MA571 Problem Set 1

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Problem 1.1 (Munkres §2, 1(a,b).)

Let $f\colon A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$.

(a) Show that $A_0 \subset f^{-1}(f(A_0))$ and that equality holds if f is injective.
(b) Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

Proof. (a). First, we will show $A_0 \subset f^{-1}(f(A_0))$. Let $x \in A_0$. Then $f(x) \in f(A_0)$. By definition, $f^{-1}(f(A_0))$ is the set of those points $x_0 \in A$ such that $f(x_0) \in f(A_0)$ and in particular we see that the containment $A_0 \subset f^{-1}(f(A_0))$ holds. Thus, $x \in f^{-1}(f(A_0))$.

Now, let us suppose the map f is injective. By our former argument, we have that $A_0 \subset f^{-1}(f(A_0))$ therefore, we will show the reverse containment. If $y \in f(A_0)$, then $f(x) = y$ for some $x \in A_0$. By the injectivity of f, if $f(x_0) = y$ for some $x_0 \in A$, then we must have that $x_0 = x$. In particular, $x_0 \in A_0$. Thus $f^{-1}(f(A_0)) \subset A_0$ and equality $A_0 = f^{-1}(f(A_0))$ holds.

(b). First, we will show that $f(f^{-1}(B_0)) \subset B_0$. Consider the preimage $f^{-1}(B_0)$ of B_0. Let $x \in f^{-1}(B_0)$. Then $f(x) = y$ for some $y \in B_0$. Since $f(f^{-1}(B_0))$ is, by definition, the set of all points $f(x) \in B$ where $x \in f^{-1}(B_0)$ and $f(x) = y$ for $y \in B_0$, we have that $f(f^{-1}(B_0)) \subset B_0$.

Now, let us suppose the map f is surjective. Let $y \in B_0$, then there exists $x \in A$ such that $f(x) = y$. Thus, $x \in f^{-1}(B_0)$. Then $y = f(x) \in f(f^{-1}(B_0))$ (in particular $B_0 \subset f(f^{-1}(B_0))$) and we have equality $B_0 = f(f^{-1}(B_0))$. ∎

Problem 1.2 (Munkres, §2, 2(g).)

Let $f\colon A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusion, unions, intersections, and differences of sets:

(g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that equality holds if f is injective.

Proof of (g). The claim is evident if A_0 and A_1 are disjoint subsets. Suppose $A_0 \cap A_1 \neq \emptyset$. Let $y \in f(A_0 \cap A_1)$. Then $y = f(x)$ for some $x \in A_0$, $x \in A_1$. Then $f(x) \in f(A_0)$ and $f(x) \in f(A_1)$ so $y \in f(A_0) \cap f(A_1)$. Thus, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

Now, suppose f is injective. Then, if $f(x) = f(x') = y$ for some $y \in B$, then $x = x'$. Let $y \in f(A_0) \cap f(A_1)$. Then $y = f(x_0)$, $y = f(x_1)$ for some $x_0 \in A_0$, $x_1 \in A_1$. But, by the injectivity of f, $x_0 = x_1$ so $x_0 \in A_0 \cap A_1$. Hence, $y \in f(A_0 \cap A_1)$ and the equality $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$ holds. ■

Problem 1.3 (Munkres, §13, 3.)

Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{\, U \mid X \setminus U \text{ is infinite or empty or all of } X \,\}$$

a topology on X?

Proof. Recall that $\mathcal{T}_c$ is the collection of all subsets U of X such that $X \setminus U$ is either countable or is all of X. Let us verify that $\mathcal{T}_c$ defines a topology on X. First, $\emptyset \in \mathcal{T}_c$ since $X \setminus \emptyset = X$ and $X \in \mathcal{T}_c$ since $X \setminus X = \emptyset$ is countable. Second, let $\{U_\alpha\}$, $\alpha \in A$, be an indexed family of nonempty elements of $\mathcal{T}_c$, then $X \setminus U_\alpha$ is countable for all α. Thus, by DeMorgan's laws, we have that

$$X \setminus \bigcup U_\alpha = \bigcap X \setminus U_\alpha$$

is countable (this follows from Corollary 7.3, since $\bigcap_\alpha X \setminus U_\alpha$ is a subset of U_β for all $\beta \in A$, hence it is countable). Thus, the union $\bigcup U_\alpha$ is in $\mathcal{T}_c$. Lastly, let $U_1, ..., U_n$ be nonempty elements of $\mathcal{T}_c$, then by DeMorgan's laws, we have that

$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i)$$

is countable by Theorem 7.5 since $\bigcup_{i=1}^{n}(X \setminus U_i)$ is a countable union of countable sets. So the finite intersection $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_c$. Therefore, $\mathcal{T}_c$ satisfies all the properties to define a topology on X.

Now, let us consider the collection of subsets of X, $\mathcal{T}_\infty$, given above. We will show that arbitrary unions of elements of $\mathcal{T}_\infty$ are, in general, not in $\mathcal{T}_\infty$. Let $X = \mathbf{Z}_+$ and suppose that $\mathcal{T}_\infty$ defines a topology on X. Consider the collection of subsets $\{\{i\}\}_{i=1}^{\infty}$. $\mathbf{Z}_+ \setminus \{i\} = \{1, ..., i-1, i+1, ...\}$ is infinite hence, $\{i\} \in \mathcal{T}_\infty$ for all $i \in \mathbf{N}$. However, $\mathbf{Z}_+ \setminus \bigcup_{i=1}^{\infty}\{i\} = \{0\}$ is finite so $\bigcup_{i=1}^{\infty}\{i\} \notin \mathcal{T}_\infty$, this is a contradiction. Therefore, $\mathcal{T}_\infty$ does not define a topology on X. ■

Problem 1.4 (Munkres, §13, 5.)

Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Proof. Let $\mathcal{T}$ be the topology generated by $\mathcal{A}$ and let $\mathcal{S}$ be the collection of all topologies $\mathcal{T}'$ that contain $\mathcal{A}$. By Lemma 13.3, it suffices to check that $\mathcal{T} = \bigcap \mathcal{T}'$. First we will show that the intersection $\bigcap \mathcal{T}'$ indeed defines a topology on X. To that end we shall prove the following lemma:

Lemma 1. *Let X be a nonempty set and let $\{\mathcal{T}_\alpha\}$ be an indexed collection of topologies on X. Then $\bigcap \mathcal{T}_\alpha$ defines a topology on X.*

Proof of Lemma 1. Let $\mathcal{T} = \bigcap \mathcal{T}_\alpha$. First, since $\emptyset \in \mathcal{T}_\alpha$ and $X \in \mathcal{T}_\alpha$ for all $\alpha \in A$, $\emptyset$ and X are in $\mathcal{T}$. Second, let $\{U_\beta\}$, $\beta \in B$, be an indexed family of nonempty elements of $\mathcal{T}$. Then, $U_\beta \in \mathcal{T}_\alpha$ for all $\beta \in B$ for all $\alpha \in A$ so $\bigcup U_\beta \in \mathcal{T}_\alpha$ for all $\alpha \in A$. Hence, $\bigcup U_\beta \in \mathcal{T}$. Lastly, let $U_1, ..., U_n$ be nonempty elements of $\mathcal{T}$. Then, $U_1, ..., U_n \in \mathcal{T}_\alpha$ for all $\alpha \in A$ so $\bigcap_{i=1}^n U_i \in \mathcal{T}_\alpha$ for all $\alpha \in A$ thus, $\bigcap_{i=1}^n U_i \in \mathcal{T}$. We see that, indeed, $\mathcal{T}$ defines a topology on X. ♦

By the Lemma 1 above, it follows that $\bigcap \mathcal{T}'$ gives a topology on X. Now, it is easy to see that $\bigcap \mathcal{T}' \subset \mathcal{T}$ since $\mathcal{T} \in \mathcal{S}$ is the coarsest topology containing $\mathcal{A}$. Let us prove this fact:

Lemma 2. *Let X be a nonempty set. Let $\mathcal{A}$ be a basis for the topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology containing $\mathcal{A}$.*

Proof of Lemma 2. This can be easily proven by contradiction for suppose $\mathcal{T}$ is not the coarsest topology containing $\mathcal{A}$. Let $\mathcal{C}$ be a strictly coarser topology that contains $\mathcal{A}$. Then there exists some open set $U \in \mathcal{T}$ not in $\mathcal{C}$. Thus, $\mathcal{C}$ is not generated by $\mathcal{A}$. ♦

On the other hand we see by Lemma 13.1 that $\mathcal{T} \subset \bigcap \mathcal{T}'$ since each $\mathcal{T}' \in \mathcal{S}$ contains the basis $\mathcal{A}$ of $\mathcal{T}$, hence contains the open sets of $\mathcal{T}$.

Suppose $\mathcal{A}$ is a subbasis for the topology on X. Then the topology $\mathcal{T}$ on X generated by $\mathcal{A}$ is the collection of unions of finite intersections. Like above, let $\mathcal{S}$ be the collection of topologies $\mathcal{T}'$ in X which contain $\mathcal{A}$. Then, $\bigcap \mathcal{T}' \subset \mathcal{T}$ since $\mathcal{T} \in \mathcal{S}$ is the coarsest topology which contains $\mathcal{A}$. To see the reverse containment, let $U \in \mathcal{T}$ then U is the union of elements $\{U_\alpha\}$ where U_α, $\alpha \in A$, is a finite intersection of elements of $\mathcal{A}$. Then, $U \in \bigcap \mathcal{T}'$ since $U_\alpha \in \mathcal{T}'$ for every $\alpha \in A$, for every topology $\mathcal{T}' \in \mathcal{S}$. ■

Problem 1.5 (Munkres, §13, 8(b).)

(b) Show that the collection

$$\mathcal{C} = \{\, [a, b) \mid a < b, \; a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbf{R}$.

Proof of (b). Let $\mathcal{T}$ denote the topology on $\mathbf{R}_\ell$, i.e, $\mathcal{T}$ is the lower limit topology on $\mathbf{R}$. It is immediate, by the definition of the lower limit topology, that $\mathcal{T}$ is finer than $\mathcal{T}'$ where $\mathcal{T}'$ denotes the topology in $\mathbf{R}$ generated by $\mathcal{C}$. Now, consider the interval $[a, b)$ for $a \in \mathbf{R} \setminus \mathbf{Q}$, $b \in \mathbf{Q}$. $[a, b)$ is in $\mathcal{T}$ however, $[a, b)$ is not in $\mathcal{T}'$ since $[a, b)$ is not expressible as a union or finite intersection of open sets $[a, b) \in \mathcal{T}$.

Proof of claim. We must show that $[a, b)$ is not expressible as a union of half closed intervals $[a_\alpha, b_\alpha)$ and as an finite intersection of half closed intervals $[a_1, b_1), ...[a_n, b_n)$. Seeking a contradiction, suppose $[a, b) = \bigcup [a_\alpha, b_\alpha)$ for α in some index A. Then $[a, b) = [a_\beta, b_\beta)$ for some $\beta \in A$. But this implies that $a_\beta = a \in \mathbf{Q}$. This is a contradiction. Similarly, if $[a, b) = \bigcap_{i=1}^n [a_i, b_i)$ then $[a, b) = [a_j, b_j)$ for some $j \in \{1, ..., n\}$ and additionally we must have $[a_j, b_j) \subset [a_k, b_k)$ for $k \neq j$. Again, this leads to a contradiction since it implies that $a = a_j \in \mathbf{Q}$ contrary to our choice of a. ◆

Thus $\mathcal{T}' \not\supset \mathcal{T}$ and so $\mathcal{T}'$ does not give the same topology as $\mathcal{T}$ on $\mathbf{R}$. ■

Problem 1.6 (Munkres, §16, 1.)

Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. Let $\mathcal{T}$ denote the topology on X and $\mathcal{S}$ denote the topology on Y inherited as a subspace of X. In addition, let $\mathcal{T}_X$ denote the topology on A viewed as a subspace of X and $\mathcal{T}_Y$ denote the topology on A viewed as a subspace of Y. Then, by definition

$$\mathcal{T}_X = \{\, A \cap U \mid U \in \mathcal{T} \,\} \quad \mathcal{T}_Y = \{\, A \cap U \mid U \in \mathcal{S} \,\} \quad \text{and} \quad \mathcal{S} = \{\, Y \cap U \mid U \in \mathcal{T} \,\}.$$

We claim $\mathcal{T}_Y = \mathcal{T}_X$.

First, we write $\mathcal{T}_X$ in a more illuminating fashion namely, (noting that $A \cap Y = A$ and that $\cap$ is associative)

$$\mathcal{T}_X = \{\, (A \cap Y) \cap U \mid U \in \mathcal{T} \,\} = \{\, A \cap (Y \cap U) \mid U \in \mathcal{T} \,\}.$$

(It is an exercise in triviality to show that the above sets are in fact equivalent.) At once one containment becomes obvious, namely if $U \in \mathcal{T}_Y$ then $U = A \cap V$ for some $V \in \mathcal{S}$, but $V = Y \cap W$ for some $W \in \mathcal{T}$ so $U = A \cap (Y \cap W)$ which, by the associativity of $\cap$, is just $U = (A \cap Y) \cap W = A \cap W$. Hence $U \in \mathcal{T}_X$ so $\mathcal{T}_Y \subset \mathcal{T}_X$. To see the reverse containement let $U \in \mathcal{T}_X$ then $U = A \cap V$ for $V \in \mathcal{T}$ and we note that, since $A \cap Y = A$, we have $U = (A \cap Y) \cap V = A \cap (Y \cap V)$ and $Y \cap V \in \mathcal{S}$ so $U \in \mathcal{T}_Y$. Thus, the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ are equivalent. ∎

Problem 1.7 (Munkres, §16, 4.)

A map $f\colon X \to Y$ is said to be an *open map* if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are open maps.

Proof. Let $\mathcal{T}$ denote the topology on X and $\mathcal{S}$ the topology on Y and give the Cartesian product $X \cap Y$ the product topology. Let U be open in $X \times Y$. Then $U = \bigcup_\alpha V_\alpha \times W_\alpha$ for $V_\alpha \in \mathcal{T}$, $W_\alpha \in \mathcal{S}$, $\alpha \in A$. First, we shall prove the following lemma:

Lemma 3. *Let X be a nonempty set. Let A_0 and A_1 be subsets of X and $f\colon X \to Y$. Then $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.*

Proof of Lemma 3. Let $y \in f(A_0 \cup A_1)$. Then $y = f(x)$ for some $x \in A_0$ or $x \in A_1$. Thus $f(x) \in f(A_0)$ or $f(x) \in f(A_1)$ so $y = f(x) \in f(A_0) \cup f(A_1)$. So $f(A_0 \cup A_1) \subset f(A_0) \cup f(A_1)$. To see the reverse containment, let $y \in f(A_0) \cup f(A_1)$, then $y = f(x)$ for $x \in A_0$ or $x \in A_1$. Hence $x \in A_0 \cup A_1$ so $f(x) = y \in f(A_0 \cup A_1)$ and we see that $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$ holds. ♦

By Lemma 1 and the definition of the projection maps, we have

$$\begin{aligned}
\pi_1\left(\bigcup_\alpha U_\alpha \times V_\alpha\right) &= \pi_1\left(\bigcup_{\beta \neq \alpha_0} U_\beta \times V_\beta\right) \cup \pi_1\left(U_{\alpha_0} \times V_{\alpha_0}\right)\\
&= \bigcup_\alpha \pi_1(U_\alpha \times V_\alpha)\\
&= \bigcup_\alpha U_\alpha
\end{aligned}$$

and

$$\begin{aligned}
\pi_2\left(\bigcup_\alpha U_\alpha \times V_\alpha\right) &= \pi_2\left(\bigcup_{\beta \neq \alpha_0} U_\beta \times V_\beta\right) \cup \pi_2\left(U_{\alpha_0} \times V_{\alpha_0}\right)\\
&= \bigcup_\alpha \pi_2(U_\alpha \times V_\alpha)\\
&= \bigcup_\alpha V_\alpha
\end{aligned}$$

both of which are open in X and Y, respectively. ∎

Problem 1.8 (Munkres, §16, 6.)

Show that the countable collection

$$\{\, (a,b)\times(c,d) \mid a<b \text{ and } c<d, \text{ and } a,b,c,d \text{ are rational} \,\}$$

is a basis for $\mathbf{R}^2$.

Proof. Let $\mathcal{B}$ denote the collection

$$\{\, (a,b)\times(c,d) \mid a<b \text{ and } c<d, \text{ and } a,b,c,d \text{ are rational} \,\}.$$

Then, for every $p=(x,y)\in\mathbf{R}^2$, by the density of the rationals in $\mathbf{R}$, there exists rational a and b, c and d such that $a<x<b$ and $c<y<d$, so $p\in(a,b)\times(c,d)$ which is in $\mathcal{B}$. Next, suppose $p=(x,y)\in((a,b)\times(c,d))\cap((a',b')\times(c',d'))$. Then $a<x<b$, $c<y<d$ and $a'<x<b'$, $c'<y<d'$. Let

$$\begin{aligned} \alpha &= \min\{a,a'\}, & \beta &= \min\{b,b'\},\\ \gamma &= \min\{c,c'\}, & \delta &= \min\{d,d'\}. \end{aligned}$$

Then,

$$x\in(\alpha,\beta)\times(\gamma,\delta)\subset((a,b)\times(c,d))\cap((a',b')\times(c',d')).$$

Thus, $\mathcal{B}$ is a basis. ∎

Problem 1.9 (Munkres, §16, 9.)

Show that the dictionary order topology on the set $\mathbf{R} \times \mathbf{R}$ is the same as the product topology $\mathbf{R}_d \times \mathbf{R}$, where $\mathbf{R}_d$ denotes $\mathbf{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbf{R}^2$.

Proof. Let $\mathcal{B}_1$ denote a basis for the dictionary topology on $\mathbf{R} \times \mathbf{R}$ and let $\mathcal{B}_2$ denote a basis for the product topology on $\mathbf{R}_d \times \mathbf{R}$ where, as in the problem prompt, $\mathbf{R}_d$ denotes the set $\mathbf{R}$ equipped with the discrete topology. We want to show that the topologies on $\mathbf{R} \times \mathbf{R}$ and $\mathbf{R}_d \times \mathbf{R}$ are equivalent. We will proceed by Lemma 13.3. Let $x \in \mathbf{R} \times \mathbf{R}$ and let $U \in \mathcal{B}_1$ be a neighborhood of x. Then ■