Purdue University, Fall 2016 Prof. Petrosyan

MA52300 Introduction to Partial Differential Equations

Midterm Exam

Date: October 17, 2016

Duration: 120 min

Name: *Solutions*

PUID:

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. Consider the Cauchy problem [20pt]

$$u_{x_1}^2 + u_{x_2}^2 = u \quad \text{in } \mathbb{R}^2, \quad u(x_1, 0) = ax_1^2 \quad \text{for } x_1 \in \mathbb{R}.$$

(a) For what positive constants a is there a (classical) solution? Is it unique?

(b) Find all solutions of the Cauchy problem above.

Solution: (a) This is a fully nonlinear PDE of first order $F(Du, u, x) = 0$, where $F(p, z, x) = p_1^2 + p_2^2 - z$. The characteristics $(\mathbf{p}(s), z(s), \mathbf{x}(s))$ then satisfy

$$\begin{aligned}
\dot{p}^1 &= -F_{x_1} - F_z p^1 = p^1, & p^1(0) &= p_1^0 \\
\dot{p}^2 &= -F_{x_2} - F_z p^2 = p^2, & p^2(0) &= p_2^0 \\
\dot{z} &= p \cdot F_p = 2(p^1)^2 + 2(p^2)^2 = 2z, & z(0) &= a(x^0)^2 \\
\dot{x}^1 &= F_{p_1} = 2p^1, & x^1(0) &= x^0 \\
\dot{x}^2 &= F_{p_2} = 2p^2 & x^2(0) &= 0
\end{aligned}$$

where p_1^0, p_2^0 must satisfy the *admissibility conditions*

$$p_1^0 = (ax_1^2)_{x_1}\big|_{x_1 = x^0} = 2ax^0, \quad (p_1^0)^2 + (p_2^0)^2 = a(x^0)^2.$$

Plugging p_1^0 into the second equation gives

$$(p_2^0)^2 = (a - 4a^2)(x^0)^2$$

which can be satisfied in an open neighborhood of x^0 iff $a - 4a^2 \geq 0$. Since $a > 0$, this gives

$$0 < a \leq 1/4.$$

Moreover, we obtain

$$p_2^0 = \pm\sqrt{a - 4a^2}\, x^0.$$

This will produce two admissible triples $(\mathbf{p}^0, z^0, \mathbf{x}^0)$ if $0 < a < 1/4$, but only one if $a = 1/4$. Moreover, if $0 < a < 1/4$ and $x_0 \neq 0$, then $F_{p_2} = 2p_2^0 \neq 0$ and thus the local existence theorem will imply that we have at least two solutions near $(x^0, 0)$. When $a = 1/4$, the local existence theorem is not applicable. However, the unique solution exists by explicit computations in part (b).

(b) We next solve the characteristic system. We readily have

$$p^1(s) = 2ax^0 e^s, \quad p^2(s) = \pm\sqrt{a - 4a^2}\, x^0 e^s, \quad z(s) = a(x^0)^2 e^{2s}$$

and hence also

$$x^1(s) = 4ax^0 e^s + (1 - 4a)x^0, \quad x^2(s) = \pm 2\sqrt{a - 4a^2}\, x^0 (e^s - 1).$$

We then notice that we only need to express $x^0 e^s$ in terms of $x_1 = x^1(s)$ and $x_2 = x^2(s)$ in order to find u. We have

$$x^0 e^s = \frac{\mp x^1(s) 2\sqrt{a - 4a^2} - x^2(s)(1 - 4a)}{\mp 8a\sqrt{a - 4a^2} \mp 2\sqrt{a - 4a^2}(1 - 4a)} = x^1(s) \pm x^2(s)\sqrt{1/(4a) - 1}$$

This gives

$$u(x_1, x_2) = a\left(x_1 \pm x_2\sqrt{1/(4a) - 1}\right)^2,$$

which can indeed be verified to be a classical solution (including the case $a = 1/4$).

2. Let L be a positive number and consider the initial boundary value problem [20pt]

$$\begin{aligned} u_{tt} - u_{xx} = 0 &\quad \text{in } (0,L) \times (0,\infty), \\ u(x,0) = \phi,\ u_t(x,0) = \psi &\quad \text{for } x \in [0,L], \\ u(0,t) = 0,\ u_x(L,t) = 0 &\quad \text{for } t > 0. \end{aligned}$$

(a) Find the compatibility condition and prove the existence of a C^2 solution under such a condition.

(b) Prove that this solution is $4L$-periodic in t; i.e.,

$$u(x, t + 4L) = u(x,t).$$

Solution: (a) We first note that we need $\phi \in C^2([0,L])$ and $\psi \in C^1([0,L])$. Next, if we have a C^2 solution, then we must have

$$u(0,t) = 0,\ u_t(0,t) = 0,\ u_{tt}(0,t) = u_{xx}(0,t),\ u_x(L,t) = 0,\ u_{xt}(L,t) = 0$$

and thus, letting $t \to 0+$ we obtain

$$\phi(0) = 0,\ \psi(0) = 0,\ \phi_{xx}(0) = 0,\ \phi_x(L) = 0,\ \psi_x(L) = 0.$$

These are necessary conditions. We will next show that they are also sufficient. Indeed, extend first ϕ and ψ to by even symmetry w.r.t. $x = L$, to $[L, 2L]$ and then, by odd symmetry w.r.t. $x = 0$ to $[-2L, 0]$, and finally by $4L$-periodicity to all of $\mathbb{R}$:

$$\tilde{\phi}(x) = \begin{cases} \phi(2L - x) & x \in [L, 2L] \\ -\tilde{\phi}(-x) & x \in [-2L, 0] \\ \tilde{\phi}(x - 4kL) & x \in [(4k-2)L, (4k+2)L] \end{cases}$$

and similarly for ψ. Note the resulting extensions $\tilde{\phi} \in C^2(\mathbb{R})$ and $\tilde{\psi} \in C^1(\mathbb{R})$ because of the compatibility conditions above (matching derivatives up to the required order). The solution is then given by the D'Alembert's formula

$$u(x,t) = \frac{\tilde{\phi}(x+t) + \tilde{\phi}(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \tilde{\psi}(\xi)d\xi,$$

which will be C^2.

(b) To show $4L$-periodicity, simply notice that the extensions $\tilde{\phi}$ and $\tilde{\psi}$ are $4L$-periodic, as well as

$$\int_{-2L}^{2L} \tilde{\psi}(\xi)d\xi = 0,$$

implying also that $\Psi(x) = \int_0^x \tilde{\psi}(\xi)d\xi$ is $4L$-periodic in x. Hence $u(x,t)$ is also $4L$-periodic in t, by the D'Alembert's formula.

3. Consider the Cauchy problem in $\mathbb{R}^2$ [20pt]

$$u_{xx} + uu_{yy} - u_y = u^2, \quad u\big|_\Gamma = 1, \quad u_y\big|_\Gamma = x,$$

where $\Gamma = \{(x, \sin x) : x \in \mathbb{R}\}$.

(a) Show that it has a real-analytic solution near the origin.

(b) Compute the second order partial derivatives u_{xx}, u_{xy}, u_{yy} at $(0, 0)$.

Solution: (a) As Γ, initial data, and fully nonlinear operator are real analytic, we only need to verify the non-characteristic condition at the origin – the existence of the real analytic solution will follow from the Cauchy-Kovalevskaya theorem.

Since Γ is given by $w = y - \sin x = 0$, the non-characteristic condition is readily satisfied:

$$\sum_{|\alpha|=2} a_\alpha \nu^\alpha = \sum_{|\alpha|=2} a_\alpha (Dw)^\alpha = w_x^2 + uw_y^2 = \cos^2 x + 1 \neq 0.$$

(b) We differentiate u along Γ, which is equivalent to differentiating the identity

$$u(x, \sin x) = 1.$$

w.r.t. x. We obtain

$$u_x + \cos x u_y = 0 \quad \Rightarrow \quad u_x = -x \cos x \text{ on } \Gamma$$

Differentiating one more time we obtain

$$u_{xx} + u_{xy} \cos x = -\cos x + x \sin x.$$

Differentiating $u_y(x, \sin x) = x$, we have

$$u_{xy} + u_{yy} \cos x = 1$$

Combining these two equations with the PDE, we have the following three identities at $(0, 0)$:

$$u_{xx} + u_{xy} = -1, \quad u_{xy} + u_{yy} = 1, \quad u_{xx} + u_{yy} = 1.$$

This gives

$$u_{xx} = -1/2, \quad u_{xy} = -1/2, \quad u_{yy} = 3/2$$

at $(0, 0)$.

4. Let Ω be *unbounded* open set and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω. Show that if $\lim_{\substack{|x|\to\infty \\ x\in\Omega}} u(x) = 0$ then [20pt]

$$\sup_{\overline{\Omega}} |u| = \sup_{\partial\Omega} |u|.$$

[*Hint.* Apply the maximum principle in the open set $\Omega_R = \Omega \cap B_R$ and let $R \to \infty$.]

Solution: Let $\Omega_R = \Omega \cap B_R$ for a large $R > 0$. The boundary $\partial\Omega_R$ is then decomposed into the union of $\Gamma_R = \partial\Omega \cap B_R$ and $\gamma_R = \overline{\Omega} \cap \partial B_R$.

Next, let $x \in \Omega$ be arbitrary and $R > |x|$. Then, by the maximum principle (for bounded open sets)

$$|u(x)| \le \sup_{\Omega_R} |u| \le \sup_{\partial\Omega_R} |u| = \max\left\{\sup_{\Gamma_R} |u|, \sup_{\gamma_R} |u|\right\} \le \max\left\{\sup_{\partial\Omega} |u|, \sup_{\gamma_R} |u|\right\} \to \sup_{\partial\Omega} |u|$$

as $\sup_{\gamma_R} |u| \to 0$ from the given limit of u at infinity. Thus,

$$\sup_{\Omega} |u| \le \sup_{\partial\Omega} |u|,$$

which is equivalent to the desired equality.

5. (a) Let u be a positive harmonic function in a ball $B_R(x_0)$. Prove that for any multi-index α [20pt]

$$|D^\alpha u(x_0)| \le \frac{C_\alpha u(x_0)}{R^{|\alpha|}}.$$

[*Hint:* Use the interior estimates in $B_{R/2}(x_0)$ combined with the Harnack inequality in $B_R(x_0)$]

(b) Use the estimate in part (a) to prove the *Liouville theorem for positive harmonic functions*: if u is a positive harmonic function in $\mathbb{R}^n$, then u is constant in $\mathbb{R}^n$.

Solution: By the interior estimates, applied in $B_{R/2}(x_0)$, we have

$$|D^\alpha u(x_0)| \le \frac{C_{|\alpha|}}{(R/2)^{n+|\alpha|}} \|u\|_{L^1(B_{R/2}(x_0))}.$$

On the other hand

$$\|u\|_{L^1(B_{R/2}(x_0))} = \int_{B_{R/2}(x_0)} |u| = \int_{B_{R/2}(x_0)} u = \alpha_n (R/2)^n u(x_0),$$

by the Mean Value Theorem (one can use the Harnack inequality here instead of MVT). Hence,

$$|D^\alpha u(x_0)| \le \frac{C_\alpha}{R^{|\alpha|}} u(x_0).$$

(b) Applying (a) with $|\alpha| = 1$ gives

$$|Du(x_0)| \le \frac{C}{R} u(x_0)$$

for any $R > 0$. Letting $R \to \infty$, we obtain

$$Du(x_0) = 0.$$

Since $x_0 \in \mathbb{R}^n$ is arbitrary, this is equivalent to $u \equiv$ *const*.