MA 523: Homework 2

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Problem 2.1

Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ▶ First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 because in such case we have $\nu(x^0) = (0, \ldots, 0, 1)$ so

$$
\begin{aligned}
0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \ldots, 0, 1) \\
&= F_{p_n}(p^0, z^0, x^0).
\end{aligned}
$$

We shall verify the noncharacteristic condition (2.1) by first flattening the boundary near x^0 and then applying the noncharacteristic boundary conditions to the flattened region. Assuming some degree of regularity near x^0, e.g., that the boundary of U be smooth, we may express Γ near x^0 as the graph of a smooth function $f \colon \mathbb{R}^{n-1} \to \mathbb{R}$, i.e., $x = (x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1}))$ on Γ and $x_n \geq f(y)$ after reorienting the coordinate axes. Then we flatten out Γ via the map $\Phi(x) \colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$
\begin{cases}
y_1 = x_1 = \Phi^1(x), \\
\quad \vdots \\
y_{n-1} = x_{n-1} = \Phi^{n-1}(x), \\
y_n = x_n - f(x_1, \ldots, x_{n-1}) = \Phi^n(x)
\end{cases}
$$

and write $y = \Phi(x)$. Let $\Psi = \Phi^{-1}$ and rewrite our PDE F in terms of y as follows,

$$0 = F\big(Du(\Psi(y)), u(\Psi(y)), \Psi(y)\big). \tag{2.2}$$

Since $\Delta = \Phi(\Gamma)$ is flat near $y^0 = \Phi(x^0) = (y_1^0, \ldots, y_{n-1}^0, 0)$, we may apply the standard noncharacteristic condition on (2.2) and get

$$0 \neq F_{u_{y_n}}\big(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)\big).$$

Before we move on to finding an expression for this derivative, let us consider the gradient $Du(\Psi(y))$. By the chain rule, we have

$$
\begin{aligned}
u_{y_i}(\Psi(y)) &= \sum_{j=1}^{n} u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_i} \\
&= u_{x_i}(\Psi(y)) + u_{x_n}(\Psi(y)) f_{y_i}(y_1, \ldots, y_{n-1}), \\
u_{y_n}(\Psi(y)) &= \sum_{j=1}^{n} u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_n} \\
&= u_{x_n}(\Psi(y)),
\end{aligned}
$$

Then, substituting u_{y_n} for u_{x_n}, we have

$$u_{y_i}(\Psi(y)) = u_{x_i}(\Psi(y)) + u_{y_n}(\Psi(y))f_{y_i}(y_1, \ldots, y_{n-1}),$$

Now, by the chain rule on (2.2), we have

$$
\begin{aligned}
0 \neq F_{u_{y_n}} & \left(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)\right) \\
&= F_{u_{y_n}} \left(u_{x_1} + u_{y_n}f_{y_1}, \ldots, u_{x_{n-1}} + u_{y_n}f_{y_{n-1}}, u_{y_n}, z^0, x^0\right) \\
&= F_{u_{x_1}}f_{y_1} + \cdots + F_{u_{x_{n-1}}}f_{y_{n-1}} + F_{u_{x_n}} \\
&= D_p F(p^0, z^0, x^0) \cdot (Df(x^0), 1) \\
&= D_p F(p^0, z^0, x^0) \cdot \nu(x^0),
\end{aligned}
$$

as we set out to show.   ◄

Problem 2.2

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x,0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. ▶ The characteristic ODEs of this PDE are

$$\dot{t} = 1, \qquad \dot{x} = a(z), \qquad \dot{z} = 0. \tag{2.3}$$

with initial conditions $t_0 = 0$, $x_0 = x(0)$ and $z(x_0, 0) = g(x_0)$ with $(x_0, 0) \in \mathbb{R} \times (0, \infty)$. Hence, we have

$$t(s) = s, \qquad x(s) = a(g(x_0))s + x_0, \qquad z(s) = g(x_0).$$

Thus, solving for x_0 and s in terms of t, x and z, we have

$$x = a(g(x_0))s + x_0$$
$$= a(z)t + x_0,$$

so, moving x_0 to the left-hand side

$$x_0 = x - a(z)t$$

hence,

$$z = g(x - a(z)t),$$

i.e.,

$$u = g(x - a(u)t),$$

as desired.

For the latter half of the problem, write

$$u\big(x + a(g(x))t, t\big) = g(x).$$

Suppose that $a(g(x))$ is not a nondecreasing function of x. Then, there exists $0 < x_1 < x_2$ such that $a(g(x_1)) > a(g(x_2))$. Define

$$y = -\frac{x_1 - x_2}{a(g(x_1)) - a(g(x_2))} > 0. \tag{2.4}$$

Then, we have

$$t_0 = x_1 + a(g(x_1))y = x_2 + a(g(x_2))y.$$

Thus,

$$
\begin{aligned}
u(x, t_0) &= g(x_1) \\
&= u\big(x_1 + a(g(x_1))t_0, t_0\big) \\
&= g(x_2) \\
&= u\big(x_2 + a(g(x_2))t_0, t_0\big).
\end{aligned}
$$

However, $g(x_1) \neq g(x_2)$ since $a(g(x_1)) > a(g(x_2))$. ◄

Problem 2.3

Show that the function $u(x, t)$ defined for $t \geq 0$ by

$$u(x, t) = \begin{cases} -\dfrac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burgers' equation*).

Solution. ▶ The shock occurs along the curve C given by $s(t) = -t^2/4$. First, we verify that the equation given by u above is in fact a solution to the inviscid Burgers' equation to the right and to the left of C: to the left of C, $4x + t^2 < 0$, the equation is trivially satisfied whereas to the right, $4x + t^2 > 0$, we have,

$$-\frac{2}{3}\left(1 + \frac{t}{\sqrt{3x + t^2}}\right) + \frac{2}{9}\left(3 + \frac{3t}{\sqrt{3x + t^2}}\right) = 0.$$

So u is indeed a solution to the inviscid Burgers' equation.

Now we examine the behavior of u along the curve C. First, we have

$$
\begin{aligned}
\sigma &= \dot{s}(t), & [\![u]\!] &= u_\ell - u_r, & [\![F]\!] &= F(u_\ell) - F(u_r) \\
&= -\frac{t}{2} & &= 0 + \frac{2}{3}\left(t + \sqrt{-\tfrac{3}{4}t^2 + t^2}\right) & &= 0 - \frac{[\![u_r]\!]^2}{2} \\
& & &= 0 + \frac{2}{3}\left(\frac{3}{2}t\right) & &= 0 - \frac{t^2}{2} \\
& & &= t & &= -\frac{t^2}{2}.
\end{aligned}
$$

Thus,

$$[\![F]\!] = -\frac{t^2}{2} = \left(-\frac{t}{2}\right)t = \sigma[\![u]\!]$$

so the PDE satisfies the Rankine–Hugoniot condition and hence, is an integral solution.

Lastly, we verify that u satisfies the entropy condition, i.e., the one-sided jump estimate, Eq. (36) from [E, §3.4.3]. We break this test into two cases, (a) u is to the left of C and (b) u is to the right of C.

First we deal with case (b). When $x > -t^2/4$, $t > 0$, $z > 0$, using linear interpolation, we have

$$u(x + z, t) - u(x, t) \leq \sup_{x > -t^2/4}\{u_x(x, t)\}(x + z - x)$$

$$\leq \sup_{x > -t^2/4}\left\{\frac{1}{\sqrt{3x + t^2}}\right\}z$$

$$=$$

◀