Math 527 - Homotopy Theory
Spring 2013
Homework 4 Solutions

In Problems 1 and 2, let **Top** denote the usual category of all topological spaces and continuous maps between them.

Problem 1. Let $U\colon \mathbf{Top} \to \mathbf{Set}$ denote the underlying set functor.

a. Show that U has a left adjoint, and describe it explicitly.

Solution. Let $D\colon \mathbf{Set} \to \mathbf{Top}$ denote the discrete space functor. We claim that D is left adjoint to U.

Let S be a set and Y a space. Since DS is a discrete space, every function $f\colon DS \to Y$ is continuous. This provides a bijection

$$\mathrm{Hom}_{\mathbf{Top}}(DS, Y) \cong \mathrm{Hom}_{\mathbf{Set}}(S, UY) \tag{1}$$

which is natural in S and Y, as one readily checks. Alternately, consider the natural transformation in **Top** given by the identity function $\epsilon\colon X_{\mathrm{dis}} \to X$, where $X_{\mathrm{dis}} = DUX$ denotes the set X equipped with the discrete topology. Consider the identity natural transformation $\eta\colon S \xrightarrow{=} S = UDS$ in **Set**. Note that the bijection (1) is induced by ϵ and η, which are thus respectively the counit and unit of the adjunction $D \dashv U$. $\square$

b. Show that U has a right adjoint, and describe it explicitly.

Solution. Let $T\colon \mathbf{Set} \to \mathbf{Top}$ denote the trivial space functor, where the trivial topology on X is $\{\emptyset, X\}$. We claim that T is right adjoint to U.

Let X be a space and S a set. Since TS is a trivial space, every function $f\colon X \to TS$ is continuous. This provides a bijection

$$\mathrm{Hom}_{\mathbf{Top}}(X, TS) \cong \mathrm{Hom}_{\mathbf{Set}}(UX, S) \tag{2}$$

which is natural in X and S, as one readily checks. Alternately, consider the natural transformation in **Top** given by the identity function $\eta\colon X \to X_{\mathrm{triv}}$, where $X_{\mathrm{triv}} = TUX$ denotes the set X equipped with the trivial topology. Consider the identity natural transformation $\epsilon\colon UTS = S \xrightarrow{=} S$ in **Set**. Note that the bijection (2) is induced by η and ϵ, which are thus respectively the unit and counit of the adjunction $U \dashv T$. $\square$

Problem 2. Show that the category **Top** is complete (i.e. has all small limits).

Solution. Let I be a small category and $F\colon I \to \mathbf{Top}$ an I-diagram. Write $X_i := F(i)$ and by abuse of notation $\lim_i X_i := \lim_I F$ (if it exists).

Consider the diagram UF of underlying sets UX_i. Since **Set** is complete, this diagram admits a limit $S = \lim_i UX_i$. We will equip S with a topology (the "limit topology") making it into the limit of F in **Top**.

Denote by $p_i\colon S \to UX_i$ the "projection" maps in the limiting cone of S. Let $\mathcal{T}$ be the topology on S generated by all subsets of the form $p_i^{-1}(O_i)$ for $O_i \subseteq X_i$ open and i any object of I. Write $X := (S, \mathcal{T})$ for the resulting space.

By construction, all projection maps $p_i\colon X \to X_i$ are continuous. Moreover, a function $f\colon W \to X$ is continuous if and only if all the projections $p_i \circ f\colon W \to X_i$ are continuous.

Now let W be a space with a cone over F, i.e. compatible continuous maps $f_i\colon W \to X_i$. By the universal property of S in **Set**, there is a unique function $f\colon UW \to S$ satisfying $p_i \circ f = f_i$ for all i. Because each f_i is continuous, so is f (with respect to the topology $\mathcal{T}$). Therefore there is a unique continuous map $f\colon W \to X$ satisfying $p_i \circ f = f_i$ for all i. This proves $X = \lim_i X_i$ in **Top**. $\square$

Problem 2'. (Not on the homework) Show that the category **Top** is cocomplete (i.e. has all small colimits).

Solution. Let I be a small category and $F\colon I \to \mathbf{Top}$ an I-diagram. Write $X_i := F(i)$ and by abuse of notation $\operatorname{colim}_i X_i := \operatorname{colim}_I F$ (if it exists).

Consider the diagram UF of underlying sets UX_i. Since **Set** is cocomplete, this diagram admits a colimit $S = \operatorname{colim}_i UX_i$. We will equip S with a topology (the "colimit topology") making it into the colimit of F in **Top**.

Denote by $\iota_i\colon UX_i \to S$ the "inclusion" maps in the colimiting cocone of S. Let $\mathcal{T}$ be the collection of subsets of S

$$\mathcal{T} = \{O \subseteq S \mid \iota_i^{-1}(O_i) \text{ is open in } X_i \text{ for all } i \in \mathrm{Ob}(I)\}$$

which is already a topology on S. Write $X := (S, \mathcal{T})$ for the resulting space.

By construction, all inclusion maps $\iota_i\colon X_i \to X$ are continuous. Moreover, a function $f\colon X \to Y$ is continuous if and only if all the restrictions $f \circ \iota_i\colon X_i \to Y$ are continuous.

Now let Y be a space with a cocone under F, i.e. compatible continuous maps $f_i\colon X_i \to Y$. By the universal property of S in **Set**, there is a unique function $f\colon S \to UY$ satisfying $f \circ \iota_i = f_i$ for all i. Because each f_i is continuous, so is f (with respect to the topology $\mathcal{T}$). Therefore there is a unique continuous map $f\colon X \to Y$ satisfying $f \circ \iota_i = f_i$ for all i. This proves $X = \operatorname{colim}_i X_i$ in **Top**. $\square$

Problem 3. Let $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $X = I = [0,1]$. Show that the inclusion $i\colon A \hookrightarrow X$ is *not* a cofibration.

Solution. Consider the mapping cylinder $M(i) \cong X \times \{0\} \cup A \times I$ and let $r\colon X \times I \to X \times \{0\} \cup A \times I$ be a continuous map. We want to show that r cannot be a retraction.

Assume $r(0,1) = (0,1)$. Pick a small neighborhood of $(0,1)$ in $X \times \{0\} \cup A \times I$, say of the form $V = N \times (0.9, 1]$ where $N = (-0.1, 0.1) \cap A$ is a small neighborhood of 0 in A. Note that the path components of V are the "vertical segments" $\{a\} \times (0.9, 1]$ for $a \in N$.

Since r is continuous, there is a neighborhood U of $(0,1)$ in $X \times I$ satisfying $r(U) \subseteq V$. Shrinking U if necessary, we may assume U is path-connected, since $X \times I$ is locally path-connected. Therefore $r(U)$ lives in one path component of V, which must be $\{0\} \times (0.9, 1]$ since $r(0,1) = (0,1)$ is in that component. But any neighborhood of $(0,1)$ in $X \times I$ contains points of the form $(a,1)$ for some $a \in A$ with $a \neq 0$. For such points, we have

$$r(a,1) \in r(U) \subseteq \{0\} \times (0.9, 1]$$

which implies $r(a,1) = (0,t) \neq (a,1)$. Therefore r is not a retraction. $\square$

Problem 4. Let (X, x) be a well-pointed space. Show that the quotient map $SX \twoheadrightarrow \Sigma X$ from the unreduced suspension of X to the reduced suspension of X is a homotopy equivalence.

Solution. Consider the two successive quotient maps

$$X \times I \overset{q_1}{\twoheadrightarrow} SX \overset{q_2}{\twoheadrightarrow} \Sigma X.$$

Note that the composite

$$I \cong \{x\} \times I \subseteq X \times I \xrightarrow{q_1} SX$$

is an embedding, so that the subspace $q_1(\{x\} \times I) \subseteq SX$ being quotiented by q_2 is contractible. Therefore it suffices to show that the inclusion $q_1(\{x\} \times I) \subseteq SX$ is a cofibration, by Hatcher Proposition 0.17.

By applying Proposition 1 twice – once to the bottom part $X \times \{0\}$ and once to the top part $X \times \{1\}$ – it suffices to show that the inclusion

$$X \times \{0, 1\} \cup \{x\} \times I \subseteq X \times I$$

is a cofibration. This is a special case of Proposition 2. □

Proposition 1. *Let $B \subseteq A \subseteq X$ be inclusions of subspaces, and assume the inclusion $i \colon A \subseteq X$ is a cofibration. Then the induced map $i' \colon A/B \to X/B$ is a cofibration.*

Proof. Consider the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, hook] & * \arrow[d] \\ A \arrow[r] \arrow[d, hook, "i"'] & A/B \arrow[d, "i'"] \\ X \arrow[r] & X/B \end{tikzcd}$$

where both squares are pushouts. Since $i \colon A \to X$ is a cofibration, so is $i' \colon A/B \to X/B$. □

Proposition 2. *Let $i\colon A \hookrightarrow X$ be a (closed) cofibration. Then the map*

$$X \times \{0,1\} \cup A \times I \hookrightarrow X \times I$$

is a cofibration.

Proof. WLOG i is an inclusion. Since $i\colon A \to X$ is a cofibration, so is the map $i \times \mathrm{id}\colon A \times I \to X \times I$. Therefore the inclusion

$$X \times I \times \{0\} \cup A \times I \times I \subseteq X \times I \times I \tag{3}$$

admits a retraction.

We want to show that the inclusion

$$X \times I \times \{0\} \cup (X \times \{0,1\} \cup A \times I) \times I \subseteq X \times I \times I \tag{4}$$

admits a retraction. The subspace in question can be written as

$$\begin{aligned} &X \times I \times \{0\} \cup (X \times \{0,1\} \cup A \times I) \times I \\ &= X \times I \times \{0\} \cup X \times \{0,1\} \times I \cup A \times I \times I \\ &= X \times (I \times \{0\} \cup \{0,1\} \times I) \cup A \times I \times I. \end{aligned}$$

There is a homeomorphism of pairs

$$\varphi\colon\ (I \times I, I \times \{0\}) \cong (I \times I, I \times \{0\} \cup \{0,1\} \times I)\,.$$

Taking the product with X, we obtain a homeomorphism of pairs

$$\mathrm{id}_X \times \varphi\colon\ (X \times I \times I, X \times I \times \{0\}) \cong (X \times I \times I, X \times (I \times \{0\} \cup \{0,1\} \times I))\,.$$

Via this homeomorphism of pairs, a retraction of (3) provides a retraction of (4). □