MA557 Homework 12

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Problem 12.1

Let R be a Noetherian domain. Show that the following are equivalent:

(i) R is a unique factorization domain

(ii) every prime ideal of R of height one is principal

(iii) R is normal with $\mathrm{Cl}(R) = 0$.

Proof. (i) $\implies$ (ii) Suppose R is a Noetherian domain. Let $\mathfrak{p}$ be a height one prime. Then there exists at least one nonzero element $x \in \mathfrak{p}$. Let $x = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the factorization of x into irreducible (prime) elements of R. Set $p := p_i$ for any prime in the factorization of x. Then the ideal generated by p is a prime ideal contained in $\mathfrak{p}$, i.e., $\langle p \rangle \subset \mathfrak{p}$. But $\mathrm{ht}(\mathfrak{p}) = 1$. Thus, $\langle p \rangle = \mathfrak{p}$.

For the following implication we need the theorem:

Theorem (Krull's Principal Ideal Theorem)**.** *In a Noetherian ring, every minimal prime ideal of a principal ideal has height at most* 1*.*

(ii) $\implies$ (ii) Suppose that every height one prime ideal in R is principal. To show that R is a UFD, it suffices to show that every irreducible element p is a prime element, that is, $\langle p \rangle$ is a prime ideal. Let $\mathfrak{p}$ be the minimal prime containing p. By Krull's principal ideal theorem, $\mathrm{ht}\,\mathfrak{p} \leq 1$. Since $\mathfrak{p}$ is principal, $\mathfrak{p} = \langle x \rangle$ for some $x \in \mathfrak{p}$. Thus, $p = xy$ for some $y \in R$. But p is prime hence, irreducible so either x or y is a unit. If x is a unit, then $\mathfrak{p} = R$, which is a contradiction. Thus, y must be a unit and we see that $\langle p \rangle = \langle xy \rangle = \mathfrak{p}$ is prime.

Now, for the following implications we need to know a couple of denfinitions: Let $D(R)$ denote the set of divisional fractional R-ideals and $F(R)$ denote the set of all principal fractional ideals. Then the *divisor class group of* R is the quotient $\mathrm{Cl}(R) := D(R)/F(R)$.

(i) $\implies$ (iii) It suffices to show that $D(R) \subset F(R)$. Let $I \in D(R)$ be nonzero. Then I is a divisional fractional R-ideal, i.e., there exist a fractional ideal J such that $IJ = R$. Now, recall that a fractional ideal I is invertible in a UFD if and only if I is finitely generated and locally principal. Hence, I is of the form $(I' : I'')$ where $I', I'' \subset R$ are nonzero. ■

PROBLEM 12.2

Let R be a ring with total ring of quotients K, M an R-module, and

$$\operatorname{Tor}(M) = \{\, x \in M \mid ax = 0 \text{ for some non zero-divisor } a \text{ of } R \,\}.$$

The submodule $\operatorname{Tor}(M)$ is called the *torsion of* M, and M is called *torsion free* if $\operatorname{Tor}(M) = 0$. Show

(a) $\operatorname{Tor}(M) = \ker(M \to K \otimes_R M)$

(b) $M/\operatorname{Tor}(M)$ is torsion free.

Proof. (a) Let S denote the set of all regular elements of R and let $\varphi\colon R \to K$, where $K \coloneqq S^{-1}R$, be the canonical localization map $a \mapsto a/1$. We show, by way of double inclusion, that $\operatorname{Tor}(M) = \ker \Phi$, where $\Phi\colon M \to K \otimes_R M$ is the canonical map $x \mapsto 1 \otimes x$. Note that this map, Φ, is well defined by the UMP of the tensor product (HW 2). Now let us show the containment $\operatorname{Tor}(M) \subset \ker \Phi$: Let $x \in \operatorname{Tor}(M)$, then x is a non-zero divisor of R such that $ax = 0$. Since a is a non-zero divisior, $a \in S$ so $a/1 = 0/1$ in K. Thus, we have

$$\Phi(xm) = 1 \otimes x = a/1 \otimes x = 0 \otimes x = 0,$$

so $x \in \ker \Phi$. Conversely, suppose that $x \in \ker(\Phi)$. By some theorem from the localization section[1] we have $K \otimes_R M \cong S^{-1}M$. Thus $1 \otimes x = 0$ implies that $x = 0$ in the localization $S^{-1}M$. This is true if and only if $ax = 0$ for some non-zero divisor a of R. Thus, $x \in \ker \Phi$ and equality holds.

(b) We prove the statement elementwise. Let $x \coloneqq x' + \operatorname{Tor}(M)$ be in $M/\operatorname{Tor}(M)$. Then $ax = 0$ for some non zero-divisor $a \in R$. This implies that $ax' + \operatorname{Tor}(M) = 0 + \operatorname{Tor}(M)$ or $ax' \in \operatorname{Tor}(M)$. Then $b(ax') = 0$ for some non zero-divisor $b \in R$. Since both a and b are non-zero divisors, and $(ba)x' = 0$ then $x' \in \operatorname{Tor}(M)$. Thus, $\operatorname{Tor}(M) = 0$. ■

[1] Sorry! I misplaced my notebook and I've been taking notes on sheets of computer paper so I hate going through the mess.

Problem 12.3

Let R be a Dedekind domain and M a finitely generated R-module of rank r. Show that:

(a) If M is torsion free then M is projective (hint: induct on r).

(b) $M \cong \mathrm{Tor}(M) \oplus P$ with P projective.

(c) If $M \neq 0$ is projective then $M \cong R^{r-1} \oplus I$ with $I \neq 0$ an ideal.

(d) If M is torsion (i.e., $M = \mathrm{Tor}(M)$) then

$$M \cong R/I_1 \oplus \cdots \oplus R/I_n \qquad \text{with} \qquad I_1 \supset \cdots \supset I_n \neq 0$$

ideals (hint: for $p_1, ..., p_s$ the minimal primes of $\mathrm{ann}(M)$ and $S = R \smallsetminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s)$, show that $S^{-1}R$ is a PID).

Proof. (a) First, we shall prove the following useful lemma:

Lemma. *Let R be an integral domain and M and R-module. Then M is torsionfree if and only if $M_\mathfrak{m}$ is a torsionfree $R_\mathfrak{m}$-module for every $\mathfrak{m} \in \mathfrak{m}\text{-Spec}\, R$.*

Proof of lemma. $\Longrightarrow$ Suppose that $(r/s)(x/t) = 0$ for some $s, t \in R \smallsetminus \mathfrak{m}$, $r \in R$. Then there exists some $u \in R$ such that $urx = 0$. If $x \neq 0$, then $ur = 0$. But, since R is an integral domain it follows that $r = 0$. Thus, $\mathrm{Tor}_{R_\mathfrak{m}}(M_\mathfrak{m}) = 0$.

$\Longleftarrow$ Conversely, if M is not torsionfree, there exists a nonzero $x \in M$ with $\mathrm{ann}_R\, x \neq 0$. Let $\mathfrak{m} \in \mathfrak{m}\text{-Spec}\, R$ contain $\mathrm{ann}_R\, x$. Then, localizing at $\mathfrak{m}$, we have $\bar{a}\bar{x} = 0$ in $M_\mathfrak{m}$. ♣

Now, by induction, let M, generated by x, be a torsionfree R module. Let $\mathfrak{m} \in \mathfrak{m}\text{-Spec}\, R$. Then $M_\mathfrak{m}$ is a torsionfree $R_\mathfrak{m}$-module.

Scratch that, let us proceed from the following: Suppose M has rank $\mathrm{rk} = 1$. Then $M \otimes_R K \cong K$ via the R-linear isomorphism φ. Hence, M embeds into K, i.e., $M' \coloneqq \varphi(M \otimes_R 1)$ is a finitely generated R-submodule of K. Then we claim that M' is a fractional ideal.

Proof of claim. ♣

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