MA544: Qual Problems

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April 8, 2016

1 Notes

Notes based off of Wheeden and Zygmund's *Measure and Integral* book.

1.1 Exam 1 Review

This is all of the material we covered before exam 1.

Introductory material I should have known from 504.

If $\mathcal{F}$ is a countable (i.e., finite or countably infinite), it will be called a *sequence of sets* and denoted $\mathcal{F} \coloneqq \{ E_k : k = 1, 2, ... \}$. The corresponding union and intersection will be written $\bigcup_k E_k$ and $\bigcap_k E_k$. A sequence $\{E_k\}$ of sets is said to *increase* to $\bigcup_k E_k$ if $E_k \subset E_{k+1}$ for all k and to *decrease* to $\bigcap_k E_k$ if $E_k \supset E_{k+1}$ for all k; we use the notation $E_k \nearrow \bigcap_k E_k$ and $E_k \searrow \bigcap_k E_k$ to denote these two possibilities. If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets, we define

$$\limsup E_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k, \qquad \liminf E_k = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k, \tag{1}$$

noting that the sets $U_j \coloneqq \bigcup_{k=j}^{\infty} E_k$ and $V_j \coloneqq \bigcap_{k=j}^{\infty} E_k$ satisfy $U_j \searrow \limsup E_k$ and $V_j \nearrow \liminf E_k$. Then $\limsup E_k$ consists of those points of $\mathbf{R}^n$ that belong to infinitely many E_k and $\liminf E_k$ to those that belong to all E_k for $k \geq k_0$ (where k_0 may vary from point to point). Thus $\liminf E_k \subset \limsup E_k$.

If E_1 and E_2 are two sets, we define $E_1 \setminus E_2$ by $E_1 \setminus E_2 \coloneqq E_1 \cap \complement E_2$ and call it the *difference* of E_1 and E_2 or the *relative complement* of E_2 in E_1. We will often have occasion to use *de Morgan laws*, which govern relations between complements, unions, and intersections; these state that

$$\complement\left(\bigcup_{E\in\mathcal{F}} E\right) = \bigcap_{E\in\mathcal{F}} \complement E, \qquad \complement\left(\bigcap_{E\in\mathcal{F}} E\right) = \bigcup_{E\in\mathcal{F}} \complement E, \tag{2}$$

and are easily verified.

If $\mathbf{x} \in \mathbf{R}^n$, we say that a sequence $\{\mathbf{x}_k\}$ *converges* to $\mathbf{x}$, or that $\mathbf{x}$ is the *limit point* of $\{\mathbf{x}_k\}$, if $\|\mathbf{x} - \mathbf{x}_k\| \to 0$ as $k \to \infty$. We denote this by writing either $\mathbf{x} = \lim_{k\to\infty} \mathbf{x}_k$ or $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$. A point $\mathbf{x} \in \mathbf{R}^n$ is called a *limit point of a set* E if it is the limit point of a sequence of distinct points of E. A point $\mathbf{x} \in E$ is called a *isolated point* of E if it is not the limit point of any sequence in E (excluding the trivial sequence $\{\mathbf{x}_k\}$ where $\mathbf{x}_k = \mathbf{x}$ for all $k \in \mathbf{N}$). It follows that $\mathbf{x}$ is isolated if and only if there is a $\delta > 0$ such that $\|\mathbf{x} - \mathbf{y}\| > \delta$ for every $\mathbf{y} \in E$, $\mathbf{y} \neq \mathbf{x}$.

For sequences $\{x_k\}$ in $\mathbf{R}$, we will write $\lim_{k\to\infty} x_K = \infty$, or $x_k \to \infty$ as $k \to \infty$, if given $M > 0$ there is an integer N such that $x_k \geq M$ whenever $k \geq M$.

A sequence $\{\mathbf{x}_k\}$ in $\mathbf{R}^n$ is called a *Cauchy sequence* if given $\varepsilon > 0$ there exists an integer N such that $\|\mathbf{x}_k - \mathbf{x}_\ell\| < \varepsilon$ for all $k, \ell \geq N$. $\mathbf{R}^n$ is a complete metric space, i.e., every Cauchy sequence in $\mathbf{R}^n$ converges to a point of $\mathbf{R}^n$.

A set $E \subset E_1$ is said to be *dense* in E_1 if for every $\mathbf{x}_1 \in E_1$ and $\varepsilon > 0$ there is a point $\mathbf{x} \in E$ such that $0 < \|\mathbf{x} - \mathbf{x}_1\| < \varepsilon$. Thus, E is dense in E_1 if every point of E_1 is a limit point of E. If $E = E_1$, we say E is *dense in itself.* As an example, the set of limit points of $\mathbf{R}^n$ each of whose coordinates is a rational number is dense in $\mathbf{R}^n$. Since this set is also countable, it follows that $\mathbf{R}^n$ is *separable*, by which we mean that $\mathbf{R}^n$ has a countable dense subset.

For nonempty subsets E of $\mathbf{R}$, we use the standard notation $\sup E$ and $\inf E$ for the *supremum* (*least upper bound*) and *infimum* (*greatest lower bound*) of E. In case $\sup E$ belong to E, it will be called $\max E$; similarly, $\inf E$ will be called $\min E$ if it belongs to E.

If $\{a_k\}_{k=1}^{\infty}$ is a sequence of points in $\mathbf{R}$, let $b_j = \sup_{k\geq j} a_k$ and $c_j = \inf_{k\geq j} a_k$, $j = 1, 2, \ldots$. Then $-\infty \leq c_j \leq b_j \leq \infty$ and $\{b_j\}$ and $\{c_j\}$ are monotone decreasing and increasing, respectively; that is, $b_j \geq b_{j+1}$ and $c_j \leq c_{j+1}$. Define $\limsup_{k\to\infty} a_k$ and $\liminf_{k\to\infty} a_k$ by

$$\begin{aligned} \limsup_{k\to\infty} a_j &= \lim_{j\to\infty} b_j = \lim_{j\to\infty} \left\{ \lim_{k\geq j} a_k \right\}, \\ \liminf_{k\to\infty} a_k &= \lim_{j\to\infty} C_j = \lim_{j\to\infty} \left\{ \lim_{k\geq j} a_k \right\}. \end{aligned} \tag{3}$$

Theorem 1 (1.4). (a) *$L := \limsup_{k\to\infty} a_k$ if and only if (i) there is a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ that converges to L and (ii) if $L' > L$, there is an integer N such that $a_k < L'$ for $k \geq N$.*

(b) *$\ell := \liminf_{k\to\infty} a_k$ if and only if (i) there is a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ that converges to ℓ and (ii) if $\ell' < \ell$, there is an integer N such that $a_k > \ell'$ for $k \geq N$.*

Thus, when they are finite, $\limsup a_k$ and $\liminf a_k$ are the largest and smallest limit points of $\{a_k\}$, respectively.

We can also use the metric on $\mathbf{R}$ to define the *diameter of a set E* by letting

$$\operatorname{diam} E := \{ \|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in E \}. \tag{4}$$

If the diameter of E is finite, E is said to be *bounded*. Equivalently, E is bounded if there is a finite constant M such that $\|\mathbf{x}\| \leq M$ for all $\mathbf{x} \in E$. If E_1 and E_2 are two sets, the *distance between E_1 and E_2* is defined by

$$d(E_1, E_2) := \inf\{ \|\mathbf{x} - \mathbf{y}\| : \mathbf{x} \in E_1, \mathbf{y} \in E_2 \}. \tag{5}$$

For $\mathbf{x} \in \mathbf{R}^n$ and $\delta > 0$, the set

$$B(\mathbf{x}, \delta) := \{ \mathbf{y} : \|bfx - \mathbf{y}\| < \delta \} \tag{6}$$

is called the *open ball with center $\mathbf{x}$ and radius δ*. A point $\mathbf{x}$ of a set E is called an *interior point* of E if there exists $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset E$. The collection of all interior points of E is called the *interior* of E and denoted E°. A set E is said to be *open* if $E^\circ = E$; that is, E is open if for each $\mathbf{x} \in E$ there exists $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset E$. The empty set $\emptyset$ is open by convention. The whole space $\mathbf{R}^n$ is clearly open and $B(\mathbf{x}, \delta)$ is evidently open. We will generally denote open sets by the letter G.

A set E is called *closed* if $\complement E$ is open. Note that $\emptyset$ and $\mathbf{R}^n$ are closed. Closed sets will generally be denoted by the letter F. The union of a set E and all its limit points is called the *closure* of E and written $\bar{E}$. By the *boundary* of E, we mean $\partial E := \bar{E} \setminus E^\circ$.

Theorem 2 (1.5). (i) $B(\bar{\mathbf{x}}, \delta) = \{ \mathbf{y} : \|\mathbf{x} - \mathbf{y}\| \leq \delta \}$

(ii) *E is closed if and only if $E = \bar{E}$; that is, E is closed if and only if it contains all of its limit points.*

(iii) *$\bar{E}$ is closed, and $\bar{E}$ is the smallest closed set containing E; that is, F is closed and $E \subset F$, then $\bar{E} \subset F$.*

(The rest of this is a bunch of theorems that can be expressed in more generality from a more topological perspective. At any rate, they are very basic.)

Consider a collection $\{A\}$ of sets A. A set is said to be of *type* A_δ if it can be written as a countable intersection of sets A and of *type* A_σ if it can be written as a countable union of sets A. The most common uses of this notation are G_δ and F_σ, where $\{G\}$ denotes open sets in $\mathbf{R}^n$ and $\{F\}$ closed sets. Hence, H is of *type* G_δ if

$$H = \bigcap_k G_k, \qquad G_k \text{ open}, \tag{7}$$

and is of *type* F_σ if

$$H = \bigcap_k F_k, \qquad F_k \text{ closed}. \tag{8}$$

The complement of a G_δ set is an F_σ and vice-a-versa.

Another type of set that we have the occasion to use is the *perfect set*, by which we mean a closed set C each of whose points is a limit point of C. Thus, a perfect set is a closed set that is dense in itself.

Theorem 3 (1.9)**.** *A perfect set is uncountable.*

Other special sets that will be important are n-dimensional intervals. When $n = 1$ and $a < b$, we will use the usual notations $[a, b] := \{\, x : a \le x \le b \,\}$, $(a, b) := \{\, x : a < x < b \,\}$, etc. Whenever we use just the word interval, we generally mean closed interval. An *n-dimensional interval I* is a subset of $\mathbf{R}^n$ of the form $I := \{\, \mathbf{x} = (x_1, \ldots, x_n) : a_k \le x_k \le b_k,\ k = 1, \ldots, n \,\}$, where $a_k < b_k$, $k = 1, \ldots, n$. An interval is thus closed, and we say it has edges parallel to the coordinate axes. If the edge lengths $b_k - a_k$ are all equal, I will be called an *n-dimensional cube* with edges parallel to the coordinate axes. Cubes will usually be denoted by the letter Q. Two intervals are said to be *nonoverlapping* if their interiors are disjoint, that is, if the most they have in common is some part of their boundaries. A set equal to an interval minus will be called a *partly-open interval.* By definition, the *volume* $\mathrm{vol}(I)$ of the interval $I := \{\, (x_1, \ldots, x_n) : a_k \le x_k \le b_k,\ k = 1, \ldots, n \,\}$ is

$$\mathrm{vol}(I) := \prod_{k=1}^{n} (b_k - a_k). \tag{9}$$

More generally, if $\{\mathbf{e}_k\}_{k=1}^n$ is any given set of n vectors emanating from a point in $\mathbf{R}^n$, we will consider the closed *parallelepiped*

$$P := \left\{ \mathbf{x} : \mathbf{x} = \sum_{k=1}^{n} t_k \mathbf{e}_k,\ 0 \le t_k \le 1 \right\}. \tag{10}$$

Note that the edges of P are parallel translates of $\mathbf{e}_k$. Thus, P is an interval if the $\mathbf{e}_k$ are parallel to the coordinate axes. The *volume* $\mathrm{vol}\, P$ of P is *by definition* the absolute value of the $n \times n$ determinant having $\mathbf{e}_1, \ldots, \mathbf{e}_n$ as rows. In case P is an interval, this definition agrees with the one given earlier. A linear transformation T of $\mathbf{R}^n$ transforms a parallelapiped P into a parallelapiped P' with volume $\mathrm{vol}\, P' = |\det T|\, \mathrm{vol}\, P$. In particular, rotation of the axes in $\mathbf{R}^n$ does not change the volume of a parallelapiped.

Theorem 4 (1.10)**.** *Every open set in* $\mathbf{R}$ *can be written as a countable union of disjoint open intervals.*

Proof. Let G be an open subset of $\mathbf{R}$. For each x in G, by Zorn's lemma, we may choose a maximal interval $I_x \subset G$. Now, if $x, x' \in G$ are distinct points, then, by maximality, either $I_x = I_{x'}$ or $I_x \cap I_{x'} = \emptyset$. Clearly, $G = \bigcup_{x \in G} I_x$. Since each I_x contains a rational number, the number of distinct I_x must be countable, and the theorem follows. ■

Theorem 5 (1.11)**.** *Every open set in* $\mathbf{R}^n$*,* $n \geq 1$*, can be written as a countable union of nonoverlapping (closed) cubes. It can also be written as a countable union of disjoint partly open cubes.*

Proof. The proof is analogous to that of Theorem 1.10, but more general. Consider a lattice of points of $\mathbf{R}^n$ with integral coordinates and the corresponding net K_0 of cubes with edge length 1 and vertices. Bisecting each edge of a cube in K_0, we obtain from it 2^n subcubes of edge length 1/2. The total collection of these subcubes for every cube in K_0 forms a net K_1 of cubes. If we continue bisecting, we obtain finer and finer nets K_j of cubes such that each cube in K_j has edge length 2^{-j} and is the union of 2^n nonoverlapping cubes in K_{j+1}.

Now let G be any open set in $\mathbf{R}^n$. Let S_0 be the collection of all cubes K_0 that lie entirely in G. Let S_1 be those cubes in K_1 that lie in G but are not subcubes of any cube in S_0. More generally, for $j \geq 1$, let S_j be the cubes in K_j that lie in G but that are not subcubes of any cube in $S_0, \ldots, S_{j-1}$. If S denotes the total collection of cubes from all the S_j, then S is countable since each K_j is countable, and the cubes in S are nonoverlapping by construction. Hence, $G = \bigcup_{Q \in S} Q$, which proves the first statement.

The second part of the statement is left as an exercise to me, but I'm not interested in solving it; there is nothing to be gained from attempting a solution to it. ■

The collection $\{ Q : Q \in K_j, j = 1, 2, \ldots \}$ constructed above is called a family of dyadic cubes. In general, by *dyadic cubes*, we mean the family of cubes obtained from repeated bisection of any initial net of cubes in $\mathbf{R}^n$.

It follows from Theorem 1.10 that any closed set in $\mathbf{R}$ can be constructed by deleting a countable number of open disjoint intervals from $\mathbf{R}$.

By cover of a set E, we mean a family $\mathcal{F}$ of sets A such that $E \subset \bigcup_{A \in \mathcal{F}} A$. A *subcover* $\mathcal{F}'$ of a cover $\mathcal{F}$ is a cover with the property that $A' \in \mathcal{F}$ whenever $A' \in \mathcal{F}'$. A cover $\mathcal{F}$ is called an *open cover* if each set in $\mathcal{F}$ is open.

Theorem 6 (1.12)**.** (a) *(The Heine–Borel theorem) A set* $E \subset \mathbf{R}^n$ *is compact if and only if it is closed and bounded.*

(b) *A set* $E \subset \mathbf{R}^n$ *is compact if and only if every sequence of points in* E *has a subsequence that converges to a point of* E*.*

By a function f defined for $\mathbf{x}$ in a set $E \subset \mathbf{R}^n$, we will always mean a *real-valued* function, unless explicitly stated otherwise. By *real-valued*, we generally mean *extended real-valued*, i.e., f may take the values $\pm\infty$; if $|f(\mathbf{x})| < \infty$ for all $\mathbf{x} \in E$, we say f is *finite* (or *finite-valued*) on E. A finite function f is said to be *bounded* on E if there is a finite number M such that $|f(\mathbf{x})| \leq M$ for $\mathbf{x} \in E$; that is, f is bounded on E if $\sup |f(\mathbf{x})|$, where $\mathbf{x} \in E$, is finite. A sequence $\{f_k\}$ of functions is said to be *uniformly bounded* on E if there is a finite M such that $|f_k(\mathbf{x})| \leq M$ for $\mathbf{x} \in E$ and all k.

By the *support* of f, we mean the closure of the set where f is not zero. Thus, the support of a function is always closed. It follows that a function defined in $\mathbf{R}^n$ has *compact support* if and only if it vanishes outside some bounded set.

A function f defined onan interval I in $\mathbf{R}$ is called *monotone increasing (decreasing)* if $f(x) \leq f(y)$ $[f(x) \geq f(y)]$ whenever $x < y$ and $x, y \in I$. By *strictly* monotone increasing (decreasing), we mean that $f(x) < f(y)$ $[f(x) > f(y)]$ if $x < y$ and $x, y \in I$.

Let f be defined on $E \subset \mathbf{R}^n$ and let $\mathbf{x}_0$ be a limit point of E. Let $B'(\mathbf{x}_0, \delta) := B(\mathbf{x}_0, \delta) \setminus \{\mathbf{x}_0\}$ denote the punctured ball with center $\mathbf{x}_0$ and radius δ, and let

$$M(\mathbf{x}_0, \delta) := \sup_{\mathbf{x} \in B'(\mathbf{x}_0, \delta) \cap E} f(\mathbf{x}), \qquad m(\mathbf{x}_0, \delta) := \inf_{\mathbf{x} \in B'(\mathbf{x}_0, \delta) \cap E} f(\mathbf{x}). \tag{11}$$

As $\delta \searrow 0$, $M(\mathbf{x}_0, \delta)$ decreases and $m(\mathbf{x}_0, \delta)$ increases, and we define

$$\begin{aligned} \limsup_{\substack{\mathbf{x} \to \mathbf{x}_0 \\ \mathbf{x} \in E}} f(\mathbf{x}) &= \lim_{\delta \to 0} M(\mathbf{x}_0, \delta) \\ \liminf_{\substack{\mathbf{x} \to \mathbf{x}_0 \\ \mathbf{x} \in E}} f(\mathbf{x}) &= \lim_{\delta \to 0} m(\mathbf{x}_0, \delta). \end{aligned} \tag{12}$$

Theorem 7 (1.14)**.** (a) $M = \limsup_{\mathbf{x} \to \mathbf{x}_0; \mathbf{x} \in E} f(\mathbf{x})$ *if and only if* (i) *there exist* $\mathbf{x}_k$ *in* $E \setminus \{\mathbf{x}_0\}$ *such that* $\mathbf{x}_k \to \mathbf{x}_0$ *and* $f(\mathbf{x}_k) \to M$ *and* (ii) *if* $M' > M$, *there exists* $\delta > 0$ *such that* $f(\mathbf{x}) < M'$ *for* $\mathbf{x} \in B'(\mathbf{x}_0, \delta) \cap E$.

(b) $m = \liminf_{\mathbf{x} \to \mathbf{x}_0; \mathbf{x} \in E} f(\mathbf{x})$ *if and only if* (i) *there exist* $\mathbf{x}_k$ *in* $E \setminus \{\mathbf{x}_0\}$ *such that* $\mathbf{x}_k \to \mathbf{x}_0$ *and* $f(\mathbf{x}_k) \to m$ *and* (ii) *if* $m' < m$, *there exists* $\delta > 0$ *such that* $f(\mathbf{x}) > m'$ *for* $\mathbf{x} \in B'(\mathbf{x}_0, \delta) \cap E$.

A function f defined on a neighborhood of $\mathbf{x}_0$ is said to be *continuous* at $\mathbf{x}_0$ if $f(\mathbf{x}_0)$ is finite and $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$. If f is not continuous at $\mathbf{x}_0$, it follows that unless $f(\mathbf{x}_0)$ is infinite, either $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ does not exist or is different from $f(\mathbf{x}_0)$.

For functions on $\mathbf{R}$, we will use the notation

$$f(x_0+) := \lim_{\substack{x \to x_0 \\ x > x_0}} f(x) \qquad f(x_0-) := \lim_{\substack{x \to x_0 \\ x < x_0}} f(x). \tag{13}$$

for the *right-* and *left-hand limits* of f at x_0, when they exist. If $f(x_0+)$, $f(x_0-)$, and $f(x_0)$ exist and are finite, but f is not continuous at x_0, then either $f(x_0+) \neq f(x_0-)$ or $f(x_0+) = f(x_0-) \neq f(x_0)$. In the first case, x_0 is called a *jump discontinuity* of f and in the second, a *removable discontinuity* of f (since by changing the value of f at x_0, we can make it continuous there). Such discontinuities are said to be of the *first kind*, as distinguished from those of the *second kind*, for which either $f(x_0+)$ or $f(x_0-)$ does not exist or for which $f(x_0+)$, $f(x_0-)$ or $f(x_0)$ are infinite.

If f is defined only in a set E containing $\mathbf{x}_0$, $E \subset \mathbf{R}^n$, then f is said to be *continuous at* $\mathbf{x}_0$ *relative to to* E if $f(\mathbf{x}_0)$ is finite and either $\mathbf{x}_0$ is an isolated point of E or $\mathbf{x}_0$ is a limit point of E and $\lim_{\mathbf{x} \to \mathbf{x}_0; \mathbf{x} \in E} f(\mathbf{x}) = f(\mathbf{x}_0)$. If $E' \subset E$, a function is said to be *continuous in* E' *relative to* E if it is continuous relative to E at every point of E'.

Theorem 8 (1.15)**.** *Let* E *be a compact set in* $\mathbf{R}^n$ *and* f *be continuous in* E *relative to* E. *Then the following are true:*

(i) f *is bounded on* E, $\sup_{\mathbf{x} \in E} |f(\mathbf{x})| < \infty$.

(ii) *f attains its supremum and infimum on E; i.e., there exists $\mathbf{x}_1, \mathbf{x}_2 \in E$ such that $f(\mathbf{x}_1) = \sup_{\mathbf{x}\in E} f(\mathbf{x})$, $f(\mathbf{x}_2) = \inf_{\mathbf{x}\in E} f(\mathbf{x})$.*

(iii) *f is uniformly continuous on E relative to E; i.e., given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$ if $\|\mathbf{x} - \mathbf{y}\| < \delta$ and $\mathbf{x}, \mathbf{y} \in E$.*

Theorem 9 (1.16)**.** *Let $\{f_k\}$ be a sequence of functions defined on E that are continuous in E relative to E and that converge uniformly on E to a finite f. Then f is continuous in E relative to E.*

A *transformation T of a set $E \subset \mathbf{R}^n$ into $\mathbf{R}^n$* is a mapping $\mathbf{y} = T\mathbf{x}$ that carries points $\mathbf{x} \in E$ into points $\mathbf{y} \in \mathbf{R}^n$. If $\mathbf{y} = (y_1, \dots, y_n)$, then T can be identified with the collection of coordinate functions $y_k = f_k(\mathbf{x})$, $k = 1, \dots, n$, which are induced by T. The *image* of E under T is the set $\{\, \mathbf{y} : \mathbf{y} = T\mathbf{x} \text{ for some } \mathbf{x} \in E, \}$. T is continuous at $\mathbf{x}_0 \in E$ relative to E.

Theorem 10 (1.17)**.** *Let $\mathbf{y} = T\mathbf{x}$ be a transformation of $\mathbf{R}^n$ that is continuous in E relative to E. If E is compact, then so is the image TE.*

If f is defined and bounded on an interval $I \coloneqq \{\, \mathbf{x} : \mathbf{x} = (x_1, \dots, x_n),\ a_k \le x_k \le b_k,\ k = 1, \dots, n \,\}$ in $\mathbf{R}^n$, its Riemann integral will be denoted

$$(R) \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n) dx_1 \cdots dx_n \qquad \text{or} \qquad (R) \int_I f(\mathbf{x}) d\mathbf{x} \tag{14}$$

and is defined as follows. Partition I into a finite collection Γ of nonoverlapping intervals, $\Gamma = \{I_k\}_{k=1}^N$, and define the *norm* $\|\Gamma\|$ of Γ to by $\|\Gamma\| \coloneqq \max_k \operatorname{diam} I_k$. Select a point $\vec{\xi_k}$ in I_k for $k \ge 1$, and let

$$\begin{aligned} R_\Gamma(\vec{\xi}_1, \dots, \vec{\xi}_n) &\coloneqq \sum_{k=1}^N f(\vec{\xi}_k) \operatorname{vol}(I_k) \\ U_\Gamma(\vec{\xi}_1, \dots, \vec{\xi}_n) &\coloneqq \sum_{k=1}^N \left[\sup_{\mathbf{x}\in I} f(\mathbf{x}) \right] \operatorname{vol}(I_k) \\ L_\Gamma(\vec{\xi}_1, \dots, \vec{\xi}_n) &\coloneqq \sum_{k=1}^N \left[\inf_{\mathbf{x}\in I_k} f(\mathbf{x}) \right] \operatorname{vol}(I_k). \end{aligned} \tag{15}$$

We define the Riemann integral by saying that $A \coloneqq (R) \int_I f(\mathbf{x}) d\mathbf{x}$ if $\lim_{\|\Gamma\|\to 0} R_\Gamma$ exists and equals A; that is, if given $\varepsilon > 0$, there exists $\varepsilon > 0$ such that $|A - R_\Gamma| < \varepsilon$ for any Γ and any chosen $\{\vec{\xi_k}\}$, provided only that $\|\Gamma\| < \delta$. This definition is actually equivalent to the statement that

$$\inf_\Gamma U_\Gamma = \sup_\Gamma L_\Gamma = A. \tag{16}$$

The integral of course exists if f is continuous on I.

Let f be a a real-valued function that is defined and finite for all x in a closed bounded interval $a \le x \le b$. Let $\Gamma = \{x_0, \dots, x_m\}$ be a *partition* of $[a, b]$; that is, Γ is a collection of points x_i, $i = 0, 1, \dots m$, satisfying $x_0 = a$, $x_m = b$, and $x_{i-1} < x_i$ for $i = 1, \dots, m$. With each partition Γ we associate the sum

$$S_\Gamma = S_\Gamma[f; a, b] \coloneqq \sum_{i=1}^m |f(x_i) - f(x_{i-1})|. \tag{17}$$

The *variation of f over* $[a,b]$ is defined as

$$V = V[f;a,b] = \sup_{\Gamma} S_\Gamma, \tag{18}$$

where the supremum is taken over all partitions Γ of $[a,b]$. The variation of $V[f;a,b]$ will sometimes also be denoted by $V[a,b]$ or $V(f)$. Since $0 \le S_\Gamma < \infty$, we have $0 \le V \le \infty$. If $V < \infty$, f is said to be of *bounded variation on* $[a,b]$; if $V = \infty$, f is of *unbounded variation on* $[a,b]$.

Here are several examples

Examples 1. Suppose f is monotone in $[a,b]$. Then, clearly, each S_Γ equals $|f(b) - f(a)|$, and therefore $V = |f(b) - f(a)|$.

Examples 2. Suppose the graph of f can be split into a finite number of monotone arcs; that is, suppose $[a,b] = \bigcup_{i=1}^{k}[a_i, a_{i+1}]$ and f monotone in each $[a_i, a_{i+1}]$. Then $V = \sum_{i=1}^{k} |f(a_{i+1}) - f(a_i)|$. To see this, we use the result of the previous example and the fact, to be proved, in Theorem 2.2, that $V = \sum_{i=1}^{k} V[a_i, a_{i+1}]$.

Examples 3. Let f be defined by $f(x) \coloneqq 0$ when $x \neq 0$ and $f(0) \coloneqq 1$, and let $[a,b]$ be any interval containing 0 in its interior. Then S_Γ is either 2 or 0 depending on whether $x = 0$ is a partition point or not. Thus, $V[a,b] = 2$.

If $\Gamma = \{x_0, x_1, \ldots, x_m\}$ is a partition of $[a,b]$, let $\|\Gamma\|$, called the *norm of* Γ, be defined as the longest subinterval of Γ:

$$\|\Gamma\| \coloneqq \max_{i=1,\ldots,m} x_i - x_{i-1}. \tag{19}$$

If f is continuous on $[a,b]$ and $\{\Gamma_j\}$ is a sequence of partitions $[a,b]$ with $\|\Gamma_j\| \to 0$, we shall see in Theorem 2.9 that $V = \lim_{j\to\infty} S_{\Gamma_j}$. The example above shows that this may fail for functions that are discontinuous even at a single point: if we take f and $[a,b]$ is in the example above and choose Γ_j such that $x = 0$ is never a partition in the point, then $\lim S_{\Gamma_j} = 0$, while if we choose the Γ_j such that $x = 0$ alternatively is and is not a point, then $\lim S_{\Gamma_j}$ does not exist.

Examples 4. Let f be the *Dirichlet function*, defined by $f(x) \coloneqq 1$ for x rational and $f(x) \coloneqq 0$ for x irrational. Then, clearly, $V[a,b] = \infty$ for any interval $[a,b]$.

Examples 5. A function that is continuous on an interval is not necessarily of bounded variation on the interval. To see this, let $\{a_j\}$ and $\{d_j\}$, $j = 1, 2, \ldots$, be two monotone decreasing sequences in $(0,1]$ with $a_1 = 1$, $\lim_{j\to\infty} a_j = \lim_{j\to\infty} d_j = 0$ and $\sum d_j = \infty$. Construct a continuous function f as follows. On each subinterval $[a_{j+1}, a_j]$, the graph of f consists of sides of the isosceles with base $[a_{j+1}, a_j]$ and height d_j. Thus, $f(a_j) = 0$, and if m_j denotes the midpoint of $[a_{j+1}, a_j]$, then $f(m_j) = d_j$. If we further define $f(0) = 0$, then f is continuous on $[0,1]$. Taking Γ_k to the be the partition defined by the points 0, $\{a_j\}_{j=1}^{k+1}$, and $\{m_j\}_{j=1}^{k}$, we see that $S_\Gamma = 2\sum_{j=1}^{k} d_j$. Hence, $V[f;0,1] = \infty$.

Examples 6. A function f defined on $[a,b]$ is said to satisfy the *Lipschitz condition* on $[a,b]$, or be a *Lipschitz function* on $[a,b]$, if there is a constant C such that

$$|f(x) - f(y)| \le C|x - y|$$

for all $x, y \in [a,b]$. Such a function is clearly of bounded variation, with $V[f;a,b] \le C(b-a)$. For example, if f has a continuous derivative on $[a,b]$, or even just a bounded derivative, then (by the mean-value theorem) f satisfies the Lipschitz condition on $[a,b]$.

Theorem 11 (2.1)**.** (i) *If f is of bounded variation on $[a,b]$, then f is bounded on $[a,b]$.*

(ii) *Let f and g be of bounded variation on $[a,b]$. Then cf (for any real constant c), $f+g$, and fg are of bounded variation on $[a,b]$. Moreover, f/g is of bounded variation if there exist some $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a,b]$.*

Proof by Carlos. And the proof of these two is rather clear.

For (i), we proceed by contradiction. Suppose that f is of bounded variation on the interval $[a,b]$. Then the variation V of f over $[a,b]$ is finite. However, if f is unbounded on $[a,b]$, for every positive real number M, there exists some $x \in [a,b]$ such that $|f(x)| > M$. In particular, for any $x' \in [a,b]$ we have

$$|f(x) - f(x')| > M.$$

In turn, this tells us that $V > |f(x) - f(x')| > M$ for any partition Γ containing x, so $V = \infty$. This yields a contradiction.

For (ii), the proofs are simple. Suppose f is of bounded variation on $[a,b]$ with variation V_f. Let c be a real constant, then

$$\begin{aligned} V[cf;a,b] &= \sup_{\Gamma} \sum_{i=1}^{n} |cf(x_i) - f(x_{i-1})| \\ &= |c| \sup_{\Gamma} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &= |c| V_f \\ &< \infty. \end{aligned}$$

Hence, cf is of bounded variation on $[a,b]$. Suppose f and g are of bounded variation on $[a,b]$ with variation V_f and V_g, respectively. Then

$$\begin{aligned} V[f+g;a,b] &= \sup_{\Gamma} \sum_{i=1}^{n} |(f+g)(x_i) - (f+g)(x_{i-1})| \\ &= \sup_{\Gamma} \sum_{i=1}^{n} |(f(x_i) - f(x_{i-1})) + (g(x_i) - g(x_{i-1}))| \\ &\leq \sup_{\Gamma} \sum_{i=1}^{n} \big[|f(x_i) - f(x_{i-1})| + |g(x_i) - g(x_{i-1})|\big] \\ &= \sup_{\Gamma} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sup_{\Gamma} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \\ &= V_f + V_g \\ &< \infty. \end{aligned}$$

Hence, $f+g$ is of bounded variation. Suppose f and g are of bounded variation on $[a,b]$ with variation V_f and V_g, respectively. Then

$$\begin{aligned}
V[fg;a,b] &= \sup_{\Gamma} \sum_{i=1}^{n} |(fg)(x_i) - (fg)(x_{i-1})| \\
&= \sup_{\Gamma} \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sup_{\Gamma} \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_i) + f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sup_{\Gamma} \sum_{i=1}^{n} |(f(x_i)g(x_i) - f(x_{i-1})g(x_i)) - (f(x_{i-1})g(x_{i-1}) - f(x_{i-1})g(x_i))| \\
&= \sup_{\Gamma} \sum_{i=1}^{n} |g(x_i)||f(x_i) - f(x_{i-1})| + \sup_{\Gamma} \sum_{i=1}^{n} |f(x_{i-1})||g(x_i) - g(x_{i-1})|
\end{aligned}$$

by part (i), since f and g are b.v. on $[a,b]$, they are bounded so there exists M and N such that $|f(x)| < M$, $|g(x)| < M$ for all $x \in [a,b]$

$$\begin{aligned}
&= MV_f + NV_g \\
&< \infty.
\end{aligned}$$

Hence, fg is of bounded variation on $[a,b]$. Suppose that f and g are of bounded variation on $[a,b]$ with variation V_f and V_g, respectively. Suppose, additionally that there exists $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for all $x \in [a,b]$. Then, we have

$$\begin{aligned}
V[f/g;a,b] &= \sup_{\Gamma} \sum_{i=1}^{n} |(f/g)(x_i) - (f/g)(x_{i-1})| \\
&= \sup_{\Gamma} \sum_{i=1}^{n} \left| \frac{f(x_i)}{g(x_i)} - \frac{f(x_{i-1})}{g(x_{i-1})} \right| \\
&= \sup_{\Gamma} \sum_{i=1}^{n} \left| \frac{g(x_{i-1})f(x_i) - g(x_i)f(x_{i-1})}{g(x_i)g(x_{i-1})} \right|
\end{aligned}$$

and since we have $g(x) > \varepsilon$ for any $x \in [a,b]$, $1/g(x) < 1/\varepsilon$ for any $x \in [a,b]$, so

$$\begin{aligned}
&\leq \sup_{\Gamma} \left[\frac{1}{\varepsilon^2} \sum_{i=1}^{n} |g(x_{i-1})f(x_i) - g(x_i)f(x_{i-1})| \right] \\
&\leq \sup_{\Gamma} \Bigg[\frac{1}{\varepsilon^2} \sum_{i=1}^{n} |g(x_{i-1})f(x_i) - g(x_{i-1})f(x_{i-1}) \\
&\qquad\qquad - (g(x_i)f(x_{i-1}) - g(x_{i-1})f(x_{i-1}))| \Bigg] \\
&=
\end{aligned}$$

etc. etc. etc. ■

Theorem 12 (2.2)**.** (i) *If $[a', b']$ is a subinterval of $[a, b]$, then $V[a', b'] \leq V[a, b]$; that is, variation increases with interval.*

(ii) *If $a < c < b$, then $V[a, b] = V[a, c] + V[c, b]$; that is, variation is additive on adjacent intervals.*

Carlos's proof. (i) follows from (ii). By recursively applying part (ii), we have

$$V[a, b] = V[a, a'] + V[a', b'] + V[b', b].$$

Hence,

$$\begin{aligned} V[a', b'] &= V[a, b] - V[a, a'] - V[b', b] \\ &\leq V[a, b] \end{aligned}$$

as desired.

To see part (ii) let f be a real-valued function defined on $[a, b]$. If $V[f; a, b] = \infty$, there is nothing to show so suppose $V[f; a, b] < \infty$. Let c be a point in $[a, b]$ not equal to either endpoint a or b. ■

Proof of (ii). Let $I := [a, b]$, $I_I := [a, c]$, $I_2 := [c, b]$, $V := V[a, b]$, $V_1 := V[a, c]$, and $V_2 := V[c, b]$. If Γ_1 and Γ_2 are any partitions of I_1 and I_2, respectively, then $\Gamma = \Gamma_1 \cup \Gamma_2$ is one of I, and $S_\Gamma[I] = S_{\Gamma_1}[I_1] + S_{\Gamma_2}[I_2]$. Thus, $S_{\Gamma_1}[I_1] + S_{\Gamma_2}[I_2] \leq V$. Therefore, taking the supremum over Γ_1 and Γ_2 separately, we obtain $V_1 + V_2 \leq V$.

To show the opposite inequality, let Γ be any partition I, and let $\bar{\Gamma}$ be Γ with c adjoined. Then $S_\Gamma[I] \leq S_{\bar{\Gamma}}[I]$, and $\bar{\Gamma}$ splits into partitions Γ_1 of I_1 and Γ_2 of I_2. Thus, we have

$$S_\Gamma[I] \leq S_{\bar{\Gamma}}[I] = S_{\Gamma_1}[I_1] + S_{\Gamma_2}[I_2] \leq V_1 + V_2.$$

Therefore, $V \leq V_1 + V_2$, which completes the proof of (ii). ■

For any real number x, define

$$x^+ := \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \qquad x^- := \begin{cases} 0 & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}.$$

These are called the *positive* and *negative parts of* x, respectively, and satisfy the relations

$$x^+, x^- \geq ; \quad |x| = x^+ + x^-; \quad x = x^+ - x^-.$$

Given a finite function f on $[a, b]$ and a partition $\Gamma = \{x_i\}_{i=0}^\infty$ of $[a, b]$, define

$$P_\Gamma = P_\Gamma[f; a, b] = \sum_{i=1}^m [f(x_i) - f(x_{i-1})]^+$$

$$N_\Gamma = N_\Gamma[f; a, b] = \sum_{i=1}^m [f(x_i) - f(x_{i-1})]^-$$

Thus, P_Γ is the sum of the positive terms of S_Γ, and $-N_\Gamma$ is the sum of the negative terms of S_Γ. In particular, we have $P_\Gamma \geq 0$, $N_\Gamma \geq 0$,

$$\begin{aligned} P_\Gamma + N_\Gamma &= S_\Gamma, \\ P_\Gamma - N_\Gamma &= f(b) - f(a). \end{aligned} \tag{20}$$

1.2 Exam 2 Review

This is all of the material we covered before exam 2.

Let f be defined on E, and let $\mathbf{x}_0$ be a limit point of E in E. Then f is said to be *upper semicontinuous at* $\mathbf{x}_0$ if

$$\limsup_{\substack{\mathbf{x}\to\mathbf{x}_0\\ \mathbf{x}\in E}} f(\mathbf{x}) \le f(\mathbf{x}_0). \tag{21}$$

Note that if $f(\mathbf{x}_0)=\infty$, then f is usc at $\mathbf{x}_0$ automatically; otherwise, the statement that f is usc at $\mathbf{x}_0$ means that given any $M > f(\mathbf{x}_0)$, there exists $\delta > 0$ such that $f(\mathbf{x}) < M$ for all $\mathbf{x} \in E$ that lie in the ball $B_\delta(\mathbf{x}_0)$.

Similarly, f is said to be *lower semicontinuous at* $\mathbf{x}_0$ if $-f$ is usc at $\mathbf{x}_0$.

Theorem (4.14)**.** *A function f is usc relative to E if and only if $\{\, \mathbf{x}\in E : f(\mathbf{x}) > a \,\}$ is relatively closed (equivalently, if $\{\, \mathbf{x}\in E : f(\mathbf{x}) < a \,\}$ is relatively open) for all finite a*

Proof of theorem 4.14. Suppose that f is usc relative to E. Given a, let $\mathbf{x}_0$ be a limit point of $\{\, \mathbf{x}\in E : f(\mathbf{x}) > a \,\}$ in E. Then there exists $\mathbf{x}_k \in E$ such that $\mathbf{x}_k \to \mathbf{x}_0$ and $f(\mathbf{x}_k) \ge a$. Since f is usc at $\mathbf{x}_0$, we have $f(\mathbf{x}_0) \ge \limsup_{k\to\infty} f(\mathbf{x}_k)$. Therefore, $f(\mathbf{x}_0) \ge a$, so $\mathbf{x}_0 \in \{\, \mathbf{x}\in E : f(\mathbf{x}) > a \,\}$. Hence, $\{\, \mathbf{x}\in E : f(\mathbf{x}) > a \,\}$ is relatively closed.

Conversely, let $\mathbf{x}_0$ be a limit point of E that is in E. If f is not usc at $\mathbf{x}_0$, then $f(\mathbf{x}_0) < \infty$, and there exists M and $\{\mathbf{x}_k\}$ such that $f(\mathbf{x}_0) < M$, $\mathbf{x}_k \in E$, $\mathbf{x}_k \to \mathbf{x}_0$, and $f(\mathbf{x}_k) \ge M$. Hence, $\{\, \mathbf{x}\in E : f(\mathbf{x}) > a \,\}$ is not relatively closed since it does not contain all its limit points in $E>$ ∎

Theorem (4.17, Egorov's theorem)**.** *Suppose that $\{f_k\}$ is a sequence of measurable functions that converge a.e. in a set E of finite measure to a finite limit f. Then given $\varepsilon > 0$ there exits a closed subset F of E such that $|E \setminus F| < \varepsilon$ and $f_k \to f$ uniformly on F.*

A function f defined on a measurable set E has *property* $\mathcal{C}$ on E if given $\varepsilon > 0$, there is a closed set $F \subset E$ such that

(i) $|E \setminus F| < \varepsilon$

(ii) f is continuous relative to F.

Theorem (4.20, Lusin's theorem)**.** *Let f be defined and finite on a measurable set E. Then f is measurable if and only if it has property $\mathcal{C}$ on E.*

We start with a nonnegative function f defined on a measurable subset E of $\mathbf{R}^n$. Let's

$$\begin{aligned}\Gamma(f,E) &\coloneqq \left\{\, (\mathbf{x}, f(\mathbf{x})) \in \mathbf{R}^{n+1} : \mathbf{x}\in E,\ f(\mathbf{x}) < \infty \,\right\},\\ R(f,E) &\coloneqq \left\{\, (\mathbf{x}, y) \in \mathbf{R}^{n+1} : \mathbf{x}\in E,\ 0 \le y \le f(\mathbf{x}) \text{ if } f(\mathbf{x}) < \infty \text{ and } 0 \le y < \infty \text{ if } f(\mathbf{x}) = \infty \,\right\}.\end{aligned} \tag{22}$$

$\Gamma(f,E)$ is called the *graph of f over E* and $R(f,E)$ the *region under f over E*.

If $R(f,E)$ is measurable (as a subset of $\mathbf{R}^{n+1}$), its measure $|R(f,E)|_{\mathbf{R}^{n+1}}$ is called the *Lebesgue integral over E*, and we write

$$\int_E f(\mathbf{x})d\mathbf{x} \coloneqq |R(f,E)|_{\mathbf{R}^{n+1}}. \tag{23}$$

This is sometimes written as

$$\int_E f$$

or at times the lengthy notation

$$\int \cdots \int_E f(x_1, \dots, x_n) dx_1 \cdots dx_n$$

is convenient.

Theorem (5.1)**.** *Let f be a nonnegative function defined on a measurable set E. Then $\int_E f$ exists if and only if f is measurable.*

Lemma (5.3)**.** *If f is a nonnegative measurable function on E, $0 \leq |E| \leq \infty$, then $|\Gamma(f, E)| = 0$.*

Theorem (5.5)**.** (i) *If f and g are measurable and if $0 \leq g \leq f$ on E, $\int_E g \leq \int_E f$. In particular, $\int_E \inf f \leq \int_E f$.*

(ii) *If f is nonnegative and measurable on E and if $\int_E f$ is finite, then $f < \infty$ a.e. in E.*

(iii) *Let E_1 and E_2 be measurable and $E_1 \subset E_2$. If f is nonnegative and measurable on E_2, then $\int_{E_1} f \leq \int_{E_2} f$.*

Theorem (5.6, the monotone convergence theorem for nonnegative functions)**.** *If $\{f_k\}$ is a sequence of nonnegative functions such that $f_k \nearrow f$ on E, then*

$$\int_E f \to \int_E f.$$

Proof. By Theorem 4.12, f is measurable since it is the limit of a sequence of measurable functions. Since $R(f_k, E) \cup \Gamma(f, E) \nearrow R(f, E)$ and $|\Gamma(f, E)| = 0$, the result follows by Theorem 3.26 on the measure of a monotone convergent sequences of measurable sets. ■

Theorem (5.9)**.** *Let f be nonnegative on E. If $|E| = 0$, then $\int_E f = 0$.*

Theorem (5.10)**.** *If f and g are nonnegative and measurable on E and if $g \leq f$ a.e. in E, then $\int_E g \leq \int_E f$.*

In particular, if $f = g$ a.e. in E, then $\int_E f = \int_E g$.

Theorem (5.11)**.** *Let f be nonnegative and measurable on E. Then $\int_E f = 0$ if and only if $f = 0$ a.e. in E.*

Corollary (5.12, Chebyshev's inequality)**.** *Let f be nonnegative and measurable on E. If $a > 0$, then*

$$\frac{1}{a} \int_E f \geq |\{ \mathbf{x} \in E : f(\mathbf{x}) > a \}|.$$

Theorem (5.13)**.** *If f is nonnegative and measurable, and if c is any nonnegative constant, then*

$$\int_E cf = c \int_E f.$$

Theorem (5.14)**.** *If f and g are nonnegative and measurable, then*

$$\int_E (f+g) = \int_E f + \int_E g.$$

Corollary. *Suppose that f and φ are measurable on E, $0 \le f \le \varphi$, and $\int_E f$ is finite. Then*

$$\int_E (\varphi - f) = \int_E \varphi - \int_E f.$$

Theorem (5.16)**.** *If f_k, $k = 1, 2, \dots,$ are nonnegative and measurable, then*

$$\int_E \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_E f_k.$$

Theorem (5.17, Fatou's lemma)**.** *If $\{f_k\}$ is a sequence of nonnegative measurable functions on E, then*

$$\int_E \liminf_{k\to\infty} f_k \le \liminf_{k\to\infty} \int_E f_k.$$

Proof of Fatou's lemma. ■

Theorem (5.19, Lebesgue's dominated convergence theorem for nonnegative functions)**.** *Let $\{f_k\}$ be a sequence of nonnegative measurable functions on E such that $f_k \to f$ a.e. in E. If there exists a measurable function φ such that $f_k \le \varphi$ a.e. for all k and if $\int_E \varphi$ is finite, then*

$$\int_E f_k \longrightarrow \int_E f.$$

Theorem (5.21)**.** *Let f be measurable in E. Then f is integrable over E if and only if $|f|$ is.*

Theorem (5.22)**.** *If $f \in L^1(E)$, then f is finite a.e. in E.*

Theorem (5.24)**.** *If $\int_E f$ exists and $E = \bigcup_{k\in\mathbf{N}} E_k$ is the countable union of disjoint measurable sets E_k, then*

$$\int_E f = \sum_{k\in\mathbf{N}} \int_{E_k} f.$$

Theorem (5.25)**.** *If $|E| = 0$ or if $f = 0$ a.e. in E, then $\int_E f = 0$.*

Theorem (5.32, monotone convergence theorem)**.** *Let $\{f_k\}$ be a sequence of measurable functions on E:*

(i) *If $f_k \nearrow f$ a.e. on E and there exists $\varphi \in L^1(E)$ such that $f_k \ge \varphi$ a.e. on E for all k, then $\int_E f_k \to \int_E f$.*

(ii) *If $f_k \searrow f$ a.e. on E and there exists $\varphi \in L^1(E)$ such that $f_k \le \varphi$ a.e. on E for all k, then $\int_E f_k \to \int_E f$.*

Theorem (5.33, uniform convergence theorem)**.** *Let $f_k \in L^1(E)$ for $k \in \mathbf{N}$ and let $\{f_k\}$ converge uniformly to f on E, $|E| < \infty$. Then $f \in L^1(E)$ and $\int_E f_k \to \int_E f$.*

Theorem (5.34, Fatou's lemma). *Let $\{f_k\}$ be a sequence of measurable functions on E. If there exists $\varphi \in L^1(E)$ such that $f_k \geq \varphi$ a.e. on E for all k, then*

$$\int_E \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int_E f_k.$$

Corollary (5.35, reverse Fatou's lemma). *Let $\{f_k\}$ be a sequence of measurable functions on E. If there exits $\varphi \in L^1(E)$ such that $f_k \leq \varphi$ a.e. on E for all k, then*

$$\int_E \limsup_{k\to\infty} f_k \geq \limsup_{k\to\infty} \int_E f_k.$$

Theorem (5.36, Lebesgue's dominated convergenge theorem). *Let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If there exists $\varphi \in L^1(E)$ such that $|f_k| \leq \varphi$ such that $|f_k| \leq \varphi$ a.e. in E for all $k \in \mathbf{N}$, then $\int_E f_k \to \int_E f$.*

Corollary (5.37, bounded convergence theeorem). *Let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If $|E| < \infty$ there is a finite constant M such that $|f_k| \leq M$ a.e. in E, then $\int_E f_k \to \int_E f$.*

Theorem (6.1 Fubini's theorem). *Let $f(\mathbf{x}, \mathbf{y}) \in L^1(I)$, $I := I_1 \times I_2$. Then*

(i) *For almost every $\mathbf{x} \in I_1$, $f(\mathbf{x}, \mathbf{y})$ is measurable and integrable on I_2 as a function of $\mathbf{y}$;*

(ii) *As a function of $\mathbf{x}$, $\int_{I_2} f(\mathbf{x}, \mathbf{y})d\mathbf{y}$ is measurable and integrable on I_1, and*

$$\iint_I f(\mathbf{x}, \mathbf{y})d\mathbf{x}d\mathbf{y} = \int_{I_1} \left[\int_{I_2} f(\mathbf{x}, \mathbf{y})d\mathbf{y}\right]d\mathbf{x}.$$

Theorem (6.8). *Let $f(\mathbf{x}, \mathbf{y})$ be a measurable function defined on a measurable subset E of $\mathbf{R}^{n+m}$, and let $E_{\mathbf{x}} := \{\, \mathbf{y} : (\mathbf{x}, \mathbf{y}) \in E \,\}$.*

(i) *For almost every $\mathbf{x} \in \mathbf{R}^n$, $f(\mathbf{x}, \mathbf{y})$ is a measurable function of $\mathbf{y}$ on $E_{\mathbf{x}}$.*

(ii) *If $f(\mathbf{x}, \mathbf{y}) \in L^1(E)$, then for almost every $\mathbf{x} \in \mathbf{R}^n$, $f(\mathbf{x}, \mathbf{y})$ is an integrable on $E_{\mathbf{x}}$ with respect to $\mathbf{y}$; moreover $\int_{E_{\mathbf{x}}} f(\mathbf{x}, \mathbf{y})d\mathbf{y}$ is an integrable function of $\mathbf{x}$ and*

$$\iint_E f(\mathbf{x}, \mathbf{y})d\mathbf{x}d\mathbf{y} = \int_{\mathbf{R}^n} \left[\int_{E_{\mathbf{x}}} f(\mathbf{x}, \mathbf{y})d\mathbf{y}\right]d\mathbf{x}.$$

Theorem (6.10, Tonelli's theorem). *Let $f(\mathbf{x}, \mathbf{y})$ be nonnegative and measurable on an interval $I = I_1 \times I_2$ of $\mathbf{R}^{n+m}$. Then, for almost every $\mathbf{x} \in I_1$, $f(\mathbf{x}, \mathbf{y})$ is a measurable function of $\mathbf{y}$ on I_2. Moreover, as a function of $\mathbf{x}$, $\int_{I_2} f(\mathbf{x}, \mathbf{y})d\mathbf{y}$ is measurable on I_1, and*

$$\iint_I f(\mathbf{x}, \mathbf{y})d\mathbf{x}d\mathbf{y} = \int_{I_1} \left[\int_{I_2} f(\mathbf{x}, \mathbf{y})d\mathbf{y}\right]d\mathbf{x}$$

If f and g are measurable in $\mathbf{R}^n$, their *convolution* $(f * g)(\mathbf{x})$ is defined by

$$(f * g)(\mathbf{x}) := \int_{\mathbf{R}^n} f(\mathbf{x} - \mathbf{y})g(\mathbf{y})d\mathbf{y},$$

provided the integral exists.

Theorem (6.14)**.** *If $f \in L^1(\mathbf{R}^n)$ and $g \in L^1(\mathbf{R}^n)$, then $(f * g)(\mathbf{x})$ exists for almost every $\mathbf{x} \in \mathbf{R}^n$ and is measurable. Moreover, $f* \in L^1(\mathbf{R}^n)$ and*

$$\int_{\mathbf{R}^n} |f * g| d\mathbf{x} \leq \Big(\int_{\mathbf{R}^n} |f| d\mathbf{x} \Big) \Big(\int_{\mathbf{R}^n} |g| d\mathbf{x} \Big)$$
$$\int_{\mathbf{R}^n} (f * g)(\mathbf{x}) d\mathbf{x} = \Big(\int_{\mathbf{R}^n} f d\mathbf{x} \Big) \Big(\int_{\mathbf{R}^n} g d\mathbf{x} \Big).$$

Corollary (6.16)**.** *If f and g are nonnegative and measurable on $\mathbf{R}^n$, then $f * g$ is measurable on $\mathbf{R}^n$ and*

$$\int_{\mathbf{R}^n} (f * g) d\mathbf{x} = \Big(\int_{\mathbf{R}^n} f d\mathbf{x} \Big) \Big(\int_{\mathbf{R}^n} g d\mathbf{x} \Big).$$

Theorem (6.17, Marcinkiewicz)**.** *Let F be a closed subset of a bounded open interval (a, b), and let $\delta(x) := \delta(x, F)$ be the corresponding distance function. Then, given $\lambda > 0$, the integral*

$$M_\lambda(x) := \int_a^b \frac{\delta(y)^\lambda}{|x - y|^{1+\lambda}} dy$$

is finite a.e. in F. Moreover, $M_\lambda \in L^1(F)$ and

$$\int_F M_\lambda dx \leq 2\lambda^{-1} |G|,$$

where $G := (a, b) \setminus F$.

1.3 Final Exam Review

Material covered since exam 2.

If f is a Riemann integrable function on an interval $[a, b]$ in $\mathbf{R}$, then the familiar definition of its indefinite integral is

$$F(x) := \int_a^x f(y)dy, \qquad a \le x \le b.$$

The fundamental theorem of calculus asserts that $F' = f$ if f is continuous. We will study an analogue of this result for Lebesgue integrable f and higher dimensions.

We must first find an appropriate definition of the indefinite integral. In two dimensions, for example, we might choose

$$F(x_1, x_2) := \int_{a_1}^{x_1} \int_{a_2}^{x_2} f(y_1, y_2)dy_1 dy_2.$$

It turns out, however, to be better to abandon the notion that the indefinite integral be a function of point and adopt the idea that it be a function of set. Thus, given $f \in L^1(A)$, where A is a measurable subset of $\mathbf{R}^n$, we define the *indefinite integral of* f to be the function

$$F(E) := \int_E f,$$

where E is any measurable subset of A.

F is an example of a *set function*, by which we mean any real-valued function F defined on a σ-algebra Σ of measurable sets such that

(i) $F(E)$ is finite for every $E \in \Sigma$.

(ii) F is *countably additive*; that is, if $E = \bigcup_k E_k$ is a union of disjoint $E_k \in \Sigma$, then

$$F(E) = \sum_k F(E_k).$$

By Theorem 5.5 and 5.24, the indefinite integral of $f \in L^1(A)$ satisfies (i) and (ii) for the σ-algebra of measurable subsets of A.

Recall that the diameter of a set E is the value

$$\sup\{\, \|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in E \,\}.$$

A set function $F(E)$ is called *continuous* if $F(E)$ tends to zero as the diameter of E tends to zero; i.e., $F(E)$ is continuous if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|F(E)| < \varepsilon$ whenever the diameter of E is less than δ. An example of a function that is *not* continuous can be obtained by setting $F(E) = 1$ for any measurable set that contains the origin, and $F(E) = 0$ otherwise.[1]

A set function F is called *absolutely continuous with respect to the Lebesgue measure*, or simply *absolutely continuous* if $F(E)$ tends to zero as the measure of E tends to zero. Thus, F is absolutely

[1] Why is this function not continuous. Consider the following argument: Let $\varepsilon = 1/2$ and let $B_k := B(\mathbf{0}, 1/k)$. Then as the diameter of B_k goes to zero, $F(B_k) = 1$ for all k so $F(B_k) \to 1 > 1/2$.

continuous if given a $\varepsilon > 0$ there exists $\delta > 0$ such that $|F(E)| < \varepsilon$ whenever the measure of E is less than δ.

A set function that is absolutely continuous is clearly continuous[2], however, the converse is false, as shown in the following example. Let A be the unit square in $\mathbf{R}^2$, let D be the diagonal of A, and consider the σ-algebra of measurable subsets E of A for which $E \cap D$ is linearly measurable. For such E, let $F(E)$ be the linear measure of $E \cap D$. Then F is a continuous set function. However, it is not absolutely continuous since the sets E containing a fixed segment of D whose $\mathbf{R}^2$-measures are arbitrarily small.

Theorem 13 (7.1)**.** *If $f \in L(A)$, its definite integral is absolutely continuous.*

Proof. We may assume that $f \geq 0$ by considering f^+ and f^-. Fix k and write $f = g+h$, where $g = f$ whenever $f \leq k$ and $g = k$ otherwise. Given $\varepsilon > 0$, we may choose k so large that $0 \leq \int_A h < \varepsilon/2$ and, *a fortiori*, $0 \leq \int_E f < \varepsilon/2$ ■

[2]Suppose F is absolutely continuous. Then, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|F(E)| < \varepsilon$ whenever $|E| < \delta$.

2 MA 544 Spring 2016

2.1 Exam 1 Prep

Problem 2.1. Let $E \subset \mathbf{R}^n$ be a measurable set, $r \in \mathbf{R}$ and define the set $rE = \{ r\mathbf{x} : \mathbf{x} \in E \}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Define a linear map $T\colon \mathbf{R}^n \to \mathbf{R}^n$ by $\mathbf{x} \mapsto r\mathbf{x}$. Using the standard basis for $\mathbf{R}^n$, this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \tag{24}$$

which has determinant $\det T = r^n$. By 3.35, we have $|E| = |T(E)| = r^n|E| = |rE|$. ∎

Problem 2.2. Let $\{E_k\}$, $k \in \mathbf{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k\to\infty} E_k \right| \leq \liminf_{k\to\infty} |E_k|.$$

Proof. If the $\liminf |E_k| = \infty$ the inequality holds trivially. Hence, we may, without loss of generality, assume that $\liminf |E_k| < \infty$. By 3.20, the set $\liminf E_k$ is measurable and we have

$$\left| \liminf_{k\to\infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \tag{25}$$

where $F_k := \bigcap_{n=k}^{\infty} E_n$. Now, note that the collection of sets $F'_k := \bigcup_{\ell=1}^{k} F_\ell$ forms an increasing sequence of measurable sets $F'_k \nearrow F'$, where $F' = \bigcup_{k=1}^{\infty} F_k = \liminf E_k$. Then, by 3.26 (i), we have

$$\lim_{k\to\infty} |F'_k| = |F'| = \left| \liminf_{k\to\infty} E_k \right|. \tag{26}$$

Hence, it suffices to show that $|F'_k| \leq |E_k|$ for all k, but this follows by monotonicity of the outer measure, 3.3, since $F'_k \subset E_k$. Thus, we have the desired inequality

$$\left| \liminf_{k\to\infty} E_k \right| \leq \liminf_{k\to\infty} |E_k|. \tag{27}$$

∎

Problem 2.3. Consider the function

$$F(x) := \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) := \{ \mathbf{y} \in \mathbf{R}^n : |\mathbf{y}| < r \}$. Prove that F is monotonic increasing and continuous.

Proof. That F is increasing is immediate from the monotonicity of the outer measure since for $x < x'$ we have $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$ so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \leq |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that F is continuous, we will prove the following lemma

Lemma 14. *For any $x > 0$, $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$.*

Proof of lemma. If $\mathbf{y} \in xB(\mathbf{0}, 1)$ then $\mathbf{y} = x\mathbf{y'}$ for $\mathbf{y'} \in B(\mathbf{0}, 1)$. Thus, $|\mathbf{y'}| = |\mathbf{y}|/x < 1$ so $|\mathbf{y}| < x$ implies that $\mathbf{y} \in B(\mathbf{0}, x)$. Hence, we have the containment $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$.

On the other hand, if $\mathbf{y} \in B(\mathbf{0}, x)$ then $|\mathbf{y}| < x$ so $|\mathbf{y}/x| < 1$. Hence, $\mathbf{y}/x \in B(\mathbf{0}, 1)$ so $x(\mathbf{y}/x) = \mathbf{y} \in B(\mathbf{0}, x)$. Thus, $B(\mathbf{0}, x) \subset xB(\mathbf{0},)$ and equality holds. ♣

In light of Lemma 14 and 3.35, for $x > 0$, we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n|B(\mathbf{0}, 1)|. \tag{28}$$

It is clear that F is continuous on the interval $[0, \infty)$ since F is a polynomial in x. ■

Problem 2.4. Let $f \colon \mathbf{R} \to \mathbf{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. From the topological definition of continuity, f is continuous at $x \in C$ if and only if for every neighborhood U of $f(x)$, the preimage $f^{-1}(U)$ is a neighborhood of x. Now, ■

Let $x \in C$. Then, by the definition of continuity, for every natural number $n > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f(x) - f(x')| < \frac{1}{2n}. \tag{29}$$

Let $x'', x' \in B(x, \delta)$. Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x)''| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \tag{30}$$

In view of these estimates, define the set

$$A_n := \left\{ x \in \mathbf{R} : \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \tag{31}$$

Good Lord, that was a long definition! We claim that $C = \bigcap_{n=1}^\infty A_n$ and that A_n is open for all n.

First, let us show that $C = \bigcap_{n=1}^\infty A_n$. Let $x \in C$. Then for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Thus, $x \in A_n$ for all n so $x \in \bigcap A_n$. On the other hand, if $x \in \bigcap A_n$ for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$.

Fix $\varepsilon > 0$. By the Archimedean principle, there exists $N > 0$ such that $\varepsilon > 1/N$. Then, since $x \in A_N$ it follows that for some $\delta' > 0$, $|x - x'| < \delta'$ implies $|f(x) - f(x')| < 1/N < \varepsilon$. Thus, $x \in C$ and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$.

Lastly, we show that A_n is open. Let $x \in A_n$. Then there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. In particular, this means that $B(x, \delta) \subset A_n$ for any $x' \in B(x, \delta)$ satisfies $|f(x) - f(x')| < 1/n$. Thus, A_n is open and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$ is a G_δ set.

Problem 2.5. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbf{R}$, then f is measurable?

Proof. No. Recall that, by definition, or 4.1, f is measurable if and only if $\{ f > a \}$ for all $a \in \mathbf{R}$. ∎

Problem 2.6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbf{R}$. Prove that the set $\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \}$ is measurable.

Proof. The idea here should be to rewrite

$$E \coloneqq \left\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \right\} \tag{32}$$

as a countable union/intersection of measurable sets. Let $x \in E$. By the Cauchy criterion, for every $N > 0$ there exists a positive integer M such that $m, n \geq M$ implies $|f_n(x) - f_m(x)| < 1/N$. With this in mind, define

$$E_N \coloneqq \left\{ x : \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \tag{33}$$

Then, like for Problem 1.4, it is not too hard to see that the E_n's are open and that $E = \bigcap_{n=1}^{\infty} E_n$. Thus, E is a G_δ set and therefore measurable. ∎

Problem 2.7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, one has $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. Suppose $f\colon [a, b] \to \mathbf{R}$ is absolutely continuous. Then for fixed $\varepsilon = 1$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, we have $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Let $\Gamma \coloneqq \{x_k\}_{k=1}^{N}$ be a partition of $[a, b]$ into closed intervals such that $x_{k+1} - x_k < \delta$, then by absolute continuity we have

$$\begin{aligned} V[f; \Gamma] = \sum_{k=1}^{N} &|f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \tag{34}$$

Thus, f is b.v. on $[a, b]$. ∎

Problem 2.8. Let f be a continuous function from $[a, b]$ into $\mathbf{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, i.e., $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(a) & \text{if } c = b \end{cases}.$$

Proof. ■

3 Exam 1

3.1 Exam 2 Prep

Problem 3.1. Define for $\mathbf{x} \in \mathbf{R}^n$,

$$f(\mathbf{x}) \coloneqq \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B_\varepsilon(\mathbf{0})$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbf{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x}) d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f \colon \mathbf{R}^n \to \mathbf{R}$ is (Lebesgue) integrable over a subset E of $\mathbf{R}^n$ (or, alternatively, f belongs to $L^1(E)$) if

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty.$$

Put $E \coloneqq \mathbf{R}^n \setminus B_\varepsilon(\mathbf{0})$. Then, to show that f belongs to $L^1(E)$ it suffices to prove the inequality

$$\int_E f(\mathbf{x}) d\mathbf{x} < \frac{C}{\varepsilon} \tag{35}$$

for some appropriate constant C. We proceed by directly computing the Lebesgue integral of f and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x}) d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{({x_1}^2 + \cdots + {x_n}^2)^{(n+1)/2}} \end{aligned}$$

let E_i denote the projection of E onto its i-th coordinate and make the trigonometric substitution $x_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \tan\theta$, $dx_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \sec^2\theta d\theta$ with $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$ giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[\frac{\cos^{n-1}\theta}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1}\theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1}\theta d\theta < \infty. \tag{36}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned}\int \cdots \int_E f(\mathbf{x}) d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{{x_n}^2} \\ &= 2C' \int_{\varepsilon}^{\infty} \frac{dx_n}{{x_n}^2} \\ &= \frac{C}{\varepsilon}\end{aligned} \tag{37}$$

as desired. ■

Problem 3.2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbf{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbf{R}^n} f = \lim_{k\to\infty} \int_{\mathbf{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbf{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbf{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit f of $\{f_k\}$ must be nonnegative a.e. in $\mathbf{R}^n$. For assume otherwise. Then there exists a collection of points $\mathbf{x}$ in $\mathbf{R}^n$ of nonzero $\mathbf{R}^n$-Lebesgue measure such that $f(\mathbf{x}) < 0$. But $f_k(\mathbf{x}) \geq 0$ for all $k \in \mathbf{N}$. Set $0 < \varepsilon < |f(\mathbf{x})|$ then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{38}$$

for all k which contradicts our assumption that $f_k \to f$ a.e. on $\mathbf{R}^n$. Therefore, the set of points $\mathbf{x} \in \mathbf{R}^n$ where $f(\mathbf{x}) < 0$ must have measure zero.

Now, based on pointwise convergence a.e. to f, given $\varepsilon > 0$ for a.e. $\mathbf{x} \in \mathbf{R}^n$ we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{39}$$

for sufficiently large k; say k greater than or equal to some index $N \in \mathbf{N}$. Moreover, we are given convergence in $L^1(\mathbf{R}^n)$ of f_k to f

$$\int_{\mathbf{R}^n} f_k \to \int_{\mathbf{R}^n} f < \infty. \tag{40}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbf{R}^n} f < \infty \tag{41}$$

and

$$\int_E f_k \leq \int_{\mathbf{R}^n} f_k < \infty \tag{42}$$

for all $k \in \mathbf{N}$. By Theorem 5.5(ii), f and the f_k's are finite a.e. in $\mathbf{R}^n$ so for some sufficiently large real number M, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in \mathbf{R}^n$. In particular, for any measurable subset E of $\mathbf{R}^n$, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in E$ so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \to \int_E f < \infty. \tag{43}$$

However, if f does not belong to $L^1(\mathbf{R}^n)$, i.e., its integral over $\mathbf{R}^n$ is infinity, there is no guarantee that f will be finite a.e. in $\mathbf{R}^n$. This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset E of $\mathbf{R}^n$. Let us demonstrate this with an example. Consider the sequence of functions ■

Problem 3.3. Assume that E is a measurable set of $\mathbf{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of $\mathbf{R}^n$, then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \tag{44}$$

Set $E_k \coloneqq \{\, \mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k \,\}$ and $F_k \coloneqq \{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{45}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \tag{46}$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \tag{47}$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (47) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \tag{48}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (49) \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \qquad (50) \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned}$$

Thus, from (49) and (50)

$$\sum_{k=1}^{\infty} |F_k| \le \int_E f(\mathbf{x}) d\mathbf{x} \le \sum_{k=0}^{\infty} |F_k| \qquad (51)$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ■

Problem 3.4. Suppose that E is a measurable subset of $\mathbf{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f, g) := \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. $\Longrightarrow$: First note that ρ is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that $\rho(f_k, f) \to 0$ as $k \to \infty$. Then, given $\varepsilon > 0$ there exist an sufficiently large index N such that for every $k \geq N$ we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \tag{52}$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in E which happens if $|f_k - f| = 0$ a.e. in E.

$\Longleftarrow$: Suppose that $f_k \to f$ as $k \to \infty$.

I don't know how to solve this. This is the intended solution:

$\Longrightarrow$: Given $\varepsilon > 0$, $\rho(f_k, f) \to 0$ implies that

$$\int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{|f_k - f|}{1 + |f_k - f} dx \longrightarrow 0.$$

Observe that the function $\Phi \colon \mathbf{R}^+ \to \mathbf{R}$ given by $\Phi(x) \coloneqq x/(1+x)$ is increasing on $\mathbf{R}^+$ and $0 < \Psi(x) < 1$, hence

$$\begin{aligned}\int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1 + \varepsilon} |\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}|.\end{aligned}$$

Therefore,

$$|\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}| \leq \frac{1 + \varepsilon}{\varepsilon} \int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \longrightarrow 0$$

as $k \to \infty$.

$\Longleftarrow$: Conversely, given $\delta > 0$, we have

$$\begin{aligned}\rho(f_k, f) = &\int_{\{\, x \in E : |f_k(x) - f(x)| > \delta \,\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &+ \int_{\{\, x \in E : |f_k(x) - f(x)| \leq \delta \,\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\leq |\{\, x \in E : |f_k(x) - f(x)| > \delta \,\}| + \frac{\delta}{1 + \delta} |E|.\end{aligned}$$

Since $|E| < \infty$ and $\delta/(1 + \delta) \searrow 0$, then for any $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\frac{\delta'}{1 + \delta'} |E| < \frac{\varepsilon}{2}.$$

If $f_k \to f$ as $k \to \infty$ in measure, then for the above δ' there is an index $N > 0$ such that $k \geq N$ implies

$$|\{\, x \in E : |f_k(x) - f(x)| > \delta' \,\}| < \frac{\varepsilon}{2}.$$

Therefore, $f_k \to f$ in measure implies $\rho(f_k, f) \to 0$ as $k \to \infty$. ■

Problem 3.5. Define the *gamma function* $\Gamma\colon \mathbf{R}^+ \to \mathbf{R}$ by

$$\Gamma(y) \coloneqq \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function* $\beta\colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$ by

$$\beta(x,y) \coloneqq \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbf{R}^+)$ for all $y \in \mathbf{R}^+$.

(b) Show that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. (a) Fix $y \in \mathbf{R}^+$. Then we must show that $\Gamma(y) < \infty$. First, since $(0,1)$ and $[1,\infty)$ are disjoint measurable subsets of $\mathbf{R}$, by Theorem 5.7 we can split the integral $\Gamma(y)$ into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u}u^{y-1}du}_{I_1} + \underbrace{\int_1^\infty e^{-u}u^{y-1}du}_{I_2}. \tag{53}$$

We will show, separately, that I_1 and I_2 are finite.

To see that I_1 is finite, note that

$$\begin{aligned} e^{-u}u^{y-1} &= e^{-u}e^{(y-1)\log u} \\ &= e^{-u+(y-1)\log u} \\ &\le e^{(y-1)\log u} \\ &= u^{y-1} \end{aligned} \tag{54}$$

since $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u}u^{y-1}du \\ &\le \int_0^1 u^{y-1}du \\ &= \left[\frac{u^y}{y}\right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \tag{55}$$

To see that I_2 is finite, note that

$$e \tag{56}$$

Intended solution:

(b) ∎

Problem 3.6. Let $f \in L^1(\mathbf{R}^n)$ and for $\mathbf{h} \in \mathbf{R}^n$ define $f_{\mathbf{h}} : \mathbf{R}^n \to \mathbf{R}$ be $f_{\mathbf{h}}(\mathbf{x}) := f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \to \mathbf{0}} \int_{\mathbf{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbf{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{57}$$

■

Problem 3.7. (a) If $f_k, g_k, f, g \in L^1(\mathbf{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbf{R}^n$, $|f_k| \leq g_k$ and

$$\int_{\mathbf{R}^n} g_k \to \int_{\mathbf{R}^n} g,$$

prove that

$$\int_{\mathbf{R}^n} f_k \to \int_{\mathbf{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L^1(\mathbf{R}^n)$ and $f_k \to f$ a.e. in $\mathbf{R}^n$, then

$$\int_{\mathbf{R}^n} |f_k - f| \to 0 \qquad \text{as} \qquad k \to \infty$$

if and only if

$$\int_{\mathbf{R}^n} |f_k| \to \int_{\mathbf{R}^n} |f| \qquad \text{as} \qquad k \to \infty.$$

Proof. (a) Since $f_k \to f$ and $g_k \to g$ a.e. and $|f_k| \leq g_k$, then by Fatou's theorem,

$$\begin{aligned} \int_{\mathbf{R}^n} (g - f) = \int_{\mathbf{R}^n} \liminf_{k \to \infty} g_k - f_k \leq \liminf_{k \to \infty} \int_{\mathbf{R}^n} g_k - f_k, \\ \int_{\mathbf{R}^n} g + f \int_{\mathbf{R}^n} \liminf_{k \to \infty} g_k + f_k \leq \liminf_{k \to \infty} \int_{\mathbf{R}^n} g_k + f_k. \end{aligned}$$

Since $f_k, g_k, f, g \in L^1(\mathbf{R}^n)$ and $\int_{\mathbf{R}^n} g_k \to \int_{\mathbf{R}^n} g$, then using the similar argument as problem 2, we have

$$\begin{aligned} \int_{\mathbf{R}^n} f \geq \limsup_{k \to \infty} \int_{\mathbf{R}^n} f_k, \\ \int_{\mathbf{R}^n} f \leq \liminf_{k \to \infty} \int_{\mathbf{R}^n} f_k. \end{aligned}$$

Therefore, $\int_{\mathbf{R}^n} f_k \to \int_{\mathbf{R}^n} f$.

(b) $\implies$: This direction is obvious by the inequality

$$\left| \int_{\mathbf{R}^n} |f_k| - |f| \right| \leq \int_{\mathbf{R}^n} ||f_k| - |f|| \leq \int_{\mathbf{R}^n} |f_k - f|.$$

$\Longleftarrow$: Let $g_k := |f_k| + |f|$ and $g := 2|f|$. Since $f_k, f \in L^1(\mathbf{R}^n)$ and $f_k \to f$ a.e., then $g_k, g \in L^1(\mathbf{R}^n)$ and $g_k \to g$ a.e. in $\mathbf{R}^n$. By the assumption, $\int_{\mathbf{R}^n} g_k \to \int_{\mathbf{R}^n} g$.

Let $\tilde{f}_k := |f_k - f|$. Then $\tilde{f}_k \to 0$ a.e. in $\mathbf{R}^n$ and $\tilde{f}_k \leq g_k$. Applying part (a) to $\tilde{f}_k$ we have

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} \tilde{f}_k = \lim_{k\to\infty} \int_{\mathbf{R}^n} |f_k - f| = 0.$$

$\blacksquare$

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Problem 3.8. Assume that $f \in L^1(\mathbf{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbf{R}^n \setminus B} |f| < \varepsilon.$$

Proof. Recall that $f \in L^1(\mathbf{R}^n)$ if and only if $|f| \in L^1(\mathbf{R}^n)$. Let $B_k := B(\mathbf{0}, k)$ for $k \in \mathbf{N}$ and χ_{B_k} be the indicator function associated with B_k. Then, the sequence of maps $\{|f_k|\}$ defined $f_k := f\chi_{B_k}$ converge pointwise to $|f_k|$. Since $|f| \in L^1(\mathbf{R}^n)$, by the monotone convergence theorem, we have

$$\int_{\mathbf{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbf{R}^n} |f|. \tag{58}$$

But this means, exactly, that for every $\varepsilon > 0$ there exists sufficiently large $N \in \mathbf{N}$ such that

$$\begin{aligned} \varepsilon &> \left| \int_{\mathbf{R}^n} |f_k| - \int_{\mathbf{R}^n} |f| \right| \\ &= -\int_{\mathbf{R}^n} |f_k| + \int_{\mathbf{R}^n} |f| \\ &= -\int_{\mathbf{R}^n} |f| + \int_{\mathbf{R}^n} |f| \\ &= -\int_{B_k} |f| + \int_{\mathbf{R}^n} |f| \\ &= \int_{\mathbf{R}^n \setminus B_k} |f| \end{aligned} \tag{59}$$

as desired. ∎

Problem 3.9. Let $f \in L^1(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. First, since the E_j's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \tag{60}$$

Let χ_{E_j} be the characteristic function of the subset E_j of E and define $f_j := f\chi_{E_j}$ for $j \in \mathbf{N}$. Note that, since both f and χ_{E_j} are measurable on E, f_j is measurable on E and $\sum_{j=1}^{\infty} f_j = f$. Moreover, since $E_j \subset E$, by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \setminus E_j} f = \int_E f_j + \int_{E \setminus E_j} f. \tag{61}$$

Hence, because the E_j's are disjoint $(E \setminus E_k) \setminus E_\ell = (E \setminus E_\ell) \setminus E_k$ so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \tag{62}$$

as desired. ∎

Problem 3.10. Let $\{f_k\}$ be a family in $L^1(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbf{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Proof. Let $\varepsilon > 0$ be given. Then, by the hypothesis, there exists $\delta > 0$ such that such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon \tag{63}$$

for all $k \in \mathbf{N}$. By Egorov's theorem, there exists a closed subset F of E such that $|E \setminus F| < \delta$ and $f_k \to f$ uniformly on F. Then, by the uniform convergence theorem,

$$\int_F f_k \longrightarrow \int_F f \tag{64}$$

as $k \to \infty$. But by hypothesis, we have

$$\int_{E \setminus F} |f_k| < \varepsilon. \tag{65}$$

Letting $\varepsilon \to 0$, we achieved the desired convergence. ∎

Problem 3.11. Let $I := [0,1]$, $f \in L^1(I)$, and define $g(x) := \int_x^1 t^{-1} f(t)dt$ for $x \in I$. Prove that $g \in L^1(I)$ and

$$\int_I g = \int_I f.$$

Proof. By Lusin's theorem, there exists a closed subset F of I with $|I \setminus F| < \varepsilon$ such that the restriction of f to $F := I \setminus E$ is continuous. Now, since F is closed in I and I is compact, it follows that I is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials $\{p_k\}$ such that $p_k \to f$ uniformly on F. Then, by the uniform convergence theorem, we have

$$\int_F p_k \longrightarrow \int_F f \tag{66}$$

so

$$\begin{aligned}\int_F \left[\int_x^1 t^{-1} p_k(t)dt\right] dx &= \int_F \left[\int_x^1 at^{-1} + q_k(t)dt\right] dx \\ &= \int_F q_k'(x) - a\log(x)dx \\ &< \infty\end{aligned} \tag{67}$$

for all k and converges uniformly to g so $g \in L^1(I)$. I don't know how to show that in fact $\int_I g = \int_I f$. Perhaps you show that the places where they differ is a set of measure zero. ■