

# MA 519: Homework 12

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November 17, 2016



## PROBLEM 12.1 (HANDOUT 15, # 10)

Consider the experiment of picking one word at random from the sentence

*All is well in the newell family*

Let  $X$  be the length of the word selected and  $Y$  the number of Ls in it. Find in a tabular form the joint PMF of  $(X, Y)$ , their marginal PMFs, means, and variances, and the correlation between  $X$  and  $Y$ .

*SOLUTION.*



## PROBLEM 12.2 (HANDOUT 15, # 11)

Consider the joint PMF  $p(x, y) = cxy$ ,  $1 \leq x \leq 3$ ,  $1 \leq y \leq 3$ .

- (a) Find the normalizing constant  $c$ .
- (b) Are  $X$  and  $Y$  independent? Prove your claim.
- (c) Find the expectations of  $X$ ,  $Y$ , and  $XY$ .

*SOLUTION.*

■

## PROBLEM 12.3 (HANDOUT 15, # 12)

A fair die is rolled twice. Let  $X$  be the maximum and  $Y$  the minimum of the two rolls. By using the joint PMF of  $X$  and  $Y$  worked out in the text, find the PMF of  $\frac{X}{Y}$ , and hence the mean of  $\frac{X}{Y}$ .

*SOLUTION.* ■

## PROBLEM 12.4 (HANDOUT 15, # 13)

Two random variables have the joint PMF  $p(x, x+1) = \frac{1}{n+1}$ ,  $x = 0, \dots, n$ . Answer the following question with as little calculation as possible.

- (a) Are  $X$  and  $Y$  independent?
- (b) What is the variance of  $Y - X$ ?
- (c) What is  $\text{Var}(Y \mid X = 1)$ ?

*SOLUTION.*

■

## PROBLEM 12.5 (HANDOUT 15, # 14)

(*Binomial Conditional Distribution*). Suppose  $X$  and  $Y$  are independent random variables, and  $X \sim \text{Bin}(m, p)$ ,  $Y \sim \text{Bin}(n, p)$ . Show that the conditional distribution of  $X$  given by  $X + Y = t$  is a hypergeometric distribution; identify the parameters of this hypergeometric distribution.

SOLUTION. ■

## PROBLEM 12.6 (HANDOUT 15, # 15)

Suppose a fair die is rolled twice. Let  $X$  and  $Y$  be the two rolls. Find the following with as little calculation as possible.

- (a)  $E(X + Y | Y = y)$ .
- (b)  $E(XY | Y = y)$ .
- (c)  $\text{Var}(X^2Y | Y = y)$ .
- (d)  $\rho_{X+Y, X-Y}$ .

SOLUTION. ■



## PROBLEM 12.7 (HANDOUT 15, # 16)

(A Standard Deviation Inequality). Let  $X$  and  $Y$  be two random variables. Show that  $\sigma_{X+Y} \leq \sigma_X + \sigma_Y$ .

*SOLUTION.* Suppose  $\sigma_X$  and  $\sigma_Y$  exist and are finite. We want to show

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y;$$

this is the same as showing that

$$\begin{aligned}\sigma_{X+Y}^2 &\leq \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y \\ \text{Var}(X+Y) &\leq \text{Var}(X) + \text{Var}(Y) + 2[\text{Var}(X)\text{Var}(Y)]^{\frac{1}{2}}.\end{aligned}$$

First, let us expand  $\text{Var}(X+Y)$  using the definition of variance, we have

$$\begin{aligned}\text{Var}(X+Y) &= E((X+Y)^2) - E(X+Y)^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 \\ &= (E(X^2) - E(X)^2) + (E(Y^2) - E(Y)^2) + 2[E(XY) - E(X)E(Y)] \\ &= \text{Var}(X) + \text{Var}(Y) + 2[E(XY) - E(X)E(Y)].\end{aligned}$$

Therefore, it suffices to show that

$$E(XY) - E(X)E(Y) \leq [\text{Var}(X)\text{Var}(Y)]^{\frac{1}{2}}.$$

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}E(XY) - E(X)E(Y) &\leq [E(X^2)E(Y^2)]^{\frac{1}{2}} - [E(X)^2E(Y)^2]^{\frac{1}{2}} \\ &= \end{aligned}$$

■

## PROBLEM 12.8 (HANDOUT 15, # 17)

Seven balls are distributed randomly in seven cells. Let  $X_k$  be the number of cells containing exactly  $k$  balls. Using the probabilities tabulated in II, 5, write down the joint distribution of  $X_2, X_3$ .

*SOLUTION.* The table referenced in this problem is on p. 40 of Feller. Let us write down a table of our own for the joint distribution of  $(X_2, X_3)$ :

$X_3 \setminus X_2$	0	1	2	3
0	0.048	0.156	0.321	0.107
1	0.109	0.214	0.027	0
2	0.018	0	0	0

Let us do a sanity check by summing over all of the entries in the table above

$$0.048 + 0.156 + 0.321 + 0.107 + 0.109 + 0.214 + 0.027 + 0 + 0.018 + 0 + 0 + 0 \approx 1. \quad \blacksquare$$

## PROBLEM 12.9 (HANDOUT 15, # 18)

Two ideal dice are thrown. Let  $X$  be the score on the first die and  $Y$  be the larger of two scores.

- (a) Write down the joint distribution of  $X$  and  $Y$ .
- (b) Find the means, the variances, and the covariance.

*SOLUTION.* For part (a): The random variable  $X$  takes on integer values between zero and six and so does  $Y$ . Moreover, the dependence of  $Y$  on  $X$  tells us that  $P(\{X = k\} \cap \{Y = \ell\}) = 0$  if  $\ell < k$ ; this allows us to fill in a significant portion of the joint distribution table:

$Y \setminus X$	1	2	3	4	5	6
1	$\frac{1}{36}$	0	0	0	0	0
2			0	0	0	0
3				0	0	0
4					0	0
5						0
6						

■

## PROBLEM 12.10 (HANDOUT 15, # 19)

Let  $X_1$  and  $X_2$  be independent and have the common geometric distribution  $\{q^k p\}$  (as in problem 4). Show without calculations that the *conditional distribution of  $X_1$  given  $X_1 + X_2$  is uniform*, that is,

$$P(X_1 = k \mid X_1 + X_2 = n) = \frac{1}{n+1}, \quad k = 0, \dots, n. \quad (12.1)$$

SOLUTION. ■

PROBLEM 12.11 (HANDOUT 15, # 20)

If two random variables  $X$  and  $Y$  assume only two values each, and if  $\text{Cov}(X, Y) = 0$ , then  $X$  and  $Y$  are independent.

*SOLUTION.*

