

MA544: Qual Problems

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1.1 Exam 1 Prep

Problem 1.1. Let $E \subset \mathbf{R}^n$ be a measurable set, $r \in \mathbf{R}$ and define the set $rE = \{r\mathbf{x} \mid \mathbf{x} \in E\}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Define a linear map $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ by $\mathbf{x} \mapsto r\mathbf{x}$. Using the standard basis for \mathbf{R}^n , this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \quad (1)$$

which has determinant $\det T = r^n$. By 3.35, we have $|E| = |T(E)| = r^n |E| = |rE|$. ■

Problem 1.2. Let $\{E_k\}$, $k \in \mathbf{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k \rightarrow \infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|.$$

Proof. If the $\liminf_{k \rightarrow \infty} |E_k| = \infty$ the inequality holds trivially. Hence, we may, without loss of generality, assume that $\liminf_{k \rightarrow \infty} |E_k| < \infty$. By 3.20, the set $\liminf_{k \rightarrow \infty} E_k$ is measurable and we have

$$\left| \liminf_{k \rightarrow \infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \quad (2)$$

where $F_k := \bigcap_{n=k}^{\infty} E_n$. Now, note that the collection of sets $F'_k := \bigcup_{\ell=1}^k F_\ell$ forms an increasing sequence of measurable sets $F'_k \nearrow F'$, where $F' = \bigcup_{k=1}^{\infty} F_k = \liminf_{k \rightarrow \infty} E_k$. Then, by 3.26 (i), we have

$$\lim_{k \rightarrow \infty} |F'_k| = |F'| = \left| \liminf_{k \rightarrow \infty} E_k \right|. \quad (3)$$

Hence, it suffices to show that $|F'_k| \leq |E_k|$ for all k , but this follows by monotonicity of the outer measure, 3.3, since $F'_k \subset E_k$. Thus, we have the desired inequality

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|. \quad (4)$$

■

Problem 1.3. Consider the function

$$F(x) := \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) := \{\mathbf{y} \in \mathbf{R}^n \mid |\mathbf{y}| < r\}$. Prove that F is monotonic increasing and continuous.

Proof. That F is increasing is immediate from the monotonicity of the outer measure since for $x < x'$ we have $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$ so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \leq |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that F is continuous, we will prove the following lemma

Lemma 1. *For any $x > 0$, $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$.*

Proof of lemma. If $\mathbf{y} \in xB(\mathbf{0}, 1)$ then $\mathbf{y} = x\mathbf{y}'$ for $\mathbf{y}' \in B(\mathbf{0}, 1)$. Thus, $|\mathbf{y}'| = |\mathbf{y}|/x < 1$ so $|\mathbf{y}| < x$ implies that $\mathbf{y} \in B(\mathbf{0}, x)$. Hence, we have the containment $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$.

On the other hand, if $\mathbf{y} \in B(\mathbf{0}, x)$ then $|\mathbf{y}| < x$ so $|\mathbf{y}|/x < 1$. Hence, $\mathbf{y}/x \in B(\mathbf{0}, 1)$ so $x(\mathbf{y}/x) = \mathbf{y} \in xB(\mathbf{0}, 1)$. Thus, $B(\mathbf{0}, x) \subset xB(\mathbf{0}, 1)$ and equality holds. ♣

In light of Lemma 1 and 3.35, for $x > 0$, we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n |B(\mathbf{0}, 1)|. \quad (5)$$

It is clear that F is continuous on the interval $[0, \infty)$ since F is a polynomial in x . ■

Problem 1.4. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ .

Proof. From the topological definition of continuity, f is continuous at $x \in C$ if and only if for every neighborhood U of $f(x)$, the preimage $f^{-1}(U)$ is a neighborhood of x . Now, ■

Let $x \in C$. Then, by the definition of continuity, for every natural number $n > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f(x) - f(x')| < \frac{1}{2n}. \quad (6)$$

Let $x'', x' \in B(x, \delta)$. Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x'')| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \quad (7)$$

In view of these estimates, define the set

$$A_n := \left\{ x \in \mathbf{R} \mid \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \quad (8)$$

Good Lord, that was a long definition! We claim that $C = \bigcap_{n=1}^{\infty} A_n$ and that A_n is open for all n .

First, let us show that $C = \bigcap_{n=1}^{\infty} A_n$. Let $x \in C$. Then for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Thus, $x \in A_n$ for all n so $x \in \bigcap A_n$. On the other hand, if $x \in \bigcap A_n$ for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$.

Fix $\varepsilon > 0$. By the Archimedean principle, there exists $N > 0$ such that $\varepsilon > 1/N$. Then, since $x \in A_N$ it follows that for some $\delta' > 0$, $|x - x'| < \delta'$ implies $|f(x) - f(x')| < 1/N < \varepsilon$. Thus, $x \in C$ and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$.

Lastly, we show that A_n is open. Let $x \in A_n$. Then there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. In particular, this means that $B(x, \delta) \subset A_n$ for any $x' \in B(x, \delta)$ satisfies $|f(x) - f(x')| < 1/n$. Thus, A_n is open and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$ is a G_δ set.

Problem 1.5. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a function. Is it true that if the sets $\{f = r\}$ are measurable for all $r \in \mathbf{R}$, then f is measurable?

Proof. No. Recall that, by definition, or 4.1, f is measurable if and only if $\{f > a\}$ for all $a \in \mathbf{R}$. ■

Problem 1.6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on \mathbf{R} . Prove that the set $\{x \mid \lim_{k \rightarrow \infty} f_k(x) \text{ exists}\}$ is measurable.

Proof. The idea here should be to rewrite

$$E := \left\{ x \mid \lim_{k \rightarrow \infty} f_k(x) \text{ exists} \right\} \quad (9)$$

as a countable union/intersection of measurable sets. Let $x \in E$. By the Cauchy criterion, for every $N > 0$ there exists a positive integer M such that $m, n \geq M$ implies $|f_n(x) - f_m(x)| < 1/N$. With this in mind, define

$$E_N := \left\{ x \mid \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \quad (10)$$

Then, like for Problem 1.4, it is not too hard to see that the E_n 's are open and that $E = \bigcap_{n=1}^{\infty} E_n$. Thus, E is a G_δ set and therefore measurable. ■

Problem 1.7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. Suppose $f: [a, b] \rightarrow \mathbf{R}$ is absolutely continuous. Then for fixed $\varepsilon = 1$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, we have $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Let $\Gamma := \{x_k\}_{k=1}^N$ be a partition of $[a, b]$ into closed intervals such that $x_{k+1} - x_k < \delta$, then by absolute continuity we have

$$\begin{aligned} V[f; \Gamma] &= \sum_{k=1}^N |f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \quad (11)$$

Thus, $f \in \text{BV}[a, b]$. ■

Problem 1.8. Let f be a continuous function from $[a, b]$ into \mathbf{R} . Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, i.e., $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f \, d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}.$$

Proof.

■

2 Exam 1

2.1 Exam 2 Prep

Problem 2.1. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) := \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B_\varepsilon(\mathbf{0})$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x}) \, d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is (Lebesgue) integrable over a subset E of \mathbb{R}^n (or f belongs to $L(E)$) if

$$\int_E f(\mathbf{x}) \, d\mathbf{x} < \infty.$$

Put $E := \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$. Then to show that f belongs to $L(E)$ it suffices to show the inequality

$$\int_E f(\mathbf{x}) \, d\mathbf{x} < \frac{C}{\varepsilon} \tag{12}$$

for some appropriate constant C . We proceed by directly computing the Lebesgue integral of f :

$$\begin{aligned} \int_E f(\mathbf{x}) \, d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{(x_1^2 + \cdots + x_n^2)^{(n+1)/2}} \end{aligned}$$

let E_i denote the projection of E onto its i -th coordinate and make the trigonometric substitution $x_1 = \sqrt{x_2^2 + \cdots + x_n^2} \tan \theta$, $dx_1 = \sqrt{x_2^2 + \cdots + x_n^2} \sec^2 \theta \, d\theta$ with $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$ giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[\frac{\cos^{n-1} \theta}{(x_2^2 + \cdots + x_n^2)^{n/2}} \, d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{(x_2^2 + \cdots + x_n^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1} \theta \, d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1} \theta \, d\theta < \infty. \tag{13}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned} \int \cdots \int_E f(\mathbf{x}) \, d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{x_n^2} \\ &= 2C' \int_\varepsilon^\infty \frac{dx_n}{x_n^2} \\ &= \frac{C}{\varepsilon} \end{aligned} \tag{14}$$

as desired. ■

Problem 2.2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on \mathbb{R}^n , and assume that f_k converges pointwise almost everywhere to a function f . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets E of \mathbb{R}^n . Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$.

Proof. This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit f of $\{f_k\}$ must be nonnegative a.e. in \mathbb{R}^n . For assume otherwise. Then there exists a collection of points \mathbf{x} in \mathbb{R}^n of nonzero \mathbb{R}^n -Lebesgue measure such that $f(\mathbf{x}) < 0$. But $f_k(\mathbf{x}) \geq 0$ for all $k \in \mathbb{N}$. Set $0 < \varepsilon < |f(\mathbf{x})|$ then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \quad (15)$$

for all k which contradicts our assumption that $f_k \rightarrow f$ a.e. on \mathbb{R}^n . Therefore, the set of points $\mathbf{x} \in \mathbb{R}^n$ where $f(\mathbf{x}) < 0$ must have measure zero.

Now, based on pointwise convergence a.e. to f , given $\varepsilon > 0$ for a.e. $\mathbf{x} \in \mathbb{R}^n$ we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \quad (16)$$

for sufficiently large k ; say k greater than or equal to some index $N \in \mathbb{N}$. Moreover, we are given convergence in $L^1(\mathbb{R}^n)$ of f_k to f

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f < \infty. \quad (17)$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \quad (18)$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \quad (19)$$

for all $k \in \mathbb{N}$. By Theorem 5.5(ii), f and the f_k 's are finite a.e. in \mathbb{R}^n so for some sufficiently large real number M , $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in \mathbb{R}^n$. In particular, for any measurable subset E of \mathbb{R}^n , $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in E$ so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \rightarrow \int_E f < \infty. \quad (20)$$

However, if f does not belong to $L^1(\mathbb{R}^n)$, i.e., its integral over \mathbb{R}^n is infinity, there is no guarantee that f will be finite a.e. in \mathbb{R}^n . This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset E of \mathbb{R}^n . Let us demonstrate this with an example. Consider the sequence of functions ■

Problem 2.3. Assume that E is a measurable set of \mathbb{R}^n , with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}| < \infty.$$

Proof. \implies : Suppose that f is integrable over a measurable subset E of \mathbb{R}^n . Then

$$\int_E f(\mathbf{x}) \, d\mathbf{x} < \infty. \quad (21)$$

Since f is nonnegative, by Theorem 5.5(ii), this means that f is finite a.e. in E . Hence, the subset $\{f = \infty\} \subset E$ has measure zero. Set $E_k := \{\mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k\}$ and $F_k := \{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}$. Note that the E_k 's are pairwise disjoint and that $E = \bigcup_{k \in \mathbb{N}} E_k$ and $F_k = \bigcup_{\ell=k}^{\infty} E_\ell$. By Theorem 3.23, we have

$$|E| = \sum_{k \in \mathbb{N}} |E_k| < \infty. \quad (22)$$

and

$$|F_k| = \sum_{\ell=k}^{\infty} |E_\ell|. \quad (23)$$

In particular, by Chebyshev's inequality, we have

$$\int_E f(\mathbf{x}) \, d\mathbf{x} \geq k|F_k| = k \sum_{\ell=k}^{\infty} |E_\ell| \quad (24)$$

\Leftarrow :

■

Problem 2.4. Suppose that E is a measurable subset of \mathbb{R}^n , with $|E| < \infty$. If f and g are measurable functions on E , define

$$\rho(f, g) := \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, f) \rightarrow 0$ as $k \rightarrow \infty$ if and only if f_k converges to f as $k \rightarrow \infty$.

Proof. \implies : First note that ρ is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that $\rho(f_k, f) \rightarrow 0$ as $k \rightarrow \infty$. Then, given $\varepsilon > 0$ there exist an sufficiently large index N such that for every $k \geq N$ we have

$$\rho(f_k, f) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \quad (25)$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in E which happens if $|f_k - f| = 0$ a.e. in E .

\Leftarrow : Suppose that $f_k \rightarrow f$ as $k \rightarrow \infty$.

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Problem 2.5. Define the *gamma function* $\Gamma: [0, \infty) \rightarrow \mathbb{R}$ by

$$\Gamma(y) := \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function* $\beta: [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ by

$$\beta(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

- (a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u} u^{y-1}$ is in $L([0, \infty))$ for all $y \in [0, \infty)$.
- (b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. ■

Problem 2.6. Let $f \in L^1(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}}: \mathbb{R}^n \rightarrow \mathbb{R}$ be $f_{\mathbf{h}}(x) := f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. ■

Problem 2.7. (a) If $f_k, g_k, f, g \in L^1(\mathbb{R}^n)$, $f_k \rightarrow f$ and $g_k \rightarrow g$ a.e. in \mathbb{R}^n , $|f_k| \leq g_k$ and

$$\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f.$$

- (b) Using part (a) show that if $f_k, f \in L^1(\mathbb{R}^n)$ and $f_k \rightarrow f$ a.e. in \mathbb{R}^n , then

$$\int_{\mathbb{R}^n} |f_k - f| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \rightarrow \int_{\mathbb{R}^n} |f| \quad \text{as} \quad k \rightarrow \infty.$$

Proof. ■