

MA553: Qual Preparation

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1 Ulrich

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Problem 1. Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G of finite index with $N \subset H$.

Solution. ►

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Problem 2. Show that every group of order $992 (= 2^5 \cdot 31)$ is solvable.

Solution. ►

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Problem 3. Let G be a group of order 56 with a normal 2-Sylow subgroup Q , and let P be a 7-Sylow subgroup of G . Show that either $G \simeq P \times Q$ or $Q \simeq \mathbb{Z}/(2) \times \mathbb{Z}/(2) \times \mathbb{Z}/(2)$.

[Hint: P acts on $Q \setminus \{e\}$ via conjugation. Show that this action is either trivial or transitive.]

Solution. ►

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Problem 4. Let R be a commutative ring and $\text{Rad}(R)$ the intersection of all maximal ideals of R .

- (a) Let $a \in R$. Show that $a \in \text{Rad}(R)$ if and only if $1 + ab$ is a unit for every $b \in R$.
- (b) Let R be a domain and $R[X]$ the polynomial ring over R . Deduce that $\text{Rad}(R[X]) = 0$.

Solution. ►

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Problem 5. Let R be a unique factorization domain and P a prime ideal of $R[X]$ with $P \cap R = 0$.

- (a) Let n be the smallest possible degree of a nonzero polynomial in P . Show that P contains a primitive polynomial f of degree n .
- (b) Show that P is the principal ideal generated by f .

Solution. ►

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Problem 6. Let k be a field of characteristic zero. assume that every polynomial in $k[X]$ of odd degree and every polynomial in $k[X]$ of degree two has a root in k . Show that k is algebraically closed.

Solution. ►

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Problem 7. Let $k \subset K$ be a finite Galois extension with Galois group $\text{Gal}(K/k)$, let L be a field with $k \subset L \subset K$, and set $H = \{ \sigma \in \text{Gal}(K/k) : \sigma(L) = L \}$.

- (a) Show that H is the normalizer of $\text{Gal}(K/L)$ in $\text{Gal}(K/k)$.
- (b) Describe the group $H/\text{Gal}(K/L)$ as an automorphism group.

Solution. ►

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