MA 544: Homework 1

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Problem 1.1 (Wheeden & Zygmund §2, Ex. 1)

Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0,1]$, but that $V[f;0,1] = +\infty$.

Proof. By properties of continuous functions, we have f is continuous on $(0,1]$ since it is the product of continuous functions on $(0,1]$. To see that f is continuous at 0 is suffices to show that $f(0+) = f(0) = 0$. To that end, let $\{x_n\} \subset [0,1]$ be a sequence such that $x_n \to 0$ and consider $\lim_{n\to\infty} f(x_n)$. Since $x_n \to 0$, for every $\varepsilon > 0$, there exists a natural number N such that $n \geq N$ implies $|0 - x_n| < \varepsilon$. Thus, for $n \geq N$ we have

$$|0 - f(x_n)| = |f(x_n)| = |x_n||\sin(1/x_n)| \leq \varepsilon|\sin(1/\varepsilon)| \leq \varepsilon.$$

Thus, $f(x_n) \to 0$ and we see that $f(0+) = 0$. Hence, f is continuous on $[0,1]$.

It is easy to see that f is bounded since $|\sin(1/x)| \leq 1$ for all $x \in (0,1]$. More explicitly, we have

$$|f(x)| \leq |x\sin(1/x)| = |x| \cdot |\sin(1/x)| \leq 1 \cdot 1.$$

Thus, $|f(x)| \leq 1$ and we see that f is bounded.

Moreover, f is continuous on $(0,1]$ since it is the product of continuous functions on $(0,1]$. To see that f is continuous at 0, it suffices to show that $f(0+) = 0$. To that end, we shall use the following limiting argument: Let $\varepsilon > 0$ and consider the limit (from the right) of $f(\varepsilon)$ as $\varepsilon \to 0$. This is

$$\lim_{\varepsilon\to 0} f(\varepsilon) \lim_{\varepsilon\to 0} \varepsilon\sin(1/\varepsilon) \leq \lim_{\varepsilon\to 0}|\varepsilon||\sin(1/\varepsilon)| \leq \lim_{\varepsilon\to 0}|\varepsilon| \cdot 1 = 0.$$

Thus, $f(0+) = 0$ and we see that f is continuous on $[0,1]$.

Last but not least, we show that f is BV. As of now, I've not been able to find the correct partition to show that the variation of f blows up, but the informal idea is the following: For any M, we (should be able to) find a partition Γ of $[0,1]$ such that $\sin(1/x_i) = 1$ for every $x_i \in \Gamma$, so we have

$$S_\Gamma = \sum_{i=0}^{m} x_i \sin(1/x_i) > M$$

and hence, $V[f;a,b] = +\infty$. I read online and they show, for the very same problem, that the partition $\Gamma := \{x_n\} = \left\{((2n+1)\pi/2)^{-1}\right\}$ as $n \to \infty$ gives $\int_0^1 f\ \mathrm{d}\phi \to \infty$, but I cannot see how Γ is a partition of the interval $[0,1]$. Anyway, it is fairly clear that if we have such a partition then

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{2}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi}{2}\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{2}{(2n+1)\pi} = +\infty,$$

by the ratio test since

$$\frac{2/((2n+1)\pi)}{2/((2(n+1)+1)\pi)} = \frac{2n+1}{2n+3} = \frac{n+1}{n+3/2}\cdots$$

which converges to 1 as $n \to \infty$. ■

Problem 1.2 (Wheeden & Zygmund §2, Ex. 2)

Prove theorem (2.1).

Proof. Recall the statement of theorem (2.1):

Theorem (Wheeden & Zygmund, 2.1). *(a) If f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$.*

(b) Let f and g be of bounded variation on $[a, b]$. Then cf (for any real constant c), $f + g$, and fg are of bounded variation on $[a, b]$. Moreover, f/g is of bounded variation on $[a, b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a, b]$.

(a) We shall proceed by contradiction. Suppose that f is not bounded, i.e., for every positive real number $M > 0$, there exists $x \in [a, b]$ such that $|f(x)| > M$. In particular, if V is the variation of f, then $|f(x_0)| > V + (f(a) + f(b))/2$ for some $x_0 \in [a, b]$. Then, putting $\Gamma = \{a, x_0, b\} \subset [a, b]$, we have

$$\begin{aligned} S_\Gamma &= |f(b) - f(x_0)| + |f(x_0) - f(a)| \\ &= |f(x_0) - f(b)| + |f(x_0) - f(a)| \\ &\geq |2f(x_0) - f(a) - f(b)| \\ &= |2(V + (f(a) + f(b))/2) - f(a) - f(b)| \\ &= |2V + f(a) + f(b) - f(a) - f(b)| \\ &= 2V \\ &> V. \end{aligned}$$

This is a contradiction since V is the supremum over all such sums.

(b) We shall prove these in the order in which they are listed above.

(i) The constant map $g(x) \coloneqq c$ for some real number c is of BV on $[a, b]$ and this is easy to see: take any two partitions $\Gamma = \{x_0, ..., x_m\}$, and $\Gamma' = \{y_0, ..., y_n\}$ of $[a, b]$, then

$$S_\Gamma = \sum_{i=0}^{m} |g(x_i) - g(x_{i-1})| = \sum_{i=0}^{m} |ct - c| = 0 = \sum_{i=0}^{m} |c - c| = \sum_{i=0}^{n} |g(y_i) - g(y_{i-1})| = S_{\Gamma'}.$$

It takes just a few more steps in logic to see that $V[g; a, b] = 0$. Therefore, by (iii) $gf = cf$ is of BV.

(ii) This result follows quite effortlessly from Jordan's theorem, so we shall not trouble ourselves with picking partitions. By Jordan's theorem, there exist bounded increasing functions f_1, f_2, and g_1, g_2 such that $f = f_1 - f_2$ and $g = g_1 - g_2$. Now, since since f_1, f_2, g_1, g_2 are bounded and increasing, the sums $h_1 = f_1 + g_1$ and $h_2 = f_2 + g_2$ are bounded and increasing. Thus,

$$f + g = f_1 - f_2 + g_1 - g_2 = (f_1 + g_1) - (f_2 + g_2) = h_1 - h_2,$$

so by Jordan's theorem $f + g$ is BV on $[a, b]$.

(iii) For this result, Jordan's theorem is not very helpful so we rely on the definition of BV. First, note that by the triangle inequality, for any $x < y$ in $[a, b]$, we have

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |(f(x)g(x) - f(x)g(y)) + (f(x)g(y) - f(y)g(y))| \\
&\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\
&\leq M|g(x) - g(y)| + N|f(x) - f(y)|, \qquad (1)
\end{aligned}$$

by part (a), where $|f| \leq M$ and $|g| \leq M$ for all $x \in [a, b]$. By (1), it follows that for any partition Γ of $[a, b]$, we have

$$S_\Gamma[fg; a, b] \leq MS_\Gamma[g; a, b] + NS_\Gamma[f; a, b].$$

Thus, passing to the supremum, we see that

$$V[fg; a, b] \leq MV[g; a, b] + NV[f; a, b] < +\infty,$$

so fg is BV on $[a, b]$.

(iv) Suppose $|g(x)| > \varepsilon$ for some $\varepsilon > 0$ for all $x \in [a, b]$. Then, by the triangle inequality, the following estimate holds

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right| &\leq \left|\frac{g(y)f(x) - g(x)f(y)}{g(x)g(y)}\right| \\
&= \frac{1}{|g(x)g(y)|}|g(y)f(x) - g(x)f(y)| \\
&< \frac{1}{\varepsilon^2}|g(y)f(x) - g(x)f(y)| \\
&< \frac{1}{\varepsilon^2}|g(y)f(x) - g(y)f(y) + g(y)f(y) - g(x)f(y)| \\
&= \frac{1}{\varepsilon^2}|(g(y)f(x) - g(y)f(y)) - (g(x)f(y) - g(y)f(y))| \\
&\leq \frac{1}{\varepsilon^2}(|g(y)||f(x) - f(y)| + |f(y)||g(x) - g(y)|) \\
&\leq \frac{1}{\varepsilon^2}(|g(y)||f(x) - f(y)| + |f(y)|(|g(x)| - |g(y)|)) \\
&\leq \frac{1}{\varepsilon^2}(N|f(x) - f(y)| + M|g(x) - g(y)|). \qquad (2)
\end{aligned}$$

Hence, for any partition Γ of $[a, b]$, we have

$$S_\Gamma[f/g; a, b] \leq \frac{1}{\varepsilon^2}(NS_\Gamma[f; a, b] + MS_\Gamma[g; a, b]).$$

Thus, passing to the supremum, we see that

$$V[f/g; a, b] \leq \frac{1}{\varepsilon^2}(NV[f; a, b] + MV[g; a, b]) < +\infty,$$

so f/g is BV on $[a, b]$.

■

Problem 1.3 (Wheeden & Zygmund §2, Ex. 3)

If $[a', b']$ is a subinterval of $[a, b]$ show that $P[a', b'] \leq P[a, b]$ and $N[a', b'] \leq N[a, b]$.

Proof. Let $f\colon [a, b] \to \mathbf{R}$. If f is unbounded, then $V[f; a, b] = +\infty$ and, by theorem 2.6, the result holds trivially.

Suppose f is BV on $[a, b]$. Then $V[f; a, b] < +\infty$. Hence, by theorem 2.2, we have

$$V[f; a', b'] \leq V[f; a, b]. \tag{3}$$

By theorem 2.6, we have

$$N[f; a', b'] = \tfrac{1}{2}(V[f; a', b'] + f(b') - f(a')) \qquad P[f; a', b'] = \tfrac{1}{2}(V[f; a', b'] - f(b') + f(a'))$$

which, by theorem 2.2, are bounded by

$$\begin{aligned} &\leq \tfrac{1}{2}(V[f; a, b] - f(b) + f(a)) \\ &= N[f; a, b] \end{aligned} \qquad \begin{aligned} &\leq \tfrac{1}{2}(V[f; a, b] - f(b) + f(a)) \\ &= P[f; a, b], \end{aligned}$$

as desired. ∎

Problem 1.4 (Wheeden & Zygmund §2, Ex. 11)

Show that $\int_a^b f \, \mathrm{d}\phi$ exists if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|, |\Gamma'| < \delta$.

Proof. $\Longrightarrow$ Suppose that $I \coloneqq \int_a^b f \, \mathrm{d}\phi$ exists. Then, for every $\varepsilon > 0$ there exits $\delta > 0$ such that for any partition Γ'' of $[a, b]$ with $|\Gamma''| < \delta/2$, $|I - R_{\Gamma''}| < \varepsilon$. Let Γ and Γ' be a partitions with $|\Gamma|, |\Gamma'| < \delta/2$. Then, for the given ε, we have $|I - R_\Gamma| < \varepsilon$ and $|I - R_{\Gamma'}| < \varepsilon$ from which we have the estimates

$$\begin{aligned}
|R_\Gamma - R_{\Gamma'}| &= |-(I - R_\Gamma) + (I - R_{\Gamma'})| \\
&\leq |-(I - R_\Gamma)| + |I - R_{\Gamma'}| \\
&= |I - R_\Gamma| + |I - R_{\Gamma'}| \\
&\leq \delta/2 + \delta/2 \\
&= \delta,
\end{aligned}$$

as desired.

$\Longleftarrow$ Conversely, suppose that given $\varepsilon > 0$ there exists $\delta > 0$ such that for any two partitions Γ, Γ' with $|\Gamma|, |\Gamma'| < \delta$ we have $|R_\Gamma - R_{\Gamma'}| < \varepsilon/2$. Put $I \coloneqq \int_a^b f \, \mathrm{d}\phi$. Let $\Gamma'' \coloneqq \Gamma \cup \Gamma'$. By Lemma 2.23, we have the following estimates on $R_{\Gamma''}$

$$L_\Gamma \leq L_{\Gamma''} \leq R_{\Gamma''} \leq U_{\Gamma''} \leq U_\Gamma.$$

Since $L_{\Gamma''} - L_\Gamma < \varepsilon$ and $U_{\Gamma''} - L_\Gamma < \varepsilon$, then

$$R_{\Gamma''} - U_{\Gamma''} \leq U_\Gamma - U_{\Gamma''} \leq \varepsilon$$

and

$$\varepsilon \geq L_{\Gamma''} - L_\Gamma \geq L_{\Gamma''} - R_{\Gamma''} L_{\Gamma''}$$

it follows that $R_{\Gamma''} \to I$ as $|\Gamma''| \to 0$... No, this is not clear at all. ∎

Problem 1.5 (Wheeden & Zygmund §2, Ex. 13)

Prove theorem (2.16).

Proof.

Theorem (Wheeden & Zygmund, 2.16). *(i) If $\int_a^b f \, \mathrm{d}\phi$ exists, then so do $\int_a^b cf \, \mathrm{d}\phi$ and $\int_a^b f \, \mathrm{d}(c\phi)$ for any constant c, and*

$$\int_a^b cf \, \mathrm{d}\phi = \int_a^b f \, \mathrm{d}(c\phi) = c\int_a^b f \, \mathrm{d}\phi.$$

(ii) If $\int_a^b f_1 \, \mathrm{d}\phi$ and $\int_a^b f_2 \, \mathrm{d}\phi$ both exist, so does $\int_a^b (f_1 + f_2) \, \mathrm{d}\phi$, and

$$\int_a^b (f_1 + f_2) \, \mathrm{d}\phi = \int_a^b f_1 \, \mathrm{d}\phi + \int_a^b f_2 \, \mathrm{d}\phi.$$

(iii) If $\int_a^b f \, \mathrm{d}\phi_1$ and $\int_a^b f \, \mathrm{d}\phi_2$ both exist, so does $\int_a^b f \, \mathrm{d}(\phi_1 + \phi_2)$, and

$$\int_a^b f \, \mathrm{d}(\phi_1 + \phi_2) = \int_a^b f \, \mathrm{d}\phi_1 + \int_a^b f \, \mathrm{d}\phi_2.$$

(i) Suppose that $I := \int_a^b f \, \mathrm{d}\phi$ exists and let c be a constant. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\Gamma| < \delta$ implies $|I - R_\Gamma| < \varepsilon/|c|$. Then, we have

$$R_\Gamma[cf; a, b] = \sum_{i=1}^n cf(\xi_i)[\phi(x_i) - \phi(x_{i-1})] = c\left(\sum_{i=1}^n f(\xi_i)[\phi(x_i) - \phi(x_{i-1})]\right) = cR_\Gamma \tag{4}$$

and

$$R_\Gamma[f; c\phi; a, b] = \sum_{i=1}^n f(\xi_i)[c\phi(x_i) - c\phi(x_{i-1})] = c\left(\sum_{i=1}^n f(\xi_i)[\phi(x_i) - \phi(x_{i-1})]\right) = cR_\Gamma \tag{5}$$

for $\Gamma = \{x_0, ..., x_n\}$.[1] Hence, we have the estimates

$$\begin{aligned} |cI - R_\Gamma[cf; a, b]| &= |cI - cR_\Gamma| \\ &= |c(I - R_\Gamma)| \\ &= |c||I - R_\Gamma| \\ &\leq |c|(\varepsilon/|c|) \\ &= \varepsilon \end{aligned}$$

for δ as given. A similar argument (in fact, the same) works for $R[f; c\phi; a, b]$. Thus, we have

$$\int_a^b cf \, \mathrm{d}\phi = \int_a^b f \, \mathrm{d}(c\phi) = c\int_a^b f \, \mathrm{d}\phi,$$

[1] The $R_\Gamma[f; c\phi; a, b]$ is just made up notation. I can't think of what else to call it.

as desired.

(ii) Suppose that $I_1 \coloneqq \int_a^b f_1 \, \mathrm{d}\phi$ and $I_2 \coloneqq \int_a^b f_2 \, \mathrm{d}\phi$ exits. Then, for every $\varepsilon > 0$ there exists δ such that if Γ is a partition of $[a,b]$ with $|\Gamma| < \delta$ then $|I_1 - R_\Gamma[f_1; a, b]| < \varepsilon/2$ and $|I_2 - R_\Gamma[f_2; a, b]| < \varepsilon/2$. Now, note that

$$\begin{aligned}
R_\Gamma[f_1 + f_2; a, b] &= \sum_{i=0}^{m} (f_1(\xi_i) + f_2(\xi_i))[\phi(x_i) - \phi(x_{i-1})] \\
&= \sum_{i=0}^{m} (f_1(\xi_i)[\phi(x_i) - \phi(x_{i-1})] + f_2(\xi_i)[\phi(x_i) - \phi(x_{i-1})]) \\
&= \sum_{i=0}^{m} f_1(\xi_i)[\phi(x_i) - \phi(x_{i-1})] + \sum_{i=0}^{m} f_2(\xi_i)[\phi(x_i) - \phi(x_{i-1})] \\
&= R_\Gamma[f_1; a, b] + R_\Gamma[f_2; a, b]. \qquad (6)
\end{aligned}$$

Thus, by (6), we have the following estimates

$$\begin{aligned}
|(I_1 + I_2) - R_\Gamma[f_1 + f_2; a, b]| &= |(I_1 + I_2) - R_\Gamma[f_1 + f_2; a, b]| \\
&= |(I_1 + I_2) - (R_\Gamma[f_1; a, b] + R_\Gamma[f_2; a, b])| \\
&= |(I_1 - R_\Gamma[f_1; a, b]) + (I_2 - R_\Gamma[f_2; a, b])|
\end{aligned}$$

which, by the triangle inequality, is

$$\begin{aligned}
&\leq |(I_1 - R_\Gamma[f_1; a, b])| + |(I_2 - R_\Gamma[f_2; a, b])| \\
&\leq \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}$$

or δ as given. Thus, $\int_a^b f_1 + f_2 \, \mathrm{d}\phi$ exists and is equal to $\int_a^b f_1 \, \mathrm{d}\phi + \int_a^b f_2 \, \mathrm{d}\phi$.

(iii) Suppose $I_1 \coloneqq \int_a^b f \, \mathrm{d}\phi_1$ and $I_2 \coloneqq \int_a^b f \, \mathrm{d}\phi_2$ exist then for every $\varepsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that for every partition Γ_1, Γ_2 of $[a,b]$ with $|\Gamma_1| < \delta_1$ and $|\Gamma_2| < \delta_2$ we have $|I_1 - R_{\Gamma_1}[f; \phi_1; a, b]| < \varepsilon/2$ and $|I_2 - R_{\Gamma_2}[f; \phi_2; a, b]| < \varepsilon/2$. Put $\delta \coloneqq \min\{\delta_1, \delta_2\}$. Now, note that

$$\begin{aligned}
R_\Gamma[f; \phi_1 + \phi_2; a, b] &= \sum_{i=0}^{m} f[(\phi_1(x_i) + \phi_2(x_i)) - (\phi_1(x_{i-1}) + \phi_2(x_{i-1}))] \\
&= \sum_{i=0}^{m} f[(\phi_1(x_i) - \phi_1(x_{i-1})) + (\phi_2(x_i) - \phi_2(x_{i-1}))] \\
&= \sum_{i=0}^{m} f[(\phi_1(x_i) - \phi_1(x_{i-1}))] + \sum_{i=0}^{m} f[(\phi_1(x_i) - \phi_1(x_{i-1}))] \\
&= R_\Gamma[f; \phi_1; a, b] + R_\Gamma[f; \phi_2; a, b]. \qquad (7)
\end{aligned}$$

Hence, we have the following estimates

$$\begin{aligned}
|(I_1 + I_2) - R_\Gamma[f; \phi_1 + \phi_2; a, b]| &= |(I_1 + I_2) - (R_\Gamma[f; \phi_1; a, b] + R_\Gamma[f; \phi_2; a, b])| \\
&= |(I_1 - R_\Gamma[f; \phi_1; a, b]) + (I_2 - R_\Gamma[f; \phi_2; a, b])|
\end{aligned}$$

which, by the triangle inequality, is

$$
\begin{aligned}
&\le |I_1 - R_\Gamma[f;\phi_1;a,b]| + |I_2 - R_\Gamma[f;\phi_2;a,b]| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}
$$

Thus, $\int_a^b f \, \mathrm{d}(\phi_1 + \phi_2)$ exists and it is equal to the sum $\int_a^b f \, \mathrm{d}\phi_1 + \int_a^b f \, \mathrm{d}\phi_2$. ■