

MA 523: Homework 7

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PROBLEM 7.1

Solve the Dirichlet problem for the Laplace equation in \mathbb{R}^2

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ u = x_1 & \text{on } |x| = 1, \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

(*Hint:* Use Laurent series.)

SOLUTION. ■

PROBLEM 7.2

Let Ω be a bounded domain with a C^1 boundary, $g \in C^2(\partial\Omega)$ and $f \in C(\bar{\Omega})$. Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \quad (*)$$

where ν is the outer normal on $\partial\Omega$. Show that the solution of $(*)$ in $C^2(\Omega) \cap C^1(\bar{\Omega})$ is unique up to a constant; i.e., if u_1 and u_2 are both solutions of $(*)$, then $u_2 = u_1 + \text{const.}$ in Ω .

(*Hint:* Look at the proof of the uniqueness for the Dirichlet problem by energy methods [E, 2.2.5a].)

SOLUTION. Suppose u_1 and u_2 are solutions to the Neumann problem $(*)$. Set $v := u_1 - u_2$. Then v is harmonic in Ω and $\frac{\partial v}{\partial \nu} = 0$ on the boundary of Ω . Now, consider the energy functional

$$I[v] = \int_{\Omega} \frac{1}{2} |Dv|^2 dx$$

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PROBLEM 7.3

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c \in \mathbb{R}$.

(*Hint:* Rewrite the problem in terms of $v(x, t) := e^{ct}u(x, t)$.)

SOLUTION.

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