# MA553: Spring 2016 Homework

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# 1 Homework 1

**Problem 1.1.** Let $G$ be a group, $a \in G$ an element of finite order $m$, and $n$ a positive integer. Prove that
$$|a^n| = \frac{m}{\gcd(m, n)}.$$

*Proof.* ∎

**Problem 1.2.** Let $G$ be a group, and let $a$, $b$ be elements of finite order $m$, $n$ respectively. Show that if $ba = ab$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$, then $|ab| = \text{lcm}(m, n)$.

*Proof.* ∎

**Problem 1.3.** Let $G$ be a group and $H$, $K$ normal subgroups with $H \cap K = \{e\}$. Show that

(a) $hk = kh$ for every $h \in H$, $k \in K$.

(b) $HK$ is a subgroup of $G$ with $HK \cong H \times K$.

*Proof.* ∎

**Problem 1.4.** Show that $A_4$ has no subgroup of order 6 (although $6 \mid 12 = |A_4|$).

*Proof.* ∎

## 2  Homework 2

**Problem 2.1.** Let $G$ be the group of order $2^3 \cdot 3$, $n \geq 2$. Show that $G$ has a normal 2-subgroup $\neq \{e\}$.

*Proof.* ∎

**Problem 2.2.** Let $G$ be a group of order $p^2 q$, $p$ and $q$ primes. Show that the Sylow $p$-Sylow subgroup or the $q$-Sylow subgroup of $G$ is normal in $G$.

*Proof.* ∎

**Problem 2.3.** Let $G$ be a subgroup of order $pqr$, $p < q < r$ primes. Show that the $r$-Sylow subgroup of $G$ is normal in $G$.

*Proof.* ∎

**Problem 2.4.** Let $G$ be a group of order $n$ and let $\varphi \colon G \to S_n$ be given by the action of $G$ on $G$ via translation.

(a) For $a \in G$ determine the number and the lengths of the disjoint cycles of the permutation $\phi(a)$.

(b) Show that $\varphi(G) \not\subset A_n$ if and only if $n$ is even and $G$ has a cyclic 2-Sylow subgroup.

(c) If $n = 2m$, $m$ odd, show that $G$ has a subgroup of index 2.

*Proof.* ∎

**Problem 2.5.** Show that the only simple groups $\neq \{e\}$ of order $< 60$ are the groups of prime order.

*Proof.* ∎

# 3 Homework 3

**Problem 3.1.** Let $G$ be a finite group, $p$ a prime number, $N$ the intersection of all $p$-Sylow subgroups of $G$. Show that $N$ is a normal $p$-subgroup of $G$ and that every normal $p$-subgroup of $G$ is contained in $N$.

*Proof.* ∎

**Problem 3.2.** Let $G$ be a group of order 231 and let $H$ be an 11-Sylow subgroup of $G$. Show that $H \subset Z(G)$.

*Proof.* ∎

**Problem 3.3.** Let $G = \{e, a_1, a_2, a_3\}$ be a non-cyclic group of order 4 and define $\varphi\colon S_3 \to \mathrm{Aut}(G)$ by $\varphi(\sigma)(e) = e$ and $\varphi(\sigma)(a_1) = a_{\sigma(i)}$. Show that $\varphi$ is well-defined and an isomorphism of groups.

*Proof.* ∎

**Problem 3.4.** Determine all groups of order 18.

*Proof.* ∎

# 4 Homework 4

**Problem 4.1.** Let $p$ be a prime and let $G$ be a nonAbelian group of order $p^3$. Show that $G' = Z(G)$.

*Proof.* ∎

**Problem 4.2.** Let $p$ be an odd prime and let $G$ be a nonAbelian group of order $p^3$ having an element of order $p^2$. Show that there exists an element $b \notin \langle a \rangle$ of order $p$.

*Proof.* ∎

**Problem 4.3.** Let $p$ be an odd prime. Determine all groups of order $p^3$.

*Proof.* ∎

**Problem 4.4.** Show that $(S_n)' = A_n$.

*Proof.* ∎

**Problem 4.5.** Show that every group of order $< 60$ is solvable.

*Proof.* ∎

**Problem 4.6.** Show that every group of order 60 that is simple (or not solvable) is isomorphic to $A_5$.

*Proof.* ∎

# 5 Homework 5

**Problem 5.1.** Find all composition series and the composition factors of $D_6$.

*Proof.* ∎

**Problem 5.2.** Let $T$ be the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of all upper triangular invertible matrices. Show that $T$ is solvable.

*Proof.* ∎

**Problem 5.3.** Let $p \in \mathbb{Z}$ be a prime number. Show:

(a) $(p-1)! \equiv -1 \mod p$.

(b) If $p \equiv 1 \mod 4$ then $x^2 \equiv -1 \mod p$ for some $x \in \mathbb{Z}$.

*Proof.* ∎

**Problem 5.4.**    (a) Show that the following are equivalent for an odd prime number $p \in \mathbb{Z}$:

    (i) $p \equiv 1 \mod 4$.

    (ii) $p = a^2 + b^2$ for some $a$, $b$ in $\mathbb{Z}$.

    (iii) $p$ is not prime in $\mathbb{Z}[i]$.

(b) Determine all prime ideals of $\mathbb{Z}[i]$.

*Proof.* ∎

# 6 Homework 6

**Problem 6.1.** Let $R$ be a domain. Show that $R$ is a UFD if and only if every nonzero nonunit in $R$ is a product of irreducible elemnets and the intersection of any two principal ideals is again principal.

*Proof.* ∎

**Problem 6.2.** Let $R$ be a PID and $p$ a prime ideal of $R[x]$. Show that $p$ is principal or $p = (a, f)$ for some $a \in R$ and some monic $f \in R[x]$.

*Proof.* ∎

**Problem 6.3.** Let $k$ be a field and $n \geq 1$. Show that $z^n + y^3 + x^2 \in k(x, y)[z]$ is irreducible.

*Proof.* ∎

**Problem 6.4.** Let $k$ be a field of characteristic zero and $n \geq 1$, $m \geq 2$. Show that $x_1{}^n + \cdots + x_m{}^n - 1 \in k[x_1, \ldots, x_m]$ is irreducible.

*Proof.* ∎

**Problem 6.5.** Show that $x^{3^n} + 2 \in \mathbb{Q}(i)[x]$ is irreducible.

*Proof.* ∎

# 7    Homework 7

**Problem 7.1.** Let $k \subset K$ and $k \subset L$ be finite field extensions contained in some field. Show that:

(a) $[KL : L] \leq [K : k]$.

(b) $[KL : k] \leq [K : k][L : k]$.

(c) $K \cap L = k$ if equality holds in (b).

*Proof.*                                                                                   ∎

**Problem 7.2.** Let $k$ be a field of characteristic $\neq 2$ and $a, b$ elements of $k$ so that $a$, $b$, $ab$ are not squares in $k$. Show that $\left[k(\sqrt{a}, \sqrt{b}) : k\right] = 4$.

*Proof.*                                                                                   ∎

**Problem 7.3.** Let $R$ be a UFD, but not a field, and write $K = \mathrm{Quot}(R)$. Show that $[\bar{K} : k] = \infty$.

*Proof.*                                                                                   ∎

**Problem 7.4.** Let $k \in K$ be an algebraic field extension. Show that every $k$-homomorphism $\delta \colon K \to K$ is an isomorphism.

*Proof.*                                                                                   ∎

**Problem 7.5.** Let $K$ be the splitting field of $x^6 - 4$ over $\mathbb{Q}$. Determine $K$ and $[K : \mathbb{Q}]$.

*Proof.*                                                                                   ∎

# 8 Homework 8

**Problem 8.1.** Let $k$ be a field, $f \in k[x]$ a polynomial of degree $n \geq 1$, and $K$ the splitting field of $f$ over $k$. Show that $[K : k] \mid n!$.

*Proof.* ∎

**Problem 8.2.** Let $k$ be a field and $n \geq 0$. Define a map $\Delta_n \colon k[x] \to k[x]$ by $\Delta_n\left(\sum a_i x^i\right) := \sum a_i \binom{i}{n} x^{i-n}$. Show that

(a) $\Delta_n$ is $k$-linear, and for $f, g \in k[x]$, $\Delta_n(fg) = \sum_{j=0}^{n} \Delta_j(f)\Delta_{n-j}(g)$.

(b) $f^{(n)} = n!\Delta_n(f)$.

(c) $f(x+a) = \sum \Delta_n(f)(a)x^n$.

(d) $a \in k$ is a root of $f$ of multiplicity $n$ if and only if $\Delta_i(f)(a) = 0$ for $0 \leq i \leq n-1$ and $\Delta_n(f)(a) \neq 0$.

*Proof.* ∎

**Problem 8.3.** Let $k \subset K$ be a finite field extension. Show that $k$ is perfect if and only if $K$ is perfect.

*Proof.* ∎

**Problem 8.4.** Let $K$ be the splitting field of $x^p - x - 1$ over $k = \mathbb{Z}/p\mathbb{Z}$. Show that $k \subset K$ is normal, separable, of degree $p$.

*Proof.* ∎

**Problem 8.5.** Let $k$ be a field of characteristic $p > 0$, and $k(x, y)$ the field of rational functions in two variables.

(a) Show that $\left[k(x, y) : k(x^p, y^p)\right] = p^2$.

(b) Show that the extension $k(x^p, y^p) \subset k(x, y)$ is not simple.

(c) Find infinitely many distinct fields $L$ with $k(x^p, y^p) \subset L \subset k(x, y)$.

*Proof.* ∎

# 9 Homework 9

**Problem 9.1.** Let $k \subset K$ be a finite extension of fields of characteristic $p > 0$. Show that if $p \nmid [K : k]$, then $k \subset K$ is separable.

*Proof.* ■

**Problem 9.2.** Let $k \subset K$ be an algebraic extension of fields of characteristic $p > 0$, let $L$ be an algebraically closed field containing $K$, and let $\delta \colon k \to L$ be an embedding. Show that $k \subset K$ is purely inseparable if and only if there exists exactly one embedding $\tau \colon K \to L$ extending $\delta$.

*Proof.* ■

**Problem 9.3.** Let $k \subset K = k(\alpha, \beta)$ be an algebraic extension of fields of characteristic $p > 0$, where $\alpha$ is separable over $k$ and $\beta$ is purely inseparable over $k$. Show that $K = k(\alpha + \beta)$.

*Proof.* ■

**Problem 9.4.** Let $f(x) \in \mathbb{F}_q[x]$ be irreducible. Show that $f(x) \mid x^{q^n} - x$ if and only if $\deg f(x) \mid n$.

*Proof.* ■

**Problem 9.5.** Show that $\mathrm{Aut}_{\mathbb{F}_q}(\bar{\mathbb{F}}_q)$ is an infinite Abelian group which is torsionfree (i.e., $\delta^n = \mathrm{id}$ implies $\delta = \mathrm{id}$ or $n = 0$).

*Proof.* ■

**Problem 9.6.** Show that in a finite field, every element can be written as a sum of two perfect squares.

*Proof.* ■