MA571: Qual Preparation

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Chapter 1

Gepner

1.1 Gepner's homework

Homework 1

Exercise 1.1. Let $\{X_i : i \in I\}$ be an I-indexed family of topologictal spaces. Show that the Cartesian product

$$X = \prod_{i \in I} X_i,$$

equipped with the product topology, has the property that for each $i \in I$ the projection $\pi_i \colon X \to X_i$ is continuous, and moreover, that X has the following universal property: for any other topological space Y, the function

$$\mathrm{Hom}_{\mathbf{Top}}(Y, X) \longrightarrow \prod_{i \in I} \mathrm{Hom}_{\mathbf{Top}}(Y, X_i),$$

induced by the projections $\pi_i \colon X \to X_i$, is a bijection.

Solution. ∎

Exercise 1.2. Let X be the set equipped with a topology and let $\{\mathcal{U}_i : i \in I\}$ a family of topologies on X. Show that

$$\mathcal{U} = \bigcap_{i \in I} \mathcal{U}_i$$

is a topology on X. Show that if $\mathcal{B}$ is a basis for a topology on X, then the topology $\mathcal{U}$ on X generated by $\mathcal{B}$ is the intersection of all topologies on X which contain $\mathcal{B}$, and that this holds even if we only require that $\mathcal{B}$ be a subbasis.

Solution. ∎

Exercise 1.3. A topological space X is said to be *Hausdorff* if, for every pair of points $x_0, x_1 \in X$ with $x_0 \neq x_1$, there exists open subsets U_0, U_1 of X such that $x_0 \ni U_0$, $x_1 \in U_1$, and $U_0 \cap U_1 = \emptyset$. Show that a topological space X is Hausdorff if and only if the diagonal inclusion $X \to X \times X$ is closed.

Solution. ■

Exercise 1.4. Let X be a topological space and let $Y \subseteq X$ be a subset of X. Show that if Y is equipped with the subspace topology then the inclusion function $\iota\colon Y \to X$ is continuous. Show that if there exists a continuous function $q\colon X \to Y$ such that $q \circ \iota = \mathrm{id}_Y$ then q is a quotient map (that is, Y is also a quotient topology). Give an example of such a situation.

Solution. ■

Exercise 1.5. A *topological group* is a group G with a topology $\mathcal{U}$ such that the multiplication $\mu\colon G \times G \to G$ and inversion $\iota\colon G \to G$ are continuous (it is standard to also assume that the topology $\mathcal{U}$ on G is Hausdorff, which we shall do). Let H be a subgroup of G, and let G/H denote the quotient of G by the action of H, equipped with the quotient topology. Show that G/H is a homogeneous space and that the quotient map $q\colon G \to G/H$ is open. If, moreover, H is a closed subset of G, show that G/H has the property that points are closed. Finally, show that if H is a normal subgroup of G, then G/H is a topological group. (Optional: is it Hausdorff?)

Solution. ■

1.2 Homework 2