MA 572: Homework 5

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Problem 5.1 (Hatcher §2.2, Ex. 3)

Let $f\colon S^n \to S^n$ be a map of degree zero. Show that there exists points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbf{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D where F points radially outward and another point on ∂D^n where F points radially inward.

Proof. Since $\deg f = 0 \neq (-1)^n = \deg a$, then $f \not\simeq a$ and so must have a fixed point $x \in S^n$. Now, consider the map $g \coloneqq a \circ f$. Since $\deg g = \deg a \circ f = (\deg a)(\deg f) = 0$, g must have a fixed point $y \in S^n$. Since $g(y) = a \circ f(y) = y$, then $f(y) = -y$.

Suppose F is a continuous nonzero vector field on S^n, i.e., a map $S^n \to \mathbf{R}^n$ which assigns to each point $x \in S^n$ a tangent vector $\mathbf{v}(x)$ at x. Then, the map $f\colon \partial D^n \to \mathbf{R}^n$ given by $f(\mathbf{v}(x)) = \mathbf{v}(x)/\|\mathbf{v}(x)\|$ is well defined and nowhere zero. ■

Problem 5.2 (Hatcher §2.2, Ex. 7)

For an invertible linear transformation $f\colon \mathbf{R}^n \to \mathbf{R}^n$ show that the induced map $H_n(\mathbf{R}^n, \mathbf{R}^n \smallsetminus \{\mathbf{0}\}) \cong \widetilde{H}_{n-1}(\mathbf{R}^n \smallsetminus \{\mathbf{0}\}) \cong \mathbf{Z}$ is id or $-$ id according to whether the determinant of f is positive or negative. [Use Gaußian elimination to show that the matrix of f can be joined by a path of invertible matrices to a diagonal matrix with ± 1's on the diagonal.]

Proof. Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be an invertible linear map. Then $\det f \neq 0$. Following the hint, we aim to construct a chain homotopy $P\colon C_*(\mathbf{R}^n) \to C_*(\mathbf{R}^n)$ such that $\partial \circ P + P \circ \partial = f_\# + g_\#$ where $g_\#$ has a matrix representation with ± 1's in the diagonal. ∎

Problem 5.3 (Hatcher §2.2, Ex. 13)

Let X be the 2-complex obtained from S^1 with its usual cell structure by attaching two 2-cells by maps of degrees 2 and 3, respectively.

(a) Compute the homology groups of all the subcomplexes $A \subset X$ and the corresponding quotient complexes X/A.

(b) Show that $X \simeq S^2$ and that the only subcomplex $A \subset X$ for which the quotient map $X \to X/A$ is a homotopy equivalence is the trivial subcomplex, the 0-cell.

Proof. ■

Problem 5.4

Proof. ■