MA 572: Homework 1

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Problem 1.1 (Hatcher §2.1, Ex. 11)

Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Proof. Suppose that A is a retract of X. Then there exists a continuous map $r\colon X \to A$ such that $r(X) = A$ and $r \mid A = \mathrm{id}_A$. Let $i\colon A \hookrightarrow X$ denote the inclusion map and $i_*\colon H_n(A) \to H_n(X)$ denote the induced homomorphism on the homology groups of A and X; do the same for r, $r_*\colon H_n(X) \to H_n(X)$. Then $r \circ i = \mathrm{id}_A$ which induces the endomorphism $(r \circ i)_* = r_* \circ i_* = \mathrm{id}_{H_n(A)}$ on $H_n(A)$. Thus, the inclusion map i_* is injective (since it has a left inverse). ■

Problem 1.2 (Hatcher §2.1, Ex. 12)

Show that chain homotopy of chain maps is an equivalence relation.

Proof. Let X and Y be topological spaces and $f, g, h \colon X \to Y$ be continuous maps. Then $f_\#, g_\#, h_\# \colon C_n(X) \to C_n(Y)$ denote the induced chain maps. We show that chain homotopy of chain maps is an equivalence relation:

(i) Let P be the 0 homomorphsim. Then, we have

$$\partial 0 + 0\partial = 0 = f_\# - f_\#.$$

Thus, $f_\#$ is chain homotopic to itself.

(ii) Suppose $f_\#$ is chain homotopic to $g_\#$. Then there exist a homomorphism $P \colon C_n(X) \to C_{n+1}(Y)$ such hat $\partial P + P\partial = g_\# - f_\#$. Put $Q \coloneqq -P$. Then, we have

$$\partial(-P) + (-P)\partial = -(\partial P + P\partial) = -(g_\# - f_\#) = f_\# - g_\#.$$

Thus, $g_\#$ is chain homotopic to $f_\#$.

(iii) Suppose that $f_\#$ is chain homotopic to $g_\#$ and $g_\#$ is chain homotopic to $h_\#$. Then there exists homomorphism $P \colon C_n(X) \to C_{n+1}(Y)$ and a homomorphism $Q \colon C_n(X) \to C_{n+1}(Y)$ such that $\partial P + P\partial = g_\# - f_\#$ and $\partial Q + Q\partial = h_\# - g_\#$. Put $R \coloneqq P + Q$. Then, we have

$$\begin{aligned}\partial(P+Q) + (P+Q)\partial &= \partial P + \partial Q + P\partial + Q\partial \\ &= (\partial Q + Q\partial) + (\partial P + P\partial) \\ &= (h_\# - g_\#) + (g_\# - f_\#) \\ &= h_\# - f_\#.\end{aligned}$$

Thus, $f_\#$ is chain homotopic to $h_\#$.

We conclude that ‘chain homotopy’ is an equivalence relation. ■

Problem 1.3 (Hatcher §2.1, Ex. 16)

(a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.

(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof. (a) $\implies$ Suppose that the relative 0th homology of X with respect to A, $H_0(X, A)$, is trivial. Let $\{X_\alpha\}$ be the set of path-components of X. We aim to show that $A \cap X_\alpha \neq \emptyset$ for all α. Let $i\colon A \hookrightarrow X$ denote the canonical inclusion map $A \subset X$. Now, the map i can be extended to a chain map between chain complexes which, by proposition 2.9, induces a homomorphism $i_*\colon H_n(A) \to H_n(X)$ between the homology groups of A and X. Similarly, the map $j\colon C_n(X) \to C_n(X, A)$ induces a map $j_*\colon H_n(X) \to H_n(X, A)$ so, by theorem 2.16, we have a long exact sequence

$$\cdots \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0. \tag{1}$$

In particular, the short exact sequence

$$0 \xrightarrow{0} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0. \tag{2}$$

But $H_0(X, A) = 0$ so the map $j_* = 0$. By short exactness of (2) we have $\operatorname{im} i_* = \ker j_* = H_0(X)$, so i_* is surjective.

(b) ■

Problem 1.4 (Hatcher §2.1, Ex. 17)

(a) Compute the homology groups $H_n(X, A)$ when X is $\mathbf{S}^2$ or $\mathbf{S}^1 \times \mathbf{S}^1$ and A is a finite set of points in X.

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown. [What are X/A and X/B?]

Proof. (a) Since A is a finite collection of points in $\mathbf{S}^2$, let us enumerate the set A via $\{a_1, ..., a_n\}$ and denote by A_k the subset $\{a_1, ..., a_k\}$ of A, where $k \leq n$. Now, by the generalization of theorem 2.16 to triples, we have the long exact sequence

$$\cdots \longrightarrow H_m(A_n, A_{n-1}) \longrightarrow H_m(\mathbf{S}^2, A_{n-1}) \longrightarrow H_m(\mathbf{S}^2, A_n) \longrightarrow H_{m-1}(A_n, A_{n-1}) \longrightarrow \cdots. \quad (3)$$

Exactness of (3) tells us that for $m \geq 2$ we have $H(\mathbf{S}^2, A_{n-1}) \cong H(\mathbf{S}^2, A_n)$ since

$$H_m(A_n, A_{n-1}) = 0 \longrightarrow H_m(\mathbf{S}^2, A_{n-1}) \longrightarrow H_m(\mathbf{S}^2, A_n) \longrightarrow 0 = H_{m-1}(A_n, A_{n-1})$$

is exact. Evidently, $H_m(A_n, A_{n-1}) = 0$ for $m > 1$.[1]

(b) ■

[1] I will prove this if time permits.