An Elementary Treatment of the Construction of the Free Product of Groups

James E. McClure and Alec McGail

Abstract. We consider the elementary part of the theory of free products of groups. We review the history and then give an elementary proof that the multiplication in the free product is associative. This proof originated in a suggestion by the second author and has not been noticed, to our knowledge, by the experts.

The free product $G * H$ of two groups G and H is an important construction in group theory. Intuitively, $G * H$ is a group which contains G and H as subgroups and is generated by them with no relations (in contrast to this, the Cartesian product $G \times H$ has commutativity relations $g \cdot h = h \cdot g$).

It turns out to be nontrivial to give an explicit construction of $G * H$.[1] It's clear from the intuitive description that, as a set, $G * H$ should consist of elements of five types:

$$e, \quad g_1 h_1 \cdots g_k, \quad g_1 h_1 \cdots h_k, \quad h_1 g_1 \cdots h_k, \quad h_1 g_1 \cdots g_k,$$

where $g_i \in G - \{e\}$ and $h_j \in H - \{e\}$; because there are to be no relations, each element of $G * H$ should occur exactly once on this list. It's also clear that to multiply two elements from this list one should juxtapose them and simplify as much as possible. For example, the product $g_1 h_1 g_2 h_2 g_3 \cdot g_3^{-1} h_2^{-1} g_2^{-1} g_4$ should be $g_1 h_1 g_4$. However, as this example shows, the amount of cancellation that can occur is unlimited, so it's not at all straightforward to show that the multiplication is associative; it isn't even clear how to list all the possible cases and subcases. The paper [**4**] gave a very complicated inductive proof of associativity[2] and thereby showed that the description of $G * H$ just given is rigorous.

For what follows it's convenient to introduce some terminology. Let us say that a *word* is a finite (possibly empty) sequence of elements of $(G - \{e\}) \cup (H - \{e\})$; it is *reduced* if no two adjacent elements are in G or in H. Thus $G * H$, as a set, should be the set of reduced words, with the empty word corresponding to e.

Twenty years after [**4**], Emil Artin ([**1**]) gave a simpler (but still complicated) proof that the multiplication in $G * H$ is associative. He began from the observation that juxtaposition of words (without simplification) is an associative operation. He then defined an equivalence relation on words by letting w and w' be equivalent if the juxtaposed word $w'w^{-1}$ (where w^{-1} has the obvious meaning) is in a certain set Δ defined inductively as follows: $w \in \Delta$ if and only if w can be written in the form $w_0 a_1 w_1 a_2 \cdots a_n w_n$, where all w_i are in Δ, all a_i are in G or all are in H, and $a_1 \cdots a_n = e$. He then showed that each equivalence class contains a unique reduced word and that juxtaposition induces a well-defined operation on equivalence classes.

http://dx.doi.org/10.4169/amer.math.monthly.122.7.690
MSC: Primary 20E06

[1] From a category-theoretic point of view, $G * H$ is the coproduct of G and H, and one can give a formal argument to show that coproducts exist in the category of groups. However, this argument doesn't give an explicit description of $G * H$, and such a description is needed for many applications.

[2] van der Waerden ([**5**]) says that this proof is due to Emil Artin and Schreier.

A year later, van der Waerden ([**5**]) gave a one page indirect proof of associativity. He showed that the permutation group on the set of reduced words contains a subgroup which is in bijective correspondence with the set of reduced words and that the correspondence takes multiplication to juxtaposition followed by simplification. van der Waerden's proof is the standard proof used in textbooks.[3]

Recently, the first author was explaining to his introductory topology class that for the construction of free products one needs to use van der Waerden's indirect approach, quoting Munkres ([**3**]) that a direct proof of associativity would be "horrendous." The second author (who was a student in the class) said "Why can't we show associativity by comparing $(xy)z$ and $x(yz)$ to xyz?" This leads to the following proof, which is no harder than van der Waerden's and (we think) more natural.

First, we need some definitions and notation. We will denote a word with commas:

$$w = a_1, a_2, \ldots, a_n.$$

If a and b are both in G or both in H, let $\overline{a, b}$ denote ab if $ab \neq e$ and the empty word otherwise. By an *immediate descendent* of w we mean a word obtained by replacing a subword a, b by $\overline{a, b}$. By a *descendent* of w we mean either w itself or a word which can be reached from w by a chain of immediate descendents.

Proposition. *If x and y are reduced descendents of w then $x = y$.*

Assuming the proposition, we define the product ww' of reduced words w and w' to be the reduced descendent of the word w, w'. This is obviously associative, since $(ww')w''$ and $w(w'w'')$ are reduced descendents of the word w, w', w'' and must therefore be equal.

The proof of the proposition is by induction on the length of w. Let the chain from w to x (resp., y) begin with x_1 (resp., y_1), and let x_1 (resp., y_1) be obtained from w by replacing a, b by $\overline{a, b}$ (resp., c, d by $\overline{c, d}$).

First we observe that if x_1 and y_1 have a common descendent z then we are done, because if u is the reduced descendent of z, then u and x are reduced descendents of x_1, so they are equal by the inductive hypothesis, and similarly $u = y$, so $x = y$.

If a, b and c, d are the same subword, then $x_1 = y_1$.

If a, b and c, d don't overlap, we obtain a common descendent z by replacing c, d in x_1 by $\overline{c, d}$.

If a, b and c, d overlap, but are not the same subword, we may assume that a is to the left of c in w, and then $b = c$ and the triple a, b, d is in G or in H. There are four cases. If $ab \neq e$ and $bd \neq e$, then the word z obtained from x_1 by replacing ab, d by $\overline{ab, d}$ is a common descendent of x_1 and y_1. If $ab = e$ and $bd \neq e$, then x_1 is an immediate descendent of y_1. If $ab \neq e$ and $bd = e$, then y_1 is an immediate descendent of x_1. Finally, if $ab = e$ and $bd = e$, then $x_1 = y_1$. This completes the proof.

Remark. The key difference between our proof and the proof in [**4**] is that we investigate the reduction process first and then consider multiplication, where [**4**] treats the reduction process separately in the various cases that arise in the proof of associativity.

Remark. For the special case of a free group (that is, a free product of $\mathbb{Z}$'s) Michael Artin gives a proof ([**2**, Section 6.7]) which is similar to the corresponding special case

[3]Some textbooks use the categorical argument mentioned in footnote 1 to prove existence of the coproduct and then use a version of van der Waerden's argument to show that the coproduct consists of the reduced words with multiplication given by juxtaposition and simplification.

of our proof but more complicated (because, as in [**1**], he defines an element of the free group to be an equivalence class of words rather than a reduced word).

ACKNOWLEDGMENT. We'd like to thank Dave Benson, Tom Hunter, and Ron Solomon for helpful correspondence. The first author was partially supported by a grant from the Simons Foundation (#279092 to James McClure). He thanks the Lord for making his work possible.

REFERENCES

1. E. Artin, The free product of groups, *Amer. J. Math.* **69** (1947) 1–4.
2. M. Artin, *Algebra.* Prentice Hall, Upper Saddle River, NJ, 1991.
3. J. Munkres, *Topology.* Second edition. Prentice Hall, Upper Saddle River, NJ, 2000.
4. O. Schreier, Die Untergruppen der freien Gruppen, *Abh. Math. Semin. Univ. Hambg.* **5** (1927) 161–183.
5. B. L. van der Waerden, Free products of groups, *Amer. J. Math.* **70** (1948) 527–528.

Department of Mathematics, Purdue University, West Lafayette, IN 47907
mcclure@math.purdue.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907
amcgail@purdue.edu

The Paul R. Halmos–Lester R. Ford Awards for 2014

The Paul R. Halmos–Lester R. Ford Awards, established in 1964, are made annually to authors of outstanding expository papers in the MONTHLY. The awards are named for Paul R. Halmos and Lester R. Ford, Sr., both distinguished mathematicians and former editors of the MONTHLY. Winners of the Halmos–Ford Awards for expository papers appearing in Volume 121 (2014) of the MONTHLY are as follows.

- Mario Ponce and Patricio Santibáñez, "On Equidistant Sets and Generalized Conics: The Old and the New," pp. 18–32.
- Daniel Velleman, "A Drug-Induced Radom Walk," pp. 283–298.
- Allison Henrich and Louis Kaufman, "Unknotting Unknots," pp. 379–390.
- Erwan Brugallé and Kristin Shaw, "A Bit of Tropical Geometry," pp. 563–589.