MA 519: Homework 11

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Problem 11.1 (DasGupta 7.2 (a), (b), (c), (d), (e))

(a) Suppose $E|X_n - c|^\alpha \to 0$, where $0 < \alpha < 1$. Does X_n necessarily converge in probability to c?
(b) Suppose $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0,1)$. Under what condition on a_n can we conclude that $X_n \xrightarrow{\mathcal{P}} \theta$?
(c) $\mathrm{o}_p(1) + \mathrm{O}_p(1) =$?
(d) $\mathrm{o}_p(1)\mathrm{O}_p(1) =$?
(e) $\mathrm{o}_p(1) + \mathrm{o}_p(1)\mathrm{O}_p(1) =$?

Solution. First let us tackle part (a) of this problem. We suspect that the statement *"X_n converges in probability to c"* is false but first let us attempt to prove this to see where the problem in the implication may lie. Fix $0 < \alpha < 1$ and suppose

$$E(|X_n - c|^\alpha) \longrightarrow 0 \text{ as } n \to \infty. \tag{11.1}$$

Set $\varphi(x) := x - c$ and let g_n denote the PDF of $X_n - c$. By the Jacobian formula

$$g_n(x) = \frac{f_n(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))} = f_n(x + c)$$

where we denote by f_n the PDF of X_n. Then equation (11.1) now reads

$$\left(\int_{-\infty}^{\infty} x f_n(x + c)\, dx\right)^\alpha \longrightarrow 0 \text{ as } n \to \infty.$$

Fix $\varepsilon > 0$. By Markov's inequality, we have

$$P(|X_n - c| > \varepsilon) \le \frac{E(|X_n - c|^\alpha)}{\varepsilon^\alpha}$$

Consider the following (rather contrived) counterexample. Let $X \sim U[\mathrm{e}^{f(\alpha,n)}, 1]$ where

$$f(\alpha, n) := \frac{\ln\left(n(\frac{\alpha+1}{n+1}) - \alpha\right)}{\alpha + 1}.$$

A short calculation shows that

$$E(|X_n - 0|^\alpha) = 1 - \frac{n}{n+1}$$

so $E(|X_n - 0|^\alpha) \to 0$ as $n \to \infty$. But does $X_n \to 0$ in probability? The answer to that appears to be yes. Indeed, by Markov's inequality we have

$$\begin{aligned} P(|X_n - 0| \ge \varepsilon) &= P(|X_n - 0|^\alpha > \varepsilon^\alpha) \\ &\le \frac{E(|X - 0|^\alpha)}{||} \end{aligned}$$

For part (c), suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n = \mathrm{o}_p(1)$ and $b_n = \mathrm{O}_p(1)$, then for the sequence $\{c_n := a_n + b_n\}$ the most we can expect is $c_n = \mathrm{O}_p(1)$. Indeed, we know that if a sequence is $\mathrm{o}_p(1)$ then it is also $\mathrm{O}_p(1)$ therefore there exists K_1 and K_2 such that $|a_n| \le K_1$, $|b_n| \le K_2$ for all $n \ge 1$. Therefore, $|c_n| \le K_1 + K_2$ for all $n \ge 1$. ■

Problem 11.2 (DasGupta 7.3 [Monte Carlo])

Consider the purely mathematical problem of finding a definite integral $f(x)\,dx$ for some (possibly complicated) function $f(x)$. Show that the SLLN provides a method for approximately finding the value of the integral by using appropriate averages $\frac{1}{n}\sum_{k=1}^{n} f(X_k)$.

Numerical analysts call this Monte Carlo integration.

Solution. ∎

Problem 11.3 (DasGupta 7.4 (a), (b))

Suppose $X_1, \dots,$ are i.i.d. and that $E(X_1) = \mu \neq 0$, $\operatorname{Var}(X_1) = \sigma^2 < \infty$. Let $S_{m,p} = \sum_{k=1}^{m} X_k^p$, $m \geq 1$, $p = 1, 2$.

(a) Identify with proof the almost sure limit of $S_{m,1}/S_{n,1}$ for fixed m, and $n \to \infty$.
(b) Identify with proof the almost sure limit of $S_{n-m,1}/S_{n,1}$ for fixed m, and $n \to \infty$.

Solution. ■

Problem 11.4 (DasGupta 7.5 (a))

Let A_n, $n \geq 1$, A be events with respect to a common sample space Ω.

(a) Prove that $I_{A_n} \xrightarrow{\mathcal{L}} I_A$ if and only if $P(A_n) \to P(A)$.

Solution. ∎

Problem 11.5 (DasGupta 7.11 [Sample Maximum])

Let X_k, $k \geq 1$, be an i.i.d. sequence, and $X_{(n)}$ the maximum of $X_1, \ldots, X_n$. Let $\xi(F) = \sup\{\, x : F(x) < 1 \,\}$, where F is the common CDF of the X_k. Prove that $X_{(n)} \xrightarrow{\text{a.s.}} \xi(F)$.

Solution. ∎

Problem 11.6 (DasGupta 7.14 (a))

Suppose X_k are i.i.d. standard Cauchy. Show that

(a) $P(|X_n| > n \text{ infinitely often}) = 1$.

Solution. ∎

Problem 11.7 (DasGupta 7.16 [Coupon Collection])

Cereal boxes contain independently and with equal probability exactly one of n different celebrity pictures. Someone having the entire set of n pictures can cash them in for money. Let W_n be the minimum number of cereal boxes one would need to purchase to own a complete set of the pictures. Find a sequence a_n such that $W_n/a_n \xrightarrow{\mathcal{P}} 1$.
(*Hint:* Approximate the mean of W_n.)

Solution. ∎

Problem 11.8 (DasGupta 7.17)

Let $X \sim \text{Bin}(n,p)$. Show that $(X_n/n)^2$ and $X_n(X_n-1)/(n(n-1))$ both converging in probability to p^2. Do they converge almost surely?

Solution. ■

Problem 11.9 (DasGupta 7.21)

Let $X_1, X_2, \ldots,$ be i.i.d. $U[0,1]$. Let

$$G_n = (X_1 \cdots X_n)^{1/n}.$$

Find c such that $G_n \xrightarrow{\mathcal{P}} c$.

Solution. ∎

Problem 11.10 (DasGupta 7.30 [Conceptual])

Suppose $X_n \xrightarrow{\mathcal{L}} X$, and also $Y_n \xrightarrow{\mathcal{L}} X$. Does this mean that $X_n - Y_n$ converge in distribution to (the point mass at) zero?

Solution. ■

Problem 11.11 (DasGupta 7.31 (a))

(a) Suppose $a_n(X_n - \theta) \to N(0, \tau^2)$; what can be said about the limiting distribution of $|X_n|$, when $\theta \neq 0$, $\theta = 0$?

Solution. ■