## Math 527 - Homotopy Theory Spring 2013 Homework 14 Solutions

**Problem 1.** Let  $n \geq 2$ . For any  $f \in \pi_{2n-1}(S^n)$ , denote its Hopf invariant by  $H(f) \in \mathbb{Z}$ .

**a.** Show that if n is odd, then H(f) = 0 holds for all  $f \in \pi_{2n-1}(S^n)$ .

**Solution.** Let  $\alpha_f \in H^n(C(f)) \simeq \mathbb{Z}$  and  $\beta_f \in H^{2n}(C(f)) \simeq \mathbb{Z}$  denote the standard generators. By graded commutativity of the cup product, we have

$$\alpha_f^2 = (-1)^{|\alpha_f||\alpha_f|} \alpha_f^2$$
$$= (-1)^{n^2} \alpha_f^2$$
$$= -\alpha_f^2$$

so that  $2\alpha_f^2 = 0$  holds, which implies  $\alpha_f^2 = 0$  since  $H^{2n}(C(f))$  is torsionfree.

**b.** Show that if n is not a power of 2, then there is no  $f \in \pi_{2n-1}(S^n)$  satisfying H(f) = 1.

**Solution.** Consider reduction of coefficients mod 2

$$\varphi \colon H^*(C(f); \mathbb{Z}) \to H^*(C(f); \mathbb{Z}/2)$$

induced by the map of rings  $\mathbb{Z} \to \mathbb{Z}/2$ , and denote the reduced cohomology class by  $\varphi(x) = \overline{x}$ . By the universal coefficient theorem (or cellular cohomology), the mod 2 cohomology of C(f) is given by

$$H^k(C(f); \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } k = 0, n, 2n \\ 0 & \text{otherwise} \end{cases}$$

and  $\varphi$  is just the usual reduction  $\mathbb{Z} \to \mathbb{Z}/2$  mod 2 in each degree. Hence, to prove the result, it suffices to show that  $\overline{\alpha_f^2} = 0 \in H^{2n}(C(f); \mathbb{Z}/2)$  holds for all f. Now we have

$$\overline{\alpha_f^2} = \overline{\alpha_f}^2 \text{ since } \varphi \text{ is a ring map}$$
$$= \operatorname{Sq}^n \overline{\alpha_f}$$

where  $Sq^i$  denotes Steenrod squares. Since n is not a power of 2,  $Sq^n$  is decomposable, i.e. it can be written as a sum

$$\operatorname{Sq}^n = \sum \operatorname{Sq}^{I^l}$$

where each  $I^l = (i_1^l, i_2^l, \dots, i_{m_l}^l)$  is a multi-index where all indices satisfy  $i_j^l < n$ . Since  $H^k(C(f); \mathbb{Z}/2) = 0$  holds for n < k < 2n, we have

$$\operatorname{Sq}^{I^{l}} \overline{\alpha_{f}} = \operatorname{Sq}^{i_{1}^{l}} \operatorname{Sq}^{i_{2}^{l}} \dots \operatorname{Sq}^{i_{m_{l}}^{l}} \overline{\alpha_{f}}$$

$$= \operatorname{Sq}^{i_{1}^{l}} \operatorname{Sq}^{i_{2}^{l}} \dots \operatorname{Sq}^{i_{m_{l}-1}^{l}}(0)$$

$$= 0$$

for all such multi-indices  $I^l$ , and therefore:

$$\operatorname{Sq}^{n} \overline{\alpha_{f}} = \sum \operatorname{Sq}^{I^{l}} \overline{\alpha_{f}}$$

$$= \sum 0$$

$$= 0. \quad \square$$

**c.** Let  $\eta: S^3 \to S^2$  and  $\nu: S^7 \to S^4$  denote the Hopf bundles. Show that these two maps satisfy  $H(\eta) = \pm 1$  and  $H(\nu) = \pm 1$ .

**Solution.** The cofiber of  $\eta$  is complex projective space  $C(\eta) \cong \mathbb{C}P^2$ , whose cohomology is

$$H^*(\mathbb{C}P^2) \cong \mathbb{Z}[\alpha]/\alpha^3$$

where  $\alpha = \alpha_{\eta}$  is a generator of  $H^2(\mathbb{C}P^2) \simeq \mathbb{Z}$ . Thus we have  $\beta_{\eta} = \pm \alpha^2$ .

Likewise, the cofiber of  $\eta$  is quaternionic projective space  $C(\nu) \cong \mathbb{H}P^2$ , whose cohomology is

$$H^*(\mathbb{H}P^2) \cong \mathbb{Z}[\alpha]/\alpha^3$$

where  $\alpha = \alpha_{\nu}$  is a generator of  $H^4(\mathbb{H}P^2) \simeq \mathbb{Z}$ . Thus we have  $\beta_{\nu} = \pm \alpha^2$ .

**Problem 2.** Let n be a positive *even* integer. Let  $\iota \in \pi_n(S^n)$  denote the class of the identity map, and consider the Whitehead product  $[\iota, \iota] \in \pi_{2n-1}(S^n)$ . Show that its Hopf invariant  $H([\iota, \iota])$  is equal to 2.

**Solution.** Consider the map of cofiber sequences

where  $\nabla$  denotes the fold map. The horizontal maps in the square

$$S^{n} \vee S^{n} \longrightarrow S^{n} \times S^{n}$$

$$\nabla \downarrow \qquad \qquad \varphi \downarrow$$

$$S^{n} \longrightarrow C([\iota, \iota])$$

induce isomorphisms on  $H^n$ , and the fold map induces the diagonal map on  $H^n$ 

$$\nabla^* \colon H^n(S^n) \to H^n(S^n \vee S^n) \cong H^n(S^n) \oplus H^n(S^n)$$
$$\alpha \mapsto \alpha_1 + \alpha_2$$

where  $\alpha \in H^n(S^n) \simeq \mathbb{Z}$  denotes the standard generator and  $\alpha_1, \alpha_2 \in H^n(S^n \vee S^n) \simeq \mathbb{Z} \oplus \mathbb{Z}$  denote the two generators.

By abuse of notation, we use the same notation for the corresponding classes in  $H^n(C([\iota, \iota])) \cong H^n(S^n)$  and  $H^n(S^n \times S^n) \cong H^n(S^n \vee S^n)$ . Thus the map induced by  $\varphi$  on  $H^n$  satisfies  $\varphi^*(\alpha) = \alpha_1 + \alpha_2$ .

The horizontal maps in the square

$$S^{n} \times S^{n} \longrightarrow S^{2n}$$

$$\varphi \downarrow \qquad \qquad \text{id} \downarrow$$

$$C([\iota, \iota]) \longrightarrow S^{2n}$$

induce isomorphisms on  $H^{2n}$ , and so does id, and hence  $\varphi$  does as well. Denote by  $\beta \in H^{2n}(S^{2n}) \simeq \mathbb{Z}$  the standard generator and, by abuse of notation, the corresponding class in  $H^{2n}(C([\iota, \iota])) \cong H^{2n}(S^{2n})$ . Then the square implies the relation  $\varphi^*(\beta) = \beta \in H^{2n}(S^{2n} \times S^{2n})$ .

The class  $\alpha^2 \in H^{2n}\left(C([\iota, \iota])\right)$  pulls back to  $H^{2n}(S^n \times S^n)$  via  $\varphi$  to the class:

$$\varphi^*(\alpha^2) = \varphi^*(\alpha)^2$$

$$= (\alpha_1 + \alpha_2)^2$$

$$= \alpha_1^2 + \alpha_1 \alpha_2 + \alpha_2 \alpha_1 + \alpha_2^2$$

$$= \alpha_1 \alpha_2 + \alpha_2 \alpha_1 \text{ by the Kunneth formula } H^*(S^n \times S^n) \cong H^*(S^n) \otimes_{\mathbb{Z}} H^*(S^n)$$

$$= \alpha_1 \alpha_2 + (-1)^{|\alpha_1||\alpha_2|} \alpha_1 \alpha_2$$

$$= \alpha_1 \alpha_2 + (-1)^{n^2} \alpha_1 \alpha_2$$

$$= 2\alpha_1 \alpha_2$$

$$= 2\beta$$

$$= 2\varphi^*(\beta)$$

$$= \varphi^*(2\beta).$$

Since  $\varphi^* \colon H^{2n}\left(C([\iota, \iota])\right) \xrightarrow{\cong} H^{2n}(S^n \times S^n)$  is an isomorphism, the equation

$$\alpha^2 = 2\beta$$

holds in  $H^{2n}\left(C([\iota,\iota])\right)$ , which proves  $H([\iota,\iota])=2$ .

**Problem 3.** Compute the following rational cohomology algebras.

a. 
$$H^*(K(\mathbb{Z},3);\mathbb{Q})$$

**Solution.** First note that by Hurewicz, given  $n \geq 2$ , we have

$$H_k(K(\mathbb{Z}, n); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0\\ 0 & \text{if } 0 < k < n\\ \mathbb{Z} & \text{if } k = n. \end{cases}$$

By the universal coefficient theorem, we have

$$H^{k}(K(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 0\\ 0 & \text{if } 0 < k < n\\ \mathbb{Q} & \text{if } k = n. \end{cases}$$

Recall that  $H^*(K(\mathbb{Z},2);\mathbb{Q}) \cong \mathbb{Q}[\iota_2]$  is a polynomial algebra on a class  $\iota_2 \in H^2(K(\mathbb{Z},2);\mathbb{Q})$  which is the image in  $\mathbb{Q}$ -coefficients of the fundamental class in  $H^2(K(\mathbb{Z},2);\mathbb{Z})$ .

Consider the path loop fibration

$$K(\mathbb{Z},2) \to PK(\mathbb{Z},3) \to K(\mathbb{Z},3)$$

where the base space  $K(\mathbb{Z},3)$  is simply-connected. Consider its cohomology Serre spectral sequence with rational coefficients

$$E_2^{p,q} = H^p(K(\mathbb{Z},3); H^q(K(\mathbb{Z},2);\mathbb{Q})) \Rightarrow H^{p+q}(\operatorname{pt};\mathbb{Q}).$$

Observe that the  $E_2$ -term is concentrated in even rows, and therefore all differentials  $d_r$  with r even are zero (since they go down r-1 steps).

Also note that the columns  $E_2^{1,*}$  and  $E_2^{2,*}$  are zero. In particular, the entry  $E_2^{4,0} = H^4(K(\mathbb{Z},3);\mathbb{Q})$  could only be hit by the differential  $d_4 \equiv 0$ , and thus  $E_2^{4,0}$  is never hit. We deduce:

$$H^4(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{4,0} = E_\infty^{4,0} = 0$$

and therefore the column  $E_2^{4,*}$  is also zero. The  $E_2$ -term looks like this:

6	$\mathbb{Q}\iota_2^3$			$\mathbb{Q}\iota_3\iota_2^3$		?
5						
4	$\mathbb{Q}\iota_2^2$			$\mathbb{Q}\iota_3\iota_2^2$		?
3						
2	$\mathbb{Q}\iota_2$			$\mathbb{Q}\iota_3\iota_2$		?
1						
0	$\mathbb{Q}1$			$\mathbb{Q}\iota_3$		?
	0	1	2	3	4	5

where elements denote generators (as Q-vector spaces), and blank entries are zero.

Since  $d_2 \equiv 0$  is zero, we have  $E_3 = E_2$ . The class  $\iota_3 \in E_3^{3,0}$  can only be hit by  $d_3$ , and therefore

$$d_3: \mathbb{Q} \simeq E_3^{0,2} \to E_3^{3,0} \simeq \mathbb{Q}$$

is an isomorphism, or equivalently:

$$d_3(\iota_2) = c\iota_3$$

for some scalar  $c \neq 0$ . By the (graded) Leibniz rule and graded commutativity, we have

$$d_3(\iota_2^k) = d_3(\iota_2)\iota_2^{k-1} + (-1)^{|\iota_2|}\iota_2 d_3(\iota_2)\iota_2^{k-2} + \dots + (-1)^{(k-1)|\iota_2|}\iota_2^{k-1} d_3(\iota_2)$$

$$= kd_3(\iota_2)\iota_2^{k-1}$$

$$= kc \,\iota_3 \iota_2^{k-1}.$$

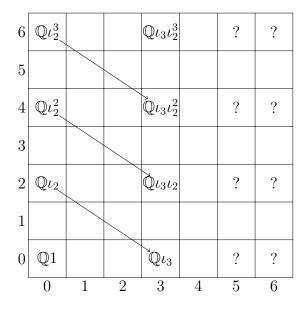
Therefore the differential

$$d_3 \colon \mathbb{Q} \simeq E_3^{0,2k} \to E_3^{3,2k-2} \simeq \mathbb{Q}$$

is an isomorphism for all  $k \geq 1$ . It follows that  $d_3$  from the column 3 to the column 6 is zero:

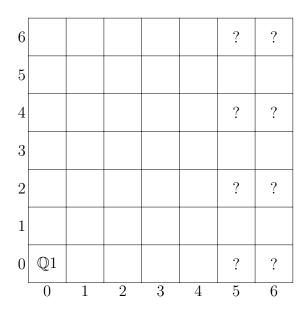
$$d_3 \colon E_3^{3,q} \xrightarrow{0} E_3^{6,q-2}.$$

The  $E_3$ -term looks like this:



where the drawn differentials  $d_3$  are isomorphisms. Therefore the  $E_4$  and  $E_5$  terms look like

this:



Note that  $E_2^{5,0} = H^5(K(\mathbb{Z},3);\mathbb{Q})$  could only be hit by the differential  $d_2 \equiv 0$  and by  $d_5$ , but now we learn from the  $E_5$ -term that  $E_5^{5,0} = E_2^{5,0}$  is not hit by  $d_5$ , because of the equality  $E_5^{0,4} = 0$ . We deduce:

$$H^5(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{5,0} = E_\infty^{5,0} = 0$$

and therefore the column  $E_2^{5,*}$  was zero to begin with.

Likewise,  $E_2^{6,0} = H^6(K(\mathbb{Z},3);\mathbb{Q})$  could only be hit by the differential  $d_3$ , which we showed does not happen, and by  $d_6 \equiv 0$ . We deduce:

$$H^6(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{6,0} = E_\infty^{6,0} = 0$$

and therefore the column  $E_2^{6,*}$  was zero to begin with.

But now  $E_2^{7,0} = H^7(K(\mathbb{Z},3);\mathbb{Q})$  could not have been hit by  $d_3$ , because of the equality  $E_3^{4,2} = 0$ , nor can it be hit by further differentials, because of the zeroes in the  $E_4$ -term. We deduce:

$$H^7(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{7,0} = E_\infty^{7,0} = 0$$

and therefore the column  $E_2^{7,*}$  was zero to begin with. Repeating this argument inductively, we deduce:

$$H^p(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{p,0} = E_{\infty}^{p,0} = 0$$

for all  $p \geq 7$  and therefore the  $E_2$ -term was concentrated in the columns 0 and 3. We conclude

$$H^p(K(\mathbb{Z},3);\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } p = 0,3\\ 0 & \text{otherwise} \end{cases}$$

and the multiplicative structure is determined by the relation  $\iota_3^2 = 0$  given by graded commutativity.

In summary, the cohomology algebra

$$H^*(K(\mathbb{Z},3);\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}[\iota_3]$$

is an exterior algebra over  $\mathbb{Q}$  on the generator  $\iota_3 \in H^3(K(\mathbb{Z},3);\mathbb{Q})$ .

**b.**  $H^*(K(\mathbb{Z},4);\mathbb{Q})$ 

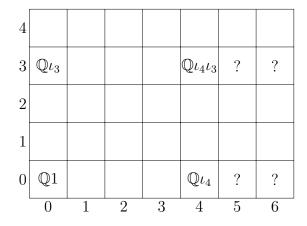
**Solution.** Consider the path loop fibration

$$K(\mathbb{Z},3) \to PK(\mathbb{Z},4) \to K(\mathbb{Z},4)$$

where the base space  $K(\mathbb{Z},4)$  is simply-connected. Consider its cohomology Serre spectral sequence with rational coefficients

$$E_2^{p,q} = H^p(K(\mathbb{Z},4); H^q(K(\mathbb{Z},3);\mathbb{Q})) \Rightarrow H^{p+q}(\mathrm{pt};\mathbb{Q}).$$

By part (a), the  $E_2$ -term is concentrated in two rows q = 0, 3 and looks like this:



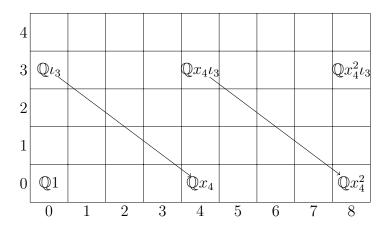
There is only room for one non-trivial differential, namely  $d_4$ , and it must destroy everything, except the  $\mathbb{Q}$  in the corner (0,0). It follows that

$$d_4: \mathbb{Q} \simeq E_4^{0,3} \to E_4^{4,0} \simeq \mathbb{Q}$$

is an isomorphism, or equivalently:

$$d_4(\iota_3) = x_4 = c\iota_4$$

for some scalar  $c \neq 0$ . The  $E_4$ -term looks like this:



Indeed, we have  $E_4^{4,3} \simeq \mathbb{Q} x_4 \iota_3$  and the differential

$$d_4 \colon E_4^{4,3} \xrightarrow{\simeq} E_4^{8,0}$$

is an isomorphism, sending the generator  $x_4\iota_3$  to a generator

$$d_4(x_4\iota_3) = d_4(x_4)\iota_3 + (-1)^{|x_4|}x_4d_4(\iota_3)$$
$$= 0 + x_4x_4$$
$$= x_4^2.$$

Repeating this argument inductively, we obtain for all  $k \geq 0$  the isomorphism  $E_4^{4k,3} \simeq \mathbb{Q} x_4^k \iota_3$  and the differential

$$d_4 \colon E_4^{4k,3} \xrightarrow{\simeq} E_4^{4(k+1),0}$$

is an isomorphism, sending the generator  $x_4^k \iota_3$  to a generator

$$d_4(x_4^k \iota_3) = d_4(x_4^k) \iota_3 + (-1)^{|x_4^k|} x_4^k d_4(\iota_3)$$
$$= x_4^{k+1}$$

and those are the only non-trivial differentials. We conclude

$$H^p(K(\mathbb{Z},4);\mathbb{Q}) = \begin{cases} \mathbb{Q}x_4^k \cong \mathbb{Q}\iota_4^k & \text{if } p = 4k\\ 0 & \text{otherwise} \end{cases}$$

so that the cohomology algebra

$$H^*(K(\mathbb{Z},4);\mathbb{Q}) \cong \mathbb{Q}[\iota_4]$$

is a polynomial algebra over  $\mathbb{Q}$  on the generator  $\iota_4 \in H^4(K(\mathbb{Z},4);\mathbb{Q})$ .