Math 527 - Homotopy Theory
Spring 2013
Homework 3 Solutions

Problem 1. An **H-space** (named after Hopf) is a pointed space (X, e) equipped with a "multiplication" map $\mu \colon X \times X \to X$ such that the basepoint e is a two-sided unit up to pointed homotopy. In other words, both maps

$$\mu(e, -) \colon X \to X$$
$$\mu(-, e) \colon X \to X$$

and pointed-homotopic to the identity map id_X. Note that μ is not assumed to be associative, not even up to homotopy.

Show that the fundamental group $\pi_1(X, e)$ of an H-space is abelian.

Solution. Let us show a stronger result: The $\pi_1(X, e)$-action on $\pi_n(X, e)$ is trivial for all $n \geq 1$.

Let $[\gamma] \in \pi_1(X, e)$ and $[\theta] \in \pi_n(X, e)$ be represented by pointed maps $\gamma \colon S^1 \to X$ and $\theta \colon S^n \to X$ (by abuse of notation), or equivalently, maps of pairs $\gamma \colon (I, \partial I) \to (X, e)$ and $\theta \colon (D^n, \partial D^n) \to (X, e)$.

Consider the continuous map $H \colon D^n \times I \to X$ defined by

$$H(z, s) = \mu\left(\theta(z), \gamma(s)\right).$$

Then H restricted to the bottom face is

$$H|_{D^n \times \{0\}} = \theta e$$

whereas H restricted to the remaining faces is

$$H|_{\partial D^n \times I \cup D^n \times \{1\}} = (e\gamma) \cdot (\theta e)$$

so that H provides a pointed homotopy between θe and $(e\gamma) \cdot (\theta e)$.

Because right multiplication by e is pointed-homotopic to the identity of X, the composite

$$\begin{tikzcd} S^n \arrow[r, "\theta"] \arrow[rr, bend right, "\theta e"'] & X \arrow[r, "\mu(-,e)"] & X \end{tikzcd}$$

is pointed-homotopic to θ, yielding the equality $[\theta e] = [\theta]$ in $\pi_n(X, e)$. Likewise, the equality $[e\gamma] = [\gamma]$ holds in $\pi_1(X, e)$. We obtain the equality

$$\begin{aligned} [\gamma] \cdot [\theta] &= [e\gamma] \cdot [\theta e] \\ &= [\theta e] \\ &= [\theta] \end{aligned}$$

in $\pi_n(X, e)$. $\square$

Problem 2. Let $f\colon X \to Y$ be a map of spaces, and $x \in X$ any basepoint. Show that the induced map

$$\pi_n f\colon \pi_n(X,x) \to \pi_n(Y, f(x))$$

for $n \geq 1$ is a map of π_1-modules, in the sense that it is $\pi_1 f$-equivariant. More precisely, for any $\gamma \in \pi_1(X,x)$ and $\theta \in \pi_n(X,x)$ the equation

$$(\pi_n f)(\gamma \cdot \theta) = (\pi_1 f)(\gamma) \cdot (\pi_n f)(\theta)$$

holds in $\pi_n(Y, f(x))$.

Solution. The equation to be proved can be written as the commutative diagram

$$\begin{tikzcd}
\pi_1(X,x) \times \pi_n(X,x) \arrow[r, "\bullet"] \arrow[d, "\pi_1 f \times \pi_n f"'] & \pi_n(X,x) \arrow[d, "\pi_n f"] \\
\pi_1(Y,f(x)) \times \pi_n(Y,f(x)) \arrow[r, "\bullet"] & \pi_n(Y,f(x))
\end{tikzcd} \tag{1}$$

Recall that the action map $\pi_1(X,x) \times \pi_n(X,x) \xrightarrow{\bullet} \pi_n(X,x)$ is obtained by applying the functor $[-,X]_*\colon \mathbf{Top}_*^{\mathrm{op}} \to \mathbf{Set}_*$ to the coaction map $c\colon S^n \to S^1 \vee S^n$.

The map $f\colon (X,x) \to (Y,f(x))$ in $\mathbf{Top}_*$ yields the postcomposition natural transformation $f_*\colon [-,X]_* \to [-,Y]_*$. Applying f_* to the coaction map c yields the commutative right-hand square of the diagram

$$\begin{tikzcd}
{[S^1,(X,x)]_* \times [S^n,(X,x)]_*} \arrow[d, "f_* \times f_*"] & {[S^1 \vee S^n,(X,x)]_*} \arrow[l, "\cong"'] \arrow[r, "c^*"] \arrow[d, "f_*"] & {[S^n,(X,x)]_*} \arrow[d, "f_*"] \\
{[S^1,(Y,f(x))]_* \times [S^n,(Y,f(x))]_*} & {[S^1 \vee S^n,(Y,f(x))]_*} \arrow[l, "\cong"'] \arrow[r, "c^*"] & {[S^n,(Y,f(x))]_*}
\end{tikzcd} \tag{2}$$

where the left-hand square also commutes, since the wedge is the coproduct in $\mathbf{Top}_*$ and in $\mathrm{Ho}(\mathbf{Top}_*)$. But the outer diagram in (2) is precisely the diagram (1). $\square$

Problem 3. Let X be the topologist's sine curve:

$$X = \{0\} \times [-1, 1] \cup \{(x, \sin\frac{1}{x}) \mid 0 < x \leq 1\}.$$

Consider the map $f\colon S^0 \to X$ which picks out the points $(0, 1)$ and $(1, \sin 1)$. Show that this map f is a weak homotopy equivalence but not a homotopy equivalence.

Solution. Write $A = \{0\} \times [-1, 1]$ and $B = \{(x, \sin\frac{1}{x}) \mid 0 < x \leq 1\}$ with $X = A \cup B$ where the union is disjoint. Recall that X is connected (being the closure of the connected subset $B \subset \mathbb{R}^2$), but not path-connected. The two path components of X are A and B.

f is a weak homotopy equivalence. Write $a := (0, 1) \in A$ and $b := (1, \sin 1) \in B$, and $S^0 = \{*_a, *_b\}$, with $f(*_a) = a$ and $f(*_b) = b$. The map $f\colon S^0 \to X$ induces a bijection on the sets of path components $\pi_0 f\colon \pi_0(S^0) \xrightarrow{\simeq} \pi_0(X) = \{[a], [b]\}$.

Since S^n is path-connected for all $n \geq 1$, any pointed map $\alpha\colon S^n \to (X, a)$ lands inside the path component $A \subseteq X$. But A is contractible, so that $\pi_n(A, a) = 0$ and $\alpha\colon S^n \to (A, a)$ is pointed-null-homotopic. This proves $\pi_n(X, a) = 0$. Therefore $\pi_n f\colon \pi_n(S^0, *_a) \xrightarrow{\simeq} \pi_n(X, a)$ is an isomorphism (between trivial groups!) for all $n \geq 1$.

Likewise, B is contractible, so that $\pi_n f\colon \pi_n(S^0, *_b) \xrightarrow{\simeq} \pi_n(X, b)$ is also an isomorphism (between trivial groups) for all $n \geq 1$.

f is not a homotopy equivalence. Let $g\colon X \to S^0$ be any continuous map. Since X is connected, the image $g(X)$ is connected and is therefore a singleton $\{*_a\}$ or $\{*_b\}$. Hence g cannot induce a bijection on π_0, and thus f has no homotopy inverse. $\square$