

# MA544: Qual Problems

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# 1 MA 544 Spring 2016

## 1.1 Exam 1 Prep

**Problem 1.1.** Let  $E \subset \mathbf{R}^n$  be a measurable set,  $r \in \mathbf{R}$  and define the set  $rE = \{r\mathbf{x} \mid \mathbf{x} \in E\}$ . Prove that  $rE$  is measurable, and that  $|rE| = |r|^n |E|$ .

*Proof.* Define a linear map  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  by  $\mathbf{x} \mapsto r\mathbf{x}$ . Using the standard basis for  $\mathbf{R}^n$ , this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \quad (1)$$

which has determinant  $\det T = r^n$ . By 3.35, we have  $|E| = |T(E)| = r^n |E| = |rE|$ . ■

**Problem 1.2.** Let  $\{E_k\}$ ,  $k \in \mathbf{N}$  be a collection of measurable sets. Define the set

$$\liminf_{k \rightarrow \infty} E_k = \bigcup_{k=1}^{\infty} \left( \bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|.$$

*Proof.* If the  $\liminf_{k \rightarrow \infty} |E_k| = \infty$  the inequality holds trivially. Hence, we may, without loss of generality, assume that  $\liminf_{k \rightarrow \infty} |E_k| < \infty$ . By 3.20, the set  $\liminf_{k \rightarrow \infty} E_k$  is measurable and we have

$$\left| \liminf_{k \rightarrow \infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \quad (2)$$

where  $F_k := \bigcap_{n=k}^{\infty} E_n$ . Now, note that the collection of sets  $F'_k := \bigcup_{\ell=1}^k F_\ell$  forms an increasing sequence of measurable sets  $F'_k \nearrow F'$ , where  $F' = \bigcup_{k=1}^{\infty} F_k = \liminf_{k \rightarrow \infty} E_k$ . Then, by 3.26 (i), we have

$$\lim_{k \rightarrow \infty} |F'_k| = |F'| = \left| \liminf_{k \rightarrow \infty} E_k \right|. \quad (3)$$

Hence, it suffices to show that  $|F'_k| \leq |E_k|$  for all  $k$ , but this follows by monotonicity of the outer measure, 3.3, since  $F'_k \subset E_k$ . Thus, we have the desired inequality

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|. \quad (4)$$

■

**Problem 1.3.** Consider the function

$$F(x) := \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here  $B(\mathbf{0}, r) := \{\mathbf{y} \in \mathbf{R}^n \mid |\mathbf{y}| < r\}$ . Prove that  $F$  is monotonic increasing and continuous.

*Proof.* That  $F$  is increasing is immediate from the monotonicity of the outer measure since for  $x < x'$  we have  $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$  so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \leq |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that  $F$  is continuous, we will prove the following lemma

**Lemma 1.** *For any  $x > 0$ ,  $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$ .*

*Proof of lemma.* If  $\mathbf{y} \in xB(\mathbf{0}, 1)$  then  $\mathbf{y} = x\mathbf{y}'$  for  $\mathbf{y}' \in B(\mathbf{0}, 1)$ . Thus,  $|\mathbf{y}'| = |\mathbf{y}|/x < 1$  so  $|\mathbf{y}| < x$  implies that  $\mathbf{y} \in B(\mathbf{0}, x)$ . Hence, we have the containment  $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$ .

On the other hand, if  $\mathbf{y} \in B(\mathbf{0}, x)$  then  $|\mathbf{y}| < x$  so  $|\mathbf{y}|/x < 1$ . Hence,  $\mathbf{y}/x \in B(\mathbf{0}, 1)$  so  $x(\mathbf{y}/x) = \mathbf{y} \in xB(\mathbf{0}, 1)$ . Thus,  $B(\mathbf{0}, x) \subset xB(\mathbf{0}, 1)$  and equality holds. ♣

In light of Lemma 1 and 3.35, for  $x > 0$ , we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n |B(\mathbf{0}, 1)|. \quad (5)$$

It is clear that  $F$  is continuous on the interval  $[0, \infty)$  since  $F$  is a polynomial in  $x$ . ■

**Problem 1.4.** Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a function. Let  $C$  be the set of all points at which  $f$  is continuous. Show that  $C$  is a set of type  $G_\delta$ .

*Proof.* From the topological definition of continuity,  $f$  is continuous at  $x \in C$  if and only if for every neighborhood  $U$  of  $f(x)$ , the preimage  $f^{-1}(U)$  is a neighborhood of  $x$ . Now, ■

Let  $x \in C$ . Then, by the definition of continuity, for every natural number  $n > 0$  there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies

$$|f(x) - f(x')| < \frac{1}{2n}. \quad (6)$$

Let  $x'', x' \in B(x, \delta)$ . Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x'')| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \quad (7)$$

In view of these estimates, define the set

$$A_n := \left\{ x \in \mathbf{R} \mid \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \quad (8)$$

Good Lord, that was a long definition! We claim that  $C = \bigcap_{n=1}^{\infty} A_n$  and that  $A_n$  is open for all  $n$ .

First, let us show that  $C = \bigcap_{n=1}^{\infty} A_n$ . Let  $x \in C$ . Then for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . Thus,  $x \in A_n$  for all  $n$  so  $x \in \bigcap A_n$ . On the other hand, if  $x \in \bigcap A_n$  for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ .

Fix  $\varepsilon > 0$ . By the Archimedean principle, there exists  $N > 0$  such that  $\varepsilon > 1/N$ . Then, since  $x \in A_N$  it follows that for some  $\delta' > 0$ ,  $|x - x'| < \delta'$  implies  $|f(x) - f(x')| < 1/N < \varepsilon$ . Thus,  $x \in C$  and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$ .

Lastly, we show that  $A_n$  is open. Let  $x \in A_n$ . Then there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . In particular, this means that  $B(x, \delta) \subset A_n$  for any  $x' \in B(x, \delta)$  satisfies  $|f(x) - f(x')| < 1/n$ . Thus,  $A_n$  is open and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$  is a  $G_\delta$  set.

**Problem 1.5.** Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a function. Is it true that if the sets  $\{f = r\}$  are measurable for all  $r \in \mathbf{R}$ , then  $f$  is measurable?

*Proof.* No. Recall that, by definition, or 4.1,  $f$  is measurable if and only if  $\{f > a\}$  for all  $a \in \mathbf{R}$ . ■

**Problem 1.6.** Let  $\{f_k\}_{k=1}^{\infty}$  be a sequence of measurable functions on  $\mathbf{R}$ . Prove that the set  $\{x \mid \lim_{k \rightarrow \infty} f_k(x) \text{ exists}\}$  is measurable.

*Proof.* The idea here should be to rewrite

$$E := \left\{ x \mid \lim_{k \rightarrow \infty} f_k(x) \text{ exists} \right\} \quad (9)$$

as a countable union/intersection of measurable sets. Let  $x \in E$ . By the Cauchy criterion, for every  $N > 0$  there exists a positive integer  $M$  such that  $m, n \geq M$  implies  $|f_n(x) - f_m(x)| < 1/N$ . With this in mind, define

$$E_N := \left\{ x \mid \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \quad (10)$$

Then, like for Problem 1.4, it is not too hard to see that the  $E_n$ 's are open and that  $E = \bigcap_{n=1}^{\infty} E_n$ . Thus,  $E$  is a  $G_\delta$  set and therefore measurable. ■

**Problem 1.7.** A real valued function  $f$  on an interval  $[a, b]$  is said to be *absolutely continuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Show that an absolutely continuous function on  $[a, b]$  is of bounded variation on  $[a, b]$ .

*Proof.* Suppose  $f: [a, b] \rightarrow \mathbf{R}$  is absolutely continuous. Then for fixed  $\varepsilon = 1$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , we have  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Let  $\Gamma := \{x_k\}_{k=1}^N$  be a partition of  $[a, b]$  into closed intervals such that  $x_{k+1} - x_k < \delta$ , then by absolute continuity we have

$$\begin{aligned} V[f; \Gamma] &= \sum_{k=1}^N |f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \quad (11)$$

Thus,  $f \in \text{BV}[a, b]$ . ■

**Problem 1.8.** Let  $f$  be a continuous function from  $[a, b]$  into  $\mathbf{R}$ . Let  $\chi_{\{c\}}$  be the characteristic function of a singleton  $\{c\}$ , i.e.,  $\chi_{\{c\}}(x) = 0$  if  $x \neq c$  and  $\chi_{\{c\}}(c) = 1$ . Show that

$$\int_a^b f \, d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}.$$

*Proof.*

■

## 2 Exam 1

## 2.1 Exam 2 Prep

**Problem 2.1.** Define for  $\mathbf{x} \in \mathbb{R}^n$ ,

$$f(\mathbf{x}) := \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that  $f$  is integrable outside any ball  $B_\varepsilon(\mathbf{0})$ , and that there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x}) \, d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

*Proof.* Recall that a real-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is (Lebesgue) integrable over a subset  $E$  of  $\mathbb{R}^n$  (or, alternatively,  $f$  belongs to  $L^1(E)$ ) if

$$\int_E f(\mathbf{x}) \, d\mathbf{x} < \infty.$$

Put  $E := \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$ . Then, to show that  $f$  belongs to  $L^1(E)$  it suffices to prove the inequality

$$\int_E f(\mathbf{x}) \, d\mathbf{x} < \frac{C}{\varepsilon} \tag{12}$$

for some appropriate constant  $C$ . We proceed by directly computing the Lebesgue integral of  $f$  and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x}) \, d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{(x_1^2 + \cdots + x_n^2)^{(n+1)/2}} \end{aligned}$$

let  $E_i$  denote the projection of  $E$  onto its  $i$ -th coordinate and make the trigonometric substitution  $x_1 = \sqrt{x_2^2 + \cdots + x_n^2} \tan \theta$ ,  $dx_1 = \sqrt{x_2^2 + \cdots + x_n^2} \sec^2 \theta \, d\theta$  with  $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$  giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[ \frac{\cos^{n-1} \theta}{(x_2^2 + \cdots + x_n^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{(x_2^2 + \cdots + x_n^2)^{n/2}} \left[ \int_{E_\theta} \cos^{n-1} \theta \, d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1} \theta \, d\theta < \infty. \tag{13}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned}
\int \cdots \int_E f(\mathbf{x}) \, d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{x_n^2} \\
&= 2C' \int_{\varepsilon}^{\infty} \frac{dx_n}{x_n^2} \\
&= \frac{C}{\varepsilon}
\end{aligned} \tag{14}$$

as desired. ■

**Problem 2.2.** Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $\mathbb{R}^n$ , and assume that  $f_k$  converges pointwise almost everywhere to a function  $f$ . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets  $E$  of  $\mathbb{R}^n$ . Moreover, show that this is not necessarily true if  $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$ .

*Proof.* This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit  $f$  of  $\{f_k\}$  must be nonnegative a.e. in  $\mathbb{R}^n$ . For assume otherwise. Then there exists a collection of points  $\mathbf{x}$  in  $\mathbb{R}^n$  of nonzero  $\mathbb{R}^n$ -Lebesgue measure such that  $f(\mathbf{x}) < 0$ . But  $f_k(\mathbf{x}) \geq 0$  for all  $k \in \mathbb{N}$ . Set  $0 < \varepsilon < |f(\mathbf{x})|$  then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{15}$$

for all  $k$  which contradicts our assumption that  $f_k \rightarrow f$  a.e. on  $\mathbb{R}^n$ . Therefore, the set of points  $\mathbf{x} \in \mathbb{R}^n$  where  $f(\mathbf{x}) < 0$  must have measure zero.

Now, based on pointwise convergence a.e. to  $f$ , given  $\varepsilon > 0$  for a.e.  $\mathbf{x} \in \mathbb{R}^n$  we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{16}$$

for sufficiently large  $k$ ; say  $k$  greater than or equal to some index  $N \in \mathbb{N}$ . Moreover, we are given convergence in  $L^1(\mathbb{R}^n)$  of  $f_k$  to  $f$

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f < \infty. \tag{17}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{18}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{19}$$



for all  $k \in \mathbb{N}$ . By Theorem 5.5(ii),  $f$  and the  $f_k$ 's are finite a.e. in  $\mathbb{R}^n$  so for some sufficiently large real number  $M$ ,  $|f|, |f_k| \leq M$  for a.e.  $\mathbf{x} \in \mathbb{R}^n$ . In particular, for any measurable subset  $E$  of  $\mathbb{R}^n$ ,  $|f|, |f_k| \leq M$  for a.e.  $\mathbf{x} \in E$  so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \rightarrow \int_E f < \infty. \quad (20)$$

However, if  $f$  does not belong to  $L^1(\mathbb{R}^n)$ , i.e., its integral over  $\mathbb{R}^n$  is infinity, there is no guarantee that  $f$  will be finite a.e. in  $\mathbb{R}^n$ . This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset  $E$  of  $\mathbb{R}^n$ . Let us demonstrate this with an example. Consider the sequence of functions ■

**Problem 2.3.** Assume that  $E$  is a measurable set of  $\mathbb{R}^n$ , with  $|E| < \infty$ . Prove that a nonnegative function  $f$  defined on  $E$  is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}| < \infty.$$

*Proof.* If  $f$  is integrable over a measurable subset  $E$  of  $\mathbb{R}^n$ , then

$$\int_E f(\mathbf{x}) \, d\mathbf{x} < \infty. \quad (21)$$

Set  $E_k := \{\mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k\}$  and  $F_k := \{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}$ . Note the following properties about the sets we have just defined: first, the  $E_k$ 's are pairwise disjoint and the  $F_k$ 's are nested in the following way  $F_{k+1} \subset F_k$ ; second,  $E = \bigcup_{k=1}^{\infty} E_k$  and  $E_k = F_k \setminus F_{k+1}$ . By Theorem 3.23, since the  $E_k$ 's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \quad (22)$$

Now, since  $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$  on  $E_k$ , we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) \, d\mathbf{x} \leq (k+1)|E_k|. \quad (23)$$

Then we have the following upper and lower estimates on the integral of  $f$  over  $E$

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) \, d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \quad (24)$$

But note that  $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$  by Corollary 3.25 since the measures of  $E_k$ ,  $F_k$ , and  $F_{k+1}$  are all finite. Hence, (24) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) \, d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \quad (25)$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned} \tag{27}$$

Thus, from (26) and (27)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x}) \, d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \tag{28}$$

so the integral  $\int_E f$  converges if and only if the sum  $\sum_{k=0}^{\infty} |F_k|$  converges. ■

**Problem 2.4.** Suppose that  $E$  is a measurable subset of  $\mathbb{R}^n$ , with  $|E| < \infty$ . If  $f$  and  $g$  are measurable functions on  $E$ , define

$$\rho(f, g) := \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$  if and only if  $f_k$  converges to  $f$  as  $k \rightarrow \infty$ .

*Proof.*  $\implies$  : First note that  $\rho$  is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$ . Then, given  $\varepsilon > 0$  there exist an

sufficiently large index  $N$  such that for every  $k \geq N$  we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \quad (29)$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in  $E$  which happens if  $|f_k - f| = 0$  a.e. in  $E$ .

$\Leftarrow$  : Suppose that  $f_k \rightarrow f$  as  $k \rightarrow \infty$ . ■

**Problem 2.5.** Define the *gamma function*  $\Gamma: \mathbb{R}_{>0} \rightarrow \mathbb{R}$  by

$$\Gamma(y) := \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function*  $\beta: \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}$  by

$$\beta(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function  $u \mapsto e^{-u} u^{y-1}$  is in  $L(\mathbb{R}_{>0})$  for all  $y \in \mathbb{R}_{>0}$ .

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

*Proof.* (a) Fix  $y \in \mathbb{R}_{>0}$ . Then we must show that  $\Gamma(y) < \infty$ . First, since  $(0, 1)$  and  $[1, \infty)$  are disjoint measurable subsets of  $\mathbb{R}$ , by Theorem 5.7 we can split the integral  $\Gamma(y)$  into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u} u^{y-1} du}_{I_1} + \underbrace{\int_1^\infty e^{-u} u^{y-1} du}_{I_2}. \quad (30)$$

We will show, separately, that  $I_1$  and  $I_2$  are finite.

To see that  $I_1$  is finite, note that

$$\begin{aligned} e^{-u} u^{y-1} &= e^{-u} e^{(y-1) \log u} \\ &= e^{-u + (y-1) \log u} \\ &\leq e^{(y-1) \log u} \\ &= u^{y-1} \end{aligned} \quad (31)$$

since  $0 < u < 1$

$$\begin{aligned}
I_1 &= \int_0^1 e^{-u} u^{y-1} du \\
&\leq \int_0^1 u^{y-1} du \\
&= \left[ \frac{u^y}{y} \right]_0^1 \\
&= \frac{1}{y} \\
&< \infty.
\end{aligned} \tag{32}$$

To see that  $I_2$  is finite, note that

$$e \tag{33}$$

(b) ■

**Problem 2.6.** Let  $f \in L^1(\mathbb{R}^n)$  and for  $\mathbf{h} \in \mathbb{R}^n$  define  $f_{\mathbf{h}}: \mathbb{R}^n \rightarrow \mathbb{R}$  be  $f_{\mathbf{h}}(\mathbf{x}) := f(\mathbf{x} - \mathbf{h})$ . Prove that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

*Proof.* Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{34}$$

■

**Problem 2.7.** (a) If  $f_k, g_k, f, g \in L^1(\mathbb{R}^n)$ ,  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ ,  $|f_k| \leq g_k$  and

$$\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if  $f_k, f \in L^1(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e. in  $\mathbb{R}^n$ , then

$$\int_{\mathbb{R}^n} |f_k - f| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \rightarrow \int_{\mathbb{R}^n} |f| \quad \text{as} \quad k \rightarrow \infty.$$

*Proof.* ■

## Review of concepts

To conclude this review sheet, here are some important lemmas, theorems, and corollaries from the book:

Let  $f$  be defined on  $E$ , and let  $\mathbf{x}_0$  be a limit point of  $E$  in  $E$ . Then  $f$  is said to be *upper semicontinuous* at  $\mathbf{x}_0$  if

$$\limsup_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \\ \mathbf{x} \in E}} f(\mathbf{x}) \leq f(\mathbf{x}_0). \quad (35)$$

Note that if  $f(\mathbf{x}_0) = \infty$ , then  $f$  is usc at  $\mathbf{x}_0$  automatically; otherwise, the statement that  $f$  is usc at  $\mathbf{x}_0$  means that given any  $M > f(\mathbf{x}_0)$ , there exists  $\delta > 0$  such that  $f(\mathbf{x}) < M$  for all  $\mathbf{x} \in E$  that lie in the ball  $B_\delta(\mathbf{x}_0)$ .

Similarly,  $f$  is said to be *lower semicontinuous* at  $\mathbf{x}_0$  if  $-f$  is usc at  $\mathbf{x}_0$ .

**Theorem (4.14).** *A function  $f$  is usc relative to  $E$  if and only if  $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$  is relatively closed (equivalently, if  $\{\mathbf{x} \in E : f(\mathbf{x}) < a\}$  is relatively open) for all finite  $a$*

*Proof of theorem 4.14.* Suppose that  $f$  is usc relative to  $E$ . Given  $a$ , let  $\mathbf{x}_0$  be a limit point of  $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$  in  $E$ . Then there exists  $\mathbf{x}_k \in E$  such that  $\mathbf{x}_k \rightarrow \mathbf{x}_0$  and  $f(\mathbf{x}_k) > a$ . Since  $f$  is usc at  $\mathbf{x}_0$ , we have  $f(\mathbf{x}_0) \geq \limsup_{k \rightarrow \infty} f(\mathbf{x}_k)$ . Therefore,  $f(\mathbf{x}_0) > a$ , so  $\mathbf{x}_0 \in \{\mathbf{x} \in E : f(\mathbf{x}) > a\}$ . Hence,  $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$  is relatively closed.

Conversely, let  $\mathbf{x}_0$  be a limit point of  $E$  that is in  $E$ . If  $f$  is not usc at  $\mathbf{x}_0$ , then  $f(\mathbf{x}_0) < \infty$ , and there exists  $M$  and  $\{\mathbf{x}_k\}$  such that  $f(\mathbf{x}_0) < M$ ,  $\mathbf{x}_k \in E$ ,  $\mathbf{x}_k \rightarrow \mathbf{x}_0$ , and  $f(\mathbf{x}_k) \geq M$ . Hence,  $\{\mathbf{x} \in E : f(\mathbf{x}) > a\}$  is not relatively closed since it does not contain all its limit points in  $E$ . ■

**Theorem (4.17, Egorov's theorem).** *Suppose that  $\{f_k\}$  is a sequence of measurable functions that converge a.e. in a set  $E$  of finite measure to a finite limit  $f$ . Then given  $\varepsilon > 0$  there exists a closed subset  $F$  of  $E$  such that  $|E \setminus F| < \varepsilon$  and  $f_k \rightarrow f$  uniformly on  $F$ .*

A function  $f$  defined on a measurable set  $E$  has *property C* on  $E$  if given  $\varepsilon > 0$ , there is a closed set  $F \subset E$  such that

$$(i) \quad |E \setminus F| < \varepsilon$$

$$(ii) \quad f \text{ is continuous relative to } F.$$

**Theorem (4.20, Lusin's theorem).** *Let  $f$  be defined and finite on a measurable set  $E$ . Then  $f$  is measurable if and only if it has property C on  $E$ .*

We start with a nonnegative function  $f$  defined on a measurable subset  $E$  of  $\mathbb{R}^n$ . Let's

$$\begin{aligned} \Gamma(f, E) &:= \{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{R}^{n+1} : \mathbf{x} \in E, f(\mathbf{x}) < \infty\}, \\ R(f, E) &:= \{(\mathbf{x}, y) \in \mathbb{R}^{n+1} : \mathbf{x} \in E, 0 \leq y \leq f(\mathbf{x}) \text{ if } f(\mathbf{x}) < \infty \text{ and } 0 \leq y < \infty \text{ if } f(\mathbf{x}) = \infty\}. \end{aligned} \quad (36)$$

$\Gamma(f, E)$  is called the *graph* of  $f$  over  $E$  and  $R(f, E)$  the *region under  $f$  over  $E$* .

If  $R(f, E)$  is measurable (as a subset of  $\mathbb{R}^{n+1}$ ), its measure  $|R(f, E)|_{\mathbb{R}^{n+1}}$  is called the *Lebesgue integral* over  $E$ , and we write

$$\int_E f(\mathbf{x}) \, d\mathbf{x} := |R(f, E)|_{\mathbb{R}^{n+1}}. \quad (37)$$

This is sometimes written as

$$\int_E f$$

or at times the lengthy notation

$$\int \cdots \int_E f(x_1, \dots, x_n) dx_1 \cdots dx_n$$

is convenient.

**Theorem (5.1).** *Let  $f$  be a nonnegative function defined on a measurable set  $E$ . Then  $\int_E f$  exists if and only if  $f$  is measurable.*

**Lemma (5.3).** *If  $f$  is a nonnegative measurable function on  $E$ ,  $0 \leq |E| \leq \infty$ , then  $|\Gamma(f, E)| = 0$ .*

**Theorem (5.5).** (i) *If  $f$  and  $g$  are measurable and if  $0 \leq g \leq f$  on  $E$ ,  $\int_E g \leq \int_E f$ . In particular,  $\int_E \inf f \leq \int_E f$ .*

(ii) *If  $f$  is nonnegative and measurable on  $E$  and if  $\int_E f$  is finite, then  $f < \infty$  a.e. in  $E$ .*

(iii) *Let  $E_1$  and  $E_2$  be measurable and  $E_1 \subset E_2$ . If  $f$  is nonnegative and measurable on  $E_2$ , then  $\int_{E_1} f \leq \int_{E_2} f$ .*

**Theorem (5.6, the monotone convergence theorem for nonnegative functions).** *If  $\{f_k\}$  is a sequence of nonnegative functions such that  $f_k \nearrow f$  on  $E$ , then*

$$\int_E f \rightarrow \int_E f.$$

*Proof.* By Theorem 4.12,  $f$  is measurable since it is the limit of a sequence of measurable functions. Since  $R(f_k, E) \cup \Gamma(f, E) \nearrow R(f, E)$  and  $|\Gamma(f, E)| = 0$ , the result follows by Theorem 3.26 on the measure of a monotone convergent sequences of measurable sets. ■

**Theorem (5.9).** *Let  $f$  be nonnegative on  $E$ . If  $|E| = 0$ , then  $\int_E f = 0$ .*

**Theorem (5.10).** *If  $f$  and  $g$  are nonnegative and measurable on  $E$  and if  $g \leq f$  a.e. in  $E$ , then  $\int_E g \leq \int_E f$ .*

*In particular, if  $f = g$  a.e. in  $E$ , then  $\int_E f = \int_E g$ .*

**Theorem (5.11).** *Let  $f$  be nonnegative and measurable on  $E$ . Then  $\int_E f = 0$  if and only if  $f = 0$  a.e. in  $E$ .*

**Corollary (5.12, Chebyshev's inequality).** *Let  $f$  be nonnegative and measurable on  $E$ . If  $a > 0$ , then*

$$\frac{1}{a} \int_E f \geq |\{\mathbf{x} \in E : f(\mathbf{x}) > a\}|.$$

**Theorem (5.13).** *If  $f$  is nonnegative and measurable, and if  $c$  is any nonnegative constant, then*

$$\int_E cf = c \int_E f.$$

**Theorem (5.14).** *If  $f$  and  $g$  are nonnegative and measurable, then*

$$\int_E (f + g) = \int_E f + \int_E g.$$

**Corollary.** *Suppose that  $f$  and  $\varphi$  are measurable on  $E$ ,  $0 \leq f \leq \varphi$ , and  $\int_E \varphi$  is finite. Then*

$$\int_E (\varphi - f) = \int_E \varphi - \int_E f.$$

**Theorem (5.16).** *If  $f_k$ ,  $k = 1, 2, \dots$ , are nonnegative and measurable, then*

$$\int_E \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_E f_k.$$

**Theorem (5.17, Fatou's lemma).** *If  $\{f_k\}$  is a sequence of nonnegative measurable functions on  $E$ , then*

$$\int_E \liminf_{k \rightarrow \infty} f_k \leq \liminf_{k \rightarrow \infty} \int_E f_k.$$

*Proof of Fatou's lemma.* ■

**Theorem (5.19, Lebesgue's dominated convergence theorem for nonnegative functions).** *Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $E$  such that  $f_k \rightarrow f$  a.e. in  $E$ . If there exists a measurable function  $\varphi$  such that  $f_k \leq \varphi$  a.e. for all  $k$  and if  $\int_E \varphi$  is finite, then*

$$\int_E f_k \rightarrow \int_E f.$$

**Theorem (5.21).** *Let  $f$  be measurable in  $E$ . Then  $f$  is integrable over  $E$  if and only if  $|f|$  is.*

**Theorem (5.22).** *If  $f \in L^1(E)$ , then  $f$  is finite a.e. in  $E$ .*

**Theorem (5.24).** *If  $\int_E f$  exists and  $E = \bigcup_{k \in \mathbb{N}} E_k$  is the countable union of disjoint measurable sets  $E_k$ , then*

$$\int_E f = \sum_{k \in \mathbb{N}} \int_{E_k} f.$$

**Theorem (5.25).** *If  $|E| = 0$  or if  $f = 0$  a.e. in  $E$ , then  $\int_E f = 0$ .*

**Theorem (5.32, monotone convergence theorem).** *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$ :*

- (i) *If  $f_k \nearrow f$  a.e. on  $E$  and there exists  $\varphi \in L^1(E)$  such that  $f_k \leq \varphi$  a.e. on  $E$  for all  $k$ , then  $\int_E f_k \rightarrow \int_E f$ .*
- (ii) *If  $f_k \searrow f$  a.e. on  $E$  and there exists  $\varphi \in L^1(E)$  such that  $f_k \leq \varphi$  a.e. on  $E$  for all  $k$ , then  $\int_E f_k \rightarrow \int_E f$ .*

**Theorem (5.33, uniform convergence theorem).** *Let  $f_k \in L^1(E)$  for  $k \in \mathbb{N}$  and let  $\{f_k\}$  converge uniformly to  $f$  on  $E$ ,  $|E| < \infty$ . Then  $f \in L^1(E)$  and  $\int_E f_k \rightarrow \int_E f$ .*

**Theorem** (5.34, Fatou's lemma). *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$ . If there exists  $\varphi \in L^1(E)$  such that  $f_k \geq \varphi$  a.e. on  $E$  for all  $k$ , then*

$$\int_E \liminf_{k \rightarrow \infty} f_k \leq \liminf_{k \rightarrow \infty} \int_E f_k.$$

**Corollary** (5.35, reverse Fatou's lemma). *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$ . If there exists  $\varphi \in L^1(E)$  such that  $f_k \leq \varphi$  a.e. on  $E$  for all  $k$ , then*

$$\int_E \limsup_{k \rightarrow \infty} f_k \geq \limsup_{k \rightarrow \infty} \int_E f_k.$$

**Theorem** (5.36, Lebesgue's dominated convergence theorem). *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$  such that  $f_k \rightarrow f$  a.e. in  $E$ . If there exists  $\varphi \in L^1(E)$  such that  $|f_k| \leq \varphi$  a.e. in  $E$  for all  $k \in \mathbb{N}$ , then  $\int_E f_k \rightarrow \int_E f$ .*

**Corollary** (5.37, bounded convergence theorem). *Let  $\{f_k\}$  be a sequence of measurable functions on  $E$  such that  $f_k \rightarrow f$  a.e. in  $E$ . If  $|E| < \infty$  there is a finite constant  $M$  such that  $|f_k| \leq M$  a.e. in  $E$ , then  $\int_E f_k \rightarrow \int_E f$ .*

**Theorem** (6.1 Fubini's theorem). *Let  $f(\mathbf{x}, \mathbf{y}) \in L^1(I)$ ,  $I := I_1 \times I_2$ . Then*

- (i) *For almost every  $\mathbf{x} \in I_1$ ,  $f(\mathbf{x}, \mathbf{y})$  is measurable and integrable on  $I_2$  as a function of  $\mathbf{y}$ ;*
- (ii) *As a function of  $\mathbf{x}$ ,  $\int_{I_2} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$  is measurable and integrable on  $I_1$ , and*

$$\iint_I f(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = \int_{I_1} \left[ \int_{I_2} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right] d\mathbf{x}.$$

**Theorem** (6.8). *Let  $f(\mathbf{x}, \mathbf{y})$  be a measurable function defined on a measurable subset  $E$  of  $\mathbb{R}^{n+m}$ , and let  $E_{\mathbf{x}} := \{\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in E\}$ .*

- (i) *For almost every  $\mathbf{x} \in \mathbb{R}^n$ ,  $f(\mathbf{x}, \mathbf{y})$  is a measurable function of  $\mathbf{y}$  on  $E_{\mathbf{x}}$ .*
- (ii) *If  $f(\mathbf{x}, \mathbf{y}) \in L^1(E)$ , then for almost every  $\mathbf{x} \in \mathbb{R}^n$ ,  $f(\mathbf{x}, \mathbf{y})$  is an integrable on  $E_{\mathbf{x}}$  with respect to  $\mathbf{y}$ ; moreover  $\int_{E_{\mathbf{x}}} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$  is an integrable function of  $\mathbf{x}$  and*

$$\iint_E f(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = \int_{\mathbb{R}^n} \left[ \int_{E_{\mathbf{x}}} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right] d\mathbf{x}.$$

**Theorem** (6.10, Tonelli's theorem). *Let  $f(\mathbf{x}, \mathbf{y})$  be nonnegative and measurable on an interval  $I = I_1 \times I_2$  of  $\mathbb{R}^{n+m}$ . Then, for almost every  $\mathbf{x} \in I_1$ ,  $f(\mathbf{x}, \mathbf{y})$  is a measurable function of  $\mathbf{y}$  on  $I_2$ . Moreover, as a function of  $\mathbf{x}$ ,  $\int_{I_2} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$  is measurable on  $I_1$ , and*

$$\iint_I f(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = \int_{I_1} \left[ \int_{I_2} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right] d\mathbf{x}$$

If  $f$  and  $g$  are measurable in  $\mathbb{R}^n$ , their *convolution*  $(f * g)(\mathbf{x})$  is defined by

$$(f * g)(\mathbf{x}) := \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y})g(\mathbf{y}) d\mathbf{y},$$

provided the integral exists.



**Theorem (6.14).** *If  $f \in L^1(\mathbb{R}^n)$  and  $g \in L^1(\mathbb{R}^n)$ , then  $(f * g)(\mathbf{x})$  exists for almost every  $\mathbf{x} \in \mathbb{R}^n$  and is measurable. Moreover,  $f * g \in L^1(\mathbb{R}^n)$  and*

$$\begin{aligned} \int_{\mathbb{R}^n} |f * g| \, d\mathbf{x} &\leq \left( \int_{\mathbb{R}^n} |f| \, d\mathbf{x} \right) \left( \int_{\mathbb{R}^n} |g| \, d\mathbf{x} \right) \\ \int_{\mathbb{R}^n} (f * g)(\mathbf{x}) \, d\mathbf{x} &= \left( \int_{\mathbb{R}^n} f \, d\mathbf{x} \right) \left( \int_{\mathbb{R}^n} g \, d\mathbf{x} \right). \end{aligned}$$

**Corollary (6.16).** *If  $f$  and  $g$  are nonnegative and measurable on  $\mathbb{R}^n$ , then  $f * g$  is measurable on  $\mathbb{R}^n$  and*

$$\int_{\mathbb{R}^n} (f * g) \, d\mathbf{x} = \left( \int_{\mathbb{R}^n} f \, d\mathbf{x} \right) \left( \int_{\mathbb{R}^n} g \, d\mathbf{x} \right).$$

**Theorem (6.17, Marcinkiewicz).** *Let  $F$  be a closed subset of a bounded open interval  $(a, b)$ , and let  $\delta(x) := \delta(x, F)$  be the corresponding distance function. Then, given  $\lambda > 0$ , the integral*

$$M_\lambda(x) := \int_a^b \frac{\delta(y)^\lambda}{|x - y|^{1+\lambda}} \, dy$$

*is finite a.e. in  $F$ . Moreover,  $M_\lambda \in L^1(F)$  and*

$$\int_F M_\lambda \, dx \leq 2\lambda^{-1}|G|,$$

*where  $G := (a, b) \setminus F$ .*