

MRC 2016 Report: Tropicalization of Character Varieties

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1 Overview

Tropical geometry is a new and exciting area of mathematics that is best described as piece-wise linear algebraic geometry. In tropical geometry the sum of two real numbers $x \oplus y$ is their maximum and the product $x \odot y$ their sum. This, together with a minimum element $-\infty$, gives us the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$. In the tropical setting, polynomials become piece-wise linear functions and algebraic varieties give way to tropical varieties—which are in some sense “skeletons” of the original variety.

During the introductory talks by Manon, we decided to try our hand at “tropicalizing” some of the $\mathrm{SL}(2, \mathbb{C})$ -character varieties that were presented Lawton’s talk and, further, tropicalize $\mathrm{PSL}(2, \mathbb{C})$ -, $\mathrm{SL}(3, \mathbb{C})$ -, an $\mathrm{Sp}(4)$ -character varieties. We broke out—loosely—into two groups: one concerned with the tropicalization of $\mathrm{PSL}(2, \mathbb{C})$ -character varieties and the other with visualization of the Newton polytopes that came about from tropicalization.

Here is a summary of the observations made by the group.

2 Tropicalization of $\mathfrak{X}(F_3, \mathrm{SL}(2, \mathbb{C}))$

From Lawton [1], we deduced that the character variety $\mathfrak{X}(F_3, \mathrm{SL}(2, \mathbb{C}))$ is cut out by the polynomial in 7 variables

$$\begin{aligned} f = & X_1 X_2 X_3 X_7 - X_4^2 + X_5^2 + X_1^2 \\ & + X_2^2 + X_3^2 + X_7^2 \\ & + X_1 X_6 X_7 + X_2 X_5 X_7 + X_3 X_4 X_7 \\ & + X_1 X_2 X_4 + X_1 X_3 X_5 + X_2 X_3 X_6 - 4. \end{aligned} \tag{1}$$

With the help of **gfan**—a software package for computing Gröbner fans and tropical varieties [1]—and **Mathematica** we were able to find $\mathrm{Trop}(\mathfrak{X}(F_3, \mathrm{SL}(2, \mathbb{C})))$. Its tropicalization is the codimension 1-cones of the dual fan to the Newton polytope $\mathrm{Newt}(f)$. Figure 1 is a picture made using the *TikZ* language to draw the edge graph of $\mathrm{Newt}(f)$

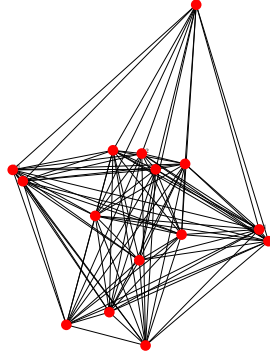


Figure 1: Edge graph of the Newton polytope of $\mathfrak{X}(F_3, \mathrm{SL}(2, \mathbb{C}))$.

References

- [1] JENSEN, A. N. Gfan, a software system for Gröbner fans and tropical varieties. Available at <http://home.imf.au.dk/jensen/software/gfan/gfan.html>.