

# MA544: Qual Preparation

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## MA 544 Spring 2016

This is material from the course MA 544 as taught in the spring of 2016.

## 1.1 Homework

These exercises were assigned from Wheeden and Zygmund's *Measure and Integral*. Therefore, most of the theorems I reference will be from [? ]. Other resources include [? ] and [? ]. For more elementary results, I cite [? ].

## Homework 1

**Problem 1** (Wheeden & Zygmund Ch. 2, Ex. 1). Let  $f(x) = x \sin(1/x)$  for  $0 < x \leq 1$  and  $f(0) = 0$ . Show that  $f$  is bounded and continuous on  $[0, 1]$ , but that  $V[f; 0, 1] = +\infty$ .

*Proof.* It is clear that the function  $f(x) = x \sin(1/x)$  is *bounded* on  $[0, 1]$  since  $|\sin(1/x)| \leq 1$  and  $|x| \leq 1$  on  $[0, 1]$ . Moreover, by properties of *continuous functions* on  $\mathbb{R}$ , it is obvious that  $f$  is continuous on  $(0, 1)$ .<sup>\*</sup> What is not obvious is continuity at 0. To show that  $f$  is continuous at 0, by Theorem 4.6 from [? , Ch. 4, p. 86], it suffices to show that  $\lim_{x \rightarrow 0} f(x) = 0$ . Consider the sequence  $\{1/k\}$ . This sequence converges to 0. Moreover, given  $\varepsilon > 0$ , by the Archimedean principle, for sufficiently large  $K$ , the inequality  $1/K < \varepsilon$  holds so for every  $k \geq K$  we have

$$|(1/k) \sin(k) - 0| \leq |1/k| < \varepsilon. \quad (1)$$

Thus,  $\lim_{k \rightarrow \infty} f(1/k) = 0$ . Thus,  $f$  is continuous on all of  $[0, 1]$ .

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<sup>\*</sup>You can, for example, take a look at Theorem 4.9 from [? , Ch. 4, p. 87].

Nevertheless,  $f$  is not of *bounded variation* on  $[0, 1]$ . By Corollary 2.10 from [?, Ch. 2, p. 23], the *total variation*  $V$  of  $f$  on  $[0, 1]$  is given by

$$\begin{aligned} V &= \int_0^1 |f'| dx \\ &= \int_0^1 |\sin(1/x) - (1/x) \cos(1/x)| dx \\ &= \int_1^\infty \frac{1}{u^2} |\sin u - u \cos u| dx \\ &\geq \int_M^\infty \frac{1}{2u} du \\ &= \infty, \end{aligned} \tag{2}$$

where, for sufficiently large  $M$ , for  $u \geq M$  we have  $|\sin u - u \cos u| > u/2$ . Thus,  $f$  is not of bounded variation. ■

**Problem 2** (Wheeden & Zygmund Ch. 2, Ex. 2). Prove theorem (2.1).

*Proof.* Recall the statement of theorem (2.1):

- (a) If  $f$  is of bounded variation on  $[a, b]$ , then  $f$  is bounded on  $[a, b]$ .
- (b) Let  $f$  and  $g$  be of bounded variation on  $[a, b]$ . Then  $cf$  (for any real constant  $c$ ),  $f + g$ , and  $fg$  are of bounded variation on  $[a, b]$ . Moreover,  $f/g$  is of bounded variation on  $[a, b]$  if there exists an  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for  $x \in [a, b]$ .

(a) Recall that  $f$  is of b.v. on  $[a, b]$  if the total variation  $V$  of  $f$  on  $[a, b]$  is finite, where  $V$  is defined to be the supremum of the sum  $\sum_{i=1}^m |f(x_i) - f(x_{i-1})|$  over all *partitions*  $\Gamma = \{x_0, \dots, x_m\}$  of  $[a, b]$  of the sum.

(b) ■

**Problem 3** (Wheeden & Zygmund Ch. 2, Ex. 3). If  $[a', b']$  is a subinterval of  $[a, b]$  show that  $P[a', b'] \leq P[a, b]$  and  $N[a', b'] \leq N[a, b]$ .

*Proof.* ■

**Problem 4** (Wheeden & Zygmund Ch. 2, Ex. 11). Show that  $\int_a^b f d\varphi$  exists if and only if given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|R_\Gamma - R_{\Gamma'}| < \varepsilon$  if  $|\Gamma|, |\Gamma'| < \delta$ .

*Proof.* ■

**Problem 5** (Wheeden & Zygmund Ch. 2, Ex. 13). Prove theorem (2.16).

*Proof.* Recall the statement of Theorem 2.16:

- (i) If  $\int_a^b f d\varphi$  exists, then so do  $\int_a^b cf d\varphi$  and  $\int_a^b f d(c\varphi)$  for any constant  $c$ , and

$$\int_a^b cf d\varphi = \int_a^b f d(c\varphi) = c \int_a^b f d\varphi.$$

(ii) If  $\int_a^b f_1 d\varphi$  and  $\int_a^b f_2 d\varphi$  both exist, so does  $\int_a^b (f_1 + f_2) d\varphi$ , and

$$\int_a^b (f_1 + f_2) d\varphi = \int_a^b f_1 d\varphi + \int_a^b f_2 d\varphi.$$

(iii) If  $\int_a^b f d\varphi_1$  and  $\int_a^b f d\varphi_2$  both exist, so does  $\int_a^b f d(\varphi_1 + \varphi_2)$ , and

$$\int_a^b f d(\varphi_1 + \varphi_2) = \int_a^b f d\varphi_1 + \int_a^b f d\varphi_2.$$

■

## **Homework 2**

### Homework 3

## Homework 4



## Homework 5

## Homework 6

## Homework 7

## Homework 8

## Homework 2

## Homework 10

## Homework 11

## Homework 12



## 1.2 Exam 1 Prep

**Problem 1.** Let  $E \subset \mathbb{R}^n$  be a measurable set,  $r \in \mathbb{R}$  and define the set  $rE = \{r\mathbf{x} : \mathbf{x} \in E\}$ . Prove that  $rE$  is measurable, and that  $|rE| = |r|^n|E|$ .

*Proof.* Define a map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $T\mathbf{x} := r\mathbf{x}$ . Note that  $T$  is *Lipschitz continuous* since for any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , the equality

$$|T\mathbf{x} - T\mathbf{y}| = |r\mathbf{x} - r\mathbf{y}| = |r||\mathbf{x} - \mathbf{y}| \quad (1)$$

is satisfied. By Theorem 3.33 from [? , Ch. 3, p.55], the image of  $E$  under  $T$  is measurable. Moreover, by Theorem 3.35 [? , Ch. 3, p. 56], since  $T$  is linear, it follows that  $|T(E)| = |\det T||E|$  where  $\det T = |r|^n$ . Lastly, we note that the image of  $E$  under  $T$  is precisely the set  $rE$  so that  $|T(E)| = |rE| = |r|^n|E|$ , as was to be shown. ■

**Problem 2.** Let  $\{E_k\}$ ,  $k \in \mathbb{N}$  be a collection of measurable sets. Define the set

$$\liminf_{k \rightarrow \infty} E_k = \bigcup_{k=1}^{\infty} \left( \bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|.$$

*Proof.* Following the style of [? , Ch. 1, p. 2], particularly, the sets defined after the introduction of equation (1.1), set

$$V_k := \bigcap_{\ell=k}^{\infty} E_{\ell}. \quad (2)$$

Note that the collection of sets  $\{V_k\}$  forms an increasing sequence, that is, if  $\mathbf{x} \in V_k$  then, by (2),  $\mathbf{x}$  is in the intersection  $E_k \cap \left( \bigcap_{\ell=k+1}^{\infty} E_{\ell} \right)$ , but, by (2),  $\bigcap_{\ell=k+1}^{\infty} E_{\ell} = V_{k+1}$  thus,  $\mathbf{x}$  is in  $V_{k+1}$  so  $V_{k+1} \supset V_k$ . Hence, we have  $V_k \nearrow \liminf E_k$ .

Now, consider the sequence  $\{|V_k|\}$  formed by the Lebesgue measure of the  $V_k$ . By Theorem 3.26 from [? , Ch. 3, p. 51], since  $V_k \nearrow \liminf E_k$ ,

$$\lim_{k \rightarrow \infty} |V_k| = \lim_{k \rightarrow \infty} \left| \bigcap_{\ell=k}^{\infty} E_{\ell} \right| = \left| \liminf_{k \rightarrow \infty} E_k \right|. \quad (3)$$

But note that, by the monotonicity of the Lebesgue measure, we have

$$\left| \bigcap_{\ell=k}^{\infty} E_{\ell} \right| \leq |E_k|, \quad (4)$$

so, by properties of the  $\liminf$ , in particular, by Theorem 19(v) from [? , Ch. 1, p. 23], we have

$$\limsup_{k \rightarrow \infty} |V_k| \leq \liminf_{k \rightarrow \infty} |E_k|. \quad (5)$$

Hence, by (3) and Proposition 19 (iv), since the sequence  $\{|V_k|\}$  converges and is equal to the measure of  $\liminf E_k$ , by (5), we have

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k| \quad (6)$$

as was to be shown. ■

**Problem 3.** Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here  $B(\mathbf{0}, r) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r\}$ . Prove that  $F$  is monotonic increasing and continuous.

*Proof.* Define the linear map  $T: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $T(r)\mathbf{x} := r\mathbf{x}$ . We claim that  $B(\mathbf{0}, r) = T(r, B(\mathbf{0}, 1))$ . To reduce notation, set  $B_1 := B(\mathbf{0}, 1)$  and  $B_r := B(\mathbf{0}, r)$ .

*Proof of claim.*  $\subset$ : Let  $\mathbf{x} \in B_r$ . Then  $|\mathbf{x}| < r$  so  $|\mathbf{x}|/r < 1$ . Thus,  $|\mathbf{x}|/r \in B_1$  so it is in the image of  $B_1$  under the map  $T(r, -)$ .

$\supset$ : On the other hand, suppose  $\mathbf{x} \in T(r, B_1)$ . Then  $\mathbf{x} = r\mathbf{y}$  for some  $\mathbf{y} \in B_1$ . Then, since  $|\mathbf{y}| < 1$ ,  $|\mathbf{x}| = r|\mathbf{y}| < r$  so  $\mathbf{x} \in B_r$ . ♣

From the claim, we see that  $F(x) = |T(x, B(\mathbf{0}, 1))|$  which, by Problem 1, is nothing more than the polynomial  $|B_1|x^n$ . It is clear, from this equivalence, that  $F$  is monotonically increasing: Take  $x, y \in [0, \infty)$  such that  $x < y$ , then  $x^n < y^n$  so

$$F(x) = |B_1|x^n < |B_1|y^n = F(y). \quad (7)$$

Thus,  $F$  is monotonically increasing.

In the argument above, since  $F(x) = |B_1|x^n$  is a polynomial in  $[0, \infty)$  (and polynomials are continuous on  $\mathbb{R}$ )  $F$  is continuous on  $[0, \infty)$ . ■

**Problem 4.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $C$  be the set of all points at which  $f$  is continuous. Show that  $C$  is a set of type  $G_\delta$ .

*Proof.* (Without much motivation) let us consider the collection of sets  $\{E_k\}$  defined by

$$E_k := \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } y, z \in B(x, \delta) \text{ implies } |f(y) - f(z)| < \frac{1}{k} \right\}. \quad (8)$$

We claim that  $C = \bigcap_{k=1}^{\infty} E_k$  and that each  $E_k$  is open.

*Proof of claim.* First, we demonstrate equality.  $\subset$ : Suppose  $x \in C$ . Then, by the definition of continuity, for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $y \in B(x, \delta)$  implies  $|f(x) - f(y)| < \varepsilon$ . In particular, for every  $k$ , there exists  $\delta > 0$  such that for  $y \in B(x, \delta)$  the inequality  $|f(x) - f(y)| < 1/k$  holds. Thus,  $x$  is in  $\bigcap_{k=1}^{\infty} E_k$ .

$\supset$ : On the other hand, suppose that  $x \in \bigcap_{k=1}^{\infty} E_k$ . Then, given  $\varepsilon > 0$ , by the Archimedean property, there exists a positive integer  $N$  such that  $1/N < \varepsilon$ . Then, since  $x \in \bigcap_{k=1}^{\infty} E_k$ ,  $x \in E_N$  so

$$|f(x) - f(y)| < \frac{1}{N} < \varepsilon. \quad (9)$$

Thus,  $x$  is in  $C$  and  $C = \bigcap_{k=1}^{\infty} E_k$ .

All that remains to be shown is that the  $E_k$  are open. But this is clear by the way we defined  $E_k$  in (8): Let  $x \in E_k$ , then there exists  $\delta > 0$  such that for any  $y, z \in B(x, \delta)$ ,  $|f(y) - f(z)| < 1/k$ ; Let  $x' \in B(x, \delta)$  and set  $\delta' := \min\{|(x + \delta) - x'|, |(x - \delta) - x'|\}$ . Then, since  $B(x', \delta') \subset B(x, \delta)$ , for every  $y, z \in B(x', \delta')$ , we have  $|f(y) - f(z)| < 1/k$ . Hence,  $x' \in E_k$  for any  $x' \in B(x, \delta)$  so  $B(x, \delta) \subset E_k$ . ♣

Since  $C$  can be expressed as the countable intersection of open sets  $E_k$ , it follows that  $C$  is a  $G_\delta$  set. ■

**Problem 5.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Is it true that if the sets  $\{f = r\}$  are measurable for all  $r \in \mathbb{R}$ , then  $f$  is measurable?

*Proof.* If  $\{f = r\}$  are measurable for all  $r \in \mathbb{R}$ , it is not necessarily the case that  $f$  is measurable. Consider the following construction: Let  $E \subset (0, 1)$  be an unmeasurable set.<sup>†</sup> Define a map  $f: \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{R} \setminus ((0, 1) \setminus E), \\ x + 1 & \text{if } x \in (0, 1) \setminus E. \end{cases} \quad (10)$$

By the definition, it is clear that  $\{f = r\}$  is measurable and  $|\{f = r\}| = 0$  since  $\{f = r\}$  contains at most two elements. However, the set  $\{0 < f < 1\} = E$  is not measurable. Thus,  $f$  is not measurable. ■

**Problem 6.** Let  $\{f_k\}_{k=1}^{\infty}$  be a sequence of measurable functions on  $\mathbb{R}$ . Prove that the set  $\{x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists}\}$  is measurable.

*Proof.* By Theorem 4.12 from [? , Ch. 4, p. 67],  $\liminf_{k \rightarrow \infty} f_k$  and  $\limsup_{k \rightarrow \infty} f_k$  are measurable. By Theorem 4.7 from [? , Ch. 4, p. 66]

$$\left\{ \liminf_{k \rightarrow \infty} f_k < \limsup_{k \rightarrow \infty} f_k \right\} \quad (11)$$

is measurable. Since

$$\left\{ \lim_{k \rightarrow \infty} f_k \text{ exists} \right\} = \left\{ \limsup_{k \rightarrow \infty} f_k = \liminf_{k \rightarrow \infty} f_k \right\} = \mathbb{R} \setminus \left\{ \liminf_{k \rightarrow \infty} f_k < \limsup_{k \rightarrow \infty} f_k \right\}, \quad (12)$$

by Theorem 3.17 from [? , Ch. 3, p. 48], the set  $\{\lim_{k \rightarrow \infty} f_k \text{ exists}\}$  is measurable. ■

**Problem 7.** A real valued function  $f$  on an interval  $[a, b]$  is said to be *absolutely continuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Show that an absolutely continuous function on  $[a, b]$  is of bounded variation on  $[a, b]$ .

<sup>†</sup>It's construction does not concern us. The interested reader such direct their refer to Theorem 3.38 from [? , Ch. 3, p. 57-58] or Theorem 17 from [? , Ch. 2§7, p. 48].

*Proof.* Suppose  $f$  is absolutely continuous on  $[a, b]$ . Let  $\varepsilon := 1$ . Then, there exists  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < 1$ . Let  $N := \lceil (b-a)/\delta \rceil$ , that is,  $N$  is the smallest integer greater than  $(b-a)/\delta$ , and consider the partition  $\Gamma = \{x_k\}$  where  $x_k := a + k(b-a)/N$ , for  $k = 0, \dots, N$ . Then  $x_k - x_{k-1} < (b-a)/N < \delta$  so, by Theorem 2.2(i) from [?, Ch. 2, p. 19], we have  $V[f; x_{k-1}, x_k] < 1$  for  $k = 0, \dots, N$ . It follows by Theorem 2.2(ii) that

$$V[f; a, b] = \sum_{k=1}^N V[f; x_{k-1}, x_k] < N. \quad (13)$$

Thus,  $f$  is b.v. on  $[a, b]$ . ■

**Problem 8.** Let  $f$  be a continuous function from  $[a, b]$  into  $\mathbb{R}$ . Let  $\chi_{\{c\}}$  be the characteristic function of a singleton  $\{c\}$ , that is,  $\chi_{\{c\}}(x) = 0$  if  $x \neq c$  and  $\chi_{\{c\}}(c) = 1$ . Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}$$

*Proof.* The result follows quite easily from more sophisticated measure theoretic arguments. At this point, however, such language has not been discussed so we shall prove this using nothing but the definition of the Riemann–Stieltjes integral and properties thereof.

Let us consider each case  $c \in (a, b)$ ,  $c = a$ , and  $c = b$  separately.

Recall that the given a partition  $\Gamma = \{x_0, \dots, x_m\}$  of  $[a, b]$ , the Riemann–Stieltjes sum of  $f$  with respect to  $\varphi$  is

$$R_\Gamma := \sum_{k=1}^m f(\xi_k) [\varphi(x_k) - \varphi(x_{k-1})]. \quad (14)$$

The Riemann–Stieltjes integral is defined as the limit

$$\int_a^b f d\varphi := \lim_{|\Gamma| \rightarrow 0} R_\Gamma \quad (15)$$

if it exists.

Suppose  $c \in (a, b)$ . Then, for any partition  $\Gamma$  of  $[a, b]$ , either  $c \in \Gamma$  or  $c \notin \Gamma$ . In the latter case,  $R_\Gamma = 0$ . In the former case  $c$  is one of the  $x_k$ , say  $c = x_\ell$  for  $0 < \ell < m$ . Then

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^m f(\xi_k) [\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= 0 + \dots + 0 + f(\xi_{\ell-1}) - f(\xi_\ell) + 0 + \dots + 0 \\ &= f(\xi_{\ell-1}) - f(\xi_\ell). \end{aligned} \quad (16)$$

Since  $f$  is continuous, given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|\xi_\ell - \xi_{\ell-1}| < \delta$  implies  $|f(\xi_\ell) - f(\xi_{\ell-1})| < \varepsilon$ . It follows that the quantity in (16) approaches 0 as  $|\Gamma|$  approaches 0. Therefore,  $\int_a^b f d\chi_{\{c\}} = 0$ .

Suppose  $c = a$ . Then, since any partition  $\Gamma$  of  $[a, b]$  must contain the point  $a$ , we have

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^m f(\chi_k) [\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= f(\xi_1) [\chi_{\{c\}}(x_1) - \chi_{\{c\}}(x_0)] + f(\xi_2) [\chi_{\{c\}}(x_2) - \chi_{\{c\}}(x_1)] \\ &\quad + \cdots + f(\xi_m) [\chi_{\{c\}}(x_m) - \chi_{\{c\}}(x_{m-1})] \\ &= -f(\xi_1) + 0 + \cdots + 0 \\ &= -f(\xi_1) \end{aligned} \tag{17}$$

Taking the limit as  $|\Gamma| \rightarrow 0$ ,  $\xi_1 \rightarrow a$  so, by continuity of  $f$ ,  $f(\xi_1) \rightarrow f(a)$ . Thus,  $\int_a^b f d\chi_{\{c\}} = -f(a)$ .

A similar argument to the one above shows that, if  $c = b$ , the Riemann-Stieltjes integral  $\int_a^b f d\chi_{\{c\}} = f(b)$ . ■

**1.3 Exam 1****Problem 1.***Proof.* ■**Problem 2.***Proof.* ■**Problem 3.**

- (i) Show that if  $B_r := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < r\}$ , then there exists a constant  $C$  such that  $|B_r| = Cr^n$ .

(Hint: Think of  $B_r$  as  $\{r\mathbf{x} : \mathbf{x} \in B_1\}$ .)

- (ii) Let  $E \subset \mathbb{R}^n$  be a measurable set and let  $\varphi_E: \mathbb{R}^n \rightarrow \mathbb{R}$  be defined  $\varphi_E(\mathbf{x}) := |E \cap B_{|\mathbf{x}|}|$ . Use part (i) to prove that  $\varphi_E$  is continuous.

*Proof.* (i) To prove this result, we use the map constructed in Problem 1 of the review sheet for Exam 1, the map  $T: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Set  $T_r: \mathbb{R}^n \rightarrow \mathbb{R}^n$  to be  $T_r := T(r)$ . Then, we claim  $B_r = T_r(B_1)$  and  $|B_r| = |T_r(B_1)|$ , which, as we saw in Problem 1 of the review sheet, has measure  $|B_1||r|^n$ . Setting  $C := |B_1|$ , we have  $|B_r| = C|r|^n$  as desired.

- (ii) To prove that  $\varphi_E$  is continuous, we provide an  $(\varepsilon, \delta)$ -argument. Let  $\varepsilon > 0$  be given. We must show that there exists  $\delta > 0$  such that  $\mathbf{y} \in B(\mathbf{x}, \delta)$  implies

$$|\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| < \varepsilon. \quad (1)$$

First, note that since  $\mathbf{x} \mapsto |\mathbf{x}|$  is continuous and polynomials  $p: \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous, then the composition  $\mathbf{x} \mapsto |\mathbf{x}|^n$  is continuous. Therefore, there exists  $\delta > 0$  such that  $\mathbf{y} \in B(\mathbf{x}, \delta)$  implies

$$||\mathbf{x}|^n - |\mathbf{y}|^n| < \frac{\varepsilon}{C}, \quad (2)$$

where  $C := |B_1|$ .

Now, let  $x \in \mathbb{R}^n$  and  $\mathbf{y} \in B(\mathbf{x}, \delta)$  as above. Then, by (2) we have

$$\begin{aligned} |\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| &= ||E \cap B_{|\mathbf{x}|}| - |E \cap B_{|\mathbf{y}|}|| \\ &\leq ||B_{|\mathbf{x}|}| - |B_{|\mathbf{y}|}|| \\ &= C||\mathbf{x}|^n - |\mathbf{y}|^n| \\ &\leq C \left[ \frac{\varepsilon}{C} \right] \\ &= \varepsilon. \end{aligned} \quad (3)$$

It follows that  $\varphi_E$  is continuous. ■

**Problem 4.** Assume that  $f: [a, b] \rightarrow \mathbb{R}$  is of bounded variation on  $[a, b]$ . Prove that  $f$  is measurable.

*Proof.* By Jordan's theorem (Corollary 2.7 from [?, Ch. 2, p. 21]), the function  $f$  is of bounded variation on  $[a, b]$  if and only if it can be written as the difference  $f_1 - f_2$  of two bounded functions  $f_1$  and  $f_2$  that are monotone increasing on  $[a, b]$ . Then,  $f_1$  and  $f_2$  are continuous a.e. on  $[a, b]$  and hence, are measurable. ■

## 1.4 Exam 2 Prep

**Problem 1.** Define for  $\mathbf{x} \in \mathbb{R}^n$ ,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that  $f$  is integrable outside any ball  $B(\mathbf{0}, \varepsilon)$ , and that there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n \setminus B(\mathbf{0}, \varepsilon)} f(\mathbf{x}) d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

*Proof.* Recall that a real-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is Lebesgueintegrable on a subset  $E$  of  $\mathbb{R}^n$  if

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \quad (1)$$

Let  $f$  be as given in the statement of the problem and set  $B_\varepsilon := B(\mathbf{0}, \varepsilon)$ . Consider the change of variables to *hyperspherical coordinates*  $(x_1, \dots, x_n) \mapsto (r, \Theta)$  where  $\Theta = (\theta_1, \dots, \theta_{n-1})$ .<sup>‡</sup> By Theorem 7.26(iii) from [?, Ch. 7, p. 123], we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\varepsilon} f(\mathbf{x}) d\mathbf{x} &= \int_{\mathbb{R}^n \setminus B_\varepsilon} f(\mathbf{x}) d\mathbf{x} \\ &= \int_{\mathbb{R}^n \setminus B_\varepsilon} \frac{1}{|\mathbf{x}|^{n+1}} d\mathbf{x}. \\ &= \int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}} dr dV, \end{aligned} \quad (2)$$

where  $S_r^{n-1}$  is the  $(n-1)$ -sphere centered at  $\mathbf{0}$  with radius  $r$ , that is, the subset  $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = r\}$  of  $\mathbb{R}^n$  and  $dV$  is the *volume element* of  $S_r^{n-1}$ . Since  $1/|r|^{n+1}$  is nonnegative, by Tonelli's theorem the iterated integrals in (2) may be exchange, that is,

$$\int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}} dr dV = \int_\varepsilon^\infty \left( \int_{S_r^{n-1}} 1 dV \right) \frac{1}{|r|^{n+1}} dr. \quad (3)$$

Now, note that from Problem 1 of the review sheet for Exam 1, we have

$$\int_{S_r^{n-1}} 1 dV = |S_r^{n-1}|_{\mathbb{R}^{n-1}} = |S^{n-1}|_{\mathbb{R}^{n-1}} |r|^{n-1}. \quad (4)$$

Set  $C := |S^{n-1}|_{\mathbb{R}^{n-1}}$ . Putting equations (2), (3), and (4) together, we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\varepsilon} f(\mathbf{x}) d\mathbf{x} &= \int_\varepsilon^\infty C |r|^{n-1} \frac{1}{|r|^{n+1}} dr \\ &= \int_\varepsilon^\infty \frac{C}{|r|^2} dr \\ &= \lim_{x \rightarrow \infty} \left[ -\frac{C}{x} - \left( -\frac{C}{\varepsilon} \right) \right] \\ &= \frac{C}{\varepsilon}, \end{aligned} \quad (5)$$

---

<sup>‡</sup>The explicit construction of the map  $(x_1, \dots, x_n) \mapsto (r, \Theta)$  is of no concern to us for now. What is important is that it exists.

as was to be shown. ■

**Problem 2.** Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $\mathbb{R}^n$ , and assume that  $f_k$  converges pointwise almost everywhere to a function  $f$ . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets  $E$  of  $\mathbb{R}^n$ . Moreover, show that this is not necessarily true if  $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$ .

*Proof.* Let  $E \subset \mathbb{R}^n$  be a measurable subset of  $\mathbb{R}^n$ . Then, since  $f_k \rightarrow f$  pointwise a.e. on  $\mathbb{R}^n$ , then  $f_k \rightarrow f$  pointwise a.e. on  $E$  and  $\mathbb{R}^n \setminus E$ . To prove that the limit of the sequence of integrals  $\{\int_E f_k\}$  exist and is equal to  $\int_E f$ , it suffices to prove that

$$\int_E f \leq \liminf_{k \rightarrow \infty} \int_E f_k \leq \limsup_{k \rightarrow \infty} \int_E f_k \leq \int_E f. \quad (6)$$

The lower bound in (6) follows from an application of Fatou's lemma:

$$\int_E f = \int_E \liminf_{k \rightarrow \infty} f \leq \liminf_{k \rightarrow \infty} \int_E f_k. \quad (7)$$

Also by Fatou's lemma, we have

$$\int_{\mathbb{R}^n \setminus E} f = \int_{\mathbb{R}^n \setminus E} \liminf_{k \rightarrow \infty} f \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus E} f_k. \quad (8)$$

Now, since  $f \in L^1(\mathbb{R}^n)$ , by equation (8) and properties of the  $\liminf$  and  $\limsup$ <sup>§</sup> we have

$$\begin{aligned} \int_E f &= \int_{\mathbb{R}^n} f - \int_{\mathbb{R}^n \setminus E} f \geq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} f - \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus E} f_k \\ &\geq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k - \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus E} f_k \\ &= \limsup_{k \rightarrow \infty} \left[ \int_{\mathbb{R}^n} f_k - \int_{\mathbb{R}^n \setminus E} f_k \right] \\ &= \limsup_{k \rightarrow \infty} \int_E f_k. \end{aligned} \quad (9)$$

By equations (7) and (9) it follows that  $\lim_{k \rightarrow \infty} \int_E f_k$  exists and is equal to  $\int_E f$ .

---

<sup>§</sup>Namely, for any sequence of positive real numbers  $\{a_k\}$  the inequality  $\liminf a_k \leq \limsup a_k$  holds



To see that the result need not be true if  $\int_E f = \infty$ , consider the following example: Let  $f_k : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f_k(x) := \begin{cases} k^2/2 & \text{if } x \in (-1/k, 1/k), \\ 1 & \text{otherwise} \end{cases} \quad (10)$$

and  $f = 1$ .

It is easy to see that  $f_k \rightarrow f$  a.e. in  $\mathbb{R}$  and that both  $\int_{\mathbb{R}} f = \infty$  and  $\lim_{k \rightarrow \infty} \int_{\mathbb{R}} f_k = \infty$ . However, if  $E := (-1, 1)$  then  $\int_E f = 1$ , but  $\lim_{k \rightarrow \infty} \int_E f_k = \infty$ . ■

**Problem 3.** Assume that  $E$  is a measurable set of  $\mathbb{R}^n$ , with  $|E| < \infty$ . Prove that a nonnegative function  $f$  defined on  $E$  is integrable if and only if

$$\sum_{k=0}^{\infty} |\{ \mathbf{x} \in E : f(\mathbf{x}) \geq k \}| < \infty.$$

*Proof.* If  $f$  is integrable over a measurable subset  $E$  of  $\mathbb{R}^n$ , then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \quad (11)$$

Set  $E_k = \{ \mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k \}$  and  $F_k = \{ \mathbf{x} \in E : f(\mathbf{x}) \geq k \}$ . Note the following properties about the sets we have just defined: first, the  $E_k$ 's are pairwise disjoint and the  $F_k$ 's are nested in the following way  $F_{k+1} \subset F_k$ ; second,  $E = \bigcup_{k=1}^{\infty} E_k$  and  $E_k = F_k \setminus F_{k+1}$ . By Theorem 3.23, since the  $E_k$ 's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \quad (12)$$

Now, since  $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$  on  $E_k$ , we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \quad (13)$$

Then we have the following upper and lower estimates on the integral of  $f$  over  $E$

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \quad (14)$$

But note that  $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$  by Corollary 3.25 since the measures of  $E_k$ ,  $F_k$ , and  $F_{k+1}$  are all finite. Hence, (14) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \quad (15)$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned} \tag{17}$$

Thus, from (16) and (17)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \tag{18}$$

so the integral  $\int_E f$  converges if and only if the sum  $\sum_{k=0}^{\infty} |F_k|$  converges. ■

**Problem 4.** Suppose that  $E$  is a measurable subset of  $\mathbb{R}^n$ , with  $|E| < \infty$ . If  $f$  and  $g$  are measurable functions on  $E$ , define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$  if and only if  $f_k$  converges to  $f$  as  $k \rightarrow \infty$ .

*Proof.* ■

**Problem 5.** Define the *gamma function*  $\Gamma: \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function*  $\beta: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function  $u \mapsto e^{-u} u^{y-1}$  is in  $L(\mathbb{R}^+)$  for all  $y \in \mathbb{R}^+$ .

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

*Proof.* ■

**Problem 6.** Let  $f \in L(\mathbb{R}^n)$  and for  $\mathbf{h} \in \mathbb{R}^n$  define  $f_{\mathbf{h}}: \mathbb{R}^n \rightarrow \mathbb{R}$  be  $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$ . Prove that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

*Proof.* ■

**Problem 7.** (a) If  $f_k, g_k, f, g \in L(\mathbb{R}^n)$ ,  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ ,  $|f_k| \leq g_k$  and

$$\int_{\mathbb{R}^n} g_k \longrightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if  $f_k, f \in L(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e. in  $\mathbb{R}^n$ , then

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \quad \text{as } k \rightarrow \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \longrightarrow \int_{\mathbb{R}^n} |f| \quad \text{as } k \rightarrow \infty.$$

*Proof.* (a)  $\implies$  (b): Assume part (a) then  $\implies$  if

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \tag{19}$$

as  $k \rightarrow \infty$ , we have

(b): ■

**1.5 Exam 2 (2010)**

**Problem 1.** Suppose  $f \in L^1(\mathbb{R}^n)$ . Show that for every  $\varepsilon > 0$  there exists a ball  $B$ , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

*Hint:* Use the monotone convergence theorem.

*Proof.* ■

**Problem 2.** (a) Prove the following generalization of *Chebyshev's inequality*: Let  $0 < p < \infty$  and  $E \subset \mathbb{R}^n$  be measurable. assume that  $|f|^p \in L^1(E)$ . Then

$$|\{x \in E : f(\mathbf{x}) > \alpha\}| \leq \frac{1}{\alpha^p} \int_{\{f > \alpha\}} f^p,$$

for  $\alpha > 0$ .

(b) Let  $p$ ,  $E$ , and  $f$  be as in part (a). In addition, assume that  $\{f_k\}$  is a sequence such that  $\int_E |f_k - f|^p \rightarrow 0$  as  $k \rightarrow \infty$ . Show that  $f_k \rightarrow f$  in measure on  $E$ .

Recall that  $f_k \rightarrow f$  in measure on  $E$  if and only if for every  $\varepsilon > 0$

$$\lim_{k \rightarrow \infty} |\{\mathbf{x} \in E : |f_k(\mathbf{x}) - f(\mathbf{x})| > \varepsilon\}| = 0.$$

*Proof.* ■

**Problem 3.** Let  $f \in L^1(\mathbb{R})$ , and define

$$F(\xi) := \int_{\mathbb{R}} f(x) \cos(2\pi x \xi) dx.$$

Prove that  $F$  is continuous and bounded on  $\mathbb{R}$ .

*Proof.* ■

**Problem 4.** Use repeated integration techniques to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} d\mathbf{x} = \pi^{n/2}.$$

*Hint:* Start from the case  $n = 1$  by using the polar coordinates in

$$\left[ \int_{\mathbb{R}} e^{-x^2} dx \right]^2 = \left[ \int_{\mathbb{R}} e^{-x^2} dx \right] \left[ \int_{\mathbb{R}} e^{-y^2} dy \right]$$

*Proof.* ■

**Problem 5.**

*Proof.* ■

## 1.6 Midterm 2

**Problem 1.** Assume that  $f \in L(\mathbb{R}^n)$ . Show that for every  $\varepsilon > 0$  there exists a ball  $B$ , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

*Proof.* ■

**Problem 2.** Let  $f \in L(E)$ , and let  $\{E_j\}$  be a countable collection of pairwise disjoint measurable subsets of  $E$ , such that  $E = \bigcup_{j=1}^{\infty} E_j$ . Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

*Proof.* ■

**Problem 3.** Let  $\{f_k\}$  be a family in  $L(E)$  satisfying the following property: For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|A| < \delta$  implies

$$\int_A |f_k| < \varepsilon$$

for all  $k \in \mathbb{N}$ . Assume  $|E| < \infty$ , and  $f_k(x) \rightarrow f(x)$  as  $k \rightarrow \infty$  for a.e.  $x \in E$ . Show that

$$\lim_{k \rightarrow \infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

*Proof.* ■

**Problem 4.** Let  $I = [0, 1]$ ,  $f \in L(I)$ , and define  $g(x) = \int_x^1 t^{-1} f(t) dt$  for  $x \in I$ . Prove that  $g \in L(I)$  and

$$\int_I g = \int_I f.$$

*Proof.* ■

### 1.7 Final Practice

**Problem 1.** Suppose  $f \in L^1(\mathbb{R}^n)$  and that  $x$  is a point in the Lebesgue set of  $f$ . For  $r > 0$ , let

$$A(r) := \frac{1}{|r|^n} \int_{B(0,r)} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| d\mathbf{y}.$$

Show that:

- (a)  $A(r)$  is a continuous function of  $r$ , and  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ ;
- (b) there exists a constant  $M > 0$  such that  $A(r) \leq M$  for all  $r > 0$ .

*Proof.* (a) Without loss of generality, we may assume  $r < s$ . Then, we want to show that as  $r \rightarrow s$ , the quantity

$$|A(s) - A(r)| \rightarrow 0.$$

Set  $F(\mathbf{y}) := |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})|$  and consider said quantity

$$\begin{aligned} |A(s) - A(r)| &= \left| \frac{1}{|s|^n} \int_{B_s} F(\mathbf{y}) d\mathbf{y} - \frac{1}{|r|^n} \int_{B_r} F(\mathbf{y}) d\mathbf{y} \right| \\ &= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) d\mathbf{y} + \frac{1}{|s|^n} \int_{B_r} F(\mathbf{y}) d\mathbf{y} - \frac{1}{|r|^n} \int_{B_r} F(\mathbf{y}) d\mathbf{y} \right| \\ &= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) d\mathbf{y} + \left( \frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(\mathbf{y}) d\mathbf{y} \right| \\ &\leq \underbrace{\frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) d\mathbf{y}}_{I_1} + \underbrace{\left( \frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(\mathbf{y}) d\mathbf{y}}_{I_2}. \end{aligned}$$

Hence, we must show that the quantities  $I_1, I_2 \rightarrow 0$  as  $r \rightarrow s$ .

To see that  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ , note that  $x$  is a point of the Lebesgue set of  $f$  and that

$$0 = \lim_{B_r \searrow \mathbf{x}} \frac{1}{|B_1||r|^n} \int_{B_r} |f(\mathbf{y}) - f(\mathbf{x})| d\mathbf{y} = \frac{1}{|B_1|} \lim_{B_r \searrow \mathbf{x}} \frac{1}{|r|^n} \int_{B_r} |f(\mathbf{t}) - f(\mathbf{x})| d\mathbf{t} = \lim_{r \rightarrow 0} A(r).$$

by making the change of variables  $\mathbf{t} = \mathbf{x} - \mathbf{y}$ .

(b) ■

**Problem 2.** Let  $E \subset \mathbb{R}^n$  be a measurable set,  $1 \leq n < \infty$ . Assume  $\{f_k\}$  is a sequence in  $L^p(E)$  converging pointwise a.e. on  $E$  to a function  $f \in L^p(E)$ . Prove that

$$\|f_k - f\|_p \rightarrow 0$$

if and only if

$$\|f_k\|_p \rightarrow \|f\|_p$$

as  $k \rightarrow \infty$ .

*Proof.* ■

**Problem 3.** Let  $1 < p < \infty$ ,  $f \in L^p(E)$ ,  $g \in L^{p'}(E)$ .

- (a) Prove that  $f * g \in C(\mathbb{R}^n)$ .
- (b) Does this conclusion continue to be valid when  $p = 1$  and  $p = \infty$ ?

*Proof.* ■

**Problem 4.** Let  $f \in L(\mathbb{R})$ , and let  $F(t) := \int_{\mathbb{R}} f(x) \cos(tx) dx$ .

- (a) Prove that  $F(t)$  is continuous for  $t \in \mathbb{R}$ .
- (b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

*Proof.* ■

**Problem 5.** Let  $f$  be of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular. Show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh$$

for all functions  $\varphi$  continuous on  $[a, b]$ .

*Proof.* ■

**1.8 Final (2010)**

**Problem 1.** Suppose that  $f \in L^1(\mathbb{R}^n)$ , and that  $\mathbf{x}$  is a point in the Lebesgue set of  $f$ . For  $r > 0$ , let

$$A(r) := \frac{1}{r^n} \int_{B_r} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| d\mathbf{y},$$

where  $B_r := B(\mathbf{0}, r)$ .

Show that

- (a)  $A(r)$  is a continuous function of  $r$ , and  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ .
- (b) There exists a constant  $M > 0$  such that  $A(r) \leq M$  for all  $r > 0$ .

*Proof.* (a)

(b) ■

**Problem 2.** Let  $E \subset \mathbb{R}^n$  be a measurable set,  $1 \leq p < \infty$ . assume that  $\{f_k\}$  is a sequence in  $L^p(E)$  converging pointwise a.e. on  $E$  to a function  $f \in L^p(E)$ . Prove that

$$\|f_k - f\|_p \longrightarrow 0 \iff \|f_k\|_p \longrightarrow \|f\|_p$$

*Hint:* To prove one of the implications, you can use the following fact without proving it:

$$\left| \frac{a-b}{2} \right| \leq \frac{|a|^p + |b|^p}{2}$$

for all  $a, b \in \mathbb{R}$ .

*Proof.* ■

**Problem 3.** Let  $0 < p < q < r \leq \infty$ ,  $E \subset \mathbb{R}^n$  be a measurable set. Show that each  $f \in L^q(E)$  is the sum of a function  $g \in L^p(E)$  and a function  $h \in L^r(E)$ .

*Proof.* ■

**Problem 4.** Prove that  $f: [a, b] \rightarrow \mathbb{R}$  is Lipschitz continuous if and only if  $f$  is absolutely continuous and there exists a constant  $M > 0$  such that  $|f'| < M$  a.e. on  $[a, b]$ .

*Proof.* ■

**Problem 5.** Let  $1 < p < \infty$ ,  $f \in L^p(\mathbb{R}^n)$ ,  $g \in L^{p'}(\mathbb{R}^n)$ .

- (a) Prove that  $f * g \in C(\mathbb{R}^n)$ .
- (b) Does this conclusion continue to be valid when  $p = 1$  or  $p = \infty$ ?

*Proof.* ■



## 1.9 Final

## MA 544 Past Quals

## 2.1 Danielli: Winter 2012

**Problem 1.** Let  $f(x, y)$ ,  $0 \leq x, y \leq 1$ , satisfy the following conditions: for each  $x$ ,  $f(x, y)$  is an integrable function of  $y$ , and  $\partial f(x, y)/\partial x$  is a bounded function of  $(x, y)$ . Prove that  $\partial f(x, y)/\partial x$  is a measurable function of  $y$  for each  $x$  and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial f(x, y)}{\partial x} dy.$$

*Proof.* ■

**Problem 2.** Let  $f$  be a function of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular, show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh.$$

*Hint:* A function  $h$  is said to be singular if  $h' = 0$ .

*Proof.* ■

**Problem 3.** Let  $E \subset \mathbb{R}$  be a measurable set, and let  $K$  be a measurable function on  $E \times E$ . Assume that there exists a positive constant  $C$  such that

$$\int_E K(x, y) dx \leq C \tag{1}$$

for a.e.  $y \in E$ , and

$$\int_E K(x, y) dy \leq C \tag{2}$$

for a.e.  $x \in E$ .

Let  $1 < p < \infty$ ,  $f \in L^p(E)$ , and define

$$T_f(x) := \int_E K(x, y) f(y) dy.$$

(a) Prove that  $T_f \in L^p(E)$  and

$$\|T_f\|_p \leq C\|f\|_p. \quad (3)$$

(b) Is (3) still valid if  $p = 1$  or  $\infty$ ? If so, are assumptions (1) and (2) needed?

*Proof.* ■

**Problem 4.** Let  $f$  be a nonnegative measurable function on  $[0, 1]$  satisfying

$$|\{x \in [0, 1] : f(x) > \alpha\}| < \frac{1}{1 + \alpha^2} \quad (4)$$

for  $\alpha > 0$ .

(a) Determine values of  $p \in [1, \infty)$  for which  $f \in L^p[0, 1]$ .

(b) If  $p_0$  is the minimum value of  $p$  for which  $p$  may fail to be in  $L^p$ , give an example of a function which satisfies (4), but which is not in  $L^{p_0}[0, 1]$ .

*Proof.* ■

**2.2 Danielli: Summer 2011**

**Problem 1.** Let  $f \in L^1(\mathbb{R})$ , and let  $F(t) := \int_{\mathbb{R}} f(x) \cos(tx) dx$ .

- (a) Prove that  $F(t)$  is continuous for  $t \in \mathbb{R}$ .
- (b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

*Hint:* Start by proving the statement for  $f = \chi_{[a,b]}$ .

*Proof.* ■

**Problem 2.** (a) Suppose that  $f_k, f \in L^2(E)$ , with  $E$  a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E f g \tag{1}$$

as  $k \rightarrow \infty$  for all  $g \in L^2(E)$ . If, in addition,  $\|f_k\|_2 \rightarrow \|f\|_2$  show that  $f_k$  converges to  $f$  in  $L^2$ , i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as  $k \rightarrow \infty$ .

- (b) Provide an example of a sequence  $f_k$  in  $L^2$  and a function  $f$  in  $L^2$  satisfying (1), but such that  $f_k$  does *not* converge to  $f$  in  $L^2$ .

*Proof.* ■

**Problem 3.** A bounded function  $f$  is said to be of bounded variation on  $\mathbb{R}$  if it is of bounded variation on any finite subinterval  $[a, b]$ , and moreover  $A := \sup_{a,b} V[a, b; f] < \infty$ . Here,  $V[a, b; f]$  denotes the total variation of  $f$  over the interval  $[a, b]$ . Show that:

- (a)  $\int_{\mathbb{R}} |f(x+h) - f(x)| dx \leq A|h|$  for all  $h \in \mathbb{R}$ .

*Hint:* For  $h > 0$ , write

$$\int_{\mathbb{R}} |f(x+h) - f(x)| dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)| dx.$$

- (b)  $\left| \int_{\mathbb{R}} f(x) \varphi'(x) dx \right| \leq A$ , where  $\varphi$  is any function of class  $C^1$ , of bounded variation, compactly supported, with  $\sup_{x \in \mathbb{R}} |\varphi(x)| \leq 1$ .

*Proof.* ■

**Problem 4.** (a) Prove the *generalized Hölder's inequality*: Assume  $1 \leq p \leq \infty$ ,  $j = 1, \dots, n$ , with  $\sum_{j=1}^{\infty} 1/p_j = 1/r \leq 1$ . If  $E$  is a measurable set and  $f_j \in L^{p_j}(E)$  for  $j = 1, \dots, n$ , then  $\prod_{j=1}^n f_j \in L^r(E)$  and

$$\|f_1 \cdots f_n\|_r \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if  $1 \leq p, q, r \leq \infty$ , with  $1/p + 1/q = 1/r + 1$ ,  $f \in L^p(\mathbb{R})$ , and  $g \in L^q(\mathbb{R})$ , then

$$|(f * g)(x)| \leq \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q dy.$$

(Recall that  $(f * g)(x) := \int f(y)g(x-y) dy$ .)

(c) Prove *Young's convolution theorem*: Assume that  $p, q, r, f$ , and  $g$  are as in part (b). Then  $f * g \in L^r(\mathbb{R})$  and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

*Proof.*

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