MA 523: Homework 2

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Problem 2.1

Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 because in such case we have $\nu(x^0) = (0, \dots, 0, 1)$ so

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall verify the noncharacteristic condition (2.1) by first flattening the boundary near x^0 and then applying the noncharacteristic boundary conditions to the flattened region. Make a change of variables $(y_1, \dots, y_n) = y(x_1, \dots, x_n)$ where

$$\begin{cases} y_1 = x_1, \\ \quad\vdots \\ y_{n-1} = x_{n-1}, \\ y_n = x_n - \varphi(x_1, \dots, x_{n-1}), \end{cases}$$

with φ a sufficiently regular map $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$. Then, note that $y^0 = y(x_1^0, \dots, x_n^0) = (y_1, \dots, y_{n-1}, 0)$ and hence $\Delta = y(\Gamma)$ is flat near y^0 so we can apply the standard noncharacteristic boundary conditions on the transformed PDE,

$$0 \neq G_{p_n}(p, z, y), \tag{2.2}$$

where G is the PDE F after applying the transformation Φ, i.e., the PDE

$$G(Dv, v, y) = F(Dv(y) D\Phi(\Psi(y)), v(y), \Psi(y)).$$

We are done after we relate (2.2) to the original PDE F. Note that by the equation above we have

$$p(y) D\Phi(\Psi(y)) = \begin{cases} p_1(y) - p_n(y)\varphi_{x_1}(y_1, \dots, y_{n-1}), \\ \qquad\vdots \\ p_{n-1} - p_n(y)\varphi_{x_{n-1}}(y_1, \dots, y_{n-1}), \\ p_n(y). \end{cases}$$

Thus,

$$G_{p_n}(p, z, \tilde{y}) = F_{p_n}(p D\Phi(\Psi(\tilde{y})), z, \Psi(\tilde{y}))$$

which, by the chain rule, is equal to

$$
\begin{aligned}
&= -F_{p_1}\varphi_{x_1}(\tilde{y}) - \cdots - F_{p_{n-1}}\varphi_{x_{n-1}} + F_{p_n} \\
&= DF(p^0, z^0, x^0) \cdot \left(-D\varphi(x_1^0, \ldots, x_{n-1}^0), 1\right) \\
&= DF(p^0, z^0, x^0) \cdot \nu(x^0)
\end{aligned}
$$

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Problem 2.2

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. ►

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Problem 2.3

Show that the function $u(x,t)$ defined by $t \geq 0$ by

$$u(x,t) = \begin{cases} -\dfrac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burger's equation*).

Solution. ► ◄