MA571 Homework 10

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Problem 10.1 (Munkres §52, Ex. 2)

Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. By Theorem 52.1, the paths α and β induce a group homomorphism $\hat{\alpha}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $\hat{\beta}\colon \pi_1(X, x_1) \to \pi_1(X, x_2)$, respectively. We want to show therefore that the induced homomorphism $\hat{\gamma} = \widehat{\alpha * \beta}$ is in fact equivalent to the composition $\hat{\beta} \circ \hat{\alpha}$. Let $[f]$ be a loop based at x_0 then

$$\begin{aligned}
\hat{\gamma}([f]) &= \widehat{\alpha * \beta}([f]) \\
&= \left[\, \overline{\alpha * \beta}\, \right] * [f] * [\alpha * \beta] \\
&= \left[\bar{\beta} * \bar{\alpha}\right] * [f] * [\alpha] * [\beta]
\end{aligned}$$

by the well-definedness of the path product operation, we have

$$= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta]$$

by associativity of the path product,

$$\begin{aligned}
&= [\bar{\beta}] * ([\bar{\alpha}] * [f] * [\alpha]) * [\beta] \\
&= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta]
\end{aligned}$$

where $\alpha([f])$ is a loop based at x_1 so

$$\begin{aligned}
&= \hat{\beta}(\hat{\alpha}([f])) \\
&= \big(\hat{\beta} \circ \hat{\alpha}\big)([f]).
\end{aligned}$$

Thus, the following diagram commutes

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\hat{\alpha}"] \arrow[dr, "\hat{\gamma} = \widehat{\alpha * \beta}"'] & \pi_1(X, x_1) \arrow[d, "\hat{\beta}"] \\
& \pi_1(X, x_2).
\end{tikzcd}$$

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Problem 10.2 (Munkres §52, Ex. 3)

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is Abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. $\implies$ Suppose that $\pi_1(X, x_0)$ is Abelian. Then for any class of loops about x_0, say $[f]$ and $[g]$, the product $[f] * [g] = [g] * [f]$. Let α and β be paths from x_0 to x_1. Then the induced map on fundamental groups $\hat{\alpha}$ and $\hat{\beta}$ yield isomorphism by Theorem 52.1 so that the map $\hat{\bar{\beta}} \circ \hat{\alpha}$ is an automorphism of $\pi_1(X, x_0)$. Moreover, we have

$$\begin{aligned}
\hat{\bar{\beta}} \circ \hat{\alpha}([f]) &= \hat{\bar{\beta}}\big(\hat{\alpha}([f])\big) \\
&= \hat{\bar{\beta}}\big([\bar{\alpha}] * [f] * [\alpha]\big) \\
&= [\beta] * \big([\bar{\alpha}] * [f] * [\alpha]\big) * [\bar{\beta}]
\end{aligned}$$

by associativity of the path product, we may rewrite the above expression as

$$= ([\beta] * [\bar{\alpha}]) * [f] * ([\alpha] * [\bar{\beta}])$$

noting that $[\beta] * [\bar{\alpha}]$ and $[\alpha] * [\bar{\beta}]$ are loops based at x_0, since $\pi_1(X, x_0)$ is Abelian, we have

$$\begin{aligned}
&= ([\beta] * [\bar{\alpha}]) * ([\alpha] * [\bar{\beta}]) * [f] \\
&= [e_{x_0}] * [f] \\
&= [f].
\end{aligned}$$

Thus, $\hat{\bar{\beta}} \circ \hat{\alpha} = \mathrm{id}_{\pi_1(X, x_0)}$, i.e., $\hat{\alpha} = \hat{\beta}$.

$\impliedby$ Let f and g be loops about x_0. Then, since X is path connected, we claim that f and g are homotopic to the path product $\alpha_1 * \bar{\beta}_1$ and $\alpha_2 * \bar{\beta}_2$ where α_i, β_i are paths from x_0 to x_1. More precisely, split f into the paths $f_1 = f(t/2)$ and $f_2 = f((t+1)/2)$; it is clear that $f = f_1 * f_2$. Let $x_2 \coloneqq f_1(1)$ then there exists a path α from x_2 to x_1 since X is path connected. Now we claim that the following

$$H(x, t) \coloneqq f_1(x) * \alpha(tx) * \bar{\alpha}((1-t)x) * f_2(x)$$

is a homotopy from $f = f_1 * f_2$ to the extended loop $\tilde{f} = f_1 * \alpha * \bar{\alpha} * f_2$.

Proof of claim. It is clear that H is continuous since it is a path products and multiplication on the unit interval I is continuous so tx is continuous. Lastly, $H(x, 0) = f_1(x) * \alpha(0) * \bar{\alpha}(0) * f_2(x)$ and $H(x, 1) = f_1(x) * \alpha(x) * \bar{\alpha}(x) * f_2(x)$. ♣

Now, let $f \simeq_p \alpha_1 * \bar{\beta}_1$ and $g \simeq_p \alpha_2 * \bar{\beta}_2$ where α_i, β_i are paths from x_0 to x_1. Then we have

$$\begin{aligned}
[f] * [g] * [\bar{f}] * [\bar{g}] &= [\alpha_1 * \bar{\beta}_1] * [\alpha_2 * \bar{\beta}_2] * [\overline{\alpha_1 * \bar{\beta}_1}] * [\overline{\alpha_2 * \bar{\beta}_2}] \\
&= [\alpha_1 * \bar{\beta}_1] * [\alpha_2 * \bar{\beta}_2] * [\beta_1 * \bar{\alpha}_1] * [\beta_2 * \bar{\alpha}_2] \\
&= [\alpha_1] * [\bar{\beta}_1] * [\alpha_2] * [\bar{\beta}_2] * [\beta_1] * [\bar{\alpha}_1] * [\beta_2] * [\bar{\alpha}_2] \\
&= \hat{\bar{\alpha}}_1\big([\bar{\beta}_1] * [\alpha_2] * [\bar{\beta}_2] * [\beta_1]\big) * [\beta_2] * [\alpha_2] \\
&= \hat{\bar{\alpha}}_1\big(\hat{\beta}_2([\alpha_2] * [\bar{\beta}_2])\big) * [\beta_2] * [\bar{\alpha}_2] \\
&= [\alpha_2] * [\bar{\beta}_2] * [\beta_2] * [\bar{\alpha}_2]
\end{aligned}$$

$$\begin{aligned} &= [\alpha_2] * [e_{x_0}] * [\bar{\alpha}_2] \\ &= [\alpha_2] * [\bar{\alpha}_2] \\ &= [e_{x_0}]. \end{aligned}$$

Thus, $\pi_1(X, x_0)$ is Abelian. ■

Problem 10.3 (Munkres §52, Ex. 4)

Let $A \subset X$; suppose $r\colon X \to A$ is continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A.) If $a_0 \in A$, show that

$$r_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

Proof. Suppose f is a loop in A based at a. Then, extending the codomain of f to X, f is a loop in X based at a. Then, since $r(a) = a$ for all a and $f(I) \subset A$, $r_*([f]) = [p(f)] = [f]$ so r_* is surjective. ∎

Problem 10.4 (Munkres §53, Ex. 6)

Show that if X is path connected, the homomorphism induced by a continuous map is independent of the base point, up to isomorphisms of the groups involved. More precisely, let $h\colon X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps "commutes"

$$\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\
{\scriptstyle\hat{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle\hat{\beta}} \\
\pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1).
\end{array}$$

Proof. Unpacking the expression on the left, we have the following sequence of equalities: Let f be a loop in X based at x_0 then

$$\begin{aligned}
\big(\hat{\beta} \circ (h_{x_0})_*\big)([f]) &= \hat{\beta}\big((h_{x_0})_*([f])\big) \\
&= \hat{\beta}\big([h(f)]\big) \\
&= [\bar{\beta}] * [h(f)] * [\beta] \\
&= \big[\, \overline{h \circ \alpha} \,\big] * [h(f)] * [h \circ \alpha] \\
&= [h \circ \bar{\alpha}] * [h(f)] * [h \circ \alpha]
\end{aligned}$$

but since $(h_{x_1})_*$ is a homomorphism

$$\begin{aligned}
&= (h_{x_0})_*(\bar{\alpha} * f * \alpha) \\
&= (h_{x_1})_*\big(\hat{\alpha}([f])\big) \\
&= \big((h_{x_1})_* \circ \hat{\alpha}\big)([f]).
\end{aligned}$$

■

Problem 10.5 (Munkres §55, Ex. 1)

Show that if A is a retract of B^2, then every continuous map $f\colon A \to A$ has a fixed point.

Proof. Suppose that A is a retract of B^2. Let $r\colon B^2 \to A$ one such retraction. If $f\colon A \to A$ is a continuous map, then $f \circ r$ is a continuous map, by Theorem 18.2(c), from B^2 to A. Expanding the codomain of f to B^2, i.e., composing with the canonical injection $\iota\colon A \hookrightarrow B^2$, we have a continuous mapping $\tilde{f}\colon B^2 \to B^2$ that coincides with f in A. Then, by Theorem 55.6, $\tilde{f}$ has a fixed point, i.e., $\tilde{f}(x) = x$ for some $x \in B^2$. By the Brouwer fixed-point theorem for the disc, there exists a point $x \in B^2$ such that $\tilde{f}(x) = x$, but $\operatorname{im} \tilde{f} = \operatorname{im} f \subset A$ so $x \in A$. It follows that f has a fixed point. ∎

Problem 10.6 (Munkres §55, Ex. 2)

Show that if $h\colon S^1 \to S^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof. ∎

Problem 10.7 ((A))

Prove that every m-manifold is locally path-connected.

Proof. Suppose M is an m-manifold. Let $x \in M$ and U' be an arbitrary neighborhood of x. Then, since M is a manifold, there exists an open neighborhood U of x that is homeomorphic to an open subset, say V, of $\mathbf{R}^m$. Let $h\colon U \to V$ be a homeomorphism. Then $h(U \cap U')$ is open in U so by Theorem 16.2, $h(U \cap U')$ is open in $\mathbf{R}^m$. Therefore, for sufficiently small values of $\delta > 0$, we have the inclusion $B(h(x), \delta) \subset h(U \cap U')$. We claim that $W \coloneqq h^{-1}(B(h(x), \delta))$ is a path-connected neighborhood of x contained in U'.

Containment is clear for h is a bijection and we have that $W \subset U \cap U' \subset U'$. By Example 3 in Munkres §24 we know that open balls in $\mathbf{R}^m$ are path-connected therefore given $y_0 = h(x_0), y_1 = h(x_1) \in h(W)$, there exists a path $p\colon I \to h(W)$ with $p(0) = y_0$ and $p(1) = y_1$. Then $q \coloneqq \left((h^{-1})\big|_{h(W)} \circ p\right)\colon I \to W$ is a path in W from x_0 to x_1. It is clear that q is continuous by Theorem 18.2(c) since it is a composition of continuous functions (where $(h^{-1})\big|_{h(W)}$ is continuous by Theorem 18.2(d) since it is the restriction of a continuous function). Lastly, $q(0) = x_0$ and $q(1) = x_1$. Since x_0 and x_1 were arbitrary, it follows that W is path-connected. Therefore, M is locally path-connected. ∎

Problem 10.8 ((B))

Prove that every m-manifold is regular.

Proof. Let $x \in M$ and U' be an arbitrary neighborhood of x. Then, since M is a manifold, there exists an open neighborhood U of x that is homeomorphic to an open subset, say V, of $\mathbf{R}^m$. Let $h\colon U \to V$ be a homeomorphism. Then $h(U \cap U')$ is open in U so by Theorem 16.2, $h(U \cap U')$ is open in $\mathbf{R}^m$. Since $\mathbf{R}^m$ is regular, by Lemma 31.1(a), there exist an neighborhood W of $h(x)$ such that $\overline{W} \subset h(U \cap U')$. We claim that $h^{-1}(W) \subset U'$ is a neighborhood of x such that $\overline{h^{-1}(W)} \subset U'$.

That $h^{-1}(W)$ is contained in U' is clear since h is a homeomorphism and W is contained in the image of $U \cap U'$. It is also easy to see that $h^{-1}(\overline{W}) \subset U'$ since, again h is a homeomorphism and $\overline{W} \subset U \cap U'$. Now, since h is a homeomorphism it is a closed map so $h^{-1}(\overline{W})$ is a closed subset of U containing $h^{-1}(W)$. Therefore, by Lemma B, $\overline{h^{-1}(W)} \subset h^{-1}(\overline{W}) \subset U'$. Thus, by Lemma 31.1, M is regular. ∎

Problem 10.9 ((C))

Prove that there is no 1-1 continuous function $\iota\colon S^1 \to \mathbf{R}$. You may assume any fact about trigonometric functions. (Note: this shows in particular that there is no $\iota\colon S^1 \to \mathbf{R}$ with $p \circ \iota$ equal to the identity map, where p is the map in the note on the Fundamental Group of the Circle.)

Proof. Seeking a contradiction, suppose that $\iota\colon S^1 \to \mathbf{R}$ is a continuous injection. Then ι cannot be a surjection since Theorem 26.6 would imply $S^1 \approx \mathbf{R}$, but S^1 is compact whereas $\mathbf{R}$ is not, contradicting Theorem 26.5. Therefore, by homework Problem 2.8 (Munkres §18, Ex. 4) ι is an imbedding of S^1 into $\mathbf{R}$ and $\operatorname{im}\iota = [a, b]$ for some $a, b \in \mathbf{R}$ with $a < b$, since S^1 is compact and connected. Define $\tilde{\iota} = \iota|_{[a,b]}$ then $\tilde{\iota}$ is a homeomorphism (it is continuous by Theorem 18.2(e), and bijective since $[a, b]$ is the image of S^1 under ι so is a homeomorphism by Theorem 26.6 since S^1 is compact and $[a, b]$ is Hausdorff). Now, take the point $x \coloneqq (a + b)/2$ in the interval $[a, b]$. By Lemma A, $S^1 \smallsetminus \tilde{\iota}(x)$ and $[a, b] \smallsetminus x$ are homeomorphic. But $[a, b] \smallsetminus x$ is disconnected, in particular $[a, x), (x, b]$ are open and closed and $[a, x) \cup (x, b] = [a, b] \setminus x$ hence, form a disconnection, but $S^1 \smallsetminus \tilde{\iota}(x)$ is connected (the map $(x, y, z) \mapsto (x \cos z + y \sin z, -x \sin z + y \cos z)$, i.e., the rotation map, yields a homomorphism). ■

Problem 10.10 ((D))

Prove Proposition C from the note on the Fundamental Group of the Circle.

Proof. ∎