4.3: 16, 17, 19, 28, 33, 34
Matlab 6.1: 1, 2, 4, 8

4.3.16| Which of the given subsets of the vector space P_2 are subspaces?
(a) $a_2t^2+a_1t+a_0$, where $a_1=0$ and $a_0=0$
(b) $a_2t^2+a_1t+a_0$, where $a_1=2a_0$
(c) $a_2t^2+a_1t+a_0$, where $a_2+a_1+a_0=2$.

(a) By Thm 4.3 as $(a_2t^2+a_1t+a_0)+(a_2't^2+a_1't+a_0')=(a_2+a_2')t^2+(a_1+a_1')t+(a_0+a_0')$ has $a_1+a_1'=0$ and $a_0+a_0'=0$ and $c(a_2t^2+a_1t+a_0)=ca_2t^2+ca_1t+ca_0$ has $ca_1=0$ and $ca_0=0$ then this is a subspace of P_2.
(b) By Thm 4.3 as $(a_2t^2+a_1t+a_0)+(a_2't^2+a_1't+a_0')=(a_2+a_2')t^2+(a_1+a_1')t+(a_0+a_0')$ has $(a_1+a_1')=a_1+a_1'=2a_0+2a_0'=2(a_0+a_0')$ and $c(a_2t^2+a_1t+a_0)=ca_2t^2+ca_1t+ca_0$ has $ca_1=c(2a_0)=2(ca_0)$ then this is a subspace of P_2.
(c) This fails Thm 4.3 b for $2(a_2t^2+a_1t+a_0)=2a_2t^2+2a_1t+2a_0$ has $2a_2+2a_1+2a_0=2(a_2+a_1+a_0)=2\cdot 2=4$, not 2.

4.3.17| Which of the following subsets of the vector space M_{nn} are subspaces?
(a) The set of all $n\times n$ symmetric matrices
(b) The set of all $n\times n$ diagonal matrices
(c) The set of all $n\times n$ nonsingular matrices.

(a) These are matrices $A=A^T$ so matrices $A=[a_{ij}]=[a_{ji}]=A^T$. By Thm 4.3 this has (a) $[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$ with $a_{ij}+b_{ij}=a_{ji}+b_{ji}$ and (b) $c[a_{ij}]=[ca_{ij}]$ has $ca_{ij}=ca_{ji}$ as needed to be a subspace.
(b) These are matrices with $a_{ij}=0$ for $j<i$ or $i<j$ (upper or lower). By Thm 4.3 this has (a)
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & b_{nn} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn}+b_{nn} \end{bmatrix}$$ is diagonal
and (b)
$$c\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}=\begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ 0 & ca_{22} & \cdots & ca_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & ca_{nn} \end{bmatrix}$$ is diagonal so is a subspace.
(c) This is not a subspace by 4.3a as $I=\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ and $-I=\begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{bmatrix}$ are nonsingular but $I+-I=0$ is singular.

4.3.19 | Which of the following subsets are subspaces of the vector space $C(-\infty,\infty)$?
(a) All nonnegative functions (b) All constant functions (c) All functions f such that $f(0)=0$
(d) All functions f such that $f(0)=5$ (e) All differentiable functions.

Check conditions of Thm 4.3.

(a) Not closed under scalar multiplication for if f is a nonnegative function $-f$ is a nonpositive function.

(b) Closed under addition: if $f=a$ and $g=b$, then $f+g=a+b$ is constant.
Closed under scalar multiplication: if $f=a$ and c a constant, $cf=ca$ is constant.
Nonempty.

(c) Closed under addition: if $f(0)=0$, $g(0)=0$, then $f(0)+g(0)=0+0=0$ so $f+g$ is in the set.
Closed under scalar mult.: if $f(0)=0$, c a constant, $cf(0)=c0=0$ so cf is in the set.
The set is nonempty so is a subspace.

(d) Not closed under scalar multiplication for $f(x)=5$ and $c=2$, then $2f(0)=2\cdot5=10$.

(e) Closed under addition: f,g differentiable, then $f+g$ is differentiable.
Closed under scalar mult.: f differentiable and c a constant, then cf is differentiable.
Nonempty.

4.3.28 | Show that the only subspaces of $\mathbb{R}^1$ are $\{0\}$ and $\mathbb{R}^1$ itself.

Let W be a subspace of $\mathbb{R}^1$. Then by Defn 4.5, 0 is in W. If there is a $x\neq0$ in W, then for any r in $\mathbb{R}^1$, rx is in W. Now any real number can be written as rx for some r, so then $\mathbb{R}^1$ is contained in W forcing W to be $\mathbb{R}^1$. Otherwise W is $\{0\}$.

4.3.33 Which of the following vectors in $\mathbb{R}^3$ are linear combinations of

$$v_1 = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}?$$

$$\text{(a)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

Following the pattern $av_1 + bv_2 + cv_3 = w$ the vector we turn this into a system. $[v_1\ v_2\ v_3]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = w$

(a)
$$\left[\begin{array}{ccc|c} 4 & 2 & -2 & 1 \\ 2 & 1 & -1 & 1 \\ -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{+r_3+r_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 2 & 1 & -1 & 1 \\ -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow[3r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -3 \\ 0 & -2 & -6 & 7 \end{array}\right] \xrightarrow{2r_2+r_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$
which is inconsistant

and thus not a linear combination. Note $-2v_1 + 3v_2 = v_3$, so we ignore this essentially.

(b)
$$\left[\begin{array}{ccc|c} 4 & 2 & -2 & 4 \\ 2 & 1 & -1 & 2 \\ -3 & -2 & 0 & -6 \end{array}\right] \xrightarrow{r_3+r_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 2 & 1 & -1 & 2 \\ -3 & -2 & 0 & -6 \end{array}\right] \xrightarrow[3r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & 3 & 6 \\ 0 & -2 & -6 & -12 \end{array}\right] \xrightarrow{2r_2+r_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which gives $-2v_1 + 6v_2 = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$

(c)
$$\left[\begin{array}{ccc|c} 4 & 2 & -2 & -2 \\ 2 & 1 & -1 & -1 \\ -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{r_3+r_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 2 & 1 & -1 & -1 \\ -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow[3r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & -6 & -2 \end{array}\right] \xrightarrow{2r_2+r_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which gives $-v_1 + v_2 = v_3$.

(d)
$$\left[\begin{array}{ccc|c} 4 & 2 & -2 & -1 \\ 2 & 1 & -1 & 2 \\ -3 & -2 & 0 & 3 \end{array}\right] \xrightarrow{r_3+r_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 2 & 1 & -1 & 2 \\ -3 & -2 & 0 & 3 \end{array}\right] \xrightarrow[3r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -2 \\ 0 & -2 & -6 & 9 \end{array}\right] \xrightarrow{2r_2+r_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$
which is inconsistent.

4.3.34| Which of the following vectors in $\mathbb{R}^4$ are linear combinations of

$v_1 = [1\ 2\ 1\ 0]$, $v_2 = [4\ 1\ -2\ 3]$, $v_3 = [1\ 2\ 6\ -5]$, v_4 $[-2\ 3\ -1\ 2]$?

(a) $[3\ 6\ 3\ 0]$ (b) $[1\ 0\ 0\ 0]$ (c) $[3\ 6\ -2\ 5]$ (d) $[0\ 0\ 0\ 1]$

The system is $\left[v_1^T\ v_2^T\ v_3^T\ v_4^T\right]\begin{bmatrix}?\\?\\?\\?\end{bmatrix} = w^T$ the vector we want to check.

(a)
$$\left[\begin{array}{cccc|c}1&4&1&-2&3\\2&1&2&3&6\\1&-2&6&-1&3\\0&3&-5&2&0\end{array}\right]\xrightarrow[-r_1+r_3]{-2r_1+r_2}\left[\begin{array}{cccc|c}1&4&1&-2&3\\0&-7&0&7&0\\0&-6&5&1&0\\0&3&-5&2&0\end{array}\right]\xrightarrow{-1/7 r_2}\left[\begin{array}{cccc|c}1&4&1&-2&3\\0&1&0&-1&0\\0&-6&5&1&0\\0&3&-5&2&0\end{array}\right]\xrightarrow[-3r_2+r_4]{6r_2+r_3}\left[\begin{array}{cccc|c}1&4&1&-2&3\\0&1&0&-1&0\\0&0&5&-5&0\\0&0&-5&5&0\end{array}\right]\xrightarrow[r_3+r_4]{1/5 r_3}$$

$$\left[\begin{array}{cccc|c}1&4&1&-2&3\\0&1&0&-1&0\\0&0&1&-1&0\\0&0&0&0&0\end{array}\right]\xrightarrow{-4r_2+r_1}\left[\begin{array}{cccc|c}1&0&1&2&3\\0&1&0&-1&0\\0&0&1&-1&0\\0&0&0&0&0\end{array}\right]\xrightarrow{-r_3+r_1}\left[\begin{array}{cccc|c}1&0&0&3&3\\0&1&0&-1&0\\0&0&1&-1&0\\0&0&0&0&0\end{array}\right]$$
gives $v_4 = 3v_1 - v_2 - v_3$ and $3v_1 = [3\ 6\ 3\ 0]$.

Note if $A = [v_1^T\ v_2^T\ v_3^T\ v_4^T]$ and R is the matrix formed by applying the row operations to I, then RA is in reduced row echelon form. If we want to check if w is a linear combination of the vectors, then it comes down to checking $[RA \mid Rw]$ is consistent.

Finding R:
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow[-r_1+r_3]{-2r_1+r_2}\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-1&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow{-1/7 r_2}\begin{bmatrix}1&0&0&0\\2/7&-1/7&0&0\\-1&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow[-3r_2+r_4]{6r_2+r_3}\begin{bmatrix}1&0&0&0\\2/7&-1/7&0&0\\5/7&-6/7&1&0\\-6/7&3/7&0&1\end{bmatrix}\xrightarrow[r_3+r_4]{1/5 r_3}\begin{bmatrix}1&0&0&0\\2/7&-1/7&0&0\\1/7&-6/35&1/5&0\\-1/7&-3/7&1&1\end{bmatrix}\xrightarrow{-4r_2+r_1}$$

$$\begin{bmatrix}-1/7&4/7&0&0\\2/7&-1/7&0&0\\1/7&-6/35&1/5&0\\-1/7&-3/7&1&1\end{bmatrix}\xrightarrow{-r_3+r_1}\begin{bmatrix}-2/7&26/35&-1/5&0\\2/7&-1/7&0&0\\1/7&-6/35&1/5&0\\-1/7&-3/7&1&1\end{bmatrix} = R \text{ so } RA = \begin{bmatrix}1&0&0&3\\0&1&0&-1\\0&0&1&-1\\0&0&0&0\end{bmatrix}.$$

(b) $R\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}-2/7\\2/7\\1/7\\-1/7\end{bmatrix}$ which is inconsistent with RA.

(c) $R\begin{bmatrix}3\\6\\-2\\5\end{bmatrix} = \begin{bmatrix}-6/7 + 156/35 + 2/5\\6/7 - 6/7\\3/7 - 36/35 - 2/5\\-3/7 - 18/7 - 2 + 5\end{bmatrix} = \begin{bmatrix}4\\0\\-1\\0\end{bmatrix}$ so $[3\ 6\ -2\ 5] = 4v_1 - v_3$

(d) $R\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ which is inconsistent with RA.

Matlab 6.1

Matlab 6.1.1 Let $v1 = [4\ 2\ 1]$, $v2 = [-2\ 3\ 1]$, and $v3 = [2\ -11\ -4]$. Determine if each of the following vectors u is a linear combination of v_1, v_2, and v_3. If it is, then display the linear combination by supplying the coefficients and appropriate operations.

(a) $u = [6\ 5\ 5]$

$V = [v1; v2; v3]$, $A = V'$, $\text{rref}([A\ u']) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

the system is inconsistent, so is not a linear combination.

(b) $u = [10\ -15\ -5]$

$\text{rref}([A, u']) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is inconsistent, so not a linear combination

(c) $u = [9\ -17.5\ -6]$

$\text{rref}([A, u']) = \begin{bmatrix} 1 & 0 & -1 & -1/2 \\ 0 & 1 & -3 & -9/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so $-\frac{1}{2}v_1 + -\frac{9}{2}v_2 = u$

(further $-v_1 - 3v_2 = v_3$).

Matlab 6.1.2 Let $v1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. Same as (1).

(a) $u = \begin{bmatrix} 11 \\ -1 \\ 3 \\ 13 \end{bmatrix}$ $\text{rref}([A\ u]) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so $u = 2v_1 - v_2 + 3v_3$.

(b) $u = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ $\text{rref}([A\ u]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is inconsistent, so not a linear combination.

Matlab 6.1.4 Let $v_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$. Same.

(a) $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $v_i = \text{reshape}(v_i, 4, 1)$. $A = [v_1\ v_2\ v_3]$ $u = \text{reshape}(u, 4, 1)$.

$\text{rref}([A\ u]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which is inconsistent, so not a linear combination.

(b) $u = \begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix}$ $\text{rref}([A\ u]) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so $u = v_1 + v_2 - 2v_3$

(c) $u = \begin{bmatrix} 1 & -2 \\ -3 & -3 \end{bmatrix}$ $\text{rref}([A\ u]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so $u = v_2 - 2v_3$

Matlab 6.1.8 Let $S = \{v_1, v_2\}$ where $v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -i \\ 1+i \end{bmatrix}$.

(a) Write vector $x_1 = \begin{bmatrix} 0 \\ 2+i \end{bmatrix}$ as a linear combination of the elements of S.

$x_1 = v_1 + v_2$.

(b) Write $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of the elements of S.

$A = [v_1 \ v_2]$ $\quad$ rref$([A \ x_2]) = \begin{bmatrix} 1 & 0 & .6-.8i \\ 0 & 1 & .6+.2i \end{bmatrix}$ so

$x_2 = (.6-.8i)v_1 + (.6+.2i)v_2$