

## MA 572: Homework 5

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**PROBLEM 5.1 (HATCHER §2.2, EX. 3)**

Let  $f: S^n \rightarrow S^n$  be a map of degree zero. Show that there exists points  $x, y \in S^n$  with  $f(x) = x$  and  $f(y) = -y$ . Use this to show that if  $F$  is a continuous vector field defined on the unit ball  $D^n$  in  $\mathbf{R}^n$  such that  $F(x) \neq 0$  for all  $x$ , then there exists a point on  $\partial D$  where  $F$  points radially outward and another point on  $\partial D^n$  where  $F$  points radially inward.

*Proof.* Since  $\deg f = 0 \neq (-1)^n = \deg a$ , then  $f \not\approx a$  and so must have a fixed point  $x \in S^n$ . Now, consider the map  $g := a \circ f$ . Since  $\deg g = \deg a \circ f = (\deg a)(\deg f) = 0$ ,  $g$  must have a fixed point  $y \in S^n$ . Since  $g(y) = a \circ f(y) = y$ , then  $f(y) = -y$ .

Suppose  $F$  is a continuous nonzero vector field on  $S^n$ , i.e., a map  $S^n \rightarrow \mathbf{R}^n$  which assigns to each point  $x \in S^n$  a tangent vector  $\mathbf{v}(x)$  at  $x$ . Then, the map  $f: \partial D^n \rightarrow \mathbf{R}^n$  given by  $f(\mathbf{v}(x)) = \mathbf{v}(x)/\|\mathbf{v}(x)\|$  is well defined and nowhere zero. ■

**PROBLEM 5.2 (HATCHER §2.2, EX. 7)**

For an invertible linear transformation  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  show that the induced map  $H_n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\}) \cong \tilde{H}_{n-1}(\mathbf{R}^n \setminus \{0\}) \cong \mathbf{Z}$  is  $\text{Id}$  or  $-\text{Id}$  according to whether the determinant of  $f$  is positive or negative. [Use Gaußian elimination to show that the matrix of  $f$  can be joined by a path of invertible matrices to a diagonal matrix with  $\pm 1$ 's on the diagonal.]

*Proof.* We show that  $O_n(\mathbf{R})$  is a deformation retraction of  $\text{GL}_n(\mathbf{R})$  and prove the result there. This procedure is adapted from a hint in *Элементарная топология* by Виро, Нецветаев и Харламов, стр. 338, номер 39.11. Suppose  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is an invertible linear transformation. Let  $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$  be the set of columns vectors of the matrix representation  $F$  of  $f$ . By Gram–Schmidt orthogonalization construct the vectors

$$\begin{aligned} \mathbf{e}_1 &:= \lambda_{11}\mathbf{f}_1 \\ \mathbf{e}_2 &:= \lambda_{21}\mathbf{f}_1 + \lambda_{22}\mathbf{f}_2 \\ &\vdots \\ \mathbf{e}_n &:= \lambda_{n1}\mathbf{f}_1 + \dots + \lambda_{nn}\mathbf{f}_n \end{aligned} \tag{5.1}$$

where the  $\lambda_{kk} > 0$  for  $k = 1, \dots, n$ . Now set

$$\mathbf{g}_k(t) := t(\lambda_{n1}\mathbf{f}_1 + \lambda_{n2}\mathbf{f}_2 + \dots + \lambda_{kk-1}\mathbf{f}_{k-1}) + (t\lambda_{kk} + 1 - t)\mathbf{f}_k. \tag{5.2}$$

Let  $g(t, A)$  be the matrix whose columns are the vectors  $\mathbf{g}_k(t)$  and define a homotopy  $f_t: I \times \text{GL}_n(\mathbf{R}) \rightarrow \text{GL}_n(\mathbf{R})$  by mapping the pair  $(t, A) \mapsto g(t, A)$ . Continuity of  $H$  follows from the fact that  $H$  is multiplication in  $\mathbf{R}^n$  followed by a linear mapping. It's not hard to see that  $f_t$  stays in  $\text{GL}_n(\mathbf{R})$  for all  $t$  and  $f_1(A)$  is in  $O_n(\mathbf{R})$ .

Last but not least, we show that  $O_n(\mathbf{R})$  consists of two connected components and that membership of  $f$  to one of these components is determined by  $\det f$ . First note that  $\det(O_n(\mathbf{R})) = \{-1, 1\}$  which is disconnected in  $\mathbf{R}$ . Hence,  $O_n(\mathbf{R})$  is disconnected. Now, if  $f \in O_n(\mathbf{R})$ , either  $\det f = 1$  or  $\det f = -1$ . Without loss of generality, we may assume  $\det f = 1$  since if  $r$  is a reflection.

Constructing the homotopy is hard. I can't think of a way of doing it and I don't have the time right now, so I'll skip this part. There are other ways to prove this indirectly, but I'm afraid I'm not familiar with Lie groups and I am not willing to state a bunch of results from that subject.

Now that we have established that either  $f \simeq \text{Id}$  or  $-\text{Id}$ , the map  $f$  on  $\mathbf{R}^n$  induces a map  $f_* = \pm \text{Id}_*$  on the homology groups  $H_n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\})$ . ■

**PROBLEM 5.3 (HATCHER §2.2, EX. 13)**

Let  $X$  be the 2-complex obtained from  $S^1$  with its usual cell structure by attaching two 2-cells by maps of degrees 2 and 3, respectively.

- (a) Compute the homology groups of all the subcomplexes  $A \subset X$  and the corresponding quotient complexes  $X/A$ .
- (b) Show that  $X \simeq S^2$  and that the only subcomplex  $A \subset X$  for which the quotient map  $X \rightarrow X/A$  is a homotopy equivalence is the trivial subcomplex, the 0-cell.

*Proof.* (a) Write  $X$  as the union  $e^0 \cup e^1$  of a 0-cell and a 1-cell. Let  $e_1^2, e_2^2$  be 2-cells attached to  $X$  via maps  $f_1, f_2: S^2 \rightarrow X$  of degrees 2 and 3, respectively. We use Lemma 2.34 to compute the cellular homology of the new CW complex  $X'$ , it then follows from Theorem 2.35 that the cellular homology is isomorphic to the singular homology of  $X'$ . First, we write down the cellular chain complex for  $X' = X^2$

$$\cdots \longrightarrow H_3^{\text{CW}}(X^3) \longrightarrow H_2^{\text{CW}}(X^2) \longrightarrow H_1^{\text{CW}}(X^1) \longrightarrow H_0^{\text{CW}}(X^0) \longrightarrow 0. \quad (5.3)$$

Filling in some of the values for  $H_n^{\text{CW}}(X^n)$  we have the chain

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z} \longrightarrow 0. \quad (5.4)$$

Now recall that by definition a subcomplex of  $X'$ ,  $A$ , is a closed subspace that is the union of cells in  $X$ . Since we have the following inclusion  $e^0 \subset e^1 \subset e_1^2, e_2^2$  this makes for the following candidates  $A_0 := e^0$ ,  $A_1 := e^0 \cup e^1$ ,  $A_{12} := e^0 \cup e^1 \cup e_1^2$ ,  $A_{22} := e^0 \cup e^1 \cup e_2^2$ ,  $X'$ . Let's compute the homology of these spaces.

- Case  $A_0$ : The cellular homology of  $A_0$  is easy enough since it is a 0-cell. It's homology will be that of a point  $H_n^{\text{CW}}(A_0) = \mathbf{Z}$  for  $n = 0$  and  $H_n^{\text{CW}}(A_0) = 0$  otherwise.
- Case  $A_1$ : The subcomplex  $A_1$  is homeomorphic to a circle  $S^1$  so its cellular homology is isomorphic to that of a circle, i.e.,  $H_n^{\text{CW}}(A_1) = \mathbf{Z}$  if  $n = 0, 1$  and  $H_n^{\text{CW}}(A_1) = 0$  otherwise.
- Case  $A_{21}$ : The cellular homology of  $A_{21}$  is more interesting since we have the attaching map of degree 2. This map  $f_1$  induces a map on homology  $f_{1*} = 2$  giving us the chain complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{Z} \xrightarrow{2} \mathbf{Z} \longrightarrow \mathbf{Z} \longrightarrow 0 \quad (5.5)$$

- Case  $A_{22}$ :
- Case  $X$ :

That concludes this part of the problem.

(b) ■

**PROBLEM 5.4**

*Proof.*

