MA 523: Homework 2

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Problem 2.1

Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 because in such case we have $\nu(x^0) = (0, \ldots, 0, 1)$ so

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \ldots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall verify the noncharacteristic condition (2.1) by first flattening the boundary near x^0 and then applying the noncharacteristic boundary conditions to the flattened region. Make a change of variables $(y_1, \ldots, y_n) = \mathbf{y}(x_1, \ldots, x_n)$ where $\mathbf{y} \colon \mathbb{R}^n \to \mathbb{R}^n$ is the change of coordinates

$$\begin{cases} y_1 = x_1, \\ \quad \vdots \\ y_{n-1} = x_{n-1}, \\ y_n = x_n - \varphi(x_1, \ldots, x_{n-1}), \end{cases}$$

with φ a sufficiently regular map $\varphi \colon \mathbb{R}^{n-1} \to \mathbb{R}$. Let $\mathbf{x} = \mathbf{y}^{-1}$. Now, note that $y^0 = y(x_1^0, \ldots, x_n^0) = (y_1, \ldots, y_{n-1}, 0)$ and hence $\Delta = \mathbf{y}(\Gamma)$ is flat near y^0 so we can apply the standard noncharacteristic boundary conditions on the transformed PDE,

$$0 \neq F_{u_n}\big(Du(\mathbf{x}(y^0)), u(\mathbf{x}(y^0)), \mathbf{x}(y^0)\big).$$

First, consider the gradient $D(\mathbf{x}(y))$. Looking at the i-th coordinate of this function, by the chain rule, we have

$$\begin{aligned} u_{x_i}(x) &= \sum_{j=1}^{n} u_{x_i} \frac{\partial x_i}{\partial y_j} \\ &= u_{x_i}(x) + u_{x_n}(x)\varphi_{y_i}(y), \\ u_{x_n}(x) &= \sum_{j=1}^{n} u_{x_n}(x) \frac{\partial x_n}{\partial y_j} \\ &= \sum_{j=1}^{n} u_{x_n}(x)\varphi_{y_j}(y). \end{aligned}$$

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Problem 2.2

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. ►

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Problem 2.3

Show that the function $u(x,t)$ defined by $t \geq 0$ by

$$u(x,t) = \begin{cases} -\dfrac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burger's equation*).

Solution. ► ◄