

# MA 544: Homework 1

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January 25, 2016



**PROBLEM 1.1 (WHEEDEN & ZYGMUND §2, EX. 1)**

Let  $f(x) = x \sin(1/x)$  for  $0 < x \leq 1$  and  $f(0) = 0$ . Show that  $f$  is bounded and continuous on  $[0, 1]$ , but that  $V[f; 0, 1] = +\infty$ .

*Proof.* By properties of continuous functions, we have  $f$  is continuous on  $(0, 1]$  since it is the product of continuous functions on  $(0, 1]$ . To see that  $f$  is continuous at 0 it suffices to show that  $f(0+) = f(0) = 0$ . To that end, let  $\{x_n\} \subset [0, 1]$  be a sequence such that  $x_n \rightarrow 0$  and consider  $\lim_{n \rightarrow \infty} f(x_n)$ . Since  $x_n \rightarrow 0$ , for every  $\varepsilon > 0$ , there exists a natural number  $N$  such that  $n \geq N$  implies  $|0 - x_n| < \varepsilon$ . Thus, for  $n \geq N$  we have

$$|0 - f(x_n)| = |f(x_n)| = |x_n| |\sin(1/x_n)| \leq \varepsilon |\sin(1/\varepsilon)| \leq \varepsilon.$$

Thus,  $f(x_n) \rightarrow 0$  and we see that  $f(0+) = 0$ . Hence,  $f$  is continuous on  $[0, 1]$ .

It is easy to see that  $f$  is bounded since  $|\sin(1/x)| \leq 1$  for all  $x \in (0, 1]$ . More explicitly, we have

$$|f(x)| \leq |x \sin(1/x)| = |x| \cdot |\sin(1/x)| \leq 1 \cdot 1.$$

Thus,  $|f(x)| \leq 1$  and we see that  $f$  is bounded.

Moreover,  $f$  is continuous on  $(0, 1]$  since it is the product of continuous functions on  $(0, 1]$ . To see that  $f$  is continuous at 0, it suffices to show that  $f(0+) = 0$ . To that end, we shall use the following limiting argument: Let  $\varepsilon > 0$  and consider the limit (from the right) of  $f(\varepsilon)$  as  $\varepsilon \rightarrow 0$ . This is

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = \lim_{\varepsilon \rightarrow 0} \varepsilon \sin(1/\varepsilon) \leq \lim_{\varepsilon \rightarrow 0} |\varepsilon| |\sin(1/\varepsilon)| \leq \lim_{\varepsilon \rightarrow 0} |\varepsilon| \cdot 1 = 0.$$

Thus,  $f(0+) = 0$  and we see that  $f$  is continuous on  $[0, 1]$ .

Last but not least, we show that  $f$  is BV. As of now, I've not been able to find the correct partition to show that the variation of  $f$  blows up, but the informal idea is the following: For any  $M$ , we (should be able to) find a partition  $\Gamma$  of  $[0, 1]$  such that  $\sin(1/x_i) = 1$  for every  $x_i \in \Gamma$ , so we have

$$S_\Gamma = \sum_{i=0}^m x_i \sin(1/x_i) > M$$

and hence,  $V[f; a, b] = +\infty$ . I read online and they show, for the very same problem, that the partition  $\Gamma := \{x_n\} = \{((2n+1)\pi/2)^{-1}\}$  as  $n \rightarrow \infty$  gives  $\int_0^1 f \, d\phi \rightarrow \infty$ , but I cannot see how  $\Gamma$  is a partition of the interval  $[0, 1]$ . Anyway, it is fairly clear that if we have such a partition then

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{2}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi}{2}\right) = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{2}{(2n+1)\pi} = +\infty,$$

by the ratio test since

$$\frac{2/((2n+1)\pi)}{2/((2(n+1)+1)\pi)} = \frac{2n+1}{2n+3} = \frac{n+1}{n+3/2} \cdots$$

which converges to 1 as  $n \rightarrow \infty$ . ■

**PROBLEM 1.2 (WHEEDEN & ZYGMUND §2, EX. 2)**

Prove theorem (2.1).

*Proof.* Recall the statement of theorem (2.1):

**Theorem** (Wheeden & Zygmund, 2.1). (a) If  $f$  is of bounded variation on  $[a, b]$ , then  $f$  is bounded on  $[a, b]$ .

(b) Let  $f$  and  $g$  be of bounded variation on  $[a, b]$ . Then  $cf$  (for any real constant  $c$ ),  $f + g$ , and  $fg$  are of bounded variation on  $[a, b]$ . Moreover,  $f/g$  is of bounded variation on  $[a, b]$  if there exists an  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for  $x \in [a, b]$ .

(a) We shall proceed by contradiction. Suppose that  $f$  is not bounded, i.e., for every positive real number  $M > 0$ , there exists  $x \in [a, b]$  such that  $|f(x)| > M$ . In particular, if  $V$  is the variation of  $f$ , then  $|f(x_0)| > V + (f(a) + f(b))/2$  for some  $x_0 \in [a, b]$ . Then, putting  $\Gamma = \{a, x_0, b\} \subset [a, b]$ , we have

$$\begin{aligned} S_\Gamma &= |f(b) - f(x_0)| + |f(x_0) - f(a)| \\ &= |f(x_0) - f(b)| + |f(x_0) - f(a)| \\ &\geq |2f(x_0) - f(a) - f(b)| \\ &= |2(V + (f(a) + f(b))/2) - f(a) - f(b)| \\ &= |2V + f(a) + f(b) - f(a) - f(b)| \\ &= 2V \\ &> V. \end{aligned}$$

This is a contradiction since  $V$  is the supremum over all such sums.

(b) We shall prove these in the order in which they are listed above.

(i) The constant map  $g(x) := c$  for some real number  $c$  is of BV on  $[a, b]$  and this is easy to see: take any two partitions  $\Gamma = \{x_0, \dots, x_m\}$ , and  $\Gamma' = \{y_0, \dots, y_n\}$  of  $[a, b]$ , then

$$S_\Gamma = \sum_{i=0}^{m-1} |g(x_{i+1}) - g(x_i)| = \sum_{i=0}^{m-1} |c - c| = 0 = \sum_{i=0}^{n-1} |c - c| = \sum_{i=0}^{n-1} |g(y_{i+1}) - g(y_i)| = S_{\Gamma'}.$$

It takes just a few more steps in logic to see that  $V[g; a, b] = 0$ . Therefore, by (iii)  $gf = cf$  is of BV.

(ii) This result follows quite effortlessly from Jordan's theorem, so we shall not trouble ourselves with picking partitions. By Jordan's theorem, there exist bounded increasing functions  $f_1, f_2$ , and  $g_1, g_2$  such that  $f = f_1 - f_2$  and  $g = g_1 - g_2$ . Now, since  $f_1, f_2, g_1, g_2$  are bounded and increasing, the sums  $h_1 = f_1 + g_1$  and  $h_2 = f_2 + g_2$  are bounded and increasing. Thus,

$$f + g = f_1 - f_2 + g_1 - g_2 = (f_1 + g_1) - (f_2 + g_2) = h_1 - h_2,$$

so by Jordan's theorem  $f + g$  is BV on  $[a, b]$ .

- (iii) For this result, Jordan's theorem is not very helpful so we rely on the definition of BV. First, note that by the triangle inequality, for any  $x < y$  in  $[a, b]$ , we have

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |(f(x)g(x) - f(x)g(y)) + (f(x)g(y) - f(y)g(y))| \\ &\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &\leq M|g(x) - g(y)| + N|f(x) - f(y)|, \end{aligned} \quad (1)$$

by part (a), where  $|f| \leq M$  and  $|g| \leq M$  for all  $x \in [a, b]$ . By (1), it follows that for any partition  $\Gamma$  of  $[a, b]$ , we have

$$S_\Gamma[f g; a, b] \leq M S_\Gamma[g; a, b] + N S_\Gamma[f; a, b].$$

Thus, passing to the supremum, we see that

$$V[f g; a, b] \leq M V[g; a, b] + N V[f; a, b] < +\infty,$$

so  $f g$  is BV on  $[a, b]$ .

- (iv) Suppose  $|g(x)| > \varepsilon$  for some  $\varepsilon > 0$  for all  $x \in [a, b]$ . Then, by the triangle inequality, the following estimate holds

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \right| &\leq \left| \frac{g(y)f(x) - g(x)f(y)}{g(x)g(y)} \right| \\ &= \frac{1}{|g(x)g(y)|} |g(y)f(x) - g(x)f(y)| \\ &< \frac{1}{\varepsilon^2} |g(y)f(x) - g(x)f(y)| \\ &< \frac{1}{\varepsilon^2} |g(y)f(x) - g(y)f(y) + g(y)f(y) - g(x)f(y)| \\ &= \frac{1}{\varepsilon^2} |(g(y)f(x) - g(y)f(y)) - (g(x)f(y) - g(y)f(y))| \\ &\leq \frac{1}{\varepsilon^2} (|g(y)||f(x) - f(y)| + |f(y)||g(x) - g(y)|) \\ &\leq \frac{1}{\varepsilon^2} (|g(y)||f(x) - f(y)| + |f(y)|(|g(x)| - |g(y)|)) \\ &\leq \frac{1}{\varepsilon^2} (N|f(x) - f(y)| + M|g(x) - g(y)|). \end{aligned} \quad (2)$$

Hence, for any partition  $\Gamma$  of  $[a, b]$ , we have

$$S_\Gamma[f/g; a, b] \leq \frac{1}{\varepsilon^2} (N S_\Gamma[f; a, b] + M S_\Gamma[g; a, b]).$$

Thus, passing to the supremum, we see that

$$V[f/g; a, b] \leq \frac{1}{\varepsilon^2} (N V[f; a, b] + M V[g; a, b]) < +\infty,$$

so  $f/g$  is BV on  $[a, b]$ . ■

**PROBLEM 1.3 (WHEEDEN & ZYGMUND §2, EX. 3)**

If  $[a', b']$  is a subinterval of  $[a, b]$  show that  $P[a', b'] \leq P[a, b]$  and  $N[a', b'] \leq N[a, b]$ .

*Proof.* Let  $f: [a, b] \rightarrow \mathbf{R}$ . If  $f$  is unbounded, then  $V[f; a, b] = +\infty$  and, by theorem 2.6, the result holds trivially.

Suppose  $f$  is BV on  $[a, b]$ . Then  $V[f; a, b] < +\infty$ . Hence, by theorem 2.2, we have

$$V[f; a', b'] \leq V[f; a, b]. \quad (3)$$

By theorem 2.6, we have

$$N[f; a', b'] = \frac{1}{2}(V[f; a', b'] + f(b') - f(a')) \quad P[f; a', b'] = \frac{1}{2}(V[f; a', b'] - f(b') + f(a'))$$

which, by theorem 2.2, are bounded by

$$\begin{aligned} &\leq \frac{1}{2}(V[f; a, b] - f(b) + f(a)) && \leq \frac{1}{2}(V[f; a, b] - f(b) + f(a)) \\ &= N[f; a, b] && = P[f; a, b], \end{aligned}$$

as desired. ■

**PROBLEM 1.4 (WHEEDEN & ZYGMUND §2, EX. 11)**

Show that  $\int_a^b f \, d\phi$  exists if and only if given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|R_\Gamma - R_{\Gamma'}| < \varepsilon$  if  $|\Gamma|, |\Gamma'| < \delta$ .

*Proof.*  $\Rightarrow$  Suppose that  $I := \int_a^b f \, d\phi$  exists. Then, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any partition  $\Gamma''$  of  $[a, b]$  with  $|\Gamma''| < \delta/2$ ,  $|I - R_{\Gamma''}| < \varepsilon$ . Let  $\Gamma$  and  $\Gamma'$  be a partitions with  $|\Gamma|, |\Gamma'| < \delta/2$ . Then, for the given  $\varepsilon$ , we have  $|I - R_\Gamma| < \varepsilon$  and  $|I - R_{\Gamma'}| < \varepsilon$  from which we have the estimates

$$\begin{aligned} |R_\Gamma - R_{\Gamma'}| &= |-(I - R_\Gamma) + (I - R_{\Gamma'})| \\ &\leq |-(I - R_\Gamma)| + |I - R_{\Gamma'}| \\ &= |I - R_\Gamma| + |I - R_{\Gamma'}| \\ &\leq \delta/2 + \delta/2 \\ &= \delta, \end{aligned}$$

as desired.

$\Leftarrow$  Conversely, suppose that given  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any two partitions  $\Gamma, \Gamma'$  with  $|\Gamma|, |\Gamma'| < \delta$  we have  $|R_\Gamma - R_{\Gamma'}| < \varepsilon/2$ . Put  $I := \int_a^b f \, d\phi$ . Let  $\Gamma'' := \Gamma \cup \Gamma'$ . By Lemma 2.23, we have the following estimates on  $R_{\Gamma''}$

$$L_\Gamma \leq L_{\Gamma''} \leq R_{\Gamma''} \leq U_{\Gamma''} \leq U_\Gamma.$$

Since  $L_{\Gamma''} - L_\Gamma < \varepsilon$  and  $U_{\Gamma''} - L_\Gamma < \varepsilon$ , then

$$R_{\Gamma''} - U_{\Gamma''} \leq U_\Gamma - U_{\Gamma''} \leq \varepsilon$$

and

$$\varepsilon \geq L_{\Gamma''} - L_\Gamma \geq L_{\Gamma''} - R_{\Gamma''} L_{\Gamma''}$$

it follows that  $R_{\Gamma''} \rightarrow I$  as  $|\Gamma''| \rightarrow 0$ ... No, this is not clear at all. ■

**PROBLEM 1.5 (WHEEDEN & ZYGMUND §2, EX. 13)**

Prove theorem (2.16).

*Proof.*

**Theorem** (Wheeden & Zygmund, 2.16). (i) If  $\int_a^b f \, d\phi$  exists, then so do  $\int_a^b cf \, d\phi$  and  $\int_a^b f \, d(c\phi)$  for any constant  $c$ , and

$$\int_a^b cf \, d\phi = \int_a^b f \, d(c\phi) = c \int_a^b f \, d\phi.$$

(ii) If  $\int_a^b f_1 \, d\phi$  and  $\int_a^b f_2 \, d\phi$  both exist, so does  $\int_a^b (f_1 + f_2) \, d\phi$ , and

$$\int_a^b (f_1 + f_2) \, d\phi = \int_a^b f_1 \, d\phi + \int_a^b f_2 \, d\phi.$$

(iii) If  $\int_a^b f \, d\phi_1$  and  $\int_a^b f \, d\phi_2$  both exist, so does  $\int_a^b f \, d(\phi_1 + \phi_2)$ , and

$$\int_a^b f \, d(\phi_1 + \phi_2) = \int_a^b f \, d\phi_1 + \int_a^b f \, d\phi_2.$$

(i) Suppose that  $I := \int_a^b f \, d\phi$  exists and let  $c$  be a constant. Then, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|\Gamma| < \delta$  implies  $|I - R_\Gamma| < \varepsilon/|c|$ . Then, we have

$$R_\Gamma[cf; a, b] = \sum_{i=1}^n cf(\xi_i)[\phi(x_i) - \phi(x_{i-1})] = c \left( \sum_{i=1}^n f(\xi_i)[\phi(x_i) - \phi(x_{i-1})] \right) = cR_\Gamma \quad (4)$$

and

$$R_\Gamma[f; c\phi; a, b] = \sum_{i=1}^n f(\xi_i)[c\phi(x_i) - c\phi(x_{i-1})] = c \left( \sum_{i=1}^n f(\xi_i)[\phi(x_i) - \phi(x_{i-1})] \right) = cR_\Gamma \quad (5)$$

for  $\Gamma = \{x_0, \dots, x_n\}$ .<sup>1</sup> Hence, we have the estimates

$$\begin{aligned} |cI - R_\Gamma[cf; a, b]| &= |cI - cR_\Gamma| \\ &= |c(I - R_\Gamma)| \\ &= |c||I - R_\Gamma| \\ &\leq |c|(\varepsilon/|c|) \\ &= \varepsilon \end{aligned}$$

for  $\delta$  as given. A similar argument (in fact, the same) works for  $R[f; c\phi; a, b]$ . Thus, we have

$$\int_a^b cf \, d\phi = \int_a^b f \, d(c\phi) = c \int_a^b f \, d\phi,$$

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<sup>1</sup>The  $R_\Gamma[f; c\phi; a, b]$  is just made up notation. I can't think of what else to call it.



as desired.

(ii) Suppose that  $I_1 := \int_a^b f_1 d\phi$  and  $I_2 := \int_a^b f_2 d\phi$  exists. Then, for every  $\varepsilon > 0$  there exists  $\delta$  such that if  $\Gamma$  is a partition of  $[a, b]$  with  $|\Gamma| < \delta$  then  $|I_1 - R_\Gamma[f_1; a, b]| < \varepsilon/2$  and  $|I_2 - R_\Gamma[f_2; a, b]| < \varepsilon/2$ . Now, note that

$$\begin{aligned}
 R_\Gamma[f_1 + f_2; a, b] &= \sum_{i=0}^m (f_1(\xi_i) + f_2(\xi_i))[\phi(x_i) - \phi(x_{i-1})] \\
 &= \sum_{i=0}^m (f_1(\xi_i)[\phi(x_i) - \phi(x_{i-1})] + f_2(\xi_i)[\phi(x_i) - \phi(x_{i-1})]) \\
 &= \sum_{i=0}^m f_1(\xi_i)[\phi(x_i) - \phi(x_{i-1})] + \sum_{i=0}^m f_2(\xi_i)[\phi(x_i) - \phi(x_{i-1})] \\
 &= R_\Gamma[f_1; a, b] + R_\Gamma[f_2; a, b].
 \end{aligned} \tag{6}$$

Thus, by (6), we have the following estimates

$$\begin{aligned}
 |(I_1 + I_2) - R_\Gamma[f_1 + f_2; a, b]| &= |(I_1 + I_2) - R_\Gamma[f_1 + f_2; a, b]| \\
 &= |(I_1 + I_2) - (R_\Gamma[f_1; a, b] + R_\Gamma[f_2; a, b])| \\
 &= |(I_1 - R_\Gamma[f_1; a, b]) + (I_2 - R_\Gamma[f_2; a, b])|
 \end{aligned}$$

which, by the triangle inequality, is

$$\begin{aligned}
 &\leq |(I_1 - R_\Gamma[f_1; a, b])| + |(I_2 - R_\Gamma[f_2; a, b])| \\
 &\leq \varepsilon/2 + \varepsilon/2 \\
 &= \varepsilon
 \end{aligned}$$

or  $\delta$  as given. Thus,  $\int_a^b f_1 + f_2 d\phi$  exists and is equal to  $\int_a^b f_1 d\phi + \int_a^b f_2 d\phi$ .

(iii) Suppose  $I_1 := \int_a^b f d\phi_1$  and  $I_2 := \int_a^b f d\phi_2$  exist then for every  $\varepsilon > 0$  there exists  $\delta_1, \delta_2 > 0$  such that for every partition  $\Gamma_1, \Gamma_2$  of  $[a, b]$  with  $|\Gamma_1| < \delta_1$  and  $|\Gamma_2| < \delta_2$  we have  $|I_1 - R_{\Gamma_1}[f; \phi_1; a, b]| < \varepsilon/2$  and  $|I_2 - R_{\Gamma_2}[f; \phi_2; a, b]| < \varepsilon/2$ . Put  $\delta := \min\{\delta_1, \delta_2\}$ . Now, note that

$$\begin{aligned}
 R_\Gamma[f; \phi_1 + \phi_2; a, b] &= \sum_{i=0}^m f[(\phi_1(x_i) + \phi_2(x_i)) - (\phi_1(x_{i-1}) + \phi_2(x_{i-1}))] \\
 &= \sum_{i=0}^m f[(\phi_1(x_i) - \phi_1(x_{i-1})) + (\phi_2(x_i) - \phi_2(x_{i-1}))] \\
 &= \sum_{i=0}^m f[(\phi_1(x_i) - \phi_1(x_{i-1}))] + \sum_{i=0}^m f[(\phi_2(x_i) - \phi_2(x_{i-1}))] \\
 &= R_\Gamma[f; \phi_1; a, b] + R_\Gamma[f; \phi_2; a, b].
 \end{aligned} \tag{7}$$

Hence, we have the following estimates

$$\begin{aligned}
 |(I_1 + I_2) - R_\Gamma[f; \phi_1 + \phi_2; a, b]| &= |(I_1 + I_2) - (R_\Gamma[f; \phi_1; a, b] + R_\Gamma[f; \phi_2; a, b])| \\
 &= |(I_1 - R_\Gamma[f; \phi_1; a, b]) + (I_2 - R_\Gamma[f; \phi_2; a, b])|
 \end{aligned}$$

which, by the triangle inequality, is

$$\begin{aligned} &\leq |I_1 - R_\Gamma[f; \phi_1; a, b]| + |I_2 - R_\Gamma[f; \phi_2; a, b]| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Thus,  $\int_a^b f \, d(\phi_1 + \phi_2)$  exists and it is equal to the sum  $\int_a^b f \, d\phi_1 + \int_a^b f \, d\phi_2$ . ■