MA571 Homework 11

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Problem 11.1 (Munkres §53, Ex. 7(abcd))

Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]

(d) Show that $\pi_1(G, x_0)$ is Abelian.

Proof. For part (a) we need to show that the operation (0) $\otimes$ is associative, (1) $\Omega(G, x_0)$ is closed under $\otimes$, (2) $\Omega(G, x_0)$ contains an identity element e and (3) for every $f \in \Omega(G, x_0)$ there exists an element $\bar{f} \in \Omega(G, x_0)$ such that $f \otimes \bar{f} = \bar{f} \otimes f = e$. We shall proceed in order: (0) Let $f, g, h \in \Omega(G, x_0)$. Then $(f \otimes g) \otimes h = f \otimes (g \otimes f)$ since the multiplication $\cdot$ is associative in G, i.e., since given $t \in I$ we have $(f(t) \cdot g(t)) \cdot h(t) = f(t) \cdot (g(t) \cdot h(t))$, in particular this holds for all $\in I$. (1) Let f and g be loops at x_0 then $f \otimes g = f(s) \cdot g(s)$ ■

PROBLEM 11.2 ((A))

Prove Proposition F from the note on the Fundamental Group of the Circle.

Proof. Recall proposition F:

Proposition F. (i) *W takes the class of the path $f_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ to n (and therefore W is onto).*

(ii) *W is one-to-one.*

(iii) *W is a homomorphism.*

(i) Now, recall that $W \colon \pi_1(S^1, x_0) \to \mathbf{Z}$ defined by $W([f]) \coloneqq w(f)$ where $w(f) = \tilde{f}(1)$ where $\tilde{f} \colon I \to \mathbf{R}$ is the lift of f, i.e. $p \circ \tilde{f} = f$. Now, let f_n be a path as above. Now, by Proposition C, since

$$f_n(s) = (\cos(2\pi ns), \sin(2\pi ns)) = (\cos(2\pi \tilde{f}_n(s)), \sin(2\pi \tilde{f}_n(s)))$$

and $\tilde{f}_n(0) = 0 = n \cdot 0$, by Proposition C, it follows that $f_n(s) = ns$. Thus, $\tilde{f}(1) = n$.

(ii) Suppose $f_1, f_2 \colon I \to S^1$ and $\tilde{f}_1(1) = \tilde{f}_2(1)$

(iii) ■

Problem 11.3 ((B))

Prove Lemma G from the note on the Fundamental Group of the Circle. (Hint: one way to do this is to use the fact, which you don't have to prove, that if $\sim$ is the equivalence relation on $[a, a+1]$ which identifies a and $a+1$ then the restriction of p induces a homeomorphism $[a, a+1]/{\sim} \to S^1$.)

Proof. Recall the statement of Lemma G:

Lemma G. *For each $a \in \mathbf{R}$, the map*

$$p_a \colon (a, a+1) \longrightarrow S^1 - p(a)$$

given by $p_a(u) = p(u)$ is a homomorphism.

We shall proceed by the hint. ■

Problem 11.4 ((C))

Show that for every point $x \in S^n$ the space $S^n - x$ is homeomorphic to $\mathbf{R}^n$. You may use the fact, shown in Step 1 of the proof of Theorem 59.3, that S^n with the *north pole* removed is homeomorphic to $\mathbf{R}^n$. (Hint: linear algebra.)

Proof. ■

Problem 11.5 ((D))

Show that every loop in S^n which is not onto is path-homotopic to a constant path. (Hint: use Problem C).

Proof. ■

Problem 11.6 ((E))

Let X be a topological space and let $A \subset X$ be a deformation retract. In the space X/A, the set A is a point (because it is an equivalence class). Show that this point is a deformation retract of X/A. (Hint: use p. 289 # 9.)

Proof. Let $H\colon X \times I \to X$ be a deformation retraction from X to A, that is, $H(0,x) = \mathrm{id}_X$ and $H(1,x) = r(x)$ where $r\colon X \to A$ is a retraction of X onto A and $\iota\colon A \hookrightarrow X$ is the inclusion of A into X. Let $p\colon X \to X/A$ be a quotient map. Now, we want to construct a deformation retraction $h\colon X/A \times I \to X/A$ from the quotient X/A to $*$, which we shall use to denote the image of A in X/A under p, and what better candidate than the map induced by $p \circ H\colon X \times I \to X/A$ on the quotient $X/A \times I$ into X/A. Consider the map $(p, \mathrm{id}_I)\colon X \times I \to X/A \times I$. This map is a quotient map by Problem 9.2 (Munkres §46, x. 9). Moreover, the map $p \circ H$ preserves the equivalence relation on $X/A \times I$ since for any two representatives (x_1, t) and (x_2, t) of $[(x,t)]$ in $X/A \times I$, we have $H(x_1,t) = H(x_2,t)$ if $x \in X - A$ and $H(x_1,t) = H_2(x_2,t)$ so $p(H(x_1,t)) = p(H(x_2,t))$ and if $x_1, x_2 \in A$ then $H(x_1,t), H(x_2,t) \in A$ so $p(H(x_1,t)) = p(H(x_2,t))$. Thus, by Theorem Q.3 the map $h\colon X/A \times I \to X/A$ induced by H, i.e., the map defined by $h(x,t) \coloneqq [H(x,t)]$, is continuous and the diagram

$$\begin{tikzcd}
X \times I \arrow[r, "H"] \arrow[d, "{(p,\mathrm{id}_I)}"'] & X \arrow[d, "p"] \\
X/A \times I \arrow[r, "h"'] & X/A
\end{tikzcd}$$

commutes. We claim that h is a deformation retraction from X/A to $*$. To that end, it suffices to show that $h(x,0) = \mathrm{id}_{X/A}$ and, using suggestive notation, $h(x,1) = \bar{r}$ where $\bar{r}\colon X/A \to *$ is a retraction of X/A onto A and $\bar{\iota}\colon * \hookrightarrow X/A$ is the inclusion of $*$ into X/A. The first is easy to verify since $h(x,0) = [H(x,0)] = [x] = \mathrm{id}_{X/A}$. Next, $h(x,1) = [H(x,1)] = [r(x)]$ and we claim that $\bar{r}(x) \coloneqq [r(x)]$ is a retraction of X/A into $*$. The map $\bar{r}$ is continuous since h is continuous (by Lemma 1 from Hw. #9 Munkres §18, Ex. 11) and $\bar{r}\colon X/A \to *$ since $r(x) \in A$ for every $x \in X$. It follows that $*$ is a deformation retract of X/A. ■