MA571 Problem Set 7

Carlos Salinas

October 17, 2015

Problem 7.1 (Munkres §26, Ex. 8)

Theorem. *Let $f\colon X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the* graph *of f,*

$$G_f = \{\, (x, f(x)) \mid x \in X \,\},$$

is closed in $X \times Y$.

[*Hint:* If G_f is closed and V is a neighborhood of $f(x_0)$, then the intersection of G_f and $X \times (Y - V)$ is closed. Apply Exercise 7.]

Proof. As we demonstrated in Problem 2.7 (Munkres §18, Ex. 17) Y is Hausdorff if and only if the diagonal, $\Delta_Y = \{\, (y, y) \mid y \in Y \,\}$, is a closed subset of $Y \times Y$. Consider the map $F\colon X \times Y \to Y \times Y$ defined by $(x, y) \mapsto (f(x), y)$. This map is continuous by Theorem 18.4 as f is, by assumption, continuous and id_Y is continuous by 18.2(b) (since it is the inclusion $Y \hookrightarrow Y$). Then

$$\begin{aligned} F^{-1}(\Delta_Y) &= \{\, (x, y) \mid F(x, y) \in \Delta_Y,\ x \in X,\ y \in Y \,\} \\ &= \{\, (x, y) \mid (f(x), y) \in \Delta_Y,\ x \in X,\ y \in Y \,\} \\ &= \{\, (x, y) \mid f(x) = y,\ x \in X,\ y \in Y \,\} \\ &= \{\, (x, f(x)) \mid x \in X,\ y \in Y \,\} \\ &= G_f \end{aligned}$$

is closed by Theorem 18.1(3).

Conversely, suppose G_f is closed in $X \times Y$. Fix a point $x_0 \in X$ and let $V \subset Y$ be an arbitrary neighborhood of $f(x_0)$. Then $Y - V$ is a closed subset of Y so, by Problem 2.1 (Munkres §17, Ex. 3), the product $X \times (Y - V)$ is closed in $Y \times Y$. In particular, by Theorem 17.1(2), the intersection $B = G_f \cap X \times (Y - V)$ is closed in $X \times Y$. Thus, by Problem 6.5 (Munkres §26, Ex. 7), since Y is a compact Hausdorff space, the projection $\pi_1(B)$ onto X is a closed subset of X. But

$$\begin{aligned} B &= \{\, (x, y) \mid (x, y) \in G_f \text{ and } (x, y) \in X \times (Y - V) \,\} \\ &= \{\, (x, y) \mid y = f(x) \text{ and } (x, y) \in X \times (Y - V)\} \\ &= \{\, (x, f(x)) \mid f(x) \in Y - V \,\} \end{aligned}$$

so we have that $\pi_1(B) = f^{-1}(Y - V) = X - f^{-1}(V)$. One containment is easy to see, namely "$\subset$": if $x \in B$ then $x = \pi_1(x, f(x))$ for at least one $f(x) \in Y - V$. To see the reverse inclusion, take $x \in f^{-1}(Y - V)$, then $f(x) \in Y - V$ so $(x, f(x)) \in B$, hence $x \in \pi_1(B)$. Thus, $X - \pi_1(B) = f^{-1}(V)$ is open so f is continuous. ■

PROBLEM 7.2 (MUNKRES §26, EX. 9)

Generalize the tube lemma as follows:

Theorem. *Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that*

$$A \times B \subset U \times V \subset N.$$

Proof. We first prove the theorem for the case that $A = \{a\}$.

■

Problem 7.3 (Munkres §26, Ex. 12)

Theorem. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then*

$$Y = \bigcap_{A \in \mathcal{A}} A.$$

Proof. ■

Problem 7.4 (Munkres §27, Ex. 2(b,d))

Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ε-neighborhood of A is contained in U.

Proof. ■

Problem 7.5 (Munkres §27, Ex. 5)

Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X. [*Hint:* Imitate the proof of Theorem 27.7.]

This is a special case of the *Baire category theorem*, which we shall study in Chapter 8.

Proof. ∎

Problem 7.6 (Munkres §28, Ex. 2(a))

Let $\{X_\alpha\}$ be a nindexed family of nonempty spaces.

(a) Show that if $\prod X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α.

Proof. ■

Problem 7.7 (Munkres §28, Ex. 10)

Show that if X is a Hausdorff space that is locally compact at the point x, then for each neighborhood U of x, there is a neighborhood V of x such that V is compact and $\overline{V} \subset U$.

Proof. ∎

Problem 7.8

Proof. ■

Problem 7.9 (A)

Let S^1 denote the circle

$$S^1 = \{ (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1 \}$$

and let B^2 denote the closed disk

$$B^2 = \{ (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1 \}.$$

Prove that the quotient space $(S^1 \times [0, 1])/(S^1 \times 0)$ (see HW #4 for the notation) is homeomorphic to B^2.

Proof. ■