# MA 544: Homework 7

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## Problem 7.1 (Wheeden & Zygmund §4, Ex. 9)

(a) Show that the limit of a decreasing (increasing) sequence of functions usc (lsc) at $\mathbf{x}_0$ is usc (lsc) at $\mathbf{x}_0$. In particular, the limit of a decreasing (increasing) sequence of functions continuous at $\mathbf{x}_0$ is usc (lsc) at $\mathbf{x}_0$.

(b) Let $f$ be usc and less than $\infty$ on $[a, b]$. Show that there exists continuous $f_k$ on $[a, b]$ such that $f_k \searrow f$.

*Proof.* ∎

## Problem 7.2 (Wheeden & Zygmund §4, Ex. 11)

Let $f$ be defined on $\mathbf{R}^n$ and let $B(\mathbf{x})$ denote the open ball $\{\, \mathbf{y} \mid |\mathbf{x} - \mathbf{y}| < r \,\}$ with center $\mathbf{x}$ and fixed radius $r$. Show that the function $g(\mathbf{x}) = \sup\{\, f(\mathbf{y}) \mid \mathbf{y} \in B(\mathbf{x}) \,\}$ is usc on $\mathbf{R}^n$. Is the same true for the closed ball $\{\, \mathbf{y} \mid |\mathbf{x} - \mathbf{y}| \le r \,\}$?

*Proof.* ■

## Problem 7.3 (Wheeden & Zygmund §4, Ex. 15)

Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set $E$ with $|E| < \infty$. If $|f_k(M)| \leq M < \infty$ for all $k$ for each $\mathbf{x} \in E$, show that given $\varepsilon > 0$, there is closed $F \subset E$ and finite $M$ such that $|E \setminus F| < \varepsilon$ and $|f - k(\mathbf{x})| \leq M$ for all $\mathbf{x} \in F$.

*Proof.* ■

## Problem 7.4 (Wheeden & Zygmund §4, Ex. 18)

If $f$ is measurable on $E$, define $\omega_f(a) = |\{ f > a \}|$ for $-\infty < a < \infty$. If $f_k \nearrow f$, show that $\omega_{f_k} \nearrow \omega_f$. If $f_k \to f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of $\omega_f$. [For the second part, show that if $f_k \to f$, then $\overline{\lim}_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$ and $\underline{\lim}_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$ for every $\varepsilon > 0$.]

*Proof.* ∎

## Problem 7.5 (Wheeden & Zygmund §5, Ex. 1)

If $f$ is a simple measurable function (not necessarily positive) taking values $a_j$ on $E_j$, $j = 1, ..., N$, show that $\int_E f = \sum_{j=1}^N a_j |E_j|$. [Use (5.24)].

*Proof.* ■

## Problem 7.6 (Wheeden & Zygmund §5, Ex. 2)

Show that the conclusion of (5.32) are not true without the assumption that $\varphi \in L(E)$. [In part (ii), for example, take $f_k = \chi_{(k,\infty)}$.]

*Proof.* ■

## Problem 7.7 (Wheeden & Zygmund §5, Ex. 3)

Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on $E$. If $f_k \to f$ and $f_k \leq f$ a.e. on $E$, show that $\int_E f_k \to \int_E f$.

*Proof.* ■

## Problem 7.8 (Wheeden & Zygmund §5, Ex. 4)

If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, ...$, and $\int_0^1 x^k f(x)\, dx \to 0$.

*Proof.* ■