MA557 Homework 6

Carlos Salinas

October 31, 2015

Problem 6.1

Let R be a Noetherian ring and I, J R-ideals. Write $I^{\langle J\rangle} = \bigcup_{n\geq 1}(I : J^n)$, which is called the *saturation* of I with respect to J. Show:

(a) If $I = \bigcap_{i=1}^m \mathfrak{q}_i$ with $\mathfrak{q}_i$ $\mathfrak{p}_i$-primary, then $I^{\langle J\rangle} = \bigcap_{J\not\subset\mathfrak{p}_i} \mathfrak{q}_i$.

(b) $I^{\langle J\rangle}$ is the unique largest R-ideal that coincides with I locally on the open set $\operatorname{Spec}(R) \smallsetminus V(J)$.

Proof. (a) We first prove the following lemma

Lemma 1 (Atiyah & Macdonald, Ex. 1.12)**.** (i) $I \subset (I : J)$
(iv) $(\bigcap_{i=1}^m I_i : J) = \bigcap_{i=1}^m (I : J)$.

Proof of Lemma 1. Both are very short proofs: (i) If $x \in I$, then $xJ \subset I$ hence, $x \in (I : J)$. (ii) $x \in (\bigcap_{i=1}^m I : J)$ if and only if $xJ \subset I_i$ for all i if and only if $x \in \bigcap_{i=1}^m (I : J)$. ♣

Now, suppose that $\bigcap_{i=1}^m \mathfrak{q}_i$ is a primary decomposition of I. Then by Lemma 1(i), $I \subset I^{\langle J\rangle}$ since $I \subset (I : J^n)$ for all $n \geq 1$. Moreover, by Lemma 1(iv) we have that $(\bigcap_{i=1}^m \mathfrak{q}_i : J^n) = \bigcap_{i=1}^m (\mathfrak{q}_i : J^n)$ so that

$$I^{\langle J\rangle} = \bigcup_{n\geq 1}\left(\bigcap_{i=1}^m (\mathfrak{q}_i : J^n)\right)$$

(b) ■

Problem 6.2

Let R be a Noetherian ring. Show that R is reduced if and only if $\mathrm{Quot}(R)$ is a finite direct product of fields.

Proof. ■

Problem 6.3

Let R be a Noetherian ring and $x \in R$ an R-regular element. Show that $\operatorname{Ass}_R(R/(x^n)) = \operatorname{Ass}_R(R/(x))$ for every $n \geq 1$.

Proof. ■

Problem 6.4

Let $\varphi\colon R \to T$ be a homomorphism of rings where T is Noetherian, let ${}^a\varphi$ be the induced map on the spectra, and let N be a T-module. Show:

(a) $\operatorname{Ass}_R(N) = {}^a\varphi(\operatorname{Ass}_T(N))$.

(b) If N is finitely generated as a T-module then $\operatorname{Ass}_R(N)$ is finite.

Proof. ■

Problem 6.5

Let K be a field that is a finitely generated $\mathbb{Z}$-algebra. Show that K is a finite field.

Proof. $\blacksquare$

Problem 6.6

Let k be a Noetherian ring, R a finitely generated k-algebra, and $\mathrm{Aut}_k(R)$ the group of k-algebra automorphisms of R. For a subgroup G of $\mathrm{Aut}_k(R)$ write $R^G = \{\, x \in R \mid \sigma(x) = x \text{ for every } \sigma \in G \,\}$, which is called the *ring of invariants* of G. Show that if G is finite then R^G is a finitely generated k-algebra (and hence a Noetherian ring).

Proof. ∎