

# MA544: Qual Problems

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# 1 MA 544 Spring 2016

## 1.1 Exam 1 Prep

**Problem 1.1.** Let  $E \subset \mathbb{R}^n$  be a measurable set,  $r \in \mathbb{R}$  and define the set  $rE = \{r\mathbf{x} \mid \mathbf{x} \in E\}$ . Prove that  $rE$  is measurable, and that  $|rE| = |r|^n|E|$ .

*Proof.* Define a linear map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $\mathbf{x} \mapsto r\mathbf{x}$ . Using the standard basis for  $\mathbb{R}^n$ , this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \quad (1)$$

which has determinant  $\det T = r^n$ . By 3.35, we have  $|E| = |T(E)| = r^n|E| = |rE|$ . ■

**Problem 1.2.** Let  $\{E_k\}$ ,  $k \in \mathbb{N}$  be a collection of measurable sets. Define the set

$$\liminf_{k \rightarrow \infty} E_k = \bigcup_{k=1}^{\infty} \left( \bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|.$$

*Proof.* If the  $\liminf_{k \rightarrow \infty} |E_k| = \infty$  the inequality holds trivially. Hence, we may, without loss of generality, assume that  $\liminf_{k \rightarrow \infty} |E_k| < \infty$ . By 3.20, the set  $\liminf_{k \rightarrow \infty} E_k$  is measurable and we have

$$\left| \liminf_{k \rightarrow \infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \quad (2)$$

where  $F_k := \bigcap_{n=k}^{\infty} E_n$ . Now, note that the collection of sets  $F'_k := \bigcup_{\ell=1}^k F_\ell$  forms an increasing sequence of measurable sets  $F'_k \nearrow F'$ , where  $F' = \bigcup_{k=1}^{\infty} F_k = \liminf_{k \rightarrow \infty} E_k$ . Then, by 3.26 (i), we have

$$\lim_{k \rightarrow \infty} |F'_k| = |F'| = \left| \liminf_{k \rightarrow \infty} E_k \right|. \quad (3)$$

Hence, it suffices to show that  $|F'_k| \leq |E_k|$  for all  $k$ , but this follows by monotonicity of the outer measure, 3.3, since  $F'_k \subset E_k$ . Thus, we have the desired inequality

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|. \quad (4)$$

■

**Problem 1.3.** Consider the function

$$F(x) := \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here  $B(\mathbf{0}, r) := \{\mathbf{y} \in \mathbb{R}^n \mid |\mathbf{y}| < r\}$ . Prove that  $F$  is monotonic increasing and continuous.

*Proof.* That  $F$  is increasing is immediate from the monotonicity of the outer measure since for  $x < x'$  we have  $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$  so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \leq |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that  $F$  is continuous, we will prove the following lemma

**Lemma 1.** *For any  $x > 0$ ,  $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$ .*

*Proof of lemma.* If  $\mathbf{y} \in xB(\mathbf{0}, 1)$  then  $\mathbf{y} = x\mathbf{y}'$  for  $\mathbf{y}' \in B(\mathbf{0}, 1)$ . Thus,  $|\mathbf{y}'| = |\mathbf{y}|/x < 1$  so  $|\mathbf{y}| < x$  implies that  $\mathbf{y} \in B(\mathbf{0}, x)$ . Hence, we have the containment  $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$ .

On the other hand, if  $\mathbf{y} \in B(\mathbf{0}, x)$  then  $|\mathbf{y}| < x$  so  $|\mathbf{y}|/x < 1$ . Hence,  $\mathbf{y}/x \in B(\mathbf{0}, 1)$  so  $x(\mathbf{y}/x) = \mathbf{y} \in xB(\mathbf{0}, 1)$ . Thus,  $B(\mathbf{0}, x) \subset xB(\mathbf{0}, 1)$  and equality holds. ♣

In light of Lemma 1 and 3.35, for  $x > 0$ , we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n |B(\mathbf{0}, 1)|. \quad (5)$$

It is clear that  $F$  is continuous on the interval  $[0, \infty)$  since  $F$  is a polynomial in  $x$ . ■

**Problem 1.4.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $C$  be the set of all points at which  $f$  is continuous. Show that  $C$  is a set of type  $G_\delta$ .

*Proof.* From the topological definition of continuity,  $f$  is continuous at  $x \in C$  if and only if for every neighborhood  $U$  of  $f(x)$ , the preimage  $f^{-1}(U)$  is a neighborhood of  $x$ . Now, ■

Let  $x \in C$ . Then, by the definition of continuity, for every natural number  $n > 0$  there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies

$$|f(x) - f(x')| < \frac{1}{2n}. \quad (6)$$

Let  $x'', x' \in B(x, \delta)$ . Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x'')| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \quad (7)$$

In view of these estimates, define the set

$$A_n := \left\{ x \in \mathbb{R} \mid \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \quad (8)$$

Good Lord, that was a long definition! We claim that  $C = \bigcap_{n=1}^{\infty} A_n$  and that  $A_n$  is open for all  $n$ .

First, let us show that  $C = \bigcap_{n=1}^{\infty} A_n$ . Let  $x \in C$ . Then for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . Thus,  $x \in A_n$  for all  $n$  so  $x \in \bigcap A_n$ . On the other hand, if  $x \in \bigcap A_n$  for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ .

Fix  $\varepsilon > 0$ . By the Archimedean principle, there exists  $N > 0$  such that  $\varepsilon > 1/N$ . Then, since  $x \in A_N$  it follows that for some  $\delta' > 0$ ,  $|x - x'| < \delta'$  implies  $|f(x) - f(x')| < 1/N < \varepsilon$ . Thus,  $x \in C$  and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$ .

Lastly, we show that  $A_n$  is open. Let  $x \in A_n$ . Then there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . In particular, this means that  $B(x, \delta) \subset A_n$  for any  $x' \in B(x, \delta)$  satisfies  $|f(x) - f(x')| < 1/n$ . Thus,  $A_n$  is open and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$  is a  $G_\delta$  set.

**Problem 1.5.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Is it true that if the sets  $\{f = r\}$  are measurable for all  $r \in \mathbb{R}$ , then  $f$  is measurable?

*Proof.* No. Recall that, by definition, or 4.1,  $f$  is measurable if and only if  $\{f > a\}$  for all  $a \in \mathbb{R}$ . ■

**Problem 1.6.** Let  $\{f_k\}_{k=1}^{\infty}$  be a sequence of measurable functions on  $\mathbb{R}$ . Prove that the set  $\{x \mid \lim_{k \rightarrow \infty} f_k(x) \text{ exists}\}$  is measurable.

*Proof.* The idea here should be to rewrite

$$E := \left\{ x \mid \lim_{k \rightarrow \infty} f_k(x) \text{ exists} \right\} \quad (9)$$

as a countable union/intersection of measurable sets. Let  $x \in E$ . By the Cauchy criterion, for every  $N > 0$  there exists a positive integer  $M$  such that  $m, n \geq M$  implies  $|f_n(x) - f_m(x)| < 1/N$ . With this in mind, define

$$E_N := \left\{ x \mid \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \quad (10)$$

Then, like for Problem 1.4, it is not too hard to see that the  $E_n$ 's are open and that  $E = \bigcap_{n=1}^{\infty} E_n$ . Thus,  $E$  is a  $G_\delta$  set and therefore measurable. ■

**Problem 1.7.** A real valued function  $f$  on an interval  $[a, b]$  is said to be *absolutely continuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Show that an absolutely continuous function on  $[a, b]$  is of bounded variation on  $[a, b]$ .

*Proof.* Suppose  $f: [a, b] \rightarrow \mathbb{R}$  is absolutely continuous. Then for fixed  $\varepsilon = 1$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , we have  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Let  $\Gamma := \{x_k\}_{k=1}^N$  be a partition of  $[a, b]$  into closed intervals such that  $x_{k+1} - x_k < \delta$ , then by absolute continuity we have

$$\begin{aligned} V[f; \Gamma] &= \sum_{k=1}^N |f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \quad (11)$$

Thus,  $f \in \text{BV}[a, b]$ . ■

**Problem 1.8.** Let  $f$  be a continuous function from  $[a, b]$  into  $\mathbb{R}$ . Let  $\chi_{\{c\}}$  be the characteristic function of a singleton  $\{c\}$ , i.e.,  $\chi_{\{c\}}(x) = 0$  if  $x \neq c$  and  $\chi_{\{c\}}(c) = 1$ . Show that

$$\int_a^b f \, d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}.$$

*Proof.*

■

## 2 Exam 1

## 2.1 Exam 2 Prep

**Problem 2.1.** Define for  $\mathbf{x} \in \mathbb{R}^n$ ,

$$f(\mathbf{x}) := \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that  $f$  is integrable outside any ball  $B(0, \varepsilon)$ , and that there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n \setminus B(0, \varepsilon)} f(x) \, dx \leq \frac{C}{\varepsilon}.$$

*Proof.* ■

**Problem 2.2.** Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $\mathbb{R}^n$ , and assume that  $f_k$  converges pointwise almost everywhere to a function  $f$ . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets  $E$  of  $\mathbb{R}^n$ . Moreover, show that this is not necessarily true if  $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$ .

*Proof.* ■

**Problem 2.3.** Assume that  $E$  is a measurable set of  $\mathbb{R}^n$ , with  $\lambda(E) < \infty$ . Prove that a nonnegative function  $f$  defined on  $E$  is integrable if and only if

$$\sum_{k=0}^{\infty} \lambda(\{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}) < \infty.$$

*Proof.* ■

**Problem 2.4.** Suppose that  $E$  is a measurable subset of  $\mathbb{R}^n$ , with  $\lambda(E) < \infty$ . If  $f$  and  $g$  are measurable functions on  $E$ , define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that  $\rho(f_k, g) \rightarrow 0$  as  $k \rightarrow \infty$  if and only if  $f_k$  converges to  $f$  as  $k \rightarrow \infty$ .

*Proof.* ■

**Problem 2.5.** Define the *gamma function*  $\Gamma: [0, \infty) \rightarrow \mathbb{R}$  by

$$\Gamma(y) := \int_0^\infty e^{-u} u^{y-1} \, du,$$

and the *beta function*  $\beta: [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  by

$$\beta(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} \, dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function  $u \mapsto e^{-u} u^{y-1}$  is in  $L([0, \infty))$  for all  $y \in [0, \infty)$ .

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

*Proof.*

■

**Problem 2.6.**

*Proof.*

■

**Problem 2.7.**

*Proof.*

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