MA 544: Homework 6

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Problem 6.1 (Wheeden & Zygmund §4, Ex. 4)

Let f be defined and measurable in $\mathbf{R}^n$. If T is a nonsingular linear transformation of $\mathbf{R}^n$, show that $f(T\mathbf{x})$ is measurable. [If $E_1 = \{\, \mathbf{x} \mid f(\mathbf{x}) > a \,\}$ and $E_2 = \{\, \mathbf{x} \mid f(T\mathbf{x}) > a \,\}$, show $E_2 = T^{-1}E_1$.]

Proof. We shall proceed as in the hint. Suppose f is measurable. Then for every finite a in $\mathbf{R}$, the set $\{f > a\}$ is measurable. Since $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is nonsingular it is invertible. Let $E_1 \coloneqq \{\, \mathbf{x} \mid f(\mathbf{x}) > a \,\}$ and $E_2 \coloneqq \{\, \mathbf{x} \mid f(T\mathbf{x}) > a \,\}$. Then, we show that $E_2 = T^{-1}E_1$.

Let $\mathbf{x} \in E_2$. Then $f(T\mathbf{x}) > a$. Since T is nonsingular, it is surjective so for some $\mathbf{x}' \in \mathbf{R}^n$, $\mathbf{x} = T^{-1}\mathbf{x}'$ so $f\big(TT^{-1}\mathbf{x}'\big) = f(\mathbf{x}') > a$. Thus, $\mathbf{x} \in T^{-1}E_1$. Similarly, if $\mathbf{x} \in T^{-1}E_1$ then $\mathbf{x} = T^{-1}\mathbf{x}'$ for some $\mathbf{x}' \in E_2$. Then $f(T\mathbf{x}) = f\big(TT^{-1}\mathbf{x}'\big) = f(\mathbf{x}') > a$. Thus, $\mathbf{x} \in E_2$. It follows that since E_1 is measurable for all a and T^{-1} is Lipschitz, by 3.33, E_2 is measurable for all a so $f \circ T$ is measurable. ∎

Problem 6.2 (Wheeden & Zygmund §4, Ex. 7)

Let f be usc and less that $+\infty$ on a compact set E. Show that f is bounded above on E. Show also that f assumes its maximum on E, i.e., that there exists $\mathbf{x}_0 \in E$ such that $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x} \in E$.

Proof. First we shall demonstrate boundedness. Suppose that f is usc on E. By 4.14 (i), the set $\{\, \mathbf{x} \mid f(\mathbf{x}) < a \,\}$ is relatively open. Set $\mathcal{G} := \{G_k\}_{k=1}^{\infty}$ where $G_k := \{\, \mathbf{x} \mid f(\mathbf{x}) < k \,\}$. Then $\mathcal{G}$ is a cover of E by relativeley open sets so there exists a finite subcover $\{G_{k_\ell}\}_{\ell=1}^{N}$ of E by relatively open sets. Set $M := \max\{k_1, ..., k_N\}$. Then $f(\mathbf{x}) < M$ for all $\mathbf{x} \in E$.

Also by 4.14, the set $\{\, \mathbf{x} \mid f(\mathbf{x}) \geq a \,\}$ is relatively closed in E. Consider the collection of relatively closed sets $\{H_{\mathbf{x}}\}_{\mathbf{x} \in E}$ where $H_{\mathbf{x}} := \{\, \mathbf{x}' \mid f(\mathbf{x}') \geq f(\mathbf{x}) \,\}$. Moreover, if we take any finite collection of the $H_{\mathbf{x}}$'s, say $H_{\mathbf{x}_1}, ..., H_{\mathbf{x}_N}$, then $\bigcap_{k=1}^{N} H_{\mathbf{x}_k} \neq \emptyset$ since $f(\mathbf{x}) \geq M$ for all $\mathbf{x} \in H_{\mathbf{x}_i}$ where $M := \min\{f(\mathbf{x}_1), ..., f(\mathbf{x}_N)\}$. Thus, $\{H_{\mathbf{x}}\}$ has the finite intersection property, so, since E is compact, the intersection $H := \bigcap_{\mathbf{x} \in E} H_{\mathbf{x}} \neq \emptyset$. Let $\mathbf{x}_0 \in H$. Then $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x} \in E$, as desired. ■

Problem 6.3 (Wheeden & Zygmund §4, Ex. 8)

(a) Let f and g be two functions which are usc at $\mathbf{x}_0$. Show that $f+g$ is usc at $\mathbf{x}_0$. Is $f-g$ usc at $\mathbf{x}_0$? When is fg usc at $\mathbf{x}_0$?

(b) If $\{f_k\}$ is a sequence of functions are usc at $\mathbf{x}_0$, show that $\inf f_k(\mathbf{x})$ is usc at $\mathbf{x}_0$.

(c) If $\{f_k\}$ is a sequence of functions which are usc at $\mathbf{x}_0$ and which converge uniformly near $\mathbf{x}_0$, show that $\lim f_k$ is usc at $\mathbf{x}_0$.

Proof. (a) Suppose that f and g are usc at $\mathbf{x}_0$. Then, by definition, given $M > f(\mathbf{x}_0), g(\mathbf{x}_0)$ there exists $\delta_f, \delta_g > 0$ such that $f(\mathbf{x}_1), g(\mathbf{x}_2) < M/2$ for all $|\mathbf{x}_1 - \mathbf{x}_0| < \delta_f, |\mathbf{x}_2 - \mathbf{x}_0| < \delta_g$. Set $\delta := \min\{\delta_f, \delta_g\}$. Then for any $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ we have

$$\begin{aligned}|f(\mathbf{x}) + g(\mathbf{x}) - (f(\mathbf{x}_0) + g(\mathbf{x}_0))| &= |(f(\mathbf{x}) - f(\mathbf{x}_0)) + (g(\mathbf{x}) - g(\mathbf{x}_0))| \\ &\le |f(\mathbf{x}) - f(\mathbf{x}_0)| + |g(\mathbf{x}) - g(\mathbf{x}_0)| \\ &< \frac{M}{2} + \frac{M}{2} = M.\end{aligned}$$

Thus, $f+g$ is usc.

For the second part of this problem, we provide a counter example. Recall the functions u_1 and u_2 from the text

$$u_1(x) := \begin{cases} 0 & \text{if } x < x_0 \\ 1 & \text{if } x \ge x_0 \end{cases} \qquad u_2(x) := \begin{cases} 0 & \text{if } x \ne x_0 \\ 1 & \text{if } x = x_0 \end{cases}. \tag{1}$$

Consider the difference $u := u_1 - u_2$ which is piecewise defined to be

$$u(x) := \begin{cases} 0 & \text{if } x \le x_0 \\ 1 & \text{if } x > x_0 \end{cases}. \tag{2}$$

Now, note that u being usc at x_0 implies that for $1/2 > f(x_0) = 0$ there exists $\delta > 0$ such that $f(x) < 1/2$ for all $x \in B_\delta(x_0)$. But for any $x = x_0 + \delta'$ in $B_\delta(x_0)$, for some $0 < \delta' < \delta$, $u(x) = 1 > 1/2$. Thus, u is not usc at x_0.

A sufficient condition is that $f, g \ge 0$ for all $\mathbf{x}$. For if f and g are usc, then $\overline{\lim}\, f_k(\mathbf{x}) \le f(\mathbf{x}_0)$ and $\overline{\lim}\, g_k(\mathbf{x}) \le g(\mathbf{x}_0)$ for all sequences $\mathbf{x} \to \mathbf{x}_0$. Thus, we have

$$\overline{\lim}\, fg(\mathbf{x}) \le \big(\overline{\lim}\, f\big)\big(\overline{\lim}\, g\big) \le fg(\mathbf{x}_0).$$

Thus, fg is usc at $\mathbf{x}_0$.

(b) Suppose that $\{f_k\}$ is a sequence of functions that are usc at $\mathbf{x}_0$. Set $f := \inf f_k$. Then for every $\varepsilon > 0$, $f_k(\mathbf{x}_0) < f(\mathbf{x}_0) + \varepsilon$. Since f_k is usc, for every sequence $\mathbf{x} \to \mathbf{x}_0$, we have $\overline{\lim}\, f_k(\mathbf{x}) \le f(\mathbf{x}_0)$ and $\overline{\lim}\, g_k(\mathbf{x}) \le g(\mathbf{x}_0)$.

$$f(\mathbf{x}) \le f_k(\mathbf{x}) \le f(\mathbf{x}_0) + \varepsilon. \tag{3}$$

Now, let $\varepsilon \to 0$ and we see that f is usc at $\mathbf{x}_0$.

(c) By near, we will assume topological proximity, i.e., on some open ball $B_\delta(\mathbf{x}_0)$ where $\delta > 0$. Suppose $f_k \to f$ uniformly on $B_\delta(\mathbf{x}_0)$. Then, for every $\varepsilon > 0$ there exists a positive integer N such that $n \geq N$ implies $|f_k(\mathbf{x}) - f(\mathbf{x})| < \varepsilon$. Fix $\varepsilon > 0$. Then, by uniform convergence of f_k on $B_\delta(\mathbf{x}_0)$, we have

$$f_k(\mathbf{x}) - \frac{\varepsilon}{2} < f(\mathbf{x}) < f_k(\mathbf{x}) + \frac{\varepsilon}{2}. \tag{4}$$

Since the f_k's are usc on $\mathbf{x}_0$, for each k there exists $\delta_k > 0$ such that $f_k(\mathbf{x}) < f_k(\mathbf{x}_0) + \varepsilon/2$ whenever $\mathbf{x} \in B_{\delta_k}(\mathbf{x}_0)$. Thus, we have

$$f(\mathbf{x}) < f_k(\mathbf{x}) + \frac{\varepsilon}{2} < \Big(f_k(\mathbf{x}_0) + \frac{\varepsilon}{2}\Big) + \frac{\varepsilon}{2} = f_k(\mathbf{x}_0) + \varepsilon \tag{5}$$

whenever $\mathbf{x} \in B_{\delta'}(\mathbf{x}_0)$ where $\delta' \coloneqq \min\{\delta, \delta_k\}$. Now, taking the lim sup on both sides of the inequality (5), we have

$$\overline{\lim}\, f(\mathbf{x}) \leq f(\mathbf{x}_0) + \varepsilon.$$

Thus, letting $\varepsilon \to 0$, we see that f is usc at $\mathbf{x}_0$. ■