

# MA 523: Homework 6

Carlos Salinas

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## PROBLEM 6.1

For  $n = 2$  find Green's function for the quadrant  $U := \{x_1, x_2 > 0\}$  by repeated reflection.

*SOLUTION.* Taking the hit, set  $x' := (x_1, -x_2)$ ,  $x'' := (-x_1, x_2)$ ,  $x''' := (-x_1, -x_2)$ , and define

$$\varphi^x(y) := \Phi(y - x') + \Phi(y - x'') - \Phi(y - x'''). \quad (6.1)$$

We claim that  $\varphi^x$ , as defined above, solves

$$\begin{cases} \Delta \varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(y - x) & \text{on } \partial U. \end{cases}$$

It is clear that  $\Delta \varphi^x = 0$  since it is built up from the fundamental solutions on  $\mathbb{R}^n$  (this follows from the linearity of the Laplace operator). To see that  $\varphi^x(y) = \Phi(y - x)$  on  $\partial U$ , we do a case by case analysis.

Note that on  $\{x_1 = 0\} \subset \partial U$ , we have

$$\varphi^x(y_1, 0) = \Phi(-x_1, y_2 + x_2) + \Phi(-x_1, y_2 - x_2) - \Phi(x_1, y_2 + x_2),$$

where, since the fundamental solution is radial, we have  $\Phi(-x_1, y_2 + x_2) = \Phi(x_1, y_2 + x_2)$ , and hence the above equals

$$\begin{aligned} &= \Phi(-x_1, y_2 - x_2) \\ &= \Phi(y - x) \end{aligned}$$

and on  $\{x_2 = 0\} \subset \partial U$ , we have

$$\varphi^x(0, y_2) = \Phi(y_1 - x_1, x_2) + \Phi(y_1 + x_1, -x_2) - \Phi(y_1 + x_1, x_2)$$

where, again because  $\Phi$  is radial,  $\Phi(y_1 + x_1, -x_2) = \Phi(y_1 + x_1, x_2)$ , thus the above equals

$$\begin{aligned} &= \Phi(y_1 - x_1, x_2) \\ &= \Phi(y - x). \end{aligned}$$

Thus,  $\varphi^x(y) = \Phi(y - x)$  on  $\partial U$ .

Therefore, Green's function on  $U$  is

$$G(x, y) = \Phi(y - x) - \Phi(y - x') - \Phi(y - x'') + \Phi(y - x''').$$

■

## PROBLEM 6.2

(Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}}u(0)$$

whenever  $u$  is positive and harmonic in  $B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$ .

*SOLUTION.*

■

## PROBLEM 6.3

Let  $P_k(x)$  and  $P_m(x)$  be homogeneous harmonic polynomials in  $\mathbb{R}^n$  of degrees  $k$  and  $m$  respectively; i.e.,

$$\begin{cases} P_k(\lambda x) = \lambda^k P_k(x), & P_m(\lambda x) = \lambda^m P_m(x) & \text{for every } x \in \mathbb{R}^n, \lambda > 0, \\ \Delta P_k = 0, & \Delta P_m = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

(a) Show that

$$\begin{cases} \frac{\partial P_k}{\partial \nu} = k P_k(x), & \frac{\partial P_m}{\partial \nu} = m P_m(x) & \text{on } \partial B(0, 1), \end{cases}$$

where  $B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$  and  $\nu$  is the outward normal on  $\partial B(0, 1)$ .

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x) d\sigma = 0, \quad \text{if } k \neq m.$$

SOLUTION. ■