MA52300 FALL 2016

HOMEWORK ASSIGNMENT 5 – *Solutions*

1. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(Ox), \quad x \in \mathbb{R}^n,$$

then $\Delta v = 0$.

Solution. Proof by direct computation:

$$v_{x_i}(x) = \sum_{k=1}^{n} u_{x_k}(Ox) O_{ki}$$

$$v_{x_i x_i}(x) = \sum_{k,\ell=1}^{n} u_{x_k x_\ell}(Ox) O_{ki} O_{\ell i}$$

Now, using that $\sum_{i=1}^{n} O_{ki} O_{\ell i} = \delta_{k\ell}$ (Kronecker's symbol), we will obtain

$$\begin{aligned}\Delta v(x) &= \sum_{k,\ell=1}^{n} u_{x_k x_\ell}(Ox) \sum_{i=1}^{n} O_{ki} O_{\ell i} \\ &= \sum_{k,\ell=1}^{n} u_{x_k x_\ell}(Ox) \delta_{k\ell} = \sum_{k=1}^{n} u_{x_k x_k}(Ox) = \Delta u(Ox) = 0.\end{aligned}$$

□

2. Let $n = 2$ and U be the halfplane $\{x_2 > 0\}$. Prove that

$$\sup_U u = \sup_{\partial U} u$$

for $u \in C^2(U) \cap C(\overline{U})$ which are harmonic in U, under the additional assumption that u is bounded from above in $\overline{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.) [*Hint:* Take for $\epsilon > 0$ the harmonic function

$$u(x_1, x_2) - \epsilon \log \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region $\{x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0\}$ with large a. Let $\epsilon \to 0$.]

Proof. Let $u \le M$ in U. Following the hint, consider

$$U_a := \{x_1^2 + (x_2+1)^2 < a^2, x_2 > 0\}$$

for $a > 0$ and

$$u_\epsilon(x) := u(x) - \epsilon \log \sqrt{x_1^2 + (x_2+1)^2}$$

for $\epsilon > 0$. Since $\log \sqrt{x_1^2 + (x_2+1)^2} = \log |x - (0,-1)|$ is harmonic and nonnegative in U, then u_ϵ is harmonic in a U. Moreover, we will use later that $u_\epsilon \le u$ in U. Now, since U_a is bounded, by the maximum principle

$$\max_{\overline{U}_a} u_\epsilon = \max_{\partial U_a} u_\epsilon. \tag{1}$$

The boundary ∂U_a consists of two pieces: $S_a := \partial U_a \cap \{x_2 > 0\}$ and $J_a := \partial U_a \cap \{x_2 = 0\}$. We claim that if ϵ is fixed and $a > a(\epsilon)$ is sufficiently large, then

$$\max_{\partial U_a} u_\epsilon = \max_{J_a} u_\epsilon. \tag{2}$$

Indeed, we have

$$\sup_{S_a} u_\epsilon = \sup_{S_a} u - \epsilon \log a \le M - \epsilon \log a$$

and if a is large enough ($\log a > \frac{M - u(0)}{\epsilon}$) we will have

$$\sup_{S_a} u_\epsilon < u(0) = u_\epsilon(0) \le \max_{J_a} u_\epsilon,$$

which implies (2). Now combining (1)–(2) we obtain

$$\max_{\overline{U}_a} u_\epsilon = \max_{J_a} u_\epsilon, \quad \text{for } a > a(\epsilon)$$

and letting $a \to \infty$

$$\sup_{\overline{U}} u_\epsilon = \sup_{\partial U} u_\epsilon. \tag{3}$$

Finally, letting $\epsilon \to 0$, we conclude

$$\sup_{\overline{U}} u = \sup_{\partial U} u. \tag{4}$$

Note, however, that the passage to the limit is nontrivial and has to be carefully justified. We must use that $u_\epsilon \le u$ in $\overline{U}$ and $u_\epsilon \nearrow u$ uniformly on compact subsets of $\overline{U}$. Let us show for instance, that

$$\sup_{\overline{U}} u_\epsilon \to \sup_{\overline{U}} u \quad \text{as } \epsilon \to 0.$$

Indeed, we readily have $\sup_{\overline{U}} u_\epsilon \le \sup_{\overline{U}} u$. On the other hand, for any given $\delta > 0$ we can find $\hat{a} = \hat{a}(\delta)$ large so that

$$\sup_{\overline{U}_{\hat{a}}} u \ge \sup_{\overline{U}} u - \delta.$$

Since $u_\epsilon \to u$ uniformly on $U_{\hat{a}}$, we will have

$$\sup_{\overline{U}} u_\epsilon \ge \sup_{\overline{U}_{\hat{a}}} u_\epsilon \ge \sup_{\overline{U}_{\hat{a}}} u - \delta \ge \sup_{\overline{U}} u - 2\delta,$$

for $0 < \epsilon < \epsilon(\delta)$. This justifies the passage to the limit on the left hand side of (3). Similarly, one justifies the passage to the limit on the right hand side and we arrive at (4). $\square$

3. Let $U \subset \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if

$$-\Delta v \le 0 \quad \text{in } U$$

(a) Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \dots, u^m$ are harmonic in U and

$$v := \phi(u^1, \dots, u^m).$$

Prove v is subharmonic. [*Hint:* Convexity for a smooth function $\phi(z)$ is equivalent to $\sum_{j,k=1}^m \phi_{z_j z_k}(z)\xi_j\xi_k \ge 0$ for any $\xi \in \mathbb{R}^m$.]

(b) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. (Assume that harmonic functions are C^∞.)

Solution. (a) Proof by a direct computation:

$$\begin{aligned} v_{x_i} &= \sum_{j=1}^m \phi_{z_j}(u^1, \dots, u^m) u^j_{x_i} \\ v_{x_i x_i} &= \sum_{j,k=1}^m \phi_{z_j z_k}(u^1, \dots, u^m) u^j_{x_i} u^k_{x_i} \\ &\quad + \sum_{j=1}^m \phi_{z_j}(u^1, \dots, u^m) u^j_{x_i x_i} \\ &\ge \sum_{j=1}^m \phi_{z_j}(u^1, \dots, u^m) u^j_{x_i x_i}, \end{aligned}$$

where in the last inequality we have used that

$$\sum_{j,k=1}^m \phi_{z_j z_k}(z)\xi_j\xi_k \ge 0,$$

with $z = (u^1, \dots, u^m)$ and $\xi_j = u^j_{x_i}$, $j = 1, \dots, m$. Now, summing over $i = 1, \dots, m$, we obtain

$$\Delta v \geq \sum_{j=1}^{m} \phi_{z_j}(u^1, \dots, u^m) \Delta u^j = 0.$$

Hence, v is subharmonic.

(b) Apply (a) with $m = n$, $u^j = u_{x_j}$ and $\phi(z) = |z|^2$.

$\square$