MA 523: Homework 2

Carlos Salinas

September 9, 2016

Problem 2.1

Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 because in such case we have $\nu(x^0) = (0, \dots, 0, 1)$ so

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall verify the noncharacteristic condition (2.1) by first flattening the boundary near x^0 and then applying the noncharacteristic boundary conditions to the flattened region. Assuming some degree of regularity near x^0, e.g., that the boundary of U be smooth, we may express Γ near x^0 as the graph of a smooth function $f \colon \mathbb{R}^{n-1} \to \mathbb{R}$, i.e., $x = (x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}))$ on Γ and $x_n \geq f(y)$ after reorienting the coordinate axes. Then we flatten out Γ via the map $\Phi(x) \colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\begin{cases} y_1 = x_1 = \Phi^1(x), \\ \quad \vdots \\ y_{n-1} = x_{n-1} = \Phi^{n-1}(x), \\ y_n = x_n - f(x_1, \dots, x_{n-1}) = \Phi^n(x) \end{cases}$$

and write $y = \Phi(x)$. Let $\Psi = \Phi^{-1}$ and rewrite our PDE F in terms of y as follows,

$$0 = F\big(Du(\Psi(y)), u(\Psi(y)), \Psi(y)\big). \tag{2.2}$$

Since $\Delta = \Phi(\Gamma)$ is flat near $y^0 = \Phi(x^0) = (y_1^0, \dots, y_{n-1}^0, 0)$, we may apply the standard noncharacteristic condition on (2.2) and get

$$0 \neq F_{u_{y_n}}\big(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)\big).$$

Before we move on to finding an expression for this derivative, let us consider the gradient $Du(\Psi(y))$. By the chain rule, we have

$$\begin{aligned} u_{y_i}(\Psi(y)) &= \sum_{j=1}^n u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_i} \\ &= u_{x_i}(\Psi(y)) + u_{x_n}(\Psi(y)) f_{y_i}(y_1, \dots, y_{n-1}), \\ u_{y_n}(\Psi(y)) &= \sum_{j=1}^n u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_n} \\ &= u_{x_n}(\Psi(y)), \end{aligned}$$

Then, substituting u_{y_n} for u_{x_n}, we have

$$u_{y_i}(\Psi(y)) = u_{x_i}(\Psi(y)) + u_{y_n}(\Psi(y)) f_{y_i}(y_1, \ldots, y_{n-1}),$$

Now, by the chain rule on (2.2), we have

$$\begin{aligned}
0 &\neq F_{u_{y_n}}\left(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)\right) \\
&= F_{u_{y_n}}\left(u_{x_1} + u_{y_n} f_{y_1}, \ldots, u_{x_{n-1}} + u_{y_n} f_{y_{n-1}}, u_{y_n}, z^0, x^0\right) \\
&= F_{u_{x_1}} f_{y_1} + \cdots + F_{u_{x_{n-1}}} f_{y_{n-1}} + F_{u_{x_n}} \\
&= D_p F(p^0, z^0, x^0) \cdot (Df(x^0), 1) \\
&= D_p F(p^0, z^0, x^0) \cdot \nu(x^0),
\end{aligned}$$

as we set out to show. ◀

Problem 2.2

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. ► Using the method of characteristics, the characteristic ODEs are

$$\begin{aligned} \dot{t} &= 1, \\ \dot{x} &= a(z), \\ \dot{z} &= 1. \end{aligned}$$

◄

Problem 2.3

Show that the function $u(x,t)$ defined by $t \geq 0$ by

$$u(x,t) = \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burger's equation*).

Solution. ► ◄