

# MA 519: Homework 13

Max Jeter, Carlos Salinas

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## PROBLEM 13.1 (HANDOUT 17, # 16)

Suppose  $X \sim \text{Exp}(1)$ ,  $Y \sim U[0, 1]$ , and  $X, Y$  are independent.

- (a) Find the density of  $X + Y$ .
- (b) Find the density of  $XY$ .

*SOLUTION.* For part (a): Since  $X$  and  $Y$  are independent, the distribution of  $X + Y$  is given by the convolution

$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_X(x-y)f_Y(y) dy,$$

where

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad f_Y(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, a straight forward calculation gives us

$$\begin{aligned} f_{X+Y}(x) &= \int_{-\infty}^{\infty} \chi_{[0,\infty)}(x-y)e^{-(x-y)}\chi_{[0,1]}(y) dy \\ &= e^{-x} \int_{-\infty}^{\infty} e^y \chi_{[0,\infty)}(x-y)\chi_{[0,1]}(y) dy \\ &= \begin{cases} 0 & \text{for } x < 0, \\ 1 - e^{-x} & \text{for } 0 \leq x \leq 1, \\ (e-1)e^{-x} & \text{for } x > 1. \end{cases} \end{aligned}$$

Now let us run a sanity check by demonstrating that  $\int_{-\infty}^{\infty} f_{X+Y}(x) dx = 1$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} f_{X+Y}(x) dx &= \int_0^1 [1 - e^{-x}] dx + (e-1) \int_1^{\infty} e^{-x} dx \\ &= [1 + e^{-1} - 1 - 0] + (e-1)[0 - (-e^{-1})] \\ &= e^{-1} + 1 - e^{-1} \\ &= 1. \end{aligned}$$

For part (b): Since  $X$  and  $Y$  are independent, we have

$$F_{XY}(z) = \iint_{\{(x,y):xy \leq z\}} f_X(x)f_Y(y) dx dy.$$

Let us find the CDF of  $XY$ . By a direct computation

$$\begin{aligned} F_{XY}(z) &= \iint_{\{(x,y):xy \leq z\}} f_X(x)f_Y(y) \, dx \, dy \\ &= \iint_{\{(x,y):xy \leq z\}} e^{-x} \chi_{[0,\infty)}(x) \chi_{[0,1]}(y) \, dx \, dy \\ &= \end{aligned}$$

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## PROBLEM 13.2 (HANDOUT 17, # 18)

Two points  $A, B$  are chosen at random from the unit circle. Find the probability that the circle centered at  $A$  with radius  $AB$  is fully contained within the original unit circle.

*SOLUTION.* The probability that a circle centered at  $A$  with radius  $AB$  is contained in the original circle is zero. What the professor means is “two points  $A, B$  are chosen at random from *inside* the unit circle”. We can think of choosing  $A$  as choosing a random variable  $0 < R < 1$  representing the distance of  $A$  from the origin and we ask what is the probability that the point  $B$  lands inside the circle of radius  $1 - R$  centered at  $A$ .

First, let us find the distribution for the radius  $R$ . We can find the CDF of  $R$  as the ratio of the area of the circle centered at the origin with radius  $x$  and the unit circle; i.e.,

$$P(R \leq x) = \frac{\pi x^2}{\pi \cdot 1^2} = x^2 \quad \text{for } 0 < x < 1.$$

Thus the PDF of  $R$  is

$$f_R(x) = 2x \quad \text{for } 0 < x < 1.$$

Now, the probability that  $B$  lands in the set

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PROBLEM 13.3 (HANDOUT 17, # 19)

Let  $X, Y$  be i.i.d.  $U[0, 1]$  random variables. Find the correlation between  $\max\{X, Y\}$  and  $\min\{X, Y\}$ .

SOLUTION. ■