Math 535 - General Topology
Fall 2012
Homework 3 Solutions

Problem 1. Let $S^1 \subset \mathbb{R}^2$ be unit circle in the plane, with the subspace topology. Consider the "winding" map

$$\begin{aligned} f \colon \mathbb{R} &\to S^1 \\ t &\mapsto (\cos t, \sin t). \end{aligned}$$

Show that f induces a homeomorphism $\mathbb{R}/{\sim} \cong S^1$, where the equivalence relation on $\mathbb{R}$ is $t \sim t'$ if and only if $t - t' = 2k\pi$ for some integer $k \in \mathbb{Z}$.

Solution. The map $f \colon \mathbb{R} \to S^1$ is clearly continuous and surjective, and it induces the given equivalence relation on $\mathbb{R}$, namely

$$\begin{aligned} f(t) = f(t') &\Leftrightarrow (\cos t, \sin t) = (\cos t', \sin t') \\ &\Leftrightarrow t - t' = 2k\pi \end{aligned}$$

for some integer $k \in \mathbb{Z}$.

It remains to check that a subset $U \subseteq S^1$ is open whenever $f^{-1}(U) \subseteq \mathbb{R}$ is open. Since f is surjective, we have $f(f^{-1}(U)) = U$, hence it suffices to show that f is an open map.

An open interval $I = (t_1, t_2) \subseteq \mathbb{R}$ is sent to an open "arc interval" $f(I) \subseteq S^1$, which is open in S^1. Indeed it can be written as $f(I) = V \cap S^1$, where $V \subseteq \mathbb{R}^2$ is the open region of $\mathbb{R}^2$ described in polar coordinates as

$$V = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid r \in (0.9, 1.1) \text{ and } \theta \in (t_1, t_2)\}. \quad \square$$

Problem 2. Let $f\colon X \twoheadrightarrow Y$ be a surjective continuous map.

a. If f is an open map, show that f is a quotient map.

Solution. It suffices to check that a subset $U \subseteq Y$ is open whenever $f^{-1}(U) \subseteq X$ is open. Since f is surjective, we have $f(f^{-1}(U)) = U$. Since f is an open map, U is open. $\square$

b. If f is a closed map, show that f is a quotient map.

Solution. Assume $f^{-1}(U) \subseteq X$ is open, so that $f^{-1}(U^c) = f^{-1}(U)^c$ is closed in X. Since f is surjective, we have $f(f^{-1}(U^c)) = U^c$. Since f is a closed map, U^c is closed, i.e. U is open. $\square$

Problem 3. Find an example of a *metric* space X and a *quotient* map $q\colon X \twoheadrightarrow Y$ which is neither an open map nor a closed map.

Note: For the purposes of the homework, Y is not required to be a metric space, although such examples can be found.

Solution. Consider the quotient map $\pi\colon \mathbb{R} \twoheadrightarrow \mathbb{R}/{\sim}$ where the equivalence relation $\sim$ in $\mathbb{R}$ identifies all points of $[0,5)$ together, i.e.

$$x \sim y \text{ if } x = y \text{ or } x, y \in [0,5).$$

Then the interval $(1,2) \subset \mathbb{R}$ is open but $\pi\left((1,2)\right) \subset \mathbb{R}/{\sim}$ is not open, since $\pi^{-1}\pi\left((1,2)\right) = [0,5)$ is not open in $\mathbb{R}$. Thus π is not an open map.

Likewise, $\{2\} \subset \mathbb{R}$ is closed but $\pi\left(\{2\}\right) \subset \mathbb{R}/{\sim}$ is not closed, since $\pi^{-1}\pi\left(\{2\}\right) = [0,5)$ is not closed in $\mathbb{R}$. Thus π is not a closed map. $\square$

Problem 4. (Bredon Exercise I.13.6) Consider the quotient space $\mathbb{R}/\mathbb{Q}$, where the equivalence relation on $\mathbb{R}$ is $x \sim x'$ if and only if $x - x' \in \mathbb{Q}$. Show that the topology on $\mathbb{R}/\mathbb{Q}$ is anti-discrete, i.e. only the empty set $\emptyset$ and all of $\mathbb{R}/\mathbb{Q}$ are open.

Solution. Let $U \subseteq \mathbb{R}/\mathbb{Q}$ be a non-empty open subset. We want to show $U = \mathbb{R}/\mathbb{Q}$.

Since U is open in $\mathbb{R}/\mathbb{Q}$, its preimage $\pi^{-1}(U)$ is open in $\mathbb{R}$. Pick $x \in U$ and a representative $\widetilde{x} \in \pi^{-1}(U)$ of x. Since $\pi^{-1}(U)$ is open in $\mathbb{R}$, there is an open interval I satisfying $\widetilde{x} \in I \subseteq \pi^{-1}(U)$.

For any real number $z \in \mathbb{R}$, there is a rational number $r \in \mathbb{Q}$ satisfying $z - r \in I \subseteq \pi^{-1}(U)$. Therefore we have

$$\pi(z) = \pi(z - r) \in U$$

so that $U = \mathbb{R}/\mathbb{Q}$ as claimed. □

Problem 5. (Bredon Exercise I.3.1) (Munkres Exercise 2.17.6) Let X be a topological space.

a. Let A and B be subsets of X. Show the equality $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Solution. ($\supseteq$) The inclusion $A \subseteq A \cup B$ implies $\overline{A} \subseteq \overline{A \cup B}$, and likewise $\overline{B} \subseteq \overline{A \cup B}$. Thus we have $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

($\subseteq$) Assume $x \notin \overline{A} \cup \overline{B}$. The condition $x \notin \overline{A}$ means that there is a neighborhood N of x that does not touch A, i.e. $N \cap A = \emptyset$. Likewise, $x \notin \overline{B}$ means that there is a neighborhood N' of x that does not touch B, i.e. $N' \cap B = \emptyset$.

Now $N \cap N'$ is a neighborhood of x that does not touch $A \cup B$, i.e.

$$\begin{aligned}(N \cap N') \cap (A \cup B) &= ((N \cap N') \cap A) \cup ((N \cap N') \cap B) \\ &= \emptyset \cup \emptyset \\ &= \emptyset\end{aligned}$$

which proves $x \notin \overline{A \cup B}$. $\square$

Cleaner solution for ($\subseteq$). The inclusions $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ yield $A \cup B \subseteq \overline{A} \cup \overline{B}$, where the latter is closed. Taking closures yields

$$\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}. \quad \square$$

b. Let $\{A_\alpha\}$ be a family of subsets of X. Show the inclusion $\bigcup_\alpha \overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha}$.

Solution. For every index β, the inclusion $A_\beta \subseteq \bigcup_\alpha A_\alpha$ implies $\overline{A_\beta} \subseteq \overline{\bigcup_\alpha A_\alpha}$. Thus we have $\bigcup_\alpha \overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha}$. $\square$

c. Find an example where the inclusion in part (b) is strict, and X is a *metric* space.

Solution. Consider the singletons $\{r\} \subset \mathbb{R}$ for every rational number $r \in \mathbb{Q}$. Then the union of closures is

$$\bigcup_{r \in \mathbb{Q}} \overline{\{r\}} = \bigcup_{r \in \mathbb{Q}} \{r\} = \mathbb{Q}$$

whereas the closure of the union is

$$\overline{\bigcup_{r \in \mathbb{Q}} \{r\}} = \overline{\mathbb{Q}} = \mathbb{R}.$$

The inclusion $\mathbb{Q} \subset \mathbb{R}$ is (very!) strict. $\square$

Another similar example. Consider the singletons $\{\frac{1}{n}\} \subset \mathbb{R}$ for every $n \in \mathbb{N}$. Then the union of the closures is

$$\bigcup_{n\in\mathbb{N}} \overline{\{\frac{1}{n}\}} = \bigcup_{n\in\mathbb{N}} \{\frac{1}{n}\} = \{\frac{1}{n} \mid n \in \mathbb{N}\}$$

whereas the closure of the union is

$$\overline{\bigcup_{n\in\mathbb{N}} \{\frac{1}{n}\}} = \overline{\{\frac{1}{n} \mid n \in \mathbb{N}\}} = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}. \quad \square$$

Problem 6. Let X be a metric space and $A \subseteq X$ a subset. The **distance** from a point $x \in X$ to the subset A is

$$d(x, A) := \inf_{a \in A} d(x, a).$$

Show the equivalence $x \in \overline{A}$ if and only if $d(x, A) = 0$.

Solution. Consider the equivalences

$$\begin{aligned} x \in \overline{A} &\Leftrightarrow \forall \epsilon > 0,\ B_\epsilon(x) \cap A \neq \emptyset \\ &\Leftrightarrow \forall \epsilon > 0,\ \exists a \in A \text{ such that } d(x, a) < \epsilon \\ &\Leftrightarrow \forall \epsilon > 0,\ \inf_{a \in A} d(x, a) < \epsilon \\ &\Leftrightarrow \inf_{a \in A} d(x, a) = 0 = d(x, A). \quad \square \end{aligned}$$

Problem 7. (Munkres Exercise 2.17.13) The **diagonal** of a space X is the set

$$\Delta := \{(x, x) \mid x \in X\} \subseteq X \times X.$$

Show that X is Hausdorff if and only if the diagonal Δ is closed in $X \times X$.

Solution. Consider the equivalent statements:

The diagonal $\Delta \subseteq X \times X$ is closed.

$\Leftrightarrow$ For every $(x, y) \notin \Delta$, there is a (basic) open neighborhood $U \times V$ of (x, y) such that $(U \times V) \cap \Delta = \emptyset$.

$\Leftrightarrow$ For every distinct points $x, y \in X$, there are open subsets $U, V \subset X$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$, i.e. X is Hausdorff. $\square$

Problem 8. Show that a countable product of first-countable topological spaces is first-countable. In other words, if the spaces $X_1, X_2, X_3, \ldots$ are first-countable, then their product $\prod_{i\in\mathbb{N}} X_i$ (with the product topology) is also first-countable.

Solution. Let $x = (x_1, x_2, \ldots) \in \prod_{i\in\mathbb{N}} X_i$. We want to find a countable neighborhood basis of x.

Because each space X_i is first-countable, there is a countable neighborhood basis $\mathcal{B}_i$ of $x_i \in X_i$. Without loss of generality, assume $X_i \in \mathcal{B}_i$. Consider the collection of subsets of $\prod_{i\in\mathbb{N}} X_i$

$$\mathcal{B} := \{\prod_{i\in\mathbb{N}} B_i \mid B_i \in \mathcal{B}_i \text{ and } B_i \neq X_i \text{ for at most finitely many indices } i\}.$$

Note that each $B \in \mathcal{B}$ is a neighborhood of x. We claim that $\mathcal{B}$ is a countable neighborhood basis of x.

$\mathcal{B}$ is a neighborhood basis of x. Any neighborhood of x contains a basic open neighborhood $\prod_i U_i$ of x, where $U_i \subseteq X_i$ is open, and $U_i \neq X_i$ for at most finitely many indices i.

Since $\mathcal{B}_i$ is a neighborhood basis of $x_i \in X_i$, there is some $B_i \in \mathcal{B}_i$ satisfying $B_i \subseteq U_i$, where we pick $B_i = X_i$ for every index i such that $U_i = X_i$. By construction, we have $B := \prod_i B_i \in \mathcal{B}$ and $B \subseteq \prod_i U_i$.

$\mathcal{B}$ is countable. Rewrite the collection $\mathcal{B}$ as

$$\begin{aligned}
\mathcal{B} &= \{\prod_{i\in\mathbb{N}} B_i \mid B_i \in \mathcal{B}_i \text{ and } B_i \neq X_i \text{ for at most finitely many indices } i\} \\
&= \bigcup_{n\in\mathbb{N}} \{\prod_{i\in\mathbb{N}} B_i \mid B_i \in \mathcal{B}_i \text{ and } B_i \neq X_i \text{ possibly for indices } i \leq n \text{ but } B_i = X_i \text{ for } i > n\} \\
&=: \bigcup_{n\in\mathbb{N}} \mathcal{B}^{(n)}.
\end{aligned}$$

Each of those "finitely supported" subcollections $\mathcal{B}^{(n)}$ is in bijection with

$$\mathcal{B}^{(n)} \simeq \prod_{i=1}^{n} \mathcal{B}_i$$

where the latter is a finite product of countable sets, hence countable. Therefore $\mathcal{B}$ is a countable union of countable sets, hence countable. □

Problem 9. Let X be a topological space. A subset $A \subseteq X$ is called **dense** in X if its closure is all of X, i.e. $\overline{A} = X$.

Show that A is dense in X if and only if every non-empty open subset of X contains a point of A.

Solution. Consider the equivalent statements

$$\begin{aligned}
\overline{A} = X &\Leftrightarrow \forall x \in X, x \in \overline{A} \\
&\Leftrightarrow \forall x \in X, \forall \text{ open neighborhood } U \text{ of } x, U \cap A \neq \emptyset \\
&\Leftrightarrow \forall U \text{ non-empty and open}, U \cap A \neq \emptyset. \quad \square
\end{aligned}$$

Problem 10. A topological space X is called **separable** if it contains a countable dense subset.

a. Show that a second-countable space is always separable.

Solution. Let $\mathcal{B} = \{B_i\}_{i\in\mathbb{N}}$ be a countable basis for the topology of X. Pick a point $b_i \in B_i$ in each basic open, and consider the set $A := \{b_i \mid i \in \mathbb{N}\}$. Then A is countable, and moreover it is dense in X.

Indeed, any non-empty open $U \subseteq X$ is a union of basic open subsets $U = \bigcup_{j\in J} B_j$, so that U contains the points $b_j \in U \cap A$ for all $j \in J$. By problem 9, A is dense in X. $\square$

Now we will show that the converse statement does not hold.

b. Let X be an *uncountable* set (e.g. the real numbers $\mathbb{R}$) endowed with the *cofinite* topology. Show that X is separable.

Solution. The closed subsets of X are precisely the finite subsets and X itself. Therefore, the closure of any *infinite* subset $S \subseteq X$ must be $\overline{S} = X$, i.e. any infinite subset is dense in X.

In particular, pick any countably infinite subset $S \subset X$ (e.g. the integers $\mathbb{Z} \subset \mathbb{R}$ in the real line). Then S is countable and dense in X. $\square$

Remark. A countable space X is always separable, since X is dense in itself.

c. Show that X from part (b) is not first-countable (let alone second-countable).

Solution. Let $x \in X$ and let $\mathcal{B} = \{B_i\}_{i\in\mathbb{N}}$ be a countable collection of neighborhoods of x. We will show that $\mathcal{B}$ is *not* a neighborhood basis of x.

Every neighborhood in a cofinite space contains a non-empty open (i.e. cofinite) subset and is thus itself cofinite as well. Therefore each neighborhood B_i can be written as

$$B_i = X \setminus F_i$$

for some finite subset $F_i \subset X$ of excluded points. Because the union $\bigcup_{i\in\mathbb{N}} F_i$ is (at most) countable whereas X is uncountable, we can pick a point $y \in X \setminus \left(\bigcup_{i\in\mathbb{N}} F_i\right)$ and satisfying $y \neq x$.

The set $U := X \setminus \{y\}$ is cofinite (hence open) in X, and it contains x, so U is a neighborhood of x.

However, no neighborhood $B_i \in \mathcal{B}$ satisfies $B_i \subseteq U$, because the point $y \notin U$ is in all the B_i:

$$y \in \left(\bigcup_{i\in\mathbb{N}} F_i\right)^c = \bigcap_{i\in\mathbb{N}} F_i^c = \bigcap_{i\in\mathbb{N}} B_i. \quad \square$$