Conrad's Differential Geometry Notes

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1 Interior, closure and boundary

We wish to develop some basic geometric concepts in metric spaces which make precise intuitive ideas centered on the themes of "interior" and "boundary" of a subset of a metric space. One warning must be given. Although there are a number of results proven in this handout, none of it is particularly deep. If you carefully study the proofs, then you'll see that none of this requires going much beyond the basic definitions. We will certainly encounter some serious ideas and nontrivial proofs in due course, but at this point the central aim is to acquire some linguistic ability when discussing some basic geometric ideas in a metric space. Thus, the main goal is to familiarize ourselves with some very convenient geometric terminology in terms of which we can discuss more sophisticated ideas later on.

1.1 Interior and closure

Let X be a metric space and $A \subseteq X$ a subset. We define the *interior* of A to be the set

$$A^\circ = \operatorname{Int} A = \left\{ a \in A : \text{there exists } r > 0 \text{ such that } B(a,) \subseteq A \right\}$$

consisting of points for which A is a "neighborhood". We define the *closure* of A to be the set

$$\bar{A} = \operatorname{Cls} A = \left\{ x \in X : x = \lim_{n\to\infty} a_n \text{ with } a_n \in A \text{ for all } n \in \mathbb{N} \right\}.$$

In words, the interior consists of points in A for which all nearby points of X are also in A whereas the closure allows for "points on the edge of A". Note that obviously

$$A^\circ \subseteq \bar{A}.$$

We will see shortly (after some examples) that A° is the largest open set inside of A — that is, it is open and contains any open lying inside of A (so in fact A is open if and only if $A = A^\circ$) — while $\bar{A}$ is the smallest closed set containing A; i.e., $\bar{A}$ is closed and lies inside any closed set containing A (so in fact A is closed if and only if $\bar{A} = A$).

Beware that we have to prove that the closure is actually closed! Just because we call something the "closure" does not mean the concept is automatically endowed with linguistically similarly sounding properties. The proof won't be particularly deep, as we'll see.

Examples 1. Let's work out the interior and closure of the "half-open" square

$$A = \left\{ (x, y) \in \mathbb{R}^2 : -1 \le x \le 1, -1 < y < 1 \right\} = [-1, 1] \times (-1, 1)$$

inside the metric space $X = \mathbb{R}^2$ (the phrase "half-open" is purely intuitive; it has no precise meaning, but the picture should make it clear why we use this terminology). Intuitively, this is a square region whose horizontal edges are "left out". The interior of A should be $(-1, 1) \times (-1, 1)$ and the closure should be $[-1, 1] \times [-1, 1]$, as drawing a picture should convince you. Of course, we want to see that such conclusions really do follow from our precise definitions.

First we check that A° is correctly described. If $-1 < x < 1$ and $-1 < y < 1$ then for

$$r = \min\{|-1-x|, |1-x|, |-1-y|, |1-y|\} > 0$$

it is easy to check that $B((x,y), r) \subseteq (-1,1)\times(-1,1)$ (since a square box with side-length r contains the disc of radius r with the same center). Thus, $(-1,1)\times(-1,1) \subseteq A$ is an open subset of $X = \mathbb{R}^2$. To check it is the full interior of A, we just have to show that the "missing points" of the form $\pm 1, y$ do not lie in the interior. But for any such point $p = (\pm 1, y) \in A$, for any positive small $r > 0$ there is always a point in $B(p, r)$ with the same y-coordinate but with the x-coordinate either slightly larger than 1 or slightly less than -1. Such a point is not in A. Thus, $p \notin A^\circ$.

Now we check that $\bar{A} = [-1,1]\times[-1,1]$. Since convergence in $\mathbb{R}^2$ forces convergence in coordinates, to see

$$\bar{A} \subseteq [-1,1]\times[-1,1]$$

it suffices to check that $[-1,1]$ is closed in $\mathbb{R}$ (since certainly $A \subseteq [-1,1]\times[-1,1]$). But this is clear (either by using sequences or by explicitly showing its complement in $\mathbb{R}$ to be open). To see that $\bar{A}$ fills up all of $[-1,1]\times[-1,1]$, we have to show that each point in $[-1,1]\times[-1,1]$ can be obtained as a limit of a sequence in A. We just have to deal with points not in $A = [-1,1]\times(-1,1)$ since points in A are limits of constant sequences. That is, we're faced with studying points of the form $(x, \pm 1)$ with $x \in [-1,1]$. Such a point is a limit of a sequence (x, q_n) with $q_n \in (-1,1)$ having limit ± 1.

Examples 2. What happens if we work with the same set A but view it inside of a metric space $X = A$ (with the Euclidean metric)? In this case, $A^\circ = A$ and $\bar{A} = A$! Indeed, quite generally for any metric space X we have $X^\circ = X$ and $\bar{X} = X$. These are easy consequences of thee definitions. Likewise, the empty subset $\emptyset$ in any metric space has interior and closure equal to the subset $\emptyset$.

The moral is that one has to always keep in mind what ambient metric space one is working in when forming interiors and closures! One could imagine that perhaps our notation for interior and closure should somehow incorporate a designation of the ambient metric space. But just as we freely use the same symbols "+" and "0" to denote the addition and additive identity in any vector space, even when working with several spaces at once, it would simply make life too cumbersome (and the notation too cluttered) to always write things like $\mathrm{Int}_X A$ or $\mathrm{Cls}_X A$. One just has to pay careful attention to what is going on so as to keep track of the ambient metric space with respect to which one is forming interiors and closures. The context will usually make it obvious what one is using as the ambient metric space, though if considering several ambient spaces at once it is sometimes helpful to use more precise notation such as $\mathrm{Int}_X A$.

Theorem 1.1. *Let A be a subset of a metric space X. Then A° is open and is the largest open set of X inside of A (i.e., it contains all others).*

Proof. We first show that A° is open. By definition, if $x \in A^\circ$, then some $B(x,r) \subseteq A$. But then since $B(x,r)$ is itsefl an open set we see that any $y \in B(x,r)$ has some $B(y,s) \subseteq B(x,r) \subseteq A$, which forces $y \in A^\circ$. That is, we have shown $B(x,r) \subseteq A^\circ$, whence A° is open.

If $U \subseteq A$ is an open set in X, then for each $u \in U$ there is some $r > 0$ such that $B(u,r) \subseteq U$ whence $B(u,r) \subseteq A$, so $u \in A^\circ$. This is true for all $u \in U$, so $U \subseteq A^\circ$. ■

Corollary 1.2. *A subset A in a metric space X is open if and only if $A = A^\circ$.*

Proof. By the theorem, A° is the unique largest open subset of X contained in A. But obviously A is open if anf only if such a unique maximal open subset of X lying in A is actually equal to A. This establishes the corollary. ■

We next want to show that the closure of a subset A in X is related to closed subsets of X containing A in a manner very similar to the way in which the interior of A is related to open subsets of X which lie inside of A. This goes along with the general idea that openness and closedness are "complementary" points to view (recall that a subset S in a metric space X is open (resp. closed) if and only if its complement $X \smallsetminus S$ is closed (resp. open)). It is actually more convenient for us to first show that closures and interiors have complementary relationship, and to then use this to deduce our desired properties of closure from already-established properties of interior.

Theorem 1.3. *Let A be a subset of a metric space X. Then $X \smallsetminus \bar{A} = (X \smallsetminus A)^\circ$ and $X \smallsetminus A^\circ = \overline{X \smallsetminus A}$.*

Before proving this theorem, we illustrate with an example. Consider $X = \mathbb{R}^2$ with the usual metric, and let $A = [-1, 1] \times (-1, 1)$ be the "half-open" square as considered above. By drawing pictures of $X \smallsetminus A$ and of the complements of $\bar{A}$ and A°, you should convince yourself intuitively that the assertions in this theorem make sense in this case.

Now we prove Theorem 1.3.

Proof. We begin by proving $X \smallsetminus \bar{A} = (X \smallsetminus A)^\circ$. If $x \in X$ is not in $\bar{A}$, there must exist some $B(x, 1/2^n)$ not meeting A, for otherwise we'd have some $x_n \in B(x, 1/2^n) \cap A$ for all $n \in \mathbb{N}$, so clearly $x_n \to x$, contrary to the fact that $x \notin \bar{A}$ is not a limit of a sequence of elements of A. This shows

$$X \smallsetminus \bar{A} \subseteq (X \smallsetminus A)^\circ.$$

Conversely, if x is in the interior of $X \smallsetminus A$ then some $B(x, r)$ lies in $X \smallsetminus A$ and hence is disjoint from A. It follows that no sequnece in A can possibly converge to x because for $\varepsilon = r > 0$ we'd run into problems (i.e., there's nothing in A within a distance of less that ε from x, since $B(x, \varepsilon) \subseteq X \smallsetminus A$).

Applying the *general* equality

$$X \smallsetminus \bar{A} = (X \smallsetminus A)^\circ$$

for arbitrary subsets A to X to the subset $X \smallsetminus A$ in the role of A, we get

$$X \smallsetminus \overline{X \smallsetminus A} = A^\circ.$$

Taking complements of both sides within X yields

$$\overline{X \smallsetminus A} = X \smallsetminus A^\circ,$$

as desired. ■

Corollary 1.4. *Let A be a subset of a metric space X. Then $\bar{A}$ is closed and is contained inside of any closed subset of X which contains A.*

Proof. Since the complement of $\bar{A}$ is equal to $(X \smallsetminus A)^\circ$, which we know to be open, it follows that $\bar{A}$ is closed. If Z is any closed set containing A, we want to prove that Z contains $\bar{A}$ (so $\bar{A}$ is "minimal" among closed sets containing A). But this is clear for several reasons. On the one hand, by definition every point $x \in \bar{A}$ is the limit of a sequnece of elements in $A \subseteq Z$, so by closedness of Z such limit points x are also in Z. This shows $\bar{A} \subseteq Z$. On the other hand, one can argue by noting that passage to complement takes Z to an open set $X \smallsetminus Z$ contained inside of $X \smallsetminus A$, so by maximality this open $X \smallsetminus Z$ must lie inside the interior of $X \smallsetminus A$, which we have seen is the complement $X \smallsetminus \bar{A}$ of $\bar{A}$. Passage back to complements then gives

$$\bar{A} = X \smallsetminus (X \smallsetminus \bar{A}) = X \smallsetminus (X \smallsetminus A)^\circ \subseteq X \smallsetminus (X \smallsetminus Z) \subseteq Z$$

as desired. ■

Corollary 1.5. *For subsets $A_1, \dots, A_n$ in a metric space X, the closure of $A_1 \cup \cdots \cup A_n$ is equal to $\bigcup_{i=1}^n \bar{A}_i$; that is, the formation of a finite union commutes with the formation of a closure.*

Proof. A closed set Z contains $\bigcup_{i=1}^n A_i$ if and only if it contains each A_i, and so if and only if it contains $\bar{A}_i$ for every i. Since $\bigcup_{i=1}^n \bar{A}_i$ is a finite union of closed sets, it is closed. We conclude that this closed set is minimal among all closed sets containing $\bigcup_{i=1}^n A_i$, so it is the closure of $\bigcup_{i=1}^n A_i$. ■

1.2 Further aspects of interior and closure

The "interior" and "closure" constructions have been seen to be well-behaved with respect to the formation of complements within a metric space. However, these notions are not well-behaved with respect intersections within a metric space. Also, one cannot capture the closure of a set just from knowing its interior. For example, a set can have empty interior and yet the closure equal to the whole space: think about the subset $\mathbb{Q}$ in $\mathbb{R}$.

Here is one mildly positive result.

Theorem 1.6. *The formation of closures is local in the sense that if U is open in a metric space X and A is an arbitrary subset of X, then the closure of $A \cap U$ in X meets U in $\bar{A} \cap U$ (where $\bar{A}$ denotes the closure of A in X). In particular, if Z is closed in X then $U \cap \overline{Z \cap U} = Z \cap U$.*

Also if U is the interior of a closed set Z in X, then $\bar{U}^\circ = U$.

After proving the theorem, we'll present an interesting example of an open subset of a metric space which is *not* equal to the interior of its closure (and hence, by the second part of the theorem, cannot be expressed as the interior of any closed set at all). It is probably not immediately obvious to you how to find such open sets, since typical open sets one writes down in $\mathbb{R}$ or $\mathbb{R}^2$ tend to be the interior of their closures.

Proof. Since $\bar{A} \cap U$ is a closed set in U that contains $A \cap U$, for the first part of the theorem we need to prove that every point $x \in \bar{A} \cap U$ is a limit of a sequence of points $x_n \in A \cap U$. Since $x \in \bar{A}$ we can write $x = \lim_{n\to\infty} x_n$ with $x_n \in A$. By hypothesis $x \in U$, so by the openness of U we must have some $B(x, r) \subseteq U$, and so since $x_n \to x$ by considering just sufficiently large n we have $x_n \in U$. Thus, for large n the sequence $\{x_n\}$ lies in $A \cap U$ and converges to x.

Now we assume that U is the interior of a closed set Z and we wish to prove U is the interior of $\bar{U}$. Since Z is a closed set containing U, it also contains the closure of U, and by openness of U the open subset U inside of $\bar{U}$ must lie inside the interior of $\bar{U}$. To summarize we have

$$U \subseteq \bar{U}^{\circ} \subseteq Z^{\circ} = U,$$

so equality is forced throughout. Check my basis $\mathcal{B}$, $\mathscr{B}$ and maximal ideals $\mathfrak{m}$, $\mathfrak{M}$ ∎