MA557 Problem Set 5

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Problem 5.1

For I an R-ideal consider the multiplicatively closed set $S = 1 + I$. Show that

(a) $\widetilde{S} = R \smallsetminus \bigcup \mathfrak{m}$, where the union is taken over all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R) \cap V(I)$.
(b) $\mathfrak{m}\text{-Spec}(R/I)$ are homeomorphic.

Proof. (a) We will show double inclusion. Recall from 4.18 that

(b) ∎

Problem 5.2

Show that the following are equivalent for a ring R:

(a) there exist rings $R_1 \neq 0$ and $R_2 \neq 0$ so that $R \cong R_1 \times R_2$;
(b) there exist an idempotent $e \in R$ with $e \neq 0$ and $e \neq 1$;
(c) $\operatorname{Spec}(R)$ is disconnected.

Proof. (a) $\iff$ (b) is immediate for suppose $R \cong R_1 \times R_2$ by $\varphi\colon R \to R_1 \times R_2$. Then, since φ is a bijection, there exist an $r \in R$ that maps to the idempotent element $(1, 0) \in R_1 \times R_2$.

Conversely, suppose $e \in R$ is idempotent. Then $e' = 1 - e$ is also idempotent since

$$(e')^2 = (1-e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e.$$

Moreover

$$ee' = e(1-e) = e - e^2 = e - e = 0.$$

Let R_1 and R_2 be the subrings of R generated by e and e', respectively. Then we claim that $R \cong R_1 \times R_2$ via $\varphi(r) = (re, re')$. It is clear that φ is a ring homomorphism: take $r_1, r_2 \in R$ then

$$\begin{aligned} \varphi(r_1 + r_2) &= ((r_1 + r_2)e, (r_1 + r_2)e') \\ &= (r_1 e + r_2 e, r_1 e' + r_2 e') \\ &= (r_1 e, r_1 e') + (r_2 e, r_2 e') \\ &= \varphi(r_1) + \varphi(r_2) \end{aligned} \qquad \begin{aligned} \varphi(r_1 r_2) &= (r_1 r_2 e, r_1 r_2 e') \\ &= \big(r_1 r_2 e^2, r_1 r_2 (e')^2\big) \\ &= (r_1 e, r_1 e')(r_2 e, r_2 e') \\ &= \varphi(r_1)\varphi(r_2). \end{aligned}$$

To prove surjective take $(r, s) \in R_1 \times R_2$ then, $r = r_1 e$ and $s = r_2 e'$ for $r_1, r_2 \in R$ then

$$\begin{aligned} \varphi(r_1 e + r_2 e') &= \varphi(r_1 e) + \varphi(r_2 e') \\ &= (r_1 e, r_1 e e') + (r_2 e' e, r_2 e e') \\ &= (r_1 e, 0) + (0, r_2 e') \\ &= (r_1 e, r_2 e') \\ &= (r, s). \end{aligned}$$

To prove injectivity take $r \in \ker \varphi$. Then $\varphi(r) = (re, re') = (0, 0)$. Then $re - re' = r(e - e') = r \cdot 1 = 0$ so $r = 0$.

(a) $\implies$ (c) Recall that a topological space X is disconnected if there exist disjoint open sets A, B with $X = A \cup B$. Suppose $R \cong R_1 \times R_2$. Then $\operatorname{Spec}(R) \approx \operatorname{Spec}(R_1 \times R_2)$: Keeping the notation as before, φ is a set bijection so it induces a bijection, call it φ^*, on $\operatorname{Spec}(R) \to \operatorname{Spec}(R_1 \times R_2)$ by sending $\operatorname{Spec}(I) \mapsto \operatorname{Spec}(\varphi(I))$; Now let $I \subset R$ be an ideal, then

$$\varphi^*(V(I)) = \varphi^*(V(eI + e'I)) = V(\varphi(eI) + \varphi(e'I)) = V(eI \times e'I)$$

is closed. Thus, φ^* is a homeomorphism. Now, we claim that the sets $A = V(R_1 \times 0)$ and $B = V(0 \times R_2)$ constitute a separation of R. First note by 4.20(2) that

$$A \cup B = V(R_1 \times 0) \cup V(0 \times R_2) = V((R_1 \times 0) \cap (0 \times R_2)) = V(0) = \operatorname{Spec}(R).$$

Moreover
$$A \cap B = V(R_1 \times 0) \cap V(0 \times R_2) = V(R_1 \times 0 + 0 \times R_2) = V(R) = \emptyset.$$

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Problem 5.3

A topological space is called *Noetherian* if the set of closed sets satisfies the dcc. Show that if a ring R is Noetherian then so is $\mathrm{Spec}(R)$, but that the converse does not hold.

Proof. Suppose R is Noetherian, then for any ascending chain of ideals

$$I_1 \subset I_2 \subset \cdots \subset I_N = I_{N+1} = \cdots$$

the chain is stationary for some positive integer N. To show that closed sets of $\mathrm{Spec}(R)$ satisfy the dcc, it suffices to show that basic closed sets for the topology on $\mathrm{Spec}(R)$ satisfy the dcc. Consider the chain

$$V(I_1) \supset V(I_2) = V(I_1 + I_2) = V(I_1) \cap V(I_2) \supset V(I_1) \cap V(I_2) \cap V(I_3) \supset \cdots .$$

This chain, like before, stabilizes at N so that we have

$$V(I_1) \supset V(I_2) \supset \cdots \supset V(I_N) = V(I_{N+1}) = \cdots .$$

■

Problem 5.4

A nonempty closed subset V of a topological space is called *irreducible* if $V = V_1 \cup V_2$, V_1 and V_2 closed subset, implies $V_1 = V$ or $V_2 = V$.

(a) Show that in a Noetherian topological space every nonempty closed subset is a finite union of irreducible closed subsets.
(b) Show that $V(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec}(R)$, are exactly the irreducible closed subsets of $\operatorname{Spec}(R)$.

Proof. ■

Problem 5.5

Show that a Noetherian ring has only finitely many minimal prime ideals.

Proof. ■

Problem 5.6

Show that a nonzero ring has at least one minimal prime ideal.

Proof. ■