MA 519: Homework 11

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Problem 11.1 (DasGupta 7.2 (a), (b), (c), (d), (e))

(a) Suppose $E|X_n - c|^\alpha \to 0$, where $0 < \alpha < 1$. Does X_n necessarily converge in probability to c?
(b) Suppose $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0,1)$. Under what condition on a_n can we conclude that $X_n \xrightarrow{\mathcal{P}} \theta$?
(c) $o_p(1) + O_p(1) =$?
(d) $o_p(1)O_p(1) =$?
(e) $o_p(1) + o_p(1)O_p(1) =$?

Solution. For part (a) we show that indeed $E(|X_n - c|^\alpha) \to 0$ implies $X_n \xrightarrow{\mathcal{P}} c$. Let $\varepsilon > 0$ be given. By Markov's inequality, we have

$$P(|X_n - c| > \varepsilon) = P(|X_n - c|^\alpha > \varepsilon^\alpha) \leq \frac{E(|X_n - c|^\alpha)}{\varepsilon^\alpha}.$$

Since $E(|X_n - c|^\alpha) \to 0$ as $n \to \infty$, it follows that $P(|X_n - c| > \varepsilon)$ as $n \to \infty$; i.e., X_n converges to c in probability.

For part (b), suppose $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0,1)$; i.e., $P(|a_n(X_n - \theta)| \leq x) \to \Phi(x)$ as $n \to \infty$. In words $X_n \xrightarrow{\mathcal{P}} \theta$ means that for every $\varepsilon > 0$ and every $\eta > 0$ there exists a positive integer N depending on ε and η such that $n \geq N$ implies

$$P(|X_n - \theta| \geq \varepsilon) < \eta.$$

First, let us find the PDF of the sequence $a_n(X_n - \theta)$. Let f_n denote the PDF of X_n, then the CDF of $a_n(X_n + \theta)$ is

$$\begin{aligned}
P(|a_n(X_n - \theta)| \leq x) &= P(-x \leq a_n(X_n - \theta) \leq x) \\
&= P\left(-\frac{x}{a_n} + \theta \leq X_n \leq \frac{x}{a_n} + \theta\right) \\
&= \int_{-x/a_n+\theta}^{x/a_n+\theta} f(y)\, dy \\
&= f(x/a_n + \theta) - f(-x/a_n + \theta),
\end{aligned}$$

therefore its PDF is

$$\begin{aligned}
\frac{dP}{dx}(|a_n(X_n - \theta)| \leq x) &= \frac{d}{dx}\left[f\left(\frac{x}{a_n} + \theta\right) - f\left(-\frac{x}{a_n} + \theta\right)\right] \\
&= \frac{1}{a_n}\big(f(x/a_n + \theta) + f(-x/a_n + \theta)\big)
\end{aligned}$$

For part (c), suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n = o_p(1)$ and $b_n = O_p(1)$, then for the sequence $\{c_n := a_n + b_n\}$ the most we can expect is $c_n = O_p(1)$. Indeed, we know that if a sequence is $o_p(1)$ then it is also $O_p(1)$ therefore there exists K_1 and K_2 such that $|a_n| \leq K_1$, $|b_n| \leq K_2$ for all $n \geq 1$. Therefore, $|c_n| \leq K_1 + K_2$ for all $n \geq 1$.

For part (d), suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n = o_p(1)$ and $b_n = O_p(1)$, then for the sequence $\{c_n := a_n b_n\}$ the most we can expect is $c_n = O_p(1)$. Again, since $\{a_n\}$ is $o_p(1)$ it is $\mathrm{O}_p(1)$ so there exists a constant $K_1 \geq 0$ such that $|a_n| \leq K_1$ for all $n \geq 1$ and similarly for $\{b_n\}$ there exists a constant K_2 such that $|b_n| \leq K_2$ for all $n \geq 1$. Therefore, $|c_n| \leq K_1 K_2$ for all $n \geq 1$ so $c_n = \mathrm{O}_p(1)$.

For part (e), suppose $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences such that $a_n, b_n = \mathrm{o}_p(1)$ and $c_n = \mathrm{O}_p(1)$, then for the sequence $\{d_n := a_n + b_n c_n\}$ the most we can expect is $d_n = \mathrm{O}_p(1)$ since there exists contstants K_1, K_2, and K_3 such that $|a_n| \leq K_1$, $|b_n| \leq K_2$, and $|c_n| \leq K_3$ for all $n \geq 1$. This implies that $|d_n| \leq K_1 + K_2 K_3$ for all $n \geq 1$. Thus, $d_n = \mathrm{O}_p(1)$. ∎

Problem 11.2 (DasGupta 7.3 [Monte Carlo])

Consider the purely mathematical problem of finding a definite integral $f(x)\,dx$ for some (possibly complicated) function $f(x)$. Show that the SLLN provides a method for approximately finding the value of the integral by using appropriate averages $\frac{1}{n}\sum_{k=1}^{n} f(X_k)$.

Numerical analysts call this Monte Carlo integration.

Solution. Let X_k, for $1 \le k \le n$, be independent and identically distributed $U[a,b]$ random variables and let $f\colon [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Moreover, let us denote the integral of f on $[a,b]$ by

$$I := \int_a^b f\,dx$$

and the average of n random sample points from $[a,b]$ by

$$I_n := \frac{1}{n}\sum_{k=1}^{n} f(X_k).$$

By the strong law of large numbers, we immediately have

$$I_n \longrightarrow E(f(X_1)) = \int_{-\infty}^{\infty} f(x)\chi_{[a,b]}(x)\,dx = \int_a^b f(x)\,dx,$$

as desired. ∎

Problem 11.3 (DasGupta 7.4 (a), (b))

Suppose $X_1, \ldots,$ are i.i.d. and that $E(X_1) = \mu \neq 0$, $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Let $S_{m,p} = \sum_{k=1}^{m} X_k^p$, $m \geq 1$, $p = 1, 2$.

(a) Identify with proof the almost sure limit of $S_{m,1}/S_{n,1}$ for fixed m, and $n \to \infty$.
(b) Identify with proof the almost sure limit of $S_{n-m,1}/S_{n,1}$ for fixed m, and $n \to \infty$.

Solution. For part (a), by the strong law of large numbers the average $\bar{X}_n = S_{n,1}/n \xrightarrow{\text{a.s.}} \mu$ as $n \to \infty$, so $S_{n,1} \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$. Therefore, since $S_{m,1}$ is a fixed, $S_{m,1}/S_{n,1} \xrightarrow{\text{a.s.}} 0$.

For part (b), we have

$$\begin{aligned} \frac{S_{n-m,1}}{S_{n,1}} &= \frac{S_{n,1} - S_{m,1}}{S_{n,1}} \\ &= 1 - \frac{S_{m,1}}{S_{n,1}} \end{aligned}$$

which converges a.s. to 1 since $S_{m,1}/S_{n,1} \xrightarrow{\text{a.s.}} 0$. ■

Problem 11.4 (DasGupta 7.5 (a))

Let A_n, $n \geq 1$, A be events with respect to a common sample space Ω.

(a) Prove that $I_{A_n} \xrightarrow{\mathcal{L}} I_A$ if and only if $P(A_n) \to P(A)$.

Solution. One direction of this is obvious; namely, since I_{A_n} and I_A are indicator random variables $E(I_{A_n}) = P(A_n)$ and $E(I_A) = P(A)$ so $E(I_{A_n}) = P(A_n) \to P(A) = E(I_A)$ implies $I_{A_n} \xrightarrow{\mathcal{L}} I_A$.

On the other hand, if $I_{A_n} \xrightarrow{\mathcal{L}} I_A$, then $P(I_{A_n} \leq x) \to P(I_{A_n} \leq x)$ so letting $x \to \infty$, $P(A_n) = P(I_{A_n} \leq \infty) \to P(I_A \leq \infty) = P(A)$. ∎

Problem 11.5 (DasGupta 7.11 [Sample Maximum])

Let X_k, $k \geq 1$, be an i.i.d. sequence, and $X_{(n)}$ the maximum of $X_1, \ldots, X_n$. Let $\xi(F) = \sup\{\, x : F(x) < 1 \,\}$, where F is the common CDF of the X_k. Prove that $X_{(n)} \xrightarrow{\text{a.s.}} \xi(F)$.

Solution. We point out that this is an immediate extension of Example 7.7 in DasGupta's book. Let $\varepsilon > 0$. Set $\xi = \xi(F)$. Then

$$\begin{aligned}
P(|\forall n \geq m, \xi - X_{(n)}| \leq \varepsilon|) &= P(\forall n \geq m, \xi - X_{(n)} \leq \varepsilon) \\
&= P(\forall n \geq m, X_{(n)} \geq \xi - \varepsilon) \\
&= P(X_{(m)} \geq \xi - \varepsilon) \\
&= 1 - P(X_{(m)} < \xi - \varepsilon) \\
&= 1 - P(X_i < \xi - \varepsilon)^m \\
&\to 1
\end{aligned}$$

with convergence above being as $m \to \infty$.

That is, by definition, $X_{(n)} \to \xi$ almost surely. ∎

Problem 11.6 (DasGupta 7.14 (a))

Suppose X_k are i.i.d. standard Cauchy. Show that

(a) $P(|X_n| > n \text{ infinitely often}) = 1$.

Solution. Recall that the sum of two independent Cauchy variables is again Cauchy. Therefore, $\bar{X}_n = \sum_{k=1}^n X_n$ is Cauchy. Now by the weak law of large numbers, we have

$$P(\limsup |\bar{X}_n| = \infty) = 1.$$

This says that the average for any n, the average $\bar{X}_n$ eventually exceeds n with probability 1. Therefore, $X_n > n$ infinitely often. ∎

Problem 11.7 (DasGupta 7.16 [Coupon Collection])

Cereal boxes contain independently and with equal probability exactly one of n different celebrity pictures. Someone having the entire set of n pictures can cash them in for money. Let W_n be the minimum number of cereal boxes one would need to purchase to own a complete set of the pictures. Find a sequence a_n such that $W_n/a_n \xrightarrow{\mathcal{P}} 1$.
(*Hint:* Approximate the mean of W_n.)

Solution. Let $X_n \sim \text{Geom}(\frac{n-k}{n})$ ■

Problem 11.8 (DasGupta 7.17)

Let $X_n \sim \text{Bin}(n,p)$. Show that $(X_n/n)^2$ and $X_n(X_n-1)/(n(n-1))$ both converge in probability to p^2. Do they converge almost surely?

Solution. First note that $(X_n/n)^2 \sim X_n(X_n-1)/(n(n-1))$ so it suffices to show that $(X_n/n)^2 \to p^2$. We show this explicitly. Let ε be given then we show that

$$P\left(\left|\left(\frac{X_n}{n}\right)^2 - p^2\right| \geq \varepsilon\right) \longrightarrow 0.$$

That is, given $\eta > 0$ there exists N such that $n \geq N$ implies

$$\begin{aligned} P\left(\left|\left(\frac{X_n}{n}\right)^2 - p^2\right| \geq \varepsilon\right) &= P\left(\left(\frac{X_n}{n}\right)^2 - p^2 \geq \varepsilon\right) + P\left(\left(\frac{X_n}{n}\right)^2 - p^2 \leq -\varepsilon\right) \\ &< \eta. \end{aligned}$$

From the calculations above, we have

$$\begin{aligned} P\left(\left(\frac{X_n}{n}\right)^2 - p^2 \geq \varepsilon\right) &= P\Big(X_n \geq n\sqrt{\varepsilon + p^2}\Big) \\ &= 1 - P\Big(X_n < n\sqrt{\varepsilon + p^2}\Big) \\ &\approx \frac{1}{\sqrt{2\pi np(1-p)}} \int_{-\infty}^{n\sqrt{\varepsilon+p^2}} \mathrm{e}^{-(x-np)^2/(2np(1-p))}\, dx \\ &\sim C\mathrm{e}^{-C''n^2} \int_{-\infty}^{C'n} \mathrm{e}^{-C'''x^2}\, dx \end{aligned}$$

since both sequences in n above are convergent and the limit of the product of convergent sequences is the product of the limits, then the limit above equals 0. ∎

Problem 11.9 (DasGupta 7.21)

Let $X_1, X_2, \ldots,$ be i.i.d. $U[0,1]$. Let

$$G_n = (X_1 \cdots X_n)^{1/n}.$$

Find c such that $G_n \xrightarrow{\mathcal{P}} c$.

Solution. Note that

$$\begin{aligned}
\ln(G_n) &= \frac{1}{n}\ln(X_1 X_2 \ldots X_n) \\
&= \frac{1}{n}\sum_{i=1}^{n}\ln(X_i) \\
&\xrightarrow{\mathcal{P}} \int_0^1 \ln(x)dx \\
&= -1
\end{aligned}$$

So that $\ln(G_n) \xrightarrow{\mathcal{P}} -1$; that is, $G_n \xrightarrow{\mathcal{P}} e^{-1}$.

■

Problem 11.10 (DasGupta 7.30 [Conceptual])

Suppose $X_n \xrightarrow{\mathcal{L}} X$, and also $Y_n \xrightarrow{\mathcal{L}} X$. Does this mean that $X_n - Y_n$ converge in distribution to (the point mass at) zero?

Solution. No. Pick $X_n = U(\{-1, 1\})$, and $Y_n = -X_n$. Then $X_n - Y_n = 2$ for all $n \in \mathbb{N}$, but X_n and Y_n are both uniformly distributed on $\{-1, 1\}$, so they both converge (in distribution) to $U(\{-1, 1\})$.

■

Problem 11.11 (DasGupta 7.31 (a))

(a) Suppose $a_n(X_n - \theta) \to N(0, \tau^2)$; what can be said about the limiting distribution of $|X_n|$, when $\theta \neq 0$, $\theta = 0$?

Solution. ■