

MA557 Homework 6

Carlos Salinas

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PROBLEM 6.1

Let R be a Noetherian ring and I, J R -ideals. Write $I^{\langle J \rangle} = \bigcup_{n \geq 1} (I : J^n)$, which is called the *saturation* of I with respect to J . Show:

- (a) If $I = \bigcap_{i=1}^m i$ with i i -primary, then $I^{\langle J \rangle} = \bigcap_{J \not\subseteq i} i$.
- (b) $I^{\langle J \rangle}$ is the unique largest R -ideal that coincides with I locally on the open set $\text{Spec}(R) \setminus V(J)$.

Proof.

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PROBLEM 6.2

Let R be a Noetherian ring. Show that R is reduced if and only if $\text{Quot}(R)$ is a finite direct product of fields.

Proof.



PROBLEM 6.3

Let R be a Noetherian ring and $x \in R$ an R -regular element. Show that $\text{Ass}_R(R/(x^n)) = \text{Ass}_R(R/(x))$ for every $n \geq 1$.

Proof.

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PROBLEM 6.4

Let $\varphi: R \rightarrow T$ be a homomorphism of rings where T is Noetherian, let ${}^a\varphi$ be the induced map on the spectra, and let N be a T -module. Show:

- (a) $\text{Ass}_R(N) = {}^a\varphi(\text{Ass}_T(N))$.
- (b) If N is finitely generated as a T -module then $\text{Ass}_R(N)$ is finite.

Proof.

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PROBLEM 6.5

Let K be a field that is a finitely generated \mathbb{Z} -algebra. Show that K is a finite field.

Proof.

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PROBLEM 6.6

Let k be a Noetherian ring, R a finitely generated k -algebra, and $\text{Aut}_k(R)$ the group of k -algebra automorphisms of R . For a subgroup G of $\text{Aut}_k(R)$ write $R^G = \{x \in R \mid \sigma(x) = x \text{ for every } \sigma \in G\}$, which is called the *ring of invariants* of G . Show that if G is finite then R^G is a finitely generated k -algebra (and hence a Noetherian ring).

Proof.

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