MA166: Exam 2 Solutions

Carlos Salinas

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Exam 2 Solutions

Problem 1 (# 1, #)**.** Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{4^n}{5(3^{2n-1})}$$

Solution. This is a geometric series and it's not hard to see that. The first thing you should do is factor it

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{4^n}{5(3^{2n-1})} &= \frac{4}{15}\sum_{n=1}^{\infty} \frac{4^{n-1}}{3^{2n-2}} \\ &= \frac{4}{15}\sum_{n=1}^{\infty} \frac{4^{n-1}}{3^{2(n-1)}}\end{aligned}$$

now shift it back to turn it into a geometric series

$$\begin{aligned}&= \frac{4}{15}\sum_{n=0}^{\infty} \frac{4^{n}}{3^{2n}} \\ &= \frac{4}{15}\sum_{n=0}^{\infty} \left(\frac{4}{3^2}\right)^n \\ &= \frac{4}{15}\sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n\end{aligned}$$

since $|4/9| < 1$, this sequence converges and it converges to

$$\begin{aligned}&= \frac{4}{15}\left(\frac{1}{1-\frac{4}{9}}\right) \\ &= \frac{4}{15}\left(\frac{1}{\frac{5}{9}}\right) \\ &= \boxed{\frac{12}{25}.}\end{aligned}$$

Answer: D, ☺

Problem 2 (# 2, #)**.** Evaluate the integral

$$\int_0^1 \frac{x^2+1}{(x+1)^2}dx.$$

Solution. Rewrite the integral and use partial fractions

$$\int_0^1 \frac{x^2+1}{(x+1)^2}dx = \int_0^1 \frac{(x^2+1+2x)-2x}{(x+1)^2}dx$$

$$
\begin{aligned}
&= \int_0^1 \left[\frac{(x^2+1+2x)}{(x+1)^2} - \frac{2x}{(x+1)^2}\right]dx \\
&= \int_0^1 \left[\frac{(x+1)^2}{(x+1)^2} - \frac{2x}{(x+1)^2}\right]dx \\
&= \int_0^1 \left[1 - \frac{2x}{(x+1)^2}\right]dx \\
&= \underbrace{\int_0^1 1dx}_{I_1} - \underbrace{\int_0^1 \frac{2x}{(x+1)^2}dx}_{I_2}.
\end{aligned}
$$

Now, rewrite $I_1 = 1$ and that's easy. To find I_2 we use partial fractions

$$
\begin{aligned}
\frac{2x}{(x+1)^2} &= \frac{A}{x+1}\frac{B}{(x+1)^2} \\
2x &= A(x+1) + B \\
&= Ax + (A+B).
\end{aligned}
$$

So $A + B = 0$ and $A = 2$ so $B = -2$. Now we compute I_2

$$
\begin{aligned}
I_2 &= \int_0^1 \frac{2x}{(x+1)^2}dx \\
&= \int_0^1 \left[\frac{2}{x+1} - \frac{2}{(x+1)^2}\right]dx \\
&= \int_0^1 \frac{2}{x+1}dx - \int_0^1 \frac{2}{(x+1)^2}dx \\
&= \left[2\ln|x+1| + \frac{2}{x+1}\right]_0^1 \\
&= 2\ln 2 - 1.
\end{aligned}
$$

Hence the integral is

$$
I_1 - I_2 = 1 - (2\ln 2 - 1) = \boxed{2 - 2\ln 2.}
$$

Answer: E. ☺

Problem 3 (# 3, #)**.** Evaluate the integral

$$
\int_0^1 \frac{x^2}{x^2+1}dx.
$$

Solution. Factor and use partial fractions

$$
\begin{aligned}
\int_0^1 \frac{x^2}{x^2+1}dx &= \int_0^1 \frac{x^2+1-1}{x^2+1}dx \\
&= \int_0^1 \frac{\left(x^2+1\right)-1}{x^2+1}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 \left[\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right] dx \\
&= \int_0^1 \left[1 - \frac{1}{x^2+1}\right] dx \\
&= \underbrace{\int_0^1 1 dx}_{I_1} - \underbrace{\int_0^1 \frac{1}{x^2+1} dx}_{I_2}.
\end{aligned}
$$

It's easy to compute $I_1 = 1$. To compute I_2 you can either use a trig substitution or realize that the integral of $1/(x^2+1)$ is $\tan^{-1}(x)$.

Using the trig substitution, let $x = \tan\theta$, $dx = \sec^2\theta d\theta$ we have

$$
\begin{aligned}
\int_0^{/pi/4} \frac{1}{x^2+1} dx &= \int_0^{\pi/4} \sec^2\theta \cos^2\theta d\theta \\
&= \int_0^1 1 d\theta \\
&= [\theta]_0^{\pi/4} \\
&= \frac{\pi}{4}.
\end{aligned}
$$

Then the integral is

$$
I_1 - I_2 = \boxed{1 - \frac{\pi}{4}.}
$$

Answer: B. ☺

Problem 4 (#, #)**.** Which of the following statements are true?

(I) The sequence $a_n = \sin(n\pi)$ is convergent.

(II) The sequence $a_n = \dfrac{2n^3+1}{n-n^3}$ is divergent.

(III) The sequence $a_n = e^{\left(\frac{2n}{n+2}\right)}$ is convergent.

Solution. (II) clearly converges. First rewrite

$$
\frac{2n^3+1}{n-n^3} = -\frac{2n^3+1}{n^3-n}
$$

make the substitution $n = x$ and use l'Hôpital's rule

$$
\begin{aligned}
&= -\frac{6x^2}{3x^2-1} \\
&= -\frac{12x}{6x} \\
&= -2.
\end{aligned}
$$

(III) converges because the sequence $2n/(n+2)$ converges to 2, so $a_n \to e^2$.

(I) is well known to not converge since $\sin \pi x$ changes value from -1 to 1 and as we get closer and closer to infinity, it keeps on moving between these two values.

Answer: E. ☺

Problem 5 (# 5, #). Which of the following statements are true?

(I) Every positive bounded sequence is convergent.

(II) The sequence $a_n = \dfrac{n \cos n}{n^2+3}$ is convergent.

(III) The sequence $a_n = \dfrac{3^n}{2^{n+1}}$ is convergent.

Solution. (I) is false. Just consider $|\sin(\pi n/2)|$. This sequence goes from 0 to 1, 0 to 1, 0 to 1 indefinitely. This sequence is positive and bounded, but it does not converge.

(II) By l'Hôpital's as $n \to \infty$, $1 + 3/n^2 \to 1$ and $n(1+3/n^2) \to \infty$ as $n \to \infty$ so

$$\lim_{n\to\infty} \frac{\cos n}{n(1+\frac{3}{n^2})} \to 0.$$

☺

Problem 6 (# 6, #). Evaluate the integral $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx$. Hint: $\cos(2t) = 1 - 2\sin^2 t$.

Solution. Use a trigonometric substitution $\sin t = x$, $\cos t\ dt = x$ so $0 \le t \le \pi/2$

$$\begin{aligned}
\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx &= \int_0^{\pi/2} \frac{\sin^2 t}{\cos t} \cos t\ dt \\
&= \int_0^{\pi/2} \sin^2 t\ dt \\
&= \frac{1}{2}\left[\int_0^{\pi/2} 1 - \cos 2t\right] dt \\
&= \frac{1}{2}\left[\int_0^{\pi/2} 1\ dt - \int_0^{\pi/2} \cos 2t\ dt\right] \\
&= \frac{1}{2}\left[t - \frac{1}{2}\cos 2t\right]_0^{\pi/2} \\
&= \frac{1}{2}\left[\frac{\pi}{2} - (-1) - (0-1)\right]_0^{\pi/2}
\end{aligned}$$

$$= \boxed{\frac{\pi}{4}.}$$

Answer: E. ☺

Problem 7 (# 7, #)**.** Evaluate the integral

$$\int_4^9 \frac{\sqrt{x}}{x-1} dx.$$

Hints: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, $\frac{2}{u^2-1} = \frac{1}{u-1} - \frac{1}{u+1}$.

Solution. Make the substitution $u^2 = x$, $2u\ du = dx$. Then

$$\begin{aligned}
\int_4^9 \frac{\sqrt{x}}{x-1} dx &= \int_2^3 \frac{u}{u^2-1} 2u\ du \\
&= \int_2^3 \frac{2u^2}{u^2-1} du \\
&= 2\int_2^3 \frac{u^2}{u^2-1} du \\
&= 2\int_2^3 \frac{u^2-1+1}{u^2-1} du \\
&= 2\left[\int_2^3 \left(1 + \frac{1}{u^2-1}\right)\ du\right] \\
&= 2\int_2^3 1\ du + \int_2^3 \frac{2}{u^2-1} du \\
&= 2\int_2^3 1\ du + \int_2^3 \left[\frac{1}{u-1} - \frac{1}{u+1}\right] du \\
&= \left[2u + \ln\left|\frac{u-1}{u+1}\right|\right]_2^3 \\
&= \left[6 + \ln\left|\frac{2}{4}\right| - 4 - \ln\left|\frac{1}{3}\right|\right] \\
&= \boxed{2 + \ln(3/2)}.
\end{aligned}$$

Answer: A. ☺

Problem 8 (# 8, #)**.** Find the arc length of the curve $y = 2x^{3/2}$, $0 \le x \le 3$.

Solution. Find the derivative

$$\frac{dy}{dx} = 3\sqrt{x}.$$

Then

$$\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_0^3 \sqrt{1 + \left(3\sqrt{x}\right)^2}\ dx$$

$$= \int_0^3 \sqrt{1+9x}\,dx$$

make the substitution $u = 1+9x$, $du = 9\,dx$, $1 \le u \le 28$

$$\begin{aligned}
&= \frac{1}{9}\int_1^{28} \sqrt{u}\,du \\
&= \int_1^{28} u^{1/2}\,du \\
&= \frac{2}{27}\left[u^{3/2}\right]_1^{28} \\
&= \boxed{\frac{2}{27}\left(28^{3/2}-1\right).}
\end{aligned}$$

Answer: E. ☺

Problem 9 (# 9, #)**.** The curve

$$x = \frac{1}{3}\left(y^2+2\right)^{3/2}, \qquad 1 \le y \le 2,$$

is rotated about the y-axis. The area of the resulting surface is

$$\int_1^2 \frac{2\pi}{3}\left(y^2+2\right)^{3/2}\left(y^2+k\right)\,dy$$

for some constant k. What is k?

Solution. What we are really looking for is the simplification of

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}.$$

We need to find

$$\frac{dx}{dy} = y\sqrt{y^2+1}$$

so

$$\begin{aligned}
\sqrt{1+\left(\frac{dx}{dy}\right)^2} &= \sqrt{1+\left(y\sqrt{y^2+2}\right)^2} \\
&= \sqrt{1+y^2(y^2+2)} \\
&= \sqrt{1+y^4+2y^2} \\
&= \sqrt{y^4+2y^2+1} \\
&= \sqrt{\left(y^2+1\right)^2} \\
&= y^2+1.
\end{aligned}$$

If we compare this to $\int_1^2 \frac{2\pi}{3}\left(y^2+2\right)^{3/2}\left(y^2+k\right)$ we see that $k=1$.

Answer: C. ☺

Problem 10 (# 10, #). Find the x-coordinate, $\bar{x}$, of the centroid of the region bounded by $y=-2x+3$, $y=0$, $x=0$ and $x=1$.

Solution. First we compute the area of the region

$$\begin{aligned} A &= \int_0^1 -2x+3 \\ &= \left[-x^2+3x\right]_0^1 \\ &= 2. \end{aligned}$$

Then the mass is 2ρ and the moment about the y-axis is

$$\begin{aligned} M_y &= \rho \int_0^1 x(-2x+3)\,dx \\ &= \rho \int_0^1 -2x^2+3x\,dx \\ &= \rho\left[-\frac{2}{3}x^3+\frac{3}{2}x^2\right]_0^1 \\ &= \rho\left[-\frac{2}{3}+\frac{3}{2}\right]_0^1 \\ &= \frac{5}{6}\rho. \end{aligned}$$

So

$$\bar{x} = \frac{M_y}{m} = \frac{(5/6)rho}{2\rho} = \boxed{\frac{5}{12}.}$$

Answer: D. ☺

Problem 11 (#, #). Evaluate the integral

$$\int_0^{\pi/3} \tan^3 x \sec x\,dx.$$

Solution. Use the following trig identity

$$\sec^2 x - \tan^2 x = 1.$$

Rewrite the integral

$$\int_0^{\pi/3} \tan^3 x \sec x\,dx = \int_0^{\pi/3} \left(\tan^2 x\right) \tan x \sec x\,dx$$

$$= \int_0^{\pi/3} \left(\sec^2 x - 1\right) \tan x \sec x \, dx$$

make the substitution $u = \sec x$, $du = \tan x \sec x \, dx$

$$\begin{aligned}
&= \int_1^2 \left(u^2 - 1\right) \tan x \sec x \frac{du}{\tan x \sec x} \\
&= \int_1^2 u^2 - 1 \, du \\
&= \left[\frac{1}{3}u^3 - u\right]_1^2 \\
&= \frac{8}{3} - 2 - \frac{1}{3} + 1 \\
&= \frac{7}{3} - 1 \\
&= \boxed{\frac{4}{3}.}
\end{aligned}$$

Answer: C. ☺

Problem 12 (# 12, #)**.** Evaluate the integral $\displaystyle\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}} \, dt$ using the table of integrals formula $\displaystyle\int \frac{du}{1+u^2} = \ln\left(u + \sqrt{1+u^2}\right) + C$.

Solution. Set $u = \sin t$, $du = \cos t \, dt$, then we have the integral

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}} \, dt &= \int_0^1 \frac{1}{1+u^2} \, dt \\
&= \left[\ln\left(u + \sqrt{1+u^2}\right)\right]_0^1 \\
&= \ln\left(1 + \sqrt{2}\right) - \ln\left(0 + \sqrt{1+0}\right) \\
&= \boxed{\ln\left(1 + \sqrt{2}\right).}
\end{aligned}$$

Answer: A. ☺