MA 544: Homework 1

Carlos Salinas

January 15, 2016

Problem 1.1 (Wheeden & Zygmund §2, Ex. 1)

Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0, 1]$, but that $V[f; 0, 1] = +\infty$.

Proof. It is straightforward to see that f is bounded and continuous on $[0, 1]$. To see that f is continuous we will appeal to an ε-δ argument. Let $\varepsilon > 0$, then for $\delta >$ for any $x \in [0, 1]$, any $y \in (\delta - x, x + \delta)$ we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |x\sin(1/x) - y\sin(1/y)| \\
&= |(x - y)(\sin(1/x) - \sin(1/y))| \\
&= |x - y||\sin(1/x) - \sin(1/y)| \\
&\leq \delta \cdot |\sin(1/x) - \sin(1/y)
\end{aligned}$$

program@epstopdf

now recall the Taylor expansion of sin about 0, $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$, then

■

Problem 1.2 (Wheeden & Zygmund §2, Ex. 2)

Prove theorem (2.1).

Proof. Recall the statement of theorem (2.1):

Theorem (Wheeden & Zygmund, 2.1). *(a) If f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$.*

(b) Let f and g be of bounded variation on $[a, b]$. Then cf (for any real constant c), $f + g$, and fg are of bounded variation on $[a, b]$. Moreover, f/g is of bounded variation on $[a, b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a, b]$.

■

Problem 1.3 (Wheeden & Zygmund §2, Ex. 3)

If $[a', b']$ is a subinterval of $[a, b]$ show that $P[a', b'] \leq P[a, b]$ and $N[a', b'] \leq N[a, b]$.

Proof. ■

Problem 1.4 (Wheeden & Zygmund §2, Ex. 11)

Show that $\int_a^b f \, \mathrm{d}\phi$ exists if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|, |\Gamma'| < \delta$.

Proof. ∎

Problem 1.5 (Wheeden & Zygmund §2, Ex. 13)

Prove theorem (2.16).

Proof.

Theorem (Wheeden & Zygmund, 2.16). *(i) If $\int_a^b f\ \mathrm{d}\phi$ exists, then so do $\int_a^b cf\ \mathrm{d}\phi$ and $\int_a^b f\ \mathrm{d}(c\phi)$ for any constant c, and*

$$\int_a^b cf\ \mathrm{d}\phi = \int_a^b f\ \mathrm{d}(c\phi) = c\int_a^b f\ \mathrm{d}\phi.$$

(ii) If $\int_a^b f_1\ \mathrm{d}\phi$ and $\int_a^b f_2\ \mathrm{d}\phi$ both exist, so does $\int_a^b (f_1+f_2)\ \mathrm{d}\phi$, and

$$\int_a^b (f_1+f_2)\ \mathrm{d}\phi = \int_a^b f_1\ \mathrm{d}\phi + \int_a^b f_2\ \mathrm{d}\phi.$$

(iii) If $\int_a^b f\ \mathrm{d}\phi_1$ and $\int_a^b f\ \mathrm{d}\phi_2$ both exist, so does $\int_a^b f\ \mathrm{d}(\phi_1+\phi_2)$, and

$$\int_a^b f\ \mathrm{d}(\phi_1+\phi_2) = \int_a^b f\ \mathrm{d}\phi_1 + \int_a^b f\ \mathrm{d}\phi_2.$$

∎