MA557 Homework 9

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Problem 9.1

Let R be a Noetherian ring, $R \subset S$ an extension of rings, and $x \in S$. Show that x is integral over R if and only if for every minimal prime $\mathfrak{q}$ of S, the image of x in $S/\mathfrak{q}$ is integral over $R/\mathfrak{q} \cap R$.

Proof. $\Longrightarrow$ Suppose that x is integral over R. Then x satisfies a monic polynomial of degree n, say $f(X) = X^n + a_1 X^{n-1} + \cdots + a_n$. Let $\mathfrak{q}$ be a minimal prime of S and consider the quotient ring $S/\mathfrak{q}$. If $x \in \mathfrak{q}$ there is nothing to show as $\bar{x} = \bar{0}$ hence satisfies the polynomial X over $R/\mathfrak{q} \cap S$. Suppose $x \notin \mathfrak{q}$. Then

$$\bar{0} = \overline{x^n + a_1 x^{n-1} + \cdots + a_n} = \bar{x}^n + \bar{a}_1 \bar{x}^{n-1} + \cdots + \bar{a}_n$$

so $\bar{x}$ satisfies the polynomial $\bar{f}(X)$. Hence, $\bar{x}$ is integral over $R/\mathfrak{q} \cap S$.

$\Longleftarrow$ Conversely, suppose that for $x \in S$ the image of x in $S/\mathfrak{q}$ is integral over $R/\mathfrak{q} \cap S$. Then we shall show that x is integral over R. For this, it suffices to show that $R[x]$ is a finite R-module.

Since I've not been successful at showing my assertion let us make an extra assumption on S. In particular, we shall assume that S is Noetherian. Since S is Noetherian, S contains finitely many minimal primes $\mathfrak{q}_1, ..., \mathfrak{q}_n$. Let $f_i(X) \in R[X]$ be the minimal polynomial of x in $S/\mathfrak{q}_i$, i.e., $f_i(x)\mathfrak{q}_i$. Then

$$f(x) = f_1(x) \cdots f_n(x) \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = \operatorname{nil} S.$$

Since $\operatorname{nil} S$ is nilpotent, $f(x)^m = 0$ for some positive integer m. Thus, x is integral over R. ∎

Problem 9.2

Let d be a square-free integer and R the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{d})$. Show that

$$R = \begin{cases} \mathbf{Z}[\sqrt{d}] & \text{if } d \ncong 1 \mod 4 \\ \mathbf{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d \cong 1 \mod 4 \end{cases}.$$

Proof. ■

Problem 9.3

Let $R \subset S$ be an integral extension of rings and I and R-ideal. Show that

(a) $\operatorname{ht} IS \leq \operatorname{ht} I$

(b) $\operatorname{ht} IS = \operatorname{ht} I$ if S is a domain and R is normal.

Proof. ■