

# MA 523: Homework 4

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## PROBLEM 4.1 (LEGENDRE TRANSFORM)

Let  $u(x_1, x_2)$  be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of  $\mathbb{R}^2$ , where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u(\mathbf{x}(p)),$$

where  $\mathbf{x} = (x^1, x^2)$ ,  $p = (p_1, p_2)$ . Show that  $v$  satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29)).

*SOLUTION.* Assuming the regularity on  $v$  prescribed above, we compute  $v_{p_1p_1}$ ,  $v_{p_1p_2}$  and  $v_{p_2p_2}$ .

First, we compute  $v_{p_1p_2}$  since in the case of  $v_{p_1p_1}$  and  $v_{p_2p_2}$ , there is some symmetry we can exploit. Taking the first partial with respect to  $p^1$ , we have

$$\begin{aligned} v_{p_1} &= \frac{\partial}{\partial p_1} (x^1(p)p^1 + x^2(p)p^2 - u(\mathbf{x}(p))) \\ &= x^1(p) + x_{p_1}^1(p)p^1 + x_{p_1}^2(p)p^2 - u_{x_1}(\mathbf{x}(p))x_{p_1}^1(p) - u_{x_2}(\mathbf{x}(p))x_{p_1}^2(p) \\ (4.1) \quad &= x^1 + x_{p_1}^1p^1 + x_{p_1}^2p^2 - p^1x_{p_1}^1 - p^2x_{p_1}^2 \\ &= x^1, \end{aligned}$$

since  $u_{x_1} = p^1$  and  $u_{x_2} = p^2$ .

Similarly, for  $v_{p_2}$ , we have

$$\begin{aligned} v_{p_2} &= \frac{\partial}{\partial p_2} (x^1(p)p^1 + x^2(p)p^2 - u(\mathbf{x}(p))) \\ &= x_{p_2}^1(p)x^1(p) + x^2(p) + x_{p_2}^2(p)p^2 - u_{x_1}(\mathbf{x}(p))x_{p_2}^1(p) - u_{x_2}(\mathbf{x}(p))x_{p_2}^2(p) \\ (4.2) \quad &= x_{p_2}^1x^1 + x^2 + x_{p_2}^2p^2 - p^1x_{p_2}^1 - p^2x_{p_2}^2 \\ &= x^2. \end{aligned}$$

Now, taking the partial of (4.1) with respect to  $p_1$  and then  $p_2$ , we have

$$v_{p_1p_1} = x_{p_1}^1 = x_{u_{x_1}}^1, \quad v_{p_1p_2} = x_{p_2}^1 = x_{u_{x_2}}^1,$$

and similarly for (4.2),

$$v_{p_1p_2} = x_{p_1}^2 = x_{u_{x_1}}^2, \quad v_{p_2p_2} = x_{p_2}^2 = x_{u_{x_2}}^2.$$

By the inverse function theorem, we have

$$\begin{aligned} \begin{bmatrix} v_{p_1 p_1} & v_{p_1 p_2} \\ v_{p_1 p_2} & v_{p_2 p_2} \end{bmatrix} &= \begin{bmatrix} x_{u_{x_1}}^1 & x_{u_{x_2}}^1 \\ x_{u_{x_1}}^2 & x_{u_{x_2}}^2 \end{bmatrix} \\ &= \begin{bmatrix} u_{x_1 x_1} & u_{x_1 x_2} \\ u_{x_1 x_2} & u_{x_2 x_2} \end{bmatrix}^{-1} \\ &= \frac{1}{J} \begin{bmatrix} u_{x_2 x_2} & -u_{x_1 x_2} \\ -u_{x_1 x_2} & u_{x_1 x_1} \end{bmatrix}. \end{aligned}$$

Hence,

$$(4.3) \quad \begin{cases} u_{x_1 x_1} = J v_{p_2 p_2} \\ u_{x_1 x_2} = -J v_{p_1 p_2} \\ u_{x_2 x_2} = J v_{p_1 p_1}, \end{cases}$$

which verifies Equation (29) from [E, 4.4.3b]. Substituting (4.3) into the original equation,

$$\begin{aligned} 0 &= J a^{11}(p) v_{p_2 p_2} - J a^{12}(p) v_{p_1 p_2} + J a^{22}(p) v_{p_1 p_1} \\ &= a^{22}(p) v_{p_1 p_1} - a^{12}(p) v_{p_1 p_2} + a^{11}(p) v_{p_2 p_2}, \end{aligned}$$

as was to be shown. ■

## PROBLEM 4.2

Find the solution  $u(x, t)$  of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant  $x > 0, t > 0$  for which

$$\begin{cases} u(x, 0) = f(x), & u_t(x, 0) = g(x), & \text{for } x > 0, \\ u_t(0, t) = \alpha u_x(0, t), & & \text{for } t > 0, \end{cases}$$

where  $\alpha \neq -1$  is a given constant. Show that generally no solution exists when  $\alpha = -1$ . (*Hint:* Use a representation  $u(x, t) = F(x - t) + G(x + t)$  for the solution.)

*SOLUTION.* Suppose  $u(x, t) = F(x - t) + G(x + t)$  is a classical solution to the one-dimensional wave equation. From Evans, we know d'Alembert's formula, i.e.,

$$u(x, t) = \frac{f(x + t) + f(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds,$$

is a solution to the PDE with the prescribed initial conditions.

Using the method of reflection, extend  $f$ ,  $g$ , and  $h(x + t) := \int_0^{x+t} g(s) ds$  to all of  $\mathbb{R}^2$ . Now, using the initial conditions prescribed above, particularly, the restriction on the derivative of  $u$  at  $x = 0$ , we obtain the following relations on  $G$  and  $F$ ,

$$\begin{aligned} u(x, 0) &= F(x) + G(x) \\ &= f(x), \\ u_t(x, 0) &= -F'(x) + G'(x) \\ &= g(x), \\ u_t(0, t) &= \alpha u_x(0, t) \\ -F'(-t) + G'(t) &= \alpha(F'(-t) + G'(t)), \end{aligned}$$

which, more concisely, reads

$$(4.4) \quad \begin{cases} F(x) = f(x) - G(x) \\ F'(x) = G'(x) - g(x) \\ F'(-t) = -\left(\frac{\alpha-1}{\alpha+1}\right)G'(t). \end{cases}$$

This gives us, after a bit of calculation,

$$\begin{aligned} f'(-t) - g(-t) &= -\left(\frac{\alpha-1}{\alpha+1}\right)(f'(t) + g(t)) \\ f(x + t) - f(x - t) - \int_{x-t}^{x+t} g(s) ds &= -\left(\frac{\alpha-1}{\alpha+1}\right) \left[ f(x + t) - f(x - t) + \int_{x-t}^{x+t} g(s) ds \right] \\ \frac{f(x + t) + f(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds &= -\frac{1}{2} \left(\frac{\alpha+1}{\alpha-1}\right) \left[ f(x + t) - f(x - t) - \int_{x-t}^{x+t} g(s) ds \right] + f(x - t). \end{aligned}$$

Thus,

$$(4.5) \quad u(x, t) = -\frac{1}{2} \left( \frac{\alpha + 1}{\alpha - 1} \right) \left[ f(x + t) - f(x - t) - \int_{x-t}^{x+t} g(s) ds \right] + f(x - t)$$

is a solution to the PDE with the prescribed initial conditions.

Suppose  $\alpha = -1$ . Then (4.5) tells us that

$$u(x, t) = f(x - t).$$

Thus,  $f'(x) = g(x)$  and

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## PROBLEM 4.3

- (a) Let  $u$  be a solution of the wave equation  $u_{tt} - c^2 u_{xx} = 0$  for  $0 < x < \pi$ ,  $t > 0$  such that  $u(0, t) = u(\pi, t) = 0$ . Show that the *energy*

$$E(t) = \frac{1}{2} \int_0^\pi (u_t^2 + c^2 u_x^2) dx, \quad t > 0$$

is independent of  $t$ ; i.e.,  $\frac{d}{dt}E = 0$  for  $t > 0$ . Assume that  $u$  is  $C^2$  up to the boundary.

- (b) Express the energy  $E$  of the Fourier series solution

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(nct) + b_n \sin(nct)) \sin(nx)$$

in terms of coefficients  $a_n$ ,  $b_n$ .

*SOLUTION.* For part (a), suppose that  $u$  is, as above, a solution to the wave equation which is  $C^2$  up to the boundary. We show that its energy is independent of  $t$ , i.e., that  $\frac{d}{dt}E = 0$ . Assuming the energy is bounded, the dominated convergence theorem allows us to permute the order of integration and differentiation like so

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{d}{dt} \left( \frac{1}{2} \int_0^\pi (u_t^2 + c^2 u_x^2) dx \right) \\ &= \frac{1}{2} \int_0^\pi \frac{\partial}{\partial t} (u_t^2 + c^2 u_x^2) dx \\ &= \frac{1}{2} \int_0^\pi 2u_t u_{tt} + 2c^2 u_x u_{xt} dx \end{aligned}$$

which, after using the relation  $u_{tt} = c^2 u_{xx}$ , becomes

$$\begin{aligned} &= c^2 \int_0^\pi u_t u_{xx} + u_x u_{xt} dx \\ &= c^2 \int_0^\pi \frac{\partial}{\partial x} (u_x u_t) dx \\ &= c^2 (u_x(\pi, t) u_t(\pi, t) - u_x(0, t) u_t(0, t)) \\ &= 0 \end{aligned}$$

since the boundary conditions, i.e.,  $u = 0$ , implies  $u_x = u_t = 0$  at the boundary.

For part (b), suppose  $u$  is a Fourier series solution to the wave equation, i.e.,

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(nct) + b_n \sin(nct)) \sin(nx).$$

First we compute  $u_t$  and  $u_x$ . They are

$$\begin{aligned} u_t(x, t) &= \frac{\partial}{\partial t} u(x, t) \\ &= \sum_{n=1}^{\infty} cn(b_n \cos(nct) - a_n \sin(nct)) \sin(nx) \end{aligned}$$

and

$$\begin{aligned} u_x(x, t) &= \frac{\partial}{\partial x} u(x, t) \\ &= \sum_{n=1}^{\infty} n(a_n \cos(nct) + b_n \sin(nct)) \cos(nx). \end{aligned}$$

Thus,

$$\begin{aligned} E(t) &= \frac{1}{2} \int_0^\pi \left[ \left( \sum_{n=1}^{\infty} cn(b_n \cos(nct) - a_n \sin(nct)) \sin(nx) \right)^2 \right. \\ &\quad \left. + c^2 \left( \sum_{n=1}^{\infty} n(a_n \cos(nct) + b_n \sin(nct)) \cos(nx) \right)^2 \right] \end{aligned}$$

which, after expanding and using the fact that  $\cos(nct)$ ,  $\sin(nct)$ ,  $\cos(nx)$ , and  $\sin(nx)$  are orthogonal, becomes

$$\begin{aligned} &= \frac{1}{2} \int_0^\pi \left[ \sum_{n,m=1}^{\infty} c^2 nm(b_n b_m \cos(nct) \cos(mct) + a_n a_m \sin(nct) \sin(mct) \right. \\ &\quad \left. - a_n b_m \cos(mct) \sin(nct) - a_m b_n \cos(nct) \sin(mct)) \sin(nx) \sin(mx) \right. \\ &\quad \left. - c^2 \sum_{n,m=1}^{\infty} n^2 (a_n a_m \cos(nct) \cos(mct) + b_n b_m \sin(nct) \sin(mct) \right. \\ &\quad \left. + a_n b_m \cos(nct) \sin(mct) + a_m b_n \cos(mct) \sin(nct)) \cos(nx) \cos(mx) \right] \\ &= \frac{1}{2} \int_0^\pi \sum_{n=1}^{\infty} \left( cn^2 (b_n^2 \cos^2(nct) + a_n^2 \sin^2(nct)) \sin^2(nx) - cn^2 (a_n^2 \cos^2(nct) + b_n^2 \sin^2(nct)) \cos^2(nx) \right) \end{aligned}$$

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