MA553 Past Qualifying Examinations

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1 Spring 2008

Problem 1.1. Let $(G, \cdot)$ be a group, $(H, +)$ be an Abelian group, and $\varphi\colon G \to H$ be a group homomorphism. If N is a subgroup such that $\ker\varphi < N < G$, show that $N \lhd G$ is a normal subgroup.

Proof. Let N be a subgroup of G containing $\ker\varphi$. Then we must show that for any $g \in G$, $gNg^{-1} \subset N$. First we observe that, since $\ker\varphi \lhd G$, then $\ker\varphi \lhd N$ since for any $g \in N$, g is also in G so that $g(\ker\varphi)g^{-1} = \ker\varphi \subset N$. Thus, $\ker\varphi \lhd N$. By the first isomorphism theorem[1], $G/\ker\varphi \cong H$ hence, $G/\ker\varphi$ is Abelian. Moreover, $N/\ker\varphi < G/\ker\varphi$ hence, $N/\ker\varphi \lhd G/\ker\varphi$. It follows immediately from the lattice isomorphism theorem[2] (this is essentially the UMP of the quotient by a group) that $N \lhd G$. ■

Problem 1.2. Let $(G, \cdot)$ be a finite Abelian group of even order, i.e., $|G| = 2k$ for some $k \in \mathbf{N}$.

(a) For k odd, show that G has exactly one element of order 2.
(b) Does the same happen for k even? Prove or give a counterexample.

Proof. (a) This problem can be solved immediately by the fundamental theorem of finitely generated Abelian groups, i.e., if G is Abelian of finite order, then

$$G \cong (\mathbf{Z}/n_1\mathbf{Z}) \times \cdots \times (\mathbf{Z}/n_s\mathbf{Z})$$

for some positive integers $n_1, ..., n_s$ satisfying some conditions[3], the most important of which are, (i) $|G| = n_1 \cdots n_s$ and $n_{i+1} \mid n_i$. Now, consider the following

(b) ■

Problem 1.3.

Proof. ■

Problem 1.4.

Proof. ■

Problem 1.5.

Proof. ■

Problem 1.6.

Proof. ■

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[1]Theorem 16 of D. & F. §3, p. 99.
[2]Theorem 20 of D. & F. §3, p. 99.
[3]You can check Theorem 3 of D. & F. §5, p. 158 to see what exactly those conditions are.

2 August, 2015

Problem 2.1.

Proof. ■

2.1 August 2010