# Math 535 - General Topology
# Fall 2012
# Homework 7 Solutions

**Problem 1.** Let $X$ be a topological space and $(Y, d)$ a metric space. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \colon X \to Y$ **converges uniformly** to a function $f \colon X \to Y$ if for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ satisfying

$$d(f_n(x), f(x)) < \epsilon \text{ for all } n \geq N \text{ and all } x \in X.$$

Note in particular that uniform convergence implies pointwise convergence (but not the other way around).

Assume each function $f_n \colon X \to Y$ is *continuous*, and the sequence converges *uniformly* to a function $f \colon X \to Y$. Show that $f$ is continuous.

**Solution.** Let $x_0 \in X$. We want to show that $f$ is continuous at $x_0$.

Let $\epsilon > 0$. By uniform convergence, there is an $N \in \mathbb{N}$ satisfying

$$d(f_n(x), f(x)) < \frac{\epsilon}{3} \text{ for all } n \geq N \text{ and all } x \in X.$$

By continuity of $f_N$ at $x_0$, there is a neighborhood $U \subseteq X$ of $x_0$ satisfying $d(f_N(x_0), f_N(x)) < \frac{\epsilon}{3}$ for all $x \in U$. Therefore, the following inequalities hold for all $x \in U$:

$$\begin{aligned} d(f(x_0), f(x)) &\leq d(f(x_0), f_N(x_0)) + d(f_N(x_0), f_N(x)) + d(f_N(x), f(x)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \quad \square \end{aligned}$$

**Problem 2.** Let $X$ be a *compact* topological space. Consider the set of all real-valued continuous functions on $X$

$$C(X) := \{f \colon X \to \mathbb{R} \mid f \text{ is continuous}\}$$

which is a real vector space via pointwise addition and scalar multiplication.

Consider the function $\|\cdot\| \colon C(X) \to \mathbb{R}$ defined by

$$\|f\| := \sup_{x \in X} |f(x)|.$$

**a.** Show that $\|\cdot\|$ is a norm on $C(X)$. (First check that $\|\cdot\|$ is well-defined.)

This norm is sometimes called the **uniform norm** or **supremum norm**.

**Solution.** The number $\|f\|$ is well-defined, since $X$ is compact, and thus real-valued continuous functions on $X$ are bounded.

We check the three properties of a norm.

1. Positivity:
$$\|f\| = \sup_{x \in X} |f(x)| \geq 0$$
since $|f(x)| \geq 0$ for all $x \in X$.
$$\begin{aligned} \|f\| = 0 &\Leftrightarrow \sup_{x \in X} |f(x)| = 0 \\ &\Leftrightarrow |f(x)| = 0 \ \text{ for } x \in X \\ &\Leftrightarrow f(x) = 0 \ \text{ for } x \in X \\ &\Leftrightarrow f = 0. \end{aligned}$$

2. Homogeneity:
$$\begin{aligned} \|\alpha f\| &= \sup_{x \in X} |\alpha f(x)| \\ &= \sup_{x \in X} |\alpha||f(x)| \\ &= |\alpha| \sup_{x \in X} |f(x)| \\ &= |\alpha| \|f\|. \end{aligned}$$

3. Triangle inequality:
$$\begin{aligned} \|f + g\| &= \sup_{x \in X} |f(x) + g(x)| \\ &\leq \sup_{x \in X} (|f(x)| + |g(x)|) \\ &\leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)| \\ &= \|f\| + \|g\|. \quad \square \end{aligned}$$

**b.** Show that a sequence $(f_n)_{n\in\mathbb{N}}$ in $C(X)$ converges to $f$ in the uniform norm (meaning $\|f_n - f\| \to 0$) if and only if the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f$.

**Solution.** Consider the equivalent conditions:

$$\begin{aligned}
&(f_n)_{n\in\mathbb{N}} \text{ converges uniformly to } f\\
\Leftrightarrow &\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } d(f_n(x), f(x)) < \epsilon \,\forall n \geq N,\ \forall x \in X\\
\Leftrightarrow &\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } d(f_n(x), f(x)) \leq \epsilon \,\forall n \geq N,\ \forall x \in X\\
\Leftrightarrow &\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \sup_{x\in X} d(f_n(x), f(x)) \leq \epsilon \,\forall n \geq N\\
\Leftrightarrow &\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \|f_n - f\| \leq \epsilon \,\forall n \geq N\\
\Leftrightarrow &\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \|f_n - f\| < \epsilon \,\forall n \geq N\\
\Leftrightarrow &\|f_n - f\| \xrightarrow{n\to\infty} 0\\
\Leftrightarrow &f_n \xrightarrow{n\to\infty} f \text{ in the uniform norm.} \quad \square
\end{aligned}$$

**c.** Show that $C(X)$ endowed with the uniform norm is complete (i.e. with respect to the metric induced by the norm).

**Solution.** Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C(X)$. Then evaluating at any $x \in X$ yields a Cauchy sequence $(f_n(x))_{n\in\mathbb{N}}$ in $\mathbb{R}$, by the inequality $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|$.

Since $\mathbb{R}$ is complete, the Cauchy sequence $f_n(x)$ converges to some unique limit, which we call $f(x)$. In other words, the function $f\colon X \to \mathbb{R}$ is defined by

$$f(x) := \lim_{n\to\infty} f_n(x).$$

We show that the sequence $f_n$ converges uniformly to $f$. For all $\epsilon > 0$, there is an $N \in \mathbb{N}$ satisfying

$$\|f_m - f_n\| < \epsilon \text{ for all } m, n \geq N$$

which implies

$$\begin{aligned}
&|f_m(x) - f_n(x)| < \epsilon \text{ for all } m, n \geq N \text{ and all } x \in X\\
\Rightarrow &|f(x) - f_n(x)| \leq \epsilon \text{ for all } n \geq N \text{ and all } x \in X
\end{aligned}$$

and proves uniform convergence.

Since each $f_n\colon X \to \mathbb{R}$ is continuous, the limit function $f\colon X \to \mathbb{R}$ is continuous (by Problem 1), and thus an element of $C(X)$.

By part (b), $f_n$ converges to $f$ in the uniform metric on $C(X)$, so that $C(X)$ is complete. $\square$

**Problem 3.** Show that a topological space $X$ is Tychonoff (a.k.a. $T_{3\frac{1}{2}}$) if and only if $X$ is homeomorphic to a subspace of a cube

$$[0,1]^I \cong \prod_{i \in I} [0,1]$$

where $I$ is an arbitrary indexing set.

**Solution.** ($\Rightarrow$) Consider the set of all continuous functions on $X$ with values in $[0,1]$

$$C := \{f \colon X \to [0,1] \mid f \text{ is continuous}\}.$$

Since $X$ is completely regular, functions in $C$ separate points from closed subsets of $X$. Since $X$ is moreover $T_1$, the embedding lemma guarantees that the evaluation map

$$\begin{aligned} e \colon X &\to \prod_{f \in C} [0,1] \cong [0,1]^C \\ x &\mapsto (f(x))_{f \in C} \end{aligned}$$

is an embedding, so that $X$ is homeomorphic to its image $e(X) \subseteq \prod_{f \in C} [0,1]$.

($\Leftarrow$) The interval $[0,1]$ is metrizable, hence $T_4$, hence Tychonoff.

An arbitrary product of Tychonoff spaces is Tychonoff, hence the cube $\prod_{i \in I} [0,1]$ is Tychonoff.

A subspace of a Tychonoff space is Tychonoff, hence $X$ is Tychonoff. (Note that being Tychonoff is invariant under homeomorphism.) $\square$

**Problem 4.** For parts (a) and (b), let $X$ and $Y$ be topological spaces, where $Y$ is *Hausdorff*.

**a.** Let $f, g\colon X \to Y$ be two continuous maps. Show that the subset

$$E := \{x \in X \mid f(x) = g(x)\}$$

where the two maps agree is closed in $X$.

**Solution.** Since $Y$ is Hausdorff, the diagonal $\Delta_Y \subseteq Y \times Y$ is closed.

Consider the unique continuous map $(f, g)\colon X \to Y \times Y$ whose components are $f$ and $g$. The subset $E$ is

$$\begin{aligned} E &= \{x \in X \mid f(x) = g(x)\} \\ &= \{x \in X \mid (f(x), g(x)) \in \Delta_Y\} \\ &= (f, g)^{-1}(\Delta_Y) \end{aligned}$$

which is closed in $X$, since $(f, g)\colon X \to Y \times Y$ is continuous. $\square$

**b.** Let $f, g\colon X \to Y$ be two continuous maps and assume $D \subseteq X$ is a dense subset on which the two maps agree, i.e. $f|_D = g|_D$. Show that the two maps agree everywhere, i.e. $f = g$.

**Solution.** The subset $E \subseteq X$ where $f$ and $g$ agree is closed, by part (a), and contains $D$ by assumption. This implies

$$\begin{aligned} D \subseteq E &\Rightarrow \overline{D} \subseteq \overline{E} = E \\ &\Rightarrow X \subseteq E \text{ since } \overline{D} = X \end{aligned}$$

so that $E = X$, i.e. $f$ and $g$ agree on all of $X$. $\square$

**c.** Find an example of a *metric* space $X$ along with a dense subset $D \subset X$ and a continuous map $f\colon D \to [0, 1]$ that does *not* admit a continuous extension to all of $X$.

**Solution.** Consider $X = \mathbb{R}$ with its standard metric, and the dense subset $D = \mathbb{R} \setminus \{0\}$. Consider the "jump" function $f\colon D \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Then $f$ is continuous, since it is locally constant. However, $f$ does not admit a continuous extension to $\mathbb{R}$, because the limits from the left and from the right at 0 do not agree:

$$\lim_{x \to 0^-} f(x) = 0 \neq 1 = \lim_{x \to 0^+} f(x). \quad \square$$

**d.** Let $X$ be a *separable* topological space. Show that the set $C(X,\mathbb{R})$ of all continuous real-valued functions on $X$ satisfies the cardinality bound

$$|C(X,\mathbb{R})| \leq |\mathbb{R}|^{\aleph_0}$$

where $\aleph_0 = |\mathbb{N}|$ is the countably infinite cardinal.

Recall: A topological space is **separable** if it has a countable dense subset.

**Solution.** Let $D \subseteq X$ be a countable dense subset of $X$, so that its cardinality satisfies $|D| \leq \aleph_0$. By part (b) and the fact that $\mathbb{R}$ is Hausdorff, the restriction map

$$C(X,\mathbb{R}) \to C(D,\mathbb{R})$$

is injective, which proves the cardinality bound

$$\begin{aligned} |C(X,\mathbb{R})| &\leq |C(D,\mathbb{R})| \\ &\leq |\mathbb{R}^D| \text{ where } \mathbb{R}^D \text{ is the set of all functions } D \to \mathbb{R} \\ &= |\mathbb{R}|^{|D|} \\ &\leq |\mathbb{R}|^{\aleph_0}. \quad \square \end{aligned}$$

**Problem 5.** (Munkres Exercise 29.1) Show that the space $\mathbb{Q}$ of rational numbers, with its standard topology, is *not* locally compact.

**Solution.** We will show that every compact subset of $\mathbb{Q}$ has empty interior, and thus cannot be a neighborhood of any point.

Let $A \subseteq \mathbb{Q}$ be a subset with non-empty interior. Then we have

$$(a, b) \cap \mathbb{Q} \subseteq A$$

for some real numbers $a < b$. Pick an *irrational* number $z \in (a, b)$. Then there is a Cauchy sequence $(r_n)_{n\in\mathbb{N}}$ in $(a, b) \cap \mathbb{Q}$ converging to $z$ when viewed as a sequence in $\mathbb{R}$. Therefore the Cauchy sequence $(r_n)$ in $A$ does not converge in $A$, so that $A$ is not complete, hence not compact. $\square$

**Problem 6.** Let $X$ be a set. The **particular point topology** on $X$ with "particular point" $p \in X$ is defined as

$$\mathcal{T} = \{S \subseteq X \mid p \in S \text{ or } S = \emptyset\}.$$

One readily checks that $\mathcal{T}$ is indeed a topology.

**a.** Show that $X$ (endowed with the particular point topology) is locally compact.

**Solution.** For any $x \in X$, the subset $\{x, p\}$ is open and contains $x$, hence is a neighborhood of $x$. Moreover $\{x, p\}$ is finite, hence compact. □

**b.** Show that $X$ is compact if and only if $X$ is finite.

**Solution.** ($\Leftarrow$) Every finite space is compact.

($\Rightarrow$) Consider the open cover $X = \bigcup_{x \in X} \{x, p\}$. Since $X$ is compact, there is a finite subcover

$$X = \{x_1, p\} \cup \ldots \cup \{x_n, p\}$$

so that $X$ is finite. □

**c.** Show that $X$ is Lindelöf if and only if $X$ is countable.

**Solution.** ($\Leftarrow$) Every countable space is Lindelöf.

($\Rightarrow$) Consider the open cover $X = \bigcup_{x \in X} \{x, p\}$. Since $X$ is Lindelöf, there is a countable subcover

$$X = \bigcup_{i \in \mathbb{N}} \{x_i, p\}$$

so that $X$ is countable. □

**d.** Assuming $X$ is uncountable, find a *compact* subspace $K \subseteq X$ whose closure $\overline{K}$ is not compact, in fact not even Lindelöf.

**Solution.** Consider the subset $K = \{p\}$ which is compact, since it is finite.

Note that the closed subsets of $X$ are:

$$\{F \subseteq X \mid p \in F^c \text{ or } F^c = \emptyset\} = \{F \subseteq X \mid p \notin F \text{ or } F = X\}.$$

In particular, $X$ is the only closed subset containing $p$, which implies

$$\overline{\{p\}} = \bigcap_{\substack{F \subseteq X \text{ closed} \\ p \in F}} F = X.$$

Since $X$ is uncountable, part (c) says that $\overline{\{p\}} = X$ is not Lindelöf, in particular not compact. □