MA 166: Quiz 6 Solutions

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You have **15 minutes** to complete this quiz. You may work in groups, but you are not allowed to use any other resources.

Problem 1. Compute **two** of the following integrals of your choice

(a) $\displaystyle\int \frac{x^2}{x^2-1}\,dx$

(b) $\displaystyle\int \frac{x}{x^2+6x+9}\,dx$

(c) $\displaystyle\int \frac{dx}{x^3+x}$.

Solutions

(a) First, rewrite the integral as

Solution.

$$\int \frac{x^2}{x^2-1}\,dx = \int \frac{(x^2-1)+1}{x^2-1}\,dx = \underbrace{\int 1\,dx}_{I_1} - \underbrace{\int \frac{dx}{x^2-1}}_{I_2}. \tag{1}$$

It's easy to calculate $I_1 = x + C_1$. To calculate I_2 we need to use partial fractions. Write

$$\begin{aligned}\frac{1}{x^2-1} &= \frac{1}{(x-1)(x+1)}\\ &= \frac{A}{x-1} + \frac{B}{x+1}.\end{aligned}$$

Now we clear denominators

$$\begin{aligned}1 &= \frac{(x-1)(x+1)}{(x-1)(x+1)}\\ &= \frac{A}{x-1}(x-1)(x+1) + \frac{B}{(x+1)}(x-1)(x+1)\\ &= \frac{A}{x-1}(x-1)(x+1) + \frac{B}{(x+1)}(x-1)(x+1)\\ &= A(x+1) + B(x-1)\\ 0x+1 &= (A+B)x + (A-B)\end{aligned}$$

and we have $A+B=0$, $A-B=1$ so $A=-B$ and $B=-1/2$, $A=1/2$. Hence, we have

$$\begin{aligned}I_2 &= \int \frac{dx}{x^2-1}\\ &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx\\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C_2\\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C_2.\end{aligned}$$

Writing $C = C_1 - C_2$ and putting I_1 and I_2, i.e, taking the difference as in (1) we have

$$\boxed{I_1 - I_2 = x - \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.}$$

By using log rules, you can also write this as

$$x + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C = x + \ln\left|\sqrt{\frac{x+1}{x-1}}\right| + C$$

and so on.

(b) Begin by factoring the denominator

$$x^2 + 6x + 9 = (x+3)^2. \tag{2}$$

Now use partial fractions

$$\begin{aligned}\frac{x}{(x+3)^2} &= \frac{A}{x+3} + \frac{B}{(x+3)^2}\\ x &= A(x+3) + B\\ x + 0 &= Ax + B + 3.\end{aligned}$$

This tells us that $B = -3$ and $A = 1$ so the integral turns into

$$\begin{aligned}\int \frac{x}{x^2+6x+9}\,dx &= \int \frac{1}{x+3} - \frac{3}{(x+3)^2}\,dx\\ &= \boxed{\ln|x+3| + \frac{3}{x+3} + C.}\end{aligned}$$

Another way you could have done this problem is by noting that

$$\frac{x}{(x+3)^2} = \frac{(x+3)-3}{(x+3)^2} = \frac{1}{x+3} - \frac{3}{(x+3)^2}$$

and you immediately have the partial fraction decomposition, but you can't always pull this trick on quotients of polynomials. I should be careful with what I am saying, you can do this, but many times it's much much messier than this systematic method of partial fractions.

(c) Factor the denominator into

$$x^3 + x = x(x^2 + 1). \tag{3}$$

Since we cannot factor x^2+1 into a product of real numbers, we must have a numerator of $Bx + C$ over its portion of the partial fraction so, using partial fractions, we have

$$\begin{aligned} \frac{1}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \\ 1 &= A(x^2 + 1) + (Bx + C)x \\ &= (A + B)x^2 + Cx + A. \end{aligned}$$

Hence, $A = 1$, $C = 0$ and $A + B = 0$ so $B = -1$ and our integral turns into

$$\begin{aligned} \int \frac{dx}{x^3 + x} &= \int \frac{1}{x} - \frac{x}{x^2+1}\, dx \\ &= \boxed{\ln|x| - \frac{1}{2}\ln\bigl(x^2 + 1\bigr) + C.} \end{aligned}$$

That last bit, the integral of $x/(x^2 + 1)$, can be computed by using the u-substitution $u = x^2 + 1$. Then $du = 2x\ dx$ and we have

$$\int \frac{x}{x^2+1}\, dx = \frac{1}{2}\int \frac{1}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln\bigl(x^2+1\bigr).$$

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