MA 523: Homework 6

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Problem 6.1

For $n = 2$ find Green's function for the quadrant $U := \{ x_1, x_2 > 0 \}$ by repeated reflection.

Solution. Taking the hit, set $x' := (x_1, -x_2)$, $x'' := (-x_1, x_2)$, $x''' := (-x_1, -x_2)$, and define

$$\varphi^x(y) := \Phi(y - x') + \Phi(y - x'') - \Phi(y - x'''). \tag{6.1}$$

We claim that φ^x, as defined above, solves

$$\begin{cases} \Delta\varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(y - x) & \text{on } \partial U. \end{cases}$$

It is clear that $\Delta\varphi^x = 0$ since it is built up from the fundamental solutions on $\mathbb{R}^n$ (this follows from the linearity of the Laplace operator). To see that $\varphi^x(y) = \Phi(x - y)$ on ∂U, we do a case by case analysis.

Note that on $\{ x_1 = 0 \} \subset \partial U$, we have

$$\varphi^x(y_1, 0) = \Phi(-x_1, y_2 + x_2) + \Phi(-x_1, y_2 - x_2) - \Phi(x_1, y_2 + x_2),$$

where, since the fundamental solution is radial, we have $\Phi(-x_1, y_2 + x_2) = \Phi(x_1, y_2 + x_2)$, and hence the above equals

$$\begin{aligned} &= \Phi(-x_1, y_2 - x_2) \\ &= \Phi(y - x) \end{aligned}$$

and on $\{ x_2 = 0 \} \subset \partial U$, we have

$$\varphi^x(0, y_2) = \Phi(y_1 - x_1, x_2) + \Phi(y_1 + x_1, -x_2) - \Phi(y_1 + x_1, x_2)$$

where, again because Φ is radial, $\Phi(y_1 + x_1, -x_2) = \Phi(y_1 + x_1, x_2)$, thus the above equals

$$\begin{aligned} &= \Phi(y_1 - x_1, x_2) \\ &= \Phi(y - x). \end{aligned}$$

Thus, $\phi^x(y) = \Phi(y - x)$ on ∂U.

Therefore, Green's function on U is

$$G(x, y) = \Phi(y - x) - \varphi^x(y) = \Phi(y - x) - \Phi(y - x') - \Phi(y - x'') + \Phi(y - x'''). \qquad \blacksquare$$

Problem 6.2

(Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}}u(0) \le u(x) \le \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B(0,r) = \{\, x \in \mathbb{R}^n : |x| < r \,\}$.

Solution. Recall Poisson's formula for the ball

$$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y), \tag{6.2}$$

where $x \in B(0,r)$ and u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,r), \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

For fixed $x \in B(0,r)$, write

$$u(x) = r^{n-2}(r+|x|)(r-|x|)\left[\frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y)\right].$$

Now, since $r + |x| \ge |x-y| \ge r - |x|$ for all $y \in \partial B(0,r)$, we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} \fint_{\partial B(0,r)} g(y)\, dS(y) \le u(x) \le \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} \fint_{\partial B(0,r)} g(y)\, dS(y). \tag{6.3}$$

Since $u = g$ on the boundary $\partial B(0,r)$, by applying the mean-value property on (6.3) we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}}u(0) \le u(x) \le \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}}u(0),$$

as desired. ■

Problem 6.3

Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$\begin{cases} P_k(\lambda x) = \lambda^k P_k(x), \quad P_m(\lambda x) = \lambda^m P_m(x) & \text{for every } x \in \mathbb{R}^n,\ \lambda > 0, \\ \quad \Delta P_k = 0, \qquad\qquad \Delta P_m = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

(a) Show that

$$\begin{cases} \dfrac{\partial P_k}{\partial \nu} = kP_k(x), \quad \dfrac{\partial P_m}{\partial \nu} = mP_m(x) \quad \text{on } \partial B(0,1), \end{cases}$$

where $B(0,1) = \{\, x \in \mathbb{R}^n : |x| < 1 \,\}$ and ν is the outward normal on $\partial B(0,1)$.

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x)\, d\sigma = 0, \qquad \text{if } k \neq m.$$

Solution. For part (a), let

$$P_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha.$$

Then, since $\nu = (x_1, \ldots, x_n)$, the derivative along ν is given by

$$\begin{aligned} \frac{\partial P_k(x)}{\partial \nu} &= \sum_{i=1}^n (P_k)_{x_i} x_i \\ &= \sum_{i=1}^n \left(\sum\nolimits_{|\alpha|=k} a_\alpha x^\alpha \right)_{x_i} x_i \\ &= \sum_{i=1}^n \left(\sum\nolimits_{j=1}^m a_\alpha x_1^{\alpha_1^j} \cdots x^{\alpha_i^j} \cdots x^{\alpha_n^j} \right)_{x_i} x_i \end{aligned}$$

where $\sum_{i=1}^n \alpha_i^j = k$ and $1 \leq j \leq \binom{n+k-1}{n} =: m$ (by the stars and bars theorem)

$$\begin{aligned} &= \sum_{i=1}^n \sum_{j=1}^m \left(\alpha_i^j a_\alpha x_1^{\alpha_1^j} \cdots x^{\alpha_i^j - 1} \cdots x^{\alpha_n^j} \right) x_i \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i^j a_\alpha x_1^{\alpha_1^j} \cdots x^{\alpha_i^j} \cdots x^{\alpha_n^j} \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i^j a_\alpha x^\alpha \end{aligned}$$

switching the order of summation, we have

$$
\begin{aligned}
&= \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_i^j a_\alpha x^\alpha \\
&= \sum_{j=1}^{m} k a_\alpha x^\alpha \\
&= k \sum_{j=1}^{m} a_\alpha x^\alpha \\
&= k P_k(x).
\end{aligned}
$$

This argument, of course, applies to every $k \in \mathbb{N}$.

For part (b), by Green's theorem, we have

$$
\begin{aligned}
\int_{B(0,r)} P_k(x)\Delta P_m(x) - (\Delta P_k(x))P_m(x)\,dx &= \int_{\partial B(0,r)} P_k(x)\tfrac{\partial}{\partial \nu} P_m(x) - \tfrac{\partial}{\partial \nu} P_k(x) P_m(x)\,dS(x) \\
&= \int_{\partial B(0,r)} (m-k)P_k(x)P_m(x)\,dS(x),
\end{aligned}
$$

where the left-hand side is equal to zero since both ΔP_k and ΔP_m are zero. Since $m \neq k$, it must be the case that

$$\int_{\partial B(0,r)} P_k(x)P_m(x)\,dS(x) = 0.$$

$\blacksquare$