MA598: Lie Groups

Carlos Salinas

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CHAPTER 1

Prologue

This summer, we will be making our way through Fulton and Harri's *Representation Theory* with applications to Lie groups, although we will also reference Knapp's *Lie Groups Beyond an Introduction* [1] and Milne's *Lie Algebras, Algebraic Groups, and Lie Groups* [**?**]. The goal is to eventually look at volume forms on

C_n is the cyclic group of order n not necessarily equal (but isomorphic) to $\mathbb{Z}/p\mathbb{Z}$
S_n is the symmetric group on $\{1, \dots, n\}$
A_n is the alternating group on $\{1, \dots, n\}$

CHAPTER 2

Representation Theory

This section of the notes correspond to the first three chapters of Fulton and Harris's book *Representation Theory*. We will quickly talk about the representation theory of finite groups and move on to the representation theory of Lie groups in the next chapter.

2.1 Representation of Finite Groups

Definitions

Throughout this section, by a *representation* of a finite group G we mean a homomorphism $\rho\colon G \to \mathrm{GL}(V)$, where V is a finite-dimensional complex vector space. A *representation* of a finite group G on a finite-dimension; we say that such a map ρ *gives V a G-module structure.* When there is little ambiguity about the map ρ, we will call V itself a representation of G; we also suppress the symbol ρ and write gv for $\rho(g)(v)$ whenever context allows us. The dimension of V is called the *degree* of ρ.

A map φ between two representations V and W of G is a vector space map $\varphi\colon V \to W$ such that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

commutes for every $g \in G$. We will call such a map a *G-linear map* when we want to distinguish it from an arbitrary linear map between the vector spaces V and W. We can thus, then define $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$, and $\operatorname{Coker}\varphi$, which naturally inherit a G-module structure from V.

A *subrperesentation* of a representation V is a vector subspace W of V which is invariant under the action by G, that is, $gw = w$ for all $w \in W$. A representation V is called *irreducible* if there is no proper nonzero invariant subspace W of V.

As it turns out, some of if V and W are representations, so are $V \oplus W$ and $V \otimes W$ with $g(v \otimes w) := gv \otimes gw$, and so are the nth tensor power $\bigotimes^n V$, the exterior power $\bigwedge^n V$ and the symmetric powers $\operatorname{Sym}^n V$. The dual $V^* = \operatorname{Hom}(V, \mathbb{C})$ of V is also a representation, though not in the most obvious way: We want the two representations of G with respect to the natural pairing between V and V^*, that is, $\langle v^*, v \rangle := v^*(v)$, so that if $\rho \colon G \to \mathrm{GL}(V)$ is a representation and $\rho^* \colon G \to \mathrm{GL}(V)$ is its dual, then we have

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$$

for all $g \in G$, $v \in V$, and $v^* \in V^*$. This in turn forces us to define the dual representation $\rho^* \colon V^* \to V^*$ by

$$\rho^*(g) := {}^{\mathrm{t}}\rho(g^{-1})$$

for all $g \in G$. Let us verify that ρ^* in fact satisfies $\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$: Let $v^* \in V^*$ and $v \in V$, then

$$\begin{aligned}
\langle \rho^*(g)(v^*), \rho(g)(v) \rangle &= \langle {}^{\mathrm{t}}\rho(g^{-1})(v^*), \rho(g)(v) \rangle \\
&= \langle v^*, \rho(g^{-1}) \circ \rho(g)(v) \rangle \\
&= \langle v^*, v \rangle.
\end{aligned}$$

Having defined the dual representation of the tensor product of two representations, it is likewise the case that if V and W are representations, then $\operatorname{Hom}(V, W)$ is also a representation, via the identification $\operatorname{Hom}(V, W) = V^* \otimes W$. Unraveling this, if we view an element of $\operatorname{Hom}(V, W)$ as a linear map φ from V to W, we have

$$(g\varphi)(v) = g\varphi(g^{-1}v)$$

for all $v \in V$. In other words, the definition is such that the diagram

$$\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\
V \arrow[r, "g\varphi"'] & W
\end{tikzcd}$$

commutes. Note that the dual representation is, in turn, a special case of this: When $W = \mathbb{C}$ is the trivial representation, i.e., $gw = w$ for all $w \in \mathbb{C}$, this makes V^* into a G-module, with $g\varphi(v) = \varphi(g^{-1}v)$, i.e., $g\varphi = {}^{\mathrm{t}}(g^{-1})$.

Exercise 1. We verify that in general the vector space of G-linear maps between two representations V and W of G is just the subspace $\operatorname{Hom}(V, W)^G$ of elements of $\operatorname{Hom}(V, G)$ fixed under the action of G. We will often denote this space by $\operatorname{Hom}_G(V, W)$.

Proof. ∎

We have taken the identification $\operatorname{Hom}(V, W) = V^* \otimes W$ as the definition of the representation $\operatorname{Hom}(V, W)$. More generally, the usual identities for vector spaces are also true for representations, e.g.,

$$V \otimes (U \oplus W) = (V \otimes U) \oplus (V \otimes W)$$

$$\bigwedge^k (V \oplus W) = \bigoplus_{a+b=k} \bigwedge^a V \otimes \bigwedge^b W$$

$$\bigwedge^k V^* = \left(\bigwedge^k V\right)^*$$

If X is any finite set and G acts on the left on X, i.e., $G \to \mathrm{Aut}(X)$ is a homomorphism to the premutation group of X, there is an associated permutation representation: Let V be the vector space with basis $\{e_x : x \in X\}$, and let G act on V by

$$g \cdot \sum a_x e_x := \sum a_x e_{gx}.$$

The regular representation, denoted R_G or simply R, corresponds to the left action of G on itself. Alternatively, R is the space of complex-valued functions on G, where an element $g \in G$ acts on a function α by $(g\alpha)(h) = \alpha(g^{-1}h)$.

Complete Reducibility; Schur's Lemma

Before we begin classifying the representations of a finite group G we should try to simplify life by restricting our search somewhat. Specifically, we have seen that representations of G can be built up out of other representations by linear algebraic operations, most simply by taking the direct sum. We should focus, then, on representations that are "atomic" with respect to this operation, i.e., that cannot be expressed as a direct sum of others; the usual term for such a representation is *indecomposable*. Happily, this situation is as nice as it could possibly be: A representation is atomic in this sense if and only if it is irreducible (i.e., contains no proper subrepresentations); and every representation is the direct sum of irreducibles, in a suitable sense uniquely so. The key to all this is

Proposition 1. *If W is a subrepresentation of a representation V of a finite group G, then there is an elementary invariant subspace W' of V, so that $V = W \oplus W'$.*

Proof. There are two ways of showing this. One can introduce a positive definite Hermitian inner product H on V which is preserved by each $g \in G$ (i.e., such that $H(gv, gw) = H(v, w)$ for all $v, w \in V$, $g \in G$). Indeed, if H_0 is any Hermitian product on V, one gets such an H by averaging over G:

$$H(v, w) := \sum_{g \in G} H_0(gv, gw). \tag{2.1}$$

Then the perpendicular subspace $W^\perp$ is complementary to W in V. Alternatively, we can simply choose an arbitrary subspace U complementary to W, let $\pi_0 \colon V \to W$ be the projection given by the direct sum decomposition $V = W \oplus U$, and average the map π_0 over G: i.e., take

$$\pi(v) := \sum_{g \in G} g(\pi_0(g^{-1}v)). \tag{2.2}$$

this will then be a G-linear map from V onto W, which is multiplication by $|G|$ on W; its kernel will, therefore, be a subspace of V invariant under G and complemnetary to W. ■

Corollary 2. *Any representation is a direct sum of irreducible representations.*

This property is called complete reducibility, or semisimplicity. We will see that, for continuous representations, the circle S^1, or any compact group, has this property; integration over the group (with respect to an invariant measure on the group) plays the role of averaging in the above proof. The (additive) group $\mathbb{R}$ does not have this property: The representation

$$a \longmapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

leaves the x axis fixed, but there is no complementary subspace. We will see other Lie groups such as $\mathrm{SL}_n(\mathbb{C})$ that are semisimple in this sense. Note also that this argument would fail if the vector space V was over a field of finite characteristic since it might then be the case that $\pi(v) = \mathbf{0}$ for for $v \in W$. The failure, of complete reducibility is one of is one of the things that makes the subject of modular representations, or representations on vector spaces over finite fields, so tricky.

The extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

Theorem 3 (Schur's lemma)**.** *If V and W are irreducible representations of G and $\varphi\colon V \to W$ is a G-module homomorphism, the*

(a) *Either φ is an isomorphism, or $\varphi = \mathbf{0}$.*
(b) *If $V = W$, then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$, I being the identity.*

Proof. The first claim follows from the fact that $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ are invariant subspaces. For the second, since $\mathbb{C}$ is algebraically closed, φ must have an eigenvalue λ, i.e., for some $\lambda \in \mathbb{C}$, $\varphi - \lambda I$ has a nonzero kernel. By Theorem 1, we must have $\varphi - \lambda I = \mathbf{0}$ so $\varphi = \lambda I$. ∎

We can summarize what we have shown thus far in

Proposition 4. *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.

Proof. It follows from Schur's lemma that if W is another representation of G, with decomposition $W = \bigoplus W_j^{\oplus b_j}$, and $\varphi\colon V \to W$ is a map of representations, then φ must map the factor $V_i^{\oplus a_i}$ into that factor $W_j^{\oplus b_j}$ for which $W_j \simeq V_i$; when applied to the identity map of V to V, the stated uniqueness follows. ∎

Occasionally, the decomposition is written

$$V = a_1 V_1 \oplus \cdots \oplus a_k V_k = a_1 V_1 + \cdots + a_k V_k,$$

especially when one is encountered only about the isomorphism classes and multiplicities of the V_i.

One more fact that will be established in the following lecture is that a finite group G admits only finitely many irreducible representations V_i up to isomorphism (in fact, we can say how many).

Our first goal, in analyzing the representations of any group, will therefore be:

(a) Described all the irreducible representations of G.

Once we have done this, there remains the problem of carrying out in practice the description of a given representation in these terms. Thus, our second goal will be:

(b) Find techniques for giving the direct sum decomposition, and in particular determining the multiplicities a_i of an arbitrary representation V.

Finally, it is the case that the representations we will most often be concerned with are those arising from simpler ones by the sort of linear- or miltilinear-algebraic operations described above. We would like, therefore, to be able to describe, in the terms above, the representation we get when we perform these operations on a know representation. This is known generally as

(c) Plethysm: Describe the decompositions, with multiplicities, of representations derived from a given representation V, such as $V \otimes V$, V^*, $\bigwedge^k V$, $\mathrm{Sym}^k V$ and $\bigwedge^k(\bigwedge^\ell V)$. Note that if V decomposes into a sum of two representations, these representatinos decompose accordingly; e.g., if $V = U \oplus W$, then

$$\bigwedge^k V = \bigoplus_{i+j=k} \bigwedge^i U \otimes \bigwedge^j W,$$

so it is enough to work out this plethysm for irreducible representations. Similarly, if V and W are two irreducible representations, we want to decompose $V \otimes W$; this is usually known as the Clebsch–Gordan problem.

Examples: Abelian Groups; S_3

One obvious place to look for examples is with Abelian groups. It does not take long, however, to deal with this case. Basically, we may obverve in general that if V is a representation of the finite group G, abelian or not, each $g \in G$ gives a map $\rho(g)\colon V \to V$; but this map is not generally a G-module homomorphism: For general $h \in G$, we will have

$$g(h(v)) \neq h(g(v)).$$

Indeed, $\rho(g)\colon V \to V$ will be G-linear for every ρ if and only if g is in the center $Z(G)$ of G. In particular, if G is Abelian, and V is an irreducible representation, then by Schur's lemma every element $g \in G$ acts on V by a scalar multiple of the identity. Every subspace of V is thus invariant; so that V must be one dimensional. The irreducible representations of an Abelian group G are thus simply elements of the dual group, i.e., homomorphism $\rho\colon G \to \mathbb{C}^*$.

We begin with the next simplest nonAbelian group, $G := S_3$. To begin with, we have two one-dimensional representations: We have the trivial representation, which we shall denote U, and the alternating representation U', defined by setting

$$gv = \mathrm{sgn}(g)v$$

for $g \in G$, $v \in \mathbb{C}$. Next, since G comes to us as a permutation group, we have a natural permutation representation, in which G acts on $\mathbb{C}^3$ by permuting the coordinates. Explicitly, if $\{e_1, e_2, e_3\}$ is the standard basis, then $g \cdot e_i = e_{g(i)}$, or equivalently,

$$g \cdot (z_1, z_2, z_3) = \left(z_{g^{-1}(1)}, z_{g^{-1}(2)}, z_{g^{-1}(3)}\right).$$

This representation, like any permutation representation, is not irreducible: The line spanned by the sum $(1, 1, 1)$ of the basis vectors is invariant, with complementary subspace

$$V := \{\, (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_2 + z_3 = 0 \,\}.$$

This two-dimensional representation of V is easily seen to be irreducible; we call it the standard representation of S_3.

Let us now turn to the problem of describing an arbitrary representation of S_3. We will see in the next lecture that a wonderful tool for doing this is called *character theory*; but, as inefficient as this may be, we would like here to adapt the more ad hoc approach.

We have just seen that the representation theory of a finite Abelian group is virtually trivial, we will start our analysis of an arbitrary representation W of S_3 by looking just at the action of the Abelian subgroup $A_3 = C_3 \subset S_3$. This yields a very simple decomposition: If we take τ to be any generator of A_3 (that is, any 3-cycle), the space W is spanned by eigenvectors v_i for the action τ, whose eigenvalues are of course all powers of the cube root of unity $\omega := e^{2\pi i/3}$. Thus,

$$W = \bigoplus V_i,$$

where

$$V_i := \mathbb{C}v_i$$

and

$$\tau v_i = \omega^{\alpha_i} v_i.$$

Next, we ask how the remaining elements of S_3 act on W in terms of this decomposition. TO see how it goes, let σ be any transposition, so that τ and σ, together generate S_3, with the relation $\sigma\tau\sigma = \tau^2$. We want to know where σ sends an eigenvector v for the action of τ, say with eigenvalue ω^i; to answer this, we look at how τ acts on $\sigma(v)$. We use the basic relation above to write

$$\begin{aligned} \tau(\sigma(v)) &= \sigma(\tau^2(v)) \\ &= \sigma(\omega^{2i} v) \\ &= \sigma^{2i}\sigma(bfv). \end{aligned}$$

The conclusion, then, is that if v is an eigenvector for τ with eigenvalue ω^i, then $\sigma(v)$ is again an eigenvector for τ, with eigenvalue ω^{2i}.

2.2 Characters

Characters

As it turns out, there is a remarkably effective tool for understanding the representations of a finite group G, called *character theory*. This is in some ways motivated by the example worked out in the last section where we saw that a representation of S_3 was determined by knowing the eigenvalues of the action of the elements τ and $\sigma \in S_3$. For a general group G, it is not clear what subgroups and or elements should play the role of A_3, τ, and σ; but the example certainly suggests that knowing all the eigenvalues of each element of G should suffice to describe the representation.

Of course, specifying all the eigenvalues of the action of each element of G is somewhat unwieldy; but fortunately, it is redundant as well. For example, if we know that the eigenvalues $\{\lambda_i\}$ of an element $g \in G$, then of course we know the eigenvalues $\{\lambda_i^k\}$ of g^k for each k as well. We can thus use this redundancy to simplify the data we have to specify.

Definition 1. If V is a representation of G, its *character* χ_V is the complex-valued function on the group defined by

$$\chi_V := \mathrm{tr}(g|_V).$$

the trace of g on V.

In particular, we have

$$\chi_V(hgh^{-1}) = \chi_V(g)$$

so that χ_V is constant on the conjugacy classes of G; such a function is called a *class function*. Note that $\chi_V(1) = \dim V$.

Proposition 5. *Let V and W be representations of G. Then*

$$\begin{aligned} \chi_{V\oplus W} &= \chi_V + \chi_W, & \chi_{V\otimes W} &= \chi_V \chi_W \\ \chi_{V^*} &= \bar{\chi}_V & \chi_{\wedge^2 V}(g) &= \tfrac{1}{2}\big[\chi_V(g)^2 - \chi_V(g^2)\big]. \end{aligned}$$

Proof. ∎

Examples 1. We compute the character table of S_3. This is easy: To begin with, the trivial representation takes the values $(1,1,1)$ no the three conjugacy classes $[(1)]$, $[(1\,2)]$, and $[(1\,2\,3)]$, whereas the alternating representation has values $(1,-1,1)$. To see the character of the standard representation, note that the permutation representation decomposes: $\mathbb{C}^3 = U \oplus V$; since the character of the permutation representation has, by Exercise 2.5, the values $(3,1,0)$, we have $\chi_V = \chi_{\mathbb{C}^3} - \chi_U = (3,1,0) - (1,1,1) = (2,0,-1)$. In sum,

S_3	$[(1)]$	$[(1\,2)]$	$[(1\,2\,3)]$
trivial U'	1	1	1
alternating U'	1	-1	1
standard V	2	0	-1

This gives us another solution of the basic problem posed in Lecture 1: If W is any representation of S_3, and we decompose W into irreducible representations $W \simeq U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$, then $\chi_W = a\chi_U + b\chi_{U'} + c\chi_V$. In particular, since the functions χ_U, $\chi_{U'}$, and χ_V are independent, we see that W is *determined up to isomorphism by its character χ_W*.

Consider, for example, $V \otimes V$. Its character is ${\chi_V}^2$, wich has values 4, 0, and 1 on the three conjugacy classes. Since $V \oplus U \oplus U'$ has the same character, this implies that $V \otimes V$ decomposes into $V \oplus U \oplus U'$, as we have seen directly. Similarly, $V \otimes U'$ has values 2, 0, and -1, so $V \otimes U' \simeq V$.

The First Projection Formula and its Consequences

In the last lecture, we asked for a way of locating explicitly the direct sum factors in the decomposition of a representation into irreducible ones. In this section we will start by giving an explicit formula for the projection of a representation onto the direct sum of the trivial factors in the decomposition; as it will turn out, this formula alone has tremendous consequences.

To start, for any representation V of a group G, we set

$$V^G := \{\, v \in V : gv = v \,\} \text{ for all } g \in G.$$

We ask for a way finding V^G explicitly. The idea behind our solution to this is already implicit in the previous lecture. We observed that for any representation V of G and any $g \in G$, the endomorphism $g\colon V \to V$ is general, not a G-module. On the other hand, if we take the average of all these endomorphisms, that is, we set

$$\varphi := \frac{1}{|G|} \sum_{g \in G} g$$

is in $\operatorname{End} V$, then the endomorphism φ will be G-linear since $\sum g = \sum hgh^{-1}$. In fact, we have

Proposition 6. *The map φ is a projection of V onto V^G.*

Proof. First, suppose $v = \varphi(w) = (1/|G|) \sum gw$. Then, for any $h \in G$

$$hv = \frac{1}{|G|} \sum hgw = \frac{1}{|G|} \sum gw$$

∎

More Projection Formulas; More Consequences

In this section, we complete the analysis of the characters of the irreducible representations of a general finite group begun in §2.2 and give a more general formula for the projection of the general representation V onto a direct sum of the factors in V isomorphic to a given irreducible representation W. The main idea for both is a generalization of the averaging of the endomorphisms $g\colon V \to V$ for both the point being that instead of simply averaging all of the g we can ask the question: What linear combinations of the endomorphisms $g\colon V \to V$ are G-linear endomorphisms? The answer is given by

Proposition 7. *Let $\alpha\colon G \to \mathbb{C}$ be any function on the group G, and for any representation V of G set $\varphi_{\alpha,V}\colon V \to V$ to be*

$$\varphi_{\alpha,V} := \sum \alpha(g) \cdot g.$$

Then $\varphi_{\alpha,V}$ is a homomorphism of G-modules for all V if and only if α is a class function.

Proof. We simply write out the condition that $\varphi_{\alpha,V}$ be G-linear, and the result falls out: We have

$$\begin{aligned} \varphi_{\alpha,V}(hv) &= \sum \alpha(g) \cdot g(hv) \\ &= \sum \alpha(hgh^{-1}) \cdot hgh^{-1}(hv) \end{aligned}$$

substituting hgh^{-1} for g

$$= h()$$

∎

Bibliography

[1] A.W. Knapp. *Lie Groups Beyond an Introduction*. Progress in Mathematics. Birkhäuser Boston, 2002.