MA 54400-MIDTERM 2 PRACTICE PROBLEMS

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1.

Proof. Use spherical coordinate,

$$\begin{aligned}\int_{B(0,\epsilon)^c} \frac{1}{|x|^{n+1}} dx &= \int_{S^{n-1}(0,r)} \int_\epsilon^\infty \frac{1}{r^{n+1}} dr d\sigma, \\ &= \int_\epsilon^\infty \frac{1}{r^{n+1}} (\int_{S^{n-1}(0,r)} d\sigma) dr \text{ (by Tonelli's Theorem)},\end{aligned}$$

Notice that

$$\int_{S^{n-1}(0,r)} d\sigma = |S^{n-1}(0,r)|_{n-1} = r^{n-1}|S^{n-1}(0,1)|_{n-1} = c_n r^{n-1},$$

Hence

$$\begin{aligned}\int_\epsilon^\infty \frac{1}{r^{n+1}} (\int_{S^{n-1}(0,r)} d\sigma) dr &= c_n \int_\epsilon^\infty \frac{1}{r^{n+1}} \cdot r^{n-1} dr, \\ &= c_n \int_\epsilon^\infty \frac{1}{r^2} dr, \\ &= \frac{c_n}{\epsilon}.\end{aligned}$$

□

2.

Proof. Because $f_k \to f$ a.e. in $\mathbb{R}^n$, then given a measurable subset $E \subset \mathbb{R}^n$ we have $f_k \to f$ a.e. in E and $f_k \to f$ a.e. in $\mathbb{R}^n \setminus E$.

$\{f_k\}$ are nonnegative, then by Fatou's theorem:

$$\int_E f = \int_E \lim_{k\to\infty} f_k \leq \liminf \int_E f_k, \tag{1}$$

$$\int_{\mathbb{R}^n \setminus E} f = \int_{\mathbb{R}^n \setminus E} \lim_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n \setminus E} f_k. \tag{2}$$

Since

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

combined with (2) we get:

$$\int_{\mathbb{R}^n} f - \int_{\mathbb{R}^n \setminus E} f \geq \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k - \liminf_{k\to\infty} \int_{\mathbb{R}^n \setminus E} f_k.$$

This implies:

$$\int_E f \geq \limsup_{k\to\infty}(\int_{\mathbb{R}^n} f_k - \int_{\mathbb{R}^n\setminus E} f_k) = \limsup_{k\to\infty}\int_E f_k. \tag{3}$$

Hence by (1) and (3) we have:

$$\int_E f \leq \liminf_{k\to\infty}\int_E f_k \leq \limsup_{k\to\infty}\int_E f_k \leq \int_E f.$$

Therefore, $\lim_{k\to\infty}\int_E f_k$ exists, and

$$\int_E f = \lim_{k\to\infty}\int_E f_k.$$

This result is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty}\int_{\mathbb{R}^n} f_k = \infty$.

For example, in $\mathbb{R}$ let

$$f_k(x) = \begin{cases} \frac{k^2}{2} & x \in (-\frac{1}{k}, \frac{1}{k}) \\ 1 & \text{otherwise} \end{cases}$$

and $f = 1$ in $\mathbb{R}$.

It's easy to see that $f_k \to f$ a.e. in $\mathbb{R}$, and $\int_{\mathbb{R}} f = \lim_{k\to\infty}\int_{\mathbb{R}} f_k = \infty$. However, if $E = (-1, 1)$ then $\int_E f = 1$, but $\lim_{k\to\infty}\int_E f_k = \infty$. □

3.

Proof. Let $E_k = \{x \in E | k \leq f(x) < k+1\}$ for $k = 0, 1, 2, \ldots$, then E_k are disjoint and $\cup_{k=0}^{\infty} E_k = E$.

Because

$$k|E_k| \leq \int_{E_k} f(x)dx \leq (k+1)|E_k|,$$

then we have:

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(x)dx \leq \sum_{k=0}^{\infty}(k+1)|E_k|. \tag{4}$$

Let $F_k = \{x \in E | f(x) \geq k\}$ for $k = 0, 1, 2, \ldots$. Notice that $E_k = F_k \setminus F_{k+1}$, then we have :

$$\sum_{k=0}^{\infty} kE_k = \sum_{k=0}^{\infty} k(F_k \setminus F_{k+1}) = \sum_{k=1}^{\infty} F_k,$$

and

$$\sum_{k=0}^{\infty}(k+1)E_k = \sum_{k=0}^{\infty}(k+1)(F_k \setminus F_{k+1}) = \sum_{k=0}^{\infty} F_k.$$

Hence, (4) can be rewritten as:

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(x)dx \leq \sum_{k=0}^{\infty} |F_k|. \tag{5}$$

Observe that $F_0 = E$, $|F_0| = |E| < \infty$. Therefore by (5) $\int_E f(x)dx < \infty$ if and only if $\sum_{k=0}^{\infty} |F_k| < \infty$. □

4.

Proof. ($\Rightarrow$) Given $\epsilon > 0$, $\rho(f_k, f) \to 0$ implies

$$\int_{\{x\in E||f_k(x)-f(x)|>\epsilon\}} \frac{|f_k - f|}{1+|f_k - f|} dx \to 0.$$

Observe that the funtion $\Phi : (0,\infty) \to \mathbb{R}$, $\Phi(x) = \frac{x}{1+x}$ is increasing on $(0,\infty)$ and $0 < \Psi(x) < 1$, hence

$$\begin{aligned} \int_{\{x\in E||f_k(x)-f(x)|>\epsilon\}} \frac{|f_k - f|}{1+|f_k - f|} dx &\geq \int_{\{x\in E||f_k(x)-f(x)|>\epsilon\}} \frac{\epsilon}{1+\epsilon} dx \\ &= \frac{\epsilon}{1+\epsilon} |\{x \in E||f_k(x) - f(x)| > \epsilon\}|. \end{aligned}$$

Therefore,

$$|\{x \in E||f_k(x) - f(x)| > \epsilon\}| \leq \frac{1+\epsilon}{\epsilon} \int_{\{x\in E||f_k(x)-f(x)|>\epsilon\}} \frac{|f_k - f|}{1+|f_k - f|} dx \to 0,$$

as $k \to \infty$.

($\Leftarrow$) By the observation on the function Φ above, given arbitrary $\delta > 0$,

$$\begin{aligned} \rho(f_k, f) &= \int_{\{x\in E||f_k(x)-f(x)|>\delta\}} \frac{|f_k - f|}{1+|f_k - f|} dx + \int_{\{x\in E||f_k(x)-f(x)|\leq\delta\}} \frac{|f_k - f|}{1+|f_k - f|} dx, \\ &\leq |\{x \in E||f_k - f| > \delta\}| + \frac{\delta}{1+\delta} |E|. \end{aligned}$$

Since $|E| < \infty$ and $\frac{\delta}{1+\delta} \searrow 0$, then for any $\epsilon > 0$, there is $\delta_0 > 0$ such that

$$\frac{\delta_0}{1+\delta_0} |E| < \frac{\epsilon}{2}.$$

If $f_k \to f$ as $k \to \infty$ in measure, then for the above δ_0 there is a $K_0 > 0$, such that for any $k > K_0$,

$$|\{x \in E||f_k(x) - f(x)| > \delta_0\}| < \frac{\epsilon}{2}.$$

Therefore, $f_k \to f$ in measure implies $\rho(f_k, f) \to 0$ as $k \to \infty$. □

5.

Proof. (a). It's not hard to show that: given $y > 0$, there exist $C = C(y)$ large enough and $\epsilon = \epsilon(y) > 0$ small enough, such that

$$u^{y-1} e^{-u} \leq C e^{-\epsilon u},$$

for $u \in (1,\infty)$. Hence $u^{y-1}e^{-u}$ is integrable on $(1,\infty)$.

On the other side, for $y > 0$ we have:

$$\int_0^1 u^{y-1} e^{-u} du \leq \int_0^1 u^{y-1} du = \frac{1}{y} < \infty.$$

Therefore, the function $u \mapsto u^{y-1}e^{-u}$ is in $L((0,\infty))$.

(b). Let $u = z^2$,

$$\int_0^\infty u^{y-1} e^{-u} du = \int_0^\infty e^{-z^2} z^{2(y-1)} 2z dz.$$

Then by Fubini theorem,

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^\infty e^{-z^2} z^{2(x-1)} 2z dz \int_0^\infty e^{-z^2} z^{2(y-1)} 2z dz, \\ &= \int_0^\infty \int_0^\infty e^{-z_1^2 - z_2^2} z_1^{2(x-1)} z_2^{2(y-1)} 4 z_1 z_2 dz_1 dz_2, \end{aligned}$$

Make change of variable: $(z_1, z_2) \mapsto (r, \theta)$, $z_1 = r\cos\theta$, $z_2 = r\sin\theta$, then by Fubini theorem:

$$\begin{aligned} \Gamma(x)\Gamma(y) &= 4 \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2x-1} (\cos\theta)^{2x-1} r^{2y-1} (\sin\theta)^{2y-1} r dr d\theta, \\ &= 2 \int_0^\infty e^{-r^2} r^{2x+2y-2} 2r dr \int_0^{\frac{\pi}{2}} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta, \\ &= 2\Gamma(x+y) \int_0^{\frac{\pi}{2}} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta. \end{aligned}$$

Let $\cos^2\theta = t$, $\theta \in (0, \frac{\pi}{2})$, then $d\theta = -\frac{1}{2} t^{-\frac{1}{2}} (1-t)^{-\frac{1}{2}} dt$. Hence,

$$\begin{aligned} 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta &= 2\int_1^0 t^{x-\frac{1}{2}} (1-t)^{y-\frac{1}{2}} (-\frac{1}{2}) t^{-\frac{1}{2}} (1-t)^{-\frac{1}{2}} dt, \\ &= \int_0^1 t^{x-1} (1-t)^{y-1} dt = B(x,y). \end{aligned}$$

Therefore,

$$\Gamma(x)\Gamma(y) = \Gamma(x+y) B(x,y).$$

□

6.

Proof. Please refer to Theorem 8.19 on page 134 in the textbook. □

7.

Proof. (a). Since $f_k \to f$, $g_k \to g$ a.e. and $|f_k| \le g_k$, then by Fatou 's theorem,

$$\int_{\mathbb{R}^n} (g - f) = \int_{\mathbb{R}^n} \liminf_{k\to\infty} (g_k - f_k) \le \liminf_{k\to\infty} \int_{\mathbb{R}^n} (g_k - f_k),$$

$$\int_{\mathbb{R}^n} (g + f) = \int_{\mathbb{R}^n} \liminf_{k\to\infty} (g_k + f_k) \le \liminf_{k\to\infty} \int_{\mathbb{R}^n} (g_k + f_k).$$

Since $f_k, g_k, f, g \in L(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$, then using the similar argument as in problem 2 we get:

$$\begin{aligned} \int_{\mathbb{R}^n} f &\geq \limsup_{k\to\infty} \int_{\mathbb{R}^n} f_k, \\ \int_{\mathbb{R}^n} f &\leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} f_k. \end{aligned}$$

Therefore, $\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f$.

(b). ($\Rightarrow$) This direction is obvious because of the inequality:

$$|\int_{\mathbb{R}^n} |f_k| - |f|| \leq \int_{\mathbb{R}^n} ||f_k| - |f|| \leq \int_{\mathbb{R}^n} |f_k - f|.$$

($\Leftarrow$) Let $g_k = |f_k| + |f|$ and $g = 2|f|$. Since $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e., then $g_k, g \in L(\mathbb{R}^n)$ and $g_k \to g$ a.e.in $\mathbb{R}^n$. By the assumption, $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$.

Let $\tilde{f}_k = |f_k - f|$. Then $\tilde{f}_k \to 0$ a.e. in $\mathbb{R}^n$ and $\tilde{f}_k \leq g_k$. Applying part (a) to $\tilde{f}_k$ we have:

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

□