Math 527 - Homotopy Theory
Spring 2013
Homework 10 Solutions

Problem 1. Let $n \geq 2$ [Sorry I forgot to write that on the homework!]. Consider the wedge $X = S^1 \vee S^n$.

a. Show that the n^{th} homotopy group of X is a free $\pi_1(X)$-module on one generator:

$$\pi_n(X) \cong \mathbb{Z}[\pi_1(X)] \cong \mathbb{Z}[t, t^{-1}].$$

Solution. By van Kampen, the fundamental group of X is the free product

$$\pi_1(X) \cong \pi_1(S^1) * \pi_1(S^n) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

The universal cover $\widetilde{X}$ of X consists of the real line $\mathbb{R}$ with a sphere S^n_i wedged at each integer $i \in \mathbb{Z} \subset \mathbb{R}$. The fundamental group $\pi_1(X) \cong \mathbb{Z}$ acts via deck transformations on $\widetilde{X}$, so that $k \in \pi_1(X)$ shifts the spheres $S^n_i \xrightarrow{\cong} S^n_{i+k}$ via the identity function, and translates the real line $\mathbb{R}$ by k, i.e. $k \cdot x = x + k$ for $x \in \mathbb{R}$.

Collapsing the line $\mathbb{R} \subset \widetilde{X}$ produces a homotopy equivalence $\widetilde{X} \xrightarrow{\simeq} \bigvee_{i \in \mathbb{Z}} S^n_i$ where the induced action of $\pi_1(X)$ shifts the wedge summands. As an abelian group, we have

$$\pi_n \left(\bigvee_{i \in \mathbb{Z}} S^n_i \right) \cong \mathbb{Z}\langle \lambda_i | i \in \mathbb{Z} \rangle$$

the free abelian group on all summand inclusions $\lambda_i \colon S^n_i \hookrightarrow \bigvee_{j \in \mathbb{Z}} S^n_j$. The $\pi_1(X)$-action is given by

$$k \cdot \lambda_i = \lambda_{i+k}$$

so that $\pi_n \left(\bigvee_{i \in \mathbb{Z}} S^n_i \right)$ is a free $\pi_1(X)$-module on one generator, say λ_0. To conclude, note that the covering map $p \colon \widetilde{X} \to X$ induces an isomorphism $p_* \colon \pi_n(\widetilde{X}) \xrightarrow{\cong} \pi_n(X)$. $\square$

b. Take a representative $f\colon S^n \to X$ of the class $2t-1 \in \pi_n(X)$ and form a space Y by attaching an $(n+1)$-cell to X via f, as illustrated in the cofiber sequence:

$$S^n \xrightarrow{f} X \xrightarrow{j} Y.$$

Show that the composite $S^1 \xrightarrow{\iota_1} X \xrightarrow{j} Y$ induces an isomorphism on integral homology:

$$H_*(S^1;\mathbb{Z}) \xrightarrow{\simeq} H_*(Y;\mathbb{Z}).$$

Here $\iota_1\colon S^1 \hookrightarrow X$ is the wedge summand inclusion.

Solution. Via the homotopy equivalence $\widetilde{X} \simeq \bigvee_{i\in\mathbb{Z}} S^n_i$, the covering map $p\colon \widetilde{X} \to X$ becomes the fold map followed by the summand inclusion

$$\textstyle\bigvee_{i\in\mathbb{Z}} S^n_i \xrightarrow{\ \nabla\ } S^n \xrightarrow{\ \iota_n\ } S^1 \vee S^n.$$

Therefore $p\colon \widetilde{X} \to X$ induces the "fold" map on integral homology

$$\begin{tikzcd}
H_n(\widetilde{X}) \arrow[r, "p_*"] \arrow[d, "\cong"'] & H_n(X) \arrow[d, equal] \\
H_n\left(\bigvee_{i\in\mathbb{Z}} S^n_i\right) \arrow[r] & H_n\left(S^1 \vee S^n\right) \\
\bigoplus_{i\in\mathbb{Z}} H_n(S^n_i) \arrow[r, "\nabla"] \arrow[u, "\cong"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

$$\textstyle\sum m_i u_i \longmapsto (\sum m_i) u$$

where $u_i \in H_n(S^n_i) \simeq \mathbb{Z}$ and $u \in H_n(S^n)$ are suitably chosen generators, u being chosen for the n-cell of Y.

Here, the isomorphism $H_n(S^n) \xrightarrow{\cong} H_n(S^1 \vee S^n)$ is induced by the summand inclusion $\iota_n\colon S^n \hookrightarrow S^1 \vee S^n$. Its inverse is induced by the summand collapse map $c_n\colon S^1 \vee S^n \twoheadrightarrow S^n$.

The (homotopy) commutative diagram

$$\begin{tikzcd}
 & \widetilde{X} \arrow[d, "p"] & \\
S^n \arrow[ur, "2\iota_1 - \iota_0"] \arrow[r, "f"'] & S^1 \vee S^n \arrow[r, "c_n"'] & S^n
\end{tikzcd}$$

induces the following diagram on integral homology:

$$
\begin{tikzcd}
& H_n(\widetilde{X}) \arrow[d, "p_*"] & \\
H_n(S^n) \arrow[ur, "2u_1 - u_0"] \arrow[r, "f_*"'] & H_n(S^1 \vee S^n) \arrow[r, "\cong", "c_{n*}"'] & H_n(S^n) \\
v \arrow[rr, mapsto] & & (2-1)u
\end{tikzcd}
$$

where again $v \in H_n(S^n)$ is a generator, chosen for the $(n+1)$-cell of Y.

With these choices of generators, the cellular chain complex of Y contains the differential

$$C_{n+1}^{CW}(Y) \simeq \mathbb{Z} \xrightarrow{1} \mathbb{Z} \simeq C_n^{CW}(Y)$$

and thus the homology of Y is isomorphic to that of S^1.

It remains to check that the map $j \circ \iota_1 \colon S^1 \to Y$ induces an isomorphism on integral homology. The only case to check is H_1. Note that the summand inclusion $S^1 \xrightarrow{\iota_1} S^1 \vee S^n$ induces an isomorphism on H_1:

$$H_1(S^1) \to H_1(S^1 \vee S^n) \cong H_1(S^1) \oplus H_1(S^n) \cong H_1(S^1).$$

Moreover, $j \colon X \hookrightarrow Y$ is the inclusion of the n-skeleton, and thus an n-connected map. Therefore j induces an isomorphism on homology H_k in dimensions $k < n$, in particular in dimension $k = 1 < n$. $\square$

Alternate solution for that last step. Since $j \circ \iota_1$ is the inclusion of a subcomplex, in particular a cellular map, it induces a map of cellular chain complexes $C_*^{CW}(S^1) \to C_*^{CW}(Y)$ which can be explicitly written as follows:

$$\begin{tikzcd}
\text{dimension} & C_*^{CW}(S^1) \arrow[r] & C_*^{CW}(Y) \\
n+1 & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, "\simeq"', "1"] \\
n & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
 & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & \vdots \arrow[d] & \vdots \arrow[d] \\
 & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
1 & \mathbb{Z} \arrow[r, "1"] \arrow[d, "0"'] & \mathbb{Z} \arrow[d, "0"] \\
0 & \mathbb{Z} \arrow[r, "1"] & \mathbb{Z}
\end{tikzcd}$$

and clearly induces an isomorphism on homology $H_*^{CW}(S^1;\mathbb{Z}) \xrightarrow{\simeq} H_*^{CW}(Y;\mathbb{Z})$. $\square$

c. Show that the same map $j \circ \iota_1 \colon S^1 \to Y$ induces an isomorphism on π_k for $k < n$ but *not* on π_n.

Solution. Since $j \circ \iota_1 \colon S^1 \to Y$ is the inclusion of the $(n-1)$-skeleton of Y, it is an $(n-1)$-connected map. Moreover, we have:

$$\begin{aligned}
\pi_{n-1}(Y) &\cong \pi_{n-1}(Y_n) \\
&= \pi_{n-1}(S^1 \vee S^n) \\
&\cong \pi_{n-1}(\widetilde{X}) \\
&= 0
\end{aligned}$$

so that $j \circ \iota_1 \colon S^1 \to Y$ induces (trivially) an isomorphism on π_{n-1}.

It remains to check $\pi_n(Y) \neq 0$.

Since X is path-connected and Y is obtained from X by attaching an $(n+1)$-cell via the attaching map $f \colon S^n \to X$, the resulting homotopy group $\pi_n(Y)$ is the quotient of $\pi_n(X) \simeq \mathbb{Z}[t, t^{-1}]$ by the $\pi_1(X)$-submodule generated by $f = 2t - 1$ (c.f. Botvinnik Theorem 11.1).

The ring map

$$\begin{aligned}
\epsilon \colon \mathbb{Z}[t, t^{-1}] &\to \mathbb{Z}[\frac{1}{2}] \\
t &\mapsto \frac{1}{2}
\end{aligned}$$

is well defined and makes $\mathbb{Z}[\frac{1}{2}]$ into a $\mathbb{Z}[t, t^{-1}]$-module. Moreover, ϵ is clearly surjective, and one readily checks the equality $\ker \epsilon = (2t-1)$ so that ϵ induces the isomorphism

$$\mathbb{Z}[t, t^{-1}]/(2t-1) \simeq \mathbb{Z}[\frac{1}{2}].$$

In particular, we conclude $\pi_n(Y) \simeq \mathbb{Z}[t, t^{-1}]/(2t-1) \neq 0$ as claimed. $\square$

Problem 2. (May § 15.2 Problems 3 and 4) Let $n \geq 1$ and let G be an abelian group.

a. Construct a connected CW complex X whose reduced integral homology is given by:

$$\widetilde{H}_i(X;\mathbb{Z}) \simeq \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Such a space is called a **Moore space** and is denoted $M(G,n)$.

Solution. Let

$$0 \to F_1 \xrightarrow{\varphi} F_0 \to G \to 0$$

be an exact sequence of abelian groups where F_i are free abelian groups. We used the fact that a subgroup of a free abelian group is always free.

Realize φ as the homology of a map between wedges of spheres. More precisely, given isomorphisms

$$F_1 \simeq \mathbb{Z}\langle g_i | i \in I\rangle \cong \bigoplus_{i\in I} \mathbb{Z}$$

$$F_0 \simeq \mathbb{Z}\langle h_j | j \in J\rangle \cong \bigoplus_{j\in J} \mathbb{Z}$$

let us build a map

$$f\colon \bigvee_{i\in I} S^n_i \to \bigvee_{j\in J} S^n_j$$

satisfying $H_n(f) = \varphi$ (via the isomorphisms above). The restriction of f to the wedge summand S^n_i is chosen so that its homotopy class satisfies

$$\begin{tikzcd}
f|_{S^n_i} \arrow[d, mapsto] & \in & \pi_n\left(\bigvee_{j\in J} S^n_j\right) \arrow[d, "\text{Hurewicz}"] \\
\varphi(g_i) & \in & H_n\left(\bigvee_{j\in J} S^n_j\right) \simeq \bigoplus_{j\in J}\mathbb{Z}
\end{tikzcd}$$

which is always possible since the Hurewicz morphism for wedges of spheres S^n is surjective (and in fact an isomorphism if $n \geq 2$).

Call the wedges of spheres A and B respectively, and let X be the cofiber of $f\colon A \to B$. Then the long exact sequence on homology of the cofiber sequence $A \xrightarrow{f} B \to X$ yields

$$\ldots \longrightarrow \widetilde{H}_k(B) \longrightarrow \widetilde{H}_k(X) \longrightarrow \widetilde{H}_{k-1}(A) \longrightarrow \ldots$$

which proves $\widetilde{H}_k(X) = 0$ for all $k \neq n, n+1$. In the critical dimensions, the exact sequence is

$$\begin{tikzcd}
\widetilde{H}_{n+1}(B) \arrow[r] & \widetilde{H}_{n+1}(X) \arrow[r] & \widetilde{H}_n(A) \arrow[r, "f_*"] & \widetilde{H}_n(B) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & \widetilde{H}_{n-1}(A) \\
0 \arrow[r] \arrow[u, equal] & \ker(\varphi) \arrow[r] \arrow[u, "\simeq"] & F_1 \arrow[r, "\varphi"] \arrow[u, "\simeq"] & F_0 \arrow[r] \arrow[u, "\simeq"] & \operatorname{coker}(\varphi) \arrow[r] \arrow[u, "\simeq"] & 0 \arrow[u, equal]
\end{tikzcd}$$

which proves $\widetilde{H}_{n+1}(X) = 0$ and $\widetilde{H}_n(X) \simeq G$. Therefore $X = M(G, n)$ is a Moore space and has been built as a connected CW complex. $\square$

b. Construct a connected CW complex Y whose homotopy groups are given by:

$$\pi_i(Y) \simeq \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Such a space is called an **Eilenberg-MacLane space** and is denoted $K(G, n)$.

Solution.

Case $n \geq 2$. The Moore space $M(G, n)$ in part (a) was explicitly built as a complex with a single 0-cell and cells in dimensions n and $n+1$, so that the its 1-skeleton is trivial and therefore so is its fundamental group $\pi_1 M(G, n)$.

By the Hurewicz theorem, the bottom homotopy groups of $M(G, n)$ are

$$\pi_i M(G, n) = 0 \text{ if } i < n$$
$$\pi_n M(G, n) \cong H_n(G, n) = G.$$

Therefore the Postnikov truncation $Y = P_n M(G, n)$ is an Eilenberg-MacLane space $K(G, n)$. Moreover, recall that the Postnikov truncation $W \to P_n W$ can be obtained by attaching cells (of dimension at least $n + 2$). This produces a model for $K(G, n)$ which is a connected CW complex.

Case $n = 1$. The construction from part (a) must be adapted when $n = 1$. Let

$$1 \to F_1 \xrightarrow{\varphi} F_0 \to G \to 1$$

be an exact sequence of groups where F_i are free groups. We used the fact that a subgroup of a free group is always free.

Realize φ as π_1 of a map between wedges of circles. Given isomorphisms

$$F_1 \simeq F\langle g_i | i \in I\rangle \cong *_{i \in I}\mathbb{Z}$$
$$F_0 \simeq F\langle h_j | j \in J\rangle \cong *_{j \in J}\mathbb{Z}$$

where $F\langle S\rangle$ denotes the free group on a set S of generators, let us build a map

$$f \colon \bigvee_{i \in I} S^1_i \to \bigvee_{j \in J} S^1_j$$

satisfying $\pi_1(f) = \varphi$ (via the isomorphisms above). This is always possible, because of the isomorphism

$$\pi_1 \left(\bigvee_{j \in J} S^1_j \right) \cong F\langle \iota_j | j \in J\rangle$$

where $\iota_j \colon S^1_j \hookrightarrow \bigvee_{k \in J} S^1_k$ denotes the summand inclusion.

Call the wedges of circles A and B respectively, and let X be the cofiber of $f\colon A \to B$. Then X is a CW complex with a single 0-cell. Hence, applying π_1 to the cofiber sequence $A \to B \to X$ yields the exact sequence

$$\begin{tikzcd}
\pi_1(A) \arrow[r] & \pi_1(B) \arrow[r] & \pi_1(X) \arrow[r] & 1 \\
F_1 \arrow[r] \arrow[u, "\simeq"] & F_0 \arrow[r] \arrow[u, "\simeq"] & G \arrow[r] \arrow[u, "\simeq"] & 1
\end{tikzcd}$$

which proves $\pi_1(X) \simeq G$. The Postnikov truncation P_1X is an Eilenberg-MacLane space $K(G,1)$, which moreover has been built as a connected CW complex. □

Remark. Applying P_1 to the Moore space $M(G,1)$ would **not** work in general, because $\pi_1 M(G,n)$ is not abelian in general. For example, with $G = \mathbb{Z} \oplus \mathbb{Z}$, a space $M(\mathbb{Z} \oplus \mathbb{Z}, 1)$ obtained in part (a) could be $S^1 \vee S^1$. However, its fundamental group $\pi_1(S^1 \vee S^1) \cong F_2$ is the free group on two generators, which is highly non abelian.

Problem 3. Let $n \geq 0$ and let X be a space with the homotopy type of a CW complex. Consider the Postnikov truncation map $t_n \colon X \to P_n X$, which may be assumed a relative CW complex.

Let $f \colon X \to Z$ be any map, where Z is an Eilenberg-MacLane space of type (G, k) for some abelian group G and $k \leq n$.

Show that there exists a map $g \colon P_n X \to Z$ satisfying $f \simeq g \circ t_n$, and this map g is **unique up to homotopy**. Here, g makes the diagram

$$\begin{tikzcd} X \arrow[r, "t_n"] \arrow[dr, "f"'] & P_n X \arrow[d, dashed, "g"] \\ & Z \end{tikzcd}$$

commute up to homotopy.

Solution. We want to show that restriction along $t_n \colon X \to P_n X$ induces a bijection on homotopy classes of maps:

$$t_n^* \colon [P_n X, Z] \to [X, Z].$$

Since the functor $[-, Z]$ is invariant under homotopy equivalences, WLOG X is a CW complex (and $t_n \colon X \to P_n X$ can still be assumed a relative CW complex).

Since X and $P_n X$ are CW complexes, mapping into an Eilenberg-MacLane space $K(G, k)$ computes cohomology:

$$\begin{tikzcd} {[P_n X, Z]} \arrow[r, "t_n^*"] \arrow[d, "\theta_{P_n X}"', "\cong"] & {[X, Z]} \arrow[d, "\cong"', "\theta_X"] \\ H^k(P_n X; G) \arrow[r, "t_n^*"'] & H^k(X; G). \end{tikzcd}$$

Here we used the fact that G is abelian to cover the case $k = 1$ as well:

$$\begin{aligned} [X, K(G,1)] &\cong H^1(X; G)/\text{conjugation in } G \\ &\cong H^1(X; G). \end{aligned}$$

Note that $t_n \colon X \to P_n X$ is an $(n+1)$-connected map, and thus induces isomorphisms on homology and cohomology with any coefficients in degrees less than $n + 1$. In particular, given $k \leq n$, the restriction map

$$t_n^* \colon H^k(P_n X; G) \xrightarrow{\cong} H^k(X; G)$$

is an isomorphism. Therefore the restriction map

$$t_n^* \colon [P_n X, Z] \xrightarrow{\cong} [X, Z]$$

is also a bijection (in fact an isomorphism of abelian groups). $\square$

Remark. The statement still holds when Z is a product of such Eilenberg-MacLane spaces. However, if Z is more complicated, but still n-truncated (i.e. $\pi_i(Z) = 0$ for $i > n$), then such a factorization $g \colon P_n X \to Z$ still exists, but its homotopy class need not be unique.