MA598: Lie Groups

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CHAPTER 1

Prologue

This summer, we will be making our way through Knapp's *Lie Groups Beyond an Introduction* [2] although, I (the writer of these notes) will occasionally refer to [1] for examples.

1.1 Lie Algebras and Lie Groups

Definitions and Examples

Let $\Bbbk$ be a field. An *algebra* $\mathfrak{g}$ (not necessarily associative) is a vector space over $\Bbbk$ with product $[X, Y]$ that is linear in each variable. The algebra is a *Lie algebra* if the product satisfies also

(a) $[X, X] = 0$ for all $X \in \mathfrak{g}$ (and hence, $[X, Y] = -[Y, X]$) and
(b) the *Jacobi identity*
$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

For any algebra $\mathfrak{g}$ we get a linear map $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{End}_{\Bbbk}\mathfrak{g}$ given by

$$(\operatorname{ad} X)(Y) := [X, Y].$$

The fact that the image is in $\operatorname{End}_{\Bbbk}\mathfrak{g}$ follows from the linearity of the bracket in the second variable and the fact that ad is linear follows from the linearity of bracket in the first variable. Whenever there is possible ambiguity in what the underlying vector space is, we write $\operatorname{ad}_{\mathfrak{g}} X$ in place of $\operatorname{ad} X$.

Suppose that (a) holds in the definition of Lie algebra. Then (b) holds if and only if

$$[Z, [X, Y]] = [X, [Z, Y]] + [[Z, X], Y]$$

which holds if and only if

$$(\operatorname{ad} Z)[X, Y] = [X, (\operatorname{ad} Z)Y] + [(\operatorname{ad} Z)X, Y]. \tag{1}$$

Any D in $\operatorname{End}_{\Bbbk}\mathfrak{g}$ for which

$$D[X, Y] = [X, DY] + [DX, Y] \tag{2}$$

is called a *derivation*.

Bibliography

[1] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Graduate Texts in Mathematics. Springer, 2003.

[2] A.W. Knapp. *Lie Groups Beyond an Introduction.* Progress in Mathematics. Birkhäuser Boston, 2002.