MA 598 (Algebraic Geometry): Homework 1

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Problem 1.1 (5-lemma)

Given a commutative diagram with exact rows

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

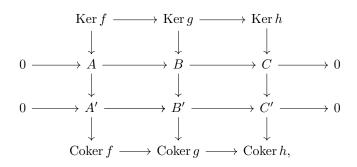
$$\downarrow^f \qquad \downarrow^g \qquad \downarrow^h$$

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0,$$

suppose f and h are isomorphisms. Prove that g is an isomorphism.

Problem 1.2 (Snake lemma)

Given the diagram with exact rows



show that the sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} h \longrightarrow 0$$

is exact.

CARLOS SALINAS PROBLEM 1.3

Problem 1.3

Given an exact sequence of modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

fix any $M \in R$ -Mod.

(a) Show that

$$0 \longrightarrow \operatorname{Hom}(M, A) \longrightarrow \operatorname{Hom}(M, B) \longrightarrow \operatorname{Hom}(M, C)$$

is exact.

(b) Show that

$$0 \longrightarrow \operatorname{Hom}(C,M) \longrightarrow \operatorname{Hom}(B,M) \longrightarrow \operatorname{Hom}(A,M)$$

is exact.

CARLOS SALINAS PROBLEM 1.4

Problem 1.4

A short exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{p} 0$$

splits if there is a homomorphism $s \colon C \to B$ called the *splitting* such that $p \circ s = \mathrm{id}_C$. In which case, we can put

$$0 \longrightarrow \operatorname{Hom}(C,M) \longrightarrow \operatorname{Hom}(B,M) \longrightarrow \operatorname{Hom}(A,M) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Hom}(M,A) \longrightarrow \operatorname{Hom}(M,B) \longrightarrow \operatorname{Hom}(M,C) \longrightarrow 0.$$

CARLOS SALINAS PROBLEM 1.5

Problem 1.5

Find an example which shows that $\operatorname{Hom}(-,M)$ is *not* exact.

Solution. Consider the following example: let $A = B = M = \mathbb{Z}$ and $C = \mathbb{Z}/2\mathbb{Z}$ then the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f=2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact, however Hom