

MA 544: Homework 3

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PROBLEM 3.1 (WHEEDEN & ZYGMUND §3, EX. 5)

Construct a subset of $[0, 1]$ in the same manner as the Cantor set, except that at the k th stage each interval removed has length $\delta 3^{-k}$, $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no interval.

Proof. Put $C_0 := [0, 1]$. We begin constructing our desired set by removing the open set $(\frac{\delta}{3}, 1 - \frac{\delta}{3})$ from the closed interval $[0, 1]$. This separates $[0, 1]$ into the union of two disjoint closed (and bounded therefore, compact) intervals $[0, \frac{\delta}{3}]$ and $[1 - \frac{\delta}{3}, 1]$ which we shall call C_1 . Next, we remove the open interval $(\frac{\delta}{9}, \frac{\delta}{3} - \frac{\delta}{9})$ from $[0, \frac{\delta}{3}]$ and the open interval $(1 - \frac{\delta}{3} + \frac{\delta}{9}, 1 - \frac{\delta}{9})$ and end up with the union C_2 of four disjoint closed intervals. Continue in this fashion ad infinitum. Note that each $C_{k+1} \subset C_k$ and each C_k is a finite union of closed subsets thus, by theorem 1.7 and the Cantor's intersection theorem, the set $C_\delta := \bigcap_{i=1}^{\infty} C_i$ is closed, compact, and nonempty.

Now we show that the set we have constructed, C_δ , is perfect. Since C_δ is closed, it remains to show that C_δ contains no isolated points. Note that, in the construction of C_δ , we never removed the endpoints the intervals which union to C_k . Thus, the endpoints of the intervals which union to C_k are in C_δ . Before continuing, we need to figure out what the length of each interval at the k th stage in the construction is and in the process, we shall prove that $|C_\delta| = 1 - \delta$.

At each stage in the construction (except for $k = 0$) we removed 2^{k-1} open intervals of length $\delta 3^{-k}$. Thus, at the k th stage of the construction, the measure of C_k will be

$$|C_k| = 1 - \sum_{i=1}^k \frac{2^{i-1}\delta}{3^i} = 1 - \frac{\delta}{3} \sum_{i=1}^k \left(\frac{2}{3}\right)^{i-1}.$$

We immediately recognize the right-hand side as a geometric sum so letting $k \rightarrow \infty$, by theorem 3.26(ii), we have

$$|C_\delta| = \lim_{k \rightarrow \infty} 1 - \frac{\delta}{3} \sum_{i=1}^k \left(\frac{2}{3}\right)^{i-1} = 1 - \lim_{k \rightarrow \infty} \frac{\delta}{3} \sum_{i=1}^k \left(\frac{2}{3}\right)^{i-1} = 1 - \frac{\delta}{3} \left(\frac{1}{1 - \frac{2}{3}}\right) = 1 - \delta.$$

Now, let $\varepsilon > 0$ be given. By the Archimedean principle, we may choose a sufficiently large natural number N so that $|C_N|2^{-N} < 2^{-N} < \varepsilon$. Let x be a point in C_δ , then $x \in C_N$ since $x \in C_k$ for all k . In particular, x is in one of the 2^N disjoint closed intervals that union to C_k , call it I . Let x' be the closest endpoint of I to x (if x is itself an endpoint, choose x' to be the opposite endpoint). Then, by the triangle inequality, we have $|x - x'| \leq |C_N|2^{-N} < \varepsilon$. Hence, the open neighborhood $B(x, \varepsilon) \setminus \{x\} \neq \emptyset$ for any ε . Thus, C_δ is perfect.

Last but not least, we show that C_δ contains no interval. Suppose that (a, b) is an interval contained in C_δ . Hence, $(a, b) \subset C_k$ for all k . (I don't know how to finish the proof without using a fact about connected 1-manifolds). Then, since (a, b) is connected, it must be contained in a connected component C of C_δ . However, the connected components of C_k , i.e., the closed intervals, have measure less than 2^{-k} so $b - a \leq 2^{-k}$. Letting $k \rightarrow \infty$, we have $b - a \leq 0$ which leads to a contradiction since the measure of an interval is strictly greater than 0. ■

PROBLEM 3.2 (WHEEDEN & ZYGMUND §3, EX. 7)

Prove (3.15).

Proof. Recall the statement of 3.15:

Lemma (Wheeden & Zygmund (3.15)). *If $\{I_k\}_k^N$ is a finite collection of nonoverlapping intervals, then $\bigcup I_k$ is measurable and $|\bigcup I_k| = \sum |I_k|$.*

Note that the proof follows exactly as corollary 3.24. By theorem 3.14, since $\bigcup I_k$ is a finite union of closed sets, $\bigcup I_k$ is measurable. To see that $|\bigcup I_k| = \sum |I_k|$. By subadditivity, we have

$$\left| \bigcup I_k \right| \leq \sum |I_k|.$$

On the other hand, since $|I_k^\circ| = |I_k^\circ| + |\partial I_k| = |I_k^\circ \cup \partial I| = |I_k|$, we have

$$\sum |I_k| = \sum |I_k^\circ| \leq \left| \bigcup I_k \right|.$$

Hence, $|\bigcup I_k| = \sum |I_k|$. ■

PROBLEM 3.3 (WHEEDEN & ZYGMUND §3, EX. 8)

Show that the Borel algebra \mathcal{B} in \mathbf{R}^n is the smallest σ -algebra containing the closed sets in \mathbf{R}^n .

Proof. We defined the Borel algebra \mathcal{B} in \mathbf{R}^n to be the smallest σ -algebra containing the open sets in \mathbf{R}^n . Since \mathcal{B} contains all the closed subsets of \mathbf{R}^n , theorem 3.17, it suffices to show that \mathcal{B} is the smallest such σ -algebra. Suppose \mathcal{B}' is another σ -algebra containing all of the closed sets in \mathbf{R}^n . Then, by theorem 3.17, \mathcal{B}' contains all of the open sets of \mathbf{R}^n and, since \mathcal{B} is the smallest σ -algebra containing the open subsets of \mathbf{R}^n , we have $\mathcal{B} \subset \mathcal{B}'$, as desired. ■

PROBLEM 3.4 (WHEEDEN & ZYGMUND §3, EX. 9)

If $\{E_k\}_{k=1}^\infty$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ (and also $\liminf E_k$) has measure zero.

Proof. Define $E := \limsup E_k$ and $E'_\ell := \bigcup_{k=\ell}^\infty E_k$. Then E'_ℓ is a decreasing (with respect to inclusion) sequence of sets with $\lim_\ell E'_\ell = E$. Then E is contained in the intersection $\bigcap_{\ell=n}^\infty E'_\ell$ for all n , so by the monotonicity of the outer measure we have

$$|E|_e \leq |E'_\ell|_e.$$

On the other hand, we also have

$$|E'_n|_e \leq \sum_{k=n}^\infty |E_k|_e$$

for all n . Since, by assumption, the sum $\sum |E_k|_e$ converges, we have, for every $\varepsilon > 0$, there exists N sufficiently large such that the sum $\sum_{k=n}^\infty |E_k|_e < \varepsilon$ for every $n \geq N$. Thus, $|E|_e \leq \varepsilon$ for every $\varepsilon > 0$. Let $\varepsilon \rightarrow 0$ and we have $|E|_e = 0$ as desired. Lastly, we note that for any sequence $\{a_k\} \subset \mathbf{R}$ we have $\liminf a_n \leq \limsup a_n$ so, naturally, $\liminf E_k = 0$. ■

PROBLEM 3.5 (WHEEDEN & ZYGMUND §3, EX. 10)

If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

Proof. Without loss of generality, we may assume that $|E_1|, |E_2| < \infty$, for otherwise the result holds trivially.

By Carathéodory, we have that

$$|E_1| = |E_1 \cap E_2| + |E_1 \setminus E_2| \quad \text{and} \quad |E_2| = |E_1 \cap E_2| + |E_2 \setminus E_1|. \quad (1)$$

Moreover, by elementary set theory, we have $(E_1 \cup E_2) \setminus E_2 = E_1 \setminus (E_1 \cap E_2)$ and $E_1 \subset E_1 \cup E_2$, and $E_1 \cap E_2 \subset E_1$ so by rearranging (1) we have

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|,$$

as desired. ■