MA52300 Fall 2016

Homework Assignment 3

Due Mon, Sep 19, 2016

1. Consider the initial value problem

$$u_t = \sin u_x; \quad u(x,0) = \frac{\pi}{4}\,x.$$

Verify that the assumptions of the Cauchy-Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

2. Consider the Cauchy problem for $u(x,y)$

$$\begin{aligned} &u_y = a(x,y,u)u_x + b(x,y,u) \\ &u(x,0) = 0 \end{aligned}$$

Let a and b be analytic functions of their arguments. Assume that $D^\alpha a(0,0,0) \geq 0$ and $D^\alpha b(0,0,0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

   (a) Show that $D^\beta u(0,0) \geq 0$ for all $|\beta| \leq 2$.

   (b) Prove that $D^\beta u(0,0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$. (*Hint.* Argue as in the proof of the Cauchy-Kovalevskaya theorem; i.e., use induction in β_2)

3. (Kovalevskaya's example) Show that the line $\{t = 0\}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution u of the heat equation in $\mathbb{R} \times \mathbb{R}$, with $u = \frac{1}{1+x^2}$ on $\{t = 0\}$. (*Hint:* Assume there is an analytic solution, compute its coefficients, and show that the resulting power series diverges in any neighborhood of $(0,0)$.)