

# MA553: Spring 2016 Homework

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## 1 Course notes

Taken from Hungerford's *Algebra*. This first section will cover the relevant group theory part.

### 1.1 Group Theory

#### Semigroups, Monoids and Groups

If  $G$  is a nonempty subset, a *binary operation* on  $G$  is a function  $G \times G \rightarrow G$ . There are several commonly noted notations for the image of  $(a, b)$  under the binary operation:  $ab$  (multiplicative notation),  $a+b$  (additive notation),  $a \cdot b$ ,  $a * b$ , etc. For convenience we shall generally use multiplicative notation throughout this chapter and refer to  $ab$  as the *product* of  $a$  and  $b$ . A set may have several binary operations defined on it (for example, addition and multiplication on  $\mathbf{Z}$  given by  $(a, b) \mapsto a+b$  or  $(a, b) \mapsto ab$  respectively).

**Definition 1.** A *semigroup* is a nonempty set  $G$  together with a binary operation on  $G$  which is

- (a) associative:  $a(bc) = (ab)c$  for all  $a, b, c \in G$ ;

a *monoid* is a semigroup  $G$  which contains a

- (b) two-sided identity element  $e \in G$  such that  $ae = ea = a$  for all  $a \in G$ .

A *group* is a monoid  $G$  such that

- (c) for every  $a \in G$  there exists a (two-sided) *inverse* element  $a^{-1} \in G$  such that  $aa^{-1} = a^{-1}a = e$ .

A semigroup  $G$  is said to be *Abelian* or *commutative* if its binary operation is

- (d) commutative:  $ab = ba$  for all  $a, b \in G$ .

Our principal interests are groups, however semigroups and monoids are convenient for stating certain theorems in the most generality. Examples are given below. The *order* of a group  $G$  is the cardinality of the set  $G$ .  $G$  is said to be finite if  $|G|$  is finite (otherwise, it is said to be infinite).

**Theorem 1 (1.2).** *If  $G$  is a monoid, then the identity element  $e$  is unique. If  $G$  is a group, then*

- (a)  $a \in G$  and  $aa = a \implies a = e$ ;
- (b) for all  $a, b, c \in G$ ,  $ab = ac \implies b = c$  and  $ba = ca \implies b = c$  (left and right cancellation);
- (c) for each  $a \in G$ , the inverse element  $a^{-1}$  is unique;
- (d) for each  $a \in G$ ,  $(a^{-1})^{-1} = a$ ;
- (e) for  $a, b \in G$ ,  $(ab)^{-1} = b^{-1}a^{-1}$ ;
- (f) for  $a, b \in G$  the equation  $ax = b$  and  $ya = b$  have unique solutions in  $G$ :  $x = a^{-1}b$  and  $y = ba^{-1}$ .

**Proposition 2** (1.3). *Let  $G$  be a semigroup. Then  $G$  is a group if and only if the following conditions hold:*

- (i) *there exists an identity element  $e \in G$  such that  $ea = a$  for all  $a \in G$  (left identity element);*
- (ii) *for each  $a \in G$ , there exists an element  $a^{-1} \in G$  such that  $a^{-1}a = e$  (left inverse).*

*Sketch of the proof.* The direction  $\implies$  is trivial.  $\impliedby$  : By Theorem 1.2(i) is true under the hypotheses.  $G \neq \emptyset$  since  $e \in G$ . If  $a \in G$ , then (ii)  $(aa^{-1})(aa^{-1}) = a(a^{-1}a)a^{-1} = aea^{-1} = aa^{-1}$  and hence  $aa^{-1} = e$  by Theorem 1.2(i). Thus  $a^{-1}$  is a two-sided inverse of  $a$ . Since  $ae = a(a^{-1}a) = (aa^{-1})a = ea = a$  for every  $a \in G$ ,  $e$  is a two-sided identity. Therefore,  $G$  is a group by Definition 1.1. ■

**Proposition 3** (1.4). *Let  $G$  be a semigroup. Then  $G$  is a group if and only if for all  $a, b \in G$  the equations  $ax = b$ ,  $ya = b$  have solutions in  $G$ .*

## 1.2 Ring Theory

### 1.3 Field Theory

## 2 Homework 1

**Problem 2.1.** Let  $G$  be a group,  $a \in G$  an element of finite order  $m$ , and  $n$  a positive integer. Prove that

$$|a^n| = \frac{m}{\gcd(m, n)}.$$

*Proof.* ■

**Problem 2.2.** Let  $G$  be a group, and let  $a, b$  be elements of finite order  $m, n$  respectively. Show that if  $ba = ab$  and  $\langle a \rangle \cap \langle b \rangle = \{e\}$ , then  $|ab| = \text{lcm}(m, n)$ .

*Proof.* ■

**Problem 2.3.** Let  $G$  be a group and  $H, K$  normal subgroups with  $H \cap K = \{e\}$ . Show that

- (a)  $hk = kh$  for every  $h \in H, k \in K$ .
- (b)  $HK$  is a subgroup of  $G$  with  $HK \cong H \times K$ .

*Proof.* ■

**Problem 2.4.** Show that  $A_4$  has no subgroup of order 6 (although  $6 \mid 12 = |A_4|$ ).

*Proof.* ■

### 3 Homework 2

**Problem 3.1.** Let  $G$  be the group of order  $2^3 \cdot 3$ ,  $n \geq 2$ . Show that  $G$  has a normal 2-subgroup  $\neq \{e\}$ .

*Proof.* ■

**Problem 3.2.** Let  $G$  be a group of order  $p^2q$ ,  $p$  and  $q$  primes. Show that the Sylow  $p$ -Sylow subgroup or the  $q$ -Sylow subgroup of  $G$  is normal in  $G$ .

*Proof.* ■

**Problem 3.3.** Let  $G$  be a subgroup of order  $pqr$ ,  $p < q < r$  primes. Show that the  $r$ -Sylow subgroup of  $G$  is normal in  $G$ .

*Proof.* ■

**Problem 3.4.** Let  $G$  be a group of order  $n$  and let  $\varphi: G \rightarrow S_n$  be given by the action of  $G$  on  $G$  via translation.

- (a) For  $a \in G$  determine the number and the lengths of the disjoint cycles of the permutation  $\phi(a)$ .
- (b) Show that  $\varphi(G) \not\subset A_n$  if and only if  $n$  is even and  $G$  has a cyclic 2-Sylow subgroup.
- (c) If  $n = 2m$ ,  $m$  odd, show that  $G$  has a subgroup of index 2.

*Proof.* ■

**Problem 3.5.** Show that the only simple groups  $\neq \{e\}$  of order  $< 60$  are the groups of prime order.

*Proof.* ■

## 4 Homework 3

**Problem 4.1.** Let  $G$  be a finite group,  $p$  a prime number,  $N$  the intersection of all  $p$ -Sylow subgroups of  $G$ . Show that  $N$  is a normal  $p$ -subgroup of  $G$  and that every normal  $p$ -subgroup of  $G$  is contained in  $N$ .

*Proof.* ■

**Problem 4.2.** Let  $G$  be a group of order 231 and let  $H$  be an 11-Sylow subgroup of  $G$ . Show that  $H \subset Z(G)$ .

*Proof.* ■

**Problem 4.3.** Let  $G = \{e, a_1, a_2, a_3\}$  be a non-cyclic group of order 4 and define  $\varphi: S_3 \rightarrow \text{Aut}(G)$  by  $\varphi(\sigma)(e) = e$  and  $\varphi(\sigma)(a_i) = a_{\sigma(i)}$ . Show that  $\varphi$  is well-defined and an isomorphism of groups.

*Proof.* ■

**Problem 4.4.** Determine all groups of order 18.

*Proof.* ■



## 5 Homework 4

**Problem 5.1.** Let  $p$  be a prime and let  $G$  be a nonAbelian group of order  $p^3$ . Show that  $G' = Z(G)$ .

*Proof.* ■

**Problem 5.2.** Let  $p$  be an odd prime and let  $G$  be a nonAbelian group of order  $p^3$  having an element of order  $p^2$ . Show that there exists an element  $b \notin \langle a \rangle$  of order  $p$ .

*Proof.* ■

**Problem 5.3.** Let  $p$  be an odd prime. Determine all groups of order  $p^3$ .

*Proof.* ■

**Problem 5.4.** Show that  $(S_n)' = A_n$ .

*Proof.* ■

**Problem 5.5.** Show that every group of order  $< 60$  is solvable.

*Proof.* ■

**Problem 5.6.** Show that every group of order 60 that is simple (or not solvable) is isomorphic to  $A_5$ .

*Proof.* ■

## 6 Homework 5

**Problem 6.1.** Find all composition series and the composition factors of  $D_6$ .

*Proof.* ■

**Problem 6.2.** Let  $T$  be the subgroup of  $\text{GL}(n, \mathbf{R})$  consisting of all upper triangular invertible matrices. Show that  $T$  is solvable.

*Proof.* ■

**Problem 6.3.** Let  $p \in \mathbf{Z}$  be a prime number. Show:

(a)  $(p-1)! \equiv -1 \pmod{p}$ .

(b) If  $p \equiv 1 \pmod{4}$  then  $x^2 \equiv -1 \pmod{p}$  for some  $x \in \mathbf{Z}$ .

*Proof.* ■

**Problem 6.4.** (a) Show that the following are equivalent for an odd prime number  $p \in \mathbf{Z}$ :

(i)  $p \equiv 1 \pmod{4}$ .

(ii)  $p = a^2 + b^2$  for some  $a, b$  in  $\mathbf{Z}$ .

(iii)  $p$  is not prime in  $\mathbf{Z}[i]$ .

(b) Determine all prime ideals of  $\mathbf{Z}[i]$ .

*Proof.* ■

## 7 Homework 6

**Problem 7.1.** Let  $R$  be a domain. Show that  $R$  is a UFD if and only if every nonzero nonunit in  $R$  is a product of irreducible elements and the intersection of any two principal ideals is again principal.

*Proof.* ■

**Problem 7.2.** Let  $R$  be a PID and  $p$  a prime ideal of  $R[x]$ . Show that  $p$  is principal or  $p = (a, f)$  for some  $a \in R$  and some monic  $f \in R[x]$ .

*Proof.* ■

**Problem 7.3.** Let  $k$  be a field and  $n \geq 1$ . Show that  $z^n + y^3 + x^2 \in k(x, y)[z]$  is irreducible.

*Proof.* ■

**Problem 7.4.** Let  $k$  be a field of characteristic zero and  $n \geq 1, m \geq 2$ . Show that  $x_1^n + \cdots + x_m^n - 1 \in k[x_1, \dots, x_m]$  is irreducible.

*Proof.* ■

**Problem 7.5.** Show that  $x^{3^n} + 2 \in \mathbf{Q}(i)[x]$  is irreducible.

*Proof.* ■

## 8 Homework 7

**Problem 8.1.** Let  $k \subset K$  and  $k \subset L$  be finite field extensions contained in some field. Show that:

- (a)  $[KL : L] \leq [K : k]$ .
- (b)  $[KL : k] \leq [K : k][L : k]$ .
- (c)  $K \cap L = k$  if equality holds in (b).

*Proof.* ■

**Problem 8.2.** Let  $k$  be a field of characteristic  $\neq 2$  and  $a, b$  elements of  $k$  so that  $a, b, ab$  are not squares in  $k$ . Show that  $[k(\sqrt{a}, \sqrt{b}) : k] = 4$ .

*Proof.* ■

**Problem 8.3.** Let  $R$  be a UFD, but not a field, and write  $K = \text{Quot}(R)$ . Show that  $[\bar{K} : k] = \infty$ .

*Proof.* ■

**Problem 8.4.** Let  $k \in K$  be an algebraic field extension. Show that every  $k$ -homomorphism  $\delta : K \rightarrow K$  is an isomorphism.

*Proof.* ■

**Problem 8.5.** Let  $K$  be the splitting field of  $x^6 - 4$  over  $\mathbf{Q}$ . Determine  $K$  and  $[K : \mathbf{Q}]$ .

*Proof.* ■

## 9 Homework 8

**Problem 9.1.** Let  $k$  be a field,  $f \in k[x]$  a polynomial of degree  $n \geq 1$ , and  $K$  the splitting field of  $f$  over  $k$ . Show that  $[K : k] \mid n!$ .

*Proof.* ■

**Problem 9.2.** Let  $k$  be a field and  $n \geq 0$ . Define a map  $\Delta_n : k[x] \rightarrow k[x]$  by  $\Delta_n(\sum a_i x^i) := \sum a_i \binom{i}{n} x^{i-n}$ . Show that

- (a)  $\Delta_n$  is  $k$ -linear, and for  $f, g \in k[x]$ ,  $\Delta_n(fg) = \sum_{j=0}^n \Delta_j(f) \Delta_{n-j}(g)$ .
- (b)  $f^{(n)} = n! \Delta_n(f)$ .
- (c)  $f(x+a) = \sum \Delta_n(f)(a) x^n$ .
- (d)  $a \in k$  is a root of  $f$  of multiplicity  $n$  if and only if  $\Delta_i(f)(a) = 0$  for  $0 \leq i \leq n-1$  and  $\Delta_n(f)(a) \neq 0$ .

*Proof.* ■

**Problem 9.3.** Let  $k \subset K$  be a finite field extension. Show that  $k$  is perfect if and only if  $K$  is perfect.

*Proof.* ■

**Problem 9.4.** Let  $K$  be the splitting field of  $x^p - x - 1$  over  $k = \mathbf{Z}/p\mathbf{Z}$ . Show that  $k \subset K$  is normal, separable, of degree  $p$ .

*Proof.* ■

**Problem 9.5.** Let  $k$  be a field of characteristic  $p > 0$ , and  $k(x, y)$  the field of rational functions in two variables.

- (a) Show that  $[k(x, y) : k(x^p, y^p)] = p^2$ .
- (b) Show that the extension  $k(x^p, y^p) \subset k(x, y)$  is not simple.
- (c) Find infinitely many distinct fields  $L$  with  $k(x^p, y^p) \subset L \subset k(x, y)$ .

*Proof.* ■

## 10 Homework 9

**Problem 10.1.** Let  $k \subset K$  be a finite extension of fields of characteristic  $p > 0$ . Show that if  $p \nmid [K : k]$ , then  $k \subset K$  is separable.

*Proof.* ■

**Problem 10.2.** Let  $k \subset K$  be an algebraic extension of fields of characteristic  $p > 0$ , let  $L$  be an algebraically closed field containing  $K$ , and let  $\delta: k \rightarrow L$  be an embedding. Show that  $k \subset K$  is purely inseparable if and only if there exists exactly one embedding  $\tau: K \rightarrow L$  extending  $\delta$ .

*Proof.* ■

**Problem 10.3.** Let  $k \subset K = k(\alpha, \beta)$  be an algebraic extension of fields of characteristic  $p > 0$ , where  $\alpha$  is separable over  $k$  and  $\beta$  is purely inseparable over  $k$ . Show that  $K = k(\alpha + \beta)$ .

*Proof.* ■

**Problem 10.4.** Let  $f(x) \in \mathbf{F}_q[x]$  be irreducible. Show that  $f(x) \mid x^{q^n} - x$  if and only if  $\deg f(x) \mid n$ .

*Proof.* ■

**Problem 10.5.** Show that  $\text{Aut}_{\mathbf{F}_q}(\overline{\mathbf{F}}_q)$  is an infinite Abelian group which is torsionfree (i.e.,  $\delta^n = \text{id}$  implies  $\delta = \text{id}$  or  $n = 0$ ).

*Proof.* ■

**Problem 10.6.** Show that in a finite field, every element can be written as a sum of two perfect squares.

*Proof.* ■