MA553: Qual Preparation

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Contents

CHAPTER 1

MA 553 Spring 2016

This is material from the course MA 533 as it was taught in the spring of 2016.

1.1 Homework

Most of the homework is Ulrich original (or as original as elementary exercises in abstract algebra can be). However, an excellent resource and one that I will often quote on these solutions is [3]. Other resources include [1] and (to a lesser extent) [2].

1.1.1 Homework 1

Problem 1. Let G be a group, $a \in G$ an element of finite order m, and n a positive integer. Prove that

$$|a^n| = \frac{m}{\gcd(m,n)}.$$

Proof. Without loss of generality, we may assume $n < m$; otherwise, by the fundamental theorem of arithmetic, there exist q and r with $r < m$ such that $n = qm + r$ so $a^n = a^{qm+r} = a^{qm}a^r = a^r$. ■

Problem 2. Let G be a group, and let a, b be elements of finite order m, n respectively. Show that if $ba = ab$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$, then $|ab| = \operatorname{lcm}(m,n)$.

Proof. ■

Problem 3. Let G be a group and H, K normal subgroups with $H \cap K = \{e\}$. Show that

(a) $hk = kh$ for every $h \in H$, $k \in K$.

(b) HK is a subgroup of G with $HK \simeq H \times K$.

Proof. ■

Problem 4. Show that A_4 has no subgroup of order 6 (although $6 \mid 12 = |A_4|$).

Proof. ■

1.1.2 Homework 2

Problem 1. Let G be the group of order $2^3 \cdot 3$, $n \geq 2$. Show that G has a normal 2-subgroup $\neq \{e\}$.

Proof. ■

Problem 2. Let G be a group of order p^2q, p and q primes. Show that the Sylow p-Sylow subgroup or the q-Sylow subgroup of G is normal in G.

Proof. ■

Problem 3. Let G be a subgroup of order pqr, $p < q < r$ primes. Show that the r-Sylow subgroup of G is normal in G.

Proof. ■

Problem 4. Let G be a group of order n and let $\varphi\colon G \to S_n$ be given by the action of G on G via translation.

(a) For $a \in G$ determine the number and the lengths of the disjoint cycles of the permutation $\varphi(a)$.

(b) Show that $\varphi(G) \not\subset A_n$ if and only if n is even and G has a cyclic 2-Sylow subgroup.

(c) If $n = 2m$, m odd, show that G has a subgroup of index 2.

Proof. ■

Problem 5. Show that the only simple groups $\neq \{e\}$ of order < 60 are the groups of prime order.

Proof. ■

1.1.3 Homework 3

Problem 1. Let G be a finite group, p a prime number, N the intersection of all p-Sylow subgroups of G. Show that N is a normal p-subgroup of G and that every normal p-subgroup of G is contained in N.

Proof. ■

Problem 2. Let G be a group of order 231 and let H be an 11-Sylow subgroup of G. Show that $H \subset Z(G)$.

Proof. ■

Problem 3. Let $G = \{e, a_1, a_2, a_3\}$ be a non-cyclic group of order 4 and define $\varphi\colon S_3 \to \mathrm{Aut}(G)$ by $\varphi(\sigma)(e) = e$ and $\varphi(\sigma)(a_1) = a_{\sigma(i)}$. Show that φ is well-defined and an isomorphism of groups.

Proof. ■

Problem 4. Determine all groups of order 18.

Proof. ■

1.1.4 Homework 4

Problem 1. Let p be a prime and let G be a nonAbelian group of order p^3. Show that $G' = Z(G)$.

Proof. ■

Problem 2. Let p be an odd prime and let G be a nonAbelian group of order p^3 having an element of order p^2. Show that there exists an element $b \notin \langle a \rangle$ of order p.

Proof. ■

Problem 3. Let p be an odd prime. Determine all groups of order p^3.

Proof. ■

Problem 4. Show that $(S_n)' = A_n$.

Proof. ■

Problem 5. Show that every group of order < 60 is solvable.

Proof. ■

Problem 6. Show that every group of order 60 that is simple (or not solvable) is isomorphic to A_5.

Proof. ■

1.1.5 Homework 5

Problem 1. Find all composition series and the composition factors of D_6.

Proof. ∎

Problem 2. Let T be the subgroup of $\mathrm{GL}(n, \mathbf{R})$ consisting of all upper triangular invertible matrices. Show that T is solvable.

Proof. ∎

Problem 3. Let $p \in \mathbf{Z}$ be a prime number. Show:

(a) $(p-1)! \equiv -1 \mod p$.

(b) If $p \equiv 1 \mod 4$ then $X^2 \equiv -1 \mod p$ for some $X \in \mathbf{Z}$.

Proof. ∎

Problem 4. (a) Show that the following are equivalent for an odd prime number $p \in \mathbf{Z}$:

(i) $p \equiv 1 \mod 4$.

(ii) $p = a^2 + b^2$ for some a, b in $\mathbf{Z}$.

(iii) p is not prime in $\mathbf{Z}[i]$.

(b) Determine all prime ideals of $\mathbf{Z}[i]$.

Proof. ∎

1.1.6 Homework 6

1.1.7 Homework 7

Problem 1. Let $k \subset K$ and $k \subset L$ be finite field extensions contained in some field. Show that:

(a) $[KL : L] \leq [K : k]$.

(b) $[KL : k] \leq [K : k][L : k]$.

(c) $K \cap L = k$ if equality holds in (b).

Proof. ■

Problem 2. Let k be a field of characteristic $\neq 2$ and a, b elements of k so that a, b, ab are not squares in k. Show that $\left[k(\sqrt{a}, \sqrt{b}) : k\right] = 4$.

Proof. ■

Problem 3. Let R be a UFD, but not a field, and write $K := \operatorname{Quot}(R)$. Show that $[\bar{K} : k] = \infty$.

Proof. ■

Problem 4. Let $k \in K$ be an algebraic field extension. Show that every k-homomorphism $\delta \colon K \to K$ is an isomorphism.

Proof. ■

Problem 5. Let K be the splitting field of $X^6 - 4$ over $\mathbf{Q}$. Determine K and $[K : \mathbf{Q}]$.

Proof. ■

1.1.8 Homework 8

1.1.9 Homework 9

Problem 1. Let $k \subset K$ be a finite extension of fields of characteristic $p > 0$. Show that if $p \nmid [K : k]$, then $k \subset K$ is separable.

Proof. ■

Problem 2. Let $k \subset K$ be an algebraic extension of fields of characteristic $p > 0$, let L be an algebraically closed field containing K, and let $\delta\colon k \to L$ be an embedding. Show that $k \subset K$ is purely inseparable if and only if there exists exactly one embedding $\tau\colon K \to L$ extending δ.

Proof. ■

Problem 3. Let $k \subset K = k(\alpha, \beta)$ be an algebraic extension of fields of characteristic $p > 0$, where α is separable over k and β is purely inseparable over k. Show that $K = k(\alpha + \beta)$.

Proof. ■

Problem 4. Let $f(X) \in \mathbf{F}_q[X]$ be irreducible. Show that $f(X) \mid X^{q^n} - X$ if and only if $\deg f(X) \mid n$.

Proof. ■

Problem 5. Show that $\mathrm{Aut}_{\mathbf{F}_q}(\bar{\mathbf{F}}_q)$ is an infinite Abelian group which is torsionfree (i.e., $\delta^n = \mathrm{id}$ implies $\delta = \mathrm{id}$ or $n = 0$).

Proof. ■

Problem 6. Show that in a finite field, every element can be written as a sum of two perfect squares.

Proof. ■

1.1.10 Homework 10

1.1.11 Homework 11

1.1.12 Homework 12

1.1.13 Homework 13

Bibliography

[1] D.S. Dummit and R.M. Foote. *Abstract Algebra*. Wiley, 2004.

[2] I.N. Herstein. *Topics in algebra*. Xerox College Pub., 1975.

[3] T.W. Hungerford. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2003.