MA 523: Homework 9

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Problem 9.1

(a) Show that for $n = 3$ the general solution to the wave equation $u_{tt} - \Delta u = 0$ with spherical symmetry about the origin has the form

$$u = \tfrac{1}{r}F(r+t) + \tfrac{1}{r}G(r-t), \quad r = |x|,$$

with suitable F and G.

(b) Show that the solution with initial data of the form

$$u(r,0) = 0, \quad u_t(r,0) = h(r)$$

(h is an even function of r) is given by

$$u = \frac{1}{2r}\int_{r-t}^{r+t} \rho h(\rho)\, d\rho.$$

Solution. For part (a): Consider the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u = g, \quad u_t = h & \text{on } \mathbb{R}^3 \times \{\, t = 0 \,\}. \end{cases} \tag{9.1}$$

Suppose that u is a solution to the initial-value problem (**??**) that is symmetric about the origin; i.e., $u(x) = u(x')$ whenever $|x| = |x'|$. Now, make the following substitution ■

Problem 9.2

Show that the solution $w(x_1, t)$ of the initial-value problem for the *Klein–Gordon equation*

$$\begin{cases} w_{tt} = w_{x_1x_1} - \lambda^2 w, & \\ w(x_1, 0) = 0, & w_t(x_1, 0) = h(x_1) \end{cases} \tag{9.2}$$

is given by

$$w(x_1, t) = \frac{1}{2}\int_{x_1 - t}^{x_1 + t} J_0(\lambda s) h(y_1)\, dy_1.$$

Here $s^2 = t^2 - (x_1 - y_1)^2$, while J_0 denotes the Bessel function defined by

$$J_0(z) := \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(z \sin\theta)\, d\theta.$$

(*Hint:* Descend to (**??**) from the two-dimensional wave equation satisfied by

$$u(x_1, x_2, t) = \cos(\lambda x_2) w(x_1, t).)$$

Solution. ■

Problem 9.3

Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u = g, \quad u_t = h & \text{on } \mathbb{R}^3 \times \{\, t = 0 \,\} \end{cases}$$

where g and h are smooth and have compact support. Show there exists a constant C such that

$$|u(x,t)| \leq C t^{-1} \quad (x \in \mathbb{R}^3,\, t > 0).$$

Solution. ∎