

MA 544: Homework 9

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Problem 9.1 (Wheeden & Zygmund §6, Ex. 1)

- (a) Let E be a measurable subset of \mathbb{R}^2 such that for almost every $x \in \mathbb{R}^1$, $\{y : (x, y) \in E\}$ has \mathbb{R}^1 -measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $\{x : (x, y) \in E\}$ has measure zero.
- (b) Let $f(x, y)$ be nonnegative and measurable in \mathbb{R}^2 . Suppose that for almost every $x \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every y . Show that for almost every $y \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every x .

Proof. (a) That E has measure zero is a consequence of Fubini's theorem. Set $E_x := \{y : (x, y) \in E\}$ and $E_y := \{x : (x, y) \in E\}$ then, by Theorem 6.8, we have

$$|E| = \iint_{\mathbb{R}^2} \chi_E \, dx \, dy = \int_{\mathbb{R}} \left[\int_{E_x} 1 \, dy \right] dx = \int_{\mathbb{R}} \left[\int_{E_y} 1 \, dx \right] dy = 0. \quad (9.1)$$

Hence, E has measure zero. Moreover, we see that $\int_{\mathbb{R}} \left[\int_{E_y} 1 \, dx \right] dy = 0$ which means that for a.e. $y \in \mathbb{R}$, E_y has \mathbb{R}^1 -measure zero.

(b) Let E be the set of all pairs $(x, y) \in \mathbb{R}^2$ such that $f(x, y)$ is not finite. By hypothesis, the set E_x has \mathbb{R}^1 -measure zero for a.e. x . Therefore, by part (a) the set E_y has measure zero. Hence, for a.e. y , $f(x, y)$ is finite for a.e. x . ■

Problem 9.2 (Wheeden & Zygmund §6, Ex. 3)

Let f be measurable and finite a.e. on $[0, 1]$. If $f(x) - f(y)$ is integrable over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, show that $f \in L[0, 1]$.

Proof. Set $I := [0, 1]$. Suppose that $f(x) - f(y) \in L(I \times I)$. Then by Fubini's theorem we have

$$\iint_{I \times I} f(x) - f(y) \, dx \, dy = \int_I \left[\int_I f(x) - f(y) \, dx \right] dy = \int_I \left[\int_I f(x) - f(y) \, dy \right] dx < \infty. \quad (9.2)$$

Hence, for a.e. $y \in \mathbb{R}$, $f(x) - f(y)$ is integrable so $f(x)$ is integrable. ■

Problem 9.3 (Wheeden & Zygmund §6, Ex. 4)

Let f be measurable and periodic with period 1: $f(t+1) = f(t)$. Suppose there is a finite c such that

$$\int_0^1 |f(a+t) - f(b+t)| dt \leq c$$

for all a and b . Show that $f \in L[0, 1]$. (Set $a = x$, $b = -x$, integrate with respect to x , and make the change of variables $\xi = x + t$, $\eta = -x + t$.)

Proof. Following the hint, write

$$c \geq \int_0^1 \int_0^1 |f(x+t) - f(-x+t)| dx dt$$

making the change of variables $\xi = x + t$, $\eta = -x + t$ and appropriate modification to the bounds of integration, i.e., $0 \leq \xi \leq 2$, $-1 \leq \eta \leq 1$ we have

$$= \int_{-1}^1 \int_0^2 |f(\xi) - f(\eta)| (\det \mathbf{J}(\xi, \eta)) d\xi d\eta$$

by Fubini's theorem

$$= \int_0^2 \int_{-1}^1 |f(\xi) - f(\eta)| (\det \mathbf{J}(\xi, \eta)) d\eta d\xi$$

where $\mathbf{J}(\xi, \eta) = \begin{bmatrix} \partial x / \partial \xi & \partial x / \partial \eta \\ \partial t / \partial \xi & \partial t / \partial \eta \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is the Jacobian of the linear transformation which sends the pair (ξ, η) to $(1/2(\xi - \eta), 1/2(\xi + \eta))$, hence we have

$$\begin{aligned} &= \frac{1}{2} \int_0^2 \int_{-1}^1 |f(\xi) - f(\eta)| d\xi d\eta \\ &= \frac{1}{2} \int_0^2 \int_{-1}^0 |f(\xi) - f(\eta)| d\xi d\eta + \frac{1}{2} \int_0^2 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta \end{aligned}$$

Here we use Theorem 3.35 to note that the translation $\eta \mapsto \eta + 1$ and the fact that f is periodic with period 1 gives us

$$= \int_0^2 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta$$

similarly, we have

$$= 2 \int_0^1 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta.$$

Hence, the inequality

$$\int_0^1 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta \leq \frac{c}{2} \tag{9.3}$$

holds so by Problem 9.2 (§6, Ex. 3), $|f| \in L[0, 1]$ hence, $f \in L[0, 1]$. ■

Problem 9.4 (Wheeden & Zygmund §6, Ex. 6)

For $f \in L(\mathbb{R}^1)$, define the *Fourier transform* \hat{f} of f by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt$$

for $x \in \mathbb{R}^1$. (For complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i \int F_1$.) Show that if f and g belong to $L(\mathbb{R}^1)$, then

$$(\widehat{f * g})(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

Proof. By direct computation we have

$$(\widehat{f * g})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(s-t) g(t) dt \right] e^{-ixs} ds$$

now do this

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s-t) g(t) e^{-ixs} dt ds$$

make the substitution $u = s - t$, then the above becomes

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) g(t) e^{-ix(u+t)} dt du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-ixu} g(t) e^{-ixt} dt du \end{aligned}$$

by Fubini's theorem, this is just

$$\begin{aligned} &= 2\pi \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-ixu} du \right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) e^{-ixt} dt \right) \\ &= 2\pi \hat{f}(x) \hat{g}(x) \end{aligned}$$

as desired. ■

Problem 9.5 (Wheeden & Zygmund §6, Ex. 7)

Let F be a closed subset of \mathbb{R}^1 and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F , prove that the function

$$\int_{\mathbb{R}^1} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}} dt$$

is integrable over F and so is finite a.e. in F . (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

Proof. Set $G := \mathbb{R} \setminus F$. By assumption, we have

$$\int_G f(x) dx < \infty. \quad (9.4)$$

By Tonelli's theorem, since $\delta(y) = 0$ for $y \in F$, we have

$$\begin{aligned} \int_F \left[\int_{\mathbb{R}} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}} dy \right] dx &= \int_F \left[\int_G \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}} dy \right] dx \\ &= \int_G \delta^\lambda(y)f(y) \left[\int_F \frac{dx}{|x-y|^{1+\lambda}} \right] dy. \end{aligned} \quad (9.5)$$

Now, by Marcinkiewicz's theorem, we have

$$\int_F \frac{dx}{|x-y|^{1+\lambda}} \leq 2\lambda^{-1} \delta(y)^{-\lambda}. \quad (9.6)$$

Then, by (9.4), we have

$$\begin{aligned} \int_F \left[\int_{\mathbb{R}} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}} dy \right] dx &\leq \int_G \delta^\lambda(y)f(y) [2\lambda^{-1} \delta(y)^{-\lambda}] dy \\ &= 2\lambda^{-1} \int_G f(y) dy \\ &< \infty \end{aligned} \quad (9.7)$$

as desired. ■

Problem 9.6 (Wheeden & Zygmund §6, Ex. 9)

- (a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.
- (b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subset \mathbb{R}^1$, and let M_α be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int M_\lambda \leq \lambda^{-1}|G|$, where $G = (a, b) - [c, d]$.

Proof. (a) Put $G := (a, b) \setminus F$. Since $\delta(y) = 0$ for $y \in F$, by Tonelli's theorem we have

$$M_\lambda(x) = \int_G \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} dy. \quad (9.8)$$

If $x \notin F$, then since G is open, there exists a sufficiently small $\varepsilon > 0$ such that $B_\varepsilon(x) \subset G$ and $m := \inf_{y \in B_\varepsilon(x)} \delta(y) > 0$. Since $\delta^\lambda(y)/|x - y|^{1+\lambda}$ is nonnegative, we have

$$\begin{aligned} \int_G \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} dy &\geq \int_{B_\varepsilon(x)} \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} dy \\ &\geq m^\lambda \int_{|x-y| < \varepsilon} \frac{1}{|x - y|^{1+\lambda}} dy \\ &= 2m^\lambda \int_0^\varepsilon \frac{1}{u^{1+\lambda}} du \\ &= [2m^\lambda \lambda^{-1} u^{-\lambda}]_0^\varepsilon \\ &= \infty. \end{aligned}$$

(b) ■