MA571 Problem Set 1

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Problem 1.1 (Munkres §2, 1(a,b).)

Let $f\colon A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$.

(a) Show that $A_0 \subset f^{-1}(f(A_0))$ and that equality holds if f is injective.
(b) Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

Proof. (a). First, we will show $A_0 \subset f^{-1}(f(A_0))$. Let $x \in A_0$. Then $f(x) \in f(A_0)$. By definition, $f^{-1}(f(A_0))$ is the set of those points $x_0 \in A$ such that $f(x_0) \in f(A_0)$ and in particular we see that the containment $A_0 \subset f^{-1}(f(A_0))$ holds. Thus, $x \in f^{-1}(f(A_0))$.

Now, let us suppose the map f is injective. By our former argument, we have that $A_0 \subset f^{-1}(f(A_0))$ therefore, we will show the reverse containment. If $y \in f(A_0)$, then $f(x) = y$ for some $x \in A_0$. By the injectivity of f, if $f(x_0) = y$ for some $x_0 \in A$, then we must have that $x_0 = x$. In particular, $x_0 \in A_0$. Thus $f^{-1}(f(A_0)) \subset A_0$ and equality $A_0 = f^{-1}(f(A_0))$ holds.

(b). First, we will show that $f(f^{-1}(B_0)) \subset B_0$. Consider the preimage $f^{-1}(B_0)$ of B_0. Let $x \in f^{-1}(B_0)$. Then $f(x) = y$ for some $y \in B_0$. Since $f(f^{-1}(B_0))$ is, by definition, the set of all points $f(x) \in B$ where $x \in f^{-1}(B_0)$ and $f(x) = y$ for $y \in B_0$, we have that $f(f^{-1}(B_0)) \subset B_0$.

Now, let us suppose the map f is surjective. Let $y \in B_0$, then there exists $x \in A$ such that $f(x) = y$. Thus, $x \in f^{-1}(B_0)$. Then $y = f(x) \in f(f^{-1}(B_0))$ (in particular $B_0 \subset f(f^{-1}(B_0))$) and we have equality $B_0 = f(f^{-1}(B_0))$. ■

Problem 1.2 (Munkres, §2, 2(g).)

Let $f \colon A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusion, unions, intersections, and differences of sets:

(g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that equality holds if f is injective.

Proof of (g). The claim is evident if A_0 and A_1 are disjoint subsets. Suppose $A_0 \cap A_1 \neq \emptyset$. Let $y \in f(A_0 \cap A_1)$. Then $y = f(x)$ for some $x \in A_0$, $x \in A_1$. Then $f(x) \in f(A_0)$ and $f(x) \in f(A_1)$ so $y \in f(A_0) \cap f(A_1)$. Thus, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

Now, suppose f is injective. Then, if $f(x) = f(x') = y$ for some $y \in B$, then $x = x'$. Let $y \in f(A_0) \cap f(A_1)$. Then $y = f(x_0)$, $y = f(x_1)$ for some $x_0 \in A_0$, $x_1 \in A_1$. But, by the injectivity of f, $x_0 = x_1$ so $x_0 \in A_0 \cap A_1$. Hence, $y \in f(A_0 \cap A_1)$ and the equality $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$ holds. ■

Problem 1.3 (Munkres, §13, 3.)

Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{ U \mid X \setminus U \text{ is infinite or empty or all of } X \}$$

a topology on X?

Proof. ∎

Problem 1.4 (Munkres, §13, 5.)

Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Proof. ■

Problem 1.5 (Munkres, §13, 8(b).)

(b) Show that the collection

$$\mathcal{C} = \{\, [a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbf{R}$.

Problem 1.6 (Munkres, §16, 1.)

Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. ∎

Problem 1.7 (Munkres, §16, 4.)

A map $f\colon X \to Y$ is said to be an *open map* if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are open maps.

Proof. ∎

Problem 1.8 (Munkres, §16, 6.)

Show that the countable collection

$$\{ (a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational} \}$$

is a basis for $\mathbf{R}^2$.

Proof. ∎

Problem 1.9 (Munkres, §16, 9.)

Show that the dictionary order topology on the set $\mathbf{R} \times \mathbf{R}$ is the same as the product topology $\mathbf{R}_d \times \mathbf{R}$, where $\mathbf{R}_d$ denotes $\mathbf{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbf{R}^2$.

Proof. ∎