

MA 519: Homework 9

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PROBLEM 9.1 (HANDOUT 13, # 7)

Let X have a *double exponential density* $f(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}$, $-\infty < x < \infty$, $\sigma > 0$.

- (a) Show that all moments exist for this distribution.
- (b) However, show that the MGF exists only for restricted values. Identify them and find a formula.

SOLUTION. For part (a), we show that the moments $m_n := E(X^n)$ for all $n \in \mathbb{N}$. By direct calculation, we have

$$\begin{aligned} m_n &= \int_{-\infty}^{\infty} x^n f(x) dx \\ &= \int_{-\infty}^{\infty} \frac{x^n}{2\sigma} e^{-\frac{|x|}{\sigma}} dx \\ &= \underbrace{\int_{-\infty}^0 \frac{x^n}{2\sigma} e^{\frac{x}{\sigma}} dx}_L + \int_0^{\infty} \frac{x^n}{2\sigma} e^{-\frac{x}{\sigma}} dx, \end{aligned}$$

making the substitution $x \mapsto -y$ to L and relabeling y to x again, the above becomes

$$\begin{aligned} &= \int_0^{\infty} \frac{x^n + (-1)x^n}{2\sigma} e^{-\frac{x}{\sigma}} dx \\ &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ I := \int_0^{\infty} \frac{x^n}{\sigma} e^{-\frac{x}{\sigma}} dx & \text{if } n = 2k \text{ is even.} \end{cases} \end{aligned}$$

To evaluate I we apply integration by parts repeatedly to arrive at

$$\begin{aligned} I &= \int_{-\infty}^0 \frac{x^n}{\sigma} e^{-\frac{x}{\sigma}} \\ &= (-0 + 0) + \int_0^{\infty} n\sigma x^{n-1} e^{-\frac{x}{\sigma}} dx \\ &= (-0 + 0) + (-0 + 0) + \int_0^{\infty} n(n-1)\sigma^2 x^{n-2} e^{-\frac{x}{\sigma}} dx & \vdots \\ &= (-0 - 0) + \cdots + (-0 + 0) + (-0 + n!\sigma^n) \\ &= n!\sigma^n. \end{aligned}$$

Therefore, m_n exist and are finite for all $n \in \mathbb{N}$.

For part (b), the MGF associated to f is given by the series

$$m(t) = \sum_{n=0}^{\infty} \frac{t^n m_n}{n!} = \sum_{k=1}^{\infty} t^{2k} \sigma^{2k}. \quad (9.1)$$

This series is geometric and, as such, converges for all $|t| < \frac{1}{\sigma}$, in which case (9.1) becomes

$$m(t) = \frac{1}{1 - t^2 \sigma^2}. \quad \blacksquare$$

PROBLEM 9.2 (HANDOUT 13, # 10)

Suppose X has Cauchy distribution as in # 6. Which of the following functions have finite expectation

$$X; \quad -X; \quad |X|; \quad \frac{1}{X}; \quad \sin X; \quad \ln |X|; \quad e^X; \quad e^{-|X|}?$$

SOLUTION. Suppose $X \sim \text{Cauchy}(0, 1)$. Then the PDF of X is given by the expression

$$f(x) = \frac{1}{\pi(x^2 + 1)}.$$

Now we proceed to find the expectations of (i) X , (ii) $-X$, (iii) $\frac{1}{X}$, (iv) $\sin X$, (v) $\ln |X|$, (vi) e^X , (vii) $e^{-|X|}$.

For (i), the expectation does not exist. Consider

$$E(X) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} dx = \lim_{x_1, x_2 \rightarrow \infty} \frac{1}{\pi} \int_{-x_1}^{x_2} \frac{x}{x^2 + 1} dx.$$

Then, making the substitution $u = x^2 + 1$, $du = 2x dx$, the integral above evaluates to

$$E(X) = \lim_{x_1, x_2 \rightarrow \infty} \frac{1}{2\pi} \ln \left(\frac{x_2^2 + 1}{x_1^2 + 1} \right).$$

However, the limit of this expression is undefined! Fix positive real number α and let $x_2 = \alpha x_1$. Then

$$E(X) = \lim_{x_1 \rightarrow \infty} \frac{1}{2\pi} \ln \left(\frac{\alpha^2 x_1^2 + 1}{x_1^2 + 1} \right) = \frac{1}{2\pi} \ln \alpha^2.$$

The value of this limit is distinct for each α . This contradicts the uniqueness of the limit and therefore, the limit must not exist.

For the rest of these ((ii) through (viii)), we use Theorem 1.33 to forgo computing the PDF.

For (ii), we have

$$E(-X) = \int_{-\infty}^{\infty} -xf(x) dx = -E(X).$$

Thus, $E(-X)$ is also undefined.

For (iii), we have

$$\begin{aligned} E(|X|) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x|}{x^2 + 1} dx \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{x}{x^2 + 1} dx \\ &= \lim_{\xi \rightarrow \infty} \frac{2}{\pi} \ln(\xi^2 + 1) \\ &= \infty. \end{aligned}$$

For (iv), we have

$$\begin{aligned} E\left(\frac{1}{X}\right) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x(x^2+1)} dx \\ &= \lim_{x_1, x_2 \rightarrow \infty} \ln\left(\frac{x_2}{\sqrt{x_1^2+1}}\right) \end{aligned}$$

which is undefined. In fact, one can show that $\frac{1}{X} \sim \text{Cauchy}(0, 1)$: First, let us find the CDF of $\frac{1}{X}$

$$\begin{aligned} F_{\frac{1}{X}}(x) &= P\left(\frac{1}{X} \leq x\right) \\ &= P\left(X \geq \frac{1}{x}\right) \\ &= 1 - P\left(X < \frac{1}{x}\right) \\ &= 1 - \frac{1}{\pi} \int_{-\infty}^{\frac{1}{x}} \frac{1}{y^2+1} dy \\ &= 1 - \tan^{-1}\left(\frac{1}{x}\right) - \frac{1}{2} \\ &= \frac{1}{2} - \tan^{-1}\left(\frac{1}{x}\right). \end{aligned}$$

Thus, the PDF of $\frac{1}{X}$ is

$$\begin{aligned} f_{\frac{1}{X}}(x) &= \frac{dF_{\frac{1}{X}}(x)}{dx} \\ &= -\left(-\frac{1}{x^2}\right)\left(\frac{1}{\left(\frac{1}{x}\right)^2+1}\right) \\ &= \frac{1}{x^2+1}. \end{aligned}$$

Thus, $\frac{1}{X} \sim \text{Cauchy}(0, 1)$ giving us another argument for why $E(\frac{1}{X})$ is undefined.

For (v), we have

$$\begin{aligned} E(\sin X) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x^2+1} dx \\ &\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\sin x|}{x^2+1} dx \end{aligned}$$

since $|\sin x| \leq 1$, we further have

$$\begin{aligned} &\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx \\ &= \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) \\ &= 1. \end{aligned}$$

Thus, $E(\sin X)$ exists and is finite.

For (vi), note that $\ln |X|$ by symmetry,

$$\begin{aligned} E(\ln |X|) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |x|}{x^2 + 1} dx \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\ln x}{x^2 + 1} dx \end{aligned}$$

making the substitution $\theta = \tan^{-1} x$, $(x^2 + 1) d\theta = dx$, we have

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \ln(\tan \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta - \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta - \int_0^{\frac{\pi}{2}} \ln\left(\sin\left(\frac{\pi}{2} - \theta\right)\right) d\theta \end{aligned}$$

making the substitution $\varphi = \frac{\pi}{2} - \theta$, $-d\varphi = d\theta$,

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta + \int_{\frac{\pi}{2}}^0 \ln(\sin \varphi) d\varphi \\ &= 0. \end{aligned}$$

For (vii), note that $E(e^X) = m(1)$ where m is the MGF of X . However, since the first term in the sequence $m_1 = E(X)$ is undefined, it follows that $m(1)$ is undefined so, in particular, $E(e^X)$ is undefined.

Lastly, for (viii), we have

$$\begin{aligned} E(e^{-|X|}) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-|x|}}{x^2 + 1} dx \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{e^{-x}}{x^2 + 1} dx \\ &\leq \frac{2}{\pi} \int_0^{\infty} \frac{1}{x^2 + 1} dx \\ &= \frac{2}{\pi} \left(\frac{\pi}{2} - 0 \right) \\ &= 1. \end{aligned}$$

Thus, $E(e^{-|X|})$ exists and is finite. ■

PROBLEM 9.3 (HANDOUT 13, # 16)

Give an example of each of the following phenomena:

- (a) A continuous random variable taking values in $[0, 1]$ with equal mean and median.
- (b) A continuous random variable taking values in $[0, 1]$ with mean equal to twice the median.
- (c) A continuous random variable for which the mean does not exist.
- (d) A continuous random variable for which the mean exists, but the variance does not exist.
- (e) A continuous random variable with a PDF that is not differentiable at zero.
- (f) A positive continuous random variable for which the mode is zero, but the mean does not exist.
- (g) A continuous random variable for which all moments exist.
- (h) A continuous random variable with median equal to zero, and 25th and 75th percentiles equal to 1.
- (i) A continuous random variable X with mean equal to median equal to mode equal to zero, and $E(\sin X) = 0$.

SOLUTION. First, note that $[0, 1]$ is a probability space under the standard Lebesgue measure on \mathbb{R} . Therefore, it makes sense to consider $X: [0, 1] \rightarrow \mathbb{R}$ random variables.

For part (a), consider the random variable $X: [0, 1] \rightarrow \mathbb{R}$ defined by $x \mapsto x$ with $X \sim \text{Uniform}[0, 1]$. Then the mean is

$$\mu = \int_{-\infty}^{\infty} xf(x) dx = \int_0^1 x dx = \frac{1}{2}$$

and the median is

$$m = F^{-1}(0.5) = \frac{1}{2}.$$

For part (b), consider again the random variable $X(x) = x$ for $x \in [0, 1]$, but this time let

$$f(x) = \begin{cases} & \\ & \end{cases},$$

be the PDF of X . Then the mean is

For part (c), in the next problem we show that if $X \sim \text{Cauchy}(0, 1)$, the mean of X does not exist.

For part (d), again consider $X \sim \text{Cauchy}(0, 1)$. Since the mean of X does not exist and $\text{Var}(X) = E(X^2) - \mu^2$, then $\text{Var}(X)$ does not exist.

For part (e), the standard exponential distribution

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

is not differentiable at 0 as the derivative from the left is 0 whereas the derivative from the right is $\lim_{x \rightarrow 0} -e^{-x} = -1$.

For part (f), $X \sim \text{Cauchy}(0, 1)$ has mode 0, but the mean does not exist.

For part (g), for Problem 9.1 part (a), we showed that the all moments of the double exponential density exists.

For part (h), define the a piecewise function F on \mathbb{R} by

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 0.5 & \text{if } x = 0 \\ 0.25x & \text{if } 0 < x < 1, \\ 0.25x + 0.5 & \text{if } 1 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

■

PROBLEM 9.4 (HANDOUT 13, # 17)

An exponential random variable with mean 4 is known to be larger than 6. What is the probability that it is larger than 8?

SOLUTION. First, since the mean of X is $\lambda := 4$, the distribution of X is given by

$$f(x) = \begin{cases} \frac{1}{4}e^{-\frac{x}{4}} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since the distribution is memoryless, we have

$$\begin{aligned} P(X > 8 | X > 6) &= P(X > 2) \\ &= \frac{1}{4} \int_2^\infty e^{-\frac{x}{4}} dx \\ &= e^{-\frac{1}{2}} \\ &\approx 0.607. \end{aligned}$$

■

PROBLEM 9.5 (HANDOUT 13, # 18)

(Sum of Gammas). Suppose X, Y are independent random variables, and $X \sim \Gamma(\alpha, \lambda)$, $Y \sim \Gamma(\beta, \lambda)$. Find the distribution of $X + Y$ by using moment-generating functions.

SOLUTION. Suppose X and Y are independent random variables. Computing the MGF of $X + Y$, we have

$$\begin{aligned} m_{X+Y}(t) &= E(e^{t(X+Y)}) \\ &= E(e^{tX})E(e^{tY}) \\ &= \left(1 - \frac{t}{\lambda}\right)^{-\alpha} \left(1 - \frac{t}{\lambda}\right)^{-\beta} \\ &= \left(1 - \frac{t}{\lambda}\right)^{-\alpha-\beta}. \end{aligned}$$

Since the MGF uniquely determines the distribution, it follows that $X + Y \sim \Gamma(\alpha + \beta, \lambda)$. ■

PROBLEM 9.6 (HANDOUT 13, # 19)

(*Product of Chi Squares*). Suppose X_1, X_2, \dots, X_n are independent chi square variables, with $X_i \sim \chi_{m_i}^2$. Find the mean and variance of $\prod_{i=1}^n X_i$.

SOLUTION. Since the X_1, \dots, X_n are independent, the mean of $X := \prod_{i=1}^n X_i$ is

$$\begin{aligned} E(X) &= \prod_{i=1}^n E(X_i) \\ &= \prod_{i=1}^n m_i \end{aligned}$$

and the variance is

$$\begin{aligned} \text{Var}(X) &= \prod_{i=1}^n [\text{Var}(X_i) + E(X_i)^2] - \prod_{i=1}^n E(X_i)^2 \\ &= \prod_{i=1}^n (2m_i + m_i^2) - \prod_{i=1}^n m_i^2. \end{aligned}$$

■

PROBLEM 9.7 (HANDOUT 13, # 20)

Let $Z \sim \text{Normal}(0, 1)$. Find

$$P\left(0.5 < \left|Z - \frac{1}{2}\right| < 1.5\right); \quad P\left(\frac{e^Z}{1+e^Z} > \frac{3}{4}\right); \quad P(\Phi(Z) < 0.5).$$

SOLUTION. Label the probabilities above (i), (ii), and (iii) from left to right.

For (i), we can decompose the probability into a sum

$$\begin{aligned} P\left(0.5 < \left|Z - \frac{1}{2}\right| < 1.5\right) &= P\left(0.5 < Z - \frac{1}{2} < 1.5\right) + P\left(0.5 < -Z + \frac{1}{2} < 1.5\right) \\ &= P(1 < Z < 2) + P(0 < -Z < 1) \\ &= P(1 < Z < 2) + P(-1 < Z < 0) \\ &= \int_1^2 \varphi(x) dx + \int_{-1}^0 \varphi(x) dx \\ &\approx 0.136 + 0.341 \\ &\approx 0.477. \end{aligned}$$

For (ii), we have

$$\begin{aligned} P\left(\frac{e^Z}{1+e^Z} > \frac{3}{4}\right) &= P\left(e^Z > \frac{3}{4}(1+e^Z)\right) \\ &= P\left(\frac{1}{4}e^Z > \frac{3}{4}\right) \\ &= P(e^Z > 3) \\ &= P(Z > \ln 3) \\ &= \int_{\ln 3}^{\infty} \varphi(x) dx \\ &\approx 0.136. \end{aligned}$$

Finally, for (iii), we have

$$P(\Phi(Z) < 0.5) = \Phi(\Phi^{-1}(0.5)) = \Phi(0) = 0.5.$$

This is true in general if the p^{th} percentile x_p of a general random variable X with CDF F is unique:

$$\begin{aligned} P(F(X) < p) &= F(F^{-1}(p)) \\ &= F(x_p) \\ &= p, \end{aligned}$$

where $F^{-1}(p)$ denotes the generalized inverse distribution. ■

PROBLEM 9.8 (HANDOUT 13, # 21)

Let $Z \sim \text{Normal}(0, 1)$. Find the density of $\frac{1}{Z}$. Is the density bounded?

SOLUTION. Since $g(x) = \frac{1}{x}$ is monotone decreasing, by the Jacobian formula

$$\begin{aligned} f_{\frac{1}{Z}}(x) &= \frac{f(g^{-1}(x))}{|g'(g^{-1}(x))|} \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2x^2}}}{\left| -\frac{1}{\left(\frac{1}{x}\right)^2} \right|} \\ &= \frac{1}{\sqrt{2\pi}x^2} e^{-\frac{1}{2x^2}}. \end{aligned}$$

The only point at which we may have a problem with the PDF of $\frac{1}{Z}$ is at $x = 0$. To this end, let us examine

$$\lim_{x \rightarrow 0} \frac{e^{-\frac{1}{2x^2}}}{x^2}.$$

By l'Hôpital's rule, we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{-\frac{1}{2x^2}}}{x^2} &= \lim_{x \rightarrow 0} \frac{\frac{1}{x^2}}{\frac{1}{2x^2}} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{2}{x^3}}{-\frac{1}{x^3} e^{\frac{1}{2x^2}}} \\ &= \lim_{x \rightarrow 0} \frac{2}{e^{\frac{1}{2x^2}}} \\ &= \lim_{x \rightarrow 0} 2e^{-\frac{1}{2x^2}} \\ &= 0. \end{aligned}$$

With this same argument, we see that $\lim_{x \rightarrow \pm\infty} f_{\frac{1}{Z}}(x) < \infty$ and indeed since $f_{\frac{1}{Z}}$ is continuous at every other point, it is bounded. ■

PROBLEM 9.9 (HANDOUT 13, # 22)

The 25th and the 75th percentile of a normally distributed random variable are -1 and 1 . What is the probability that the random variable is between -2 and 2 ?

SOLUTION. Using a table of CDF values* for the standard distribution and the corollary to Theorem 1.45, we have

$$\begin{aligned} -1 &\approx \mu - 0.67\sigma \\ 1 &\approx \mu + 0.68\sigma. \end{aligned}$$

Thus, $\mu \approx 0$ and $\sigma \approx 1.493$. With this, we have

$$P(-2 \leq X \leq 2) \approx \frac{1}{1.49 \cdot \sqrt{2\pi}} \int_{-2}^2 e^{-\frac{x^2}{4.44}} dx \approx 0.820. \quad \blacksquare$$

*Found here https://en.wikipedia.org/wiki/Standard_normal_table.