MA 523: Homework, Midterms and Practice Problems Solutions

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Contents

1 Midterms and Qualifying Exams

1.1 Qualifying Exam, August '04

Exercise 1.1. Consider the initial value problem

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = -u, \\ \qquad\qquad\qquad u = f \qquad \text{on } S^1 = \{\, x^2 + y^2 = 1 \,\}, \end{cases}$$

where a and b satisfy

$$a(x,y) + b(x,y)y > 0$$

for any $x, y \in \mathbb{R}^n \setminus \{(0,0)\}$.

(a) Show that the initial value problem has a unique solution in a neighborhood of S^1. Assume that a, b, and f are smooth.
(b) Show that the solution of the initial value problem actually exists in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Solution. ■

Exercise 1.2. Let $u \in C^2\big(\mathbb{R} \times [0,\infty)\big)$ be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty), \\ \qquad u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where f and g have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx,$$

and the potential energy by

$$P(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx.$$

Show that

(a) $K(t) + P(t)$ is constant in t,
(b) $K(t) = P(t)$ for all large enough times t.

Solution. ■

Exercise 1.3. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = e^{-t}g(x) & \text{for } x \in \mathbb{R}^3,\ t > 0, \\ \quad u(x,0) = u_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution $u = e^{-t} + t - 1$ when $g(x) = 1$.

Solution. ■

Exercise 1.4. Let Ω be a bounded open set in $\mathbb{R}^n$ and $g \in C_0^\infty(\Omega)$. Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega,\ t > 0, \\ u(x,0) = g(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } xi \in \partial\Omega,\ t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n,\ t > 0, \\ v(x,0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put $g = 0$ outside Ω.

(a) Show that
$$-v(x,t) \leq u(x,t) \leq v(x,t)$$
for any $x \in \Omega$, $t > 0$.

(b) Use (a) to conclude that
$$\lim_{t \to \infty} u(x,t) = 0,$$
for any $x \in \Omega$.

Solution. ■

Exercise 1.5. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \qquad P_m(\lambda x) = \lambda^m P_m(x),$$

for any $x \in \mathbb{R}^n$, $\lambda > 0$,

$$\Delta P_k = 0, \qquad \Delta P_m = 0$$

in $\mathbb{R}^n$.

(a) Show that
$$\frac{\partial P_k(x)}{\partial \nu} = k P_k(x), \qquad \frac{\partial P_m(x)}{\partial \nu} = m P_m(x)$$
on ∂B_1, where $B_1 = \{\, |x| < 1 \,\}$ and ν is the outward normal on ∂B_1.

(b) Use (a) and Green's second identity to prove that
$$\int_{\partial B_1} P_k(x) P_m(x)\, dS = 0,$$
if $k \neq m$.

Solution. ■

1.2 Qualifying Exam, August '05

Exercise 1.6.

(a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ \qquad u = 1 \qquad \text{on } S^1 = \{ x^2 + y^2 = 1 \}. \end{cases}$$

(b) Is the solution unique in a neighborhood of the point $(1, 0)$? Justify your answer.

Solution. ■

Exercise 1.7. Consider the second order PDE in $\{ x > 0, y > 0 \} \subset \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

(a) Classify the equation and reduce it to the canonical form.
(b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy} G(x/y).$$

Solution. ■

Exercise 1.8. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^3$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial (i.e., $f(y) = f(|y|)$) and supported in $B_R = \{ |x| < R \}$, then

$$u(x) = c\Phi(x),$$

for any $x \in \mathbb{R}^n \setminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y)\, dy.$$

[*Hint:* Use spherical (polar) coordinates and the mean value property.]

Solution. ■

Exercise 1.9. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \qquad u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where $g, h \in C_0^\infty(\mathbb{R}^2)$ and $\alpha \geq 0$ is a constant.

(a) Show that for an appropriate choice of constant λ and μ the function

$$v(x_1, x_2, x_3, t) := e^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation $v_{tt} - \Delta v = 0$.

(b) Use (a) to prove the following domain of dependence result: for any point $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$ the value $u(x_0, t_0)$ is uniquely determined by values of g and h in $\overline{B_{t_0}(x_0)} := \{\, |x - x_0| \leq t_0 \,\}$. (You may use the corresponding result for the wave equation without proof.)

Solution. ■

Exercise 1.10. Let $u(x, t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0, \infty)$ with the initial condition

$$u(x, 0) = u_0(x)$$

for $x \in \mathbb{R}$, where $u_0 \in C^\infty$ is 2π-periodic, i.e., $u_0(x + 2\pi) = u_0(x)$. Show that

$$\lim_{t \to \infty} u(x, t) = a_0,$$

uniformly in $x \in \mathbb{R}$, where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x)\, dx.$$

Solution. ■

1.3 Qualifying Exam, January '14

Exercise 1.11. Consider the first order equation in $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

(a) Find the characteristic curves of the equation.
(b) Consider the Cauchy problem for this equation prescribed on the line $x_1 = 1$:

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on f so that the proble is solvable in a neighborhood of the point $(1, 0)$.

Solution. ■

Exercise 1.12. Let u be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{ x_n > 0 \}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where g is a continuous function with compact support in $\mathbb{R}^{n-1}$. Here $n \geq 2$. Prove that

$$u(x) \longrightarrow 0, \qquad \text{as } |x| \longrightarrow \infty,$$

for $x \in \{ x_n > 0 \}$.

Solution. ■

Exercise 1.13. Let u be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \qquad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions $u(x, 0) = g(x)$, where g is a bounded continuous function on $\mathbb{R}$ satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^\alpha, \qquad x, y \in \mathbb{R}$$

with a constant $\alpha \in (0, 1]$. Show that

$$\begin{aligned} |u(x,t) - u(y,t)| &\leq M|x - y|^\alpha, & x, y \in \mathbb{R}, t > 0, \\ |u(x,t) - u(x,s)| &\leq C_\alpha M|t - s|^{\alpha/2}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of $u(x,t)$ as a convolution with the heat kernel $\Phi(y,t)$, make a change of variables $z = y/\sqrt{t}$ and use that $\left|\sqrt{t} - \sqrt{s}\right| \leq \sqrt{|t - s|}$.]

Solution. ■

Exercise 1.14. Let u be a positive harmonic function in the unit ball B_1 in $\mathbb{R}^n$. Show that

$$\big|D(\ln u)\big| \le M \qquad \text{in } B_{1/2}$$

for a constant M depending only on the dimension n.
[*Hint:* Use the interior derivative estimate $\big|Du(x)\big| \le (C_n/r)\sup_{B_r(x)}|u|$ for $B_r(x) \subset B_1$ as well as the Harnack inequality for harmonic functions.]

Solution. ∎

Exercise 1.15. Let u be a C^2 solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0,\infty), \\ \qquad u = 0, \qquad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some $k \ge 0$. Prove that there exists a function $\varphi(r)$ such that

$$u(x,t) = t^{k+2}\varphi\big(|x|/t\big).$$

[*Hint:* As one of the steps show that u is $(k+2)$-homogeneous in (x,t) variables, i.e., $u(\lambda x, \lambda t) = \lambda^{k+2}u(x,t)$ for any $\lambda > 0$.]

Solution. ∎