Math 571 Homework Assignment 1

1. Let $\{X_i\}_{i\in I}$ be an I-indexed family of topological spaces. Show that the cartesian product
$$X = \prod_{i\in I} X_i,$$
equipped with the product topology, has the property that for each $i \in I$ the projection $p_i : X \to X_i$ is continuous, and moreover, that X has the following universal property: for any other topological space Y, the function
$$\mathrm{Hom}_{\mathrm{Top}}(Y, X) \longrightarrow \prod_{i\in I} \mathrm{Hom}_{\mathrm{Top}}(Y, X_i),$$
induced by the projections $p_i : X \to X_i$, is a bijection.

2. Let X be a set equipped and let $\{\mathcal{U}_i\}_{i\in I}$ be a family of topologies on X. Show that
$$\mathcal{U} = \bigcap_{i\in I} \mathcal{U}_i$$
is a topology on X. Show that if $\mathcal{B}$ is a basis for a topology on X, then the topology $\mathcal{U}$ on X generated by $\mathcal{B}$ is the intersection of all topologies on X which contain $\mathcal{B}$, and that this holds even if we only require that $\mathcal{B}$ is a subbasis.

3. A topological space X is said to be *Hausdorff* if, for every pair of points $x_0, x_1 \in X$ with $x_0 \neq x_1$, there exist open subsets U_0, U_1 of X such that $x_0 \in U_0$, $x_1 \in U_1$, and $U_0 \cap U_1 = \emptyset$. Show that a topological space X is Hausdorff if and only if the diagonal inclusion $X \to X \times X$ is closed.

4. Let X be a topological space and let $Y \subseteq X$ be a subset of X. Show that if Y is equipped with the subspace topology then the inclusion function $i : Y \to X$ is continuous. Show that if there exists a continuous function $q : X \to Y$ such that $q \circ i = \mathrm{id}_Y$ then q is a quotient map (that is, Y is also a quotient of X, equipped with the quotient topology). Give an example of such a situation.

5. A *topological group* is group G equipped with a topology $\mathcal{U}$ such that the multiplication $\mu : G \times G \to G$ and inversion $\iota : G \to G$ functions are continuous (it is standard to also assume that the topology $\mathcal{U}$ on G is Hausdorff, which we will do). Let H be a subgroup of G, and let G/H denote the quotient of G by the action of H, equipped with the quotient topology. Show that G/H is a homogeneous space and that the quotient map $q : G \to G/H$ is open. If, moreover, H is a closed subset of G, show that G/H has the property that points are closed. Finally, show that if H is a normal subgroup of G, then G/H is a topological group. (Optional: is it Hausdorff?)