

# MA 523: Homework 4

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## PROBLEM 4.1 (LEGENDRE TRANSFORM)

Let  $u(x_1, x_2)$  be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of  $\mathbb{R}^2$ , where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u(\mathbf{x}(p)),$$

where  $\mathbf{x} = (x^1, x^2)$ ,  $p = (p_1, p_2)$ . Show that  $v$  satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29)).

SOLUTION. ■

## PROBLEM 4.2

Find the solution  $u(x, t)$  of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant  $x > 0, t > 0$  for which

$$\begin{cases} u(x, 0) = f(x), & u_t(x, 0) = g(x), & \text{for } x > 0 \\ u_t(0, t) = \alpha u_x(0, t), & & \text{for } t > 0, \end{cases}$$

where  $\alpha \neq -1$  is a given constant. Show that generally no solution exists when  $\alpha = -1$ . (*Hint:* Use a representation  $u(x, t) = F(x - t) + G(x + t)$  for the solution.)

*SOLUTION.*

■

## PROBLEM 4.3

- (a) Let  $u$  be a solution of the wave equation  $u_{tt} - c^2 u_{xx} = 0$  for  $0 < x < \pi$ ,  $t > 0$  such that  $u(0, t) = u(\pi, t) = 0$ . Show that the *energy*

$$E(t) = \frac{1}{2} \int_0^\pi (u_t^2 + c^2 u_x^2) dx, \quad t > 0$$

is independent of  $t$ ; i.e.,  $dE/dt = 0$  for  $t > 0$ . Assume that  $u$  is  $C^2$  up to the boundary.

- (b) Express the energy  $E$  of the Fourier series solution

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(nct) + b_n \sin(nct)) \sin nx$$

in terms of coefficients  $a_n$ ,  $b_n$ .

*SOLUTION.* For part (a), suppose that  $u$  is, as above, a solution to the wave equation which is  $C^2$  up to the boundary. We show that its energy is independent of  $t$ , i.e., that  $dE/dt = 0$ . Assuming the energy is bounded, the dominated convergence theorem allows us to permute the order of integration and differentiation like so

$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{d}{dt} \left( \frac{1}{2} \int_0^\pi (u_t^2 + c^2 u_x^2) dx \right) \\ &= \frac{1}{2} \int_0^\pi \frac{d}{dt} (u_t^2 + c^2 u_x^2) dx \\ &= \frac{1}{2} \int_0^\pi 2u_t u_{tt} + 2c^2 u_x u_{xt} dx \\ &= \int_0^\pi c^2 u_t u_{xx} + c^2 u_x u_{xt} dx \\ &= \end{aligned}$$

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