MA571: Qual Preparation

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1 MA 571 Fall 2015

This is material from the course MA 571 as it was taught in the fall of 2015.

1.1 Homework

Most of the homework is from [1] with a few exercises (especially those involving the quotient topology and manifolds) written by McClure. Unless otherwise stated, whenever we quote a result, e.g., Theorem 1.1, it is understood to come from Munkre's *Topology*.

Throughout these notes

$\mathbb{R}$	is the set of real numbers
$\mathbb{C}$	is the set of complex numbers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{Z}$	is the set of the integers
$\mathbb{Z}^+$	is the set of positive integers, that is, $x \in \mathbb{Z}$ with $x \geq 0$
$\mathbb{N}$	is the set of the natural numbers $1, 2, \ldots$
$A \smallsetminus B$	is the set difference of A and B, that is, the complement of $A \cap B$ in A
$X \approx Y$	means X and Y are homeomorphic
$f \simeq g$	means f is homotopic to g
$X \simeq Y$	means X and Y are homotopy equivalent
$G \cong H$	means G and H are isomorphic as groups

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2 Past Qualifying Examinations

2.1 MA 571: Qualifying Exam, August 2014

Problem 1. Let X be a topological space, let A be a subset of X, and let U be an open subset of X. Prove that $U \cap \bar{A} \subset \overline{U \cap A}$.

Proof. Let $x \in U \cap \bar{A}$. Then $x \in U$ and $x \in \bar{A}$. This means that, since U is open, by Lemma C there exist an open neighborhood V of x such that $V \subset U$. Moreover, since $x \in \bar{A}$, $V' \cap A \neq \emptyset$ for every open neighborhood V' of x. In particular, $V \cap A \neq \emptyset$. Thus, we have $V \cap U \neq \emptyset$ and $V \cap A \neq \emptyset$ so $V \cap (U \cap A) \neq \emptyset$. ■

Problem 2. Let X be the following subspace of $\mathbb{R}^2$:

$$((0, 1] \times [0, 1]) \cup ([2, 3) \times [0, 1]).$$

Let $\sim$ be the equivalence relation on X with $(1, t) \sim (2, t)$ (that is $(s, t) \sim (s', t') \iff (s, t) = (s', t')$ or $t = t'$ and $\{s, s'\} = \{1, 2\}$; you do *not* have to prove that this is an equivalence relation). Prove that $X/{\sim}$ is homeomorphic to $(0, 2) \times [0, 1]$. (*Hint*: construct maps in both directions).

Proof. We shall proceed by the hint. Let $q\colon X \to X/{\sim}$ denote the quotinet map. Then, for $(x, y) \in X$, we define the map

We shall proceed by the hint. Let $q\colon X \to X/{\sim}$ denote the quotient map. Then, for $x \in X$, we define the map

$$h(s,t) :- \begin{cases} (s,t) & \text{if } (s,t) \in (0,1] \times [0,1] \\ (s-1,t) & \text{if } (s,t) \in (2,3] \times [0,1]) \end{cases}$$

from $X \to (0, 2) \times [0, 1]$.

By the UMP of the quotient space (Theorem Q.3), if we can show that h is continuous and preserves the equivalence relation, the induced map on the quotient space, $h'\colon X/{\sim} \to (0, 2) \times [0, 1]$ will be continuous. To that end, we will use the pasting lemma. First, note that $(0, 1] \times [0, 1]$ and $[2, 3) \times [0, 1]$ are closed subsets of X since $(0, 1] \times [0, 1]$ is the complement of $((1, \infty) \times (-2, 2)) \cap X$ which is open in X (since X inherits its topology from $\mathbb{R}^2$), similarly, $[2, 3) \times [0, 1]$ is closed in X since it is the complement of $((-\infty, 2) \times (-2, 2)) \cap X$ which is open in X for the same reasons. It is clear that the maps $x \mapsto x$ and $x \mapsto x - 1$ are continuous onto their image, since the latter is nothing more than the inclusion map and the former is nothing more than subtraction, which is continuous by Theorem 21.5. Thus, by the pasting lemma, h is continuous.

Now we show that h does in fact preserve the equivalence relation. Suppose $(s, t) \sim (s', t')$. Then either $(s, t) = (s', t')$ or $t = t'$ and $s, s' \in \{1, 2\}$. In the former case, we have $h(s, t) = h(s', t')$ (whether $(s, t), (s', t') \in (0, 1] \times [0, 1]$ or its complement). In the latter case, we may, without loss of generality, assume that $(s,t) = (1,t)$ and $(s',t') = (2,t)$. Then $h(s,t) = (1,t) = (2-1,t) = h(s',t')$. Thus, by Theorem Q.3, the induced map $h' \colon X/{\sim} \to (0,2) \times [0,1]$ is continuous. Moreover, the map is bijective with inverse

$$(h')^{-1} := \begin{cases} [s,t] & \text{if } x \in (0,1] \\ [s+1,t] & \text{if } x \in [1,2) \end{cases}.$$

This is clearly an inverse as

$$h' \circ (h')^{-1} = \mathrm{id}_{X/\sim}$$

and

$$(h')^{-1} \circ h' = \mathrm{id}_{(0,2)\times[0,1]}\,.$$

Thus, by Theorem 26.6, h' is a homeomorphism. ∎

Problem 3. Prove that there is an equivalence relation $\sim$ on the interval $[0,1]$ such that $[0,1]/{\sim}$ is homeomorphic to $[0,1] \times [0,1]$. As part of your proof *explain* how you are using one or more properties of the quotient topology.

Proof. First, it suffices to find a continuous surjective map $f \colon [0,1] \to [0,1]\times[0,1]$ and quotient out by the preimage of every point $x \in [0,1] \times [0,1]$. These maps are hard to describe in general, but they exists (take for example a space-filling curve). Next, note that if C is a closed subset of $[0,1]$ then it is compact so $f(C)$ is compact. But since $[0,1]\times[0,1]$ is compact Hausdorff, then $f(C) \subset [0,1]\times[0,1]$ will be closed. It follows by that f will be a Munkres quotient map, so by Theorem Q.4, $f' \colon [0,1]/{\sim} \to [0,1]\times[0,1]$ is a homeomorphism for some equivalence relation $\sim$ on $[0,1]$. ∎

Problem 4. Let D be the closed unit disk in $\mathbb{R}^2$, that is, the set

$$\{\, (x,y) : x^2 + y^2 \le 1 \,\}.$$

Let E be the open unit disk

$$\{\, (x,y) : x^2 + y^2 < 1 \,\}.$$

Let X be the one-point compactification of E, and let $f \colon D \to X$ be the map defined by

$$f(x,y) = \begin{cases} (x,y) & \text{if } x^2+y^2 < 1 \\ \infty & \text{if } x^2+y^2 = 1. \end{cases}$$

Prove that f is continuous.

Proof. By the section on the one-point-compactification, it suffices to check two cases of open sets (1) all sets U open in E, and (2) all sets of the form $U = X \smallsetminus C$ containing the point at infinity, ∞, where C is compact. In the first case, it is clear that f is continuous since it is just the inclusion map and is in fact bijective on E. For the second case, suppose that U is a neighborhood of ∞. Then $Y - U$ is a compact subset of E, hence closed since X is a compact Hausdorff space. But since f is bijective, continuous on E, then $f^{-1}(X - U)$ is a closed subset of E. Thus, by theorem 18.2, f is continuous. ■

Problem 5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is path-connected. Prove that Y is path-connected.

Proof. First we prove the following important result:

Lemma 1. *Path-connectedness is a topological property, i.e., if X is path-connected and $f\colon X \to Y$ is a continuous map then, $f(X)$ is path connected.*

Proof. Since X is path-connected, for any pair of points $x, x' \in X$ there exists a continuous map $p\colon [0,1] \to X$ such that $p(0) = x$ and $p(1) = x'$. Since composition of continuous maps is continuous, $f \circ p\colon [0,1] \to Y$ is a path from $f(x)$ to $f(x')$. Since this property holds for any $y \in f(X)$, it follows that $f(X)$ is path-connected. ♣

Now, suppose that X is homotopy-equivalent to Y. Then there exists continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Now, since X is path-connected, by Lemma (1) we have $f(X)$ is path-connected. Thus, it suffices to show that for every point $y \in Y$ there exists a path $p\colon [0,1] \to Y$ from y to some point $y' \in f(X)$. Now, since $f \circ g \simeq \mathrm{id}_Y$, there exists a homotopy, say $H\colon Y \times [0,1] \to Y$ such that $H(s,0) = f \circ g(s)$ and $H(s,1) = s$. Consider the evaluation $H_y :- H(y,t) \circ H(y,t)$ where the map $(y,t)\colon [0,1] \to Y \times [0,1]$ is the imbedding of $[0,1]$ at y (which is continuous by Theorem 18.4) thus, H_y is continuous. Moreover, $H_y(0) = f \circ g(y) \in f(Y)$ and $H_y(1) = \mathrm{id}_Y(y) = y$ so H_y is a path from y to a point $f \circ g(y)$ in $f(X)$. Since we can do this for any point $y \in Y$, it follows, since path-connectedness is an equivalence relation, that Y is path-connected. ■

Problem 6. Let a and b denote the points $(-1,0)$ and $(1,0)$ in $\mathbb{R}^2$. Let x_0 denote the origin $(0,0)$. Use the Seifert–van Kampen theorem to calculate $\pi_1\left(\mathbb{R}^2 \smallsetminus \{a,b\}, x_0\right)$. You may not use any other method.

Proof. We'll use Theorem 70.2's version of the Seifert–van Kampen theorem. Define

$$U :- \left(-\infty, \tfrac{1}{2}\right) \times \mathbb{R} \quad \text{and} \quad V :- \left(-\tfrac{1}{2}, \infty\right) \times \mathbb{R}.$$

Then $U \cap V = (-1/2, 1/2) \times \mathbb{R}$ is clearly path-connected since it is a convex set. Moreover, note that $U \simeq \mathbb{R}^2 \smallsetminus \{x_0\}$ and $V \simeq \mathbb{R}^2 \smallsetminus \{x_0\}$ (in the case of U, first consider the homeomorphism $(x,y) \mapsto (x+1,y)$ which sends a to $(0,0)$ and then the homotopy $(x,y) \mapsto \frac{1}{t}(x,ys))$.

Once we have established the above, since the fundamental group of a space is invariant under homotopy-equivalence, $\pi_1(U,x_0) \cong \pi_1\left(\mathbb{R}^2 \smallsetminus \{x_0\}, y_0\right) \cong \mathbb{Z}$ for some arbitrary $y_0 \neq x_0$ and similarly $\pi_1(V,x_0) \cong \mathbb{Z}$. Thus, by the classical version of the Seifert–van Kampen theorem

$$\pi_1\left(\mathbb{R}^2 \smallsetminus \{a,b\}, x_0\right) \cong \frac{\mathbb{Z} * \mathbb{Z}}{N}$$

where N is the least normal subgroup ∎

Problem 7. Let $p\colon E \to B$ be a covering map with B locally connected, and let $x \in B$. Prove that x has a neighborhood W with the following property: for every connected component C of $p^{-1}(W)$, the map $p\colon C \to W$ is a homeomorphism.

Proof. Let U be an evenly covered neighborhood of x. Then $p^{-1}(U) = \bigcup_\alpha V_\alpha$ where the V_α are open in E and $V_\alpha \cap V_\beta = \emptyset$ whenever $\alpha \neq \beta$. For any α, let C be a connected component of $p^{-1}(U)$ containing $p^{-1}(x) \cap V_\alpha$ (the latter is a one point set since $p|_{V_\alpha}$ is a bijection). Then $C \subset V_\alpha$ for at most one such α for otherwise $C \cap V_\beta \neq \emptyset$ for some $\beta \neq \alpha$, so $C \cap V_\beta$ and $C \cap V_\alpha$ form a separation of (note that $C \smallsetminus (C \cap V_\beta) = C \cap V_\alpha$ and vice-versa thus, $C \cap V_\beta$ and $C \cap V_\alpha$ are open and closed in the subspace topology on C, conversely) by Lemma 23.1.

Thus, $p(C) \subset U$ is connected by Theorem 23.5. Moreover, since $V_\alpha \supset C$ is homeomorphic to U by the restriction $p|_{V_\alpha}$, $p(C)$ is a connected component of U as the following lemma shows

Lemma 2. *Suppose C is a connected component of X and $h\colon X \to Y$ is a homeomorphism. Then $h(C)$ is a connected component of Y.*

Proof of lemma. Let C be a connected component of X. By theorem 23.5, $h(C)$ is a connected subset of Y, moreover, is open. By Theorem 25.1, $h(C)$ is contained in a connected component of Y, say D. Hence, we must show that $D \subset h(C)$. Now, since h is a homeomorphism, $h^{-1}(D)$ is a connected subset of X, by Theorem 23.5, so is contained in only one component of X. But $h^{-1}(D) \cap C \neq \emptyset$ so $h^{-1}(D) \subset C$. Thus, since h is a set-bijection, $D \subset h(C)$. ♣

so by Theorem 25.3, $p(C)$ is open in B since B is locally connected. Thus, the restriction $p|_C$ is a homeomorphism onto its image $W :- p(C)$, by Lemma A, which is a neighborhood of x. ∎

2.2 MA 571: Qualifying Exam, January 2014

Problem 1. Let X be a topological space, let A be a subset of X, and let U be an open subset of X. Prove that $U \cap \bar{A} \subset \overline{U \cap A}$.

Proof. The proof is simple and we have shown this before in the August 2014 quals, it goes as follows: If $U \cap \bar{A} = \emptyset$, there is nothing to show. Let $x \in U \cap \bar{A}$. Then $x \in U$ and $x \in \bar{A}$. Since $x \in U$ and U is open, by Lemma C, there exists a neighborhood V of x such that $V \subset U$; in particular, note that $V \cap U \neq \emptyset$. But $x \in \bar{A}$ so $V \cap A \neq \emptyset$. Thus, $V \cap (U \cap A) \neq \emptyset$. Thus, $x \in \overline{U \cap A}$. ■

Problem 2. Let $\sim$ be an equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. Prove that the quotient space $\mathbb{R}^2/{\sim}$ is compact but not Hausdorff.

Proof. To show that $\mathbb{R}^2/{\sim}$ is compact, we need to show that for every open covering $\mathcal{A}$ of $\mathbb{R}^2/{\sim}$, there is a finite subcover $\mathcal{A}' \subset \mathcal{A}$. Let $q\colon \mathbb{R}^2 \to \mathbb{R}^2/{\sim}$ denote the quotient map. Then, since q is continuous and onto $\mathbb{R}^2/{\sim}$, the set $\left\{q^{-1}(A_\alpha)\right\}_{A_\alpha \in \mathcal{A}}$ is an open cover of $\mathbb{R}^2$. In particular, there exists at least one A_α such that $q^{-1}(A_\alpha)$ is a neighborhood of $(0, 0)$. By Lemma C, there exists a basic open neighborhood, i.e., an open ball $B((0, 0), \varepsilon) \subset q^{-1}(A_\alpha)$ for $\varepsilon > 0$. Now, for any point $[(x, y)] \in \mathbb{R}^2$ pick a representative $(x, y) \in \mathbb{R}^2$. Then, by the Archimedean principle, there exists a positive real numbers $t', t'' > 0$ such that $t'x < \sqrt{\varepsilon}$ and $t''y < \sqrt{\varepsilon}$. Define $t :- \min\{t', t''\}$. Then $tx < \sqrt{\varepsilon}$ and $ty < \varepsilon$. Thus, $(tx, ty) \in A_\alpha$ (since $t^2x^2 + t^2y^2 < \varepsilon$). Since we can do this for any point $[x] \in \mathbb{R}^2/{\sim}$, it follows that $A_\alpha \supset \mathbb{R}^2/{\sim}$. Thus, $\mathcal{A}' :- \{A_\alpha\}$ is a finite subset of $\mathcal{A}$ which covers $\mathbb{R}^2/{\sim}$. Thus, $\mathbb{R}^2/{\sim}$ is compact.

To show that $\mathbb{R}^2/{\sim}$ is not compact, we will employ a very similar strategy, that is, we will show that every neighborhood of the point $[0, 0] \in \mathbb{R}^2/{\sim}$, contains every point $[x, y] \in \mathbb{R}^2/{\sim}$. Let $[x, y] \in \mathbb{R}^2/{\sim}$ and let U be a neighborhood of $[0, 0]$. Then $q^{-1}(U)$ is an open neighborhood of $(0, 0)$, i.e., there exists an open ball $B((0, 0), \varepsilon) \subset q^{-1}(U)$. But as we have just shown, for sufficiently small values of $t > 0$, $(tx, ty) \in B((0, 0), \varepsilon) \subset q^{-1}(U)$. Thus, $[x, y] \in U$. In particular, for any open neighborhood V of $[x, y]$, $V \cap U \neq \emptyset$. Thus, $\mathbb{R}^2/{\sim}$ is not Hausdorff. ■

Problem 3. Let X and Y be topological spaces. Let $x_0 \in X$ and let C be a compact subset of Y. Let N be an open set in $X \times Y$ containing $\{x_0\} \times C$. Prove that there is an open set U containing x_0 and an open set V containing C such that $U \times V \subset N$.

Proof. This is a classical theorem called the tube lemma. We shall prove first in the style of Munkres and second in the style of McClure (if I can find the proof or somehow reconstruct it).

Let X, Y, x_0, N, and C be as above. Note that since C is compact and the injection $\iota_{x_0}\colon X \hookrightarrow X \times Y$ given by $\iota_{x_0}(y) :- (x_0, y)$ is continuous by Theorem

18.4 (since its components, i.e., projections to X and Y, are continuous these are $\pi_1(\iota_{x_0})(x) = x_0$ and $\pi_1(\iota_{x_0})(y) = y$ a constant map and identity map, respectively) so the image of C under ι_{x_0}, $\{x_0\} \times C$, is compact by Theorem 23.5. For every point $x \in \{x_0\} \times C$, let $U_x \times V_x$ be a basic open neighborhood of x contained in N (this can be arranged by Lemma C). Then the collection $\mathcal{A} :- \{U_x \times V_x\}_{x \in \{x_0\} \times Y}$ forms an open covering of $\{x_0\} \times C$. Thus, there exists a finite subcover $\{U_{x_i} \times V_{x_i}\}_{i=1}^n$ of $\mathcal{A}$.

Define $W :- U_{x_1} \cap \cdots \cap U_{x_n}$. This set is clearly open since it is a finite intersection of open sets and contains x_0 since every $U_{x_i} \times V_{x_i}$ intersects $\{x_0\} \times Y$. Define $W' :- \pi_2(N) \cap Y$. This set is open since it is a finite intersection of open sets in Y. The $W \times W' \subset N$. This is clear since every point $(x, y) \in W \times W'$ is in N ($x \in W \subset U_{x_i}$ for all i which in turn is a subset of $\pi_1(N)$ and $y \in W' = \pi_2(N)$). Lastly, $W \times W' \supset \{x_0\} \times C$ since $x_0 \in W$ and $W' = \pi_1(N) \supset C$. Thus, $W \times W' \subset N$ containing $\{x_0\} \times C$ as desired. ■

Problem 4. Let X be a locally compact Hausdorff space and let A be a subset with the property that $A \cap K$ is closed for every compact K. Prove that A is closed.

Proof. Here's what I have so far:

We will try to show that $\bar{A} \subset A$. Let $x \in \bar{A}$. Then, for every neighborhood U of x, $U \cap A \neq \emptyset$. Now, since X is locally compact, there exists a neighborhood V of x such that $\bar{V}$ is compact and is a subset of U. Since X is Hausdorff, $\bar{V}$ is compact so $\bar{V} \cap A$ is closed. ■

Problem 5. Let X and Y be path-connected and let $h \colon X \to Y$ be a continuous function which induces the trivial homomorphism of fundamental groups. Let $x_0, x_1 \in X$ and let f and g be paths from x_0 to x_1. Prove that $h \circ f$ and $h \circ g$ are homotopic.

Proof. Consider the path-product $\gamma :- f * \bar{g}$. γ is a loop based at x_0 since $\gamma(0) = f(0) = x_0$ and $\gamma(1) = \bar{g}(2-1) = \bar{g}(1) = x_0$. Thus, $[\gamma] \in \pi_1(X, x_0)$. Now, since $h_* \colon \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$ induces the trivial homomorphism, i.e., $h(\gamma) \simeq_p e_{x_0}$, there exists a homotopy $H \colon [0,1] \times [0,1] \to Y$ such that $H(s, 0) = h \circ \gamma(s)$ and $H(s, 1) = e_{x_0}(s)$. Now, since Y is path-connected, there exists a path $\delta \colon [0,1] \to Y$ from $h(x_0)$ to $h(x_1)$. ■

Problem 6. Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $aabbcdc^{-1}d^{-1}$.

(i) Calculate $H_1(X)$.
(ii) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?

Proof. ■

Problem 7. Let $p\colon E \to B$ be a covering map with B locally connected, and let $x \in B$. Prove that x has a neighborhood W with the following property: for every connected component C of $p^{-1}(W)$, the map $p\colon C \to W$ is a homeomorphism.

Proof. ■

2.3 MA571: Qualifying Exam, January 2012

Problem 1. Let X be a topological space. Recall that a subset of X is *dense* if its closure is X. Prove that the intersection of two dense open sets is dense.

Proof. Suppose U and V are open dense subsets of X. We will show that $U \cap V$ is dense in X, i.e., $\overline{U \cap V} = X$. To that end, we will show that for any point $x \in X$, for any neighborhood W of x, $W \cap (U \cap V) \neq \emptyset$. Therefore, let $x \in X$. Let W be a neighborhood of x. Then, since U is dense in X, $W \cap U \neq \emptyset$. Let $y \in W \cap U$. Then, since U and V are open, $U \cap V$ is open so $U \cap V$ is a neighborhood of y. Moreover, since V is dense in X, $(W \cap U) \cap V \neq \emptyset$. Now, since intersection is associative, $(W \cap U) \cap V = W \cap (U \cap V) \neq \emptyset$. Thus, $x \in \overline{U \cap V}$ and we have $\overline{U \cap V} = X$ as desired. ■

Problem 2. Let X be a set with two elements $\{a, b\}$. Give X the *indiscrete* topology. Give $X \times \mathbb{R}$ the product topology. Let $A \subset X \times \mathbb{R}$ be $(\{a\} \times [0,1]) \cup (\{b\} \times (0,1))$. Prove that A is compact.

You may use the fact that a set is compact if every covering by *basic* open sets has a finite subcovering.

Proof. Let $\mathcal{U}$ be an open cover of A by basic open sets. Then each $U \in \mathcal{U}$ is of the form $\{a, b\} \times V$ where V is an open subset of $\mathbb{R}$. Then, the V's, i.e., $\pi_2(U)$ where $\pi_2 \colon X \times \mathbb{R} \to \mathbb{R}$ is an open map by previous work, form open cover of $[0, 1]$ (since $\bigcup_{U \in \mathcal{U}} U \supset A$, we must have $\bigcup_{U \in \mathcal{U}} \pi_2(U) \supset [0, 1]$). Now, since $[0, 1]$ is compact in X there is a finite collection of the V's, say $\{V_1, \ldots, V_n\}$, that cover $[0, 1]$. Call U_i the element of $\mathcal{U}$ such that $\pi_2(U_i) = V_i$. Then the U_i's form a finite subcover of A. Thus, A is compact. ■

Problem 3. Let B^2 be the disk

$$\left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \right\}.$$

Let S^1 be the circle

$$\left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \right\}.$$

Prove that there is an equivalence relation $\sim$ such that B^2 is homeomorphic to $(S^1 \times [0, 1])/{\sim}$. As port of your proof explain how you are using one or more properties of the quotient topology.

Proof. Such an equivalence relation is called the cone of S^1. We define it as follows, let $(x, y, z), (x', y', z') \in S^1 \times [0, 1]$ then we say $(x, y, z) \sim (x', y', z')$ if and only if $(x, y) = (x', y')$ or $z = z' = 0$. We shall take it on faith that $\sim$ is in fact an equivalence relation (we may return to this if time permits).

By the UMP of the quotient space, we need to find a continuous surjection $f \colon S^1 \times [0, 1] \to B^2$ that preserves the equivalence relation $\sim$. So consider the map $f(x, y, z) :- (zx, zy)$. This map is continuous by Theorem 18.4 since $\pi_1 \circ f(x, y, z) = zx$ is multiplication on $\mathbb{R}$ and similarly for $\pi_2 \circ f(x, y, z)$. Moreover, this map preserves the equivalence relation: let $(x, y, z) \sim (x', y', z')$ then $(x, y, z) = (x', y', z')$ in which case

$$f(x, y, z) = (zx, zy) = (z'x', z'y') = f(x', y', z')$$

or $z = z' = 0$ so

$$f(x, y, 0) = (0 \cdot x, 0 \cdot y) = (0, 0) = (0 \cdot x', 0 \cdot y') = f(x', y', 0).$$

In either case, we have $f(x, y, z) = f(x', y', z')$. Thus, by the UMP of the quotient space, the induced map $f' : (S^1 \times [0, 1])/\sim \to B^2$ is continuous.

Now, since $S^1 \times [0, 1]$ is closed and bounded, by Heine–Borel, $S^1 \times [0, 1]$ is a compact subset of $\mathbb{R}^3$. Therefore, $(S^1 \times [0, 1])/\sim$ is compact. Since $B^2 \subset \mathbb{R}^2$ is Hausdorff, it suffices to show, by Theorem 26.6, that f is bijective.

It is eassy to see that f is surjective since for any point $(x, y) \neq (0, 0)$ in B^2, $\sqrt{x^2 + y^2} \leq 1$ so letting $z = \sqrt{x^2 + y^2}$, $x' = x/\sqrt{x^2 + y^2}$, and $y' = y/\sqrt{x^2 + y^2}$ we have

$$f(x', y', z) = \sqrt{x^2 + y^2} \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) = (x, y).$$

And, trivially, if $(x, y) = 0$, we have $\varphi(x, y, 0) = 0$ for any $(x, y) \in S^1$.

To see that it is injective, merely note that, by the definition of f, $f(x, y, z) = f(x', y', z')$ if and only if $(x, y, z) = (x', y', z')$ or $z = z' = 0$ which precisely means that $(x, y, z) \sim (x', y', z')$. Thus, f is injective.

It follows that $(S^1 \times [0, 1])/\sim \approx B^2$. ■

Problem 4. Let X be a set with 2 elements $\{a, b\}$. Give X the *discrete* topology. Let Y be any topological space. Recall that $\mathcal{C}(X, Y)$ denotes the set of continuous functions from X to Y, with the compact-open topology. Prove that $\mathcal{C}(X, Y)$ is homeomorphic to $Y \times Y$ (with the product topology).

Proof. Consider the map $F : \mathcal{C}(X, Y) \to Y \times Y$ given by $F(f) :- (f(a), f(b))$. This map is continuous by Theorem 18.4, since $\pi_1(F)$ and $\pi_2(F)$ are, respectively, the evaluation of f at a and the evaluation of f and b, both of which are continuous because under the compact-open topology. This map is clearly surjective since for any $(y_1, y_2) \in Y \times Y$ we may define the function $f(a) :- y_1$ and $f(b) :- y_2$ which is continuous sinced X has the discrete topology. Moreover, F is injective since if $(f(a), f(b)) = (g(a), g(b))$ then $f(x) = g(x)$ for all $x \in X$ hence, $f = g$. Therefore, to show that F is a homeomorphism, it suffices to show that F is an open map.

Now it suffices to find a continuous inverse. For any $(y_1, y_2) \in Y \times Y$, define the map $g : Y \times Y \to \mathcal{C}(X, Y)$.

$$g(y_1, y_2) :- f(y) = \begin{cases} a & \text{if } y = y_1 \\ b & \text{if } y = y_2. \end{cases}$$

■

Problem 5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is path-connected. Prove that Y is path-connected.

Proof. ■

Problem 6. Suppose that X is a wedge of two circles: that is, X is a Hausdorff space which is a union of two subspaces A_1 and A_2 such that A_1 and A_2 are each homeomorphic to S^1 and $A_1 \cap A_2$ is a single point p.

Use the Seifert–van Kampen theorem to calculate $\pi_1(X, p)$. You should state what deformation retractions you are using, but you do not have to give formulas for them.

Proof. ■

Problem 7. Let $p \colon E \to B$ be a covering map. Let A be a connected space and let $a \in A$. Prove that if two continuous functions $\alpha, \beta \colon A \to E$ have a property that $\alpha(a) = \beta(a)$ and $p \circ \alpha = p \circ \beta$ then $\alpha = \beta$.

For partial credit, you may assume that p is the standard covering map from $\mathbb{R}$ to S^1.

Proof. ■

Here's an extra problem I felt like doing since I thought it might be on the exam:

Problem.

Theorem (Munkres, Theorem 18.4)**.** *Let $f \colon A \to X \times Y$ be given by the equation $f(a) :- (f_1(a), f_2(a))$. Then f is continuous if and only if $f_1 \colon A \to X$ and $f_2 \colon A \to Y$ are continuous.*

Proof. Let $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ be projections onto the 1st and 2nd factors, respectively. These maps are continuous and open by previous work. Now, for every $a \in A$ we have

$$\pi_1(f(a)) = f_1(a) \qquad \text{and} \qquad \pi_2(f(a)) = f_2(a).$$

Therefore, if f is continuous, then f_1 and f_2 are the composites of the continuous functions above therefore, are continuous.

Conversely, suppose that f_1 and f_2 are continuous. By Lemma C, it suffices to show that for each basic open set $U \times V \subset X \times Y$, the preimage $f^{-1}(U \times V)$ is open. But $a \in f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$, if and only if $f_1(a) \in U$ and $f_2(a) \in V$. Thus, $f^{-1}(U \times V) = f^{-1}(U) \cap f^{-1}(V)$ which is open in A since U is open in X and V is open in Y and f_1, f_2 are continuous. ■

2.4 MA 571: Qualifying Exam, January 2011

Problem 1. Let A be a subset of a topological space X and let B be a subset of A. Prove that $\bar{A} \setminus \bar{B} \subset \overline{A \setminus B}$.

Proof. ■

Problem 2. Let G be a topological group (thas is, a group with a topology for which the group operations are continuous) and let H be a subgroup of G. Suppose that G is connected, that H is a normal subgroup of G, and that the subspace topology on H is discrete. Prove that $gh = hg$ for every $g \in G$, $h \in G$.

Proof. ■

Problem 3. Let X be the space with two points and the discrete topology. Let $Y = \prod_{n=1}^{\infty} X$, with the product topology. What are the connected components of Y? Prove that your answer is correct.

Proof. ■

Problem 4. Let X and Y be topological spaces. Let $x_0 \in X$ and let C be a compact subset of Y. Let N be an open set in $X \times Y$ containing $\{x_0\} \times C$. Prove that there is an open set U containing x_0 and an open set V containing C such that $U \times V \subset N$.

Proof. ■

Problem 5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is connected. Prove that Y is connected.

Proof. ■

Problem 6. Let $p\colon E \to B$ be a covering map. Let $e_0 \in E$ and $b_0 \in B$ with $p(e_0) = b_0$. Let Y be simply connected (in particular, Y is path-connected). Let $y_0 \in Y$. Let $f\colon Y \to B$ be continuous, with $f(y_0) = b_0$.

Prove that the following function $g\colon Y \to E$ is well-defined: Given $y \in Y$, choose a path γ from y_0 to y; let β be the lift of $f \circ \gamma$ to E starting at e_0; now define $g(y) = \beta(1)$.

You may use the fact (without proving it) that the lift of a path homotopy is again a path homotopy.

Proof. ■

Problem 7. Let S^2 be the 2-sphere, that is, the following subspace of $\mathbb{R}^3$:

$$\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \} .$$

Let x_0 be the point $(0, 0, 1) \in S^2$.

Use the Seifert–van Kampen theorem to prove that $\pi_1(S^2, x_0)$ is the trivial group. You may use either of the two versions of the Seifert–van Kampen theorem given in Munkres's book. You will not get credit for any other method.

Proof. ■

References

[1] J.R. Munkres. *Topology*. Featured Titles for Topology Series. Prentice Hall, Incorporated, 2000.