MA571 Homework 9

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Problem 1. Let X be a Hausdorff space and let A be a compact subset of X. Prove from the definitions that A is closed.

Proof. This is Theorem 26.3 from Munkres, p. 165.

We shall prove that $X - A$ is open, so that A is closed. Let $x_0 \in X - A$. We show there is a neighborhood of x_0 disjoint from A. For each point $a \in A$, let us choose disjoint neighborhoods U_a and V_a of the points x_0 and a, respectively (using the Hausdorff condition). The collection

$$\{ V_a \mid a \in A \}$$

is a covering of Y by sets open in X; therefore, finitely many of them $V_{a_1}, ..., V_{a_n}$ cover A. The open set $V = V_{a_1} \cup \cdots \cup V_{a_n}$ contains Y, and is disjoint from the open set $U = U_{a_1} \cap \cdots \cap U_{a_n}$ formed by taking the intersection of the corresponding neighbrohoods of x_0. For if z a point of V, then $z \in V_{a_i}$ for some i, hence $z \notin U_{a_i}$ so $z \notin U$. Then U is a neighborhood of x_0 disjoint from Y, as desired. ∎

Problem 2. Let X be a Hausdorff space and let A and B be disjoint compact subsets of X. Prove that there are open sets U and V such that U and V are disjoint, $A \subset U$ and $B \subset V$.

Proof. Suppose that A and B are disjoint compact subsets of X. By Theorem 26.4 for every $x \in B$ there exists disjoint open sets $U_x \supset A$ and V_x a neighborhood of x. ∎

Problem 3. Prove the Tube Lemma: Let X and Y be topological spaces with Y compact, let $x_0 \in X$, and let N be an open set of $X \times Y$ containing $x_0 \times Y$, then there is an open set W of X containing x_0 with $W \times Y \subset N$.

Proof. ∎

Problem 4. Show that if Y is compact, then the projection map $X \times X \to X$ is a closed map.

Proof. ∎

Problem 5. Let X be a compact space and suppose we are given a nested sequence of subsets $C_1 \supset C_2 \supset \cdots$ with all C_i closed. Let U be an open set containing $\bigcap C_i$. Prove that there is an i_0 with $C_{i_0} \subset U$.

Proof. ∎

Problem 6. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i \colon X \to \mathbf{R}$, $i = 1, ..., n$ with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. Prove that X is homeomorphic to a subspace of $\mathbf{R}^n$.

Proof. ∎

Problem 7. Let X be a compact metric space and let $\mathcal{U}$ be a covering of X by open sets. Prove that there is an $\varepsilon > 0$ such that, for each set $S \subset X$ with diameter $< \varepsilon$, there is a $U \in \mathcal{U}$ with $S \subset U$. (This fact is known as the "Lebesgue number lemma.")

Proof. ∎

Problem 8. Let S^1 denote the circle $\{ x^2 + y^2 = 1 \}$ in $\mathbf{R}^2$. Define an equivalence relation on S^1 by

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } (x, y) = (-x', -y')$$

(you do not have to prove that this is an equivalence relation). Prove that the quotient space $S^1/{\sim}$ is homeomorphic to S^1.

One way to do this is by using complex numbers.

Proof. ■

Problem 9. Let X be a nonempty compact Hausdorff space and let $f : X \to X$ be a continuous function. Suppose f is 1-1. Prove that there is a nonempty closed set A with $f(A) = A$. (The hypothesis that f is 1-1 is not actually needed, but it makes the proof a little easier.)

Proof. ■

Problem 10. Let $\sim$ be the equivalence relation on $\mathbf{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. Prove that the quotient space $\mathbf{R}^2/{\sim}$ is compact but not Hausdorff.

Proof. ■

Problem 11. Let X be a locally compact Hausdorff space. Explain how to construct the one-point compactification of X and prove that the space you construct is really compact (you do not have to prove anything else for this problem.)

Proof. ■

Problem 12. Show that if $\prod_{n=1}^{\infty} X_n$ is locally compact (and each X_n is nonempty), then each X_n is locally compact and X_n is compact for all but finitely many n.

Proof. ■

Problem 13. Let X be a locally compact Hausdorff space, let Y be any space, and let the function space $\mathcal{C}(X, Y)$ have the compact-open topology. Prove that the map

$$e : X \times \mathcal{C}(X, Y) \to Y$$

define by the equation $e(x, f) = f(x)$ is continuous.

Proof. ■

Problem 14. Let I be the unit interval, and let Y be a path-connected space. Prove that any two maps from I to Y are homotopic.

Proof. ■

Problem 15. Let X be a topological space and $f : [0, 1] \to X$ any continuous function. Define $\bar{f}$ by $\bar{f}(t) = f(1 - t)$. Prove that $f * \bar{f}$ is path-homotopic to the constant path at $f(0)$.

Proof. ■

Problem 16. LEt X be a path-connected topological space and let $x_0, x_1 \in X$. Recall that any path α from x_0 to x_1 gives an isomorphism $\hat{\alpha}$ from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ (you do not have to prove this.)

Suppose that for every pair of paths α and β from x_0 to x_1 the isomorphisms $\hat{\alpha}$ and $\hat{\beta}$ are the same. Prove that $\pi_1(X, x_0)$ is Abelian.

Proof. ■