MA 544: Homework 12

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Problem 12.1 (Wheeden & Zygmund §8, Ex. 2)

Prove the converse of Hölder's inequality for $p = 1$ and ∞. Show also that for $1 \leq p \leq \infty$, a real-valued measurable f belongs to $L^p(E)$ if $fg \in L^1(E)$ for every $g \in L^{p'}(E)$, $1/p + 1/p' = 1$. The negation is also of interest: if $f \in L^p(E)$ then there exists $g \in L^{p'}(E)$ such that $fg \notin L^1(E)$. (To verify the negation, construct g of the form $\sum a_k g_k$ satisfying $\int_E fg_k \to \infty$.)

Proof. In this problem, we finish the proof of Theorem 8.8 for the case $p = 1, \infty$. Therefore, we must show that:

> For f a measurable real-valued function on E and $p = 1, \infty$. Then
> $$\|f\|_p = \sup \int_E fg,$$
> where the supremum is taken over every real-valued g such that $\|g\|_{p'} \leq 1$ and $\int_E fg$ exists.

In both cases, $p = 1$ and $p = \infty$, we may, without loss of generality, assume $\|f\|_p \neq 0$; otherwise, by Hölder's inequality, $\|fg\|_1 \leq \|f\|_p \|g\|_{p'} = 0$ implies $\|fg\|_1 = 0$ so, by Theorem 5.11, $fg = 0$ almost everywhere on E and therefore, $f = 0$ almost everywhere on E.

Let us prove this for $p = 1$. Recall that, by convention, if $p = 1$ its conjugate exponent, p', is ∞ and vice versa. Suppose $\|g\|_\infty \leq 1$ and the integral $\int_E fg$ exists. One direction is trivial, namely, by Hölder's inequality

$$\int_E fg \leq \int_E |fg| \leq \|f\|_1 \|g\|_1 \leq \|f\|_1, \tag{12.1}$$

for all g with $\|g\|_\infty \leq 1$. Hence,

$$\sup \int_E fg \leq \|f\|_1.$$

To get the reverse inequality, consider $g := \operatorname{sgn} f$. The function g is measurable since $g = f/|f|$ for all $f(\mathbf{x}) \neq 0$ and $g = 0$ otherwise. Moreover, g is in $L^\infty(E)$ since $\|g\|_\infty \leq 1$, that is, $|g| \leq 1$ almost everywhere on E. Therefore

$$\|f\|_1 = \int_E |f| = \int_E fg \leq \sup_{\|g'\|_\infty \leq 1} \int_E fg'. \tag{12.2}$$

Thus, $\|f\|_1 = \sup \int fg$ where the supremum is taken over all $g \in L^\infty(E)$ with $\|g\| \leq 1$.

Now, consider the case where $p = \infty$. By Hölder's inequality, it is clear that

$$\sup \int_E fg \leq \|f\|_\infty \tag{12.3}$$

since $\int_E fg \leq \|f\|_\infty \|g\|_1$ for all $g \in L(E)$. To prove the reverse inequality, we consider the cases $\|f\|_\infty < \infty$ and $\|f\|_\infty = \infty$ separately.

Suppose $0 < \|f\|_\infty < \infty$; we may, without loss of generality, assume $\|f\|_\infty = 1$ by normalizing f by its essential supremum. Now, by definition

$$\|f\|_\infty = \inf\{\, \alpha : |\{\, \mathbf{x} \in E : f(\mathbf{x}) > \alpha \,\}| = 0 \,\} = 1. \tag{12.4}$$

Set $E_k \coloneqq \{\, \mathbf{x} \in E : f(\mathbf{x}) > 1 - 1/k \,\} \cap B(\mathbf{0}, k)$. Then $E_k \nearrow \bigcup E_k$ and $|E \setminus \bigcup E_k| = 0$ by Equation (12.4) and the definition of the essential supremum. Therefore, $\int_E fg = \int_{\bigcup E_k} fg$. Moreover, $|E_k| < |B(\mathbf{0}, k)| < \infty$ so we can define the sequence of functions

$$g_k(\mathbf{x}) \coloneqq \begin{cases} \dfrac{1}{|E_k|} & \text{if } x \in E_k \\ 0 & \text{otherwise} \end{cases}. \tag{12.5}$$

Note that $\|g_k\|_1 = 1$ and

$$\int_E fg_k = \int_{E_k} fg_k \geq \int_{E_k} \left(1 - \frac{1}{k}\right) g_k = \left(1 - \frac{1}{k}\right) \int_E g_k = 1 - \frac{1}{k}$$

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Problem 12.2 (Wheeden & Zygmund §8, Ex. 3)

Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when $p < 1$.

Proof. Recall the statement of Theorem 8.12

> *Suppose that* $1 \le p \le \infty$, $1/p + 1/p' = 1$, $a = \{a_k\}$, $b = \{b_k\}$, *and* $ab = \{a_k b_k\}$. *Then* $\|ab\|_1 \le \|a\|_p \|b\|_{p'}$.

The second inequality in the full statement of the theorem is straight forward since

$$\sum_k |a_k b_k| \le \sum \left|(\sup_k |a_k|)|b_k|\right| = \sup_k |a_k| \cdot \sum_k |b_k|.$$

This proves the statement for $p = 1, \infty$.

As in the proof of Hölder's inequality, we may, without loss of generality, assume $\|a\|_p = \|b\|_{p'} = 1$ (the other cases being trivial, i.e., $\|a\|_p = 0$ or $\|b\|_{p'} = 0$, or reducible to this one by, for instance, taking a_k' to be $a_k/\|a\|_p$ and b_k' to be $b_k/\|b\|_{p'}$). Now, suppose $1 < p < \infty$. By Young's inequality, we have

$$\begin{aligned}
\sum_k |a_k b_k| &\le \sum_k \left(\frac{|a_k|^p}{p} + \frac{|b_k|^{p'}}{p'} \right) \\
&= \frac{1}{p} \sum_k |a_k|^p + \frac{1}{p'} \sum_k |b_k|^{p'} \\
&= \frac{1}{p} \|a\|_p{}^p + \frac{1}{p'} \|b\|_{p'}{}^{p'} \\
&= \frac{1}{p} + \frac{1}{p'} \\
&= 1 \\
&= \|a\|_p \|b\|_{p'},
\end{aligned}$$

as was to be shown.

Recall the statement of Theorem 8.13

> *Suppose that* $1 \le p \le \infty$, $1/p + 1/p' = 1$, $a = \{a_k\}$, $b = \{b_k\}$, *and* $ab = \{a_k b_k\}$. *Then* $\|a + b\|_p \le \|a\|_p + \|b\|_p$.

For $p = 1$, Minkowski's inequality is nothing more than the triangle inequality so we are finished. For $p = \infty$, by the triangle inequality, we have

$$|a_k + b_k| \le |a_k| + |b_k| \le \sup_k |a_k| + \sup_k |b_k|.$$

By the definition of the supremum, since the right hand side of the inequality above holds for all k, the right-hand side is an upper bound for $|a_k + b_k|$, so

$$\sup_k |a_k + b_k| \le \sup_k |a_k| + \sup_k |b_k|.$$

holds.

Now, suppose $1 < p < \infty$. Then, we have

$$\begin{aligned}
{\|a+b\|_p}^p &= \sum_k |a_k + b_k|^p \\
&= \sum_k |a_k + b_k|^{p-1}|a_k + b_k| \\
&\le \sum_k |a_k + b_k|^{p-1}|a_k| + \sum_k |a_k + b_k|^{p-1}|b_k| \\
&= \sum_k \left(|a_k + b_k|^{p(p-1)}\right)^{1/p}(|a_k|^p)^{1/p} + \sum_k \left(|a_k + b_k|^{p(p-1)}\right)^{1/p}(|b_k|^p)^{1/p} \\
&\le \left[\sum_k (|a_k + b_k|^p)^{(p-1)/p}\right]\left[\sum_k (|a_k|^p)^{1/p}\right] + \left[\sum_k (|a_k + b_k|^p)^{(p-1)/p}\right]\left[\sum_k (|b_k|^p)^{1/p}\right] \\
&= {\|a+b\|_p}^{p-1}\|a\|_p + {\|a+b\|_p}^{p-1}\|b\|_p \\
&= {\|a+b\|_p}^{p-1}(\|a\|_p + \|b\|_p).
\end{aligned}$$

Now, dived both sides of the inequality above by ${\|a+b\|_p}^{p-1}$ and we achieve Minkowski's inequality for ℓ^p.

To see that Minkowski's inequality fails for $p < 1$, consider the sequences $a = (0, 1, 0, \dots)$ and $b = (1, 0, \dots)$. Then

$$\|a_k + b_k\|_p = 2^{1/p}, \qquad \|a_k\|_p = 1, \qquad \|b_k\|_p = 1.$$

Since $2^{1/p} > 2$ for $p < 1$, we have

$$\|a_k + b_k\|_P \ge \|a_k\|_p + \|b_k\|_p.$$

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Problem 12.3 (Wheeden & Zygmund §8, Ex. 4)

Let f and g be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let $1 < p < \infty$. Prove that equality holds in the inequality $\left|\int fg\right| \le \|f\|_p \|g\|_{p'}$ if and only if fg has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e.

If $\|f + g\|_p = \|f\|_p + \|g\|_p$ and $g \neq 0$ in Minkowski's inequality, show that f is a multiple of g.

Find analogues of these results for the spaces ℓ^p.

Proof. $\Longleftarrow$ Suppose fg has constant sign and $|f|^p = M|g|^{p'}$. Then, by Hölder's inequality, we have

$$\begin{aligned}\left|\int fg\right| &= \int |fg| \\ &\le \left[\int |f|^p\right]^{1/p} \left[\int |g|^{p'}\right]^{1/p'} \\ &= \left[\int M|g|^{p'}\right]^{1/p} \left[\int |g|^{p'}\right]^{1/p'} \\ &= M^{1/p}\end{aligned}$$

$\Longrightarrow$

Assuming we proved the result above, recall from Minkowski's inequality that

$$\|f + g\|_p{}^p \le \int |f + g|^{p-1}|f| + \int |f + g|^{p-1}|g|.$$

Therefore, if equality holds ■

Problem 12.4 (Wheeden & Zygmund §8, Ex. 5)

For $0 < p \leq \infty$ and $0 < |E| < \infty$, define

$$N_p[f] \coloneqq \left(\frac{1}{E}\int_E |f|^p\right)^{1/p},$$

where $N_\infty[f]$ means $\|f\|_\infty$. Prove that if $p_1 < p_2$, then $N_{p_1}[f] \leq N_{p_2}[f]$. Prove also that if $1 \leq p \leq \infty$, then $N_p[f+g] \leq N_p[f] + N_p[g]$, $(1/|E|)\int_E |fg| \leq N_p[f]N_{p'}[g]$, $1/p + 1/p' = 1$, and $\lim_{p\to\infty} N_p[f] = \|f\|_\infty$. Thus, N_p behaves like $\|\cdot\|_p$ but has the advantage of being monotone in p. Recall Exercise 28 of Chapter 5.

Proof. ∎

Problem 12.5 (Wheeden & Zygmund §8, Ex. 6)

(a) Let $1 \le p_i,\ r \le \infty$ and $\sum_{i=1}^k 1/p_i = 1/r$. Prove the following generalization of Hölder's inequality:
$$\|f_1 \cdots f_k\|_r \le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

(b) Let $1 \le p < r < q \le \infty$ and define $\theta \in (0,1)$ by $1/r = \theta/p + (1-\theta)/q$. Prove the interpolation estimate
$$\|f\|_r \le \|f\|_p{}^{\theta} \|f\|_q{}^{1-\theta}.$$
In particular, if $A \coloneqq \max\{\|f\|_p, \|f\|_q\}$, then $\|f\|_r \le A$.

Proof. (a) We will proceed by induction on k the number of measurable f_k whose p_k-norm is finite. When $k = 2$, by applying Hölder's inequality on $|fg|^r$ with $1/(p/r) + 1/(p'/r) = 1$ we have

$$\begin{aligned}
\|fg\|_r{}^r &= \left(\int_E |fg|^r\right) \\
&\le \left(\int_E |f|^{r(p/r)}\right)^{r/p} \left(\int_E |g|^{r(p'/r)}\right)^{r/p'} \\
&= \|f\|_p{}^r \|g\|_{p'}{}^r.
\end{aligned}$$

Therefore,
$$\|fg\|_r \le \|f\|_p \|g\|_{p'}. \tag{12.6}$$

Now, suppose Equation (12.6) holds for $j \le n-1$ functions measurable functions $f_j \in L^{p_j}(E)$ where $\sum_j 1/p_j = r$. Suppose $\sum_{j=1}^n 1/p_j = 1/r$ with $f_j \in L^{p_j}(E)$ and consider
$$\|f_1 f_2 \cdots f_n\|_r{}^r = \int_E |f_1 f_2 \cdots f_n|^r.$$

Set $g \coloneqq f_2 \cdots f_k$ and $p' \coloneqq \left(\sum_{j=2}^n 1/p_j\right)^{-1}$, then, by (12.6), we have

$$\begin{aligned}
\|f_1 f_2 \cdots f_n\|_r &= \|f_1 g\|_r \\
&\le \|f_1\|_{p_1} \|g\|_{p'} \\
&\le \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_n\|_{p_n}
\end{aligned}$$

as desired.

(b) Without loss of generality, assume $\|f\|_p = \|f\|_q = 1$. By what we have just shown, with

$1/r = 1/(p/\theta) + 1/(p/(1-\theta))$ and Jensen's inequality, we have

$$\begin{aligned}
\|f\|_r &= \left\||f|^\theta |f|^{1-\theta}\right\| \\
&\leq \left\||f|^\theta\right\|_p \left\||f|^{1-\theta}\right\|_q \\
&= \left[\int |f|^{\theta p}\right]^{1/p} \left[\int |f|^{(1-\theta)q}\right]^{1/q} \\
&\leq \left[\int |f|^p\right]^{\theta/p} \left[\int |f|^q\right]^{(1-\theta)/q} \\
&= |f\|_r \leq \|f\|_p{}^\theta \|f\|_q{}^{1-\theta}.
\end{aligned}$$

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Problem 12.6 (Wheeden & Zygmund §8, Ex. 9)

If f is real-valued and measurable on E, $|E| > 0$, define its essential infimum on E by

$$\operatorname{ess\,inf} f \coloneqq \sup\{\, \alpha : |\{\, x \in E : f(x) < \alpha \,\}| = 0 \,\}.$$

If $f \geq 0$, show that $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup} 1/f)^{-1}$.

Proof. First, let us deal with the edge case. Suppose the essential infimum of f is zero. Then, for every $\alpha > 0$, we have $|\{\, \mathbf{x} \in E : f(\mathbf{x}) > \alpha \,\}| > 0$. Thus, for every $0 < \beta < \infty$, $|\{\, \mathbf{x} \in E : 1/f(\mathbf{x}) > \beta \,\}| > 0$ so the essential supremum of f is ∞.

If $\operatorname{ess\,inf} f = \infty$, and we interpret $1/\infty$ to mean 0, equality holds.

Now, suppose $0 < \operatorname{ess\,inf} f < \infty$. Then, there exists $\alpha > 0$ such that $|\{\, \mathbf{x} \in E : f(\mathbf{x}) < \alpha \,\}| > 0$. Thus, we have

$$\begin{aligned}
\operatorname{ess\,inf} f &= \sup\{\, \alpha : |\{\, x \in E : f(x) < \alpha \,\}| = 0 \,\} \\
&= \sup\left\{\, \frac{1}{\beta} : |\{\, \mathbf{x} \in E : f(\mathbf{x}) < 1/\beta \,\}| \,\right\} \\
&= \sup\left\{\, \frac{1}{\beta} : |\{\, \mathbf{x} \in E : 1/f(\mathbf{x}) > \beta \,\}| \,\right\} \\
&= (\inf\{\, \beta : |\{\, \mathbf{x} \in E : 1/f(\mathbf{x}) > \beta \,\}| \,\})^{-1} \\
&= (\operatorname{ess\,sup} 1/f)^{-1}
\end{aligned}$$

as desired. ∎

Problem 12.7 (Wheeden & Zygmund §8, Ex. 11)

If $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty < M$ for all k, prove that $f_k g_k \to fg$ in L^p.

Proof. First, note that, by Minkowski's inequality, we have

$$\begin{aligned}
\|fg - f_k g_k\|_p &= \|(fg - fg_k) - (fg_k - f_k g - k)\|_p \\
&\leq \|fg - fg_k\|_p + \|fg_k - f_k g_k\|_p \\
&\leq \|fg - fg_k\|_p + M\|f - f_k\|_p.
\end{aligned}$$

Since we have complete control over the $M\|f - f_k\|_p$ term, i.e., $M\|f - f_k\|_p \to 0$ as $k \to \infty$, we need only show that $\|fg_k - f_k g_k\|_p \to 0$ as $k \to \infty$. First, note that since $g_k \to g$ pointwise and the g_k are bounded above by M a.e., then $|g| \leq M$ so by the triangle inequality, $|g - g_k| \leq |g| + |g_k \leq 2M$. Thus, we have

$$\|fg - fg_k\|_p{}^p \leq 2M\|f\|_p{}^p = 2M \int |f|^p.$$

Thus, $|fg - fg_k|^p \in L$ so by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\lim_{k\to\infty} \int |fg - fg_k|^p &= \int \lim_{k\to\infty} |fg - fg_k|^p \\
&= \int \lim_{k\to\infty} |f|^p |g - g_k|^p \\
&= 0.
\end{aligned}$$

Thus, $\|fg - fg_k\|_p \to 0$ as $k \to \infty$ so $\|fg - f_k g_k\|_p \to 0$ as $k \to \infty$, as desired. ∎