MA 523: Homework 4

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September 28, 2016

Problem 4.1 (Legendre transform)

Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1 x_1} + 2a^{12}(Du)u_{x_1 x_2} + a^{22}(Du)u_{x_2 x_2} = 0$$

in some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u\big(\mathbf{x}(p)\big),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies the *linear* equation

$$a^{22}(p)v_{p_1 p_2} - 2a^{12}(p)v_{p_1 p_2} + a^{11}(p)v_{p_1 p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29)).

Solution. Assuming the regularity on v prescribed above, we compute $v_{p_1 p_1}$, $v_{p_1 p_2}$ and $v_{p_2 p_2}$.

First, we compute $v_{p_1 p_2}$ since in the case of $v_{p_1 p_1}$ and $v_{p_2 p_2}$, there is some symmetry we can exploit. Taking the first partial with respect to p^1, we have

(4.1)
$$\begin{aligned}
v_{p_1} &= \tfrac{\partial}{\partial p_1}\big(x^1(p)p^1 + x^2(p)p^2 - u(\mathbf{x}(p))\big) \\
&= x^1(p) + x^1_{p_1}(p)p^1 + x^2_{p_1}(p)p^2 - u_{x_1}\big(\mathbf{x}(p)\big)x^1_{p_1}(p) - u_{x_2}\big(\mathbf{x}(p)\big)x^2_{p_1}(p).
\end{aligned}$$

We may as well compute v_{p_2} here

(4.2)
$$\begin{aligned}
v_{p_2} &= \tfrac{\partial}{\partial p_2}\big(x^1(p)p^1 + x^2(p)p^2 - u(\mathbf{x}(p))\big) \\
&= x^1_{p_2}(p)x^1(p) + x^2(p) + x^2_{p_2}(p)p^2 - u_{x_1}\big(\mathbf{x}(p)\big)x^1_{p_2}(p) - u_{x_2}\big(\mathbf{x}(p)\big)x^2_{p_2}(p).
\end{aligned}$$

Now, taking the partial with respect to p_2 of (4.1), we have   ■

PROBLEM 4.2

Find the solution $u(x,t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$\begin{cases} u(x,0) = f(x), \qquad u_t(x,0) = g(x), & \text{for } x > 0 \\ u_t(0,t) = \alpha u_x(0,t), & \text{for } t > 0, \end{cases}$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$. (*Hint:* Use a representation $u(x,t) = F(x-t) + G(x+t)$ for the solution.)

SOLUTION. ∎

PROBLEM 4.3

(a) Let u be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0,t) = u(\pi, t) = 0$. Show that the *energy*

$$E(t) = \frac{1}{2} \int_0^\pi \left(u_t^2 + c^2 u_x^2 \right) dx, \quad t > 0$$

is independent of t; i.e., $\frac{d}{dt} E = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x,t) = \sum_{n=1}^\infty \left(a_n \cos(nct) + b_n \sin(nct) \right) \sin(nx)$$

in terms of coefficients a_n, b_n.

SOLUTION. For part (a), suppose that u is, as above, a solution to the wave equation which is C^2 up to the boundary. We show that its energy is independent of t, i.e., that $\frac{d}{dt} E = 0$. Assuming the energy is bounded, the dominated convergence theorem allows us to permute the order of integration and differentiation like so

$$\begin{aligned}
\tfrac{d}{dt} E(t) &= \frac{d}{dt} \left(\frac{1}{2} \int_0^\pi \left(u_t^2 + c^2 u_x^2 \right) dx \right) \\
&= \frac{1}{2} \int_0^\pi \tfrac{\partial}{\partial t} \left(u_t^2 + c^2 u_x^2 \right) dx \\
&= \frac{1}{2} \int_0^\pi 2 u_t u_{tt} + 2 c^2 u_x u_{xt} \, dx
\end{aligned}$$

which, after using the relation $u_{tt} = c^2 u_{xx}$, becomes

$$\begin{aligned}
&= c^2 \int_0^\pi u_t u_{xx} + u_x u_{xt} \, dx \\
&= c^2 \int_0^\pi \tfrac{\partial}{\partial x} \left(u_x u_t \right) dx \\
&= c^2 \left(u_x(\pi, t) u_t(\pi, t) - u_x(0, t) u_t(0, t) \right) \\
&= 0
\end{aligned}$$

since the boundary conditions, i.e., $u = 0$, implies $u_x = u_t = 0$ at the boundary.

For part (b), suppose u is a Fourier series solution to the wave equation, i.e.,

$$u(x,t) = \sum_{n=1}^\infty \left(a_n \cos(nct) + b_n \sin(nct) \right) \sin(nx).$$

First we compute u_t and u_x. They are

$$u_t(x,t) = \tfrac{\partial}{\partial t} u(x,t)$$
$$= \sum_{n=1}^{\infty} cn\big(b_n \cos(nct) - a_n \sin(nct)\big) \sin(nx)$$

and

$$u_x(x,t) = \tfrac{\partial}{\partial x} u(x,t)$$
$$= \sum_{n=1}^{\infty} n\big(a_n \cos(nct) + b_n \sin(nct)\big) \cos(nx).$$

Thus,

$$E(t) = \frac{1}{2} \int_0^{\pi} \left[\left(\sum_{n=1}^{\infty} cn\big(b_n \cos(nct) - a_n \sin(nct)\big) \sin(nx) \right)^2 \right.$$
$$\left. + c^2 \left(\sum_{n=1}^{\infty} n\big(a_n \cos(nct) + b_n \sin(nct)\big) \cos(nx) \right)^2 \right]$$

which, after expanding and using the fact that $\cos(nct)$, $\sin(nct)$, $\cos(nx)$, and $\sin(nx)$ are orthogonal, becomes

$$= \frac{1}{2} \int_0^{\pi} \left[\sum_{n,m=1}^{\infty} c^2 nm\big(b_n b_m \cos(nct)\cos(mct) + a_n a_m \sin(nct)\sin(mct) \right.$$

$$- a_n b_m \cos(mct)\sin(nct) - a_m b_n \cos(nct)\sin(mct)\big) \sin(nx)\sin(mx)$$

$$- c^2 \sum_{n,m=1}^{\infty} n^2 \big(a_n a_m \cos(nct)\cos(mct) + b_n b_m \sin(nct)\sin(mct)$$

$$\left. + a_n b_m \cos(nct)\sin(mct) + a_m b_n \cos(mct)\sin(nct)\big) \cos(nx)\cos(mx) \right]$$

$$= \frac{1}{2} \int_0^{\pi} \sum_{n=1}^{\infty} \left(cn^2 \big(b_n^2 \cos^2(nct) + a_n^2 \sin^2(nct)\big) \sin^2(nx) - cn^2 \big(a_n^2 \cos^2(nct) + b_n^2 \sin^2(nct)\big) \cos^2(nx) \right)$$

Consider the map $X \leftarrow Y$ ■