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MA 26500-215 Quiz 10

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1. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map that sends

$$\begin{aligned} T(1,0,0) &= (3,2,4) \\ T(0,1,0) &= (2,0,2) \\ T(0,0,1) &= (4,2,3). \end{aligned}$$

(a) (4 points) Find the value of $T(2,1,-1)$.

Solution: Since T is a linear map, we know that $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ and $T(c\mathbf{v}) = cT(\mathbf{v})$ so

$$\begin{aligned} T(2,1,-1) &= T((2,0,0) + (0,1,0) + (0,0,-1)) \\ &= T(2,0,0) + T(0,1,0) + T(0,0,-1) \\ &= 2T(1,0,0) + T(0,1,0) - T(0,0,1) \\ &= 2(3,2,4) + (2,0,2) - (4,2,3) \\ &= (6,4,8) + (2,0,2) + (-4,-2,-3) \\ &= (6+2-4, 4+0-2, 8+2-3) \\ &= (4,2,7). \end{aligned}$$

(b) (6 points) Find the matrix representation of T with respect to the standard basis on $\mathbb{R}^3$.

Solution: Using the standard basis on $\mathbb{R}^3$ which, by the way, is the set $\{(1,0,0),(0,1,0),(0,0,1)\}$, for a general vector

$$\mathbf{x} = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1),$$

we have

$$T(\mathbf{x}) = (3x_1 + 2x_2 + 4x_3, 2x_1 + 0x_2 + 2x_3, 4x_1 + 2x_2 + 3x_3).$$

Thus,

$$\begin{aligned} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{bmatrix} \\ &= \begin{bmatrix} 3x_1 + 2x_2 + 4x_3 \\ 2x_1 + 0x_2 + 2x_3 \\ 4x_1 + 2x_2 + 3x_3 \end{bmatrix}. \end{aligned}$$

This tells us that the matrix must be

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

(c) (10 points) Using the matrix representation of T, find the characteristic polynomial. **You do not have to simplify it**.

Solution: To find the minimal polynomial of T, we find

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & 0-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix} \\ &= (3-\lambda) \det \begin{bmatrix} 0-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 2 \\ 4 & 3-\lambda \end{bmatrix} \\ &\quad + 4 \det \begin{bmatrix} 2 & 0-\lambda \\ 4 & 2 \end{bmatrix} \\ &= (3-\lambda)(\lambda^2 - 3\lambda - 4) - 2(6 - 2\lambda - 8) + 4(4 + 4\lambda) \\ &= (3\lambda^2 - 9\lambda - 12 - \lambda^3 + 3\lambda^2 + 4\lambda) \\ &\quad + (-12 + 4\lambda + 16) \\ &\quad + (16 + 16\lambda) \\ &= -\lambda^3 + 6\lambda^2 + 15\lambda + 8. \end{aligned}$$

Using this equation, we can find the eigenvalues of T. One thing you can do is first try to find a root of $\lambda^3 - 6\lambda^2 - 15\lambda - 8$. As it turns out, $\lambda = -1$ is a root since

$$(-1)^3 - 6(-1)^2 - 15(-1) - 8 = -1 - 6 + 15 - 8 = -15 + 15 = 0.$$

So, using long division, we can factor $(\lambda + 1)$ from $\lambda^3 - 6\lambda^2 - 15\lambda - 8$ and so

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda + 1)(\lambda^2 - 7\lambda - 8).$$

And you can do it again since $(-1)^2 - 7(-1) - 8 = 1 + 7 - 8 = 0$ so

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda + 1)^2(\lambda - 8).$$

Thus, the eigenvalues are -1 and 8.