MA553: Qual Preparation

Carlos Salinas

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Contents

1 Ulrich

1.1 Ulrich: Winter 2002

Problem 1. Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G of finite index with $N \subset H$.

Solution. ▶ Let $n = [G : H]$ and $X = \{ H, g_1 H, \ldots, g_{n-1} H \}$ the set of left-cosets of H in G with representatives $g_0 = e, g_1, \ldots, g_{n-1}$. Let G act on X by left multiplication, i.e., $g \mapsto g g_i H$; this is indeed an action since $e(g_i H) = e g_i H = g_i H$ for all $g_i H \in X$ and for $k_1, k_2 \in G$ $k_2(k_1 g_i H) = k_2 k_1 g_i H = (k_2 k_1) g_i H$. By Cayley's theorem, this induces a homomorphism $\varphi \colon G \to S_n$. Note that the action is not necessarily faithful. However, by the first isomorphism theorem, the kernel of φ, $N = \operatorname{Ker} \varphi$, is a normal subgroup of G with index $[G : N] \leq |S_n| = n!$ and $N \subset H$ since $g \in N$ if and only if $g g_i H = g_i H$ which, in particular, implies that $gH = H$. Thus, $N \subset H$ and $[G : N] < \infty$. ◀

Problem 2. Show that every group of order 992 $(= 32 \cdot 31)$ is solvable.

Solution. ▶ Suppose G is a group with order $|G| = 992 = 2^5 \cdot 3$. By Sylow's theorem, the number of 2-Sylow subgroups in G is either 1 or 3. If the number of 2-Sylow subgroups is 1, then $P \lhd G$ and the quotient G/P has order $[G : P] = 3$, hence, is cyclic. Moreover, since P is a p-group, it is solvable. Since P and G/P are solvable, G is solvable.

Now, suppose the number of 2-Sylow subgroups is 3. Let $\operatorname{Syl}_2(G) = \{P, P_1, P_2\}$. Then, by Sylow's theorem, the three 2-Sylow subgroups are conjugate, i.e., there exists $g_1, g_2 \in G$ such that $P_1 = g_1 P {g_1}^{-1}$ and $P_2 = g_2 P {g_2}^{-1}$. Thus, G acts on the set $\operatorname{Syl}_2(P)$ by conjugation. This actions defines a (not necessarily injective) homomorphism $\varphi \colon G \to S_3$. Now, we ask: What is the kernel of this homomorphism? By the first isomorphism theorem, we know that the index of the kernel in G divides the order of S_3, i.e., $[G : \operatorname{Ker} \varphi] \mid 6$. Since $|G| < \infty$ implies that the order of the kernel is one of the following values

$$|\operatorname{Ker} \varphi| = 2^4, 2^4 \cdot 3, 2^5, 2^5 \cdot 3.$$

Now, $|\operatorname{Ker} \varphi| \neq 2^5 \cdot 3$ since we know at least one automorphism, namely conjugation by g_1, which sends $P \mapsto P_1$. Thus, the order of the kernel is either 2^4, $2^4 \cdot 3$ or 2^5. If the $|\operatorname{Ker} \varphi| = 2^4$ or 2^5, we are done for similar reasons to the argument we gave in the previous paragraph, namely, that $\operatorname{Ker} \varphi \lhd G$ and $G/\operatorname{Ker} \varphi$ is solvable (for $|\operatorname{Ker} \varphi| = 2^4$, the quotient $G/\operatorname{Ker} \varphi$ has order 6 so is isomorphic to one of two groups, S_3 or Z_6, both of which are solvable).

Suppose $\operatorname{Ker} \varphi$ has order $2^4 \cdot 3$. Then the number of 3-Sylow subgroups is either 1, 4 or 16. If this number is 1, we are done as $Q \in \operatorname{Syl}_3(\operatorname{Ker} \varphi)$ is a normal subgroup and the quotient is a p-group. Suppose the number of 3-Sylow subgroups is 16. Then there are $16 \cdot 2 = 32$ elements of order 3 in $\operatorname{Ker} \varphi$. ◀

Problem 3. Let G be a group of order 56 with a normal 2-Sylow subgroup Q, and let P be a 7-Sylow subgroup of G. Show that either $G \simeq P \times Q$ or $Q \simeq \mathbb{Z}/(2) \times \mathbb{Z}/(2) \times \mathbb{Z}/(2)$.

[*Hint*: P acts on $Q \setminus \{e\}$ via conjugation. Show that this action is either trivial or transitive.]

Solution. ▶ First, note that, by the fundamental theorem of arithmetic, the order of G can be broken down into $56 = 2^3 \cdot 7$. Suppose G has a normal 2-Sylow subgroup Q and let $P \in \mathrm{Syl}_3(G)$. Then $|\mathrm{Syl}_3(G)| = 1, 4$. If $|\mathrm{Syl}_3(G)| = 1$, then P is the unique 3-Sylow subgroup of G, hence it is normal. Thus, $|P||Q| = |G|$ and $PQ = G$ since, if $g \in Q \cap G$, then $|g| = 3$, but $2 \mid |g|$ so $g = e$. Thus, $G \simeq P \times Q$.

Now, suppose $|\mathrm{Syl}_3(G)| = 4$. Then G contains 4 3-Sylow subgroups which, by Sylow's theorem, are conjugate, i.e., there exists $g_1, g_2, g_3 \in G$ such that $\mathrm{Syl}_p(G) = \left\{P, g_1 P {g_1}^{-1}, g_2 P {g_2}^{-1}, g_3 P {g_3}^{-1}\right\}$. ◀

Problem 4. Let R be a commutative ring and $\mathrm{Rad}(R)$ the intersection of all maximal ideals of R.

(a) Let $a \in R$. Show that $a \in \mathrm{Rad}(R)$ if and only if $1 + ab$ is a unit for every $b \in R$.
(b) Let R be a domain and $R[X]$ the polynomial ring over R. Deduce that $\mathrm{Rad}(R[X]) = 0$.

Solution. ▶ ◀

Problem 5. Let R be a unique factorization domain and P a prime ideal of $R[X]$ with $P \cap R = 0$.

(a) Let n be the smallest possible degree of a nonzero polynomial in P. Show that P contains a primitive polynomial f of degree n.
(b) Show that P is the principal ideal generated by f.

Solution. ▶ ◀

Problem 6. Let k be a field of characteristic zero. assume that every polynomial in $k[X]$ of odd degree and every polynomial in $k[X]$ of degree two has a root in k. Show that k is algebraically closed.

Solution. ▶ ◀

Problem 7. Let $k \subset K$ be a finite Galois extension with Galois group $\mathrm{Gal}(K/k)$, let L be a field with $k \subset L \subset K$, and set $H = \{\, \sigma \in \mathrm{Gal}(K/k) : \sigma(L) = L \,\}$.

(a) Show that H is the normalizer of $\mathrm{Gal}(K/L)$ in $\mathrm{Gal}(K/k)$.
(b) Describe the group $H/\mathrm{Gal}(K/L)$ as an automorphism group.

Solution. ► ◄