

MA 523: Homework 1

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PROBLEM 1.1 (TAYLOR'S FORMULA)

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$$

as $x \rightarrow \mathbf{0}$ for each $k = 1, 2, \dots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m .

Solution. ► Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Proof of claim. We shall prove this by induction on m . The case $m = 1$ is straightforward by the chain rule

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\ &= D^{(1,0,\dots,0)} f(tx) x_1 + \dots + D^{(0,\dots,0,1)} f(tx) x_n \\ &= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^\alpha f(tx) x^\alpha. \end{aligned}$$

Now, assume this for $m - 1$ and consider the m -th derivative of g we have

$$\begin{aligned} \frac{d^m}{dt^m} g(t) &= \frac{d}{dt} \left[\frac{d^{m-1}}{dt^{m-1}} g(t) \right] \\ &= \frac{d}{dt} \left[\sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} D^\alpha f(tx) x^\alpha \right] \\ &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \left[\frac{d}{dt} D^\alpha f(tx) x^\alpha \right] \\ &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} [D^{(1,0,\dots,0)} D^\alpha f(tx) x_1 + \dots + D^{(0,\dots,0,1)} D^\alpha f(tx) x_n] \\ &= \end{aligned}$$

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PROBLEM 1.2

Write down the characteristic equation for the p.d.e.

$$u_t + b \cdot Du = f \tag{*}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve (*) subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ►

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PROBLEM 1.3

Solve using the characteristics:

- (a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.
- (b) $u u_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.
- (c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ►

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PROBLEM 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with $u(x_1, 0) = (1 - x_1^2)/2$.

Solution. ►

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