

Exercises in Basic Mathematics

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Basic Mathematics Exercises

CHAPTER 2

Algebra Exercises

Algebraic Geometry Exercises

3.1 Elementary Algebraic Geometry

3.2 Affine Geometry (first level of abstraction), Zariski Topology

Definition 1. Given any ideal $\mathfrak{a} \subset k[X_1, \dots, X_q]$, define $\mathcal{V}_k(\mathfrak{a})$ by

$$\mathcal{V}_k(\mathfrak{a}) := \{ \mathbf{x} \in \mathbb{A}^q : \text{for every } f \in \mathfrak{a}, f(\mathbf{x}) = 0 \}.$$

We call $\mathcal{V}_k(\mathfrak{a})$ the *set of Ω -points of the affine k -variety determined by \mathfrak{a}* . With a slight abuse of language, we call $\mathcal{V}_k(\mathfrak{a})$ the *affine k -variety determined by \mathfrak{a}* . Similarly, given by any ideal $\mathfrak{a} \subset \bar{k}[X_1, \dots, X_q]$, defined by $\mathcal{V}_{\bar{k}}(\mathfrak{a})$ by

$$\mathcal{V}_{\bar{k}}(\mathfrak{a}) := \{ \mathbf{x} \in \mathbb{A}^q : \text{for every } f \in \mathfrak{a}, f(\mathbf{x}) = 0 \}.$$

We call $\mathcal{V}_{\bar{k}}(\mathfrak{a})$ the *set of Ω -points of the (geometric) affine \bar{k} -variety determined by \mathfrak{a}* , or for short, the *(geometric) affine variety determined by \mathfrak{a}* .

To ease the notation, we usually drop the subscript k or \bar{k} and simply write \mathcal{V} .

If A is a (commutative) ring (with unit 1), recall that the *radical*, $\sqrt{\mathfrak{b}}$, of an ideal, $\mathfrak{a} \subset A$, is defined by

$$\sqrt{\mathfrak{a}} := \{ a \in A : \text{there exists } n \geq 1, a^n \in \mathfrak{a} \}.$$

A *radical ideal* is an ideal, \mathfrak{a} , such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

The following properties are easily verified. We state them for \mathcal{V}_k , but they also hold for $\mathcal{V}_{\bar{k}}$:

$$\begin{aligned} \mathcal{V}(0) &= \mathbb{A}^n, \mathcal{V}(A) = \emptyset \\ \mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) &= \mathcal{V}(\mathfrak{a}\mathfrak{b}) = \mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) \\ \mathfrak{a} \subset \mathfrak{b} &\text{ implies that } \mathcal{V}(\mathfrak{b}) \subset \mathcal{V}(\mathfrak{a}) \\ \mathcal{V}\left(\sum_{\alpha} \mathfrak{a}_{\alpha}\right) &= \bigcap_{\alpha} \mathcal{V}(\mathfrak{a}_{\alpha}) \\ \mathcal{V}(\sqrt{\mathfrak{a}}) &= \mathcal{V}(\mathfrak{a}) \end{aligned}$$

From the relations above, it follows that the sets $\mathcal{V}(\mathfrak{a})$ can be taken as closed subsets of \mathbb{A}^q , and we obtain a topology on \mathbb{A}^q . This is the *k-topology on \mathbb{A}^q* . If we consider ideals in $\bar{k}[X_1, \dots, X_q]$ (i.e., sets of the form $\mathcal{V}_{\bar{k}}(\mathfrak{a})$), we obtain the *Zariski topology on \mathbb{A}^q* .

The Zariski topology is not necessarily Hausdorff (except when $\mathcal{V}(\mathfrak{a})$ consists of a finite set of points.)

Let us see that \mathbb{A}^q is not Hausdorff in the Zariski topology. Let $P, Q \in \mathbb{A}^q$, with $P \neq Q$. The line \overline{PQ} is isomorphic to \mathbb{A}^1 . Thus, it is enough to show that \mathbb{A}^1 is not Hausdorff. Consider any ideal $\mathfrak{a} \subset \bar{k}[X]$. Then, \mathfrak{a} is a principal ideal, and thus

$$\mathfrak{a} = (f)$$

for some polynomial f , which shows that $\mathcal{V}(\mathfrak{a}) = \mathcal{V}(f)$ is a finite set. As a consequence, the closed sets of \mathbb{A}^1 (other than \mathbb{A}^1) are finite. Then, the union of two closed sets (distinct from \mathbb{A}^1) is also finite, and thus distinct from \mathbb{A}^1 .

The topology on \mathbb{A}^q is not the product topology on $\prod_{i=1}^q \mathbb{A}^1$.

For example, when $n = 2$, the closed set in $\mathbb{A}^1 \times \mathbb{A}^1$ are those sets consisting of finitely many horizontal and vertical lines, and intersections of such sets. However

$$X^2 + Y^2 - 1 = 0$$

defines a closed set in \mathbb{A}^2 not of the previous form.

To go backwards from subsets of \mathbb{A}^q to ideals, we make the following definition.

Definition 2. Given any subset $S \subset \mathbb{A}^q$, define $\mathcal{I}_k(S)$ and $\mathcal{I}_{\bar{k}}(S)$ by

$$\mathcal{I}_k(S) := \{ f \in k[X_1, \dots, X_q] : \text{for every } s \in S, f(s) = 0 \}$$

and

$$\mathcal{I}_{\bar{k}}(S) := \{ f \in \bar{k}[X_1, \dots, X_q] : \text{for every } s \in S, f(s) = 0 \}$$

The following properties are easily shown (following our conventions, they are stated for \mathcal{I}_k , but they are easily shown for $\mathcal{I}_{\bar{k}}$).

Differential Geometry Exercises

4.1 The Matrix Exponential; Some Matrix Lie Groups

The Exponential Map

The Lie Groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathfrak{o}(n)$, $\mathrm{SO}(n)$, **the Lie Algebras** $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{o}(n)$, $\mathfrak{so}(n)$, **and the Exponential Map**

The notation $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{so}(n)$ is rather strange and deserves some explanation. The groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathfrak{o}(n)$, and $\mathrm{SO}(n)$ are more than just groups. They are also topological groups, which means that they are topological spaces (viewed as subspaces of \mathbb{R}^{n^2}) and that the multiplication and the inverse operations are smooth. In fact, they are smooth real manifolds. Such objects are called *Lie groups*. The real vector spaces $\mathfrak{sl}(n)$ and $\mathfrak{so}(n)$ are what are called *Lie algebras*. However, we have not defined the algebra structure on $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{so}(n)$ yet. The algebra structure is called the *Lie bracket*, which is defined as

$$[A, B] = AB - BA.$$

Lie algebras are associated with Lie groups. What is going on is that the Lie algebra of a Lie group is its tangent space at the identity, i.e., the space of all tangent vectors at the identity. In some sense, the Lie algebra achieves a linearization of the Lie group. The exponential map is a map from the Lie algebra to the Lie group, for example,

$$\exp: \mathfrak{so}(n) \longrightarrow \mathrm{SO}(n)$$

and

$$\exp: \mathfrak{sl}(n, \mathbb{R}) \longrightarrow \mathrm{SL}(n, \mathbb{R}).$$

The exponential map often allows a parametrization of the Lie group elements by simpler objects, the Lie algebra elements.

One might ask, “What happened to the Lie algebras $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{o}(n)$ associated with the Lie group $\mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{o}(n)$?” We will see later that $\mathfrak{gl}(n, \mathbb{R})$ is the set of all real $n \times n$ matrices and that $\mathfrak{o}(n) = \mathfrak{so}(n)$.

The properties of the exponential map play an important role in the study of Lie groups. For example, it is clear that the map

$$\exp: \mathfrak{gl}(n, \mathbb{R}) \longrightarrow \mathrm{GL}(n, \mathbb{R})$$

is well-defined, but since $\det(e^A) = e^{\mathrm{tr} A}$, every matrix of the form e^A has a positive determinant and \exp is not surjective.