Math 535 - General Topology
Fall 2012
Homework 12 Solutions

Problem 1. Consider the open cover $\mathcal{U} = \{B_1(x)\}_{x \in \mathbb{R}}$ of $\mathbb{R}$ by open balls of radius 1, i.e. open intervals $B_1(x) = (x-1, x+1)$. Find a partition of unity on $\mathbb{R}$ subordinate to $\mathcal{U}$.

Solution. We will construct a partition of unity which is in fact subordinate to the open cover $\{B_1(n)\}_{n \in \mathbb{Z}}$ of $\mathbb{R}$. Let $\rho \colon \mathbb{R} \to [0,1]$ be the "trapezoid bump" function defined by

$$\rho(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{4} \\ 1 - 2\left(|x| - \frac{1}{4}\right) & \text{if } \frac{1}{4} \leq |x| \leq \frac{3}{4} \\ 0 & \text{if } |x| \geq \frac{3}{4} \end{cases}$$

as illustrated in figure 1. Clearly ρ is continuous, and its support is $\operatorname{supp} \rho = \overline{(-\frac{3}{4}, \frac{3}{4})} = [-\frac{3}{4}, \frac{3}{4}]$.

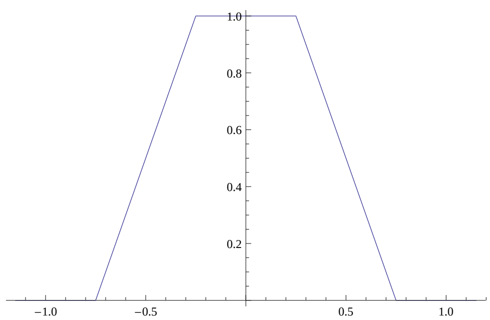

Figure 1: Graph of ρ.

For every integer $n \in \mathbb{Z}$, define $\rho_n \colon \mathbb{R} \to [0,1]$ by $\rho_n(x) = \rho(x-n)$, so that ρ_n is the shifted bump function centered at n. Their supports are

$$\operatorname{supp} \rho_n = \overline{(n - \frac{3}{4}, n + \frac{3}{4})} = [n - \frac{3}{4}, n + \frac{3}{4}] \subseteq (n-1, n+1)$$

and in particular the family $\{\operatorname{supp} \rho_n\}_{n \in \mathbb{Z}}$ is locally finite.

It remains to check that the functions ρ_n add up to 1, as illustrated in figure 2.

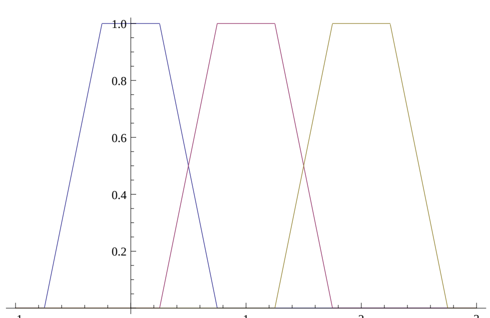

Figure 2: Graph of ρ_0, ρ_1, and ρ_2.

Every real number $x \in \mathbb{R}$ can be uniquely written as

$$x = \lfloor x \rfloor + \langle x \rangle$$

with $\lfloor x \rfloor \in \mathbb{Z}$ and $\langle x \rangle \in [0, 1)$ – respectively the floor of x and fractional part of x. Then we have

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} \rho_n(x) &= \rho_{\lfloor x \rfloor}(x) + \rho_{\lfloor x \rfloor + 1}(x) \\
&= \rho(\langle x \rangle) + \rho(\langle x \rangle - 1) \\
&= \rho(\langle x \rangle) + \rho(1 - \langle x \rangle) \\
&= 1. \quad \square
\end{aligned}$$

Problem 2. Let X be a second-countable locally compact space. Show that X is σ-compact.

Solution. Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be a countable basis for the topology of X. For every $x \in X$, pick a compact neighborhood K_x of x, and pick a basic open neighborhood $B_{n(x)}$ of x inside K_x:

$$x \in B_{n(x)} \subseteq K_x.$$

The set $I = \{n(x) \in \mathbb{N} \mid x \in X\} \subseteq \mathbb{N}$ is countable. Each index $i \in I$ is of the form $n(x)$ for at least one point $x \in X$, though possibly many. For each $i \in I$, pick one such point $x_i \in X$ satisfying $i = n(x_i)$.

We claim $X = \bigcup_{i\in I} K_{x_i}$, which exhibits X as a countable union of compact subsets. For every $x \in X$, we have

$$\begin{aligned} x &\in B_{n(x)} \\ &= B_i \text{ writing } i = n(x) \in I \\ &= B_{n(x_i)} \\ &\subseteq K_{x_i}. \quad \square \end{aligned}$$

Problem 3. Show that any closed subspace $C \subseteq X$ of a paracompact space X is paracompact.

Solution. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover C. Since each U_α is open in C, it can be written as

$$U_\alpha = \widetilde{U}_\alpha \cap C$$

for some $\widetilde{U}_\alpha \subseteq X$ open in X. Then we have the inclusion

$$C \subseteq \bigcup_{\alpha \in A} \widetilde{U}_\alpha$$

and the collection $\{\widetilde{U}_\alpha\}_{\alpha \in A} \cup \{X \setminus C\}$ is an open cover of X. Since X is paracompact, this open cover admits a locally finite open refinement $\{\widetilde{V}_\beta\}_{\beta \in B}$. Writing $V_\beta := \widetilde{V}_\beta \cap C$, the collection $\mathcal{V} = \{V_\beta\}_\beta$ is an open cover of C. We claim that $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}$.

$\mathcal{V}$ is locally finite. Let $x \in C$. There is an X-neighborhood $\widetilde{N}_x$ of x such that $\widetilde{N}_x$ intersects finitely many $\widetilde{V}_\beta$. Then $\widetilde{N}_x \cap C$ is a C-neighborhood of x which intersects finitely many $\widetilde{V}_\beta \cap C = V_\beta$.

$\mathcal{V}$ refines $\mathcal{U}$. For every index $\beta \in B$, there is an index $\alpha = \alpha(\beta) \in A$ satisfying $\widetilde{V}_\beta \subseteq \widetilde{U}_{\alpha(\beta)}$ or possibly $\widetilde{V}_\beta \subseteq X \setminus C$. In the latter case, we have $V_\beta = \widetilde{V}_\beta \cap C = \emptyset$ so that we can ignore those cases.

Using the same index $\alpha = \alpha(\beta)$, the following inclusion holds:

$$V_\beta = \widetilde{V}_\beta \cap C \subseteq \widetilde{U}_{\alpha(\beta)} \cap C = U_{\alpha(\beta)}. \quad \square$$

Problem 4. Let $\{X_i\}_{i \in I}$ be a collection of topological spaces and let $X := \coprod_{i \in I} X_i$ denote their coproduct.

a. Show that an arbitrary coproduct of paracompact spaces is paracompact. In other words, if each X_i is paracompact, then so is X.

Solution. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Each open U_α can be written as

$$U_\alpha = \bigsqcup_{i \in I} U_{\alpha,i}$$

where $U_{\alpha,i} := U_\alpha \cap X_i \subseteq X_i$ is open in X_i.

For every index $i \in I$, the collection $\{U_{\alpha,i}\}_{\alpha \in A}$ is an open cover of X_i. Since X_i is paracompact, this open cover admits a locally finite open refinement $\{V_{\beta,i}\}_{\beta \in B_i}$ for some indexing set B_i. Take the union of all those indexing sets

$$B = \bigsqcup_{i \in I} B_i$$

and the union of all the refinements

$$\mathcal{V} = \{V_\beta\}_{\beta \in B} := \bigsqcup_{i \in I} \{V_{\beta,i}\}_{\beta \in B_i}.$$

Note that $\mathcal{V}$ is an open cover of X. Indeed, each $V_{\beta,i} \subseteq X_i$ is open in X_i and thus in X since X_i is open in X. Moreover, their union is

$$\bigcup_{\beta \in B} V_\beta = \bigcup_{i \in I} \bigcup_{\beta \in B_i} V_{\beta,i} = \bigcup_{i \in I} X_i = X.$$

We claim that $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}$.

$\mathcal{V}$ is locally finite. Each point $x \in X$ lives in exactly one summand $X_i \subseteq X$. There is an X_i-neighborhood $N \subseteq X_i$ of x which intersects only finitely many $V_{\beta,i}$. For any other index $j \neq i$, we have $X_i \cap X_j = \emptyset$ and thus N intersects none of the $V_{\beta,j}$. Moreover, N is an X-neighborhood of x since X_i is open in X.

$\mathcal{V}$ refines $\mathcal{U}$. Each $V_\beta = V_{\beta,i}$ is included in some $U_{\alpha,i} = U_\alpha \cap X_i \subseteq U_\alpha$. $\square$

b. Show that the converse holds: If the coproduct X is paracompact, then so is each summand X_i.

Solution. $X_i \subseteq X$ is closed in X, hence paracompact by Problem 3. $\square$

c. Show that a coproduct of compact spaces is compact if and only if the collection is finite. In other words, assume each X_i is compact, and show that their coproduct X is compact if and only if the indexing set I is finite.

Solution. ($\Rightarrow$) Note that each X_i is open in X, and consider the open cover $\{X_i\}_{i\in I}$ of X. Since X is compact, there is a finite subcover $X = X_{i_1} \cup \ldots \cup X_{i_k}$. But since the union $X = \coprod_{i\in I} X_i$ is disjoint, the subcover must be equal to the original cover. In other words, the equality

$$\coprod_{i\in I} X_i = X_{i_1} \cup \ldots \cup X_{i_k}$$

implies the equality $I = \{i_1, \ldots, i_k\}$, so that I is finite.

($\Leftarrow$) $X = \coprod_{i\in I} X_i$ is a finite union of compact subsets, hence compact. $\square$