MA571 Problem Set 5

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Problem 5.1 (Munkres §23, Ex. 3)

Let $\{A_\alpha\}$ be a collection of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup A_n$ is connected.

Proof. ■

Problem 5.2 (Munkres §23, Ex. 6)

Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X \setminus A$, then C intersects ∂A.

Proof. ∎

Problem 5.3 (Munkres §23, Ex. 7)

Is the space $\mathbf{R}_\ell$ connected? Justify your answer.

Proof. ∎

Problem 5.4 (Munkres §23, Ex. 9)

Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that

$$(X \times Y) \setminus (A \times B)$$

is connected.

Proof. ■

Problem 5.5 (Munkres §24, Ex. 1(ac))

(a) Show that no two of the spaces $(0, 1)$, $(0, 1]$ and $[0, 1]$ are homeomorphic. [*Hint:* What happens if you remove a point from each of these spaces?]
(c) Show $\mathbf{R}^n$ and $\mathbf{R}$ are not homeomorphic if $n > 1$.

Proof. ■

Problem 5.6 (Munkres §24, Ex. 2)

Let $f\colon S^1 \to \mathbf{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

Proof. ■

Problem 5.7 (Munkres §25, Ex. 2(b))

(b) Consider $\mathbf{R}^\omega$ in the uniform topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same component of $\mathbf{R}^\omega$ if and only if the sequence

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, ...)$$

is bounded. [*Hint:* It suffices to consider the case where $\mathbf{y} = \mathbf{0}$.]

Proof. ■

Problem 5.8 (Munkres §25, Ex. 4)

Let X be locally path connected. Show that every connected open set in X is path connected.

Proof. ■

Problem 5.9 (Munkres §25, Ex. 6)

A space X is said to be *weakly locally path connected at x* if for every neighborhood U of x, there is a connected subspace of X contained in U that contains a neighborhood of x. Show that if X is weakly locally connected at each of its points, then X is locally connected. [*Hint:* H]

Proof. ■

Problem 5.10 (A)

Let X be a topological space. The quotient space $(X \times [0,1])/(X \times 0)$ is called the *cone* of X and denoted CX.

Prove that if X is homeomorphic to Y then CX is homeomorphic to CY (*Hint:* There are maps in both directions).

Proof. ∎

Problem 5.11 (Extra problem)

Notation: for positive integers i, n, I, N, let us write $(i, n) \gg (I, N)$ if $i > I$ and $n > N$.

Theorem 13. *A sequence $\{\mathbf{x}_n\}$ in $\mathbf{R}^\omega$ converges to $\mathbf{0}$ in the box topology if and only if two conditions hold:*

(i) for each k, $\lim_{n\to\infty} x_n^{(k)} = 0$, and
(ii) there is a pair (I, N) with $x_n^{(k)} = 0$ whenever $(i, n) \gg (I, N)$.

Proof. ■