MA 523: Homework, Midterms and Practice Problems Solutions

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Contents

1 Homework Solutions

These are my (corrected) solutions to Petrosyan's Math 523 homework for the fall semester of 2016.

1.1 Homework 1

Problem 1.1.1 (Taylor's formula). Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O\big(|x|^{m+1}\big)$$

as $x \to \mathbf{0}$ for each $m = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. Taking the hint, let us consider the function in one variable $g(t) := f(tx)$ for $x \in \mathbb{R}^n$ fixed. We show by induction on m that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha. \tag{1.1.1}$$

Once we have shown (1.1.1) holds, evaluating g at $t = 1$ gives us the desired equality; i.e,

$$f(x) = g(1)$$

which, by Taylor's formula in one variable, is

$$= \sum_{j=0}^{m} \frac{g^{(j)}(0)}{j!} 1^j + O\big(|x|^{m+1}\big)$$

applying (1.1.1) here gives us

$$\begin{aligned}
&= \sum_{k=0}^{m} \frac{1}{k!} \left[\sum\nolimits_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha f(tx) x^\alpha \right] + O\big(|x|^{m+1}\big) \\
&= \sum_{k=0}^{m} \left[\sum\nolimits_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha \right] + O\big(|x|^{m+1}\big) \\
&= \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O\big(|x|^{m+1}\big)
\end{aligned}$$

as desired.

Let us now show that (1.1.1) holds. To prove this we consider the algebra on the differential operator d/dt. By the chain rule, we have

$$\frac{d}{dt}(\,\cdot\,) = \sum_{k=1}^{m} x_k \frac{\partial}{\partial x_k}(\,\cdot\,).$$

Since f is smooth by Schwartz's theorem the differential operators $\partial/\partial x_k$ and $\partial/\partial x_l$ commute for all $1 \le k, l \le n$. Therefore, by the multinomial theorem,

$$\frac{d^m}{dt^m}(\,\cdot\,) = \left(\sum\nolimits_{k=1}^{m} x_k \frac{\partial}{\partial x_k}(\,\cdot\,)\right)^k = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha D^\alpha(\,\cdot\,). \qquad \blacksquare$$

Problem 1.1.2. Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition $u = g$ on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. Write

$$F(p, z, x, t) := (b, 1) \cdot p - f = 0.$$

Then the characteristic equations to the problem $(*)$ with the initial value $u(\,\cdot\,, 0) = g(\,\cdot\,)$ are given by

$$\begin{cases} \dot{p} = -D_{x,t}F - D_z F p = 0, \\ \dot{z} = D_p F \cdot p = (b, 1) \cdot p = f, \\ (\dot{x}, \dot{t}) = D_p F = (b, 1). \end{cases}$$

Now let us solve the characteristic equations above subject to the initial values $(x(0), t(0)) = (x^0, 0) \in \mathbb{R}^n \times (0, \infty)$; these are

$$\begin{cases} x(s) = x^0 + bs, \quad t(s) = s, \\ z(s) = z(0) + \displaystyle\int_0^s f(x(\tau), t(\tau))\, d\tau \\ \qquad = g(x^0) + \displaystyle\int_0^s f(x^0 + b\tau, \tau)\, d\tau. \end{cases}$$

Solving back, we have $t = s$, $x^0 = x - bs = x - bt$, and therefore

$$u(x, t) = z(s) = g(x - bt, t) + \int_0^t f(x - b\tau, \tau)\, d\tau$$

solves the transport equation $(*)$ with initial value $u(\,\cdot\,, 0) = g(\,\cdot\,)$.

This verifies formula [5] from [E, §2.1.2]. $\blacksquare$

Problem 1.1.3. Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $u u_{x_1} + u_{x_2} = 1$, $u(x_1, x_1) = \frac{x_1}{2}$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. For part (a): Write $F := (x_1^2, x_2^2)\cdot p - z^2 = 0$. We have the characteristic equations

$$\begin{cases} \dot{p} = -D_x F - D_z F p = 2\big((x_1 - z)p_1, (x_2 - z)p_2\big), \\ \dot{z} = D_p \cdot p = z^2, \\ \dot{x} = (x_1^2, x_2^2). \end{cases}$$

We can solve the characteristic equations with respect to the initial conditions $x(0) = (x^0, 2x^0)$, $z(0) = 1$ on the line $x_1 = 2x_2$; these are

$$\begin{cases} x_1(s) = \dfrac{x^0}{1 + x^0 s}, \quad x_2(s) = \dfrac{2x^0}{1 + 2x^0 s}, \\ z(s) = \dfrac{1}{1 - s}. \end{cases}$$

Now we solve these in terms of the coordinates (x_1, x_2). Assuming $x^0 \neq 0$, we have

$$s = \frac{1}{x^0} - \frac{1}{x_1} \quad \text{and} \quad s = \frac{1}{2x^0} - \frac{1}{x_2}.$$

Therefore,

$$\begin{aligned} s &= 2\left(\frac{1}{2x^0} - \frac{1}{x_2}\right) - \left(\frac{1}{x^0} - \frac{1}{x_1}\right) \\ &= \frac{1}{x_1} - \frac{2}{x_2}. \end{aligned}$$

Thus,

$$u(x_1, x_2) = \frac{1}{1 - \left(\frac{1}{x_1} - \frac{2}{x_2}\right)} = \frac{x_1 x_2}{x_1 x_2 - x_2 - 2x_1}$$

solves the PDE F for (x_1, x_2) on the line $x_1 = 2x_2$ away from the origin.

For part (b): Write $F = (z, 1)\cdot p - 1 = 0$. Then we have the characteristic equations

$$\begin{cases} \dot{p} = -D_x F - D_z p = -(p_1, 0) \\ \dot{z} = D_p \cdot p = 1 \\ \dot{x} = D_p F = (z, 1) \end{cases}$$

Next we solve the characteristic equations subject to the initial conditions $x(0) = (x^0, x^0)$, $z(0) = \frac{x^0}{2}$ on the line $x_1 = x_2$; these are

$$\begin{cases} z(s) = \frac{1}{2}x^0 + s, \\ x_1(s) = x^0 + \frac{1}{2}\big(x^0 s + s^2\big), \quad x_2(s) = x^0 + s. \end{cases}$$

Then, solving in terms of the coordinates (x_1, x_2), we have

$$x^0 = 2(x_2 - z) \quad \text{and} \quad s = 2z - x_2.$$

Therefore,

$$\begin{aligned} x_1 &= 2(x_2 - z) + (x_2 - z)(2z - x_2) + \tfrac{1}{2}(2z - x_2)^2 \\ &= -\tfrac{1}{2}x_2(x_2 - 4) + (x_2 - 2)z. \end{aligned}$$

Hence,

$$u(x_1, x_2) = \frac{2x_1 + x_2^2 - 4x_2}{2(x_2 - 2)}$$

solves the PDE F subject to the condition $u(x_1, x_1) = \frac{x_1}{2}$ provided $x_2 \neq 2$.

For part (c): Write $F := (x_1, 2x_2, 1) \cdot p - 3z = 0$. Then the characteristic equations are

$$\begin{cases} \dot{p} = -D_x F - D_z p = (2p_1, p_2, 3p_3) \\ \dot{z} = D_p F \cdot p = 3z \\ \dot{x} = D_p F = (x_1, 2x_2, 1) \end{cases}$$

Next we sole the characteristic equations subject to the initial conditions $x(0) = (x_1^0, x_2^0, 0)$, $z(s) = g(x_1^0, x_2^0)$; these are

$$\begin{cases} x_1(s) = x_1^0 \mathrm{e}^s, \quad x_2(s) = x_2^0 \mathrm{e}^{2s}, \quad x_3(s) = s, \\ z(s) = g(x_1^0, x_2^0)\mathrm{e}^{3s}. \end{cases}$$

Then, solving for u in terms of the coordinates (x_1, x_2, x_3), we have

$$s = x_3, \quad x_1^0 = x_1 \mathrm{e}^{-s}, \quad \text{and} \quad x_2^0 = x_2 \mathrm{e}^{-2s}.$$

Thus,

$$u(x_1, x_2, x_3) = g(x_1 \mathrm{e}^{-x_3}, x_2 \mathrm{e}^{-2x_3})\mathrm{e}^{3x_3}$$

solves the PDE F subject to the condition $u(x_1, x_2, 0) = g(x_1, x_2)$. ■

PROBLEM 1.1.4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \tfrac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with $u(x_1, 0) = \frac{1 - x_1^2}{2}$.

SOLUTION. The equation is nonlinear and therefore, we do not expect the method of characteristics to yield a unique solution to the PDE

$$F := x_1 p_1 + x_2 p_2 + \tfrac{1}{2}(p_1^2 + p_2^2) - z.$$

Let us find the characteristic equations for F; these are

$$\begin{cases} \dot{p} = -D_x F - D_z F p = -(p_1, p_2) - (-1)(p_1, p_2) = 0, \\ \dot{z} = D_p F \cdot p = (x_1 + p_1, x_2 + p_2) \cdot (p_1, p_2) = (x_1 + p_1)p_1 + (x_2 + p_2)p_2, \\ \dot{x} = D_p F = (x_1 + p_1, x_2 + p_2), \end{cases}$$

Next we solve the characteristic equations subject to the initial values $x(0) = (x^0, 0)$, $z(0) = \frac{1}{2}(1 - (x^0)^2)$ and, after revisiting the equation F, $p_1(0) = -x^0$ and

$$p_2(0)^2 = 2\Big(-(x^0)^2 + \tfrac{1}{2}(x^0)^2 + \tfrac{1}{2}(1 - (x^0)^2)\Big) = 1$$

so $p_2(0) = \pm 1$. Therefore, the solution to the characteristic equations subject to these initial values is

$$\begin{cases} p_1(s) = -x^0, \quad p_2(s) = \pm 1, \\ x_1(s) = x^0, \quad x_2(s) = \pm 1(\mathrm{e}^s - 1), \\ z(s) = \frac{1}{2}\big(1 - (x^0)^2\big) + (\mathrm{e}^s - 1). \end{cases}$$

Thus, solving for s and x^0 in terms of the coordinates (x_1, x_2), we have

$$u(x_1, x_2) = \tfrac{1}{2}(1 - x_1^2) \pm x_2. \qquad \blacksquare$$

1.2 Homework 2

Problem 1.2.1. Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. Throughout this, let (p^0, z^0, x^0) denote an admissible triple to the PDE F at some point x^0 in its domain. First, note that the condition

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 since in that case the normal to the hypersurface at x^0 will be $(0, \dots, 0, 1)$; i.e.,

$$\begin{aligned} 0 &\neq D_pF(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall therefore proceed to flatten the hypersurface Γ near x^0 and apply the standard noncharacteristic boundary condition.

Assuming Γ is reasonably tame, by the implicit function theorem, it can be written as the graph $\{\, x_n = \varphi(x_1, \dots, x_{n-1}) \,\}$ on some neighborhood U of x^0 for φ smooth. Now consider the smooth mapping $\Phi\colon U \to V$ given by

$$\begin{cases} y_j := \Phi^j(x) := x_j, & 1 \leq j \leq n-1, \\ y_n := \Phi^n(x) := x_n - \varphi(x_1, \dots, x_{n-1}), \end{cases}$$

where we use y to denote new coordinates on the image of Φ. Note that $\nu(x^0)$ is parallel to the gradient $D_x\Phi^n = (-\varphi_{x_1}, \dots, -\varphi_{x_{n-1}}, 1)$ so the inner product of the latter with $F_{p_n}(p^0, z^0, x^0)$ is nonzero if and only if the inner product of $\nu(x^0)$ with $F_{p_n}(p^0, z^0, x^0)$ is nonzero.

Set $\Delta := \Phi(\Gamma)$ and define $v(y) := u(\Phi^{-1}(y))$. Then $u(x) = v(\Phi(x))$. Moreover, by the chain rule we have

$$D_{x_i}u = \sum_{j=1}^{n} D_{y_j}v D_{x_j}\Phi^j, \quad 1 \leq j \leq n;$$

i.e., $D_xu = D_yvD_x\Phi$. Thus, v satisfies the PDE

$$G(D_yv, v, y) := F(D_yvD_x\Phi, v, \Phi^{-1}(y)) = 0$$

in Δ and, since Δ has been flattened near $y^0 := \Phi(x^0)$, applying the noncharacteristic condition, we have

$$\begin{aligned} D_{p_n}G &= (D_{p_1}F)(D_{x_1}\Phi^n) + \cdots + (D_{p_n}F)(D_{x_n}\Phi) \\ &= D_pF \cdot D_x\Phi^n. \end{aligned}$$

Therefore, if (p^0, z^0, x^0) is a compatible triple for F and $(q^0, z^0, y^0) = (p^0D_x\Phi(x^0), z^0, \Phi(x^0))$ is the corresponding for G, then

$$D_{p_n}G(q^0, z^0, y^0) = D_pF(p^0, z^0, x^0) \cdot D_x\Phi^n(x^0);$$

i.e.,

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$$

where $\nu(x^0)$ is the normal vector to Γ at x^0. ∎

PROBLEM 1.2.2. Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

SOLUTION. Write $F(p, z, x, t) := (a(z), 1) \cdot p = 0$. Using the method of characteristics, we have the following characteristic ODEs to solve

$$\begin{cases} \dot{p} = -D_{x,t}F - D_zFp = -a'(z)p_1(p_1, p_2), \\ \dot{z} = D_pF \cdot p = (a(z), 1) \cdot p = 0, \\ \dot{x} = D_{p_1}F = a(z), \quad \dot{t} = D_{p_2}F = 1. \end{cases}$$

Solving these subject to the initial conditions $x(0) = x^0$, $t(0) = 0$, and $z(0) = g(x^0)$, we have

$$\begin{cases} x(s) = x^0 + a(g(x^0))s, \quad t(s) = s, \\ z(s) = g(x^0). \end{cases}$$

Thus, we see that u is constant on the projected characteristics

$$x = x^0 + a(g(x^0))t; \tag{1.2.1}$$

i.e., $u = g(x^0)$.

Solving for u in terms of x and t, we have

$$x^0 = x - a(u)t$$

so

$$u = g(x - a(u)t).$$

Now, choose another starting point $y^0 < x^0$. Then we must have $u = g(y^0)$ on the curve

$$x = y^0 + a(g(y^0))t. \tag{1.2.2}$$

Thus, if $g(y^0) > g(x^0)$ the two characteristics (1.2.1) and (1.2.2) will cross at some $t^0 > 0$ and we cannot have a continuous solution up to that point. If $g(y^0) < g(x^0)$ the characteristics (1.2.1) and (1.2.2) will not cross and therefore, u solves the PDE on the upper halfplane $\{ t > 0 \}$. ∎

Problem 1.2.3. Show that the function $u(x,t)$ defined for $t \geq 0$ by

$$u(x,t) = \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x+t^2}\right) & \text{for } 4x+t^2 > 0, \\ 0 & \text{for } 4x+t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burgers' equation*).

Solution. First we show that u is in fact a weak solution of the inviscid Burgers' equation. That is, write u_l and u_r for u restricted to the domains $\{ 4x+t^2 < 0 \}$ and $\{ 4x+t^2 > 0 \}$, respectively. Then a trivial calculation shows that u_l and u_r are solutions to the inviscid Burgers' equation: It is clear that u_l is a solution of the Burgers' equation since it is the trivial solution; for u_r we must work a little harder;

$$\begin{aligned} (u_r)_t &= -\frac{2}{3}\left(1 + \frac{t}{\sqrt{3x+t^2}}\right), \\ (u_r)_x &= -\left(\frac{1}{\sqrt{3x+t^2}}\right) \end{aligned}$$

hence,

$$\begin{aligned} u_t + (u^2/2)_x &= u_t + u_x u \\ &= -\frac{2}{3}\left(1 + \frac{t}{\sqrt{3x+t^2}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{3x+t^2}}\right)\left(t + \sqrt{3x+t^2}\right) \\ &= 0. \end{aligned}$$

Lastly, we need to verify the Rankine–Hugoniot condition on the curve $\Gamma := \{ 4x+t^2 = 0 \}$. Write $s(t) = x = -t^2/4$ (the natural parametrization of the curve Γ), then

$$\begin{aligned} \sigma &= \dot{s}(t), & [\![u]\!] &= u_l - u_r, & [\![F]\!] &= F(u_l) - F(u_r) \\ &= -\frac{t}{2} & &= 0 + \frac{2}{3}\left(t + \sqrt{-\frac{3}{4}t^2 + t^2}\right) & &= 0 - \frac{[\![u_r]\!]^2}{2} \\ & & &= 0 + \frac{2}{3}\left(\frac{3}{2}t\right) & &= 0 - \frac{t^2}{2} \\ & & &= t & &= -\frac{t^2}{2}. \end{aligned}$$

Thus,

$$[\![F]\!] = -\frac{t^2}{2} = \left(-\frac{t}{2}\right)t = \sigma[\![u]\!]$$

so the PDE satisfies the Rankine–Hugoniot condition and hence, is an integral solution.

Finally, to check that u is an entropy solution we note that $u^2/2$ is strictly convex and therefore, we must show that $u_l > u_r$. But this is obvious since $u_l - u_r = t$ which is strictly greater than zero. ∎

1.3 Homework 3

PROBLEM 1.3.1. Consider the initial value problem

$$\begin{cases} u_t = \sin u_x, \\ u(x,0) = \frac{\pi}{4}x. \end{cases}$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

SOLUTION. The equation $F(p, z, x, t) := \sin p_1 - p_2 = 0$ is a fully nonlinear first-order PDE. We first verify that the curve Γ is characteristic near the origin; i.e., we must show that $F_p \cdot \nu \neq 0$ where ν is the normal vector to Γ at the origin. In this case, $\nu = (0, 1)$ and $F_p = (\cos p_1, -1)$; hence,

$$F_p \cdot \nu = (\cos p_1, -1) \cdot (0, 1) = -1 \neq 0.$$

Moreover, the curve Γ is analytic (since it is cut out by the equation $\frac{\pi}{4}x$) and the initial conditions are analytic. Therefore, the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and we can obtain an analytic solution

$$u(x,t) = \sum_{m,n} \frac{a_{m,n}}{m!n!} x^m t^n$$

about the origin.

First, we must compute the coefficients $a_{j,k}$. To this end, we must find the partial derivatives $u_{j,k}$ and potentially, relations among them which will help us to find these coefficients. Naïvely listing the partials with respect to t and x, we have

$$\begin{aligned} u(0,0) &= 0 & u_x(0,0) &= \frac{\pi}{4} \\ u_t(0,0) &= \sin u_x(0,0) = \frac{\sqrt{2}}{2} & u_{xx}(0,0) &= 0 \\ u_{tx}(0,0) &= 0 & u_{tt}(0,0) &= -\cos\bigl(u_x(0,0)\bigr) u_{xt}(0,0) = 0 \\ u_{xxx}(0,0) &= 0 & u_{ttx}(0,0) &= 0 \\ &\vdots & &\vdots \end{aligned}$$

It is not difficult to see that higher derivatives of u will be zero. Thus,

$$u(x,t) = \frac{\pi}{4}x + \frac{\sqrt{2}}{2}t.$$

Plugging this equation into F we see that

$$u_t - \sin u_x = \frac{\sqrt{2}}{2} - \sin(\pi/4) = 0;$$

i.e., $u(x,t)$, as defined above, is an analytic solution to the PDE F. ∎

PROBLEM 1.3.2. Consider the Cauchy problem for $u(x, y)$

$$\begin{cases} u_y = a(x, y, u)u_x + b(x, y, u), \\ u(x, 0) = 0 \end{cases}$$

let a and b be analytic functions of their arguments. Assume that $D^\alpha a(0, 0, 0) \geq 0$ and $D^\alpha b(0, 0, 0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

(a) Show that $D^\beta u(0, 0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $D^\beta u(0, 0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$.
Hint: Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in β_2.

SOLUTION. For part (a): We compute all partial $D^\beta u$ at $(0, 0)$ for $|\beta| \leq 2$ explicitly; these are

$$\begin{aligned} u(0,0) &= u_x(0,0) = u_{xx}(0,0) = 0, \\ u_y(0,0) &= a(0,0,0)u_x(0,0) + b(0,0,0) = b \geq 0, \\ u_{xy}(0,0) &= (a_x(0,0,0) + a_u(0,0,0)u_x(0,0)) + a(0,0,0)u_{xx}(0,0) + b_x(0,0,0) \\ &\quad + b_z(0,0,0)u_x(0,0) \geq 0, \\ u_{yy}(0,0) &= (a_y(0,0,0) + a_u(0,0,0)u_y(0,0))u_x(0,0) \\ &\quad + b_y(0,0,0) + b_u(0,0,0)u_y(0,0) \geq 0. \end{aligned}$$

For part (b): Following the proof of the Cauchy–Kovalevskaya theorem, we use induction on β_2. The case $\beta_2 = 0$ is clear as $D^{(\beta_1,0)}u = \frac{\partial^{\beta_1} u}{\partial x^{\beta_1}} = 0$ by our previous work. Now suppose the proposition is true for all $\beta_2 \leq n - 1$, we show the proposition holds for $\beta_2 = n$. From our previous work above, we have

$$\begin{aligned} D^\beta u &= D^{\beta_1, n-1} u_y \\ &= D^{\beta_1, n-1}[a(x, y, u)u_x + b(x, y, u)] \\ &= P_\beta(D^\gamma u, D^\delta a, D^\varepsilon b), \end{aligned}$$

where P_β is some polynomial with nonnegative coefficients depending only on $D^\gamma u$ with $|\gamma| \leq |\beta|$ and $|\gamma_2| \leq n - 1$. Since all partial derivatives in $D^\beta u$ are nonnegative at the origin and P_β is a polynomial with positive coefficients, it follows that $D^\beta u(0, 0) \geq 0$ for all β. ■

PROBLEM 1.3.3. (Kovalevskaya's example) show that the line $\{ t = 0 \}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution u of the heat equation in $\mathbb{R} \times \mathbb{R}$, with $u = \frac{1}{1+x^2}$ on $\{t = 0\}$.

Hint: Assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of $(0, 0)$.

SOLUTION. First we show that the line $\Gamma := \{ t = 0 \}$ is characteristic for the heat equation. With $\nu = (1, 0)$ the normal to the line Γ, the noncharacteristic condition reads

$$\sum_{|\alpha|=2} a_\alpha \nu^\alpha \neq 0.$$

However,

$$\sum_{|\alpha|=2} a_\alpha \nu^\alpha = 1 \cdot 1 + a_{0,2} \cdot 0 = 1 \neq 0.$$

Thus, Γ is characteristic for $u_t = u_{xx}$.

Now, suppose u is an analytic solution to the heat equation $u_t - u_{xx} = 0$ given by

$$u(x,t) = \sum_{m,n} \frac{a_{m,n}}{m!n!} x^m t^n.$$

Let us compute the coefficients $a_{m,n}$ near $(0,0)$. From the PDE, we have the relation

$$\begin{aligned} a_{m,n} &= D^{(m,n)} u(0,0) \\ &= D^{(m,n-1)} u_t(0,0) \\ &= D^{(m,n-1)} u_{xx}(0,0) \\ &= D^{(m+2,n-1)} u(0,0) \\ &= a_{m+2,n-1}. \end{aligned} \tag{1.3.1}$$

Form the initial condition, we have

$$u(x,0) = \sum_{k=1}^{\infty} (-1)^k x^{2k} \tag{1.3.2}$$

for a sufficiently small neighborhood about $(0,0)$, where the right-hand side is given taylor series of $\frac{1}{1+x^2}$. Taking the m^{th} x-partial derivative at $(0,0)$, with the help of Eq. (1.3.2) we find the coefficients

$$a_{m,0} = \begin{cases} 0 & \text{if } m = 2k+1 \text{ is odd} \\ (-1)^k (2k)! & \text{if } m = 2k \text{ is even.} \end{cases} \tag{1.3.3}$$

Putting all of this information together, we deduce that

$$a_{2m+1,n} = 0$$

for all m, n and, recursively,

$$a_{2m,n} = a_{2m+2,n-1} = \cdots = a_{2(m+n),0} = (-1)^{m+n} \big(2(m+n)\big)!.$$

Thus, for small $t > 0$ we have

$$u(0,t) = \sum_n a_{0,n} t^n. \tag{1.3.4}$$

However, by the ratio test, we see that the coefficients of the form $a_{0,n}$ grow very quickly; i.e.,

$$\begin{aligned} \frac{|a_{0,n+1}|}{|a_{0,n}|} &= \frac{\frac{(2n+2)!}{2n!}}{\frac{(n+1)!}{n!}} \\ &= \frac{(2n+2)(2n+1)}{n+1} \\ &= 2(2n+1) \end{aligned}$$

which approaches ∞ as $n \to \infty$. Therefore, the radius of convergence for (1.3.4) is zero. This contradicts the assumption that u is analytic. $\frac{1}{R} = \infty$ so ■

1.4 Homework 4

PROBLEM 1.4.1 (Legendre transform). Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u\big(\mathbf{x}(p)\big),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

Hint: See [Evans, 4.4.3b], prove the identities (29).

SOLUTION. ■

PROBLEM 1.4.2. Find the solution $u(x,t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$\begin{cases} u(x,0) = f(x), & u_t(x,0) = g(x), \quad \text{for } x > 0, \\ u_t(0,t) = \alpha u_x(0,t), & \qquad\qquad\qquad\ \text{for } t > 0, \end{cases}$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$.

Hint: Use a representation $u(x,t) = F(x-t) + G(x+t)$ for the solution.

SOLUTION. ■

PROBLEM 1.4.3. (a) Let u be a solution of the wave equation $u_{tt} - c^2u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0,t) = u(\pi,t) = 0$. Show that the *energy*

$$E(t) = \frac{1}{2}\int_0^\pi \left(u_t^2 + c^2u_x^2\right) dx, \quad t > 0$$

is independent of t; i.e., $dE/dt = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x,t) = \sum_{n=1}^{\infty} \big(a_n \cos(nct) + b_n \sin(nct)\big) \sin(nx)$$

in terms of coefficients a_n, b_n.

Solution. ∎

1.5 Homework 5

PROBLEM 1.5.1. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$, $x \in \mathbb{R}^n$, then $\Delta v = 0$.

SOLUTION. ∎

PROBLEM 1.5.2. Let $n = 2$ and U be the halfplane $\{ x_2 > 0 \}$. Prove that

$$\sup_U u = \sup_{\partial U} u$$

for $u \in C^2(U) \cap C(\bar{U})$ which are harmonic in U under the additional assumption that u is bounded from above in $\bar{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.)

Hint: Take for $\epsilon > 0$ the harmonic function

$$u(x_1, x_2) - \epsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region $\left\{ x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0 \right\}$ with large a. Let $\epsilon \to 0$.

SOLUTION. ∎

PROBLEM 1.5.3. Let $U \subseteq \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Let $\varphi \colon \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \ldots, u^m$ are harmonic in U and

$$v := \varphi(u_1, \ldots, u_m).$$

Prove v is subharmonic.

Hint: Convexity for a smooth function $\varphi(z)$ is equivalent to $\sum_{j,k=1}^m \varphi_{z_j, z_k}(z)\xi_j\xi_k \geq 0$ for any $\xi \in \mathbb{R}^m$.

(b) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. (Assume that harmonic functions are C^∞.)

SOLUTION. ∎

1.6 Homework 6

PROBLEM 1.6.1. For $n = 2$ find Green's function for the quadrant $U := \{ x_1, x_2 > 0 \}$ by repeated reflection.

SOLUTION. Taking the hit, set $x' := (x_1, -x_2)$, $x'' := (-x_1, x_2)$, $x''' := (-x_1, -x_2)$, and define

$$\varphi^x(y) := \Phi(y - x') + \Phi(y - x'') - \Phi(y - x'''). \tag{1.6.1}$$

We claim that φ^x, as defined above, solves

$$\begin{cases} \Delta\varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(y - x) & \text{on } \partial U. \end{cases}$$

It is clear that $\Delta\varphi^x = 0$ since it is built up from the fundamental solutions on $\mathbb{R}^n$ (this follows from the linearity of the Laplace operator). To see that $\varphi^x(y) = \Phi(x - y)$ on ∂U, we do a case by case analysis.

Note that on $\{ x_1 = 0 \} \subseteq \partial U$, we have

$$\varphi^x(y_1, 0) = \Phi(-x_1, y_2 + x_2) + \Phi(-x_1, y_2 - x_2) - \Phi(x_1, y_2 + x_2),$$

where, since the fundamental solution is radial, we have $\Phi(-x_1, y_2 + x_2) = \Phi(x_1, y_2 + x_2)$, and hence the above equals

$$\begin{aligned} &= \Phi(-x_1, y_2 - x_2) \\ &= \Phi(y - x) \end{aligned}$$

and on $\{ x_2 = 0 \} \subseteq \partial U$, we have

$$\varphi^x(0, y_2) = \Phi(y_1 - x_1, x_2) + \Phi(y_1 + x_1, -x_2) - \Phi(y_1 + x_1, x_2)$$

where, again because Φ is radial, $\Phi(y_1 + x_1, -x_2) = \Phi(y_1 + x_1, x_2)$, thus the above equals

$$\begin{aligned} &= \Phi(y_1 - x_1, x_2) \\ &= \Phi(y - x). \end{aligned}$$

Thus, $\varphi^x(y) = \Phi(y - x)$ on ∂U.

Therefore, Green's function on U is

$$G(x, y) = \Phi(y - x) - \varphi^x(y) = \Phi(y - x) - \Phi(y - x') - \Phi(y - x'') + \Phi(y - x'''). \qquad \blacksquare$$

PROBLEM 1.6.2. (Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B(0, r) = \{ x \in \mathbb{R}^n : |x| < r \}$.

Solution. Recall Poisson's formula for the ball

$$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \, dS(y), \tag{1.6.2}$$

where $x \in B(0,r)$ and u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,r), \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

For fixed $x \in B(0,r)$, write

$$u(x) = r^{n-2}(r+|x|)(r-|x|) \left[\frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \, dS(y) \right].$$

Now, since $r + |x| \geq |x-y| \geq r - |x|$ for all $y \in \partial B(0,r)$, we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} ⨍_{\partial B(0,r)} g(y) \, dS(y) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} ⨍_{\partial B(0,r)} g(y) \, dS(y). \tag{1.6.3}$$

Since $u = g$ on the boundary $\partial B(0,r)$, by applying the mean-value property on (1.6.3) we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} u(0),$$

as desired. ■

Problem 1.6.3. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$\begin{cases} P_k(\lambda x) = \lambda^k P_k(x), & P_m(\lambda x) = \lambda^m P_m(x) & \text{for every } x \in \mathbb{R}^n, \ \lambda > 0, \\ \Delta P_k = 0, & \Delta P_m = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

(a) Show that

$$\begin{cases} \dfrac{\partial P_k}{\partial \nu} = k P_k(x), \quad \dfrac{\partial P_m}{\partial \nu} = m P_m(x) \quad \text{on } \partial B(0,1), \end{cases}$$

where $B(0,1) = \{ x \in \mathbb{R}^n : |x| < 1 \}$ and ν is the outward normal on $\partial B(0,1)$.

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x) \, d\sigma = 0, \qquad \text{if } k \neq m.$$

Solution. For part (a), let

$$P_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha.$$

Then, since $\nu = (x_1, \dots, x_n)$, the derivative along ν is given by

$$\begin{aligned}
\frac{\partial P_k(x)}{\partial \nu} &= \sum_{j=1}^{n} (P_k)_{x_j} x_j \\
&= \sum_{j=1}^{n} \left(\sum\nolimits_{|\alpha|=k} a_\alpha x^\alpha \right)_{x_j} x_j \\
&= \sum_{j=1}^{n} \left(\sum\nolimits_{l=1}^{m} a_\alpha x_1^{\alpha_1^l} \cdots x^{\alpha_j^l} \cdots x^{\alpha_n^l} \right)_{x_j} x_j
\end{aligned}$$

where $\sum_{j=1}^{n} \alpha_j^l = k$ and $1 \le j \le \binom{n+k-1}{n} =: m$ (by the stars and bars theorem)

$$\begin{aligned}
&= \sum_{j=1}^{n} \sum_{l=1}^{m} \left(\alpha_j^l a_\alpha x_1^{\alpha_1^l} \cdots x^{\alpha_j^l - 1} \cdots x^{\alpha_n^l} \right) x_j \\
&= \sum_{j=1}^{n} \sum_{l=1}^{m} \alpha_j^l a_\alpha x_1^{\alpha_1^l} \cdots x^{\alpha_j^l} \cdots x^{\alpha_n^l} \\
&= \sum_{j=1}^{n} \sum_{l=1}^{m} \alpha_j^l a_\alpha x^\alpha
\end{aligned}$$

switching the order of summation, we have

$$\begin{aligned}
&= \sum_{l=1}^{m} \sum_{j=1}^{n} \alpha_j^l a_\alpha x^\alpha \\
&= \sum_{l=1}^{m} k a_\alpha x^\alpha \\
&= k \sum_{l=1}^{m} a_\alpha x^\alpha \\
&= k P_k(x).
\end{aligned}$$

This argument, of course, applies to every $k \in \mathbb{N}$.

For part (b), by Green's theorem, we have

$$\begin{aligned}
\int_{B(0,r)} P_k(x) \Delta P_m(x) - (\Delta P_k(x)) P_m(x)\, dx &= \int_{\partial B(0,r)} P_k(x) \frac{\partial}{\partial \nu} P_m(x) - \frac{\partial}{\partial \nu} P_k(x) P_m(x)\, dS(x) \\
&= \int_{\partial B(0,r)} (m-k) P_k(x) P_m(x)\, dS(x),
\end{aligned}$$

where the left-hand side is equal to zero since both ΔP_k and ΔP_m are zero. Since $m \neq k$, it must be the case that

$$\int_{\partial B(0,r)} P_k(x) P_m(x)\, dS(x) = 0.$$

■

1.7 Homework 7

PROBLEM 1.7.1. Solve the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ \quad u = x_1 & \text{on } |x| = 1, \\ \quad u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

Hint: Use Laurent series.

SOLUTION. First, let us make the change of variables $(x_1, x_2) \mapsto r\mathrm{e}^{\mathrm{i}\theta}$ to the Dirichlet problem in question:

$$\begin{cases} \Delta u = 0 & \text{in } 1 < r < 2, \\ \quad u = \dfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) & \text{on } r = 1, \\ \quad u = 1 + \dfrac{1}{i}(\mathrm{e}^{\mathrm{i}2\theta} - \mathrm{e}^{-\mathrm{i}2\theta}) & \text{on } r = 2. \end{cases} \tag{1.7.1}$$

Now, suppose u is a solution, of the form

$$u(r\mathrm{e}^{\mathrm{i}\theta}) = b \ln r + \sum_{n=-\infty}^{\infty} (a_n r^n + \overline{a_{-n}} r^{-n}) \mathrm{e}^{\mathrm{i}n\theta},$$

to the problem (1.7.1). It is easy to see that u is in fact harmonic:

$$\begin{aligned} \Delta u &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \\ &= -br^{-2} + br^{-2} + \sum_{n=-\infty}^{\infty} \Big[\big(n(n-1) + n - n^2\big) a_n r^n \\ &\qquad + \big(n(n-1) + n - n^2\big) \overline{a_{-n}} r^{-n} \Big] \mathrm{e}^{\mathrm{i}n\theta} \\ &= 0. \end{aligned}$$

Next we use the boundary data to compute the coefficients a_n, $n \in \mathbb{Z}$. Using the data (1.7.1), on $r = 1$ we have

$$\frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) = \sum_{n=-\infty}^{\infty} (a_n + \overline{a_{-n}}) \mathrm{e}^{\mathrm{i}n\theta},$$

and on $r = 2$

$$1 + \frac{1}{i}(\mathrm{e}^{\mathrm{i}2\theta} - \mathrm{e}^{-\mathrm{i}2\theta}) = b \ln 2 + \sum_{n=-\infty}^{\infty} (2^n a_n + 2^{-n} a_{-n}) \mathrm{e}^{\mathrm{i}n\theta}.$$

These equations immediately tell us that $b = 1/\ln 2$. Moreover, the following relations on the coefficients hold

$$\begin{cases} \dfrac{1}{2} = a_1 + \overline{a_{-1}} & \dfrac{1}{2} = a_{-1} + \overline{a_1}, \\ \dfrac{1}{i} = 2^2 a_2 + 2^{-2}\overline{a_{-2}}, & -\dfrac{1}{i} = 2^2 a_{-2} + 2^{-2}\overline{a_2}, \\ 0 = a_n + \overline{a_{-n}} & \qquad\qquad\qquad \text{for } n \neq \pm 1, \\ 0 = 2^n a_n + 2^{-n}\overline{a_{-n}} & \qquad\qquad\qquad \text{for } n \neq \pm 2. \end{cases}$$

A little calculation shows that

$$\begin{cases} a_1 = -\dfrac{1}{6}, \quad a_{-1} = \dfrac{2}{3}, \\ a_2 = -\dfrac{4\mathrm{i}}{15}, \quad a_{-2} = -\dfrac{4\mathrm{i}}{15}, \\ a_n = 0 \qquad\qquad\qquad \text{for } n \neq \pm 1, \pm 2. \end{cases}$$

Thus,

$$\begin{aligned} u(r\mathrm{e}^{\mathrm{i}\theta}) &= \frac{1}{\ln 2}\ln r + \left(-\frac{4\mathrm{i}}{15}r^{-2} + \frac{4\mathrm{i}}{15}r^2\right)\mathrm{e}^{-\mathrm{i}2\theta} + \left(\frac{2}{3}r^{-1} - \frac{1}{6}r\right)\mathrm{e}^{-\mathrm{i}\theta} \\ &\quad + \left(-\frac{1}{6}r + \frac{2}{3}r^{-1}\right)\mathrm{e}^{\mathrm{i}\theta} + \left(-\frac{4\mathrm{i}}{15}r^2 + \frac{4\mathrm{i}}{15}r^{-2}\right)\mathrm{e}^{\mathrm{i}2\theta} \\ &= \frac{1}{\ln 2}\ln r - \frac{8}{15}r^{-4}\left(\frac{r^2\mathrm{e}^{\mathrm{i}2\theta} - r^2\mathrm{e}^{-\mathrm{i}2\theta}}{2\mathrm{i}}\right) + \frac{8}{15}\left(\frac{r^2\mathrm{e}^{\mathrm{i}2\theta} - r^2\mathrm{e}^{-\mathrm{i}2\theta}}{2\mathrm{i}}\right) \\ &\quad + \frac{4}{3}r^{-2}\left(\frac{r\mathrm{e}^{\mathrm{i}\theta} + r\mathrm{e}^{-\mathrm{i}\theta}}{2}\right) - \frac{1}{3}\left(\frac{r\mathrm{e}^{\mathrm{i}\theta} + r\mathrm{e}^{-\mathrm{i}\theta}}{2}\right). \end{aligned}$$

Substituting back, we have

$$u(x_1, x_2) = \frac{1}{\ln 2}\ln(x_1^2 + x_2^2) - \frac{16x_1x_2}{15(x_1^2 + x_2^2)^2} + \frac{16x_1x_2}{15} + \frac{4x_1}{3(x_1^2 + x_2^2)} - \frac{x_1}{3}. \tag{1.7.2}$$

It is easy to see that (1.7.2) satisfies the boundary data at $|x| = 1$ and $|x| = 2$. ∎

PROBLEM 1.7.2. Let Ω be a bounded domain with a C^1 boundary, $g \in C^2(\partial\Omega)$ and $f \in C(\bar{\Omega})$. Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where ν is the outer normal on $\partial\Omega$. Show that the solution of $(*)$ in $C^2(\Omega) \cap C^1(\bar{\Omega})$ is unique up to a constant; i.e., if u_1 and u_2 are both solutions of $(*)$, then $u_2 = u_1 + \text{const.}$ in Ω.

Hint: Look at the proof of the uniqueness for the Dirichlet problem by energy methods [E, 2.2.5a].

Solution. Suppose u_1 and u_2 are solutions to the Neumann problem $(*)$. Define $v := u_1 - u_2$. Then v is harmonic in Ω and $\partial v/\partial\nu = 0$ on $\partial\Omega$. Consider the energy functional

$$E[v] = \frac{1}{2}\int_\Omega |Dv|^2\,dx.$$

By Green's formula version (ii),

$$\begin{aligned} E[v] &= \frac{1}{2}\int_\Omega |Dv|^2\,dx \\ &= -\frac{1}{2}\int_\Omega v\Delta v\,dx + \int_{\partial U}\frac{\partial v}{\partial\nu} v\,dS(x) \\ &= 0. \end{aligned}$$

This implies that $|Dv|^2 = Dv\cdot Dv = 0$ which, since the standard inner product on $\mathbb{R}^n$ is positive-definite, implies that $Dw \equiv 0$. It follows that $u_1 = u_2 + \text{const}$, i.e., the solution u to $(*)$ is unique up to a constant. ■

Problem 1.7.3. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta_x u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\}, \end{cases}$$

where $c \in \mathbb{R}$.

Hint: Rewrite the problem in terms of $v(x,t) := \mathrm{e}^{ct}u(x,t)$.

Solution. Taking the hint, let us rewrite the problem in terms of $v(x,t) = \mathrm{e}^{ct}u(x,t)$:

$$\begin{cases} v_t - \Delta_x v = \mathrm{e}^{ct} f & \text{in } \mathbb{R}^n \times (0,\infty), \\ v = g & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\}. \end{cases} \tag{1.7.3}$$

By Duhamel's principle, the problem (1.7.3) is solved by

$$v(x,t) = \int_{\mathbb{R}^n}\Phi(x-y,t)g(y)\,dy + \int_0^t\int_{\mathbb{R}^n}\Phi(x-y,t-s)\mathrm{e}^{cs}f(y,s)\,dyds,$$

where Φ is the fundamental solution to the heat equation. Thus, the formula

$$u(x,t) = \mathrm{e}^{-ct}v(x,t) = \mathrm{e}^{-ct}\int_{\mathbb{R}^n}\Phi(x-y,t)g(y)\,dy + \mathrm{e}^{-ct}\int_0^t\int_{\mathbb{R}^n}\Phi(x-y,t-s)\mathrm{e}^{cs}f(y,s)\,dy\,ds$$

solves the original problem. ■

1.8 Homework 8

PROBLEM 1.8.1. Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t)$$

where

$$\Phi(x,t) = \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

is positive for $|x| < 1$, $t > 0$.

Hint: Show that u satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{cases} u = 0 & \text{on } \{\,|x| = 1\,\} \times \{\,t \geq 0\,\}, \\ u = \delta_0 & \text{on } \{\,|x| \leq 1\,\} \times \{\,t = 0\,\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{\,|x| \leq 1\,\} \times \{\,\epsilon \leq t \leq T\,\}$ for small $\epsilon > 0$ and large $T > 0$ pass to the limit as $\epsilon \to 0+$ and $T \to \infty$.

SOLUTION. Taking the hint, let us verify that $u_t = u_{xx}$, for $t > 0$. By direct computation, we have

$$\begin{aligned} \Phi_x(x,t) &= \frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \\ &= -\frac{x\mathrm{e}^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}, \end{aligned} \qquad \begin{aligned} \Phi_{xx}(x,t) &= \frac{\partial}{\partial x}\left(-\frac{x\mathrm{e}^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}\right) \\ &= \frac{x^2\mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)\mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}}, \end{aligned}$$

and

$$\begin{aligned} \Phi_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \\ &= \frac{x^2\mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{\mathrm{e}^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)\mathrm{e}^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}}. \end{aligned}$$

Since $\Phi_t = \Phi_{xx}$ it follows that $u_t = u_{xx}$ (assuming uniform convergence of u).

Next we show that $u = 0$ on $\{\,|x| = 1\,\} \times \{\,t \geq 0\,\}$ and $u = \delta_0$ on $\{\,|x| = 1\,\} \times \{\,t = 0\,\}$.

To show $u = 0$ fix a $t \geq 0$ and, after relabeling if necessary, assume that $x = 1$ which gives us

$$\begin{aligned} u(1,t) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{-\frac{(1-2k)^2}{4t}}}{\sqrt{4\pi t}} \\ &= \frac{1}{\sqrt{4\pi t}} \Big(\cdots - \mathrm{e}^{-\frac{9}{4t}} + \mathrm{e}^{-\frac{1}{4t}} - \mathrm{e}^{-\frac{1}{4t}} + \mathrm{e}^{-\frac{9}{4t}} + \cdots \Big) \\ &= 0. \end{aligned}$$

Similarly for $u(-1,t) = 0$.

For $u(|x| \leq 1, 0)$, we have a

$$\begin{aligned} u(|x| \leq 1, 0) &= \sum_{k=-\infty}^{\infty} (-1)^k \lim_{t \to 0+} \left[\mathrm{e}^{-\frac{(x-2k)^2}{4t}} / \sqrt{4\pi t} \right] \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \delta_0(x - 2k) \\ &= \delta_0(x) \end{aligned}$$

since $|x| \leq 1$ and values δ_0 is zero for values $x - 2k$ outside of the interval $[-1, 1]$.

At last we show that u is positive for $|x| < 1$, $t > 0$. Seeking a contradiction, suppose u is negative on some point (x_0, t_0) in $\{\, |x| < 1 \,\} \times \{\, \epsilon \leq t \leq T \,\}$. Then by the minimum principle, u achieves its minimum somewhere on the bottom boundary $\{\, |x| = 1 \,\} \times \{\, t = \epsilon \,\}$. Therefore, there exists a sequence $(x_n, t_n+) \to (x, 0)$, where $|x_n|, |x| < 1$, such that $u(x, 0) < 0$. However, we have shown above that $u(x, 0) = \delta_0(x)$ for $|x| < 1$; i.e., either $u(x, 0) = 0$ or $u(x, 0) = +\infty$. This is a contradiction. Therefore, it must be the case that $u \geq 0$ for $|x| < 1$, $t > 0$. ■

Problem 1.8.2 (Tikhonov's example). Let

$$g(t) := \begin{cases} \mathrm{e}^{-t^{-2}} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then $g \in C^\infty(\mathbb{R})$ and we define

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that $u(x,t)$ solves the heat equation in $\mathbb{R} \times (0, \infty)$ with the initial condition $u(x, 0) = 0$, $x \in \mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

Solution. Let u be as above. Then

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k)!}x^{2k}\right) \\ &= \sum_{k=0}^{\infty}\frac{g^{(k+1)}(t)}{(2k)!}x^{2k} \\ &= \sum_{k=2}^{\infty}\frac{g^{(k)}(t)}{(2k-2)!}x^{2k-2}, \end{aligned}$$

and

$$\begin{aligned} u_x(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k)!}x^{2k}\right) \\ &= \sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k)!}2kx^{2k-1} \\ &= \sum_{k=1}^{\infty}\frac{g^{(k)}(t)}{(2k-1)!}x^{2k-1}, \end{aligned} \qquad \begin{aligned} u_{xx}(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k-1)!}x^{2k-1}\right) \\ &= \sum_{k=1}^{\infty}\frac{g^{(k)}(t)}{(2k-1)!}(2k-1)x^{2k-2} + \frac{\partial}{\partial x}g^{(0)}(t) \\ &= \sum_{k=2}^{\infty}\frac{g^{(k)}(t)}{(2k-2)!}x^{2k-2}. \end{aligned}$$

Thus, $u_t - \Delta u = 0$; i.e., u solves the heat equation. As this example shows, unless some assumptions on u such as subexponential (cf. [E §2.3], Theorem 7) growth is assumed. ■

Problem 1.8.3. Evaluate the integral

$$\int_{-\infty}^{\infty}\cos(ax)\mathrm{e}^{-x^2}\,dx, \qquad (a>0).$$

Hint: Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.

Solution. By separation of variables,

$$u(x,t) = \cos(ax)\mathrm{e}^{-a^2t}$$

is a solution to the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = \cos(ax) & \text{on } \mathbb{R}\times\{\,t=0\,\}. \end{cases}$$

However, the convolution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\cos(ay)\mathrm{e}^{-\frac{|x-y|^2}{4t}}\,dy$$

is also a solution to the Cauchy problem. Now note that

$$\begin{aligned}\int_{-\infty}^{\infty} \cos(ay)\mathrm{e}^{-y^2}\,dy &= \sqrt{\pi}\cdot u(0,\tfrac{1}{4})\\ &= \sqrt{\pi}\mathrm{e}^{-\frac{a^2}{4}}.\end{aligned}$$ ■

1.9 Homework 9

PROBLEM 1.9.1. (a) Show that for $n = 3$ the general solution to the wave equation $u_{tt} - \Delta u = 0$ with spherical symmetry about the origin has the form

$$u = \frac{1}{r}F(r+t) + \frac{1}{r}G(r-t), \quad r = |x|,$$

with suitable F and G.

(b) Show that the solution with initial data of the form

$$u(r,0) = 0, \quad u_t(r,0) = h(r)$$

(h is an even function of r) is given by

$$u = \frac{1}{2r}\int_{r-t}^{r+t} \rho h(\rho)\, d\rho.$$

Solution. ■

PROBLEM 1.9.2. Show that the solution $w(x_1, t)$ of the initial-value problem for the *Klein–Gordon equation*

$$\begin{cases} w_{tt} = w_{x_1 x_1} - \lambda^2 w, & \\ w(x_1, 0) = 0, & w_t(x_1, 0) = h(x_1) \end{cases} \tag{1.9.1}$$

is given by

$$w(x_1, t) = \frac{1}{2}\int_{x_1-t}^{x_1+t} J_0(\lambda s) h(y_1)\, dy_1.$$

Here $s^2 = t^2 - (x_1 - y_1)^2$, while J_0 denotes the Bessel function defined by

$$J_0(z) := \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(z \sin\theta)\, d\theta.$$

Hint: Descend to (1.9.1) from the two-dimensional wave equation satisfied by

$$u(x_1, x_2, t) = \cos(\lambda x_2) w(x_1, t).$$

Solution. ■

PROBLEM 1.9.3. Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \qquad u = g, \quad u_t = h & \text{on } \mathbb{R}^3 \times \{\, t = 0 \,\} \end{cases}$$

where g and h are smooth and have compact support. Show there exists a constant C such that

$$|u(x,t)| \le Ct^{-1} \quad (x \in \mathbb{R}^3,\ t > 0).$$

Solution. ■

2 Exams

2.1 Midterm Practice Problems

PROBLEM 2.1.1. Solve $u_{x_1}^2 + x_2 u_{x_2} = u$ with initial conditions $u(x_1, 1) = x_1^2/4 + 1$.

SOLUTION. By inspection, we may suspect that $v(x_1, x_2) = x_1^2/4 + x_2$ is a solution to the PDE. It certainly satisfies the boundary condition. A routine calculation shows that v is in fact a solution to the PDE. Lucky guess!

More formally, let us solve this problem using the method of characteristics. First, write

$$F(p, z, x) = \left(p^1(s)\right)^2 + x^2(s)p^2(s) - z(s) = 0.$$

Then, the characteristic ODEs are

$$\begin{cases} \left(\dot{p}^1(s), \dot{p}^2(s)\right) = -(0, p^2(s)) + (p^1(s), p^2(s)) \\ \qquad = (p^1(s), 0), \\ \dot{z}(s) = (2p^1(s), x^2(s)) \cdot (p^1(s), p^2(s)) \\ \qquad = 2p^1(s)^2 + x^2(s)p^2(s), \\ \left(\dot{x}^1(s), \dot{x}^2(s)\right) = (2p^1(s), x^2(s)). \end{cases}$$

Now, for $(x^1(0), x^2(0)) = (x^0, 1)$, integrating the characteristics, we get

$$\begin{cases} \left(p^1(s), p^2(s)\right) = \left(p_0^1 e^s, p_0^2\right), \\ \left(x^1(s), x^2(s)\right) = \left(2p_0^1 e^s + x_0^1, x_0^2 e^s\right), \\ z(s) = \dfrac{\left(x^0\right)^2}{4} e^{2s} + p_0^2 e^s + z^0 \end{cases}$$

Using the initial condition and the PDE, we find that

$$\begin{aligned} &p_0^1 = \frac{x^0}{2}, \quad p_0^2 = \frac{\left(x^0\right)^2}{4} + 1 - \frac{\left(x^0\right)^2}{4} = 1, \\ &x_0^1 = 0, \qquad x_0^2 = 1 \\ &z^0 = 0, \end{aligned}$$

and consequently

$$\begin{cases} \left(x^1(s), x^2(s)\right) = \left(x^0 e^s, e^s\right), \\ z(s) = \dfrac{\left(x^0\right)^2}{4} e^{2s} + e^s \end{cases}$$

so, rewriting z in terms of (x^1, x^2), we have

$$\begin{aligned} z(s) &= \frac{\left(x^0\right)^2}{4} e^{2s} + e^s \\ &= \frac{\left(x^1(s)\right)^2}{4} + x^2(s), \end{aligned}$$

so the solution in terms of (x_1, x_2), is

$$u(x_1, x_2) = \frac{x_1^2}{4} + x_2,$$

just as we suspected. ■

Problem 2.1.2. Find the maximal $t_0 > 0$ for which the (classical) solution of the Cauchy problem

$$\begin{cases} uu_x + u_t = 0, \\ \quad u(x,0) = \mathrm{e}^{-\frac{x^2}{2}}, \end{cases}$$

exists in $\mathbb{R} \times [0, t)$; i.e., the first time $t = t_0$ when the shock develops.

Solution. First, let us find a solution to the PDE using the method of characteristics. Write

$$F(p, z, x) = z(s)p^1(s) + p^2(s).$$

Then, the characteristic ODEs are

$$\begin{cases} \big(\dot{p}^1(s), \dot{p}^2(s)\big) = -(0,0) - p^1(p^1(s), p^2(s)) \\ \qquad\qquad\qquad = \big(-p^1(s)^2, -p^1(s)p^2(s)\big), \\ \qquad\qquad \dot{z}(s) = (z(s), 1) \cdot (p^1(s), p^2(s)) \\ \qquad\qquad\qquad = z(s)p^1(s) + p^2(s) \\ \qquad\qquad\qquad = 0, \\ \big(\dot{x}(s), \dot{t}(s)\big) = (z(s), 1). \end{cases}$$

Thus, integrating the characteristic ODEs from $(x^0, 0)$, we have

$$\begin{cases} \qquad \dot{z}(s) = z^0, \\ \big(x(s), t(s)\big) = (z^0 s + x^0, s); \end{cases}$$

since the PDE is quasilinear, we disregard (p^1, p^2).

Applying the boundary conditions, we see that

$$z^0 = u(x^0, 0) = \mathrm{e}^{-(x^0)^2/2}.$$

Here's where it gets tricky. After a little struggling, we see that there is really no way to solve for z in terms of $(x(s), t(s))$. However, we can solve for the projected characteristics:

$$(x(t, y), t) = (\mathrm{e}^{-y^2/2} t + y, t);$$

and this is really all that matters for us to find the time t_0 when the shock develops, i.e., the time when the projected characteristic fails to be injective.

A little calculation shows that this happens precisely when $t = \mathrm{e}^{-1/2}$. ■

PROBLEM 2.1.3. If ρ_0 denotes the maximum density of cars on a highway (i.e., under bumpet-to-bumper conditions), then a reasonable model for traffic density ρ is given by

$$\begin{cases} \rho_t + (F(\rho))_x = 0, \\ \qquad F(\rho) = c\rho\left(1 - \dfrac{\rho}{\rho_0}\right), \end{cases}$$

where $c > 0$ is a constant (free speed of highway). Suppose the initial density is

$$\rho(x,0) = \begin{cases} \frac{1}{2}\rho_0 & \text{if } x < 0, \\ \rho_0 & \text{if } x > 0. \end{cases}$$

Find the shock curve and describe the weak solution. (Interpret your result for the traffic flow.)

SOLUTION. First, note that

$$\begin{aligned} (F(\rho))_x &= F'(\rho)\rho_x \\ &= \left[-c\frac{\rho}{\rho_0} + c\left(1 - \frac{\rho}{\rho_0}\right)\right]\rho_x \\ &= \left(c - \frac{2c\rho}{\rho_0}\right)\rho_x. \end{aligned}$$

Let us find a solution to the PDE using the method of characteristics. Write

$$G(p,z,x) = p^2(s) + F'(z(s))p^1(s).$$

Then, the characteristic ODEs are

$$\begin{cases} \big(\dot{p}^1(s), \dot{p}^2(s)\big) = \big(-F''(z(s))p^1(s), -F''(z(s))p^2(s)\big), \\ \qquad \dot{z}(s) = F'(z(s))p^1(s) + p^2(s) \\ \qquad\quad = 0, \\ \big(\dot{x}^1(s), \dot{x}^2(s)\big) = \big(F'(z(s)), 1\big). \end{cases}$$

Now, integrating the characteristics, we have

$$\begin{cases} \qquad z(s) = z^0, \\ \big(x^1(s), x^2(s)\big) = \big(F'(z^0)s + x^0, s\big). \end{cases}$$

We have two cases to consider, $x^0 < 0$ or $x^0 > 0$. For $x^0 < 0$, $z^0 = \frac{\rho_0}{2}$ and the projected characteristics look like

$$\begin{aligned} \left(F'(\frac{\rho_0}{2})t + x^0, t\right) &= \left(\left[c - \frac{2c(\frac{\rho_0}{2})}{\rho_0}\right]t + x^0, t\right) \\ &= (0 \cdot t + x^0, t) \\ &= (x^0, t) \end{aligned}$$

(where we have replaced s with the more appropriate t). Whereas for $x^0 > 0$, we have

$$\begin{aligned}\big(F'(\rho_0)t + x^0, t\big) &= \left(\left[c - \frac{2c\rho_0}{\rho_0}\right]t + x^0, t\right)\\ &= (-ct + x^0, t).\end{aligned}$$

These characteristics intersect precisely when

$$t = \frac{x_1^0 - x_2^0}{c},$$

where $x_1^0 > 0$, $x_2^0 < 0$. ■

Problem 2.1.4. Find the characteristics of the second order equation

$$u_{xx} - (2\cos x)u_{xy} - (3 + \sin^2 x)u_{yy} - yu_y = 0,$$

and transform it to the canonical form.

Solution. First, writing the PDE in the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + 2Du_x + 2Eu_y + Fu = 0,$$

we see that $A = 1$, $B = -\cos x$, $C = -3\sin^2 x$, and $E = -\frac{y}{2}$. We solve for the characteristic curve by find a solution to the ODEs

$$\begin{aligned}\frac{dy}{dx} &= \frac{B \pm \sqrt{B^2 - AC}}{A}\\ &= -\cos x \pm \sqrt{\cos^2 x + 3 + \sin^2 x}\\ &= -\cos x \pm 2.\end{aligned}$$

The solutions give us the following ODEs

$$\begin{cases} y = -\sin x + 2x + \xi(x,y),\\ y = -\sin x - 2x + \eta(x,y).\end{cases}$$

Integrating these equations, we have

$$\begin{cases} \xi(x,y) = y + \sin x - 2x,\\ \eta(x,y) = y + \sin x + 2x.\end{cases}$$

These are the characteristic strips for the PDE.

To put this PDE in canonical form, we first compute the following partial derivatives

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x,\\ u_y &= u_\xi \xi_y + u_\eta \eta_y,\\ u_{xx} &= u_\xi \xi_{xx} + u_\eta \eta_{xx} + (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + (u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x)\eta_x\\ &= u_{\xi\xi}(\xi_x)^2 + u_{\eta\eta}(\eta_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx},\end{aligned}$$

exploiting symmetry, we can find u_{yy} by replacing x with y above

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + u_{\eta\eta}(\eta_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_\xi\xi_{yy} + u_\eta\eta_{yy},$$

the last thing we need to figure out is the mixed partial

$$\begin{aligned} u_{xy} &= u_\xi\xi_{xy} + u_\eta\eta_{xy} + (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + (u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y)\eta_x \\ &= u_{\xi\xi}\xi_x\xi_y + u_{\eta\eta}\eta_x\eta_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_\xi\xi_{xy} + u_\eta\eta_{xy}. \end{aligned}$$

Now find the partials $\xi_x, \eta_x, \xi_y, \eta_y, \xi_{xy}, \ldots$, etc.

$$\begin{aligned} \xi_x &= \cos x - 2, & \eta_x &= \cos x + 2, \\ \xi_{xx} &= -\sin x, & \eta_{xx} &= -\sin x, \\ \xi_{xy} &= 0, & \eta_{xy} &= 0, \\ \xi_y &= 1, & \eta_y &= 1, \\ \xi_{yy} &= 0, & \eta_{yy} &= 0. \end{aligned}$$

Thus,

$$\begin{cases} u_x = (\cos x - 2)u_\xi + (\cos x + 2)u_\eta, \\ u_y = u_\xi + u_\eta, \\ u_{xx} = (\cos x - 2)^2 u_{\xi\xi} + (\cos x + 2)^2 u_{\eta\eta} \\ \qquad + 2(\cos x + 2)(\cos x - 2)u_{\xi\eta} - (\sin x)u_\xi - (\sin x)u_\eta \\ \quad = (\cos^2 x - 4\cos x + 4)u_{\xi\xi} + (\cos^2 x + 4\cos x + 4)u_{\eta\eta} \\ \qquad + 2(\cos^2 x - 4)u_{\xi\eta} - (\sin x)u_\xi - (\sin x)u_\eta \\ u_{yy} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}, \\ u_{xy} = (\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}, \end{cases}$$

so the canonical form is

$$\begin{aligned} 0 &= u_{xx} - (2\cos x)u_{xy} - (3\sin^2 x)u_{yy} - yu_y \\ &= \xi^2 u_{\xi\xi} + \eta^2 u_{\eta\eta} \\ &\quad + 2\xi\eta u_{\xi\eta} - (\sin x)u_\xi - (\sin x)u_\eta \\ &\quad - (2\cos x)\big((\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}\big) \\ &\quad - (3\sin^2 x)(u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}) \\ &\quad - y(u_\xi + u_\eta) \end{aligned}$$

Who cares. ■

PROBLEM 2.1.5. Let $Lu \coloneqq u_{xx} - 4u_{yy} + \sin(y + 2x)u_x = 0$.

(a) Consider the level curve $\Gamma = \{\, (x, y) : \varphi(x, y) = C \,\}$ where $|D\varphi| \neq 0$ on Γ. Define what it means for Γ to be characteristic with respect to L at a point $(x_0, y_0) \in \Gamma$.
(b) Find the points at which the curve $x^2 + y^2 = 5$ is characteristic.

(c) Is it true that every smooth simple closed curve Γ in $\mathbb{R}^2$ has at least one point at which it is characteristic with respect to L?

Solution. ∎

Problem 2.1.6. Consider the second order equation

$$Lu := u_{xx} - 2xu_{xy} + x^2 u_{yy} - 2u_y = 0.$$

(a) Find the characteristic curves of $Lu = 0$. What is the type of this equation?
(b) Find the points on the line $\Gamma := \{ (x, y) \in \mathbb{R}^2 : x + y = 1 \}$ at which Γ is characteristic with respect to $Lu = 0$.

Solution. ∎

Problem 2.1.7. Solve the initial boundary value problem for the equation $u_{tt} = u_{xx}$ in $\{ x > 0, t > 0 \}$ satisfying

$$\begin{cases} u(x,0) = \sin^2 x, \quad u_t(x,0) = \sin x, \\ u(0,t) = 0. \end{cases}$$

Solution. ∎

Problem 2.1.8. Consider the initial/boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } 0 < x < \pi,\ t > 0, \\ u(x,0) = x, \quad u_t(x,0) = 0 & \text{for } 0 < x < \pi, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0 & \text{for } t > 0. \end{cases}$$

(a) Find a weak solution of the problem.
(b) Is the solution unique? Continuous? C^1?

Solution. ∎

Problem 2.1.9. Let B_1^+ denote the open half-ball $\{ x \in \mathbb{R}^n : |x| < 1, x_n > 0 \}$. Assume $u \in C(\bar{B}_1^+)$ is harmonic in B_1^+ with $u = 0$ on $\partial B_1^+ \cap \{ x_n = 0 \}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0, \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0, \end{cases}$$

for $x \in B_1$. Prove v is harmonic in B_1.

Hint: It will be enough to prove that $\int_B \nabla v \nabla \eta \, dx = 0$ for any test function $\eta \in C_0^\infty(B_1)$. Split $\int_{B_1} = \int_{B_1^+} + \int_{B_1^-}$ and apply the integration by parts formula to each of $\int_{B_1^\pm}$.

Solution. ■

Problem 2.1.10. Let u and v be harmonic functions in the unit ball $B_1 \subseteq \mathbb{R}^n$. What can you conclude about u and v if

(a) $D^\alpha u(0) = D^\alpha v(0)$ for every multiindex α?
(b) $u(x) \leq v(x)$ for every $x \in B_1$ and $u(0) = v(0)$?

Justify your answer in each case.

Solution. ■

Problem 2.1.11. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

is a solution to the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial, i.e., $f(y) = f(|y|)$, and supported in $B_R := \{\, |x| < R \,\}$, then

$$u(x) = c\Phi(x)$$

for any $x \in \mathbb{R}^n \setminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y)\, dy.$$

Hint: Use polar (spherical) coordinates and apply the mean value property for harmonic functions.

Solution. ■

2.2 Final Practice Problems

Problem 2.2.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Show that the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u + \alpha \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

has at most one solution in $C^2(\Omega) \cap C(\bar{\Omega})$ if $\alpha > 0$. Here ν is the outward normal on $\partial\Omega$ and f, g are assumed to be smooth.

Solution. Let us assume that Ω is also a connected subset of $\mathbb{R}^n$. We will use energy methods to show that there is only one solution to the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u + \alpha \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases} \tag{2.2.1}$$

Suppose u_1 and u_2 are two distinct solutions to the problem (2.2.1). Define $v := u_1 - u_2$. Then v solves the problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ v + \alpha \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the energy

$$E[v] = \frac{1}{2} \int_\Omega |Dv|^2 \, dx.$$

By Green's formula, we may recast the expression above as the sum

$$\begin{aligned} E[v] &= -\frac{1}{2}\left[\int_\Omega v\Delta v \, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} v \, dS(x)\right] \\ &= -\frac{\alpha}{2} \int_{\partial\Omega} v^2 \, dS(x) \\ &\geq 0. \end{aligned}$$

However, since $\alpha > 0$ and v^2 is strictly positive, it must be the case that $v \equiv 0$ on $\partial\Omega$. The maximum principle then implies that $v \equiv 0$ in Ω. It follows that $u_1 = u_2$; i.e., the solution is unique. ∎

Problem 2.2.2. Let g be a continuous function with compact support in $\mathbb{R}^n$. Write the formula for the bounded solution of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\}. \end{cases}$$

Prove that $\lim_{t\to\infty} u(x,t) = 0$, where the convergence is uniform in $x \in \mathbb{R}^n$.

Solution. From previous work on the heat equation, we know that the convolution

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|x-y|^2}{4t}} g(y)\,dy$$

the initial-value problem above. A crude estimate on the magnitude of u gives us

$$\begin{aligned} |u(x,t)| &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left| \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|x-y|^2}{4t}} g(y)\,dy \right| \\ &\le \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left| \mathrm{e}^{-\frac{|x-y|^2}{4t}} g(y) \right| dy \\ &< M t^{-\frac{n}{2}}; \end{aligned} \tag{2.2.2}$$

where $M < \infty$ is chosen such that

$$\frac{1}{(4\pi)^{\frac{n}{2}}} \int_{\operatorname{Supp} g} \left| \mathrm{e}^{-\frac{|x-y|^2}{4t}} g(y) \right| dy < M.$$

Thus, using the estimate (2.2.2) we see that $\lim_{t\to\infty} u(x,t) = 0$ uniformly. ■

Problem 2.2.3. Find an explicit solution to the problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty), \\ u = \mathrm{e}^{3x} & \text{in } \mathbb{R} \times \{\, t = 0 \,\}. \end{cases}$$

Solution. By separation of variables, suppose we can write $u(x,t)$ as the product $X(x)T(t)$. Then

$$\begin{cases} X(x)T'(t) - X''(x)T(t) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ X(x)T(0) = \mathrm{e}^{3x} & \text{on } \mathbb{R} \times \{\, t = 0 \,\}. \end{cases} \tag{2.2.3}$$

After some algebraic maneuvers, we see $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = 9$ so it suffices to solve the system of ODEs

$$\begin{cases} X''(x) - 9X(x) = 0, \\ \quad T'(t) - 9T(t) = 0. \end{cases}$$

The solution to these are

$$\begin{cases} X(x) = C_1 \mathrm{e}^{3x}, \\ \ T(t) = C_2 \mathrm{e}^{9t}. \end{cases}$$

Thus,

$$u(x,t) = X(x)T(t) = C\mathrm{e}^{3x+9t}$$

solves (2.2.3). Analyzing the initial conditions, we conclude that $C = 1$. In conclusion,

$$u(x,t) = \mathrm{e}^{3x+9t}$$

solves the original problem.

Another way to solve this problem is by computing the convolution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \mathrm{e}^{3x}\, dy.$$

Putting this through WolframAlpha gives

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \left[\sqrt{4\pi t} \mathrm{e}^{9t+3x} \right] = \mathrm{e}^{9t+3x}$$

which agrees with our 'separation of variables' solution. ■

Problem 2.2.4. Find a formula for the solution of

$$\begin{cases} u_{tt} - u_{xx} + u = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = f, \quad u_t = g & \text{on } \mathbb{R} \times \{\, t = 0 \,\} \end{cases}$$

where $f, g \in C_0^\infty(\mathbb{R})$.

Hint: Method I: Use Hadamard's method of descent. Namely, find $h(y)$ such that $v(x,y,t) := h(y)u(x,t)$ solves

$$v_{tt} - (v_{xx} + v_{yy}) = 0.$$

Method II: Use the Fourier transform.

Solution. By Method I: Set $h(y) := \cos y$ and $v(x,y,t) := h(y)u(x,t)$. Then v solves the initial-value problem

$$\begin{cases} v_{tt} - (v_{xx} + v_{yy}) = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ v = \tilde{f}, \quad v_t = \tilde{g} & \text{on } \mathbb{R}^2 \times \{\, t = 0 \,\} \end{cases}$$

where $\tilde{f} := hf$ and $\tilde{g} := hg$. The solution to this problem is given by the average integral

$$v(x,y,t) = \frac{1}{2\pi t^2} \iint_{B(x,y,t)} \left[\frac{(tf(\xi) + t^2 g(\xi)) \cos\eta + tD((\cos\eta) f(\xi)) \cdot (\xi - x, \eta - y)}{(t^2 - (\xi - x)^2 - (\eta - y)^2)^{\frac{1}{2}}} \right] d\xi\, d\eta$$

Therefore, the equation

$$v(x,0,t) = \frac{1}{2\pi t^2} \iint_{B(x,0,t)} \left[\frac{(tf(\xi) + t^2 g(\xi)) \cos\eta + t[f'(\xi)(\xi - x) - \eta \sin(\eta)]}{(t^2 - (\xi - x)^2 - \eta^2)^{\frac{1}{2}}} \right] d\xi\, d\eta$$

solves the original problem.

To simplify this, let us first compute the following integrals since the former integral is too large to work with directly,

$$I_1 = \frac{1}{2\pi t^2} \iint_{B(x,0,t)} \left[\frac{(tf(\xi) + t^2 g(\xi)) \cos \eta}{(t^2 - (\xi - x)^2 - \eta^2)^{\frac{1}{2}}} \right] d\xi \, d\eta,$$

$$I_2 = \frac{1}{2\pi t^2} \iint_{B(x,0,t)} \left[\frac{-t\eta \sin \eta}{(t^2 - (\xi - x)^2 - \eta^2)^{\frac{1}{2}}} \right] d\xi \, d\eta,$$

$$I_3 = \frac{1}{2\pi t^2} \iint_{B(x,0,t)} \left[\frac{tf'(\xi)(\xi - x)}{(t^2 - (\xi - x)^2 - \eta^2)^{\frac{1}{2}}} \right] d\xi \, d\eta.$$

Throughout the following analysis, set $s := \sqrt{t^2 - (\xi - x)^2}$. For I_1, we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi t^2} \int_{x-t}^{x+t} \left[\int_{-s}^{s} \frac{\cos \eta}{\sqrt{s^2 - \eta^2}} \, d\eta \right] (tf(\xi) + t^2 g(\xi)) \, d\xi \\ &= \frac{1}{2t} \int_{x-t}^{x+t} J_0(s)[f(\xi) + tg(\xi)] \, d\xi; \end{aligned} \tag{2.2.4}$$

for I_2, we have

$$\begin{aligned} I_2 &= \frac{1}{2\pi t^2} \int_{x-t}^{x+t} \left[\int_{-s}^{s} \frac{\eta \sin \eta}{\sqrt{s - \eta^2}} \, d\eta \right] (-t) \, d\xi \\ &= -\frac{1}{2t} \int_{x-t}^{x+t} sJ_1(s) \, d\xi; \end{aligned} \tag{2.2.5}$$

and for I_3, we have

$$\begin{aligned} I_3 &= \frac{1}{2\pi t^2} \int_{x-t}^{x+t} \left[\int_{-s}^{s} \frac{1}{\sqrt{s^2 - \eta^2}} \, d\eta \right] tf'(\xi)(\xi - x) \, d\xi \\ &= \frac{1}{2t} \int_{x-t}^{x+t} f'(\xi)(\xi - x) \, d\xi. \end{aligned} \tag{2.2.6}$$

Putting (2.2.4), (2.2.5), and (2.2.6) together, we have

$$u(x,t) = \frac{1}{2t} \int_{x-t}^{x+t} [J_0(s)(f(\xi) + tg(\xi)) + f'(\xi)(\xi - x) - sJ_1(s)] \, d\xi$$

which solves the initial-value problem in question. ∎

PROBLEM 2.2.5. Let $u \in C^2(\mathbb{R}^n \times [0, \infty))$ satisfy

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{in } \mathbb{R}^n \times \{ t = 0 \}. \end{cases}$$

Show that if both g and h are radial, then so is $u(\,\cdot\,,t)$ for any $t > 0$. (Recall that the function f is called radial if $f(x) = f(|x|)$.)

Solution. Let $A \in \mathrm{SO}(n)$ be a rotation matrix. Set $v(x) := u(Ax)$. Then v solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = g(Ax) = g(x), \quad v_t(x,0) = h(Ax) = h(x) & \text{in } \mathbb{R}^n \times \{\, t = 0 \,\}. \end{cases}$$

By the uniqueness for the wave equation there exist at most one C^2 solution to the initial-value problem above. Thus, it must be the case that $v = u$; i.e., $u(x) = u(|x|)$ since $A \in \mathrm{SO}(3)$ was arbitrary. ■

Problem 2.2.6. Find the value of the solution u of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{for } x \in \mathbb{R}^3,\ t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = \varphi(x), \end{cases}$$

where

$$\varphi(x) := \begin{cases} 1 & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a \end{cases}$$

at a point (x,t) such that $|x| + t < a$.

Solution. By Kirchhoff's formula, to solution to this initial-value problem is given by

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t\varphi(y)\, dS(y).$$

Then, since $\varphi(y) \equiv 1$ for $|y| = |x| + t < a$, the integral above becomes

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t\, dS(y) = t.$$ ■

Problem 2.2.7. Let u be a nonzero harmonic function in $B(0,R) := \{\, x \in \mathbb{R}^n : |x| < R \,\}$. Define

$$E(r) := \fint_{\partial B(0,r)} u^2(y)\, dS(y).$$

Show that $\ln E(r)$ is a convex function of $\ln r$; i.e.,

$$E\big(\sqrt{ab}\big)^2 \leq E(a)E(b), \quad \text{for } a, b > 0,$$

for any $0 < a \leq c < R$.

Hint: Use the representation of u as a uniformly convergent series

$$u(x) = \sum_{k=0}^{\infty} p_k(x), \qquad |x| < R,$$

where $p_k(x)$ is a homogeneous harmonic polynomial of order k.

Solution. Write
$$u(x) = \lim_{n\to\infty} \sum_{k=0}^{n} p_k(x), \qquad |x| < R,$$
where $p_k(x)$ is a homogeneous harmonic polynomial of order k; the limit converges uniformly. Then
$$E(r) = \fint_{\partial B(0,r)} \Big[\lim_{n\to\infty} \sum\nolimits_{k=0}^{n} p_k(x)\Big]^2 \, dS(y)$$
expand the sum by the multinomial theorem
$$= \fint_{\partial B(0,r)} \lim_{n\to\infty} \left[\sum_{k_0+\cdots+k_n=2} \binom{2}{k_0,\ldots,k_n} p_0(x)^{k_0}\cdots p_n(x)^{k_n}\right] dS(y)$$
since the limit is uniform, we may interchange the limit with the integral
$$= \lim_{n\to\infty} \left[\fint_{\partial B(0,r)} \sum_{k_0+\cdots+k_n=2} \binom{2}{k_0,\ldots,k_n} p_0(x)^{k_0}\cdots p_n(x)^{k_n} \, dS(y)\right]$$
$$= \lim_{n\to\infty} \fint_{\partial B(0,r)} p_0(x)^2 + \cdots + p_n(x)^2 \, dS(y)$$
since harmonic polynomials of distinct degrees are orthogonal
$$= \fint_{\partial B(0,r)} \Big[\sum\nolimits_{k=0}^{\infty} p_k(x)^2\Big] \, dS(y);$$
i.e.,
$$E(r) = \fint_{\partial B(0,r)} \sum_{k=0}^{\infty} p_k(x)^2 \, dS(y) \tag{2.2.7}$$
for $r < R$.

Now, applying the Cauchy–Schwartz to (2.2.7) with $r = \sqrt{ab}$ we achieve the desired inequality. ∎

Problem 2.2.8. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution to the following Cauchy problem,
$$\begin{cases} u_{tt} - \Delta u = \mathrm{e}^{-t} f(x) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u = u_t = 0, & \text{on } \mathbb{R}^3 \times \{\, t = 0 \,\}. \end{cases}$$
Verify that the integral representation reduces to the obvious solution $u = \mathrm{e}^{-t} + t - 1$ when $f(x) = 1$.

Solution. Proceeding by Duhamel's principle, define $v := u(x,t;s)$. Then v is a solution of
$$\begin{cases} v_{tt} - \Delta v = \mathrm{e}^{-t} f(x) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ v = 0, \quad v_t(\,\cdot\,;s) = \mathrm{e}^{-s} f(\,\cdot\,), & \text{on } \mathbb{R}^3 \times \{\, t = 0 \,\}. \end{cases} \tag{2.2.8}$$

By Kirchhoff's formula,

$$\begin{aligned} v(x,t;s) &= \fint_{\partial B(x,t)} t\mathrm{e}^{-s} f(y)\, dS(y) \\ &= \frac{\mathrm{e}^{-s}}{4\pi t} \int_{\partial B(x,t)} f(y)\, dS(y) \\ &= \frac{\mathrm{e}^{-s}}{4\pi(t-s)} \int_{\partial B(x,t-s)} f(y)\, dS(y) \end{aligned}$$

solves (2.2.8).

Then,

$$\begin{aligned} u(x,t) &= \int_0^t \left[\int_{\partial B(x,t-s)} f(y)\, dS(y) \right] \left(\frac{\mathrm{e}^{-s}}{4\pi(t-s)} \right) ds \\ &= \int_0^t \left[\int_{\partial B(x,t-s)} \left(\frac{f(y)}{t-s} \right) dS(y) \right] \left(\frac{\mathrm{e}^{-s}}{4\pi} \right) ds \\ &= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,r)} \frac{\mathrm{e}^{t-r} f(y)}{r}\, dS(y)\, dr \end{aligned} \tag{2.2.9}$$

solves the original problem.

In the case $f(x) = 1$, (2.2.9) becomes

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,r)} \frac{\mathrm{e}^{t-r}}{r}\, dS(y)\, dr \\ &= \frac{1}{4\pi} \int_0^t \left[\int_{\partial B(x,r)} dS(y) \right] \frac{\mathrm{e}^{-r}}{r}\, dr \\ &= \frac{1}{4\pi} \int_0^t 4\pi r^2 \left(\frac{\mathrm{e}^{-r}}{r} \right) dr \\ &= \int_0^t r\mathrm{e}^{-r}\, dr \\ &= \mathrm{e}^{-t} + t - 1. \end{aligned}$$

■

Problem 2.2.9. Let $f(x) = \mathrm{e}^{-|x|^2}$, $x \in \mathbb{R}^n$. Find $f * f$.

Hint: Use either the heat equation or the Fourier transform.

Solution. First we proceed by the heat equation. Suppose u is a solution to the initial-value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = f & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\} \end{cases} \tag{2.2.10}$$

where $f(x) := \mathrm{e}^{-|x|^2}$. Then

$$u(x,t) = f(x) * \Phi(x,t),$$

where Φ is the fundamental solution to the heat equation, solves (2.2.10). But

$$\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|x|^2}{4t}} = \frac{1}{(4\pi t)^{\frac{n}{2}}} [f(x)]^{\frac{1}{4t}}.$$

Therefore, the convolution we are after is precisely

$$(f * f)(x) = \pi^{\frac{n}{2}} u(x, \tfrac{1}{4}).$$

Solving for $u(x, \frac{1}{4})$, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-|x-y|^2} f(y)\, dy \\
&= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-|x-y|^2} \mathrm{e}^{-|y|^2}\, dy \\
&= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{e}^{-|x-y|^2-|y|^2}\, dy
\end{aligned}$$

which, by Fubini's theorem, becomes the product of integrals in one coordinate

$$\begin{aligned}
&= \frac{1}{\pi^{\frac{n}{2}}} \prod_{k=1}^{n} \left[\int_{\mathbb{R}} \mathrm{e}^{-|x_k-y_k|^2-|y_k|^2}\, dy_k \right] \\
&= \frac{1}{\pi^{\frac{n}{2}}} \prod_{k=1}^{n} \left[\int_{\mathbb{R}} \mathrm{e}^{-(x_k^2-2x_k y_k+y_k^2)-y_k^2}\, dy_k \right] \\
&= \frac{1}{\pi^{\frac{n}{2}}} \mathrm{e}^{-|x|^2} \prod_{k=1}^{n} \underbrace{\left[\int_{\mathbb{R}} \mathrm{e}^{2x_k y_k - 2y_k^2}\, dy_k \right]}_{I_k}.
\end{aligned}$$

Let us find I_k and complete the solution of u above. This is,

$$\begin{aligned}
I_k &= \int_{\mathbb{R}} \mathrm{e}^{-2\left(y_k - \frac{1}{2}x_k\right)^2 + \frac{1}{2}x_k^2}\, dy_k \\
&= \frac{\mathrm{e}^{\frac{1}{2}x_k^2}}{2} \int_{\mathbb{R}} \mathrm{e}^{-z^2}\, dz \\
&= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}x_k^2}.
\end{aligned}$$

Thus, $u(x, \frac{1}{4})$ is

$$u(x, \tfrac{1}{4}) = \frac{1}{2^{\frac{n}{2}}} \mathrm{e}^{-\frac{|x|^2}{2}}$$

so

$$f * f = \left(\frac{\pi}{2}\right)^{\frac{n}{2}} \mathrm{e}^{-\frac{|x|^2}{2}}. \qquad \blacksquare$$

Remarks. We still had to compute the convolution $f * f$ barehanded. Realizing it as the solution to the heat equation was of no help. Perhaps finding a solution through separation of variables is supposed to make this problem easier.

Problem 2.2.10. Recall that a solution to the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dt,$$

where, for $t > 0$,

$$\Phi(z,t) = (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x|^2}{4t}}.$$

Assume that g is continuous and compactly supported. Show that there exists a $C > 0$ such that

$$|Du(x,t)| \le C t^{-\frac{1}{2}} \|g\|_{L^\infty}.$$

Solution. Let us immediately jump to the partial derivative D_{x_k} of u,

$$D_{x_j} u = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(y) \left[-\frac{(x_j - y_j)}{2t} \right] \mathrm{e}^{-\frac{-|x-y|^2}{4t}}\, dy.$$

Hence,

$$\begin{aligned} |D_x u(x,t)| &\le \frac{\|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi)^{\frac{n}{2}}} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|x-y|}{2t} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, dy \\ &= \frac{\|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi)^{\frac{n}{2}}} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|z|}{2t} \mathrm{e}^{-\frac{|z|^2}{4t}}\, dz \end{aligned}$$

setting $w = |z|/\sqrt{t}$, we have

$$\begin{aligned} &= \frac{\|g\|_{L^\infty(\mathbb{R}^n)}}{2(4\pi)^{\frac{n}{2}}} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{|w|}{t^{\frac{1}{2}}} \mathrm{e}^{-\frac{w^2}{4}} t^{\frac{n}{2}}\, dw \\ &= \frac{\|g\|_{L^\infty(\mathbb{R}^n)}}{2(4\pi)^{\frac{n}{2}}} \frac{1}{\sqrt{t}} \int_{\mathbb{R}^n} |w| \mathrm{e}^{-\frac{w^2}{4}}\, dw \\ &= \frac{\|g\|_{L^\infty(\mathbb{R}^n)} C_n}{\sqrt{t}}, \end{aligned}$$

where

$$C_n := \frac{1}{2(4\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |w| \mathrm{e}^{-\frac{w^2}{4}}\, dw > 0. \qquad \blacksquare$$

3 Qualifying Exams

3.1 Qualifying Exam, August ‘04

PROBLEM 3.1.1. Consider the initial value problem

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = -u, \\ \hfill u = f \qquad \text{on } S^1 = \{\, x^2 + y^2 = 1 \,\}, \end{cases}$$

where a and b satisfy

$$a(x,y) + b(x,y)y > 0$$

for any $x, y \in \mathbb{R}^n \smallsetminus \{(0,0)\}$.

(a) Show that the initial value problem has a unique solution in a neighborhood of S^1. Assume that a, b, and f are smooth.
(b) Show that the solution of the initial value problem actually exists in $\mathbb{R}^2 \smallsetminus \{(0,0)\}$.

SOLUTION. ∎

PROBLEM 3.1.2. Let $u \in C^2\big(\mathbb{R} \times [0,\infty)\big)$ be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty), \\ \qquad u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where f and g have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx,$$

and the potential energy by

$$P(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx.$$

Show that

(a) $K(t) + P(t)$ is constant in t,
(b) $K(t) = P(t)$ for all large enough times t.

SOLUTION. ∎

PROBLEM 3.1.3. Use Kirchhoff’s formula and Duhamel’s principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = \mathrm{e}^{-t} g(x) & \text{for } x \in \mathbb{R}^3,\ t > 0, \\ \quad u(x,0) = u_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution $u = \mathrm{e}^{-t} + t - 1$ when $g(x) = 1$.

Solution. ∎

Problem 3.1.4. Let Ω be a bounded open set in $\mathbb{R}^n$ and $g \in C_0^\infty(\Omega)$. Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega,\ t > 0, \\ u(x,0) = g(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } xi \in \partial\Omega,\ t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n,\ t > 0, \\ v(x,0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put $g = 0$ outside Ω.

(a) Show that

$$-v(x,t) \leq u(x,t) \leq v(x,t)$$

for any $x \in \Omega$, $t > 0$.

(b) Use (a) to conclude that

$$\lim_{t\to\infty} u(x,t) = 0,$$

for any $x \in \Omega$.

Solution. ∎

Problem 3.1.5. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \qquad P_m(\lambda x) = \lambda^m P_m(x),$$

for any $x \in \mathbb{R}^n$, $\lambda > 0$,

$$\Delta P_k = 0, \qquad \Delta P_m = 0$$

in $\mathbb{R}^n$.

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = kP_k(x), \qquad \frac{\partial P_m(x)}{\partial \nu} = mP_m(x)$$

on ∂B_1, where $B_1 = \{\, |x| < 1 \,\}$ and ν is the outward normal on ∂B_1.

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x)\, dS = 0,$$

if $k \neq m$.

Solution. ∎

3.2 Qualifying Exam, August '05

PROBLEM 3.2.1.

(a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ \qquad\qquad u = 1 \qquad \text{on } S^1 = \{\, x^2 + y^2 = 1 \,\}. \end{cases}$$

(b) Is the solution unique in a neighborhood of the point $(1, 0)$? Justify your answer.

SOLUTION. The solution to teh first part is

$$u(x, y) = \frac{x^2 + y^2}{4} + \frac{3}{4}.$$

■

PROBLEM 3.2.2. Consider the second order PDE in $\{\, x > 0, y > 0 \,\} \subseteq \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

(a) Classify the equation and reduce it to the canonical form.
(b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy}\, G(x/y).$$

SOLUTION. ■

PROBLEM 3.2.3. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^3$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if f is radial (i.e., $f(y) = f(|y|)$) and supported in $B_R = \{\, |x| < R \,\}$, then

$$u(x) = c\Phi(x),$$

for any $x \in \mathbb{R}^n \setminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y)\, dy.$$

[*Hint:* Use spherical (polar) coordinates and the mean value property.]

SOLUTION. ■

PROBLEM 3.2.4. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \qquad u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where $g, h \in C_0^\infty(\mathbb{R}^2)$ and $\alpha \geq 0$ is a constant.

(a) Show that for an appropriate choice of constant λ and μ the function

$$v(x_1, x_2, x_3, t) := \mathrm{e}^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation $v_{tt} - \Delta v = 0$.

(b) Use (a) to prove the following domain of dependence result: for any point $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$ the value $u(x_0, t_0)$ is uniquely determined by values of g and h in $\overline{B_{t_0}(x_0)} := \{\, |x - x_0| \leq t_0 \,\}$. (You may use the corresponding result for the wave equation without proof.)

SOLUTION. ■

PROBLEM 3.2.5. Let $u(x,t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0, \infty)$ with the initial condition

$$u(x,0) = u_0(x)$$

for $x \in \mathbb{R}$, where $u_0 \in C^\infty$ is 2π-periodic, i.e., $u_0(x + 2\pi) = u_0(x)$. Show that

$$\lim_{t \to \infty} u(x,t) = a_0,$$

uniformly in $x \in \mathbb{R}$, where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x)\, dx.$$

SOLUTION. ■

3.3 Qualifying Exam, January ‘14

PROBLEM 3.3.1. Consider the first order equation in $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

(a) Find the characteristic curves of the equation.
(b) Consider the Cauchy problem for this equation prescribed on the line $x_1 = 1$:

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on f so that the proble is solvable in a neighborhood of the point $(1, 0)$.

SOLUTION. ■

PROBLEM 3.3.2. Let u be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{ x_n > 0 \}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where g is a continuous function with compact support in $\mathbb{R}^{n-1}$. Here $n \geq 2$. Prove that

$$u(x) \longrightarrow 0, \qquad \text{as } |x| \longrightarrow \infty,$$

for $x \in \{ x_n > 0 \}$.

SOLUTION. ■

PROBLEM 3.3.3. Let u be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \qquad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions $u(x, 0) = g(x)$, where g is a bounded continuous function on $\mathbb{R}$ satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^\alpha, \qquad x, y \in \mathbb{R}$$

with a constant $\alpha \in (0, 1]$. Show that

$$\begin{aligned} |u(x,t) - u(y,t)| &\leq M|x - y|^\alpha, & x, y \in \mathbb{R}, t > 0, \\ |u(x,t) - u(x,s)| &\leq C_\alpha M |t - s|^{\alpha/2}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of $u(x,t)$ as a convolution with the heat kernel $\Phi(y,t)$, make a change of variables $z = y/\sqrt{t}$ and use that $\left|\sqrt{t} - \sqrt{s}\right| \leq \sqrt{|t - s|}$.]

SOLUTION. ■

PROBLEM 3.3.4. Let u be a positive harmonic function in the unit ball B_1 in $\mathbb{R}^n$. Show that

$$\left|D(\ln u)\right| \le M \qquad \text{in } B_{1/2}$$

for a constant M depending only on the dimension n.
[*Hint:* Use the interior derivative estimate $\left|Du(x)\right| \le (C_n/r)\sup_{B_r(x)}|u|$ for $B_r(x) \subseteq B_1$ as well as the Harnack inequality for harmonic functions.]

SOLUTION. ■

PROBLEM 3.3.5. Let u be a C^2 solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0,\infty), \\ \qquad u = 0, \qquad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some $k \ge 0$. Prove that there exists a function $\varphi(r)$ such that

$$u(x,t) = t^{k+2}\varphi\big(|x|/t\big).$$

[*Hint:* As one of the steps show that u is $(k+2)$-homogeneous in (x,t) variables, i.e., $u(\lambda x, \lambda t) = \lambda^{k+2}u(x,t)$ for any $\lambda > 0$.]

SOLUTION. ■