

MA 523: Homework 8

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PROBLEM 8.1

Show that the function

$$u(x, t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x - 2k, t)$$

where

$$\Phi(x, t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

is positive for $|x| < 1$, $t > 0$.

(*Hint*: Show that u satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{cases} u = 0 & \text{on } \{|x| = 1\} \times \{t \geq 0\}, \\ u = \delta_0 & \text{on } \{|x| \leq 1\} \times \{t = 0\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{|x| \leq 1\} \times \{\varepsilon \leq t \leq T\}$ for small $\varepsilon > 0$ and large $T > 0$ pass to the limit as $\varepsilon \rightarrow 0^+$ and $T \rightarrow \infty$.)

SOLUTION. Taking the hint, let us verify that $u_t = u_{xx}$, for $t > 0$. By direct computation, we have

$$\begin{aligned} \Phi_x(x, t) &= \frac{\partial}{\partial x} \left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \right) & \Phi_{xx}(x, t) &= \frac{\partial}{\partial x} \left(-\frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}} \right) \\ &= -\frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}}, & &= \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}} \\ & & &= \frac{(x^2 - 2t) e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}}, \end{aligned}$$

and

$$\begin{aligned} \Phi_t(x, t) &= \frac{\partial}{\partial t} \left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \right) \\ &= \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}} \\ &= \frac{(x^2 - 2t) e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}}. \end{aligned}$$

Since $\Phi_t = \Phi_{xx}$ it follows that $u_t = u_{xx}$ (assuming uniform convergence of u).

Next we show that $u = 0$ on $\{|x| = 1\} \times \{t \geq 0\}$ and $u = \delta_0$ on $\{|x| = 1\} \times \{t = 0\}$. To show $u = 0$ fix a $t \geq 0$ and, after relabeling if necessary, assume that $x = 1$ which gives us

$$\begin{aligned} u(1, t) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-\frac{(1-2k)^2}{4t}}}{\sqrt{4\pi t}} \\ &= \frac{1}{\sqrt{4\pi t}} \left(\dots - e^{-\frac{9}{4t}} + e^{-\frac{1}{4t}} - e^{-\frac{1}{4t}} + e^{-\frac{9}{4t}} + \dots \right) \\ &= 0. \end{aligned}$$

Similarly for $u(-1, t) = 0$.

For $u(|x| \leq 1, 0)$, we have a

$$\begin{aligned} u(|x| \leq 1, 0) &= \sum_{k=-\infty}^{\infty} (-1)^k \lim_{t \rightarrow 0^+} \left[e^{-\frac{(x-2k)^2}{4t}} / \sqrt{4\pi t} \right] \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \delta_0(x - 2k) \\ &= \delta_0(x) \end{aligned}$$

since $|x| \leq 1$.

Now, ■

PROBLEM 8.2 (TIKHONOV'S EXAMPLE)

Let

$$g(t) := \begin{cases} e^{-t^{-2}} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then $g \in C^\infty(\mathbb{R})$ and we define

$$u(x, t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that $u(x, t)$ solves the heat equation in $\mathbb{R} \times (0, \infty)$ with the initial condition $u(x, 0) = 0$, $x \in \mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

SOLUTION. Let u be as above. Then

$$\begin{aligned} u_t(x, t) &= \frac{\partial}{\partial t} \left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} \\ &= \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}, \end{aligned}$$

and

$$\begin{aligned} u_x(x, t) &= \frac{\partial}{\partial x} \left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2k x^{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}, \\ u_{xx}(x, t) &= \frac{\partial}{\partial x} \left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1} \right) \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} (2k-1) x^{2k-2} + \frac{\partial}{\partial x} g^{(0)}(t) \\ &= \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}. \end{aligned}$$

Thus, $u_t - \Delta u = 0$; i.e., u solves the heat equation. As this example shows, unless some assumptions on u such as subexponential (cf. [E §2.3], Theorem 7) growth is assumed. ■

PROBLEM 8.3

Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax)e^{-x^2} dx, \quad (a > 0).$$

(*Hint:* Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.)

SOLUTION. By separation of variables,

$$u(x, t) = \cos(ax)e^{-a^2 t}$$

is a solution to the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = \cos(ax) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

However, the convolution

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \cos(ay)e^{-\frac{|x-y|^2}{4t}} dy$$

is also a solution to the Cauchy problem. Now note that

$$\begin{aligned} \int_{-\infty}^{\infty} \cos(ay)e^{-y^2} dy &= \sqrt{\pi} \cdot u(0, \tfrac{1}{4}) \\ &= \sqrt{\pi} e^{-\frac{a^2}{4}}. \end{aligned}$$

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