3.1: 3, 5, 11
3.2: 2, 3, 4, 7, 10, 14, 15, 17
Matlab 8.1: 5, 6

3.1.3 Determine whether each of the following permutations of $S = \{1, 2, 3, 4\}$ is even odd:
(a) 4213 (b) 1243 (c) 1234

Following example 4, (a) 4 precedes 2, 1, 3; 2 precedes 1 so the inversions needed is 4, so even.
(b) 4 precedes 3 is the only inversion, so odd. (c) nothing is inverted, so even.

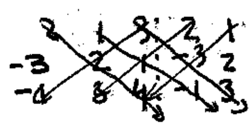
3.1.5 Determine the sign associated with each of the following permutations of the column indices of a 5×5 matrix: (a) 25431 (b) 31245 (c) 21345

(a) 5 precedes 4, 3, 1; 4 precedes 3, 1; 3 precedes 1; 2 precedes 1, so 7 inversions which makes it odd so negative sign.
(b) 3 preceds 1, 2; so even with a positive sign.
(c) 2 precedes 1; so odd with a negative sign.

3.1.11 Evaluate:
(a) $\det\left(\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)$ (b) $\begin{vmatrix} 2 & 1 & 3 \\ -3 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix}$ (c) $\det\left(\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}\right)$

(a) By Defn (3.2).

$$\det\left(\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right) = \underset{a_{11}a_{22}a_{33}}{(2)(2)(2)} + \underset{a_{12}a_{23}a_{31}}{(1)(1)(0)} + \underset{a_{13}a_{21}a_{32}}{(3)(3)(1)} - \underset{a_{11}a_{23}a_{32}}{(2)(1)(1)} - \underset{a_{12}a_{21}a_{33}}{(1)(3)(2)} - \underset{a_{13}a_{22}a_{31}}{(3)(2)(0)}$$
$$= 8 + 0 + 9 - 2 - 6 - 0 = 9$$

(b) 
$$\det\left(\begin{bmatrix} 2 & 1 & 3 \\ -3 & 2 & 1 \\ -1 & 3 & 4 \end{bmatrix}\right) = (2)(2)(4) + (1)(1)(-1) + (3)(-3)(3) - (3)(2)(-1) - (2)(1)(3) - (1)(-3)(4)$$
$$= 16 - 1 - 27 + 6 - 6 + 12 = 0$$

(c) Notice the only permutation in Defn (3.2) that is non zero is $a_{14}a_{23}a_{32}a_{41}$ and permutation 4321 has sign + as it is 6 inversions.
Thus
$$\det\left(\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}\right) = +(3)(4)(2)(6) = 144$$

3.2.2 Compute the following determinants via reduction to triangular form or by citing a particular theorem or corollary:

(a) $\begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix}$ (b) $\begin{vmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix}$ (d) $\begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix}$ (e) $\begin{vmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{vmatrix}$ (f) $\begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix}$.

(a) By example 7, $\begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix} = (2)(-1) - (-2)(3) = -2+6 = 4$

(b) By Thm (3.7) as it is upper triangular $\begin{vmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = (4)(-2)(3) = -24.$

(c) While triangular, not of proper form.

$$\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix} \underset{\text{Thm}(3.6)}{\overset{-r_2+r_1}{=}} \begin{vmatrix} 1 & -1 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix} \underset{\substack{-2r_1+r_2 \\ -3r_1+r_3}}{\overset{\text{Thm}(3.6)}{=}} \begin{vmatrix} 1 & -1 & 2 \\ 0 & 7 & -4 \\ 0 & 3 & -6 \end{vmatrix} \underset{-2r_3+r_2}{\overset{\text{Thm}(3.6)}{=}} \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 8 \\ 0 & 3 & -6 \end{vmatrix} \underset{-3r_2+r_3}{\overset{\text{Thm}(3.6)}{=}} \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 8 \\ 0 & 0 & -30 \end{vmatrix} \overset{(3.7)}{=} -30$$

(d)
$$\begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix} \underset{\text{Thm}(3.6)}{\overset{-r_1+r_2}{=}} \begin{vmatrix} 4 & -3 & 5 \\ 1 & 5 & -5 \\ 2 & 0 & 4 \end{vmatrix} \underset{\text{Thm}(3.2)}{\overset{r_1 \leftrightarrow r_2}{=}} -\begin{vmatrix} 1 & 5 & -5 \\ 4 & -3 & 5 \\ 2 & 0 & 4 \end{vmatrix} \underset{\substack{-4r_1+r_2 \\ -2r_1+r_3}}{\overset{\text{Thm}(3.6)}{=}} -\begin{vmatrix} 1 & 5 & -5 \\ 0 & -23 & 25 \\ 0 & -10 & 14 \end{vmatrix} \underset{\text{Thm}(3.6)}{\overset{2r_3+r_2}{=}} -\begin{vmatrix} 1 & 5 & -5 \\ 0 & -3 & -3 \\ 0 & -10 & 14 \end{vmatrix} \; -\tfrac{1}{3} r_2 \to r_2$$

$$\underset{\text{Thm}(3.5)}{=} 3\begin{vmatrix} 1 & 5 & -5 \\ 0 & 1 & 1 \\ 0 & -10 & 14 \end{vmatrix} \underset{10r_2+r_3}{\overset{\text{Thm}(3.6)}{=}} 3\begin{vmatrix} 1 & 5 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 24 \end{vmatrix} = 3 \cdot 24 = 72.$$

(e) This is lower triangular so by Thm (3.7) $\begin{vmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{vmatrix} = (4)(2)(-3)(5) = -120$

(f)
$$\begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} \underset{(3.2)}{\overset{r_1 \leftrightarrow r_2}{=}} -\begin{vmatrix} 3 & 2 & -4 & -2 \\ 2 & 0 & 1 & 4 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} \underset{(3.6)}{\overset{-r_2+r_1}{=}} -\begin{vmatrix} 1 & 2 & -5 & -6 \\ 2 & 0 & 1 & 4 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix} \underset{\substack{-2r_1+r_2 \\ -2r_1+r_3 \\ -11r_1+r_4}}{\overset{(3.6)}{=}} -\begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & -4 & 11 & 16 \\ 0 & -1 & 9 & 12 \\ 0 & -14 & 51 & 72 \end{vmatrix} \; r_2 \leftrightarrow r_3$$

$$\underset{(3.2)}{=} \begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & -1 & 9 & 12 \\ 0 & -4 & 11 & 16 \\ 0 & -14 & 51 & 72 \end{vmatrix} \underset{(3.5)}{\overset{-r_2}{=}} -\begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & 1 & -9 & -12 \\ 0 & -4 & 11 & 16 \\ 0 & -14 & 51 & 72 \end{vmatrix} \underset{\substack{4r_2+r_3 \\ 14r_2+r_4}}{\overset{(3.6)}{=}} -\begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & 1 & -9 & -12 \\ 0 & 0 & -25 & -32 \\ 0 & 0 & -75 & -96 \end{vmatrix} \underset{-3r_3+r_4}{\overset{(3.6)}{=}} -\begin{vmatrix} 1 & 2 & -5 & -6 \\ 0 & 1 & -9 & -12 \\ 0 & 0 & -25 & -32 \\ 0 & 0 & 0 & 0 \end{vmatrix} \overset{(3.3)}{=} 0$$

3.2.3 If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3$, find $\begin{vmatrix} a_1+2b_1-3c_1 & a_2+2b_2-3c_2 & a_3+2b_3-3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\begin{vmatrix} a_1+2b_1-3c_1 & a_2+2b_2-3c_2 & a_3+2b_3-3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \underset{(3.6)}{\overset{3r_3+r_1}{=}} \begin{vmatrix} a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \underset{(3.6)}{\overset{-2r_2+r_1}{=}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3$$

3.2.4 If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -2$, find $\begin{vmatrix} a_1 - \frac{1}{2}a_3 & a_2 & a_3 \\ b_1 - \frac{1}{2}b_3 & b_2 & b_3 \\ c_1 - \frac{1}{2}c_3 & c_2 & c_3 \end{vmatrix}$.

$$\begin{vmatrix} a_1 - \frac{1}{2}a_3 & a_2 & a_3 \\ b_1 - \frac{1}{2}b_3 & b_2 & b_3 \\ c_1 - \frac{1}{2}c_3 & c_2 & c_3 \end{vmatrix} \xrightarrow[(3.6)]{C_1 + \frac{1}{2}C_3} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -2.$$

3.2.7 Evaluate:

(a) $\begin{vmatrix} -4 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 2 & 1 & -5 \end{vmatrix}$ (c) $\begin{vmatrix} t-1 & -1 & -2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{vmatrix}$ (d) $\begin{vmatrix} t+1 & 4 \\ 2 & t-3 \end{vmatrix}$.

(a)
$$\begin{vmatrix} -4 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 3 \end{vmatrix} \; r_1 \leftrightarrow r_4 \; \overset{}{\underset{(3.2)}{=}} - \begin{vmatrix} 1 & 3 & 0 & 3 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ -4 & 2 & 0 & 0 \end{vmatrix} \begin{matrix} \\ -2r_1+r_2 \\ -3r_1+r_3 \\ 4r_1+r_4 \end{matrix} \overset{(3.6)}{=} - \begin{vmatrix} 1 & 3 & 0 & 3 \\ 0 & -3 & 1 & -6 \\ 0 & -8 & 0 & -7 \\ 0 & 14 & 0 & 12 \end{vmatrix} \; -3r_2+r_3 \; \overset{(3.6)}{=} - \begin{vmatrix} 1 & 3 & 0 & 3 \\ 0 & -3 & 1 & -6 \\ 0 & 1 & -3 & 11 \\ 0 & 14 & 0 & 12 \end{vmatrix} \; r_2 \leftrightarrow r_3 \; \underset{(3.2)}{=} \begin{vmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & -3 & 11 \\ 0 & -3 & 1 & -6 \\ 0 & 14 & 0 & 12 \end{vmatrix} \begin{matrix} \\ \\ 3r_2+r_3 \\ -14r_2+r_4 \end{matrix}$$

$$\underset{(3.6)}{=} \begin{vmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & -3 & 11 \\ 0 & 0 & -8 & 27 \\ 0 & 0 & 42 & -142 \end{vmatrix} \; 5r_3+r_4 \; \overset{(3.6)}{=} \begin{vmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & -3 & 11 \\ 0 & 0 & -8 & 27 \\ 0 & 0 & 2 & -7 \end{vmatrix} \; 4r_4+r_3 \; \overset{(3.6)}{=} \begin{vmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & -3 & 11 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -7 \end{vmatrix} \; r_3 \leftrightarrow r_4 \; \overset{(3.2)}{=} - \begin{vmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & -3 & 11 \\ 0 & 0 & 2 & -7 \\ 0 & 0 & 0 & -1 \end{vmatrix} \begin{matrix} \\ \\ \frac{1}{2}r_3 \\ -r_4 \end{matrix} \overset{(3.5)}{=} 2 \begin{vmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & -3 & 11 \\ 0 & 0 & 1 & -7/2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \overset{(3.7)}{=} 2$$

(b)
$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 2 & 1 & -5 \end{vmatrix} \overset{(3.7)}{=} (2)(3)(4)(-5) = -120$$

(c)
$$\begin{vmatrix} t-1 & -1 & -2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{vmatrix} \overset{(3.7)}{=} (t-1)(t-2)(t-3)$$

(d)
$$\begin{vmatrix} t+1 & 4 \\ 2 & t-3 \end{vmatrix} \overset{\text{Defn } 3.2}{=} (t+1)(t-3) - (2)(4) = t^2 + t - 3t - 3 - 8 = t^2 - 2t - 11.$$

3.2.10 Show that if k is a scalar and A is $n \times n$ then $\det(kA) = k^n \det(A)$.

Two ways.

1) $\det(kA) \overset{\text{Defn }(3.2)}{=} \sum (\pm) k a_{1j_1} k a_{2j_2} \cdots k a_{nj_n} = k^n \sum (\pm) a_{1j_1} \cdots a_{nj_n} \overset{\text{Defn}(3.2)}{=} k^n \det(A)$.

2) Let $r_1, \dots, r_n$ be the rows of A. Then

$$\det(kA) = \det\left(\begin{bmatrix} kr_1 \\ \vdots \\ kr_n \end{bmatrix}\right) \underset{(3.2)}{=} k \det\left(\begin{bmatrix} r_1 \\ kr_2 \\ \vdots \\ kr_n \end{bmatrix}\right) = \cdots \underset{(3.2)}{=} k^n \det\left(\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}\right) = k^n \det(A).$$

3.2.14 | Show that if $AB = I_n$, then $\det(A) \neq 0$ and $\det(B) \neq 0$.

Suppose $AB = I_n$. We know $\det(I_n) = 1$ and $\det(AB) = \det(A)\det(B)$ by (3.9). Then $1 = \det(I_n) = \det(AB) \overset{(3.9)}{=} \det(A)\det(B)$. If $\det(A) = 0$ or $\det(B) = 0$, then this would be $1 = 0$, a contradiction. Thus $\det(A) \neq 0$ and $\det(B) \neq 0$.

3.2.15 | (a) Show that if $A = A^{-1}$, then $\det(A) = \pm 1$. (b) If $A^T = A^{-1}$, what is $\det(A)$?

(a) Suppose $A = A^{-1}$. Then $1 = \det(I_n) = \det(AA^{-1}) = \det(A^2) \overset{(3.9)}{=} \det(A)\det(A) = (\det(A))^2$. Thus $\det(A) = \pm 1$.

(b) Suppose $A^T = A^{-1}$. Then $\det(A) \overset{(3.1)}{=} \det(A^T) = \det(A^{-1}) \overset{\text{Cor.}(3.2)}{=} \frac{1}{\det(A)}$ showing that $(\det(A))^2 = 1$ so $\det(A) = \pm 1$.

3.2.17 | If A is a nonsingular matrix such that $A^2 = A$, what is $\det(A)$?

$\det(A) = \det(A^2) \overset{(3.9)}{=} \det(A)\det(A) = (\det(A))^2$ so that $(\det(A))^2 - \det(A)$ $= \det(A)(\det(A) - 1) = 0$ showing $\det(A) = 0$ or $\det(A) = 1$. As A is nonsingular, $\det(A) \neq 0$ thus $\det(A) = 1$.

Matlab 8.1

Matlab 8.1.5 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Compute and record $\det(A) = \underline{27}$

we will perform a series of row operations on A and compute the determinant of each new matrix.

Let $B = A_{R_1 \leftrightarrow R_2}$; $\det(B) = \underline{-27}$. How is $\det(B)$ related to $\det(A)$? $\underline{\det(B) = -\det(A)}$

Let $C = A_{R_2 \leftrightarrow R_3}$; $\det(C) = \underline{-27}$. How is $\det(C)$ related to $\det(A)$? $\underline{\det(C) = -\det(A)}$

Let $D = A_{2R_1+R_2}$; $\det(D) = \underline{27}$. How is $\det(D)$ related to $\det(A)$? $\underline{\det(D) = \det(A)}$

Let $E = A_{-4R_1+R_3}$; $\det(E) = \underline{27}$. How is $\det(E)$ related to $\det(A)$? $\underline{\det(E) = \det(A)}$

Let $F = A_{3R_1}$; $\det(F) = \underline{81}$. How is $\det(F)$ related to $\det(A)$? $\underline{\det(F) = 3\det(A)}$.

Let $G = A_{-2R_2}$; $\det(G) = \underline{-54}$. How is $\det(G)$ related to $\det(A)$? $\underline{\det(G) = -2\det(A)}$.

Let $H = A_{1/2 R_3}$; $\det(H) = \underline{13.5}$. How is $\det(H)$ related to $\det(A)$? $\underline{\det(H) = \frac{1}{2}\det(A)}$.

You can repeat with $A = \begin{bmatrix} 2 & -5 & 3 \\ 0 & 2 & -1 \\ 3 & 2 & 1 \end{bmatrix}$.

Conjectures.

If we inter-change rows the determinant changes sign.

If we replace one row by a linear combination of itself with another row the determinant is unchanged.

If we multiply a row by scalar K the determinant is multiplied by K.

Matlab 8.1.6 Fill in the blanks.

(a) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$; $\text{rref}(A) = \underline{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$; $\det(A) = \underline{6.6613e\text{-}16 = 0}$; $\det(\text{rref}(A)) = \underline{0}$.

(b) Let $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; $\text{rref}(B) = \underline{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}$; $\det(B) = \underline{0}$; $\det(\text{rref}(B)) = \underline{0}$.

(c) Let $C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix}$; $\text{rref}(C) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$; $\det(C) = \underline{-3.3307e\text{-}16 = 0}$; $\det(\text{rref}(C)) = \underline{0}$.

(d) Let $D = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$; $\text{rref}(D) = \underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$; $\det(D) = \underline{4}$; $\det(\text{rref}(D)) = \underline{1}$.

(e) <u>True or False</u>: For any square matrix Q; $\det(Q) = \det(\text{rref}(Q))$. <u>False, see (d).</u>

(f) Based upon the few experiments in parts (a)–(d), does there seem to be a connection between the following:

rref is I → det is not zero

rref is not I → det is zero

Draw an arrow between those that appear to be related.

<u>Conjectures</u>: Let Q be a square matrix.

If $\text{rref}(Q) = I$, then $\det(Q)$ is <u>not zero</u>.

If $\text{rref}(Q) \neq I$, then $\det(Q)$ is <u>zero</u>.

The determinant of a nonsingular matrix is <u>not zero</u>.

The determinant of a singular matrix is <u>zero</u>.