

## MA557 Problem Set 6

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**PROBLEM 6.1**

For an  $n$  by  $n$  matrix  $\varphi$  with entries in  $R$  write  $I_t(\varphi)$  for the  $R$ -ideal generated by all the  $t$  by  $t$  minors of  $\varphi$  (set  $I_t(\varphi) = R$  for  $t \leq 0$  and  $I_t(\varphi) = 0$  for  $t > \min\{m, n\}$ ). Thinking of  $\varphi$  as an  $R$ -linear map  $\varphi: R^m \rightarrow R^n$  set  $M = \text{coker}(\varphi)$  and define  $F_i(M) = \text{Fitt}_i(M) = I_{n-i}(\varphi)$ . This ideal is called the  $i$ th Fitting ideal of  $M$ . Show:

- (a)  $F_i(M)$  only depends on  $i$  and  $M$  (but not on  $m, n, \varphi$ ).
- (b)  $(\text{ann}(M))^n \subset F_0(M) \subset \text{ann}(M)$ .
- (c) In case  $R$  is local,  $F_i(M) = R$  if and only if  $\mu(M) \leq i$ .
- (d)  $V(F_i(M)) = \{ \mathfrak{p} \in \text{Spec}(R) \mid \mu_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > i \}$ .

*Proof.*

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**PROBLEM 6.2**

Let  $I$  be an ideal in a Noetherian ring. Show that either  $I$  contains an  $R$ -regular element or else  $aI = 0$  for some  $0 \neq a \in R$ .

*Proof.*

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**PROBLEM 6.3**

Let  $I \subset J$  be ideals in a Noetherian ring. Show that if  $I_{\mathfrak{p}} = J_{\mathfrak{p}}$  for every associated prime  $\mathfrak{p}$  of  $I$ , then  $I = J$ .

*Proof.*

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**PROBLEM 6.4**

Let  $R$  be a Noetherian ring and  $M$  a finite  $R$ -module. Show that  $\ell(M) < \infty$  if and only if  $(M) \subset \mathfrak{m}\text{-Spec}(R)$ .

*Proof.*

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**PROBLEM 6.5**

Let  $R$  be a Noetherian ring,  $M \neq 0$  a finite  $R$ -module, and

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

a chain of submodules with  $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ ,  $\mathfrak{p}_i \in \text{Spec}(R)$ .

- (a) Show that  $\text{Ass}(M) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$  and that the minimal elements of the two sets coincide (hence only depend on  $M$ ).
- (b) Let  $\mathfrak{p}$  be minimal in  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ . Show that in any chain as above, the multiplicity with which the factor  $R/\mathfrak{p}$  appears is  $\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$  (hence only depends on  $M$ ).

*Proof.*

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**PROBLEM 6.6**

Let  $R = k[X, Y]$  be a polynomial ring over a field and  $I = (X^2, XY) \subset R$ . Find two distinct shortest primary decompositions of  $I$ .

*Proof.*

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