MIT OCW: Solutions to courses I find interesting

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Contents

The following solutions to MIT OCW psets are arranged in the order in which I originally attempted the courses.

1 6.041: Probabilistic Systems Analysis

Because the notation A^c is too ugly to our eyes and we will often be working with set complements we shall set aside the notation $\tilde{A}$ for the complement of A.

1.1 Problem-Set 1

PROBLEM 1.1.1. Express each of the following events in terms of the events A, B, and C as well as the operations of complementation, union, and intersection:

(a) at least one of the events A, B, C occurs;

(b) at most one of the events A, B, C occurs;

(c) none of the events A, B, C occurs;

(d) exactly one of the events A, B, C occurs;

(e) events A and B occur, but not C;

(f) either event A occurs or, if not, then B also does not occur.

In each case draw the corresponding Venn diagram.

SOLUTION. We present only one of the many possible expressions for (a)-(g) and we shall omit the finer details; suffice it to say, these are all consequences of elementary set theory. We also omit the Venn diagrams the problem is asking us to draw as it would be a bad investment of our time to trace them out using PGF/Ti*k*Z.

For part (a): the event, call it E, that at least one of A, B, C occurs is the expression

$$E = A \cup B \cup C.$$

For part (b): the event E that at most one of A, B, C occurs is the expression

$$E = [(A \cap B) \cup (A \cap C) \widetilde{\cup (B} \cap C) \cup (A \cap B \cap C)].$$

For part (c): the event E that none of A, B, C occur is the expression

$$E = A \widetilde{\cup B} \cup C$$

For part (d): the event E that all three events A, B, C occur is the expression

$$E = A \cap B \cap C.$$

For part (e): the event E that exactly one of the events A, B, C occurs is the expression

$$E = (A \cup B \cup C) \setminus [(A \cap B) \cup (A \cap C) \cup (B \cap C) \cup (A \cap B \cap C)].$$

For part (f): the event E that A and B occur, but not C is the expression

$$E = (A \cup B) \cap \tilde{C}.$$

For part (g): the event E that A occurs or, if not, then B also does not occur is the expression

$$E = A \cup (C \setminus B).$$

■

Problem 1.1.2. You flip a fair coin three times, determine the probability of the below events. Assume all sequences are equally likely.

(a) Three heads: HHH.

(b) The sequence hea, tail, head: HTH.

(c) Any sequence with two heads and one tail.

(d) Any sequence where the number of heads is greater than or eqal to the number of tails.

Solution. For part (a): Under the equally likely hypothesis, the probability of getting three heads, assuming independence of each throw, is $\frac{1}{8}$. (We can justify this by writing a table with all the possible outcomes of three tosses of a coin. There are eight of them and HHH is precisely one sample point of this sample space.)

For part (b): By the same reasoning as above, HTH has a probability of $\frac{1}{8}$ of occurring.

For part (c): There are precisely 3 such sequences HHT, HTH and THH. Therefore, the probability of any sequence with two heads and one tail occurring is $\frac{3}{8}$.

For part (d): We see almost immediately that this event is a superset of the one considered in part (c) since sequences having two tails and one head are forbidden; therefore, we expect the probability of this event to be greater than or equal to $\frac{3}{8}$. In fact, the probability is $\frac{1}{2}$ which comes about froms the addition of the event HHH to part (c). ■

Problem 1.1.3. Bob has a peculiar pair of four-sided dice. When he rolls the dice, the probability of any particular outcome is proportional to the sum of the results of each die. All outcomes that result in a particular sum are equally likely.

(a) What is the probability of the sum being even?

(b) What is the probability of Bob rolling a 2 and a 3, in any order?

Solution. For part (a): Since this question concerns only the sums and we know that the sums are uniformly distributed on $\{2, \ldots, 8\}$ from the assumptions in the statement of the problem, the probability is $\frac{4}{7}$; i.e., there are 4 even numbers that could come up from rolling the pair of dice 2, 4, 6, and 8 and there are 7 possibilities.

For part (b): The probability is $\frac{1}{21}$ and we can figure this out from the statement of the problem. There are 3 ways for the sum of the faces on the two dice to sum up to 7; it can come up 2 and 5, 3 and 4, or 1 and 6. ■

Problem 1.1.4. Alice and Bob each choose at random a number in the interval $[0, 2]$. We assume a uniform probability law under which the probability of an event is proportional to its area. Consider the following events

$$\begin{aligned} A &= \left\{ \text{the magnitude of the difference of the events is greater than } \tfrac{1}{3} \right\}, \\ B &= \left\{ \text{at least one of the numbers is greater that } \tfrac{1}{3} \right\}, \\ C &= \left\{ \text{the two numbers are equal} \right\}, \\ D &= \left\{ \text{Alice's number is greater that } \tfrac{1}{3} \right\}. \end{aligned}$$

Find the probabilities $P(B)$, $P(C)$, and $P(A \cap D)$.

Solution. Let a denote Alice's choice and b Bob's. Without loss of generality, assume $b > a$.

The probability of A that $b - a > \frac{1}{3}$ is $\frac{1}{6}$ as for any nondegenerate interval $I = [c, d]$, the probability of landing on that interval is the ratio $\frac{d-c}{2-0} = \frac{1}{2}(d - c)$.

The probability of C that $b > \frac{1}{3}$ or $a > \frac{1}{3}$ is $1 - \frac{11}{36} = \frac{25}{36}$. We can break the event into the events $b \le \frac{1}{3}$ and $a \ge \frac{1}{3}$, $a \le \frac{1}{3}$ and $b \ge \frac{1}{3}$, and $a \le \frac{1}{3}$ and $b \le \frac{1}{3}$. This results in

$$2 \cdot \tfrac{1}{6}(1 - \tfrac{1}{6}) + \tfrac{1}{6} \cdot \tfrac{1}{6} = \tfrac{11}{36}.$$

(Don't ask why I did it this way; i.e., why I found the probability of the complement instead of finding the probability of the event directly. I just happen to have computed the probability of the complement reflexively and am too lazy to fix it.)

Lastly, the probability of $A \cap D$ that alice's number is greater that $\frac{1}{3}$ and the magnitude of the difference of the events is greater that $\frac{1}{3}$ is, by Bayes' theorem,

$$\begin{aligned} P(A \cap D) &= \frac{P(D \mid A)}{P(A)} \\ &= \frac{}{1/6} \end{aligned}$$

■

PROBLEM 1.1.5. Mike and John are playing a friendly game of darts where the dart board is a disk with radius 10 inches.

Whenever a dart falls within 1 inch of the center, 50 points are scored. If the point of impact is between 1 and 3 inches from the center, 30 points are scored, if it is at a distance of 3 to 5 inches, 20 points are scored and if it is further than 5 inches, 10 points are scored.

Assume that both players are skilled enough to be able to throw the dart within the boundaries of the board.

Mike can place the dart uniformly on the board (i.e., the probability of the dart falling in a given region is proportional to its area).

(a) What is the probability that Mike scores 50 points on one throw?

(b) What is the probability of him scoring 30 points on one throw?

(c) John is right handed and twice more likely to throw in the right half of the board than it the left half. Across each half, the dart falls uniformly in that region. Answer the previous questions for John's throw.

SOLUTION. ■

PROBLEM 1.1.6. Prove that for three events A, B, and C, we have

$$P(A \cap B \cap C) \geq P(A) + P(B) + P(C) - 2.$$

SOLUTION. ■

PROBLEM 1.1.7. Consider an experiment whose sample space is the real line.

(a) Let $\{a_n\}$ be an increasing sequence of numbers that converges to a and $\{b_n\}$ a decreasing sequence of numbers that converges to b. Show that

$$\lim_{n\to\infty} P([a_n, b_n]) = P([a, b]).$$

Here, the notation $[a, b]$ stands for the closed interval $\{x : a \leq x \leq b\}$.

Note: This result seems intuitively obvious. The issue is to derive it using he axioms of probability theory.

(b) Let $\{a_n\}$ be a decreasing sequence that converges to a and $\{b_n\}$ an increasing sequence that converges to b. Is it true that

$$\lim_{n\to\infty} P([a_n, b_n]) = P([a, b])?$$

Note: You may use freely the results from the problems in the text in your proofs.

SOLUTION. ■

1.2 Problem-Set 2

1.3 Problem-Set 3

1.4 Problem-Set 4

1.5 Problem-Set 5

1.6 Problem-Set 6

1.7 Problem-Set 7

1.8 Problem-Set 8

1.9 Problem-Set 9

1.10 Problem-Set 10

1.11 Problem-Set 11

2 18.440 – Introduction to Probability

Here are solutions to some of the exercises for this class.

3 18.175 – Probability Theory

4 6.002 – Electronic Circuits

4.1 Problem-Set 1

PROBLEM 4.1.1. Suppose

SOLUTION. ∎

4.2 Problem-Set 2

4.3 Problem-Set 3

4.4 Problem-Set 4

4.5 Problem-Set 5

4.6 Problem-Set 6

4.7 Problem-Set 7

4.8 Problem-Set 8

4.9 Problem-Set 9

4.10 Problem-Set 10

4.11 Problem-Set 11

5 6.003 – Signals and Systems

5.1 Problem-Set 1

5.2 Problem-Set 2

5.3 Problem-Set 3

5.4 Problem-Set 4

5.5 Problem-Set 5

5.6 Problem-Set 6

5.7 Problem-Set 7

5.8 Problem-Set 8

5.9 Problem-Set 9

5.10 Problem-Set 10

5.11 Problem-Set 11

5.12 Problem-Set 12

6 6.004 – Computation Structures

6.1 Problem-Set 1

6.2 Problem-Set 2

6.3 Problem-Set 3

6.4 Problem-Set 4

6.5 Problem-Set 5

6.6 Problem-Set 6

6.7 Problem-Set 7

6.8 Problem-Set 8

6.9 Problem-Set 9

6.10 Problem-Set 10

6.11 Problem-Set 11

6.12 Problem-Set 12

7 18.112 – Functions of a Complex Variable

8 18.102 – Introduction to Functional Analysis

9 18.755 – Differential Geometry

10 6.006: Introduction to Algorithms

10.1 Problem-Set 1

10.2 Problem-Set 2

10.3 Problem-Set 3

10.4 Problem-Set 4

10.5 Problem-Set 5

10.6 Problem-Set 6

10.7 Problem-Set 7

10.8 Problem-Set 8

10.9 Problem-Set 9

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10.11 Problem-Set 11

10.12 Problem-Set 12