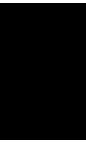


# MA553: Spring 2016 Homework

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# Course Notes

These notes roughly correspond to the three sections (by the same name) on groups, rings and fields from Dummit and Foote's *Abstract Algebra* [1]. I also make nominal use of Herstein's *Topics in Abstract Algebra* [2]

## 1.1 Group Theory

## 1.2 Rings

## 1.3 Field Theory

# Homework (Spring 2016)

## 2.1 Homework 1

**Problem 2.1.** Let  $G$  be a group,  $a \in G$  an element of finite order  $m$ , and  $n$  a positive integer. Prove that

$$|a^n| = \frac{m}{\gcd(m, n)}.$$

*Proof.* Without loss of generality, we may assume  $n < m$ ; otherwise, by the fundamental theorem of arithmetic, there exist  $q$  and  $r$  with  $r < m$  such that  $n = qm + r$  so  $a^n = a^{qm+r} = a^{qm}a^r = a^r$ . ■

**Problem 2.2.** Let  $G$  be a group, and let  $a, b$  be elements of finite order  $m, n$  respectively. Show that if  $ba = ab$  and  $\langle a \rangle \cap \langle b \rangle = \{e\}$ , then  $|ab| = \text{lcm}(m, n)$ .

*Proof.* ■

**Problem 2.3.** Let  $G$  be a group and  $H, K$  normal subgroups with  $H \cap K = \{e\}$ . Show that

- (a)  $hk = kh$  for every  $h \in H, k \in K$ .
- (b)  $HK$  is a subgroup of  $G$  with  $HK \simeq H \times K$ .

*Proof.* ■

**Problem 2.4.** Show that  $A_4$  has no subgroup of order 6 (although  $6 \mid 12 = |A_4|$ ).

*Proof.* ■

## 2.2 Homework 2

**Problem 2.5.** Let  $G$  be the group of order  $2^3 \cdot 3$ ,  $n \geq 2$ . Show that  $G$  has a normal 2-subgroup  $\neq \{e\}$ .

*Proof.* ■

**Problem 2.6.** Let  $G$  be a group of order  $p^2q$ ,  $p$  and  $q$  primes. Show that the Sylow  $p$ -Sylow subgroup or the  $q$ -Sylow subgroup of  $G$  is normal in  $G$ .

*Proof.* ■

**Problem 2.7.** Let  $G$  be a subgroup of order  $pqr$ ,  $p < q < r$  primes. Show that the  $r$ -Sylow subgroup of  $G$  is normal in  $G$ .

*Proof.* ■

**Problem 2.8.** Let  $G$  be a group of order  $n$  and let  $\varphi: G \rightarrow S_n$  be given by the action of  $G$  on  $G$  via translation.

- (a) For  $a \in G$  determine the number and the lengths of the disjoint cycles of the permutation  $\varphi(a)$ .
- (b) Show that  $\varphi(G) \not\subset A_n$  if and only if  $n$  is even and  $G$  has a cyclic 2-Sylow subgroup.
- (c) If  $n = 2m$ ,  $m$  odd, show that  $G$  has a subgroup of index 2.

*Proof.* ■

**Problem 2.9.** Show that the only simple groups  $\neq \{e\}$  of order  $< 60$  are the groups of prime order.

*Proof.* ■

### 2.3 Homework 3

**Problem 2.10.** Let  $G$  be a finite group,  $p$  a prime number,  $N$  the intersection of all  $p$ -Sylow subgroups of  $G$ . Show that  $N$  is a normal  $p$ -subgroup of  $G$  and that every normal  $p$ -subgroup of  $G$  is contained in  $N$ .

*Proof.* ■

**Problem 2.11.** Let  $G$  be a group of order 231 and let  $H$  be an 11-Sylow subgroup of  $G$ . Show that  $H \subset Z(G)$ .

*Proof.* ■

**Problem 2.12.** Let  $G = \{e, a_1, a_2, a_3\}$  be a non-cyclic group of order 4 and define  $\varphi: S_3 \rightarrow \text{Aut}(G)$  by  $\varphi(\sigma)(e) = e$  and  $\varphi(\sigma)(a_i) = a_{\sigma(i)}$ . Show that  $\varphi$  is well-defined and an isomorphism of groups.

*Proof.* ■

**Problem 2.13.** Determine all groups of order 18.

*Proof.* ■

## 2.4 Homework 4

**Problem 2.14.** Let  $p$  be a prime and let  $G$  be a nonAbelian group of order  $p^3$ . Show that  $G' = Z(G)$ .

*Proof.* ■

**Problem 2.15.** Let  $p$  be an odd prime and let  $G$  be a nonAbelian group of order  $p^3$  having an element of order  $p^2$ . Show that there exists an element  $b \notin \langle a \rangle$  of order  $p$ .

*Proof.* ■

**Problem 2.16.** Let  $p$  be an odd prime. Determine all groups of order  $p^3$ .

*Proof.* ■

**Problem 2.17.** Show that  $(S_n)' = A_n$ .

*Proof.* ■

**Problem 2.18.** Show that every group of order  $< 60$  is solvable.

*Proof.* ■

**Problem 2.19.** Show that every group of order 60 that is simple (or not solvable) is isomorphic to  $A_5$ .

*Proof.* ■



## 2.5 Homework 5

**Problem 2.20.** Find all composition series and the composition factors of  $D_6$ .

*Proof.* ■

**Problem 2.21.** Let  $T$  be the subgroup of  $\text{GL}(n, \mathbb{R})$  consisting of all upper triangular invertible matrices. Show that  $T$  is solvable.

*Proof.* ■

**Problem 2.22.** Let  $p \in \mathbb{Z}$  be a prime number. Show:

- (a)  $(p-1)! \equiv -1 \pmod{p}$ .
- (b) If  $p \equiv 1 \pmod{4}$  then  $x^2 \equiv -1 \pmod{p}$  for some  $x \in \mathbb{Z}$ .

*Proof.* ■

**Problem 2.23.** (a) Show that the following are equivalent for an odd prime number  $p \in \mathbb{Z}$ :

- (i)  $p \equiv 1 \pmod{4}$ .
- (ii)  $p = a^2 + b^2$  for some  $a, b$  in  $\mathbb{Z}$ .
- (iii)  $p$  is not prime in  $\mathbb{Z}[i]$ .

- (b) Determine all prime ideals of  $\mathbb{Z}[i]$ .

*Proof.* ■

## 2.6 Homework 6

**Problem 2.24.** Let  $R$  be a domain. Show that  $R$  is a UFD if and only if every nonzero nonunit in  $R$  is a product of irreducible elements and the intersection of any two principal ideals is again principal.

*Proof.* ■

**Problem 2.25.** Let  $R$  be a PID and  $p$  a prime ideal of  $R[X]$ . Show that  $p$  is principal or  $p = (a, f)$  for some  $a \in R$  and some monic  $f \in R[X]$ .

*Proof.* ■

**Problem 2.26.** Let  $k$  be a field and  $n \geq 1$ . Show that  $Z^n + Y^3 + X^2 \in k(X, Y)[Z]$  is irreducible.

*Proof.* ■

**Problem 2.27.** Let  $k$  be a field of characteristic zero and  $n \geq 1$ ,  $m \geq 2$ . Show that  $X_1^n + \cdots + X_m^n - 1 \in k[X_1, \dots, X_m]$  is irreducible.

*Proof.* ■

**Problem 2.28.** Show that  $X^{3^n} + 2 \in \mathbb{Q}(i)[X]$  is irreducible.

*Proof.* ■

## 2.7 Homework 7

**Problem 2.29.** Let  $k \subset K$  and  $k \subset L$  be finite field extensions contained in some field. Show that:

- (a)  $[KL : L] \leq [K : k]$ .
- (b)  $[KL : k] \leq [K : k][L : k]$ .
- (c)  $K \cap L = k$  if equality holds in (b).

*Proof.* ■

**Problem 2.30.** Let  $k$  be a field of characteristic  $\neq 2$  and  $a, b$  elements of  $k$  so that  $a, b, ab$  are not squares in  $k$ . Show that  $[k(\sqrt{a}, \sqrt{b}) : k] = 4$ .

*Proof.* ■

**Problem 2.31.** Let  $R$  be a UFD, but not a field, and write  $K := \text{Quot}(R)$ . Show that  $[\bar{K} : k] = \infty$ .

*Proof.* ■

**Problem 2.32.** Let  $k \in K$  be an algebraic field extension. Show that every  $k$ -homomorphism  $\delta : K \rightarrow K$  is an isomorphism.

*Proof.* ■

**Problem 2.33.** Let  $K$  be the splitting field of  $X^6 - 4$  over  $\mathbb{Q}$ . Determine  $K$  and  $[K : \mathbb{Q}]$ .

*Proof.* ■

## 2.8 Homework 8

**Problem 2.34.** Let  $k$  be a field,  $f \in k[X]$  a polynomial of degree  $n \geq 1$ , and  $K$  the splitting field of  $f$  over  $k$ . Show that  $[K : k] \mid n!$ .

*Proof.* ■

**Problem 2.35.** Let  $k$  be a field and  $n \geq 0$ . Define a map  $\Delta_n : k[X] \rightarrow k[X]$  by  $\Delta_n(\sum a_i X^i) := \sum a_i \binom{i}{n} X^{i-n}$ . Show that

- (a)  $\Delta_n$  is  $k$ -linear, and for  $f, g \in k[X]$ ,  $\Delta_n(fg) = \sum_{j=0}^n \Delta_j(f) \Delta_{n-j}(g)$ .
- (b)  $f^{(n)} = n! \Delta_n(f)$ .
- (c)  $f(x+a) = \sum \Delta_n(f)(a) X^n$ .
- (d)  $a \in k$  is a root of  $f$  of multiplicity  $n$  if and only if  $\Delta_i(f)(a) = 0$  for  $0 \leq i \leq n-1$  and  $\Delta_n(f)(a) \neq 0$ .

*Proof.* ■

**Problem 2.36.** Let  $k \subset K$  be a finite field extension. Show that  $k$  is perfect if and only if  $K$  is perfect.

*Proof.* ■

**Problem 2.37.** Let  $K$  be the splitting field of  $X^p - X - 1$  over  $k = \mathbb{Z}/p\mathbb{Z}$ . Show that  $k \subset K$  is normal, separable, of degree  $p$ .

*Proof.* ■

**Problem 2.38.** Let  $k$  be a field of characteristic  $p > 0$ , and  $k(X, Y)$  the field of rational functions in two variables.

- (a) Show that  $[k(X, Y) : k(X^p, Y^p)] = p^2$ .
- (b) Show that the extension  $k(X^p, Y^p) \subset k(X, Y)$  is not simple.
- (c) Find infinitely many distinct fields  $L$  with  $k(X^p, Y^p) \subset L \subset k(X, Y)$ .

*Proof.* ■

## 2.9 Homework 9

**Problem 2.39.** Let  $k \subset K$  be a finite extension of fields of characteristic  $p > 0$ . Show that if  $p \nmid [K : k]$ , then  $k \subset K$  is separable.

*Proof.* ■

**Problem 2.40.** Let  $k \subset K$  be an algebraic extension of fields of characteristic  $p > 0$ , let  $L$  be an algebraically closed field containing  $K$ , and let  $\delta: k \rightarrow L$  be an embedding. Show that  $k \subset K$  is purely inseparable if and only if there exists exactly one embedding  $\tau: K \rightarrow L$  extending  $\delta$ .

*Proof.* ■

**Problem 2.41.** Let  $k \subset K = k(\alpha, \beta)$  be an algebraic extension of fields of characteristic  $p > 0$ , where  $\alpha$  is separable over  $k$  and  $\beta$  is purely inseparable over  $k$ . Show that  $K = k(\alpha + \beta)$ .

*Proof.* ■

**Problem 2.42.** Let  $f(x) \in \mathbb{F}_q[X]$  be irreducible. Show that  $f(X) \mid X^{q^n} - X$  if and only if  $\deg f(X) \mid n$ .

*Proof.* ■

**Problem 2.43.** Show that  $\text{Aut}_{\mathbb{F}_q}(\overline{\mathbb{F}_q})$  is an infinite Abelian group which is torsionfree (i.e.,  $\delta^n = \text{id}$  implies  $\delta = \text{id}$  or  $n = 0$ ).

*Proof.* ■

**Problem 2.44.** Show that in a finite field, every element can be written as a sum of two perfect squares.

*Proof.* ■

# Bibliography

- [1] D.S. Dummit and R.M. Foote. *Abstract Algebra*. Wiley, 2004. ISBN 9780471433347. URL <https://books.google.com/books?id=KIGbCgAAQBAJ>.
- [2] I.N. Herstein. *Topics in algebra*. Xerox College Pub., 1975. URL <https://books.google.com/books?id=lgfvAAAAMAAJ>.