Homotopy of Character Varieties, Part III

...or any moduli space homotopic to one.

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Theorem (Florentino & L-, 2009, 2014; A. C. Casimiro, C. Florentino, L-, A. G. Oliveira, 2015)

Let G be a real reductive algebraic group, K a maximal compact subgroup, and Γ a (finitely generated) free group or Abelian group. Then $\mathfrak{X}_\Gamma(G)$ strongly deformation retracts onto $\mathfrak{X}_\Gamma(K)$. In particular, they have the same homotopy type.

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- One naturally wonders if this situation generalizes to arbitrary finitely generated groups Γ.
- Although particular counter-examples were known much earlier (via Atiyah-Bott 1983 and Hitchin 1987), Biswas-Florentino showed using Higgs bundle theory (2011) that for a closed surface of genus $g \geq 2$ the moduli spaces $\mathfrak{X}_\Gamma(G)$ and $\mathfrak{X}_\Gamma(K)$ are *never* homotopic.

Sketch of Proof

Go to board.

Poincaré Polynomials

The Poincaré polynomial of $\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{SU}(2))$ was calculated by T. Baird, using methods of equivariant cohomology. His result is that $P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{SU}(2))) = 1 + t - \frac{t(1+t^3)^r}{1-t^4} + \frac{t^3}{2}\left(\frac{(1+t)^r}{1-t^2} - \frac{(1-t)^r}{1+t^2}\right)$.

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- $P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{O}(3,\mathbb{C}))) = P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{O}(3))) = 2^r P_t(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{SU}(2)))$

Euler Characteristic

Theorem (S. Cavazos, L-, 2014)

Let $q = xy$. *Then the E-polynomial for* $\mathfrak{X}_r := \mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{SL}_2(\mathbb{C}))$ *is*

$$E_{\mathbb{Z}^{\star r}}(q) = (q-1)^{r-1}\left((q+1)^{r-1}-1\right)q^{r-1} + \frac{1}{2}q\left((q-1)^{r-1}+(q+1)^{r-1}\right),$$

and the E-polynomial of $\mathfrak{X}_r(\mathrm{SL}_2(\mathbb{C}))^{sing} \cong \mathfrak{X}_{\mathbb{Z}^r}(\mathrm{SL}_2(\mathbb{C}))$ *is given by*

$$E_{\mathbb{Z}^r}(q) = \frac{1}{2}\left((q-1)^r + (q+1)^r\right).$$

Consequently, the difference of these is the E-polynomial of $\mathfrak{X}_r(\mathrm{SL}_2(\mathbb{C}))^{sm}$.

Corollary

$\chi(\mathfrak{X}_r) = 2^{r-2}$, $\chi(\mathfrak{X}_r^{sm}) = -2^{r-2}$, $\chi(\mathfrak{X}_r^{sing}) = 2^{r-1}$, $\chi((\mathfrak{X}_r^{sing})^{sm}) = -2^{r-1}$, *and* $\chi((\mathfrak{X}_r^{sing})^{sing}) = 2^r$.

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Vicente Mũnoz and I have recently generalized these results to $\mathrm{SL}(3,\mathbb{C})$. Consequently, for $r \geq 2$,

$$\chi(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{SL}(3,\mathbb{C}))) = \chi(\mathfrak{X}_{\mathbb{Z}^{\star r}}(\mathrm{SU}(3))) = 2(3)^{r-2}.$$

Covering Spaces

Theorem (D. Ramras, L-, 2014)

Let G be either a connected reductive algebraic group over $\mathbb{C}$, or a compact connected Lie group, and assume that $\pi_1(G)$ is torsion-free. Let Γ be exponent-canceling of rank r. Let $q : H \to G$ be a covering homomorphism, and identify $\pi_1(H)$ with its image under the injective homomorphism $q_\# : \pi_1(H) \to \pi_1(G)$ induced by q. Then the induced map $q_* : \mathfrak{X}_\Gamma(H) \to \mathfrak{X}_\Gamma(G)$ is a surjective, normal covering map with structure group $(\pi_1(G)/\pi_1(H))^r$.

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- In particular, $\mathfrak{X}_\Gamma(\tilde{G}) = \widetilde{\mathfrak{X}_\Gamma(G)}$ in many cases.
- For instance, if $\mathfrak{X}_\Gamma([G, G])$ is simply connected.

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- However, if $\mathfrak{X}_{\mathbb{Z}^2}(\mathsf{SU}(2)) \to \mathfrak{X}_{\mathbb{Z}^2}(\mathrm{PSU}(2))$ is a covering with structure group $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^2$,

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Corollary

$\mathfrak{X}_\Gamma(H) \to \mathfrak{X}_\Gamma(G)$ induce isomorphisms on homotopy groups π_k for all $k \geq 2$ (and all compatible basepoints).

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2. $\pi_0(\mathfrak{X}_\Gamma(G)) = 0$ if the surface is open.
3. Assume Γ is Abelian and G is semisimple. Then, $\pi_0(\mathfrak{X}_\Gamma(G))) = 0$ iff Γ does not have torsion, and one of the following is true: (a) $r := \mathrm{Rank}(\Gamma) = 1$, (b) $r = 2$ and G is simply connected, or (c) $r > 2$ and G is a product of simply connected groups of type A_n or C_n.

Let $\mathfrak{X}^0_\Gamma(G)$ denote the connected component of the trivial representation in $\mathfrak{X}_\Gamma(G)$.

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Theorem (Biswas, L-, Ramras, 2014)

Let G be either a connected reductive $\mathbb{C}$-group, or a connected compact Lie group, and let Γ be one the following:

1. *a free group,*
2. *a free Abelian group, or*
3. *the fundamental group of a closed orientable surface.*

Then $\pi_1(\mathfrak{X}_\Gamma^0(G)) = \pi_1(G/[G,G])^r$, *where* $r = \mathrm{Rank}\,(\Gamma/[\Gamma,\Gamma])$.

Theorem (Florentino, L-, Ramras, 2014)

Let G_n be any of $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{SU}(n)$ or $\mathrm{U}(n)$. Then

$$\pi_2(\mathfrak{X}_r(G_n)) = 0.$$

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Theorem (Florentino, L-, Ramras, 2014)

Assume $(r-1)(n-1) \geq 2$ and $1 < k < 2(r-1)(n-1) - 1$. Then

$$\pi_k(\mathfrak{X}_r(G_n)^{irr}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & \textit{if } k = 2 \\ \mathbb{Z}^r, & \textit{if } k \textit{ is odd and } k < 2n \\ \mathbb{Z}, & \textit{if } k \textit{ is even and } 2 < k < 2n \\ (\mathbb{Z}/n!\mathbb{Z})^r \oplus \mathbb{Z}, & \textit{if } k = 2n \end{cases}$$

Moreover, $\pi_k(\mathfrak{X}_r(G_n)^{irr})$ is finite for $k > 2n$.

Sketch of Proof of π_2-triviality

Go to board.

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- Florentino and L- showed for $G = \mathrm{SL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{C})$ that

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- This result does not even generalize to $G = PSL(2, \mathbb{C})$ by Heusner and Porti.
- Florentino and L- conjectured: If $r \geq 3$, or $r \geq 2$ and the $\mathrm{Rank}(G)$ is sufficiently large, then
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- Note that $\mathfrak{X}_2(PSL(2, \mathbb{C}))$ has smooth points which are reducible and singular points which are irreducible; so a condition on the rank of G when $r = 2$ is necessary.

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- Then clearly, $[\rho \oplus \psi_1]$ has positive dimensional stabilizer and so is reducible, but yet it is in $\mathfrak{X}_2(G)^{sm} \times \mathfrak{X}_2(PSL(2, \mathbb{C}))^{sm}$ and so is a smooth point.

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- On the other hand, $[\rho \oplus \psi_2]$ has a finite stabilizer and so is irreducible (but not good), but is in $\mathfrak{X}_2(G)^{sm} \times \mathfrak{X}_2(PSL(2, \mathbb{C}))^{sing}$ and so is a singular point.

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- This shows that there are Lie groups H of arbitrarily large rank with the property that $\mathfrak{X}_2(H)$ has smooth reducibles and singular irreducibles.

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- Consider ρ with coordinate matrices:

$$\frac{1}{12}\begin{pmatrix} 37 & 35i & 0 & 0 \\ -35i & 37 & 0 & 0 \\ 0 & 0 & 13 & 5i \\ 0 & 0 & -5i & 13 \end{pmatrix},$$

and

$$\frac{1}{40}\begin{pmatrix} 401 & 399i & 0 & 0 \\ -399i & 401 & 0 & 0 \\ 0 & 0 & 41 & 9i \\ 0 & 0 & -9i & 41 \end{pmatrix}.$$

We now give an example to show that the groups H in the above example do not only arise as products with rank 1 Lie groups.

- Sikora explicitly describes $\mathfrak{X}_2(\mathrm{SO}_4(\mathbb{C}))$.
- Consider ρ with coordinate matrices:

$$\frac{1}{12}\begin{pmatrix} 37 & 35i & 0 & 0 \\ -35i & 37 & 0 & 0 \\ 0 & 0 & 13 & 5i \\ 0 & 0 & -5i & 13 \end{pmatrix},$$

and

$$\frac{1}{40}\begin{pmatrix} 401 & 399i & 0 & 0 \\ -399i & 401 & 0 & 0 \\ 0 & 0 & 41 & 9i \\ 0 & 0 & -9i & 41 \end{pmatrix}.$$

- It is clearly reducible and completely reducible.

- However, as the relations and generators are explicitly computed by Sikora for this variety, in *Mathematica* we can compute the rank of the Jacobian matrix at ρ, finding it to be 11.

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- Therefore, we have a smooth point in $\mathfrak{X}_2(\mathrm{SO}_4(\mathbb{C}))$ that is not in $\mathfrak{X}_2(\mathrm{SO}_4(\mathbb{C}))^{irr}$ arising from a reductive group of semisimple rank 2 that is not a product with a rank 1 group.

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- However, note that the simple factors are each of rank 1.

Richardson essentially proved the following theorem in the semisimple case, which we generalize (answering most of the conjecture of Florentino-L-):

Theorem

Let $r \geq 2$. If G is a connected reductive $\mathbb{C}$–group such that the Lie algebra of DG has simple factors of rank 2 or more, then:

(1) $\mathfrak{X}_r(G)^{red} \subset \mathfrak{X}_r(G)^{sing}$,

(2) $\mathfrak{X}_r(G)^{irr} - \mathfrak{X}_r(G)^{good} \subset \mathfrak{X}_r(G)^{sing}$, *and all points in* $\mathfrak{X}_r(G)^{irr} - \mathfrak{X}_r(G)^{good}$ *are orbifold singularities,*

(3) $\mathfrak{X}_r(G)^{good} = \mathfrak{X}_r(G)^{sm}$.

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The previous examples/theorems show that if the semisimple factors are rank 1, there are examples where the theorem holds and examples where it is false.

Thank you!

- References are at

  http://arxiv.org/a/lawton_s_1.

- I gratefully acknowledge that today's theorems are brought to you with the help of:

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