MA544: Qual Preparation

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Contents

CHAPTER 1

MA 544 Spring 2016

This is material from the course MA 544 as taught in the spring of 2016.

1.1 Homework

These exercises were assigned from Wheeden and Zygmund's *Measure and Integral*. Therefore, most of the theorems I reference will be from [4]. Other resources include [1] and [2]. For more elementary results, I cite [3].

Homework 1

Problem 1 (Wheeden & Zygmund Ch. 2, Ex. 1)**.** Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0, 1]$, but that $V[f; 0, 1] = +\infty$.

Proof. It is clear that the function $f(x) = x\sin(1/x)$ is *bounded* on $[0, 1]$ since $|\sin(1/x)| \leq 1$ and $|x| \leq 1$ on $[0, 1]$. Moreover, by properties of *continuous functions* on $\mathbb{R}$, it is obvious that f is continuous on $(0, 1)$.* What is not obvious is continuity at 0. To show that f is continuous at 0, by Theorem 4.6 from [3, Ch. 4, p. 86], it suffices to show that $\lim_{x\to 0} f(x) = 0$. Consider the sequence $\{1/k\}$. This sequence converges to 0. Moreover, given $\varepsilon > 0$, by the Archimedean principle, for sufficiently large K, the inequality $1/K < \varepsilon$ holds so for every $k \geq K$ we have

$$|(1/k)\sin(k) - 0| \leq |1/k| < \varepsilon. \tag{1}$$

Thus, $\lim_{k\to\infty} f(1/k) = 0$. Thus, f is continuous on all of $[0, 1]$.

*You can, for example, take a look at Theorem 4.9 from [3, Ch. 4, p. 87].

Nevertheless, f is not of *bounded variation* on $[0,1]$. By Corollary 2.10 from [4, Ch. 2, p. 23], the *total variation* V of f on $[0,1]$ is given by

$$\begin{aligned} V &= \int_0^1 |f'|\,dx \\ &= \int_0^1 |\sin(1/x) - (1/x)\cos(1/x)|\,dx \\ &= \int_1^\infty \frac{1}{u^2}|\sin u - u\cos u|\,dx \\ &\geq \int_M^\infty \frac{1}{2u}\,du \\ &= \infty, \end{aligned} \tag{2}$$

where, for sufficiently large M, for $u \geq M$ we have $|\sin u - u\cos u| > u/2$. Thus, f is not of bounded variation. ■

Problem 2 (Wheeden & Zygmund Ch. 2, Ex. 2)**.** Prove theorem (2.1).

Proof. Recall the statement of theorem (2.1):

(a) If f is of bounded variation on $[a,b]$, then f is bounded on $[a,b]$.

(b) Let f and g be of bounded variation on $[a,b]$. Then cf (for any real constant c), $f+g$, and fg are of bounded variation on $[a,b]$. Moreover, f/g is of bounded variation on $[a,b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a,b]$.

(a) Recall that f is of b.v. on $[a,b]$ if the total variation V of f on $[a,b]$ is finite, where V is defined to be the supremum of the sum $\sum_{i=1}^m |f(x_i) - f(x_{i-1})|$ over all *partitions* $\Gamma = \{x_0, \ldots, x_m\}$ of $[a,b]$ of the sum.

(b) ■

Problem 3 (Wheeden & Zygmund Ch. 2, Ex. 3)**.** If $[a',b']$ is a subinterval of $[a,b]$ show that $P[a',b'] \leq P[a,b]$ and $N[a',b'] \leq N[a,b]$.

Proof. ■

Problem 4 (Wheeden & Zygmund Ch. 2, Ex. 11)**.** Show that $\int_a^b f\,d\varphi$ exists if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|, |\Gamma'| < \delta$.

Proof. ■

Problem 5 (Wheeden & Zygmund Ch. 2, Ex. 13)**.** Prove theorem (2.16).

Proof. Recall the statement of Theorem 2.16:

(i) If $\int_a^b f\,d\varphi$ exists, then so do $\int_a^b cf\,d\varphi$ and $\int_a^b f\,d(c\varphi)$ for any constant c, and

$$\int_a^b cf\,d\varphi = \int_a^b f\,d(c\varphi) = c\int_a^b f\,d\varphi.$$

(ii) If $\int_a^b f_1\,d\varphi$ and $\int_a^b f_2\,d\varphi$ both exist, so does $\int_a^b (f_1+f_2)\,d\varphi$, and

$$\int_a^b (f_1+f_2)\,d\varphi = \int_a^b f_1\,d\varphi + \int_a^b f_2\,d\varphi.$$

(iii) If $\int_a^b f\,d\varphi_1$ and $\int_a^b f\,d\varphi_2$ both exist, so does $\int_a^b f\,d(\varphi_1+\varphi_2)$, and

$$\int_a^b f\,d(\varphi_1+\varphi_2) = \int_a^b f\,d\varphi_1 + \int_a^b f\,d\varphi_2.$$

■

1.2 Exam 1 Prep

Problem 1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{ r\mathbf{x} : \mathbf{x} \in E \}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Define a map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by $T\mathbf{x} := r\mathbf{x}$. Note that T is *Lipschitz continuous* since for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the equality

$$|T\mathbf{x} - T\mathbf{y}| = |r\mathbf{x} - r\mathbf{y}| = |r||\mathbf{x} - \mathbf{y}| \tag{1}$$

is satisfied. By Theorem 3.33 from [4, §3.3, p.55], the image of E under T is measurable. Moreover, by Theorem 3.35 [4, §3.3, p. 56], since T is linear, it follows that $|T(E)| = |\det T||E|$ where $\det T = |r|^n$. Lastly, we note that the image of E under T is precisely the set rE so that $|T(E)| = |rE| = |r|^n|E|$, as was to be shown. ■

Problem 2. Let $\{E_k\}$, $k \in \mathbb{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k\to\infty} E_k \right| \leq \liminf_{k\to\infty} |E_k|.$$

Proof. Following the style of [4, §1.1, p. 2], particularly, the sets defined after the introduction of equation (1.1), set

$$V_k := \bigcap_{\ell=k}^{\infty} E_\ell. \tag{2}$$

Note that the collection of sets $\{V_k\}$ forms an increasing sequence, that is, if $\mathbf{x} \in V_k$ then, by (2), $\mathbf{x}$ is in the intersection $E_k \cap \left(\bigcap_{\ell=k+1} E_\ell\right)$, but, by (2), $\bigcap_{\ell=k+1} E_\ell = V_{k+1}$ thus, $\mathbf{x}$ is in V_{k+1} so $V_{k+1} \supset V_k$. Hence, we have $V_k \nearrow \liminf E_k$.

Now, consider the sequence $\{|V_k|\}$ formed by the Lebesgue measure of the V_k. By Theorem 3.26 from [4, §3.3, p. 51], since $V_k \nearrow \liminf E_k$,

$$\lim_{k\to\infty} |V_k| = \lim_{k\to\infty} \left| \bigcap_{\ell=k}^{\infty} E_\ell \right| = \left| \liminf_{k\to\infty} E_k \right|. \tag{3}$$

But note that, by the monotonicity of the Lebesgue measure, we have

$$\left| \bigcap_{\ell=k}^{\infty} E_\ell \right| \leq |E_k|, \tag{4}$$

so, by properties of the $\liminf$, in particular, by Theorem 19(v) from [2, §1.5, p. 23], we have

$$\limsup_{k\to\infty} |V_k| \leq \liminf_{k\to\infty} |E_k|. \tag{5}$$

Hence, by (3) and Proposition 19 (iv), since the sequence $\{|V_k|\}$ converges and is equal to the measure of $\liminf E_k$, by (5), we have

$$\left|\liminf_{k\to\infty} E_k\right| \leq \liminf_{k\to\infty} |E_k| \tag{6}$$

as was to be shown. ■

Problem 3. Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) = \{\, \mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r \,\}$. Prove that F is monotonic increasing and continuous.

Proof. Define the linear map $T\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ by $T(r)\mathbf{x} := r\mathbf{x}$. We claim that $B(\mathbf{0}, r) = T(r, B(\mathbf{0}, 1))$. To reduce notation, set $B_1 := B(\mathbf{0}, 1)$ and $B_r := B(\mathbf{0}, r)$.

Proof of claim. $\subset$: Let $\mathbf{x} \in B_r$. Then $|\mathbf{x}| < r$ so $|\mathbf{x}|/r < 1$. Thus, $|\mathbf{x}|/r \in B_1$ so it is in the image of B_1 under the map $T(r, -)$.

$\supset$: On the other hand, suppose $\mathbf{x} \in T(r, B_1)$. Then $\mathbf{x} = r\mathbf{y}$ for some $\mathbf{y} \in B_1$. Then, since $|\mathbf{y}| < 1$, $|\mathbf{x}| = r|\mathbf{y}| < r$ so $\mathbf{x} \in B_r$. ♣

From the claim, we see that $F(x) = |T(x, B(\mathbf{0}, 1))|$ which, by Problem 1, is nothing more that the polynomial $|B_1|x^n$. It is clear, from this equivalence, that F is monotonically increasing: Take $x, y \in [0, \infty)$ such that $x < y$, then $x^n < y^n$ so

$$F(x) = |B_1|x^n < |B_1|y^n = F(y). \tag{7}$$

Thus, F is monotonically increasing.

In the argument above, since $F(x) = |B_1|x^n$ is a polynomial in $[0, \infty)$ (and polynomials are continuous on $\mathbb{R}$) F is continuous on $[0, \infty)$. ■

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. (Without much motivation) let us consider the collection of sets $\{E_k\}$ defined by

$$E_k := \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } y, z \in B(x, \delta) \text{ implies } |f(y) - f(z)| < \frac{1}{k} \right\}. \tag{8}$$

We claim that $C = \bigcap_{k=1}^{\infty} E_k$ and that each E_k is open.

Proof of claim. First, we demonstrate equality. $\subset$: Suppose $x \in C$. Then, by the definition of continuity, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $y \in B(x, \delta)$ implies $|f(x) - f(y)| < \delta$. In particular, for every k, there exists $\delta > 0$ such that for $y \in B(x, \delta)$ the inequality $|f(x) - f(y)| < 1/k$ holds. Thus, x is in $\bigcap_{k=1}^{\infty} E_k$.

$\supset$: On the other hand, suppose that $x \in \bigcap_{k=1}^{\infty} E_k$. Then, given $\varepsilon > 0$, by the Archimedean property, there exists a positive integer N such that $1/N < \varepsilon$. Then, since $x \in \bigcap_{k=1}^{\infty} E_k$, $x \in E_N$ so

$$|f(x) - f(y)| < \frac{1}{N} < \varepsilon. \tag{9}$$

Thus, x is in C and $C = \bigcap_{k=1}^{\infty} E_k$.

All that remains to be shown is that the E_k are open. But this is clear by the way we defined E_k in (8): Let $x \in E_k$, then there exists $\delta > 0$ such that for any $y, z \in B(x, \delta)$, $|f(y) - f(z)| < 1/k$; Let $x' \in B(x, \delta)$ and set $\delta' := \min\{|(x+\delta) - x'|, |(x-\delta) - x|\}$. Then, since $B(x', \delta') \subset B(x, \delta)$, for every $y, z \in B(x', \delta')$, we have $|f(y) - f(z)| < 1/k$. Hence, $x' \in E_k$ for any $x' \in B(x, \delta)$ so $B(x, \delta) \subset E_k$. ♣

Since C can be expressed as the countable intersection of open sets E_k, it follows that C is a G_δ set. ■

Problem 5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{\, f = r \,\}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Proof. If $\{\, f = r \,\}$ are measurable for all $r \in \mathbb{R}$, it is not necessarily the case that f is measurable. Consider the following construction: Let $E \subset (0,1)$ be an unmeasurable set.† Define a map $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{R} \setminus ((0,1) \setminus E), \\ x+1 & \text{if } x \in (0,1) \setminus E. \end{cases} \tag{10}$$

By the definition, it is clear that $\{\, f = r \,\}$ is measurable and $|\{\, f = r \,\}| = 0$ since $\{\, f = r \,\}$ contains at most two elements. However, the set $\{\, 0 < f < 1 \,\} = E$ is not measurable. Thus, f is not measurable. ■

Problem 6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set $\{\, x : \lim_{k\to\infty} f_k(x) \text{ exists} \,\}$ is measurable.

Proof. In a fashion similar to that of Problem 4, consider the set collection of sets $\{E_k\}$ given by

$$E_k = \left\{ x \in \mathbb{R} : \text{there exists } N \text{ such that } m, n \geq N \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{k} \right\}. \tag{11}$$

You can show that the E_k are open and that $\{\, x : \lim_{x\to\infty} f_k(x) \text{ exists} \,\} = \bigcap_{k=1}^{\infty} E_k$. Then, since open sets are measurable and, by Theorem 3.18 from [4, Ch. 3, p. 48], the countable intersection of measurable sets is measurable, $\{\, x : \lim_{x\to\infty} f_k(x) \text{ exists} \,\}$ is measurable.

There is a slicker proof. By Theorem 4.12 from [4, Ch. 4, p. 67], $\liminf_{k\to\infty} f_k$ and $\limsup_{k\to\infty} f_k$ are measurable. By Theorem 4.7 from [4, Ch. 4, p. 66]

$$\left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\} \tag{12}$$

is measurable. Since

$$\left\{ \lim_{k\to\infty} f_k \text{ exists} \right\} = \left\{ \limsup_{k\to\infty} f_k = \liminf_{k\to\infty} f_k \right\} = \mathbb{R} \setminus \left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\}, \tag{13}$$

by Theorem 3.17 from [4, Ch. 3, p. 48], the set $\{\, \lim_{k\to\infty} f_k \text{ exists} \,\}$ is measurable. ■

†It's construction does not concern us. The interested reader such direct their refer to Theorem 3.38 from [4, Ch. 3, p. 57-58] or Theorem 17 from [2, Ch. 2§7, p. 48].

Problem 7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. ∎

Problem 8. Let f be a continuous function from $[a, b]$ into $\mathbb{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, that is, $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(a) & \text{if } c = b \end{cases}.$$

Proof. ∎

1.3 Exam 1

1.4 Exam 2 Prep

Problem 1. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B_\varepsilon(\mathbf{0})$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x})\, d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f\colon \mathbb{R}^n \to \mathbb{R}$ is (Lebesgue) integrable over a subset E of $\mathbb{R}^n$ (or, alternatively, f belongs to $L(E)$) if

$$\int_E f(\mathbf{x})\, d\mathbf{x} < \infty.$$

Put $E = \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$. Then, to show that f belongs to $L(E)$ it suffices to prove the inequality

$$\int_E f(\mathbf{x})\, d\mathbf{x} < \frac{C}{\varepsilon} \tag{1}$$

for some appropriate constant C. We proceed by directly computing the Lebesgue integral of f and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x})\, d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{({x_1}^2 + \cdots + {x_n}^2)^{(n+1)/2}} \end{aligned}$$

let E_i denote the projection of E onto its i-th coordinate and make the trigonometric substitution $x_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \tan\theta$, $dx_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \sec^2\theta d\theta$ with $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$ giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[\frac{\cos^{n-1}\theta}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1}\theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1}\theta d\theta < \infty. \tag{2}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned} \int \cdots \int_E f(\mathbf{x}) d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{{x_n}^2} \\ &= 2C' \int_{\varepsilon}^{\infty} \frac{dx_n}{{x_n}^2} \\ &= \frac{C}{\varepsilon} \end{aligned} \tag{3}$$

as desired. ■

Problem 2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit f of $\{f_k\}$ must be nonnegative a.e. in $\mathbb{R}^n$. For assume otherwise. Then there exists a collection of points $\mathbf{x}$ in $\mathbb{R}^n$ of nonzero $\mathbb{R}^n$-Lebesgue measure such that $f(\mathbf{x}) < 0$. But $f_k(\mathbf{x}) \geq 0$ for all $k \in \mathbb{N}$. Set $0 < \varepsilon < |f(\mathbf{x})|$ then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{4}$$

for all k which contradicts our assumption that $f_k \to f$ a.e. on $\mathbb{R}^n$. Therefore, the set of points $\mathbf{x} \in \mathbb{R}^n$ where $f(\mathbf{x}) < 0$ must have measure zero.

Now, based on pointwise convergence a.e. to f, given $\varepsilon > 0$ for a.e. $\mathbf{x} \in \mathbb{R}^n$ we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{5}$$

for sufficiently large k; say k greater than or equal to some index $N \in \mathbb{N}$. Moreover, we are given convergence in $L(\mathbb{R}^n)$ of f_k to f

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f < \infty. \tag{6}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{7}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{8}$$

for all $k \in \mathbb{N}$. By Theorem 5.5(ii), f and the f_k's are finite a.e. in $\mathbb{R}^n$ so for some sufficiently large real number M, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in \mathbb{R}^n$. In particular, for any measurable subset E of $\mathbb{R}^n$, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in E$ so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \to \int_E f < \infty. \tag{9}$$

However, if f does not belong to $L(\mathbb{R}^n)$, i.e., its integral over $\mathbb{R}^n$ is infinity, there is no guarantee that f will be finite a.e. in $\mathbb{R}^n$. This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset E of $\mathbb{R}^n$. Let us demonstrate this with an example. Consider the sequence of functions ■

Problem 3. Assume that E is a measurable set of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of $\mathbb{R}^n$, then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \tag{10}$$

Set $E_k = \{\, \mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k \,\}$ and $F_k = \{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{11}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \tag{12}$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \tag{13}$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (13) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \tag{14}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\ &= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\ &= |F_1| + \sum_{k=1}^{\infty}(k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (15) \\ &= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\ &= \sum_{k=1}^{\infty} |F_{k+1}|\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}(k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty}(k+1)|F_k| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \\ &= |F_0| + \sum_{k=1}^{\infty}(k+1)|F_k| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \\ &= |F_0| + \sum_{k=0}^{\infty}(k+2)|F_{k+1}| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \qquad (16) \\ &= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\ &= \sum_{k=0}^{\infty} |F_k|.\end{aligned}$$

Thus, from (15) and (16)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x})d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \qquad (17)$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ∎

Problem 4. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. $\implies$: First note that ρ is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that $\rho(f_k, f) \to 0$ as $k \to \infty$. Then, given $\varepsilon > 0$ there exist an sufficiently large index N such that for every $k \geq N$ we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \tag{18}$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in E which happens if $|f_k - f| = 0$ a.e. in E.

$\Longleftarrow$: Suppose that $f_k \to f$ as $k \to \infty$.

I don't know how to solve this. This is the intended solution:

$\Longrightarrow$: Given $\varepsilon > 0$, $\rho(f_k, f) \to 0$ implies that

$$\int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f} dx \longrightarrow 0.$$

Observe that the function $\Phi \colon \mathbb{R}^+ \to \mathbb{R}$ given by $\Phi(x) = x/(1+x)$ is increasing on $\mathbb{R}^+$ and $0 < \Psi(x) < 1$, hence

$$\begin{aligned}\int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1+\varepsilon} |\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}|.\end{aligned}$$

Therefore,

$$|\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}| \leq \frac{1+\varepsilon}{\varepsilon} \int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f|} \, dx \longrightarrow 0$$

as $k \to \infty$.

$\Longleftarrow$: Conversely, given $\delta > 0$, we have

$$\begin{aligned}\rho(f_k, f) &= \int_{\{ x \in E : |f_k(x) - f(x)| > \delta \}} \frac{|f_k - f|}{1 + |f_k - f|} \, dx \\ &\quad + \int_{\{ x \in E : |f_k(x) - f(x)| \leq \delta \}} \frac{|f_k - f|}{1 + |f_k - f|} \, dx \\ &\leq |\{ x \in E : |f_k(x) - f(x)| > \delta \}| + \frac{\delta}{1+\delta} |E|.\end{aligned}$$

Since $|E| < \infty$ and $\delta/(1+\delta) \searrow 0$, then for any $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\frac{\delta'}{1+\delta'} |E| < \frac{\varepsilon}{2}.$$

If $f_k \to f$ as $k \to \infty$ in measure, then for the above δ' there is an index $N > 0$ such that $k \geq N$ implies

$$|\{ x \in E : |f_k(x) - f(x)| > \delta' \}| < \frac{\varepsilon}{2}.$$

Therefore, $f_k \to f$ in measure implies $\rho(f_k, f) \to 0$ as $k \to \infty$. ∎

Problem 5. Define the *gamma function* $\Gamma\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1}\, du,$$

and the *beta function* $\beta\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. (a) Fix $y \in \mathbb{R}^+$. Then we must show that $\Gamma(y) < \infty$. First, since $(0,1)$ and $[1,\infty)$ are disjoint measurable subsets of $\mathbb{R}$, by Theorem 5.7 we can split the integral $\Gamma(y)$ into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u}u^{y-1}\, du}_{I_1} + \underbrace{\int_1^\infty e^{-u}u^{y-1}\, du}_{I_2}. \tag{19}$$

We will show, separately, that I_1 and I_2 are finite.

To see that I_1 is finite, note that

$$\begin{aligned} e^{-u}u^{y-1} &= e^{-u}e^{(y-1)\log u} \\ &= e^{-u+(y-1)\log u} \\ &\le e^{(y-1)\log u} \\ &= u^{y-1} \end{aligned} \tag{20}$$

since $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u}u^{y-1}\, du \\ &\le \int_0^1 u^{y-1}\, du \\ &= \left[\frac{u^y}{y}\right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \tag{21}$$

To see that I_2 is finite, note that

$$e \tag{22}$$

Intended solution:

(b) ∎

Problem 6. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}} \colon \mathbb{R}^n \to \mathbb{R}$ be $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \to \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{23}$$

■

Problem 7. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \leq g_k$ and

$$\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \to 0 \qquad \text{as} \qquad k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \to \int_{\mathbb{R}^n} |f| \qquad \text{as} \qquad k \to \infty.$$

Proof. (a) Since $f_k \to f$ and $g_k \to g$ a.e. and $|f_k| \leq g_k$, then by Fatou's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} (g - f) = \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k - f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k - f_k, \\ \int_{\mathbb{R}^n} g + f \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k + f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k + f_k. \end{aligned}$$

Since $f_k, g_k, f, g \in L(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$, then using the similar argument as problem 2, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f \geq \limsup_{k \to \infty} \int_{\mathbb{R}^n} f_k, \\ \int_{\mathbb{R}^n} f \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} f_k. \end{aligned}$$

Therefore, $\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f$.

(b) $\Longrightarrow$: This direction is obvious by the inequality

$$\left| \int_{\mathbb{R}^n} |f_k| - |f| \right| \leq \int_{\mathbb{R}^n} ||f_k| - |f|| \leq \int_{\mathbb{R}^n} |f_k - f|.$$

$\Longleftarrow$: Let $g_k = |f_k| + |f|$ and $g = 2|f|$. Since $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e., then $g_k, g \in L(\mathbb{R}^n)$ and $g_k \to g$ a.e. in $\mathbb{R}^n$. By the assumption, $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$.

Let $\tilde{f}_k = |f_k - f|$. Then $\tilde{f}_k \to 0$ a.e. in $\mathbb{R}^n$ and $\tilde{f}_k \leq g_k$. Applying part (a) to $\tilde{f}_k$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

■

1.5 Midterm 2

Problem 1. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Proof. Recall that $f \in L(\mathbb{R}^n)$ if and only if $|f| \in L(\mathbb{R}^n)$. Let $B_k = B(\mathbf{0}, k)$ for $k \in \mathbb{N}$ and χ_{B_k} be the indicator function associated with B_k. Then, the sequence of maps $\{|f_k|\}$ defined $f_k = f\chi_{B_k}$ converge pointwise to $|f_k|$. Since $|f| \in L(\mathbb{R}^n)$, by the monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbb{R}^n} |f|. \tag{1}$$

But this means, exactly, that for every $\varepsilon > 0$ there exists sufficiently large $N \in \mathbb{N}$ such that

$$\begin{aligned} \varepsilon &> \left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \\ &= -\int_{\mathbb{R}^n} |f_k| + \int_{\mathbb{R}^n} |f| \\ &= -\int_{\mathbb{R}^n} |f| + \int_{\mathbb{R}^n} |f| \\ &= -\int_{B_k} |f| + \int_{\mathbb{R}^n} |f| \\ &= \int_{\mathbb{R}^n \setminus B_k} |f| \end{aligned} \tag{2}$$

as desired. ∎

Problem 2. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. First, since the E_j's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \tag{3}$$

Let χ_{E_j} be the characteristic function of the subset E_j of E and define $f_j = f\chi_{E_j}$ for $j \in \mathbb{N}$. Note that, since both f and χ_{E_j} are measurable on E, f_j is measurable on E and $\sum_{j=1}^{\infty} f_j = f$. Moreover, since $E_j \subset E$, by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \setminus E_j} f = \int_E f_j + \int_{E \setminus E_j} f. \tag{4}$$

Hence, because the E_j's are disjoint $(E \setminus E_k) \setminus E_\ell = (E \setminus E_\ell) \setminus E_k$ so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \tag{5}$$

as desired. ■

Problem 3. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbb{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Proof. Let $\varepsilon > 0$ be given. Then, by the hypothesis, there exists $\delta > 0$ such that such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon \tag{6}$$

for all $k \in \mathbb{N}$. By Egorov's theorem, there exists a closed subset F of E such that $|E \setminus F| < \delta$ and $f_k \to f$ uniformly on F. Then, by the uniform convergence theorem,

$$\int_F f_k \longrightarrow \int_F f \tag{7}$$

as $k \to \infty$. But by hypothesis, we have

$$\int_{E\setminus F} |f_k| < \varepsilon. \tag{8}$$

Letting $\varepsilon \to 0$, we achieved the desired convergence. ■

Problem 4. Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)\, dt$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Proof. By Lusin's theorem, there exists a closed subset F of I with $|I \setminus F| < \varepsilon$ such that the restriction of f to $F = I \setminus E$ is continuous. Now, since F is closed in I and I is compact, it follows that I is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials $\{p_k\}$ such that $p_k \to f$ uniformly on F. Then, by the uniform convergence theorem, we have

$$\int_F p_k \longrightarrow \int_F f \tag{9}$$

so

$$\begin{aligned}\int_F \left[\int_x^1 t^{-1} p_k(t) dt\right] dx &= \int_F \left[\int_x^1 a t^{-1} + q_k(t) dt\right] dx \\ &= \int_F q_k'(x) - a\log(x)\, dx \\ &< \infty \end{aligned} \tag{10}$$

for all k and converges uniformly to g so $g \in L(I)$. I don't know how to show that in fact $\int_I g = \int_I f$. Perhaps you show that the places where they differ is a set of measure zero. ■

1.6 Final Practice

Problem 1. Suppose $f \in L^1(\mathbb{R})$ and that x is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) := \frac{1}{r} \int_{B(0,r)} |f(x-y) - f(x)|\, dy.$$

Show that:

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$;
(b) there exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

Proof. ■

Problem 2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \leq n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0$$

if and only if

$$\|f_k\|_p \longrightarrow \|f\|_p$$

as $k \to \infty$.

Proof. ■

Problem 3. Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

Proof. ■

Problem 4. Let $f \in L(\mathbb{R})$, and let $F(t) := \int_{\mathbb{R}} f(x)\cos(tx)dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Proof. ■

Problem 5. Let f be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular. Show that

$$\int_a^b \varphi\, df = \int_a^b \varphi f' dx + \int_a^b \varphi\, dh$$

for all functions φ continuous on $[a, b]$.

Proof. ■

CHAPTER 2

MA 544 Past Quals

2.1 Danielli: Winter 2012

Problem 1. Let $f(x,y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $\partial f(x,y)/\partial x$ is a bounded function of (x,y). Prove that $\partial f(x,y)/\partial x$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial f(x,y)}{\partial x}\,dy.$$

Proof. ∎

Problem 2. Let f be a function of bounded variation on $[a,b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular, show that

$$\int_a^b \varphi\,df = \int_a^b \varphi f'\,dx + \int_a^b \varphi\,dh.$$

Hint: A function h is said to be singular if $h' = 0$.

Proof. ∎

Problem 3. Let $E \subset \mathbb{R}$ be a measurable set, and let K be a measurable function on $E \times E$. Assume that there exists a positive constant C such that

$$\int_E K(x,y)\,dx \le C \tag{1}$$

for a.e. $y \in E$, and

$$\int_E K(x,y)\,dy \le C \tag{2}$$

for a.e. $x \in E$.

Let $1 < p < \infty$, $f \in L^p(E)$, and define

$$T_f(x) := \int_E K(x,y) f(y)\, dy.$$

(a) Prove that $T_f \in L^p(E)$ and

$$\|T_f\|_p \le C\|f\|_p. \tag{3}$$

(b) Is (3) still valid if $p = 1$ or ∞? If so, are assumptions (1) and (2) needed?

Proof. ∎

Problem 4. Let f be a nonnegative measurable function on $[0,1]$ satisfying

$$|\{\, x \in [0,1] : f(x) > \alpha \,\}| < \frac{1}{1+\alpha^2} \tag{4}$$

for $\alpha > 0$.

(a) Determine values of $p \in [1, \infty)$ for which $f \in L^p[0,1]$.

(b) If p_0 is the minimum value of p for which p may fail to be in L^p, give an example of a function which satisfies (4), but which is not in $L^{p_0}[0,1]$.

Proof. ∎

2.2 Danielli: Summer 2011

Problem 1. Let $f \in L^1(\mathbb{R})$, and let $F(t) := \int_{\mathbb{R}} f(x)\cos(tx)\,dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.

(b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Hint: Start by proving the statement for $f = \chi_{[a,b]}$.

Proof. ■

Problem 2. (a) Suppose that $f_k, f \in L^2(E)$, with E a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E fg \tag{1}$$

as $k \to \infty$ for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \to \|f\|_2$ show that f_k converges to f in L^2, i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as $k \to \infty$.

(b) Provide an example of a sequence f_k in L^2 and a function f in L^2 satisfying (1), but such that f_k does *not* converge to f in L^2.

Proof. ■

Problem 3. A bounded function f is said to be of bounded variation on $\mathbb{R}$ if it is of bounded variation on any finite subinterval $[a,b]$, and moreover $A := \sup_{a,b} V[a,b;f] < \infty$. Here, $V[a,b;f]$ denotes the total variation of f over the interval $[a,b]$. Show that:

(a) $\displaystyle\int_{\mathbb{R}} |f(x+h) - f(x)|\,dx \le A|h|$ for all $h \in \mathbb{R}$.

Hint: For $h > 0$, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,dx.$$

(b) $\displaystyle\left|\int_{\mathbb{R}} f(x)\varphi'(x)\,dx\right| \le A$, where φ is any function of class C^1, of bounded variation, compactly supported, with $\sup_{x\in\mathbb{R}} |\varphi(x)| \le 1$.

Proof. ■

Problem 4. (a) Prove the *generalized Hölder's inequality*: Assume $1 \le p \le \infty$, $j = 1, \ldots, n$, with $\sum_{j=1}^{\infty} 1/p_j = 1/r \le 1$. If E is a measurable set and $f_j \in L^{p_j}(E)$ for $j = 1, \ldots, n$, then $\prod_{j=1}^{n} f_j \in L^r(E)$ and

$$\|f_1 \cdots f_n\|_r \le \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if $1 \le p, q, r \le \infty$, with $1/p + 1/q = 1/r + 1$, $f \in L^p(\mathbb{R})$, and $g \in L^p(\mathbb{R})$, then

$$|(f * g)(x)| \le \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \int |f(y)|^p |g(x-y)|^q \, dy.$$

(Recall that $(f * g)(x) := \int f(y) g(x-y) \, dy$.)

(c) Prove *Young's convolution theorem*: Assume that p, q, r, f, and g are as in part (b). Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Proof. ■

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