Rank 1 Character Varieties-Part III Relations

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MRC - Snowbird

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- Let $\mathfrak{J}$ be the ideal of relations for $\mathbb{C}[\mathfrak{Y}_r]$ and enumerate the minimal generators $t_1, ..., t_{N_r}$. Then $\mathbb{C}[\mathfrak{Y}_r] = \mathbb{C}[t_1, ..., t_{N_r}]/\mathfrak{J}$.

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- Note: $\mathfrak{J}/\mathbb{C}[t_1, ..., t_{N_r}]^+\mathfrak{J}$ is a vector space. A basis is a generatoring set for $\mathfrak{J}$.

Description of Ideal

In general,

$$\mathfrak{X}_r = \mathrm{Spec}_{max}\left(\mathbb{C}[t_1, ..., t_{\frac{r(r^2+5)}{6}}]/\mathfrak{I}_r\right)$$

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Here is the description:

Let $\mathbf{Z}_i = \mathbf{X}_i - \frac{1}{2}\mathrm{tr}(\mathbf{X}_i)\mathbf{I}$ (generic traceless matrix) and let $s_3(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) = \sum_{\sigma\in S_3} \mathrm{sign}(\sigma)\mathbf{A}_{\sigma(1)}\mathbf{A}_{\sigma(2)}\mathbf{A}_{\sigma(3)}$.

- Type 1 relations:

$$\operatorname{tr}(s_3(\mathbf{Z}_{i_1}, \mathbf{Z}_{i_2}, \mathbf{Z}_{i_3}))\operatorname{tr}(s_3(\mathbf{Z}_{j_1}, \mathbf{Z}_{j_2}, \mathbf{Z}_{j_3})) + 18\det(\operatorname{tr}((\mathbf{Z}_{i_{row}}\mathbf{Z}_{j_{column}})) = 0,$$

for $1 \leq i_1 < i_2 < i_3 \leq r$, $1 \leq j_1 < j_2 < j_3 \leq r$.

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- Type 2:

$$\sum_{k=0}^{3}(-1)^k\mathrm{tr}(\mathbf{Z}_i\mathbf{Z}_{p_k})\mathrm{tr}(s_3(\mathbf{Z}_{p_0},...,\hat{\mathbf{Z}}_{p_k},...,\mathbf{Z}_{p_3}))=0,$$

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- Note: this relation shows up only at the rank 4 case.

Sketch of Proof

Again,
$\mathbb{C}[\mathfrak{X}_r] \cong \mathbb{C}[\mathrm{SL}(2,\mathbb{C})^{\times r}/\!\!/\mathrm{SL}(2,\mathbb{C})] \cong \mathbb{C}[\mathfrak{gl}(2,\mathbb{C})^{\times r}/\!\!/\mathrm{SL}(2,\mathbb{C})]/\Delta.$

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However,

$$\begin{aligned}
\mathfrak{gl}(2,\mathbb{C})^{\times r} /\!\!/ \mathrm{SL}(2,\mathbb{C}) &= \mathfrak{gl}(2,\mathbb{C})^{\times r} /\!\!/ \mathrm{PSL}(2,\mathbb{C}) \\
&= \mathfrak{gl}(2,\mathbb{C})^{\times r} /\!\!/ \mathrm{SO}(3,\mathbb{C}) \\
&\cong \mathbb{C}\left(\frac{x_{11}+x_{22}}{2}\right)^{\times r} \bigoplus \mathfrak{so}(3,\mathbb{C})^{\times r} /\!\!/ \mathrm{SO}(3,\mathbb{C}),
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where the coordinates for $\mathfrak{gl}(2,\mathbb{C})$ are $\{x_{11}, x_{21}, x_{12}, x_{22}\}$.

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But Weyl solved the relations problem for orthogonal invariants in 1946 in his book *The Classical Groups, Their Invariants and Representations*.
Rewriting those invariants in terms of traces then gives the result.

Relations in Γ

Also, for a finitely generated Γ, $\mathfrak{X}_\Gamma(G)$ is always cut out of $\mathfrak{X}_r(G)$ by using the relations in Γ; this can be made explicit in the $G = \mathrm{SL}(2, \mathbb{C})$ case without using an (elimination ideal) algorithm.

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Theorem

Let $\Gamma = \langle \gamma_1, ..., \gamma_r \mid R_i,\ \ i \in I\rangle$*, and denote* $\gamma_0 = 1$*. Then* $\mathfrak{X}_\Gamma(\mathrm{SL}(2,\mathbb{C})$ *is given by*

$$\{[\rho] \in \mathfrak{X}_r(\mathrm{SL}(2,\mathbb{C})) \mid \mathrm{tr}(\rho(R_i\gamma_j)) - \mathrm{tr}(\rho(\gamma_j)) = 0, \forall i, j\}.$$

Notice that the relations for the free group case are defined over $\mathbb{Z}[1/2]$, we could clear denominators if we really wanted to get relations over $\mathbb{Z}$. Thus, we can make sense of $\mathfrak{X}_\Gamma$ over any commutative ring with identity; in particular over finite fields.

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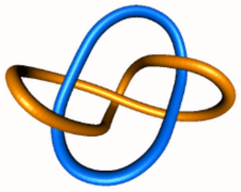
- Experimentally, the solution sets have all orders (coming in pairs), and so we conjecture that all dimension 0 varieties arise this way (up to isomorphism).

Whitehead Link



- Recall, $\mathfrak{X}_2(\mathrm{SL}(2,\mathbb{C}) = \mathbb{C}^3$ and so for all $w \in F_2 = \langle a, b\rangle$, there is a unique $P_w \in \mathbb{C}[x,y,z]$ so

$$P_w(\mathrm{tr}(a), \mathrm{tr}(b), \mathrm{tr}(ab)) = \mathrm{tr}(w).$$

- The fundamental group of the complement in S^3 admits the presentation

$$\Gamma = \left\langle a, b \mid \overbrace{a^{-1}b^{-1}aba^{-1}bab^{-1}aba^{-1}b^{-1}ab^{-1}a^{-1}b}^{w} \right\rangle.$$

- So the character variety $\mathfrak{X}_\Gamma(\mathrm{SL}(2,\mathbb{C}))$ is given by $\{(x,y,z)\in\mathbb{C}^3 \mid P_w(x,y,z)-2=0,\ P_{aw}(x,y,z)-x=0,\ P_{bw}(x,y,z)-y=0, P_{abw}(x,y,z)-z=0\}$.

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- Using a Groebner Basis algorithm, we then get $\{(x,y,z)\in\mathbb{C}^3 \mid x^5y-2x^4y^2z-x^4z+x^3y^3z^2+2x^3y^3+4x^3yz^2-7x^3y-2x^2y^4z-3x^2y^2z^3+5x^2y^2z-2x^2z^3+6x^2z+xy^5+4xy^3z^2-7xy^3+3xyz^4-13xyz^2+12xy-y^4z-2y^2z^3+6y^2z-z^5+6z^3-8z=0\}$.

- So the character variety $\mathfrak{X}_\Gamma(SL(2,\mathbb{C}))$ is given by $\{(x,y,z)\in\mathbb{C}^3 \mid P_w(x,y,z)-2=0,\ P_{aw}(x,y,z)-x=0,\ P_{bw}(x,y,z)-y=0, P_{abw}(x,y,z)-z=0\}$.
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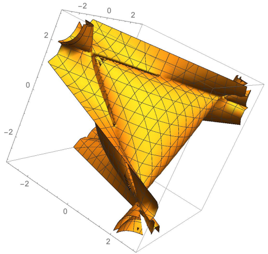

Rank 4 Case

- The fundamental group of the 5-holed sphere is a free group on four letters with the following presentation:

$$\pi = \langle a, b, c, d, e \mid abcde = 1 \rangle \cong \langle a, b, c, d \rangle,$$

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- The character variety is by definition $\mathrm{Hom}(\pi, \mathsf{SL}(2, \mathbb{C}))/\!\!/\mathsf{SL}(2, \mathbb{C}) \cong \mathsf{SL}(2, \mathbb{C})^{\times 4}/\!\!/\mathsf{SL}(2, \mathbb{C})$, given by

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- So this is the moduli space of (polystable) flat $\mathsf{SL}(2,\mathbb{C})$-bundles over the 5-holed sphere.

- The coordinate ring has the following presentation:

$$\mathbb{C}[\mathrm{SL}(2,\mathbb{C})^{\times 4}/\!\!/\mathrm{SL}(2,\mathbb{C})] = \mathbb{C}[r_1,\ldots,r_9][t_1,\ldots,t_5]/(f_1,\ldots,f_{14}),$$

where $\{r_1,\ldots,r_9\}$ is a minimal generating set for the rational function field, $\{r_1,\ldots,r_9,t_1,\ldots,t_5\}$ is a minimal generating set for the coordinate ring, and $\{f_1,\ldots,f_{14}\}$ is a minimal generating set for the ideal of relations in terms of the generators.

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- Consequences are that $\mathrm{SL}(2,\mathbb{C})^{\times 4}/\!\!/\mathrm{SL}(2,\mathbb{C})$ embeds in $\mathbb{C}^{14}$ and its dimension is 9. Since it has 14 relations it is very far from a complete intersection like the rank $1, 2, 3$ cases.

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- More still, at a generic smooth point $[\rho]$, $\{dr_1, \dots, dr_9\}$ generates $T^*_{[\rho]}\left(\mathrm{SL}(2,\mathbb{C})^{\times 4}/\!\!/\mathrm{SL}(2,\mathbb{C})\right) \cong \mathbb{C}^9$.

Here are the formulas for the generators:
$r_1 = \mathrm{tr}(\mathbf{A})$, $r_2 = \mathrm{tr}(\mathbf{B})$, $r_3 = \mathrm{tr}(\mathbf{C})$, $r_4 = \mathrm{tr}(\mathbf{D})$, $r_5 = \mathrm{tr}(\mathbf{AB})$, $r_6 = \mathrm{tr}(\mathbf{AC})$, $r_7 = \mathrm{tr}(\mathbf{AD})$, $r_8 = \mathrm{tr}(\mathbf{BC})$, $r_9 = \mathrm{tr}(\mathbf{BD})$
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Here are the formulas for the ideal of relations (f_1 through f_4 are all of one type, f_5 through f_8 are the rank 3 relation for each set of 3, and f_9 through f_{14} are a generalized relation of the same type as f_5 through f_8).

$$
\begin{aligned}
f_1 = {} & 3\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})^2 - 3\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{A})^2 - \\
& 3\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{A})^2 + 3\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{A})^2 + \\
& 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A}) - 3\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{A}) + \\
& 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A}) + 3\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{A}) - \\
& 3\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{A}) - 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{A}) + \\
& 3\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{A}) - 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{A}) - \\
& 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{ABC}) + 6\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{ABD}) - 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{ACD}) - \\
& 12\mathrm{tr}(\mathbf{BCD}) + 6\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B}) + 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C}) + \\
& 6\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{C}) + 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D}) - 6\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})
\end{aligned}
$$

$$f_2 = -3\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})^2 + 3\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2 + 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{B})^2 - 3\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{B})^2 - 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{B}) + 3\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{B}) - 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B}) - 3\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - 3\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{B}) + 3\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{B}) + 3\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{B}) + 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{B}) - 6\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ABC}) + 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ABD}) + 12\mathrm{tr}(\mathbf{ACD}) + 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BCD}) - 6\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A}) - 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C}) - 6\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{D}) - 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D}) + 6\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})$$

$$
\begin{aligned}
f_3 = {} & 3\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})^2 - 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})^2 - \\
& 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{C})^2 + 3\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{C})^2 + \\
& 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{C}) - 3\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{C}) + \\
& 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{C}) - 3\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) + \\
& 3\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 3\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + \\
& 3\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{C}) - 3\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{C}) - \\
& 6\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABC}) - 12\mathrm{tr}(\mathbf{ABD}) - 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ACD}) + \\
& 6\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BCD}) + 6\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{A}) + 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A}) + \\
& 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B}) + 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{D}) - 6\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})
\end{aligned}
$$

$$f_4 = -3\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})^2 + 3\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})^2 + 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 - 3\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 - 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D}) + 3\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{D}) - 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{D}) - 3\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) + 3\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 3\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 3\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 12\mathrm{tr}(\mathbf{ABC}) + 6\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABD}) - 6\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ACD}) + 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BCD}) - 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A}) - 6\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{B}) - 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B}) - 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C}) + 6\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})$$

$$f_5 = 36\mathrm{tr}(\mathbf{AB})^2 + 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{AB}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AB}) - 36\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AB}) + 36\mathrm{tr}(\mathbf{AC})^2 + 36\mathrm{tr}(\mathbf{BC})^2 + 36\mathrm{tr}(\mathbf{ABC})^2 + 36\mathrm{tr}(\mathbf{A})^2 + 36\mathrm{tr}(\mathbf{B})^2 + 36\mathrm{tr}(\mathbf{C})^2 - 36\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A}) - 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{B}) - 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) - 36\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 36\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 144$$

$$f_6 = 36\mathrm{tr}(\mathbf{AC})^2 + 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{AC}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AC}) - 36\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AC}) + 36\mathrm{tr}(\mathbf{AD})^2 + 36\mathrm{tr}(\mathbf{CD})^2 + 36\mathrm{tr}(\mathbf{ACD})^2 + 36\mathrm{tr}(\mathbf{A})^2 + 36\mathrm{tr}(\mathbf{C})^2 + 36\mathrm{tr}(\mathbf{D})^2 - 36\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A}) - 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{C}) - 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 36\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) - 144$$

$$f_7 = 36\mathrm{tr}(\mathbf{BC})^2 + 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{BC}) - 36\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{BC}) - 36\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BC}) + 36\mathrm{tr}(\mathbf{BD})^2 + 36\mathrm{tr}(\mathbf{CD})^2 + 36\mathrm{tr}(\mathbf{BCD})^2 + 36\mathrm{tr}(\mathbf{B})^2 + 36\mathrm{tr}(\mathbf{C})^2 + 36\mathrm{tr}(\mathbf{D})^2 - 36\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{B}) - 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{C}) - 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 36\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) - 144$$

$$f_8 = 36\mathrm{tr}(\mathbf{AB})^2 + 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{AB}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AB}) - 36\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AB}) + 36\mathrm{tr}(\mathbf{AD})^2 + 36\mathrm{tr}(\mathbf{BD})^2 + 36\mathrm{tr}(\mathbf{ABD})^2 + 36\mathrm{tr}(\mathbf{A})^2 + 36\mathrm{tr}(\mathbf{B})^2 + 36\mathrm{tr}(\mathbf{D})^2 - 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A}) - 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{B}) - 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 36\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 144$$

$$
\begin{aligned}
f_9 = {} & 18\operatorname{tr}(\mathbf{BD})\operatorname{tr}(\mathbf{AC})^2 - 18\operatorname{tr}(\mathbf{AD})\operatorname{tr}(\mathbf{BC})\operatorname{tr}(\mathbf{AC}) - \\
& 18\operatorname{tr}(\mathbf{AB})\operatorname{tr}(\mathbf{CD})\operatorname{tr}(\mathbf{AC}) - 18\operatorname{tr}(\mathbf{ACD})\operatorname{tr}(\mathbf{B})\operatorname{tr}(\mathbf{AC}) + \\
& 18\operatorname{tr}(\mathbf{CD})\operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{B})\operatorname{tr}(\mathbf{AC}) - 18\operatorname{tr}(\mathbf{BD})\operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{C})\operatorname{tr}(\mathbf{AC}) + \\
& 18\operatorname{tr}(\mathbf{AD})\operatorname{tr}(\mathbf{B})\operatorname{tr}(\mathbf{C})\operatorname{tr}(\mathbf{AC}) - 18\operatorname{tr}(\mathbf{ABC})\operatorname{tr}(\mathbf{D})\operatorname{tr}(\mathbf{AC}) + \\
& 18\operatorname{tr}(\mathbf{BC})\operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{D})\operatorname{tr}(\mathbf{AC}) + 18\operatorname{tr}(\mathbf{AB})\operatorname{tr}(\mathbf{C})\operatorname{tr}(\mathbf{D})\operatorname{tr}(\mathbf{AC}) - \\
& 18\operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{B})\operatorname{tr}(\mathbf{C})\operatorname{tr}(\mathbf{D})\operatorname{tr}(\mathbf{AC}) + 18\operatorname{tr}(\mathbf{BD})\operatorname{tr}(\mathbf{A})^2 + \\
& 18\operatorname{tr}(\mathbf{BC})\operatorname{tr}(\mathbf{CD})\operatorname{tr}(\mathbf{A})^2 + 18\operatorname{tr}(\mathbf{AB})\operatorname{tr}(\mathbf{AD})\operatorname{tr}(\mathbf{C})^2 + \\
& 18\operatorname{tr}(\mathbf{BD})\operatorname{tr}(\mathbf{C})^2 - 18\operatorname{tr}(\mathbf{AD})\operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{B})\operatorname{tr}(\mathbf{C})^2 - 36\operatorname{tr}(\mathbf{AB})\operatorname{tr}(\mathbf{AD}) - \\
& 72\operatorname{tr}(\mathbf{BD}) - 36\operatorname{tr}(\mathbf{BC})\operatorname{tr}(\mathbf{CD}) + 36\operatorname{tr}(\mathbf{ABC})\operatorname{tr}(\mathbf{ACD}) - \\
& 18\operatorname{tr}(\mathbf{CD})\operatorname{tr}(\mathbf{ABC})\operatorname{tr}(\mathbf{A}) - 18\operatorname{tr}(\mathbf{BC})\operatorname{tr}(\mathbf{ACD})\operatorname{tr}(\mathbf{A}) + \\
& 18\operatorname{tr}(\mathbf{AD})\operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{B}) - 18\operatorname{tr}(\mathbf{AD})\operatorname{tr}(\mathbf{ABC})\operatorname{tr}(\mathbf{C}) \cdots
\end{aligned}
$$

$$\cdots - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})^2\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})^2\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})^2\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 36\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})$$

$$f_{10} = -18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{AB})^2 + 18\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AB})^2 + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{AB}) + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{AB}) + 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AB}) - 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AB}) - 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AB}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AB}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})^2 - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})^2 + 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{AD}) + 36\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{BD}) + 72\mathrm{tr}(\mathbf{CD}) + 36\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{ABD}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A}) - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})$$

$$f_{11} = -18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})^2 + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BC})^2 + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{BC}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{BC}) - 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{BC}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{BC}) - 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BC}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BC}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})^2 - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C})^2 + 72\mathrm{tr}(\mathbf{AD}) + 36\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BD}) + 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{CD}) + 36\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{BCD}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})^2\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})$$

$$
\begin{aligned}
f_{12} = {} & -18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{AD})^2 + 18\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AD})^2 + \\
& 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{AD}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{AD}) - \\
& 18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AD}) - 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AD}) + \\
& 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AD}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AD}) - \\
& 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})^2 - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D})^2 + 18\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 + \\
& 36\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{AC}) + 72\mathrm{tr}(\mathbf{BC}) + 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{CD}) + \\
& 36\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{ACD}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A}) - \\
& 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - \\
& 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 36\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - \\
& 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{D}) - \\
& 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - \\
& 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})
\end{aligned}
$$

$$f_{13} = 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})^2 - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{BD}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{BD}) - 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{BD}) + 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{BD}) - 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{BD}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{BD}) + 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BD}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BD}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BD}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BD}) + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{B})^2 + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})^2 + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{D})^2 + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D})^2 - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})^2 + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 - 72\mathrm{tr}(\mathbf{AC}) - 36\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BC}) - 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{CD}) + 36\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{BCD}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{B}) + 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{C}) + 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{D}) \cdots$$

$$\cdots - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})$$

$$f_{14} = -18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})^2 + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{CD})^2 + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{CD}) + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{CD}) - 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{CD}) - 18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{CD}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{CD}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{CD}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})^2 + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})^2 - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{D})^2 + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})^2 + 72\mathrm{tr}(\mathbf{AB}) + 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BC}) + 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BD}) + 36\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{BCD}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})$$

The above remarks can be generalized to any rank free group (any n-holed sphere). I used a *Mathematica* notebook to perform routine computations, but at no point was an (elimination ideal) algorithm used to generate relations.

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- From results of Florentino-Lawton (2012), we know that the singular locus is exactly the reducible representations (and so corresponds to the free Abelian character variety).

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- So we not only know the local structure but also know the Betti numbers of as well.

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- All of these statements generalize to arbitrary free groups explicitly.

Exercises

1. For simple classes of Γ, work out $\mathfrak{X}_\Gamma(\mathrm{SL}(2,\mathbb{C}))$ using the above algorithm.
2. Once general formulas are known, determine the counting polynomials for interesting finite rings R and Γ's (example: torus knots and links and finite fields).
3. Fix a curve C in $\mathbb{C}^2$. Is there a Γ that makes $\mathfrak{X}_\Gamma(\mathrm{SL}(2,\mathbb{C})) \cong C$? Try two generator groups.
4. For the 1-holed torus and the 4-holed sphere, what are the counting polynomials for the relative character varieties over interesting finite rings R?
5. $\mathrm{Out}(F_r)$ acts on the $\mathbb{F}_q$-points of $\mathfrak{X}_r$. For a fixed $\alpha \in \mathrm{Out}(F_r)$ what is the growth of the length of the maximal orbit? Can the growth rates be used to determine if α is pseudo-anosov?