MA571 Homework 10

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Problem 10.1 (Munkres §52, Ex. 2)

Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. By Theorem 52.1, the paths α and β induce a group homomorphism $\hat{\alpha}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $\hat{\beta}\colon \pi_1(X, x_1) \to \pi_1(X, x_2)$, respectively. We want to show therefore that the induced homomorphism $\hat{\gamma} = \widehat{\alpha * \beta}$ is in fact equivalent to the composition $\hat{\beta} \circ \hat{\alpha}$. Let $[f]$ be a loop based at x_0 then

$$\begin{aligned}\hat{\gamma}([f]) &= \widehat{\alpha * \beta}([f])\\ &= \left[\,\overline{\alpha * \beta}\,\right] * [f] * [\alpha * \beta]\\ &= \left[\bar{\beta} * \bar{\alpha}\right] * [f] * [\alpha] * [\beta]\end{aligned}$$

by the well-definedness of the path product operation, we have

$$= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta]$$

by associativity of the path product,

$$\begin{aligned}&= [\bar{\beta}] * ([\bar{\alpha}] * [f] * [\alpha]) * [\beta]\\ &= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta]\end{aligned}$$

where $\alpha([f])$ is a loop based at x_1 so

$$\begin{aligned}&= \hat{\beta}(\hat{\alpha}([f]))\\ &= \big(\hat{\beta} \circ \hat{\alpha}\big)([f]).\end{aligned}$$

Thus, the following diagram commutes

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\hat{\alpha}"] \arrow[dr, "\hat{\gamma} = \widehat{\alpha * \beta}"'] & \pi_1(X, x_1) \arrow[d, "\hat{\beta}"] \\ & \pi_1(X, x_2). \end{tikzcd}$$

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Problem 10.2 (Munkres §52, Ex. 3)

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is Abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. $\Longrightarrow$ Suppose that $\pi_1(X, x_0)$ is Abelian. Then for any class of loops about x_0, say $[f]$ and $[g]$, the product $[f] * [g] = [g] * [f]$. Let α and β be paths from x_0 to x_1. Then the induced map on fundamental groups $\hat{\alpha}$ and $\hat{\beta}$ yield isomorphism by Theorem 52.1 so that the map $\hat{\bar{\beta}} \circ \hat{\alpha}$ is an automorphism of $\pi_1(X, x_0)$. Moreover, we have

$$\begin{aligned}
\hat{\bar{\beta}} \circ \hat{\alpha}([f]) &= \hat{\bar{\beta}}\big(\hat{\alpha}([f])\big) \\
&= \hat{\bar{\beta}}\big([\bar{\alpha}] * [f] * [\alpha]\big) \\
&= [\beta] * \big([\bar{\alpha}] * [f] * [\alpha]\big) * [\bar{\beta}]
\end{aligned}$$

by associativity of the path product, we may rewrite the above expression as

$$= ([\beta] * [\bar{\alpha}]) * [f] * ([\alpha] * [\bar{\beta}])$$

noting that $[\beta] * [\bar{\alpha}]$ and $[\alpha] * [\bar{\beta}]$ are loops based at x_0, since $\pi_1(X, x_0)$ is Abelian, we have

$$\begin{aligned}
&= ([\beta] * [\bar{\alpha}]) * ([\alpha] * [\bar{\beta}]) * [f] \\
&= [e_{x_0}] * [f] \\
&= [f].
\end{aligned}$$

Thus, $\hat{\bar{\beta}} \circ \hat{\alpha} = \mathrm{id}_{\pi_1(X, x_0)}$, i.e., $\hat{\alpha} = \hat{\beta}$.

$\Longleftarrow$ Let f and g be loops about x_0. Then, since X is path connected, we claim that f and g are homotopic to the path product $\alpha_1 * \bar{\beta}_1$ and $\alpha_2 * \bar{\beta}_2$ where α_i, β_i are paths from x_0 to x_1. More precisely, split f into the paths $f_1 = f(t/2)$ and $f_2 = f((t+1)/2)$; it is clear that $f = f_1 * f_2$. Let $x_2 \coloneqq f_1(1)$ then there exists a path α from x_2 to x_1 since X is path connected. Now we claim that the following

$$H(x, t) \coloneqq f_1(x) * \alpha(tx) * \bar{\alpha}((1-t)x) * f_2(x)$$

is a homotopy from $f = f_1 * f_2$ to the extended loop $\tilde{f} = f_1 * \alpha * \bar{\alpha} * f_2$.

Proof of claim. It is clear that H is continuous since it is a path products and multiplication on the unit interval I is continuous so tx is continuous. Lastly, $H(x, 0) = f_1(x) * \alpha(0) * \bar{\alpha}(0) * f_2(x)$ and $H(x, 1) = f_1(x) * \alpha(x) * \bar{\alpha}(x) * f_2(x)$. ♣

Now, let $f \simeq_p \alpha_1 * \bar{\beta}_1$ and $g \simeq_p \alpha_2 * \bar{\beta}_2$ where α_i, β_i are paths from x_0 to x_1. Then we have

$$\begin{aligned}
[f] * [g] * [\bar{f}] * [\bar{g}] &= [\alpha_1 * \bar{\beta}_1] * [\alpha_2 * \bar{\beta}_2] * [\overline{\alpha_1 * \bar{\beta}_1}] * [\overline{\alpha_2 * \bar{\beta}_2}] \\
&= [\alpha_1 * \bar{\beta}_1] * [\alpha_2 * \bar{\beta}_2] * [\beta_1 * \bar{\alpha}_1] * [\beta_2 * \bar{\alpha}_2] \\
&= [\alpha_1] * [\bar{\beta}_1] * [\alpha_2] * [\bar{\beta}_2] * [\beta_1] * [\bar{\alpha}_1] * [\beta_2] * [\bar{\alpha}_2] \\
&= \hat{\bar{\alpha}}_1\big([\bar{\beta}_1] * [\alpha_2] * [\bar{\beta}_2] * [\beta_1]\big) * [\beta_2] * [\alpha_2] \\
&= \hat{\bar{\alpha}}_1\big(\hat{\beta}_2([\alpha_2] * [\bar{\beta}_2])\big) * [\beta_2] * [\bar{\alpha}_2] \\
&= [\alpha_2] * [\bar{\beta}_2] * [\beta_2] * [\bar{\alpha}_2]
\end{aligned}$$

$$
\begin{aligned}
&= [\alpha_2] * [e_{x_0}] * [\bar{\alpha}_2] \\
&= [\alpha_2] * [\bar{\alpha}_2] \\
&= [e_{x_0}].
\end{aligned}
$$

Thus, $\pi_1(X, x_0)$ is Abelian. ■

Problem 10.3 (Munkres §52, Ex. 4)

Let $A \subset X$; suppose $r\colon X \to A$ is continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A.) If $a_0 \in A$, show that

$$r_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

Proof. Suppose f is a loop in A based at a. Then, extending the codomain of f to X, f is a loop in X based at a. Then, since $r(a) = a$ for all a and $f(I) \subset A$, $r_*([f]) = [p(f)] = [f]$ so r_* is surjective. ■

Problem 10.4 (Munkres §53, Ex. 6)

Show that if X is path connected, the homomorphism induced by a continuous map is independent of the base point, up to isomorphisms of the groups involved. More precisely, let $h\colon X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps "commutes"

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ {\scriptstyle \hat{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\beta}} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1). \end{array}$$

Proof. Unpacking the expression on the left, we have the following sequence of equalities: Let f be a loop in X based at x_0 then

$$\begin{aligned} \big(\hat{\beta} \circ (h_{x_0})_*\big)([f]) &= \hat{\beta}\big((h_{x_0})_*([f])\big) \\ &= \hat{\beta}\big([h(f)]\big) \\ &= [\bar{\beta}] * [h(f)] * [\beta] \\ &= \big[\, \overline{h \circ \alpha} \,\big] * [h(f)] * [h \circ \alpha] \\ &= [h \circ \bar{\alpha}] * [h(f)] * [h \circ \alpha] \end{aligned}$$

but since $(h_{x_1})_*$ is a homomorphism

$$\begin{aligned} &= (h_{x_0})_*(\bar{\alpha} * f * \alpha) \\ &= (h_{x_1})_*\big(\hat{\alpha}([f])\big) \\ &= \big((h_{x_1})_* \circ \hat{\alpha}\big)([f]). \end{aligned}$$

■

Problem 10.5 (Munkres §55, Ex. 1)

Show that if A is a retract of B^2, then every continuous map $f\colon A \to A$ has a fixed point.

Proof. Suppose that A is a retract of B^2. Let $r\colon B^2 \to A$ one such retraction. If $f\colon A \to A$ is a continuous map, then $f \circ r$ is a continuous map, by Theorem 18.2(c), from B^2 to A. Expanding the codomain of f to B^2, i.e., composing with the canonical injection $\iota\colon A \hookrightarrow B^2$, we have a continuous mapping $\tilde{f}\colon B^2 \to B^2$ that coincides with f in A. Then, by Theorem 55.6, $\tilde{f}$ has a fixed point, i.e., $\tilde{f}(x) = x$ for some $x \in B^2$. By the Brouwer fixed-point theorem for the disc, there exists a point $x \in B^2$ such that $\tilde{f}(x) = x$, but $\operatorname{im} \tilde{f} = \operatorname{im} f \subset A$ so $x \in A$. It follows that f has a fixed point. ■

Problem 10.6 (Munkres §55, Ex. 2)

Show that if $h\colon S^1 \to S^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof. By Lemma 55.3 $h\colon S^1 \to X$ is nulhomotopic if and only if h extends to a continuous function $k\colon B^2 \to S^1$. By Theorem 18.2(e), we may expand the codomain of k to B^2 giving us a map $\tilde{k}\colon B^2 \to B^2$. By the Brouwer fixed-point theorem, there exists a point $x \in B^2$ such that $\tilde{k}(x) = x$. But $x \in \tilde{k}(B^2) \subset S^1$ so $x \in S^1$. Since k restricts to h on S^1, it follows that $h(x) = x$.

Since $S^1 \subset \mathbf{R}^2$, write $h(x) = (h_1(x), h_2(x))$ where $h_1 = \pi_1 \circ h$ and $h_2 = \pi_2 \circ h$. Let $H(x,t)\colon S^1 \times I \to S^1$ be the map

$$H(x,t) := \begin{bmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{bmatrix} \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}$$

Note that $H(x,t)$ is continuous since multiplication is continuous in $\mathbf{R}$ by Theorem 21.5 and each component is continuous so by Theorem 18.4 together with 18.2(e) $H(x,t)$ is continuous. Moreover, we have

$$\begin{aligned} H(x,0) &= ((\cos 0)h_1(x) - (\sin 0)h_2(x), (\sin 0)h_1(x) + (\cos 0)h_2(x)) \\ &= h(x) \\ H(x,1) &= ((\cos \pi)h_1(x) - (\sin \pi)h_2(x), (\sin \pi)h_1(x) + (\cos \pi)h_2(x)) \\ &= (-h_1(x), -h_2(x)) \\ &= -(h_1(x), h_2(x)) \\ &= -h(x). \end{aligned}$$

Therefore, $h \simeq -h$. By the previous argument, there exists $x \in S^1$ such that $-h(x) = x$, i.e., $h(x) = -x$. ■

Problem 10.7 ((A))

Prove that every m-manifold is locally path-connected.

Proof. Suppose M is an m-manifold. Let $x \in M$ and U' be an arbitrary neighborhood of x. Then, since M is a manifold, there exists an open neighborhood U of x that is homeomorphic to an open subset, say V, of $\mathbf{R}^m$. Let $h\colon U \to V$ be a homeomorphism. Then $h(U \cap U')$ is open in U so by Theorem 16.2, $h(U \cap U')$ is open in $\mathbf{R}^m$. Therefore, for sufficiently small values of $\delta > 0$, we have the inclusion $B(h(x), \delta) \subset h(U \cap U')$. We claim that $W \coloneqq h^{-1}(B(h(x), \delta))$ is a path-connected neighborhood of x contained in U'.

Containment is clear for h is a bijection and we have that $W \subset U \cap U' \subset U'$. By Example 3 in Munkres §24 we know that open balls in $\mathbf{R}^m$ are path-connected therefore given $y_0 = h(x_0), y_1 = h(x_1) \in h(W)$, there exists a path $p\colon I \to h(W)$ with $p(0) = y_0$ and $p(1) = y_1$. Then $q \coloneqq \left((h^{-1})\big|_{h(W)} \circ p\right)\colon I \to W$ is a path in W from x_0 to x_1. It is clear that q is continuous by Theorem 18.2(c) since it is a composition of continuous functions (where $(h^{-1})\big|_{h(W)}$ is continuous by Theorem 18.2(d) since it is the restriction of a continuous function). Lastly, $q(0) = x_0$ and $q(1) = x_1$. Since x_0 and x_1 were arbitrary, it follows that W is path-connected. Therefore, M is locally path-connected. ∎

Problem 10.8 ((B))

Prove that every m-manifold is regular.

Proof. Let $x \in M$ and U' be an arbitrary neighborhood of x. Then, since M is a manifold, there exists an open neighborhood U of x that is homeomorphic to an open subset, say V, of $\mathbf{R}^m$. Let $h\colon U \to V$ be a homeomorphism. Then $h(U \cap U')$ is open in U so by Theorem 16.2, $h(U \cap U')$ is open in $\mathbf{R}^m$. Since $\mathbf{R}^m$ is regular, by Lemma 31.1(a), there exist an neighborhood W of $h(x)$ such that $\overline{W} \subset h(U \cap U')$. We claim that $h^{-1}(W) \subset U'$ is a neighborhood of x such that $\overline{h^{-1}(W)} \subset U'$.

That $h^{-1}(W)$ is contained in U' is clear since h is a homeomorphism and W is contained in the image of $U \cap U'$. It is also easy to see that $h^{-1}(\overline{W}) \subset U'$ since, again h is a homeomorphism and $\overline{W} \subset U \cap U'$. Now, since h is a homeomorphism it is a closed map so $h^{-1}(\overline{W})$ is a closed subset of U containing $h^{-1}(W)$. Therefore, by Lemma B, $\overline{h^{-1}(W)} \subset h^{-1}(\overline{W}) \subset U'$. Thus, by Lemma 31.1, M is regular. ∎

Problem 10.9 ((C))

Prove that there is no 1-1 continuous function $\iota\colon S^1 \to \mathbf{R}$. You may assume any fact about trigonometric functions. (Note: this shows in particular that there is no $\iota\colon S^1 \to \mathbf{R}$ with $p \circ \iota$ equal to the identity map, where p is the map in the note on the Fundamental Group of the Circle.)

Proof. Seeking a contradiction, suppose that $\iota\colon S^1 \to \mathbf{R}$ is a continuous injection. Then ι cannot be a surjection since Theorem 26.6 would imply $S^1 \approx \mathbf{R}$, but S^1 is compact whereas $\mathbf{R}$ is not, contradicting Theorem 26.5. Therefore, by homework Problem 2.8 (Munkres §18, Ex. 4) ι is an imbedding of S^1 into $\mathbf{R}$ and $\operatorname{im}\iota = [a, b]$ for some $a, b \in \mathbf{R}$ with $a < b$, since S^1 is compact and connected. Define $\tilde{\iota} = \iota|_{[a,b]}$ then $\tilde{\iota}$ is a homeomorphism (it is continuous by Theorem 18.2(e), and bijective since $[a, b]$ is the image of S^1 under ι so is a homeomorphism by Theorem 26.6 since S^1 is compact and $[a, b]$ is Hausdorff). Now, take the point $x \coloneqq (a + b)/2$ in the interval $[a, b]$. By Lemma A, $S^1 \smallsetminus \tilde{\iota}^{-1}(x)$ and $[a, b] \smallsetminus x$ are homeomorphic. But $[a, b] \smallsetminus x$ is disconnected, in particular $[a, x), (x, b]$ are open and closed and $[a, x) \cup (x, b] = [a, b] \setminus x$ hence, form a disconnection, but $S^1 \smallsetminus \tilde{\iota}(x)$ is (path) connected: Let $(x_0, y_0) \in S^1$. We rotate the circle S^1 so that (x_0, y_0) gets moved to the point $(1, 0)$ clockwise via the the map from $\mathbf{R}^2 \to \mathbf{R}^2$

$$R_\theta(x, y) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{where } \theta = \arctan(y_0/x_0).$$

Note that

$$\begin{aligned} R_\theta(x_0, y_0) &= (\cos\theta x_0 + \sin\theta y_0, -\sin\theta x_0 + \cos\theta y_0) \\ &= \left(\frac{x_0^2}{\sqrt{x_0^2 + y_0^2}} + \frac{y_0^2}{\sqrt{x_0^2 + y_0^2}}, -\frac{x_0 y_0}{\sqrt{x_0^2 + y_0^2}} + \frac{x_0 y_0}{\sqrt{x_0^2 + y_0^2}} \right) \\ &= (1, 0) \end{aligned}$$

The map is continuous since multiplication is continuous by Theorem 21.5 and since R_θ is bijective (in particular, the matrix $\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]$ has determinant equal to 1 so is invertible), R_θ is a homeomorphism by Theorem 26.6 so by Lemma A, its restriction to S^1 is a homeomorphism onto its image $R_\theta(S^1)$. Without loss of generality, we may assume that $\iota^{-1}(x) = (1, 0)$. Take any $(x_1, y_1), (x_2, y_2) \in S^1 \smallsetminus (0, 1)$. Let $\phi_1 = \arctan(y_1/x_1)/2\pi$ and $\phi_2 = \arctan(y_2/x_2)/2\pi$. Then the map $p(t) = (\cos 2\pi q(t), \sin 2\pi q(t))$ where $q(t) = (1 - t)\phi_1 + t\phi_2$ (counterclockwise, avoiding $(0, 1)$) is a path from (x_1, y_1) to (x_2, y_2). Thus $S^1 \smallsetminus (0, 1)$ is path connected so is connected (as demonstrated by Munkres in p. 155 although he does not actually assign the result to a corollary as he should).

To recap, if ι is an injection of S^1 into $\mathbf{R}$ Theorem 26.6 tells us that $S^1 \approx \iota(S^1)$. But we have just shown that the circle S^1 minus a point is (path) connected whereas the closed interval $[a, b]$ minus a point is not contradicting Lemma A. Therefore, no such imbedding may exists for supposing so has led to a contradiction. ∎

Problem 10.10 ((D))

Prove Proposition C from the note on the Fundamental Group of the Circle.

Proof. The statement of Proposition C is as follows:

Proposition C. *Let A be a connected space and let $a \in A$. If two continuous functions $\alpha, \beta \colon A \to \mathbf{R}$ have the property that $\alpha(a) = \beta(a)$ and $p \circ \alpha = p \circ \beta$ (where $p(u) = (\cos 2\pi u, \sin 2\pi u)$) then $\alpha = \beta$.*

First note that

$$\begin{aligned} p \circ \alpha(x) &= p \circ \beta(x) \\ p(\alpha(x)) &= p(\beta(x)) \\ (\cos 2\pi\alpha(x), \sin 2\pi\alpha(x)) &= (\cos 2\pi\beta(x), \sin 2\pi\beta(x)) \end{aligned}$$

by the sum-to-product identity, we have

$$\begin{aligned} \cos 2\pi\alpha(x) - \cos 2\pi\beta(x) &= -2\sin(\pi(\alpha(x) - \beta(x)))\sin(\pi(\alpha(x) + \beta(x))) \\ &= 0 \\ \sin 2\pi\alpha(x) - \sin 2\pi\beta(x) &= 2\sin(\pi(\alpha(x) - \beta(x)))\cos(\pi(\alpha(x) + \beta(x))) \\ &= 0. \end{aligned}$$

These in turn imply that $\sin(\pi(\alpha(x) - \beta(x))) = 0$ which happens if and only if $\alpha(x) - \beta(x) \in \mathbf{Z}$. By Theorem 21.5, $(\alpha - \beta)(x)$ is continuous so $(\alpha - \beta)(A) \subset \mathbf{Z} \subset \mathbf{R}$ is connected. Since $\mathbf{Z}$ the only connected subsets of $\mathbf{Z}$ are singleton sets (for suppose $B \subset \mathbf{Z}$ has two elements, say x and y with $x < y$, then $\{ z \in B \mid z \leq x \}$ and $\{ z \in B \mid z \geq y \}$ are open since $\mathbf{Z}$ has the discrete topology and union up to B hence form a separation of B). Then, since $\alpha(a) = \beta(a)$ for some $a \in A$, $0 \in (\alpha - \beta)(A)$ so $(\alpha - \beta)(A) = \{0\}$. It follows that $\alpha = \beta$. ∎