MA557 Problem Set 2

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September 21, 2015

Problem 2.1

Let $\mathfrak{a}$ be an R-ideal and M a finite R-module. Show that

$$\sqrt{\operatorname{ann}(M/\mathfrak{a}M)} = \sqrt{\operatorname{ann}(M) + \mathfrak{a}}.$$

Proof. One inclusion is immediate, namely,

$$\sqrt{\operatorname{ann}(M/\mathfrak{a}M)} \subset \sqrt{\operatorname{ann}(M) + \mathfrak{a}}$$

since $x \in \sqrt{\operatorname{ann}(M/\mathfrak{a}M)}$ if $x^n \in \operatorname{ann}(M/\mathfrak{a}M)$ if $x^n M \subset \mathfrak{a}M$, i.e., $x^n m = \sum y_i m_i$ for $y_i \in \mathfrak{a}$, $m_i \in M$. But $x' \in \sqrt{\operatorname{ann}(M) + \mathfrak{a}}$ if $x'^n = n + y$ for $n \in \operatorname{ann}(M)$, $y \in \mathfrak{a}$, or $x'^n m = (n+y)m = nm + ym = ym$, in particular, $x'^n m \in \mathfrak{a}M$ so $x' \in \sqrt{\operatorname{ann}(M/\mathfrak{a}M)}$. To see the reverse inclusion note that [cf.Atiyah & MacDonald, Proposition 2.4 or Matsumura, Theorem 2.1] if $x^n \in \operatorname{ann}(M/\mathfrak{a}M)$ then there exists a $y \in \mathfrak{a}$ such that $(x^n + y)M = 0$ or $x^n M = -yM \subset \mathfrak{a}M$ so $x \in \sqrt{\operatorname{ann}(M/\mathfrak{a}M)}$. Thus, $\sqrt{\operatorname{ann}(M) + \mathfrak{a}} \subset \sqrt{\operatorname{ann}(M/\mathfrak{a}M}$ and we have equality. ■

Problem 2.2

Let R be a local ring and M, N finite R-modules. Show that $M \otimes N = 0$ if and only if $M = 0$ or $N = 0$.

Proof. $\Longleftarrow$ If either $M = 0$ or $N = 0$ it is immediate that $M \otimes N = 0$.

$\Longrightarrow$ To see the forward direction, we take Atiyah and MacDonald's hint and let $\mathfrak{m}$ be the maximal ideal of R and let $k = R/\mathfrak{m}$ denote its residue field. Let $M_k = k \otimes M \cong M/\mathfrak{m}M$ by Theorem 2.13. But $M \otimes N = 0$ implies $M_k \otimes_k N_k = 0$ as vector spaces so $M_k = 0$ or $N_k = 0$. Thus, by Nakayama's lemma, $M = 0$ or $N = 0$ since $M_k = 0$ or $N_k = 0$, in other words, since $\mathfrak{m}M = M$ or $\mathfrak{m}N = N$ and $\mathfrak{m} = \mathrm{rad}(R)$. ∎

Problem 2.3

Show that $R^n \cong R^m$ if and only if $n = m$.

Proof. $\Longleftarrow$ If $n = m$ then the isomorphism $R^n \cong R^m$ is canonical.

$\Longrightarrow$ Suppose that $R^n \cong R^m$. Let $\varphi\colon R^n \to R^m$ be a R-linear isomorphism. Let $\mathfrak{m}$ be a maximal ideal of R and let $k = R/\mathfrak{m}$ be its residue field. Then there is an induced k-linear isomorphism $\varphi^*\colon k^n \to k^m$. By the Rank-Nullity theorem and since φ^* is a bijection, we have that the dimension of k^n and k^m are equal, i.e., $n = m$. ∎

Problem 2.4

Prove 2.7.

Proof. Recall the statement of Theorem 2.7:

Theorem. *(a)* $M \otimes N \cong N \otimes M$ *via* $x \otimes y \mapsto y \otimes x$.
(b) $(M \otimes N) \otimes P \cong M \otimes N \otimes P \cong M \otimes (N \otimes P)$ *via* $(x \otimes y) \otimes z \mapsto x \otimes y \otimes z \mapsto x \otimes (y \otimes z)$.
(c) $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$ *via* $(x + y) \otimes z \mapsto x \otimes z + y \otimes z$.
(d) $R \otimes M \cong M$ *via* $r \otimes x \mapsto rx$.

In all cases we must show that the prescribed mapping is well defined.

(a) The map $\varphi \colon M \times N \to N \times M$ given by $(x, y) \mapsto (y, x)$ is a homomorphism. Therefore φ induces a homomorphism $R^{\bigoplus(M\times N)} \xrightarrow{\cong} R^{\bigoplus(N\times M)}$ which preserves the elements in the tensor quotient. Thus, we conclude that the map $x \otimes y \to y \otimes x$ is well defined.

(b) Fix an element $z \in P$. Then the mapping $(x, y) \mapsto x \otimes y \otimes z$ is bilinear in x and y and so induces a homomorphism $\varphi_z \colon M \otimes N \to M \otimes P$ such that $\varphi_z(x \otimes y) = x \otimes y \otimes z$. Next we consider the mapping $(t, z) \mapsto \varphi_z(t)$ of $(M \otimes N) \times P \to M \otimes N \otimes P$. This is bilinear and therefore induces a homomorphism $\varphi \colon (M \otimes N) \otimes P \to M \otimes N \otimes P$ such that $\varphi((x \otimes y) \otimes z) = x \otimes y \otimes z$.

Similarly we can construct a map $\psi_0 \colon M \times N \times P \mapsto (M \otimes N) \otimes P$ which sends $(x, y, z) \mapsto (x \otimes y) \otimes z$. This is a trilinear map and so it induces a homomorphism $\psi \colon M \otimes N \otimes P \to (M \otimes N) \otimes P$ such that $\psi(x \otimes y \otimes z) = (x \otimes y) \otimes z$. It is clear that φ and ψ are inverses of each other. Therefore, we have that $(M \otimes N) \otimes P \cong M \otimes N \otimes P$.

The proof that $M \otimes (N \otimes P) \cong M \otimes N \otimes P$ is analogous, fixing an element $x \in M$ and repeating the whole process above.

(c) Consider the map $\varphi \colon (M \oplus N) \times P \to (M \otimes P) \oplus (N \otimes P)$ which sends an element $(x + y, z) \mapsto x \otimes z + y \otimes z$.

(d) ∎

Problem 2.5

Prove 2.8.

Proof. Recall the statement of Proposition 2.8:

Proposition. *Let M be an R-module, N an R-S-bimodule and P an S-module. Then:*

(a) $M \otimes N$ *is an R-S-bimodule via* $(\sum m_i \otimes n_i)s = \sum m_i \otimes (sn_i)$.
(b) *The free module* $(M \otimes N) \otimes_S P \cong M \otimes (N \otimes_S P)$ *as R-S-bimodules via* $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$.

(i)

(ii) ∎

Problem 2.6

Prove 2.9.

Proof. Recall the statement of Theorem 2.9:

Theorem. *Let $\psi\colon R \to S$ be a ring map and M and R module. Then $S \otimes M$ is an S-module (by Proposition 2.8) and $\mu\colon M \to S \otimes M$ with $\mu(m) = 1 \otimes m$ is an R-linear map. Moreover, for every R-linear map $\varphi\colon M \to N$, where N is any S-module, there exists a unique S-linear map f so that $\varphi = f \circ \mu$, i.e, the diagram commmutes*

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Problem 2.7

Prove 2.10.

Proof. Recall the statement of Proposition 2.10:

Proposition. *Let S and T be R-algebras. Then there is an R-algebra structure on $S \otimes T$ with $(s_1 \otimes t_1)(s_2 \otimes t_2) = (s_1 s_2) \otimes (t_1 t_2)$.*

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