

## MA 523: Homework 1

Carlos Salinas

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**PROBLEM 1.1 (TAYLOR'S FORMULA)**

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be smooth,  $n \geq 2$ . Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$$

as  $x \rightarrow 0$  for each  $k = 1, 2, \dots$ , assuming that you know this formula for  $n = 1$ .

*Hint:* Fix  $x \in \mathbb{R}^n$  and consider the function of one variable  $g(t) := f(tx)$ . Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on  $m$ .

**Solution.** ► Taking the hint, fix  $x \in \mathbb{R}^n$  and consider the function of one variable  $g(t) := f(tx)$ . We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

*Proof of claim.* We shall prove this by induction on  $m$ . The case  $m = 1$  follows from Taylor's formula in 1 dimension as

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\ &= \frac{d}{dt} f(tx_1, \dots, tx_n) \\ &= \end{aligned}$$

□

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**PROBLEM 1.2**

Write down the characteristic equation for the p.d.e.

$$u_t + b \cdot Du = f \tag{*}$$

on  $\mathbb{R}^n \times (0, \infty)$ , where  $b \in \mathbb{R}^n$ . Using the characteristic equation, solve (\*) subject to the initial condition

$$u = g$$

on  $\mathbb{R}^n \times \{t = 0\}$ . Make sure the answer agrees with formula (5) in §2.1.2 of [E].

**Solution.** ►

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**PROBLEM 1.3**

Solve using the characteristics:

- (a)  $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$ ,  $u = 1$  on the line  $x_2 = 2x_1$ .
- (b)  $uu_{x_1} + u_{x_2} = 1$ ,  $u(x_1, x_2) = x_1/2$ .
- (c)  $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$ ,  $u(x_1, x_2, 0) = g(x_1, x_2)$ .

**Solution.** ►

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**PROBLEM 1.4**

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with  $u(x_1, 0) = (1 - x_1^2)/2$ .

**Solution.** ►

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