INTRODUCTION TO KAUFFMAN BRACKET SKEIN MODULES. MRC WORKSHOP INTRODUCTORY NOTES VER. June 1, 2016

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ABSTRACT. We introduce Kauffman bracket skein modules and skein algebras and describe briefly their relation to character varieties, topological quantum field theory, quantum Teichmuller spaces, A-polynomial, colored Jones polynomials, AJ-conjecture.

1. SKEIN MODULES

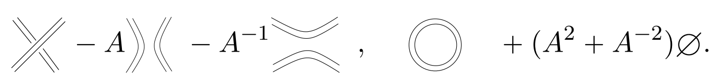
We assume that the reader is familiar with rudimentary knot theory – the definition of tame links, Reidemeister moves, Kauffman bracket and Jones polynomial. There is a large number of great references for that, including online notes, for example [Ro], Sections. 1.1,1.2,1.8, 2.1-4, 3.1-2, 4.1-5.

Let M be a 3-dimensional manifold, possibly with boundary. A framed link in M is a tame embedding of a finite number of disjoint annuli into M,

$$L = S^1 \times I \cup ... \cup S^1 \times I \hookrightarrow M,$$

where $I = [0, 1]$. Framed links are considered up to isotopy (i.e. homotopy within the space of framed links). We denote their set, including the empty link, $\emptyset$, by $\mathcal{L}(M)$.

Let R be a commutative ring with identity and a fixed invertible element $A \in R$. (The most natural choice is $\mathbb{Z}[A^{\pm 1}]$ since it is the initial object in the category of such rings.) The *Kauffman bracket skein module $\mathcal{S}(M; R, A)$ of M* is the quotient of the free R-module $R\mathcal{L}(M)$ by the submodule generated by the Kauffman bracket skein relations (1).

(1) 

We will call $\mathcal{S}(M; R, A)$ a *skein module* for simplicity. We should point out though that the general concept of a skein module, introduced by J. Przytycki and V. Turaev, [Pr1, Tu], encompasses other types of skein modules as well, eg. [HP1, Ka, Pr2, S3, SW].

Denoting the class of a framed link $L \subset M$ in $\mathcal{S}(M; R, A)$ by $[L]$, we obtain a framed link invariant

$$[\cdot] : \mathcal{L}(M) \to \mathcal{S}(M; R, A) \tag{2}$$

satisfying the Kauffman bracket skein relations. (Note that for a link L composed of components K_1, K_2, $[L]$ is usually not equal $[K_1]+[K_2]$.)

Exercise 1. *Prove that $\mathcal{S}(\mathbb{R}^3; R, A) = S(S^3; R, A) = R$. (This statement is equivalent to the fact that the Kauffman bracket is the unique framed link invariant satisfying the above skein relations.)*

The above observations imply that $[\cdot]$ is the natural generalization of the Kauffman bracket to links in 3-manifolds other than $\mathbb{R}^3, S^3$.

Note that the skein relations involve links with coinciding values in $H_1(M, \mathbb{Z}/2)$. Consequently, $\mathcal{S}(M; R, A)$ is a $H_1(M, \mathbb{Z}/2)$-graded module.

Having defined skein modules $\mathcal{S}(M; R, A)$, it is naturally tempting to compute their explicit presentations for all M and R. This has been accomplished in some instances only – see below. No algorithm for computing skein modules in general is known so far!

2. Basic Properties of Skein Modules

Proposition 2 (Universal coefficients property, [PS1])**.** *Let $f : R \to R'$ be a homomorphism of rings mapping the distinguished element of R to the distinguished element of R'. Then for any 3-manifold M there exists an isomorphism of R'-modules*

$$f_* : \mathcal{S}(M; R', A) \to \mathcal{S}(M; R, A) \otimes_R R'$$

such that $f_*([L]) = [L]$ for any framed link L in M.

Proof. Let $SkeinRel(M; R, A) \subset R\mathcal{L}(M)$ denote the submodule generated by the Kauffman bracket skein relations. Tensoring the exact sequence

$$0 \to SkeinRel(M; R, A) \to R\mathcal{L}(M) \to \mathcal{S}(M; R, A) \to 0$$

by R' yields the exact sequence

$$SkeinRel(M; R, A) \otimes_R R' \to R'\mathcal{L}(M) \to \mathcal{S}(M; R, A) \otimes_R R' \to 0.$$

Since the image of the first map is $SkeinRel(M; R')$, the statement follows. □

Consequently, $\mathcal{S}(M; R, A)$ is determined by $\mathcal{S}(M; \mathbb{Z}[A^{\pm 1}], A)$ for all rings R.

Skein modules are easy to describe for oriented I-bundles over surfaces[1].

A *multi-loop* in a surface F is an embedded collection of (unoriented) simple loops in F with no contractible components.

Let M be an orientable I-bundle over F. Every orientation preserving loop in F defines a framed knot in M with the framing parallel to F. With every orientation reversing loop in F associate a framed knot in M by adding a half-twist to it. (Choose one of the two possible.) In this way, every multi-loop in F defines a framed link in M.

[1] That is $F \times I$, for F oriented. For every non-orientable F, there is an orientable I-bundle over it. Such bundle is unique for any surface other than the Klein bottle.

Theorem 3 ([Pr1, HP1, SW]). *For every oriented I-bundle over a surface F, the skein module $\mathcal{S}(F; R, A)$ is free with a basis given by multi-loops in F.*

As we have mentioned earlier, there is no algorithm for finding an exact algebraic description of $\mathcal{S}(M; R, A)$ for an arbitrary 3-manifold M. Free bases of skein modules were also found for lens spaces, [HP2], twist knot complements, [BL], torus knot complements, [Ma], some prism manifolds, [Mr2], and (for some R) for the quaternionic manifold, [GH].

However, skein modules are not always free. Torsion in skein modules was found for $M = S^2 \times S^1$, [HP3], $\mathbb{RP}^3 \# \mathbb{RP}^3$, [Mr1], and for some manifolds with a separating torus, [PV]. Among those, $\mathcal{S}(\mathbb{RP}^3 \# \mathbb{RP}^3; \mathbb{Z}[A^{\pm 1}], A)$ has non-cyclic torsion.

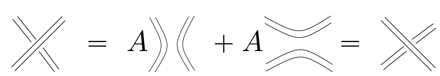
The known computations of skein modules suggest the following:

Conjecture 4 (J. Przytycki). *$\mathcal{S}(M; R, A)$ is free if M has no separating sphere or torus.*

We describe skein modules for unoriented manifolds in the addendum.

3. $A = \pm 1, \pm i$ and skein algebras of groups

Assume now that $A = \pm 1$. Skein modules are particularly simple in that case since



then.

As a consequence of the above observation, one can define the product on $\mathcal{S}(M; R, A)$ as follows:

Exercise 5. *Prove that there is a unique R-linear product on $\mathcal{S}(M; R, \pm 1)$ such that that $[L_1] \cdot [L_2] = [L_1 \cup L_2]$, where $L_1 \cup L_2$ is any disjoint union of links $L_1, L_2 \subset M$.*

Note that the above product is commutative and associative. Soon we will see that the R-algebra $\mathcal{S}(M; R, \pm 1)$ depends on $\pi_1(M)$ only. For that consider the following:

Definition 6. *The skein algebra $\mathcal{S}(\Gamma; R)$ of a group Γ is the quotient of the polynomial ring $R[x_g : g \in \Gamma]$ (i.e. the symmetric tensor algebra over $R\Gamma$) by the ideal generated by the relations $x_e = 2$, $x_g x_h = x_{gh} + x_{gh^{-1}}$.*

Note that $2x_g = x_e x_g = x_g + x_{g^{-1}}$, implying that

$$x_{g^{-1}} = x_g \tag{3}$$

for every $g \in \Gamma$. Furthermore,

$$0 = x_g x_h - x_h x_g = x_{gh} + x_{gh^{-1}} - x_{hg} - x_{hg^{-1}} = x_{gh} - x_{hg},$$

by (3). Consequently,

$$x_{hgh^{-1}} = x_{hg^{-1}h^{-1}} = x_g.$$

Hence, for any topological space X, $\mathcal{S}(\pi_1(X);R)$ is generated by free homotopy classes of unoriented loops in X.

Theorem 7. *(Proof in* [PS1]*.)If $A=-1$ then there is a unique R-algebra homomorphism $\xi:\mathcal{S}(M;R,A)\to\mathcal{S}(\pi_1(M);R)$ sending each framed knot K in M to $-x_g$, where $g\in\pi_1(M)$ is any representative of the free homotopy class of K.*

The case of $A=1$ is more complicated because $[K]\in\mathcal{S}(M;R,A)$ depends on the framing of K then. That complication can be resolved by considering spin structures on M. For that, consider a Riemannian metric on a 3-manifold M. A spin structure on M is a lift of its tangent $SO(3)$-bundle TM to an $SU(2)$-bundle. Since the tangent bundle of every 3-manifold is trivial, there are $H^1(M,\mathbb{Z}/2)$ of such spin structures on M. Each of them defines a "sign" function, $Spin(K)\in\mathbb{Z}/2$, depending on whether the framing of K lifts to a section of $SU(2)$-bundle over K. Let $Spin(L)$ be the product of $Spin(K)$ over all connected components K of L.

Theorem 8 ([Ba])**.** *$\phi:R\mathcal{L}(M)\to R\mathcal{L}(M)$ sending L to $Spin(L)\cdot L$ descends to an isomorphism $\phi:\mathcal{S}(M;R,A)\to\mathcal{S}(M;R,-A)$.*

Consequently, $\mathcal{S}(M;R,1)$ is isomorphic to $\mathcal{S}(\pi_1(M);R)$ as well.

4. Skein algebras of groups and character varieties

For an algebraically closed field $\mathbb{K}$, let $X(\Gamma,\mathbb{K})$ be the $SL(2,\mathbb{K})$-character variety of Γ, thought as a (possibly nonreduced) algebraic scheme, cf. for example [S3] and the references within. That means that its coordinate ring, $\mathbb{K}[X(\Gamma,\mathbb{K})]$, may have non-zero nilpotents. (Formally speaking, $\mathbb{K}[X(\Gamma,\mathbb{K})]$ denotes here the algebra of the global sections of the structure sheaf on $X(\Gamma,\mathbb{K})$.)

Observe that the relations

$$x_e=2 \quad \text{and} \quad x_gx_h=x_{gh}+x_{gh^{-1}}$$

of skein algebras resemble the $SL(2)$-trace relations

$$Tr(I)=2, \quad \text{and} \quad Tr(A)Tr(B)=Tr(AB)+Tr(AB^{-1}).$$

Indeed, there is an obvious algebra homomorphism

$$\psi:\mathcal{S}(\Gamma;\mathbb{K})\to\mathbb{K}[X(\Gamma,\mathbb{K})], \quad \psi(x_g)([\rho])=tr(\rho(g)).$$

Theorem 9 ([PS2])**.** *If $\chi(\mathbb{K})\neq 2$, ψ is an isomorphism of $\mathbb{K}$-algebras.*

Independently, D. Bullock proved a somewhat weaker version, "up to nilpotents", in [Bu].

Consequently, $\mathcal{S}(M;\mathbb{K},A)$ is a deformation of $\mathbb{K}[X(\Gamma,\mathbb{K})]$.

5. Skein algebras of surfaces

Let $M = F \times I$ now, where F is an oriented surface, and let R be an arbitrary commutative ring. The trivial I-bundle structure of M allows for a notion of a product on $\mathcal{S}(M; R, A)$ defined for framed links $L_1, L_2 \subset M$ as the union of these two links with L_1 is positioned in $F \times (0, 1/2)$ and L_2 in $F \times (1/2, 1)$. This operation extends linearly to a multiplication operation on the entire module $\mathcal{S}(F, B)$ with the identity $\varnothing$. We call the resulted R-algebra *skein algebra of F* and we denote it for simplicity by $\mathcal{S}(F; R, A)$. (This product coincides with the one defined earlier for $A = \pm 1$.)

Example 10. *(1) $\mathcal{S}(\mathbb{R}^2; R, A) = \mathcal{S}(S^2; R, A) = R$ (see Exercise 1 above).*
(2) $\mathcal{S}(S^1 \times I; R, A) = R[x]$, where x is the core of the annulus $S^1 \times I$ with the flat framing.
(3) For a three punctured sphere (pair of pants) P, $\mathcal{S}(P; R, A) = R[a, b, c]$ where a, b, c represent knots parallel to the three boundary components of P.
(4) $\mathcal{T} = \mathbb{C}[A^{\pm 1}]\langle l^{\pm 1}, m^{\pm 1}\rangle/(lm - Aml)$ is called a quantum torus. Skein algebra of the torus is isomorphic to $\mathcal{T}^\tau$, i.e. the invariant part of an involution τ on $\mathcal{T}$ such that $\tau(l) = l^{-1}, \tau(m) = m^{-1}$, [FG, Sa].
(5) Skein algebras of trice-punctured disk and punctured torus were computed in [BP].

2-disk, annulus, and the pair of pants are the only surfaces with commutative skein algebras for any R. Skein algebras have known purely algebraic descriptions for surfaces with non-trivial boundary. Such descriptions are not known for closed surfaces though. Some basic properties of them are known – for example for any F and any integral domain R, $\mathcal{S}(F; R, A)$ has no (non-zero) zero divisors. If A is not a root of unity, then the center of $\mathcal{S}(F; R, A)$ is a polynomial ring in variables corresponding to the boundary components of F, [PS3].

Taking $R = \mathbb{C}[[h]]$ and $A = e^h$, the skein algebra multiplication is commutative mod h, but generally speaking not mod h^2.

$$\{x, y\} = \frac{xy - yx}{h} \bmod h$$

is a pairing on $\mathcal{S}(F; \mathbb{C}, A) \otimes \mathbb{C} = \mathbb{C}[X(\pi_1(F))]$. It is easy to see that it is a Poisson bracket, i.e. a Lie algebra bracket such that

$$[xy, z] = x[y, z] + y[x, z].$$

[BFK] proved that this bracket coincides with the Goldman bracket on $X(\pi_1(F), \mathbb{C})$ of [Go]. In other words, skein algebras are quantum deformations of character varieties of surfaces "in the direction of" Goldman bracket.

6. Skein algebras and quantum topology

Skein modules and skein algebras play an important role in quantum topology.

6.1. **Skein algebras and TQFT.** In his seminal work [Wi], Witten postulated that for each semi-simple G and every "level" $r \in \mathbb{Z}_+$ there exist a "Topological Quantum Field Theory" (TQFT), which associates with every closed oriented surface F with marked points P (and some extra data) a complex vector space $V(F, P)$ called, a *state space*. Furthermore, for every 3-manifold M with a properly embedded 1-mfld $L \hookrightarrow M$ (which may be $\varnothing$) whose components are labeled by representations of G, TQFT associates $I(M, L) \in V(\partial M, \partial K)$. A rigorous construction of these theories was achieved by Reshetikhin and Turaev.

$V(S^2, \varnothing) = \mathbb{C}$ for every G and r. For the Lie group $SU(2)$, the invariant of $L \subset D^3$ at level r, $I(D^3, L) \in V(S^2, \varnothing) = \mathbb{C}$ is the colored Jones polynomial of L at r-th root of 1. In particular, labeling all components of L by the defining 2-dim representation of $SU(2)$ yields the Jones polynomial of L.

It turns out that $End(V(F, \varnothing))$ coincides with the skein algebra of F for A $4r$-th root of 1 quotiented by the "Jones-Wenzl idempotent". (That is partially proved in [S2].)

6.2. **Skein algebras and quantum Teichmüller theory.** The skein algebra of a punctured surface (almost) coincides with the quantum Teichmüller space of Chekhov-Fock, through the "quantum trace" embedding of Bonahon-Wang, [BW1, BW2]; see also [Le, Mu].

6.3. **Skein modules, the non-commutative A-polynomial, the colored Jones polynomial, and the AJ-conjecture.** For a knot $K \subset S^3$, let $\mathcal{N}(K)$ denote an open tubular neighborhood of K and let

$$T = \partial(S^3 - \mathcal{N}(K))$$

be the boundary torus. Then $\mathcal{S}(M - \mathcal{N}(K); R, A)$ is a module over the skein algebra $\mathcal{S}(T; R, A)$. (It can be set up as a left or right module. Let us say, left module.)

$$\mathcal{O}(K) = \{v \in \mathcal{S}(T; \mathbb{C}[A^{\pm 1}], A) : \mathcal{S}(T; \mathbb{C}[A^{\pm 1}], A) \cdot v \cdot \varnothing = 0\}$$

is called the orthogonal ideal of K.

Consider colored Jones polynomials of K now, which we will denote by $J_K(n)$ here, normalized so that $J(0) = 0$, $J(1) = 1$, $J(2) =$ Jones polynomial of K. Then setting $J_K(-n) = -J_K(n)$, makes the colored Jones polynomial belong to the space of functions $\mathcal{F} = Fun(\mathbb{Z}, \mathbb{C}[q^{\pm 1}])$.

Consider operators E and Q on $\mathcal{F}$:

$$E(f)(n) = f(n + 1), \quad Q(f) = q^n f.$$

Since E and Q q-commute, $E^{\pm 1}, Q^{\pm 1}$ generate subalgebra

$$\mathcal{T} = \mathbb{C}[q]\langle E^{\pm 1}, Q^{\pm 1}\rangle/(EQ = qQE)$$

of the algebra of linear transformations of $\mathcal{F}$. (Quantum torus again!)

$$I_K = \{p \in \mathcal{T} : p \cdot J_K = 0\}$$

is the *recursive ideal* of K. Its name stems from the fact that its elements define recursive relations on the values of J_K. (Such relations are called "q-holonomic".) As alluded before, [FG] constructed an isomorphism

$$\Psi : \mathcal{S}(T; \mathbb{C}[A^{\pm 1}) \to \mathcal{T}^{\tau}$$

sending A to q and the (a, b)-curve on T to

$$(-1)^{a+b} q^{-ab/2}(E^a Q^b + E^{-a} Q^{-b}),$$

where $\tau(E) = E^{-1}, \tau(Q) = Q^{-1}$. Garoufalidis proved that under this isomorphism the orthogonal ideal $\mathcal{O}(K)$ corresponds to $I_K \cap \mathcal{T}^{\tau}$. The quantum torus $\mathcal{T}$ can be localized to an algebra $\mathcal{T}_{loc}$ isomorphic as vector space with $\mathbb{C}(q, Q)[E]$, with the monomial multiplication

$$a(q, Q)E^k \cdot b(q, Q)E^l = a(q, Q)b(q, q^k Q)E^{k+l}.$$

This localized ring is a principal ideal domain. The non-commutative A-polynomial is the generator of the localization of I_K in $\mathcal{T}_{loc}$.

Let $\varepsilon : \mathcal{T}_{loc} \to \mathbb{C}[Q^{\pm 1}, E^{\pm 1}]$ be the map sending q to 1.

Conjecture 11 (AJ conjecture, [Ga])**.** *For every knot K, $\varepsilon(A_q)(L, M)$ is the A-polynomial $A(L, M)$ of K.*

The above conjecture implies that $\varepsilon(I_K)$ and the ideal in $\mathbb{C}[L^{\pm 1}, M^{\pm 1}]$ generated by the A-polynomial of K are "essentially" equal.

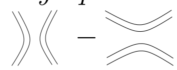
A similar idea was considered earlier by [FGL], where the authors considered the lift of the kernel of the map

$$\mathcal{S}(T; \mathbb{C}[A^{\pm 1}, A) \to \mathcal{S}(S^3 - \mathcal{N}(K); \mathbb{C}[A^{\pm 1}, A)$$

to $\mathcal{S}(T; \mathbb{C}[A^{\pm 1}], A)$ as a non-commutative analog of the ideal in $\mathbb{C}[L^{\pm 1}, M^{\pm 1}]$ generated by the A-polynomial.

7. Addendum: Skein modules of nonorientable 3-manifolds

Let us finish the notes by discussing skein modules of nonorientable 3-manifolds. Let R be any commutative ring.

Exercise 12. *(1) In any nonorientable 3-manifold M, $[L] \in \mathcal{S}(M; R, A)$ is invariant under crossing changes in L. In particular, the brackets of any two homotopically equivalent links coincide in $\mathcal{S}(M; R, A)$.*
(2) $(A^2 - 1)\Big($  $\Big) = 0$ *in* $\mathcal{S}(M; R, A)$.

Assign to each $H_1(M, \mathbb{Z}/2)$ a framed link in M representing it. Now extend this assignment linearly to an R-module map

$$\psi : RH_1(M, \mathbb{Z}/2) \to \mathcal{S}(M; R, A).$$

Exercise 13.

$$(A^2 - 1)\mathcal{S}(M) + Im\, \psi = \mathcal{S}(M).$$

Therefore, for $\mathbb{Z}/2$-homology spheres, $\mathcal{S}(M; R, A)$ is a torsion module, annihilated by $A^2 - 1$. On the other hand, since for an algebraically closed field $\mathbb{K}$ with $char(\mathbb{K}) \neq 2$ we have $\mathcal{S}(M; \mathbb{K}) = \mathcal{X}[X(\pi_1(M))]$, the skein module $\mathcal{S}(M; \mathbb{K}, \pm 1)$ may have an arbitrarily large torsion part.

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