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MA 26500-215 Quiz 11

July 29, 2016

Problem 1: (6 points)

Find the least squares solution $\bar{\mathbf{x}}$ of the system $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$ where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \qquad \bar{\mathbf{b}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

Solution: First, compute all the necessary matrices and vectors

$$\begin{aligned} A^{\mathrm{T}}A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A^{\mathrm{T}}\bar{\mathbf{b}} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -2 \end{bmatrix}. \end{aligned}$$

Then, to find $\bar{\mathbf{x}}$ we compute

$$\begin{aligned} \bar{\mathbf{x}} &= (A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}\bar{\mathbf{b}}) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -2 \end{bmatrix}. \end{aligned}$$

Problem 2: (4 points)

Suppose that A and B are conjugate matrices. Show that if λ is an eigenvalue of A then it is an eigenvalue of B.

Solution: Suppose that λ is an eigenvalue of A and that A is conjugate to B. Then, λ is an eigenvalue of A means that there exists a vector (the associated eigenvector) $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$; while A is conjugate to B means that there exists an invertible matrix P such that $A = PBP^{-1}$. Thus,

$$PBP^{-1}\mathbf{x} = A\mathbf{x} = \lambda\mathbf{x}$$

so

$$\begin{aligned} PBP^{-1}\mathbf{x} &= \lambda\mathbf{x} \\ BP^{-1}\mathbf{x} &= P^{-1}\lambda\mathbf{x} \\ &= \lambda P^{-1}\mathbf{x} \end{aligned}$$

now let $\mathbf{y} = P^{-1}\mathbf{x}$ and we have

$$B\mathbf{y} = \lambda\mathbf{y}.$$

So λ is an eigenvalue of B with associated eigenvector $\mathbf{y} = P^{-1}\mathbf{x}$.

Problem 3: (8 points)

Suppose that P is an idempotent matrix, i.e., $P^2 = P$. Show that the only possible eigenvalues for P are $\lambda = 0$ and $\lambda = 1$.

Solution: Suppose that P is an idempotent matrix and λ is an eigenvalue of P. Then $P\mathbf{x} = \lambda\mathbf{x}$ for some eigenvector $\mathbf{x} \neq \mathbf{0}$. Now, since we have

$$P^2\mathbf{x} = P\mathbf{x}$$

then

$$\begin{aligned} P^2\mathbf{x} &= P(P\mathbf{x}) \\ P\mathbf{x} &= \lambda P\mathbf{x} \\ \lambda\mathbf{x} &= \lambda^2\mathbf{x}. \end{aligned}$$

Thus, $\lambda^2 = \lambda$ so $\lambda^2 - \lambda = \lambda(\lambda - 1) = 0$. Thus, $\lambda = 0$ or $\lambda = 1$.