

MA 544: Homework 4

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PROBLEM 4.1 (WHEEDEN & ZYGMUND §3, EX. 12)

If E_1 and E_2 are measurable sets in \mathbf{R}^1 , show $E_1 \times E_2$ is a measurable subset of \mathbf{R}^2 and $|E_1 \times E_2| = |E_1||E_2|$. (Interpret $0 \cdot \infty$ as 0.) [HINT: Use a characterization of measurability.]

Proof. By (3.28) (i) we may write E_1 and E_2 as the set difference $H_1 \setminus Z_1$ and $H_2 \setminus Z_2$, respectively, where H_1 and H_2 are G_δ and Z_1 and Z_2 are measure zero. Now, by elementary set theory, the Cartesian product $E_1 \times E_2$ can then be written as

$$E_1 \times E_2 = (H_1 \setminus Z_1) \times (H_2 \setminus Z_2) = \underbrace{(H_1 \times H_2)}_H \setminus \underbrace{(Z_1 \times H_2 \cup H_1 \times Z_2 \cup Z_1 \times Z_2)}_Z \quad (1)$$

Hence, we win by (3.28) (i) if we can show that the Cartesian product of two G_δ sets is an G_δ set and if the Cartesian product of a measurable set with a set of measure zero is measure zero.

First, we prove the former, since the argument to be made is little more than elementary set theory.

Lemma 1. *The Cartesian product of G_δ sets is again G_δ .*

Proof of lemma 1. Let G_1 and G_2 be G_δ . Write $G_1 = \bigcap G'_k$ and $G_2 = \bigcap G''_\ell$ where the G'_k 's and the G''_ℓ 's are open subsets of \mathbf{R} . Then, $G'_k \times G''_\ell$ are open subsets of \mathbf{R}^2 by the definition of the product topology. Moreover, $G'_k \times G''_\ell \subset G_1 \times G_2$ hence, $\bigcap_{k,\ell} G'_k \times G''_\ell \subset G_1 \times G_2$. Thus, it suffices to show that $\bigcap_{k,\ell} G'_k \times G''_\ell \supset G_1 \times G_2$. Let $(x, y) \in G_1 \times G_2$. Then $x \in G_1$ and $y \in G_2$. But since $G_1 = \bigcap G'_k$ and $G_2 = \bigcap G''_\ell$ then $x \in G'_k$ and $y \in G''_\ell$ for some k, ℓ . In other words, $(x, y) \in G'_k \times G''_\ell$ so (x, y) is in the intersection $\bigcap_{k,\ell} G'_k \times G''_\ell$. Hence, we have $G_1 \times G_2 = \bigcap_{k,\ell} G'_k \times G''_\ell$. We conclude that if G_1 and G_2 are G_δ , then so is their Cartesian product $G_1 \times G_2$. ♣

Lemma 2. *Let E be measurable and Z be measure zero. Then $E \times Z$ is measure zero.*

Proof of lemma 2. Let E be a measurable set with $|E| < \infty$ and Z a set of measure zero. Then, for every $\varepsilon > 0$ there exists a countable collection of intervals $\{I_k\}$ containing Z such that $\sum \text{vol}(I_k) < \varepsilon$. Similarly, we can find a collection $\{I'_k\}$ of intervals containing E such that $\sum \text{vol}(I'_k) < |E| + \varepsilon$. Then, $\{I'_k \times I_\ell\}$ is a countable collection of 2-intervals containing $E \times Z$ with

$$\begin{aligned} \sum_{k,\ell} \text{vol}(I'_k \times I_\ell) &= \sum_{k,\ell} \text{vol}(I'_k) \text{vol}(I_\ell) \\ &= \sum_k \sum_\ell \text{vol}(I'_k) \text{vol}(I_\ell) \\ &= \left(\sum_k \text{vol}(I'_k) \right) \left(\sum_\ell \text{vol}(I_\ell) \right) \\ &= (|E| + \varepsilon) \varepsilon \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we have $E \times Z$ is measure zero. If $|E| = \infty$, partition E into disjoint finite measure subsets of \mathbf{R} by taking the following intersection

$$E_k = E \cap (B(0, k) \setminus B(0, k-1))$$

for $k \in \mathbf{N}$.¹ By our previous argument, $E_k \times Z$ is measure zero so $\{E_k \times Z\}$ is a cover of $E \times Z$

¹In fact, it might be quicker from now on to quote the fact that \mathbf{R}^n is σ -finite.

hence, by (3.24), we have

$$\begin{aligned}
 |E \times Z| &= \left| \left(\bigcup_k E_k \right) \times Z \right| \\
 &= \left| \bigcup_k E_k \times Z \right| \\
 &= \sum_k |E_k \times Z| \\
 &= 0.
 \end{aligned}$$

Thus, $E \times Z$ is measure zero. ♣

By lemma 1 and ??, $E_1 \times E_2$ is measurable with $|E_1 \times E_2| = |H_1 \times H_2|$. It's left to show is that $|H_1 \times H_2| = |H_1||H_2|$.

Lemma 3. *If G_1 and G_2 are G_δ then $|G_1 \times G_2| = |G_1||G_2|$.*

But before that, we need to prove the above for the case where G_1 and G_2 are open sets.

Lemma 4. *If G_1 and G_2 are open then $|G_1 \times G_2| = |G_1||G_2|$.*

Proof of lemma 4. Let G_1 and G_2 be open with $|G_1|, |G_2| < \infty$. By (1.11), we may write G_1 and G_2 as the countable intersection of a collection of nonoverlapping closed intervals $\{I_k\}$ and $\{I'_k\}$, respectively. Therefore, we have

$$|G_1| = \sum_k \text{vol}(I_k) \quad \text{and} \quad |G_2| = \sum_k \text{vol}(I'_k).$$

Moreover the collection $\{I_k \times I'_\ell\}$ is a cover of $G_1 \times G_2$ of nonoverlapping closed 2-intervals,² so by (3.2) we have

$$\begin{aligned}
 |G_1 \times G_2| &= \sum_{k,\ell} \text{vol}(I_k \times I'_\ell) \\
 &= \sum_{k,\ell} \text{vol}(I_k) \text{vol}(I'_\ell) \\
 &= \left(\sum_k \text{vol}(I_k) \right) (\text{vol}(I'_\ell)) \\
 &= |G_1||G_2|
 \end{aligned}$$
♣

²They are closed because of elementary topology: the Cartesian product of two closed sets is again closed in the product topology; and they are nonoverlapping because if $(x, y) \in I_k \times I'_\ell \cap I_{k'} \times I'_{\ell'} \neq \emptyset$ then $x \in I_k \cap I_{k'}$ and $y \in I'_\ell \cap I'_{\ell'}$ a contradiction.

Proof of lemma 3. Now that we have the result of lemma 4 we may easily proceed to the countable case. Let G_1 and G_2 be G_δ . Then by lemma 1 $G_1 \times G_2$ is G_δ and we may write $G_1 \times G_2$ as the intersection of a countable collection of open sets $\{G'_k\}$. In particular, if $\{G'_k\}$ is a collection of open sets covering $G_1 \times G_2$ that intersects to $G_1 \times G_2$ then the collection $\{G''_k\}$, where $G''_k := \bigcap_{\ell=1}^k G'_\ell$, also intersects to $G_1 \times G_2$ and has the property that $G''_{k+1} \subset G''_k$. Thus, we may as well assume that $\{H_k\}$ is decreasing so, by (3.26), we have

$$|G_1 \times G_2| = \lim_{k \rightarrow \infty} |H_k|,$$

but H_k is open in the product topology so $H_k = H'_k \times H''_k$ for open subsets $H'_k, H''_k \subset \mathbf{R}$, giving us

$$= \lim_{k \rightarrow \infty} |H_k \times H''_k|,$$

which, by lemma 4, is just

$$\begin{aligned} &= \lim_{k \rightarrow \infty} |H'_k| |H''_k| \\ &= |E_1| |E_2|, \end{aligned}$$

since $H'_k \supset E_1$ and $H''_k \supset E_2$ are open so $\bigcap H'_k \supset E_1$ and $\bigcap H''_k \supset E_2$ and their outer measure approach the outer measure of E_1 and E_2 as $k \rightarrow \infty$. ♣

Putting together our results, by lemma 2 and 3, we have $|E_1 \times E_2| = |E_1| |E_2|$, as desired. ■

PROBLEM 4.2 (WHEEDEN & ZYGMUND §3, EX. 13)

Motivated by (3.7), define the *inner measure* of E by $|E|_i = \sup|F|$, where the supremum is taken over all closed subsets F of E . Show that

- (i) $|E|_i \leq |E|_e$, and
- (ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$.

[Use (3.22).]

Proof. (i) If the outer measure of E is infinite, the inequality holds trivially. Suppose $|E|_e < \infty$. Since closed sets are measurable and their outer measure is equal to their Lebesgue measure, then we may replace $|F|$ by $|F|_e$ to mirror the definition of the outer-measure and, by the monotonicity of the outer measure, we have

$$|F| = |F|_e \leq |E|_e. \quad (2)$$

Taking the supremum on both sides of (2), we obtain the desired inequality

$$|E|_i \leq |E|_e. \quad (3)$$

(ii) \implies Suppose E is measurable with $|E| < \infty$. Then for every $\varepsilon > 0$ there exists an open set $G \supset E$ such that $|G|_e < |E|_e + \varepsilon$.

\Leftarrow Suppose that $|E|_i = |E|_e$. Then ■

PROBLEM 4.3 (WHEEDEN & ZYGMUND §3, EX. 14)

Show that the conclusion of part (ii) of Exercise 13 is false if $|E|_e = +\infty$.

Proof.

■

PROBLEM 4.4 (WHEEDEN & ZYGMUND §3, EX. 15)

If E is measurable and A is any subset of E , show that $|E| = |A|_i + |E - A|_e$. (See Exercise 13 for the definition of $|A|_i$.)

Proof.

■