MA52300 FALL 2016

HOMEWORK ASSIGNMENT 7 – *Solutions*

1. Solve the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2 \\ u = x_1 & \text{on } |x| = 1 \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

Solution. We look for the solution in the form of Laurent series

$$u(r,\theta) = c_0 \log r + \sum_{k=-\infty}^{\infty} (a_k \cos k\theta + b_k \sin k\theta) r^k,$$

where (r,θ) are the polar coordinates in the plane. The boundary condition can be written in polar coordinates as

$$\begin{aligned} u(1,\theta) &= \cos\theta \\ u(2,\theta) &= 1 + 4\cos\theta\sin\theta = 1 + 2\sin 2\theta. \end{aligned}$$

Since the boundary functions are trigonometric polynomials of degree 2, then in the Laurent series we may assume that $a_k = b_k = 0$ for $|k| > 2$. We next choose the remaining coefficients so that the boundary conditions are satisfied.

1) On $r = 1$, equating the coefficients, we have

$$\begin{aligned} a_2 + a_{-2} &= 0 \qquad & b_2 - b_{-2} &= 0 \\ a_1 + a_{-1} &= 1 \qquad & b_1 - b_{-1} &= 0 \\ a_0 &= 0 \end{aligned}$$

2) On $r = 2$, equating the coefficients, we have

$$\begin{aligned} 4a_2 + \frac{1}{4} a_{-2} &= 0 \qquad & 4b_2 - \frac{1}{4} b_{-2} &= 2 \\ 2a_1 + \frac{1}{2} a_{-1} &= 0 \qquad & 2b_1 - \frac{1}{2} b_{-1} &= 0 \\ a_0 + c_0 \log 2 &= 1 \end{aligned}$$

□

This gives

$$a_2 = a_{-2} = 0 \qquad\qquad b_2 = b_{-2} = \frac{8}{15}$$
$$a_1 = -\frac{1}{3}, \quad a_{-1} = 4/3, \qquad\qquad b_1 = b_{-1} = 0,$$
$$a_0 = 0, \quad c_0 = \frac{1}{\log 2}.$$

Hence, in polar coordinates, the solution is

$$u(r,\theta) = \frac{8}{15} r^2 \sin 2\theta - \frac{8}{15} r^{-2} \sin 2\theta - \frac{1}{3} r \cos\theta + \frac{4}{3} r^{-1} \cos\theta + \frac{\log r}{\log 2}.$$

In the Cartesian coordinates, we have

$$u(x) = \frac{16}{15} x_1 x_2 - \frac{16}{15} \frac{x_1 x_2}{|x|^4} - \frac{1}{3} x_1 + \frac{4}{3} \frac{x_2}{|x|^2} + \frac{\log |x|}{\log 2}.$$

2. Let Ω be a bounded domain with C^1 boundary, $g \in C(\partial\Omega)$ and $f \in C(\overline{\Omega})$. Consider then the so-called *Neumann problem*

$$(*) \qquad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= g \quad \text{on } \partial\Omega, \end{aligned}$$

where ν is the outer normal on $\partial\Omega$. Show that the solution of $(*)$ in $C^2(\Omega) \cap C^1(\overline{\Omega})$ is unique up to a constant; i.e., if u_1 and u_2 are both solutions of $(*)$, then $u_2 = u_1 + \mathit{const}$ in Ω.

Hint: Look at the proof of the uniqueness for the Dirichlet problem by energy methods, [E, 2.2.5a].

Solution. The proof is almost a verbatim repetition of that of Theorem 16 in [E, 2.2.5a].

Set $w = u_2 - u_1$. Then $\Delta w = 0$ in Ω and $\partial w/\partial\nu = 0$ on $\partial\Omega$. Integrating by parts, we obtain

$$\int_\Omega |Dw|^2\, dx = -\int_\Omega w \Delta w\, dx + \int_{\partial\Omega} w \frac{\partial w}{\partial \nu} dS = 0.$$

Thus, $Dw \equiv 0$ in Ω and, since Ω is connected[1], we deduce $w = u_2 - u_1 \equiv \mathit{const}$ in Ω. This completes the proof. $\square$

3. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c \in \mathbb{R}$.

[1]By definition, a domain is a connected open set

[*Hint*: Rewrite the problem in terms of $v(x,t) := e^{ct}u(x,t)$]

Solution. Following the hint let $v(x,t) = e^{ct}u(x,t)$. Then also

$$u(x,t) = e^{-ct}v(x,t).$$

Furthermore,

$$u_t = -ce^{-ct}v + e^{-ct}v_t$$
$$\Delta u = e^{-ct}\Delta v$$

Then the equation

$$u_t - \Delta u + cu = f$$

takes the form

$$e^{-ct}(-cv + v_t - \Delta v + cv) = f$$

or, equivalently,

$$v_t - \Delta v = e^{ct}f. \tag{1}$$

Besides, at the initial time we have

$$v(x,0) = u(x,0) = g(x). \tag{2}$$

A solution to problem (1)–(2) is given then by the formula

$$v(x,t) = \int_{\mathbb{R}^n} g(y)\Phi(x-y,t)dy + \int_0^t\int_{\mathbb{R}^n} e^{cs}f(y,s)\Phi(x-y,t-s).$$

This gives

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} g(y)e^{-ct}\Phi(x-y,t)dy + \int_0^t\int_{\mathbb{R}^n} f(y,s)e^{-c(t-s)}\Phi(x-y,t-s) \\ &= \int_{\mathbb{R}^n} g(y)\Phi_c(x-y,t)dy + \int_0^t\int_{\mathbb{R}^n} f(y,s)\Phi_c(x-y,t-s), \end{aligned}$$

where

$$\Phi_c(x,t) = e^{-ct}\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-ct-\frac{|x|^2}{4t}}.$$

(The latter function can be actually considered as the fundamental solution of the equation $u_t - \Delta u + cu = 0$.) □