1.5.16 Find a 2x2 matrix $B \neq 0$ and $B \neq I_2$ such that $AB = BA$, where $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. How many such matrices are there?

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$ and

$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix}$ so we get from $AB = BA$ that

$a+2c = a$, $b+2d = 2a+b$, $c = c$, $2c+d = d$ so $2c = 0$ or $c = 0$

and $a = d$ so B is of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Thus as long as $b \neq 0$ with $a=0$ and $a \neq 1$ with $b=0$ and such matrix $B = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ works. Thus there are infinitely such matrices.

1.5.30 Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 6 & 2 \\ 5 & 1 & 3 \end{bmatrix}$. Find the matrices S and K described in Exercise 29.

Exercise 29 gives $A = S + K$ where $S = \frac{1}{2}(A + A^T)$ is symmetric and $K = \frac{1}{2}(A - A^T)$ is skew symmetric. Then

$$S = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 1 & 3 & -2 \\ 4 & 6 & 2 \\ 5 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 5 \\ 3 & 6 & 1 \\ -2 & 2 & 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & 7 & 3 \\ 7 & 12 & 3 \\ 3 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 7/2 & 3/2 \\ 7/2 & 6 & 3/2 \\ 3/2 & 3/2 & 3 \end{bmatrix}$$ and

$$K = \frac{1}{2}(A - A^T) = \frac{1}{2}\left(\begin{bmatrix} 1 & 3 & -2 \\ 4 & 6 & 2 \\ 5 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 5 \\ 3 & 6 & 1 \\ -2 & 2 & 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & -1 & -7 \\ 1 & 0 & 1 \\ 7 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & -7/2 \\ 1/2 & 0 & 1/2 \\ 7/2 & -1/2 & 0 \end{bmatrix}.$$

1.5.32 If $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find D^{-1}.

If $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} a & b & c \\ e & f & g \\ h & i & j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\begin{bmatrix} 4a & 4b & 4c \\ -2e & -2f & -2g \\ 3h & 3i & 3j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So $a = 1/4$, $f = -1/2$, $j = 1/3$

and $b = c = e = g = h = i = 0$ So $D^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$.

1.5.35 If $A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix}$, find $(AB)^{-1}$.

By Thm 1.6 $(AB)^{-1} = B^{-1}A^{-1}$ as they are invertible (they are inverses as stated).

Then $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 2 & 5 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 19 \\ 7 & 0 \end{bmatrix}$

1.5.36 Suppose that $A^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. Solve the linear system $A\vec{x} = \vec{b}$ for each of the following matrices $\vec{b}$: (a) $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$.

Note if A is invertible, then the solutions $\vec{x} = A^{-1}\vec{b}$.

(a) $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 22 \end{bmatrix}$ (b) $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 15 \end{bmatrix} = \begin{bmatrix} 38 \\ 53 \end{bmatrix}$.

2.1.2 Find a row echelon form for each of the given matrices. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} -1 & 1 & -1 & 0 & 3 \\ -3 & 4 & 1 & 1 & 10 \\ 4 & -6 & -4 & -2 & -14 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & -4 \\ -2 & -1 & 10 \\ 4 & 3 & -12 \end{bmatrix}$.

(a)
$$A = \begin{bmatrix} -1 & 1 & -1 & 0 & 3 \\ -3 & 4 & 1 & 1 & 10 \\ 4 & -6 & -4 & -2 & -14 \end{bmatrix} \xrightarrow{-r_1 \to r_1} \begin{bmatrix} 1 & -1 & 1 & 0 & -3 \\ -3 & 4 & 1 & 1 & 10 \\ 4 & -6 & -4 & -2 & -14 \end{bmatrix} \xrightarrow[-4r_1 + r_3 \to r_3]{3r_1 + r_2 \to r_2} \begin{bmatrix} 1 & -1 & 1 & 0 & -3 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & -2 & -8 & -2 & -2 \end{bmatrix} \xrightarrow{2r_2 + r_3 \to r_3}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & -3 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ which is in row echelon form in Defn 2.1 (a)-(c).

(b)
$$A = \begin{bmatrix} 1 & 1 & -4 \\ -2 & -1 & 10 \\ 4 & 3 & -12 \end{bmatrix} \xrightarrow[-4r_1 + r_3 \to r_3]{2r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix} \xrightarrow{r_2 + r_3 \to r_3} \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$
which is in row echelon form.

2.1.6 Find the reduced row echelon form of each of the given matrices. Record the row operations you perform, using the notation for elementary row operations.

(a) $A = \begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 2 & -2 & 7 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & -1 \\ 5 & 6 & -3 \\ -2 & -2 & 2 \end{bmatrix}$.

(a)
$$A = \begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 2 & -2 & 7 \end{bmatrix} \xrightarrow{-r_1 \to r_1} \begin{bmatrix} 1 & -2 & 5 \\ 2 & -1 & 6 \\ 2 & -2 & 7 \end{bmatrix} \xrightarrow[-2r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow{-r_3 + r_2 \to r_2} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -1 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow[-2r_2 + r_3 \to r_3]{2r_2 + r_1 \to r_1}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{-r_3 \to r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[r_3 + r_2 \to r_2]{-r_3 + r_1 \to r_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b)
$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & -1 \\ 5 & 6 & -3 \\ -2 & -2 & 2 \end{bmatrix} \xrightarrow[\substack{-5r_1 + r_3 \to r_3 \\ 2r_1 + r_4 \to r_4}]{-3r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-r_2 + r_3 \to r_3} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-r_2 + r_1 \to r_1} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.1.8 Let x, y, z, and w be nonzero real numbers. Label each of the following matrices REF if it is in row echelon form, RREF if it is in reduced row echelon form, or N if it is not REF and not RREF.

(a) $\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & y & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) REF — Defn 2.1 Satisfies (a), (b), (c)

(b) RREF — Defn 2.1 Satisfies (a) – (d)

(c) N — Satisfies none of Defn 2.1

2.1.10 Prove:

(a) Every matrix is row equivalent to itself.

(b) If B is row equivalent to A, then A is row equivalent to B

(c) If C is row equivalent to B and B is row equivalent to A, then C is row equivalent to A.

(a) Every matrix is row equivalent to itself as it is produced by no elementary row operations (0 is finite).

(b) Note each row operation has an inverse operation.

I: $r_i \leftrightarrow r_j$ is undone by $r_j \leftrightarrow r_i$

II: $kr_i \rightarrow r_i$ is undone by $\frac{1}{k} r_i \rightarrow r_i$ for $k \neq 0$.

III: $kr_i + r_j \rightarrow r_j$ is undone by $-kr_i + r_j \rightarrow r_j$.

Let $O_1, O_2, \ldots, O_k$ be the operations taking B to A applied in the order listed and let O_i^{-1} be their inverse operation. Then $O_k^{-1}, O_{k-1}^{-1}, \ldots, O_1^{-1}$ applied to A in this order undoes each operation on B giving the matrix B. Hence A is row equivalent to B.

(c) Suppose C is row equivalent to B by operations $O_1, \ldots, O_k$ in this order and B is row equivalent to A by operations $O_1', \ldots, O_\ell'$ applied in this order. Then the operations $O_1, \ldots, O_k, O_1', \ldots, O_\ell'$ applied in this order take C to A so C is row equivalent to A.

2.1.12 Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & -1 & 2 \\ 3 & 1 & 2 & 4 & 1 \end{bmatrix}$.

(a) Find a matrix in column echelon form that is column equivalent to A.

(b) Find a matrix in reduced column echelon form that is column equivalent to A.

(a)
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & -1 & 2 \\ 3 & 1 & 2 & 4 & 1 \end{bmatrix} \xrightarrow[\substack{-4c_1 + c_4 \to c_4 \\ -5c_1 + c_5 \to c_5}]{\substack{-2c_1 + c_2 \to c_2 \\ -3c_1 + c_3 \to c_3}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & -3 & -9 & -8 \\ 3 & -5 & -7 & -8 & -14 \end{bmatrix} \xrightarrow{-c_4 + c_5 \to c_5} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & -3 & -9 & 1 \\ 3 & -5 & -7 & -8 & -6 \end{bmatrix} \xrightarrow{c_2 \leftrightarrow c_5}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & -3 & -9 & -3 \\ 3 & -6 & -7 & -8 & -5 \end{bmatrix} \xrightarrow[\substack{3c_2 + c_5 \to c_5}]{\substack{3c_2 + c_3 \to c_3 \\ 9c_2 + c_4 \to c_4}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & -6 & -25 & -62 & -23 \end{bmatrix} \xrightarrow{-\frac{1}{25}c_3 \to c_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & -6 & 1 & -62 & -23 \end{bmatrix} \xrightarrow[\substack{23c_3 + c_5 \to c_5}]{\substack{62c_3 + c_4 \to c_4}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & -6 & 1 & 0 & 0 \end{bmatrix}$$ is in column echelon form.

(b) continuing.

$$\xrightarrow[\substack{6c_3 + c_2 \to c_2}]{\substack{-3c_3 + c_1 \to c_1}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{-2c_2 + c_1 \to c_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$ is in reduced column echelon form.

Matlab 3.1

Matlab 3.1.1 With matrices from routine matdat1 compute and record the results of the following matrix expressions. If an operation is not defined, state why.

$A+B = \begin{bmatrix} 7 & 0 & 4 \\ 0 & 4 & 5 \\ 2 & 4 & 2 \end{bmatrix}$ $B-D$ is not defined as B is a 3x3 while D is a 2x3

$A*B = \begin{bmatrix} 17 & -1 & 13 \\ 22 & -10 & 3 \\ -3 & 16 & -2 \end{bmatrix}$ $B*A = \begin{bmatrix} 3 & 17 & 16 \\ -4 & 9 & 17 \\ 22 & -10 & -7 \end{bmatrix}$ $D*C = \begin{bmatrix} -16 & 12 & 24 \\ -25 & 19 & 46 \end{bmatrix}$

$C' = \begin{bmatrix} 1 & 0 & -5 \\ -1 & 1 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ $C*x = \begin{bmatrix} -3 \\ 7 \\ 25 \end{bmatrix}$ $x*x$ is not defined as x is a 3x1

$x'*x = 14$ $((A-B)*x)' = \begin{bmatrix} -20 & -13 & 48 \end{bmatrix}$ $A^2 = \begin{bmatrix} 20 & -3 & -1 \\ -7 & 26 & 9 \\ -14 & 20 & 29 \end{bmatrix}$

$A*A = \begin{bmatrix} 20 & -3 & -1 \\ -7 & 26 & 9 \\ -14 & 20 & 29 \end{bmatrix}$ $6*D = \begin{bmatrix} -6 & 12 & 18 \\ 0 & 24 & 30 \end{bmatrix}$ $5*A-3*B = \begin{bmatrix} 19 & -16 & -4 \\ 8 & -12 & 17 \\ -30 & 44 & 10 \end{bmatrix}$

Matlab 3.1.2 $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 4 & -2 \\ 7 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 5 \\ -5 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 5 \\ 2 & -1 & 6 \end{bmatrix}$.

Perform the following matrix algebra computations in Matlab. Record your results.

(a) $A+B = \begin{bmatrix} 0 & 5 \\ 6 & 2 \\ 10 & 0 \end{bmatrix}$ (b) $B+C$ dimensions must agree   (c) $D*A = \begin{bmatrix} 4 & 22 \\ 16 & 8 \\ 18 & 8 \end{bmatrix}$

(d) $2*A-3*B = \begin{bmatrix} 5 & 0 \\ -8 & 14 \\ -15 & 5 \end{bmatrix}$ (e) $A' = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \end{bmatrix}$ (f) $C^2 = \begin{bmatrix} -24 & 20 \\ -20 & -16 \end{bmatrix}$

Matlab 3.1.5 Let A and X be the matrices defined below. $A = \begin{bmatrix} 6 & -1 & 1 \\ 0 & 13 & -16 \\ 0 & 8 & -11 \end{bmatrix}$, $X = \begin{bmatrix} 10.5 \\ 21.0 \\ 10.5 \end{bmatrix}$

(a) Determine a scalar r s.t. $AX = rX$. $r = 5$

$A*X = \begin{bmatrix} 52.5 \\ 105.0 \\ 52.5 \end{bmatrix} = rX$ $52.5 = r10.5$ so $r = 5$

(b) Compute $AX - rX$ for the value of r from part (a). $AX - rX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) Is it true $A'X = rX$ for the value of r determined in part (a)? No

$A'X - 5X = \begin{bmatrix} 10.5 \\ 241.5 \\ -493.5 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$