4.6: 14, 15, 16, 17, 19, 20, 23, 24

4.6.14 Let $S=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}, \begin{bmatrix}-1&1\\1&-1\end{bmatrix}\right\}$. Find a basis for the subspace $W=\text{Span}(S)$ of M_{22}.

As S spans W by definition, we just need to check which subset is L.I.

$$\left[\begin{array}{cccc|c}1&0&1&-1&0\\0&1&1&1&0\\0&1&1&1&0\\1&0&1&-1&0\end{array}\right]\begin{array}{l}\\ \\ -r_2+r_3\\ -r_1+r_4\end{array} \left[\begin{array}{cccc|c}1&0&1&-1&0\\0&1&1&1&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right]$$

shows $S'=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ are L.I. and $W=\text{Span}(S')$ so form a basis.

4.6.15 Find all values of a for which $\{[a^2\ 0\ 1], [0\ a\ 2], [1\ 0\ 1]\}$ is a basis for R_3.

We need to find a s.t. this spans R_3 and are L.I.

Say $a_1[a^2\ \ 0\ \ 1] + a_2[0\ \ a\ \ 2] + a_3[1\ \ 0\ \ 1] = [x\ y\ z]$. Then

$$\left[\begin{array}{ccc|c}a^2&0&1&x\\0&a&0&y\\1&2&1&z\end{array}\right]\begin{array}{c}r_1\\ \updownarrow\\ r_3\end{array} \left[\begin{array}{ccc|c}1&2&1&z\\0&a&0&y\\a^2&0&1&x\end{array}\right]_{-a^2r_1+r_3} \left[\begin{array}{ccc|c}1&2&1&z\\0&a&0&y\\0&-2a^2&1-a^2&x-a^2z\end{array}\right]_{-2ar_2+r_3} \left[\begin{array}{ccc|c}1&2&1&z\\0&a&0&y\\0&0&1-a^2&x-ay-a^2z\end{array}\right]$$

we need $a\neq 0$ (or middle row is zero) and $1-a^2\neq 0$ or $a\neq 1,-1$. Thus this is a basis if $a\neq 0,-1,1$.

4.6.16 Find a basis for the subspace W of M_{33} consisting of all symmetric matrices.

Let A be in W. Then A is of the form $\begin{bmatrix}a&b&c\\b&d&e\\c&e&f\end{bmatrix}$. Split A as

$$\begin{bmatrix}a&b&c\\b&d&e\\c&e&f\end{bmatrix} = a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + b\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} + c\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix} + d\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + e\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix} + f\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

$S=\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$ clearly spans W and are linearly independent, for if $A=0$, then by the decomposition above, $a=b=c=d=e=f=0$.

4.6.17 Find a basis for the subspace of M_{33} consisting of all diagonal matrices.

$\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}$ is a general matrix in the subspace W of all diagonal matrices. Then decompose it as $a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + b\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + c\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}$. Set $S=\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$.

S spans W by the decomposition, and is L.I. Thus S forms a basis.

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Find a basis for the given subspaces of $\mathbb{R}^3$ and $\mathbb{R}^4$.

4.6.19 (a) All vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $b = a + c$.

Let W be this subspace. W is a V.S. by checking closures.

Now $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ a+c \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

S forms a basis. S spans W by construction. S is L.I. for

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{-r_1+r_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{-r_2+r_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ as needed.

(b) All vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $b = a$.

Let W be this subspace. W is a V.S. by checking closures.

Now $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ a \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ so $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

S spans W by construction and $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{-r_1+r_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ shows L.I.

Thus S forms a basis.

(c) All vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $2a + b - c = 0$.

Let W be this subspace. W is a V.S. by checking closures. Now $c = 2a + b$ so

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 2a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. S spans W

by construction and $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{-2r_1+r_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{-r_2+r_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ shows L.I. Hence is a basis.

4.6.20 (a) All vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a = 0$.

Let W be this subspace. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ so let $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

S is a basis for it is L.I. and span W by construction.

(b) All vectors of the form $\begin{bmatrix} a+c \\ a-b \\ b+c \\ -a+b \end{bmatrix}$.

Let W be this subspace.

$\begin{bmatrix} a+c \\ a-b \\ b+c \\ -a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ Span W and

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[r_1+r_4]{-r_1+r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow[-r_3+r_4]{r_3+r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow{-1/2 r_4} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[-r_4+r_3]{-r_4+r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Shows L.I.. Thus forms a basis.

(c) All vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a-b+5c=0$.

Let W be this subspace. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b-5c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$.

$S = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}\right\}$. S Spans W. $\left[\begin{array}{cc|c} 1 & -5 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{5r_3+r_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{-r_1+r_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

So S is L.I. Thus it forms a basis.

Find the dimensions of the given subspaces of R_4.

4.6.23 (a) All vectors of the form $[a \ b \ c \ d]$, where $d=a+b$.

Let W be this subspace. $[a \ b \ c \ d] = [a \ b \ c \ a+b] = a[1\,0\,0\,1] + b[0\,1\,0\,1] + c[0\,0\,1\,0]$.

$S = \{[1\,0\,0\,1], [0\,1\,0\,1], [0\,0\,1\,0]\}$ spans W, and $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{-r_1-r_2+r_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ shows they are L.I.

So S is a basis and $\dim(W)=3$.

(b) All vectors of the form $[a \ b \ c \ d]$, where $c=a-b$ and $d=a+b$.

Let W be this subspace. $[a \ b \ c \ d] = [a \ b \ a-b \ a+b] = a[1\,0\,1\,1] + b[0\,1\,-1\,1]$.

$S = \{[1\,0\,1\,1][0\,1\,-1\,1]\}$ spans W.

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right] \begin{matrix} \\ \\ -r_1+r_3 \\ -r_1+r_4 \end{matrix} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right] \begin{matrix} \\ \\ r_2+r_3 \\ -r_2+r_4 \end{matrix} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ So they are L.I. and thus form a basis.

So $\dim(W) = 2$.

4.6.24 (a) All vectors of the form $[a \ b \ c \ d]$, where $a=b$.

Let W be this subspace. $[a \ b \ c \ d] = [a \ a \ c \ d] = a[1\,1\,0\,0] + c[0\,0\,1\,0] + d[0\,0\,0\,1]$.

$\mathrm{Span}(S)=W$ and $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-r_1+r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ So they are L.I. Thus form a basis.

$\dim(W)=3$

(b) All vectors of the form $[a+c \ \ a-b \ \ b+c \ \ -a+b]$.

Let W be this subspace. Then

$[a+c \ \ a-b \ \ b+c \ \ -a+b] = a[1\,1\,0\,-1] + b[0\,-1\,1\,1] + c[1\,0\,1\,0]$.

Let S be these three vectors. Then

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \begin{matrix} \\ -r_1+r_2 \\ \\ r_1+r_4 \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{matrix} -r_2 \\ r_2+r_3 \\ r_2+r_4 \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $[1\,1\,0\,-1], [0\,-1\,1\,1]$

Span W and $\dim(W)=2$