Math 527 - Homotopy Theory Spring 2013 Homework 9 Solutions

Problem 1. Consider the Hopf map $\eta: S^3 \to S^2$.

a. Describe the cofiber $C(\eta)$. It is a familiar space.

Solution. Recall that $\eta: S^3 \to \mathbb{C}P^1$ was defined as the quotient by the action of $S^1 \subset \mathbb{C}^{\times}$ (via scalar multiplication) on the unit sphere $S^3 \subset \mathbb{C}^2 \setminus \{0\}$. Its cofiber $C(\eta)$ is obtained by attaching a 4-cell to $\mathbb{C}P^1$ using η as attaching map. This CW structure on $C(\eta)$ is the standard CW structure on $\mathbb{C}P^2$.

b. Consider the canonical comparison map $\varphi \colon F(\eta) \to \Omega C(\eta)$ from the homotopy fiber to the loop space of the cofiber. Find the lowest dimension k such that $\pi_k F(\eta)$ is not isomorphic to $\pi_k \Omega C(\eta)$ (and thus φ cannot possibly induce an isomorphism on π_k).

Solution. Since η is a fibration, its homotopy fiber $F(\eta)$ is homotopy equivalent to its strict fiber $\eta^{-1}(*) = S^1$. Thus the comparison map φ can be viewed as a map

$$\varphi \colon S^1 \to \Omega \mathbb{C}P^2$$
.

Recall the natural isomorphism $\pi_i(\Omega X) \cong \pi_{i+1}(X)$ for all $i \geq 0$. To study the homotopy groups of $\mathbb{C}P^2$, note that the long exact sequence on homotopy of the fibration

$$S^1 \to S^5 \xrightarrow{q} \mathbb{C}P^2$$

yields the isomorphisms

$$\partial \colon \pi_2(\mathbb{C}P^2) \xrightarrow{\cong} \pi_1(S^1) \cong \mathbb{Z}$$

$$q_* \colon \pi_i(S^5) \xrightarrow{\cong} \pi_i(\mathbb{C}P^2)$$

for all $i \geq 3$. In particular, we obtain:

$$\pi_0(\Omega \mathbb{C}P^2) \cong \pi_1(\mathbb{C}P^2) = 0$$

$$\pi_1(\Omega \mathbb{C}P^2) \cong \pi_2(\mathbb{C}P^2) \cong \mathbb{Z}$$

$$\pi_2(\Omega \mathbb{C}P^2) \cong \pi_3(\mathbb{C}P^2) \cong \pi_3(S^5) = 0$$

$$\pi_3(\Omega \mathbb{C}P^2) \cong \pi_4(\mathbb{C}P^2) \cong \pi_4(S^5) = 0$$

$$\pi_4(\Omega \mathbb{C}P^2) \cong \pi_5(\mathbb{C}P^2) \cong \pi_5(S^5) \cong \mathbb{Z}.$$

Comparing with the homotopy groups of S^1 , we conclude $\pi_i(S^1) \simeq \pi_i(\Omega \mathbb{C}P^2)$ for all i < 4 whereas $\pi_4(S^1) \not\simeq \pi_4(\Omega \mathbb{C}P^2)$.

Problem 2. Show that a (strictly) commutative diagram

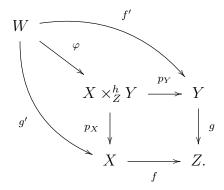
$$\begin{array}{ccc}
W & \xrightarrow{f'} & Y \\
g' \downarrow & & \downarrow g \\
X & \xrightarrow{f} & Z
\end{array}$$

is homotopy k-Cartesian if and only if for all $x \in X$, the induced map on homotopy fibers

$$f''\colon F_x(g')\to F_{f(x)}(g)$$

over the respective basepoints $x \in X$ and $f(x) \in Z$ is k-connected. Here we have $k \geq 0$ or $k = \infty$.

Solution. Consider the comparison map φ from W to the homotopy pullback



Because the homotopy pullback square induces a homotopy equivalence on homotopy fibers $F(p_X) \xrightarrow{\simeq} F(g)$, the condition on homotopy fibers in the statement is equivalent to the following: The map on homotopy fibers $\varphi'': F(g') \to F(p_X)$ induced by the diagram

$$W \xrightarrow{\varphi} X \times_Z^h Y$$

$$g' \downarrow \qquad \qquad \downarrow p_X$$

$$X \xrightarrow{\text{id}} X$$

is k-connected. For any basepoint $x \in X$ (which we omit from the notation), consider the diagram

$$F(g') \xrightarrow{\varphi''} F(p_X) \tag{1}$$

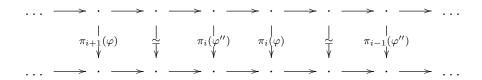
$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$W \xrightarrow{\varphi} X \times_Z^h Y$$

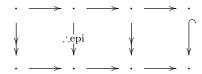
$$g' \downarrow \qquad \qquad \downarrow p_X$$

$$X \xrightarrow{\text{id}} X$$

where both columns are fiber sequences. Now consider the induced map of long exact sequences on homotopy



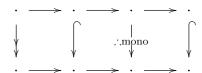
Recall the four-lemma for epimorphisms, schematically summarized here:



This implies the following:

- 1. If $\pi_i(\varphi)$ is an epimorphism, then $\pi_i(\varphi'')$ is an epimorphism.
- 2. If $\pi_i(\varphi'')$ is an epimorphism and $\pi_{i-1}(\varphi'')$ is a monomorphism, then $\pi_i(\varphi)$ is an epimorphism.

Now recall the four-lemma for monomorphisms, schematically summarized here:



This implies the following:

1. If $\pi_i(\varphi)$ is an monomorphism and $\pi_{i+1}(\varphi)$ is an epimorphism, then $\pi_i(\varphi'')$ is a monomorphism.

2. If $\pi_i(\varphi'')$ is a monomorphism, then $\pi_i(\varphi)$ is a monomorphism.

From both statements 1., we deduce: If φ is k-connected, then φ'' is k-connected.

From both statements 2., we deduce: If φ'' is k-connected, then φ is k-connected.

Therefore, φ is k-connected if and only if φ'' is k-connected.

Alternate solution. Taking horizontal fibers of the two bottom rows in (1) yields the diagram

$$F(g') \xrightarrow{\varphi''} F(p_X)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$F(\varphi) \longrightarrow W \xrightarrow{\varphi} X \times_Z^h Y$$

$$g'' \downarrow \qquad \qquad \downarrow^{p_X}$$

$$* \simeq F(\mathrm{id}) \longrightarrow X \xrightarrow{\mathrm{id}} X.$$

It is a fact (c.f. Strom, Theorem 8.57) that filling in the top left corner of the 3-by-3 diagram using either the vertical or the horizontal homotopy fiber will yield equivalent results:

$$F(g'') \simeq F(\varphi'') \longrightarrow F(g') \xrightarrow{\varphi''} F(p_X)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$F(\varphi) \longrightarrow W \longrightarrow X \times_Z^h Y$$

$$g'' \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$* \simeq F(\mathrm{id}) \longrightarrow X \longrightarrow X$$

from which we deduce the equivalence $F(\varphi'') \simeq F(g'') \simeq F(\varphi)$.

To conclude, consider the equivalent conditions:

$$\varphi$$
 is k-connected $\Leftrightarrow F(\varphi)$ is $(k-1)$ -connected $\Leftrightarrow F(\varphi'')$ is $(k-1)$ -connected $\Leftrightarrow \varphi''$ is k-connected. \square

Problem 3. Let $f: X \to Y$ be an *n*-connected map between spaces, and assume X is *m*-connected.

a. Using Blakers-Massey, show that the canonical comparison map

$$\varphi \colon F(f) \to \Omega C(f)$$

from the homotopy fiber to the loop space of the cofiber of f is (m+n)-connected.

Solution. Consider the homotopy pushout diagram

$$\begin{array}{ccc} X & \stackrel{f}{\longrightarrow} & Y \\ \downarrow & \stackrel{\text{ho}}{\vdash} & \downarrow \\ * & \longrightarrow & C(f). \end{array}$$

Since X is m-connected, the map $X \to *$ is (m+1)-connected. By Blakers-Massey, the diagram is (m+1+n-1=m+n)-Cartesian. By Problem 2, the map induced on (horizontal) homotopy fibers $F(f) \to \Omega C(f)$ is (m+n)-connected.

- **b.** Looking back (c.f. Problem 1) at the example of the Hopf map $\eta: S^3 \to S^2$, conclude that:
 - The connectivity estimate m + n in part (a) cannot be improved in general;
 - The map $\varphi_{\eta} \colon F(\eta) \to \Omega C(\eta)$ does in fact induce isomorphisms on homotopy groups below the least dimension k satisfying $\pi_k F(\eta) \not\simeq \pi_k \Omega C(\eta)$.

Solution. The map $\eta: S^3 \to S^2$ is 1-connected, and its source S^3 is 2-connected. By part (a), the comparison map

$$\varphi \colon F(\eta) \to \Omega C(\eta)$$

is 3-connected. In particular, φ induces isomorphisms on homotopy groups π_i for i < 4. In the cases i = 0, 2, and 3, this is automatic since both groups are trivial. In the remaining case i = 1, the fact that φ is 3-connected guarantees that it induces an isomorphism on π_1 .

However, φ cannot be 4-connected, because the induced map on π_4

$$0 = \pi_4(S^1) \xrightarrow{\varphi_*} \pi_4(\Omega \mathbb{C}P^2) \cong \mathbb{Z}$$

cannot be surjective.