MA 519: Homework 7

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Problem 7.1 (Handout 10, # 4)

(*Poisson Approximation.*) One hundred people will each toss a fair coin 200 times. Approximate the probability that at least 10 of the 100 people would each have obtained exactly 100 heads and 100 tails.

Solution. Let X denote the number of people who obtain exactly 100 heads and (consequently) 100 tails. First, we compute the probability that any one given person obtains exactly 100 heads. There are 2^{200} possible outcomes for 200 tosses of a fair coin, and $\binom{200}{100}$ possible ways of obtaining exactly 100 heads. Thus, the probability that any one person obtains exactly 100 head in 200 tosses of a fair coin is

$$p = \frac{\binom{200}{100}}{2^{200}} \approx 0.056.$$

Now, assuming $X \sim \text{Poisson}(5.635)$, the probability that at least 10 of the 100 people have each obtained exactly 100 heads and 100 tails is

$$\begin{aligned} P(X \geq 10) &= 1 - P(X < 10) \\ &= 1 - \sum_{i=1}^{9} P(X = i) \\ &= 1 - e^{-5.635} \sum_{i=0}^{9} \frac{5.635^i}{i!} \\ &= 1 - 0.883 \\ &\approx 0.117. \end{aligned}$$

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Problem 7.2 (Handout 10, # 5)

(*A Pretty Question.*) Suppose X is a Poisson distributed random variable. Can three different values of X have an equal probability?

Solution. No. Let $X \sim \text{Poisson}(\lambda)$. First, we show that given any two values $k_1, k_2 \in \mathbb{Z}_{\geq 0}$, there exists λ such that $p(k_1) = p(k_2)$.

Observe that for $p(k_1) = p(k_2)$ we must have

$$\begin{aligned} e^{-\lambda}\frac{\lambda^{k_1}}{k_1!} &= e^{-\lambda}\frac{\lambda^{k_2}}{k_2!} \\ \lambda^{k_1-k_2} &= \frac{k_1!}{k_2!} \end{aligned}$$

this implies that, given k_1 and k_2,

$$\begin{aligned} (k_1 - k_2)\ln\lambda &= \ln(k_1!/k_2!) \\ \lambda(k_1, k_2) &= e^{\ln(k_1!/k_2!)/(k_1-k_2)}. \end{aligned}$$

For example, $\lambda(3, 5) \approx 4.472$ and

$$p(3) \approx 0.170 \approx p(5).$$

We now prove our original claim. Assume for a moment that X is continuous. We will show that the PMF p of X has at most one critical point. Write the PMF of p as its continuous analogue

$$p(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{\Gamma(x)}.$$

Then, taking the derivative of p, we have

$$\begin{aligned} p'(x) &= e^{-\lambda}\frac{\Gamma(x)\lambda^x\ln\lambda - \lambda^x\Gamma(x)}{\Gamma(x)^2} \\ &= \frac{e^{-\lambda}\lambda^x}{\Gamma(x)}\left[\ln\lambda - \frac{\Gamma'(x)}{\Gamma(x)}\right] \\ &= p(x)\left[\ln\lambda - \frac{\Gamma'(x)}{\Gamma(x)}\right]. \end{aligned}$$

Since $\Gamma, \Gamma' > 0$ for all $x \in \mathbb{R}_{\geq 0}$, $p'(x) = 0$ if and only if

$$\ln\lambda = \frac{\Gamma'(x)}{\Gamma(x)}$$

which happens at most once since the quotient. ■

Problem 7.3 (Handout 10, # 6)

(*Poisson Approximation.*) There are 20 couples seated at a rectangular table, husbands on one side and the wives on the other, in a random order. Using a Poisson approximation, find the probability that exactly two husbands are seated directly across from their wives; at least three are; at most three are.

Solution. Let X count the number of husbands that have been seated directly across from their wives. First, we find the probability that exactly one man has been seated across from his wife. To this end, we compute the probability of the complement, i.e., the probability that either A_1, no men are paired with their wives, or A_2, at least two men are paired with their wives, occur. The latter, $p_2 = P(A_2)$, is easy to compute: There are 20! possible arrangements and, fixing two men with their respective wives, 18! arrangements for the other couples, giving us a probability of

$$p_2 = \frac{18!}{20!} = \frac{1}{19 \cdot 20}.$$

To compute the probability $p_1 = P(A_1)$ we employ the inclusion-exclusion principle: Let B_i, $1 \leq i \leq 20$, denote the event that the i^{th} man is not paired with his wife. Since all the probabilities $B_i = B_j$, $B_i \cap B_j = B_k \cap B_\ell$, etc., are equivalent, we need only find $P(B_1), P(B_1 \cap B_2), P(B_1 \cap B_2 \cap B_3)$, etc., as follows

$$\begin{aligned} P(B_1) &= \frac{19!}{20!} \\ P(B_1 \cap B_2) &= \frac{}{20!} \\ &\vdots \\ P\Big(\textstyle\bigcap_{i=1}^{20} B_i\Big) &= \qquad\qquad\qquad\qquad \vdots \\ P\Big(\textstyle\bigcap_{i=1}^{20} B_i\Big) &= \end{aligned}$$

■

Problem 7.4 (Handout 10, # 7)

(*Poisson Approximation.*) There are 5 coins on a desk, with probabilities 0.05, 0.1, 0.05, 0.01, and 0.04 for heads. By using a Poisson approximation, find the probability of obtaining at least one head when the five coins are each tossed once. Is the number of heads obtained binomially distributed in this problem?

Solution. ■

Problem 7.5 (Handout 10, # 8)

A book of 500 pages contains 500 misprints. Estimate the chances that a given page contains at least three misprints.

Solution. ∎

Problem 7.6 (Handout 10, # 9)

Estimate the number of raisins which a cookie should contain on the average if it is desired that not more than one cookie out of a hundred should be without raisin.

Solution. ■

Problem 7.7 (Handout 10, # 10)

The terms Poisson$(k; X)$ of the Poisson distribution reach their maximum when k is the largest integer not exceeding X.

Solution. ∎

Problem 7.8 (Handout 10, # 11)

Prove

$$\text{Poisson}(0,\lambda)+\cdots+\text{Poisson}(n,\lambda)=\frac{1}{n!}\int_\lambda^\infty e^{-x}x^n\,dx.$$

Solution. ■

Problem 7.9 (Handout 10, # 12)

There is a random number N of coins in your pocket, where N has a Poisson distribution with mean μ. Each one is tossed once.
Let X be the number of times a head shows.
Find the distribution of X.

Solution. ■

Problem 7.10 (Handout 10, # 14)

Find the MGF of a general Poisson distribution, and hence prove that the mean and the variance of an arbitrary Poisson distribution are equal.

Solution. ■

Problem 7.11 (Handout 10, # 17 (a))

(*Poisson approximations.*) 20 couples are seated in a rectangular table, husbands on one side and the wives on the other. First, find the expected number of husbands that sit directly across from their wives. Then, using a Poisson approximation, find the probability that two do; three do; at most five do.

Solution. ■