

## MA 523: Homework 2

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September 9, 2016



**Problem 2.1**

Verify assertion (36) in [E, §3.2.3], that when  $\Gamma$  is not flat near  $x^0$  the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here  $\nu(x^0)$  denotes the normal to the hypersurface  $\Gamma$  at  $x^0$ ).

**Solution.** ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the standard noncharacteristic boundary condition if  $\Gamma$  is flat near  $x^0$  because in such case we have  $\nu(x^0) = (0, \dots, 0, 1)$  so

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall verify the noncharacteristic condition (2.1) by first flattening the boundary near  $x^0$  and then applying the noncharacteristic boundary conditions to the flattened region. Make a change of variables  $(y_1, \dots, y_n) = \mathbf{y}(x_1, \dots, x_n)$  where  $\mathbf{y}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the change of coordinates

$$\begin{cases} y_1 = x_1, \\ \vdots \\ y_{n-1} = x_{n-1}, \\ y_n = x_n - \varphi(x_1, \dots, x_{n-1}), \end{cases}$$

with  $\varphi$  a sufficiently regular map  $\varphi: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  such that  $x_n^0 = \varphi(x_1^0, \dots, x_{n-1}^0)$ . Let  $\mathbf{x} = \mathbf{y}^{-1}$ . Now, note that  $y^0 = y(x_1^0, \dots, x_n^0) = (y_1, \dots, y_{n-1}, 0)$  and hence  $\Delta = \mathbf{y}(\Gamma)$  is flat near  $y^0$  so we can apply the standard noncharacteristic boundary conditions on the transformed PDE,

$$0 \neq F_{u_{y_n}}(Du(\mathbf{x}(y^0)), u(\mathbf{x}(y^0)), \mathbf{x}(y^0)).$$

First, consider the gradient  $D(\mathbf{x}(y))$ . Looking at the  $i$ -th coordinate of this function, by the chain rule, we have

$$\begin{aligned} u_{y_i}(\mathbf{x}(y)(y)) &= \sum_{j=1}^n u_{x_i} \frac{\partial x_i}{\partial y_j} \\ &= u_{x_i}(\mathbf{x}(y)) + u_{x_n}(\mathbf{x}(y)) \varphi_{y_i}(y), \\ u_{y_n}(\mathbf{x}(y)(y)) &= \sum_{j=1}^n u_{x_n}(\mathbf{x}(y)) \frac{\partial x_n}{\partial y_j} \\ &= \sum_{j=1}^n u_{x_n}(\mathbf{x}(y)) \varphi_{y_j}(y). \end{aligned}$$

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**Problem 2.2**

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions  $u(x, 0) = g(x)$  is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some  $t > 0$ , unless  $a(g(x))$  is a nondecreasing function of  $x$ .

**Solution.** ►

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**Problem 2.3**

Show that the function  $u(x, t)$  defined by  $t \geq 0$  by

$$u(x, t) = \begin{cases} -\frac{2}{3} \left( t + \sqrt{3x + t^2} \right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law  $u_t + (u^2/2)_x = 0$  (*inviscid Burger's equation*).

**Solution.** ►

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