MA571 Midterm 2: Practice Problems

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Problem 1. Let X be a Hausdorff space and let A be a compact subset of X. Prove from the definitions that A is closed.

Proof. This is Theorem 26.3 from Munkers §26, p. 165; we shall paraphrase it.

We show that $X - A$ is open. To that end we will show that, given a point $x_0 \in X - A$, there is neighborhood U of x_0 disjoint from A. For each point $a \in A$, by the Hausdorff property of X, choose disjoint neighborhoods U_a and V_a of x_0 and a, respectively. Then the collection $\{ V_a \mid a \in A \}$ forms an open cover of A so, by Lemma 26.1, only finitely many of the V_a's cover A, say $V_{a_1}, ..., V_{a_n}$. Define $U := U_{a_1} \cap \cdots \cap U_{a_n}$. We claim that U is a neighborhood of x_0 disjoint from A. First, it is clear that U is a neighborhood of x_0 since each U_a contains x_0 and U is an intersection of finitely many of these. Second, note that if $z \in U \cap A$ then $z \in U_{a_i}$ for all i and $z \in V_{a_j}$ for some $j \in \{1, ..., n\}$, but $U_{a_j} \cap V_{a_j} = \emptyset$. Therefore, $U \cap A = \emptyset$. By Lemma C, it follows that $X - A$ is open. ∎

Problem 2. Let X be a Hausdorff space and let A and B be disjoint compact subsets of X. Prove that there are open sets U and V such that U and V are disjoint, $A \subset U$ and $B \subset V$.

Proof. This is Ex. 5 from Munkres §26, p. 171.

Suppose A and B are disjoint compact subspaces of X. Since X is Hausdorff, by Theorem 26.4, for every $x \in B$ there exists disjoint open sets U_x and V_x where $U_x \supset A$ and V_x is a neighborhood of x. Then the collection $\{ V_x \mid x \in B\}$ is an open cover of B so by Lemma 26.1, only finitely many of the V_x's cover B, say $V_{x_1}, ..., V_{x_n}$. Define $U := U_{x_1} \cap \cdots \cap U_{x_n}$ and $V := V_{x_1} \cup \cdots \cup V_{x_n}$. We claim that U and V are disjoint neighborhood containing A and B, respectively. It is clear that U and V are open since U is a finite intersection of open sets and V is a union of open sets and that they contain A and B, respectively, since each of the U_x's contain A and $V_{x_1}, ..., V_{x_n}$ is an open cover of B. Lastly, U and V are disjoint since intersection distributes over union, i.e., we have

$$U \cap V = \left(\bigcap_{i=1}^{n} U_{x_i} \right) \cap \left(\bigcup_{j=1}^{n} V_{x_j} \right) = \bigcup \left(\bigcap_{i=1}^{n} U_{x_i} \cap V_{x_j} \right) = \emptyset$$

since $U_{x_i} \cap V_{x_i} = \emptyset$ so $\left(\bigcap_{i=1}^{n} U_{x_j} \right) \cap V_{x_i} = \emptyset$. ∎

Problem 3. Prove the Tube Lemma: Let X and Y be topological spaces with Y compact, let $x_0 \in X$, and let N be an open set of $X \times Y$ containing $x_0 \times Y$, then there is an open set W of X containing x_0 with $W \times Y \subset N$.

Proof. This is Lemma 26.8 from Munkres §26, p. 168, but is proved in *Step 1* in the process of showing Theorem 26.7; we paraphrase the proof here.

Let $x_0 \in X$, and let N be an open set of $X \times Y$ containing $x_0 \times Y$. Cover $x_0 \times Y$ by basic open sets $U \times V$ lying in N. Note that $x_0 \times Y$ is compact, since it is an imbedding of Y given by the map $y \mapsto (x_0, y)$ from Y into $X \times Y$ therefore, by Lemma 26.1, only finitely many of the $U \times V$'s, say $U_1 \times V_1, ..., U_n \times V_n$, cover $x_0 \times Y$. Define $W := U_1 \cap \cdots \cap U_n$. We claim that W is a neighborhood of x_0 such that $W \times Y \subset N$. First, it is clear that W is a neighborhood of x_0 since it is the finite intersection of open sets and each $U_i \times V_i$ intersects $x_0 \times Y$ hence contains a point of the form (x_0, y) so $U_i = \pi_1(U_i \times V_i)$ contains x_0. Lastly, $W \times Y \subset N$ since $W \times Y \subset \bigcup_{i=1}^{n} U_i \times V_i$.

To see this let $(x, y) \in W \times Y$ and consider the point $(x_0, y) \in x_0 \times Y$. Since (x_0, y) is in $U_i \times V_i$ for some i, we have $y \in V_i$. But $x \in U_j$ for every j since $x \in W$. Thus $(x, y) \in U_i \times V_i$ as desired. It follows that, W is a neighborhood of x_0 with $W \times Y \subset N$ as desired. ■

Problem 4. Show that if Y is compact, then the projection map $X \times X \to X$ is a closed map.

Proof. We shall proceed by the tube lemma, i.e, Theorem 26.8. Let C be a closed subset of $X \times Y$ then $N = (X \times Y) - C$ is open. Choose $x_0 \in X - \pi_1(C)$. Then $x_0 \times Y$ is contained in N so by the tube lemma, there exists a neighborhood W of x_0 such that $W \times Y \subset N$. In particular, $W \subset X - \pi_1(C)$ otherwise if $x \in W \cap \pi_1(C)$ then $x \times Y \subset N$ and $(x, y) \in C$ for some $y \in Y$, but $N \cap C = \emptyset$. It follows by Lemma C that $X - \pi_1(C)$ is open so $\pi_1(C)$ is closed. Since C was chosen arbitrarily we see that π_1 is a closed map. ■

Problem 5. Let X be a compact space and suppose we are given a nested sequence of subsets $C_1 \supset C_2 \supset \cdots$ with all C_i closed. Let U be an open set containing $\bigcap C_i$. Prove that there is an i_0 with $C_{i_0} \subset U$.

Proof. Consider the family of open sets $U_i \coloneqq X - C_i$. Since U is open $X - U$ is closed so by Theorem 26.2 is compact. We claim that U_i forms an open cover of $X - U$. To see note that by De Morgan's laws

$$\bigcup_{i \in \mathbf{N}} U_i = \bigcup_{i \in \mathbf{N}} X - C_i = X - \bigcap_{i \in \mathbf{N}} C_i \supset X - U$$

since $\bigcap_{i \in \mathbf{N}} C_i \subset U$. Therefore by Lemma 26.1 only finitely many of the U_i's cover $X - U$, say $U_{i_1}, ..., U_{i_n}$. Thus, we have that $X - U \subset \bigcup_{i=1}^n U_i$ so $U \supset \bigcap_{j=1}^n C_{i_j} = C_{i_n}$ as desired. ■

Problem 6. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i \colon X \to \mathbf{R}$, $i = 1, ..., n$ with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. Prove that X is homeomorphic to a subspace of $\mathbf{R}^n$.

Proof. Consider the map $f \colon X \to \mathbf{R}^n$ defined by $f \coloneqq (f_1, ..., f_n)$. This map is continuous by Theorem 18.4 since each component f_i is continuous. We claim that $X \approx f(X)$. To prove this it suffices to show that f is injective so that its restriction to $f(X)$ will be surjective and lastly invoke Theorem 26.6. Suppose $f(x) = f(y)$ but $x \neq y$. Then $f_i(x) \neq f_i(y)$ for some i, but this implies that $f(x) \neq f(y)$. This is a contradiction therefore, $x = y$. It follows that f is a bijection from a compact space X into $f(X) \subset \mathbf{R}^n$ so by Theorem 26.6, we have $X \approx f(X)$. ■

Problem 7. Let X be a compact metric space and let $\mathcal{U}$ be a covering of X by open sets. Prove that there is an $\varepsilon > 0$ such that, for each set $S \subset X$ with diameter $< \varepsilon$, there is a $U \in \mathcal{U}$ with $S \subset U$. (This fact is known as the "Lebesgue number lemma.")

Proof. This is Lemma 27.5 from Munkres §27, p. 175; we will paraphrase the proof.

Let $\mathcal{U}$ be an open cover of X. If $X \in \mathcal{U}$, then any positive number is a Lebesgue number for $\mathcal{U}$. Suppose $X \notin \mathcal{U}$. Choose a finite subcollection $U_1, ..., U_n$ of $\mathcal{U}$ that covers X. For each i, set $C_i \coloneqq X - U_i$ and define the map $f \colon X \to \mathbf{R}$ via $f(x) \coloneqq \frac{1}{n} \sum_{i=1}^n d(x, C_i)$. We show that $f(x) > 0$ for all x. Given $x \in X$, choose i so that $x \in U_i$. Then choose ε so that the ε-neighborhood of x lies in U_i. Then $d(x, C_i) \geq \varepsilon$, so that $f(x) \geq \varepsilon / n$.

Since f is continuous, it has a minimum value δ; we show that δ is our required Lebesgue number. Let B be a subset of X of diameter less that δ. Choose a point x_0 of B; then B lies in a δ-neighborhood of x_0. Now $\delta \leq f(x_0) \leq d(x_0, C_m)$, where $d(x_0, C_m)$ is the largest of the numbers $d(x_0, C_i)$. Then the δ-neighborhood of x_0 is contained in the element $U_m = X - C_m$ of the covering $\mathcal{U}$. ■

Problem 8. Let S^1 denote the circle $\left\{ x^2 + y^2 = 1 \right\}$ in $\mathbf{R}^2$. Define an equivalence relation on S^1 by

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } (x, y) = (-x', -y')$$

(you do not have to prove that this is an equivalence relation). Prove that the quotient space $S^1/\sim$ is homeomorphic to S^1.

One way to do this is by using complex numbers.

Proof. Since Dr. McClure said that we can assume anything from complex analysis (and we don't need much) to begin with we shall assume that $S^1 \subset \mathbf{C}$. Now, the situation is as follows we want to find a map $f\colon S^1 \to S^1$ which preserves $\sim$ that makes the following diagram commute

$$\begin{tikzcd} S^1 \arrow[d, "q"] \arrow[dr, "f"] & \\ S^1/\sim \arrow[r, "\bar{f}"'] & S^1. \end{tikzcd}$$

Define $f(z) \coloneqq z^2$. We claim that f is continuous and preserves $\sim$. First, it is clear that $f(x + iy) = f(x + iy)$ and if $x' + iy' = -x - iy$ then

$$\begin{aligned} f(x + iy) &= (x + iy)^2 \\ &= (-x - iy)^2 \\ &= f(-x - iy) \\ &= f(x' + iy') \end{aligned}$$

so f preserves $\sim$. Since z^2 is multiplication on $\mathbf{C}$ by Theorem 21.5 f is continuous (or at least the argument can be extended to make this operation continuous). Thus, by Theorem Q.3 the induced map on the quotient $\bar{f}\colon S^1/\sim \to S^1$ is continuous. By Theorem 26.6 it suffices to show that $\bar{f}$ is bijective. It is clear that $\bar{f}$ is surjective since f is surjective; that is, take an element $x + iy \in S^1$ then by elementary properties of the complex numbers we have

$$f\Big(\|x + iy\| e^{i\pi\theta/2}\Big) = x + iy$$

where $\theta = \arg(x + iy)$. To see that this map is injective simply note that if $f(x + iy) = f(x' + iy')$ then

$$x^2 - y^2 - ((x')^2 + (y')^2) = i2(x'y' - xy)$$

if and only if $x' = x$ and $y' = y$ or $x' = -x$ and $y' = -y$ so $\bar{f}$ is injective. It follows that $\bar{f}$ is a homeomorphism so $S^1/\sim \approx S^1$. ■

Problem 9. Let X be a nonempty compact Hausdorff space and let $f\colon X \to X$ be a continuous function. Suppose f is 1-1. Prove that there is a nonempty closed set A with $f(A) = A$. (The hypothesis that f is 1-1 is not actually needed, but it makes the proof a little easier.)

Proof. We prove the more general case. First, we will show that the f is a closed map. Suppose C is a closed subset of X then, since X is compact, by Theorem 26.2 C is compact. Then since f is continuous $f(C)$ is compact in X so $f(C)$ is closed by Theorem 26.3. Thus, f is a closed map. Now consider the countable collection of nested closed subsets $X \supset f(X) \supset f^2(X) \supset \cdots$. Indeed, $f^i(X) \supset f^{i+1}(X)$ since if $x \in f^{i+1}(X)$ then there exists $y \in X$ such that $f^{i+1}(y) = x$. Let $z \coloneqq f(y)$ then $f^i(z) = f^{i+1}(y) = x \in f^i(X)$. We claim that $f\left(\bigcap_{i\in\mathbf{N}} f^i(X)\right) = \bigcap_{i\in\mathbf{N}} f^i(X)$ is the set we are looking for. First, since f is a closed map and each $f^i(X)$ is closed (since X is compact Hausdorff) then the intersection $A \coloneqq \bigcap_{i\in\mathbf{N}} f^i(X)$ is closed. By the finite intersection property, Theorem 26.9, F is nonempty since X is nonempty and f is a function (for recall that a function from X to X is an element of the set X^X and if the codomain of such an element is empty then $X^X = \emptyset$, but that would imply $X = \emptyset$) and for any finite subcollection $\left\{f^i(X)\right\}_{i\in I}$ the intersection $\bigcap_{i\in I} f^i(X) = f^m(X)$ where $m = \max\{i \in I\}$. Lastly, we show that $f(A) = A$. One containment is clear, namely $f(A) \subset A$ for if $x \in f(A)$ then $x = f(y)$ for some $y \in A$, i.e., $y \in f^i(X)$ for all i so $x \in A$. To see the reverse take $x \in A$ then $x \in f^i(X)$ for all i. Thus, $f^{-1}(x) \subset f^i(X)$ for all i so $f^{-1}(x) \in A$, i.e., $x \in f(A)$. ■

Problem 10. Let $\sim$ be the equivalence relation on $\mathbf{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. Prove that the quotient space $\mathbf{R}^2/\sim$ is compact but not Hausdorff.

Proof. We first show that the quotient space is not Hausdorff. Let $q\colon \mathbf{R}^2 \to \mathbf{R}^2/\sim$ denote the quotient map. We show that for any point $[(x, y)]$ in the quotient, for any neighborhood V of $[(x, y)]$, for any neighborhood U of $[(0, 0)]$ the intersection $U \cap V \neq \emptyset$. Let U be a neighborhood of $[(0, 0)]$ and V be a neighborhood of $[(x, y)]$. Then $p^{-1}(U)$ is a neighborhood of $(0, 0)$ and $p^{-1}(V) \supset \{\, (tx, ty) \mid t \neq 0 \,\}$ is a neighborhood of (x, y). But since $p^{-1}(U)$ is open, it contains an ε-ball about $(0, 0)$, say $B((0, 0), \varepsilon)$ for $\varepsilon > 0$. But for sufficiently small values of $|t|$, $(tx, ty) \in B((0, 0), \varepsilon)$ for any $\varepsilon > 0$ (for example $t^2x^2 + t^2y^2 \leq \varepsilon$ if $|t| \leq \sqrt{\varepsilon/(x^2 + y^2)}$ so $(tx, ty) \in B((0, 0), \varepsilon)$). Hence $[(x, y)] \in U$ so $U \cap V \neq \emptyset$. Since U and V were arbitrary, we conclude that $\mathbf{R}^2/\sim$ is not Hausdorff.

To see that $\mathbf{R}^2/\sim$ is in fact compact let $\mathcal{U}$ be an open cover of $\mathbf{R}^2/\sim$. Then at least one $U \in \mathcal{U}$ contains the equivalence class of $(0, 0)$. Thus, by the previous argument $q^{-1}(U)$ contains an open ball $B((0, 0), \varepsilon)$ for $\varepsilon > 0$ and this open ball contains (tx, ty) for sufficiently small values of $|t|$, hence U contains every equivalence class of $\mathbf{R}^2/\sim$. Thus, $\mathbf{R}^2/\sim$ is compact. ■

Problem 11. Let X be a locally compact Hausdorff space. Explain how to construct the one-point compactification of X and prove that the space you construct is really compact (you do not have to prove anything else for this problem.)

Proof. This is Theorem 29.1 from Munkres §29, p. 183. We will summarize his argument.

Munkres's construction really only begins in step 2 of his argument. Let Y denote the one-point compactification of X. We topologize Y by defining the topology on Y to be (1) all sets U open in X and (2) all sets of the form $U = Y - C$, where C is a compact subspace of X.

To prove compactness, let $\mathcal{U}$ be an open cover of Y. Isolate an open set of type (2) in the cover, say U, which must exist for otherwise $\infty \notin \bigcup_{U_\alpha\in\mathcal{U}} U_\alpha$ so $\mathcal{U}$ does not cover Y. Given U, let $C \coloneqq Y - U$. Then C is a compact subset of X and is covered by the union of all open sets of type 1 in $\mathcal{U}$. By Lemma 26.1, only finitely many of these U_α's cover C, say $U_1, \ldots, U_n$. Then $U_1, \ldots, U_n, U$

is an open cover of Y since $C \subset \bigcup_{i=1}^{n} U_i$ and $C \cup (Y - C) = Y$ so $Y \subset (\bigcup_{i=1}^{n} U_i) \cup U$. Therefore, Y is compact. ∎

Problem 12. Show that if $\prod_{n=1}^{\infty} X_n$ is locally compact (and each X_n is nonempty), then each X_n is locally compact and X_n is compact for all but finitely many n.

Proof. Define $X \coloneqq \prod_{n=1}^{\infty} X_n$ and let $\mathbf{x} \in X$. Since X is locally compact, there exists a compact set C and an open neighborhood U of $\mathbf{x}$ such that $C \supset U$. Without loss of generality, we may assume that $U = \prod U_n$ where U_n is open in X_n and $U_n = X_n$ for all but finitely many n's. Now, since the projection maps, $\pi_n \colon X \to X_n$, are continuous and by Theorem 26.5, $\pi_n(C) \supset U_n$ is compact. Since $U_n = X_n$ for all but finitely many n's, $\pi_n(C) = X_n$ is compact for all but finitely many n's. Otherwise, $\pi_n(C) \supset U_n$ so X_n is locally compact. ∎

Problem 13. Let X be a locally compact Hausdorff space, let Y be any space, and let the function space $\mathcal{C}(X, Y)$ have the compact-open topology. Prove that the map

$$e \colon X \times \mathcal{C}(X, Y) \to Y$$

define by the equation $e(x, f) = f(x)$ is continuous.

Proof. This is Theorem 46.10 from Munkres §46, p. 286. We paraphrase the proof here.

Given a pair $(x, f) \in \mathcal{C}(X, Y)$ and an open set V in Y containing $e(x, f) = f(x)$, by Theorem 18.1(4) we wish to find an open set about (x, f) that e maps into V. First, using the continuity of f and the fact that X is locally compact Hausdorff, we can choose an open set U about x having compact closure $\overline{U}$, such that f carries $\overline{U}$ into V. Then, consider the open set $U \times S(\overline{U}, V) \subset X \times \mathcal{C}(X, Y)$. It is an open set containing (x, f) and if (x', f') is in this set, then $e(x', f') = f'(x') \in V$, as desired. ∎

Problem 14. Let I be the unit interval, and let Y be a path-connected space. Prove that any two maps from I to Y are homotopic.

Proof. This is Ex. 2(b) from Munkres §51.

It suffices to show that a map $f \colon I \to Y$ is homotopic to a constant map $e_{y_0} \colon I \to Y$. Set $y_0 \coloneqq f(0)$. Consider the map $H(x, t) \coloneqq f((1 - t)x)$. This map is continuous since it is the composition of continuous map f (by hypothesis) and $(1 - t)x$ (by Theorem 25.1, multiplication is continuous in $\mathbf{R}$ hence in I). Moreover, $H(x, 0) = f(x)$ and $H(x, 1) = f(0 \cdot x) = y_0 = e_{y_0}$. Thus, $f \simeq e_{y_0}$. Now, let $g \colon I \to Y$ be continuous. Then by the argument we presented above, $g \simeq e_{y_1}$. But Y is path connected so there exists a path $p \colon I \to Y$ such that $p(0) = y_0$ and $p(1) = y_1$. Then the map $K(x, t) = p(t)$ is a homotopy from e_{y_0} to e_{y_1}. Since $\simeq$ is an equivalence relation, it follows by symmetry that $f \simeq g$. ∎

Problem 15. Let X be a topological space and $f \colon [0, 1] \to X$ any continuous function. Define $\bar{f}$ by $\bar{f}(t) = f(1 - t)$. Prove that $f * \bar{f}$ is path-homotopic to the constant path at $f(0)$.

Proof. By the definition of the product path, we have

$$f * \bar{f} = \begin{cases} \bar{f}(2s) & \text{for } s \in [0, 1/2] \\ f(2s-1) & \text{for } s \in [1/2, 1] \end{cases}$$
$$= \begin{cases} f(1-2s) & \text{for } s \in [0, 1/2] \\ f(2s-1) & \text{for } s \in [1/2, 1] \end{cases}.$$

Now, consider the map $H(x, t) := (f * \bar{f})(tx)$. This map is continuous since tx is continuous in I and $f * \bar{f}$ is continuous. Now, note that

$$\begin{aligned} H(x, 0) &= (f * \bar{f})(0) \\ &= f(0) \\ H(x, 1) &= (f * \bar{f})(x). \end{aligned}$$

Hence $f * \bar{f} \simeq f(0)$. ■

Problem 16. Let X be a path-connected topological space and let $x_0, x_1 \in X$. Recall that any path α from x_0 to x_1 gives an isomorphism $\hat{\alpha}$ from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ (you do not have to prove this.)

Suppose that for every pair of paths α and β from x_0 to x_1 the isomorphisms $\hat{\alpha}$ and $\hat{\beta}$ are the same. Prove that $\pi_1(X, x_0)$ is Abelian.

Proof. Let $[f], [g] \in \pi_1(X, x_0)$. Let $\alpha \colon I \to X$ be a path from x_0 to x_1 and define $\beta = g * \alpha$. Then, by assumption $\hat{\alpha} = \hat{\beta}$ so

$$\begin{aligned} \hat{\beta}([f]) &= [\bar{\alpha} * \bar{g}] * [f] * [g * \alpha] \\ &= [\bar{\alpha}] * [\bar{g} * f * g] * [\alpha] \\ &= [\bar{\alpha}] * [f] * [\alpha] \\ &= \hat{\alpha}([f]) \end{aligned}$$

so $[\bar{g}] * [f] * [g] = [f]$. Hence $\pi_1(X, x_0)$ is Abelian. ■