

# MA 523: Homework 7

Carlos Salinas

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## PROBLEM 7.1

Solve the Dirichlet problem for the Laplace equation in  $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ u = x_1 & \text{on } |x| = 1, \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

(Hint: Use Laurent series.)

*SOLUTION.* First, let us make the change of variables  $(x_1, x_2) \mapsto re^{i\theta}$  to the Dirichlet problem in question:

$$\begin{cases} \Delta u = 0 & \text{in } 1 < r < 2, \\ u = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) & \text{on } r = 1, \\ u = 1 + \frac{1}{i}(e^{i2\theta} - e^{-i2\theta}) & \text{on } r = 2. \end{cases} \quad (7.1)$$

Now, suppose  $u$  is a solution, of the form

$$u(re^{i\theta}) = b \ln r + \sum_{n \in \mathbb{Z}} (a_n r^n + \overline{a_{-n}} r^{-n}) e^{in\theta},$$

to the problem (7.1). It is easy to see that  $u$  is in fact harmonic:

$$\begin{aligned} \Delta u &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \\ &= -br^{-2} + br^{-2} + \sum_{n \in \mathbb{Z}} [(n(n-1) + n - n^2) a_n r^n \\ &\quad + (n(n-1) + n - n^2) \overline{a_{-n}} r^{-n}] e^{in\theta} \\ &= 0. \end{aligned}$$

Next we use the boundary data to compute the coefficients  $a_n$ ,  $n \in \mathbb{Z}$ . Using the data (7.1) on  $r = 1$ , we have

$$\frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \sum_{n \in \mathbb{Z}} (a_n + \overline{a_{-n}}) e^{in\theta} \quad (7.2)$$

and on  $r = 2$ ,

$$1 + \frac{1}{i}(e^{i2\theta} - e^{-i2\theta}) = b \ln 2 + \sum_{n \in \mathbb{Z}} (2^n a_n + 2^{-n} \overline{a_{-n}}) e^{in\theta}. \quad (7.3)$$

These equations immediately tell us that  $b = 1/\ln 2$ . Moreover, the following relations on the coefficients hold

$$\begin{cases} \frac{1}{2} = a_1 + \overline{a_{-1}} & \frac{1}{2} = a_{-1} + \overline{a_1}, \\ \frac{1}{i} = 2^2 a_2 + 2^{-2} \overline{a_{-2}}, & -\frac{1}{i} = 2^2 a_{-2} + 2^{-2} \overline{a_2}, \\ 0 = a_n + \overline{a_{-n}} & \text{for } n \neq \pm 1, \\ 0 = 2^n a_n + 2^{-n} \overline{a_{-n}} & \text{for } n \neq \pm 2. \end{cases}$$

A little calculation shows that

$$\begin{cases} a_1 = -\frac{1}{6}, & a_{-1} = \frac{2}{3}, \\ a_2 = -\frac{4i}{15}, & a_{-2} = -\frac{4i}{15}, \\ a_n = 0 & \text{for } n \neq \pm 1, \pm 2. \end{cases}$$

Thus,

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{\ln 2} \ln r + \left(-\frac{4i}{15}r^{-2} + \frac{4i}{15}r^2\right)e^{-i2\theta} + \left(\frac{2}{3}r^{-1} - \frac{1}{6}r\right)e^{-i\theta} \\ &\quad + \left(-\frac{1}{6}r + \frac{2}{3}r^{-1}\right)e^{i\theta} + \left(-\frac{4i}{15}r^2 + \frac{4i}{15}r^{-2}\right)e^{i2\theta} \\ &= \frac{1}{\ln 2} \ln r - \frac{8}{15}r^{-4} \left(\frac{r^2e^{i2\theta} - r^2e^{-i2\theta}}{2i}\right) + \frac{8}{15} \left(\frac{r^2e^{i2\theta} - r^2e^{-i2\theta}}{2i}\right) \\ &\quad + \frac{4}{3}r^{-2} \left(\frac{re^{i\theta} + re^{-i\theta}}{2}\right) - \frac{1}{3}r \left(\frac{re^{i\theta} + re^{-i\theta}}{2}\right). \end{aligned}$$

Substituting back, we have

$$u(x_1, x_2) = \frac{1}{\ln 2} \ln(x_1^2 + x_2^2) - \frac{16x_1x_2}{15(x_1^2 + x_2^2)^2} + \frac{16x_1x_2}{15} + \frac{4x_1}{3(x_1^2 + x_2^2)} - \frac{(x_1^2 + x_2^2)^{\frac{1}{2}}x_1}{3}. \quad (7.4)$$

It is easy to see that (??) satisfies the boundary data at  $|x| = 1$  and  $|x| = 2$ . ■

## PROBLEM 7.2

Let  $\Omega$  be a bounded domain with a  $C^1$  boundary,  $g \in C^2(\partial\Omega)$  and  $f \in C(\bar{\Omega})$ . Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \quad (*)$$

where  $\nu$  is the outer normal on  $\partial\Omega$ . Show that the solution of  $(*)$  in  $C^2(\Omega) \cap C^1(\bar{\Omega})$  is unique up to a constant; i.e., if  $u_1$  and  $u_2$  are both solutions of  $(*)$ , then  $u_2 = u_1 + \text{const.}$  in  $\Omega$ . (*Hint:* Look at the proof of the uniqueness for the Dirichlet problem by energy methods [E, 2.2.5a].)

*SOLUTION.* Suppose  $u_1$  and  $u_2$  are solutions to the Neumann problem  $(*)$ . Define  $v := u_1 - u_2$ . Then  $v$  is harmonic in  $\Omega$  and  $\partial v / \partial \nu = 0$  on  $\partial\Omega$ . Consider the energy functional

$$E[v] = \frac{1}{2} \int_{\Omega} |Dv|^2 dx.$$

By Green's formula version (ii),

$$\begin{aligned} E[v] &= \frac{1}{2} \int_{\Omega} |Dv|^2 dx \\ &= -\frac{1}{2} \int_{\Omega} v \Delta v dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} v dS(x) \\ &= 0. \end{aligned}$$

This implies that  $|Dv|^2 = Dv \cdot Dv = 0$  which, since the standard inner product on  $\mathbb{R}^n$  is positive-definite, implies that  $Dv \equiv 0$ . It follows that  $u_1 = u_2 + \text{const.}$ , i.e., the solution  $u$  to  $(*)$  is unique up to a constant. ■

## PROBLEM 7.3

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta_x u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where  $c \in \mathbb{R}$ .

(*Hint:* Rewrite the problem in terms of  $v(x, t) := e^{ct}u(x, t)$ .)

*SOLUTION.* Taking the hint, let us rewrite the problem in terms of  $v(x, t) = e^{ct}u(x, t)$ :

$$\begin{cases} v_t - \Delta_x v = e^{ct}f & \text{in } \mathbb{R}^n \times (0, \infty), \\ v = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \quad (7.5)$$

By remark (17) from [E, §2.3.c], the solution to this problem has the form

$$v(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t)g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s)e^{cs}f(y, s) dy ds,$$

where  $\Phi$  is the fundamental solution to the heat equation. Thus, the formula

$$u(x, t) = e^{-ct} \int_{\mathbb{R}^n} \Phi(x - y, t)g(y) dy + e^{-ct} \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s)e^{cs}f(y, s) dy ds$$

solves the original problem. ■