MA557 Homework 10

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November 16, 2015

Problem 10.1

Let $\varphi\colon R \to S$ be a homomorphism of rings, ${}_a\varphi\colon \operatorname{Spec} S \to \operatorname{Spec} R$ the induced map of the spectra, and $\mathfrak{p} \in \operatorname{Spec} R$. Show that the fiber $({}_a\varphi)^{-1}(\mathfrak{p})$ is homeomorphic to $\operatorname{Spec}(S \otimes_R k(\mathfrak{p}))$.

Proof. ■

Problem 10.2

Let $R \subset S$ be an integral extension of rings with S a Noetherian ring, and let $\mathfrak{p} \in \operatorname{Spec} R$. Show that there are only finitely many primes in S lying over $\mathfrak{p}$.

Proof. ■

PROBLEM 10.3

Let $\varphi\colon R \to S$ be a homeomorphism of rings with S a Noetherian ring. Show that the following are equivalent:

(i) φ satisfies going up.

(ii) ${}_a\varphi\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is a closed map.

(iii) for every $\mathfrak{q} \in \operatorname{Spec} S$, the induced map $\operatorname{Spec}(S/\mathfrak{q}) \to \operatorname{Spec}(R/\mathfrak{q} \cap S)$ is surjective.

Proof. ■

Problem 10.4

Let $R \subset S$ be an integral extension of domains with R normal, $K = \operatorname{Quot} R$, $\alpha \in S$, $X^n + a_1 X^{n-1} + \cdots + a_n$ the minimal polynomial of α over K (recall $a_i \in R$). Show that for any R-ideal I, $\alpha \in \sqrt{IS}$ if and only if $a_i \in \sqrt{I}$ for $1 \leq i \leq n$.

Proof. ■

Problem 10.5

Let k be a field and $R = k[X_1, ..., X_n]$ a k-algebra. Show that the following are equivalent:

(i) R is a domain with $\dim R = n - 1$

(ii) $R \cong k[X_1, ..., X_n]/(f)$, where $k[X_1, ..., X_n]$ is a polynomial ring and f is an irreducible polynomial.

Proof. ■