MA557 Homework 8

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November 9, 2015

PROBLEM 8.1

Let R be a ring such that $R_{\mathfrak{m}}$ is Noetherian for every $\mathfrak{m} \in \mathfrak{m}\text{-Spec}\, R$ and for every $f \in R$, $f \neq 0$, there exists at most finitely many $\mathfrak{m} \in \mathfrak{m}\text{-Spec}\, R$ with $f \in \mathfrak{m}$. Show that R is Noetherian.

Proof. As suggested by Atiyah and MacDonald, we will attempt to show that if I is a proper ideal of R it is finitely generated. Since $I \subsetneq R$ is proper, there are finitely many maximal ideals which contain it since given any point $f \in I$ there are only finitely many ideals which contain f. Let $\mathfrak{m}_1, ..., \mathfrak{m}_r$ be the maximal ideals containing I and let $\mathfrak{m}_1, ..., \mathfrak{m}_{r+s}$ be the finitely many maximal ideals which contain $f \in I$. Supposing $\mathfrak{m}_{r+1}, ..., \mathfrak{m}_{r+s}$ do not contain I, then take $x_j \in I \setminus \mathfrak{m}_j$ for $r < j \leq r+s$. Since each $R_{\mathfrak{m}_i}$ is Noetherian, the extension $I_{\mathfrak{m}_i}$ is finitely generated in $R_{\mathfrak{m}_i}$, say by x_i^j for $1 \leq i \leq r$ and $1 \leq j \leq n_i$ where n_i depends on i. Then the ideal $J \coloneqq (x_0, \left\{x_i^j\right\}, x_{r+1}, ..., x_{r+s})$ and I coincide on $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}\, R$. Therefore $J = I$. ∎

Problem 8.2

Let k be a field, $T = k\big[\{X_j \mid j \in \mathbf{N}\}\big]$ a polynomial ring, n_i a sequence in $\mathbf{N}$ with $0 < n_i - n_{i-1} < n_{i+1} - n_i$ for every i, $\mathfrak{p}_i = (X_j \mid n_i \leq j \leq n_{i+1} - 1)$, $W = T \smallsetminus \bigcup_{i\in\mathbf{N}} \mathfrak{p}_i$ and $R = W^{-1}T$. Show that

(a) $\mathfrak{m}\text{-Spec}\, R = \{ W^{-1}\mathfrak{p}_i \mid i \in \mathbf{N} \}$ and $R_{W^{-1}\mathfrak{p}_i}$ is a Noetherian ring of dimension $n_{i+1} - n_i$.

(b) R is a Noetherian ring and $\dim R = \infty$.

Proof. ∎

Problem 8.3

Let k be a field, $R = k[X_1, ..., X_n]$ a polynomial ring and $f \in R \setminus k$. Show that $\dim R/(f) = n - 1$.

Proof. ■

Problem 8.4

Let R be a Noetherian semilocal ring and M a finite R-module.

(a) Write $n = \max\{\, \mu_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \in \mathfrak{m}\text{-}\operatorname{Spec} R \,\}$. Show that M can be generated by n elements.

(b) Assume R is a domain. Show that if M is projective then M is free.

Proof. ∎

Problem 8.5

Let R be a Noetherian ring. Show that the following are equivalent:

(i) R is a normal ring

(ii) R is a finite direct product of normal domains

(iii) R is a reduced and integrally closed in its total ring of quotients.

Proof. ■