MA 519: Homework 5

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Problem 5.1 (Handout 7, # 6(d, f))

Find the variance of the following random variables

(d) $X = \#$ of tosses of a fair coin necessary to obtain a head for the first time.
(f) $X = \#$ matches observed in random sitting of 4 husbands and their wives in opposite sides of a linear table.
This is an example of the *matching problem.*

Solution. Recall that the variance of a random variable can be computed as

$$\mathrm{Var}(X) = E(X^2) - E(X)^2.$$

For part (d), let X be as above. First, note that X takes every value on $\mathbb{N}$. Thus, its PMF is

$$p(n) = P(X = n) = \frac{1}{2^n}$$

and its expectation the value of the series

$$E(X) = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Using a little bit of analysis we can find the value of $E(X)$, e.g., by considering the function $f(x) := \sum_{n=1}^{\infty} nx^{n-1}$, taking its indefinite integral, and noting that it is a geometric series sans the first term. Concretely,

$$\int f(x)\,dx = \sum_{n=1}^{\infty} x^n = -1 + \sum_{n=0}^{\infty} x^n,$$

which, for $|x| < 1$, converges to the value $x/(1-x)$. Taking the derivative of this, we have $1/(1-x)^2$. Thus,

$$\begin{aligned} E(X) &= \sum_{n=1}^{\infty} \frac{n}{2^n} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \\ &= \frac{1/2}{\big(1-(1/2)\big)^2} \\ &= 2. \end{aligned}$$

This is the mean of X.

Next we must compute the mean of X^2. We have already computed the PMF of X hence,

$$E(X^2) = \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

To find the limit of this series, we can use a similar method to the one in the last paragraph. That is, consider the function $g(x) := \sum_{n=1}^{\infty} n^2 x^{n-1}$. Taking its integral, we have

$$xG(x) = \int g(x)\,dx = \sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1}$$

and repeat this on G, giving us

$$\int G(x)\,dx = \sum_{n=1}^{\infty} x^n = -1 + \sum_{n=0}^{n} x^n = \frac{x}{1-x}.$$

Tracing back our steps,

$$\int g(x) = \frac{x}{(1-x)^2}$$

so

$$g(x) = \frac{1-x^2}{(1-x)^4}.$$

Thus,

$$\begin{aligned} E(X) &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \\ &= \frac{(1/2)\big(1-(1/2)^2\big)}{\big(1-(1/2)\big)^4} \\ &= 6. \end{aligned}$$

Putting all of this together, the variance is

$$\boxed{\mathrm{Var}(X) = 6 - (2)^2 = 2.}$$

For part (f), again, we let X be as above. The PMF of X is given by

$$P(X=1) = \frac{1}{12} \qquad P(X=2) = \frac{1}{24}$$
$$P(X=3) = 0 \qquad P(X=4) = \frac{1}{24}.$$

Then, the mean is

$$\begin{aligned} \mu &= \frac{1}{12} + 2\cdot\frac{1}{24} + 3\cdot 0 + 4\cdot\frac{1}{24} \\ &= \frac{1}{3} \\ &= 0.333. \end{aligned}$$

The second moment is

$$\begin{aligned} E(X^2) &= \frac{2}{12} + \frac{4}{24} + \frac{16}{24} \\ &= \frac{2+4+16}{24} \\ &= \frac{22}{24} \\ &= \frac{11}{12} \\ &\approx 0.917. \end{aligned}$$

Thus, the variance is

$$\boxed{\operatorname{Var}(X) = \frac{11}{12} - \left(\frac{1}{3}\right)^2 \approx 0.806.}$$

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Problem 5.2 (Handout 7, # 8)

(*Nonexistence of variance*).

(a) Show that for a suitable positive constant c, the function $p(x) = c/x^3$, $x = 1, \ldots$, is a valid probability mass function (PMF).
(b) Show that in this case, the expectation of the underlying random variable exists, but the variance does not!

Solution. For part (a), note that $p(x)$ given above satisfies the requirements to be a probability mass function. First, set $1/c = \sum_{x=1}^{\infty} 1/x^3$, and note that indeed c is well defined (because the relevant series converges, by the p-test.)

This means that $1 = \sum_{x=1}^{\infty} c/x^3 = \sum p(x)$, by definition. Moreover, because $p(x) = c/x^3 > 0$ for all x in our domain, $p(x) \in [0, 1]$. That is, p is a valid probability mass function.

Set X equal to the random variable described by p. Next, note that

$$\begin{aligned} E(X) &= \sum_{n=1}^{\infty} n \frac{c}{n^3} \\ &= \sum_{n=1}^{\infty} \frac{c}{n^2} \end{aligned}$$

which converges (and thus exists), again by the p-test.

However,

$$\begin{aligned} E(X^2) &= \sum_{n=1}^{\infty} n^2 \frac{c}{n^3} \\ &= \sum_{n=1}^{\infty} \frac{c}{n} \end{aligned}$$

which does not converge, again by the p-test. That is, the variance $E(X^2) - E(X)^2$ does not exist. ∎

Problem 5.3 (Handout 7, # 9)

In a box, there are 2 black and 4 white balls. These are drawn out one by one at random (without replacement).

(a) Let X be the draw at which the first black ball comes out. Find the mean the variance of X.
(b) Let X be the draw at which the second black ball comes out. Find the meman the variance of X.

Solution. For part (a), we must first find the PMF of X. This we do explicitly,

$$P(X=1)=\frac{2}{6}=\frac{1}{3}, \qquad P(X=2)=\frac{2}{5}\cdot\frac{4}{6}=\frac{4}{15},$$
$$P(X=3)=\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}=\frac{1}{5}, \qquad P(X=4)=\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}=\frac{2}{15},$$
$$P(X=5)=1\cdot\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}=\frac{1}{15}.$$

Thus,

$$\boxed{\begin{aligned} E(X)&=1\cdot\frac{1}{3}+2\cdot\frac{4}{15}+3\cdot\frac{1}{5}+4\cdot\frac{2}{15}+5\cdot\frac{1}{15}\\ &=\frac{7}{3}\\ &=2.333.\end{aligned}}$$

Similarly, we have

$$\begin{aligned} E(X^2)&=1^2\cdot\frac{1}{3}+2^2\cdot\frac{4}{15}+3^2\cdot\frac{1}{5}+4^2\cdot\frac{2}{15}+5^2\cdot\frac{1}{15}\\ &=7.\end{aligned}$$

Hence,

$$\boxed{\operatorname{Var}(X)=7-\left(\frac{7}{3}\right)^2\approx 1.556.}$$

For part (b) we have a similar setup. We compute the PMF of X explicitly

$$P(X=2)=\frac{2}{6}\cdot\frac{1}{5}=\frac{1}{15}, \qquad P(X=3)=\frac{4}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}+\frac{2}{6}\cdot\frac{4}{5}\cdot\frac{1}{4}=\frac{2}{15},$$
$$P(X=4)=3\cdot\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}=\frac{3}{15}, \qquad P(X=5)=4\cdot\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{4}{15},$$
$$P(X=6)=5\cdot\frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}\cdot\frac{2}{2}\cdot 1=\frac{5}{15}.$$

Thus,

$$\boxed{\begin{aligned} E(X)&=\frac{2+3\cdot 2+4\cdot 3+5\cdot 4+6\cdot 5}{15}\\ &=\frac{14}{3}\\ &\approx 4.667.\end{aligned}}$$

Similarly,

$$\begin{aligned} E(X^2) &= \frac{2^2 + 3^2 \cdot 2 + 4^2 \cdot 3 + 5^2 \cdot 4 + 6^2 \cdot 5}{15} \\ &= \frac{70}{3} \\ &\approx 23.333. \end{aligned}$$

Thus,

$$\boxed{\mathrm{Var}(X) \approx \frac{70}{3} - \left(\frac{14}{3}\right)^2 \approx 1.556.}$$

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Problem 5.4 (Handout 7, # 10)

Suppose X has a *discrete uniform distribution* on the set $\{1, \ldots, N\}$.

Find formulas for the mean and the variance of X.

Solution. First, we find the mean:

$$\begin{aligned} E(X) &= \sum_{n=1}^{N} n\frac{1}{N} \\ &= \frac{1}{N}\frac{N(N+1)}{2} \\ &= \frac{(N+1)}{2} \end{aligned}$$

Next, we find the variance:

$$\begin{aligned} E(X^2) - E(X)^2 &= \sum_{n=1}^{N} n^2\frac{1}{N} - \left[\frac{(N+1)}{2}\right]^2 \\ &= \frac{N^2}{3} + \frac{N}{2} + \frac{1}{6} - \left[\frac{(N+1)}{2}\right]^2 \\ &= \frac{N^2-1}{12} \end{aligned}$$

$\blacksquare$

Problem 5.5 (Handout 7, # 11)

(*Be Original*) Give an example of a random variable with mean 1 and variance 100.

Solution. Let X be the random variable whose PMF is given by

$$\begin{aligned} P(X = -10 - 1) &= 0.5 \\ P(X = 10 - 1) &= 0.5 \\ P\big(X \neq \pm\sqrt{10} - 1\big) &= 0 \end{aligned}$$

(Note that those expressions are very easy to simplify (-10-1 =-11, 10-1=9), but leaving them in that form makes the arithmetic more obvious.)

Then we see that the mean of X is given by

$$\begin{aligned} E(X) &= 0.5(-10 - 1 + 10 - 1) \\ &= 1 \end{aligned}$$

and the variance of X is given by

$$\begin{aligned} E((X - E(X))^2) &= E((X - 1)^2) \\ &= 0.5(10^2 + (-10)^2) \\ &= 0.5(10^2 + (-10)^2) \\ &= 100 \end{aligned}$$

so that X is such a random variable as described in the problem. ■

Problem 5.6 (Handout 7, # 13)

(*Be Original*). Suppose a random variable X has the property that its second and fourth moment are both 1.

What can you say about the nature of X?

Solution. By Hölder's inequality,

$$\mu = E(X) \leq E(X^2) < \infty.$$

Similarly, we can show that for any $n \in \mathbb{N}$, X^n has finite mean. ■

Problem 5.7 (Handout 7, # 14)

(*Be Original*). One of the following inequalities is true in general for all nonnegative random variables. Identify which one!

$$E(X)E(X^4) \geq E(X^2)E(X^3);$$
$$E(X)E(X^4) \leq E(X^2)E(X^2).$$

Solution. We show that the first of these inequalities is not true in general. Consider the probability space $\Omega = \{0, 1\}$ and the random variable $X(0) = -1$, $X(1) = 2$ with PMF $P(X = 0) = P(X = 1) = 1/2$. Then,

$$E(X) = -\frac{1}{2} + 1 = \frac{1}{2}, \qquad E(X^2) = \frac{1}{2} + 2 = \frac{5}{2},$$
$$E(X^3) = -\frac{1}{2} + 4 = \frac{7}{2}, \qquad E(X^4) = \frac{1}{2} + 8 = \frac{17}{2},$$

so

$$\frac{17}{4} = \frac{1}{2} \cdot \frac{17}{2} \not\geq \frac{5}{2} \cdot \frac{7}{2} = \frac{35}{4}.$$

That leaves the second inequality. How do we see that the second inequality is true? By the Cauchy–Schwartz inequality,

$$E(X^4) \leq E(X^2).$$

Moreover, ∎

Problem 5.8 (Handout 7, # 15)

Suppose X is the number of heads obtained in 4 tosses of a fair coin.

Find the expected value of the weird function

$$\log\bigl(2+\sin(\tfrac{\pi}{4}x)\bigr).$$

Solution. First, note that

$$P(X=0)=\frac{1}{16},\qquad P(X=1)=\frac{4}{16},$$
$$P(X=2)=\frac{6}{16},\qquad P(X=3)=\frac{4}{16},$$
$$P(X=4)=\frac{1}{16}.$$

Thus, computing the expected value of the function, we get

$$\begin{aligned}
E\bigl[\log\bigl(2+\sin(\tfrac{\pi}{4}X)\bigr)\bigr] &= \sum_{x=0}^{4} p(X=x)\log\bigl(2+\sin(\tfrac{\pi}{4}x)\bigr)\\
&= \frac{1}{16}\Bigl[\log(2)+4\log\bigl(2+\tfrac{\sqrt{2}}{2}\bigr)+6\log(2+1)+4\log\bigl(2+\tfrac{\sqrt{2}}{2}\bigr)+\log(2)\Bigr]\\
&= \frac{1}{16}\Bigl[2\log(2)+8\log\bigl(2+\tfrac{\sqrt{2}}{2}\bigr)+6\log(3)\Bigr]\\
&\approx 0.9966.
\end{aligned}$$

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Problem 5.9 (Handout 7, # 16)

In a sequence of Bernoulli trials let X be the length of the run (of either successes or failures) started by the first trial.

(a) Find the distribution of X, $E(X)$, $\operatorname{Var}(X)$.

Solution. From DasGupta's book, X has the PMF

$$P(X=n)=\binom{N}{n}p^n(1-p)^{N-n}$$

where N is the number of trials.

The mean can be easily computed by taking the series

$$\begin{aligned}
E(X)&=\sum_{n=0}^{N}\left[\binom{N}{n}p^n(1-p)^{N-n}\right]n\\
&=\sum_{n=1}^{N}\left[n\binom{N}{n}\right]p^n(1-p)^{N-n}\\
&=N\sum_{n=1}^{N}\binom{N-1}{n-1}p^n(1-p)^{N-n}\\
&=Np\sum_{n=1}^{N}\binom{N-1}{n-1}p^{n-1}(1-p)^{(N-1)-(n-1)}
\end{aligned}$$

now we can use the binomial identity $(x+y)^n=\sum_{n=0}^{N}\binom{N}{n}x^ny^{N-n}$ to obtain

$$\begin{aligned}
&=Np\big(p+(1-p)\big)^{N-1}\\
&=Np.
\end{aligned}$$

To find the variance, we must first compute the second moment of X. But first, note that

$$\begin{aligned}
n^2\binom{N}{n}&=Nn\binom{N-1}{n-1}\\
&=N(n-1)\binom{N-1}{n-1}+N\binom{N-1}{n-1}\\
&=N(N-1)\binom{N-2}{n-2}+N\binom{N-1}{n-1};
\end{aligned}$$

this will be helpful in computing the second moment. Now,

$$\begin{aligned} E(X) &= \sum_{n=0}^{N}\left[\binom{N}{n}p^n(1-p)^{N-n}\right]n^2 \\ &= \underbrace{N(N-1)\sum_{n=0}^{N}\binom{N-2}{n-2}p^n(1-p)^{N-n}}_{S_1} + \underbrace{N\sum_{n=0}^{N}\binom{N-1}{n-1}p^n(1-p)^{N-n}}_{S_2}. \end{aligned}$$

Note that S_2 is exactly the mean of X and hence $S_2 = Np$. To find S_1 we use a similar trick

$$\begin{aligned} S_1 &= N(N-1)\sum_{n=0}^{N}\binom{N-2}{n-2}p^n(1-p)^{N-n} \\ &= N(N-1)p^2\sum_{n=2}^{N}\binom{N-2}{n-2}p^{n-2}(1-p)^{(N-2)-(n-2)} \\ &= N(N-1)p^2. \end{aligned}$$

Thus, the second moment is

$$E(X^2) = N(N-1)p^2 + Np = N^2p^2 + Np(1-p).$$

Lastly, the variance of X is

$$\mathrm{Var}(X) = N^2p^2 + Np(1-p) - (Np)^2 = Np(1-p).$$

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Problem 5.10 (Handout 7, # 17)

A man with n keys wants to open his door and tries the keys independently and at random. Find the mean and variance of the number of trials

(a) if unsuccessful keys are not eliminated from further selections;
(b) if they are.

(Assume that only one key fits the door. The exact distributions are given in II, 7, but are not required for the present problem.)

Solution. For part (a), the mean is

$$\begin{aligned} E(X) &= \sum_{k=1}^{n} \frac{k}{n} \\ &= \frac{1}{n}\left[\frac{n(n+1)}{2}\right] \\ &= \frac{n+1}{2}. \end{aligned}$$

The second moment is

$$\begin{aligned} E(X^2) &= \frac{1}{n}\sum_{k=1}^{n} k^2 \\ &= \frac{1}{n}\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right] \\ &= \frac{n^2}{3} + \frac{n}{2} + \frac{1}{6}. \end{aligned}$$

Thus, the variance is

$$\mathrm{Var}(X) = \frac{n^2}{3} + \frac{n}{2} + \frac{1}{6} - \left[\frac{n-1}{2}\right]^2.$$

For part (b), the mean is

$$\begin{aligned} E(X) &= \sum_{k=1}^{n} \frac{k}{n-k+1} \\ &= \end{aligned}$$

■