MA 519: Homework 9

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Problem 9.1 (Handout 13, # 7)

Let X have a *double exponential* density $f(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$, $-\infty < x < \infty$, $\sigma > 0$.

(a) Show that all moments exist for this distribution.
(b) However, show that the MGF exists only for restricted values. Identify them and find a formula.

Solution. For part (a), we show that the moments $m_n := E(X^n) < \infty$ for all $n \in \mathbb{N}$. By direct calculation, we have

$$\begin{aligned} m_n &= \int_{-\infty}^{\infty} x^n f(x)\,dx \\ &= \int_{-\infty}^{\infty} \frac{x^n}{2\sigma}e^{-\frac{|x|}{\sigma}}\,dx \\ &= \underbrace{\int_{-\infty}^{0} \frac{x^n}{2\sigma}e^{\frac{x}{\sigma}}\,dx}_{L} + \int_{0}^{\infty} \frac{x^n}{2\sigma}e^{-\frac{x}{\sigma}}\,dx, \end{aligned}$$

making the substitution $x \mapsto -y$ to L and relabeling y to x again, the above becomes

$$\begin{aligned} &= \int_{0}^{\infty} \frac{x^n + (-1)x^n}{2\sigma}e^{-\frac{x}{\sigma}}\,dx \\ &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ I := \int_0^\infty \frac{x^n}{\sigma}e^{-\frac{x}{\sigma}}\,dx & \text{if } n = 2k \text{ is even.} \end{cases} \end{aligned}$$

To evaluate I we use integration by parts recursively to arrive at

$$\begin{aligned} I &= \int_{-\infty}^{0} \frac{x^n}{\sigma}e^{-\frac{x}{\sigma}} \\ &= (-0+0) + \int_0^\infty n\sigma x^{n-1}e^{-\frac{x}{\sigma}}\,dx \\ &= (-0+0) + (-0+0) + \int_0^\infty n(n-1)\sigma^2 x^{n-1}e^{-\frac{x}{\sigma}}\,dx \\ &\vdots \\ &= (-0-0) + \cdots + (-0+0) + (-0+n!\sigma^n) \\ &= n!\sigma^n. \end{aligned}$$

Therefore, $m_n < \infty$ for all $n \in \mathbb{N}$, i.e., all moments of this distribution exist.

Set $m_0 := 1$. Then for part (b), the MGF associated to f is

$$m(t) = \sum_{n=0}^{\infty} \frac{t^n m_n}{n!} = \sum_{k=1}^{\infty} t^{2k}\sigma^{2k}. \tag{9.1}$$

This is a geometric series and so converges for all $-\frac{1}{\sigma} < t < \frac{1}{\sigma}$, in which case (9.1) becomes

$$m(t) = \frac{1}{1 - t^2\sigma^2}.$$

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Problem 9.2 (Handout 13, # 10)

Suppose X has Cauchy distribution as in # 6. Which of the following functions have finite expectation

$$X; \quad -X; \quad |x|; \quad \frac{1}{X}; \quad \sin X; \quad \ln|X|; \quad \mathrm{e}^{X}; \quad \mathrm{e}^{-|X|}?$$

Solution. Recall from Handout 13, # 6, that X is the horizontal distance along the wall when a flashlight is aimed at random with angle $\Theta \sim \text{Uniform}[-\pi, \pi]$. First, let us find the PDF of X. Let y be the vertical distance from the light source to the wall. Then, by elementary trigonometry, we have the following relation between X and Θ:

$$X = y \tan\Theta. \tag{9.2}$$

Using (9.2), we can figure out the CDF of X and consequently its PDF:

$$\begin{aligned}
F(x) &= P(X \leq x) \\
&= P(y\tan\Theta \leq x) \\
&= P\left(\Theta \leq \tan^{-1}\left(\frac{x}{y}\right)\right) \\
&= P\left(-\frac{\pi}{2} \leq \Theta \leq \tan^{-1}\left(\frac{x}{y}\right)\right) + P \\
&= \\
&= \frac{1}{2\pi}\int_{-\pi}^{\tan^{-1}(\frac{x}{y})} 1\, dy \\
&= \frac{1}{2\pi}\left[\tan^{-1}\left(\frac{x}{y}\right) + \pi\right] \\
&= \frac{1}{2\pi}\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2},
\end{aligned}$$

therefore

$$\begin{aligned}
f(x) &= \frac{dF(x)}{dx} \\
&=
\end{aligned}$$

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Problem 9.3 (Handout 13, # 16)

Give an example of each of the following phenomena:

(a) A continuous random variable taking values in $[0, 1]$ with equal mean and median.
(b) A continuous random variable taking values in $[0, 1]$ with mean equal to twice the median.
(c) A continuous random variable for which the mean does not exist.
(d) A continuous random variable for which the mean exists, but the variance does not exist.
(e) A continuous random variable with a PDF that is not differentiable at zero.
(f) a positive continuous random variable for which the mode is zero, but the mean does not exist.
(g) A continuous random variable for which all moments exist.
(h) A continuous random variable with median equal to zero, and 25^{th} and 75^{th} percentiles equal to 1.
(i) A continuous random variable X with mean equal to median equal to mode equal to zero, and $E(\sin X) = 0$.

Solution. First, note that $[0, 1]$ is a probability space under the standard Lebesgue measure on $\mathbb{R}$. Therefore, it makes sense to consider $X\colon [0, 1] \to \mathbb{R}$ random variables.

For part (a), consider the random variable $X\colon [0, 1] \to \mathbb{R}$ defined by $x \mapsto x$ with $X \sim \text{Uniform}[0, 1]$. Then the mean is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

and the median is

$$m = \inf\{\, x : F(x) = x \geq 0.5 \,\} = \frac{1}{2}.$$

For part (b), consider again the random variable $X(x) = x$ for $x \in [0, 1]$, but this time let

$$f(x) = \begin{cases} & , \\ & . \end{cases}$$

be the PDF of X. Then the mean is ∎

Problem 9.4 (Handout 13, # 17)

An exponential random variable with mean 4 is known to be larger than 6. What is the probability that it is larger than 8?

Solution. ∎

Problem 9.5 (Handout 13, # 18)

(Sum of Gammas). Suppose X, Y are independent random variables, and $X \sim \Gamma(\alpha, \lambda)$, $Y \sim \Gamma(\beta, \lambda)$. Find the distribution of $X + Y$ by using moment-generating functions.

Solution. ∎

Problem 9.6 (Handout 13, # 19)

(Product of Chi Squares). Suppose $X_1, X_2, \ldots, X_n$ are independent chi square variables, with $X_i \sim \chi^2_{m_i}$. Find the mean and variance of $\prod_{i=1}^n X_i$.

Solution. ■

Problem 9.7 (Handout 13, # 20)

Let $Z \sim \text{Normal}(0,1)$. Find

$$P\left(0.5 < \left|Z - \frac{1}{2}\right| < 1.5\right); \quad P\left(\frac{e^Z}{1+e^Z} > \frac{3}{4}\right); \quad P(\Phi(Z) < 0.5).$$

Solution. ■

Problem 9.8 (Handout 13, # 21)

Let $Z \sim \text{Normal}(0, 1)$. Find the density of $\frac{1}{Z}$. Is the density bounded?

Solution. ■

Problem 9.9 (Handout 13, # 22)

The 25^{th} and the 75^{th} percentile of a normally distributed random variable are -1 and 1. What is the probability that the random variable is between -2 and 2?

Solution. ■