MA 562: Notes

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1 Preliminaries

These set of notes are based off of Boothby's *Differential Geometry* book, chapters 1 through 6.

Definition 1. A *topological space* M is a pair $(X, \mathcal{T})$, where X is a set, $\mathcal{T}$ is a collection of subsets of X such that

(a) $\emptyset, X \in \mathcal{T}$.

(b) The union of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

$$\{U_\alpha\} \subset \mathcal{T} \implies \bigcup_\alpha U_\alpha \in \mathcal{T}.$$

(c) Intersection of a finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

$$\{U_1, ..., U_k\} \subset \mathcal{T} \implies \bigcap_{j=1}^{k} U_j \in \mathcal{T}.$$

$\mathcal{T}$ is called the *topology* of M. Elements of $\mathcal{T}$ are called the *open sets* of M. By abuse of notation, we sometimes refer to X as M.

Definition 2. (a) A *metric* on X is a function $d\colon X \times X \to \mathbf{R}$ such that

(1) $d(x, y) \geq 0\ \forall x, y \in X$ and $d(x, y) = 0 \iff x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) + d(y, z) \geq d(x, z)$ (the triangle inequality).

(b) $B_d(x, r) = \{\, y \in X \mid d(x, y) < r \,\}$.

(c) A topological space M is a *metric space* if the set of balls $B_d(x, r)$ form a *basis* of M, i.e., any open set of M can be written as a union of open balls $B_d(x, r)$ for some $x \in X$, $r > 0$.

Definition 3. A topological space X is *Hausdorff* if for any $x_1 \neq x_2$ in X, there exist open sets $U_1 \ni x_1$, $U_2 \ni x_2$ such that $U_1 \cap U_2 = \emptyset$.

Definition 4. A *topological manifold* M of dimension n is a topological space such that

(a) M is Hausdorff.

(b) locally Euclidean, i.e., $\forall x \in M$ there exists a neighborhood U of X which is homeomorphic to $V \subset \mathbf{R}^n$ (there exists a map $f\colon U_x \to V \subset \mathbf{R}^n$ such that f is bijective, continuous and f^{-1} is continuous).

(c) M has a countable basis of open sets.

Theorem 1 (Boothby I.3.6)**.** *A* topological manifold *is metrizable (also locally connected, locally compact, and normal).*

Definition 5. (a) A *covering* of a topological manifold is a collection of open sets $\{U_\alpha\}$ such that any $x \in M$ is contained in some U_α.

(b) A manifold is *compact* if every open cover contains a finite subcover.

Definition 6. (1) Half space
$$\mathbf{H}^n = \{\, x \in \mathbf{R}^n \mid x_n \geq 0 \,\}.$$

(2) Manifold with boundary. (Similar to definition 4)

  (a) M is Hausdorff.

  (b) M has a countable basis of open sets.

  (c) For any $x \in M$, there exists U open, $x \in U$ such that:

    (i) $\varphi\colon U \to V \subset \mathbf{R}^n$ is a homeomorphism, or

    (ii) $\varphi\colon U \to V \subset \mathbf{H}^n$ is a homeomorphism with x such that $\varphi(x) \in \partial \mathbf{H}^n$ referred to as *boundary points.*

(3)

Example 1 (Unit Quaternions and Rotations in $\mathbf{R}^3$)**.**

$$f(v) = z \wedge z^{-1} \qquad v = (v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$$

where $z = \cos(\alpha/2) + \sin(\alpha/2)\hat{v}$. $\hat{v} = v/\|v\|$. Quaternion multplication:

$$\begin{aligned} ij &= k & ji &= -k \\ jk &= i & kj &= -i \\ ki &= j & ik &= -j. \end{aligned}$$

Can check that z and $-z$ correspond to the same rotation.

Topologically, unit quaternions $\simeq S^3 = \left\{\, x \in \mathbf{R}^4 \;\middle|\; \|x\| = 1 \right\}$ and rotations $\simeq S^3/{\sim}$, $z \sim -z$

$$\mathbf{R}P^3 \approx \big(\mathbf{R}^4 \setminus \{0\}\big)/x \sim \lambda x.$$

for all $\mathbf{R}^{n+1} \setminus \{0\}$ can always find λ such that λx has norm 1. There are precisely 2 such λ which differ by a sign. Therefore, $\mathbf{R}P^n$ can be constructed by identifying antipodal points of S^n in $\mathbf{R}^{n+1}$.