Mathematics Notes

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1 Group Theory

1.1 The Sign of a Permutation

Summary of Keith Conrad's blurb by the same name.

Throughout this discussion $n \geq 2$. A cycle in S_n is the (non-unique) product of transpositions: the identity (1), (1 2)(1 2), and a k-cycle with $k \geq 2$ can be written as

$$(i_1\ i_2\ \cdots\ i_k) = (i_1\ i_2)(i_2\ i_3)\cdots(i_{k-1}\ i_k).$$

For example a 3-cycle $(a\ b\ c)$ – which means a, b and c are distict – can be written as

$$(a\ b\ c) = (a\ b)(b\ c).$$

This is not the only way to write $(a\ b\ c)$ using transpositions, e.g., $(a\ b\ c) = (b\ c)(a\ c) = (a\ c)(a\ b)$.

Since any permutation in S_n is the product of cycles and any cycle is a product of transpositions, any permutation in S_n is a product of transpositions. Unlike the unique decomposition of a permutation into products of disjoint cycles, the decomposition of a permutation is

2 McMullen's Notes on Real Analysis

3 McMullen's Complex Analysis Notes

3.1 Basic Complex Analysis

Some Notation

The complex numbers will be denoted $\mathbb{C}$. We let Δ, $\mathbb{H}$, and $\widehat{\mathbb{C}}$ denote the unit disk $|z| < 1$, the upper half-plane $\Im z > 1$ and the Riemann sphere $\mathbb{C} \cup \{\infty\}$. We write $S^1(r)$ for the circle $|z| = r$ and S^1 for the unit circle, each oriented counter-clockwise. We also set $\Delta^* = \Delta \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Algebra and complex numbers

The complex numbers are formally defined as the field $\mathbb{C} = \mathbb{R}[i]$, where $i^2 = -1$. They are represented in the Euclidean plane by $z = (x, y) = x + iy$. There are two square-roots of -1 in $\mathbb{C}$; the number i is the one with positive imaginary part.

An important role is played by the Galois involution $z \mapsto \bar{z}$. We define $|z|^2 = N(z) = z\bar{z} = x^2+y^2$. Compatibility of $|z|$ with the Euclidean metric justifies the identification of $\mathbb{C}$ and $\mathbb{R}^2$. We also see that $\mathbb{C}$ is a field as $1/z = \bar{z}/|z|$.

It is also convenient to describe complex numbers by polar coordinates

$$z = [r, \theta] = r(\cos\theta + i\sin\theta).$$

Here $r = |z|$ and $\theta = \arg z$ in $\mathbb{R}/2\pi\mathbb{Z}$. (The multivaluedness of $\arg z$ requires care but is also the source of powerful results like Cauchy's integral formula.) We then have

$$[r_1, \theta_1][r_2, \theta_2] = [r_1 r_2, \theta_1 + \theta_2].$$

In particular, the linear maps $f(z) = az + b$, $a \neq 0$, of $\mathbb{C}$ to itself, preserve angles and orientations.

This formula should be proved geometrically; in fact, it is a consequence of the formula $|ab| = |a||b|$ and similar triangles. It can then be used to derive the addition formulas for sine and cosine.

Algebraic closure

A critical feature of the complex numbers is that they are *algebraically closed*, i.e., every polynomial has a root.

Classically, the complex numbers were introduced in the course of solving real cubic equations. Starting with $x^3 + ax + b = 0$ one can make a Tschirnhaus transformation so $a = 0$. This is done by introducing a new variable, $y = cx^2 + d$ such that $\sum y_i = \sum {y_i}^2 = 0$; even when a and b are real, it may be necessary to choose c complex (the discriminant of the equation for c is $27b^2 + 4a^3$). It is negative when the cubic has only one real root; this can be checked by looking at the product of of the values of the cubic at its min and max.

Polynomials and rational functions

Using addition and multiplication we obtain naturally the polynomial functions $f(z) = \sum_0^n a_n z^2 \colon \mathbb{C} \to \mathbb{C}$. The ring of polynomials $\mathbb{C}[z]$ is an integral domain and a unique factorization domain, since $\mathbb{C}$ is a field. Indeed, since $\mathbb{C}$ is algebraically closed, every polynomial factors into linear terms.

It is useful to add the allowed value ∞ to obtain the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then the rational functions determine rational maps $f\colon \mathbb{C} \to \mathbb{C}$. The rational functions $\mathbb{C}(z)$ are the same as the field of fractions for the domain $\mathbb{C}[z]$. We set $f(z) = \infty$ if $q(z) = 0$; these points are called the *poles* of f.

Analysis functions

Let U be an open set in $\mathbb{C}$ and $f\colon U \to \mathbb{C}$ a function. We say that f is *analytic* if

$$f'(z) = \lim_{t \to \infty} \frac{f(z+) - f(z)}{t}$$

exists for all $z \in U$. It is important that t approaches 0 through arbitrary values of $\mathbb{C}$. Remarkably, this condition implies that $f \in C^{\infty}$.

4 McMullen's Topology Notes