MA571: Qual Problems

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December 28, 2015

1 Covering Space Problems

Compiled from Prof. McClure's old quals.

Problem 1.1.

Proof. ∎

Problem 1.2.

Proof. ∎

Problem 1.3.

Proof. ∎

Problem 1.4.

Proof. ∎

Problem 1.5.

Proof. ∎

Problem 1.6.

Proof. ∎

Problem 1.7.

Proof. ∎

1.1 Kyle's Stuff

Problem 1.8 (No. 5)**.** Let X be a topological space and let $x_0 \in X$. Let U and V be open sets containing x_0, and suppose that the hypotheses of the seifert–van Kampen theorem are satisfied. Let $i_1 \colon U \cap V \to U$, $i_2 \colon U \cap V \to V$, $j_1 \colon U \to X$, and $j_2 \colon V \to X$ be the inclusion maps. Suppose that $(i_1)_* \colon \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is onto. Prove, using the Seifert–van Kampen theorem, that $(j_2)_* \colon \pi_1(V, x_0) \to \pi_1(X, x_0)$ is onto.

Proof. We use the classical Seifert–van Kampen theorem (Theorem 70.2). Suppose $(i_1)_* \colon \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is onto. Then for every element $\gamma \in \pi_1(U, x_0)$, $\gamma = (i_1)_*(\gamma')$ for some element $\gamma' \in \pi_1(U \cap V, x_0)$. Now, let $\gamma'' \in \pi_1(X, x_0)$. By the classical Seifert–van Kapmen theorem, the map

$$j \colon \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is surjective and its kernel is the leans normal subgroup N of the free product that contanis all elements represented by words of the form $(i_1(g)^{-1}, i_2(g))$. ∎

Problem 1.9 (No. 6)**.** As in 5., but instead suppose that $(i_1)_* \colon \pi_1(U \cap V, x_0) \to \pi_1(X, x_0)$ is an isomorphism. Prove, using the Seifert–van Kampen theorem, that there is a homomorphism $\Phi \colon \pi_1(X, x_0) \to \pi_1(V, x_0)$ for which $\Phi \circ (j_2)_*$ is the identity map of $\pi_1(V, x_0)$.

Proof. ∎

2 August 2014

Problem 2.1. Let X be a topological space, let A be a subset of X, and let U be an open subset of X. Prove that $U \cap \bar{A} \subset \overline{U \cap A}$.

Proof. Let $x \in U \cap \bar{A}$. Then $x \in U$ and $x \in \bar{A}$. This means that, since U is open, by Lemma C there exist an open neighborhood V of x such that $V \subset U$. Moreover, since $x \in \bar{A}$, $V' \cap A \neq \emptyset$ for every open neighborhood V' of x. In particular, $V \cap A \neq \emptyset$. Thus, we have $V \cap U \neq \emptyset$ and $V \cap A \neq \emptyset$ so $V \cap (U \cap A) \neq \emptyset$. ∎

Problem 2.2. Let X be the following subspace of $\mathbf{R}^2$:

$$((0,1] \times [0,1]) \cup ([2,3) \times [0,1]).$$

Let $\sim$ be the equivalence relation on X with $(1,t) \sim (2,t)$ (that is $(s,t) \sim (s',t') \iff (s,t) = (s',t')$ or $t = t'$ and $\{s,s'\} = \{1,2\}$; you do *not* have to prove that this is an equivalence relation). Prove that $X/{\sim}$ is homeomorphic to $(0,2) \times [0,1]$. (*Hint*: construct maps in both directions).

Proof. ∎

Problem 2.3. Prove that there is an equivalence relation $\sim$ on the interval $[0,1]$ such that $[0,1]/{\sim}$ is homeomorphic to $[0,1] \times [0,1]$. As part of your proof *explain* how you are using one or more properties of the quotient topology.

Proof. ∎

Problem 2.4. Let D be the closed unit disk in $\mathbf{R}^2$, that is, the set

$$\{\, (x,y) \mid x^2 + y^2 \leq 1 \,\}.$$

Let E be the open unit disk

$$\{\, (x,y) \mid x^2 + y^2 < 1 \,\}.$$

Let X be the one-point compactification of E, and let $f \colon D \to X$ be the map defined by

$$f(x,y) = \begin{cases} (x,y) & \text{if } x^2 + y^2 < 1 \\ \infty & \text{if } x^2 + y^2 = 1. \end{cases}$$

Prove that f is continuous.

Proof. ∎

Problem 2.5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is path-connected. Prove that Y is path-connected.

Proof. ∎

Problem 2.6. Let a and b denote the points $(-1,0)$ and $(1,0)$ in $\mathbf{R}^2$. Let x_0 denote the origin $(0,0)$. Use the Seifert–van Kampen theorem to calculate $\pi_1\big(\mathbf{R}^2 - \{a,b\}, x_0\big)$. You may not use any other method.

Proof. ∎

Problem 2.7. Let $p\colon E \to B$ be a covering map with B locally connected, and let $x \in B$. Prove that x has a neighborhood W with the following property: for every connected component C of $p^{-1}(W)$, the map $p\colon C \to W$ is a homeomorphism.

Proof. ∎