

MA557 Problem Set 6

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PROBLEM 6.1

For an n by n matrix φ with entries in R write $I_t(\varphi)$ for the R -ideal generated by all the t by t minors of φ (set $I_t(\varphi) = R$ for $t \leq 0$ and $I_t(\varphi) = 0$ for $t > \min\{m, n\}$). Thinking of φ as an R -linear map $\varphi: R^m \rightarrow R^n$ set $M = \text{coker}(\varphi)$ and define $F_i(M) = \text{Fitt}_i(M) = I_{n-i}(\varphi)$. This ideal is called the i th Fitting ideal of M . Show:

- (a) $F_i(M)$ only depends on i and M (but not on m, n, φ).
- (b) $(\text{ann}(M))^n \subset F_0(M) \subset \text{ann}(M)$.
- (c) In case R is local, $F_i(M) = R$ if and only if $\mu(M) \leq i$.
- (d) $V(F_i(M)) = \{ \mathfrak{p} \in \text{Spec}(R) \mid \mu_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > i \}$.

Proof.

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PROBLEM 6.2

Let I be an ideal in a Noetherian ring. Show that either I contains an R -regular element or else $aI = 0$ for some $0 \neq a \in R$.

Proof.

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PROBLEM 6.3

Let $I \subset J$ be ideals in a Noetherian ring. Show that if $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ for every associated prime \mathfrak{p} of I , then $I = J$.

Proof.

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PROBLEM 6.4

Let R be a Noetherian ring and M a finite R -module. Show that $\ell(M) < \infty$ if and only if $\text{Supp}(M) \subset \mathfrak{m}\text{-Spec}(R)$.

Proof.

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PROBLEM 6.5

Let R be a Noetherian ring, $M \neq 0$ a finite R -module, and

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

a chain of submodules with $M_i/M_{i-1} \cong R/\mathfrak{p}_i$, $\mathfrak{p}_i \in \text{Spec}(R)$.

- (a) Show that $\text{Ass}(M) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and that the minimal elements of the two sets coincide (hence only depend on M).
- (b) Let \mathfrak{p} be minimal in $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Show that in any chain as above, the multiplicity with which the factor R/\mathfrak{p} appears is $\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ (hence only depends on M).

Proof.

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PROBLEM 6.6

Let $R = k[X, Y]$ be a polynomial ring over a field and $I = (X^2, XY) \subset R$. Find two distinct shortest primary decompositions of I .

Proof.

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