

MA 544: Homework 5

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February 17, 2016

PROBLEM 5.1 (WHEEDEN & ZYGMUND §3, EX. 14)

Show that the conclusion of part (ii) of Exercise 13 (Problem) is false if $|E|_e = +\infty$.

Proof. Let $V \subset [0, 1]$ denote the Vitali set defined in 3.38 and consider the union $E := V \cup (2, \infty)$. It is clear that the inner and outer measure of E is ∞ . However, E itself is unmeasurable since otherwise $E \cap [0, 1] = V \cap [0, 1] = V$ would be measurable. ■

PROBLEM 5.2 (WHEEDEN & ZYGMUND §3, EX. 16)

Prove (3.34).

Proof.

Lemma. $|P| = v(P)$.

The result is trivial by 3.36, but then again, it is used to prove 3.36.

Let $\{\mathbf{e}_k\}_{k=1}^n$ be a set of orthogonal vectors emanating from a point in \mathbb{R}^n . The closed parallelepiped corresponding to $\{\mathbf{e}_k\}_{k=1}^n$ is the set

$$P = \left\{ \mathbf{x} \mid \mathbf{x} = \sum_{k=1}^n t_k \mathbf{e}_k, 0 \leq t_k \leq 1 \right\}. \quad (1)$$

Let's do it this way. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the map which sends the standard basis of \mathbb{R}^n to $\{\mathbf{e}_k\}_{k=1}^n$. This map has determinant not equal to ± 1 . ■

PROBLEM 5.3 (WHEEDEN & ZYGMUND §3, EX. 18)

Prove that outer measure is *translation invariant*; that is, if $E_{\mathbf{h}} := \{\mathbf{x} + \mathbf{h} \mid \mathbf{x} \in E\}$ is the translate of E by \mathbf{h} , $\mathbf{h} \in \mathbb{R}^n$, show that $|E_{\mathbf{h}}|_e = |E|_e$. If E is measurable, show that $E_{\mathbf{h}}$ is also measurable. [This fact was used in proving (3.37).]

Proof. By 3.6, given $\varepsilon > 0$, there exists an open set $G \supset E$ with $|G|_e \leq |E|_e + \varepsilon$. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the linear transformation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{h}$, $\mathbf{h} \in \mathbb{R}^n$. By 3.35 we have $|G|_e = |G| = |T(G)| = |T(G)|_e$ and $T(G)$ is an open set containing $E_{\mathbf{h}}$. Hence, we have an upper bound on the outer measure of $E_{\mathbf{h}}$ given by the inequality

$$|E_{\mathbf{h}}|_e \leq |T(G)|_e = |G|_e \leq |E|_e + \varepsilon. \quad (2)$$

On the other hand, by 3.6 there exists an open set $H \supset E_{\mathbf{h}}$ with $|H|_e \leq |E_{\mathbf{h}}|_e + \varepsilon$. Then by 3.35, we get the inequality

$$|E|_e \leq |T^{-1}(H)|_e = |H|_e \leq |E_{\mathbf{h}}|_e + \varepsilon. \quad (3)$$

Putting (2) and (3) we have

$$|E|_e - \varepsilon \leq |E_{\mathbf{h}}|_e \leq |E|_e + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we have $|E|_e = |E_{\mathbf{h}}|_e$. It then follows that if E is measurable then $E_{\mathbf{h}}$ is measurable since $E_{\mathbf{h}} = T(E)$ and T is a Lipschitz transformation and $|E| = |E_{\mathbf{h}}|$. ■

PROBLEM 5.4 (WHEEDEN & ZYGMUND §4, EX. 1)

Prove corollary (4.2) and theorem (4.8)

Proof.

Corollary (Wheeden & Zygmund, 4.2). *If f is measurable, then $\{f > -\infty\}$, $\{f < +\infty\}$, $\{f = +\infty\}$, $\{a \leq f \leq b\}$, $\{f = a\}$, etc., are all measurable. Moreover f is measurable if and only if $\{a < f < +\infty\}$ is measurable for every finite a .*

Suppose that f is measurable. By 4.1, we have $\{f \geq a\}$ and $\{f \leq a\}$ are measurable so

$$\{f = a\} = \{f \geq a\} \cap \{f \leq a\} \quad (4)$$

is measurable and for $b > a$

$$\{a \leq f \leq b\} = \{f \geq a\} \cap \{f \leq b\}. \quad (5)$$

Proof of corollary 4.2. Now, consider the sequence of measurable sets $\{E_k\}_{k=0}^{\infty}$ where $E_k := \{f < a + k\}$. Then $\{f < \infty\} = \bigcup_{k=0}^{\infty} E_k$ and since $E_k \nearrow \{f < \infty\}$ (take $\mathbf{x} \in E_k$ then $f(\mathbf{x}) < a + k$ so $f(\mathbf{x}) < a + k + 1 \implies \mathbf{x} \in E_{k+1}$), by 3.26, we have $\{f < \infty\}$ is measurable.

Similarly for $\{f > -\infty\}$ we may consider the family $\{E_k\}_{k=0}^{\infty}$ where $E_k := \{f > a - k\}$ (take $\mathbf{x} \in E_k$ then $f(\mathbf{x}) > a - k$ so $f(\mathbf{x}) > a - k - 1 \implies \mathbf{x} \in E_{k+1}$) and taking the limit as $k \rightarrow \infty$ we have $\{f > -\infty\}$ is measurable.

Last but not least, since $\{f < \infty\}$ is measurable, $\{f = \infty\} = \{f < \infty\}^c$ is measurable.

Now, \implies suppose f is measurable. Then $\{a < f < b\} = \{a \leq f \leq b\} \cap \{f = a\}^c \cap \{f = b\}^c$ is measurable for all finite $a < b$. Moreover, the family $\{E_k\}_{k=0}^{\infty}$ of sets $\{E_k\}_{k=0}^{\infty}$ where $E_k := \{a \leq f < b + k\}$ is measurable for all k so, by 3.26, $\{a \leq f < \infty\}$ is measurable since $E_k \nearrow \{a \leq f < \infty\}$.

\Leftarrow On the other hand, suppose that $\{a \leq f < \infty\}$ is measurable for every finite a . Then, for fixed $a \in \mathbb{R}$ the family $\{E_k\}_{k=0}^{\infty}$ where $E_k := \{a - k \leq f < \infty\}$ is measurable. By 3.26, $\{f < \infty\}$ is measurable so $\{f = \infty\} = \{f < \infty\}^c$ is measurable. Thus,

$$\{f > a\} = \{a < f < \infty\} \cup \{f = \infty\}$$

is measurable so f is measurable. ♣

Theorem (Wheeden & Zygmund, 4.8). *If f is measurable and λ is any real number, then $f + \lambda$ and λf are measurable.*

Proof of theorem 4.8. If f is measurable, then $\{f > a\}$ is measurable for all a so $\{f > a - \lambda\} = \{f + \lambda > a\}$ is measurable for all a . Hence, $f + \lambda$ is measurable.

If $\lambda \neq 0$, then $\{f > a/\lambda\}$ is measurable for all a so λf is measurable. If $\lambda = 0$ then $\lambda f = 0$ is clearly measurable since $\{0 > a\} = (a, 0)$ is open for all a (possibly empty if $a \geq 0$, but still an open set).

Thus, $f + \lambda$ and λf are measurable. ♣

■

PROBLEM 5.5 (WHEEDEN & ZYGMUND §4, EX. 2)

Let f be a simple function, taking its distinct values on disjoint sets E_1, \dots, E_N . Show that f is measurable if and only if E_1, \dots, E_N are measurable.

Proof. \implies Suppose f is a simple function taking distinct values on disjoint sets E_1, \dots, E_N . Then $f = \sum_{k=1}^N a_k \chi_{E_k}$. If f is measurable, $\{f > a\}$ is measurable for all finite a . In particular, $\{f > a_k\} = E_k$ is measurable.

\impliedby On the other hand, suppose that E_k is measurable for all $1 \leq k \leq N$. Then χ_{E_k} is measurable and by Problem 5.4, the sum $f = \sum_{k=1}^N a_k \chi_{E_k}$ is measurable. ■