MA 166: Quiz 5 Solutions

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You have **15 minutes** to complete this quiz. You may work in groups, but you are not allowed to use any other resources.

Problem 1. Evaluate the integral

$$\int_0^{\pi/2} \cos^2 x \, dx.$$

Problem 2. Evaluate the integral

$$\int \frac{x^2}{\sqrt{4 - x^2}} \, dx.$$

(Use C for the constant of integration.)

Problem 3. Evaluate the integral

$$\int \frac{e^x}{1 - e^{2x}} \, dx$$

[Hint: First use a substitution and then partial fractions.]

Solutions

Solution to Problem 1. Use the double-angle identity

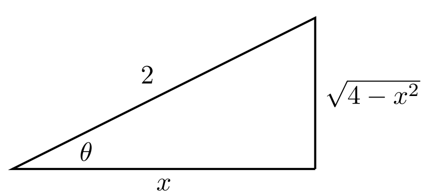
$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \tag{1}$$

to rewrite the integral

$$\begin{aligned}
\int_0^{\pi/2} \cos^2 x \, dx &= \int_0^{\pi/2} \frac{1+\cos 2x}{2}\, dx \\
&= \frac{1}{2}\int_0^{\pi/2} 1+\cos 2x \, dx \\
&= \frac{1}{2}\int_0^{\pi/2} 1\, dx + \frac{1}{2}\int_0^{\pi/2} \cos 2x\, dx \\
&= \frac{1}{2}\left(x\Big|_0^{\pi/2}\right) + \frac{1}{2}\left(\frac{\sin 2x}{2}\Big|_0^{\pi/2}\right) \\
&= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) + \frac{1}{2}\left(\frac{1}{2}\sin\pi - \frac{1}{2}\sin 2\cdot 0\right) \\
&= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) + \frac{1}{2}(0-0) \\
&= \frac{1}{2}\frac{\pi}{2} + \frac{1}{2}\cdot 0 \\
&= \boxed{\frac{\pi}{4}.}
\end{aligned}$$

☺

Solution to Problem 2. Since we have something of the form $\sqrt{4-x^2}$ in the denominator, the best approach to this problem is to make a trigonometric substitution. Draw the triangle



(2)

from which we can deduce that

$$\cos\theta = \frac{x}{2} \qquad\qquad \sin\theta = \frac{\sqrt{4-x^2}}{2}. \tag{3}$$

Hence, $2\cos\theta = x$ and $-2\sin\theta\; d\theta = dx$ so substituting this into our original integral, we have

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4-x^2}}\; dx &= \int \frac{(2\cos\theta)^2}{2\sin\theta}(-2\sin\theta)\; d\theta \\
&= -4\int \cos^2\theta\; d\theta
\end{aligned}$$

here, using the double-angle formula (1), we have

$$\begin{aligned}
&= -4\int \frac{1}{2}(1+\cos 2\theta)\; d\theta \\
&= -2\int 1+\cos 2\theta\; d\theta \\
&= -2\int 1\; d\theta + \int \cos 2\theta\; d\theta \\
&= -2\theta - 2\left(\frac{1}{2}\sin 2\theta\right) + C \\
&= -2\theta - \sin 2\theta + C.
\end{aligned}$$

Substituting back in, we have

$$\theta = \cos^{-1}(x/2)$$

and

$$\sin 2\theta = 2\cos\theta\sin\theta = 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) = \frac{x\sqrt{4-x^2}}{2}$$

so our integral is

$$\boxed{-2\cos^{-1}(x/2) - \frac{x\sqrt{4-x^2}}{2} + C.}$$

Note that if you used a different substitution, say you labeled the adjacent side with $\sqrt{4-x^2}$, then $\sin\theta = x/2$ and you would get

$$\boxed{2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C'}$$

where $C' = C - \pi$. This is because $\cos^{-1}\theta = \pi/2 - \sin^{-1}\theta$. ☺

Solution to Problem 3. Make the substitution $u = e^x$, then $du = e^x\,dx = u\,dx$ and our integral turns into

$$\begin{aligned}\int \frac{e^x}{1-e^{2x}}\,dx &= \int \frac{u}{1-u^2}\frac{du}{u}\\ &= \int \frac{1}{1-u^2}\,du\\ &= \int \frac{1}{(1-u)(1+u)}\,du.\end{aligned}$$

Now we find the partial fraction decomposition

$$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$$

so, clearing denominators, we have

$$1 = A(1+u) + B(1-u) = (A-B)u + A + B.$$

so we must have $A - B = 0$ and $A + B = 1$. This tells us that $A = B$ so substituting this into the former equation $A + B = A + A = 2A = 1$ so $A = B = 1/2$. Hence, our integral turns into

$$\begin{aligned}\int \frac{1}{(1-u)(1+u)}\,du &= \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}\,du\\ &= \frac{1}{2}\int \frac{1}{1-u}\,du + \frac{1}{2}\int \frac{1}{1+u}\,du\\ &= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C\end{aligned}$$

substituting back our value of u, we have

$$\begin{aligned}&= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C\\ &= \boxed{-\frac{1}{2}\ln|1-e^x| + \frac{1}{2}\ln|1+e^x| + C.}\end{aligned}$$

Note that this problem can also be done using a trig substitution. Looking back at the original integral after we made a substitution

$$\int \frac{1}{1-u^2}\,du = \int \frac{1}{\left(\sqrt{1-u^2}\right)^2}\,du$$

and making the trig substitution $\cos\theta = x$, $-\sin\theta\, d\theta = dx$ we have

$$\begin{aligned}
\int \frac{1}{\left(\sqrt{1-u^2}\right)^2}\, du &= \int \frac{-\sin\theta}{\sin^2\theta}\, d\theta \\
&= -\int \csc\theta\, d\theta \\
&= -\ln|\csc\theta - \cot\theta| + C
\end{aligned}$$

where $\csc\theta = 1/\sqrt{1-u^2}$ and $\cot\theta = u/\sqrt{1-u^2}$ so

$$\begin{aligned}
&= -\ln|\csc\theta - \cot\theta| + C \\
&= -\ln\left|\frac{1}{\sqrt{1-u^2}} - \frac{u}{\sqrt{1-u^2}}\right| + C \\
&= -\ln\left|\frac{1-u}{\sqrt{1-u^2}}\right| + C \\
&= -\ln\left|\frac{1-u}{\sqrt{1-u^2}}\right| + C
\end{aligned}$$

by properties of the logarithm, namely, $\ln(a/b) = \ln a - \ln b$, we have

$$\begin{aligned}
&= -\ln|1-u| + \ln\left|\sqrt{1-u^2}\right| + C \\
&= -\ln|1-u| + \frac{1}{2}\ln\left|1-u^2\right| + C \\
&= -\ln|1-u| + \frac{1}{2}\ln|(1-u)(1+u)| + C \\
&= -\ln|1-u| + \frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|(1+u)| + C \\
&= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1-u| + C
\end{aligned}$$

lastly, we substitute our original value of u

$$= \boxed{-\frac{1}{2}\ln|1-e^x| + \frac{1}{2}|1+e^x| + C.} \qquad ☺$$

This integral was much tougher to compute than the partial fractions as it required you to know the integral of $\csc\theta$ (or $\sec\theta$ if your trig substitution was $\sin\theta = x$), which is why I wanted you to do the partial fractions. Still, some students took this approach. There's more than one way to skin a cat, I suppose.