MA557 Homework 7

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PROBLEM 7.1

Let R be a Noetherian ring and I, J R-ideals. Write $I^{\langle J\rangle} = \bigcup_{n\geq 1}(I : J^n)$, which is called the *saturation of I with respect to J*. Show:

(a) If $I = \bigcap_{i=1}^m \mathfrak{q}_i$ with $\mathfrak{q}_i$ $\mathfrak{p}_i$-primary, then $I^{\langle J\rangle} = \bigcap_{J\not\subset\mathfrak{p}_i}\mathfrak{q}_i$.

(b) $I^{\langle J\rangle}$ is the unique largest R-ideal that coincides with I locally on the open set $\operatorname{Spec}(R)\smallsetminus V(J)$.

Proof. (a) We shall demonstrate double inclusion: Let $\bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal decomposition of I into primary ideals where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. $\implies$ Suppose $x\in I^{\langle J\rangle}$ then $xJ^n\subset I$ for some $n\geq 1$. Given i such that $\mathfrak{p}_i\not\supset J^*$ take $y\in J\smallsetminus\mathfrak{p}_i$. Then $xy^n\in\mathfrak{q}_i$ so $x\in\mathfrak{q}_i$ since $\mathfrak{q}_i$ is primary and $y\notin\mathfrak{p}_i$. Hence, $I^{\langle J\rangle}\subset\bigcap_{J\not\subset\mathfrak{p}_i}\mathfrak{q}_i$. $\impliedby$ Conversely, suppose that $x\in\bigcap_{J\not\subset\mathfrak{p}_i}\mathfrak{q}_i$ then $x\in\mathfrak{q}_i$ for all $\mathfrak{q}_i\not\supset J$. Take any $\mathfrak{p}_j$ containing J. Then $\mathfrak{p}_j = \operatorname{nil}(R/\mathfrak{q}_j)^c$ (this is easily seen from the fact that $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, i.e., $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary and the correspondence theorem for ideals) so there exists n_j with $xJ^{n_j}\subset\mathfrak{q}_j$ (since, in the quotient, $\bar{J}$ is nilpotent). Let n be the maximum of all such n_j then $xJ^n\mathfrak{q}_i$ for all i, i.e, $x\in(I:J^n) = \bigcap_i^m(\mathfrak{q}_i : J^n)$. Thus, $x\in I^{\langle J\rangle}$.

(b) We will prove that $I^{\langle J\rangle}$ is precisely the set of all $x\in R$ such that $x/1$ vanishes in $R_\mathfrak{p}$ for all $\mathfrak{p}\not\supset J$. $\implies$ Given $x\in I^{\langle J\rangle}$, $xJ^n\subset I$ for some $n\geq 1$. Let $\mathfrak{p}$ be a prime ideal not containing J and let $y\in J\smallsetminus\mathfrak{p}$. Then $xy^n\in I$ and $y^n\notin\mathfrak{p}$ so $x/1 = 0$ in $R_\mathfrak{p}$. $\impliedby$ Conversely, suppose that $x/1$ vanishes in $R_\mathfrak{p}$ for some prime ideal $\mathfrak{p}\subset R$. Then $xy = 0$ for some $y\in R\smallsetminus\mathfrak{p}$. Since $\mathfrak{p} = \sqrt{\mathfrak{q}_i}$ for some i, $y^n\in\mathfrak{q}_i$ for some $n\geq 1$. Let $\bigcap_{i=1}^m\mathfrak{q}_i$ be a minimal decomposition of 0 (one exists since R is Noetherian) where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. By part (a), it suffices to show that ■

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*Why does such an ideal exist? Well, suppose that $\mathfrak{p}_i\supset J$ for all $1\leq i\leq m$. Then $J\subset\bigcap_{i=1}^m\mathfrak{p}_i = \bigcap_{i=1}^m\sqrt{\mathfrak{q}_i} = \sqrt{\bigcap_{i=1}^m\mathfrak{q}_i} = \sqrt{I}$. What next?

Problem 7.2

Let R be a Noetherian ring. Show that R is reduced if and only if $\mathrm{Quot}(R)$ is a finite direct product of fields.

Proof. $\Longrightarrow$ Suppose that R is reduced. Then $\sqrt{0} = 0$. Since R is Noetherian, by 5.13 every ideal has a primary decomposition. Let $\bigcap_{i=1}^n \mathfrak{q}_\mathfrak{i}$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, be such a decomposition for 0. Then we have that $0 = \sqrt{0} = \sqrt{\bigcap_{i=1}^n \mathfrak{q}_i} = \bigcap_{i=1}^n \sqrt{\mathfrak{q}_i} = \bigcap_{i=1}^n \mathfrak{p}_i$ so that $\mathfrak{q}_i = \mathfrak{p}_i$. In particular, $\mathfrak{p}_i$ are minimal over 0 so by 5.15 the decomposition is unique. Thus, by the Chinese remainder theorem we have an exact sequence

$$0 \longrightarrow R \overset{\varphi}{\hookrightarrow} \prod_{i=1}^n R/\mathfrak{p}_i \overset{\pi}{\twoheadrightarrow} \operatorname{coker}\varphi \longrightarrow 0.$$

Now, consider $\mathrm{Quot}(R) = S^{-1}R$ where $S = R \setminus \bigcup_{i=1}^n \mathfrak{p}_i$. By 4.6, since $S^{-1}R$ is a flat R-module, we have an exact sequence

$$0 \longrightarrow S^{-1}R \overset{\varphi}{\hookrightarrow} S^{-1}\left(\prod_{i=1}^n R/\mathfrak{p}_i\right) \cong \prod_{i=1}^n S^{-1}R/S^{-1}\mathfrak{p}_i \overset{\pi}{\twoheadrightarrow} S^{-1}\operatorname{coker}\varphi \longrightarrow 0.$$

In fact, φ is an R-linear isomorphism since $\operatorname{coker}\varphi$ is annihilated by an element of S.[†] It follows that $\mathrm{Quot}(R) \cong \prod_{i=1}^n k_1$ where $k_i = \mathrm{Quot}(R_{\mathfrak{p}_i})$.

$\Longleftarrow$ Conversely, suppose that $\mathrm{Quot}(R) \cong \prod_{i=1}^n k_i$ where k_i is a field. ■

[†] Why? We need to show that the set $\mathrm{ann}_R(\operatorname{coker}\varphi)$ is nonempty.

Problem 7.3

Let R be a Noetherian ring and $x \in R$ an R-regular element. Show that $\operatorname{Ass}_R(R/(x^n)) = \operatorname{Ass}_R(R/(x))$ for every $n \geq 1$.

Proof. Suppose that x is an R-regular element then $x \notin \operatorname{nil}(R)$, i.e., x is not a nilpotent element. Recall that the associated of (x^n) are precisely $\operatorname{Ass}_R(R/(x^n))$. We will show double inclusion: One direction is easy namely $\implies$ suppose that $\mathfrak{p} \in \operatorname{Ass}_R(R/(x^n))$. Then $\mathfrak{p} = \operatorname{ann}(\bar{y}) = ((x^n) : y)$ for some $y \in R$. Then $\sqrt{\mathfrak{p}} = \sqrt{((x^n) : y)} = \left(\sqrt{(x^n)} : y\right) = ((x) : y)$ so $\mathfrak{p} \in \operatorname{Ass}_R(R/(x))$.

$\impliedby$ Conversely, and this idea we owe to Matsumura, since $(x)/(x^n) \cong R/(x^{n-1})$ (just take the map $\varphi \colon y \mapsto xy/(x^n)$ from R into $(x)/(x^n)$; it is clear that $(x^{n-1}) \subset \ker \varphi$; now take an element $z \in \ker \varphi$, $\phi(z) = xz = \bar{0}$ so $xz = x^n y$ for some y, but since x is regular $z = x^{n-1}y$ hence, $z \in (x^n)$) we have the short exact sequence of R-modules

$$0 \longrightarrow R/(x^{n-1}) \longrightarrow R/(x^n) \longrightarrow R/(x) \longrightarrow 0,$$

i.e., $R/(x^n)$ splits. Then $\operatorname{Ass}(x^n) = \operatorname{Ass}(x^{n-1}) \cup \operatorname{Ass}(x)$. In particular, by induction on n, we have $\operatorname{Ass}(x^n) = \bigcup_{i=1}^n \operatorname{Ass}(x) = \operatorname{Ass}(x)$. ■

Problem 7.4

Let $\varphi\colon R \to T$ be a homomorphism of rings where T is Noetherian, let ${}^a\varphi$ be the induced map on the spectra, and let N be a T-module. Show:

(a) $\operatorname{Ass}_R(N) = {}^a\varphi(\operatorname{Ass}_T(N))$.

(b) If N is finitely generated as a T-module then $\operatorname{Ass}_R(N)$ is finite.

Proof. (a)

(b) ■

Problem 7.5

Let K be a field that is a finitely generated $\mathbf{Z}$-algebra. Show that K is a finite field.

Proof. Write $K \coloneqq \mathbf{Z}[X_1, ..., X_n]$. It suffices to show that the characteristic of K is finite. It is clear that the characteristic of K is not zero, for otherwise we may embed $\mathbf{Q}$ into K, but $\mathbf{Q}$ is infinitely generated as a $\mathbf{Z}$-algebra. ■

Problem 7.6

Let k be a Noetherian ring, R a finitely generated k-algebra, and $\mathrm{Aut}_k(R)$ the group of k-algebra automorphisms of R. For a subgroup G of $\mathrm{Aut}_k(R)$ write $R^G = \{\, x \in R \mid \sigma(x) = x \text{ for every } \sigma \in G \,\}$, which is called the *ring of invariants* of G. Show that if G is finite then R^G is a finitely generated k-algebra (and hence a Noetherian ring).

Proof. First, we note that R is integral over R^G since, given $x \in R$, x is the root of a monic polynomial $p(X) = \prod_{\sigma \in G}(X - \sigma(x))$. Then $p(X) \in R^G[X]$ since the elements of G act trivially on R^G, hence the coefficients of $p(X)$ must be in R^G. It follows that $R = R^G[X_1, ..., X_n]$ is a finitely generated R^G-module hence a finitely generated k-algebra so is Noetherian. ■