Math 527 - Homotopy Theory
Spring 2013
Homework 8 Solutions

Problem 1. Let X be an n-connected space, for some $n \geq 0$. Show that X admits a CW approximation with a single 0-cell and cells in dimensions greater than n.

Solution. Since X is n-connected, the inclusion of any point $\gamma_n \colon * \to X$ is n-connected. Take $X'_n := *$. As in the proof of the CW approximation theorem, we can form X'_{n+1} by attaching $(n+1)$-cells to X'_n and extend γ_n to a map $\gamma_{n+1} \colon X'_{n+1} \to X$ which is $(n+1)$-connected.

Repeating the process, we inductively build a CW complex $X' := \operatorname{colim}_i X'_i$ with a weak homotopy equivalence $\gamma \colon X' \xrightarrow{\sim} X$. Note that X' has a single 0-cell and cells of dimension greater than n, as its n-skeleton is $X'_n = *$. $\square$

Problem 2. (Hatcher § 4.1 Exercise 17) Let X and Y be CW-complexes where X is m-connected and Y is n-connected, for some $m, n \geq 0$.

a. Show that the inclusion $X \vee Y \to X \times Y$ is $(m+n+1)$-connected.

Solution. By Problem 1, let $\gamma_X \colon X' \xrightarrow{\sim} X$ and $\gamma_Y \colon Y' \xrightarrow{\sim} Y$ be CW approximations of X (resp. Y) with a single 0-cell and cells in dimensions greater than m (resp. n).

Since X and Y are CW complexes, γ_X and γ_Y are homotopy equivalences, by Whitehead. Since X' and X are well-pointed, γ_X is homotopic to a pointed map $\gamma'_X \colon X' \xrightarrow{\simeq} X$, and the latter is a pointed homotopy equivalence; likewise for $\gamma'_Y \colon Y' \xrightarrow{\simeq} Y$. Let $\lambda_X \colon X' \xrightarrow{\simeq} X$ and $\lambda_Y \colon Y' \xrightarrow{\simeq} Y$ be pointed homotopy inverses of γ'_X and γ'_Y respectively.

Because γ'_X and γ'_Y are pointed maps, they define together the map

$$\gamma'_X \vee \gamma'_Y \colon X' \vee Y' \to X \vee Y$$

which is still a pointed homotopy equivalence. Indeed, the map

$$\lambda_X \vee \lambda_Y \colon X \vee Y \to X' \vee Y'$$

is a pointed homotopy inverse of $\gamma'_X \vee \gamma'_Y$.

Moreover, the map $\gamma'_X \times \gamma'_Y \colon X' \times Y' \to X \times Y$ is also a pointed homotopy equivalence, with pointed homotopy inverse $\lambda_X \times \lambda_Y \colon X \times Y \to X' \times Y'$. Thus the connectivity of $X \vee Y \to X \times Y$ is the same as that of $X' \vee Y' \to X' \times Y'$.

By construction, all cells in $X' \times Y'$ of dimension less than $m+n+2$ are in the subcomplex $X' \vee Y' \subseteq X' \times Y'$. Therefore the $(m+n+1)$ skeleta of $X' \vee Y'$ and $X' \times Y'$ agree, which implies that the inclusion is $(m+n+1)$-connected. $\square$

b. Show that the smash product $X \wedge Y$ is $(m+n+1)$-connected.

Solution. As in part (a), the map $\gamma_X \wedge \gamma_Y \colon X' \wedge Y' \xrightarrow{\simeq} X \wedge Y$ is a pointed homotopy equivalence, so that the connectivity of $X \wedge Y$ equals that of $X' \wedge Y'$. Quotienting a subcomplex yields a CW structure on the quotient

$$X' \wedge Y' = X' \times Y' / X' \vee Y'$$

with a 0-cell corresponding to $X' \vee Y'$, plus a cell for each cell of $X' \times Y'$ that was not in $X' \vee Y'$. Thus $X' \wedge Y'$ has a single 0-cell and cells of dimension at least $m+n+2$, so that it is $(m+n+1)$-connected. $\square$

Problem 3. (Whitehead products) For each $n \geq 1$, consider the sphere S^n with its CW-structure having one 0-cell and one n-cell. For any positive integers $p, q \geq 1$, the product $S^p \times S^q$ inherits a CW-structure with four cells, in dimensions 0, p, q, and $p+q$ respectively. The $(p+q-1)$-skeleton of $S^p \times S^q$ is $S^p \vee S^q$ so that the attaching map of the top cell has the form

$$w \colon S^{p+q-1} \to S^p \vee S^q.$$

For any pointed space X, precomposition by w defines an operation

$$\pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X)$$

called the **Whitehead product**, denoted by brackets $[\alpha, \beta] \in \pi_{p+q-1}(X)$.

a. For $p = q = 1$, the Whitehead product takes the form $\pi_1(X) \times \pi_1(X) \to \pi_1(X)$. What is this map?

Solution. The characteristic map of the $(p+q)$-cell of $S^p \times S^q$ is the product of two quotient maps

$$\varphi_{p+q} \colon D^p \times D^q \xrightarrow{\varphi_p \times \varphi_q} D^p/\partial D^p \times D^q/\partial D^q \cong S^p \times S^q$$

whose restriction to the boundary of $D^p \times D^q \cong D^{p+q}$ is the attaching map

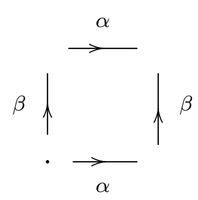
$$\partial(D^p \times D^q) = (\partial D^p \times D^q) \cup (D^p \times \partial D^q) \xrightarrow{w} S^p \vee S^q \subset S^p \times S^q$$

explicitly given by

$$\begin{aligned} w(x, y) &= (\overline{x}, \overline{y}) \\ &= \begin{cases} (\overline{x}, *) & \text{if } y \in \partial D^q \\ (*, \overline{y}) & \text{if } x \in \partial D^p \end{cases} \end{aligned}$$

where $\overline{x} = \varphi_p(x) \in D^p/\partial D^p$ denotes the class of $x \in D^p$.

In the case $p = q = 1$, given loops $\alpha, \beta \in \pi_1(X)$, the loop $[\alpha, \beta] \in \pi_1(X)$ is obtained by going around the boundary of the disk $D^2 \cong D^1 \times D^1$ pictured here:



from the basepoint $(0, 0) \in D^2$. Going around counterclockwise – there is an orientation convention here – we obtain:

$$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} \in \pi_1(X)$$

the commutator of α and β. $\square$

b. (Hatcher § 4.2 Exercise 36) More generally, for $p = 1$ and $q \geq 1$, describe the Whitehead product $\pi_1(X) \times \pi_q(X) \to \pi_q(X)$.

Solution. Given $\gamma \in \pi_1(X)$ and $\beta \in \pi_q(X)$, the Whitehead product $[\gamma, \beta] \in \pi_q(X)$ is represented by

$$\begin{aligned} \partial(D^1 \times D^q) = (\partial D^1 \times D^q) \cup (D^1 \times \partial D^q) &\to X \\ (0, x) &\mapsto \beta(x) \\ (t, z) &\mapsto \gamma(t) \ \text{ for } z \in \partial D^q \\ (1, x) &\mapsto \beta(x). \end{aligned}$$

Thus, its restriction to the faces $(D^1 \times \partial D^q) \cup (\{1\} \times D^q)$ represents $\gamma \cdot \beta$, by definition of the π_1-action on π_q. The restriction to the remaining face $\{0\} \times D^q$ represents β. However, the two pieces $(D^1 \times \partial D^q) \cup (\{1\} \times D^q)$ and $\{0\} \times D^q$ have opposite orientations. There are identifications making the diagram

$$\begin{tikzcd} \partial(D^1 \times D^q) = (\partial D^1 \times D^q) \cup (D^1 \times \partial D^q) \arrow[rr, "\cong"] \arrow[d, "\text{quotient}"'] & & S^q \arrow[dd, "\text{pinch}"] \\ (\partial D^1 \times D^q) \cup (D^1 \times \partial D^q)/(\{0\} \times \partial D^q) \arrow[d, "\cong"'] & & \\ S^q \vee S^q \arrow[rr, "\tau \vee \mathrm{id}"'] & & S^q \vee S^q \end{tikzcd}$$

commute up to pointed homotopy. Here $\tau \colon S^q \to S^q$ is an orientation-reversing homeomorphism (e.g. multiplying a coordinate by -1 or permuting two coordinates).

Using this orientation convention, we conclude:

$$[\gamma, \beta] = (\gamma \cdot \beta) - \beta. \quad \square$$

c. (Hatcher § 4.2 Exercise 37) Show that a path-connected H-space (c.f. Homework 3 Problem 1) has trivial Whitehead products.

Solution. Let X be a path-connected H-space, with multiplication map $\mu\colon X \times X \to X$ and unit e. Let $p, q \geq 1$. Consider the cofiber sequence

$$S^{p+q-1} \xrightarrow{w} S^p \vee S^q \xrightarrow{i} S^p \times S^q$$

and apply the functor $[-, X]_*$ to obtain an exact sequence of pointed sets

$$[S^p \times S^q, X]_* \xrightarrow{i^*} [S^p \vee S^q, X]_* \xrightarrow{w^*} [S^{p+q-1}, X]_*.$$

We want to show that w^* is the trivial map, or equivalently, the restriction i^* is surjective. Given a pointed map $(\alpha, \beta)\colon S^p \vee S^q \to X$, consider the composite

$$\begin{tikzcd} S^p \times S^q \arrow[r, "\alpha\times\beta"] \arrow[rr, bend right, "\mu(\alpha,\beta)"'] & X \times X \arrow[r, "\mu"] & X. \end{tikzcd}$$

Its restriction to $S^p \vee S^q \subset S^p \times S^q$ satisfies

$$\mu(\alpha, \beta)|_{S^p} = \alpha e = \mu(-, e) \circ \alpha \simeq \alpha$$
$$\mu(\alpha, \beta)|_{S^q} = e\beta = \mu(e, -) \circ \beta \simeq \beta$$

from which we conclude

$$i^*\mu(\alpha, \beta) = \mu(\alpha, \beta)|_{S^p \vee S^q} \simeq (\alpha, \beta). \quad \square$$

Problem 4. Let $f\colon X' \xrightarrow{\sim} X$ and $g\colon Y' \xrightarrow{\sim} Y$ be pointed maps between well-pointed spaces, and assume that f and g are weak homotopy equivalences.

a. Show that the map $f \vee g\colon X' \vee Y' \to X \vee Y$ is a weak homotopy equivalence.

Solution. Since the inclusion of the basepoint $* \to X$ is a cofibration, the strict pushout

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \vee Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is also a homotopy pushout. Likewise for $X' \vee Y'$. Now the map of diagrams

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y' & & * \arrow[r] \arrow[d] & Y \\ X' & \arrow[rr, squiggly] & & X & \end{tikzcd}$$

is an objectwise weak homotopy equivalence. By weak homotopy invariance, the induced map on homotopy pushouts $f \vee g\colon X' \vee Y' \xrightarrow{\sim} X \vee Y$ is a weak homotopy equivalence. $\square$

b. Show that the map $f \wedge g\colon X' \wedge Y' \to X \wedge Y$ is a weak homotopy equivalence.

Solution. Since we are working in CGWH spaces, every cofibration is a closed cofibration. Since the inclusions of basepoints $\{x_0\} \hookrightarrow X$ and $\{y_0\} \hookrightarrow Y$ are cofibrations, then so is the inclusion

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \xrightarrow{\iota} X \times Y$$

by the Product Theorem for cofibrations (c.f. Tom Dieck, Theorem 5.4.6). Likewise, the inclusion $X' \vee Y' \xrightarrow{\iota'} X' \times Y'$ is a cofibration.

Thus, the strict cofiber of ι

$$X \vee Y \xrightarrow{\iota} X \times Y \twoheadrightarrow X \wedge Y$$

is also a homotopy cofiber. Likewise, $X' \wedge Y'$ is a homotopy cofiber of ι'.

The map $f \times g\colon X' \times Y' \xrightarrow{\sim} X \times Y$ is a weak homotopy equivalence, by the natural isomorphism $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$. We know from part (a) that $f \vee g\colon X' \vee Y' \xrightarrow{\sim} X \vee Y$ is a weak homotopy equivalence. Therefore, the map of diagrams

$$\begin{tikzcd}
X' \vee Y' \arrow[r] \arrow[d, "f\vee g"', "\sim"] & X' \times Y' \arrow[d, "f\times g"', "\sim"] \\
X \vee Y \arrow[r] & X \times Y2
\end{tikzcd}$$

is an objectwise weak homotopy equivalence, and thus induces a weak homotopy equivalence on homotopy cofibers

$$\begin{tikzcd}
X' \vee Y' \arrow[r] \arrow[d, "f\vee g"', "\sim"] & X' \times Y' \arrow[r] \arrow[d, "f\times g"', "\sim"] & X' \wedge Y' \arrow[d, "f\wedge g"', "\sim"] \\
X \vee Y \arrow[r] & X \times Y \arrow[r] & X \wedge Y.
\end{tikzcd} \qquad \square$$

Remark. One cannot remove the assumption of well-pointedness in general. There are even examples where f and g are homotopy equivalences, yet $f \vee g$ is not a weak homotopy equivalence. See for example:

http://mathoverflow.net/questions/116980/is-the-wedge-sum-of-two-cones-over-the-hawaiian-earring-contractible