MA 523: Homework 1

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Problem 1.1 (Taylor's formula)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to \mathbf{0}$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. ▸ Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Proof of claim. We shall proceed by induction on m. The case $m = 1$ follows easily from the chain rule:

$$\begin{aligned}
\frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\
&= D^{(1,0,\ldots,0)} f(tx) x_1 + \cdots + D^{(0,\ldots,0,1)} f(tx) x_n \\
&= \big(D^{(1,0,\ldots,0)} x_1 + \cdots + D^{(0,\ldots,0,1)} x_n\big) f(tx)
\end{aligned}$$

which we can write compactly as

$$= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Now, assume the result for $n \leq m - 1$. Then

$$\begin{aligned}
\frac{d^m}{dt^m} g(t) &= \frac{d}{dt}\left[\frac{d^{m-1}}{dt^{m-1}} g(t)\right] \\
&= \frac{d}{dt}\left[\sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} D^\alpha f(tx) x^\alpha\right]
\end{aligned}$$

since the derivative is a linear operator, we have

$$\begin{aligned}
&= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \frac{d}{dt}\big[D^\alpha f(tx) x^\alpha\big] \\
&= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \sum_{|\beta|=1} D^{\alpha+\beta} f(tx) x^{\alpha+\beta}
\end{aligned}$$

but since f is smooth, the order in which we take derivatives does not matter and, hence the operators commute giving us

$$= \left[\sum_{|\beta|=1} (Dx)^\beta \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha \right] f(tx). \tag{1.1}$$

From here it suffices to do some combinatorics on the operators and reduce it to the desired expression. By the multinomial theorem, we have

$$\left(\sum\nolimits_{|\alpha'|=1} (Dx)^{\alpha'} \right)^{m-1} = \sum_{|\alpha|=m-1} \binom{|\alpha|}{\alpha} (Dx)^\alpha = \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha.$$

Thus (1.1) becomes

$$\begin{aligned} \left[\sum_{|\beta|=1} (Dx)^\beta \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha \right] f(tx) &= \left[\sum_{|\beta|=1} (Dx)^\beta \left(\sum\nolimits_{|\alpha'|=1} (Dx)^{\alpha'} \right)^{m-1} \right] f(tx) \\ &= \left[\left(\sum\nolimits_{|\beta|=1} (Dx)^\beta \right)^m \right] f(tx) \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} (Dx)^\alpha f(tx) \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha, \end{aligned}$$

as desired. ■

Now, applying Taylor's formula in 1 variable to $g(t)$ and evaluating at $t = 1$ we have

$$\begin{aligned} f(x) &= g(1) \\ &= \sum_{i=0}^{k} \frac{g^{(i)}(0)}{i!} 1^i + \mathrm{O}\left(|x|^{k+1}\right) \\ &= \sum_{i=0}^{k} \frac{1}{i!} \sum_{|\alpha|=i} \frac{i!}{\alpha!} D^\alpha f(tx) x^\alpha + \mathrm{O}\left(|x|^{k+1}\right) \\ &= \sum_{i=0}^{k} \sum_{|\alpha|=i} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\left(|x|^{k+1}\right) \\ &= \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\left(|x|^{k+1}\right) \end{aligned}$$

as desired. ◂

Problem 1.2

Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ▸ For reference, formula (5) in §2.1.2 of [E] is the solution to the nonhomogeneous problem

$$u(x,t) = g(x - tb) + \int_0^1 f\big(x + (s-t)b, s\big)\, ds$$

where $x \in \mathbb{R}^n$, $t > 0$.

Using the structure of characteristic ODE, we have

$$\begin{aligned}
\dot{p}(s) &= -D_x F\big(p(s), z(s), x(s)\big) - D_z F\big(p(s), z(s), x(s)\big) p(s) \\
\dot{z}(s) &= D_p F\big(p(s), z(s), x(s)\big) \cdot p(s) \\
\dot{x}(s) &= D_p F\big(p(s), z(s), x(s)\big).
\end{aligned}$$

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PROBLEM 1.3

Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ►

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Problem 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\left(u_{x_1}^2 + u_{x_2}^2\right)$$

find a solution with $u(x_1, 0) = \left(1 - x_1^2\right)/2$.

Solution. ►

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