MA 544: Homework 11

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Problem 11.1 (Wheeden & Zygmund §7, Ex. 11)

Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt.$$

(Consider the case when f is the characteristic function of an interval, an open set, etc.)

Proof. ∎

Problem 11.2 (Wheeden & Zygmund §7, Ex. 15)

Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\varphi(x) = \int_a^x f(t)dt + \varphi(a)$ in (a, b) and f is monotone increasing, then φ is convex in (a, b). (Use Exercise 14.)

Proof. ∎

Problem 11.3 (Wheeden & Zygmund §5, Ex. 8)

Prove (5.49).

Proof. Recall the content of equation 5.49: For f measurable, we have

$$\omega(\alpha) \le \frac{1}{\alpha^p} \int\limits_{\{ f > \alpha \}} f^p, \quad \alpha > 0. \tag{11.1}$$

■

Problem 11.4 (Wheeden & Zygmund §5, Ex. 11)

For which p does $1/x \in L^p(0,1)$? $L^p(1,\infty)$? $L^p(0,\infty)$?

Proof. ■

Problem 11.5 (Wheeden & Zygmund §5, Ex. 12)

Give an example of a bounded continuous f on $(0, \infty)$ such that $\lim_{x \to \infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

Proof. ∎

Problem 11.6 (Wheeden & Zygmund §5, Ex. 17)

If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < \infty$. (Use Exercise 16.)

Proof. ∎