MA 26500-215   Quiz 10

July 26, 2016

1. Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map that sends

$$T(1,0,0) = (3,2,4)$$
$$T(0,1,0) = (2,0,2)$$
$$T(0,0,1) = (4,2,3).$$

(a) (4 points) Find the value of $T(2,1,-1)$.

Solution: Since T is a linear map, we know that $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ and $T(c\mathbf{v}) = cT(\mathbf{v})$ so

$$
\begin{aligned}
T(2,1,-1) &= T((2,0,0) + (0,1,0) + (0,0,-1)) \\
&= T(2,0,0) + T(0,1,0) + T(0,0,-1) \\
&= 2T(1,0,0) + T(0,1,0) - T(0,0,1) \\
&= 2(3,2,4) + (2,0,2) - (4,2,3) \\
&= (6,4,8) + (2,0,2) + (-4,-2,-3) \\
&= (6+2-4, 4+0-2, 8+2-3) \\
&= (4,2,7).
\end{aligned}
$$

(b) (6 points) Find the matrix representation of T with respect to the standard basis on $\mathbb{R}^3$.

Solution: Using the standard basis on $\mathbb{R}^3$ which, by the way, is the set $\{(1,0,0), (0,1,0), (0,0,1)\}$, for a general vector

$$\mathbf{x} = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1),$$

we have

$$T(\mathbf{x}) = (3x_1 + 2x_2 + 4x_3, 2x_1 + 0x_2 + 2x_3, 4x_1 + 2x_2 + 3x_3).$$

Thus,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3x_1 + 2x_2 + 4x_3 \\ 2x_1 + 0x_2 + 2x_3 \\ 4x_1 + 2x_2 + 3x_3 \end{bmatrix}.$$

This tells us that the matrix must be

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

(c) (10 points) Using the matrix representation of T, find the characteristic polynomial. **You do not have to simplify it.**

Solution: To find the minimal polynomial of T, we find

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 2 & 4 \\ 2 & 0 - \lambda & 2 \\ 4 & 2 & 3 - \lambda \end{bmatrix}$$

$$= (3 - \lambda)\det \begin{bmatrix} 0 - \lambda & 2 \\ 2 & 3 - \lambda \end{bmatrix} - 2\det \begin{bmatrix} 2 & 2 \\ 4 & 3 - \lambda \end{bmatrix}$$

$$+ 4\det \begin{bmatrix} 2 & 0 - \lambda \\ 4 & 2 \end{bmatrix}$$

$$=$$

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