

MA544: Qual Problems

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## MA 544 (Spring 2016)

## 1.1 Exam 1 Prep

**Problem 1.** Let  $E \subset \mathbb{R}^n$  be a measurable set,  $r \in \mathbb{R}$  and define the set  $rE = \{r\mathbf{x} : \mathbf{x} \in E\}$ . Prove that  $rE$  is measurable, and that  $|rE| = |r|^n|E|$ .

*Proof.* Define a map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $T\mathbf{x} := r\mathbf{x}$ . Note that  $T$  is *Lipschitz continuous* since for any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , the equality

$$|T\mathbf{x} - T\mathbf{y}| = |r\mathbf{x} - r\mathbf{y}| = |r||\mathbf{x} - \mathbf{y}| \quad (1)$$

is satisfied. By Theorem 3.33 from [? , §3.3, p.55], the image of  $E$  under  $T$  is measurable. Moreover, by Theorem 3.35 [? , §3.3, p. 56], since  $T$  is linear, it follows that  $|T(E)| = |\det T||E|$  where  $\det T = |r|^n$ . Lastly, we note that the image of  $E$  under  $T$  is precisely the set  $rE$  so that  $|T(E)| = |rE| = |r|^n|E|$ , as was to be shown. ■

**Problem 2.** Let  $\{E_k\}$ ,  $k \in \mathbb{N}$  be a collection of measurable sets. Define the set

$$\liminf_{k \rightarrow \infty} E_k = \bigcup_{k=1}^{\infty} \left( \bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|.$$

*Proof.* Following the style of [? , §1.1, p. 2], particularly, the sets defined after the introduction of equation (1.1), set

$$V_k := \bigcap_{\ell=k}^{\infty} E_{\ell}. \quad (2)$$

Note that the collection of sets  $\{V_k\}$  forms an increasing sequence, that is, if  $\mathbf{x} \in V_k$  then, by (1.1.0.2),  $\mathbf{x}$  is in the intersection  $E_k \cap (\bigcap_{\ell=k+1}^{\infty} E_{\ell})$ , but, by (1.1.0.2),  $\bigcap_{\ell=k+1}^{\infty} E_{\ell} = V_{k+1}$  thus,  $\mathbf{x}$  is in  $V_{k+1}$  so  $V_{k+1} \supset V_k$ . Hence, we have  $V_k \nearrow \liminf E_k$ .

Now, consider the sequence  $\{|V_k|\}$  formed by the Lebesgue measure of the  $V_k$ . By Theorem 3.26 from [? , §3.3, p. 51], since  $V_k \nearrow \liminf E_k$ ,

$$\lim_{k \rightarrow \infty} |V_k| = \lim_{k \rightarrow \infty} \left| \bigcap_{\ell=k}^{\infty} E_{\ell} \right| = \left| \liminf_{k \rightarrow \infty} E_k \right|. \quad (3)$$

But note that, by the monotonicity of the Lebesgue measure, we have

$$\left| \bigcap_{\ell=k}^{\infty} E_{\ell} \right| \leq |E_k|, \quad (4)$$

so, by properties of the  $\liminf$ , in particular, by Theorem 19(v) from [? , §1.5, p. 23], we have

$$\limsup_{k \rightarrow \infty} |V_k| \leq \liminf_{k \rightarrow \infty} |E_k|. \quad (5)$$

Hence, by (1.1.0.3) and Proposition 19 (iv), since the sequence  $\{|V_k|\}$  converges and is equal to the measure of  $\liminf E_k$ , by (1.1.0.5), we have

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k| \quad (6)$$

as was to be shown. ■

**Problem 3.** Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here  $B(\mathbf{0}, r) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r\}$ . Prove that  $F$  is monotonic increasing and continuous.

*Proof.* Define the linear map  $T: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $T(r)\mathbf{x} := r\mathbf{x}$ . We claim that  $B(\mathbf{0}, r) = T(r, B(\mathbf{0}, 1))$ . To reduce notation, set  $B_1 := B(\mathbf{0}, 1)$  and  $B_r := B(\mathbf{0}, r)$ .

*Proof of claim.*  $\subset$ : Let  $\mathbf{x} \in B_r$ . Then  $|\mathbf{x}| < r$  so  $|\mathbf{x}|/r < 1$ . Thus,  $|\mathbf{x}|/r \in B_1$  so it is in the image of  $B_1$  under the map  $T(r, -)$ .

$\supset$ : On the other hand, suppose  $\mathbf{x} \in T(r, B_1)$ . Then  $\mathbf{x} = r\mathbf{y}$  for some  $\mathbf{y} \in B_1$ . Then, since  $|\mathbf{y}| < 1$ ,  $|\mathbf{x}| = r|\mathbf{y}| < r$  so  $\mathbf{x} \in B_r$ . ♣

From the claim, we see that  $F(x) = |T(x, B(\mathbf{0}, 1))|$  which, by Problem 1, is nothing more than the polynomial  $|B_1|x^n$ . It is clear, from this equivalence, that  $F$  is monotonically increasing: Take  $x, y \in [0, \infty)$  such that  $x < y$ , then  $x^n < y^n$  so

$$F(x) = |B_1|x^n < |B_1|y^n = F(y). \quad (7)$$

Thus,  $F$  is monotonically increasing.

In the argument above, since  $F(x) = |B_1|x^n$  is a polynomial in  $[0, \infty)$  (and polynomials are continuous on  $\mathbb{R}$ )  $F$  is continuous on  $[0, \infty)$ . ■

**Problem 4.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $C$  be the set of all points at which  $f$  is continuous. Show that  $C$  is a set of type  $G_\delta$ .

*Proof.* From the topological definition of continuity,  $f$  is continuous at  $x \in C$  if and only if for every neighborhood  $U$  of  $f(x)$ , the preimage  $f^{-1}(U)$  is a neighborhood of  $x$ . Now, ■

Let  $x \in C$ . Then, by the definition of continuity, for every natural number  $n > 0$  there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies

$$|f(x) - f(x')| < \frac{1}{2n}. \quad (8)$$

Let  $x'', x' \in B(x, \delta)$ . Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x'')| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \quad (9)$$

In view of these estimates, define the set

$$A_n = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \quad (10)$$

Good Lord, that was a long definition! We claim that  $C = \bigcap_{n=1}^{\infty} A_n$  and that  $A_n$  is open for all  $n$ .

First, let us show that  $C = \bigcap_{n=1}^{\infty} A_n$ . Let  $x \in C$ . Then for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . Thus,  $x \in A_n$  for all  $n$  so  $x \in \bigcap A_n$ . On the other hand, if  $x \in \bigcap A_n$  for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . Fix  $\varepsilon > 0$ . By the Archimedean principle, there exists  $N > 0$  such that  $\varepsilon > 1/N$ . Then, since  $x \in A_N$  it follows that for some  $\delta' > 0$ ,  $|x - x'| < \delta'$  implies  $|f(x) - f(x')| < 1/N < \varepsilon$ . Thus,  $x \in C$  and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$ .

Lastly, we show that  $A_n$  is open. Let  $x \in A_n$ . Then there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . In particular, this means that  $B(x, \delta) \subset A_n$  for any  $x' \in B(x, \delta)$  satisfies  $|f(x) - f(x')| < 1/n$ . Thus,  $A_n$  is open and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$  is a  $G_\delta$  set.

**Problem 5.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Is it true that if the sets  $\{f = r\}$  are measurable for all  $r \in \mathbb{R}$ , then  $f$  is measurable?

*Proof.* No. Recall that, by definition, or 4.1,  $f$  is measurable if and only if  $\{f > a\}$  for all  $a \in \mathbb{R}$ . ■

**Problem 6.** Let  $\{f_k\}_{k=1}^{\infty}$  be a sequence of measurable functions on  $\mathbb{R}$ . Prove that the set  $\{x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists}\}$  is measurable.

*Proof.* The idea here should be to rewrite

$$E = \left\{ x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists} \right\} \quad (11)$$

as a countable union/intersection of measurable sets. Let  $x \in E$ . By the Cauchy criterion, for every  $N > 0$  there exists a positive integer  $M$  such that  $m, n \geq M$  implies  $|f_n(x) - f_m(x)| < 1/N$ . With this in mind, define

$$E_N = \left\{ x : \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \quad (12)$$

Then, like for Problem 1.4, it is not too hard to see that the  $E_n$ 's are open and that  $E = \bigcap_{n=1}^{\infty} E_n$ . Thus,  $E$  is a  $G_\delta$  set and therefore measurable. ■

**Problem 7.** A real valued function  $f$  on an interval  $[a, b]$  is said to be *absolutely continuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Show that an absolutely continuous function on  $[a, b]$  is of bounded variation on  $[a, b]$ .

*Proof.* Suppose  $f: [a, b] \rightarrow \mathbb{R}$  is absolutely continuous. Then for fixed  $\varepsilon = 1$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , we have  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Let  $\Gamma = \{x_k\}_{k=1}^N$  be a partition of  $[a, b]$  into closed intervals such that  $x_{k+1} - x_k < \delta$ , then by absolute continuity we have

$$V[f; \Gamma] = \sum_{k=1}^N |f(x_{k+1}) - f(x_k)| < 1. \quad (13)$$

Thus,  $f$  is b.v. on  $[a, b]$ . ■

**Problem 8.** Let  $f$  be a continuous function from  $[a, b]$  into  $\mathbb{R}$ . Let  $\chi_{\{c\}}$  be the characteristic function of a singleton  $\{c\}$ , that is,  $\chi_{\{c\}}(x) = 0$  if  $x \neq c$  and  $\chi_{\{c\}}(c) = 1$ . Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}.$$

*Proof.* ■

## 1.2 Exam 1

### 1.3 Exam 2 Prep

**Problem 1.** Define for  $\mathbf{x} \in \mathbb{R}^n$ ,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that  $f$  is integrable outside any ball  $B_\varepsilon(\mathbf{0})$ , and that there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x}) d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

*Proof.* Recall that a real-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is (Lebesgue) integrable over a subset  $E$  of  $\mathbb{R}^n$  (or, alternatively,  $f$  belongs to  $L(E)$ ) if

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty.$$

Put  $E = \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$ . Then, to show that  $f$  belongs to  $L(E)$  it suffices to prove the inequality

$$\int_E f(\mathbf{x}) d\mathbf{x} < \frac{C}{\varepsilon} \quad (1)$$

for some appropriate constant  $C$ . We proceed by directly computing the Lebesgue integral of  $f$  and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x}) d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{(x_1^2 + \cdots + x_n^2)^{(n+1)/2}} \end{aligned}$$

let  $E_i$  denote the projection of  $E$  onto its  $i$ -th coordinate and make the trigonometric substitution  $x_1 = \sqrt{x_2^2 + \cdots + x_n^2} \tan \theta$ ,  $dx_1 = \sqrt{x_2^2 + \cdots + x_n^2} \sec^2 \theta d\theta$  with  $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$  giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[ \frac{\cos^{n-1} \theta}{(x_2^2 + \cdots + x_n^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{(x_2^2 + \cdots + x_n^2)^{n/2}} \left[ \int_{E_\theta} \cos^{n-1} \theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1} \theta d\theta < \infty. \quad (2)$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned} \int \cdots \int_E f(\mathbf{x}) d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{x_n^2} \\ &= 2C' \int_{\varepsilon}^{\infty} \frac{dx_n}{x_n^2} \\ &= \frac{C}{\varepsilon} \end{aligned} \tag{3}$$

as desired. ■

**Problem 2.** Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $\mathbb{R}^n$ , and assume that  $f_k$  converges pointwise almost everywhere to a function  $f$ . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets  $E$  of  $\mathbb{R}^n$ . Moreover, show that this is not necessarily true if  $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$ .

*Proof.* This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit  $f$  of  $\{f_k\}$  must be nonnegative a.e. in  $\mathbb{R}^n$ . For assume otherwise. Then there exists a collection of points  $\mathbf{x}$  in  $\mathbb{R}^n$  of nonzero  $\mathbb{R}^n$ -Lebesgue measure such that  $f(\mathbf{x}) < 0$ . But  $f_k(\mathbf{x}) \geq 0$  for all  $k \in \mathbb{N}$ . Set  $0 < \varepsilon < |f(\mathbf{x})|$  then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{4}$$

for all  $k$  which contradicts our assumption that  $f_k \rightarrow f$  a.e. on  $\mathbb{R}^n$ . Therefore, the set of points  $\mathbf{x} \in \mathbb{R}^n$  where  $f(\mathbf{x}) < 0$  must have measure zero.

Now, based on pointwise convergence a.e. to  $f$ , given  $\varepsilon > 0$  for a.e.  $\mathbf{x} \in \mathbb{R}^n$  we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{5}$$

for sufficiently large  $k$ ; say  $k$  greater than or equal to some index  $N \in \mathbb{N}$ . Moreover, we are given convergence in  $L(\mathbb{R}^n)$  of  $f_k$  to  $f$

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f < \infty. \tag{6}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{7}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{8}$$



for all  $k \in \mathbb{N}$ . By Theorem 5.5(ii),  $f$  and the  $f_k$ 's are finite a.e. in  $\mathbb{R}^n$  so for some sufficiently large real number  $M$ ,  $|f|, |f_k| \leq M$  for a.e.  $\mathbf{x} \in \mathbb{R}^n$ . In particular, for any measurable subset  $E$  of  $\mathbb{R}^n$ ,  $|f|, |f_k| \leq M$  for a.e.  $\mathbf{x} \in E$  so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \rightarrow \int_E f < \infty. \quad (9)$$

However, if  $f$  does not belong to  $L(\mathbb{R}^n)$ , i.e., its integral over  $\mathbb{R}^n$  is infinity, there is no guarantee that  $f$  will be finite a.e. in  $\mathbb{R}^n$ . This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset  $E$  of  $\mathbb{R}^n$ . Let us demonstrate this with an example. Consider the sequence of functions ■

**Problem 3.** Assume that  $E$  is a measurable set of  $\mathbb{R}^n$ , with  $|E| < \infty$ . Prove that a nonnegative function  $f$  defined on  $E$  is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}| < \infty.$$

*Proof.* If  $f$  is integrable over a measurable subset  $E$  of  $\mathbb{R}^n$ , then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \quad (10)$$

Set  $E_k = \{\mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k\}$  and  $F_k = \{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}$ . Note the following properties about the sets we have just defined: first, the  $E_k$ 's are pairwise disjoint and the  $F_k$ 's are nested in the following way  $F_{k+1} \subset F_k$ ; second,  $E = \bigcup_{k=1}^{\infty} E_k$  and  $E_k = F_k \setminus F_{k+1}$ . By Theorem 3.23, since the  $E_k$ 's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \quad (11)$$

Now, since  $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$  on  $E_k$ , we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \quad (12)$$

Then we have the following upper and lower estimates on the integral of  $f$  over  $E$

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \quad (13)$$

But note that  $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$  by Corollary 3.25 since the measures of  $E_k$ ,  $F_k$ , and  $F_{k+1}$  are all finite. Hence, (1.3.0.13) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \quad (14)$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned} \tag{16}$$

Thus, from (1.3.0.15) and (1.3.0.16)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \tag{17}$$

so the integral  $\int_E f$  converges if and only if the sum  $\sum_{k=0}^{\infty} |F_k|$  converges. ■

**Problem 4.** Suppose that  $E$  is a measurable subset of  $\mathbb{R}^n$ , with  $|E| < \infty$ . If  $f$  and  $g$  are measurable functions on  $E$ , define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$  if and only if  $f_k$  converges to  $f$  as  $k \rightarrow \infty$ .

*Proof.*  $\implies$  : First note that  $\rho$  is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$ . Then, given  $\varepsilon > 0$  there exist an

sufficiently large index  $N$  such that for every  $k \geq N$  we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \quad (18)$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in  $E$  which happens if  $|f_k - f| = 0$  a.e. in  $E$ .

$\Leftarrow$  : Suppose that  $f_k \rightarrow f$  as  $k \rightarrow \infty$ .

I don't know how to solve this. This is the intended solution:

$\Rightarrow$  : Given  $\varepsilon > 0$ ,  $\rho(f_k, f) \rightarrow 0$  implies that

$$\int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \rightarrow 0.$$

Observe that the function  $\Phi: \mathbb{R}^+ \rightarrow \mathbb{R}$  given by  $\Phi(x) = x/(1+x)$  is increasing on  $\mathbb{R}^+$  and  $0 < \Phi(x) < 1$ , hence

$$\begin{aligned} \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1 + \varepsilon} |\{x \in E : |f_k(x) - f(x)| > \varepsilon\}|. \end{aligned}$$

Therefore,

$$|\{x \in E : |f_k(x) - f(x)| > \varepsilon\}| \leq \frac{1 + \varepsilon}{\varepsilon} \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \rightarrow 0$$

as  $k \rightarrow \infty$ .

$\Leftarrow$  : Conversely, given  $\delta > 0$ , we have

$$\begin{aligned} \rho(f_k, f) &= \int_{\{x \in E : |f_k(x) - f(x)| > \delta\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\quad + \int_{\{x \in E : |f_k(x) - f(x)| \leq \delta\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\leq |\{x \in E : |f_k(x) - f(x)| > \delta\}| + \frac{\delta}{1 + \delta} |E|. \end{aligned}$$

Since  $|E| < \infty$  and  $\delta/(1+\delta) \searrow 0$ , then for any  $\varepsilon > 0$ , there exists  $\delta' > 0$  such that

$$\frac{\delta'}{1 + \delta'} |E| < \frac{\varepsilon}{2}.$$

If  $f_k \rightarrow f$  as  $k \rightarrow \infty$  in measure, then for the above  $\delta'$  there is an index  $N > 0$  such that  $k \geq N$  implies

$$|\{x \in E : |f_k(x) - f(x)| > \delta'\}| < \frac{\varepsilon}{2}.$$

Therefore,  $f_k \rightarrow f$  in measure implies  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$ . ■

**Problem 5.** Define the *gamma function*  $\Gamma: \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function*  $\beta: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function  $u \mapsto e^{-u} u^{y-1}$  is in  $L(\mathbb{R}^+)$  for all  $y \in \mathbb{R}^+$ .

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

*Proof.* (a) Fix  $y \in \mathbb{R}^+$ . Then we must show that  $\Gamma(y) < \infty$ . First, since  $(0, 1)$  and  $[1, \infty)$  are disjoint measurable subsets of  $\mathbb{R}$ , by Theorem 5.7 we can split the integral  $\Gamma(y)$  into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u} u^{y-1} du}_{I_1} + \underbrace{\int_1^\infty e^{-u} u^{y-1} du}_{I_2}. \quad (19)$$

We will show, separately, that  $I_1$  and  $I_2$  are finite.

To see that  $I_1$  is finite, note that

$$\begin{aligned} e^{-u} u^{y-1} &= e^{-u} e^{(y-1) \log u} \\ &= e^{-u+(y-1) \log u} \\ &\leq e^{(y-1) \log u} \\ &= u^{y-1} \end{aligned} \quad (20)$$

since  $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u} u^{y-1} du \\ &\leq \int_0^1 u^{y-1} du \\ &= \left[ \frac{u^y}{y} \right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \quad (21)$$

To see that  $I_2$  is finite, note that

$$e \quad (22)$$

**Intended solution:**

(b) ■

**Problem 6.** Let  $f \in L(\mathbb{R}^n)$  and for  $\mathbf{h} \in \mathbb{R}^n$  define  $f_{\mathbf{h}}: \mathbb{R}^n \rightarrow \mathbb{R}$  be  $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$ . Prove that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

*Proof.* Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{23}$$

■

**Problem 7.** (a) If  $f_k, g_k, f, g \in L(\mathbb{R}^n)$ ,  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ ,  $|f_k| \leq g_k$  and

$$\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if  $f_k, f \in L(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e. in  $\mathbb{R}^n$ , then

$$\int_{\mathbb{R}^n} |f_k - f| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \rightarrow \int_{\mathbb{R}^n} |f| \quad \text{as} \quad k \rightarrow \infty.$$

*Proof.* (a) Since  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. and  $|f_k| \leq g_k$ , then by Fatou's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} (g - f) &= \int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} g_k - f_k \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_k - f_k, \\ \int_{\mathbb{R}^n} g + f &= \int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} g_k + f_k \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_k + f_k. \end{aligned}$$

Since  $f_k, g_k, f, g \in L(\mathbb{R}^n)$  and  $\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g$ , then using the similar argument as problem 2, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f &\geq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k, \\ \int_{\mathbb{R}^n} f &\leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k. \end{aligned}$$

Therefore,  $\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f$ .

(b)  $\implies$  : This direction is obvious by the inequality

$$\left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \leq \int_{\mathbb{R}^n} ||f_k| - |f|| \leq \int_{\mathbb{R}^n} |f_k - f|.$$

$\Leftarrow$  : Let  $g_k = |f_k| + |f|$  and  $g = 2|f|$ . Since  $f_k, f \in L(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e., then  $g_k, g \in L(\mathbb{R}^n)$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ . By the assumption,  $\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g$ .

Let  $\tilde{f}_k = |f_k - f|$ . Then  $\tilde{f}_k \rightarrow 0$  a.e. in  $\mathbb{R}^n$  and  $\tilde{f}_k \leq g_k$ . Applying part (a) to  $\tilde{f}_k$  we have

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

■

## 1.4 Midterm 2

**Problem 1.** Assume that  $f \in L(\mathbb{R}^n)$ . Show that for every  $\varepsilon > 0$  there exists a ball  $B$ , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

*Proof.* Recall that  $f \in L(\mathbb{R}^n)$  if and only if  $|f| \in L(\mathbb{R}^n)$ . Let  $B_k = B(\mathbf{0}, k)$  for  $k \in \mathbb{N}$  and  $\chi_{B_k}$  be the indicator function associated with  $B_k$ . Then, the sequence of maps  $\{|f_k|\}$  defined  $f_k = f\chi_{B_k}$  converge pointwise to  $|f|$ . Since  $|f| \in L(\mathbb{R}^n)$ , by the monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbb{R}^n} |f|. \quad (1)$$

But this means, exactly, that for every  $\varepsilon > 0$  there exists sufficiently large  $N \in \mathbb{N}$  such that

$$\begin{aligned} \varepsilon &> \left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \\ &= - \int_{\mathbb{R}^n} |f_k| + \int_{\mathbb{R}^n} |f| \\ &= - \int_{\mathbb{R}^n} |f| + \int_{\mathbb{R}^n} |f| \\ &= - \int_{B_k} |f| + \int_{\mathbb{R}^n} |f| \\ &= \int_{\mathbb{R}^n \setminus B_k} |f| \end{aligned} \quad (2)$$

as desired. ■

**Problem 2.** Let  $f \in L(E)$ , and let  $\{E_j\}$  be a countable collection of pairwise disjoint measurable subsets of  $E$ , such that  $E = \bigcup_{j=1}^{\infty} E_j$ . Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

*Proof.* First, since the  $E_j$ 's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \quad (3)$$

Let  $\chi_{E_j}$  be the characteristic function of the subset  $E_j$  of  $E$  and define  $f_j = f\chi_{E_j}$  for  $j \in \mathbb{N}$ . Note that, since both  $f$  and  $\chi_{E_j}$  are measurable on  $E$ ,  $f_j$  is measurable on  $E$  and  $\sum_{j=1}^{\infty} f_j = f$ . Moreover, since  $E_j \subset E$ , by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \setminus E_j} f = \int_E f_j + \int_{E \setminus E_j} f. \quad (4)$$

Hence, because the  $E_j$ 's are disjoint  $(E \setminus E_k) \setminus E_\ell = (E \setminus E_\ell) \setminus E_k$  so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \quad (5)$$

as desired. ■

**Problem 3.** Let  $\{f_k\}$  be a family in  $L(E)$  satisfying the following property: For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|A| < \delta$  implies

$$\int_A |f_k| < \varepsilon$$

for all  $k \in \mathbb{N}$ . Assume  $|E| < \infty$ , and  $f_k(x) \rightarrow f(x)$  as  $k \rightarrow \infty$  for a.e.  $x \in E$ . Show that

$$\lim_{k \rightarrow \infty} \int_E f_k = \int_E f.$$

(Hint: Use Egorov's theorem.)

*Proof.* Let  $\varepsilon > 0$  be given. Then, by the hypothesis, there exists  $\delta > 0$  such that  $|A| < \delta$  implies

$$\int_A |f_k| < \varepsilon \quad (6)$$

for all  $k \in \mathbb{N}$ . By Egorov's theorem, there exists a closed subset  $F$  of  $E$  such that  $|E \setminus F| < \delta$  and  $f_k \rightarrow f$  uniformly on  $F$ . Then, by the uniform convergence theorem,

$$\int_F f_k \rightarrow \int_F f \quad (7)$$

as  $k \rightarrow \infty$ . But by hypothesis, we have

$$\int_{E \setminus F} |f_k| < \varepsilon. \quad (8)$$

Letting  $\varepsilon \rightarrow 0$ , we achieved the desired convergence. ■

**Problem 4.** Let  $I = [0, 1]$ ,  $f \in L(I)$ , and define  $g(x) = \int_x^1 t^{-1} f(t) dt$  for  $x \in I$ . Prove that  $g \in L(I)$  and

$$\int_I g = \int_I f.$$

*Proof.* By Lusin's theorem, there exists a closed subset  $F$  of  $I$  with  $|I \setminus F| < \varepsilon$  such that the restriction of  $f$  to  $F = I \setminus E$  is continuous. Now, since  $F$  is closed in  $I$  and  $I$  is compact, it follows that  $F$  is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials  $\{p_k\}$  such that  $p_k \rightarrow f$  uniformly on  $F$ . Then, by the uniform convergence theorem, we have

$$\int_F p_k \rightarrow \int_F f \quad (9)$$



so

$$\begin{aligned}\int_F \left[ \int_x^1 t^{-1} p_k(t) dt \right] dx &= \int_F \left[ \int_x^1 at^{-1} + q_k(t) dt \right] dx \\ &= \int_F q'_k(x) - a \log(x) dx \\ &< \infty\end{aligned}\tag{10}$$

for all  $k$  and converges uniformly to  $g$  so  $g \in L(I)$ . I don't know how to show that in fact  $\int_I g = \int_I f$ . Perhaps you show that the places where they differ is a set of measure zero. ■

**1.5 Final Practice**

**Problem 1.** Suppose  $f \in L^1(\mathbb{R})$  and that  $x$  is a point in the Lebesgue set of  $f$ . For  $r > 0$ , let

$$A(r) := \frac{1}{r} \int_{B(0,r)} |f(x-y) - f(x)| dy.$$

Show that:

- (a)  $A(r)$  is a continuous function of  $r$ , and  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ ;
- (b) there exists a constant  $M > 0$  such that  $A(r) \leq M$  for all  $r > 0$ .

*Proof.* ■

**Problem 2.** Let  $E \subset \mathbb{R}^n$  be a measurable set,  $1 \leq n < \infty$ . Assume  $\{f_k\}$  is a sequence in  $L^p(E)$  converging pointwise a.e. on  $E$  to a function  $f \in L^p(E)$ . Prove that

$$\|f_k - f\|_p \rightarrow 0$$

if and only if

$$\|f_k\|_p \rightarrow \|f\|_p$$

as  $k \rightarrow \infty$ .

*Proof.* ■

**Problem 3.** Let  $1 < p < \infty$ ,  $f \in L^p(E)$ ,  $g \in L^{p'}(E)$ .

- (a) Prove that  $f * g \in C(\mathbb{R}^n)$ .
- (b) Does this conclusion continue to be valid when  $p = 1$  and  $p = \infty$ ?

*Proof.* ■

**Problem 4.** Let  $f \in L(\mathbb{R})$ , and let  $F(t) := \int_{\mathbb{R}} f(x) \cos(tx) dx$ .

- (a) Prove that  $F(t)$  is continuous for  $t \in \mathbb{R}$ .
- (b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

*Proof.* ■

**Problem 5.** Let  $f$  be of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular. Show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh$$

for all functions  $\varphi$  continuous on  $[a, b]$ .

*Proof.* ■

# MA 544 Past Quals

## 2.1 Danielli

### Winter 2012

**Problem 1.** Let  $f(x, y)$ ,  $0 \leq x, y \leq 1$ , satisfy the following conditions: for each  $x$ ,  $f(x, y)$  is an integrable function of  $y$ , and  $\partial f(x, y)/\partial x$  is a bounded function of  $(x, y)$ . Prove that  $\partial f(x, y)/\partial x$  is a measurable function of  $y$  for each  $x$  and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial f(x, y)}{\partial x} dy.$$

*Proof.* ■

**Problem 2.** Let  $f$  be a function of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular, show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh.$$

*Hint:* A function  $h$  is said to be singular if  $h' = 0$ .

*Proof.* ■

**Problem 3.** Let  $E \subset \mathbb{R}$  be a measurable set, and let  $K$  be a measurable function on  $E \times E$ . Assume that there exists a positive constant  $C$  such that

$$\int_E K(x, y) dx \leq C \tag{1}$$

for a.e.  $y \in E$ , and

$$\int_E K(x, y) dy \leq C \tag{2}$$

for a.e.  $x \in E$ .

Let  $1 < p < \infty$ ,  $f \in L^p(E)$ , and define

$$T_f(x) := \int_E K(x, y) f(y) dy.$$

(a) Prove that  $T_f \in L^p(E)$  and

$$\|T_f\|_p \leq C \|f\|_p. \tag{3}$$

(b) Is (2.1.0.3) still valid if  $p = 1$  or  $\infty$ ? If so, are assumptions (2.1.0.1) and (2.1.0.2) needed?

*Proof.* ■

**Problem 4.** Let  $f$  be a nonnegative measurable function on  $[0, 1]$  satisfying

$$|\{x \in [0, 1] : f(x) > \alpha\}| < \frac{1}{1 + \alpha^2} \tag{4}$$

for  $\alpha > 0$ .

(a) Determine values of  $p \in [1, \infty)$  for which  $f \in L^p[0, 1]$ .

(b) If  $p_0$  is the minimum value of  $p$  for which  $p$  may fail to be in  $L^p$ , give an example of a function which satisfies (2.1.0.4), but which is not in  $L^{p_0}[0, 1]$ .

*Proof.* ■

**Summer 2011**

**Problem 5.** Let  $f \in L^1(\mathbb{R})$ , and let  $F(t) := \int_{\mathbb{R}} f(x) \cos(tx) dx$ .

- (a) Prove that  $F(t)$  is continuous for  $t \in \mathbb{R}$ .
- (b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

*Hint:* Start by proving the statement for  $f = \chi_{[a,b]}$ .

*Proof.* ■

**Problem 6.** (a) Suppose that  $f_k, f \in L^2(E)$ , with  $E$  a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E f g \quad (5)$$

as  $k \rightarrow \infty$  for all  $g \in L^2(E)$ . If, in addition,  $\|f_k\|_2 \rightarrow \|f\|_2$  show that  $f_k$  converges to  $f$  in  $L^2$ , i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as  $k \rightarrow \infty$ .

- (b) Provide an example of a sequence  $f_k$  in  $L^2$  and a function  $f$  in  $L^2$  satisfying (2.1.0.5), but such that  $f_k$  does *not* converge to  $f$  in  $L^2$ .

*Proof.* ■

**Problem 7.** A bounded function  $f$  is said to be of bounded variation on  $\mathbb{R}$  if it is of bounded variation on any finite subinterval  $[a, b]$ , and moreover  $A := \sup_{a,b} V[a, b; f] < \infty$ . Here,  $V[a, b; f]$  denotes the total variation of  $f$  over the interval  $[a, b]$ . Show that:

- (a)  $\int_{\mathbb{R}} |f(x+h) - f(x)| dx \leq A|h|$  for all  $h \in \mathbb{R}$ .

*Hint:* For  $h > 0$ , write

$$\int_{\mathbb{R}} |f(x+h) - f(x)| dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)| dx.$$

- (b)  $\left| \int_{\mathbb{R}} f(x) \varphi'(x) dx \right| \leq A$ , where  $\varphi$  is any function of class  $C^1$ , of bounded variation, compactly supported, with  $\sup_{x \in \mathbb{R}} |\varphi(x)| \leq 1$ .

*Proof.* ■

**Problem 8.** (a) Prove the *generalized Hölder's inequality*: Assume  $1 \leq p \leq \infty$ ,  $j = 1, \dots, n$ , with  $\sum_{j=1}^{\infty} 1/p_j = 1/r \leq 1$ . If  $E$  is a measurable set and  $f_j \in L^{p_j}(E)$  for  $j = 1, \dots, n$ , then  $\prod_{j=1}^n f_j \in L^r(E)$  and

$$\|f_1 \cdots f_n\|_r \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if  $1 \leq p, q, r \leq \infty$ , with  $1/p + 1/q = 1/r + 1$ ,  $f \in L^p(\mathbb{R})$ , and  $g \in L^q(\mathbb{R})$ , then

$$|(f * g)(x)| \leq \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q dy.$$

(Recall that  $(f * g)(x) := \int f(y)g(x-y) dy$ .)

(c) Prove *Young's convolution theorem*: Assume that  $p, q, r, f$ , and  $g$  are as in part (b). Then  $f * g \in L^r(\mathbb{R})$  and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

*Proof.*

■