Hwk 14 P.1

4.6: 2, 3, 7, 8, 11, 12

4.6.2) Which of the following sets of vectors are bases for $\mathbb{R}^3$?

(a) $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}2\\3\\4\end{bmatrix},\begin{bmatrix}4\\1\\-1\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}3\\2\\2\end{bmatrix},\begin{bmatrix}-1\\3\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\-1\end{bmatrix},\begin{bmatrix}3\\4\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$.

Need to show $\left[\vec{v}_1\,\vec{v}_2 \cdots \middle|\, \begin{matrix}a\\b\\c\end{matrix}\right]$ is L.I. and span $\mathbb{R}^3$.

(a) No. $\left[\begin{array}{cc|c}1&0&a\\2&1&b\\0&-1&c\end{array}\right] \xrightarrow{-2r_1+r_2} \left[\begin{array}{cc|c}1&0&a\\0&1&b-2a\\0&-1&c\end{array}\right] \xrightarrow{r_2+r_3} \left[\begin{array}{cc|c}1&0&a\\0&1&b-2a\\0&0&b-2a+c\end{array}\right]$ needs $b-2a+c=0$

but are L.I.

(b) $\left[\begin{array}{cccc|c}1&2&4&0&a\\1&3&1&1&b\\-1&4&-1&-1&c\end{array}\right] \xrightarrow[r_1+r_3]{-r_1+r_2} \left[\begin{array}{cccc|c}1&2&4&0&a\\0&1&-3&1&b-a\\0&6&3&-1&a+c\end{array}\right] \xrightarrow[-6r_2+r_3]{-2r_2+r_1} \left[\begin{array}{cccc|c}1&0&10&-2&3a-2b\\0&1&-3&1&b-a\\0&0&21&-7&7a-6b+c\end{array}\right] \xrightarrow{\frac{1}{21}r_3}$

$\left[\begin{array}{cccc|c}1&0&10&-2&3a-2b\\0&1&-3&1&b-a\\0&0&1&-1/3&\frac{1}{21}(7a-6b+c)\end{array}\right]$ which is consistent, so span $\mathbb{R}^3$ (well, first three needed).

but not L.I. so No.

(c) $\left[\begin{array}{ccc|c}3&-1&0&a\\2&3&1&b\\2&1&0&c\end{array}\right] \xrightarrow{-r_3+r_1} \left[\begin{array}{ccc|c}1&-2&0&a-c\\2&2&1&b\\2&1&0&c\end{array}\right] \xrightarrow{-r_3+r_2} \left[\begin{array}{ccc|c}1&-2&0&a-c\\0&1&1&b-c\\2&1&0&c\end{array}\right] \xrightarrow{-2r_1+r_3}$

$\left[\begin{array}{ccc|c}1&-2&0&a-c\\0&1&1&b-c\\0&5&0&-2a+3c\end{array}\right] \xrightarrow{-5r_2+r_3} \left[\begin{array}{ccc|c}1&-2&0&a-c\\0&1&1&b-c\\0&0&-5&-2a-5b+8c\end{array}\right]$ which is consistent

and shows L.I., thus forms a basis for $\mathbb{R}^3$.

(d) $\left[\begin{array}{cccc|c}1&0&3&0&a\\0&2&4&1&b\\0&-1&1&0&c\end{array}\right] \xrightarrow{r_3+r_2} \left[\begin{array}{cccc|c}1&0&3&0&a\\0&1&5&1&b+c\\0&-1&1&0&c\end{array}\right] \xrightarrow{r_2+r_3} \left[\begin{array}{cccc|c}1&0&3&0&a\\0&1&5&1&b+c\\0&0&6&0&b+2c\end{array}\right]$ which is consistent

but not L.I., so not a basis for $\mathbb{R}^3$.

4.6.31 Which of the following sets of vectors are bases for R_4?

(a) $\{[1\,0\,0\,1], [0\,1\,0\,0], [1\,1\,1\,1], [0\,1\,1\,1]\}$ (b) $\{[1\,-1\,0\,2], [3\,-1\,2\,1], [1\,0\,0\,1]\}$

(c) $\{[-2\,4\,6\,4], [0\,1\,2\,0], [-1\,2\,3\,2], [-3\,2\,5\,6], [-2\,-1\,0\,4]\}$

(d) $\{[0\,0\,1\,1], [-1\,1\,1\,2], [1\,1\,0\,0], [2\,1\,2\,1]\}$

(a)
$$\left[\begin{array}{cccc|c} 1&0&1&0&a\\ 0&1&1&1&b\\ 0&0&1&1&c\\ 1&0&1&1&d \end{array}\right] \xrightarrow{-r_1+r_4} \left[\begin{array}{cccc|c} 1&0&1&0&a\\ 0&1&1&1&b\\ 0&0&1&1&c\\ 0&0&0&1&-a+d \end{array}\right] \xrightarrow{\substack{-r_4+r_2\\ -r_4+r_3}} \left[\begin{array}{cccc|c} 1&0&1&0&a\\ 0&1&1&0&a+b-d\\ 0&0&1&0&a+c-d\\ 0&0&0&1&-a+d \end{array}\right] \xrightarrow{\substack{-r_3+r_1\\ -r_3+r_2}} \left[\begin{array}{cccc|c} 1&0&0&0&d-c\\ 0&1&0&0&b-c\\ 0&0&1&0&a+c-d\\ 0&0&0&1&-a+d \end{array}\right]$$

So is L.I. and spans R_4. Hence a basis.

(b)
$$\left[\begin{array}{ccc|c} 1&3&1&a\\ -1&-1&0&b\\ 0&2&0&c\\ 2&1&1&d \end{array}\right] \xrightarrow{\substack{r_1+r_2\\ \frac{1}{2}r_3\\ -2r_1+r_4}} \left[\begin{array}{ccc|c} 1&3&1&a\\ 0&2&1&a+b\\ 0&1&0&c/3\\ 0&-5&1&d-2a \end{array}\right] \xrightarrow{\substack{-2r_3+r_2\\ 5r_3+r_4}} \left[\begin{array}{ccc|c} 1&3&1&a\\ 0&0&1&a+b-\frac{2}{3}c\\ 0&1&0&c/3\\ 0&0&1&d+\frac{5}{3}c-2a \end{array}\right] \xrightarrow{-r_2+r_4}$$

$$\left[\begin{array}{ccc|c} 1&3&1&a\\ 0&0&1&a+b-\frac{2}{3}c\\ 0&1&0&c/3\\ 0&0&0&-3a-b+\frac{7}{3}c+d \end{array}\right]$$ which is consistent if $-3a-b+\frac{7}{3}c+d=0$,

So does not span R_4.

(c)
$$\left[\begin{array}{ccccc|c} -2&0&-1&-3&-2&a\\ 4&1&2&2&-1&b\\ 6&2&3&5&0&c\\ 4&0&2&6&4&d \end{array}\right] \xrightarrow{\substack{2r_1+r_2\\ 3r_1+r_3\\ 2r_1+r_4}} \left[\begin{array}{ccccc|c} -2&0&-1&-3&-2&a\\ 0&1&0&-4&-5&2a+b\\ 0&2&0&-4&-6&2a+c\\ 0&0&0&0&0&2a+d \end{array}\right]$$ which is consistent if $2a+d=0$

So ~~does~~ is not a basis or span R_4.

(d)
$$\left[\begin{array}{cccc|c} 0&-1&1&2&a\\ 0&1&1&1&b\\ 1&1&0&2&c\\ 1&2&0&1&d \end{array}\right] \xrightarrow{r_3+r_1} \left[\begin{array}{cccc|c} 1&0&1&4&a+c\\ 0&1&1&1&b\\ 1&1&0&2&c\\ 1&2&0&1&d \end{array}\right] \xrightarrow{\substack{-r_1+r_3\\ -r_1+r_4}} \left[\begin{array}{cccc|c} 1&0&1&4&a+c\\ 0&1&1&1&b\\ 0&1&-1&-2&-a+c\\ 0&2&-1&-3&-a+d \end{array}\right] \xrightarrow{\substack{-r_2+r_3\\ -2r_2+r_4}} \left[\begin{array}{cccc|c} 1&0&1&4&a+c\\ 0&1&1&1&b\\ 0&0&-2&-3&-a-b+c\\ 0&0&-3&-5&-a-b+d \end{array}\right] \xrightarrow{-r_4+r_3}$$

$$\left[\begin{array}{cccc|c} 1&0&1&4&a+c\\ 0&1&1&1&b\\ 0&0&1&2&c-d\\ 0&0&-3&-5&-a-b-d \end{array}\right] \xrightarrow{3r_3+r_4} \left[\begin{array}{cccc|c} 1&0&1&4&a+c\\ 0&1&1&1&b\\ 0&0&1&2&c-d\\ 0&0&0&1&-a-b+3c-4d \end{array}\right]$$ which is consistent and L.I.

thus spans R_4.

Determine which of the given subsets form a basis for $\mathbb{R}^3$. Express $\begin{bmatrix}2\\1\\3\end{bmatrix}$ in terms of the subset that does form a basis.

4.6.7 (a) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}3\\1\\3\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$

(a) $$\left[\begin{array}{ccc|c}1&1&0&a\\1&2&1&b\\1&3&0&c\end{array}\right]\begin{array}{l}-r_1+r_2\\-r_1+r_3\end{array}\left[\begin{array}{ccc|c}1&1&0&a\\0&1&1&-a+b\\0&2&0&-a+c\end{array}\right]\begin{array}{l}-r_2+r_3\\-2r_2+r_3\end{array}\left[\begin{array}{ccc|c}1&0&-1&2a-b\\0&1&1&-a+b\\0&0&-2&a-2b+c\end{array}\right]-\tfrac{1}{2}r_3\left[\begin{array}{ccc|c}1&0&-1&2a-b\\0&1&1&-a+b\\0&0&1&-\tfrac12(a-2b+c)\end{array}\right]$$

$$\begin{array}{l}r_3+r_1\\-r_3+r_2\end{array}\left[\begin{array}{ccc|c}1&0&0&\tfrac32 a-\tfrac12 c\\0&1&0&-\tfrac12 a+\tfrac12 c\\0&0&1&-\tfrac12 a+b-\tfrac12 c\end{array}\right]$$ So this forms a basis and $\begin{bmatrix}2\\1\\3\end{bmatrix}=\frac32\begin{bmatrix}1\\1\\1\end{bmatrix}+\frac12\begin{bmatrix}1\\2\\3\end{bmatrix}-\frac32\begin{bmatrix}0\\1\\0\end{bmatrix}$

(b) Not a basis, for the set contains $\begin{bmatrix}0\\0\\0\end{bmatrix}$, so dropping it

$$\left[\begin{array}{cc|c}1&2&a\\2&1&b\\3&3&c\end{array}\right]\begin{array}{l}-2r_1+r_2\\-3r_1+r_3\end{array}\left[\begin{array}{cc|c}1&2&a\\0&-3&-2a+b\\0&-3&-3a+c\end{array}\right]-r_2+r_3\left[\begin{array}{cc|c}1&2&a\\0&-3&-2a+b\\0&0&-a-b+c\end{array}\right]-\tfrac13 r_2\left[\begin{array}{cc|c}1&2&a\\0&1&\tfrac23 a-\tfrac13 b\\0&0&-a-b+c\end{array}\right]$$

$$-2r_2+r_1\left[\begin{array}{cc|c}1&0&-\tfrac13 a+\tfrac23 b\\0&1&\tfrac23 a-\tfrac13 b\\0&0&-a-b+c\end{array}\right]$$ So as $-(2)-(1)+(3)=0$, $\begin{bmatrix}2\\1\\3\end{bmatrix}=0\begin{bmatrix}1\\2\\3\end{bmatrix}+1\begin{bmatrix}2\\1\\3\end{bmatrix}$

4.6.8 (a) $\left\{\begin{bmatrix}2\\1\\3\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}1\\1\\4\end{bmatrix},\begin{bmatrix}1\\5\\1\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}2\\2\\0\end{bmatrix},\begin{bmatrix}3\\4\\-1\end{bmatrix}\right\}$

(a) $$\left[\begin{array}{cccc|c}2&1&1&1&a\\1&2&1&5&b\\3&1&4&1&c\end{array}\right]\xrightarrow{\text{RREF}}\left[\begin{array}{cccc|c}1&0&0&-\tfrac98&\tfrac78 a-\tfrac38 b-\tfrac18 c\\0&1&0&\tfrac{23}{8}&-\tfrac18 a+\tfrac58 b-\tfrac18 c\\0&0&1&\tfrac38&-\tfrac58 a+\tfrac18 b+\tfrac38 c\end{array}\right]$$ So $\begin{bmatrix}2\\1\\3\end{bmatrix}=1\begin{bmatrix}2\\1\\3\end{bmatrix}+0\begin{bmatrix}1\\2\\1\end{bmatrix}+0\begin{bmatrix}1\\1\\4\end{bmatrix}$

Not a basis unless dropping 4th vector.

(b) $$\left[\begin{array}{ccc|c}1&2&3&a\\1&2&4&b\\2&0&-1&c\end{array}\right]\xrightarrow[\text{Getting Lazy}]{\text{RREF}}\left[\begin{array}{ccc|c}1&0&0&-2a+2b+\tfrac12 c\\0&1&0&3a-\tfrac52 b-\tfrac14 c\\0&0&1&-a+b\end{array}\right]$$ So $\begin{bmatrix}2\\1\\3\end{bmatrix}=-\frac12\begin{bmatrix}1\\1\\2\end{bmatrix}+\frac{11}{4}\begin{bmatrix}2\\2\\0\end{bmatrix}-1\begin{bmatrix}3\\4\\-1\end{bmatrix}$

also forms a basis

4.6.11 Find a basis for the subspace W of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}11\\10\\7\end{bmatrix}, \begin{bmatrix}7\\6\\4\end{bmatrix} \right\}$.
What is the dimension of W?

Form

$$\left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & a \\ 2 & 2 & 10 & 6 & b \\ 2 & 1 & 7 & 4 & c \end{array}\right] \begin{array}{l} \\ -2r_1+r_2 \\ -2r_1+r_3 \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & a \\ 0 & -4 & -12 & -8 & -2a+b \\ 0 & -5 & -15 & -10 & -2a+c \end{array}\right] -r_3+r_2 \left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & a \\ 0 & 1 & 3 & 2 & b-c \\ 0 & -5 & -15 & -10 & -2a+c \end{array}\right]$$

$$\begin{array}{l} -3r_2+r_1 \\ \\ 5r_2+r_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & a-3b+3c \\ 0 & 1 & 3 & 2 & b-c \\ 0 & 0 & 0 & 0 & -2a+5b-4c \end{array}\right]$$

So W is spanned by $\begin{bmatrix}1\\2\\2\end{bmatrix}$ and $\begin{bmatrix}3\\2\\1\end{bmatrix}$ which has dimension 2. It is the plane $-2a+5b-4c=0$.

4.6.12 Find a basis for the subspace W of R_4 spanned by
$\{[1\ 1\ 0\ -1], [0\ 1\ 2\ 1], [1\ 0\ 1\ -1], [1\ 1\ -6\ -3], [-1\ -5\ 1\ 0]\}$.
What is the dimension of W?

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & a \\ 1 & 1 & 0 & 1 & -5 & b \\ 0 & 2 & 1 & -6 & 1 & c \\ -1 & 1 & -1 & -3 & 0 & d \end{array}\right] \begin{array}{l} \\ -r_1+r_2 \\ \\ r_1+r_4 \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & a \\ 0 & 1 & -1 & 0 & -4 & -a+b \\ 0 & 2 & 1 & -6 & 1 & c \\ 0 & 1 & 0 & -2 & -1 & a+d \end{array}\right] \begin{array}{l} -2r_2+r_3 \\ -r_2+r_4 \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & a \\ 0 & 1 & -1 & 0 & -4 & -a+b \\ 0 & 0 & 3 & -6 & 9 & 2a-2b+c \\ 0 & 0 & 1 & -2 & 3 & 2a-b+d \end{array}\right]$$

$$\begin{array}{l} -r_4+r_1 \\ r_4+r_2 \\ -3r_4+r_3 \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & -4 & -a+b-d \\ 0 & 1 & 0 & -2 & -1 & a+d \\ 0 & 0 & 0 & 0 & 0 & -4a+b+c-3d \\ 0 & 0 & 1 & -2 & 3 & 2a-b+d \end{array}\right] \begin{array}{c} r_3 \\ \Updownarrow \\ r_4 \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & -4 & -a+b-d \\ 0 & 1 & 0 & -2 & -1 & a+d \\ 0 & 0 & 1 & -2 & 3 & 2a-b+d \\ 0 & 0 & 0 & 0 & 0 & -4a+b+c-3d \end{array}\right]$$

So W is the space $-4a+b+c-3d=0$ which is spanned by
$[1\ 1\ 0\ -1]$, $[0\ 1\ 2\ 1]$, $[1\ 0\ 1\ -1]$ of dimension 3