MA571 Homework 14

Carlos Salinas

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Problem 14.1 (Munkres §74, Ex. 6)

If $n > 1$, show that the fundamental group of the n-fold torus is not Abelian. [*Hint:* Let G be a free group on the set $\{\alpha_1, \beta_1, ..., \alpha_n, \beta_n\}$; let F be the free group on the set $\{\gamma, \delta\}$. Consider the homomorphism of G onto F that sends α_1 and β_1 to γ and all other α_i and β_i to δ.]

Proof. Let $\mathbf{T}^n$ denote the n-fold torus and let us fix a base-point $x_0 \in \mathbf{T}^n$. By Theorem 74.3, the fundamental group of $\mathbf{T}^n$, $\pi_1(\mathbf{T}^n, x_0)$, is isomorphic to the quotient of the free group G on the set $\{\alpha_1, \beta_1, ..., \alpha_n, \beta_n\}$, by the least normal subgroup N containing

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1}. \tag{1}$$

Now we proceed by the hint. Let F be the free group on the set $\{\gamma, \delta\}$. We define a homomorphism $\varphi\colon G \to F$ by the rule $\alpha_1 \mapsto \gamma$, $\beta_1 \mapsto \gamma$ and $\alpha_i \mapsto \delta$ and $\beta_i \mapsto \delta$ for all $i \neq 1$. By Lemma 69.1, φ determines a homomorphism $G \to F$. Moreover, note that φ is surjective so by the 1st isomorphism theorem, $G/\ker\varphi \cong F$. Now, our next goal is to use the universal mapping property of the group quotient which guarantees the existence and uniqueness of a map $\bar{\varphi}\colon G/N \to F$.[1] To that end, we need to show that $N < \ker\varphi$. But N is the intersection of all normal subgroups of G containing (1) hence, it suffices to show that $\ker\varphi$ contains (1). But this is immediate since

$$\begin{aligned}\varphi(\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1}) &= \varphi(\alpha_1)\varphi(\beta_1)\varphi(\alpha_1^{-1})\varphi(\beta_1)^{-1}\cdots\varphi(\alpha_n)\varphi(\beta_n)\varphi(\alpha_n^{-1})\varphi(\beta_n)^{-1}\\ &= \delta\delta\delta^{-1}\delta^{-1}\gamma\gamma\gamma^{-1}\gamma^{-1}\cdots\gamma\gamma\gamma^{-1}\gamma^{-1}\\ &= 1.\end{aligned}$$

Thus, there exists a map $\bar{\varphi}\colon G/N \to F$ such that $\varphi = \bar{\varphi}\circ\pi_N$ where π is the canonical (group) projection map $\pi\colon G \to G/N$ defined by the rule $g \mapsto g + N$. Since $\varphi(G) = F$, $\bar{\varphi}(G/N) = F$ which is non-Abelian so G/N.[2] ■

[1] Munkres never explicitly calls it this in his short exposition of group theory or, indeed, the UMP of the free product, quotient topology, product topology. These concepts are very illuminating and makes the whole process of writing thinking about a particular algebraic/geometric object much easier. In my opinion of course. tl;dr I don't know where this is stated in Munkres; we all know some group theory—this is true; please don't take off points.

[2] This is by elementary group theory: Say G is Abelian and $\varphi\colon G \to F$ is an homomorphism. Then $\varphi(G) < F$ is Abelian since, by the properties of the homomorphism, for any g_1, g_2, $\varphi(g_1g_2) = \varphi(g_2g_1)$ so $\varphi(g_1)\varphi(g_2) = \varphi(g_2)\varphi(g_2)$.

Problem 14.2 (Munkres §75, Ex. 1)

Calculate $H_1(\mathbf{P}^2\#\mathbf{T})$. Assuming that the list of compact surfaces given in Theorem 75.5 is a complete list, to which of these surfaces is $\mathbf{P}^2\#\mathbf{T}$ homeomorphic?

Proof. We shall begin by setting up, but not actually computing the fundamental group, $\pi_1(\mathbf{P}^2\#\mathbf{T})$. On Munkres §74, p. 453, Munkres gives us the labeling scheme

$$aabcb^{-1}c^{-1} \tag{2}$$

for the connected sum of $\mathbf{P}^2$ and $\mathbf{T}$; we shall not prove that this, indeed, determines the quotient space $\mathbf{P}^2\#\mathbf{T}$ (unless we have time), but instead we will use the labeling scheme to do our computations. Now, given (2), by Theorem 74.3, the fundamental group of $\mathbf{P}^2\#\mathbf{T}$ is the quotient of the free group on the set $\{a, b, c\}$, G, by the least normal subgroup N containing (2). Now, by Corollary 75.2, the Abelianization of $\pi_1(\mathbf{P}^2\#bfT)$ is isomorphic to

$$H_1(\mathbf{P}^2\#\mathbf{T}) \cong \frac{\mathbf{Z}(p(a)) \times \mathbf{Z}(p(b)) \times \mathbf{Z}(p(c))}{p(\langle aabcb^{-1}c^{-1}\rangle)} \tag{3}$$

and simplifying the quotient above taking note that p is a homomorphism whose image is lies inside an Abelian group, we have

$$= \frac{\mathbf{Z}(p(a)) \times \mathbf{Z}(p(b)) \times \mathbf{Z}(p(c))}{\langle 2p(a)\rangle} \tag{4}$$

where we use the module notation $\mathbf{Z}(x)$ to denote the free Abelian group generated by x, because why not; these happen to be $\mathbf{Z}$-modules after all. By the 1st isomorphism theorem, (4) is isomorphic to

$$\mathbf{Z}/(2) \times \mathbf{Z} \times \mathbf{Z} \tag{5}$$

by the obvious homomorphism, i.e., the one sending $(p(a), 0, 0) \mapsto (1, 0, 0)$, $(0, p(b), 0) \mapsto (0, 1, 0)$, and $(0, 0, p(c)) \mapsto (0, 0, 1))$. This map is bijective since it has an inverse, the map sending $(1, 0, 0) \mapsto (p(a), 0, 0)$, $(0, 1, 0) \mapsto (0, p(b), 0)$, and $(0, 0, 1) \mapsto (0, 0, p(c))$. It follows from Lemma 67.7 that both of the maps described above are homomorphisms. From the list given in Theorem 75.5, $\mathbf{P}^2\#\mathbf{T}$ must be homeomorphic to $\mathbf{P}^2\#\mathbf{P}^2$. ■

Problem 14.3 (Munkres §75, Ex. 2)

If $\mathbf{K}$ is the Klein bottle, calculate $H_1(\mathbf{K})$ directly.

Proof. ∎

Problem 14.4 (Munkres §75, Ex. 3(a,b,c))

Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $acadbcb^{-1}d$.

(a) Check that all vertices of P are mapped to the same point of the quotient space X by the pasting map.

(b) Calculate $H_1(X)$.

(c) Assuming X is homeomorphic to one of the surfaces given in Theorem 75.5 (which it is), which surface is it ?

Proof. ∎

Problem 14.5 (A)

Define P^n to be the space $\mathbf{S}^n/\!\sim$ where $z \sim z'$ if and only if $z = z'$ or $z = -z'$. Use the Seifert–van Kampen Theorem to calculate $\pi_1(\mathbf{P}^n)$. (Hint: induction starting from the case $n = 2$ that was done in class.)

Proof. ∎

Problem 14.6 (B)

A topological space X is called *homogeneous* if for every pair of points $x, y \in X$ there is a homeomorphism $\varphi\colon X \to X$ with $\varphi(x) = y$. Prove that every connected 2-manifold is homogeneous. (Hint: use the optional problem from the previous assignment.)

Proof. ■

Problem 14.7 (Optional problem)

(i) Let $x \subset \mathbf{R}^3$ be the cylinder

$$\left\{ (x, y, z) \;\middle|\; x^2 + y^2 = \tfrac{1}{\sqrt{2}} \text{ and } |z| \leq \tfrac{1}{\sqrt{2}} \right\}$$

and let $f\colon X \to \mathbf{R}^3$ be the map

$$f(x, y, z) = \left(2^{1/4} x\sqrt{1 - z^2}, 2^{1/4}, y\sqrt{1 - z^2}, z\right).$$

Prove that f is a homeomorphism from X to the subspace

$$Y = \mathbf{S}^2 \cap \left\{ (x, y, z) \;\middle|\; |z| \leq \tfrac{1}{\sqrt{2}} \right\}.$$

(ii) Prove that the Möbius band is homeomorphic to P^2 with an open disk removed (think of $\mathbf{P}^2$ as $\mathbf{S}^2/{\sim}$ and use part (i)).

Proof. ■