

MA557 Problem Set 3

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September 21, 2015

Problem 3.1

Let R be a domain and Γ the set of all principal ideals in R . Show that R is a unique factorization domain if and only if Γ satisfies the ascending chain condition and every irreducible element of R is prime.

Proof.

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Problem 3.2

Let M be an Artinian R -module. Show that every injective R -linear map $\varphi: M \rightarrow M$ is an isomorphism.

Proof.

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Problem 3.3

Let M be a finitely generated Artinian module. Show that M is Noetherian.

Proof.

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Problem 3.4

Let R be a ring that is Artinian or Noetherian, and $x \in R$. Show that for some $n > 0$, the image of x in $R/(0 : x)^n$ is a nonzero-divisor on that ring.

Proof.

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Problem 3.5

Let R be an Artinian ring. Show that $R \cong R_1 \times \cdots \times R_n$ with R_i Artinian local rings.

Proof.

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Problem 3.6

Let R be an Artinian ring all of whose maximal ideals are principal. Show that every ideal in R is principal.

Proof.

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Problem 3.7

Prove 2.12.

Proof. Recall the statement of

Theorem (2.12). (a) *The following are equivalent:*

- (1) $0 \rightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M''$ is exact
- (2) $0 \rightarrow \text{hom}(N, M') \xrightarrow{\text{hom}(N, \varphi)} \text{hom}(N, M) \xrightarrow{\text{hom}(N, \psi)} \text{hom}(N, M'')$ is exact for all modules N .

(b) *The following are equivalent:*

- (1) $M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \rightarrow 0$ is exact.
- (2) $0 \rightarrow \text{hom}(M'', N) \xrightarrow{\text{hom}(\psi, N)} \text{hom}(M, N) \xrightarrow{\text{hom}(\varphi, N)} \text{hom}(M', N)$ is exact for all modules N .
- (3) $M' \otimes N \xrightarrow{\varphi \otimes N} M \otimes N \xrightarrow{\psi \otimes N} M'' \otimes N \rightarrow 0$ is exact for all modules N .

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