1. Given n and m, positive integers, compute the number of elements in the following set:

$$S = \{(x_1, x_2, \dots x_m) : 1 \leq x_1 \leq x_2 \leq \dots \leq x_m \leq n, \text{ where } x_i \text{ are integers.}\}$$

$\boxed{\text{Ross p.18 \#15, See also p.17, \#6}}$

$$\underbrace{1 \leq x_1}_{y_1} \leq \underbrace{x_2 \leq}_{y_2} x_3 \leq \cdots \underbrace{\leq x_m}_{y_m} \underbrace{\leq n}_{y_{m+1}}$$

$\boxed{\text{Answer Key} \quad \text{MA519, Fall 2016 (Yip)}}$

Let
$$y_1 = x_1 - 1 \geq 0$$
$$y_2 = x_2 - x_1 \geq 0$$
$$y_3 = x_3 - x_2 \geq 0$$
$$\vdots$$
$$y_m = x_m - x_{m-1} \geq 0$$
$$y_{m+1} = n - x_m \geq 0$$

$$y_1 + y_2 + \cdots + y_m + y_{m+1} = n - 1, \quad y_i \geq 0$$

$$\text{Number of solution} = \binom{(n-1) + (m+1) - 1}{(m+1) - 1} = \binom{n+m-1}{m}.$$

(Proposition 6.2, p. 13)

2. Consider the matching problem for n people with n hats. All the people throw their hats into a container and then each of them picks one at random. Let S be the number of people getting back their own hats. Compute $\mathrm{E}(S)$ and $\mathrm{Var}(S)$.

(Hint: introduce random variables $X_i, i = 1, 2, \ldots n$, such that $X_i = 1$ if the i-th person gets back his own hat and $X_i = 0$ if not. Express S in terms of the X_i's.)

$$S = X_1 + X_2 + \cdots + X_n, \quad X_i = \begin{cases} 1 & \text{if } i \text{ matches} \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{\text{Ross, p.285, eg. 2h}}$

$$\begin{aligned} ES &= E(X_1 + X_2 + \cdots + X_n) \\ &= EX_1 + EX_2 + \cdots + EX_n \\ &= P(X_1 = 1) + P(X_2 = 1) + \cdots + P(X_n = 1) \\ &= \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \\ &= 1 \end{aligned}$$

$$\begin{aligned} E(S^2) &= E\left[(X_1 + \cdots + X_n)^2\right] \\ &= E\left(\sum_{i,j} X_i X_j\right) \\ &= E\left[\sum_i X_i^2 + \sum_{i \neq j} X_i X_j\right] \\ &= \sum_i E(X_i^2) + \sum_{i \neq j} E(X_i X_j) \end{aligned}$$

$X_i^2 = X_i$

$$= \sum_i E(X_i) + \sum_{i \neq j} E(X_i X_j)$$

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$$= \sum_i P(X_i = 1) + \sum_{i \neq j} P(X_i = 1, X_j = 1)$$

$$= n \times \frac{1}{n} + n(n-1)\left(\frac{1}{n}\,\frac{1}{n-1}\right)$$

$$= 2$$

Hence $Var(S) = E(S^2) - (ES)^2$

$$= 2 - 1$$

$$= 1$$

Note

① S is NOT Binomial, even though S is the sum of Bernoulli r.v. X_i. (the X_i's are NOT independent.)

② We in fact know the exact pdf of S

$$P(S = k) = \frac{1}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-k}}{(n-k)!}\right)$$

Hence, we have $ES = \sum_{k=0}^{n} k P(S = k)$

But such a formula is not quite useful.

(3) We can also "understand" the result in terms of large n, $n \gg 1$ or as $n \to \infty$

$$P(S=k) = \frac{1}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-k}}{(n-k)!}\right)$$

$$\Big\downarrow n \to \infty$$

$$\frac{e^{-1}}{k!} \sim \text{Poisson}, \lambda = 1.$$

Hence $\begin{cases} ES = \lambda = 1 \\ Var(S) = \lambda = 1 \end{cases}$

$\uparrow$

But the above results in fact hold even for finite n.

3. Mr. Smith, new in the neighborhood, is known to have two children. He is seen walking with one of his child who is a boy. Under each of the following assumptions about Mr. Smith, compute the probability that both of his children are boys.

   (a) Regardless of gender, Mr. Smith chooses the elder child with probability p;

   (b) In the event that Mr. Smith's two children are of different gender, he chooses boy with probability p.

[Ross, p.71, q.3m]

Mr Smith's ~~family~~ children:

elder	B	B	G	G
younger	B	G	B	G

$E = \{\text{see Mr. Smith with a boy}\}$

$$P(BB|E) = \frac{P(BB \cap E)}{P(E)} = \frac{P(BB)}{P(E)}$$

$$= \frac{P(BB)}{\underbrace{P(E|BB)}_{1} P(BB) + P(E|BG)P(BG) + P(E|GB)P(GB) + \underbrace{P(E|GG)}_{0} P(GG)}$$

$$= \frac{1/4}{\frac{1}{4}\underbrace{P(E|BB)}_{1} + \frac{1}{4}P(E|BG) + \frac{1}{4}P(E|GB)}$$

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$$= \frac{1}{1 + P(E|BG) + P(E|GB)}$$

(a)

$$= \frac{1}{1 + p + (1-p)}$$

$$= \frac{1}{2}$$

[This does make sense as the gender of the elder (& younger) child is equally likely to be boy & girl. Hence the answer does not depend on p.]

(b)

$$= \frac{1}{1 + p + p} = \frac{1}{1+2p}$$

[eg. $P(BB|E) = 1$ if $p = 0$;

$P(BB|E) = \frac{1}{3}$ if $p = 1$.]

4. Consider N interviewees arriving at a company for interviews. Each person is then directed to the first room with probability p or the second room with probability $q = 1 - p$. The decision is done independently for each person. Let X and Y be the number of persons in the first and second room. In the following two situations, find the probability distribution functions of X and Y and determine also if they are independent.

   (a) N is some fixed deterministic number, i.e. each day the company will only interview certain fixed number of people, say, $N = 23$.

   (b) N is Poisson random variable with paramter λ.

(a)

$$\begin{tikzcd} & X \\ N \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & Y \end{tikzcd}$$

$\underline{\text{fixed}}$

$$P(X=i) = \binom{N}{i} p^i q^{N-i}$$

$$P(Y=j) = \binom{N}{j} p^{N-j} q^j$$

X & Y <u>must be dependent</u>, since $X+Y=N$
or $Y=N-X$.

Or, Analytically,

$$P(X=i, Y=j) = \begin{cases} \binom{N}{i} p^i q^j & \text{if } i+j=N \\ 0 & \text{otherwise} \end{cases}$$

$$\neq P(X=i)P(Y=j)$$

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(b)

$$\begin{tikzcd} & X \\ N \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & Y \end{tikzcd}$$

$N \sim$ Poisson, λ

$$X \sim \text{Poisson}(\lambda p), \quad P(X=i) = \frac{e^{-\lambda p}(\lambda p)^i}{i!}$$

$$Y \sim \text{Poisson}(\lambda q), \quad P(Y=j) = \frac{e^{-\lambda q}(\lambda q)^j}{j!}$$

$$P(X=i, Y=j) = \sum_{n=0}^{\infty} P(X=i, Y=j \mid N=n) P(N=n)$$

$$= P(X=i, Y=j \mid N=i+j) P(N=i+j)$$

$$= \binom{i+j}{i} p^i q^j \frac{e^{-\lambda} \lambda^{i+j}}{(i+j)!} = \frac{p^i q^j}{i!\, j!} e^{-\lambda} \lambda^{i+j}$$

$$= \left[\frac{e^{-\lambda p}(\lambda p)^i}{i!}\right] \left[\frac{e^{-\lambda q}(\lambda q)^j}{j!}\right] = P(X=i) P(Y=j)$$

Hence X & Y are independent.

5. Let X be a Poisson random variable with paremeter λ.

(a) Show that $E(X^n) = \lambda E((X+1)^{n-1})$, for $n = 1, 2, \ldots$. [Ross, p 171, #19]

(b) Compute $E(X^3)$.

(c) Compute $E\left(\frac{1}{X+1}\right)$.

(a) $$E(X^n) = \sum_{i=0}^{\infty} i^n \frac{e^{-\lambda}\lambda^i}{i!}$$ $n=1, 2, 3, \ldots$

$$= \sum_{i=1}^{\infty} i^{n-1} \frac{e^{-\lambda}\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} i^{n-1} \frac{e^{-\lambda}\lambda^{i-1}}{(i-1)!} \quad j = i-1$$

$$= \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} \frac{e^{-\lambda}\lambda^j}{j!}$$

$$= \lambda E[(X+1)^{n-1}]$$

(b) $$E[X^3] = \lambda E[(X+1)^2]$$

$$= \lambda E[X^2 + 2X + 1]$$

$$= \lambda [E(X^2) + 2EX + 1]$$

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$$= \lambda\left[\lambda E(X+1) + 2EX + 1\right]$$

$$= \lambda\left[\lambda^2 + \lambda + 2\lambda + 1\right]$$

$$= \lambda\left[\lambda^2 + 3\lambda + 1\right]$$

(c) $E\left(\frac{1}{X+1}\right)$ (Note: cannot use (a) for $n=0$)

$$= \sum_{i=0}^{\infty} \frac{1}{i+1} \frac{e^{-\lambda}\lambda^i}{i!} = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{(i+1)!}$$

$$= \frac{1}{\lambda} \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^{i+1}}{(i+1)!} \quad (j = i+1) =$$

$$= \frac{1}{\lambda} \sum_{j=1}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} = \frac{e^{-\lambda}}{\lambda}\left(\sum_{j=1}^{\infty} \frac{\lambda^j}{j!}\right)$$

$$= \frac{e^{-\lambda}}{\lambda}\left[e^{\lambda} - 1\right] = \boxed{\frac{1-e^{-\lambda}}{\lambda}}$$