MA571 Problem Set 1

Carlos Salinas

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Problem 1.1 (Munkres §2, 1(a,b).)

Let $f\colon A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$.

(a) Show that $A_0 \subset f^{-1}(f(A_0))$ and that equality holds if f is injective.
(b) Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

Proof. (a). First, we will show $A_0 \subset f^{-1}(f(A_0))$. Let $x \in A_0$. Then $f(x) \in f(A_0)$. By definition, $f^{-1}(f(A_0))$ is the set of those points $x_0 \in A$ such that $f(x_0) \in f(A_0)$ and in particular we see that the containment $A_0 \subset f^{-1}(f(A_0))$ holds. Thus, $x \in f^{-1}(f(A_0))$.

Now, let us suppose the map f is injective. By our former argument, we have that $A_0 \subset f^{-1}(f(A_0))$ therefore, we will show the reverse containment. If $y \in f(A_0)$, then $f(x) = y$ for some $x \in A_0$. By the injectivity of f, if $f(x_0) = y$ for some $x_0 \in A$, then we must have that $x_0 = x$. In particular, $x_0 \in A_0$. Thus $f^{-1}(f(A_0)) \subset A_0$ and equality $A_0 = f^{-1}(f(A_0))$ holds.

(b). First, we will show that $f(f^{-1}(B_0)) \subset B_0$. Consider the preimage $f^{-1}(B_0)$ of B_0. Let $x \in f^{-1}(B_0)$. Then $f(x) = y$ for some $y \in B_0$. Since $f(f^{-1}(B_0))$ is, by definition, the set of all points $f(x) \in B$ where $x \in f^{-1}(B_0)$ and $f(x) = y$ for $y \in B_0$, we have that $f(f^{-1}(B_0)) \subset B_0$.

Now, let us suppose the map f is surjective. Let $y \in B_0$, then there exists $x \in A$ such that $f(x) = y$. Thus, $x \in f^{-1}(B_0)$. Then $y = f(x) \in f(f^{-1}(B_0))$ (in particular $B_0 \subset f(f^{-1}(B_0))$) and we have equality $B_0 = f(f^{-1}(B_0))$. ■

Problem 1.2 (Munkres, §2, 2(g).)

Let $f\colon A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusion, unions, intersections, and differences of sets:

(g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that equality holds if f is injective.

Proof of (g). The claim is evident if A_0 and A_1 are disjoint subsets. Suppose $A_0 \cap A_1 \neq \emptyset$. Let $y \in f(A_0 \cap A_1)$. Then $y = f(x)$ for some $x \in A_0$, $x \in A_1$. Then $f(x) \in f(A_0)$ and $f(x) \in f(A_1)$ so $y \in f(A_0) \cap f(A_1)$. Thus, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

Now, suppose f is injective. Then, if $f(x) = f(x') = y$ for some $y \in B$, then $x = x'$. Let $y \in f(A_0) \cap f(A_1)$. Then $y = f(x_0)$, $y = f(x_1)$ for some $x_0 \in A_0$, $x_1 \in A_1$. But, by the injectivity of f, $x_0 = x_1$ so $x_0 \in A_0 \cap A_1$. Hence, $y \in f(A_0 \cap A_1)$ and the equality $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$ holds. ∎

Problem 1.3 (Munkres, §13, 3.)

Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{ U \mid X \setminus U \text{ is infinite or empty or all of } X \}$$

a topology on X?

Proof. Recall that $\mathcal{T}_c$ is the collection of all subsets U of X such that $X \setminus U$ is either countable or is all of X. Let us verify that $\mathcal{T}_c$ defines a topology on X. First, $\emptyset \in \mathcal{T}_c$ since $X \setminus \emptyset = X$ and $X \in \mathcal{T}_c$ since $X \setminus X = \emptyset$ is countable. Second, let $\{U_\alpha\}$, $\alpha \in A$, be an indexed family of nonempty elements of $\mathcal{T}_c$, then $X \setminus U_\alpha$ is countable for all α. Thus, by DeMorgan's laws, we have that

$$X \setminus \bigcup U_\alpha = \bigcap X \setminus U_\alpha$$

is countable (this follows from Corollary 7.3, since $\bigcap_\alpha X \setminus U_\alpha$ is a subset of U_β for all $\beta \in A$, hence it is countable). Thus, the union $\bigcup U_\alpha$ is in $\mathcal{T}_c$. Lastly, let $U_1, ..., U_n$ be nonempty elements of $\mathcal{T}_c$, then by DeMorgan's laws, we have that

$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i)$$

is countable by Theorem 7.5 since $\bigcup_{i=1}^n (X \setminus U_i)$ is a countable union of countable sets. So the finite intersection $\bigcap_{i=1}^n U_i \in \mathcal{T}_c$. Therefore, $\mathcal{T}_c$ satisfies all the properties to define a topology on X.

Now, let us consider the collection of subsets of X, $\mathcal{T}_\infty$, given above. We will show that arbitrary unions of elements of $\mathcal{T}_\infty$ are, in general, not in $\mathcal{T}_\infty$. Let $X = \mathbf{Z}_+$ and suppose that $\mathcal{T}_\infty$ defines a topology on X. Consider the collection of subsets $\{\{i\}\}_{i=1}^\infty$. $\mathbf{Z}_+ \setminus \{n\} = \{1, ..., n-1, n+1, ...\}$ is infinite hence, $\{i\} \in \mathcal{T}_\infty$ for all $i \in \{1, ...\}$. However, $\mathbf{Z}_+ \setminus \bigcup_{i=1}^\infty \{i\} = \{0\}$ is finite so $\bigcup_{i=1}^\infty \{i\} \notin \mathcal{T}_\infty$, this is a contradiction. Therefore, $\mathcal{T}_\infty$ does not define a topology on X. ∎

Problem 1.4 (Munkres, §13, 5.)

Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Proof. Let $\mathcal{T}$ be the topology generated by $\mathcal{A}$ and let $\mathcal{S}$ be the collection of all topologies $\mathcal{T}'$ that contain $\mathcal{A}$. By Lemma 13.3, it suffices to check that $\mathcal{T} = \bigcap \mathcal{T}'$. First we will show that the intersection $\bigcap \mathcal{T}'$ indeed defines a topology on X. To that end we shall prove the following lemma:

Lemma 1. *Let X be a nonempty set and let $\{\mathcal{T}_\alpha\}$ be an indexed collection of topologies on X. Then $\bigcap \mathcal{T}_\alpha$ defines a topology on X.*

Proof of Lemma 1. Let $\mathcal{T} = \bigcap \mathcal{T}_\alpha$. First, since $\emptyset \in \mathcal{T}_\alpha$ and $X \in \mathcal{T}_\alpha$ for all $\alpha \in A$, $\emptyset$ and X are in $\mathcal{T}$. Second, let $\{U_\beta\}$, $\beta \in B$, be an indexed family of nonempty elements of $\mathcal{T}$. Then, $U_\beta \in \mathcal{T}_\alpha$ for all $\beta \in B$ for all $\alpha \in A$ so $\bigcup U_\beta \in \mathcal{T}_\alpha$ for all $\alpha \in A$. Hence, $\bigcup U_\beta \in \mathcal{T}$. Lastly, let $U_1, ..., U_n$ be nonempty elements of $\mathcal{T}$. Then, $U_1, ..., U_n \in \mathcal{T}_\alpha$ for all $\alpha \in A$ so $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha$ for all $\alpha \in A$ thus, $\bigcap_{i=1}^n U_i \in \mathcal{T}$. We see that, indeed, $\mathcal{T}$ defines a topology on X. ♦

By the Lemma 1 above, it follows that $\bigcap \mathcal{T}'$ gives a topology on X. Now, it is easy to see that $\bigcap \mathcal{T}' \subset \mathcal{T}$ since $\mathcal{T} \in \mathcal{S}$ is the coarsest topology containing $\mathcal{A}$. Let us prove this fact:

Lemma 2. *Let X be a nonempty set. Let $\mathcal{A}$ be a basis for the topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology containing $\mathcal{A}$.*

Proof of Lemma 2. This can be easily proven by contradiction for suppose $\mathcal{T}$ is not the coarsest topology containing $\mathcal{A}$. Let $\mathcal{C}$ be a strictly coarser topology that contains $\mathcal{A}$. Then there exists some open set $U \in \mathcal{T}$ not in $\mathcal{C}$. Thus, $\mathcal{C}$ is not generated by $\mathcal{A}$. ♦

On the other hand we see by Lemma 13.1 that $\mathcal{T} \subset \bigcap \mathcal{T}'$ since each $\mathcal{T}' \in \mathcal{S}$ contains the basis $\mathcal{A}$ of $\mathcal{T}$, hence contains the open sets of $\mathcal{T}$.

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Problem 1.5 (Munkres, §13, 8(b).)

(b) Show that the collection

$$\mathcal{C} = \{\, [a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbf{R}$.

Proof of (b). ∎

Problem 1.6 (Munkres, §16, 1.)

Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. ■

Problem 1.7 (Munkres, §16, 4.)

A map $f\colon X \to Y$ is said to be an *open map* if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are open maps.

Proof. ■

Problem 1.8 (Munkres, §16, 6.)

Show that the countable collection

$$\{\, (a,b) \times (c,d) \mid a < b \text{ and } c < d, \text{ and } a,b,c,d \text{ are rational} \,\}$$

is a basis for $\mathbf{R}^2$.

Proof. ∎

Problem 1.9 (Munkres, §16, 9.)

Show that the dictionary order topology on the set $\mathbf{R} \times \mathbf{R}$ is the same as the product topology $\mathbf{R}_d \times \mathbf{R}$, where $\mathbf{R}_d$ denotes $\mathbf{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbf{R}^2$.

Proof. ■