MA52300 FALL 2016

Homework Assignment 9

Due Wed, Nov 30

1. (a) Show that for $n = 3$ the general solution of the wave equation $u_{tt} - \Delta u = 0$ with spherical symmetry about the origin has the form

$$u = \frac{F(r+t) + G(r-t)}{r}, \quad r = |x|$$

with suitable F, G.

(b) Show that the solution with initial data of the form

$$u = 0, \quad u_t = h(r)$$

($h =$ even function of r) is given by

$$u = \frac{1}{2r} \int_{r-t}^{r+t} \rho h(\rho)\, d\rho.$$

2. Show that the solution $w(x_1, t)$ of the initial-value problem for the *Klein-Gordon equation*

$$w_{tt} = w_{x_1 x_1} - \lambda^2 w \tag{1}$$

$$w(x_1, 0) = 0, \quad w_t(x_1, 0) = h(x_1) \tag{2}$$

is given by

$$w(x_1, t) = \frac{1}{2} \int_{x_1 - t}^{x_1 + t} J_0(\lambda s)\, h(y_1)\, dy_1.$$

Here

$$s^2 = t^2 - (x_1 - y_1)^2,$$

while J_0 denotes the Bessel function defined by

$$J_0(z) = \frac{2}{\pi} \int_0^{\pi/2} \cos(z \sin\theta)\, d\theta.$$

Hint: "Descend" to (1) from the two-dimensional wave equation satisfied by

$$u(x_1, x_2, t) = \cos(\lambda x_2) w(x_1, t).$$

3. Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

where g, h, are smooth and have compact support. Show there exists a constant C such that

$$|u(x,t)| \le C/t \quad (x \in \mathbb{R}^3, t > 0).$$