MA 519: Homework 11

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PROBLEM 11.1 (DASGUPTA 7.2 (A), (B), (C), (D), (E))

(a) Suppose $E|X_n - c|^\alpha \to 0$, where $0 < \alpha < 1$. Does X_n necessarily converge in probability to c?

(b) Suppose $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0,1)$. Under what condition on a_n can we conclude that $X_n \xrightarrow{\mathcal{P}} \theta$?

(c) $o_p(1) + O_p(1) =$?

(d) $o_p(1)O_p(1) =$?

(e) $o_p(1) + o_p(1)O_p(1) =$?

SOLUTION. First let us tackle part (a) of this problem. We suspect that the statement "X_n *converges in probability to* c" is false but first let us attempt to prove this to see where the problem in the implication may lie. Fix $0 < \alpha < 1$ and suppose

$$E(|X_n - c|^\alpha) \longrightarrow 0 \text{ as } n \to \infty. \tag{11.1}$$

Set $\varphi(x) := x - c$ and let g_n denote the PDF of $X_n - c$. By the Jacobian formula

$$g_n(x) = \frac{f_n(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))} = f_n(x + c)$$

where we denote by f_n the PDF of X_n. Then equation (11.1) now reads

$$\left(\int_{-\infty}^{\infty} x f_n(x + c) \, dx \right)^\alpha \longrightarrow 0 \text{ as } n \to \infty.$$

Fix $\varepsilon > 0$. By Markov's inequality, we have

$$P(|X_n - c| > \varepsilon) \leq \frac{E(|X_n - c|^\alpha)}{\varepsilon^\alpha}$$

For part (c), suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n = o_p(1)$ and $b_n = O_p(1)$, then for the sequence $\{c_n := a_n + b_n\}$ the most we can expect is $c_n = O_p(1)$. Indeed, we know that if a sequence is $o_p(1)$ then it is also $O_p(1)$ therefore there exists K_1 and K_2 such that $|a_n| \leq K_1$, $|b_n| \leq K_2$ for all $n \geq 1$. Therefore, $|c_n| \leq K_1 + K_2$ for all $n \geq 1$. ∎

Problem 11.2 (DasGupta 7.3 [Monte Carlo])

Consider the purely mathematical problem of finding a definite integral $f(x)\,dx$ for some (possibly complicated) function $f(x)$. Show that the SLLN provides a method for approximately finding the value of the integral by using appropriate averages $\frac{1}{n}\sum_{k=1}^{n} f(X_k)$.

     Numerical analysts call this Monte Carlo integration.

Solution.   ∎

PROBLEM 11.3 (DASGUPTA 7.4 (A), (B))

Suppose $X_1, \ldots,$ are i.i.d. and that $E(X_1) = \mu \neq 0$, $\text{Var}(X_1) = \sigma^2 < \infty$. Let $S_{m,p} = \sum_{k=1}^{m} X_k^p$, $m \geq 1$, $p = 1, 2$.

(a) Identify with proof the almost sure limit of $S_{m,1}/S_{n,1}$ for fixed m, and $n \to \infty$.
(b) Identify with proof the almost sure limit of $S_{n-m,1}/S_{n,1}$ for fixed m, and $n \to \infty$.

SOLUTION.   ∎

PROBLEM 11.4 (DASGUPTA 7.5 (A))

Let A_n, $n \geq 1$, A be events with respect to a common sample space Ω.

(a) Prove that $I_{A_n} \xrightarrow{\mathcal{L}} I_A$ if and only if $P(A_n) \to P(A)$.

SOLUTION.   ∎

PROBLEM 11.5 (DASGUPTA 7.11 [SAMPLE MAXIMUM])

Let X_k, $k \geq 1$, be an i.i.d. sequence, and $X_{(n)}$ the maximum of $X_1, \ldots, X_n$. Let $\xi(F) = \sup\{\, x : F(x) < 1 \,\}$, where F is the common CDF of the X_k. Prove that $X_{(n)} \xrightarrow{\text{a.s.}} \xi(F)$.

SOLUTION.   ∎

Problem 11.6 (DasGupta 7.14 (a))

Suppose X_k are i.i.d. standard Cauchy. Show that

  (a) $P(|X_n| > n \text{ infinitely often}) = 1$.

Solution.   ∎

Problem 11.7 (DasGupta 7.16 [Coupon Collection])

Cereal boxes contain independently and with equal probability exactly one of n different celebrity pictures. Someone having the entire set of n pictures can cash them in for money. Let W_n be the minimum number of cereal boxes one would need to purchase to own a complete set of the pictures. Find a sequence a_n such that $W_n/a_n \xrightarrow{\mathcal{P}} 1$.

(*Hint:* Approximate the mean of W_n.)

Solution.   ∎

PROBLEM 11.8 (DASGUPTA 7.17)

Let $X \sim \text{Bin}(n, p)$. Show that $(X_n/n)^2$ and $X_n(X_n - 1)/(n(n - 1))$ both converging in probability to p^2. Do they converge almost surely?

SOLUTION.   ∎

PROBLEM 11.9 (DASGUPTA 7.21)

Let $X_1, X_2, \ldots,$ be i.i.d. $U[0,1]$. Let

$$G_n = (X_1 \cdots X_n)^{1/n}.$$

Find c such that $G_n \xrightarrow{\mathcal{P}} c$.

SOLUTION.   ∎

PROBLEM 11.10 (DASGUPTA 7.30 [CONCEPTUAL])

Suppose $X_n \xrightarrow{\mathcal{L}} X$, and also $Y_n \xrightarrow{\mathcal{L}} X$. Does this mean that $X_n - Y_n$ converge in distribution to (the point mass at) zero?

SOLUTION.   ∎

PROBLEM 11.11 (DASGUPTA 7.31 (A))

(a) Suppose $a_n(X_n - \theta) \to N(0, \tau^2)$; what can be said about the limiting distribution of $|X_n|$, when $\theta \neq 0$, $\theta = 0$?

SOLUTION.   ∎