Sheaf theory

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Chapter 1

The statement of de Rham's theorem

Before doing anything fancy, let's start at the beginning. Let $U \subseteq \mathbb{R}^3$ be an open set. In calculus class, we learn about operations

$$\{\text{functions}\} \xrightarrow{\nabla} \{\text{vector fields}\} \xrightarrow{\nabla\times} \{\text{vector fields}\} \xrightarrow{\nabla\cdot} \{\text{functions}\}$$

such that $(\nabla\times)(\nabla) = 0$ and $(\nabla\cdot)(\nabla\times) = 0$. This is the prototype of a *complex*. An obvious question: does $\nabla \times v = 0$ imply that v is gradient? Answer: sometimes yes (e.g. if $U = \mathbb{R}^3$) and sometimes no (e.g. if $U = \mathbb{R}^3$ minus a line). To quantify the failure introduce the first de Rham cohomology

$$H^1_{dR}(U) = \frac{\{v \text{ a vector field on } U \mid \nabla \times v = 0\}}{\{\nabla f\}}$$

Contrary to first appearances, for reasonable U this is finite dimensional and computable. This follows from de Rham's theorem, which we now explain. First, let's generalize this to an open set $U \subset \mathbb{R}^n$. Once $n > 3$ vector calculus is useless, but there is a good replacement. A differential form of degree p, or p-form, is an expression

$$\alpha = \sum f_{i_1,\ldots i_p}(x_1, \ldots, x_n) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

such that the x_i are coordinates, the f's are C^∞ functions, $dx_i \wedge \ldots$ are symbols where $\wedge$ is an anticommutative product. Let $\mathcal{E}^p(U)$ denote the vector space of p-forms. Define the exterior derivative by

$$d\alpha = \sum \sum_j \frac{\partial f_{i_1,\ldots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

This is a $p+1$-form.

Lemma 1.0.1. $d^2 = 0$

Proof for $p = 0$.

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

$$d(df) = \sum_i \sum_j \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i$$

Using anticommutivity, we can rewrite this as

$$\sum_{j<i} \left(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dx_j \wedge dx_i = 0$$

□

A (cochain) complex is a collection of abelian groups M^i and homomorphisms $d : M^i \to M^{i+1}$ such that $d^2 = 0$. We define the pth cohomology of this by

$$H^p(M^\bullet, d) = \frac{\ker d : M^p \to M^{p+1}}{\operatorname{im} d : M^{p-1} \to M^p}$$

So we have an example of a complex $(\mathcal{E}^\bullet(U), d)$ called the de Rham complex of U. It's cohomology is the de Rham cohomology $H^p_{dR}(U) = H^p(\mathcal{E}^\bullet(U), d)$. Here is a basic computation.

Theorem 1.0.2 (Poincaré's lemma)**.**

$$H^p_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof for $n \le 2$. . We first treat the case $n = 1$. Clearly $H^0_{dR}(\mathbb{R})$ consists of constant functions. If $\alpha = f(x)dx$, then $d(\int_0^x f(x)dx) = \alpha$. There are no p-forms for $p > 1$.

Next we treat $n = 2$ which is contains all the ideas of the general case. Let x, y be the coordinates. We define some operators

$$\mathcal{E}^\bullet(\mathbb{R}^2) \underset{\pi^*}{\overset{s^*}{\rightleftarrows}} \mathcal{E}^\bullet(\mathbb{R})$$

π^* is pullback along the projection $\mathbb{R}^2 \to \mathbb{R}$. It takes a form in x and treats it as a form in x, y. The pullback along the zero section s^* sets y and dy to zero. Note that $s^* \circ \pi^*$ is the identity. Although $\pi^* \circ s^*$ is not the identity, we will show that it induces the identity on cohomology. This will show that $H^*_{dR}(\mathbb{R}^2) \cong H^*_{dR}(\mathbb{R})$, which is all we need. This involves a new concept. We introduce an operator $H : \mathcal{E}^p(\mathbb{R}^2) \to \mathcal{E}^{p-1}(\mathbb{R}^2)$ of degree -1 called a *homotopy*. It integrates y as follows:

$$\begin{aligned} H(f(x,y)) &= 0 \\ H(f(x,y)dx) &= 0 \\ H(f(x,y)dy) &= \int_0^y f(x,y)dy \\ H(f(x,y)dx \wedge dy) &= (\int_0^y f(x,y)dy)dx \end{aligned}$$

A computation using nothing more than the fundamental theorem of calculus shows that

$$1 - \pi^* s^* = \pm(Hd - dH)$$

This implies the left side induces 0 on $H^*_{dR}(\mathbb{R}^2)$, or equivalently that $\pi^* \circ s^*$ acts like the identity.

$\square$

Before describing de Rham's theorem, we have to say what's happening at the other end. The standard n dimensional simplex, or n-simplex, $\Delta^n \subset \mathbb{R}^{n+1}$ is the convex hull of the unit vectors $(1, 0, \ldots, 0), (0, 1, \ldots), \ldots$. The convex hull of a subset of these is called a face. This is homeomorphic to a simplex of smaller dimension. Omitting all but the ith vertex is called the ith face of Δ^n. We have a standard homeomorphism

$$\delta_i : \Delta^{n-1} \to i\text{th face of } \Delta^n$$

A geometric simplicial complex is given by a collection of simplices glued along faces. Historically, the first (co)homology theory was defined for simplical complexes. A bit later singular cohomology was developed, which is a bit more flexible and convenient for our purposes. Here we start with an arbitrary topological space X. A (real, complex) singular p-cochain α is an integer (real, complex) valued function on the set of all continuous maps $f : \Delta^p \to X$. It might help to think of $\alpha(f)$ as a combinatorial integral $\int_f \alpha$. Let $S^p(X)$ $(S^p(X, \mathbb{R}), S^p(X, \mathbb{C}))$ denote the group of these cochains. Define $\delta : S^p(X) \to S^{p+1}(X)$ by

$$\delta(\alpha)(f) = \sum (-1)^i \alpha(f \circ \delta_i)$$

Lemma 1.0.3. $\delta^2 = 0$

Proof for $p = 0$. Let $\alpha \in S^0$. Fix $f : \Delta^2 \to X$. Label the restriction of f to the vertices by $0, 1, 2$ and faces $01, 02, 12$. Then

$$\delta^2(f) = \delta\alpha(12) - \delta\alpha(02) + \delta\alpha(01)$$

$$= \alpha(1) - \alpha(2) - \alpha(0) + \alpha(2) + \alpha(0) - \alpha(1) = 0$$

$\square$

Thus we have a complex. Singular cohomology is defined by $H^p(X, \mathbb{Z}) = H^p(S^\bullet(X), \delta)$, and similarly for real or complex valued singular cohomology. These groups are highly computable.

Theorem 1.0.4 (de Rham)**.** *If $X \subset \mathbb{R}^n$ is open, or more generally a manifold, then $H^p_{dR}(X, \mathbb{R}) \cong H^p(X, \mathbb{R})$ for all p.*

We will give a proof of this later on as an easy application of sheaf theory. Sheaf methods will help obtain parallel theorems

Theorem 1.0.5 (Holomorphic de Rham)**.** *If $X \subset \mathbb{C}^n$ is a complex manifold, then $H^p(X, \mathbb{C})$ can be computed using holomorphic differential forms.*

Theorem 1.0.6 (Algebraic de Rham)**.** *If $X \subset \mathbb{C}^n$ is a nonsingular algebraic variety, then $H^p(X, \mathbb{C})$ can be computed using algebraic differential forms.*

The last theorem is due to Grothendieck. The proof is a lot harder, so we"ll try to give the proof by the end of the semester but there's no guarantee.