MA598: Lie Groups

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Chapter 1

Introduction

In any algebra textbook, the study of group theory is usually concerned about the theory of finite (or at least finitely generated) groups. However, most groups which appear as groups of symmetries of various geometric objects are not finite: for example, the group $\mathrm{SO}_3 \mathbb{R}$ of all rotations of three-dimensional space is not finite and is not even finitely generated. Thus, much of the material learned in a basic algebra course does not apply here; for exampel, it is not clear, whether, say, the set of all morphisms between such groups can be explicitly described.

The theory of Lie groups answers these questions by replacing the notion of a finitely generated group by that of a Lie group — a group which at the same time is a finite dimensional manifold. It turns out that in many ways such groups can be described and studied as easily as finitely generated groups — or even easier. The key role is played by the notion of a Lie algebra, the tangent space to G at the identity. It turns out that the group operation on G defines a certain skew-symmetric bilinear form on $\mathfrak{g} := T_1 G$; axiomatizing the properties of this operation gives a definition of the Lie algebra.

The fundamental result of the theory of Lie groups is that many properties of Lie groups are completely determined by the properties of corresponding Lie algebras. For example, the set of morphisms between two (connected and simply connected) Lie groups is the same as the set of morphisms between the corresponding Lie algebras; thus, describing them is essentially reduced to a linear algebra problem.

Similarly, Lie algebras also provide a key to the study of the structure of Lie groups and their representations. In particular, this allows one to get a complete classification of Lie groups (semisimple and more generally, reductive Lie groups; this includes all compact Lie groups and all classical Lie groups such as $\mathrm{SO}_n \mathbb{R}$) in terms of relatively simple geometric objects so-called root systems. This result is considered by many mathematicians to be one of the most beautiful achievements in all mathematics.

To conclude this introduction, we will give a simple example which shows that Lie groups naturally appear as groups of symmetries of various objects — and how one can use the theory of Lie grousp and Lie algebras to make use of these symmetries.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Define the Laplace operator $\Delta\colon C^\infty(S^2) \to C^\infty(S^2)$ by $\Delta_{\mathrm{sph}} f := \Delta(\tilde{f})\big|_{S^2}$, where $\tilde{f}$ is the result of extending f to $\mathbb{R}^3 \quad \{(0,0,0)\}$ (constant along each ray), and Δ is the usual Laplace operator in $\mathbb{R}^3$. It is easy to see that Δ_{sph} is a second order differential operator on the sphere; one can write explicit formulas for it in the spherical coordinates, but they are not particularly nice.

For many applications, it is important to know that the eigenvalues and eigenfunctions of Δ_{sph}. In particular, this problem arises in quantum mechanics: the eigenvalues are related to to the energy levels of a hydrogen atom in quantum mechanical description. Unfortunately, trying to find the eigenfunctions by brute force gives a second-order differential equation which is very difficult to solve.

However, it is easy to notice that this problem has some symmetry — namely, the group $\mathrm{SO}_3\,\mathbb{R}$ acting on the sphere by rotations. How can one us this symmetry?

If we had just one symmetry, given by some rotation $R\colon S^2 \to S^2$, we could consider its action on the space of complex-valued functions $C^\infty(S^2, \mathbb{C})$. If we could diagonalize this operator, this would help us study Δ_{sph}: it is a general result of linear algebra that if A and B are two linear operators, and A is diagonalizable, then B must preserve eigenspaces for A. Applying this to the pair R, $\Delta_{textsph}$, we get that Δ_{sph} preserves eigenspaces for R, so we can diagonalize Δ_{sph} independently in each of the eigenspaces.

HOwever, this will not solve the problem: for each individual rotation R, the eigenspace will still be too large (in fact, infinite-dimensional), so diagonalizing Δ_{sph} in each of them is not very easy either. This is not surprising: after all, we only used one of many symmetries. Can we use all of the rotations in $\mathrm{SO}_3\,\mathbb{R}$ simultaneously?

This however presents two problems

- $\mathrm{SO}_3\,\mathbb{R}$ is not a finitely generated group, so apparently we will need to use infinitely (in fact, uncountably) many different symmetries and diagonalize each of them.

- $\mathrm{SO}_3\,\mathbb{R}$ is not commutative, so different operators from $\mathrm{SO}_3\,\mathbb{R}$ cannot be diagonalized simultaneously.

The goal of the theory of Lie groups is to give tools to deal with these (and similar) problems. IN short, the answer to the first problem is that $\mathrm{SO}_3\,\mathbb{R}$ is in a certain sense finitely generated — namely, it is generated by three generators, infinitesimal rotations around the x-, y-, z-axes.

The answer to the second problem is that instead of decomposing the $C^\infty(S^2, \mathbb{C})$ into adirect sum of common eigenspaces for operators $R \in \mathrm{SO}_3\,\mathbb{R}$, we need to decompose it into irreducible representations of $\mathrm{SO}_3\,\mathbb{R}$. In order to do this, we need to develop the theory of representations of $\mathrm{SO}_3\,\mathbb{R}$. WE will do this an complete the analysis of this example in a couple of sections.

Chapter 2

Lie Groups: Basic Definitions

2.1 Differential geometry review

This book assumes that the reader is familiar with basic notions of differential geometry. For the reader's convenience, in this section, we briefly remind you of some of the definitions and fix notation for further use.

Unless otherwise specified, all manifolds considered will be real smooth manifolds. All manifolds we will consider will have at most countably many connected components.

For a manifold M and a point $p \in M$, we denote by T_pM the tangent space to M at point p, and by TM the tangent bundle to M. The space of vector fields on M (i.e, global sections of TM) is denoted by $\operatorname{Vect} M$. For a mrphism $f\colon M \to N$ and a point $p \in M$, $q := f(p)$, $df\colon T_pM \to T_qN$ is the corresponding map of map of tangent spaces.

Recall that a morphism $f\colon M \to N$ is called an *immersion* if $\operatorname{rk} df = \dim M$ for every point $p \in M$; in this case, one can choose local coordinates in a neighborhood of p and in a neighborhood of q such that f is given by $f(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$.

An *immersed submanifold* in a manifold M is a subset $N \subset M$ with a structure of a manifold (not necessarily the one inherited from M) such that the inclusion map $\iota\colon N \hookrightarrow M$ is an immersion. Note that the manifold structure on N is part of the data: in general it is not unique. However, it is usually suppressed in the notation. Note also that for any point $p \in N$, the tangent space to N is naturally a subspace of the tangent space to M at p, i.e., $T_pN \subset T_pM$.

An *embedded submanifold* $N \subset M$ is an immersed submanifold such that the inclusion map $\iota\colon N \hookrightarrow M$ is a homeomorphism. In this case the smooth structure on N is uniquely determined by the smooth structure on M.

Following Spivak, we will use the word submanifold for *embedded submanifolds*.

All of the notions above have complex analogs, in which manifolds are replaced by complex analytic manifolds and smooth maps by holomorphic maps.

2.2 Lie groups, subgroups and cosets

Definition 1. A Lie group is a set G with two structures: G is a group and G is a manifold. These structures agree in the following sense: the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are smooth maps.

A morphism of Lie groups is a smooth map which also preserves the group operation: $f(gh) = f(g)f(h)$, $f(1) = 1$. We will use the standard notation $\operatorname{Im} f$ and $\operatorname{Ker} f$ for the image and the kernel of the morphism.

The word real is used to distinguish these Lie groups from complex Lie groups. However, it is frequently omitted.

One can also consider other classes of manifolds: C^1, C^2, analytic (denoted C^ω). It turns out that all of them are equivalent: every C^0 Lie group has a unique analytic structure. This is a higly non-trivial result (it was one of Hilber's 20 problems), and we are not going to prove it. Proof of a weaker result, that C^2 implies analyticity, is much easier.

In a similar way, one defines complex Lie groups.

Definition 2. A complex Lie group is a set G with two structures: G is a group and G is an analytic manifold. These structures agree in the following sense: multiplication map $G \times G \to G$ and inversion map $G \to G$ are analytic maps.

A moprhism of complex Lie groups is an analytic map which also preserves the group operation, $f(gh) = f(g)f(h)$, $f(1) = 1$.

Through out this book, we try to treat both real and complex cases simultaneously. Thus, most theorems in this book apply both to real and complex Lie groups.

When talking about complex Lie groups, submanifolds will mean complex analytic submanifold, tangent spaces will be considered as complex vector spaces, all morphisms between manifolds will be assumed holomorphic, etc.

Examples 1. The following are examples of Lie groups

(1) $\mathbb{R}^n$, with the group operation given by addition

(2) $\mathbb{R}^\times := (\mathbb{R} \setminus \{0\}, \cdot)$
$\mathbb{R}^+ := (\{\, x \in \mathbb{R} : x > 0 \,\}, +)$

(3) $S^1 := (\{\, z \in \mathbb{C} : |z| = 1 \,\}, \cdot)$

(4) $\mathrm{GL}_n \mathbb{R} \subset \mathbb{R}^{n^2}$. Many of the groups we will consider are subgroups of $\mathrm{GL}_n \mathbb{R}$ or $\mathrm{GL}_n \mathbb{C}$.

(5) $\mathrm{SU}_2 := \{\, A \in \mathrm{GL}_2 \mathbb{C} : A A^{+} = I, \det A = 1 \,\}$. Indeed, one can easily see that

$$\mathrm{SU}_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C},\, |a|^2 + |b|^2 = 1 \right\}.$$

Writing $a = x_1 + \mathrm{i}x_2$, $b = x_3 + \mathrm{i}x_4$, $x_i \in \mathbb{R}$, we see that SU_2 is diffeomorphic to $S^3 = \{\, (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = 1 \,\}$.

(6) In fact, all usual groups of linear algebra, such as $\mathrm{GL}_n \mathbb{R}$, $\mathrm{SL}_n \mathbb{R}$, O_n, U_n, SO_n, SU_n, Sp_{2n} are all Lie groups.

Note that this definition of a Lie group does not require G to be connected. Thus, any finite group is a 0-dimensional Lie group. Since the theory of finite groups is complicated enough, it makes sense to separate the finite part. It can be done as follows.

Theorem 1. *Let G be a real or complex Lie group. Denote by G^0 the connected component of the identity. Then G^0 is a normal subgroup of G and is a Lie group itself (real or complex, respectively). The quotient group G/G^0 is discrete.*

Proof. We need to show that G^0 is closed under the operations of multiplication and inversion. Since the image of a connected topological space under a continuous map is connected, the inversion map i must take G^0 to one component of G, that which contains $i(1) = 1$, namely G^0. In a similar way one shows that G^0 is closed under multiplication.

To check that this is a normal subgroup, we must show that if $g \in G$ and $h \in G^0$, then $ghg^{-1} \in G^0$. Conjugation by g is continuous and thus will take G^0 to some connected component of G; since it fixes 1, this component is G^0.

The fact that the quotient is discrete is obvious. ■

This theorem mostly reduces the study of arbitrary Lie groups to the study of finite groups and connected Lie groups. In fact, one can go further and reduce the study of connected Lie groups to the study of connected simply-connected Lie groups.

Theorem 2. *If G is a connected Lie group, then its universal cover $\tilde{G}$ has a canonical structure of a Lie group such that the covering map $p\colon \tilde{G} \to G$ is a marphism of a Lie groups whose kernel is isomorphic to the fundamental group of G; $\operatorname{Ker} p = \pi_1(G)$ as a group. Moreover, in this case $\operatorname{Ker} p$ is a discrete central subgroup in $\tilde{G}$.*

Proof. The proof follows from the following general result of topology: if M, N are manifolds (or, more generally, nice enough topological spaces), then any continuous mapping $f\colon M \to N$ can be lifted to a map of universal covers $\tilde{f}\colon \tilde{M} \to \tilde{N}$. Moreover, if we choose $m \in M$, $n \in N$ such that $f(m) = n$ and choose liftings $\tilde{m} \in M$, $\tilde{n} \in \tilde{N}$ such that $p(\tilde{m}) = m$, $p(\tilde{n}) = n$, there is a unique lifting $\tilde{f}$ of f such that $\tilde{f}(\tilde{m}) = \tilde{n}$.

Now let us choose some element $\tilde{1} \in \tilde{G}$ such that $p(\tilde{1}) = 1$ in G. Then, by the above theorem, there is a unique map $\tilde{\imath}\colon \tilde{G} \to \tilde{G}$ which lifts the inversion map $i\colon G \to G$ and satisfies $\tilde{\imath}(\tilde{1}) = \tilde{1}$. In a similar way, one constructs the multiplication map $\tilde{G} \times \tilde{G} \to \tilde{G}$.

Finally, the fact that $\operatorname{Ker} p$ is central follows from the results of Exercise 2.2. (Whatever that exercise may be.) ■

Definition 3. A closed Lie subgroup H of a Lie group G is a subgroup which is also a submanifold.

Theorem 3. (1) *Any closed LIe subgroup is closed in G.*

(2) *Any closed subgroup of a Lie goup is a closed real Lie subgroup.*

Proof. The proof o fthe first part is given in Exercise 2.1. The second part is much harder and will not be proved here. Thep roof uses the technique of Lie algebras and can be found, for example, in 10, Corollary 1.10.7. ■

Corollary 4.

(1) *If G is a connected LIe group and U is a neighborhood of 1, then U generates G.*

(2) *Let $f\colon G_1 \to G_2$ be a morphism of Lie groups, with G_2 connected, such that $df\colon T_1G_1 \to T_2G_2$ is surjective. Then f is surjective.*

Proof. (1) Let H be the subgroup generated by U. Then H is open in G: for any element $h \in H$, the set $h \cdot U$ is a neighborhoof of h in G. Since it is an open subset of a manifold, it is a submanifold, so H is a closed Lie subgroup. Therefore, by Theorem 2.9 it is closed and is nonempty, so $H = G$.

(2) Given the assumptions, the inverse function theorem says that f is surjective onto some neighborhood U of $1 \in G_2$. Since an image of a group morphism is a subgroup, and U generates G_2, f is surjective. ■

As in the theory of discrete groups, given a closed Lie subgroup $H < G$, we can define the notion of cosets and define the coset space G/H as the set of equivalence classes. The following theorem shows that the coset space is actually a manifold.

Theorem 5.

(1) *Let G be a Lie group of dimension n and $H < G$ a closed Lie subgroup of dimension k. Then the coset space G/H has a natural structure of a manifold of dimension $n - k$ such that the canonical map $p\colon G \to G/H$ is a fiber bundle, with fiber diffeomorphic to H. The tangent space at $\tilde{1} = p(1)$ is given by $T_{\tilde{1}}G/H = T_1G/T_1H$.*

(2) *If H is a normal closed Lie subgroup then G/H has a canonical structure of a Lie group.*

Proof. Denote by $p\colon G \to G/H$ the canonical map. Let $g \in G$ and $\bar{g} = p(g) \in G/H$. Then the set $g \cdot H$ is a submanifold in G as it is an image of H under diffeomorphism $x \mapsto gx$. Choose a submanifold $M \subset G$ such that $g \in M$ and M is transversal to the manifold gH, i.e., $T_gG = T_ggH \oplus T_gM$. Let $U \subset M$ be a sufficiently small neighborhood of g in M. Then the set $UH := \{\, uh : u \in U, h \in H \,\}$ is open in G (which easily follows from the inverse function theorem applied to the map $U \times H \to G$). Consider $\bar{U} = p(U)$; since $p^{-1}(\bar{U}) = UH$ is open, $\bar{U}$ is an open neighborhood of $\bar{g}$ in G/H and the map $U \to \bar{U}$ is a homeomorphism. This gives a local chart for G/H and at the same time shows that $G \to G/H$ is a fiber bundle with fiber H. Now we just need to show that the transition functions between such charts are smooth and that the smooth structure does not depend on the choice of g, M.

This argument also shows that the kernel of the projection $\pi\colon T_gG \to T_{\bar{g}}G/H$ is equal to T_ggH. In particular, for $g = 1$ this gives us an isomorphism $T_{\tilde{1}}G/H \simeq T_1G/T_1H$. ■

Corollary 6. *Let H be a closed Lie subgroup of a Lie group G.*

(1) *If H is connected, then the set of connected components $\pi_0 G = \pi_0 G/H$. In particular, if H, G/H are connected, then so is G.*

(2) *If G, H are connected, then there is an exact sequence of fundamental groups*

$$\pi_2 G/H \longrightarrow \pi_1 H \longrightarrow \pi_1 G/H \longrightarrow 1.$$

This corollary follows from more general long exact sequence of homotopy groups associated with any fiber bundle.