MA 572: Homework 4

Carlos Salinas

February 19, 2016

Problem 4.1 (Hatcher §2.1, Ex. 20)

Show that $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Proof. First note that the reduced suspension of X, ΣX, which is homotopy equivalent to SX, can be realized as the quotient space CX/X. Given the imbedding $X \hookrightarrow CX$, by 2.16 and excision (or 2.22) we have the long exact sequence in

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_n(X) \arrow[r] & \widetilde{H}_n(CX) \arrow[r] & \widetilde{H}_n(CX,X) \arrow[dll, out=0, in=180] \\ & \widetilde{H}_{n-1}(X) \arrow[r] & \widetilde{H}_{n-1}(CX) \arrow[r] & \widetilde{H}_{n-1}(CX,X) \arrow[r] & \cdots, \end{tikzcd} \tag{1}$$

where $\widetilde{H}_n(CX,X) \cong \widetilde{H}_n(CX/X) \cong \widetilde{H}_n(SX)$. Since CX is contractible, we have $H_n(CX) = 0$ for all n and the long exact sequence (1) yields an isomorphism

$$\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X).$$ ■

Problem 4.2 (Hatcher §2.1, Ex. 22)

Prove by induction on the dimension the following facts about the homology of a finite dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:

(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.

(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.

(c) If X is has k n-cells, then $H_n(X)$ is generated by at most k elements.

Proof. To make Hatcher's comment more precise, if $X = \bigcup_{k=0}^n X^k$ is a n-dimensional CW-complex, by excision the relative homology

$$H_i\big(X^k, X^{k-1}\big) \cong \widetilde{H}_i\big(X^k/X^{k-1}\big) \cong \widetilde{H}_i\big(\bigvee_{\alpha_k} S^k\big) \cong \bigoplus_{\alpha_k} \widetilde{H}_i\big(S^k\big) \cong \begin{cases} \bigoplus_{\alpha_k} \mathbf{Z} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where α_k is the index over the k-dimensional cells of X.

(a) We proceed by induction on n the dimension of X. If $n = 0$ then X^0 is a collection of points and we know that $H_i(X^0) = 0$ for all $i > 0$ and $H_0 = \oplus_\alpha \mathbf{Z}$ where α runs over all X^0.

Now for the inductive step, suppose $H_i(X^{n-1}) = 0$ for all $i > n-1$. Then, we have the long exact sequence

$$\begin{array}{l} \cdots \longrightarrow H_{n+1}\big(X^{n-1}\big) \longrightarrow H_{n+1}(X^n) \longrightarrow H_{n+1}\big(X^n/X^{n-1}\big) \longrightarrow \\ \longrightarrow H_n\big(X^{n-1}\big) \longrightarrow H_n(X^n) \longrightarrow H_n\big(X^n/X^{n-1}\big) \longrightarrow \cdots . \end{array} \tag{3}$$

But by the induction hypothesis, $H_i\big(X^{n-1}\big) = 0$ for all $i > n-1$ and $H_{n-1}\big(X^{n-1}\big) = \bigoplus_\alpha \mathbf{Z}$ over some index α. Then (3) becomes

$$\begin{array}{l} \cdots \longrightarrow 0 \longrightarrow H_{n+1}(X^n) \longrightarrow 0 \longrightarrow \\ \longrightarrow 0 \longrightarrow H_n(X^n) \longrightarrow \bigoplus_{\alpha_n} \mathbf{Z} \longrightarrow \cdots . \end{array} \tag{4}$$

It's easy to see, if we extend (4) to the left, i.e., by adding H_{n+1} etc., that $H_i(X^n) = 0$ since it is sandwiched between two 0's. Moreover, by exactness at $H_n\big(X^n/X^{n-1}\big)$, we have an injection $H_n(X^n) \hookrightarrow \bigoplus_{\alpha_n} \mathbf{Z}$ so $H_n(X^n)$ is free abelian.

Now, note that if $m > i$ the $H_i(X) \cong H_i(X^m)$.

Proof of observation. Since X is n-dimensional all we need to show is that $H_i(X^m) \cong H_i\big(X^{i+1}\big)$. By induction on m, the base case $m = i+1$ is clear. Now for the inductive step, suppose $H_i\big(X^{m'}\big) \cong H_i\big(X^{i+1}\big)$ for $m > i$. Then we have

$$\cdots \longrightarrow H_{i+1}\big(X^{m+1}, X^m\big) \longrightarrow H_i\big(X^m\big) \longrightarrow H_i\big(X^{m+1}\big) \longrightarrow H_i\big(X^{m+1}, X^m\big) \longrightarrow \cdots . \tag{5}$$

But $H_{i+1}\big(X^{m+1}, X^m\big) \cong H_{i+1}\big(X^{m+1}, X^m\big) \cong 0$ so $H_i\big(X^m\big) \cong H_i\big(X^{m+1}\big)$, as desired. ♣

(b) If $n < 1$ in the case $n = 0$ there are no 1-cells and in the case $n = 1$ there are no 0-cells so we cannot say anything. Therefore, we begin at $n > 1$. By our observation, we have $H_n(X) \cong H_n(X^n)$ and by part **(a)** we have

$$\cdots \longrightarrow H_n\big(X^{n-2}\big) \longrightarrow H_n\big(X^n\big) \longrightarrow H_n\big(X^n, X^{n-2}\big) \longrightarrow H_{n-1}\big(X^{n-2}\big) \longrightarrow \cdots , \tag{6}$$

where $H_n\big(X^{n-2}\big) \cong H_{n-1}\big(X^{n-2}\big) \cong 0$. Hence, by exactness at $H_n(X^n)$, we have $H_n(X^n) \cong H_n\big(X^n, X^{n-2}\big)$ so by 2.34 (a), we have

$$H_n(X^n) \cong H_n\big(X^n, X^{n-1}\big) \cong H_n\big(X^n, X^{n-2}\big) \cong \bigoplus_\alpha \mathbf{Z}$$

where α is an index over the n-cells of X.

(c) By the observation $H_n(X) \cong H_n\big(X^{n+1}\big)$. Moreover, we have the following exact sequence

$$\cdots \longrightarrow H_n\big(X^{n+1}, X^n\big) \longrightarrow H_n(X^n) \longrightarrow H_n\big(X^{n+1}\big) \longrightarrow H_n\big(X^{n+1}, X^n\big) \longrightarrow \cdots \tag{7}$$

where $H_{n+1}\big(X^{n+1}, X^n\big) \cong H_n\big(X^{n+1}, X^n\big) \cong 0$ giving us an injection $H_n(X^n) \hookrightarrow H_n\big(X^{n+1}\big)$. By part **(a)**, $H_n(X^n)$ is free abelian and has rank at most the the number of n-cells of X. Thus, $H_n(X)$ is free abelian and has rank at most the number of n-cells of X. ■

Problem 4.3 (Hatcher §2.2, Ex. 2)

Given a map $f\colon S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbf{R}\mathrm{P}^{2n} \to \mathbf{R}\mathrm{P}^{2n}$ has a fixed point. Construct maps $\mathbf{R}\mathrm{P}^{2n-1} \to \mathbf{R}\mathrm{P}^{2n-1}$ without fixed points from linear transformations $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ without eigenvectors.

Proof. Given a map $f\colon S^{2n} \to S^{2n}$, either f has a fixed point or it does not. If f does not have a fixed point then $f \simeq a$ so $\deg f = (-1)^{2n+1} = -1$ so $\deg(-f) = \deg(-\operatorname{Id} \circ f) = 1$ has a fixed point for otherwise we may homotope f to the antipodal map a and so, by transitivity, $\operatorname{Id} \simeq a$, but this is nonsense.

Note that, by example 1.43, $p\colon S^{2n} \to \mathbf{R}\mathrm{P}^{2n}$ is a cover of $\mathbf{R}^{2n}$. Let $f\colon \mathbf{R}\mathrm{P}^{2n} \to \mathbf{R}\mathrm{P}^{2n}$. By the lifting property, there exists a map $\tilde{f}$ making the diagram below commute

$$\begin{array}{ccc} S^{2n} & \xrightarrow{\tilde{f}} & S^{2n} \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p\circ\tilde{f}} \\ \mathbf{R}\mathrm{P}^{2n} & \xrightarrow{f} & \mathbf{R}\mathrm{P}^{2n}. \end{array} \tag{8}$$

Now, by the first part to this problem, we know that for some $x \in S^{2n}$, $f(x) = x$ or $f(x) = -x$. But under the quotient map p, $[x] = [-x]$. Hence, every map $f\colon \mathbf{R}\mathrm{P}^{2n} \to \mathbf{R}\mathrm{P}^{2n}$ has a fixed point.

For this last part, all we need to do is find a matrix with nonreal eigenvalues like the linear map

$$T = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}. \tag{9}$$

Then the characteristic polynomial of T is $x^2 + 1$ which has nonreal roots so T has no invariant subspaces besides $\{0\}$ and $\mathbf{R}^{2n}$. Since $\mathbf{R}\mathrm{P}^{2n-1}$ can be realized as a quotient of $\mathbf{R}^{2n}$, by the UMP of the quotient topology, this map induces a continuous map $[T]\colon \mathbf{R}\mathrm{P}^{2n-1} \to \mathbf{R}\mathrm{P}^{2n-1}$ which fixes no 1-dimensional subspaces. Thus $[T]$ has no fixed points. ■