

MA557 Problem Set 3

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PROBLEM 3.1

Find an example of a finitely generated ring extension $R \subset S$ where S is a Noetherian ring, but R is not.

Proof.

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PROBLEM 3.2

Consider the homomorphism of rings

$$\begin{array}{ccc} & S & \\ & \downarrow \psi & \\ R & \xrightarrow{\varphi} & T. \end{array}$$

The *fiber product* of R and S over T is the subring $R \times_T S = \{ (r, s) \mid \varphi(r) = \psi(s) \}$ of $R \times S$. Assume φ and ψ are surjective. Show that if R and S are Noetherian rings then so is $R \times_T S$.

Proof. Suppose that R and S are Noetherian rings with surjective ring maps $\varphi: R \rightarrow T$ and $\psi: S \rightarrow T$. Then, by (3.5), the product $R \times S$ is Noetherian. Define the ring map $\Phi: R \times S \rightarrow T \times T$ by $\Phi = (\varphi, \psi)$. Then the diagonal, $\Delta_T = \{ (t, t) \mid t \in T \}$, of $T \times T$ is exactly the image of the fiber product of R and S under the ring map Φ . And this is not terribly difficult to see: It is clear, by the definition of the fiber product, that $\Phi(R \times_T S) \subset \Delta_T$. To show the reverse containment, take an element $(t, t) \in \Delta_T$. Then, since φ and ψ are surjective, there are corresponding elements r and s of the rings R and S , respectively, such that $\varphi(r) = t$ and $\psi(s) = t$. Hence, (t, t) are in the image $R \times_T S$ under Φ .

Now, it is clear that $R \times S$ and $T \times T$ have an $R \times S$ -module structure ($R \times S$ by the usual ring multiplication and $T \times T$ by $(r, s)(t, t') = (\varphi(r)t, \psi(s)t')$) so they have an $R \times_T S$ -module structure by restriction to the subring $R \times_T S$ of $R \times S$. Consider the quotient module $T \times T / \Delta_T$. $T \times T / \Delta_T$ also inherits an $R \times_T S$ -module structure from $T \times T$. Note that the map $\Phi: R \times S \rightarrow T \times T$ is an $R \times_T S$ -linear map: It is clear that Φ is linear with respect to “+”, what is not so obvious is that multiplication by scalars is preserved so take $(r', s') \in R \times_T S$ and $(r, s) \in R \times S$, then

$$\begin{aligned} \Phi((r', s')(r, s)) &= \Phi(r'r, s's) \\ &= (\varphi(r'r), \psi(s's)) \\ &= (\varphi(r')\varphi(r), \psi(s')\psi(s)) \\ &= (\varphi(r'), \psi(s'))(\varphi(r), \psi(s)) \\ &= \Phi(r', s')\Phi(r, s) \end{aligned}$$

as desired. Therefore, Φ induces an $R \times_T S$ -linear map $\Phi^*: R \times S \rightarrow T \times T / \Delta_T$ via composition with the quotient map, i.e., $\Phi^* = \pi \circ \Phi$ and we have the following exact sequence of $R \times_T S$ -modules

$$0 \longrightarrow R \times_T S \xhookrightarrow{\iota} R \times S \xrightarrow{\Phi^*} \frac{T \times T}{\Delta_T} \longrightarrow 0.$$

By (3.4), $R \times_T S$ are Noetherian. ■

PROBLEM 3.3

Let M be an R -module. Show that M is a flat R -module if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$ -module for every maximal ideal \mathfrak{m} of R .

Proof. \implies : Suppose that M is a flat R -module.

\impliedby :

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PROBLEM 3.4

Let M be an R -module and \mathfrak{a} an R -ideal.

- (a) Show that if $M_{\mathfrak{m}} = 0$ for every maximal ideal \mathfrak{m} containing \mathfrak{a} , then $M = \mathfrak{a}M$.
- (b) Show that the converse holds in case M is finite.

Proof. (a) Suppose that $M_{\mathfrak{m}} = 0$ for every maximal ideal \mathfrak{m} containing \mathfrak{a} . ■

PROBLEM 3.5

Prove that every power of a maximal ideal is primary.

Proof.



PROBLEM 3.6

- (a) Show that the radical of a primary ideal is prime.
- (b) Find an example of a power of a prime ideal that is not primary.
- (c) Let \mathfrak{p} be a prime ideal of a ring R and $n \in \mathbf{N}$. The R -ideal $\mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$ is called the *n th symbolic power of \mathfrak{p}* . Show that $\mathfrak{p}^{(n)}$ is primary.

Proof.

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