

MA557 Homework 10

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PROBLEM 10.1

Let $\varphi: R \rightarrow S$ be a homomorphism of rings, ${}^a\varphi: \operatorname{Spec} S \rightarrow \operatorname{Spec} R$ the induced map of the spectra, and $\mathfrak{p} \in \operatorname{Spec} R$. Show that the fiber $({}^a\varphi)^{-1}(\mathfrak{p})$ is homeomorphic to $\operatorname{Spec}(S \otimes_R k(\mathfrak{p}))$.

Proof. This is demonstrated (to some extent) by Matsumura, following Theorem 7.2 on p. 47, we shall attempt to supply the missing details here. Recall, from the definition of the pre-image, that

$$({}^a\varphi)^{-1}(\mathfrak{p}) = \{ \mathfrak{q} \in \operatorname{Spec} S \mid \mathfrak{q} \cap R = \mathfrak{p} \}.$$

Define a ring homomorphism $\psi: S \rightarrow S \otimes k(\mathfrak{p})$ via $\psi(s) := s \otimes 1$. We claim that the induced map on the spectra, i.e., the map ${}^a\psi: \operatorname{Spec}(S \otimes k(\mathfrak{p})) \rightarrow \operatorname{Spec} S$, is a homeomorphism onto its image and that $\operatorname{im} {}^a\psi = ({}^a\varphi)^{-1}$. First we will show the latter, that is,

$$\begin{aligned} \operatorname{im} {}^a\psi &= \{ \mathfrak{q} \in \operatorname{Spec} S \mid \mathfrak{q} = \mathfrak{r} \cap S \text{ for } \mathfrak{r} \in \operatorname{Spec}(S \otimes k(\mathfrak{p})) \} \\ &= \end{aligned}$$

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PROBLEM 10.2

Let $R \subset S$ be an integral extension of rings with S a Noetherian ring, and let $\mathfrak{p} \in \operatorname{Spec} R$. Show that there are only finitely many primes in S lying over \mathfrak{p} .

Proof. This is the same as Exercise 9.3 from Matsumura, where he suggests the following approach: Taking a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ and localizing at \mathfrak{p} we may assume that \mathfrak{p} is maximal. ■

PROBLEM 10.3

Let $\varphi: R \rightarrow S$ be a homeomorphism of rings with S a Noetherian ring. Show that the following are equivalent:

- (i) φ satisfies going up.
- (ii) ${}^a\varphi: \operatorname{Spec} S \rightarrow \operatorname{Spec} R$ is a closed map.
- (iii) for every $\mathfrak{q} \in \operatorname{Spec} S$, the induced map $\operatorname{Spec}(S/\mathfrak{q}) \rightarrow \operatorname{Spec}(R/\mathfrak{q} \cap S)$ is surjective.

Proof.

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PROBLEM 10.4

Let $R \subset S$ be an integral extension of domains with R normal, $K = \text{Quot } R$, $\alpha \in S$, $X^n + a_1X^{n-1} + \cdots + a_n$ the minimal polynomial of α over K (recall $a_i \in R$). Show that for any R -ideal I , $\alpha \in \sqrt{IS}$ if and only if $a_i \in \sqrt{I}$ for $1 \leq i \leq n$.

Proof.

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PROBLEM 10.5

Let k be a field and $R = k[X_1, \dots, X_n]$ a k -algebra. Show that the following are equivalent:

- (i) R is a domain with $\dim R = n - 1$
- (ii) $R \cong k[X_1, \dots, X_n]/(f)$, where $k[X_1, \dots, X_n]$ is a polynomial ring and f is an irreducible polynomial.

Proof.

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