Math 527 - Homotopy Theory
Spring 2013
Homework 12 Solutions

Problem 1. Consider the standard inclusions $\mathbb{C}^0 \to \mathbb{C}^1 \to \ldots \to \mathbb{C}^n \to \mathbb{C}^{n+1} \to \ldots$ given by appending a zero in the last coordinate:

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ 0 \end{bmatrix}.$$

These give rise to inclusions $\ldots \to U(n) \to U(n+1) \to \ldots$ described in terms of matrices by:

$$M \mapsto \left[\begin{array}{ccc|c} & & & 0 \\ & M & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

where $U(n)$ denotes the Lie group of $n \times n$ unitary matrices with complex coefficients.

a. Show that the connectivity of the map $U(n) \to U(n+1)$ goes to infinity as n goes to infinity.

Solution. For each $n \geq 1$, consider the evaluation map

$$\begin{aligned} p \colon U(n) &\to S^{2n-1} \\ M &\mapsto M(e_n) \end{aligned}$$

which picks out the last column of the matrix M, viewed as a unit vector in $\mathbb{C}^n$. Here $\{e_1, e_2, \ldots, e_n\}$ denotes the standard basis of $\mathbb{C}^n$.

The map p is clearly surjective, and is moreover a fibration (in fact a fiber bundle), with strict fiber $U(n-1)$, yielding the fiber sequence:

$$U(n-1) \hookrightarrow U(n) \xrightarrow{p} S^{2n-1}.$$

The homotopy fiber ΩS^{2n-1} of the inclusion $U(n-1) \hookrightarrow U(n)$ is $(2n-3)$-connected, so that the inclusion $U(n-1) \hookrightarrow U(n)$ is $(2n-2)$-connected. $\square$

b. Denote the infinite union $U := \mathrm{colim}_n U(n)$. Show that its homotopy groups satisfy

$$\pi_k U \cong \operatorname*{colim}_n \pi_k U(n)$$

and using part (a), find n large enough (as a function of k) to guarantee that the map $U(n) \to U$ induces an isomorphism $\pi_k U(n) \xrightarrow{\cong} \pi_k U$.

Solution. Since each inclusion $U(n-1) \hookrightarrow U(n)$ is a closed embedding, Corollary 2.5.6 of May-Ponto applies, providing the desired isomorphism

$$\pi_k U \cong \operatorname*{colim}_n \pi_k U(n).$$

Note that $U(1) = S^1$ is path-connected, and thus so are all subsequent $U(n)$ for $n \geq 1$.

The connectivity estimate of part (a) guarantees the following. For $k < 2n$, not only is $\pi_k U(n) \xrightarrow{\cong} \pi_k U(n+1)$ an isomorphism, but so are all subsequent induced maps on π_k:

$$\pi_k U(n) \xrightarrow{\cong} \pi_k U(n+1) \xrightarrow{\cong} \pi_k U(n+2) \xrightarrow{\cong} \ldots$$

which proves the isomorphism $\pi_k U(n) \xrightarrow{\cong} \mathrm{colim}_m \pi_k U(m) \cong \pi_k U$.

Therefore the following condition guarantees that n is large enough:

$$\begin{aligned} k \leq 2n - 1 &\Leftrightarrow k + 1 \leq 2n \\ &\Leftrightarrow \frac{k+1}{2} \leq n \\ &\Leftrightarrow n \geq \left\lceil \frac{k+1}{2} \right\rceil. \end{aligned}$$

For example, the low-dimensional homotopy groups $\pi_k U$ are achieved at the following stages:

$$\begin{aligned} &\pi_0 U \cong \pi_0 U(1) = * \\ &\pi_1 U \cong \pi_1 U(1) \\ &\pi_2 U \cong \pi_2 U(2) \\ &\pi_3 U \cong \pi_3 U(2) \\ &\pi_4 U \cong \pi_4 U(3) \\ &\pi_5 U \cong \pi_5 U(3) \end{aligned}$$

etc. $\square$

c. Compute $\pi_k U$ for $0 \leq k \leq 3$.

Solution. From part (b), we already know $\pi_0 U = *$ and $\pi_1 U \cong \pi_1 U(1) = \pi_1 S^1 \cong \mathbb{Z}$.

Since the inclusion $U(1) \hookrightarrow U(2)$ is 2-connected, it induces a surjection $0 = \pi_2 U(1) \twoheadrightarrow \pi_2 U(2)$ which proves $\pi_2 U(2) = 0$. From part (b), we obtain $\pi_2 U \cong \pi_2 U(2) = 0$.

In fact, we can extract more information from the fiber sequence $U(1) \hookrightarrow U(2) \twoheadrightarrow S^3$. The long exact sequence on homotopy

$$\ldots \to \pi_k S^1 \to \pi_k U(2) \to \pi_k S^3 \to \pi_{k-1} S^1 \to \ldots$$

provides the isomorphism $\pi_k U(2) \xrightarrow{\cong} \pi_k S^3$ for all $k \geq 3$. In particular, we obtain $\pi_3 U(2) \cong \pi_3 S^3 \cong \mathbb{Z}$. From part (b), we obtain $\pi_3 U \cong \pi_3 U(2) \cong \mathbb{Z}$.

In summary, the first few homotopy groups of U are:

$$\begin{aligned}
\pi_0 U &= * \\
\pi_1 U &= \mathbb{Z} \\
\pi_2 U &= 0 \\
\pi_3 U &= \mathbb{Z}. \quad \square
\end{aligned}$$

Problem 2. Let (X, e) be a pointed space. The **James construction** on X is the pointed space obtained by taking words in the elements of X and declaring that e is a unit. Formally, it is the quotient space:

$$J(X) := \coprod_{k \geq 1} X^k / \sim$$

where $\sim$ is the equivalence relation generated by identifications of the form:

$$(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_k) \sim (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k).$$

a. Show that $J(X)$ is a topological monoid (under concatenation of words).

Solution. $J(X)$ under concatenation of words is the free monoid on the underlying pointed set of (X, e). It remains to check that it is a *topological* monoid, i.e. that the multiplication map

$$\mu \colon J(X) \times J(X) \to J(X)$$

is continuous. Note that the unit map $* \to J(X)$ is automatically continuous.

The multiplication map on $J(X)$ is induced from the multiplication on the free semigroup $\coprod_{k \geq 1} X^k$ on X, i.e. before declaring that $e \in X$ is a unit. This is illustrated in the commutative diagram

$$\begin{tikzcd}
\left(\coprod_{n \geq 1} X^n\right) \times \left(\coprod_{m \geq 1} X^m\right) \arrow[r, "\mu"] \arrow[d, "q \times q"'] & \left(\coprod_{k \geq 1} X^k\right) \arrow[d, "q"] \\
J(X) \times J(X) \arrow[r, "\mu"'] & J(X)
\end{tikzcd}$$

where $q \colon \coprod_{k \geq 1} X^k \twoheadrightarrow J(X)$ denotes the quotient map.

In convenient **Top** (say, compactly generated weakly Hausdorff spaces), a product of two quotient maps is still a quotient map, so that $q \times q$ is a quotient map. Therefore, to prove continuity of the bottom map μ, it suffices to prove continuity of the top map μ.

In naive **Top** as well as in convenient **Top**, the functor $X \times -$ preserves arbitrary coproducts. Therefore, the top map μ is naturally isomorphic to the top map in the commutative diagram:

$$\begin{tikzcd}
\coprod_{n,m \geq 1} X^n \times X^m \arrow[r, "\mu"] \arrow[dr, "\coprod \mu_{n,m}"'] & \coprod_{k \geq 1} X^k \\
& \coprod_{n,m \geq 1} X^{n+m} \arrow[u]
\end{tikzcd}$$

The map $\coprod \mu_{n,m}$ is continuous (in fact a homeomorphism) since each $\mu_{n,m} X^n \times X^m \to X^{n+m}$ is continuous (in fact a homeomorphism). The upward map

$$\coprod_{n,m \geq 1} X^{n+m} \to \coprod_{k \geq 1} X^k$$

is continuous, since its restriction to any summand $X^{n+m} \to \coprod_{k \geq 1} X^k$ is continuous (being just a summand inclusion). $\square$

b. Let M be a topological monoid and $f\colon X \to M$ a pointed map. Show that there is a unique continuous map of monoids $\widetilde{f}\colon J(X) \to M$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\iota_1"'] & M \\
J(X) \arrow[ur, "\widetilde{f}"'] &
\end{tikzcd}$$

commute. Here $\iota_1\colon X \to J(X)$ denotes the canonical "inclusion of single-letter words", i.e. the composite

$$X = X^1 \hookrightarrow \coprod_{k\geq 1} X^k \twoheadrightarrow J(X).$$

Solution. Since $J(X)$ is the free monoid on the underlying pointed set (X, e), there is a unique map of monoids $\widetilde{f}\colon J(X) \to M$ making the diagram commute. Explicitly, it is given by

$$\widetilde{f}(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n)$$

which is indeed well defined since f is pointed, that is, $f(e) = 1_M$.

It remains to show that $\widetilde{f}$ is continuous. Since $J(X)$ has a quotient topology, it suffices to show that the composite

$$\begin{tikzcd}
\coprod_{n\geq 1} X^n \arrow[d, "q"'] \arrow[dr, "\widetilde{f}\circ q"] & \\
J(X) \arrow[r, "\widetilde{f}"'] & M
\end{tikzcd}$$

is continuous. But restricted to each summand X^n, the composite $\widetilde{f} \circ q|_{X^n}\colon X^n \to M$ is the map given by

$$\widetilde{f} \circ q(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n)$$

which is the composite

$$X^n \xrightarrow{f^n} M^n \xrightarrow{\mu_n} M$$

of two continuous maps. Here $\mu_n\colon M^n \to M$ is the multiplication map of n inputs, which is unambiguously defined since M is strictly associative, and moreover μ_n is continuous. $\square$

Upshot. This shows that $J(X)$ is in fact the free topological monoid on X. In other words, let $U\colon \mathbf{TopMon} \to \mathbf{Top}_*$ denote the forgetful functor from topological monoids to pointed spaces. Then the functor $J\colon \mathbf{Top}_* \to \mathbf{TopMon}$ is left adjoint to U, and $\iota_1\colon X \to J(X)$ is the unit map of the adjunction.

Definition. Let (X, x_0) be a pointed space. The space of **Moore loops** $\Omega_M X$ in X is the space of pairs (γ, τ) with $\tau \in [0, \infty)$ and $\gamma \colon [0, \tau] \to X$ a loop at the basepoint, i.e. a continuous map satisfying $\gamma(0) = \gamma(\tau) = x_0$. It is topologized as the subspace:

$$\begin{aligned}\Omega_M X &= \{(\gamma, \tau) \in \mathrm{Map}\,([0, \infty), X) \times [0, \infty) \mid \gamma(0) = x_0 \text{ and } \gamma(t) = x_0 \text{ for all } t \geq \tau\} \\ &\subseteq \mathrm{Map}\,([0, \infty), X) \times [0, \infty).\end{aligned}$$

The basepoint of $\Omega_M X$ is the "instantaneous loop" $c_0 := (\gamma, 0)$.

Concatenation of Moore loops is defined as follows: $(\gamma_1, \tau_1) * (\gamma_2, \tau_2) \in \Omega_M X$ is the Moore loop $(\gamma, \tau_1 + \tau_2)$ given by

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } 0 \leq t \leq \tau_1 \\ \gamma_2(t - \tau_1) & \text{if } \tau_1 \leq t \leq \tau_1 + \tau_2 \end{cases}$$

also denoted $\gamma = \gamma_1 *_M \gamma_2$ by abuse of notation.

Concatenation makes $\Omega_M X$ into a (strict) monoid with unit c_0, and moreover one can check that it is a topological monoid, i.e. the concatenation map

$$* \colon \Omega_M X \times \Omega_M X \to \Omega_M X$$

is continuous.

Problem 3.

a. Show that the usual loop space ΩX and the Moore loop space $\Omega_M X$ are naturally homotopy equivalent, by an equivalence $\varphi\colon \Omega X \xrightarrow{\simeq} \Omega_M X$ which is moreover an H-map, i.e. such that the diagram

$$\begin{tikzcd}
\Omega X \times \Omega X \arrow[r, "\varphi\times\varphi"] \arrow[d, "\text{concatenation}"'] & \Omega_M X \times \Omega_M X \arrow[d, "\text{concatenation}"] \\
\Omega X \arrow[r, "\varphi"'] & \Omega_M X
\end{tikzcd} \tag{1}$$

commutes up to homotopy.

Solution. Define $\varphi\colon \Omega X \to \Omega_M X$ by

$$\varphi(\gamma) = (\gamma, 1)$$

where $\gamma\colon [0,1] \to X$ is a loop in X based at x_0, with standard parametrization by the unit interval. Clearly φ is continuous, and is natural in X.

Note that φ is not a pointed map, but it does send the basepoint to the basepoint component.

Define $\psi\colon \Omega_M X \to \Omega X$ by

$$\psi(\gamma, \tau) = \gamma_\tau$$

where the latter denotes the loop $\gamma_\tau\colon [0,1] \to X$ rescaled by a factor of τ:

$$\gamma_\tau(t) := \gamma(\tau t).$$

Clearly ψ is continuous, and is natural in X.

One composite is the identity, namely $\psi\varphi = \mathrm{id}_{\Omega X}\colon \Omega X \to \Omega X$.

The other composite $\varphi\psi\colon \Omega_M X \to \Omega_M X$ is homotopic to the identity, via the homotopy

$$H(\gamma, \tau, s) = \left(\gamma_{(1-s)+s\tau}, \frac{\tau}{(1-s)+s\tau}\right)$$

for $s \in [0,1]$. Indeed, H is continuous and satisfies

$$\begin{aligned}
H(\gamma, \tau, 0) &= (\gamma_1, \tau) = (\gamma, \tau) \\
H(\gamma, \tau, 1) &= (\gamma_\tau, 1) = \varphi\psi(\gamma, \tau).
\end{aligned}$$

It remains to show that $\varphi\colon \Omega X \to \Omega_M X$ preserves concatenation up to homotopy. Let $\alpha, \beta \in \Omega X$. The two ways of going around the diagram (1) yield:

$$\varphi(\alpha * \beta) = (\alpha * \beta, 1)$$

$$\begin{aligned}
\varphi(\alpha) * \varphi(\beta) &= (\alpha, 1) * (\beta, 1) \\
&= (\alpha *_M \beta, 2).
\end{aligned}$$

Consider the homotopy $G\colon \Omega X \times \Omega X \times I \to \Omega_M X$ given by

$$G(\alpha, \beta, s) = \left(\alpha_{1+s}, \frac{1}{1+s}\right) * \left(\beta_{1+s}, \frac{1}{1+s}\right).$$

Then G is indeed continuous, and it satisfies

$$\begin{aligned} G(\alpha, \beta, 0) &= (\alpha_1, 1) * (\beta_1, 1) \\ &= \varphi(\alpha) * \varphi(\beta) \\ G(\alpha, \beta, 1) &= \left(\alpha_2, \frac{1}{2}\right) * \left(\beta_2, \frac{1}{2}\right) \\ &= (\alpha * \beta, 1) \\ &= \varphi(\alpha * \beta). \quad \square \end{aligned}$$

b. Deduce that the canonical map $\eta\colon X \to \Omega\Sigma X$ naturally extends up to homotopy to an H-map $\widetilde{\eta}\colon : J(X) \to \Omega\Sigma X$. Here $J(X)$ denotes the James construction on X (c.f. Problem 2).

"Extension up to homotopy" means that $\widetilde{\eta}$ makes the following diagram commute up to homotopy:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\iota_1"'] & \Omega\Sigma X. \\ J(X) \arrow[ur, "\widetilde{\eta}"'] & \end{tikzcd}$$

Solution. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\iota_1"'] & \Omega\Sigma X \arrow[r, shift left, "\varphi"] & \Omega_M \Sigma X. \arrow[l, shift left, "\psi"] \\ J(X) \arrow[ur, dashed, "\widetilde{\eta}"'] \arrow[urr, dashed, bend right, "\eta'"'] & & \end{tikzcd}$$

Since $\Omega_M\Sigma X$ is a (strict) topological monoid, there is a unique continuous map of monoids $\eta'\colon J(X) \to \Omega_M\Sigma X$ satisfying $\eta' \circ \iota_1 = \varphi \circ \eta$.

Take $\widetilde{\eta} := \psi \circ \eta'$. Since $\psi\colon \Omega_M X \xrightarrow{\simeq} \Omega X$ is a homotopy equivalence, the left triangle commutes up to homotopy:

$$\begin{aligned} \widetilde{\eta} \circ \iota_1 &= \psi \circ \eta' \circ \iota_1 \\ &= \psi \circ \varphi \circ \eta \\ &\simeq \mathrm{id}_{\Omega\Sigma X} \circ \eta \\ &= \eta. \end{aligned}$$

Since η' is a map of monoids, it is in particular an H-map. Since ψ is also an H-map, the composite $\widetilde{\eta} = \psi \circ \eta'$ is also an H-map.

Since φ and ψ are both natural in X, then so is $\eta'\colon J(X) \to \Omega\Sigma X$. $\square$