MA 523: Homework 7

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Problem 7.1

Solve the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ u = x_1 & \text{on } |x| = 1, \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

(*Hint:* Use Laurent series.)

Solution. Suppose u is a solution to the Dirichlet problem above with the form

$$u(x_1, x_2) = a \ln |x| + \operatorname{Re}\bigl(L(x_1, x_2)\bigr)$$

where

$$L(x_1, x_2) = \sum_{n \in \mathbb{Z}} a_n (x_1 + \mathrm{i} x_2)^n$$

is an honest Laurent series. Make a change of variables $x_1 + \mathrm{i}x_2 \mapsto r\mathrm{e}^{\mathrm{i}n\theta}$ and rewrite the solution u in terms of our new variables

$$u(r\mathrm{e}^{\mathrm{i}n\theta}) = b \ln r + \sum_{n \in \mathbb{Z}} (a_n r^n + \overline{a_{-n}} r^{-n}) \mathrm{e}^{\mathrm{i}n\theta}. \tag{7.1}$$

Since u is harmonic, applying Laplace's equation to (7.1)

$$\begin{aligned} \Delta u &= u_{rr} + \tfrac{1}{r} u_r + \tfrac{1}{r^2} u_{\theta\theta} \\ &= b \ln r + \sum_{n \in \mathbb{Z}} (a_n r^n + \overline{a_{-n}} r^{-n}) \mathrm{e}^{\mathrm{i}n\theta} \\ &= -br^{-2} + br^{-2} \sum_{n \in \mathbb{Z}} \Bigl(\bigl(n(n-1) + n - n^2\bigr) a_n r^n + \bigl(n(n-1) + n - n^2\bigr) \overline{a_{-n}} r^{-n} \Bigr) \mathrm{e}^{\mathrm{i}n\theta} \\ &= 0, \end{aligned}$$

which tells us nothing about the coefficients. ■

Problem 7.2

Let Ω be a bounded domain with a C^1 boundary, $g \in C^2(\partial\Omega)$ and $f \in C(\bar{\Omega})$. Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where ν is the outer normal on $\partial\Omega$. Show that the solution of $(*)$ in $C^2(\Omega) \cap C^1(\bar{\Omega})$ is unique up to a constant; i.e., if u_1 and u_2 are both solutions of $(*)$, then $u_2 = u_1 + \text{const.}$ in Ω.
(*Hint:* Look at the proof of the uniqueness for the Dirichlet problem by energy methods [E, 2.2.5a].)

Solution. Suppose u_1 and u_2 are solutions to the Neumann problem $(*)$. Define $v := u_1 - u_2$. Then v is harmonic in Ω and $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$. Consider the energy functional

$$E[v] = \frac{1}{2}\int_\Omega |Dv|^2\, dx.$$

By Green's formula version (ii),

$$\begin{aligned} E[v] &= \frac{1}{2}\int_\Omega |Dv|^2\, dx \\ &= -\frac{1}{2}\int_\Omega v\Delta v\, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} v\, dS(x) \\ &= 0. \end{aligned}$$

This implies that $|Dv|^2 = Dv \cdot Dv = 0$ which, since the standard inner product on $\mathbb{R}^n$ is positive-definite, implies that $Dw \equiv 0$. It follows that $u_1 = u_2 + \text{const}$, i.e., the solution u to $(*)$ is unique up to a constant. ■

Problem 7.3

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{ t = 0 \}, \end{cases}$$

where $c \in \mathbb{R}$.
(*Hint:* Rewrite the problem in terms of $v(x,t) \coloneqq \mathrm{e}^{ct} u(x,t)$.)

Solution. ■