MA 523: Homework, Midterms and Practice Problems Solutions

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Contents

1 Homework Solutions

These are my (corrected) solutions to Petrosyan's Math 523 homework for the fall semester of 2016.

1.1 Homework 1

Problem 1.1.1 (Taylor's formula). Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to 0$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.
Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha| = m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. Taking the hint, apply Taylor's formula to the function $g(t) = f(tx)$,

$$g(t) = \sum_{k=0}^{\infty}$$

■

Problem 1.1.2. Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ■

Problem 1.1.3. Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $u u_{x_1} + u_{x_2} = 1$, $u(x_1, x_1) = \frac{x_1}{2}$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ■

Problem 1.1.4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\left(u_{x_1}^2 + u_{x_2}^2\right)$$

find a solution with $u(x_1, 0) = \frac{1-x_1^2}{2}$.

Solution. ∎

1.2 Homework 2

Problem 1.2.1. Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ∎

Problem 1.2.2. Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. ∎

Problem 1.2.3. Show that the function $u(x, t)$ defined for $t \geq 0$ by

$$u(x,t) = \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (\frac{u^2}{2})_x = 0$ (*inviscid Burgers' equation*).

Solution. ∎

1.3 Homework 3

Problem 1.3.1. Consider the initial value problem

$$\begin{cases} u_t = \sin u_x, \\ u(x,0) = \frac{\pi}{4}x. \end{cases}$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the taylor series of the solution about the origin.

Solution. ∎

Problem 1.3.2. Consider the Cauchy problem for $u(x, y)$

$$\begin{cases} u_y = a(x,y,u)u_x + b(x,y,u), \\ u(x,0) = 0, \end{cases}$$

let a and b be analytic functions of their arguments. Assume that $D^\alpha a(0,0,0) \geq 0$ and $D^\alpha b(0,0,0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

(a) Show that $D^\beta u(0,0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $D^\beta u(0,0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$. (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in β_2)

Solution. ∎

Problem 1.3.3. (Kovalevskaya's example) show that the line $\{ t = 0 \}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution u of the heat equation in $\mathbb{R} \times \mathbb{R}$, with $u = \frac{1}{1+x^2}$ on $\{t = 0\}$. (*Hint:* assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of $(0,0)$.)

Solution. ∎

1.4 Homework 4

PROBLEM 1.4.1 (LEGENDRE TRANSFORM). Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u\big(\mathbf{x}(p)\big),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29)).

Solution. ■

PROBLEM 1.4.2. Find the solution $u(x, t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$\begin{cases} u(x,0) = f(x), & u_t(x,0) = g(x), \quad \text{for } x > 0, \\ u_t(0,t) = \alpha u_x(0,t), & \qquad\qquad\qquad\;\; \text{for } t > 0, \end{cases}$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$. (*Hint:* Use a representation $u(x, t) = F(x - t) + G(x + t)$ for the solution.)

Solution. ■

PROBLEM 1.4.3. (a) Let u be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0, t) = u(\pi, t) = 0$. Show that the *energy*

$$E(t) = \frac{1}{2}\int_0^\pi \left(u_t^2 + c^2 u_x^2\right) dx, \quad t > 0$$

is independent of t; i.e., $\frac{d}{dt}E = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x,t) = \sum_{n=1}^{\infty}\big(a_n \cos(nct) + b_n \sin(nct)\big)\sin(nx)$$

in terms of coefficients a_n, b_n.

Solution. ■

1.5 Homework 5

Problem 1.5.1. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$, $x \in \mathbb{R}^n$, then $\Delta v = 0$.

Solution. ∎

Problem 1.5.2. Let $n = 2$ and U be the halfplane $\{ x_2 > 0 \}$. Prove that

$$\sup_U u = \sup_{\partial U} u$$

for $u \in C^2(U) \cap C(\bar{U})$ which are harmonic in U under the additional assumption that u is bounded from above in $\bar{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.)
[*Hint:* Take for $\varepsilon > 0$ the harmonic function

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region $\left\{ x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0 \right\}$ with large a. Let $\varepsilon \to 0$.]

Solution. ∎

Problem 1.5.3. Let $U \subset \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \dots, u^m$ are harmonic in U and

$$v := \varphi(u_1, \dots, u_m).$$

Prove v is subharmonic.

[*Hint:* Convexity for a smooth function $\varphi(z)$ is equivalent to $\sum_{j,k=1}^m \varphi_{z_j,z_k}(z)\xi_j\xi_k \geq 0$ for any $\xi \in \mathbb{R}^m$.]

(b) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. (Assume that harmonic functions are C^∞.)

Solution. ∎

1.6 Homework 6

Problem 1.6.1. For $n = 2$ find Green's function for the quadrant $U := \{ x_1, x_2 > 0 \}$ by repeated reflection.

Solution. Taking the hit, set $x' := (x_1, -x_2)$, $x'' := (-x_1, x_2)$, $x''' := (-x_1, -x_2)$, and define

$$\varphi^x(y) := \Phi(y - x') + \Phi(y - x'') - \Phi(y - x'''). \tag{1}$$

We claim that φ^x, as defined above, solves

$$\begin{cases} \Delta\varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(y - x) & \text{on } \partial U. \end{cases}$$

It is clear that $\Delta\varphi^x = 0$ since it is built up from the fundamental solutions on $\mathbb{R}^n$ (this follows from the linearity of the Laplace operator). To see that $\varphi^x(y) = \Phi(x - y)$ on ∂U, we do a case by case analysis.

Note that on $\{ x_1 = 0 \} \subset \partial U$, we have

$$\varphi^x(y_1, 0) = \Phi(-x_1, y_2 + x_2) + \Phi(-x_1, y_2 - x_2) - \Phi(x_1, y_2 + x_2),$$

where, since the fundamental solution is radial, we have $\Phi(-x_1, y_2 + x_2) = \Phi(x_1, y_2 + x_2)$, and hence the above equals

$$\begin{aligned} &= \Phi(-x_1, y_2 - x_2) \\ &= \Phi(y - x) \end{aligned}$$

and on $\{ x_2 = 0 \} \subset \partial U$, we have

$$\varphi^x(0, y_2) = \Phi(y_1 - x_1, x_2) + \Phi(y_1 + x_1, -x_2) - \Phi(y_1 + x_1, x_2)$$

where, again because Φ is radial, $\Phi(y_1 + x_1, -x_2) = \Phi(y_1 + x_1, x_2)$, thus the above equals

$$\begin{aligned} &= \Phi(y_1 - x_1, x_2) \\ &= \Phi(y - x). \end{aligned}$$

Thus, $\phi^x(y) = \Phi(y - x)$ on ∂U.

Therefore, Green's function on U is

$$G(x, y) = \Phi(y - x) - \varphi^x(y) = \Phi(y - x) - \Phi(y - x') - \Phi(y - x'') + \Phi(y - x'''). \qquad \blacksquare$$

Problem 1.6.2. (Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \le u(x) \le \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B(0, r) = \{ x \in \mathbb{R}^n : |x| < r \}$.

Solution. Recall Poisson's formula for the ball

$$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y), \tag{2}$$

where $x \in B(0,r)$ and u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,r), \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

For fixed $x \in B(0,r)$, write

$$u(x) = r^{n-2}(r+|x|)(r-|x|)\left[\frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y)\right].$$

Now, since $r + |x| \geq |x-y| \geq r - |x|$ for all $y \in \partial B(0,r)$, we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} ⨍_{\partial B(0,r)} g(y)\, dS(y) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} ⨍_{\partial B(0,r)} g(y)\, dS(y). \tag{3}$$

Since $u = g$ on the boundary $\partial B(0,r)$, by applying the mean-value property on (3) we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} u(0),$$

as desired. ■

Problem 1.6.3. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$\begin{cases} P_k(\lambda x) = \lambda^k P_k(x), & P_m(\lambda x) = \lambda^m P_m(x) & \text{for every } x \in \mathbb{R}^n,\ \lambda > 0, \\ \Delta P_k = 0, & \Delta P_m = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

(a) Show that

$$\begin{cases} \dfrac{\partial P_k}{\partial \nu} = k P_k(x), \quad \dfrac{\partial P_m}{\partial \nu} = m P_m(x) \quad \text{on } \partial B(0,1), \end{cases}$$

where $B(0,1) = \{\, x \in \mathbb{R}^n : |x| < 1 \,\}$ and ν is the outward normal on $\partial B(0,1)$.

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x)\, d\sigma = 0, \qquad \text{if } k \neq m.$$

Solution. For part (a), let

$$P_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha.$$

Then, since $\nu = (x_1, \ldots, x_n)$, the derivative along ν is given by

$$\begin{aligned}
\frac{\partial P_k(x)}{\partial \nu} &= \sum_{i=1}^{n} (P_k)_{x_i} x_i \\
&= \sum_{i=1}^{n} \left(\sum\nolimits_{|\alpha|=k} a_\alpha x^\alpha \right)_{x_i} x_i \\
&= \sum_{i=1}^{n} \left(\sum\nolimits_{j=1}^{m} a_\alpha x_1^{\alpha_1^j} \cdots x^{\alpha_i^j} \cdots x^{\alpha_n^j} \right)_{x_i} x_i
\end{aligned}$$

where $\sum_{i=1}^{n} \alpha_i^j = k$ and $1 \leq j \leq \binom{n+k-1}{n} =: m$ (by the stars and bars theorem)

$$\begin{aligned}
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\alpha_i^j a_\alpha x_1^{\alpha_1^j} \cdots x^{\alpha_i^j - 1} \cdots x^{\alpha_n^j} \right) x_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i^j a_\alpha x_1^{\alpha_1^j} \cdots x^{\alpha_i^j} \cdots x^{\alpha_n^j} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i^j a_\alpha x^\alpha
\end{aligned}$$

switching the order of summation, we have

$$\begin{aligned}
&= \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_i^j a_\alpha x^\alpha \\
&= \sum_{j=1}^{m} k a_\alpha x^\alpha \\
&= k \sum_{j=1}^{m} a_\alpha x^\alpha \\
&= k P_k(x).
\end{aligned}$$

This argument, of course, applies to every $k \in \mathbb{N}$.

For part (b), by Green's theorem, we have

$$\begin{aligned}
\int_{B(0,r)} P_k(x) \Delta P_m(x) - (\Delta P_k(x)) P_m(x)\, dx &= \int_{\partial B(0,r)} P_k(x) \tfrac{\partial}{\partial \nu} P_m(x) - \tfrac{\partial}{\partial \nu} P_k(x) P_m(x)\, dS(x) \\
&= \int_{\partial B(0,r)} (m-k) P_k(x) P_m(x)\, dS(x),
\end{aligned}$$

where the left-hand side is equal to zero since both ΔP_k and ΔP_m are zero. Since $m \neq k$, it must be the case that

$$\int_{\partial B(0,r)} P_k(x) P_m(x)\, dS(x) = 0.$$

$\blacksquare$

Problem 1.6.4. Solve the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ u = x_1 & \text{on } |x| = 1, \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

(*Hint:* Use Laurent series.)

Solution. ■

Problem 1.6.5. Let Ω be a bounded domain with a C^1 boundary, $g \in C^2(\partial\Omega)$ and $f \in C(\bar{\Omega})$. Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where ν is the outer normal on $\partial\Omega$. Show that the solution of $(*)$ in $C^2(\Omega) \cap C^1(\bar{\Omega})$ is unique up to a constant; i.e., if u_1 and u_2 are both solutions of $(*)$, then $u_2 = u_1 + \text{const.}$ in Ω.
(*Hint:* Look at the proof of the uniqueness for the Dirichlet problem by Energy methods [E, 2.2.5a].)

Solution. ■

Problem 1.6.6. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{ t = 0 \}, \end{cases}$$

where $c \in \mathbb{R}$.
(*Hint:* Rewrite the problem in terms of $v(x,t) := \mathrm{e}^{ct} u(x,t)$.)

Solution. ■

2 Exams

2.1 Midterm Practice Problems

PROBLEM 2.1.1. Solve $u_{x_1}^2 + x_2 u_{x_2} = u$ with initial conditions $u(x_1, 1) = \frac{x_1^2}{4} + 1$.

Solution. By inspection, we may suspect that $v(x_1, x_2) = \frac{x_1^2}{4} + x_2$ is a solution to the PDE. It certainly satisfies the boundary condition. A routine calculation shows that v is in fact a solution to the PDE. Lucky guess!

More formally, let us solve this problem using the method of characteristics. First, write

$$F(p, z, x) = \left(p^1(s)\right)^2 + x^2(s)p^2(s) - z(s) = 0.$$

Then, the characteristic ODEs are

$$\begin{cases} \left(\dot{p}^1(s), \dot{p}^2(s)\right) = -(0, p^2(s)) + (p^1(s), p^2(s)) \\ \qquad\qquad\qquad\;\; = (p^1(s), 0), \\ \dot{z}(s) = (2p^1(s), x^2(s)) \cdot (p^1(s), p^2(s)) \\ \qquad\;\; = 2p^1(s)^2 + x^2(s)p^2(s), \\ \left(\dot{x}^1(s), \dot{x}^2(s)\right) = (2p^1(s), x^2(s)). \end{cases}$$

Now, for $(x^1(0), x^2(0)) = (x^0, 1)$, integrating the characteristics, we get

$$\begin{cases} \left(p^1(s), p^2(s)\right) = \left(p_0^1 e^s, p_0^2\right), \\ \left(x^1(s), x^2(s)\right) = \left(2p_0^1 e^s + x_0^1, x_0^2 e^s\right), \\ z(s) = \dfrac{\left(x^0\right)^2}{4} e^{2s} + p_0^2 e^s + z^0 \end{cases}$$

Using the initial condition and the PDE, we find that

$$\begin{aligned} p_0^1 &= \frac{x^0}{2}, \quad p_0^2 = \frac{\left(x^0\right)^2}{4} + 1 - \frac{\left(x^0\right)^2}{4} = 1, \\ x_0^1 &= 0, \qquad x_0^2 = 1 \\ z^0 &= 0, \end{aligned}$$

and consequently

$$\begin{cases} \left(x^1(s), x^2(s)\right) = \left(x^0 e^s, e^s\right), \\ z(s) = \dfrac{\left(x^0\right)^2}{4} e^{2s} + e^s \end{cases}$$

so, rewriting z in terms of (x^1, x^2), we have

$$\begin{aligned} z(s) &= \frac{\left(x^0\right)^2}{4} e^{2s} + e^s \\ &= \frac{\left(x^1(s)\right)^2}{4} + x^2(s), \end{aligned}$$

so the solution in terms of (x_1, x_2), is

$$u(x_1, x_2) = \frac{x_1^2}{4} + x_2,$$

just as we suspected. ∎

Problem 2.1.2. Find the maximal $t_0 > 0$ for which the (classical) solution of the Cauchy problem

$$\begin{cases} uu_x + u_t = 0, \\ u(x,0) = \mathrm{e}^{-\frac{x^2}{2}}, \end{cases}$$

exists in $\mathbb{R} \times [0, t)$; i.e., the first time $t = t_0$ when the shock develops.

Solution. First, let us find a solution to the PDE using the method of characteristics. Write

$$F(p, z, x) = z(s)p^1(s) + p^2(s).$$

Then, the characteristic ODEs are

$$\begin{cases} \big(\dot{p}^1(s), \dot{p}^2(s)\big) = -(0,0) - p^1(p^1(s), p^2(s)) \\ \qquad\qquad\qquad = \big(-p^1(s)^2, -p^1(s)p^2(s)\big), \\ \dot{z}(s) = (z(s), 1) \cdot (p^1(s), p^2(s)) \\ \qquad\quad = z(s)p^1(s) + p^2(s) \\ \qquad\quad = 0, \\ \big(\dot{x}(s), \dot{t}(s)\big) = (z(s), 1). \end{cases}$$

Thus, integrating the characteristic ODEs from $(x^0, 0)$, we have

$$\begin{cases} \dot{z}(s) = z^0, \\ \big(x(s), t(s)\big) = (z^0 s + x^0, s); \end{cases}$$

since the PDE is quasilinear, we disregard (p^1, p^2).
Applying the boundary conditions, we see that

$$z^0 = u(x^0, 0) = \mathrm{e}^{-\frac{(x^0)^2}{2}}.$$

Here's where it gets tricky. After a little struggling, we see that there is really no way to solve for z in terms of $(x(s), t(s))$. However, we can solve for the projected characteristics:

$$(x(t, y), t) = (\mathrm{e}^{-\frac{y^2}{2}} t + y, t);$$

and this is really all that matters for us to find the time t_0 when the shock develops, i.e., the time when the projected characteristic fails to be injective.
A little calculation shows that this happens precisely when $t = \mathrm{e}^{-\frac{1}{2}}$. ∎

PROBLEM 2.1.3. If ρ_0 denotes the maximum density of cars on a highway (i.e., under bumpet-to-bumper conditions), then a reasonable model for traffic density ρ is given by

$$\begin{cases} \rho_t + (F(\rho))_x = 0, \\ F(\rho) = c\rho\left(1 - \dfrac{\rho}{\rho_0}\right), \end{cases}$$

where $c > 0$ is a constant (free speed of highway). Suppose the initial density is

$$\rho(x,0) = \begin{cases} \frac{1}{2}\rho_0 & \text{if } x < 0, \\ \rho_0 & \text{if } x > 0. \end{cases}$$

Find the shock curve and describe the weak solution. (Interpret your result for the traffic flow.)

Solution. First, note that

$$\begin{aligned} (F(\rho))_x &= F'(\rho)\rho_x \\ &= \left[-c\frac{\rho}{\rho_0} + c\left(1 - \frac{\rho}{\rho_0}\right)\right]\rho_x \\ &= \left(c - \frac{2c\rho}{\rho_0}\right)\rho_x. \end{aligned}$$

Let us find a solution to the PDE using the method of characteristics. Write

$$G(p,z,x) = p^2(s) + F'(z(s))p^1(s).$$

Then, the characteristic ODEs are

$$\begin{cases} \left(\dot{p}^1(s), \dot{p}^2(s)\right) = \left(-F''(z(s))p^1(s), -F''(z(s))p^2(s)\right), \\ \dot{z}(s) = F'(z(s))p^1(s) + p^2(s) \\ \qquad = 0, \\ \left(\dot{x}^1(s), \dot{x}^2(s)\right) = \left(F'(z(s)), 1\right). \end{cases}$$

Now, integrating the characteristics, we have

$$\begin{cases} z(s) = z^0, \\ \left(x^1(s), x^2(s)\right) = \left(F'(z^0)s + x^0, s\right). \end{cases}$$

We have two cases to consider, $x^0 < 0$ or $x^0 > 0$. For $x^0 < 0$, $z^0 = \frac{\rho_0}{2}$ and the projected characteristics look like

$$\begin{aligned} \left(F'(\tfrac{\rho_0}{2})t + x^0, t\right) &= \left(\left[c - \frac{2c(\frac{\rho_0}{2})}{\rho_0}\right]t + x^0, t\right) \\ &= (0 \cdot t + x^0, t) \\ &= (x^0, t) \end{aligned}$$

(where we have replaced s with the more appropriate t). Whereas for $x^0 > 0$, we have

$$\begin{aligned}\left(F'(\rho_0)t + x^0, t\right) &= \left(\left[c - \frac{2c\rho_0}{\rho_0}\right]t + x^0, t\right)\\ &= (-ct + x^0, t).\end{aligned}$$

These characteristics intersect precisely when

$$t = \frac{x_1^0 - x_2^0}{c},$$

where $x_1^0 > 0$, $x_2^0 < 0$. ■

Problem 2.1.4. Find the characteristics of the second order equation

$$u_{xx} - (2\cos x)u_{xy} - (3 + \sin^2 x)u_{yy} - yu_y = 0,$$

and transform it to the canonical form.

Solution. First, writing the PDE in the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + 2Du_x + 2Eu_y + Fu = 0,$$

we see that $A = 1$, $B = -\cos x$, $C = -3\sin^2 x$, and $E = -\frac{y}{2}$. We solve for the characteristic curve by find a solution to the ODEs

$$\begin{aligned}\frac{dy}{dx} &= \frac{B \pm \sqrt{B^2 - AC}}{A}\\ &= -\cos x \pm \sqrt{\cos^2 x + 3 + \sin^2 x}\\ &= -\cos x \pm 2.\end{aligned}$$

The solutions give us the following ODEs

$$\begin{cases} y = -\sin x + 2x + \xi(x, y),\\ y = -\sin x - 2x + \eta(x, y).\end{cases}$$

Integrating these equations, we have

$$\begin{cases} \xi(x, y) = y + \sin x - 2x,\\ \eta(x, y) = y + \sin x + 2x.\end{cases}$$

These are the characteristic strips for the PDE.

To put this PDE in canonical form, we first compute the following partial derivatives

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x,\\ u_y &= u_\xi \xi_y + u_\eta \eta_y,\\ u_{xx} &= u_\xi \xi_{xx} + u_\eta \eta_{xx} + (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + (u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x)\eta_x\\ &= u_{\xi\xi}(\xi_x)^2 + u_{\eta\eta}(\eta_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx},\end{aligned}$$

exploiting symmetry, we can find u_{yy} by replacing x with y above

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + u_{\eta\eta}(\eta_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_\xi\xi_{yy} + u_\eta\eta_{yy},$$

the last thing we need to figure out is the mixed partial

$$\begin{aligned} u_{xy} &= u_\xi\xi_{xy} + u_\eta\eta_{xy} + (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + (u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y)\eta_x \\ &= u_{\xi\xi}\xi_x\xi_y + u_{\eta\eta}\eta_x\eta_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_\xi\xi_{xy} + u_\eta\eta_{xy}. \end{aligned}$$

Now find the partials $\xi_x, \eta_x, \xi_y, \eta_y, \xi_{xy}, \dots$, etc.

$$\begin{aligned} \xi_x &= \cos x - 2, & \eta_x &= \cos x + 2, \\ \xi_{xx} &= -\sin x, & \eta_{xx} &= -\sin x, \\ \xi_{xy} &= 0, & \eta_{xy} &= 0, \\ \xi_y &= 1, & \eta_y &= 1, \\ \xi_{yy} &= 0, & \eta_{yy} &= 0. \end{aligned}$$

Thus,

$$\begin{cases} u_x = (\cos x - 2)u_\xi + (\cos x + 2)u_\eta, \\ u_y = u_\xi + u_\eta, \\ u_{xx} = (\cos x - 2)^2 u_{\xi\xi} + (\cos x + 2)^2 u_{\eta\eta} \\ \qquad + 2(\cos x + 2)(\cos x - 2)u_{\xi\eta} - (\sin x)u_\xi - (\sin x)u_\eta \\ \quad = (\cos^2 x - 4\cos x + 4)u_{\xi\xi} + (\cos^2 x + 4\cos x + 4)u_{\eta\eta} \\ \qquad + 2(\cos^2 x - 4)u_{\xi\eta} - (\sin x)u_\xi - (\sin x)u_\eta \\ u_{yy} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}, \\ u_{xy} = (\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}, \end{cases}$$

so the canonical form is

$$\begin{aligned} 0 &= u_{xx} - (2\cos x)u_{xy} - (3\sin^2 x)u_{yy} - yu_y \\ &= \xi^2 u_{\xi\xi} + \eta^2 u_{\eta\eta} \\ &\quad + 2\xi\eta u_{\xi\eta} - (\sin x)u_\xi - (\sin x)u_\eta \\ &\quad - (2\cos x)\big((\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}\big) \\ &\quad - (3\sin^2 x)(u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}) \\ &\quad - y(u_\xi + u_\eta) \end{aligned}$$

Who cares. ∎

PROBLEM 2.1.5. Let $Lu \coloneqq u_{xx} - 4u_{yy} + \sin(y + 2x)u_x = 0$.

(a) Consider the level curve $\Gamma = \{ (x, y) : \varphi(x, y) = C \}$ where $|D\varphi| \neq 0$ on Γ. Define what it means for Γ to be characteristic with respect to L at a point $(x_0, y_0) \in \Gamma$.
(b) Find the points at which the curve $x^2 + y^2 = 5$ is characteristic.

(c) Is it true that every smooth simple closed curve Γ in $\mathbb{R}^2$ has at least one point at which it is characteristic with respect to L?

Solution. ■

Problem 2.1.6. Consider the second order equation

$$Lu := u_{xx} - 2xu_{xy} + x^2 u_{yy} - 2u_y = 0.$$

(a) Find the characteristic curves of $Lu = 0$. What is the type of this equation?
(b) Find the points on the line $\Gamma := \{ (x, y) \in \mathbb{R}^2 : x + y = 1 \}$ at which Γ is characteristic with respect to $Lu = 0$.

Solution. ■

Problem 2.1.7. Solve the initial boundary value problem for the equation $u_{tt} = u_{xx}$ in $\{ x > 0, t > 0 \}$ satisfying

$$\begin{cases} u(x,0) = \sin^2 x, \quad u_t(x,0) = \sin x, \\ u(0,t) = 0. \end{cases}$$

Solution. ■

Problem 2.1.8. Consider the initial/boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & & \text{for } 0 < x < \pi, t > 0, \\ u(x,0) = x, & u_t(x,0) = 0 & \text{for } 0 < x < \pi, \\ u_x(0,t) = 0, & u_x(\pi,t) = 0 & \text{for } t > 0. \end{cases}$$

(a) Find a weak solution of the problem.
(b) Is the solution unique? Continuous? C^1?

Solution. ■

Problem 2.1.9. Let B_1^+ denote the open half-ball $\{ x \in \mathbb{R}^n : |x| < 1, x_n > 0 \}$. Assume $u \in C(\bar{B}_1^+)$ is harmonic in B_1^+ with $u = 0$ on $\partial B_1^+ \cap \{ x_n = 0 \}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0, \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0, \end{cases}$$

for $x \in B_1$. Prove v is harmonic in B_1.
Hint: It will be enough to prove that $\int_B \nabla v \nabla \eta \, dx = 0$ for any test function $\eta \in C_0^\infty(B_1)$. Split $\int_{B_1} = \int_{B_1^+} + \int_{B_1^-}$ and apply the integration by parts formula to each of $\int_{B_1^\pm}$.

Solution. ■

PROBLEM 2.1.10. Let u and v be harmonic functions in the unit ball $B_1 \subset \mathbb{R}^n$. What can you conclude about u and v if

(a) $D^\alpha u(0) = D^\alpha v(0)$ for every multiindex α?
(b) $u(x) \le v(x)$ for every $x \in B_1$ and $u(0) = v(0)$?

Justify your answer in each case.

Solution. ■

PROBLEM 2.1.11. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

is a solution to the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial, i.e., $f(y) = f(|y|)$, and supported in $B_R := \{\, |x| < R \,\}$, then

$$u(x) = c\Phi(x)$$

for any $x \in \mathbb{R}^n \setminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y)\, dy.$$

[*Hint:* Use polar (spherical) coordinates and apply the mean value property for harmonic functions.]

Solution. ■

3 Qualifying Exams

3.1 Qualifying Exam, August ‘04

PROBLEM 3.1.1. Consider the initial value problem

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = -u, \\ \qquad\qquad\qquad\quad u = f \qquad \text{on } S^1 = \{\, x^2 + y^2 = 1 \,\}, \end{cases}$$

where a and b satisfy

$$a(x,y) + b(x,y)y > 0$$

for any $x, y \in \mathbb{R}^n \setminus \{(0,0)\}$.

(a) Show that the initial value problem has a unique solution in a neighborhood of S^1. Assume that a, b, and f are smooth.
(b) Show that the solution of the initial value problem actually exists in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Solution. ■

PROBLEM 3.1.2. Let $u \in C^2\big(\mathbb{R} \times [0,\infty)\big)$ be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty), \\ \qquad u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where f and g have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx,$$

and the potential energy by

$$P(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx.$$

Show that

(a) $K(t) + P(t)$ is constant in t,
(b) $K(t) = P(t)$ for all large enough times t.

Solution. ■

PROBLEM 3.1.3. Use Kirchhoff’s formula and Duhamel’s principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = e^{-t} g(x) & \text{for } x \in \mathbb{R}^3,\ t > 0, \\ \ u(x,0) = u_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution $u = e^{-t} + t - 1$ when $g(x) = 1$.

Solution. ∎

Problem 3.1.4. Let Ω be a bounded open set in $\mathbb{R}^n$ and $g \in C_0^\infty(\Omega)$. Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega,\, t > 0, \\ u(x,0) = g(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } xi \in \partial\Omega,\, t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n,\, t > 0, \\ v(x,0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put $g = 0$ outside Ω.

(a) Show that

$$-v(x,t) \leq u(x,t) \leq v(x,t)$$

for any $x \in \Omega$, $t > 0$.

(b) Use (a) to conclude that

$$\lim_{t\to\infty} u(x,t) = 0,$$

for any $x \in \Omega$.

Solution. ∎

Problem 3.1.5. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \qquad P_m(\lambda x) = \lambda^m P_m(x),$$

for any $x \in \mathbb{R}^n$, $\lambda > 0$,

$$\Delta P_k = 0, \qquad \Delta P_m = 0$$

in $\mathbb{R}^n$.

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = kP_k(x), \qquad \frac{\partial P_m(x)}{\partial \nu} = mP_m(x)$$

on ∂B_1, where $B_1 = \{\, |x| < 1 \,\}$ and ν is the outward normal on ∂B_1.

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x)P_m(x)\, dS = 0,$$

if $k \neq m$.

Solution. ∎

3.2 Qualifying Exam, August '05

Problem 3.2.1.

(a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ \qquad\qquad u = 1 \qquad \text{on } S^1 = \{ x^2 + y^2 = 1 \}. \end{cases}$$

(b) Is the solution unique in a neighborhood of the point $(1, 0)$? Justify your answer.

Solution. The solution to teh first part is

$$u(x, y) = \frac{x^2 + y^2}{4} + \frac{3}{4}.$$

∎

Problem 3.2.2. Consider the second order PDE in $\{ x > 0, y > 0 \} \subset \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

(a) Classify the equation and reduce it to the canonical form.
(b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy}\, G(\tfrac{x}{y}).$$

Solution. ∎

Problem 3.2.3. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^3$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial (i.e., $f(y) = f(|y|)$) and supported in $B_R = \{ |x| < R \}$, then

$$u(x) = c\Phi(x),$$

for any $x \in \mathbb{R}^n \setminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y)\, dy.$$

[*Hint:* Use spherical (polar) coordinates and the mean value property.]

Solution. ∎

Problem 3.2.4. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \qquad u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where $g, h \in C_0^\infty(\mathbb{R}^2)$ and $\alpha \geq 0$ is a constant.

(a) Show that for an appropriate choice of constant λ and μ the function

$$v(x_1, x_2, x_3, t) := e^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation $v_{tt} - \Delta v = 0$.

(b) Use (a) to prove the following domain of dependence result: for any point $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$ the value $u(x_0, t_0)$ is uniquely determined by values of g and h in $\overline{B_{t_0}(x_0)} := \{ |x - x_0| \leq t_0 \}$. (You may use the corresponding result for the wave equation without proof.)

Solution. ■

Problem 3.2.5. Let $u(x, t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0, \infty)$ with the initial condition

$$u(x, 0) = u_0(x)$$

for $x \in \mathbb{R}$, where $u_0 \in C^\infty$ is 2π-periodic, i.e., $u_0(x + 2\pi) = u_0(x)$. Show that

$$\lim_{t \to \infty} u(x, t) = a_0,$$

uniformly in $x \in \mathbb{R}$, where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x)\, dx.$$

Solution. ■

3.3 Qualifying Exam, January '14

PROBLEM 3.3.1. Consider the first order equation in $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

(a) Find the characteristic curves of the equation.
(b) Consider the Cauchy problem for this equation prescribed on the line $x_1 = 1$:

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on f so that the proble is solvable in a neighborhood of the point $(1, 0)$.

Solution. ∎

PROBLEM 3.3.2. Let u be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{ x_n > 0 \}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where g is a continuous function with compact support in $\mathbb{R}^{n-1}$. Here $n \geq 2$. Prove that

$$u(x) \longrightarrow 0, \qquad \text{as } |x| \longrightarrow \infty,$$

for $x \in \{ x_n > 0 \}$.

Solution. ∎

PROBLEM 3.3.3. Let u be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \qquad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions $u(x, 0) = g(x)$, where g is a bounded continuous function on $\mathbb{R}$ satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^{\alpha}, \qquad x, y \in \mathbb{R}$$

with a constant $\alpha \in (0, 1]$. Show that

$$\begin{aligned} |u(x, t) - u(y, t)| &\leq M|x - y|^{\alpha}, & x, y \in \mathbb{R}, t > 0, \\ |u(x, t) - u(x, s)| &\leq C_{\alpha} M|t - s|^{\frac{\alpha}{2}}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of $u(x, t)$ as a convolution with the heat kernel $\Phi(y, t)$, make a change of variables $z = \frac{y}{\sqrt{t}}$ and use that $\left|\sqrt{t} - \sqrt{s}\right| \leq \sqrt{|t - s|}$.]

Solution. ∎

PROBLEM 3.3.4. Let u be a positive harmonic function in the unit ball B_1 in $\mathbb{R}^n$. Show that

$$\left|D(\ln u)\right| \leq M \qquad \text{in } B_{\frac{1}{2}}$$

for a constant M depending only on the dimension n.
[*Hint:* Use the interior derivative estimate $\left|Du(x)\right| \leq (\frac{C_n}{r}) \sup_{B_r(x)} |u|$ for $B_r(x) \subset B_1$ as well as the Harnack inequality for harmonic functions.]

Solution. ∎

PROBLEM 3.3.5. Let u be a C^2 solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = 0, \qquad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some $k \geq 0$. Prove that there exists a function $\varphi(r)$ such that

$$u(x,t) = t^{k+2} \varphi\left(\frac{|x|}{t}\right).$$

[*Hint:* As one of the steps show that u is $(k+2)$-homogeneous in (x,t) variables, i.e., $u(\lambda x, \lambda t) = \lambda^{k+2} u(x,t)$ for any $\lambda > 0$.]

Solution. ∎