MA166: Recitation 7 Prep

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February 25, 2016

1 Homework Problems

Section 1.1: Homework 15

Problem 1.1 (WebAssign, HW 15, # 1)**.** Use the table of integrals to evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{\cos x}{\sin^2 x - 36}\, dx.$$

Solution. First make the substitution $u = \sin x$. Then from the table, we have the formula

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| + C. \tag{1}$$

So

$$\begin{aligned}
\int \frac{\cos x}{\sin^2 x - 36} &= \int \frac{du}{u^2 - 6^2}\, du \\
&= \frac{1}{12} \ln\left|\frac{u-6}{u+6}\right| + C \\
&= \boxed{\frac{1}{12} \ln\left|\frac{\sin x - 6}{\sin x + 6}\right| + C.}
\end{aligned}$$

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Problem 1.2 (WebAssign, HW 15, # 2)**.** Use the table of integrals to evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{1}{x^2\sqrt{81x^2 + 4}}\, dx.$$

Solution. First make the substitution $u = 9x$. Then from the table, we have the formula

$$\int \frac{du}{u^2\sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C. \tag{2}$$

So

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{81x^2 + 4}}\, dx &= \frac{1}{9} \int \frac{1}{\frac{1}{81}u^2\sqrt{u^2 + 2^2}}\, dx \\
&= 9 \int \frac{1}{u^2\sqrt{u^2 + 2^2}}\, du \\
&= -\frac{9\sqrt{u^2 + 4}}{4u} + C \\
&= \boxed{-\frac{\sqrt{81x^2 + 4}}{4x} + C.}
\end{aligned}$$

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Problem 1.3 (WebAssign, HW 15, # 3)**.** Use the table of integrals to evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{\tan^3(6/z)}{z^2}\, dz.$$

Solution. First make the substitution $u = 6/z$. Then from the table, we have the formula

$$\int \tan^3 u\, du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C. \tag{3}$$

So

$$\begin{aligned}\int \frac{\tan^3(6/z)}{z^2}\, dz &= -\frac{1}{6}\int \tan^3 u\, du \\ &= -\frac{1}{12}\tan^2 u - \frac{1}{6}\ln|\cos u| + C \\ &= \boxed{-\frac{1}{12}\tan^2(6/z) - \frac{1}{6}\ln|\cos(6/z)| + C.}\end{aligned}$$

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Problem 1.4 (WebAssign, HW 15, # 4)**.** Use the table of integrals to evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{e^{2x}}{13 - e^{4x}}\, dx.$$

Solution. Make the substitution $u = e^{2x}$. Then, from the table of integrals we have the formula

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C. \tag{4}$$

So

$$\begin{aligned}\int \frac{e^{2x}}{13 - e^{4x}}\, dx &= \frac{1}{2}\int \frac{1}{13 - u^2}\, du \\ &= \frac{1}{4\sqrt{13}}\ln\left|\frac{u+\sqrt{13}}{u-\sqrt{13}}\right| + C. \\ &= \boxed{\frac{1}{4\sqrt{13}}\ln\left|\frac{e^{2x}+\sqrt{13}}{e^{2x}-\sqrt{13}}\right| + C.}\end{aligned}$$

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Problem 1.5 (WebAssign, HW 15, # 5)**.** Use the trapezoidal rule, the midpoint rule, and Simpson's rule to approximate the given integral with specified value n. (Round your anwser to six decimal places).

$$\int_1^4 2\sqrt{\ln}\, dx, \qquad n = 6$$

Solution. These problems take too long, just remember the formulas

$$\int_a^b f(x)\,dx \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right] \tag{5}$$

for the trapezoidal rule,

$$\int_a^b \approx \frac{b-a}{N}\sum_{n=0}^{N-1} f\Big(a+\tfrac{n(b-a)}{N}\Big) \tag{6}$$

for the midpoint rule and

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\big[f(a)+4f\big(\tfrac{a+b}{2}\big)+f(b)\big]. \tag{7}$$

for Simpson's rule. ☺

Problem 1.6 (WebAssign, HW 15, # 6)**.** Use the trapezoidal rule, the midpoint rule, and Simpson's rule to approximate the given integral with specified value n. (Round your anwser to six decimal places).

$$\int_0^4 e^{2\sqrt{t}}\,dt, \qquad n=8.$$

Solution. Look at my comment above. ☺

Section 1.2: Homework 16

Problem 1.7 (WebAssign, HW 16, # 1)**.** Determine whether the integral is convergent or divergent.

$$\int_{-\infty}^0 \frac{dx}{4-7x}.$$

Solution. Let's develop some strategies for attacking these problems. First of all, we like to work with positive numbers whenever possible, so let's make the substitution $u=-x$. This changes the bounds from $(-\infty,0)$ to $(\infty,0)$. Now, remember that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \tag{8}$$

Thus, our integral turns into

$$\int_{-\infty}^0 \frac{dx}{4-7x} = -\int_0^\infty \frac{-du}{4+7u} = \int_0^\infty \frac{du}{4+7u}.$$

Finding the integral of this, we have

$$\int_0^\infty \frac{du}{4+7u} = \frac{1}{7}\ln|4+7u|\Big|_0^\infty.$$

Now, what happens as $u\to\infty$? The value of $\frac{1}{7}\ln|4+7u|$ gets bigger and bigger so the integral diverges. ☺

Problem 1.8 (WebAssign, HW 16, # 2)**.** Determine whether the integral is convergent or divergent.

$$\int_2^\infty e^{-9p}\,dp.$$

Solution. Compute the integral

$$\int_2^\infty e^{-9p}\,dp = -\frac{1}{9}e^{-9p}\Big|_2^\infty = -\frac{1}{9}e^{-9p} + \frac{1}{9}e^{-18}.$$

What happens as $p \to \infty$? The value of $-\frac{1}{9}e^{-9p}$ gets closer and closer to 0 so the integral converges and is equal to

$$\boxed{\frac{1}{9e^{18}}}.$$

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Problem 1.9 (WebAssign, HW 16, # 3)**.** Determine whether the integral is convergent or divergent.

$$\int_{-\infty}^\infty 3xe^{-x^2}\,dx.$$

Solution. Remember that if we have three points $a < c < b$ in the real line, we can rewrite the integral of $f(x)$ as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{9}$$

Rewrite the integral above as

$$\int_{-\infty}^\infty 3xe^{-x^2}\,dx = \int_{-\infty}^0 3xe^{-x^2}\,dx + \int_0^\infty 3e^{-x^2}\,dx - \underbrace{\int_0^\infty 3ue^{-u^2}\,du}_{I_1} + \underbrace{\int_0^\infty 3xe^{-x^2}\,dx}_{I_2}$$

where $u = -x$. For the same reasons as the previous problem, I_1 and I_2 converge. Moreover, it is easy to see that $I_1 = I_2$ so the integral $\boxed{-I_1 + I_2 = 0}$. ☺

Problem 1.10 (WebAssign, HW 16, # 4)**.** Determine whether the integral is convergent or divergent.

$$\int_1^\infty 37\frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx.$$

Solution. Compute the integral by using the substitution $u = \sqrt{x}$

$$\int_1^\infty 37\frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^\infty 74e^{-u}\,du = -74e^{-u}\Big|_1^\infty$$

which converges for similar reasons as problem 1 from this homework. Thus, the integral is $\boxed{74/e}$. ☺

Problem 1.11 (WebAssign, HW 16, # 5)**.** Determine whether the integral is convergent or divergent.

$$\int_{-\infty}^{\infty} 31\cos\pi t\; dt.$$

Solution. ☺

Problem 1.12 (WebAssign, HW 16, # 6)**.** Determine whether the integral is convergent or divergent.

$$\int_{2}^{\infty} \frac{dv}{v^2+5v-6}.$$

Solution. Factor and use partial fractions

$$\begin{aligned}
\int_{2}^{\infty} \frac{dv}{v^2+5v-6} &= \int_{2}^{\infty} \frac{dv}{(v+6)(v-1)} \\
&= \int_{2}^{\infty} \left(\frac{-\frac{1}{7}}{v+6} + \frac{\frac{1}{7}}{v-1}\right) dv \\
&= -\frac{1}{7}\ln|v+6| + \frac{1}{7}\ln|v-1|\Big|_{2}^{\infty} \\
&= \frac{1}{7}\ln\left|\frac{v-1}{v+6}\right|\Bigg|_{2}^{\infty} \\
&= -\frac{1}{7}\ln\left|\frac{1}{8}\right| \\
&= \boxed{\frac{1}{7}\ln 8.}
\end{aligned}$$

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Problem 1.13 (WebAssign, HW 16, # 7)**.** Determine whether the integral is convergent or divergent.

$$\int_{1}^{\infty} 25\frac{\ln x}{x}\; dx.$$

Solution. Use the substitution $u = \ln x$ then rewrite the integral and compute

$$\begin{aligned}
\int_{1}^{\infty} 25\frac{\ln x}{x}\; dx &= 25\int_{0}^{\infty} u\; du \\
&= 25\,\frac{1}{2}u^2\Big|_{0}^{\infty} \\
&= \frac{25}{2}\,u^2\big|_{0}^{\infty}
\end{aligned}$$

which clearly diverges as $u \to \infty$ since u^2 keeps getting bigger and bigger. ☺

Problem 1.14 (WebAssign, HW 16, # 8)**.** Determine whether the integral is convergent or divergent.

$$\int_{-2}^{3} \frac{45}{x^4}\, dx.$$

Solution. Rewrite the integral as

$$\int_{-2}^{3} \frac{45}{x^4}\, dx = \int_{-2}^{0} \frac{45}{x^4}\, dx + \int_{0}^{3} \frac{45}{x^4}\, dx = \underbrace{45\int_{-2}^{0} \frac{dx}{x^4}}_{I_1} + \underbrace{45\int_{3}^{0} \frac{du}{u^4}}_{I_2}$$

where we let $u = -x$. Now, computing I_1 and I_2 we have

$$I_1 = -\frac{45}{3}\, x^{-3}\Big|_{-2}^{0}.$$

It is clear that as $x \to 0$, the integral grows closer and closer to $-\infty$. The same is true of I_2 so the integral diverges. ☺

Problem 1.15 (WebAssign, HW 16, # 9)**.** Determine whether the integral is convergent or divergent.

$$\int_{0}^{9} \frac{7}{\sqrt[3]{x-1}}\, dx.$$

Solution. By straight calculation

$$\begin{aligned}
\int_{0}^{9} \frac{7}{\sqrt[3]{x-1}}\, dx &= 7\int_{0}^{0} (x-1)^{-1/3}\, dx \\
&= \frac{21}{2}\sqrt{x-1}\Big|_{0}^{9} \\
&= \frac{21}{2}(4-0) \\
&= \boxed{48.}
\end{aligned}$$

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Section 1.3: Homework 17

Problem 1.16 (WebAssign, HW 17, # 1)**.** Find the exact length of the curve.

$$y = 2 + 2x^{3/2}, \qquad 0 \le x \le 1.$$

Solution. Remember the arclength formula

$$L(a,b) = \int_{a}^{b} \sqrt{1 + \frac{dy}{dx}^2}\, dx. \tag{10}$$

So let's find dy/dx, $dy/dx = 3x^{1/2}$ by the power rule so we have

$$\begin{aligned}
L(0,1) &= \int_0^1 \sqrt{1 + \left(3x^{1/2}\right)^2}\, dx \\
&= \int_0^1 \sqrt{1+9x}\, dx \\
&= \int_0^1 (1+9x)^{1/2}\, dx \\
&= \frac{2}{3}(1+9x)^{3/2}\Big|_0^1 \\
&= \boxed{\frac{2}{3}\left(10\sqrt{10}+1\right).}
\end{aligned}$$

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Problem 1.17 (WebAssign, HW 17, # 2)**.** Find the exact length of the curve.

$$x = \frac{\sqrt{y}(y-3)}{3}, \qquad 9 \le y \le 25.$$

Solution. By straight computation,

$$\frac{dx}{dy} = \frac{1}{3}\left(\tfrac{3}{2}y^{1/2} - \frac{3}{2}y^{-1/2}\right) = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}.$$

Thus, the archlength is

$$\begin{aligned}
L(9,25) &= \int_9^{25} \sqrt{1 + \left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}\right)^2}\, dy \\
&= \frac{1}{2}\int_9^{25} \left(y^{1/2} + y^{-1/2}\right) dy \\
&= \frac{1}{3}y^{3/2} + y^{1/2}\Big|_9^{25} \\
&= \boxed{\frac{104}{3}.}
\end{aligned}$$

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Problem 1.18 (WebAssign, HW 17, # 3)**.** Find the exact length of the curve.

$$y = \ln|\sec x|, \qquad 0 \le x \le \frac{\pi}{3}.$$

Solution. Compute

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x.$$

So

$$\begin{aligned}
L(0,\pi/3) &= \int_0^{\pi/3} \sqrt{1+\tan^2 x}\; dx \\
&= \int_0^{\pi/3} \sec x \; dx \\
&= \ln|\sec x + \tan x|\Big|_0^{\pi/3} \\
&= \boxed{\ln 3}.
\end{aligned}$$

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Problem 1.19 (WebAssign, HW 17, # 4)**.** Find the exact length of the curve.

$$y = \ln\left(1-x^2\right), \qquad 0 \le x \le \frac{1}{3}.$$

Solution. Find

$$\frac{dy}{dx} = -\frac{2x}{1-x^2}.$$

Then, by partial fractions etc., we have

$$\begin{aligned}
L(0,1/3) &= \int_0^{1/3} \sqrt{1+\frac{4x^2}{\left(1-x^2\right)^2}}\; dx \\
&= \int_0^{1/3} \frac{1+x^2}{1-x^2}\; dx \\
&= \int_0^{1/3} \left(-1+\frac{2}{1-x^2}\right) \\
&= \int_0^{1/3} -1+\frac{1}{1+x}+\frac{1}{1-x}\; dx \\
&= -x+\ln\left|\frac{1+x}{1-x}\right|\Bigg|_0^{1/3} \\
&= \boxed{\ln(2)-\frac{1}{3}.}
\end{aligned}$$

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Problem 1.20 (WebAssign, HW 17, # 5)**.** Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y = x^3, \qquad 0 \le x \le 3.$$

Solution. Using the cylinder method, set $y = x^3$ to your length and the change in the arc will be $\sqrt{1+(dy/dx)^2}\; dx = \sqrt{1+3x^2}$ so our surface area will be

$$S(0,3) = \int_0^3 2\pi x^3 \sqrt{1+9x^4}\; dx$$

make the substitution $u = 1 + 9x^4$

$$= \frac{2\pi}{36} \int_1^{730} \sqrt{u}\, du$$
$$= \frac{\pi}{18} \frac{2}{3} u^{3/2} \Big|_1^{730}$$
$$= \boxed{\frac{\pi}{27}(730\sqrt{730} - 1).}$$

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Problem 1.21 (WebAssign, HW 17, # 6)**.** Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y = \sin\Big(\frac{\pi x}{3}\Big), \qquad 0 \le x \le 3.$$

Solution. Find

$$\frac{dy}{dx} = \frac{\pi}{3} \cos \frac{\pi x}{3}.$$

So

$$S(0,3) = 2\pi \int_0^3 y\sqrt{1 + (dy/dx)^2}\, dx$$
$$= 2\pi \int_0^3 \sin\Big(\frac{\pi x}{3}\Big) \sqrt{1 + \frac{\pi^2}{9} \cos^2\Big(\frac{\pi x}{3}\Big)}\, dx$$

make the substitution $u = (\pi/3)\cos(\pi x/3)$

$$= -\frac{18}{\pi} \int_{\pi/3}^{-\pi/3} \sqrt{1 + u^2}\, du$$
$$= \frac{18}{\pi} \int_{-\pi/3}^{\pi/3} \sqrt{1 + u^2}\, du$$
$$= \frac{36}{\pi} \int_0^{\pi/3} \sqrt{1 + u^2}\, du$$
$$= \frac{36}{\pi} \int_0^{\pi/3} \sqrt{1 + u^2}\, du$$

use a trig substitution

$$= \frac{36}{\pi} \left[\frac{u\sqrt{1 + u^2}}{2} - \frac{1}{2} \ln\Big(u + \sqrt{1 + u^2}\Big) \right]_0^{\pi/3}$$
$$= \boxed{6\sqrt{1 + \frac{\pi^2}{9}} + \frac{18}{\pi}\left(\frac{\pi}{3} + \sqrt{1 + \frac{\pi^2}{9}}\right).}$$

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Problem 1.22 (WebAssign, HW 17, # 7)**.** The given curve is rotated about the y-axis. Find the area of the resulting surface.

$$y = \sqrt[3]{x}, \qquad 2 \le y \le 4.$$

Solution. Express x in terms of y, $x = y^3$ and find

$$\frac{dx}{dy} = 3y^2.$$

Then

$$\begin{aligned} S &= 2\pi \int_2^4 x\sqrt{1 + (dx/dy)^2}\, dy \\ &= 2\pi \int_2^4 y^3\sqrt{1 + 9y^4}\, dy \\ &= \frac{2\pi}{36} \int_2^4 (36y^3)\sqrt{1 + 9y^4}\, dy \end{aligned}$$

make the substitution $u = 1 + 9y^4$, then

$$\begin{aligned} &= \frac{\pi}{18} \int_{145}^{2305} \sqrt{1 + u}\, du \\ &= \frac{\pi}{27} \left[u^{3/2}\right]_{145}^{2305} \\ &= \boxed{\frac{\pi}{2}\left(2305\sqrt{2305} - 145\sqrt{145}\right).} \end{aligned}$$

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Problem 1.23 (WebAssign, HW 17, # 8)**.** The given curve is rotated about the y-axis. Find the area of the resulting surface.

$$y = 4 - x^2, \qquad 0 \le x \le 5.$$

Solution. Fird

$$\frac{dy}{dx} = 2x.$$

Then

$$S(0, 5) = 2\pi \int_0^5 x\sqrt{1 + 4x^2}\, dx$$

make the substitution $u = 1 + 4x^2$, then

$$\begin{aligned} &= \frac{\pi}{4} \int_1^{101} \sqrt{u}\, du \\ &= \frac{3\pi}{8} \left[u^{3/2}\right]_1^{101} \\ &= \boxed{\frac{3\pi}{8}(101\sqrt{101} - 1).} \end{aligned}$$

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2 Exam II Problems

Relevant exam problems

Problem 2.1 (Spring 2014, # 8)**.** Which of the following improper integrals converge?

I. $\int_0^\infty xe^{-x^2}\,dx$

II. $\int_{-\infty}^{\infty} \frac{dx}{x}$

III. $\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}}$.

Solution. First, let's compute the integrals I, II and III. Here's I

$$\begin{aligned}
I_1 &= \int_0^\infty xe^{-x^2}\,dx \\
&= \frac{1}{2}\int_0^\infty e^{-u}\,du \\
&= \left[-\frac{1}{2}e^{-u}\right]_0^\infty \\
&= \frac{1}{2}.
\end{aligned}$$

Here's II

$$\begin{aligned}
I_2 &= \int_{-\infty}^{\infty} \frac{dx}{x} \\
&= \int_{-\infty}^{0} \frac{dx}{x} + \int_0^\infty \frac{dx}{x} \\
&= -\int_{\infty}^{0} \frac{du}{u} + \int_0^\infty \frac{dx}{x} \\
&= \int_0^\infty \frac{du}{u} + \int_0^\infty \frac{dx}{x} \\
&= [\ln u]_0^\infty + [\ln x]_0^\infty
\end{aligned}$$

this clearly divereges since $\ln x \to \infty$ as $x \to 0$ and $lnx \to \infty$ as $x \to \infty$. The same goes for $\ln u$. You can't win. Here's III

$$\begin{aligned}
I_3 &= \int_{-1}^{1} \frac{dx}{\sqrt[3]{x}} \\
&= \int_{-1}^{1} x^{1/3}\,dx \\
&= \frac{3}{4}\left[x^{4/3}\right]_{-1}^{1}
\end{aligned}$$

$$= 0.$$

Hence, I and III converge, but III does not. ☺

Problem 2.2 (Spring 2014, # 9)**.** Find the exact length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/3$.

Solution. First find the derivative with respect to x

$$\frac{dy}{dx} = \tan x.$$

Then

$$\begin{aligned}
S(0, \pi/3) &= \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\; dx \\
&= \int_0^{\pi/3} \sec x \; dx \\
&= [\ln|\sec x + \tan x|]_0^{\pi/3} \\
&= \ln\left(2 + \sqrt{3}\right) - \ln(1 - 0) \\
&= \boxed{\ln\left(2 + \sqrt{3}\right).}
\end{aligned}$$

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Problem 2.3 (Spring 2015, # 9)**.** If the upper part of the ellipse $y^2/4 + x^2/16 = 1$ is revolved around the x-axis to generate an ellipsoid S, then the surface area of S is given by?

Solution. Let's just consider the portion on that lies on the first quadrant and double the result. Using the cylindrical method, we have to rewrite

$$y = \sqrt{4 - \frac{x^2}{4}}$$

and

$$\frac{dy}{dx} = \frac{x}{4\sqrt{4 - \frac{x^2}{4}}}.$$

$$\begin{aligned}
S(0, 1) &= \int_0^1 \sqrt{1 + \left(\frac{x}{4\sqrt{4 - \frac{x^2}{4}}}\right)^2}\; dx \\
&= \int_0^1 \sqrt{1 + \frac{x^2}{16(4 - x^2/4)^2}}\; dx \\
&= \int_0^1 \sqrt{1 + \frac{x^2}{8(16 - x^2)^2}}\; dx
\end{aligned}$$

$$= \int_0^1 \sqrt{\frac{8(16-x^2)^2+x^2}{8(16-x^2)^2}}\ dx$$

$$=$$

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