k \to fg$ in $L^p$.

**Solution**. ► ◄

# 2 Danielli

## 2.1 Danielli: Practice Exams Spring 2016

### 2.1.1 Exam 1 Practice

**Exercise 1.** Let $E \subseteq \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{ rx : x \in E \}$. Prove that $rE$ is measurable, and that $|rE| = |r|^n |E|$.

**Solution.** ▸ Define a map a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = rx$. Since a the image of a measurable set $E$ under linear map is measurable and $m(T(E)) = |\det T| m(E) = |r|^n m(E)$, it suffices to show that $T(E) = rE$.

Let $y \in T(E)$ then $y = rx$ for some $x \in E$. Thus, $y \in rE$. Let $y \in rE$. Then, $y = rx = T(x)$ for some $x \in E$. Thus, $y \in T(E)$. It follows that $m(rE) = |r|^n m(E)$. ◂

**Exercise 2.** Let $\{E_n\}$, $n \in \mathbb{N}$ be a collection of measurable sets. Define the set

$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} E_k \right).$$

Show that

$$m\left( \liminf_{n\to\infty} E_n \right) \leq \liminf_{n\to\infty} m(E_n).$$

**Solution.** ▸ Here's a quick and dirty way of proving this: let $\mathbb{1}_{E_n}$ be the characteristic function of $E_n$. Then, by Fatou's lemma,

$$\int \liminf_{n\to\infty} \mathbb{1}_{E_n}(x)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int \mathbb{1}_{E_n}(x)\,\mathrm{d}x. \tag{1}$$

By definition of the characteristic function, it is easy to see that the right hand-side of the Equation (1) is

$$\liminf_{k\to\infty} m(E_k).$$

But what about the left-hand side of (1)? We claim that

$$\liminf_{n\to\infty} \mathbb{1}_{E_n} = \mathbb{1}_E$$

where $E = \liminf_{n\to\infty} E_n$.

*Proof of claim.* Suppose $x \in E$. We must show that $\liminf_{n\to\infty} \mathbb{1}_{E_n}(x) = 1$. By definition

$$\liminf_{n\to\infty} \mathbb{1}_{E_n} = \lim_{n\to\infty} \left[ \inf_{k\geq n} \mathbb{1}_{E_k} \right].$$

Now $x \in E$ if and only if $x \in \bigcap_{k=N}^{\infty} E_k$ for some $N \in \mathbb{N}$. Then for $k \geq N$

$$\inf_{k\geq n} \mathbb{1}_{E_k}(x) = 1$$

so $\liminf_{n\to\infty} \mathbb{1}_{E_n}(x) = 1$.

On the other hand, if $x \notin E$ then $x \notin \bigcap_{k=n}^{\infty} E_k$ for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$,

$$\inf_{k\geq n} \mathbb{1}_{E_k}(x) = 0$$

so $\liminf_{n\to\infty} \mathbb{1}_{E_k} = 0$. ∎

Having established this equivalence, we have

$$m\Big(\liminf_{n\to\infty} E_n\Big) = \int \liminf_{n\to\infty} \mathbb{1}_{E_n}(x)\,\mathrm{d}x \le \liminf_{n\to\infty} \int \mathbb{1}_{E_n}(x)\,\mathrm{d}x = \liminf_{n\to\infty} m(E_n).$$

◂

**Exercise 3.** Consider the function

$$F(x) = \begin{cases} m\big(B(\mathbf{0}, x)\big) & x > 0, \\ 0 & x = 0. \end{cases}$$

Here $B(\mathbf{0}, r) = \{\, y \in \mathbb{R}^n : |y| < r \,\}$. Prove that $F$ is monotonic increasing and continuous.

**Solution.** ▸ Let $T\colon \mathbb{R}^n \times [0, x) \to \mathbb{R}^n$ be the linear map given by $T(x, r) = rx$. By Problem 1, we know that $T\big(B(\mathbf{0}, 1), r\big) = B(\mathbf{0}, r)$ and consequently, $m\big(B(\mathbf{0}, 1)\big) = |r|^n m\big(B(\mathbf{0}, 1)\big)$. Interpreting $B(\mathbf{0}, 0) = \emptyset$, we have $F(x) = |r|^n m\big(B(\mathbf{0}, 1)\big)$ and it is easy to see that $F$ is both monotonically increasing and continuous since it is a polynomial in $r$. ◂

**Exercise 4.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Let $C$ be the set of all points at which $f$ is continuous. Show that $C$ is a set of type $G_\delta$.

**Solution.** ▸ Let $C$ be the subset of $\mathbb{R}$ where $f$ is continuous, i.e., the set

$$C = \big\{\, x \in \mathbb{R} : \text{given } \varepsilon > 0 \text{ there exist } \delta > 0 \text{ such that } |f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \,\big\}.$$

In light of the latter equality, for each $n \in \mathbb{N}$ define the following family of subsets of $C$,

$$G_n = \Big\{\, x \in \mathbb{R} : \text{there exists } \delta_n > 0 \text{ such that } |f(x) - f(y)| < \frac{1}{n} \text{ whenever } |x - y| < \delta_n \,\Big\}.$$

We claim that (i) the $G_n$ are open and (ii) $C = \bigcap_{n\in\mathbb{N}} G_n$.

The proof of (i) is easy: let $x \in G_n$ then there exists $\delta_n > 0$ such that

$$|f(x) - f(y)| < \frac{1}{n}.$$

Then $B(x, \delta_n) \subseteq G_n$ since $x' \in B(x, \delta_n)$ implies that $|x - x'| < \delta$ so

$$|f(x) - f(x')| < \frac{1}{n}.$$

The proof of (ii) is also straight forward: let $x \in C$ then given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $|x - y| < \delta$. In particular, if $\varepsilon = 1/n$ then there exists $\delta_n$ such that $|x - y| < \delta_n$ implies

$$|f(x) - f(y)| < \frac{1}{n}$$

for ever $n \in \mathbb{N}$. Thus, $x \in \bigcap_{n\in\mathbb{N}} G_n$. On the other hand, if $x \in \bigcap_{ni\in\mathbb{N}} G_n$, then $x \in G_n$ for all $n \in \mathbb{N}$. Thus, given $\varepsilon > 0$, by the Archimedean property of the real numbers, there exists a positive integer $N$ such that $1/N < \varepsilon$ and hence for $\delta = \delta_N > 0$ we have

$$|f(x) - f(y)| < \frac{1}{N}$$

whenever $|x - y| < \delta_N$. Thus, $x \in C$.

It follows that $C = \bigcap_{n\in\mathbb{N}} G_n$ and hence is a $G_\delta$ set. ◂

**Exercise 5.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, then $f$ is measurable?

**Solution**. ▸ The statement is false and, of course, the counterexample involves existence of nonmeasurable sets. Let $V \subseteq [0, 1]$ be a Vitali set and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} x & \text{if } x \in V, \\ -x & \text{if } x \in \mathbb{R} \setminus V. \end{cases}$$

Then, $\{ f = r \}$ is measurable for all $r \in \mathbb{R}$ since the set either consists of a single point or is the empty set. However, $\{ f \geq 0 \} = V$ is not measurable. ◂

**Exercise 6.** Let $\{f_k\}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set

$$\left\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \right\}$$

is measurable.

**Solution**. ▸ Suppose $\{f_n\}$, $n \in \mathbb{N}$, is a sequence of measurable functions and let

$$E = \left\{ x : \lim_{n\to\infty} f_n(x) \text{ exists} \right\}.$$

Then, by general properties of the limit supremum and the limit infimum, we know that $\lim_{n\to\infty} f_n(x)$ exists if and only if

$$\limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x).$$

Both of these functions are measurable so the set

$$E = \left\{ x : \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x) \right\}.$$

is measurable. ◂

**Exercise 7.** A real valued function $f$ on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in $(a, b)$ satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

**Solution**. ▸ Let $\varepsilon = 1$ then, since $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $y - x < \delta$ (assuming $x < y$). Partition the closed interval $[a,b]$ into subintervals $\{[a_n, b_n] : 1 \le n \le N\}$ of length less than or equal to $\delta$. Then

$$\operatorname{var}(f; [a_n, b_n]) \le 1.$$

Thus,

$$\operatorname{var}(f; [a,b]) \le N$$

for every partition $\Gamma$ of $[a,b]$. ◂

**Exercise 8.** Let $f$ be a continuous function from $[a,b]$ into $\mathbb{R}$. Let $\mathbb{1}_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, that is, $\mathbb{1}_{\{c\}}(x) = 0$ if $x \ne c$ and $\mathbb{1}_{\{c\}}(c) = 1$. Show that

$$\int_a^b f\,\mathrm{d}\mathbb{1}_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a,b), \\ -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

**Solution**. ▸ There are three cases to consider (1) $c \in (a,b)$, (2) $c = a$ and (3) $c = b$. Cases (2) and (3) can be handled easily: if $c = a$ then the Rieman–Stieltjes integral of $f$ with respect to $\mathbb{1}_{\{c\}}$ is the supremum over all sums

$$\sum_{n=1}^{N} f(\xi_n)\big[\mathbb{1}_{\{c\}}(x_n) - \mathbb{1}_{\{c\}}(x_{n-1})\big]$$

where $x_0 = a$ and $x_N = b$ for all partitions $\Gamma = \{x_0, \ldots, x_N\}$ of $[a,b]$. Thus, the sum

$$\sum_{n=1}^{N} f(\xi_n)\big[\mathbb{1}_{\{c\}}(x_n) - \mathbb{1}_{\{c\}}(x_{n-1})\big] = \begin{cases} -f(\xi_0) & \text{if } c = a, \\ f(\xi_N) & \text{if } c = b. \end{cases}$$

Letting $\Delta(\Gamma) \to 0$, $\xi_0 \to a$ and $\xi_N \to b$ giving us

$$\int_a^b f\,\mathrm{d}\mathbb{1}_{\{c\}} = \begin{cases} -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

It remains to show that

$$\int_a^b f\,\mathrm{d}\mathbb{1}_{\{c\}} = 0$$

if $c \in (a,b)$. To that end, note that if $\Gamma_c$ is a partition containing the point $c$, say, $x_m = c$ for some $1 \le m \le N$, the partial sums yield

$$\sum_{n=1}^{N} f(\xi_n)\big[\mathbb{1}_{\{c\}}(x_n) - \mathbb{1}_{\{c\}}(x_{n-1})\big] = f(\xi_{m+1}) - f(\xi_m).$$

Letting $\Delta(\Gamma_c) \to 0$, since $f$ is continuous, $f(\xi_{m+1}) \to f(\xi_m)$. Thus,

$$\int_a^b f\,\mathrm{d}\mathbb{1}_{\{c\}} = 0.$$

◂

### 2.1.2 Exam 1

I lost this exam. These are the questions I could recall explicitly. For the first problem, we were asked to show that the Dichlet function $\mathbb{1}_{\mathbb{Q}}(x)$ is not Riemann integrable and prove something about $\mathbb{Q}$. For the second question, we were asked to show that the measure of countable union of disjoint measurable sets $\{E_n : n \in \mathbb{N}\}$, is equal to the sum of their individual measures (or something to that effect).

**Exercise 1.**

**Exercise 2.**

**Exercise 3.**

(i) Show that if $B_r = \{\, x \in \mathbb{R}^n : |x| < r \,\}$, then there exists a constant $C$ such that $|B_r| = Cr^n$. (*Hint*: Think of $B_r$ as $\{\, rx : x \in B_1 \,\}$.)

(ii) Let $E \subseteq \mathbb{R}^n$ be a measurable set and let $\varphi_E \colon \mathbb{R}^n \to \mathbb{R}$ be defined $\varphi_E(x) = m(E \cap B_{|x|})$. Use part (i) to prove that $\varphi_E$ is continuous.

**Solution**. ▸ For part (i), as in the practice problems, define the linear map $T \colon \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = rx$. Note that this map is Lipschitz so the image of a measurable set $E$ under $T$ is measurable and $m(T(E)) = |\det T| m(E) = |r|^n m(E)$. It is not too difficult to see that

$$T(B_1) = B_r$$

as sets, so $m(B_r) = |r|^n m(B_1)$. Now, let $C = m(B_1)$.

For part (ii), note that for any $|x|, |y| \in \mathbb{R}$, by part (i), we have

$$\begin{aligned} |\varphi_E(x) - \varphi_E(y)| &\leq \big|C|x| - C|y|\big| \\ &= C\big||x| - |y|\big|. \end{aligned}$$

In particular, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\big||x| - |y|\big| < \frac{\varepsilon}{C}.$$

Thus,

$$\begin{aligned} |\varphi_E(x) - \varphi_E(y)| &< C\left(\frac{\varepsilon}{C}\right) \\ &= \varepsilon \end{aligned}$$

and $\varphi_E$ is continuous. Check this $\emptyset$ out ◂

**Exercise 4.** Assume that $f \colon [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$. Prove that $f$ is measurable.

**Solution**. ▸ Suppose that $f \colon [a, b] \to \mathbb{R}$ is b.v. on $[a, b]$. Then $f$, by Jordan's theorem, $f = g - h$ where $g$ and $h$ are monotone increasing functions. Since monotone functions are a.e. continuous, $g$ and $h$ are measurable functions. Thus, $f$ is measurable. ◂

### 2.1.3 Exam 2 Practice Problems

**Exercise 1.** Define for $x \in \mathbb{R}^n$,

$$f(x) = \begin{cases} |x|^{-(n+1)} & \text{if } x \neq \mathbf{0}, \\ 0 & \text{if } x = \mathbf{0}. \end{cases}$$

Prove that $f$ is integrable outside any ball $B(\mathbf{0}, \varepsilon)$, and that there exists a constant $C > 0$ such that

$$\int\limits_{\mathbb{R}^n \setminus B(\mathbf{0},\varepsilon)} f(x)\,\mathrm{d}x \leq \frac{C}{\varepsilon}.$$

**Solution.** ▸ Danielli gave a wonderful solution to this problem by using spherical coordinates to compute the integral. However, she did not justify the use of polar coordinates, or even make it clear what exactly the meaning of $\mathrm{d}x$ and $\mathrm{d}\sigma$ mean in this context. The justification comes from the polar decomposition of the Lebesgue measure (we shall not prove this here). Changing the integral in question to polar coordinates, we have

$$\int\limits_{\mathbb{R}^n \setminus B(\mathbf{0},\varepsilon)} f(x)\,\mathrm{d}x = \int\limits_{S^{n-1}(\mathbf{0},r)} \int_{\varepsilon}^{\infty} \frac{1}{r^{n+1}}\,\mathrm{d}r\mathrm{d}\sigma$$

which, by Tonelli's theorem, becomes

$$= \int_{\varepsilon}^{\infty} \frac{1}{r^{n+1}} \left[ \int_{S^{n-1}(\mathbf{0},r)} \mathrm{d}\sigma \right] \mathrm{d}r$$

which, by a problem from the homework, becomes

$$\begin{aligned} &= C_n \int_{\varepsilon}^{\infty} r^{n-1} \left[ \frac{1}{r^{n+1}} \right] \mathrm{d}r \\ &= C_n \int_{\varepsilon}^{\infty} \frac{1}{r^2}\,\mathrm{d}r \\ &= \frac{C_n}{\varepsilon}, \end{aligned}$$

as desired. ◂

**Exercise 2.** Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that $f_k$ converges pointwise almost everywhere to a function $f$. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets $E$ of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k = \infty$.

**Solution**. ▸ Let $E$ be a measurable subset of $\mathbb{R}^n$. First, note that since $f_n \to f$ a.e. in $\mathbb{R}^n$, we clearly have $f_n\restriction_E \to f\restriction_E$ and $f_n\restriction_{\mathbb{R}^n\setminus E} \to f\restriction_{\mathbb{R}^n\setminus E}$. By the monotonicity of the Lebesgue integral,

$$\lim_{n\to\infty}\int_{\mathbb{R}^n} f\,\mathrm{d}x = \lim_{n\to\infty}\left[\int_E f_n\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus E} f_n\,\mathrm{d}x\right]$$

and by basic properties of the limit supremum, we have

$$\begin{aligned}\int_E f\,\mathrm{d}x &= \int_{\mathbb{R}^n} f\,\mathrm{d}x - \int\limits_{\mathbb{R}^n\setminus E} f\,\mathrm{d}x \\ &\geq \lim_{n\to\infty}\int_{\mathbb{R}^n} f_n\,\mathrm{d}x - \liminf_{n\to\infty}\int\limits_{\mathbb{R}^n\setminus E} f_n\,\mathrm{d}x \qquad (1)\\ &= \limsup_{n\to\infty}\left[\int_{\mathbb{R}^n} f_n\,\mathrm{d}x - \int_{\mathbb{R}^n\setminus E} f_n\,\mathrm{d}x\right] \quad = \limsup_{n\to\infty}\int_E f_n\,\mathrm{d}x.\end{aligned}$$

Now, by Fatou's lemma, we have

$$\begin{aligned}\int_E f\,\mathrm{d}x &= \int_E \liminf_{n\to\infty} f_n\,\mathrm{d}x \\ &\leq \liminf_{n\to\infty}\int_E f_n\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int\limits_{\mathbb{R}^n\setminus E} f\,\mathrm{d}x &= \int\limits_{\mathbb{R}^n\setminus E} \liminf_{n\to\infty} f_n\,\mathrm{d}x \\ &\leq \liminf_{n\to\infty}\int\limits_{\mathbb{R}^n\setminus E} f_n\,\mathrm{d}x,\end{aligned}$$

both of which are finite since $f \in L^1(\mathbb{R}^n)$ and $\lim_{n\to\infty}\int_{\mathbb{R}^n} f_n\,\mathrm{d}x < \infty$. Thus,

$$\limsup_{n\to\infty}\int_E f_n\,\mathrm{d}x \leq \int_E f\,\mathrm{d}x \leq \liminf_{n\to\infty}\int_E f_n\,\mathrm{d}x \leq .$$

Therefore,

$$\int_E f_n\,\mathrm{d}x \longrightarrow \int_E f\,\mathrm{d}x$$

as $n \to \infty$.

This is not true in general as the sequence

$$f_n(x) = \begin{cases} k^2/2 & \text{if } x \in (-1/k, 1/k), \\ 1 & \text{otherwise.}\end{cases}$$

Then, $f \to 1$ a.e. in $\mathbb{R}$ and

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n\,\mathrm{d}x = \int_{\mathbb{R}} f\,\mathrm{d}x = \infty,$$

but for $E = (0, 1)$,

$$\int_E f \, \mathrm{d}x = 1$$

while

$$\lim_{n \to \infty} \int_E f_n \, \mathrm{d}x = \infty.$$

◂

**Exercise 3.** Assume that $E$ is a measurable set of $\mathbb{R}^n$, with $m(E) < \infty$. Prove that a nonnegative function $f$ defined on $E$ is integrable if and only if

$$\sum_{k=0}^{\infty} m\{ x \in E : f(x) \geq k \} < \infty.$$

**Solution.** ▸ $\implies$ Suppose that $f$ is integrable on $E$. Let

$$E_n = \{ x \in E : f(x) \geq n \}.$$

Define the sequence of measurable sets $\{E_n' : n \in \mathbb{N}\}$ where

$$E_n' = \{ x \in E : n + 1 > f(x) \geq n \}.$$

Now, we note that

$$E_n = \bigcup_{k=n}^{\infty} E_k'.$$

and since the $E_k'$ are disjoint,

$$m(E_n) = \sum_{k=n}^{\infty} m(E_k').$$

Moreover,

$$E = \bigcup_{n=1}^{\infty} E_n'$$

so

$$m(E) = \sum_{n=1}^{\infty} m(E_n').$$

Thus,

$$\sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} \left[ \sum_{k=n}^{\infty} m(E_k') \right]$$

by reordering the latter sum,

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n m\big(E_n'\big) \\
&\leq \sum_{n=1}^{\infty} \int_{E_n'} f \,\mathrm{d}x \\
&= \int_E f \,\mathrm{d}x \\
&< \infty.
\end{aligned}
$$

Thus, $\sum_{n=1}^{\infty} m(E_n) < \infty$.

$\Longleftarrow$ On the other hand, suppose that

$$\sum_{n=1}^{\infty} m(E_n) < \infty.$$

Then, using the sequence $\{E_n : n \in \mathbb{N}\}$ above together with

$$E_0 = \big\{ x \in E : 1 > f(x) \geq 0 \big\},$$

we have

$$
\begin{aligned}
\int_X f \,\mathrm{d}x &= \lim_{n\to\infty} \sum_{k=0}^{n} \int_{E_k'} f \,\mathrm{d}x \\
&\leq \lim_{n\to\infty} \sum_{k=0}^{n} (n+1) m\big(E_n'\big) \\
&= m(E) + m(E_1) + \cdots \\
&< \infty.
\end{aligned}
$$

Thus, $f$ is integrable. ◂

**Exercise 4.** Suppose that $E$ is a measurable subset of $\mathbb{R}^n$, with $m(E) < \infty$. If $f$ and $g$ are measurable functions on $E$, define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if $f_k$ converges to $f$ as $k \to \infty$.

**Solution.** ▸ $\Longrightarrow$ Suppose that $\rho(f_n, f) \to 0$ as $n \to \infty$. Then, given $\varepsilon > 0$ there exist an index $N \in \mathbb{N}$ such that

$$\int_E \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \,\mathrm{d}x < \frac{\varepsilon^2}{1 + \varepsilon}$$

whenever $n \geq N$. This implies that the measure of the set

$$E_\varepsilon = \big\{ x \in E : |f_n(x) - f(x)| > \varepsilon \big\}$$

goes to 0 as $n \to \infty$ since

$$\begin{aligned}
\frac{\varepsilon^2}{1+\varepsilon} &> \int_E \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|}\,\mathrm{d}x \\
&\geq \int_{E_\varepsilon} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|}\,\mathrm{d}x \\
&\geq \int_{E_\varepsilon} \frac{\varepsilon}{1+\varepsilon}\,\mathrm{d}x \\
&= \frac{\varepsilon m(E_\varepsilon)}{1+\varepsilon}
\end{aligned}$$

which implies that

$$m(E_\varepsilon) < \varepsilon.$$

Thus, $f_n \to f$ for a.e. $x \in E$.

$\Longleftarrow$ On the other hand, suppose that $f_n \to f$ a.e. on $E$. Then, for every $\varepsilon > 0$

$$\lim_{n\to\infty} m\big\{\, x \in E : |f(x) - f_n(x)| > \varepsilon \,\big\} = 0.$$

Then, we note that

$$\begin{aligned}
\rho(f_n, f) &= \int_E \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|}\,\mathrm{d}x \\
&= \int_{E\setminus E_\varepsilon} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|}\,\mathrm{d}x + \int_{E_\varepsilon} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|}\,\mathrm{d}x \\
&\leq m(E_\varepsilon) + \frac{1+\varepsilon}{\varepsilon} m(E_\varepsilon) \\
&= \left(1 + \frac{1+\varepsilon}{\varepsilon}\right) m(E_\varepsilon) \\
&= \left(\frac{1+2\varepsilon}{\varepsilon}\right) m(E_\varepsilon).
\end{aligned}$$

Taking the limit on both sides of this inequality, we see that $\rho(f_n, f) \to 0$ as $n \to \infty$ since

$$\left(\frac{1+2\varepsilon}{\varepsilon}\right) m(E_\varepsilon) \longrightarrow 0$$

as $n \to \infty$. ◂

**Exercise 5.** Define the *gamma function* $\Gamma\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty \mathrm{e}^{-u} u^{y-1}\,\mathrm{d}u,$$

and the *beta function* $\beta\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto \mathrm{e}^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that
$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

**Solution**. ▸ For part (a), we break the proof into two cases: fix $y_0 \in \mathbb{R}^+$ then (1) $0 \le y_0 \le 1$, or (2) $y > 1$. In case (1), we have

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-u}u^{y_0-1}\,\mathrm{d}x &\le \int_0^\infty \mathrm{e}^{-u}\,\mathrm{d}u \\ &< \infty. \end{aligned}$$

In case (2), ◂

**Exercise 6.** Let $f \in L(\mathbb{R}^n)$ and for $h \in \mathbb{R}^n$ define $f_h \colon \mathbb{R}^n \to \mathbb{R}$ be $f_h(x) = f(x - h)$. Prove that

$$\lim_{h\to\mathbf{0}} \int_{\mathbb{R}^n} |f_h - f| = 0.$$

**Solution**. ▸ Since $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, there exist a sequence of compactly supported continuous functions $\{g_n : n \in \mathbb{N}\}$ such that $g_n \to f$ as $n \to \infty$, i.e., given $\varepsilon > 0$, there exists an index $N \in \mathbb{N}$ such that $n \ge N$ implies
$$|f(x) - g_n(x)| < \frac{\varepsilon}{3}$$
for all $x \in \mathbb{R}^n$. Moreover, for any sequence $\{h_n : n \in \mathbb{N}\}$ such that $h_n \to \mathbf{0}$, by the uniform continuity of $g_n$ (since $g_n$ is continuous on a compact set), there exist an index $N' \in \mathbb{N}$ such that

$$|g_n(x + h_k) - g_n(x)| < \frac{\varepsilon}{3}$$

whenever $k \ge N'$. Thus, we have

$$\begin{aligned} |f(x + h_n) - f(x)| &= \Big|\big(f(x + h_n) - g_n(x + h_n)\big) + \big(g_n(x + h_n) - g_n(x)\big) + \big(-g_n(x) - f(x)\big)\Big| \\ &\le |f(x + h_n) - g_n(x + h_n)| + |g_n(x + h_n) - g_n(x)| + |f(x) - g_n(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Thus,
$$\lim_{h\to\mathbf{0}} \int_{\mathbb{R}^n} |f_h - f| = 0.$$

◂

**Exercise 7.** (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \le g_k$ and

$$\int_{\mathbb{R}^n} g_k \longrightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \qquad \text{as } k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \longrightarrow \int_{\mathbb{R}^n} |f| \qquad \text{as } k \to \infty.$$

**Solution**. ▸ For part (a), suppose that $|f_n| \le g_n$, then $g_n(x) - f_n(x), g_n(x) + f_n(x) \ge 0$ for all $x \in \mathbb{R}^n$. Thus, by Fatou's lemma, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \liminf_{n\to\infty} \big(g_n(x) - f_n(x)\big) \mathrm{d}x &\le \liminf_{n\to\infty} \left[\int_{\mathbb{R}^n} g_n(x) - f_n(x)\,\mathrm{d}x\right] \\
\int_{\mathbb{R}^n} g(x) - f(x)\,\mathrm{d}x &\le \liminf_{n\to\infty} \int_{\mathbb{R}^n} g_n(x) + \liminf_{n\to\infty} \int_{\mathbb{R}^n} -f_n(x)\,\mathrm{d}x \\
\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x &\le \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x - \limsup_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\,\mathrm{d}x \\
\limsup_{n\to\infty} \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x &\le \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n} \liminf_{n\to\infty} \big(g_n(x) + f_n(x)\big) \mathrm{d}x &\le \liminf_{n\to\infty} \left[\int_{\mathbb{R}^n} g_n(x) + f_n(x)\,\mathrm{d}x\right] \\
\int_{\mathbb{R}^n} g(x) + f(x)\,\mathrm{d}x &\le \liminf_{n\to\infty} \int_{\mathbb{R}^n} g_n(x)\,\mathrm{d}x + \liminf_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\,\mathrm{d}x \\
\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x + \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x &\le \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x + \liminf_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\,\mathrm{d}x \\
\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x & \liminf_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\,\mathrm{d}x.
\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\,\mathrm{d}x \le \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\,\mathrm{d}x$$

which implies that

$$\int_{\mathbb{R}^n} f_n(x)\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$$

as $n \to \infty$.

For part (b), $\implies$ suppose that

$$\int_{\mathbb{R}^n} |f_n(x) - f(x)| \, \mathrm{d}x \longrightarrow 0$$

as $n \to \infty$. Then, by the reverse triangle inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} |f_n(x)| - |f(x)| \, \mathrm{d}x \right| &\leq \int_{\mathbb{R}^n} \big| |f_n(x)| - |f(x)| \big| \, \mathrm{d}x \\ &\leq \int_{\mathbb{R}^n} \big| f_n(x) - f(x) \big| \, \mathrm{d}x. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} |f_n(x)| \, \mathrm{d}x = \int_{\mathbb{R}^n} |f(x)| \, \mathrm{d}x.$$

$\impliedby$ On the other hand, suppose that

$$\int_{\mathbb{R}^n} |f_n(x)| \, \mathrm{d}x \longrightarrow \int_{\mathbb{R}^n} |f(x)| \, \mathrm{d}x$$

as $n \to \infty$. Then $|f_n(x) - f(x)| \leq |f_n(x)|$ so by the generalized Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} |f_n(x) - f(x)| \, \mathrm{d}x = \int_{\mathbb{R}^n} \left[ \lim_{n\to\infty} |f_n(x) - f(x)| \right] \mathrm{d}x = 0.$$

◂

### 2.1.4 Exam 2 (2010)

**Exercise 1.** Suppose $f \in L^1(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball $B$, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

*Hint*: Use the monotone convergence theorem.

**Solution**. ▸ Consider the sequence of functions $\{f_n(x) : n \in \mathbb{N}\}$ where

$$f_n(x) = |f(x)| \mathbb{1}_{B(\mathbf{0},n)}(x).$$

Then, $f_n \uparrow |f|$ so by the monotone convergence theorem, given $\varepsilon > 0$, there exists an index $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)| \, \mathrm{d}x - \int_{\mathbb{R}^n} f_n(x) \, \mathrm{d}x &= \int_{\mathbb{R}^n} |f(x)| - |f(x)| \mathbb{1}_{B(\mathbf{0},n)} \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n \setminus B(\mathbf{0},n)} |f(x)| \, \mathrm{d}x \\ &< \varepsilon. \end{aligned}$$

Let $B = B(\mathbf{0}, N+1)$. ◂

**Exercise 2.**

(a) Prove the following generalization of *Chebyshev's inequality*: Let $0 < p < \infty$ and $E \subseteq \mathbb{R}^n$ be measurable. Assume that $|f|^p \in L^1(E)$. Then

$$m\{ x \in E : f(x) > \alpha \} \leq \frac{1}{\alpha^p} \int_{\{f > \alpha\}} f^p,$$

for $\alpha > 0$.

(b) Let $p$, $E$, and $f$ be as in part (a). In addition, assume that $\{f_k\}$ is a sequence such that $\int_E |f_k - f|^p \to 0$ as $k \to \infty$. Show that $f_k \to f$ in measure on $E$.

Recall that $f_k \to f$ in measure on $E$ if and only if for every $\varepsilon > 0$

$$\lim_{k \to \infty} m\{ x \in E : |f_k(x) - f(x)| > \varepsilon \} = 0.$$

**Solution**. ▸ Part (a) is almost trivial. Let

$$E_\alpha = \{ x \in E : f(x) > \alpha \}.$$

Then, $|f|^p \geq \alpha^p$ for all $x \in E_\alpha$. Thus,

$$\int_{E_\alpha} \alpha^p \, \mathrm{d}x = \alpha^p m(E_\alpha) \leq \int_{E_\alpha} |f|^p \, \mathrm{d}x,$$

as was to be shown.

Part (b) follows directly from Chebyshev's inequality, as

$$\lim_{n\to\infty} m(E_\varepsilon) < \lim_{n\to\infty} \left[ \frac{1}{\varepsilon^p} \int_{E_\varepsilon} |f_n(x) - f(x)|^p \, \mathrm{d}x \right] = \frac{1}{\varepsilon^p} \lim_{n\to\infty} \int_{E_\varepsilon} |f_n(x) - f(x)| \, \mathrm{d}x = 0,$$

where $E_\varepsilon = \{ x \in E : |f_n(x) - f(x)| > \varepsilon \}$. ◂

**Exercise 3.** Let $f \in L^1(\mathbb{R})$, and define

$$F(\xi) = \int_{\mathbb{R}} f(x) \cos(2\pi x \xi) \, \mathrm{d}x.$$

Prove that $F$ is continuous and bounded on $\mathbb{R}$.

**Solution**. ▸ It is easy to see that $F$ is bounded as $|\cos(2\pi x\xi)| < 1$ for all $\chi \in \mathbb{R}$ so

$$|F(\xi)| = \left| \int_{\mathbb{R}} f(x) \cos(2\pi x \xi) \, \mathrm{d}x \right. \leq \left| \int_{\mathbb{R}} f(x) \, \mathrm{d}x \right| \leq \|f\|_1.$$

To see that $F$ is in fact continuous, note that since $\cos(2\pi x\xi)$ is continuous, as a function of $\xi$, given $\varepsilon > 0$ there exist $\delta > 0$ such that $|\xi - \chi| < \delta$ implies

$$|\cos(2\pi x\xi) - \cos(2\pi x \chi)| < \frac{\varepsilon}{\|f\|_1}.$$

Thus, for $|\xi - \chi| < \delta$, we have

$$\begin{aligned}
|F(\xi) - F(\xi)| &= \left| \int_{\mathbb{R}} f(x) \cos(2\pi x\xi) \, \mathrm{d}x - \int_{\mathbb{R}} f(x) \cos(2\pi x \chi) \, \mathrm{d}x \right| \\
&\leq \int_{\mathbb{R}} \left| f(x) \big( \cos(2\pi x \xi) - \cos(2\pi x \chi) \big) \right| \mathrm{d}x \\
&\leq \frac{\varepsilon}{\|f\|_1} \int_{\mathbb{R}} |f(x)| \, \mathrm{d}x \\
&= \varepsilon.
\end{aligned}$$

Thus, $F$ is continuous. ◂

**Exercise 4.** Use repeated integration techniques to prove that

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2} \, \mathrm{d}x = \pi^{n/2}.$$

*Hint*: Start from the case $n = 1$ by using the polar coordinates in

$$\left[\int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}x\right]^2 = \left[\int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}x\right]\left[\int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}y\right]$$

**Solution**. ► We shall proceed by induction. For the case $n = 1$, note that

$$\begin{aligned}\left[\int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}x\right]^2 &= \left[\int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}x\right]\left[\int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}y\right] \\ &= \int\limits_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{-x^2+y^2}\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Making a smooth change of variables via polar coordinates, we have

$$\left[\int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}x\right]^2 = \int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = \pi.$$

Thus,

$$\int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

Now, suppose

$$\int_{\mathbb{R}^k} \mathrm{e}^{-x_1{}^2-\cdots-x_k{}^2}\,\mathrm{d}x_1\cdots\mathrm{d}x_k = \sqrt[k]{\pi}$$

for all $k \le n-1$. Then, for the case $n$, we have

$$\int_{\mathbb{R}^n} \mathrm{e}^{-x_1{}^2\cdots-x_{n-1}{}^2-x_n{}^2}\,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{-x_1{}^2\cdots-x_{n-1}{}^2}\mathrm{e}^{-x_n{}^2}\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\mathrm{d}x_n$$

◄

### 2.1.5 Exam 2

**Exercise 1.** Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball $B$, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

**Solution.** ► ◄

**Exercise 2.** Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of $E$, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

**Solution.** ► ◄

**Exercise 3.** Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbb{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

**Solution.** ► ◄

**Exercise 4.** Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)\,\mathrm{d}t$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

**Solution.** ► ◄

### 2.1.6 Final Exam Practice Problems

**Exercise 1.** Suppose $f \in L^1(\mathbb{R}^n)$ and that $x$ is a point in the Lebesgue set of $f$. For $r > 0$, let

$$A(r) = \frac{1}{|r|^n} \int_{B(0,r)} |f(x-y) - f(x)|\, \mathrm{d}y.$$

Show that:

(a) $A(r)$ is a continuous function of $r$, and $A(r) \to 0$ as $r \to 0$;

(b) there exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

**Solution**. ▸ (a) Without loss of generality, we may assume $r < s$. Then, we want to show that as $r \to s$, the quantity

$$|A(s) - A(r)| \longrightarrow 0.$$

Set $F(y) = |f(x-y) - f(x)|$ and consider said quantity

$$\begin{aligned}
|A(s) - A(r)| &= \left| \frac{1}{|s|^n} \int_{B_s} F(y)\, \mathrm{d}y - \frac{1}{|r|^n} \int_{B_r} F(y)\, \mathrm{d}y \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y)\, \mathrm{d}y + \frac{1}{|s|^n} \int_{B_r} F(y)\, \mathrm{d}y - \frac{1}{|r|^n} \int_{B_r} F(y)\, \mathrm{d}y \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y)\, \mathrm{d}y + \left( \frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(y)\, \mathrm{d}y \right| \\
&\leq \underbrace{\frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y)\, \mathrm{d}y}_{I_1} + \underbrace{\left( \frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(y)\, \mathrm{d}y}_{I_2}.
\end{aligned}$$

Hence, we must show that the quantities $I_1, I_2 \to 0$ as $r \to s$.

To see that $A(r) \to 0$ as $r \to 0$, note that $x$ is a point of the Lebesgue set of $f$ and that

$$0 = \lim_{B_r \downarrow x} \frac{1}{|B_1||r|^n} \int_{B_r} |f(y) - f(x)|\, \mathrm{d}y = \frac{1}{|B_1|} \lim_{B_r \downarrow x} \frac{1}{|r|^n} \int_{B_r} |f(t) - f(x)|\, \mathrm{d}t = \lim_{r \to 0} A(r).$$

by making the change of variables $t = x - y$.

(b) ◂

**Exercise 2.** Let $E \subseteq \mathbb{R}^n$ be a measurable set, $1 \leq n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on $E$ to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0$$

if and only if

$$\|f_k\|_p \longrightarrow \|f\|_p$$

as $k \to \infty$.

**Solution.** ► ◄

**Exercise 3.** Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.

(b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

**Solution.** ► ◄

**Exercise 4.** Let $f \in L(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x)\cos(tx)dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.

(b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

**Solution.** ► ◄

**Exercise 5.** Let $f$ be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with $g$ absolutely continuous and $h$ singular. Show that

$$\int_a^b \varphi\,\mathrm{d}f = \int_a^b \varphi f' dx + \int_a^b \varphi\,\mathrm{d}h$$

for all functions $\varphi$ continuous on $[a, b]$.

**Solution.** ► ◄

### 2.1.7 Final Exam 2010

**Exercise 1.** Suppose that $f \in L^1(\mathbb{R}^n)$, and that $x$ is a point in the Lebesgue set of $f$. For $r > 0$, let

$$A(r) = \frac{1}{r^n} \int_{B_r} |f(x-y) - f(x)| \, \mathrm{d}y,$$

where $B_r = B(\mathbf{0}, r)$.

Show that

(a) $A(r)$ is a continuous function of $r$, and $A(r) \to 0$ as $r \to 0$.

(b) There exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

**Solution.** ▸ (a)

(b) ◂

**Exercise 2.** Let $E \subseteq \mathbb{R}^n$ be a measurable set, $1 \leq p < \infty$. assume that $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on $E$ to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0 \iff \|f_k\|_p \longrightarrow \|f\|_p$$

*Hint*: To prove one of the implications, you can use the following fact without proving it:

$$\left|\frac{a-b}{2}\right| \leq \frac{|a|^p + |b|^p}{2}$$

for all $a, b \in \mathbb{R}$.

**Solution.** ▸ ◂

**Exercise 3.** Let $0 < p < q < r \leq \infty$, $E \subseteq \mathbb{R}^n$ be a measurable set. Show that each $f \in L^q(E)$ is the sum of a function $g \in L^p(E)$ and a function $h \in L^r(E)$.

**Solution.** ▸ ◂

**Exercise 4.** Prove that $f : [a, b] \to \mathbb{R}$ is Lipschitz continuous if and only if $f$ is absolutely continuous and there exists a constant $M > 0$ such that $|f'| < M$ a.e. on $[a, b]$.

**Solution.** ▸ ◂

**Exercise 5.** Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.

(b) Does this conclusion continue to be valid when $p = 1$ or $p = \infty$?.

**Solution.** ▸ ◂

### 2.1.8 Final Exam

Never went to get it.

**Exercise 1.**

**Exercise 2.**

**Exercise 3.**

**Exercise 4.**

## 2.2 Danielli: Summer 2011

**Exercise 1.** Let $f \in L^1(\mathbb{R})$, and let $\hat{f}(x) = \int_{\mathbb{R}} f(t)\cos(xt)\,\mathrm{d}t$.

(a) Prove that $\hat{f}(x)$ is continuous for $x \in \mathbb{R}$.

(b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{x\to\infty} \hat{f}(x) = 0.$$

*Hint*: Start by proving the statement for $f = \mathbb{1}_{[a,b]}$.

**Solution**. ▸ For part (a): let $\varepsilon > 0$ be given. Then, since $\cos(xt)$ is continuous there exists $\delta' > 0$ such that $|x - y| < \delta$ implies

$$|\cos(xt) - \cos(yt)| < \frac{\varepsilon}{\|f\|_1}.$$

Now, let $\delta = \delta'$. Then we have

$$\begin{aligned}
|\hat{f}(x) - \hat{f}(y)| &= \left| \int_{\mathbb{R}} f(t)\cos(xt)\,\mathrm{d}t - \int_{\mathbb{R}} f(t)\cos(yt)\,\mathrm{d}t \right| \\
&\leq \int_{\mathbb{R}} |f(t)||\cos(xt) - \cos(yt)|\,\mathrm{d}t \\
&< \frac{\varepsilon}{\|f\|_1} \int_{\mathbb{R}} |f(t)|\,\mathrm{d}t \\
&= \frac{\varepsilon}{\|f\|_1} \|f\|_1 \\
&= \varepsilon.
\end{aligned}$$

Since this can be done for any $x \in \mathbb{R}$, $\hat{f}$ is continuous on $\mathbb{R}$.

For part (b): since simple functions are dense in $L^1(\mathbb{R})$, $f$ there exists a sequence of simple functions $\{s_n\}$, $n \in \mathbb{N}$, such that $\int_{\mathbb{R}} s_n \to \|f\|_1$. Therefore, it suffices to prove the result for characteristic functions. Let $f = \mathbb{1}_{[a,b]}$ and consider the limit

$$\lim_{x\to\infty} \hat{f}(x) = \lim_{x\to\infty} \int_{\mathbb{R}} f(t)\cos(xt)\,\mathrm{d}t.$$

Since $f = \mathbb{1}_{[a,b]}$, we have

$$\begin{aligned}
\lim_{x\to\infty} \int_{\mathbb{R}} f(t)\cos(xt)\,\mathrm{d}t &= \lim_{x\to\infty} \int_a^b \cos(xt)\,\mathrm{d}t \\
&= \lim_{x\to\infty} \left[ \frac{1}{x}(\sin(xa) - \sin(xb)) \right] \\
&= \lim_{x\to\infty} \left[ \frac{\sin(xa)}{x} - \frac{\sin(xb)}{x} \right] \\
&= \left[ \lim_{x\to\infty} \frac{\sin(xa)}{x} \right] - \left[ \lim_{x\to\infty} \frac{\sin(xb)}{x} \right] \\
&= 1 - 1 \\
&= 0,
\end{aligned}$$

as we set out to show. ◂

**Exercise 2.**

(a) Suppose that $f_k, f \in L^2(E)$, with $E$ a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E fg \tag{$\star$}$$

as $k \to \infty$ for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \to \|f\|_2$ show that $f_k$ converges to $f$ in $L^2$, i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as $k \to \infty$.

(b) Provide an example of a sequence $f_k$ in $L^2$ and a function $f$ in $L^2$ satisfying ($\star$), but such that $f_k$ does *not* converge to $f$ in $L^2$.

**Solution**. ▸ For part (a): expand the limit

$$\begin{aligned}\lim_{n\to\infty}\int_E |f - f_n|^2\,\mathrm{d}x &= \lim_{n\to\infty}\left[\int_E \left(|f|^2 - 2|f f_n| + |f|_n^2\right)\mathrm{d}x\right] \\ &= \lim_{n\to\infty}\left[\|f_n\|_2 + \|f\|_2 - 2\int_E f_n f\,\mathrm{d}x\right] \\ &= \lim_{n\to\infty}\|f_n\|_2 + \lim_{n\to\infty}\|f\|_2 - 2\lim_{n\to\infty}\int_E f_n f\,\mathrm{d}x.\end{aligned} \tag{1}$$

Since

$$\int_E f_n g\,\mathrm{d}x \longrightarrow \int_E fg\,\mathrm{d}x$$

for every $g \in L^p(E)$,

$$\int_E f_n f\,\mathrm{d}x \longrightarrow \int_E f^2\,\mathrm{d}x = \|f\|_2{}^2.$$

Moreover, $\|f_n\|_2 \to \|f\|_2$ so the limit in (1) converges to

$$\lim_{n\to\infty}\|f_n\|_2 + \lim_{n\to\infty}\|f\|_2 - 2\lim_{n\to\infty}\int_E f_n f\,\mathrm{d}x = \|f\|_2 + \|f\|_2 - 2\|f\|_2 = 0$$

as $n \to \infty$.

For part (b), consider the sequence $\{f_n\}$, $n \in \mathbb{N}$, where $f_n(x) = \log(n)\exp(-nx)$. Then, we claim that

$f_n \xrightarrow{L^2[0,1]} 0$, but that $f_n \nrightarrow 0$ pointwise. To see the former, first note that

$$\begin{aligned}
\lim_{n\to\infty}\left[\int_0^1 f_n(x)\,\mathrm{d}x\right] &= \lim_{n\to\infty}\left[\int_0^1 \log(n)\exp(-nx)\,\mathrm{d}x\right] \\
&= \lim_{n\to\infty}\left[\log(n)\exp(-nx)\big|_0^1\right] \\
&= \lim_{n\to\infty}\left[\frac{1}{n}\log(n) - \frac{1}{n}\log(n)\exp(-n)\right] \\
&= \lim_{n\to\infty}\left[\left(\frac{1-\exp(-n)}{n}\right)\log(n)\right] \\
&= 0.
\end{aligned}$$

However, $f_n$ does not converge to 0 a.e. since, for $x = 0$ there exist no $N \in \mathbb{N}$ such that

$$|\log(n)| < 1.$$

for all $n \geq N$. ◂

**Exercise 3.** A bounded function $f$ is said to be of bounded variation on $\mathbb{R}$ if it is of bounded variation on any finite subinterval $[a, b]$, and moreover $A := \sup_{a,b} V[a, b; f] < \infty$. Here, $V[a, b; f]$ denotes the total variation of $f$ over the interval $[a, b]$. Show that:

(a) $\displaystyle\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x \leq A|h|$ for all $h \in \mathbb{R}$.

*Hint*: For $h > 0$, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x = \sum_{n=-\infty}^{\infty}\int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,\mathrm{d}x.$$

(b) $\displaystyle\left|\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x\right| \leq A$, where $\varphi$ is any function of class $C^1$, of bounded variation, compactly supported, with $\sup_{x\in\mathbb{R}} |\varphi(x)| \leq 1$.

**Solution**. ▸ For part (a), it suffices to consider only positive $h$ as, making the change of variables $u = x + h$ yields

$$\int_{\mathbb{R}} |f(u) - f(u-h)|\,\mathrm{d}u = \int_{\mathbb{R}} |f(u + (-h)) - f(u)|\,\mathrm{d}u$$

where $-h$ is positive (and letting $h = 0$, we have a trivial inequality). Now, taking the hint, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x = \sum_{n=-\infty}^{\infty}\int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,\mathrm{d}x.$$

Now, since $|f((n+1)h)-f(nh)|$ is a sum in the total variation of $f$ on the interval $[nh,(n+1)h]$, $|f(x+h)-f(x)|$ is bounded by $V[nh,(n+2)h;f]$. Thus, we have

$$\begin{aligned}\int_{\mathbb{R}} |f(x+h)-f(x)|\,\mathrm{d}x &= \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h)-f(x)|\,\mathrm{d}x \\ &\leq \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} V[nh,(n+2)h;f]\,\mathrm{d}x \\ &= \sum_{n=-\infty}^{\infty} V[nh,(n+2)h;f] \int_{nh}^{(n+1)h} \mathrm{d}x \\ &= \sum_{n=-\infty}^{\infty} V[nh,(n+2)h;f]|h| \\ &= 2A|h|.\end{aligned}$$

I suspect there is an error here as the most obvious bound we can get is $2A|h|$ and not the stricter $A|h|$.

For part (b), $f$ is absolutely continuous since it is of bounded variation and $\varphi$ is absolutely continuous since it is Lipschitz ($\varphi$ is differentiable on a compact set, thus, by the mean value theorem $|\varphi(x)-\varphi(y)| \leq \varphi'(\xi)|x-y|$ for some $\xi \in \operatorname{Supp}\varphi$). Assuming $\operatorname{Supp}\varphi$ has nonempty interior, $\operatorname{Supp}\varphi$ contains a closed interval $I=[a,b]$ (in fact, $\operatorname{Supp}\varphi$ is of the form $[a,b] \setminus \bigcup_{n\in A} I_n$, $A \subseteq \mathbb{N}$, where $I_n=(a_n,b_n)$ with $a_n, b_n \in \operatorname{Supp}\varphi$) and thus, by integration by parts, we have

$$\begin{aligned}\int_a^b f\varphi'\,\mathrm{d}x &= f(b)\varphi(b)-f(a)\varphi(a) - \int_a^b f'\varphi\,\mathrm{d}x \\ &\leq f(b)-f(a)-\int_a^b f'\,\mathrm{d}x \\ &= 2(f(b)-f(a)) \\ &\leq 2V[a,b;f]\end{aligned}$$

Thus, summing over every

$$\sum_{n=0}^{\infty} \int_{a_n}^{b_n} f\varphi'\,\mathrm{d}x \leq 2|A|.$$

◂

**Exercise 4.**

(a) Prove the *generalized Hölder's inequality*: Assume $1 \leq p_j \leq \infty$, $j=1,\dots,n$, with $\sum_{j=1}^n 1/p_j = 1/r \leq 1$. If $E$ is a measurable set and $f_j \in L^{p_j}(E)$ for $j=1,\dots,n$, then $\prod_{j=1}^n f_j \in L^r(E)$ and

$$\|f_1\cdots f_n\|_r \leq \|f_1\|_{p_1}\cdots\|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if $1 \leq p,q,r \leq \infty$, with $1/p+1/q=1/r+1$, $f \in L^p(\mathbb{R})$, and $g \in L^q(\mathbb{R})$, then

$$|(f*g)(x)|^r \leq \|f\|_p{}^{r-p}\|g\|_q{}^{r-q}\int |f(y)|^p|g(x-y)|^q\,\mathrm{d}y.$$

(Recall that $(f * g)(x) = \int f(y)g(x-y)\,\mathrm{d}y$.)

(c) Prove *Young's convolution theorem*: Assume that $p, q, r, f$, and $g$ are as in part (b). Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

**Solution**. ▸ For (a) we shall proceed by induction on $n$ the number of measurable functions $f_j \in L^{p_j}(E)$, $1 \leq j \leq n$. The case $n = 2$ holds by using Hölder's inequality on the exponents $r/p + r/q = 1$,

$$\begin{aligned}\left[\int_E |f_1 f_2|^r\right]^{1/r} \mathrm{d}x &= \|{f_1}^r {f_2}^r\|_1 \\ &\leq \|{f_1}^r\|_{p/r}\|{f_2}^r\|_{q/r} \\ &= \|f_1\|_p \|f_2\|_q.\end{aligned}$$

Now, suppose this holds for $n-1$ measurable functions $f_j \in L^{p_j}(E)$, $1 \leq j \leq n-1$. Then for $f_j \in L^{p_j}(E)$ with $\sum_{j=1}^n 1/p_j = 1/r$, we have $r' = \sum_{j=1}^{n-1} 1/p_j = 1/r - 1/p_n$ so by the inductive step

$$\|f_1 \cdots f_{n-1}\|_{r'} \leq \|f_1\|_{p_1} \cdots \|f_{n-1}\|_{p_{n-1}}$$

hence, $f_1 \cdots f_{n-1} \in L^{r'}(E)$. Thus,

$$\begin{aligned}\|f_1 \cdots f_{n-1} f_n\|_r &\leq \|f_1 \cdots f_{n-1}\|_{r'} \|f_n\|_{p_n} \\ &\leq \|f_1\|_{p_1} \cdots \|f_{n-1}\|_{p_{n-1}} \|f_n\|_{p_n},\end{aligned}$$

as we set out to show.

For part (b), applying the generalized Hölder's inequality we proved in part (a),

$$\begin{aligned}|f * g| &= \left|\int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y\right| \\ &\leq \int_{\mathbb{R}} \big|f(y)g(x-y)\big|\,\mathrm{d}y \\ &= \int_{\mathbb{R}} |f(y)|^{1+p/r-p/r}|g(x-y)|^{1+q/r-q/r}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} |f(y)|^{p/r}|g(x)|^{q/r}|f(y)|^{1-p/r}|g(x-y)|^{1-q/r}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} |f(y)|^{p/r}|g(x)|^{q/r}|f(y)|^{(r-p)/r}|g(x-y)|^{(r-q)/r}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \big(|f(y)|^p|g(x)|^q\big)^{1/r}|f(y)|^{(r-p)/r}|g(x-y)|^{(r-q)/r}\,\mathrm{d}y \\ &\leq \big\|\big(|f(y)|^p|g(x)|^q\big)^{1/r}\big\|_r \big\||f(y)|^{(r-p)/r}\big\|_{pr/(r-p)} \big\||g(x-y)|^{(r-q)/r}\big\|_{qr/(r-q)} \\ &= \|f\|_p{}^{(r-p)/r}\|g\|_q{}^{(r-q)/r}\left[\int_{\mathbb{R}} |f(y)|^p|g(x-y)|^q\,\mathrm{d}y\right]^{1/r}.\end{aligned}$$

Raising both sides to the power $r$, we have

$$|(f * g)(x)|^r \leq \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \int |f(y)|^p |g(x-y)|^q \, \mathrm{d}y,$$

as desired.

For part (c), using the estimate we worked out in part (b) together with Tonelli's theorem, we have

$$\begin{aligned}
\|f * g\|_r{}^r &= \int_{\mathbb{R}} |f * g(x)|^r \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}} \left[ \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \int_{\mathbb{R}} |f(y)|^p |g(x-y)|^q \, \mathrm{d}y \right] \\
&= \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \iint\limits_{\mathbb{R}\times\mathbb{R}} |f(y)|^p |g(x-y)|^q \, \mathrm{d}y \\
&= \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \int_{\mathbb{R}} |f(y)|^p \left[ \int_{\mathbb{R}} |g(x-y)|^q \, \mathrm{d}y \right] \mathrm{d}x \\
&\leq \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \|f\|_p{}^p \|g_q\|^q \\
&= \|f\|_p{}^r \|g\|_q{}^r.
\end{aligned}$$

Taking the $r$th root on each side, we achieve the desired estimate

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

◂

## 2.3 Danielli: Winter 2012

**Exercise 1.** Let $f(x, y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each $x$, $f(x, y)$ is an integrable function of $y$, and $\partial f(x, y)/\partial x$ is a bounded function of $(x, y)$. Prove that $\partial f(x, y)/\partial x$ is a measurable function of $y$ for each $x$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 f(x, y)\,\mathrm{d}y = \int_0^1 \frac{\partial f(x, y)}{\partial x}\,\mathrm{d}y.$$

**Solution**. ▸ The end points can be dealt with separately. Fix a point $x_0 \in (0, 1)$ and consider the sequence of measurable functions $\{f_n'\}$ where

$$f_n'(y) = \frac{f(x_0 + h_n, y) - f(x_0, y)}{h_n}$$

where $\{h_n\}$ is a sequence of numbers converging to 0. Since $f$ is differentiable as a function of $x$, the sequence $\{f_n'(x_0, y)\}$ converges to $\partial f/\partial x(x_0, y)$. Now, since $|\partial f/\partial x(x, y)| \le M$ for some $M \in \mathbb{R}^+$ for all $(x, y) \in [0, 1] \times [0, 1]$, by the bounded convergence theorem

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 f_n'(y)\,\mathrm{d}y &= \int_0^1 \lim_{n\to\infty} f_n'(y)\,\mathrm{d}y \\ &= \int_0^1 \frac{\partial f(x_0, y)}{\partial x}\,\mathrm{d}y.\end{aligned} \tag{1}$$

It remains to show that the left side of (1) is the derivative of the integral of $f(x_0, y)$ as a function of $y$. But this is exactly

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 f_n'(y)\,\mathrm{d}y &= \lim_{n\to\infty}\int_0^1 \frac{f(x_0 + h_n, y) - f(x_0, y)}{h_n}\mathrm{d}x \\ &= \lim_{n\to\infty}\frac{\int_0^1 f(x_0 + h_h, y) - f(x_0, y)}{h_n}\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 f(x_0, y)\,\mathrm{d}y.\end{aligned}$$

It follows that for any $x \in [0, 1]$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 f(x, y)\,\mathrm{d}y = \int_0^1 \frac{\partial f(x, y)}{\partial x}\,\mathrm{d}y$$

so $\partial f/\partial x(x, y)$ is a measurable function of $y$. ◂

**Exercise 2.** Let $f$ be a function of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with $g$ absolutely continuous and $h$ singular, show that

$$\int_a^b \varphi\,\mathrm{d}f = \int_a^b \varphi f'\,\mathrm{d}x + \int_a^b \varphi\,\mathrm{d}h.$$

*Hint*: A function $h$ is said to be singular if $h' = 0$.

**Solution.** ▸ Let ◂

**Exercise 3.** Let $E \subseteq \mathbb{R}$ be a measurable set, and let $K$ be a measurable function on $E \times E$. Assume that there exists a positive constant $C$ such that

$$\int_E K(x, y)\,\mathrm{d}x \leq C \tag{$\star$}$$

for a.e. $y \in E$, and

$$\int_E K(x, y)\,\mathrm{d}y \leq C \tag{$\clubsuit$}$$

for a.e. $x \in E$.

Let $1 < p < \infty$, $f \in L^p(E)$, and define

$$T_f(x) = \int_E K(x, y) f(y)\,\mathrm{d}y.$$

(a) Prove that $T_f \in L^p(E)$ and

$$\|T_f\|_p \leq C\|f\|_p. \tag{$\spadesuit$}$$

(b) Is ($\spadesuit$) still valid if $p = 1$ or $\infty$? If so, are assumptions ($\star$) and ($\clubsuit$) needed?

**Solution.** ▸ ◂

**Exercise 4.** Let $f$ be a nonnegative measurable function on $[0, 1]$ satisfying

$$m\{\, x \in [0, 1] : f(x) > \alpha \,\} < \frac{1}{1 + \alpha^2} \tag{$\blacklozenge$}$$

for $\alpha > 0$.

(a) Determine values of $p \in [1, \infty)$ for which $f \in L^p[0, 1]$.

(b) If $p_0$ is the minimum value of $p$ for which $p$ may fail to be in $L^p$, give an example of a function which satisfies ($\blacklozenge$), but which is not in $L^{p_0}[0, 1]$.

**Solution.** ▸ ◂

# 3 Bañuelos

## 3.1 Bañuelos: Summer 2000

**Exercise 1.** Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions with the property that for all $n \geq 1$

$$\mu(\{ x \in X : |f_n(x)| \geq \lambda \}) \leq C \exp(-\lambda^2/n)$$

for all $\lambda > 0$. (Here $C$ is a constant independent of $n$.) Let $n_k = 2^k$. Prove that

$$\limsup_{k\to\infty} \frac{|f_{n_k}|}{\sqrt{n_k \log(\log(n_k))}} \leq 1 \quad \text{a.e.}$$

**Solution.** ▸ Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions such that

$$\mu(\{ x \in X : |f_n(x)| \geq \lambda \}) \leq C \exp(-\lambda^2/n) \tag{1}$$

for all $\lambda$. Now, consider the subsequence $\{f_{2^k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$. We aim to show that

$$\limsup_{k\to\infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} \leq 1$$

almost everywhere. To that end, it suffices to show that the set

$$E = \left\{ x \in X : \limsup_{k\to\infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} > 1 \right\}$$

has measure zero. Let $x \in E$ then

$$\limsup_{k\to\infty} \frac{|f_{2^k}(x)|}{\sqrt{2^k \log(\log(2^k))}} > 1.$$

This means that there exists some subsequence $\{k_m\}_{m=1}^{\infty} \subseteq \{k\}_{n=1}^{\infty}$ such that

$$\lim_{m\to\infty} \frac{|f_{2^{k_m}}(x)|}{\sqrt{2^{k_m} \log(\log(2^{k_m}))}} > 1.$$

This means that, for sufficiently large $N$

$$|f_{2^{k_n}}(x)| > \sqrt{2^{k_n} \log(\log(2^{k_n}))}$$

for all $n \geq N$. But by Equation (1) we have

$$\begin{aligned} \mu\left(\left\{ x \in X : \frac{|f_{2^{k_n}}(x)|}{\sqrt{2^{k_n} \log(\log(2^{k_n}))}} \geq 1 \right\}\right) &\leq C \exp\left(-\left(\sqrt{2^{k_n} \log(\log(2^{k_n}))}\right)^2 \Big/ 2^{k_n}\right) \\ &= C \exp\left(-2^{k_n} \log(\log(2^{k_n}))\Big/ 2^{k_n}\right) \\ &= C \exp\left(-\log(\log(2^{k_n}))\right) \\ &= C \exp\left(\log(1/\log(2^{k_n}))\right) \\ &= \frac{C}{\log(2^{k_n})}. \end{aligned} \tag{2}$$

Letting $n \to \infty$, we see that the measure of the set on the left-hand side of Equation (2) must go to 0 so $\mu(E) = 0$. ◂

**Exercise 2.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n$ be a sequence of measurable functions with $f_1 \in L^1(\mu)$ and with the property that

$$\mu(\{\, x \in X : |f_n(x)| > \lambda \,\}) \leq \mu(\{\, x \in X : |f_1(x)| > \lambda \,\})$$

for all $n$ and all $\lambda > 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_X \left[ \max_{1\leq j\leq n} |f_j| \right] \mathrm{d}\mu = 0.$$

[*Hint*: You may use the fact that $\|f\|_1 = \int_0^\infty \mu(\{\, |f(x)| > \lambda \,\})\,\mathrm{d}\lambda$.]

**Solution**. ▸ Define $g_n, h_n \colon \mathcal{F} \to [0, \infty]$ for $n \in \mathbb{N}$ by

$$g_n(\lambda) = \mu(\{\, x \in X : |f_n(x)| > \lambda \,\}), \qquad h_n(\lambda) = \mu\left(\left\{\, x \in X : \max_{1\leq i\leq n} |f_i(x)| > \lambda \,\right\}\right).$$

Now, note that, by the monotonicity of $\mu$, we have

$$h_n(\lambda) \leq \sum_{i=1}^{n} g_n(\lambda) \leq n g_1(\lambda).$$

Thus,

$$\frac{h_n(\lambda)}{n} \leq g_1(\lambda).$$

Since $\|f_1\|_1 = \int_0^\infty g_1(\lambda)\,\mathrm{d}\lambda$, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \int_X \left[ \max_{1\leq j\leq n} |f_j| \right] \mathrm{d}\mu &= \lim_{n\to\infty} \int_X \frac{h_n(x)}{n}\,\mathrm{d}\mu \\
&= \int_X \lim_{n\to\infty} \frac{h_n(x)}{n}\,\mathrm{d}\mu \\
&\leq \int_X \lim_{n\to\infty} \frac{\mu(X)}{n} \\
&= 0
\end{aligned}$$

as we wanted to show. ◂

**Exercise 3.**

(i) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions. Prove that $f_n \to f$ is measurable if and only if every subsequence $\{f_{n_k}\}$ contains a further subsequence $\{f_{n_{k_j}}\}$ that converges a.e. to $f$.

(ii) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $F \colon \mathbb{R} \to \mathbb{R}$ be continuous and $f_n \to f$ in measure. Prove that $F(f_n) \to F(f)$ in measure. (You may assume, of course, that $f_n$, $F$, $F(f_n)$, and $F(f)$ are all measurable.)

**Solution**. ▸ Recall that a sequence of measurable functions $\{f_n\}$ converge in measure to a limit $f$ if for every $\varepsilon > 0$ the limit

$$\lim_{n\to\infty} \mu(\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\}) = 0.$$

For part (i) $\Longrightarrow$ suppose that $f_n \to f$ in measure. Then given $\varepsilon > 0$ and $\delta > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\mu(\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\}) < \delta.$$

In particular, given $\varepsilon = k^{-1}$ and $\delta = 2^{-k}$, consider the countable collection of measurable sets $\{E_k\}_{k=1}^{\infty}$ given by

$$E_k = \left\{ x \in X : |f(x) - f_{n_k}(x)| \geq \frac{1}{k} \right\},$$

where $n_k \geq N(k)$ (which depends on our choice of $k$) such that

$$\mu(E_k) < \frac{1}{2^k}.$$

Now, by the Borel–Cantelli lemma, since

$$\sum_{k=1}^{\infty} \mu(E_k) < \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty,$$

for almost every $x \in X$, there exists $N_x \in \mathbb{N}$ such that $x \notin E_k$ for $k \geq N_x$. This means that for $k \geq N_x$, we have

$$|f(x) - f_{n_k}(x)| < \frac{1}{k}.$$

Let $\{f_{n_{k+1}}\}$ be the subsequence of $\{f_{n_k}\}$. Then

$$\lim_{k\to\infty} f_{n_{k+1}} = f$$

as desired.

$\Longleftarrow$ On the other hand, suppose that every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ contains a subsequence $\{f_{n_{k_j}}\}$ that converges to $f$. Seeking a contradiction, suppose that given $\varepsilon > 0$ there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$M = \mu\big(\{\, x \in X : |f(x) - f_{n_k}(x)| \geq \varepsilon \,\}\big) > 0.$$

But by assumption there exists a subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ that converges almost everywhere to $f$. We claim that this implies that $f_{n_{k_j}} \to f$ in measure.

*Proof of claim.* This is adapted from a proof in Royden, Proposition 3, Ch. 5.

First note that $f$ is measurable since it is the pointwise limit almost everywhere of a sequence of measurable functions. Let $\varepsilon, \delta > 0$ be given.Here is where the assumption that $\mu(X) < \infty$ is essential! By Egorov's theorem, there is a measurable subset $E \subseteq X$ with $\mu(X \setminus E) < \delta$ such that $f_n \to f$ uniformly on $E$. Thus, there is an index $N$ such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. Thus, for $n \geq N$,

$$\{ x \in X : |f(x) - f_n(x)| \geq \varepsilon \} \subseteq X \setminus E$$

so

$$\mu(\{ x \in X : |f(x) - f_n(x)| \geq \varepsilon \}) < \varepsilon.$$

Thus, we have

$$\lim_{n\to\infty} \mu(\{ x \in X : |f(x) - f_n(x)| \geq \varepsilon \}) = 0,$$

i.e., $f_n \to f$ in measure. ∎

Hence, since $f_{n_{k_j}} \to f$ in measure, but $M > 0$ we have a contradiction.

For (ii) since $F$ is continuous given $\varepsilon > 0$ there exist $\delta > 0$ such that $|x - x'| < \delta$ implies $|F(x) - F(x')| < \varepsilon$. By part (i), $f_n \to f$ in measure if and only if every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ contains a subsequence $\{f_{n_{k_j}}\}$ that converges to $f$ almost everywhere, i.e., given $\delta > 0$ there exists an index $N$ such that $n_{k_j} \geq N$ implies

$$|f(x) - f_{n_{k_j}}(x)| < \delta$$

for almost every $x \in X$. Thus,

$$\left|F(f(x)) - F(f_{n_{k_j}}(x))\right| < \varepsilon$$

and we see that for every subsequence $\{F \circ f_{n_k}\}$ of $\{F \circ f_n\}$ we can find a subsequence $\{F \circ f_{n_{k_j}}\}$ that converges almost everywhere to $F \circ f$. ◂

**Exercise 4.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space and suppose $f \in L^1(\mu)$ is nonnegative. Suppose $1 < p < \infty$ and let $1 < q < \infty$ be its conjugate exponent, i.e., $1/p + 1/q = 1$. Suppose $f$ has the property that

$$\int_E f \, \mathrm{d}\mu \leq \mu(E)^{1/q}$$

for all measurable sets $E$. Prove that $f \in L^r(\mu)$ for any $1 \leq r < p$.

[*Hint*: Consider $\left\{ x \in X : 2^n \leq f(x) < 2^{n+1} \right\}$, if you like.]

**Solution**. ▸ By previous problems, we know that if $\mu(X) < \infty$ and $f \in L^p(X)$, then $f \in L^r(X)$ for $1 \leq r < p$, so it suffices to show that $\|f\|_p < \infty$.

Instead of following the hint, consider the set

$$E_t = \{ x \in X : f(x) \geq t \}$$

and let

$$\omega(t) = \mu(E_t),$$

i.e., the distribution function of $f$. Then, we have

$$\int_0^\infty \omega(t)\,\mathrm{d}t = \int_X f\,\mathrm{d}\mu.$$

In particular, if we make the substitution $\alpha = t^{1/p}$, $\mathrm{d}\alpha = t^{1/q}/p\,\mathrm{d}t = \alpha^{p/q}/p\,\mathrm{d}t$, we have

$$\int_X f^r\,\mathrm{d}\mu = \int_0^\infty p\alpha^{-p/q}\omega(\alpha)\,\mathrm{d}\alpha.$$

Now, by Chebyshev's inequality, we have

$$t\omega(t) \le \int_{E_t} f\,\mathrm{d}\mu \le \omega(t)^{1/q}$$

so

$$\omega(t) \le t^{-p}.$$

Thus,

$$\int_X f^r\,\mathrm{d}\mu = \int_0^\infty p\alpha^{-p/q}\omega(\alpha)\,\mathrm{d}\alpha \le \int_0^\infty p\alpha^{-p-p/q}\,\mathrm{d}\alpha.$$

Since $p + p/q > 1$, the integral above is finite. Thus, $f \in L^p(X)$ and we have $f \in L^r(X)$ for all $1 \le r < p$. ◂

**Exercise 5.** Let $f$ be a continuous function on $[-1, 1]$. Find

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|)\,\mathrm{d}x.$$

**Solution**. ▸ To find the limit of the integral

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|)\,\mathrm{d}x$$

we first make the following substitutions: Let $y = nx$, $\mathrm{d}y = n\,\mathrm{d}x$. Then

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|)\,\mathrm{d}x = \frac{1}{n}\int_{-1}^{1} f(y/n)(1 - |y|)\,\mathrm{d}y.$$

By the extreme value theorem, since $f$ is continuous and $[-1, 1]$ is compact $f$ is bounded on $[-1, 1]$ by, say $M$. Let $g(x) = M$. Then $g \in L^1(X)$ since $\|g\|_1 = 2M$. Thus, by the Lebesgue dominated convergence theorem, since

$$|f(y/n)(1 - |y|)| \le M$$

on $[-1, 1]$ and $g \in L^1([-1, 1])$ it follows that

$$\begin{aligned}
\lim_{n\to\infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|)\,\mathrm{d}x &= \lim_{n\to\infty} \frac{1}{n}\int_{-1}^{1} f(y/n)(1 - |y|)\,\mathrm{d}y \\
&= \int_{-1}^{1} \lim_{n\to\infty}\left[\frac{f(y/n)(1 - |y|)}{n}\right]\mathrm{d}y \\
&= \int_{-1}^{1} \lim_{n\to\infty}\left[\frac{f(y/n)}{n} - \frac{|y|}{n}\right]\mathrm{d}y \\
&= 0.
\end{aligned}$$

◂

**Exercise 6.** Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $f \in L^p(\mu)$, $1 \leq p < \infty$. Suppose $E_n$ is a sequence of measurable sets satisfying $\mu(E_n) = 1/n$ for all $n$. Prove that

$$\lim_{n\to\infty}\left[n^{(p-1)/p}\int_{E_n}|f|\,\mathrm{d}\mu\right] = 0.$$

**Solution**. ▸ The result follows immediately by Hölder's inequality. Let $C = \|f\|_p$. Since $f \in L^p(X)$, then $f \in L^p(E_n)$ for all $n \in \mathbb{N}$. Thus, by Hölder's inequality

$$\begin{aligned}\|f\|_{L^1(E_n)} &\leq \|f\|_{L^p(E_n)}\mu(E)^{1/q}\\ &\leq C\mu(E)^{1/q}\\ &= C\mu(E)^{p/(p-1)}\\ &= Cn^{-p/(p-1)}\\ &= Cn^{p/(1-p)}.\end{aligned}$$

Hence, the integral is bounded above by

$$\begin{aligned}0 \leq n^{(p-1)/p}\int_{E_n}|f|\,\mathrm{d}\mu &\leq Cn^{(p-1)/p+p/(1-p)}\\ &= Cn^{(2p-1)/(p(1-p))}.\end{aligned}$$

Since $p > 1$, $1 - p < 0$ and $2p - 1 > 0$ so the exponent $(2p-1)/(p(1-p)) < 0$. Thus, as $n \to \infty$

$$Cn^{(2p-1)/(p(1-p))} \longrightarrow 0.$$

It follows that

$$\lim_{n\to\infty}\left[n^{(p-1)/p}\int_{E_n}|f|\,\mathrm{d}\mu\right] = 0.$$

◂

**Exercise 7.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{g_n\}$ be a sequence of nonnegative measurable functions with the property that $g_n \in L^1(\mu)$ for every $n$ and $g_n \to g$ in $L^1(\mu)$. Let $\{f_n\}$ be another sequence of nonnegative measurable functions on $(X, \mathcal{F}, \mu)$.

(i) If $f_n \leq g_n$ almost everywhere for every $n$, prove that

$$\limsup_{n\to\infty}\int_X f_n\,\mathrm{d}\mu \leq \int_X \limsup_{n\to\infty} f_n\,\mathrm{d}\mu.$$

[*Hint*: Start by considering a subsequence $\{f_{n_k}\}$ such that

$$\lim_{n_k\to\infty}\int_X f_{n_k}\,\mathrm{d}\mu = \limsup_{n\to\infty}\int_X f_n\,\mathrm{d}\mu$$

and let $\{g_{n_{k_j}}\}$ be a subsequence of $\{g_{n_k}\}$ such that $g_{n_{k_j}} \to g$ almost everywhere.]

(ii) If $f_n \to f$ almost everywhere and if $f_n \le g_n$ almost everywhere for all $n$, then $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

**Solution**. ▸ Part (i) is a generalization of what is colloquially known as the reverse Fatou's lemma. Consider the sequence of measurable functions $\{h_n\}$ where $g_n - f_n$. Note that $g_n - f_n \ge 0$ for all $x \in X$ since $g_n \ge f$ for all $x \in X$. Thus, by Fatou's lemma, we have

$$\begin{aligned}
\liminf_{n\to\infty} \int_X g_n - f_n \,\mathrm{d}\mu &\le \int_X \liminf_{n\to\infty} \int_X g_n - f_n \,\mathrm{d}\mu \\
\liminf_{n\to\infty} \left[\int_X g_n \,\mathrm{d}\mu - \int_X f_n \,\mathrm{d}\mu\right] &\le \int_X \left[\liminf_{n\to\infty} g_n + \liminf_{n\to\infty} -f_n\right] \mathrm{d}\mu \\
\liminf_{n\to\infty} \int_X g_n \,\mathrm{d}\mu + \liminf\left[-\int_X f_n \,\mathrm{d}\mu\right] &\le \int_X \liminf_{n\to\infty} g_n + \int_X \liminf_{n\to\infty}(-f_n) \,\mathrm{d}\mu \\
\liminf_{n\to\infty} \int_X g_n \,\mathrm{d}\mu - \limsup \int_X f_n \,\mathrm{d}\mu &\le \int_X \liminf_{n\to\infty} g_n - \int_X \limsup_{n\to\infty} f_n \,\mathrm{d}\mu \\
\int_X g \,\mathrm{d}\mu - \limsup \int_X f_n \,\mathrm{d}\mu &\le \int_X \liminf_{n\to\infty} g_n - \int_X \limsup_{n\to\infty} f_n \,\mathrm{d}\mu.
\end{aligned}$$

Now, let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ such that

$$\int_X f_{n_k} \,\mathrm{d}\mu \longrightarrow \limsup_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

Since $g_n \to g$ in $L^1(X)$, for every subsequence $\{g_{n_k}\}$ there exists a subsequence $\{g_{n_{k_j}}\}$ that converges to $g$. ◂

**Exercise 8.** Let $f \in L^1(\mathbb{R})$. Consider the function

$$F(x) = \int_{\mathbb{R}} \exp(\mathrm{i}xt) f(t) \,\mathrm{d}t.$$

(i) Show that $F \in L^\infty(\mathbb{R})$ and that $F$ is continuous at every $x \in \mathbb{R}$. Moreover, if $|t|^k f(t) \in L^\infty(\mathbb{R})$ for all $k \ge 1$, show that $F$ is infinitely differentiable, i.e., $F \in C^\infty(\mathbb{R})$.

(ii) Suppose $f$ is continuous as well as in $L^1(\mathbb{R})$. Show that $\lim_{|x|\to\infty} F(x) = 0$.

[*Hint*: Using $\exp(-\mathrm{i}\pi) = -1$, write $F(x) = \left(\int_{\mathbb{R}} (\exp(\mathrm{i}xt) - \exp(\mathrm{i}xt - \mathrm{i}\pi))\right)/2$.]

**Solution**. ▸ For part (a), we claim that

$$|F(x)| \le \|f\|_1.$$

Since $|-|\colon \mathbb{C} \to \mathbb{R}$ is convex, by Jensen's inequality, we have

$$\begin{aligned}
|F(x)| &= \left|\int_X \exp(\mathrm{i}xt) f(t) \,\mathrm{d}t\right| \\
&\le \int_X |\exp(\mathrm{i}xt) f(t)| \,\mathrm{d}t \\
&\le \int_X |f(t)| \,\mathrm{d}t \\
&= \|f\|_1.
\end{aligned}$$

Thus, $\operatorname{ess\,sup}_{x\in\mathbb{R}} F \le \|f_1\|$ so $F \in L^\infty(\mathbb{R})$. To see that $F$ is continuous, take let $\varepsilon > 0$ be given. Consider the sequence of functions $f_n = f\chi_{\{|t|\le n\}}$. Then $f_n \to f$ and, by Lebesgue's dominated convergence theorem, there exists an index $N$ such that for every $n \ge N$ we have

$$\int_{\mathbb{R}} |f|\,\mathrm{d}t - \int_{\mathbb{R}} |f_n|\,\mathrm{d}t = \int_{\{|t|>n\}} |f|\,\mathrm{d}t < \frac{\varepsilon}{4}.$$

Let $\delta < \varepsilon/$ ◂

## 3.2 Bañuelos: Summer 2000

**Exercise 1.** For any two subsets $A$ and $B$ of $\mathbb{R}$ define $A + B = \{ a + b : a \in A, b \in B \}$.

(i) Suppose $A$ is closed and $B$ is compact. Prove that $A + B$ is closed.
(ii) Give an example that shows that (i) may be false if we only assume that $A$ and $B$ are closed.

**Solution**. ▸ For part (i): Let $x \in \overline{A+B}$. Then there is a sequence $\{a_n + b_n\}$ in $A+B$, with $a \in A$, $b_n \in B$, that converges to $x$. Now, consider the sequences $\{a_n\}$, $\{b_n\}$ in $A$ and $B$, respectively. Since $A$ is closed $a_n \to a$ for some $a \in A$. Moreover, since $\mathbb{R}$ is a complete metric space and $B$ is compact, there is a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ that converges to a point $b$ in $B$. Thus, taking the subsequence $\{a_{n_k} + b_{n_k}\}$ of $\{a_n + b_n\}$, $a_{n_k} + b_{n_k} \to a + b$ which is in $A + B$. But $a_n + b_n \to x$ so every subsequence of $\{a_n + b_n\}$ converges to $x$. By the uniqueness of the limit, we must have $x = a + b$.

For part (ii): Consider the subsets $A = \mathbb{Z}$ and $B = \sqrt{2}\mathbb{Z}$ of $\mathbb{R}$. Both $A$ and $B$ are discrete subsets of $\mathbb{R}$, hence closed. However, we claim that $A + B$ is dense in $\mathbb{R}$; in particular, $\overline{A+B} \neq A + B$. To see this, choose a point $x \in \mathbb{R}$; we may as well assume that $x \in [0, 1]$ since $0 \leq x - n < 1$ where $n = \lceil x \rceil$ if $x > 0$ (or if $x < 0$, let $n = \lfloor n \rfloor$), i.e., the largest integer smaller than $x$. Thus, it suffices to show that 0 is a limit point of $A + B$.

Suppose there exists $\varepsilon > 0$ such that $(0, \varepsilon) \cap (A + B) = \emptyset$. Let $\alpha = \inf(A + B) \cap (0, \infty)$. We claim that $\alpha \in A + B$. Seeking a contradiction, suppose that $\alpha \notin (A + B) \cap (0, \infty)$. Then there exists elements $x, y \in (A + B) \cap (0, \infty)$ such that

$$\alpha \leq x \leq y \leq \left(1 + \frac{1}{4}\right)\alpha$$

(by the infimum property). Then $x - y \in A + B$ (since $A + B$ is a ring), such that

$$0 \leq x - y \leq \tfrac{1}{4}\alpha < \alpha.$$

This contradicts that $\alpha$ is the infimum of $A + B$. Thus, $\alpha \in A + B$.

Now, let $x \in (A + B) \cap (0, \infty)$ and let $n = \lfloor x/\alpha \rfloor$ be the largest integer smaller than $x/\alpha$. Then

$$n\alpha \leq x < (n + 1)\alpha$$

so that $0 \leq x - n\alpha < \alpha$. This tells us that $x - n\alpha \in [0, \alpha)$, but since $\alpha$ is the infimum of $(A + B) \cap (0, \infty)$, we must have $x - n\alpha = 0$ so $x = n\alpha$. It follows that every element in $A + B$ is a multiple of $\alpha$.

In particular, since $1, \sqrt{2} \in A + B$, there exists integers $n$ and $m$ such that

$$\sqrt{2} = \frac{\sqrt{2}}{1} = \frac{m\alpha}{n\alpha}.$$

Since $\sqrt{2}$ is irrational, this is a contradiction. Thus, it must be that $(0, \varepsilon) \cap (A + B)$ is nonempty for every $\varepsilon > 0$ and it follows that we can reasonably approximate any $x \in [0, 1]$ and by extension, any $x \in \mathbb{R}$. Thus, $A + B$ is dense in $\mathbb{R}$ but it is not $\mathbb{R}$. ◂

**Exercise 2.** Suppose $f : [0, 1] \to \mathbb{R}$ is differentiable at every $x \in [0, 1]$ where by differentiability at 0 and 1 we mean right and left differentiability, respectively. Prove that $f'$ is continuous if and only if $f$ is uniformly differentiable. That is, if and only if for all $\varepsilon > 0$ there is an $h_0 > 0$ such that

$$\left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

whenever $0 \leq x, x + h \leq 1$, $0 < |h| < h_0$.

**Solution**. ► $\Longleftarrow$ If $f$ is uniformly differentiable then given $\varepsilon > 0$, there exists $h_0$ such that $|x - y| < h_0$ implies

$$\left| f'(x) - \frac{f(x) - f(y)}{x - y} \right| < \frac{\varepsilon}{2}$$
$$\left| f'(y) - \frac{f(y) - f(x)}{x - y} \right| < \frac{\varepsilon}{2}.$$

Then, for $|x - y| < h_0$, we have

$$\begin{aligned} |f'(x) - f'(y)| &= \left| f'(x) - \frac{f(x) - f(y)}{x - y} - \left( f'(y) - \frac{f(y) - f(x)}{x - y} \right) \right| \\ &\leq \left| f'(x) - \frac{f(x) - f(y)}{x - y} \right| + \left| f'(y) - \frac{f(y) - f(x)}{x - y} \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, $f'$ is continuous on $[0, 1]$.

$\Longrightarrow$ Conversely, if $f'$ is continuous on $[0, 1]$ then given $\varepsilon > 0$ there exists $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f'(y) - f'(x)| < \varepsilon.$$

Now, consider the estimate

$$0 < \left| \frac{f(x + h) - f(x)}{h} - f'(x) \right|$$

for $|h| < \delta$. By the mean value theorem, there exists $\xi \in (x - \delta, x + \delta)$ such that

$$\frac{f(x + h) - f(x)}{h} = f'(\xi).$$

Thus,

$$\begin{aligned} \left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| &= |f'(\xi) - f'(x)| \\ &< \varepsilon \end{aligned}$$

since $|\xi - x| < \delta$. Thus, $f$ is uniformly differentiable. ◄

**Exercise 3.** Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$ and let $F_1, \ldots, F_{17}$ be seventeen measurable subsets of $X$ with $\mu(F_j) = 1/4$ for every $j$.

(i) Prove that five of these subsets must have an intersection of positive measure. That is, if $E_1, \ldots, E_k$ denotes the collection of all nonempty intersections of the $F_j$ taken five at a time ($k \leq 6188$), show that at least one of these sets must have positive measure.
(ii) Is the conclusion in (i) true if we take sixteen sets instead of seventeen?

**Solution**. ▸ For part (i), recall that $\mu(F_i) = \int_X \chi_{F_i}\,\mathrm{d}x$ and that $\chi_{F_i \cap F_j} = \chi_{F_i}\chi_{F_j}$. Thus, we need only consider characteristic functions for this problem. Seeking a contradiction, suppose that $F_{i_1} \cap \cdots \cap F_{i_5}$ has measure zero for all $\{i_1, \ldots, i_5\} \subseteq \{1, \ldots, 17\}$. Then, for every $\varepsilon > 0$,

$$\int_X \chi_{F_{i_1}} \ldots \chi_{F_{i_5}}\,\mathrm{d}x < \varepsilon.$$

For part (ii) we provide a counter example. Consider the interval $I = [0,1]$. Partition the interval into subintervals of measure $1/4$ by

$$I_i = \left[\frac{i}{20}, \frac{i+5}{20}\right]$$

for $0 \leq i \leq 15$. Then, at best if we take 5 consecutive intervals

$$I_j \cap I_{j+1} \cap \cdots \cap I_{j+4} = \left[\frac{i}{20}, \frac{i+5}{20}\right] \cap \left[\frac{i+1}{20}, \frac{i+6}{20}\right] \cap \cdots \cap \left[\frac{i+4}{20}, \frac{i+9}{20}\right]$$

◂

**Exercise 4.** Let $f_n \colon X \to [0,\infty)$ be a sequence of measurable functions on the measure space $(X, \mathcal{F}, \mu)$. Suppose there is a positive constant $M$ such that the functions $g_n(x) = f_n(x)\chi_{\{f_n \leq M\}}(x)$ satisfy $\|g_n\|_1 \leq An^{-4/3}$ and for which $\mu\{f_n(x) > M\} \leq Bn^{-5/4}$, where $A$ and $B$ are positive constants independent of $n$. Prove that

$$\sum_{n=1}^{\infty} f_n < \infty$$

almost everywhere.

**Solution**. ▸ It suffices to show that

$$\sum_{n\in\mathbb{N}} \int_X f_n\,\mathrm{d}x < \infty$$

for then we may interchange the order of the sum and the integral and by Lebesgue's dominated convergence theorem

$$\int_X \Big[\sum_{n\in\mathbb{N}} f_n(x)\Big]\mathrm{d}x = \sum_{n\in\mathbb{N}} \Big[\int_X f_n(x)\Big]\mathrm{d}x$$

and $\|\sum_{n\in\mathbb{N}} f_n\|_1 < \infty$ implies that $\sum_{n\in\mathbb{N}} f_n(x) < \infty$ almost everywhere.

First, since there exists an $M$ such that

$$\int\limits_{\{f_n \leq M\}} f(x)\,\mathrm{d}x \leq An^{-4/3}$$

and

$$\mu\{f_n(x) > M\} \leq Bn^{-5/4}.$$

Hence, a good place to estimate the sum of integrals $\sum_{n\in\mathbb{N}} \int_X f_n \, \mathrm{d}x < \infty$ is by splitting the integrals of $f_n$ along this $M$, i.e., to do the following:

$$\begin{aligned}\int_X f_n \, \mathrm{d}x &= \int\limits_{\{f_n \le M\}} f_n(x) \, \mathrm{d}x + \int\limits_{\{f_n > M\}} f_n(x) \, \mathrm{d}x \\ &\le An^{-4/3} + \int\limits_{\{f_n > M\}} f_n(x) \, \mathrm{d}x \\ &= \end{aligned}$$

◂

**Exercise 5.** Let $\{g_n\}$ be a bounded sequence of functions on $[0,1]$ which is uniformly Lipschitz. That isthere is a constant $M$ (independent of $n$) such that for all $n$, $|g_n(x) - g_n(y)| \le M|x - y|$ for all $x, y \in [0,1]$ and $|g_n(x)| \le M$ for all $x \in [0,1]$.

(i) Prove that for any $0 \le a \le b \le 1$,

$$\lim_{n\to\infty} \int_a^b g_n(x) \sin(2n\pi x) \, \mathrm{d}x = 0.$$

(ii) Prove that for any $f \in L^1[0,1]$,

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x) \sin(2n\pi x) \, \mathrm{d}x = 0.$$

**Solution**. ▸ ◂

**Exercise 6.** Let $\{f_n\}$ be a sequence of nonnegative functions in $L^1[0,1]$ with the property that $\int_0^1 f_n(t) \, \mathrm{d}t = 1$ and $\int_{1/n}^1 f_n(t) \, \mathrm{d}t \le 1/n$ for all $n$. Define $h(x) = \sup_n f_n(x)$. Prove that $h \notin L^1[0,1]$.

**Solution**. ▸ ◂

### 3.3 Bañuelos: Winter 2007

**Exercise 1.** Let $f\colon [0,1] \to \mathbb{R}$.

(i) Define what it means for $f$ to be absolutely continuous.
(ii) Define what it means for $f$ to be of bounded variation.
(iii) Let $V(f;0,x)$ be the total variation of $f$ on $[0,x]$. Prove that if $f$ is absolutely continuous on $[0,1]$ so is $V(f;0,x)$.

**Solution.** ▸ ◂

**Exercise 2.**

(i) Suppose that $f\colon [0,1] \to \mathbb{R}$ is nondecreasing with $f(0) = 0$ and $f(1) = 1$. For $a > 0$, let $A$ be set of all $x \in (0,1)$ for which

$$\limsup_{h\to 0} \frac{f(x+h) - f(x)}{h} > a.$$

Prove that $m^*(A) < 1/a$, where $m^*$ denotes the Lebesgue outer measure.

(ii) Prove that there is no Lebesgue measurable set $A$ in $[0,1]$ with the property that $m(A \cap I) = m(I)/4$ for every interval $I$.

[*Hint*: Consider the function $f(x) = \chi_A(x)$.]

**Solution.** ▸ ◂

**Exercise 3.** Let $\{E_j\}_{j=1}^{\infty}$ be Lebesgue measurable sets in $[0,1]$ and let $E = \bigcup_{j=1}^{\infty} E_j$ and suppose there is an $\varepsilon > 0$ such that $\sum_{j=1}^{\infty} m(E_j) \le m(E) + \varepsilon$.

(i) Show that for all measurable sets $A \subseteq [0,1]$

$$\sum_{j=1}^{\infty} m(A \cap E_j) \le m(A \cap E) + \varepsilon.$$

(ii) Let $A$ be the set of all $x \in [0,1]$ which are in at least two of $E_j'$. Prove that $m(A) \le \varepsilon$.

**Solution.** ▸ ◂

**Exercise 4.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n\colon X \to [0,\infty)$ be a sequence measurable functions and suppose that $\|f_n\|_p \le 1$, $1 < p < \infty$, and that $f_n \to f$ almost everywhere. Prove

(i) $f \in L^p(\mu)$.
(ii) $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

**Solution.** ▸

◂

**Exercise 5.**

**Solution.** ► ◄

**Exercise 6.**

**Solution.** ► ◄

## 3.4 Bañuelos: Winter 2013

**Exercise 1.**

(a)

(i) Define almost uniform convergence on the measure space $(X, \mathcal{F}, \mu)$.
(ii) Let $f_n$ be a sequence of nonnegative measurable functions converging almost uniformly to the nonnegative function $f$. Prove that $\sqrt{f_n}$ converges almost uniformly to $\sqrt{f}$.

(b)

(i) Suppose $f_n$ has the property that $\int_X |f_n| \, d\mu \to 0$.
(ii) Does it follow that $f_n \to 0$ almost everywhere? Justify your answer.
(iii) Does it follow that $f_n \to 0$ almost uniformly? Justify your answer.

**Solution.** ► ◄

**Exercise 2.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1 \le p \le \infty$ and $q$ be its conjugate exponent. Suppose $f_n \to f$ in $L^p$ and $g_n \to g$ in $L^q$. Prove that $f_n g_n \to fg$ in $L^1$.

**Solution.** ► ◄

**Exercise 3.** Let $\{a_k\}$ be a sequence of positive numbers converging to infinity. Prove that the following limit exists

$$\lim_{k\to\infty} \int_0^\infty \frac{\exp(-x)\cos x}{a_k x^2 + (1/a_k)} \, dx$$

and find it. Make sure to justify all steps.

**Solution.** ► ◄

**Exercise 4.** Let $(X, \mathcal{F}, \mu)$ be $\sigma$-finite and $f$ be measurable such that for all $\lambda > 0$

$$\mu(\{ x \in X : |f(x)| > \lambda \}) \le \frac{20}{\lambda^p}$$

where $1 < p < \infty$. Let $q$ be the conjugate exponent of $p$. Prove that there is a constant $C$ depending only on $p$ such that

$$\int_E |f(x)| \, d\mu \le C m(E)^{1/q},$$

for all measurable sets $E$ with $0 < \mu(E) < \infty$. (The inequality holds trivially when $\mu(E) = 0$ or $\mu(E) = \infty$.)
[*Hint*: Recall $\int_E |f(x)| \, d\mu = \int_0^\infty ? \, d\lambda$ and "break it" at the right place!]

**Solution.** ► ◄

**Exercise 5.** Suppose $f\colon [0,1] \to \mathbb{R}$ is of bounded variation with $V(f; 0, 1) = \alpha$. For any $\beta > \alpha$, set

$$A = \left\{ x \in (0,1) : \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} > \beta \right\}.$$

Prove that for any $0 < p < 1$, $m(A) \le (\alpha/\beta)^p$, where $m$ denotes the Lebesgue measure.

**Solution.** ► ◄

**Exercise 6.** Let $f \in L^1(0,1)$ and for $x \in (0,1)$, define

$$h(x) = \int_x^1 \frac{f(t)}{t} \, \mathrm{d}t.$$

(i) Prove that $h$ is continuous on $(0,1)$.
(ii) Show that

$$\int_0^1 h(t) \, \mathrm{d}t = \int_0^1 f(t) \, \mathrm{d}t.$$

**Solution.** ► ◄

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