Conrad's Differential Geometry Notes

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1 Interior, closure and boundary

We wish to develop some basic geometric concepts in metric spaces which make precise intuitive ideas centered on the themes of "interior" and "boundary" of a subset of a metric space. One warning must be given. Although there are a number of results proven in this handout, none of it is particularly deep. If you carefully study the proofs, then you'll see that none of this requires going much beyond the basic definitions. We will certainly encounter some serious ideas and nontrivial proofs in due course, but at this point the central aim is to acquire some linguistic ability when discussing some basic geometric ideas in a metric space. Thus, the main goal is to familiarize ourselves with some very convenient geometric terminology in terms of which we can discuss more sophisticated ideas later on.

1.1 Interior and closure

Let X be a metric space and $A \subseteq X$ a subset. We define the *interior* of A to be the set

$$A^\circ = \operatorname{Int} A = \left\{ a \in A : \text{there exists } r > 0 \text{ such that } B(a,) \subseteq A \right\}$$

consisting of points for which A is a "neighborhood". We define the *closure* of A to be the set

$$\bar{A} = \operatorname{Cls} A = \left\{ x \in X : x = \lim_{n\to\infty} a_n \text{ with } a_n \in A \text{ for all } n \in \mathbb{N} \right\}.$$

In words, the interior consists of points in A for which all nearby points of X are also in A whereas the closure allows for "points on the edge of A". Note that obviously

$$A^\circ \subseteq \bar{A}.$$

We will see shortly (after some examples) that A° is the largest open set inside of A — that is, it is open and contains any open lying inside of A (so in fact A is open if and only if $A = A^\circ$) — while $\bar{A}$ is the smallest closed set containing A; i.e., $\bar{A}$ is closed and lies inside any closed set containing A (so in fact A is closed if and only if $\bar{A} = A$).

Beware that we have to prove that the closure is actually closed! Just because we call something the "closure" does not mean the concept is automatically endowed with linguistically similarly sounding properties. The proof won't be particularly deep, as we'll see.

Example 1. Let's work out the interior and closure of the "half-open" square

$$A = \left\{ (x, y) \in \mathbb{R}^2 : -1 \le x \le 1, -1 < y < 1 \right\} = [-1, 1] \times (-1, 1)$$

inside the metric space $X = \mathbb{R}^2$ (the phrase "half-open" is purely intuitive; it has no precise meaning, but the picture should make it clear why we use this terminology). Intuitively, this is a square region whose horizontal edges are "left out". The interior of A should be $(-1, 1) \times (-1, 1)$ and the closure should be $[-1, 1] \times [-1, 1]$, as drawing a picture should convince you. Of course, we want to see that such conclusions really do follow from our precise definitions.

First we check that A° is correctly described. If $-1 < x < 1$ and $-1 < y < 1$ then for

$$r = \min\big\{|-1-x|, |1-x|, |-1-y|, |1-y|\big\} > 0$$

it is easy to check that $B\big((x,y),r\big) \subseteq (-1,1)\times(-1,1)$ (since a square box with side-length r contains the disc of radius r with the same center). Thus, $(-1,1)\times(-1,1) \subseteq A$ is an open subset of $X = \mathbb{R}^2$. To check it is the full interior of A, we just have to show that the "missing points" of the form $\pm 1, y$ do not lie in the interior. But for any such point $p = (\pm 1, y) \in A$, for any positive small $r > 0$ there is always a point in $B(p,r)$ with the same y-coordinate but with the x-coordinate either slightly larger than 1 or slightly less than -1. Such a point is not in A. Thus, $p \notin A^\circ$.

Now we check that $\bar{A} = [-1,1]\times[-1,1]$. Since convergence in $\mathbb{R}^2$ forces convergence in coordinates, to see

$$\bar{A} \subseteq [-1,1]\times[-1,1]$$

it suffices to check that $[-1,1]$ is closed in $\mathbb{R}$ (since certainly $A \subseteq [-1,1]\times[-1,1]$). But this is clear (either by using sequences or by explicitly showing its complement in $\mathbb{R}$ to be open). To see that $\bar{A}$ fills up all of $[-1,1]\times[-1,1]$, we have to show that each point in $[-1,1]\times[-1,1]$ can be obtained as a limit of a sequence in A. We just have to deal with points not in $A = [-1,1]\times(-1,1)$ since points in A are limits of constant sequences. That is, we're faced with studying points of the form $(x, \pm 1)$ with $x \in [-1,1]$. Such a point is a limit of a sequence (x, q_n) with $q_n \in (-1,1)$ having limit ± 1.

Example 2. What happens if we work with the same set A but view it inside of a metric space $X = A$ (with the Euclidean metric)? In this case, $A^\circ = A$ and $\bar{A} = A$! Indeed, quite generally for any metric space X we have $X^\circ = X$ and $\bar{X} = X$. These are easy consequences of thee definitions. Likewise, the empty subset $\emptyset$ in any metric space has interior and closuer equal to the subset $\emptyset$.

The moral is that one has to always.