

## MA 572: Homework 5

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**PROBLEM 5.1 (HATCHER §2.2, EX. 3)**

1 Let  $f: S^n \rightarrow S^n$  be a map of degree zero. Show that there exists points  $x, y \in S^n$  with  $f(x) = x$   
 2 and  $f(y) = -y$ . Use this to show that if  $F$  is a continuous vector field defined on the unit ball  $D^n$  in  
 3  $\mathbf{R}^n$  such that  $F(x) \neq 0$  for all  $x$ , then there exists a point on  $\partial D$  where  $F$  points radially outward  
 4 and another point on  $\partial D^n$  where  $F$  points radially inward.

5 *Proof.* Since  $\deg f = 0 \neq (-1)^n = \deg a$ , then  $f \not\approx a$  and so must have a fixed point  $x \in S^n$ . Now,  
 6 consider the map  $g := a \circ f$ . Since  $\deg g = \deg a \circ f = (\deg a)(\deg f) = 0$ ,  $g$  must have a fixed point  
 7  $y \in S^n$ . Since  $g(y) = a \circ f(y) = y$ , then  $f(y) = -y$ .

8 Suppose  $F$  is a continuous nonzero vector field on  $S^n$ , i.e., a map  $S^n \rightarrow \mathbf{R}^n$  which assigns  
 9 to each point  $x \in S^n$  a tangent vector  $\mathbf{v}(x)$  at  $x$ . Then, the map  $f: \partial D^n \rightarrow \mathbf{R}^n$  given by  
 10  $f(\mathbf{v}(x)) = \mathbf{v}(x)/\|\mathbf{v}(x)\|$  is well defined and nowhere zero. ■

**PROBLEM 5.2 (HATCHER §2.2, EX. 7)**

11 For an invertible linear transformation  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  show that the induced map  $H_n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\}) \cong$   
 12  $\tilde{H}_{n-1}(\mathbf{R}^n \setminus \{0\}) \cong \mathbf{Z}$  is id or  $-\text{id}$  according to whether the determinant of  $f$  is positive or negative.  
 13 [Use Gaußian elimination to show that the matrix of  $f$  can be joined by a path of invertible matrices  
 14 to a diagonal matrix with  $\pm 1$ 's on the diagonal.]

15 *Proof.* We show that  $O_n(\mathbf{R})$  is a deformation retraction of  $GL_n(\mathbf{R})$  and prove the result there. This  
 16 procedure is adapted from a hint in *Элементарная топология* by Виро, Нецветаев и Харламов,  
 17 стр. 338, номер 39.11. Suppose  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is an invertible linear transformation. Let  $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$  be  
 18 the set of columns vectors of the matrix representation  $F$  of  $f$ . By Gram–Schmidt orthogonalization  
 19 construct the vectors

$$\begin{aligned} \mathbf{e}_1 &:= \lambda_{11}\mathbf{f}_1 \\ \mathbf{e}_2 &:= \lambda_{21}\mathbf{f}_1 + \lambda_{22}\mathbf{f}_2 \\ &\vdots \\ \mathbf{e}_n &:= \lambda_{n1}\mathbf{f}_1 + \dots + \lambda_{nn}\mathbf{f}_n \end{aligned} \tag{5.1}$$

20 where the  $\lambda_{kk} > 0$  for  $k = 1, \dots, n$ . Now set

$$\mathbf{g}_k(t) := t(\lambda_{n1}\mathbf{f}_1 + \lambda_{n2}\mathbf{f}_2 + \dots + \lambda_{k,k-1}\mathbf{f}_{k-1}) + (t\lambda_{kk} + 1 - t)\mathbf{f}_k. \tag{5.2}$$

21 Let  $g(t, A)$  be the matrix whose columns are the vectors  $\mathbf{g}_k(t)$  and define a homotopy  $H: I \times$   
 22  $GL_n(\mathbf{R}) \rightarrow GL_n(\mathbf{R})$  by mapping the pair  $(t, A) \mapsto g(t, A)$ . Continuity of  $H$  follows from the fact  
 23 that  $H$  it is multiplication in  $\mathbf{R}^n$  followed by a linear mapping. It's not hard to see that  $H$  stays in  
 24  $GL_n(\mathbf{R})$  for all  $t$  and  $H(1, A)$  is in  $O_n(\mathbf{R})$ .

25 Last but not least, we show that  $O_n(\mathbf{R})$  consists of two connected components and that  
 26 membership of  $f$  to one of these components is determined by  $\det f$ . First note that  $\det(O_n(\mathbf{R})) =$   
 27  $\{-1, 1\}$  which is disconnected in  $\mathbf{R}$ . Hence,  $O_n(\mathbf{R})$  is disconnected. If  $f \in O_n(\mathbf{R})$ , either  $\det f = 1$   
 28 or  $-1$ . Suppose it is the former. ■

**PROBLEM 5.3 (HATCHER §2.2, EX. 13)**

29 Let  $X$  be the 2-complex obtained from  $S^1$  with its usual cell structure by attaching two 2-cells by  
30 maps of degrees 2 and 3, respectively.

31 (a) Compute the homology groups of all the subcomplexes  $A \subset X$  and the corresponding quotient  
32 complexes  $X/A$ .

33 (b) Show that  $X \simeq S^2$  and that the only subcomplex  $A \subset X$  for which the quotient map  $X \rightarrow X/A$   
34 is a homotopy equivalence is the trivial subcomplex, the 0-cell.

35 *Proof.*

■

**PROBLEM 5.4**

36 *Proof.*

