# MA557 Homework 12

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December 6, 2015

## Problem 12.1

Let $R$ be a Noetherian domain. Show that the following are equivalent:

(i) $R$ is a unique factorization domain

(ii) every prime ideal of $R$ of height one is principal

(iii) $R$ is normal with $\mathrm{Cl}(R) = 0$.

*Proof.* (i) $\implies$ (ii) Suppose $R$ is a Noetherian domain. Let $\mathfrak{p}$ be a height one prime. Then there exists at least one nonzero element $x \in \mathfrak{p}$. Let $x = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the factorization of $x$ into irreducible (prime) elements of $R$. Set $p := p_i$ for any prime in the factorization of $x$. Then the ideal generated by $p$ is a prime ideal contained in $\mathfrak{p}$, i.e., $\langle p \rangle \subset \mathfrak{p}$. But $\mathrm{ht}(\mathfrak{p}) = 1$. Thus, $\langle p \rangle = \mathfrak{p}$.

(ii) $\implies$ (ii) Suppose that every height one prime ideal in $R$ is principal. To show that $R$ is a UFD, it suffices to show that every irreducible element $p$ is a prime element, that is, $\langle p \rangle$ is a prime ideal. Let $\mathfrak{p}$ be the minimal prime containing $p$. Since $\mathfrak{p}$ is principal, $\mathfrak{p} = \langle x \rangle$ for some $x \in \mathfrak{p}$. Thus, $p = xy$ for some $y \in R$. But $p$ is prime hence, irreducible so either $x$ or $y$ is a unit. If $x$ is a unit, then $\mathfrak{p} = R$, which is a contradiction. Thus, $y$ must be a unit and we see that $\langle p \rangle = \langle xy \rangle = \mathfrak{p}$ is prime. $\blacksquare$

## Problem 12.2

Let $R$ be a ring with total ring of quotients $K$, $M$ an $R$-module, and

$$\mathcal{T}(M) = \{\, x \in M \mid RR \text{ of } a \text{ zero-divisor non some for } ax = 0 \,\}.$$

The submodule $\mathcal{T}(M)$ is called the *torsion of $M$*, and $M$ is called *torsion free* if $\mathcal{T}(M) = 0$. Show

(a) $\mathcal{T}(M) = \ker(M \to K \otimes_R M)$

(b) $M/\mathcal{T}(M)$ is torsion free.

*Proof.* ∎

## PROBLEM 12.3

Let $R$ be a Dedekind domain and $M$ a finitely generated $R$-module of rank $r$. Show that:

(a) If $M$ is torsion free then $M$ is projective (hint: induct on $r$).

(b) $M \cong \mathscr{T}(M) \oplus P$ with $P$ projective.

(c) If $M \neq 0$ is projective then $M \cong R^{r-1} \oplus I$ with $I \neq 0$ an ideal.

(d) If $M$ is torsion (i.e., $M = \mathscr{T}(M)$) then

$$M \cong R/I_1 \oplus \cdots \oplus R/I_n \qquad \text{with} \qquad I_1 \supset \cdots \supset I_n \neq 0$$

ideals (hint: for $p_1, \ldots, p_s$ the minimal primes of $\mathrm{ann}(M)$ and $S = R \smallsetminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s)$, show that $S^{-1}R$ is a PID).

*Proof.* ∎