

# MA557 Problem Set 3

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**PROBLEM 3.1**

Find an example of a finitely generated ring extension  $R \subset S$  where  $S$  is a Noetherian ring, but  $R$  is not.

*Proof.*



**PROBLEM 3.2**

Consider the homomorphism of rings

$$\begin{array}{ccc} & S & \\ & \downarrow \psi & \\ R & \xrightarrow{\varphi} & T. \end{array}$$

The *fiber product* of  $R$  and  $S$  over  $T$  is the subring  $R \times_T S = \{ (r, s) \mid \varphi(r) = \psi(s) \}$  of  $R \times S$ . Assume  $\varphi$  and  $\psi$  are surjective. Show that if  $R$  and  $S$  are Noetherian rings then so is  $R \times_T S$ .

*Proof.* Suppose that  $R$  and  $S$  are Noetherian rings with surjective ring maps  $\varphi: R \rightarrow T$  and  $\psi: S \rightarrow T$ . Then, by (3.5), the product  $R \times S$  is Noetherian. Define the ring map  $\Phi: R \times S \rightarrow T \times T$  by  $\Phi = (\varphi, \psi)$ . Then the diagonal,  $\Delta_T = \{ (t, t) \mid t \in T \}$ , of  $T \times T$  is exactly the image of the fiber product of  $R$  and  $S$  under the ring map  $\Phi$ . And this is not terribly difficult to see: It is clear, by the definition of the fiber product, that  $\Phi(R \times_T S) \subset \Delta_T$ . To show the reverse containment, take an element  $(t, t) \in \Delta_T$ . Then, since  $\varphi$  and  $\psi$  are surjective, there are corresponding elements  $r$  and  $s$  of the rings  $R$  and  $S$ , respectively, such that  $\varphi(r) = t$  and  $\psi(s) = t$ . Hence,  $(t, t)$  are in the image  $R \times_T S$  under  $\Phi$ .

Now, consider the quotient  $R \times S$ -module  $T \times T / \Delta_T$ . Define the homomorphism  $\Phi^*: R \times S \rightarrow T \times T / \Delta_T$  given by the composition  $\Phi^* = \pi \circ \Phi$  where  $\pi$  is the canonical projection map of  $T \times T$  onto the quotient module  $T \times T / \Delta_T$ . This map is an  $R \times S$ -linear map, i.e., take  $(r, s) \in R \times S$  as a ring and  $(r_1, s_1), (r_2, s_2) \in R \times S$  as an  $R \times S$ -module, then

$$\begin{aligned} \Phi^*((r_1, s_1) + (r, s)(r_2, s_2)) &= \Phi^*((r_1, s_1) + (rr_2, ss_2)) \\ &= \Phi^*(r_1 + rr_2, s_1 + ss_2) \\ &= \pi(\varphi(r_1 + rr_2), \psi(s_1 + ss_2)) \\ &= \pi(\varphi(r_1) + \varphi(r)\varphi(r_2), \psi(s_1) + \psi(s)\psi(s_2)) \\ &= (\pi \circ \varphi(r_1) + (\pi \circ \varphi(r))(\pi \circ \varphi(r_2)), \pi \circ \psi(s_1) + (\pi \circ \psi(s))(\pi \circ \psi(s_2))) \\ &= \pi \circ \varphi((r_1, s_1)) + \pi \circ \varphi((r, s))\pi \circ \varphi((r_2, s_2)), \\ &= \Phi^*((r_1, s_1)) + \Phi^*((r, s))\Phi^*((r_2, s_2)), \end{aligned}$$

with  $\ker \Phi^* = R \times_T S$ . Therefore, we have the following exact sequence

$$0 \longrightarrow R \times_T S \xhookrightarrow{\iota} R \times S \xrightarrow{\Phi^*} \frac{T \times T}{\Delta_T} \longrightarrow 0.$$

By (3.4),  $R \times_T S$  are Noetherian. ■

**PROBLEM 3.3**

Let  $M$  be an  $R$ -module. Show that  $M$  is a flat  $R$ -module if and only if  $M_{\mathfrak{m}}$  is a flat  $R_{\mathfrak{m}}$ -module for every maximal ideal  $\mathfrak{m}$  of  $R$ .

*Proof.*  $\implies$  : Suppose that  $M$  is a flat  $R$ -module.

$\impliedby$  :

■

**PROBLEM 3.4**

Let  $M$  be an  $R$ -module and  $\mathfrak{a}$  an  $R$ -ideal.

- (a) Show that if  $M_{\mathfrak{m}} = 0$  for every maximal ideal  $\mathfrak{m}$  containing  $\mathfrak{a}$ , then  $M = \mathfrak{a}M$ .
- (b) Show that the converse holds in case  $M$  is finite.

*Proof.*

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**PROBLEM 3.5**

Prove that every power of a maximal ideal is primary.

*Proof.*



**PROBLEM 3.6**

- (a) Show that the radical of a primary ideal is prime.
- (b) Find an example of a power of a prime ideal that is not primary.
- (c) Let  $\mathfrak{p}$  be a prime ideal of a ring  $R$  and  $n \in \mathbf{N}$ . The  $R$ -ideal  $\mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$  is called the  *$n$ th symbolic power of  $\mathfrak{p}$* . Show that  $\mathfrak{p}^{(n)}$  is primary.

*Proof.*

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