MA557 Problem Set 3

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Problem 3.1

Let R be a domain and Γ the set of all principal ideals in R. Show that R is a unique factorization domain if and only if Γ satisfies the ascending chain condition and every irreducible element of R is prime.

Proof. $\Longrightarrow$ Suppose that R is a UFD and let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ be an ascending chain of ideals in Γ. Then $\mathfrak{a}_1 = \langle a_1 \rangle$ for some $a_i \in R$ since every ideal belonging to Γ is principal. Now, since R is a UFD, a_1 factors uniquely (up to associates) as the finite product $a_1 = p_1 \cdots p_k$ of (not necessarily distinct) irreducible elements $p_1, ..., p_k \in R$. If a_1 is irreducible we are done (because in such a case $a_2 \mid a_1$ if and only if $a_2 = ua_1$ where u is a unit, hence $\mathfrak{a}_1 = \mathfrak{a}_2 = \cdots$). Suppose a_1 is not irreducible. Then since $a_2 \mid a_1$, the irreducible factors of a_2 consist of some (or all) of the irreducible factors of a_1 (more precisely we can write $a_2 = p_{\sigma(1)} \cdots p_{\sigma(\ell)}$ for some injection $\sigma \colon \{1, ..., \ell\} \hookrightarrow \{1, ..., k\}$ where $\ell \leq k$). Inductively applying this argument to a_n for $n \geq 1$, we see that the process (of factoring a_n's from a_1) must terminate for some positive r for otherwise we have that

$$a_1 = a_2 b_2 = (a_3 b_3) b_2 = \cdots = (a_n b_n) b_{n-1} \cdots b_2 = \cdots ,$$

but every factorization of a_1 into irreducibles must have length k. Thus, the ascending chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_r = \mathfrak{a}_{r+1} = \cdots$ is stationary for some positive integer r and we say that Γ satisfies the acc.

$\Longleftarrow$ Conversely, suppose that Γ satisfies the acc. Let $a_1 \in R$. If a_1 is irreducible we are done. Suppose a_1 is reducible, then $a_1 = a_2 b_2$ for some non-units $a_2, b_{11} \in R$. If both a_2 and b_2 are irreducible, we are done. Without loss of generality we may assume a_2 is reducible (as the argument to follow may be applied to the b_i's in case they are not irreducible). Then $a_2 = a_3 b_3$ (and so $a_1 = a_2 b_2 = (a_3 b_3) b_2$) for some non-units $a_3, b_3 \in R$. Then we get the ascending chain of principal ideals

$$\langle a_1 \rangle \subset \langle a_2 \rangle \subset \langle a_3 \rangle \subset \cdots$$

which must stabilize for some positive integer r since Γ satisfies the acc. This argument shows that there exits a factorization of a_1 into irreducibles. We must now prove that this factorization in unique (up to associates).

Suppose $a = p_1 \cdots p_k = q_1 \cdots q_\ell$ where $p_1, ..., p_k, q_1, ..., q_\ell \in R$ are irreducibles. ■

Problem 3.2

Let M be an Artinian R-module. Show that every injective R-linear map $\varphi\colon M \to M$ is an isomorphism.

Proof. Suppose φ is not surjective. Then, there is some element $x \in M$ that is not in the image of φ. Now consider cokernels $\operatorname{coker}(\varphi^n) = M/\operatorname{im}(\varphi^n)$ ∎

Problem 3.3

Let M be a finitely generated Artinian module. Show that M is Noetherian.

Proof. Suppose that M is not Noetherian. Let Γ be the set of all non-finitely generated submodules N of M. ■

Problem 3.4

Let R be a ring that is Artinian or Noetherian, and $x \in R$. Show that for some $n > 0$, the image of x in $R/(0 : x)^n$ is a nonzero-divisor on that ring.

Proof. ■

Problem 3.5

Let R be an Artinian ring. Show that $R \cong R_1 \times \cdots \times R_n$ with R_i Artinian local rings.

Proof. ■

Problem 3.6

Let R be an Artinian ring all of whose maximal ideals are principal. Show that every ideal in R is principal.

Proof. ■

Problem 3.7

Prove 2.12.

Proof. Recall the statement of Theorem 2.12:

Theorem. *Let R be a ring, M, M' and M'' be R-modules. Then*

(a) *The following are equivalent:*

   (1) $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M''$ *is exact*
   (2) $0 \to \hom(N, M') \xrightarrow{\hom(N,\varphi)} \hom(N, M) \xrightarrow{\hom(N,\psi)} \hom(N, M'')$ *is exact for all modules N.*

(b) *The following are equivalent:*

   (1) $M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ *is exact.*
   (2) $0 \to \hom(M'', N) \xrightarrow{\hom(\psi,N)} \hom(M, N) \xrightarrow{\hom(\psi,N)} \hom(M', N)$ *is exact for all modules N.*
   (3) $M' \otimes N \xrightarrow{\varphi\otimes N} M \otimes N \xrightarrow{\psi\otimes N} M'' \otimes N \to 0$ *is exact for all modules N.*

■