

MA166: Recitation 7 Prep

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1 Homework Problems

Section 1.1: Homework 15

Problem 1.1 (WebAssign, HW 15, # 1). Use the table of integrals to evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{\cos x}{\sin^2 x - 36} dx.$$

Solution. First make the substitution $u = \sin x$. Then from the table, we have the formula

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C. \quad (1)$$

So

$$\begin{aligned} \int \frac{\cos x}{\sin^2 x - 36} dx &= \int \frac{du}{u^2 - 6^2} du \\ &= \frac{1}{12} \ln \left| \frac{u - 6}{u + 6} \right| + C \\ &= \boxed{\frac{1}{12} \ln \left| \frac{\sin x - 6}{\sin x + 6} \right| + C}. \end{aligned}$$

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Problem 1.2 (WebAssign, HW 15, # 2). Use the table of integrals to evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{1}{x^2 \sqrt{81x^2 + 4}} dx.$$

Solution. First make the substitution $u = 9x$. Then from the table, we have the formula

$$\int \frac{du}{u^2 \sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C. \quad (2)$$

So

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{81x^2 + 4}} dx &= \frac{1}{9} \int \frac{1}{\frac{1}{81} u^2 \sqrt{u^2 + 2^2}} du \\ &= 9 \int \frac{1}{u^2 \sqrt{u^2 + 2^2}} du \\ &= -\frac{9\sqrt{u^2 + 4}}{4u} + C \\ &= \boxed{-\frac{\sqrt{81x^2 + 4}}{4x} + C}. \end{aligned}$$

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Problem 1.3 (WebAssign, HW 15, # 3). Use the table of integrals to evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{\tan^3(6/z)}{z^2} dz.$$

Solution. First make the substitution $u = 6/z$. Then from the table, we have the formula

$$\int \tan^3 u \, du = \frac{1}{2} \tan^2 u + \ln |\cos u| + C. \quad (3)$$

So

$$\begin{aligned} \int \frac{\tan^3(6/z)}{z^2} dz &= -\frac{1}{6} \int \tan^3 u \, du \\ &= -\frac{1}{12} \tan^2 u - \frac{1}{6} \ln |\cos u| + C \\ &= \boxed{-\frac{1}{12} \tan^2(6/z) - \frac{1}{6} \ln |\cos(6/z)| + C.} \end{aligned}$$

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Problem 1.4 (WebAssign, HW 15, # 4). Use the table of integrals to evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{e^{2x}}{13 - e^{4x}} dx.$$

Solution. Make the substitution $u = e^{2x}$. Then, from the table of integrals we have the formula

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{u+a}{u-a} \right| + C. \quad (4)$$

So

$$\begin{aligned} \int \frac{e^{2x}}{13 - e^{4x}} dx &= \frac{1}{2} \int \frac{1}{13 - u^2} du \\ &= \frac{1}{4\sqrt{13}} \ln \left| \frac{u + \sqrt{13}}{u - \sqrt{13}} \right| + C \\ &= \boxed{\frac{1}{4\sqrt{13}} \ln \left| \frac{e^{2x} + \sqrt{13}}{e^{2x} - \sqrt{13}} \right| + C.} \end{aligned}$$

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Problem 1.5 (WebAssign, HW 15, # 5). Use the trapezoidal rule, the midpoint rule, and Simpson's rule to approximate the given integral with specified value n . (Round your answer to six decimal places).

$$\int_1^4 2\sqrt{\ln x} \, dx, \quad n = 6$$

Solution. These problems take too long, just remember the formulas

$$\int_a^b f(x) dx \approx (b-a) \left[\frac{f(a) + f(b)}{2} \right] \quad (5)$$

for the trapezoidal rule,

$$\int_a^b f(x) dx \approx \frac{b-a}{N} \sum_{n=0}^{N-1} f\left(a + \frac{n(b-a)}{N}\right) \quad (6)$$

for the midpoint rule and

$$\int_a^b f(x) dx \approx \frac{b-a}{6} [f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)]. \quad (7)$$

for Simpson's rule. ☺

Problem 1.6 (WebAssign, HW 15, # 6). Use the trapezoidal rule, the midpoint rule, and Simpson's rule to approximate the given integral with specified value n . (Round your answer to six decimal places).

$$\int_0^4 e^{2\sqrt{t}} dt, \quad n = 8.$$

Solution. Look at my comment above. ☺

Section 1.2: Homework 16

Problem 1.7 (WebAssign, HW 16, # 1). Determine whether the integral is convergent or divergent.

$$\int_{-\infty}^0 \frac{dx}{4-7x}.$$

Solution. Let's develop some strategies for attacking these problems. First of all, we like to work with positive numbers whenever possible, so let's make the substitution $u = -x$. This changes the bounds from $(-\infty, 0)$ to $(\infty, 0)$. Now, remember that

$$\int_a^b f(x) dx = - \int_b^a f(x) dx. \quad (8)$$

Thus, our integral turns into

$$\int_{-\infty}^0 \frac{dx}{4-7x} = - \int_0^{\infty} \frac{-du}{4+7u} = \int_0^{\infty} \frac{du}{4+7u}.$$

Finding the integral of this, we have

$$\int_0^{\infty} \frac{du}{4+7u} = \frac{1}{7} \ln |4+7u| \Big|_0^{\infty}.$$

Now, what happens as $u \rightarrow \infty$? The value of $\frac{1}{7} \ln |4+7u|$ gets bigger and bigger so the integral diverges. ☺

Problem 1.8 (WebAssign, HW 16, # 2). Determine whether the integral is convergent or divergent.

$$\int_2^{\infty} e^{-9p} dp.$$

Solution. Compute the integral

$$\int_2^{\infty} e^{-9p} dp = -\frac{1}{9}e^{-9p} \Big|_2^{\infty} = -\frac{1}{9}e^{-9p} + \frac{1}{9}e^{-18}.$$

What happens as $p \rightarrow \infty$? The value of $-\frac{1}{9}e^{-9p}$ gets closer and closer to 0 so the integral converges and is equal to

$$\boxed{\frac{1}{9e^{18}}}.$$

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Problem 1.9 (WebAssign, HW 16, # 3). Determine whether the integral is convergent or divergent.

$$\int_{-\infty}^{\infty} 3xe^{-x^2} dx.$$

Solution. Remember that if we have three points $a < c < b$ in the real line, we can rewrite the integral of $f(x)$ as

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx. \quad (9)$$

Rewrite the integral above as

$$\int_{-\infty}^{\infty} 3xe^{-x^2} dx = \int_{-\infty}^0 3xe^{-x^2} dx + \int_0^{\infty} 3e^{-x^2} dx - \underbrace{\int_0^{\infty} 3ue^{-u^2} du}_{I_1} + \underbrace{\int_0^{\infty} 3xe^{-x^2} dx}_{I_2}$$

where $u = -x$. For the same reasons as the previous problem, I_1 and I_2 converge. Moreover, it is easy to see that $I_1 = I_2$ so the integral $\boxed{-I_1 + I_2 = 0}$. ☺

Problem 1.10 (WebAssign, HW 16, # 4). Determine whether the integral is convergent or divergent.

$$\int_1^{\infty} 37 \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx.$$

Solution. Compute the integral by using the substitution $u = \sqrt{x}$

$$\int_1^{\infty} 37 \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \int_1^{\infty} 74e^{-u} du = -74e^{-u} \Big|_1^{\infty}$$

which converges for similar reasons as problem 1 from this homework. Thus, the integral is $\boxed{74/e}$. ☺

Problem 1.11 (WebAssign, HW 16, # 5). Determine whether the integral is convergent or divergent.

$$\int_{-\infty}^{\infty} 31 \cos \pi t \, dt.$$

Solution.

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Problem 1.12 (WebAssign, HW 16, # 6). Determine whether the integral is convergent or divergent.

$$\int_2^{\infty} \frac{dv}{v^2 + 5v - 6}.$$

Solution. Factor and use partial fractions

$$\begin{aligned} \int_2^{\infty} \frac{dv}{v^2 + 5v - 6} &= \int_2^{\infty} \frac{dv}{(v+6)(v-1)} \\ &= \int_2^{\infty} \left(\frac{-\frac{1}{7}}{v+6} + \frac{\frac{1}{7}}{v-1} \right) dv \\ &= -\frac{1}{7} \ln |v+6| + \frac{1}{7} \ln |v-1| \Big|_2^{\infty} \\ &= \frac{1}{7} \ln \left| \frac{v-1}{v+6} \right| \Big|_2^{\infty} \\ &= -\frac{1}{7} \ln \left| \frac{1}{8} \right| \\ &= \boxed{\frac{1}{7} \ln 8.} \end{aligned}$$

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Problem 1.13 (WebAssign, HW 16, # 7). Determine whether the integral is convergent or divergent.

$$\int_1^{\infty} 25 \frac{\ln x}{x} \, dx.$$

Solution. Use the substitution $u = \ln x$ then rewrite the integral and compute

$$\begin{aligned} \int_1^{\infty} 25 \frac{\ln x}{x} \, dx &= 25 \int_0^{\infty} u \, du \\ &= 25 \frac{1}{2} u^2 \Big|_0^{\infty} \\ &= \frac{25}{2} u^2 \Big|_0^{\infty} \end{aligned}$$

which clearly diverges as $u \rightarrow \infty$ since u^2 keeps getting bigger and bigger.

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Problem 1.14 (WebAssign, HW 16, # 8). Determine whether the integral is convergent or divergent.

$$\int_{-2}^3 \frac{45}{x^4} dx.$$

Solution. Rewrite the integral as

$$\int_{-2}^3 \frac{45}{x^4} dx = \int_{-2}^0 \frac{45}{x^4} dx + \int_0^3 \frac{45}{x^4} dx = 45 \underbrace{\int_{-2}^0 \frac{dx}{x^4}}_{I_1} + 45 \underbrace{\int_3^0 \frac{du}{u^4}}_{I_2}$$

where we let $u = -x$. Now, computing I_1 and I_2 we have

$$I_1 = -\frac{45}{3} x^{-3} \Big|_{-2}^0.$$

It is clear that as $x \rightarrow 0$, the integral grows closer and closer to $-\infty$. The same is true of I_2 so the integral diverges. ☹

Problem 1.15 (WebAssign, HW 16, # 9). Determine whether the integral is convergent or divergent.

$$\int_0^9 \frac{7}{\sqrt[3]{x-1}} dx.$$

Solution. By straight calculation

$$\begin{aligned} \int_0^9 \frac{7}{\sqrt[3]{x-1}} dx &= 7 \int_0^9 (x-1)^{-1/3} dx \\ &= \frac{21}{2} \sqrt{x-1} \Big|_0^9 \\ &= \frac{21}{2} (4-0) \\ &= \boxed{42}. \end{aligned}$$

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Section 1.3: Homework 17

Problem 1.16 (WebAssign, HW 17, # 1). Find the exact length of the curve.

$$y = 2 + 2x^{3/2}, \quad 0 \leq x \leq 1.$$

Solution. Remember the arclength formula

$$L(a, b) = \int_a^b \sqrt{1 + \frac{dy}{dx}^2} dx. \tag{10}$$

So let's find dy/dx , $dy/dx = 3x^{1/2}$ by the power rule so we have

$$\begin{aligned} L(0, 1) &= \int_0^1 \sqrt{1 + (3x^{1/2})^2} \, dx \\ &= \int_0^1 \sqrt{1 + 9x} \, dx \\ &= \int_0^1 (1 + 9x)^{1/2} \, dx \\ &= \frac{2}{3} (1 + 9x)^{3/2} \Big|_0^1 \\ &= \boxed{\frac{2}{3} (10\sqrt{10} + 1)}. \end{aligned}$$

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Problem 1.17 (WebAssign, HW 17, # 2). Find the exact length of the curve.

$$x = \frac{\sqrt{y}(y-3)}{3}, \quad 9 \leq y \leq 25.$$

Solution. By straight computation,

$$\frac{dx}{dy} = \frac{1}{3} \left(\frac{3}{2} y^{1/2} - \frac{3}{2} y^{-1/2} \right) = \frac{1}{2} y^{1/2} - \frac{1}{2} y^{-1/2}.$$

Thus, the arclength is

$$\begin{aligned} L(9, 25) &= \int_9^{25} \sqrt{1 + \left(\frac{1}{2} y^{1/2} - \frac{1}{2} y^{-1/2} \right)^2} \, dy \\ &= \frac{1}{2} \int_9^{25} (y^{1/2} + y^{-1/2}) \, dy \\ &= \frac{1}{3} y^{3/2} + y^{1/2} \Big|_9^{25} \\ &= \boxed{\frac{104}{3}}. \end{aligned}$$

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Problem 1.18 (WebAssign, HW 17, # 3). Find the exact length of the curve.

$$y = \ln |\sec x|, \quad 0 \leq x \leq \frac{\pi}{3}.$$

Solution. Compute

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x.$$

So

$$\begin{aligned} L(0, \pi/3) &= \int_0^{\pi/3} \sqrt{1 + \tan^2 x} \, dx \\ &= \int_0^{\pi/3} \sec x \, dx \\ &= \ln |\sec x + \tan x| \Big|_0^{\pi/3} \\ &= \boxed{\ln 3}. \end{aligned}$$

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Problem 1.19 (WebAssign, HW 17, # 4). Find the exact length of the curve.

$$y = \ln(1 - x^2), \quad 0 \leq x \leq \frac{1}{3}.$$

Solution. Find

$$\frac{dy}{dx} = -\frac{2x}{1 - x^2}.$$

Then, by partial fractions etc., we have

$$\begin{aligned} L(0, 1/3) &= \int_0^{1/3} \sqrt{1 + \frac{4x^2}{(1 - x^2)^2}} \, dx \\ &= \int_0^{1/3} \frac{1 + x^2}{1 - x^2} \, dx \\ &= \int_0^{1/3} \left(-1 + \frac{2}{1 - x^2} \right) \\ &= \int_0^{1/3} -1 + \frac{1}{1 + x} + \frac{1}{1 - x} \, dx \\ &= -x + \ln \left| \frac{1 + x}{1 - x} \right| \Big|_0^{1/3} \\ &= \boxed{\ln(2) - \frac{1}{3}}. \end{aligned}$$

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Problem 1.20 (WebAssign, HW 17, # 5). Find the exact area of the surface obtained by rotating the curve about the x -axis.

$$y = x^3, \quad 0 \leq x \leq 3.$$

Solution. Using the cylinder method, set $y = x^3$ to your length and the change in the arc will be $\sqrt{1 + (dy/dx)^2} \, dx = \sqrt{1 + 9x^4}$ so our surface area will be

$$S(0, 3) = \int_0^3 2\pi x^3 \sqrt{1 + 9x^4} \, dx$$

make the substitution $u = 1 + 9x^4$

$$\begin{aligned}
 &= \frac{2\pi}{36} \int_1^{730} \sqrt{u} \, du \\
 &= \frac{\pi}{18} \frac{2}{3} u^{3/2} \Big|_1^{730} \\
 &= \boxed{\frac{\pi}{27} (730\sqrt{730} - 1)}.
 \end{aligned}$$

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Problem 1.21 (WebAssign, HW 17, # 6). Find the exact area of the surface obtained by rotating the curve about the x -axis.

$$y = \sin\left(\frac{\pi x}{3}\right), \quad 0 \leq x \leq 3.$$

Solution. Find

$$\frac{dy}{dx} = \frac{\pi}{3} \cos \frac{\pi x}{3}.$$

So

$$\begin{aligned}
 S(0, 3) &= 2\pi \int_0^3 y \sqrt{1 + (dy/dx)^2} \, dx \\
 &= 2\pi \int_0^3 \sin\left(\frac{\pi x}{3}\right) \sqrt{1 + \frac{\pi^2}{9} \cos^2\left(\frac{\pi x}{3}\right)} \, dx
 \end{aligned}$$

make the substitution $u = (\pi/3) \cos(\pi x/3)$

$$\begin{aligned}
 &= -\frac{18}{\pi} \int_{\pi/3}^{-\pi/3} \sqrt{1 + u^2} \, du \\
 &= \frac{18}{\pi} \int_{-\pi/3}^{\pi/3} \sqrt{1 + u^2} \, du \\
 &= \frac{36}{\pi} \int_0^{\pi/3} \sqrt{1 + u^2} \, du \\
 &= \frac{36}{\pi} \int_0^{\pi/3} \sqrt{1 + u^2} \, du
 \end{aligned}$$

use a trig substitution

$$\begin{aligned}
 &= \frac{36}{\pi} \left[\frac{u\sqrt{1+u^2}}{2} - \frac{1}{2} \ln(u + \sqrt{1+u^2}) \right]_0^{\pi/3} \\
 &= \boxed{6\sqrt{1 + \frac{\pi^2}{9}} + \frac{18}{\pi} \left(\frac{\pi}{3} + \sqrt{1 + \frac{\pi^2}{9}} \right)}.
 \end{aligned}$$

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Problem 1.22 (WebAssign, HW 17, # 7). The given curve is rotated about the y -axis. Find the area of the resulting surface.

$$y = \sqrt[3]{x}, \quad 2 \leq y \leq 4.$$

Solution. Express x in terms of y , $x = y^3$ and find

$$\frac{dx}{dy} = 3y^2.$$

Then

$$\begin{aligned} S &= 2\pi \int_2^4 x \sqrt{1 + (dx/dy)^2} dy \\ &= 2\pi \int_2^4 y^3 \sqrt{1 + 9y^4} dy \\ &= \frac{2\pi}{36} \int_2^4 (36y^3) \sqrt{1 + 9y^4} dy \end{aligned}$$

make the substitution $u = 1 + 9y^4$, then

$$\begin{aligned} &= \frac{\pi}{18} \int_{145}^{2305} \sqrt{1 + u} du \\ &= \frac{\pi}{27} \left[u^{3/2} \right]_{145}^{2305} \\ &= \boxed{\frac{\pi}{2} (2305\sqrt{2305} - 145\sqrt{145})}. \end{aligned}$$

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Problem 1.23 (WebAssign, HW 17, # 8). The given curve is rotated about the y -axis. Find the area of the resulting surface.

$$y = 4 - x^2, \quad 0 \leq x \leq 5.$$

Solution. Find

$$\frac{dy}{dx} = -2x.$$

Then

$$S(0, 5) = 2\pi \int_0^5 x \sqrt{1 + 4x^2} dx$$

make the substitution $u = 1 + 4x^2$, then

$$\begin{aligned} &= \frac{\pi}{4} \int_1^{101} \sqrt{u} du \\ &= \frac{3\pi}{8} \left[u^{3/2} \right]_1^{101} \\ &= \boxed{\frac{3\pi}{8} (101\sqrt{101} - 1)}. \end{aligned}$$

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2 Exam II Problems

Relevant exam problems

Problem 2.1 (Spring 2014, # 8). Which of the following improper integrals converge?

I. $\int_0^{\infty} x e^{-x^2} dx$

II. $\int_{-\infty}^{\infty} \frac{dx}{x}$

III. $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}.$

Solution. ☺

Problem 2.2 (Spring 2014, # 9). Find the exact length of the curve $y = \ln(\sec x)$, $0 \leq x \leq \pi/3$.

Solution. ☺

Problem 2.3 (Spring 2015, # 9). If the upper part of the ellipse $y^2/4 + x^2/16 = 1$ is revolved around the x -axis to generate an ellipsoid S , then the surface area of S is given by?

Solution. ☺