

MA557 Problem Set 3

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PROBLEM 3.1

Let R be a domain and Γ the set of all principal ideals in R . Show that R is a unique factorization domain if and only if Γ satisfies the ascending chain condition and every irreducible element of R is prime.

Proof. \Rightarrow Suppose that R is a UFD and let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ be an ascending chain of ideals in Γ . Then $\mathfrak{a}_1 = \langle a_1 \rangle$ for some $a_i \in R$ since every ideal belonging to Γ is principal. Now, since R is a UFD, a_1 factors uniquely (up to associates) as the finite product $a_1 = p_1 \cdots p_k$ of (not necessarily distinct) irreducible elements $p_1, \dots, p_k \in R$. If a_1 is irreducible we are done (because in such a case $a_2 \mid a_1$ if and only if $a_2 = ua_1$ where u is a unit, hence $\mathfrak{a}_1 = \mathfrak{a}_2 = \cdots$). Suppose a_1 is not irreducible. Then since $a_2 \mid a_1$, the irreducible factors of a_2 consist of some (or all) of the irreducible factors of a_1 (more precisely we can write $a_2 = p_{\sigma(1)} \cdots p_{\sigma(\ell)}$ for some injection $\sigma: \{1, \dots, \ell\} \hookrightarrow \{1, \dots, k\}$ where $\ell \leq k$). Inductively applying this argument to a_n for $n \geq 1$, we see that the process (of factoring a_n 's from a_1) must terminate for some positive r for otherwise we have that

$$a_1 = a_2 b_2 = (a_3 b_3) b_2 = \cdots = (a_n b_n) b_{n-1} \cdots b_2 = \cdots,$$

but every factorization of a_1 into irreducibles must have length k . Thus, the ascending chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_r = \mathfrak{a}_{r+1} = \cdots$ is stationary for some positive integer r and we say that Γ satisfies the acc.

\Leftarrow Conversely, suppose that Γ satisfies the acc. Let $a_1 \in R$. If a_1 is irreducible we are done. Suppose a_1 is reducible, then $a_1 = a_2 b_2$ for some non-units $a_2, b_{11} \in R$. If both a_2 and b_2 are irreducible, we are done. Without loss of generality we may assume a_2 is reducible (as the argument to follow may be applied to the b_i 's in case they are not irreducible). Then $a_2 = a_3 b_3$ (and so $a_1 = a_2 b_2 = (a_3 b_3) b_2$) for some non-units $a_3, b_3 \in R$. Then we get the ascending chain of principal ideals

$$\langle a_1 \rangle \subset \langle a_2 \rangle \subset \langle a_3 \rangle \subset \cdots$$

which must stabilize for some positive integer r since Γ satisfies the acc. This argument shows that there exists a factorization of a_1 into irreducibles. We must now prove that this factorization is unique (up to associates).

Suppose $a = p_1 \cdots p_k = q_1 \cdots q_\ell$ where $p_1, \dots, p_k, q_1, \dots, q_\ell \in R$ are irreducibles. ■

PROBLEM 3.2

Let M be an Artinian R -module. Show that every injective R -linear map $\varphi: M \rightarrow M$ is an isomorphism.

Proof. Suppose φ is not surjective. Then, there is some element $x \in M$ that is not in the image of φ . Now consider cokernels $\text{coker}(\varphi^n) = M/\text{im}(\varphi^n)$ ■

PROBLEM 3.3

Let M be a finitely generated Artinian module. Show that M is Noetherian.

Proof. We shall induct on n the number of generators of M .

Base case: Suppose that $M = Rx$. Then M is cyclic and $M \cong R/\text{ann}(x)$ is an Artinian R -module, hence an Artinian $R/\text{ann}(x)$ -module (if $N_0 \supset N_1 \supset \cdots$ is a descending chain in M as an $R/\text{ann}(x)$ -module, then it is a descending chain in M as an R -module, hence is stationary for some n). Then M is an Artinian ring. By (3.17) M is Noetherian hence, M is Noetherian as an R -module.

Induction step: Assume that every Artinian module with a minimal generating set of cardinality $\leq n$ is Noetherian. Let $M = Rx_1 + \cdots + Rx_{n+1}$. Then

$$0 \longrightarrow Rx_1 \hookrightarrow M = Rx_1 + \cdots + Rx_n \twoheadrightarrow M/Rx_1 \rightarrow 0$$

is exact. Since Rx_1 is Noetherian and M/Rx_1 is Noetherian (since its minimal generating set has cardinality $\leq n$), by (3.5) M is Noetherian. ■

PROBLEM 3.4

Let R be a ring that is Artinian or Noetherian, and $x \in R$. Show that for some $n > 0$, the image of x in $R/(\text{ann } x)^n$ is a nonzero-divisor on that ring.

Proof. By 3.7 it is sufficient to prove this for R Noetherian. ■

PROBLEM 3.5

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PROBLEM 3.6

Let R be an Artinian ring all of whose maximal ideals are principal. Show that every ideal in R is principal.

Proof.

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PROBLEM 3.7

Prove 2.13.

Proof. Recall the statement of 2.13:

