MA557 Homework 6

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PROBLEM 6.1

Let R be a Noetherian ring and I, J R-ideals. Write $I^{\langle J \rangle} = \bigcup_{n \geq 1}(I : J^n)$, which is called the *saturation of I with respect to J*. Show:

(a) If $I = \bigcap_{i=1}^{m} \mathfrak{q}_i$ with $\mathfrak{q}_i$ $\mathfrak{p}_i$-primary, then $I^{\langle J \rangle} = \bigcap_{J \not\subset \mathfrak{p}_i} \mathfrak{q}_i$.

(b) $I^{\langle J \rangle}$ is the unique largest R-ideal that coincides with I locally on the open set $\mathrm{Spec}(R) \smallsetminus V(J)$.

Proof. (a) We shall demonstrate double inclusion: Let $\bigcap_{i=1}^{m} \mathfrak{q}_i$ be a minimal decomposition of I into primary ideals where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. $\implies$ Suppose $x \in I^{\langle J \rangle}$ then $xJ^n \subset I$ for some $n \geq 1$. Given i such that $\mathfrak{p}_i \not\supset J^*$ take $y \in J \smallsetminus \mathfrak{p}_i$. Then $xy^n \in \mathfrak{q}_i$ so $x \in \mathfrak{q}_i$ since $\mathfrak{q}_i$ is primary and $y \notin \mathfrak{p}_i$. Hence, $I^{\langle J \rangle} \subset \bigcap_{J \not\subset \mathfrak{p}_i} \mathfrak{q}_i$. $\impliedby$ Conversely, suppose that $x \in \bigcap_{J \not\subset \mathfrak{p}_i} \mathfrak{q}_i$ then $x \in \mathfrak{q}_i$ for all $\mathfrak{q}_i \not\supset J$. Take any $\mathfrak{p}_j$ containing J. Then $\mathfrak{p}_j = \mathrm{nil}(R/\mathfrak{q}_j)^c$ (this is easily seen from the fact that $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, i.e., $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary and the correspondence theorem for ideals) so there exists n_j with $xJ^{n_j} \subset \mathfrak{q}_j$ (since, in the quotient, $\bar{J}$ is nilpotent). Let n be the maximum of all such n_j then we claim that $xJ^n \subset \mathfrak{q}_i$ for all i. We have already seen what happens in case that $\mathfrak{p}_i \supset J$ so let us consider the case that $\mathfrak{p}_i \not\supset J$. ∎

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*Why does such an ideal exist? Well, suppose that $\mathfrak{p}_i \supset J$ for all $1 \leq i \leq m$. Then

Problem 6.2

Let R be a Noetherian ring. Show that R is reduced if and only if $\mathrm{Quot}(R)$ is a finite direct product of fields.

Proof. $\implies$ Suppose that R is reduced. ∎

Problem 6.3

Let R be a Noetherian ring and $x \in R$ an R-regular element. Show that $\mathrm{Ass}_R(R/(x^n)) = \mathrm{Ass}_R(R/(x))$ for every $n \geq 1$.

Proof.   ∎

PROBLEM 6.4

Let $\varphi \colon R \to T$ be a homomorphism of rings where T is Noetherian, let $^{a}\varphi$ be the induced map on the spectra, and let N be a T-module. Show:

(a) $\mathrm{Ass}_R(N) =^{a} \varphi(\mathrm{Ass}_T(N))$.

(b) If N is finitely generated as a T-module then $\mathrm{Ass}_R(N)$ is finite.

Proof.   ∎

PROBLEM 6.5

Let K be a field that is a finitely generated $\mathbb{Z}$-algebra. Show that K is a finite field.

Proof. ∎

PROBLEM 6.6

Let k be a Noetherian ring, R a finitely generated k-algebra, and $\operatorname{Aut}_k(R)$ the group of k-algebra automorphisms of R. For a subgroup G of $\operatorname{Aut}_k(R)$ write $R^G = \{\, x \in R \mid \sigma(x) = x \text{ for every } \sigma \in G \,\}$, which is called the *ring of invariants* of G. Show that if G is finite then R^G is a finitely generated k-algebra (and hence a Noetherian ring).

Proof.   ∎