MA 544: Homework 12

Carlos Salinas

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Problem 12.1 (Wheeden & Zygmund §8, Ex. 2)

Prove the converse of Hölder's inequality for $p = 1$ and ∞. Show also that for $1 \le p \le \infty$, a real-valued measurable f belongs to $L^p(E)$ if $fg \in L^1(E)$ for every $g \in L^{p'}(E)$, $1/p + 1/p' = 1$. The negation is also of interest: if $f \in L^p(E)$ then there exists $g \in L^{p'}(E)$ such that $fg \notin L^1(E)$. (To verify the negation, construct g of the form $\sum a_k g_k$ satisfying $\int_E f g_k \to \infty$.)

Proof. By the commentary at the end of the proof of Theorem 8.8 (and this `StackExchange` post `http://math.stackexchange.com/questions/37832/on-the-hölder-inequality`) we must prove the following:

If $1 \le p \le \infty$, then $f \in L^p(E)$ if $fg \in L(E)$ for every $g \in L^{p'}(E)$, with $1/p + 1/p' = 1$.

We prove this for $p = 1$ first. Recall that by convention, if $p = 1$ its conjugate exponent, p', is ∞ and vice versa. Therefore, suppose

$$\|fg\|_1 \le \|f\|_1 \cdot \|g\|_\infty \tag{12.1}$$

for every $g \in L^\infty(M)$. By the definition of the essential supremum, $|g|$ is bounded almost everywhere in E by $\|g\|_\infty$. Then, the result follows by Theorem 8.8 since

$$\|f\|_1 = \sup_E fg$$

which, by Equation (12.1), gives us

$$\sup \int_E fg \le \sup(\|f\|_1 \cdot \|g\|_\infty)$$

■

Problem 12.2 (Wheeden & Zygmund §8, Ex. 3)

Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when $p < 1$.

Proof. ■

Problem 12.3 (Wheeden & Zygmund §8, Ex. 4)

Let f and g be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let $1 < p < \infty$. Prove that equality holds in the inequality $\left|\int fg\right| \leq \|f\|_p\|g\|_{p'}$ if and only if fg has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e.

If $\|f+g\|_p = \|f\|_p + \|g\|_p$ and $g \neq 0$ in Minkowski's inequality, show that f is a multiple of g.

Find analogues of these results for the spaces ℓ^p.

Proof. ∎

Problem 12.4 (Wheeden & Zygmund §8, Ex. 5)

For $0 < p \leq \infty$ and $0 < |E| < \infty$, define

$$N_p[f] \coloneqq \left(\frac{1}{E}\int_E |f|^p\right)^{1/p},$$

where $N_\infty[f]$ means $\|f\|_\infty$. Prove that if $p_1 < p_2$, then $N_{p_1}[f] \leq N_{p_2}[f]$. Prove also that if $1 \leq p \leq \infty$, then $N_p[f+g] \leq N_p[f] + N_p[g]$, $(1/|E|)\int_E |fg| \leq N_p[f]N_{p'}[g]$, $1/p + 1/p' = 1$, and $\lim_{p\to\infty} N_p[f] = \|f\|_\infty$. Thus, N_p behaves like $\|\cdot\|_p$ but has the advantage of being monotone in p. Recall Exercise 28 of Chapter 5.

Proof. ■

Problem 12.5 (Wheeden & Zygmund §8, Ex. 6)

(a) Let $1 \leq p_i,\ r \leq \infty$ and $\sum_{i=1}^k 1/p_i = 1/r$. Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

(b) Let $1 \leq p < r < q \leq \infty$ and define $\theta \in (0, 1)$ by $1/r = \theta/p + (1-\theta)/q$. Prove the interpolation estimate

$$\|f\|_r \leq \|f\|_p{}^{\theta} \|f\|_q{}^{1-\theta}.$$

In particular, if $A := \max\{\|f\|_p, \|f\|_q\}$, then $\|f\|_r \leq A$.

Proof. ∎

Problem 12.6 (Wheeden & Zygmund §8, Ex. 9)

If f is real-valued and measurable on E, $|E| > 0$, define its essential infimum on E by

$$\operatorname{ess\,inf} f \coloneqq \sup\{\, \alpha : |\{\, x \in E : f(x) < \alpha \,\}| = 0 \,\}.$$

If $f \geq 0$, show that $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup} 1/f)^{-1}$.

Proof. ■

Problem 12.7 (Wheeden & Zygmund §8, Ex. 11)

If $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty < M$ for all k, prove that $f_k g_k \to fg$ in L^p.

Proof. ∎