

# MA571 Midterm 2: Practice Problems

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**Problem 1.** Let  $X$  be a Hausdorff space and let  $A$  be a compact subset of  $X$ . Prove from the definitions that  $A$  is closed.

*Proof.* This is Theorem 26.3 from Munkers §26, p. 165; we shall paraphrase it.

We show that  $X - A$  is open. To that end we will show that, given a point  $x_0 \in X - A$ , there is neighborhood  $U$  of  $x_0$  disjoint from  $A$ . For each point  $a \in A$ , by the Hausdorff property of  $X$ , choose disjoint neighborhoods  $U_a$  and  $V_a$  of  $x_0$  and  $a$ , respectively. Then the collection  $\{V_a \mid a \in A\}$  forms an open cover of  $A$  so, by Lemma 26.1, only finitely many of the  $V_a$ 's cover  $A$ , say  $V_{a_1}, \dots, V_{a_n}$ . Define  $U := U_{a_1} \cap \dots \cap U_{a_n}$ . We claim that  $U$  is a neighborhood of  $x_0$  disjoint from  $A$ . First, it is clear that  $U$  is a neighborhood of  $x_0$  since each  $U_a$  contains  $x_0$  and  $U$  is an intersection of finitely many of these. Second, note that if  $z \in U \cap A$  then  $z \in U_{a_i}$  for all  $i$  and  $z \in V_{a_j}$  for some  $j \in \{1, \dots, n\}$ , but  $U_{a_j} \cap V_{a_j} = \emptyset$ . Therefore,  $U \cap A = \emptyset$ . By Lemma C, it follows that  $X - A$  is open. ■

**Problem 2.** Let  $X$  be a Hausdorff space and let  $A$  and  $B$  be disjoint compact subsets of  $X$ . Prove that there are open sets  $U$  and  $V$  such that  $U$  and  $V$  are disjoint,  $A \subset U$  and  $B \subset V$ .

*Proof.* This is Ex. 5 from Munkres §26, p. 171.

Suppose  $A$  and  $B$  are disjoint compact subspaces of  $X$ . Since  $X$  is Hausdorff, by Theorem 26.4, for every  $x \in B$  there exists disjoint open sets  $U_x$  and  $V_x$  where  $U_x \supset A$  and  $V_x$  is a neighborhood of  $x$ . Then the collection  $\{V_x \mid x \in B\}$  is an open cover of  $B$  so by Lemma 26.1, only finitely many of the  $V_x$ 's cover  $B$ , say  $V_{x_1}, \dots, V_{x_n}$ . Define  $U := U_{x_1} \cap \dots \cap U_{x_n}$  and  $V := V_{x_1} \cup \dots \cup V_{x_n}$ . We claim that  $U$  and  $V$  are disjoint neighborhoods containing  $A$  and  $B$ , respectively. It is clear that  $U$  and  $V$  are open since  $U$  is a finite intersection of open sets and  $V$  is a union of open sets and that they contain  $A$  and  $B$ , respectively, since each of the  $U_x$ 's contain  $A$  and  $V_{x_1}, \dots, V_{x_n}$  is an open cover of  $B$ . Lastly,  $U$  and  $V$  are disjoint since intersection distributes over union, i.e., we have

$$U \cap V = \left( \bigcap_{i=1}^n U_{x_i} \right) \cap \left( \bigcup_{j=1}^n V_{x_j} \right) = \bigcup_{j=1}^n \left( \bigcap_{i=1}^n U_{x_i} \cap V_{x_j} \right) = \emptyset$$

since  $U_{x_i} \cap V_{x_i} = \emptyset$  so  $(\bigcap_{i=1}^n U_{x_i}) \cap V_{x_i} = \emptyset$ . ■

**Problem 3.** Prove the Tube Lemma: Let  $X$  and  $Y$  be topological spaces with  $Y$  compact, let  $x_0 \in X$ , and let  $N$  be an open set of  $X \times Y$  containing  $x_0 \times Y$ , then there is an open set  $W$  of  $X$  containing  $x_0$  with  $W \times Y \subset N$ .

*Proof.* This is Lemma 26.8 from Munkres §26, p. 168, but is proved in *Step 1* in the process of showing Theorem 26.7; we paraphrase the proof here.

Let  $x_0 \in X$ , and let  $N$  be an open set of  $X \times Y$  containing  $x_0 \times Y$ . Cover  $x_0 \times Y$  by basic open sets  $U \times V$  lying in  $N$ . Note that  $x_0 \times Y$  is compact, since it is an imbedding of  $Y$  given by the map  $y \mapsto (x_0, y)$  from  $Y$  into  $X \times Y$  therefore, by Lemma 26.1, only finitely many of the  $U \times V$ 's, say  $U_1 \times V_1, \dots, U_n \times V_n$ , cover  $x_0 \times Y$ . Define  $W := U_1 \cap \dots \cap U_n$ . We claim that  $W$  is a neighborhood of  $x_0$  such that  $W \times Y \subset N$ . First, it is clear that  $W$  is a neighborhood of  $x_0$  since it is the finite intersection of open sets and each  $U_i \times V_i$  intersects  $x_0 \times Y$  hence contains a point of the form  $(x_0, y)$  so  $U_i = \pi_1(U_i \times V_i)$  contains  $x_0$ . Lastly,  $W \times Y \subset N$  since  $W \times Y \subset \bigcup_{i=1}^n U_i \times V_i$ .

To see this let  $(x, y) \in W \times Y$  and consider the point  $(x_0, y) \in x_0 \times Y$ . Since  $(x_0, y)$  is in  $U_i \times V_i$  for some  $i$ , we have  $y \in V_i$ . But  $x \in U_j$  for every  $j$  since  $x \in W$ . Thus  $(x, y) \in U_i \times V_i$  as desired. It follows that,  $W$  is a neighborhood of  $x_0$  with  $W \times Y \subset N$  as desired. ■

**Problem 4.** Show that if  $Y$  is compact, then the projection map  $X \times X \rightarrow X$  is a closed map.

*Proof.* We shall proceed by the tube lemma, i.e, Theorem 26.8. Let  $C$  be a closed subset of  $X \times Y$  then  $N = (X \times Y) - C$  is open. Choose  $x_0 \in X - \pi_1(C)$ . Then  $x_0 \times Y$  is contained in  $N$  so by the tube lemma, there exists a neighborhood  $W$  of  $x_0$  such that  $W \times Y \subset N$ . In particular,  $W \subset X - \pi_1(C)$  otherwise if  $x \in W \cap \pi_1(C)$  then  $x \times Y \subset N$  and  $(x, y) \in C$  for some  $y \in Y$ , but  $N \cap C = \emptyset$ . It follows by Lemma C that  $X - \pi_1(C)$  is open so  $\pi_1(C)$  is closed. Since  $C$  was chosen arbitrarily we see that  $\pi_1$  is a closed map. ■

**Problem 5.** Let  $X$  be a compact space and suppose we are given a nested sequence of subsets  $C_1 \supset C_2 \supset \dots$  with all  $C_i$  closed. Let  $U$  be an open set containing  $\bigcap C_i$ . Prove that there is an  $i_0$  with  $C_{i_0} \subset U$ .

*Proof.* Consider the family of open sets  $U_i := X - C_i$ . Since  $U$  is open  $X - U$  is closed so by Theorem 26.2 is compact. We claim that  $U_i$  forms an open cover of  $X - U$ . To see note that by De Morgan's laws

$$\bigcup_{i \in \mathbb{N}} U_i = \bigcup_{i \in \mathbb{N}} X - C_i = X - \bigcap_{i \in \mathbb{N}} C_i \supset X - U$$

since  $\bigcap_{i \in \mathbb{N}} C_i \subset U$ . Therefore by Lemma 26.1 only finitely many of the  $U_i$ 's cover  $X - U$ , say  $U_{i_1}, \dots, U_{i_n}$ . Thus, we have that  $X - U \subset \bigcup_{i=1}^n U_i$  so  $U \supset \bigcap_{j=1}^n C_{i_j} = C_{i_n}$  as desired. ■

**Problem 6.** Let  $X$  be a compact space, and suppose there is a finite family of continuous functions  $f_i: X \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$  with the following property: given  $x \neq y$  in  $X$  there is an  $i$  such that  $f_i(x) \neq f_i(y)$ . Prove that  $X$  is homeomorphic to a subspace of  $\mathbb{R}^n$ .

*Proof.* Consider the map  $f: X \rightarrow \mathbb{R}^n$  defined by  $f := (f_1, \dots, f_n)$ . This map is continuous by Theorem 18.4 since each component  $f_i$  is continuous. We claim that  $X \approx f(X)$ . To prove this it suffices to show that  $f$  is injective so that its restriction to  $f(X)$  will be surjective and lastly invoke Theorem 26.6. Suppose  $f(x) = f(y)$  but  $x \neq y$ . Then  $f_i(x) = f_i(y)$  for some  $i$ , but this implies that  $f(x) = f(y)$ . This is a contradiction therefore,  $x = y$ . It follows that  $f$  is a bijection from a compact space  $X$  into  $f(X) \subset \mathbb{R}^n$  so by Theorem 26.6, we have  $X \approx f(X)$ . ■

**Problem 7.** Let  $X$  be a compact metric space and let  $\mathcal{U}$  be a covering of  $X$  by open sets. Prove that there is an  $\varepsilon > 0$  such that, for each set  $S \subset X$  with diameter  $< \varepsilon$ , there is a  $U \in \mathcal{U}$  with  $S \subset U$ . (This fact is known as the "Lebesgue number lemma.")

*Proof.* This is Lemma 27.5 from Munkres §27, p. 175; we will paraphrase the proof.

Let  $\mathcal{U}$  be an open cover of  $X$ . If  $X \in \mathcal{U}$ , then any positive number is a Lebesgue number for  $\mathcal{U}$ . Suppose  $X \notin \mathcal{U}$ . Choose a finite subcollection  $U_1, \dots, U_n$  of  $\mathcal{U}$  that covers  $X$ . For each  $i$ , set  $C_i := X - U_i$  and define the map  $f: X \rightarrow \mathbb{R}$  via  $f(x) := \frac{1}{n} \sum_{i=1}^n d(x, C_i)$ . We show that  $f(x) > 0$  for all  $x$ . Given  $x \in X$ , choose  $i$  so that  $x \in U_i$ . Then choose  $\varepsilon$  so that the  $\varepsilon$ -neighborhood of  $x$  lies in  $U_i$ . Then  $d(x, C_i) \geq \varepsilon$ , so that  $f(x) \geq \varepsilon/n$ .

Since  $f$  is continuous, it has a minimum value  $\delta$ ; we show that  $\delta$  is our required Lebesgue number. Let  $B$  be a subset of  $X$  of diameter less than  $\delta$ . Choose a point  $x_0$  of  $B$ ; then  $B$  lies in a  $\delta$ -neighborhood of  $x_0$ . Now  $\delta \leq f(x_0) \leq d(x_0, C_m)$ , where  $d(x_0, C_m)$  is the largest of the numbers  $d(x_0, C_i)$ . Then the  $\delta$ -neighborhood of  $x_0$  is contained in the element  $U_m = X - C_m$  of the covering  $\mathcal{U}$ . ■

**Problem 8.** Let  $S^1$  denote the circle  $\{x^2 + y^2 = 1\}$  in  $\mathbb{R}^2$ . Define an equivalence relation on  $S^1$  by

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } (x, y) = (-x', -y')$$

(you do not have to prove that this is an equivalence relation). Prove that the quotient space  $S^1/\sim$  is homeomorphic to  $S^1$ .

One way to do this is by using complex numbers.

*Proof.* Since Dr. McClure said that we can assume anything from complex analysis (and we don't need much) to begin with we shall assume that  $S^1 \subset \mathbb{C}$ . Now, the situation is as follows we want to find a map  $f: S^1 \rightarrow S^1$  which preserves  $\sim$  that makes the following diagram commute

$$\begin{array}{ccc} S^1 & & \\ \downarrow q & \searrow f & \\ S^1/\sim & \xrightarrow{\bar{f}} & S^1. \end{array}$$

Define  $f(z) := z^2$ . We claim that  $f$  is continuous and preserves  $\sim$ . First, it is clear that  $f(x + iy) = f(x + iy)$  and if  $x' + iy' = -x - iy$  then

$$\begin{aligned} f(x + iy) &= (x + iy)^2 \\ &= (-x - iy)^2 \\ &= f(-x - iy) \\ &= f(x' + iy') \end{aligned}$$

so  $f$  preserves  $\sim$ . Since  $z^2$  is multiplication on  $\mathbb{C}$  by Theorem 21.5  $f$  is continuous (or at least the argument can be extended to make this operation continuous). Thus, by Theorem Q.3 the induced map on the quotient  $\bar{f}: S^1/\sim \rightarrow S^1$  is continuous. By Theorem 26.6 it suffices to show that  $\bar{f}$  is bijective. It is clear that  $\bar{f}$  is surjective since  $f$  is surjective; that is, take an element  $x + iy \in S^1$  then by elementary properties of the complex numbers we have

$$f\left(\|x + iy\|e^{i\pi\theta/2}\right) = x + iy$$

where  $\theta = \arg(x + iy)$ . To see that this map is injective simply note that if  $f(x + iy) = f(x' + iy')$  then

$$x^2 - y^2 - ((x')^2 + (y')^2) = i2(x'y' - xy)$$

if and only if  $x' = x$  and  $y' = y$  or  $x' = -x$  and  $y' = -y$  so  $\bar{f}$  is injective. It follows that  $\bar{f}$  is a homeomorphism so  $S^1/\sim \approx S^1$ . ■

**Problem 9.** Let  $X$  be a nonempty compact Hausdorff space and let  $f: X \rightarrow X$  be a continuous function. Suppose  $f$  is 1-1. Prove that there is a nonempty closed set  $A$  with  $f(A) = A$ . (The hypothesis that  $f$  is 1-1 is not actually needed, but it makes the proof a little easier.)

*Proof.* We prove the more general case. First, we will show that the  $f$  is a closed map. Suppose  $C$  is a closed subset of  $X$  then, since  $X$  is compact, by Theorem 26.2  $C$  is compact. Then since  $f$  is continuous  $f(C)$  is compact in  $X$  so  $f(C)$  is closed by Theorem 26.3. Thus,  $f$  is a closed map. Now consider the countable collection of nested closed subsets  $X \supset f(X) \supset f^2(X) \supset \dots$ . Indeed,  $f^i(X) \supset f^{i+1}(X)$  since if  $x \in f^{i+1}(X)$  then there exists  $y \in X$  such that  $f^{i+1}(y) = x$ . Let  $z := f(y)$  then  $f^i(z) = f^{i+1}(y) = x \in f^i(X)$ . We claim that  $f(\bigcap_{i \in \mathbb{N}} f^i(X)) = \bigcap_{i \in \mathbb{N}} f^i(X)$  is the set we are looking for. First, since  $f$  is a closed map and each  $f^i(X)$  is closed (since  $X$  is compact Hausdorff) then the intersection  $A := \bigcap_{i \in \mathbb{N}} f^i(X)$  is closed. By the finite intersection property, Theorem 26.9,  $A$  is nonempty since  $X$  is nonempty and  $f$  is a function (for recall that a function from  $X$  to  $X$  is an element of the set  $X^X$  and if the codomain of such an element is empty then  $X^X = \emptyset$ , but that would imply  $X = \emptyset$ ) and for any finite subcollection  $\{f^i(X)\}_{i \in I}$  the intersection  $\bigcap_{i \in I} f^i(X) = f^m(X)$  where  $m = \max\{i \in I\}$ . Lastly, we show that  $f(A) = A$ . One containment is clear, namely  $f(A) \subset A$  for if  $x \in f(A)$  then  $x = f(y)$  for some  $y \in A$ , i.e.,  $y \in f^i(X)$  for all  $i$  so  $x \in A$ . To see the reverse take  $x \in A$  then  $x \in f^i(X)$  for all  $i$ . Thus,  $f^{-1}(x) \subset f^i(X)$  for all  $i$  so  $f^{-1}(x) \in A$ , i.e.,  $x \in f(A)$ . ■

**Problem 10.** Let  $\sim$  be the equivalence relation on  $\mathbb{R}^2$  defined by  $(x, y) \sim (x', y')$  if and only if there is a nonzero  $t$  with  $(x, y) = (tx', ty')$ . Prove that the quotient space  $\mathbb{R}^2/\sim$  is compact but not Hausdorff.

*Proof.* We first show that the quotient space is not Hausdorff. Let  $q: \mathbb{R}^2 \rightarrow \mathbb{R}^2/\sim$  denote the quotient map. We show that for any point  $[(x, y)]$  in the quotient, for any neighborhood  $V$  of  $[(x, y)]$ , for any neighborhood  $U$  of  $[(0, 0)]$  the intersection  $U \cap V \neq \emptyset$ . Let  $U$  be a neighborhood of  $[(0, 0)]$  and  $V$  be a neighborhood of  $[(x, y)]$ . Then  $p^{-1}(U)$  is a neighborhood of  $(0, 0)$  and  $p^{-1}(V) \supset \{(tx, ty) \mid t \neq 0\}$  is a neighborhood of  $(x, y)$ . But since  $p^{-1}(U)$  is open, it contains an  $\varepsilon$ -ball about  $(0, 0)$ , say  $B((0, 0), \varepsilon)$  for  $\varepsilon > 0$ . But for sufficiently small values of  $|t|$ ,  $(tx, ty) \in B((0, 0), \varepsilon)$  for any  $\varepsilon > 0$  (for example  $t^2x^2 + t^2y^2 \leq \varepsilon$  if  $|t| \leq \sqrt{\varepsilon/(x^2 + y^2)}$  so  $(tx, ty) \in B((0, 0), \varepsilon)$ ). Hence  $[(x, y)] \in U$  so  $U \cap V \neq \emptyset$ . Since  $U$  and  $V$  were arbitrary, we conclude that  $\mathbb{R}^2/\sim$  is not Hausdorff.

To see that  $\mathbb{R}^2/\sim$  is in fact compact let  $\mathcal{U}$  be an open cover of  $\mathbb{R}^2/\sim$ . Then at least one  $U \in \mathcal{U}$  contains the equivalence class of  $(0, 0)$ . Thus, by the previous argument  $q^{-1}(U)$  contains an open ball  $B((0, 0), \varepsilon)$  for  $\varepsilon > 0$  and this open ball contains  $(tx, ty)$  for sufficiently small values of  $|t|$ , hence  $U$  contains every equivalence class of  $\mathbb{R}^2/\sim$ . Thus,  $\mathbb{R}^2/\sim$  is compact. ■

**Problem 11.** Let  $X$  be a locally compact Hausdorff space. Explain how to construct the one-point compactification of  $X$  and prove that the space you construct is really compact (you do not have to prove anything else for this problem.)

*Proof.* This is Theorem 29.1 from Munkres §29, p. 183. We will summarize his argument.

Munkres's construction really only begins in step 2 of his argument. Let  $Y$  denote the one-point compactification of  $X$ . We topologize  $Y$  by defining the topology on  $Y$  to be (1) all sets  $U$  open in  $X$  and (2) all sets of the form  $U = Y - C$ , where  $C$  is a compact subspace of  $X$ .

To prove compactness, let  $\mathcal{U}$  be an open cover of  $Y$ . Isolate an open set of type (2) in the cover, say  $U$ , which must exist for otherwise  $\infty \notin \bigcup_{U_\alpha \in \mathcal{U}} U_\alpha$  so  $\mathcal{U}$  does not cover  $Y$ . Given  $U$ , let  $C := Y - U$ . Then  $C$  is a compact subset of  $X$  and is covered by the union of all open sets of type 1 in  $\mathcal{U}$ . By Lemma 26.1, only finitely many of these  $U_\alpha$ 's cover  $C$ , say  $U_1, \dots, U_n$ . Then  $U_1, \dots, U_n, U$

is an open cover of  $Y$  since  $C \subset \bigcup_{i=1}^n U_i$  and  $C \cup (Y - C) = Y$  so  $Y \subset (\bigcup_{i=1}^n U_i) \cup U$ . Therefore,  $Y$  is compact. ■

**Problem 12.** Show that if  $\prod_{n=1}^{\infty} X_n$  is locally compact (and each  $X_n$  is nonempty), then each  $X_n$  is locally compact and  $X_n$  is compact for all but finitely many  $n$ .

*Proof.* Define  $X := \prod_{n=1}^{\infty} X_n$  and let  $\mathbf{x} \in X$ . Since  $X$  is locally compact, there exists a compact set  $C$  and an open neighborhood  $U$  of  $\mathbf{x}$  such that  $C \supset U$ . Without loss of generality, we may assume that  $U = \prod U_n$  where  $U_n$  is open in  $X_n$  and  $U_n = X_n$  for all but finitely many  $n$ 's. Now, since the projection maps,  $\pi_n: X \rightarrow X_n$ , are continuous and by Theorem 26.5,  $\pi_n(C) \supset U_n$  is compact. Since  $U_n = X_n$  for all but finitely many  $n$ 's,  $\pi_n(C) = X_n$  is compact for all but finitely many  $n$ 's. Otherwise,  $\pi_n(C) \supset U_n$  so  $X_n$  is locally compact. ■

**Problem 13.** Let  $X$  be a locally compact Hausdorff space, let  $Y$  be any space, and let the function space  $\mathcal{C}(X, Y)$  have the compact-open topology. Prove that the map

$$e: X \times \mathcal{C}(X, Y) \rightarrow Y$$

define by the equation  $e(x, f) = f(x)$  is continuous.

*Proof.* This is Theorem 46.10 from Munkres §46, p. 286. We paraphrase the proof here.

Given a pair  $(x, f) \in \mathcal{C}(X, Y)$  and an open set  $V$  in  $Y$  containing  $e(x, f) = f(x)$ , by Theorem 18.1(4) we wish to find an open set about  $(x, f)$  that  $e$  maps into  $V$ . First, using the continuity of  $f$  and the fact that  $X$  is locally compact Hausdorff, we can choose an open set  $U$  about  $x$  having compact closure  $\overline{U}$ , such that  $f$  carries  $\overline{U}$  into  $V$ . Then, consider the open set  $U \times S(\overline{U}, V) \subset X \times \mathcal{C}(X, Y)$ . It is an open set containing  $(x, f)$  and if  $(x', f')$  is in this set, then  $e(x', f') = f'(x') \in V$ , as desired. ■

**Problem 14.** Let  $I$  be the unit interval, and let  $Y$  be a path-connected space. Prove that any two maps from  $I$  to  $Y$  are homotopic.

*Proof.* This is Ex. 2(b) from Munkres §51. ■

**Problem 15.** Let  $X$  be a topological space and  $f: [0, 1] \rightarrow X$  any continuous function. Define  $\bar{f}$  by  $\bar{f}(t) = f(1 - t)$ . Prove that  $f * \bar{f}$  is path-homotopic to the constant path at  $f(0)$ .

*Proof.* ■

**Problem 16.** Let  $X$  be a path-connected topological space and let  $x_0, x_1 \in X$ . Recall that any path  $\alpha$  from  $x_0$  to  $x_1$  gives an isomorphism  $\hat{\alpha}$  from  $\pi_1(X, x_0)$  to  $\pi_1(X, x_1)$  (you do not have to prove this.)

Suppose that for every pair of paths  $\alpha$  and  $\beta$  from  $x_0$  to  $x_1$  the isomorphisms  $\hat{\alpha}$  and  $\hat{\beta}$  are the same. Prove that  $\pi_1(X, x_0)$  is Abelian.

*Proof.* ■