MA52300 FALL 2016

Homework Assignment 2 – *Solutions*

1. Verify assertion (36) in [E,§3.2.3], that when Γ is not flat near x^0, the noncharacteristic condition is

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Proof. Let Γ be given as the graph $x_n = \gamma(x_1, \ldots, x_{n-1})$ in a neighborhood U of x^0. Consider then the invertible mapping $\Phi = (\Phi^1, \Phi^2, \ldots, \Phi^n) : U \to V \subset \mathbb{R}^n$ given by

$$\begin{aligned} y_i &= \Phi^i(x) = x_i, \quad i = 1, \ldots, n-1, \\ y_n &= \Phi^n(x) = x_n - \gamma(x_1, \ldots, x_{n-1}). \end{aligned}$$

Then $\Phi(\Gamma) \subset \{y_n = 0\}$, or, equivalently, $\Gamma \subset \{\Phi^n = 0\}$. Hence the normal ν to Γ is parallel to $D_x\Phi^n = (-\gamma_{x_1}, \ldots, -\gamma_{x_{n-1}}, 1)$. As a consequence, the condition $D_pF \cdot \nu \neq 0$ is equivalent to $D_pF \cdot D_x\Phi^n \neq 0$.

Let now $v : V \to \mathbb{R}$ be defined by $v(y) = u(\Phi^{-1}(y))$. Then, we have the identity

$$u(x) = v(\Phi(x)).$$

Apllying the chain rule, we have

$$D_{x_i}u = \sum_{j=1}^{n} D_{y_j}v D_{x_i}\Phi^j, \quad i = 1, \ldots, n, \quad \text{or} \quad D_xu = D_yvD_x\Phi.$$

Now, the equation $F(D_xu, u, x) = 0$ becomes

$$\tilde{F}(D_yv, v, y) = F(D_yvD_x\Phi, v, \Phi^{-1}(y)) = 0.$$

We next note that

$$\begin{aligned} D_{p_n}\tilde{F} &= (D_{p_1}F)(D_{x_1}\Phi^n) + (D_{p_2}F)(D_{x_2}\Phi^n) + \cdots + (D_{p_n}F)(D_{x_n}\Phi^n) \\ &= D_pF \cdot D_x\Phi^n. \end{aligned}$$

Therefore, if (p^0, z^0, x^0) is a compatible triple for F and $(q^0, z^0, x^0) = (p^0D_x\Phi(x^0)^{-1}, z^0, \Phi(x^0))$ is the corresponding compatible triple for $\tilde{F}$, then

$$D_{p_n}\tilde{F}(q^0, z^0, y^0) = D_pF(p^0, z^0, x^0) \cdot D_x\Phi^n(x^0).$$

Hence, the noncharacteristic condition becomes $D_pF(p^0, z^0, x^0) \cdot D_x\Phi^n(x^0) \neq 0$, or, as noted above $D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$. $\square$

2. Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. The characteristic ODE for this equation is

$$\dot{x} = a(z), \quad \dot{t} = 1, \quad \dot{z} = 0.$$

with initial conditions

$$x(0) = x^0, \quad t(0) = 0, \quad z(0) = z^0 := g(x^0)$$

Integrating, we obtain

$$t(s) = s, \quad z(s) = z^0, \quad x(s) = x^0 + a(z^0)s.$$

Thus, $u = g(x^0) = \mathit{const}$ on projected characteristics

$$x = x^0 + a(g(x^0))t.$$

Expressing x^0 in terms of x, t and u, we obtain

$$x^0 = x - a(u)t \Rightarrow u = g(x - a(u)t).$$

Consider now a different initial point $(\bar{x}^0, 0)$ with $\bar{x}^0 < x^0$. Then we must have $u = g(\bar{x}^0)$ on

$$x = \bar{x}^0 + a(g(\bar{x}^0))t.$$

Now, if $a(g(\bar{x}^0)) > a(g(x^0))$ the two characteristics will cross for some $t > 0$. Thus we cannot have a continuous solution up to that time. On the other hand if $a(g(\bar{x}^0)) < a(g(x^0))$ for any $\bar{x}^0 < x^0$, no two projected characteristics will cross for $t > 0$ and this method will give us a solution in the whole halfspace $t > 0$. $\square$

3. Show that the function $u(x, t)$ defined for $t \geq 0$ by

$$u(x,t) = \begin{cases} -\frac{2}{3}(t + \sqrt{3x + t^2}) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burger's equation*).

Solution. We first check that u is a weak solution. Let

$$\Omega_l = \{(x,t) : 4x + t^2 < 0, t > 0\} \quad \Omega_r = \{(x,t) : 4x + t^2 > 0, t > 0\}.$$

Then $u = u_l = 0$ in Ω_l and $u = u_r = -\frac{2}{3}(t + \sqrt{3x + t^2})$ in Ω_r. Obviously, u_l satisfies Burger's equation in Ω_l. As for u_r, observe that $3x + t^2 =$

$\frac{3}{4}(4x+t^2)+\frac{1}{4}t^2>0$ in Ω_r; thus, u_r is smooth in Ω_r and Burger's equation can be verified by a straightforward differentiation:

$$u_t = -\frac{2}{3}\left(1+\frac{t}{\sqrt{3x+t^2}}\right)$$
$$\left(\frac{u^2}{2}\right)_x = \frac{2(t+\sqrt{3x+t^2})}{3\sqrt{3x+t^2}} = -u_t.$$

So, it is left to verify that the Rankine-Hugoniot condition is verified on the interface γ between Ω_l and Ω_r. We have

$$\gamma : x = s(t) = -\frac{t^2}{4}, \quad \sigma = \dot{s} = -\frac{t}{2}$$
$$[[u]] = u_l - u_r = -u_r = \tfrac{2}{3}(t+\sqrt{-\tfrac{3}{4}t^2+t^2}) = t$$
$$[[F(u)]] = F(u_l) - F(u_r) = -\frac{u_r^2}{2} = -\frac{t^2}{2}$$
$$\frac{[[F(u)]]}{[[u]]} = -\frac{t}{2} = \sigma.$$

Finally, to check that u is an entropy solution, we note that $F(u) = u^2/2$ is strictly convex and therefore we need to verify that $u_l > u_r$. But the latter is obvious, since $u_l - u_r = t > 0$. $\square$