Lemma 26.8 (the tube lemma). Let X, Y be topological spaces with Y compact. Let $x \in X$ and let N be an open set of $X \times Y$ containing $\{x\} \times Y$. Then there is a neighborhood W of x with $W \times Y \subset X \times Y$.

Proof By definition of the product topology, for each $y \in Y$ there are neighborhoods U_y of x and V_y of y with $U_y \times V_y \subset N$. Then $\{V_y\}$ is an open covering of Y, so there exist $y_1, \ldots, y_m$ with

$$Y = V_{y_1} \cup \cdots \cup V_{y_m}.$$

Let

$$W = U_{y_1} \cap \cdots \cap U_{y_m}.$$

Then W is a neighborhood of x since each U_{y_i} is, and if $(x, y) \in W \times Y$ then $y \in V_{y_i}$ for some i and $x \in U_{y_i}$, so $(x, y) \in U_{y_i} \times V_{y_i} \subset N$; thus $W \times Y \subset N$.
QED

Theorem 26.7 Let X, Y be compact spaces. Then $X \times Y$ is compact.

Proof. Let $\{A_\alpha\}_{\alpha \in J}$ be an open cover of $X \times Y$.

Let $x \in X$. Then $\{x\} \times Y \approx Y$ (because the projection map is a continuous open bijection by Theorem 19.3 and p. 92 # 4), so $\{x\} \times Y$ is compact. By Lemma 26.1 there is a finite subset J_x of J with

$$\{x\} \times Y \subset \bigcup_{\alpha \in J_x} A_\alpha.$$

By Lemma 26.8 there is a neighborhood W_x of x with

$$W_x \times Y \subset \bigcup_{\alpha \in J_x} A_\alpha$$

Now $\{W_x\}$ is an open covering of X, so there exist $x_1, \ldots, x_n$ with

$$X = \bigcup_{i=1}^{n} W_x,$$

and we have

$$\begin{aligned} X \times Y &= (W_{x_1} \times Y) \cup \cdots \cup (W_{x_n} \times Y) \\ &\subset \Big(\bigcup_{\alpha \in J_{x_1}} A_\alpha \Big) \cup \cdots \cup \Big(\bigcup_{\alpha \in J_{x_n}} A_\alpha \Big) \\ &= \bigcup_{\alpha \in J_{x_1} \cup \cdots \cup J_{x_n}} A_\alpha, \end{aligned}$$

which shows that $X \times Y$ is covered by finitely many $\{A_\alpha\}$.
QED