# MA 544: Homework 4

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## Problem 4.1 (Wheeden & Zygmund §3, Ex. 12)

If $E_1$ and $E_2$ are measurable sets in $\mathbb{R}^1$, show $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1||E_2|$. (Interpret $0 \cdot \infty$ as 0.) [Hint: Use a characterization of measurability.]

*Proof.* By (3.28) (i) we may write $E_1$ and $E_2$ as the unions $H_1 \cup Z_1$ and $H_2 \cup Z_2$, respectively, where $H_1$ and $H_2$ are $F_\sigma$ and $Z_1$ and $Z_2$ are measure zero. Now, by elementary set theory, the Cartesian product $E_1 \times E_2$ can then be written as

$$E_1 \times E_2 = (H_1 \cup Z_1) \times (H_2 \cup Z_2) = \underbrace{H_1 \times H_2}_{H} \cup \underbrace{H_1 \times Z_2 \cup Z_1 \times H_2 \cup Z_1 \times Z_2}_{Z}. \tag{1}$$

Hence, we win by (3.28) (i) if we can show that the Cartesian of two $F_\sigma$ sets is an $F_\sigma$ set and if the Cartesian product of a measurable set with a set of measure zero is measure zero.

First, we prove the former, since the argument to be made is little more than elementary set theory. Let $F_1$ and $F_2$ be $F_\sigma$. Write $F_1 = \bigcup F_k'$ and $F_2 = \bigcup F_k''$ where the $F_k'$'s and the $F_k''$'s are closed subsets of $\mathbb{R}$. Then, $F_k' \times F_\ell''$ are closed subsets of $\mathbb{R}^2$ by elementary topology. Moreover, $F_k' \times F_\ell'' \subset F_1 \times F_2$ hence, $\bigcup_{k,\ell} F_k' \times F_\ell'' \subset F_1 \times F_2$. Thus, it suffices to show that $\bigcup_{k,\ell} F_k' \times F_\ell'' \supset F_1 \times F_2$. Let $(x, y) \in F_1 \times F_2$. Then $x \in F_1$ and $y \in F_2$. But since $F_1 = \bigcup F_k'$ and $F_2 = \bigcup F_k''$ then $x \in F_k'$ and $x \in F_\ell''$ for some $k$, $\ell$. In other words, $(x, y) \in F_k' \times F_\ell''$ so $(x, y)$ is in the union $\bigcup_{k,\ell} F_k' \times F_\ell''$. Hence, we have $F_1 \times F_2 = \bigcup_{k,\ell} F_k' \times F_\ell''$. We conclude that if $F_1$ and $F_2$ are $F_\sigma$, then so is their Cartesian product $F_1 \times F_2$.

Let $E$ be a measurable set with $|E| < \infty$ and $Z$ a set of measure zero. Then, for every $\varepsilon > 0$ there exists a countable collection of intervals $\{I_k\}$ containing $Z$ such that $\sum \operatorname{vol}(I_k) < \varepsilon$. Similarly, we can find a collection $\{I_k'\}$ of intervals containing $E$ such that $\sum \operatorname{vol}(I_k') < |E| + \varepsilon$. Then, $\{I_k' \times I_\ell\}$ is a countable collection of 2-intervals containing $E \times Z$ with

$$\begin{aligned}
\sum \operatorname{vol}(I_k' \times I_\ell) &= \sum \operatorname{vol}(I_k') \operatorname{vol}(I_\ell) \\
&= \sum_k \sum_\ell \operatorname{vol}(I_k') \operatorname{vol}(I_\ell) \\
&= \left( \sum_k \operatorname{vol}(I_k') \right) \left( \sum_\ell \operatorname{vol}(I_\ell) \right) \\
&= (|E| + \varepsilon)\varepsilon
\end{aligned}$$

Letting $\varepsilon \to 0$, we have $E \times Z$ is measure zero. If $|E| = \infty$, partition $E$ into disjoint finite measure subsets of $\mathbb{R}$ by taking the following intersection

$$E_k = E \cap (B(0, k) \setminus B(0, k - 1))$$

for $k \in \mathbf{N}$. By our previous argument, $E_k \times Z$ is measure zero so $\{E_k \times Z\}$ is a cover of $E \times Z$

Then, $E = \bigcup E_k$ so by subadditivity we have

$$|E|$$

■

## Problem 4.2 (Wheeden & Zygmund §3, Ex. 13)

Motivated by (3.7), define the *inner measure* of $E$ by $|E|_i = \sup|F|$, where the supremum is taken over all closed subsets $F$ of $E$. Show that

(i) $|E|_t \leq |E|_e$, and

(ii) if $|E|_e < +\infty$, then $E$ is measurable if and only if $|E|_i = |E|_e$.

[Use (3.22).]

*Proof.* ∎

## Problem 4.3 (Wheeden & Zygmund §3, Ex. 14)

Show that the conclusion of part (ii) of Exercise 13 is false if $|E|_e = +\infty$.

*Proof.* ■

## Problem 4.4 (Wheeden & Zygmund §3, Ex. 15)

If $E$ is measurable and $A$ is any subset of $E$, show that $|E| = |A|_i + |E - A|_e$. (See Exercise 13 for the definition of $|A|_i$.)

*Proof.* ■