MA166: Recitation 8 Prep

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1 Homework Solutions

Section 1.1: Homework 18

Problem 1.1. The masses m_i are located at the points P_i. Find the moments M_x and M_y and the center of mass of the system.

$$m_1 = 2, \qquad m_2 = 1, \qquad m_3 = 7;$$
$$P_1(2,-5), \qquad P_2(-3,1), \qquad P_3(3,5).$$

Solution. The definitions for the moment of the system about the y-axis is

$$M_y = \sum_{i=1}^{n} m_i x_i, \tag{1}$$

and for the moment of the system about the x-axis is

$$M_x = \sum_{i=1}^{n} m_i y_i. \tag{2}$$

So all you need to do for this problem is to plug in the values

$$M_y = m_1x_1 + m_2x_2 + m_3x_3 = \boxed{-4,}$$

and

$$M_x = m_1y_1 + m_2y_2 + m_3y_3 = \boxed{-2.}$$

Then the total mass is $M = 10$ so

$$(\bar{x}, \bar{y}) = \boxed{\left(-\frac{2}{5}, -\frac{1}{5}\right).}$$

☺

Problem 1.2. Sketch the region bounded by the curves, and visually estimate the location of the centroid.

$$y = 4x, \qquad y = 0, \qquad x = 1.$$

Solution. The image you can find yourself. It's at the centroid of the triangle (assuming uniform distribution of mass) and there's a very simple formula for finding the centroid of a triangle, from a purely geometric perspective, it is

$$(\bar{x}, \bar{y}) = \frac{1}{3}(x_1 + x_2 + x_3, y_1 + y_2 + y_3) \tag{3}$$

where (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of the triangle. The vertices are very clearly $(0,0)$, $(1,0)$ and $(1,4)$ hence

$$(\bar{x}, \bar{y}) = \boxed{\left(\frac{2}{3}, \frac{4}{3}\right).}$$

☺

Problem 1.3. Sketch the region bounded by the curves, and visually estimate the location of the centroid. Find the exact coordinates of the centroid.

$$y = e^x, \qquad y = 0, \qquad x = 5.$$

Find the exact coordinates of the centroid.

Solution. I'll assume you can plot this on your own. Having me do it is asking for too much this late at night :-). Now, recall the definition of the moments about the axes

$$M_y = \int_a^b x(f(x) - g(x))\, dx \tag{4}$$

and

$$M_x = \int \frac{(f(x) - g(x))^2}{2}\, dx \tag{5}$$

and of course the formula for the centroid

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{A}, \frac{M_x}{A}\right). \tag{6}$$

Now the first thing we need to do is to calculate the area

$$A = \int_0^5 e^x\, dx = e^5 - 1.$$

Next, we calculate M_y and M_x like so

$$\begin{aligned} M_x &= \int_0^5 xe^x\, dx \\ &= [xe^x - e^x]_0^5 \\ &= 4e^5 + 1 \end{aligned} \qquad\qquad \begin{aligned} M_y &= \int_0^5 \frac{e^{2x}}{2}\, dx \\ &= \frac{1}{4}\left[e^{2x}\right]_0^5 \\ &= \frac{e^{10} - 1}{4}. \end{aligned}$$

So the centroid is

$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{1 + 4e^5}{e^5 - 1}, \frac{e^5 + 1}{4}\right)}.$$

☺

Problem 1.4. Find the centroid of the region bounded by the given curves.

$$y = 6\sin 5x, \qquad y = 6\cos 5x, \qquad x = 0, \qquad x = \frac{\pi}{20}.$$

Solution. What a horrible calculation. Spare my poor fingers having to type this out in details :^).

The area is

$$A = 6\int_0^{\pi/12} \cos 3x - \sin 3x\, dx$$

$$
\begin{aligned}
&= 2[\sin 3x + \cos 3x]_0^{\pi/12} \\
&= \boxed{2(\sqrt{2}-1).}
\end{aligned}
$$

Skipping straight to the centroid, we have the following

$$
\begin{aligned}
\bar{x} &= \frac{1}{2(\sqrt{2}-1)} \int_0^{\pi/12} x\cos 3x - x\sin 3x \; dx \\
&= \frac{3}{\sqrt{2}-1} \int_0^{\pi/12} x\cos 3x - x\sin 3x \; dx \\
&= \frac{3}{\sqrt{2}-1} \int_0^{\pi/12} \left[\frac{x\sin 3x + x\cos 3x}{3} \right. \\
&\qquad \left. + \frac{\cos 3x - \sin 3x}{9} \right]_0^{\pi/12} \\
&= \frac{\pi\sqrt{2}-4}{12(\sqrt{2}-1)}
\end{aligned}
$$

$$
\begin{aligned}
\bar{y} &= \frac{1}{4(\sqrt{2}-1)} \int_0^{\pi/12} \cos^2 3x - \sin^2 3x \; dx \\
&= \frac{9}{\sqrt{2}-1} \int_0^{\pi/12} \cos^2 3x - \sin^2 3x \; dx \\
&= \frac{3}{2(\sqrt{2}-1)} [\sin 6x]_0^{\pi/12} \\
&= \frac{3}{2(\sqrt{2}-1)}.
\end{aligned}
$$

So the answer is

$$
\boxed{(\bar{x}, \bar{y}) = \left(\frac{\pi\sqrt{2}-4}{12(\sqrt{2}-1)}, \frac{3}{2(\sqrt{2}-1)}. \right).}
$$

☺

Problem 1.5. Find the centroid of the region bounded by the given curves.

$$
y = x^3, \qquad x + y = 30, \qquad y = 0.
$$

Solution. The curves are simple enough to sketch. We compute the area

$$
\begin{aligned}
A &= \int_0^3 x^3 \; dx + \int_3^{30} (30 - x) \; dx \\
&= \frac{1539}{4}.
\end{aligned}
$$

Next we compute directly $\bar{x}$ and $\bar{y}$

$$
\begin{aligned}
\bar{x} &= \frac{2}{1539} \int_0^{27} \Big((30-y)^2 - y^{2/3} \Big)^2 \; dx \\
&= \frac{2}{1539} \int_0^{27} 900 - 60y + y^2 - y^{2/3} \; dy \\
&= \frac{2}{1539} \left[900y - 30y^2 + \tfrac{1}{3}y^3 - \tfrac{3}{5}y^{5/3} \right]_0^{27} \\
&= \frac{1092}{95}
\end{aligned}
$$

$$
\begin{aligned}
\bar{y} &= \frac{4}{1539} \int_0^{27} y\Big(30 - y - y^{1/3}\Big) \; dy \\
&= \frac{4}{1539} \int_0^{27} 30y - y^2 - y^{4/3} \; dy \\
&= \frac{4}{1539} \left[15y^2 - \tfrac{1}{3}y^3 - \tfrac{3}{7}y^{7/3} \right]_0^{27} \\
&= \frac{1188}{133}.
\end{aligned}
$$

So the answer is

$$\boxed{(\bar{x},\bar{y}) = \left(\frac{1092}{95}, \frac{1188}{133}\right).}$$

☺

Problem 1.6. Calculate the moments M_x, M_y and the center of mass of the lamina with the given density and shape. $\rho = 3$.

Solution. Just a silly calculation. Using plain old geometry, we can compute the area of the region by inspection $A = \frac{1}{4}\pi^2 + \frac{1}{2}$. We still have to parameterize the quarter-circle and find the equation for the line. These are

$$f(x) = \sqrt{1-x^2} \qquad \text{and} \qquad g(x) = x - 1.$$

Hence

$$\begin{aligned} M_x &= \frac{3}{2}\int_0^1 \left(\sqrt{1-x^2}\right)^2 - (x-1)^2 \, dx \\ &= \frac{3}{2}\int_0^1 (1-x^2) - (x^2 - 2x + 1) \, dx \\ &= \frac{3}{2}\int_0^1 -2x^2 + 2x \, dx \\ &= \frac{3}{2}\left[-\tfrac{2}{3}x^3 + x^2\right]_0^1 \\ &= \frac{1}{2} \end{aligned} \qquad \begin{aligned} M_y &= 3\int_0^1 x\sqrt{1-x^2} - x(x-1) \, dx \\ &= 3\left[-\tfrac{1}{3}\left(1-x^3\right)^{3/2} - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1 \\ &= \frac{3}{2}. \end{aligned}$$

Then

$$\boxed{(\bar{x},\bar{y}) = \left(\frac{2}{\pi+2}, \frac{2}{3\pi+6}\right).}$$

☺

Problem 1.7.

Solution. ☺

Section 1.2: Homework 19

Problem 1.8 (HW 19, # 1)**.** List the first five terms of a sequence

$$a_n = \frac{(-1)^{n-1}}{6^n}.$$

Solution. Just plug in the values $n = 1, 2, 3, 4, 5$ into the equation. ☺

Problem 1.9 (HW 19, # 2)**.** List the first five terms of the sequence

$$a_1 = 4, \qquad a_{n+1} = 5a_n - 1$$

Solution. The sequence is recursive and depends on the value of the previous terms. ☺

Problem 1.10 (HW 19, # 3)**.** Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues. (Assume that n begins with 1).

Solution. The denominator of the nth is the nth odd integer; odd integers are not divisible by 2 so odd integers have the form $2n-1$. Hence, $\boxed{a_n = 1/(2n-1)}$. ☺

Problem 1.11 (HW 19, # 4)**.** Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = 2 - (0.3)^n.$$

Solution. Since the 2 part of a_n is constant, we may ignore it for the moment. What happens to 0.3^n as $n \to \infty$? The sequence is geometric, i.e., of the form r^n and we have that the limit exits if $-1 < r < 1$. ☺

Problem 1.12 (HW 19, # 5)**.** Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{n^3}{5n^3+1}$$

Solution. Do the following

$$\begin{aligned} a_n &= \frac{n^3}{5n^3+1} \\ &= \frac{n^3/n^3}{(5n^5+1)/n^3} \\ &= \frac{1}{5+\frac{1}{n^3}} \end{aligned}$$

where $5 + \frac{1}{n^3} \to 5 + 0$ as $n \to \infty$ so the limit is $\boxed{1/5}$. ☺

Problem 1.13 (HW 19, # 7)**.** Determine whether the sequence converges or diverges. If it converges, fin the limit.

Solution. As $n \to \infty$, $8n \to 0$ so $\lim_{n\to\infty} a_n = \cos(0) = 1$. ☺

Problem 1.14 (HW 19, # 8)**.** Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{(8n-1)!}{(8n+1)!}.$$

Solution. By the definition of the factorial we have $n! = n(n-1)!$, henec

$$\begin{aligned} a_n &= \frac{(8n-1)!}{(8n+1)!} \\ &= \frac{(8n-1)!}{(8n+1)8n(8n-1)!} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(8n+1)8n} \\
&= \frac{1}{64n^2+8n}
\end{aligned}
$$

which clearly goes to 0 as $n \to \infty$. ☺

Problem 1.15 (HW 19, # 9)**.** Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = n^2 e^{-3n}.$$

Solution. Write

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} n^2 e^{-3n} \\
&= \lim_{n\to\infty} \frac{n}{e^{3n}}
\end{aligned}
$$

by L'Hôpital's rule twice

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{2x}{3e^{3x}} \\
&= \lim_{n\to\infty} \frac{2}{9e^{3x}} \\
&= 0.
\end{aligned}
$$

☺

Problem 1.16 (HW 19, # 10)**.** Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \tfrac{n}{4}\sin\left(\tfrac{4}{n}\right).$$

Solution. Let $m = 4/n$ and rewrite

$$
\begin{aligned}
a_n &= \tfrac{n}{4}\sin(4/n) \\
&= \frac{\sin(4/n)}{4/n} \\
&= \frac{\sin m}{m}.
\end{aligned}
$$

Now as $n \to \infty$, $m \to 0$ so

$$\lim_{n\to\infty} \tfrac{1}{4}\sin(4/n) = 1$$

by some theorem in the book. ☺

Problem 1.17 (HW 19, # 11)**.** Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{\sin 3n}{1+\sqrt{n}}.$$

Solution. Since sin is periodic, $-1 \le \sin 3n \le 1$ so by the squeeze theorem we have

$$\frac{-1}{1+\sqrt{n}} \le \frac{\sin 3n}{1+\sqrt{n}} \le \frac{1}{1+\sqrt{n}}.$$

Letting $n \to \infty$, we see that the limit of $\sin 3n/(1+\sqrt{n})$ is 0. ☺

Section 1.3: Homework 20

Problem 1.18 (HW 20, # 1)**.** Determine whether the sequence is increasing, decreasing, or monotonic.

$$a_n = \frac{3n-7}{7n+3}.$$

Solution. Clearly increasing and bounded, just check that a_n never exceeds say 5. You can check that this is increasing by replacing n by x and taking the derivative

$$\frac{d}{dx}\left(\frac{3n-7}{7n+3}\right) = \frac{3(7x+3)-7(3x-7)}{(7x+3)^2} = \frac{58}{(7x+3)^2} > 0.$$

☺

Problem 1.19 (HW 20, # 2)**.** Determine whether the sequence is increasing, decreasing, or monotonic.

$$a_n = 6ne^{-5n}.$$

Solution. The function is decreasing since e^{-5n} is decreasing. You can check by the derivative test. Moreover the sequence is bounded by $\frac{6}{e^5}$ above and 0 below. ☺

Problem 1.20 (HW 20, # 3)**.** Determine whether the sequence is increasing, decreasing, or monotonic.

$$a_n = \frac{n}{5n^2+3}.$$

Solution. The sequence is clearly decreasing. Take the derivative and check. It is bounded since $0 < a_n \leq 1/8$. ☺

Problem 1.21 (HW 20, # 4)**.** (a) What is the difference between a sequence and a series?
(b) What is a convergent series? What is a divergent series?

Solution. (a) A sequence is an ordered list of numbers whereas a series is the sum of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent. ☺

Problem 1.22 (HW 20, # 5)**.** Determine whether the series is increasing, decreasing, or monotonic.

$$\left(7 - 9 + \frac{81}{7} - \frac{729}{49} + \cdots\right).$$

Solution. The series can be written

$$7\sum_{n=1}^{\infty}\left(-\frac{9}{7}\right)^{n-1}.$$

This is a geometric series with $|9/7| > 1$ so it diverges. ☺

Problem 1.23 (HW 20, # 6)**.** Determine whether the series is increasing, decreasing, or monotonic.

$$\sum_{n=1}^{\infty} 6\left(\frac{1}{2}\right)^{n-1}.$$

Solution. This is yet another geometric series. Note that $|1/2| < 1$ so this series converges. Remember the formula

$$\frac{a}{1-r} \tag{7}$$

for the convergence of a geometric series. Plugging in $a = 6$ and $r = 1/2$ into this equation, we get 12. ☺

Problem 1.24 (HW 20, # 7)**.** Determine whether the series is increasing, decreasing, or monotonic.

$$\sum_{n=1}^{\infty} \frac{6^n}{(-2)^{n-1}}.$$

Solution. Rewrite the sum as

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{6^n}{(-2)^{n-1}} &= 6\sum_{n=1}^{\infty} \frac{6^{n-1}}{(-2)^{n-1}} \\ &= 6\sum_{n=1}^{\infty} \left(-\frac{6}{2}\right)^{n-1} & &= 6\sum_{n=1}^{\infty} (-3)^{n-1}.\end{aligned}$$

This clearly does not converge since $|-3| > 1$. ☺

Problem 1.25 (HW 20, # 8)**.** Determine whether the series is increasing, decreasing, or monotonic.

$$\sum_{n=1}^{\infty} \frac{(-7)^{n-1}}{8^n}.$$

Solution. Rewrite the sequence as

$$\sum_{n=1}^{\infty} \frac{(-7)^{n-1}}{8^n} = \frac{1}{8}\sum_{n=1}^{\infty} \left(-\frac{7}{8}\right)^{n-1}$$

this converges since $|-7/8| < 1$ and by the formula (7) it converges to $1/15$. ☺

Problem 1.26 (HW 20, # 9)**.** Determine whether the series is increasing, decreasing, or monotonic.

$$\sum_{n=0}^{\infty} \frac{1}{(\sqrt{14})^n}.$$

Solution. Rewrite it as

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{14}}\right)^n.$$

You can see that since $\left|1/\sqrt{14}\right| < 1$, the series converges and by (7) it converges to $\left(14+\sqrt{14}\right)/13$. ☺

2 Past Exam Problems

Problem 2.1.

Solution. ☺