MA166: Exam 2 Prep

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As I promised, here are somewhat detailed solutions to two of the last Exam 2's for MA 166.

1 MA 166 Exam 2, Spring 2014

Problem 1.1. Evaluate the following integral

$$\int_0^\pi \sin^2 x \cos^2 x \, dx$$

Solution. The following trig identities are useful

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{1}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}. \tag{2}$$

With that in mind we compute the integral

$$\begin{aligned}
\int_0^\pi \sin^2 x \cos^2 x \, dx &= \int_0^\pi (\sin x \cos x)^2 \, dx \\
&= \int_0^\pi \left(\frac{1}{2}\sin 2x\right)^2 \, dx \\
&= \frac{1}{4}\int_0^\pi \sin^2 2x \, dx \\
&= \frac{1}{4}\int_0^\pi \frac{1-\cos 4x}{2} \, dx \\
&= \frac{1}{8}\int_0^\pi 1-\cos 4x \, dx \\
&= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^\pi \\
&= \frac{1}{8}[\pi - 0 - (0-0)] \\
&= \boxed{\frac{\pi}{8}.}
\end{aligned}$$

Answer: B. ☺

Problem 1.2. Evaluate the following integral

$$\int_0^{\pi/4} \sec^4 x \tan x \, dx.$$

Solution. The following identities are useful

$$\sec^2\theta - \tan^2\theta = 1. \tag{3}$$

Substitute $u = \tan x$, $du = \sec^2 x\ dx$

$$\begin{aligned}
\int_0^{\pi/4} \sec^4 x \tan x\ dx &= \int_0^{\pi/4} \sec^2 x \sec^2 x \tan x\ dx \\
&= \int_0^{\pi/4} \sec^2 x(1+\tan^2 x)\tan x\ dx \\
&= \int_0^1 (1+u^2)u\ dx \\
&= \int_0^1 u + u^3\ dx \\
&= \left[\frac{1}{2}u^2 + \frac{1}{4}u^4\right]_0^1 \\
&= \frac{1}{2} + \frac{1}{4} - 0 - 0 \\
&= \boxed{\frac{3}{4}.}
\end{aligned}$$

Answer: A. ☺

Problem 1.3. After the appropriate trigonometric substitution, the integral

$$\int_2^5 \frac{dx}{\sqrt{x^2 - 4x + 13}}$$

becomes

Solution. The general strategy for these types of problems is to complete the square in the denominator and make some sort of trig substitution

$$x^2 - 4x + 13 = \left(x^2 - 4x + 4\right) + 9 = (x-2)^2 + 9.$$

Make the u-substitution $u = (x-2)/3$, $du = dx/3$

$$\begin{aligned}
\int_2^5 \frac{dx}{\sqrt{x^2-4x+13}} &= \int_2^5 \frac{dx}{3\sqrt{(x-2)^2/9+1}} \\
&= \frac{1}{3}\int_2^5 \frac{dx}{\sqrt{\left(\frac{x-2}{3}\right)^2+1}} \\
&= \int_0^1 \frac{du}{\sqrt{u^2+1}}
\end{aligned}$$

follow it up with the trig substitution $\tan\theta = u$, $\sec^2\theta\ d\theta = du$, $0 \le \theta \le \pi/4$

$$= \int_0^{\pi/4} \sec^2\theta \cos\theta\ d\theta$$

$$
\begin{aligned}
&= \int_0^{\pi/4} \sec\theta \, d\theta \\
&= [\ln|\sec\theta + \tan\theta|]_0^{\pi/4} \\
&= \ln\left|\frac{\sec\pi/4 + \tan\pi/4}{\sec 0 + \tan 0}\right| \\
&= \ln\left|\frac{\sqrt{2}+1}{1+0}\right| \\
&= \boxed{\ln\left|\sqrt{2}+1\right|}
\end{aligned}
$$

Answer: A. ☺

Problem 1.4. Compute

$$\int_3^4 \frac{3}{x^2 - x - 2} \, dx.$$

Solution. Factor

$$x^2 - x - 2 = (x-2)(x+1)$$

and use partial fractions

$$
\begin{aligned}
\frac{3}{(x-2)(x+1)} &= \frac{A}{x-2} + \frac{B}{x+1} \\
3 &= A(x+1) + B(x-2) \\
0x + 3 &= (A+B)x + A - 2B
\end{aligned}
$$

gives you $A - 2B = 3$, $A + B = 0$ so $A = -B$, $-B - 2B = 3$, $B = -1$ and $A = 1$. Now we can compute the integral

$$
\begin{aligned}
\int_3^4 \frac{3}{x^2-x-2} \, dx &= \int_3^4 \left[\frac{1}{x-2} - \frac{1}{x+1}\right] dx \\
&= [\ln|x-2| - \ln|x+1|]_3^4 \\
&= \left[\ln\left|\frac{x-2}{x+1}\right|\right]_3^4 \\
&= \ln\left|\frac{2}{5}\right| - \ln\left|\frac{1}{4}\right| \\
&= \ln\left|\frac{2/5}{1/4}\right| \\
&= \boxed{\ln\left|\frac{8}{5}\right|}.
\end{aligned}
$$

Remember your log properties!

Answer: B. ☺

Problem 1.5. It is known that

$$\int \frac{2x-3}{x(x^2+1)} dx = a\ln x + b\ln\left(x^2+1\right) + c\tan x + C$$

for some constants a, b, c and C. What is b?

Solution. There is a typo in the original problem; instead of $c\tan x$ it should be a $c\tan^{-1} x$. One thing you can do is use the fundamental theorem of calculus

$$f(x) = \frac{d}{dt}\int_a^x f(t)\,dt. \tag{4}$$

Applying the fundamental theorem on our function, we get

$$\begin{aligned}\frac{2x-3}{x(x^2+1)} &= \frac{a}{x} + \frac{2bx}{x^2+1} + \frac{c}{x^2+1}\\ &= \frac{a}{x} + \frac{2bx+c}{x^2+1}\\ &= \frac{a\left(x^2+1\right)+(2bx+c)x}{x(x^2+1)}\\ &= \frac{(a+2b)x^2+cx+a}{x(x^2+1)}.\end{aligned}$$

Now we solve for the values in the numerator by noting that $a+2b=0$, $c=2$ and $a=-3$, so $b=3/2$.

Answer: E. ☺

Problem 1.6. Evaluate the integral

$$\int \frac{x^2+5x+1}{\left(x^2+1\right)^2} dx.$$

Solution. Remember, you cannot use the method of partial fractions if the highest term in the numerator is greater than or equal to the highest term in the denominator, here we have x^2 on the top and x^4 on the bottom, if you expand the square, so we should be okay. Then by partial fractions, we can write

$$\frac{x^2+5x+1}{\left(x^2+1\right)^2} = \frac{A}{x^2+1} + \frac{B}{\left(x^2+1\right)^2}$$

☺

Problem 1.7.

Solution. ☺

Problem 1.8.

Solution. ☺

Problem 1.9.

Solution. ☺

Problem 1.10.

Solution. ☺

Problem 1.11.

Solution. ☺

Problem 1.12.

Solution. ☺

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Problem 2.1.

Solution. ☺

Problem 2.2.

Solution. ☺

Problem 2.3.

Solution. ☺

Problem 2.4.

Solution. ☺

Problem 2.5.

Solution. ☺

Problem 2.6.

Solution. ☺

Problem 2.7.

Solution. ☺

Problem 2.8.

Solution. ☺

Problem 2.9.

Solution. ☺

Problem 2.10.

Solution. ☺

Problem 2.11.

Solution. ☺

Problem 2.12.

Solution. ☺