Math 535 - General Topology
Fall 2012
Homework 4 Solutions

Problem 1. Let $\{\Lambda_\alpha\}_{\alpha \in A}$ be a family of directed set. Show that the product $\prod_{\alpha \in A} \Lambda_\alpha$ becomes a directed set by defining the relation

$$\lambda \leq \lambda' \text{ if } \lambda_\alpha \leq \lambda'_\alpha \text{ in } \Lambda_\alpha \text{ for all } \alpha \in A$$

i.e. the componentwise preorder. (First check that this is indeed a preorder.)

Solution. We check that the relation is a preorder.

1. Reflexivity: $\lambda \leq \lambda$ for all $\lambda \in \prod_{\alpha \in A} \Lambda_\alpha$ since $\lambda_\alpha \leq \lambda_\alpha$ for all $\alpha \in A$.
2. Transitivity: $\lambda \leq \lambda'$ and $\lambda' \leq \lambda''$ implies $\lambda \leq \lambda''$. Indeed, we have $\lambda_\alpha \leq \lambda'_\alpha$ and $\lambda'_\alpha \leq \lambda''_\alpha$ for all $\alpha \in A$, so that $\lambda_\alpha \leq \lambda''_\alpha$ holds for all $\alpha \in A$.

Next, for any $\lambda, \lambda' \in \prod_{\alpha \in A} \Lambda_\alpha$, take "componentwise" upper bounds, i.e. for every index $\alpha \in A$, pick $\lambda''_\alpha \in \Lambda_\alpha$ satisfying $\lambda_\alpha \leq \lambda''_\alpha$ and $\lambda'_\alpha \leq \lambda''_\alpha$. Then the element $\lambda'' \in \prod_{\alpha \in A} \Lambda_\alpha$ with components λ''_α is an upper bound for λ and λ', i.e. it satisfies $\lambda \leq \lambda''$ and $\lambda' \leq \lambda''$. $\square$

Problem 2. Consider the space $\mathbb{R}^{\mathbb{N}}$ with the *box* topology. Consider the subset

$$Z = \{x \in \mathbb{R}^{\mathbb{N}} \mid x_i > 0 \text{ for all } i \in \mathbb{N}\}$$

and the point $\underline{0} = (0, 0, \ldots)$. We know $\underline{0} \in \overline{Z}$, but now we will find an explicit net in Z that converges to $\underline{0}$.

Consider the directed set $\Lambda := \mathbb{N}^{\mathbb{N}} \cong \prod_{i\in\mathbb{N}} \mathbb{N} = \{(n_1, n_2, \ldots) \mid n_i \in \mathbb{N}\}$ with the componentwise preorder (as in Problem 1).

Consider the net φ in Z indexed by Λ which assigns to the list $\lambda = (n_1, n_2, \ldots)$ the point

$$\varphi(\lambda) = \left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots\right) \in Z.$$

Show that this net φ converges to $\underline{0}$.

Solution. Let $U = \prod_{i\in N} U_i$ be a (basic) open neighborhood of $\underline{0}$, i.e. an "open box" around that point. Since $U_i \subseteq \mathbb{R}$ is an open neighborhood of 0, there is a small radius $\epsilon_i > 0$ satisfying $(-\epsilon_i, \epsilon_i) \subseteq U_i$. For each $i \in \mathbb{N}$, pick $M_i \in \mathbb{N}$ large enough so that $\frac{1}{M_i} < \epsilon_i$.

For every index

$$\lambda = (n_1, n_2, \ldots) \geq (M_1, M_2, \ldots) \in \Lambda$$

the net φ has value

$$\varphi(\lambda) = (\frac{1}{n_1}, \frac{1}{n_2}, \ldots) \in U.$$

Indeed, the components satisfy $\frac{1}{n_i} \leq \frac{1}{M_i} < \epsilon_i$ which guarantees $\frac{1}{n_i} \in (-\epsilon_i, \epsilon_i) \subseteq U_i$ for all $i \in \mathbb{N}$. $\square$

Problem 3. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces. Show that a net $(x_\lambda)_{\lambda\in\Lambda}$ in the product $\prod_{\alpha\in A} X_\alpha$ converges to a point x if and only if for each index $\alpha \in A$, the net $(p_\alpha(x_\lambda))_{\lambda\in\Lambda}$ in X_α converges to $p_\alpha(x)$.

Here $p_\beta \colon \prod_{\alpha\in A} X_\alpha \to X_\beta$ denotes the canonical projection.

Solution. ($\Rightarrow$) Each projection p_α is continuous, so that convergence $x_\lambda \to x$ in the product guarantees convergence $p_\alpha(x_\lambda) \to p_\alpha(x)$ in each factor.

($\Leftarrow$) Let $U = \prod_{\alpha\in A} U_\alpha$ be a (basic) open neighborhood of $x \in \prod_{\alpha\in A} X_\alpha$, which means $U_\alpha \subseteq X_\alpha$ is open, and $U_\alpha \neq X_\alpha$ for at most finitely many indices, say $\alpha_1, \ldots, \alpha_k$.

For $i = 1, \ldots, k$, the net $(p_{\alpha_i}(x_\lambda))_{\lambda\in\Lambda}$ in X_{α_i} converges to $p_{\alpha_i}(x)$. Since U_{α_i} is a neighborhood of $p_{\alpha_i}(x)$, convergence of the net guarantees that there is some $\lambda_i \in \Lambda$ satisfying $p_{\alpha_i}(x_\lambda) \in U_{\alpha_i}$ for all $\lambda \geq \lambda_i$.

Let $\lambda' \in \Lambda$ be an upper bound for $\{\lambda_1, \ldots, \lambda_k\}$. Then for all $\lambda \geq \lambda'$, we have $p_{\alpha_i}(x_\lambda) \in U_{\alpha_i}$ for $i = 1, \ldots, k$. Moreover, for all other indices $\alpha \in A$, we automatically have $p_\alpha(x_\lambda) \in U_\alpha = X_\alpha$. Therefore, we have $x_\lambda \in U$ for all $\lambda \geq \lambda'$. $\square$

Problem 4. (Brown Exercise 3.5.3) Prove that a discrete space is compact if and only if it is finite.

Solution. ($\Leftarrow$) Every finite space X is compact. Indeed, let $\{U_\alpha\}$ be an open cover of X. For each point $x \in X$, choose some $U_{\alpha(x)}$ containing x. Then we obtain a finite subcover

$$X = \bigcup_{x \in X} U_{\alpha(x)}.$$

($\Rightarrow$) Since the space X is discrete, each singleton $\{x\}$ is open in X. The equality

$$X = \bigcup_{x \in X} \{x\}$$

means that the collection $\{\{x\}\}_{x \in X}$ of all singletons is an open cover of X. Since X is compact, there is a finite subcover $\{\{x_1\}, \ldots, \{x_n\}\}$, which means

$$X = \{x_1\} \cup \ldots \cup \{x_n\} = \{x_1, \ldots, x_n\}$$

is finite. $\square$

Problem 5. (Munkres Exercise 3.26.2) Let X be a set endowed with the cofinite topology. Show that every subspace $A \subseteq X$ is compact.

Solution. Let $\{U_\alpha\}$ be a collection of open subsets of X that cover A, meaning $A \subseteq \bigcup_\alpha U_\alpha$. Pick any index α_0. Since U_{α_0} is open in X (and non-empty), it is

$$U_{\alpha_0} = X \setminus F$$

for some finite set $F \subset X$. In particular, U_{α_0} contains all of A except at most finitely many points, say $a_1, \ldots, a_k$. For $i = 1, \ldots, k$ respectively, pick an open U_{α_i} containing the point a_i. Then we have

$$A \subseteq U_{\alpha_0} \cup U_{\alpha_1} \cup \ldots \cup U_{\alpha_k}$$

so that A is compact. $\square$

Problem 6. (Munkres Exercise 3.26.5) (Willard Exercise 6.17B.5) Let X be a *Hausdorff* topological space.

a. Let $A \subset X$ be a *compact* subspace and $x_0 \in X \setminus A$ a point outside A. Show that A and x_0 can be separated by neighborhoods, i.e. there exist open subsets $U, V \subset X$ satisfying $A \subseteq U$, $x_0 \in V$, and $U \cap V = \emptyset$.

Solution. For each $a \in A$, choose open subsets U_a and V_a that separate a and x_0, i.e. $a \in U_a$, $x_0 \in V_x$ and $U_a \cap V_a = \emptyset$.

The collection of open subsets $\{U_a\}_{a \in A}$ covers A, and A is compact, so that there is a finite subcover

$$A \subseteq U_{a_1} \cup \ldots \cup U_{a_n} =: U.$$

Note that U is an open neighborhood of A.

Now take $V := V_{a_1} \cap \ldots \cap V_{a_n}$, which is an open neighborhood of x_0. The equality $V \cap U_{a_i} = \emptyset$ for all $i = 1, \ldots, n$ guarantees

$$V \cap (U_{a_1} \cup \ldots \cup U_{a_n}) = \emptyset$$

i.e. $V \cap U = \emptyset$. □

b. Let $A, B \subset X$ be disjoint *compact* subspaces. Show that A and B can be separated by neighborhoods, i.e. there exist open subsets $U, V \subset X$ satisfying $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Solution. For each $b \in B$, the point b is outside A, since A and B are disjoint. Because A is compact, part (a) applies and we can separate A and b by neighborhoods. In other words, there are open subsets U_b and V_b satisfying $A \subseteq U_b$, $b \in V_b$, and $U_b \cap V_b = \emptyset$.

The collection of open subsets $\{V_b\}_{b \in B}$ covers B, and B is compact, so that there is a finite subcover

$$B \subseteq V_{b_1} \cup \ldots \cup V_{b_n} =: V.$$

Note that V is an open neighborhood of B.

Now take $U := U_{b_1} \cap \ldots \cap U_{b_n}$, which is an open neighborhood of A. The equality $U \cap V_{b_i} = \emptyset$ for all $i = 1, \ldots, n$ guarantees

$$U \cap (V_{b_1} \cup \ldots \cup V_{b_n}) = \emptyset$$

i.e. $U \cap V = \emptyset$. □

Problem 7. Let D^n denote the unit disc in $\mathbb{R}^n$ (with the usual Euclidean norm)

$$D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

and let S^n denote the unit sphere in $\mathbb{R}^{n+1}$

$$S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

Show that there is a homeomorphism

$$(D^n \amalg D^n)/\sim \cong S^n$$

where the two discs are glued along their edges, i.e. the equivalence relation $\sim$ is generated by $x^{(1)} \sim x^{(2)}$ for all $x \in D^n$ with $\|x\| = 1$. Here the superscript denotes that $x^{(1)} \in D^n \amalg D^n$ lives in the first summand while $x^{(2)}$ lives in the second summand.

Solution. Writing $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$, consider the map $f_U \colon D^n \to S^n$ sending the disc to the upper hemisphere:

$$f_U(x) = (x, \sqrt{1 - \|x\|^2}).$$

Then f_U is continuous since its $n+1$ components are continuous, and in fact f_U is a homeomorphism onto the upper hemisphere, with inverse the projection $p_{\mathbb{R}^n} \colon S^n_{\text{upper}} \to D^n$ onto the first n coordinates. Indeed, a point $(x_1, \ldots, x_n, x_{n+1}) = (x, x_{n+1})$, where x denotes $(x_1, \ldots, x_n)$, is in the upper hemisphere S^n_{upper} if and only if it satisfies

$$\begin{cases} x_1^2 + \ldots + x_n^2 + x_{n+1}^2 = 1 = \|x\|^2 + x_{n+1}^2 \\ x_{n+1} \geq 0 \end{cases}$$

so that it is of the form $(x, \sqrt{1 - \|x\|^2})$ for some $x \in D^n$.

Likewise, consider the map $f_L \colon D^n \to S^n$ sending the disc homeomorphically onto the lower hemisphere:

$$f_L(x) = (x, -\sqrt{1 - \|x\|^2}).$$

Consider the map $f \colon D^n \amalg D^n \to S^n$ whose restrictions to the first and second summands are f_U and f_L respectively. Then f is continuous, since its restriction to each summand is continuous. Moreover, f is surjective, because of $S^n = S^n_{\text{upper}} \cup S^n_{\text{lower}}$.

However, f is not injective. Because the restrictions f_U and f_L are injective, non-injectivity can only happen when taking inputs from different summands:

$$\begin{aligned} f(x^{(1)}) = f(y^{(2)}) &\Leftrightarrow f_U(x) = f_L(y) \\ &\Leftrightarrow \left(x, \sqrt{1 - \|x\|^2}\right) = \left(y, -\sqrt{1 - \|y\|^2}\right) \\ &\Leftrightarrow x = y \text{ and } \|x\| = \|y\| = 1 \\ &\Leftrightarrow x^{(1)} \sim y^{(2)}. \end{aligned}$$

Therefore f induces a continuous map on the quotient

$$\overline{f} \colon (D^n \amalg D^n)/\sim \cong S^n$$

which is surjective (because f is) and injective (because $f(x) = f(x') \Rightarrow x \sim x'$).

Since the disc $D^n \subset \mathbb{R}^n$ is closed and bounded, it is compact. Therefore the finite union $D^n \amalg D^n$ is compact, and so is its quotient $(D^n \amalg D^n)/\sim$. Since S^n is a metric space, it is Hausdorff. Now $\overline{f}$ is a continuous bijection from a compact space to a Hausdorff space, and is therefore a homeomorphism. $\square$

Problem 8. (Munkres Exercise 3.26.4)

a. Let (X, d) be a metric space, and $K \subseteq X$ a compact subspace. Show that K is closed (in X) and bounded.

Solution. Since X is Hausdorff, K is closed in X.

For boundedness, pick any point $x \in X$ and consider the open cover by increasingly large open balls

$$K \subseteq \bigcup_{n \in \mathbb{N}} B_n(x).$$

Since K is compact, there is a finite subcover

$$K \subseteq B_{n_1}(x) \cup \ldots \cup B_{n_k}(x) = B_N(x)$$

where $N = \max\{n_1, \ldots, n_k\}$. Therefore K is bounded. □

Now we show that the converse does not hold.

b. Find a metric space (X, d) and a subset $C \subseteq X$ which is closed and bounded, but such that C is *not* compact.

Solution. Let X be an infinite set equipped with the discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

which induces the discrete topology. Consider the subset $C := X \subseteq X$, which is closed in X. Moreover X is bounded: $\operatorname{diam}(X) = 1$.

However, X is an infinite discrete space, hence not compact. □