MA557 Problem Set 3

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Problem 3.1

Find an example of a finitely generated ring extension $R \subset S$ where S is a Noetherian ring, but R is not.

Proof. ■

Problem 3.2

Consider the homomorphism of rings

$$\begin{tikzcd} & S \arrow[d, "\psi"] \\ R \arrow[r, "\varphi"] & T. \end{tikzcd}$$

The *fiber product* of R and S over T is the subring $R \times_T S = \{ (r, s) \mid \varphi(t) = \psi(s) \}$ of $R \times S$. Assume φ and ψ are surjective. Show that if R and S are Noetherian rings then so is $R \times_T S$.

Proof. ∎

Problem 3.3

Let M be an R-module. Show that M is a flat R-module if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.

Proof. ■

Problem 3.4

Let M be an R-module and $\mathfrak{a}$ an R-ideal.

(a) Show that if $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, then $M = IM$.
(b) Show that the converse holds in case M is finite.

Proof. ■

Problem 3.5

Prove that every power of a maximal ideal is primary.

Proof. ■

Problem 3.6

(a) Show that the radical of a primary ideal is prime.
(b) Find an example of a power of a prime ideal that is not primary.
(c) Let $\mathfrak{p}$ be a prime ideal of a ring R and $n \in \mathbf{N}$. The R-ideal $\mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$ s called the *nth symbolic power of* $\mathfrak{p}$. Show that $\mathfrak{p}^{(n)}$ is primary.

Proof. ∎