

MA557 Homework 12

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PROBLEM 12.1

Let R be a Noetherian domain. Show that the following are equivalent:

- (i) R is a unique factorization domain
- (ii) every prime ideal of R of height one is principal
- (iii) R is normal with $\text{Cl}(R) = 0$.

Proof. (i) \implies (ii) Suppose R is a Noetherian domain. Let \mathfrak{p} be a height one prime. Then there exists at least one nonzero element $x \in \mathfrak{p}$. Let $x = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the factorization of x into irreducible (prime) elements of R . Set $p := p_i$ for any prime in the factorization of x . Then the ideal generated by p is a prime ideal contained in \mathfrak{p} , i.e., $\langle p \rangle \subset \mathfrak{p}$. But $\text{ht}(\mathfrak{p}) = 1$. Thus, $\langle p \rangle = \mathfrak{p}$.

(ii) \implies (i) Suppose that every height one prime ideal in R is principal. To show that R is a UFD, it suffices to show that every irreducible element p is a prime element, that is, $\langle p \rangle$ is a prime ideal. Let \mathfrak{p} be the minimal prime containing p . Since \mathfrak{p} is principal, $\mathfrak{p} = \langle x \rangle$ for some $x \in \mathfrak{p}$. Thus, $p = xy$ for some $y \in R$. But p is prime hence, irreducible so either x or y is a unit. If x is a unit, then $\mathfrak{p} = R$, which is a contradiction. Thus, y must be a unit and we see that $\langle p \rangle = \langle xy \rangle = \mathfrak{p}$ is prime.

Now, for the following implications we need to know a couple of definitions: Let $D(R)$ denote the set of divisional fractional R -ideals and $F(R)$ denote the set of all principal fractional ideals. Then the *divisor class group* of R is the quotient $\text{Cl}(R) := D(R)/F(R)$. ■

PROBLEM 12.2

Let R be a ring with total ring of quotients K , M an R -module, and

$$\mathcal{T}(M) = \{ x \in M \mid ax = 0 \text{ for some non zero-divisor } a \text{ of } RR \}.$$

The submodule $\mathcal{T}(M)$ is called the *torsion of M* , and M is called *torsion free* if $\mathcal{T}(M) = 0$. Show

- (a) $\mathcal{T}(M) = \ker(M \rightarrow K \otimes_R M)$
- (b) $M/\mathcal{T}(M)$ is torsion free.

Proof.

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PROBLEM 12.3

Let R be a Dedekind domain and M a finitely generated R -module of rank r . Show that:

- (a) If M is torsion free then M is projective (hint: induct on r).
- (b) $M \cong \mathcal{T}(M) \oplus P$ with P projective.
- (c) If $M \neq 0$ is projective then $M \cong R^{r-1} \oplus I$ with $I \neq 0$ an ideal.
- (d) If M is torsion (i.e., $M = \mathcal{T}(M)$) then

$$M \cong R/I_1 \oplus \cdots \oplus R/I_n \quad \text{with} \quad I_1 \supset \cdots \supset I_n \neq 0$$

ideals (hint: for p_1, \dots, p_s the minimal primes of $\text{ann}(M)$ and $S = R \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s)$, show that $S^{-1}R$ is a PID).

Proof.

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