MA 523: Homework 3

Carlos Salinas

September 19, 2016

Problem 3.1

Consider the initial value problem

$$u_t = \sin u_x; \qquad u(x,0) = \frac{\pi}{4}x.$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

Solution. First we zero the boundary data, that is, set $v := u - (\pi/4)x$, then

$$v(x,0) = u(x,0) = \frac{\pi}{4}x - \frac{\pi}{4}x = 0$$

and

$$\begin{aligned}
0 &= u_t - \sin u_x \\
&= \Big(v + \frac{\pi}{4}x\Big)_t - \sin\Big(v + \frac{\pi}{4}x\Big)_x \\
&= v_t - \sin(v_x + \pi/4).
\end{aligned}$$

■

Problem 3.2

Consider the Cauchy problem for $u(x, y)$

$$\begin{aligned} u_y &= a(x, y, u)u_x + b(x, y, u) \\ u(x, 0) &= 0 \end{aligned}$$

Let a and b be analytic functions of their arguments. Assume that $D^\alpha a(0, 0, 0) \geq 0$ and $D^\alpha b(0, 0, 0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

(a) Show that $D^\beta u(0, 0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $D^\beta u(0, 0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$. (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in β_2)

Solution. ■

Problem 3.3

(Kovalevskaya's example) Show that the line $\{t = 0\}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution of the heat equation in $\mathbf{R} \times \mathbf{R}$, with $u = 1/(1 + x^2)$ on $\{t = 0\}$. (*Hint:* Assume there is an analytic solution, compute its coefficients, and show that the resulting power series diverges in any neighborhood of $(0, 0)$.)

Solution. ■