MA 544: Homework 11

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PROBLEM 11.1 (WHEEDEN & ZYGMUND §7, EX. 11)

Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt.$$

(Consider the case when f is the characteristic function of an interval, an open set, etc.)

Proof. As the parenthesized text suggests, we will prove the result in stages by first showing that the equality holds for characteristic functions, for simple functions, and lastly, we pass to the limit. First, note that, by Theorem 5.21, f is integrable if and only if $|f|$ is integrable therefore, it suffices to prove the result for $f \geq 0$.

Suppose f is the characteristic function of an open interval $(c, d) \subset (a, b)$, then

$$\begin{aligned}
\int_a^b f(g(t))g'(t)dt &= \int_{E_1} f(g(t))g'(t)dt + \int_{E_2} f(g(t))g'(t)dt \\
&= \int_{E_1} g'(t)dt \\
&= g(\beta) - g(\alpha) \\
&= d - c \\
&= \int_a^b f(x)dx,
\end{aligned} \tag{11.1}$$

where $E_1 := \{\, t \in [\alpha, \beta] : g(t) \in [c, d]\,\}$ and $E_2 := \{\, t \in [\alpha, \beta] : g(t) \in [a, b] \smallsetminus [c, d]\,\}$, the latter being equivalent to the complement in $[a, b]$ of the former, i.e., $E_2 = [a, b] \smallsetminus E_1$.

Now, let $G \subset [a, b]$ be open. Then G can be written as the disjoint union of intervals $\{I_k\}$. ∎

Problem 11.2 (Wheeden & Zygmund §7, Ex. 15)

Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\varphi(x) = \int_a^x f(t)dt + \varphi(a)$ in (a, b) and f is monotone increasing, then φ is convex in (a, b). (Use Exercise 14.)

Proof. ∎

PROBLEM 11.3 (WHEEDEN & ZYGMUND §5, EX. 8)

Prove (5.49).

Proof. Recall the content of equation 5.49: For f measurable, we have

$$\omega(\alpha) \leq \frac{1}{\alpha^p} \int\limits_{\{f > \alpha\}} f^p, \quad \alpha > 0. \tag{11.2}$$

∎

Problem 11.4 (Wheeden & Zygmund §5, Ex. 11)

For which p does $1/x \in L^p(0,1)$? $L^p(1,\infty)$? $L^p(0,\infty)$?

Proof. ■

PROBLEM 11.5 (WHEEDEN & ZYGMUND §5, EX. 12)

Give an example of a bounded continuous f on $(0, \infty)$ such that $\lim_{x \to \infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

Proof.   ∎

Problem 11.6 (Wheeden & Zygmund §5, Ex. 17)

If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k) < \infty$. (Use Exercise 16.)

Proof. ∎