MA571: Qual Problems

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1 MA 571: Midterm 1, Fall 2015

2 MA 571: Midterm 2, Fall 2015

3 August 2014

Problem 3.1. Let X be a topological space, let A be a subset of X, and let U be an open subset of X. Prove that $U \cap \bar{A} \subset \overline{U \cap A}$.

Proof. Let $x \in U \cap \bar{A}$. Then $x \in U$ and $x \in \bar{A}$. This means that, since U is open, by Lemma C there exist an open neighborhood V of x such that $V \subset U$. Moreover, since $x \in \bar{A}$, $V' \cap A \neq \emptyset$ for every open neighborhood V' of x. In particular, $V \cap A \neq \emptyset$. Thus, we have $V \cap U \neq \emptyset$ and $V \cap A \neq \emptyset$ so $V \cap (U \cap A) \neq \emptyset$. ■

Problem 3.2. Let X be the following subspace of $\mathbb{R}^2$:

$$((0,1] \times [0,1]) \cup ([2,3) \times [0,1]).$$

Let $\sim$ be the equivalence relation on X with $(1,t) \sim (2,t)$ (that is $(s,t) \sim (s',t') \iff (s,t) = (s',t')$ or $t = t'$ and $\{s,s'\} = \{1,2\}$; you do *not* have to prove that this is an equivalence relation). Prove that $X/{\sim}$ is homeomorphic to $(0,2) \times [0,1]$. (*Hint*: construct maps in both directions).

Proof. We shall proceed by the hint. Let $q \colon X \to X/{\sim}$ denote the quotinet map. Then, for $(x,y) \in X$, we define the map

We shall proceed by the hint. Let $q \colon X \to X/{\sim}$ denote the quotient map. Then, for $x \in X$, we define the map

$$h(s,t) := \begin{cases} (s,t) & \text{if } (s,t) \in (0,1] \times [0,1] \\ (s-1,t) & \text{if } (s,t) \in (2,3] \times [0,1]) \end{cases}$$

from $X \to (0,2) \times [0,1]$.

By the UMP of the quotient space (Theorem Q.3), if we can show that h is continuous and preserves the equivalence relation, the induced map on the quotient space, $h' \colon X/{\sim} \to (0,2) \times [0,1]$ will be continuous. To that end, we will use the pasting lemma. First, note that $(0,1] \times [0,1]$ and $[2,3) \times [0,1]$ are closed subsets of X since $(0,1] \times [0,1]$ is the complement of $((1,\infty) \times (-2,2)) \cap X$ which is open in X (since X inherits its topology from $\mathbb{R}^2$), similarly, $[2,3) \times [0,1]$ is closed in X since it is the complement of $((-\infty,2) \times (-2,2)) \cap X$ which is open in X for the same reasons. It is clear that the maps $x \mapsto x$ and $x \mapsto x-1$ are continuous onto their image, since the latter is nothing more than the inclusion map and the former is nothing more than subtraction, which is continuous by Theorem 21.5. Thus, by the pasting lemma, h is continuous.

Now we show that h does in fact preserve the equivalence relation. Suppose $(s,t) \sim (s',t')$. Then either $(s,t) = (s',t')$ or $t = t'$ and $s, s' \in \{1,2\}$. In the former case, we have $h(s,t) = h(s',t')$ (whether $(s,t), (s',t') \in (0,1] \times [0,1]$ or its complement). In the latter case, we may, without loss of generality, assume that $(s,t) = (1,t)$ and $(s',t') = (2,t)$. Then $h(s,t) = (1,t) = (2-1,t) = h(s',t')$. Thus, by Theorem Q.3, the induced map $h' \colon X/{\sim} \to (0,2) \times [0,1]$ is continuous. Moreover, the map is bijective with inverse

$$(h')^{-1} := \begin{cases} [s,t] & \text{if } x \in (0,1] \\ [s+1,t] & \text{if } x \in [1,2). \end{cases}.$$

This is clearly an inverse as

$$h' \circ (h')^{-1} = \mathrm{id}_{X/\sim}$$

and

$$(h')^{-1} \circ h' = \mathrm{id}_{(0,2)\times[0,1]}\,.$$

Thus, by Theorem 26.6, h' is a homeomorphism. ■

Problem 3.3. Prove that there is an equivalence relation $\sim$ on the interval $[0,1]$ such that $[0,1]/\sim$ is homeomorphic to $[0,1] \times [0,1]$. As part of your proof *explain* how you are using one or more properties of the quotient topology.

Proof. First, it suffices to find a continuous surjective map $f\colon [0,1] \to [0,1] \times [0,1]$ and quotient out by the preimage of every point $x \in [0,1] \times [0,1]$. These maps are hard to describe in general, but they exists (take for example a space-filling curve). Next, note that if C is a closed subset of $[0,1]$ then it is compact so $f(C)$ is compact. But since $[0,1] \times [0,1]$ is compact Hausdorff, then $f(C) \subset [0,1] \times [0,1]$ will be closed. It follows by that f will be a Munkres quotient map, so by Theorem Q.4, $f'\colon [0,1]/\sim \to [0,1] \times [0,1]$ is a homeomorphism for some equivalence relation $\sim$ on $[0,1]$. ■

Problem 3.4. Let D be the closed unit disk in $\mathbb{R}^2$, that is, the set

$$\{\, (x,y) \mid x^2 + y^2 \leq 1 \,\}.$$

Let E be the open unit disk

$$\{\, (x,y) \mid x^2 + y^2 < 1 \,\}.$$

Let X be the one-point compactification of E, and let $f\colon D \to X$ be the map defined by

$$f(x,y) = \begin{cases} (x,y) & \text{if } x^2 + y^2 < 1 \\ \infty & \text{if } x^2 + y^2 = 1. \end{cases}$$

Prove that f is continuous.

Proof. By the section on the one-point-compactification, it suffices to check two cases of open sets (1) all sets U open in E, and (2) all sets of the form $U = X - C$ containing the point at infinity, ∞, where C is compact. In the first case, it is clear that f is continuous since it is just the inclusion map and is in fact bijective on E. For the second case, suppose that U is a neighborhood of ∞. Then $Y - U$ is a compact subset of E, hence closed since X is a compact Hausdorff space. But since f is bijective, continuous on E, then $f^{-1}(X - U)$ is a closed subset of E. Thus, by theorem 18.2, f is continuous. ■

Problem 3.5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is path-connected. Prove that Y is path-connected.

Proof. First we prove the following important result:

Lemma 1. *Path-connectedness is a topological property, i.e., if X is path-connected and $f\colon X \to Y$ is a continuous map then, $f(X)$ is path connected.*

Proof. Since X is path-connected, for any pair of points $x, x' \in X$ there exists a continuous map $p\colon [0,1] \to X$ such that $p(0) = x$ and $p(1) = x'$. Since composition of continuous maps is continuous, $f \circ p\colon [0,1] \to Y$ is a path from $f(x)$ to $f(x')$. Since this property holds for any $y \in f(X)$, it follows that $f(X)$ is path-connected. ♣

Now, suppose that X is homotopy-equivalent to Y. Then there exists continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Now, since X is path-connected, by Lemma (1) we have $f(X)$ is path-connected. Thus, it suffices to show that for every point $y \in Y$ there exists a path $p\colon [0,1] \to Y$ from y to some point $y' \in f(X)$. Now, since $f \circ g \simeq \mathrm{id}_Y$, there exists a homotopy, say $H\colon Y \times [0,1] \to Y$ such that $H(s,0) = f \circ g(s)$ and $H(s,1) = s$. Consider the evaluation $H_y := H(y,t) \circ H(y,t)$ where the map $(y,t)\colon [0,1] \to Y \times [0,1]$ is the imbedding of $[0,1]$ at y (which is continuous by Theorem 18.4) thus, H_y is continuous. Moreover, $H_y(0) = f \circ g(y) \in f(Y)$ and $H_y(1) = \mathrm{id}_Y(y) = y$ so H_y is a path from y to a point $f \circ g(y)$ in $f(X)$. Since we can do this for any point $y \in Y$, it follows, since path-connectedness is an equivalence relation, that Y is path-connected. ∎

Problem 3.6. Let a and b denote the points $(-1,0)$ and $(1,0)$ in $\mathbb{R}^2$. Let x_0 denote the origin $(0,0)$. Use the Seifert–van Kampen theorem to calculate $\pi_1\big(\mathbb{R}^2 - \{a,b\}, x_0\big)$. You may not use any other method.

Proof. We'll use Theorem 70.2's version of the Seifert–van Kampen theorem. Define

$$U := \big(-\infty, \tfrac{1}{2}\big) \times \mathbb{R} \quad \text{and} \quad V := \big(-\tfrac{1}{2}, \infty\big) \times \mathbb{R}.$$

Then $U \cap V = (-1/2, 1/2) \times \mathbb{R}$ is clearly path-connected since it is a convex set. Moreover, note that $U \simeq \mathbb{R}^2 \smallsetminus \{x_0\}$ and $V \simeq \mathbb{R}^2 \smallsetminus \{x_0\}$ (in the case of U, first consider the homeomorphism $(x,y) \mapsto (x+1, y)$ which sends a to $(0,0)$ and then the homotopy $(x,y) \mapsto \frac{1}{t}(x, ys)$).

Once we have established the above, since the fundamental group of a space is invariant under homotopy-equivalence, $\pi_1(U, x_0) \cong \pi_1\big(\mathbb{R}^2 \smallsetminus \{x_0\}, y_0\big) \cong \mathbb{Z}$ for some arbitrary $y_0 \neq x_0$ and similarly $\pi_1(V, x_0) \cong \mathbb{Z}$. Thus, by the classical version of the Seifert–van Kampen theorem

$$\pi_1\big(\mathbb{R}^2 \smallsetminus \{a,b\}, x_0\big) \cong \frac{\mathbb{Z} * \mathbb{Z}}{N}$$

where N is the least normal subgroup ∎

Problem 3.7. Let $p\colon E \to B$ be a covering map with B locally connected, and let $x \in B$. Prove that x has a neighborhood W with the following property: for every connected component C of $p^{-1}(W)$, the map $p\colon C \to W$ is a homeomorphism.

Proof. Let U be an evenly covered neighborhood of x. Then $p^{-1}(U) = \bigcup_\alpha V_\alpha$ where the V_α are open in E and $V_\alpha \cap V_\beta = \emptyset$ whenever $\alpha \neq \beta$. For any α, let C be a connected component of $p^{-1}(U)$ containing $p^{-1}(x) \cap V_\alpha$ (the latter is a one point set since $p|_{V_\alpha}$ is a bijection). Then $C \subset V_\alpha$ for at most one such α for otherwise $C \cap V_\beta \neq \emptyset$ for some $\beta \neq \alpha$, so $C \cap V_\beta$ and $C \cap V_\alpha$ form a separation of (note that $C \smallsetminus (C \cap V_\beta) = C \cap V_\alpha$ and vice-versa thus, $C \cap V_\beta$ and $C \cap V_\alpha$ are open and closed in the subspace topology on C, conversely) by Lemma 23.1.

Thus, $p(C) \subset U$ is connected by Theorem 23.5. Moreover, since $V_\alpha \supset C$ is homeomorphic to U by the restriction $p|_{V_\alpha}$, $p(C)$ is a connected component of U as the following lemma shows

Lemma 2. *Suppose C is a connected component of X and $h\colon X \to Y$ is a homeomorphism. Then $h(C)$ is a connected component of Y.*

Proof of lemma. Let C be a connected component of X. By theorem 23.5, $h(C)$ is a connected subset of Y, moreover, is open. By Theorem 25.1, $h(C)$ is contained in a connected component of Y, say D. Hence, we must show that $D \subset h(C)$. Now, since h is a homeomorphism, $h^{-1}(D)$ is a connected subset of X, by Theorem 23.5, so is contained in only one component of X. But $h^{-1}(D) \cap C \neq \emptyset$ so $h^{-1}(D) \subset C$. Thus, since h is a set-bijection, $D \subset h(C)$. ♣

so by Theorem 25.3, $p(C)$ is open in B since B is locally connected. Thus, the restriction $p|_C$ is a homeomorphism onto its image $W := p(C)$, by Lemma A, which is a neighborhood of x. ■

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Problem 4.1. Let X be a topological space, let A be a subset of X, and let U be an open subset of X. Prove that $U \cap \bar{A} \subset \overline{U \cap A}$.

Proof. The proof is simple and we have shown this before in the August 2014 quals, it goes as follows: If $U \cap \bar{A} = \emptyset$, there is nothing to show. Let $x \in U \cap \bar{A}$. Then $x \in U$ and $x \in \bar{A}$. Since $x \in U$ and U is open, by Lemma C, there exists a neighborhood V of x such that $V \subset U$; in particular, note that $V \cap U \neq \emptyset$. But $x \in \bar{A}$ so $V \cap A \neq \emptyset$. Thus, $V \cap (U \cap A) \neq \emptyset$. Thus, $x \in \overline{U \cap A}$. ∎

Problem 4.2. Let $\sim$ be an equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. Prove that the quotient space $\mathbb{R}^2/{\sim}$ is compact but not Hausdorff.

Proof. To show that $\mathbb{R}^2/{\sim}$ is compact, we need to show that for every open covering $\mathcal{A}$ of $\mathbb{R}^2/{\sim}$, there is a finite subcover $\mathcal{A}' \subset \mathcal{A}$. Let $q\colon \mathbb{R}^2 \to \mathbb{R}^2/{\sim}$ denote the quotient map. Then, since q is continuous and onto $\mathbb{R}^2/{\sim}$, the set $\left\{q^{-1}(A_\alpha)\right\}_{A_\alpha \in \mathcal{A}}$ is an open cover of $\mathbb{R}^2$. In particular, there exists at least one A_α such that $q^{-1}(A_\alpha)$ is a neighborhood of $(0, 0)$. By Lemma C, there exists a basic open neighborhood, i.e., an open ball $B((0, 0), \varepsilon) \subset q^{-1}(A_\alpha)$ for $\varepsilon > 0$. Now, for any point $[(x, y)] \in \mathbb{R}^2$ pick a representative $(x, y) \in \mathbb{R}^2$. Then, by the Archimedean principle, there exists a positive real numbers $t', t'' > 0$ such that $t'x < \sqrt{\varepsilon}$ and $t''y < \sqrt{\varepsilon}$. Define $t := \min\{t', t''\}$. Then $tx < \sqrt{\varepsilon}$ and $ty < \varepsilon$. Thus, $(tx, ty) \in A_\alpha$ (since $t^2x^2 + t^2y^2 < \varepsilon$). Since we can do this for any point $[x] \in \mathbb{R}^2/{\sim}$, it follows that $A_\alpha \supset \mathbb{R}^2/{\sim}$. Thus, $\mathcal{A}' := \{A_\alpha\}$ is a finite subset of $\mathcal{A}$ which covers $\mathbb{R}^2/{\sim}$. Thus, $\mathbb{R}^2/{\sim}$ is compact.

To show that $\mathbb{R}^2/{\sim}$ is not compact, we will employ a very similar strategy, that is, we will show that every neighborhood of the point $[0, 0] \in \mathbb{R}^2/{\sim}$, contains every point $[x, y] \in \mathbb{R}^2/{\sim}$. Let $[x, y] \in \mathbb{R}^2/{\sim}$ and let U be a neighborhood of $[0, 0]$. Then $q^{-1}(U)$ is an open neighborhood of $(0, 0)$, i.e., there exists an open ball $B((0, 0), \varepsilon) \subset q^{-1}(U)$. But as we have just shown, for sufficiently small values of $t > 0$, $(tx, ty) \in B((0, 0), \varepsilon) \subset q^{-1}(U)$. Thus, $[x, y] \in U$. In particular, for any open neighborhood V of $[x, y]$, $V \cap U \neq \emptyset$. Thus, $\mathbb{R}^2/{\sim}$ is not Hausdorff. ∎

Problem 4.3. Let X and Y be topological spaces. Let $x_0 \in X$ and let C be a compact subset of Y. Let N be an open set in $X \times Y$ containing $\{x_0\} \times C$. Prove that there is an open set U containing x_0 and an open set V containing C such that $U \times V \subset N$.

Proof. This is a classical theorem called the tube lemma. We shall prove first in the style of Munkres and second in the style of McClure (if I can find the proof or somehow reconstruct it).

Let X, Y, x_0, N, and C be as above. Note that since C is compact and the injection $\iota_{x_0}\colon X \hookrightarrow X \times Y$ given by $\iota_{x_0}(y) := (x_0, y)$ is continuous by Theorem 18.4 (since its components, i.e., projections to X and Y, are continuous these are $\pi_1(\iota_{x_0})(x) = x_0$ and $\pi_1(\iota_{x_0})(y) = y$ a constant map and identity map, respectively) so the image of C under ι_{x_0}, $\{x_0\} \times C$, is compact by Theorem 23.5. For every point $x \in \{x_0\} \times C$, let $U_x \times V_x$ be a basic open neighborhood of x contained in N (this can be arranged by Lemma C). Then the collection $\mathcal{A} := \{U_x \times V_x\}_{x \in \{x_0\} \times Y}$ forms an open covering of $\{x_0\} \times C$. Thus, there exists a finite subcover $\{U_{x_i} \times V_{x_i}\}_{i=1}^n$ of $\mathcal{A}$.

Define $W := U_{x_1} \cap \cdots \cap U_{x_n}$. This set is clearly open since it is a finite intersection of open sets and contains x_0 since every $U_{x_i} \times V_{x_i}$ intersects $\{x_0\} \times Y$. Define $W' := \pi_2(N) \cap Y$. This set is open since it is a finite intersection of open sets in Y. The $W \times W' \subset N$. This is clear since every point $(x, y) \in W \times W'$ is in N ($x \in W \subset U_{x_i}$ for all i which in turn is a subset of $\pi_1(N)$

and $y \in W' = \pi_2(N)$). Lastly, $W \times W' \supset \{x_0\} \times C$ since $x_0 \in W$ and $W' = \pi_1(N) \supset C$. Thus, $W \times W' \subset N$ containing $\{x_0\} \times C$ as desired. ■

Problem 4.4. Let X be a locally compact Hausdorff space and let A be a subset with the property that $A \cap K$ is closed for every compact K. Prove that A is closed.

Proof. Here's what I have so far:

We will try to show that $\bar{A} \subset A$. Let $x \in \bar{A}$. Then, for every neighborhood U of x, $U \cap A \neq \emptyset$. Now, since X is locally compact, there exists a neighborhood V of x such that $\bar{V}$ is compact and is a subset of U. Since X is Hausdorff, $\bar{V}$ is compact so $\bar{V} \cap A$ is closed. ■

Problem 4.5. Let X and Y be path-connected and let $h\colon X \to Y$ be a continuous function which induces the trivial homomorphism of fundamental groups. Let $x_0, x_1 \in X$ and let f and g be paths from x_0 to x_1. Prove that $h \circ f$ and $h \circ g$ are homotopic.

Proof. Consider the path-product $\gamma \coloneqq f * \bar{g}$. γ is a loop based at x_0 since $\gamma(0) = f(0) = x_0$ and $\gamma(1) = \bar{g}(2-1) = \bar{g}(1) = x_0$. Thus, $[\gamma] \in \pi_1(X, x_0)$. Now, since $h_*\colon \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$ induces the trivial homomorphism, i.e., $h(\gamma) \simeq_p e_{x_0}$, there exists a homotopy $H\colon [0,1] \times [0,1] \to Y$ such that $H(s, 0) = h \circ \gamma(s)$ and $H(s,1) = e_{x_0}(s)$. Now, since Y is path-connected, there exists a path $\delta\colon [0,1] \to Y$ from $h(x_0)$ to $h(x_1)$. ■

Problem 4.6. Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $aabbcdc^{-1}d^{-1}$.

(i) Calculate $H_1(X)$.
(ii) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?

Proof. ■

Problem 4.7. Let $p\colon E \to B$ be a covering map with B locally connected, and let $x \in B$. Prove that x has a neighborhood W with the following property: for every connected component C of $p^{-1}(W)$, the map $p\colon C \to W$ is a homeomorphism.

Proof. ■