MA 519: Homework 2

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Problem 2.1 (Handout 2, # 5)

Four men throw their watches into the sea, and the sea brings each man one watch back at random. What is the probability that at least one man gets his own watch back?

Solution. ► The sample space Ω is in correspondence with S_4 the set of bijections from the set $\{1, 2, 3, 4\}$ to itself and therefore

$$\#\Omega = \#S_4 = 4! = 24. \tag{2.1}$$

Let A denote the event that at least one man gets his own watch back. This is a case where it is easier find the probability of the complement of A, i.e., the event $\Omega \setminus A$ that no man gets his own watch back.

In this case, we can explicitly list the ways in which no man gets his own watch back: We can translate, i.e., each man receives the i-th man to the left's watch (there are 3 ways to do this before the everybody receives their own watch back); and we can transpose watches between any two any two men (there are 3 ways to do this as choosing one pair of men to trade watches determines the other pair). In summary, there are 6 ways for each man to not get his own wallet back. Thus,

$$P(\Omega \setminus A) = \frac{6}{24} = 0.25$$

and hence

$$P(A) = 1 - P(\Omega \setminus A) = 0.75.$$

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Problem 2.2 (Handout 2, #7)

Calculate the probability that in Bridge, the hand of at least one player is void in a particular suit.

Solution. ► Let Ω denote the sample space and A denote the event that at least one player is void in a particular suit. As in the previous problem, it is easier to compute the probability of $\Omega \setminus A$ the event that no player is void in a particular suit. But first, we count the number of sample points in Ω

$$\#\Omega = \binom{52}{13, 13, 13, 13} = 53644737765488792839237440000.$$

Now, partition the deck into the 4 suits and from each deck draw a card to put into each player's hand. For the first player, this gives us

$$13^4 \cdot \binom{36}{9} = 2688826220080$$

potential hands for the first player

$$12^4 \cdot 279 = 97186003200$$

for the second and

$$11^4 \cdot \binom{18}{9} = 711845420 \qquad 10^4 \cdot \binom{9}{9} = 10000.$$

Therefore, we have

$$\#(\Omega \setminus A) = 2786724078700$$

total hands where each player has a card from each suit. Thus,

$$P(A) = 1 - P(\Omega \setminus A) = 1 - \frac{2786724078700}{53644737765488792839237440000} \approx 1.$$

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Problem 2.3 (Handout 2, # 12)

If n balls are placed at random into n cells, find the probability that exactly one cell remains empty.

Solution. ► Let Ω denote the sample space. Then by Feller's occupancy problem, we have

$$\#\Omega = \binom{2n-1}{n}.$$

Now, there are n ways to choose a bin to be empty and $n-1$ ways for the extra ball to go into the remaining cells. Thus, the event A that exactly one cell remains empty has probability

$$P(A) = \frac{n(n-1)}{\binom{2n-1}{n}}.$$

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Problem 2.4 (Handout 2, # 13)

Spread of rumors. In a town of $n + 1$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, etc. At each step the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor told r times without:

(a) returning to the originator,
(b) being repeated to any person.

Do the same problem when at each step the rumor told by one person to a gathering of N randomly chosen people. (The first question is the special case $N = 1$).

Solution. ►

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Problem 2.5 (Handout 2, # 14)

A family problem. In a certain family four girls take turns at washing dishes. Out of a total of four breakages, three were caused by the youngest girl, and she was thereafter called clumsy. Was she justified in attributing the frequency of breakages to chance? Discuss the connection with random placement of balls.

Solution. ► Let A denote the event that the youngest girl broke a dish and B denote the event that somebody broke a dish. By Bayes' theorem

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^{c})P(A^{c})}.$$

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Problem 2.6 (Handout 2, # 15)

A car is parked among N cars in a row, not at either end. On his return the owner finds exactly r of the N places still occupied. What is the probability that both neighboring places are empty?

Solution. ►

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Problem 2.7 (Handout 2, # 16)

Find the probability that in a random arrangement of 52 bridge card no two aces are adjacent.

Solution. ►

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Problem 2.8 (Handout 2, # 17)

Suppose $P(A) = 3/4$, and $P(B) = 1/3$.
Prove that $P(A \cap B) \geq 1/12$. Can it be equal to $1/12$?

Solution. ►

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Problem 2.9 (Handout 2, # 18)

Suppose you have infinitely many events $A_1, A_2, \ldots$, and each one is sure to occur, i.e., $P(A_i) = 1$ for any i.

Prove that $P\left(\bigcap_{i=1}^{n} A_i\right) = 1$.

Solution. ►

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Problem 2.10 (Handout 2, # 19)

There are n blue, n green, n red, and n white balls in an urn. Four balls are drawn from the urn with replacement. Find the probability that there are balls of at least three different colors among the four drawn.

Solution. ►

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