$\boxed{\text{Each problem is worth 14 points}}$

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

1. Let X be a Hausdorff space and let A be a compact subset of X. **Prove** from the definitions that A is closed.

2. Let X be a Hausdorff space and let A and B be disjoint compact subsets of X. Prove that there are open sets U and V such that U and V are disjoint, $A \subset U$ and $B \subset V$.

3. Prove the Tube Lemma: let X and Y be topological spaces with Y compact, let $x_0 \in X$, and let N be an open set of $X \times Y$ containing $\{x_0\} \times Y$, **then** there is an open set W of X containing x_0 with $W \times Y \subset N$.

4. Show that if Y is compact, then the projection map $X \times Y \to X$ is a closed map.

5. Let X be a compact space and suppose we are given a nested sequence of subsets

$$C_1 \supset C_2 \supset \cdots$$

   with all C_i closed. Let U be an open set containing $\cap C_i$.

   Prove that there is an i_0 with $C_{i_0} \subset U$.

6. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. **Prove** that X is homeomorphic to a subspace of $\mathbb{R}^n$.

7. Let X be a compact metric space and let $\mathcal{U}$ be a covering of X by open sets.

   Prove that there is an $\epsilon > 0$ such that, for each set $S \subset X$ with diameter $< \epsilon$, there is a $U \in \mathcal{U}$ with $S \subset U$. (This fact is known as the "Lebesgue number lemma.")

8. Let S^1 denote the circle

$$\{x^2 + y^2 = 1\}$$

   in $\mathbb{R}^2$. Define an equivalence relation on S^1 by

$$(x, y) \sim (x', y') \Leftrightarrow (x, y) = (x', y') \text{ or } (x, y) = (-x', -y')$$

   (you do *not* have to prove that this is an equivalence relation). **Prove** that the quotient space $S^1/\sim$ is homeomorphic to S^1.

   One way to do this is by using complex numbers.

9. Let X be a nonempty compact Hausdorff space and let $f : X \to X$ be a continuous function. Suppose f is 1-1. **Prove** that there is a nonempty closed set A with $f(A) = A$. (The hypothesis that f is 1-1 is not actually needed, but it makes the proof a little easier).

10. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. **Prove** that the quotient space $\mathbb{R}^2/{\sim}$ is compact but not Hausdorff.

11. Let X be a locally compact Hausdorff space. **Explain** how to construct the one-point compactification of X, and **prove** that the space you construct is really compact (you do not have to prove anything else for this problem).

12. Show that if $\prod_{n=1}^{\infty} X_n$ is locally compact (and each X_n is nonempty), then each X_n is locally compact and X_n is compact for all but finitely many n.

13. Let X be a locally compact Hausdorff space, let Y be any space, and let the function space $\mathcal{C}(X, Y)$ have the compact-open topology.

    Prove that the map
    $$e : X \times \mathcal{C}(X, Y) \to Y$$
    defined by the equation
    $$e(x, f) = f(x)$$
    is continuous.

14. Let I be the unit interval, and let Y be a path-connected space. Prove that any two maps from I to Y are homotopic.

15. Let X be a topological space and $f : [0, 1] \to X$ any continuous function. Define $\bar{f}$ by $\bar{f}(t) = f(1 - t)$. Prove that $f * \bar{f}$ is path-homotopic to the constant path at $f(0)$.

16. Let X be a path-connected topological space and let $x_0, x_1 \in X$. Recall that any path α from x_0 to x_1 gives an isomorphism $\hat{\alpha}$ from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ (you do not have to prove this).

    Suppose that for every pair of paths α and β from x_0 to x_1 the isomorphisms $\hat{\alpha}$ and $\hat{\beta}$ are the same. **Prove** that $\pi_1(X, x_0)$ is abelian.