MA 598 (Algebraic Geometry): Homework 1

Carlos Salinas

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Problem 1.1 (5-lemma)

Given a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0, \end{array}$$

suppose f and h are isomorphisms. Prove that g is an isomorphism.

Solution. ■

Problem 1.2 (Snake lemma)

Given the diagram with exact rows

$$\begin{tikzcd}
& \operatorname{Ker} f \arrow[r] \arrow[d] & \operatorname{Ker} g \arrow[r] \arrow[d] & \operatorname{Ker} h \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
& \operatorname{Coker} f \arrow[r] & \operatorname{Coker} g \arrow[r] & \operatorname{Coker} h, &
\end{tikzcd}$$

show that the sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} g \longrightarrow \operatorname{Coker} h \longrightarrow 0$$

is exact.

Solution. ■

Problem 1.3

Given an exact sequence of modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

fix any $M \in R$-**Mod**.

(a) Show that

$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C)$$

is exact.

(b) Show that

$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M)$$

is exact.

Solution. ∎

Problem 1.4

A short exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{p} 0$$

splits if there is a homomorphism $s\colon C \to B$ called the *splitting* such that $p \circ s = \mathrm{id}_C$. In which case, we can put

$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C) \longrightarrow 0.$$

Solution. ∎

Problem 1.5

Find an example which shows that $\mathrm{Hom}(-, M)$ is *not* exact.

Solution. Consider the following example: let $A = B = M = \mathbf{Z}$ and $C = \mathbf{Z}/2\mathbf{Z}$ then the sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{f=2} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0$$

is exact, however Hom ■