MA 523: Homework, Midterms and Practice Problems Solutions

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Contents

1 Homework Solutions

1.1 Homework 1

PROBLEM 1.1 (Taylor's formula). Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to \mathbf{0}$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbf{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha| = m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

SOLUTION. ■

PROBLEM 1.2. Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbf{R}^n \times (0, \infty)$, where $b \in \mathbf{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbf{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

SOLUTION. ■

PROBLEM 1.3. Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $u u_{x_1} + u_{x_2} = 1$, $u(x_1, x_1) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

SOLUTION. ■

PROBLEM 1.4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\big(u_{x_1}^2 + u_{x_2}^2\big)$$

find a solution with $u(x_1, 0) = \big(1 - x_1^2\big)/2$.

SOLUTION. ■

1.2 Homework 2

PROBLEM 1.5. Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

SOLUTION. First, note that the condition

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{1}$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 because in such case we have $\nu(x^0) = (0, \dots, 0, 1)$ so

$$\begin{aligned} 0 &\neq D_pF(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall verify the noncharacteristic condition (1) by first flattening the boundary near x^0 and then applying the noncharacteristic boundary conditions to the flattened region. Assuming some degree of regularity near x^0, e.g., that the boundary of U be smooth, we may express Γ near x^0 as the graph of a smooth function $f\colon \mathbf{R}^{n-1} \to \mathbf{R}$, i.e., $x = (x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}))$ on Γ and $x_n \geq f(y)$ after reorienting the coordinate axes. Then we flatten out Γ via the map $\Phi(x)\colon \mathbf{R}^n \to \mathbf{R}^n$ given by

$$\begin{cases} y_1 = x_1 = \Phi^1(x), \\ \quad\vdots \\ y_{n-1} = x_{n-1} = \Phi^{n-1}(x), \\ y_n = x_n - f(x_1, \dots, x_{n-1}) = \Phi^n(x) \end{cases}$$

and write $y = \Phi(x)$. Let $\Psi = \Phi^{-1}$ and rewrite our PDE F in terms of y as follows,

$$0 = F\big(Du(\Psi(y)), u(\Psi(y)), \Psi(y)\big). \tag{2}$$

Since $\Delta = \Phi(\Gamma)$ is flat near $y^0 = \Phi(x^0) = (y_1^0, \dots, y_{n-1}^0, 0)$, we may apply the standard noncharacteristic condition on (2) and get

$$0 \neq F_{u_{y_n}}\big(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)\big).$$

Before we move on to finding an expression for this derivative, let us consider the gradient $Du(\Psi(y))$. By the chain rule, we have

$$\begin{aligned} u_{y_i}(\Psi(y)) &= \sum_{j=1}^{n} u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_i} \\ &= u_{x_i}(\Psi(y)) + u_{x_n}(\Psi(y)) f_{y_i}(y_1, \dots, y_{n-1}), \\ u_{y_n}(\Psi(y)) &= \sum_{j=1}^{n} u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_n} \\ &= u_{x_n}(\Psi(y)), \end{aligned}$$

Then, substituting u_{y_n} for u_{x_n}, we have

$$u_{y_i}(\Psi(y)) = u_{x_i}(\Psi(y)) + u_{y_n}(\Psi(y)) f_{y_i}(y_1, \ldots, y_{n-1}),$$

Now, by the chain rule on (2), we have

$$\begin{aligned} 0 &\neq F_{u_{y_n}}\big(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)\big) \\ &= F_{u_{y_n}}\big(u_{x_1} + u_{y_n} f_{y_1}, \ldots, u_{x_{n-1}} + u_{y_n} f_{y_{n-1}}, u_{y_n}, z^0, x^0\big) \\ &= F_{u_{x_1}} f_{y_1} + \cdots + F_{u_{x_{n-1}}} f_{y_{n-1}} + F_{u_{x_n}} \\ &= D_p F(p^0, z^0, x^0) \cdot (Df(x^0), 1) \\ &= D_p F(p^0, z^0, x^0) \cdot \nu(x^0), \end{aligned}$$

as we set out to show. ■

PROBLEM 1.6. Show that the solution of the quasilinear PDE

$$u_t + a(u) u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

SOLUTION. The characteristic ODEs of this PDE are

$$\dot{t} = 1, \qquad \dot{x} = a(z), \qquad \dot{z} = 0. \tag{3}$$

with initial conditions $t_0 = 0$, $x_0 = x(0)$ and $z(x_0, 0) = g(x_0)$ with $(x_0, 0) \in \mathbf{R} \times (0, \infty)$. Hence, we have

$$t(s) = s, \qquad x(s) = a(g(x_0))s + x_0, \qquad z(s) = g(x_0).$$

Thus, solving for x_0 and s in terms of t, x and z, we have

$$\begin{aligned} x &= a(g(x_0))s + x_0 \\ &= a(z)t + x_0, \end{aligned}$$

so, moving x_0 to the left-hand side

$$x_0 = x - a(z)t$$

hence,

$$z = g(x - a(z)t),$$

i.e.,

$$u = g(x - a(u)t),$$

as desired.

For the latter half of the problem, write

$$u\big(x + a(g(x))t, t\big) = g(x).$$

Suppose that $a(g(x))$ is not a nondecreasing function of x. Then, there exists $0 < x_1 < x_2$ such that $a(g(x_1)) > a(g(x_2))$. Define

$$y = -\frac{x_1 - x_2}{a(g(x_1)) - a(g(x_2))} > 0. \tag{4}$$

Then, we have

$$t_0 = x_1 + a(g(x_1))y = x_2 + a(g(x_2))y.$$

Thus,

$$\begin{aligned} u(x, t_0) &= g(x_1) \\ &= u\big(x_1 + a(g(x_1))t_0, t_0\big) \\ &= g(x_2) \\ &= u\big(x_2 + a(g(x_2))t_0, t_0\big). \end{aligned}$$

However, $g(x_1) \neq g(x_2)$ since $a(g(x_1)) > a(g(x_2))$. ∎

PROBLEM 1.7. Show that the function $u(x,t)$ defined for $t \geq 0$ by

$$u(x,t) = \begin{cases} -\dfrac{2}{3}\Big(t + \sqrt{3x + t^2}\Big) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burgers' equation*).

SOLUTION. The shock occurs along the curve C given by $s(t) = -t^2/4$. First, we verify that the equation given by u above is in fact a solution to the inviscid Burgers' equation to the right and to the left of C: to the left of C, $4x + t^2 < 0$, the equation is trivially satisfied whereas to the right, $4x + t^2 > 0$, we have,

$$-\frac{2}{3}\left(1 + \frac{t}{\sqrt{3x + t^2}}\right) + \frac{2}{9}\left(3 + \frac{3t}{\sqrt{3x + t^2}}\right) = 0.$$

So u is indeed a solution to the inviscid Burgers' equation.

Now we examine the behavior of u along the curve C. First, we have

$$\begin{aligned} \sigma &= \dot{s}(t), \\ &= -\frac{t}{2} \end{aligned} \qquad \begin{aligned} [\![u]\!] &= u_\ell - u_r, \\ &= 0 + \frac{2}{3}\left(t + \sqrt{-\tfrac{3}{4}t^2 + t^2}\right) \\ &= 0 + \frac{2}{3}\left(\frac{3}{2}t\right) \\ &= t \end{aligned} \qquad \begin{aligned} [\![F]\!] &= F(u_\ell) - F(u_r) \\ &= 0 - \frac{[\![u_r]\!]^2}{2} \\ &= 0 - \frac{t^2}{2} \\ &= -\frac{t^2}{2}. \end{aligned}$$

Thus,

$$[\![F]\!] = -\frac{t^2}{2} = \left(-\frac{t}{2}\right)t = \sigma[\![u]\!]$$

so the PDE satisfies the Rankine–Hugoniot condition and hence, is an integral solution.

Lastly, we verify that u satisfies the entropy condition, i.e., the one-sided jump estimate, Eq. (36) from [E, §3.4.3]. We break this test into two cases, (a) u is to the left of C and (b) u is to the right of C.

First we deal with case (b). When $x > -t^2/4$, $t > 0$, $z > 0$, using linear interpolation, we have

$$\begin{aligned} u(x+z,t) - u(x,t) &\leq \sup_{x>-t^2/4} \big\{u_x(x,t)\big\}(x+z-x) \\ &= \sup_{x>-t^2/4} \left\{\frac{1}{\sqrt{3x+t^2}}\right\} z \end{aligned}$$

and since the function u_x is decreasing on $x > -t^2/4$, it achieves its maximum as $x- \to -t^2/4$

$$\begin{aligned} &= \frac{1}{\sqrt{-(3/4)t^2 + t^2}} z \\ &= \frac{2}{t} z \end{aligned}$$

for all $t > 0$, $z > 0$. Thus, u satisfies the entropy condition on $x > -t^2/4$.

For case (a), we have run into the problem of the characteristic curve of $u(x,t)$ crossing the curve C for sufficiently large values of t. But in that case, we are in the region $x > -t^2/4$ and the argument in provided for case (b), shows that u satisfies the entropy condition. On the other hand, assuming t is less than some threshold $t'(x)$ we stay in the region $x < -t^2/4$ and the entropy condition

$$u(x+z,t) - u(x,t) = u(x+z,t) \leq 0$$

is satisfied. ∎

1.3 Homework 3

PROBLEM 1.8. Consider the initial value problem

$$u_t = \sin u_x; \qquad u(x,0) = \frac{\pi}{4}x.$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the taylor series of the solution about the origin.

SOLUTION. The initial value problem certainly satisfies the assumptions of the Cauchy–Kovalevskaya theorem, that is, setting $\mathbf{u} := (u, u_x, u_t, t)$, the $\mathbf{b}$ are all identically 0, and $\mathbf{c}(\mathbf{u}, x) = \sin u_x(x,t)$ is analytic. Next we show that the Taylor series of u at $(0,0)$,

$$u(x,t) = \sum_{\alpha,\beta} \frac{a_{\alpha,\beta}}{\alpha!\beta!} x^\alpha t^\beta$$

is a solution to our PDE.

First, we must compute the coefficients $a_{\alpha,\beta}$. To this end, we must find the partial derivatives $u_{\alpha,\beta}$ and potentially, relations among them which will help us to find these coefficients. Naïvely listing the partials with respect to t and x, we have

$$\begin{aligned}
u(0,0) &= 0 \\
u_x(0,0) &= \frac{\pi}{4} \\
u_t(0,0) &= \sin u_x(0,0) = \frac{\sqrt{2}}{2} \\
u_{xx}(0,0) &= 0 \\
u_{tx}(0,0) &= 0 \\
u_{tt}(0,0) &= -\cos\big(u_x(0,0)\big) u_{xt}(0,0) = 0 \\
u_{xxx}(0,0) &= 0 \\
u_{ttx}(0,0) &= 0,
\end{aligned}$$

etc. Thus,

$$u = \frac{\pi}{4}x + \frac{\sqrt{2}}{2}t. \tag{5}$$

Plugging in Eq. (5) into our PDE, we have

$$u_t - \sin u_x = \frac{\sqrt{2}}{2} - \sin(\pi/4) = 0,$$

as desired. ■

PROBLEM 1.9. Consider the Cauchy problem for $u(x,y)$

$$\begin{aligned}
u_y &= a(x,y,u)u_x + b(x,y,u) \\
u(x,0) &= 0
\end{aligned}$$

let a and b be analytic functions of their arguments. Assume that $d^\alpha a(0,0,0) \geq 0$ and $d^\alpha b(0,0,0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

(a) Show that $D^\beta u(0,0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $D^\beta u(0,0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$. (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in β_2)

Solution. Write

$$a(x,y,u) = \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma} x^\alpha y^\beta u^\gamma, \qquad b(x,y,u) = \sum_{\alpha,\beta,\gamma} b_{\alpha,\beta,\gamma} x^\alpha y^\beta u^\gamma$$

where the right-hand side of the expressions above converge to the left-hand side for $|x|+|y|+|u| < r$ for some sufficiently small r.

For part (a) we show this explicitly by considering all cases. The case $\beta = (0,0)$ is obvious as are the cases $\beta = (0,1)$ and $\beta = (1,0)$ since $u_x(0,0) = 0$ and

$$\begin{aligned} u_y(0,0) &= a\big(0,0,u(0,0)\big)u_x(0,0) + b\big(0,0,u(0,0)\big) \\ &= a(0,0,0)u_x(0,0) + b(0,0,0) \\ &= b(0,0,0) \\ &\geq 0 \end{aligned}$$

since b is a series of strictly positive numbers. For $\beta = (2,0)$, we have $u_{xx}(0,0) = 0$. For $\beta = (1,1)$, we have

$$\begin{aligned} u_{xy}(0,0) &= a\big(0,0,u(0,0)\big)u_{xx}(0,0) + \frac{\partial}{\partial x}a\big(0,0,u(0,0)\big)u_x(0,0) + \frac{\partial}{\partial x}b\big(0,0,u(0,0)\big) \\ &= \frac{\partial}{\partial x}b(0,0,0) \\ &\geq 0. \end{aligned}$$

For $\beta = (0,2)$, we have

$$\begin{aligned} u_{yy}(0,0) &= a\big(0,0,u(0,0)\big)u_{xy}(0,0) + \frac{\partial}{\partial y}a\big(0,0,u(0,0)\big)u_x(0,0) + \frac{\partial}{\partial y}b\big(0,0,u(0,0)\big) \\ &= a(0,0,0)\frac{\partial}{\partial y}b(0,0,0) + \frac{\partial}{\partial y}b(0,0,0) \\ &\geq 0 \end{aligned}$$

since the latter is a sum of positive numbers.

For part (b), in the proof of the Cauchy–Kovalevskaya theorem, for $\beta_2 = 0$, we have

$$D^\beta u(0,0) = 0$$

since u is constant on the hypersurface $\{\, y = 0 \,\}$. In particular, $D^\beta u(0,0) \geq 0$.

Now, suppose $D^\beta u(0,0) \geq 0$ for all $\beta_2 \leq n-1$. Then, for $\beta = (m,n)$, we have

$$\begin{aligned} D^\beta u(0,0) &= d^{(m,n-1)} u_y(0,0) \\ &= d^{(m,n-1)}\big(a u_x + b\big)(0,0) \\ &= \end{aligned}$$

This is essentially proven in Evans. The expression above will be given by a polynomial in derivatives of lower order (as in equation (23)). Since these derivatives are products of positive numbers since $D^\beta a(0,0), D^\beta b(0,0) \geq 0$, this shows that $D^\beta u(0,0) \geq 0$. I tried finding an expression for this, but it was not very nice. ■

Problem 1.10. (Kovalevskaya's example) show that the line $\{ t = 0 \}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution u of the heat equation in $\mathbf{R} \times \mathbf{R}$, with $u = 1/(1+x^2)$ on $\{t = 0\}$. (*Hint:* assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of $(0,0)$.)

Solution. First we show that the line $\gamma := \{ t = 0 \}$ is characteristic for the heat equation. With $\boldsymbol{\nu} = (1,0)$ the normal to the line γ, the noncharacteristic condition reads

$$\sum_{|\alpha|=2} a_\alpha \boldsymbol{\nu}^\alpha \neq 0.$$

However,

$$\sum_{|\alpha|=2} a_\alpha \boldsymbol{\nu}^\alpha = 1 \cdot 1 + a_{0,2} \cdot 0 = 1 \neq 0.$$

Thus, γ is characteristic for $u_t = u_{xx}$.

Next suppose u is an analytic solution to the heat equation with

$$u(x,t) = \sum_{m,n} \frac{a_{m,n}}{m!n!} x^m t^n$$

on $\mathbf{R} \times \mathbf{R}$.

Let us compute the coefficients $a_{m,n}$ near $(0,0)$. From the PDE, we have the relation

$$\begin{aligned} a_{m,n} &= d^{(m,n)} u(0,0) \\ &= d^{(m,n-1)} u_t(0,0) \\ &= d^{(m,n-1)} u_{xx}(0,0) \\ &= d^{(m+2,n-1)} u(0,0) \\ &= a_{m+2,n-1}. \end{aligned} \tag{6}$$

Form the boundary condition, we have

$$u(x,0) = \sum_{k=1}^{\infty} (-1)^k x^{2k} \tag{7}$$

for a sufficiently small neighborhood about $(0,0)$, where the right-hand side is given taylor series of $1/(1+x^2)$. Taking the m^{th} x-partial derivative at $(0,0)$, with the help of Eq. (7) we find the coefficients

$$a_{m,0} = \begin{cases} 0 & \text{if } m = 2k+1 \text{ is odd} \\ (-1)^k (2k)! & \text{if } m = 2k \text{ is even.} \end{cases} \tag{8}$$

Putting all of this information together, we deduce that

$$a_{2m+1,n} = 0$$

for all m, n and, recursively,

$$a_{2m,n} = a_{2m+2,n-1} = \cdots = a_{2(m+n),0} = (-1)^{m+n}\big(2(m+n)\big)!.$$

From this we see that the coefficients of the form $a_{2n,n}$ grow very quickly, that is,

$$\begin{aligned}\frac{a_{2n,n}}{(2n)!n!} &= (-1)^{2n}\frac{\big(2(n+n)\big)!}{(2n)!n!}\\ &= \frac{(4n)!}{(2n)!n!}\end{aligned}$$

which, by Stirling's formula, is asymptotically equal to

$$\begin{aligned}&\asymp \frac{\sqrt{2\pi n}(4n/\mathrm{e})^{4n}}{\sqrt{4\pi n}(2n/\mathrm{e})^{2n}\sqrt{2\pi n}(n/\mathrm{e})^{n}}\\ &= \frac{\sqrt{2\pi n}(4n/\mathrm{e})^{4n}}{\sqrt{8\pi n^2}(2n/\mathrm{e})^{2n}(n/\mathrm{e})^{n}}\\ &= \frac{\big(\sqrt{\pi/n}\big)4^{4n}}{2\cdot 2^{2n}}\left(\frac{n}{\mathrm{e}}\right)^{4n-3n-n}\\ &= \frac{\sqrt{\pi/n}}{2}\left(\frac{16}{2}\right)^{2n}\left(\frac{n}{\mathrm{e}}\right)^{n}\\ &= \big(\sqrt{\pi/n}\big)2^{6n-1}\left(\frac{n}{\mathrm{e}}\right)^{n}\\ &= \alpha\beta_n n^{n+1/2}\end{aligned}$$

which approaches ∞ as $n \to \infty$. This shows that for $x, t > 0$, the terms $a_{2n,n}$ grow arbitrarily large.

In particular, for any small $t > 0$, we want the series

$$\sum_n a_{0,n}t^n$$

to converge to 0 but the size of the coefficients $a_{0,n}$ prevent us from doing this. This leads to a contradiction. ■

1.4 Homework 4

Problem 1.11 (Legendre transform). Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u\big(\mathbf{x}(p)\big),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29)).

Solution. Assuming the regularity on v prescribed above, we compute $v_{p_1p_1}$, $v_{p_1p_2}$ and $v_{p_2p_2}$.

First, we compute $v_{p_1p_2}$ since in the case of $v_{p_1p_1}$ and $v_{p_2p_2}$, there is some symmetry we can exploit. Taking the first partial with respect to p^1, we have

$$\begin{aligned}
v_{p_1} &= \tfrac{\partial}{\partial p_1}\big(x^1(p)p^1 + x^2(p)p^2 - u\big(\mathbf{x}(p)\big)\big) \\
&= x^1(p) + x^1_{p_1}(p)p^1 + x^2_{p_1}(p)p^2 - u_{x_1}\big(\mathbf{x}(p)\big)x^1_{p_1}(p) - u_{x_2}\big(\mathbf{x}(p)\big)x^2_{p_1}(p) \\
&= x^1 + x^1_{p_1}p^1 + x^2_{p_1}p^2 - p^1x^1_{p_1} - p^2x^2_{p_1} \\
&= x^1,
\end{aligned} \tag{9}$$

since $u_{x_1} = p^1$ and $u_{x_2} = p^2$.

Similarly, for v_{p_2}, we have

$$\begin{aligned}
v_{p_2} &= \tfrac{\partial}{\partial p_2}\big(x^1(p)p^1 + x^2(p)p^2 - u\big(\mathbf{x}(p)\big)\big) \\
&= x^1_{p_2}(p)x^1(p) + x^2(p) + x^2_{p_2}(p)p^2 - u_{x_1}\big(\mathbf{x}(p)\big)x^1_{p_2}(p) - u_{x_2}\big(\mathbf{x}(p)\big)x^2_{p_2}(p) \\
&= x^1_{p_2}x^1 + x^2 + x^2_{p_2}p^2 - p^1x^1_{p_2} - p^2x^2_{p_2} \\
&= x^2.
\end{aligned} \tag{10}$$

Now, taking the partial of (9) with respect to p_1 and then p_2, we have

$$v_{p_1p_1} = x^1_{p_1} = x^1_{u_{x_1}}, \qquad v_{p_1p_2} = x^1_{p_2} = x^1_{u_{x_2}},$$

and similarly for (10),

$$v_{p_1p_2} = x^2_{p_1} = x^2_{u_{x_1}}, \qquad v_{p_2p_2} = x^2_{p_2} = x^2_{u_{x_2}}.$$

By the inverse function theorem, we have

$$\begin{aligned}\begin{bmatrix} v_{p_1p_1} & v_{p_1p_2} \\ v_{p_1p_2} & v_{p_2p_2}\end{bmatrix} &= \begin{bmatrix} x^1_{u_{x_1}} & x^1_{u_{x_2}} \\ x^2_{u_{x_1}} & x^2_{u_{x_2}}\end{bmatrix} \\ &= \begin{bmatrix} u_{x_1x_1} & u_{x_1x_2} \\ u_{x_1x_2} & u_{x_2x_2}\end{bmatrix}^{-1} \\ &= \frac{1}{J}\begin{bmatrix} u_{x_2x_2} & -u_{x_1x_2} \\ -u_{x_1x_2} & u_{x_1x_1}\end{bmatrix}.\end{aligned}$$

Hence,

$$\begin{cases} u_{x_1x_1} = Jv_{p_2p_2} \\ u_{x_1x_2} = -Jv_{p_1p_2} \\ u_{x_2x_2} = Jv_{p_1p_1}, \end{cases} \tag{11}$$

which verifies Equation (29) from [E, 4.4.3b]. Substituting (11) into the original equation,

$$\begin{aligned} 0 &= Ja^{11}(p)v_{p_2p_2} - Ja^{12}(p)v_{p_1p_2} + Ja^{22}(p)v_{p_1p_1} \\ &= a^{22}(p)v_{p_1p_1} - a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_2p_2}, \end{aligned}$$

as was to be shown. ■

Problem 1.12. Find the solution $u(x,t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$\begin{cases} u(x,0) = f(x), \qquad u_t(x,0) = g(x), & \text{for } x > 0, \\ u_t(0,t) = \alpha u_x(0,t), & \text{for } t > 0, \end{cases}$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$. (*Hint:* Use a representation $u(x,t) = F(x-t) + G(x+t)$ for the solution.)

Solution. Suppose $u(x,t) = F(x-t) + G(x+t)$ is a classical solution to the one-dimensional wave equation with the prescribed initial conditions. Then, we want to extend the data to all of x so that we can exploit d'Alembert's formula. Suppose we gave done this by, e.g., taking the odd reflection of $\tilde{f}$, $\tilde{g}$, and $\tilde{h}(x,t) := \int_{x-t}^{x+t} \tilde{g}(s)ds$. All we need to do is use the initial data to find the relation between $\tilde{f}$, $\tilde{g}$, and $\tilde{g}$ at $x = 0$.

Using d'Alembert's formula,

$$\tilde{u}(x,t) = \frac{\tilde{f}(x+t) + \tilde{f}(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \tilde{g}(s)\, ds,$$

we compute $\tilde{u}_t(0,t)$ and $\alpha\tilde{u}_x(0,t)$ to match our initial data.

Hence,

$$\begin{aligned}
u_t(0,t) &= \tfrac{1}{2}\big(-\tilde{f}(-t) + \tilde{f}(t) + \tilde{g}(t) - \tilde{g}(-t)\big),\\
u_x(0,t) &= \tfrac{1}{2}\big(\tilde{f}(-t) + \tilde{f}(t) + \tilde{g}(t) - \tilde{g}(-t)\big),
\end{aligned}$$

so

$$0 = \tfrac{1}{2}\big(-(1+\alpha)\tilde{f}(-t) + (1-\alpha)\tilde{f}(t) + (1-\alpha)(\tilde{g}(t) - \tilde{g}(t))\big)$$

■

Problem 1.13. (a) Let u be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0,t) = u(\pi,t) = 0$. Show that the *energy*

$$E(t) = \frac{1}{2}\int_0^\pi \left(u_t^2 + c^2 u_x^2\right) dx, \quad t > 0$$

is independent of t; i.e., $\frac{d}{dt}E = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x,t) = \sum_{n=1}^{\infty}\big(a_n\cos(nct) + b_n\sin(nct)\big)\sin(nx)$$

in terms of coefficients a_n, b_n.

Solution. For part (a), suppose that u is, as above, a solution to the wave equation which is C^2 up to the boundary. We show that its energy is independent of t, i.e., that $\frac{d}{dt}E = 0$. Assuming the energy is bounded, the dominated convergence theorem allows us to permute the order of integration and differentiation like so

$$\begin{aligned}
\tfrac{d}{dt}E(t) &= \frac{d}{dt}\left(\frac{1}{2}\int_0^\pi \left(u_t^2 + c^2u_x^2\right) dx\right)\\
&= \frac{1}{2}\int_0^\pi \tfrac{\partial}{\partial t}\left(u_t^2 + c^2u_x^2\right) dx\\
&= \frac{1}{2}\int_0^\pi 2u_tu_{tt} + 2c^2u_xu_{xt}\, dx
\end{aligned}$$

which, after using the relation $u_{tt} = c^2u_{xx}$, becomes

$$\begin{aligned}
&= c^2\int_0^\pi u_tu_{xx} + u_xu_{xt}\, dx\\
&= c^2\int_0^\pi \tfrac{\partial}{\partial x}(u_xu_t)\, dx\\
&= c^2\big(u_x(\pi,t)u_t(\pi,t) - u_x(0,t)u_t(0,t)\big)\\
&= 0
\end{aligned}$$

since the boundary conditions, i.e., $u = 0$, implies $u_x = u_t = 0$ at the boundary.

For part (b), suppose u is a Fourier series solution to the wave equation, i.e.,

$$u(x,t) = \sum_{n=1}^{\infty} \big(a_n \cos(nct) + b_n \sin(nct)\big) \sin(nx).$$

First, we compute u_x and u_t, they are

$$\begin{aligned} u_x(x,t) &= \sum_{n=1}^{\infty} n\big(a_n \cos(nct) + b_n \sin(nct)\big) \cos(nx), \\ u_t(x,t) &= \sum_{n=1}^{\infty} nc\big(-a_n \sin(nct) + b_n \cos(nct)\big) \sin(nx). \end{aligned} \tag{12}$$

Let $u_x^k(x,t)$ and $u_t^k(x,t)$ be the partial sums of the equations above so $u_x^k \to u_x$ and $u_t^k \to u_t$ as $k \to \infty$. Then

$$E_k(t) = \frac{1}{2} \int_0^{\pi} (u_t^k)^2 + c^2 (u_x^k)^2 \, dt$$

taking into account orthogonality relations of cos and sin, we have

$$\begin{aligned} &= \frac{1}{2} \sum_{n=1}^{k} n^2 c^2 \Bigg[\bigg(\int_0^{\pi} \big(a_n^2 \sin(nct) + b_n^2 \cos(nct)\big) \bigg) \sin^2(nx) \\ &\qquad\qquad + \bigg(\int_0^{\pi} \big(a_n^2 \cos(nct) + a_n^2 \sin(nct)\big) \bigg) \cos^2(nx) \Bigg] \\ &= \sum_{n=1}^{k} n^2 c^2 \pi (a_n^2 + b_n^2). \end{aligned}$$

Thus,

$$E(t) = \lim_{k \to \infty} E_k(t) = \sum_{n=1}^{\infty} n^2 c^2 \pi (a_n^2 + b_n^2). \tag{13}$$

■

1.5 Homework 5

PROBLEM 1.14. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$, $x \in \mathbb{R}^n$, then $\Delta v = 0$.

SOLUTION. Let

$$O = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be an orthogonal $n \times n$ matrix. We will show that $\Delta v = 0$, where $v(x) = u(Ox)$.

First, let us compute the gradient of v,

$$\begin{aligned} Dv(x) &= Du(Ox) \\ &= Du\big(a_{11}x_1 + \cdots + a_{1n}x_n, \dots, a_{n1}x_1 + \cdots + a_{nn}x_n\big) \\ &= \Big(\sum\nolimits_{j=1}^n a_{j1}u_{x_j}, \dots, \sum\nolimits_{j=1}^n a_{jn}u_{x_j}\Big) \\ &= O^{\mathrm{T}} Du(x). \end{aligned}$$

Lastly, we compute the divergence of Dv,

$$\begin{aligned} \Delta v(x) &= \operatorname{div} Dv(x) \\ &= \operatorname{div}\Big(\sum\nolimits_{j=1}^n a_{j1}u_{x_j}, \dots, \sum\nolimits_{j=1}^n a_{jn}u_{x_j}\Big). \end{aligned}$$

Here the partial derivatives become unwieldy so we will first examine the partial $\frac{\partial}{\partial x_1}$ of the first term and proceed from there. In this case,

$$\begin{aligned} \frac{\partial}{\partial x_1} \sum_{j=1}^n a_{j1}u_{x_j} &= a_{11}(u_{x_1})_{x_1} + a_{21}(u_{x_2})_{x_1} + \cdots + a_{n1}(u_{x_n})_{x_1} \\ &= a_{11}(a_{11}u_{x_1x_1} + a_{21}u_{x_1x_2} + \cdots + a_{n1}u_{x_1x_n}) \\ &\quad + \cdots + a_{n1}(a_{11}u_{x_1x_n} + a_{21}u_{x_2x_n} + \cdots + a_{n1}u_{x_nx_n}) \\ &= a_{11}^2u_{x_1x_1} + 2a_{11}a_{21}u_{x_1x_2} + 2a_{11}a_{31}u_{x_1x_3} + \cdots + a_{21}^2u_{x_2x_2} \\ &\quad + \cdots + a_{k1}^2u_{x_kx_k} + \cdots + a_{n1}^2u_{x_nx_n}. \end{aligned}$$

Similarly, taking the k^{th} partial of the k^{th} entry of Dv, we have

$$\begin{aligned} \frac{\partial}{\partial x_k} \sum_{j=1}^n a_{jk}u_{x_j} &= a_{1k}(a_{1k}u_{x_1x_1} + \cdots + a_{nk}u_{x_1x_n}) \\ &\quad + \cdots + a_{nk}(a_{1k}u_{x_1x_n} + \cdots + a_{nk}u_{x_nx_n}) \\ &= a_{1k}^2u_{x_1x_1} + a_{2k}^2u_{x_2x_2} + \cdots + a_{kk}^2u_{x_kx_k} \\ &\quad + \cdots + a_{nk}^2u_{x_nx_n} + \{\text{mixed terms}\}. \end{aligned} \tag{14}$$

Now, since O is orthogonal, we have

$$
\begin{aligned}
OO^{\mathrm{T}} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}^2 + \cdots + a_{1n}^2 & a_{11}a_{21} + \cdots + a_{1n}a_{2n} & \cdots & a_{11}a_{n1} + \cdots + a_{1n}a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} + \cdots + a_{nn}a_{1n} & a_{n1}a_{21} + \cdots + a_{nn}a_{2n} & \cdots & a_{n1}^2 + \cdots + a_{nn}^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}.
\end{aligned}
$$

We can sum up the results of our calculation as

$$
\begin{cases} \text{(a)} \ \sum_{j=1}^n a_{kj}a_{\ell j} = \sum_{j=1}^n a_{kj}^2 = 1 & \text{if } k = \ell, \\ \text{(b)} \ \sum_{j=1}^n a_{kj}a_{\ell j} = 0 & \text{if } k \neq \ell. \end{cases} \tag{15}
$$

for $1 \leq k, \ell \leq n$.

Now, going back to (14), we have

$$
\begin{aligned}
\operatorname{div} Dv &= \sum_{k=1}^n \left[\frac{\partial}{\partial x_k} \sum_{j=1}^n a_{jk} u_{x_j} \right] \\
&= (a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2) u_{x_1x_1} + (a_{12}^2 + a_{22}^2 + \cdots + a_{2n}^2) u_{x_2x_2} \\
&\quad + \cdots + (a_{1n}^2 + \cdots + a_{nn}^2) u_{x_nx_n} + \{\text{mixed terms}\} \\
&= u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} \\
&= 0,
\end{aligned} \tag{16}
$$

as desired.

All that is left to show as that the mixed terms in the expression above actually have coefficients of the form in (15) (b). And in fact one can see, expanding (16), that the mixed terms have the form

$$
\sum_{j=1}^n a_{kj}a_{\ell j} = 0.
$$

For example, the first member in the mixed terms sequence is

$$
2(a_{11}a_{21} + a_{12}a_{22} + \cdots + a_{1n}a_{2n}) u_{x_1x_2} = 0.
$$

(Time permits, I will provide a better argument than simply expanding (16); but a little routine calculation shows that these terms in fact have the form we have described.) ■

PROBLEM 1.15. Let $n = 2$ and U be the halfplane $\{ x_2 > 0 \}$. Prove that

$$\sup_U u = \sup_{\partial U} u$$

for $u \in C^2(U) \cap C(\bar{U})$ which are harmonic in U under the additional assumption that u is bounded from above in $\bar{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.)
[*Hint:* Take for $\varepsilon > 0$ the harmonic function

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region $\left\{ x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0 \right\}$ with large a. Let $\varepsilon \to 0$.]

SOLUTION. Consider the harmonic function

$$u_\varepsilon(x_1, x_2) := u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Set $U_a := \left\{ x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0 \right\}$.

First, we note that $u_\varepsilon \uparrow u$ as $\varepsilon \to 0$ pointwise, i.e., given $\eta > 0$, for

$$0 < \varepsilon(x_1, x_2) < \eta / \ln \sqrt{x_1^2 + (x_2 + 1)^2},$$

we have

$$\begin{aligned}
|u_\varepsilon(x_1, x_2) - u(x_1, x_2)| &= \left| u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} - u(x_1, x_2) \right| \\
&= \left| \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} \right| \\
&= \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} \\
&< \eta,
\end{aligned}$$

for any $(x_1, x_2) \in U_a$.

Moreover, a simple calculation shows that u_ε is in fact harmonic. By the linearity the Laplacian, it suffices to show that the Laplacian of

$$v_\varepsilon(x) := \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}$$

is 0. First, we calculate he partial derivatives $\frac{\partial^2}{\partial x_1 \partial x_1}$ and $\frac{\partial^2}{\partial x_2 \partial x_2}$

$$\begin{aligned}
\frac{(v_\varepsilon)_{x_1}}{\varepsilon} &= -\frac{x_1}{x_1^2 + (x_2 + 1)^2} & \frac{(v_\varepsilon)_{x_2}}{\varepsilon} &= -\frac{(x_2 + 1)}{x_1^2 + (x_2 + 1)^2} \\
\frac{(v_\varepsilon)_{x_1 x_1}}{\varepsilon} &= -\frac{-x_1^2 + (x_2 + 1)^2}{\left(x_1^2 + (x_2 + 1)^2\right)^2} & \frac{(v_\varepsilon)_{x_2 x_2}}{\varepsilon} &= -\frac{x_1^2 - (x_2 + 1)^2}{\left(x_1^2 + (x_2 + 1)^2\right)^2}.
\end{aligned}$$

Thus,

$$\Delta u_\varepsilon = \Delta u + \Delta v_\varepsilon = \Delta v_\varepsilon = \varepsilon \left(-\frac{-x_1^2 + (x_2 + 1)^2}{\left(x_1^2 + (x_2 + 1)^2\right)^2} - \frac{x_1^2 - (x_2 + 1)^2}{\left(x_1^2 + (x_2 + 1)^2\right)^2} \right) = 0.$$

Now, observe that, for any a, by the maximum principle, we have

$$\max_{\bar{U}_a} u_\varepsilon = \max_{\partial U_a} u_\varepsilon$$

for any a. Choose a large enough so

$$\sup_{\partial U_a} u_\varepsilon \leq \sup_{\partial U} u.$$

Then,

$$\sup_{\bar{U}_a} u_\varepsilon \leq \sup_{\partial U} u$$

so, taking $a \to \infty$, we have

$$\sup_{\bar{U}} u_\varepsilon \leq \sup_{\partial U} u.$$

Thus, for every $x_1, x_2 \in U$,

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} < \sup_{\partial U} u.$$

Letting $\varepsilon \to 0$, we achieve the desired inequality, i.e.,

$$\sup_{\bar{U}} u \leq \sup_{\partial U} u.$$

The last inequality is obvious and stems from the fact that $\partial U \subset \bar{U}$, i.e., the inequality

$$\sup_{\partial U} u \leq \sup_{\bar{U}} u.$$

We conclude that

$$\sup_{\partial U} u = \sup_{\bar{U}} u,$$

as was to be shown. ■

PROBLEM 1.16. Let $U \subset \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \dots, u^m$ are harmonic in U and

$$v := \varphi(u_1, \dots, u_m).$$

Prove v is subharmonic.

[*Hint:* Convexity for a smooth function $\varphi(z)$ is equivalent to $\sum_{j,k=1}^m \varphi_{z_j,z_k}(z)\xi_j\xi_k \geq 0$ for any $\xi \in \mathbb{R}^m$.]

(b) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. (Assume that harmonic functions are C^∞.)

SOLUTION. For part (a), by the chain rule, we have

$$v_{x_i} = \varphi_{u_1} u^1_{x_i} + \cdots + \varphi_{u_m} u^m_{x_i}.$$

Taking another partial, we have

$$\begin{aligned} v_{x_i x_i} &= (v_{x_i})_{x_i} \\ &= \frac{\partial}{\partial x_i}\left(\varphi_{u_1} u^1_{x_i} + \cdots + \varphi_{u_m} u^m_{x_i}\right) \\ &= \varphi_{u_1} u^1_{x_i x_i} + \cdots + \varphi_{u_m} u^m_{x_i x_i} \\ &\quad + \left(\varphi_{u_1 u_1} u^1_{x_i} + \cdots + \varphi_{u_1 u_m} u^m_{x_i}\right) u^1_{x_i} \\ &\quad\quad + \cdots + \left(\varphi_{u_1 u_m} u^1_{x_i} + \cdots + \varphi_{u_m u_m} u^m_{x_i}\right) u^m_{x_i}. \end{aligned} \tag{17}$$

Now, taking the sum

$$\begin{aligned} \sum_{i=1}^n v_{x_i x_i} &= \sum_{i=1}^n \sum_{j=1}^m \varphi_{u_j} u^j_{x_i x_i} \\ &= \sum_{j=1}^m \sum_{i=1}^n \varphi_{u_j} u^j_{x_i x_i} \\ &= \sum_{j=1}^m \left(\varphi_{u_j} u^j_{x_1 x_1} + \cdots + \varphi_{u_j} u^j_{x_n x_n}\right) \\ &= \sum_{j=1}^m \varphi_{u_j}\left(u^j_{x_1 x_1} + \cdots + u^j_{x_n x_n}\right) \\ &= 0, \end{aligned}$$

since $\Delta u^j = 0$ for all j.

What about the remaining terms in (17)? These terms can be written in the form

$$\sum_{j,k=1}^m \varphi_{u_j u_k}(u)\xi_j\xi_k,$$

where $\xi_i = \big(u^1_{x_i}, \dots, u^m_{x_i}\big)(x_1, \dots, x_n) \in \mathbb{R}^m$ for any $(x_1, \dots, x_n) \in \mathbb{R}^n$. Since φ is convex, by assumption, this quantity is greater than or equal to 0.

Thus, $\Delta v \geq 0$ so v is subharmonic.

For part (b), we have
$$v = |Du|^2 = u^2_{x_1} + \cdots + u^2_{x_n}.$$
Taking the partial derivative with respect to x_i, we have
$$\begin{aligned} v_{x_i} &= \frac{\partial}{\partial x_i}\big(u^2_{x_1} + \cdots + u^2_{x_n}\big) \\ &= 2u_{x_1}u_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_n}, \end{aligned}$$
and again
$$\begin{aligned} v_{x_ix_i} &= (v_{x_i})_{x_i} \\ &= \frac{\partial}{\partial x_i}\big(2u_{x_1}u_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_n}\big) \\ &= 2u_{x_1}u_{x_1x_ix_i} + 2u^2_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_ix_n} + 2u^2_{x_ix_n} \\ &= 2\sum_{j=1}^n \big(u_{x_j}u_{x_jx_ix_i} + u^2_{x_jx_i}\big). \end{aligned}$$
Then
$$\begin{aligned} \frac{\Delta v}{2} &= \sum_{i,j=1}^n \big(u_{x_j}u_{x_jx_ix_i} + u^2_{x_jx_i}\big) \\ &= \sum_{i,j=1}^n u_{x_j}u_{x_jx_ix_i} + \sum_{i,j=1}^n u^2_{x_jx_i}, \end{aligned}$$
splitting the second term into the sum
$$\begin{aligned} &= \sum_{i,j=1}^n u_{x_j}u_{x_jx_ix_i} + \sum_{1\leq j<i\leq n} u^2_{x_jx_i} \\ &\quad + \sum_{1\leq i<j\leq n} u^2_{x_jx_i} + \sum_{1\leq i=j\leq n} u^2_{x_ix_i}, \end{aligned}$$
where the last term is 0 since u is harmonic, giving us
$$\begin{aligned} &= \sum_{i,j=1}^n u_{x_j}u_{x_jx_ix_i} + \sum_{1\leq j<i\leq n} u^2_{x_jx_i} + \sum_{1\leq i<j\leq n} u^2_{x_jx_i} \\ &= \sum_{i,j=1}^n u_{x_j}u_{x_jx_ix_i} + 2\sum_{1\leq j<i\leq n} u^2_{x_jx_i}, \end{aligned}$$

here $\sum_{j=1}^n u_{x_i x_j x_j} = \Delta u_{x_i} = 0$ since the derivatives of harmonic functions are harmonic, so

$$
\begin{aligned}
&= \sum_{j=1}^{n} u_{x_j}(\Delta u_{x_j}) + 2 \sum_{1 \le j < i \le n} u_{x_j x_i}^2 \\
&= 2 \sum_{1 \le j < i \le n} u_{x_j x_i}^2 \\
&\ge 0,
\end{aligned}
$$

as desired. That is, $\Delta v \ge 0$ so v is subharmonic. ■

1.6 Homework 6

PROBLEM 1.17. For $n = 2$ find Green's function for the quadrant $\{ x_1 > 0, x_2 > 0 \}$ by repeated reflection.

SOLUTION. ■

PROBLEM 1.18. (Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \le u(x) \le \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B(0, r) = \{ x \in \mathbb{R}^n : |x| < r \}$.

SOLUTION. ■

PROBLEM 1.19. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$\begin{aligned} P_k(\lambda x) &= \lambda^k P_k(x), & P_m(\lambda x) &= \lambda^m P_m(x) & &\text{for every } x \in \mathbb{R}^n,\ \lambda > 0, \\ \Delta P_k &= 0, & \Delta P_m &= 0 & &\text{in } \mathbb{R}^n. \end{aligned}$$

(a) Show that

$$\frac{\partial P_k}{\partial \nu} = k P_k(x), \qquad \frac{\partial P_m}{\partial \nu} = m P_m(x) \qquad \text{on } \partial B(0,1)$$

where $B(0, 1) = \{ x \in \mathbb{R}^n : |x| < 1 \}$ and ν is the outward normal on $\partial B(0, 1)$.

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x)\, d\sigma = 0, \qquad \text{if } k \ne m.$$

SOLUTION. ■

2 Midterms and Qualifying Exams

2.1 Qualifying Exam, August ‘04

PROBLEM 2.1. Consider the initial value problem

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = -u, \\ \qquad\qquad\qquad\qquad u = f \qquad \text{on } S^1 = \{\, x^2 + y^2 = 1 \,\}, \end{cases}$$

where a and b satisfy

$$a(x,y) + b(x,y)y > 0$$

for any $x, y \in \mathbb{R}^n \setminus \{(0,0)\}$.

(a) Show that the initial value problem has a unique solution in a neighborhood of S^1. Assume that a, b, and f are smooth.
(b) Show that the solution of the initial value problem actually exists in $\mathbb{R}^2 \setminus \{(0,0)\}$.

SOLUTION. ■

PROBLEM 2.2. Let $u \in C^2\big(\mathbb{R} \times [0,\infty)\big)$ be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty), \\ \qquad u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where f and g have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx,$$

and the potential energy by

$$P(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx.$$

Show that

(a) $K(t) + P(t)$ is constant in t,
(b) $K(t) = P(t)$ for all large enough times t.

SOLUTION. ■

PROBLEM 2.3. Use Kirchhoff’s formula and Duhamel’s principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = e^{-t}g(x) & \text{for } x \in \mathbb{R}^3,\ t > 0, \\ \quad u(x,0) = u_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution $u = e^{-t} + t - 1$ when $g(x) = 1$.

Solution. ∎

PROBLEM 2.4. Let Ω be a bounded open set in $\mathbb{R}^n$ and $g \in C_0^\infty(\Omega)$. Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega,\ t > 0, \\ u(x,0) = g(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } xi \in \partial\Omega,\ t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n,\ t > 0, \\ v(x,0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put $g = 0$ outside Ω.

(a) Show that

$$-v(x,t) \leq u(x,t) \leq v(x,t)$$

for any $x \in \Omega$, $t > 0$.

(b) Use (a) to conclude that

$$\lim_{t\to\infty} u(x,t) = 0,$$

for any $x \in \Omega$.

Solution. ∎

PROBLEM 2.5. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \qquad P_m(\lambda x) = \lambda^m P_m(x),$$

for any $x \in \mathbb{R}^n$, $\lambda > 0$,

$$\Delta P_k = 0, \qquad \Delta P_m = 0$$

in $\mathbb{R}^n$.

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = kP_k(x), \qquad \frac{\partial P_m(x)}{\partial \nu} = mP_m(x)$$

on ∂B_1, where $B_1 = \{\, |x| < 1 \,\}$ and ν is the outward normal on ∂B_1.

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x)\, dS = 0,$$

if $k \neq m$.

Solution. ∎

2.2 Qualifying Exam, August '05

PROBLEM 2.6.

(a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ \qquad\qquad u = 1 \qquad \text{on } S^1 = \{ x^2 + y^2 = 1 \}. \end{cases}$$

(b) Is the solution unique in a neighborhood of the point $(1, 0)$? Justify your answer.

Solution. ■

PROBLEM 2.7. Consider the second order PDE in $\{ x > 0, y > 0 \} \subset \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

(a) Classify the equation and reduce it to the canonical form.
(b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy}\, G(x/y).$$

Solution. ■

PROBLEM 2.8. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^3$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial (i.e., $f(y) = f(|y|)$) and supported in $B_R = \{ |x| < R \}$, then

$$u(x) = c\Phi(x),$$

for any $x \in \mathbb{R}^n \setminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y)\, dy.$$

[*Hint:* Use spherical (polar) coordinates and the mean value property.]

Solution. ■

PROBLEM 2.9. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \qquad u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where $g, h \in C_0^\infty(\mathbb{R}^2)$ and $\alpha \geq 0$ is a constant.

(a) Show that for an appropriate choice of constant λ and μ the function

$$v(x_1, x_2, x_3, t) := e^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation $v_{tt} - \Delta v = 0$.

(b) Use (a) to prove the following domain of dependence result: for any point $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$ the value $u(x_0, t_0)$ is uniquely determined by values of g and h in $\overline{B_{t_0}(x_0)} := \{\, |x - x_0| \le t_0 \,\}$. (You may use the corresponding result for the wave equation without proof.)

Solution. ■

Problem 2.10. Let $u(x,t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0, \infty)$ with the initial condition

$$u(x, 0) = u_0(x)$$

for $x \in \mathbb{R}$, where $u_0 \in C^\infty$ is 2π-periodic, i.e., $u_0(x + 2\pi) = u_0(x)$. Show that

$$\lim_{t \to \infty} u(x,t) = a_0,$$

uniformly in $x \in \mathbb{R}$, where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x)\, dx.$$

Solution. ■

2.3 Qualifying Exam, January '14

Problem 2.11. Consider the first order equation in $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

(a) Find the characteristic curves of the equation.
(b) Consider the Cauchy problem for this equation prescribed on the line $x_1 = 1$:

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on f so that the proble is solvable in a neighborhood of the point $(1, 0)$.

Solution. ■

Problem 2.12. Let u be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{ x_n > 0 \}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where g is a continuous function with compact support in $\mathbb{R}^{n-1}$. Here $n \geq 2$. Prove that

$$u(x) \longrightarrow 0, \qquad \text{as } |x| \longrightarrow \infty,$$

for $x \in \{ x_n > 0 \}$.

Solution. ■

Problem 2.13. Let u be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \qquad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions $u(x, 0) = g(x)$, where g is a bounded continuous function on $\mathbb{R}$ satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^{\alpha}, \qquad x, y \in \mathbb{R}$$

with a constant $\alpha \in (0, 1]$. Show that

$$\begin{aligned} |u(x,t) - u(y,t)| &\leq M|x-y|^{\alpha}, & x, y \in \mathbb{R}, t > 0, \\ |u(x,t) - u(x,s)| &\leq C_{\alpha} M |t-s|^{\alpha/2}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of $u(x,t)$ as a convolution with the heat kernel $\Phi(y,t)$, make a change of variables $z = y/\sqrt{t}$ and use that $\left|\sqrt{t} - \sqrt{s}\right| \leq \sqrt{|t-s|}$.]

Solution. ■

Problem 2.14. Let u be a positive harmonic function in the unit ball B_1 in $\mathbb{R}^n$. Show that

$$\left|D(\ln u)\right| \le M \qquad \text{in } B_{1/2}$$

for a constant M depending only on the dimension n.
[*Hint:* Use the interior derivative estimate $\left|Du(x)\right| \le (C_n/r) \sup_{B_r(x)} |u|$ for $B_r(x) \subset B_1$ as well as the Harnack inequality for harmonic functions.]

Solution. ■

Problem 2.15. Let u be a C^2 solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0,\infty), \\ \qquad u = 0, \qquad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some $k \ge 0$. Prove that there exists a function $\varphi(r)$ such that

$$u(x,t) = t^{k+2} \varphi\big(|x|/t\big).$$

[*Hint:* As one of the steps show that u is $(k+2)$-homogeneous in (x,t) variables, i.e., $u(\lambda x, \lambda t) = \lambda^{k+2} u(x,t)$ for any $\lambda > 0$.]

Solution. ■