Representation Theory

Carlos Salinas

August 20, 2016

Contents

1 What is Representation Theory?

Groups arise in nature as "sets of symmetries (of an object), which are closed under composition and under taking inverses". For example, the *symmetric group* S_n is the group of all permutations (symmetries) of $\{1, \ldots, n\}$; the *alternating group* A_n is the set of all symmetries preserving the parity of the number of ordered pairs; the *dihedral group* D_{2n} is the group of symmetries of the regular n-gon in the plane. The *orthogonal group* $\mathrm{O}(3)$ is the group of distance-preserving transformations of Euclidean space which fix the origin. There is also the group of *all* distance preserving transformations, which includes the translations along with $\mathrm{O}(3)$.*

The official definition is of course more abstract, a group is a set G with a binary operation $*$ which is associative, has a unit element e and for which inverses exist. Associativity allows a convenient abuse of notation, where we write gh for $g * h$; we have $ghk = (gh)k = g(hk)$ and parentheses are unnecessary. I will often write 1 for e, but this is dangerous on rare occasions, such that when studying the group $\mathbb{Z}$ under addition; in that case, $e = 0$.

The abstract definition notwithstanding, the interesting situation involves a group "acting" on a set. Formally, an action of a group G on a set X is an "action map" $a\colon G \times X \to X$ which is *compatible with the group law*, in the sense that

$$\begin{aligned} a\big(h, a(g, x)\big) &= a(hg, x) \\ a(e, x) &= x. \end{aligned}$$

This justifies the abuse of notation $a(g, x) = gx$, for we have $h(gx) = (hg)x$.

From this point of view, geometry asks, "Given a geometric object X, what is its group of symetries?" Representation theory reverses the quostion to "Given a group G, what objects X does it act on?" and attempts to answer this question by classifying such X up to isomorphism.

Before restricting to the linear case, our main concern, let us remember another way to describe an action of G on X. Every $g \in G$ defines a map $a(g)\colon X \to X$ by $x \mapsto gx$. This map is a bijection, with inverse map $a(g^{-1})$: indeed, $\big(a(g^{-1}) \circ a(g)\big)(x) = g^{-1}gx = ex = x$ from the properties of the action. Hence $a(g)$ belongs to the set $\operatorname{Sym} X$ of bijective self-maps of X. This set forms a group under composition, and the properties of an action imply that

Proposition 1.1. *An action of G on X "is the same as" a group homomorphism $\alpha\colon G \to \operatorname{Sym} X$.*

The formulation of Prop. 1.1 leads to the following observation. For any action a of H on X and group homomorphism $\varphi\colon G \to H$, there is defined a *restricted* or *pulled-back* action $\varphi^* a$ of G on X, as $\varphi^* a = a \circ \varphi$. In the original definition, the action sends (g, x) to $\varphi(g)(x)$.

1.1 Tautological action of $\operatorname{Sym} X$ on X

This is the obvious action, call it T, sending f, x to $f(x)$, where $f\colon X \to X$ is a bijection and $x \in X$. In this language, the action a of G on X is $\alpha^* T$ with the homomorphism α of the proposition – the pull-back under α of the tautological action.

1.2 Linearity

The question of classifying all possible X with action of G is hopeless in such generality, but one should recall that, in first approximation, mathematics is linear. So we shall take our X to be a*vector space* over some

*This group is isomorphic to the *semi-direct product* $\mathrm{O}(3) \ltimes \mathbb{R}^3$.

ground *field*, and ask that the action of G be linear, as well, in other words, that it should preserve the vector space structure. Our interest is mostly confined to the case when the field of scalars is $\mathbb{C}$, although we shall occasionally mention how the picture changes when other fields are studied.

Definition 1.2. A linear representation ρ of G on a complex vector space V is a set-theoretic action on V which preserves the linear structure, i.e.,

$$\begin{aligned} \rho(g)(\mathbf{v}_1 + \mathbf{v}_2) &= \rho(g)\mathbf{v}_1 + \rho(g)\mathbf{v}_2, && \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V, \\ \rho(g)(k\mathbf{v}) &= k\rho(g)\mathbf{v} && \text{for all } k \in \mathbb{C}, \mathbf{v} \in V. \end{aligned}$$

Unless otherwise mentioned, a *representation* will mean a *finite-dimensional complex representation*.

1.3 The general linear group

Let V be a complex vector space of dimension $n < \infty$. After choosing a basis, we can identify it with $\mathbb{C}^n$, although we shal lavoid doing so without good reason. Recall that the *endomorphism algebra* $\mathrm{End}(V)$ is the set of all linear maps (or *operators*) $L\colon V \to V$, with the natural addition of linear maps and the composition as multiplication. If V has been identified with $\mathbb{C}^n$, a linear map is uniquely representable by a matrix, and the addition of linear maps becomes the entrywise addition, while the composition becomes the matrix multiplication.

Inside $\mathrm{End}(V)$, there is contained the group $\mathrm{GL}(V)$ of invertible linear operators; the group operation, of course, is composition.

Proposition 1.3. *V is naturally a representation of* $\mathrm{GL}(V)$.

It is called the *standard* representation of $\mathrm{GL}(V)$. The following corresponds to Prop. 1.1, involvinge the same abuse of language.

Proposition 1.4. *A representation of G on V "is the same as" a group homomorphism from G to* $\mathrm{GL}(V)$.

Proof. Observe that, to give a linear action of G on V, we must assign to each $g \in G$ a linear self-map $\rho(g) \in \mathrm{End}(V)$. Compatibility of the action with the group law requires

$$\rho(h)\big(\rho(g)(\mathbf{v})\big) = \rho(hg)(\mathbf{v}), \qquad \rho(1)(\mathbf{v}) = \mathbf{v},$$

for all $\mathbf{v} \in V$, whence we conclude that $\rho(1) = \mathrm{Id}$, $\rho(hg) = \rho(h) \circ \rho(g)$. Taking $h = g^{-1}$ shows that $\rho(g)$ is invertible, hence lands in $\mathrm{GL}(V)$. The first relation then says that we are dealing with a group homomorphism. ∎

Definition 1.5. An *isomorphism* φ between two representation (ρ_1, V_1) and (ρ_2, V_2) of G is a linear isomorphism $\varphi\colon V_1 \to V_2$ which intertwines with the action of G, that is, satisfies

$$\varphi\big(\rho_1(h)(\mathbf{v})\big) = \rho_2(g)\big(\varphi(\mathbf{v})\big).$$

Note that the equality makes sense even if φ is not invertible, in which case it is just called an *intertwining operator* or *G-linear map*. However, if φ is invertible, we can write instead

$$\rho_2 = \varphi \circ \rho_1 \circ \varphi^{-1}, \tag{1}$$

meaning that we have an equality of linear maps after intserting any group element g. Observe that this relation determines ρ_2 if ρ_1 and φ are known. We can finally formulate the basic problem of representation theory: Classify all representation of a given group G, up to isomorphism.

For arbitrary G, this is very hard! We shall concentrate on finite groups, where a very good general theory exists. Later on, we shall study some examples of topological compact groups, such as U(1) and SU(2). The general theory for compact groups is also completely understood, but requires more difficult methods.

I close with a simple observation, tying in with Def. 1.2. Given any representation ρ of G no a space V of dimension n, a choice of basis in V identifies this linearly with $\mathbb{C}^n$. Call the isomorphism φ. Then, by formula (**??**), we can define a new representation ρ_2 of G on $\mathbb{C}^n$, which is isomorphic to (ρ, V). So any n-dimensional representation of G is isomorphic to a representation on $\mathbb{C}^n$. The use of an abstract vector space does not lead to 'new' representation, but it does free us from the presence of a distinguished basis.

2 Lecture 2

In this section, we shall discuss the representations of a cyclic group, and then proceed to define the important notions of irreducibility and complete reducibility.

2.1 Concrete realization of isomorphism classes

We observed last time that every m-dimensional representation of a group G was isomorphic to a representation on $\mathbb{C}^m$. This leads to a concrete realization of the set of m-dimensional isomorphism classes of representations.

Proposition 2.1. *The set of m-dimensional isomorphism classes of G-representations is in bijection with the quotient*

$$\mathrm{Hom}\big(G, \mathrm{GL}(m, \mathbb{C})\big)/\mathrm{GL}(m, \mathbb{C})$$

of the set of group homomorphisms to $\mathrm{GL}(m)$ *by the overall conjugation action on the latter.*

Proof. This feels rather tautological, but conjugation by $\varphi \in \mathrm{GL}(m)$ sends a homomorphism ρ to the new homomorphism $g \mapsto \varphi \circ \rho(g)\varphi^{-1}$. According to Def. 1.2, this has exactly the effect of identifying isomorphic representations. ∎

Remark 2.2. The proposition is not as useful as it looks (at least to us). It can be helpful in undesrtanding certain infinite discrete groups (such as the following example, $\mathbb{Z}$) in which case the set Hom can have interesting geometric structures. However, for finite groups, the set of isomorphism classes is finite so its description above is not too enlightening.

2.2 Representation of $\mathbb{Z}$

We shall classify all representations of the group $\mathbb{Z}$, with its additive structure. We must have $\rho(0) = \mathrm{Id}$. Aside from that, we must specify an invertible matrix $\rho(n)$ for every $n \in \mathbb{Z}$. However, given $\rho(1)$, we can recover $\rho(n)$ as $\rho(1 + \cdots + 1) = \rho(1)^n$. So there is no choice involved. Conversely, for any invertible map $\rho(1) \in \mathrm{GL}(m)$, we obtain a representation of $\mathbb{Z}$ this way.

Thus, m-dimensional isomorphism classes of representations of $\mathbb{Z}$ are in bijection with conjugacy classes in $\mathrm{GL}(m)$. These can be parametrized by the *Jordan canonical form* (see the next example). We will have m continuous parameters – the eigenvalues, which are nonzero complex numbers, and are defined up to reordering – and some discrete parameters whenever two or more eigenvalues coincide, specifying the Jordan block sizes.

2.3 The cyclic group of order n

Let $G = \{1, g, \ldots, g^{n-1}\}$, with the relation $g^n = 1$. A representation of G on V defines an invertible endomorphism $\rho(g) \in \mathrm{GL}(V)$. As before, $\rho(1) = \mathrm{Id}$ and $\rho(g^k) = \rho(g)^k$, so all other images of ρ are determined by the single operator $\rho(g)$.

Choosing a basis of V allows us to convert $\rho(g)$ into a matrix A, but we shall want to be careful with our choice. Recall that from general theory that there exists a *Jordan basis* in which $\rho(g)$ takes its block-diagonal

Jordan normal form

$$A = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_m \end{bmatrix}$$

where the *Jordan blocks* J_k take the form

$$J_k = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

However, we must impose the condition $A^n = \mathrm{Id}$. But A^n itself will be block-diagonal, with blocks ${J_k}^n$, so we must have ${J_k}^n = 1$. To compute that, let N be the Jordan matrix with $\lambda = 0$; then we have $J = \lambda \,\mathrm{Id} + N$, so

$$J^n = \left(\lambda \,\mathrm{Id} + N\right)^n = \lambda^n \,\mathrm{Id} + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots.$$

Notice that for the above to be Id, since N^p is the matrix of zeros and ones only, with the ones in index position (i, j) with $i = j + k$ (a line parallel to the diagonal, k steps above it). So the sum above can be Id if and only if $\lambda^n = 1$ and $N = 0$. In other words, J is a 1×1 black, and $\rho(g)$ is *diagonal* in this basis. We conclude the following.

Proposition 2.3. *If V is a representation of the cyclic group G of order n, there exists a basis in which the action of every group element is diagonal, with the n-th roots of unity on the diagonal.*

2.4 Finite Abelian groups

The discussion for cyclic groups generalizes to any finite Abelian group A. (The resulting classification of representations is more or less explicit, depending on whether we are willing to use the classification theorem for finite Abelian groups.) We recall the following fact from linear algebra:

Proposition 2.4. *Any family of commuting, separately diagonalizable $m \times m$ matrices can be simultaneously diagonalized.*

The proof of this can be found in most linear algebra books.

This implies that any representation of A is isomorphic to one where every group element acts diagonally. Each diagonal entry then determines a *one-dimensional* representation of A. So the classification reads: m-dimensional isomorphism classes of representations of A are in bijection with unordered m-tuples of 1-dimensional representations. Note that for 1-dimensional representations, viewed as homomorphisms $\rho\colon A \to \mathbb{C}^\times$, there is no distinction between identity and isomorphism (the conjugation action of $\mathrm{GL}(1, \mathbb{C})$ on itself is trivial) .

To say more, we must invoke the classification of finite Abelian groups, according to which A is isomorphic to a direct product of cyclic groups. To specify a 1-dimensional representation of A we must then specify a root of unity of the appropriate order independently for each generator.

2.5 Subrepresentations and reducibility

Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G.

Definition 2.5. A sub representation of V is a G-invariant subspace $W \subseteq V$; i.e., for every $\mathbf{w} \in W$, $g \in G$ we have $\rho(g)(\mathbf{w}) \in W$. W becomes a representation under the action $\rho(g)$.

Recall that, given a subspace $W \subseteq V$, we can form the *quotient space* V/W, the set of W-cosets $\mathbf{v} + V$. If W was G-invariant, the G-action on V *descends to* an action on V/W by setting $g(\mathbf{v} + W) = \rho(g)(\mathbf{v}) + W$. If we choose another $\mathbf{v}'$ in the same coset as $\mathbf{v}$, then $\mathbf{v} - \mathbf{v}' \in W$, so $\rho(g)(\mathbf{v} - \mathbf{v}') \in W$, and then the cosets $\rho(\mathbf{v}) + W$ and $\rho(\mathbf{v}') + W$ agree.

Definition 2.6. With this action, V/W is called the *quotient representation* of V under W.

Definition 2.7. The *direct sum* of two representations (ρ_1, V_1) and (ρ_2, V_2) is the space $V_1 \oplus V_2$ with the block-diagonal action $\rho_1 \oplus \rho_2$ of G.

In the direct sum $V_1 \oplus V_2$, V_1 is a subrepresentation and V_2 is isomorphic to the associated quotient representation. Of course the roles of 1 and 2 can be interchanged. However, one should take care that for an *arbitrary* group, it need not be the case that any representation V with subrepresentation W decomposes as $W \oplus W/V$. This will be proved for complex representations of *finite* groups.

Definition 2.8. A representation is called *irreducible* if it contains no proper invariant subspaces. It is called *completely reducible* if it decomposes as a direct sum of irreducible subrepresentation.

In particular, irreducible representations are completely reducible.

For example, 1-dimensional representations of any group are irreducible. Earlier, we thus proved that finite-dimensional complex representations of a finite Abelian group are completely reducible: indeed, we decomposed V into a direct sum of lines $\bigoplus_{i=1}^{n} L_i$, where $n = \dim V$, along the vectors in the diagonal basis. Each line is preserved by the action of the group. In the cyclic case, the possible actions of C_n on the line correspond to the n eligible roots of unity to specify for $\rho(g)$.

Proposition 2.9. *Every complex representation of a finite Abelian group is completely reducible, and every irreducible representation is* 1*-dimensional.*

It will be our goal to establish an analogous proposition for every finite group G. The result is called the *complete reducibility theorem*. For nonAbelian groups, we shall have to give up on the 1-dimensional requirement, but we shall still salvage a canonical decomposition.

3 Complete Reducibility and Unitarity

In the homework, you find an example of a complex representation of the group $\mathbb{Z}$ which is not completely reducible, and also of a representation of the cyclic group of prime order p over the finite field $\mathbb{F}_p$ which is not completely reducible. This underlines the importance of the following complete reducibility theorem for finite groups.

Theorem 3.1. *Every complex representation of a finite group is completely reducible.*

The theorem is so important that we shall give two proofs. The first uses inner products and so applies only to $\mathbb{R}$ or $\mathbb{C}$, but generalizes to *compact groups*. The more algebraic proof, on the other hand, extends to any field of scalars *whose characteristic does not divide the order of the group* (equivalently, the order of the group should not be 0 in the field).

Beautiful as it is, the result would have limited value without some supply of irreducible representations. It turns out that the following example provides an adequate supply.

3.1 The regular representation

Let $\mathbb{C}[G]$ be the vector space of complex functions on G. It has a basis $\{\mathbf{e}_g : g \in G\}$, with $\mathbf{e}_g$ representing the function equual to 1 at g and 0 elsewhere. G acts on this basis as follows:

$$\lambda(g)(\mathbf{e}_h) = \mathbf{e}_{gh}.$$

This set theoretic action extends by linearity to the vector space:

$$\lambda(g)\Big(\textstyle\sum_{h\in G} v_h \cdot \mathbf{e}_h\Big) = \displaystyle\sum_{h\in G} v_h \cdot \lambda(g)\mathbf{e}_h = \sum_{h\in G} v_h \cdot \mathbf{e}_{gh}.$$

On coordinates, the action is opposite to what you might expect: namely, the h-coordinate of $\lambda(g)(\mathbf{v})$ is $v_{g^{-1}h}$. The result is the *left regular representation* of G. Later we will decompose λ into irreducibles, and we shall see that *every* irreducible isomorphism class of Grepresentations occur in the decomposition.

Remark 3.2. If G acts on a set X, let $\mathbb{C}[X]$ be the vector space of functions on X, with obvious basis basis $\{\mathbf{e}_x : x \in X\}$. By linear extension of the permutation action $\rho(g)(\mathbf{e}_x) = \mathbf{e}_{gx}$