MA 523: Homework, Midterms and Practice Problems Solutions

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Contents

1 Homework Solutions

1.1 Homework 1

PROBLEM 1.1 (Taylor's formula). Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to \mathbf{0}$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

SOLUTION. ■

PROBLEM 1.2. Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \qquad (*)$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

SOLUTION. ■

PROBLEM 1.3. Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_1) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

SOLUTION. ■

PROBLEM 1.4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\big(u_{x_1}^2 + u_{x_2}^2\big)$$

find a solution with $u(x_1, 0) = \big(1 - x_1^2\big)/2$.

SOLUTION. ■

1.2 Homework 2

Problem 1.5. Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ■

Problem 1.6. Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. ■

Problem 1.7. Show that the function $u(x, t)$ defined for $t \geq 0$ by

$$u(x,t) = \begin{cases} -\dfrac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burgers' equation*).

Solution. ■

1.3 Homework 3

PROBLEM 1.8. Consider the initial value problem

$$u_t = \sin u_x; \qquad u(x,0) = \frac{\pi}{4}x.$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the taylor series of the solution about the origin.

SOLUTION. ■

PROBLEM 1.9. Consider the Cauchy problem for $u(x,y)$

$$\begin{aligned} u_y &= a(x,y,u)u_x + b(x,y,u) \\ u(x,0) &= 0 \end{aligned}$$

let a and b be analytic functions of their arguments. Assume that $d^\alpha a(0,0,0) \geq 0$ and $d^\alpha b(0,0,0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

(a) Show that $D^\beta u(0,0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $D^\beta u(0,0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$. (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in β_2)

SOLUTION. ■

PROBLEM 1.10. (Kovalevskaya's example) show that the line $\{\, t = 0 \,\}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution u of the heat equation in $\mathbb{R} \times \mathbb{R}$, with $u = 1/(1+x^2)$ on $\{t = 0\}$. (*Hint:* assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of $(0,0)$.)

SOLUTION. ■

1.4 Homework 4

PROBLEM 1.11 (Legendre transform). Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u\big(\mathbf{x}(p)\big),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29)).

SOLUTION. ■

PROBLEM 1.12. Find the solution $u(x, t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$\begin{cases} u(x, 0) = f(x), & u_t(x, 0) = g(x), \quad \text{for } x > 0, \\ u_t(0, t) = \alpha u_x(0, t), & \qquad\qquad\qquad \text{for } t > 0, \end{cases}$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$. (*Hint:* Use a representation $u(x, t) = F(x - t) + G(x + t)$ for the solution.)

SOLUTION. ■

PROBLEM 1.13. (a) Let u be a solution of the wave equation $u_{tt} - c^2u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0, t) = u(\pi, t) = 0$. Show that the *energy*

$$E(t) = \frac{1}{2}\int_0^\pi \left(u_t^2 + c^2u_x^2\right) dx, \quad t > 0$$

is independent of t; i.e., $\frac{d}{dt}E = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x, t) = \sum_{n=1}^{\infty} \big(a_n \cos(nct) + b_n \sin(nct)\big) \sin(nx)$$

in terms of coefficients a_n, b_n.

SOLUTION. ■

1.5 Homework 5

Problem 1.14. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$, $x \in \mathbb{R}^n$, then $\Delta v = 0$.

Solution. ■

Problem 1.15. Let $n = 2$ and U be the halfplane $\{ x_2 > 0 \}$. Prove that

$$\sup_U u = \sup_{\partial U} u$$

for $u \in C^2(U) \cap C(\bar{U})$ which are harmonic in U under the additional assumption that u is bounded from above in $\bar{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.)
[*Hint:* Take for $\varepsilon > 0$ the harmonic function

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region $\left\{ x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0 \right\}$ with large a. Let $\varepsilon \to 0$.]

Solution. ■

Problem 1.16. Let $U \subset \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \dots, u^m$ are harmonic in U and

$$v := \varphi(u_1, \dots, u_m).$$

Prove v is subharmonic.

[*Hint:* Convexity for a smooth function $\varphi(z)$ is equivalent to $\sum_{j,k=1}^m \varphi_{z_j, z_k}(z)\xi_j\xi_k \geq 0$ for any $\xi \in \mathbb{R}^m$.]

(b) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. (Assume that harmonic functions are C^∞.)

Solution. ■

1.6 Homework 6

Problem 1.17. For $n = 2$ find Green's function for the quadrant $\{ x_1 > 0, x_2 > 0 \}$ by repeated reflection.

Solution. ∎

Problem 1.18. (Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \le u(x) \le \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B(0, r) = \{ x \in \mathbb{R}^n : |x| < r \}$.

Solution. ∎

Problem 1.19. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$\begin{aligned} &P_k(\lambda x) = \lambda^k P_k(x), && P_m(\lambda x) = \lambda^m P_m(x) && \text{for every } x \in \mathbb{R}^n,\ \lambda > 0, \\ &\Delta P_k = 0, && \Delta P_m = 0 && \text{in } \mathbb{R}^n. \end{aligned}$$

(a) Show that

$$\frac{\partial P_k}{\partial \nu} = k P_k(x), \qquad \frac{\partial P_m}{\partial \nu} = m P_m(x) \qquad \text{on } \partial B(0, 1)$$

where $B(0, 1) = \{ x \in \mathbb{R}^n : |x| < 1 \}$ and ν is the outward normal on $\partial B(0, 1)$.

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x)\, d\sigma = 0, \qquad \text{if } k \ne m.$$

Solution. ∎

2 Midterms and Qualifying Exams

2.1 Qualifying Exam, August ‘04

PROBLEM 2.1. Consider the initial value problem

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = -u, & \\ \qquad\qquad\qquad\qquad u = f & \text{on } S^1 = \{\, x^2 + y^2 = 1 \,\}, \end{cases}$$

where a and b satisfy

$$a(x,y) + b(x,y)y > 0$$

for any $x, y \in \mathbb{R}^n \smallsetminus \{(0,0)\}$.

(a) Show that the initial value problem has a unique solution in a neighborhood of S^1. Assume that a, b, and f are smooth.
(b) Show that the solution of the initial value problem actually exists in $\mathbb{R}^2 \smallsetminus \{(0,0)\}$.

SOLUTION. ■

PROBLEM 2.2. Let $u \in C^2\big(\mathbb{R} \times [0,\infty)\big)$ be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty), \\ \qquad u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where f and g have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx,$$

and the potential energy by

$$P(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx.$$

Show that

(a) $K(t) + P(t)$ is constant in t,
(b) $K(t) = P(t)$ for all large enough times t.

SOLUTION. ■

PROBLEM 2.3. Use Kirchhoff’s formula and Duhamel’s principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = e^{-t} g(x) & \text{for } x \in \mathbb{R}^3,\ t > 0, \\ \ u(x,0) = u_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution $u = e^{-t} + t - 1$ when $g(x) = 1$.

Solution. ∎

Problem 2.4. Let Ω be a bounded open set in $\mathbb{R}^n$ and $g \in C_0^\infty(\Omega)$. Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega,\ t > 0, \\ u(x,0) = g(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } xi \in \partial\Omega,\ t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n,\ t > 0, \\ v(x,0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put $g = 0$ outside Ω.

(a) Show that

$$-v(x,t) \leq u(x,t) \leq v(x,t)$$

for any $x \in \Omega$, $t > 0$.

(b) Use (a) to conclude that

$$\lim_{t\to\infty} u(x,t) = 0,$$

for any $x \in \Omega$.

Solution. ∎

Problem 2.5. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \qquad P_m(\lambda x) = \lambda^m P_m(x),$$

for any $x \in \mathbb{R}^n$, $\lambda > 0$,

$$\Delta P_k = 0, \qquad \Delta P_m = 0$$

in $\mathbb{R}^n$.

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = k P_k(x), \qquad \frac{\partial P_m(x)}{\partial \nu} = m P_m(x)$$

on ∂B_1, where $B_1 = \{\, |x| < 1 \,\}$ and ν is the outward normal on ∂B_1.

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x)\, dS = 0,$$

if $k \neq m$.

Solution. ∎

2.2 Qualifying Exam, August '05

PROBLEM 2.6.

(a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ \qquad\qquad u = 1 \qquad \text{on } S^1 = \{ x^2 + y^2 = 1 \}. \end{cases}$$

(b) Is the solution unique in a neighborhood of the point $(1, 0)$? Justify your answer.

Solution. ■

PROBLEM 2.7. Consider the second order PDE in $\{ x > 0, y > 0 \} \subset \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

(a) Classify the equation and reduce it to the canonical form.
(b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy}\,G(x/y).$$

Solution. ■

PROBLEM 2.8. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^3$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial (i.e., $f(y) = f(|y|)$) and supported in $B_R = \{ |x| < R \}$, then

$$u(x) = c\Phi(x),$$

for any $x \in \mathbb{R}^n \setminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y)\, dy.$$

[*Hint:* Use spherical (polar) coordinates and the mean value property.]

Solution. ■

PROBLEM 2.9. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \qquad u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where $g, h \in C_0^\infty(\mathbb{R}^2)$ and $\alpha \geq 0$ is a constant.

(a) Show that for an appropriate choice of constant λ and μ the function

$$v(x_1, x_2, x_3, t) := e^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation $v_{tt} - \Delta v = 0$.

(b) Use (a) to prove the following domain of dependence result: for any point $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$ the value $u(x_0, t_0)$ is uniquely determined by values of g and h in $\overline{B_{t_0}(x_0)} := \{ |x - x_0| \le t_0 \}$. (You may use the corresponding result for the wave equation without proof.)

Solution. ■

Problem 2.10. Let $u(x,t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0, \infty)$ with the initial condition

$$u(x, 0) = u_0(x)$$

for $x \in \mathbb{R}$, where $u_0 \in C^\infty$ is 2π-periodic, i.e., $u_0(x + 2\pi) = u_0(x)$. Show that

$$\lim_{t\to\infty} u(x,t) = a_0,$$

uniformly in $x \in \mathbb{R}$, where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x)\, dx.$$

Solution. ■

2.3 Qualifying Exam, January '14

PROBLEM 2.11. Consider the first order equation in $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

(a) Find the characteristic curves of the equation.
(b) Consider the Cauchy problem for this equation prescribed on the line $x_1 = 1$:

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on f so that the proble is solvable in a neighborhood of the point $(1, 0)$.

SOLUTION. ■

PROBLEM 2.12. Let u be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{ x_n > 0 \}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where g is a continuous function with compact support in $\mathbb{R}^{n-1}$. Here $n \geq 2$. Prove that

$$u(x) \longrightarrow 0, \qquad \text{as } |x| \longrightarrow \infty,$$

for $x \in \{ x_n > 0 \}$.

SOLUTION. ■

PROBLEM 2.13. Let u be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \qquad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions $u(x, 0) = g(x)$, where g is a bounded continuous function on $\mathbb{R}$ satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^\alpha, \qquad x, y \in \mathbb{R}$$

with a constant $\alpha \in (0, 1]$. Show that

$$\begin{aligned} |u(x,t) - u(y,t)| &\leq M|x - y|^\alpha, & x, y \in \mathbb{R}, t > 0, \\ |u(x,t) - u(x,s)| &\leq C_\alpha M|t - s|^{\alpha/2}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of $u(x, t)$ as a convolution with the heat kernel $\Phi(y, t)$, make a change of variables $z = y/\sqrt{t}$ and use that $\left|\sqrt{t} - \sqrt{s}\right| \leq \sqrt{|t - s|}$.]

SOLUTION. ■

PROBLEM 2.14. Let u be a positive harmonic function in the unit ball B_1 in $\mathbb{R}^n$. Show that

$$\big|D(\ln u)\big| \le M \qquad \text{in } B_{1/2}$$

for a constant M depending only on the dimension n.
[*Hint:* Use the interior derivative estimate $\big|Du(x)\big| \le (C_n/r)\sup_{B_r(x)}|u|$ for $B_r(x) \subset B_1$ as well as the Harnack inequality for harmonic functions.]

SOLUTION. ■

PROBLEM 2.15. Let u be a C^2 solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0,\infty), \\ \qquad u = 0, \qquad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some $k \ge 0$. Prove that there exists a function $\varphi(r)$ such that

$$u(x,t) = t^{k+2}\varphi\big(|x|/t\big).$$

[*Hint:* As one of the steps show that u is $(k+2)$-homogeneous in (x,t) variables, i.e., $u(\lambda x, \lambda t) = \lambda^{k+2}u(x,t)$ for any $\lambda > 0$.]

SOLUTION. ■