

MA 572: Homework 4

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PROBLEM 4.1 (HATCHER §2.1, EX. 20)

Show that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ for all n , where SX is the suspension of X . More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Proof. First note that the reduced suspension of X , ΣX , which is homotopy equivalent to SX , can be realized as the quotient space CX/X . Given the imbedding $X \hookrightarrow CX$, by 2.16 and excision (or 2.22) we have the long exact sequence in

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \tilde{H}_n(X) & \longrightarrow & \tilde{H}_n(CX) & \longrightarrow & \tilde{H}_n(CX, X) & \longrightarrow & \cdots \\ & & & & & & \searrow & & \\ & & & & & & \tilde{H}_{n-1}(X) & \longrightarrow & \tilde{H}_{n-1}(CX) & \longrightarrow & \tilde{H}_{n-1}(CX, X) & \longrightarrow & \cdots \end{array} \quad (1)$$

where $\tilde{H}_n(CX, X) \cong \tilde{H}_n(CX/X) \cong \tilde{H}_n(SX)$. Since CX is contractible, we have $H_n(CX) = 0$ for all n and the long exact sequence (1) yields an isomorphism

$$\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X). \quad \blacksquare$$

Prove by induction on the dimension the following facts about the homology of a finite dimensional CW complex X , using the observation that X^n/X^{n-1} is a wedge sum of n -spheres:

- Proof.* To make Hatcher's comment more precise, if $X = \bigcup_{k=0}^n X^k$ is a n -dimensional CW-complex, by excision the relative homology

where α_k is the index over the k -dimensional cells of X .

Now for the inductive step, suppose $H_i(X^{n-1}) = 0$ for all $i > n - 1$. Then, we have the long exact sequence

But by the induction hypothesis, $H_i(X^{n-1}) = 0$ for all $i > n - 1$ and $H_{n-1}(X^{n-1}) = \bigoplus_{\alpha} \mathbf{Z}$ over some index α . Then (3) becomes

It's easy to see, if we extend (4) to the left, i.e., by adding H_{n+1} etc., that $H_i(X^n) = 0$ since it is sandwiched between two 0's. Moreover, by exactness at $H_n(X^n/X^{n-1})$, we have an injection $H_n(X^n) \hookrightarrow \bigoplus_{\alpha_n} \mathbf{Z}$ so $H_n(X^n)$ is free abelian.

2

Proof of observation. Since X is n -dimensional all we need to show is that $H_i(X^m) \cong H_i(X^{i+1})$. By induction on m , the base case $m = i + 1$ is clear. Now for the inductive step, suppose $H_i(X^{m'}) \cong H_i(X^{i+1})$ for $m > i$. Then we have

$$\cdots \longrightarrow H_{i+1}(X^{m+1}, X^m) \longrightarrow H_i(X^m) \longrightarrow H_i(X^{m+1}) \longrightarrow H_i(X^{m+1}, X^m) \longrightarrow \cdots \quad (5)$$

But $H_{i+1}(X^{m+1}, X^m) \cong H_{i+1}(X^{m+1}, X^m) \cong 0$ so $H_i(X^m) \cong H_i(X^{m+1})$, as desired. ♣

(b) If $n < 1$ in the case $n = 0$ there are no 1-cells and in the case $n = 1$ there are no 0-cells so we cannot say anything. Therefore, we begin at $n > 1$. By our observation, we have $H_n(X) \cong H_n(X^n)$ and by part **(a)** we have

$$\cdots \longrightarrow H_n(X^{n-2}) \longrightarrow H_n(X^n) \longrightarrow H_n(X^n, X^{n-2}) \longrightarrow H_{n-1}(X^{n-2}) \longrightarrow \cdots, \quad (6)$$

where $H_n(X^{n-2}) \cong H_{n-1}(X^{n-2}) \cong 0$. Hence, by exactness at $H_n(X^n)$, we have $H_n(X^n) \cong H_n(X^n, X^{n-2})$ so by 2.34 (a), we have

$$H_n(X^n) \cong H_n(X^n, X^{n-1}) \cong H_n(X^n, X^{n-2}) \cong \bigoplus_{\alpha} \mathbf{Z}$$

where α is an index over the n -cells of X .

(c) By the observation $H_n(X) \cong H_n(X^{n+1})$. Moreover, we have the following exact sequence

$$\cdots \longrightarrow H_n(X^{n+1}, X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) \longrightarrow \cdots \quad (7)$$

where $H_{n+1}(X^{n+1}, X^n) \cong H_n(X^{n+1}, X^n) \cong 0$ giving us an injection $H_n(X^n) \hookrightarrow H_n(X^{n+1})$. By part **(a)**, $H_n(X^n)$ is free abelian and has rank at most the number of n -cells of X . Thus, $H_n(X)$ is free abelian and has rank at most the number of n -cells of X . ■

PROBLEM 4.3 (HATCHER §2.2, EX. 2)

Given a map $f: S^{2n} \rightarrow S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbf{RP}^{2n} \rightarrow \mathbf{RP}^{2n}$ has a fixed point. Construct maps $\mathbf{RP}^{2n-1} \rightarrow \mathbf{RP}^{2n-1}$ without fixed points from linear transformations $\mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n}$ without eigenvectors.

Proof. Given a map $f: S^{2n} \rightarrow S^{2n}$, either f has a fixed point or it does not. If f does not have a fixed point then $f \simeq a$ so $\deg f = (-1)^{2n+1} = -1$ so $\deg(-f) = \deg(-\text{Id} \circ f) = 1$ has a fixed point for otherwise we may homotope f to the antipodal map a and so, by transitivity, $\text{Id} \simeq a$, but this is nonsense.

Realize \mathbf{RP}^{2n} as the quotient S^{2n} / \sim where $x \sim -x$. ■