Math 571 Homework Assignment 2

1. A topological space X is said to be *totally disconnected* if a subspace $Y \subset X$ is connected if and only if $Y = \{x\}$ consists of only a single point $x \in X$. Show that if X is discrete (that is, all subsets of X are open) then X is totally disconnected. Find an example of a totally disconnected space which is not discrete.

2. Let X be a simply ordered set equipped with the order topology. Show that if X is connected then X is a continuum.

3. Show that a metric $d : X \times X \to \mathbb{R}$ on a set X determines a coarsest topology $\mathfrak{U}$ on X for which the distance function $d : X \times X \to \mathbb{R}$ is continuous, and give an explicit basis for this topology. Recall that a function $f : X \to Y$ between metric spaces is said to be *continuous at* x if, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d(x, x') < \delta$ then $d(f(x), f(x')) < \varepsilon$; show that f is continuous (in the sense of topology) if and only if it is continuous at x for all $x \in X$. Finally, show that every compact subspace of a metric space is closed and bounded, and find an example of a metric space for which there exists a closed and bounded subspace which is not compact.

4. Let X be a compact space, Y a Hausdorff space, and $f : X \to Y$ a continuous function. Show that f is a closed map (that is, f sends closed sets to closed sets), and also that the projection $p : X \times Y \to Y$ is a closed map.

5. Let $f : W \to X$ and $g : W \to Y$ be continuous functions. The *pushout* $X \coprod_W Y$ of f and g is the quotient of the disjoint union $X \coprod Y$ by the equivalence relation generated by the relation $x \sim y$ if there exists a $w \in W$ such that $x = f(w)$ and $y = g(w)$. Show that $X \coprod_W Y$ comes equipped continuous functions $i : X \to X \coprod_W Y$ and $j : Y \to X \coprod_W Y$ such that $i \circ f = j \circ g$, and is universal amongst topological spaces Z equipped with continuous functions $i' : X \to Z$ and $j' : Y \to Z$ such that $i' \circ f = j' \circ g$ in the following sense: given any such space Z, there exists a unique continuous function $k : X \coprod_W Y \to Z$ such that $i' = k \circ i$ and $j' = k \circ j$.