MA557 Problem Set 3

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Problem 3.1

Let R be a domain and Γ the set of all principal ideals in R. Show that R is a unique factorization domain if and only if Γ satisfies the ascending chain condition and every irreducible element of R is prime.

Proof. $\Longrightarrow$ Suppose that R is a UFD and let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ be an ascending chain of ideals in Γ. Then $\mathfrak{a}_1 = \langle a_1 \rangle$ for some $a_i \in R$ since every ideal belonging to Γ is principal. Now, since R is a UFD, a_1 factors uniquely (up to associates) as the finite product $a_1 = p_1 \cdots p_k$ of (not necessarily distinct) irreducible elements $p_1, ..., p_k \in R$. If a_1 is irreducible we are done (because in such a case $a_2 \mid a_1$ if and only if $a_2 = ua_1$ where u is a unit, hence $\mathfrak{a}_1 = \mathfrak{a}_2 = \cdots$). Suppose a_1 is not irreducible. Then since $a_2 \mid a_1$, the irreducible factors of a_2 consist of some (or all) of the irreducible factors of a_1 (more precisely we can write $a_2 = p_{\sigma(1)} \cdots p_{\sigma(\ell)}$ for some injection $\sigma \colon \{1, ..., \ell\} \hookrightarrow \{1, ..., k\}$ where $\ell \leq k$). Inductively applying this argument to a_n for $n \geq 1$, we see that the process (of factoring a_n's from a_1) must terminate for some positive r for otherwise we have that

$$a_1 = a_2 b_2 = (a_3 b_3) b_2 = \cdots = (a_n b_n) b_{n-1} \cdots b_2 = \cdots ,$$

but every factorization of a_1 into irreducibles must have length k. Thus, the ascending chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_r = \mathfrak{a}_{r+1} = \cdots$ is stationary for some positive integer r and we say that Γ satisfies the acc.

$\Longleftarrow$ Conversely, suppose that Γ satisfies the acc. Let $a_1 \in R$. If a_1 is irreducible we are done. Suppose a_1 is reducible, then $a_1 = a_2 b_2$ for some non-units $a_2, b_{11} \in R$. If both a_2 and b_2 are irreducible, we are done. Without loss of generality we may assume a_2 is reducible (as the argument to follow may be applied to the b_i's in case they are not irreducible). Then $a_2 = a_3 b_3$ (and so $a_1 = a_2 b_2 = (a_3 b_3) b_2$) for some non-units $a_3, b_3 \in R$. Then we get the ascending chain of principal ideals

$$\langle a_1 \rangle \subset \langle a_2 \rangle \subset \langle a_3 \rangle \subset \cdots$$

which must stabilize for some positive integer r since Γ satisfies the acc. This argument shows that there exits a factorization of a_1 into irreducibles. We must now prove that this factorization in unique (up to associates).

Suppose $a = p_1 \cdots p_k = q_1 \cdots q_\ell$ where $p_1, ..., p_k, q_1, ..., q_\ell \in R$ are irreducibles. ∎

Problem 3.2

Let M be an Artinian R-module. Show that every injective R-linear map $\varphi\colon M \to M$ is an isomorphism.

Proof. Suppose, towards a contradiction, that φ is not surjective. Then $M \smallsetminus \operatorname{im}\varphi \neq \emptyset$. Let $x \in M \smallsetminus \operatorname{im}\varphi$ and consider the descending chain of submodules $\operatorname{im}\varphi \supset \operatorname{im}\varphi^2 \supset \cdots$ (the containment is clear since for any endomorphism $f\colon R \to R$ we have $f(R) \subset R$ so $f^2(R) = f(f(R)) \subset f(R)$). By the dcc on R, this chain must stabilize for some positive integer n, i.e., $\operatorname{im}\varphi^n = \operatorname{im}\varphi^{n+1} = \cdots$. Then $\varphi^n(x) \in \operatorname{im}\varphi^n$, but $\varphi^n(x) \in \operatorname{im}\varphi^{n+1}$ so that $\varphi^n(x) = \varphi^{n+1}(y)$ for some $y \in M$. However, since φ is injective $\varphi^n(x) = \varphi^{n+1}(y) = \varphi^n(\varphi(y))$ implies that $x = \varphi(y)$ contrary to our choice of x. Therefore, φ is an isomorphism. ■

Problem 3.3

Let M be a finitely generated Artinian module. Show that M is Noetherian.

Proof. We shall induct on n the number of generators of M.

Base case: Suppose that $M = Rx$. Then M is cyclic and $M \cong R/\operatorname{ann}(x)$ is an Artinian R-module, hence an Artinian $R/\operatorname{ann}(x)$-module (if $N_0 \supset N_2 \supset \cdots$ is a descending chain in M as an $R/\operatorname{ann}(x)$-module, then it is a descendig chain in M as an R-model, hence is stationary for nome n). Then M is an Artinian ring. By (3.17) M is Noetherian hence, M is Noetherian as an R-module.

Induction step: Assume that every Artinian module with a minimal generating set of cardinality $\leq n$ is Noetherian. Let $M = Rx_1 + \cdots + Rx_{n+1}$. Then

$$0 \longrightarrow Rx_1 \hookrightarrow M = Rx_1 + \cdots + Rx_n \twoheadrightarrow M/Rx_1 \to 0$$

is exact. Since Rx_1 is Noetherian and M/Rx_1 is Noetherian (since its minimal generating set has cardinality $\leq n$), by 3.5, M is Noetherian. ∎

Problem 3.4

Let R be a ring that is Artinian or Noetherian, and $x \in R$. Show that for some $n > 0$, the image of x in $R/(\operatorname{ann} x^n)$ is a non-zerodivisor on that ring.

Proof. By 3.7 it suffices to show the result for R a Noetherian module. Consider the ascending chain of submodules $\operatorname{ann} x \subset \operatorname{ann} x^2 \subset \cdots$ (this containment is clear since if $y \in \operatorname{ann} x^n$ then $yx^n = 0$ so $yx^{n+1} = 0$). By the acc on R, this chain stabilizes for some positive integer n, i.e., $\operatorname{ann} x^n = \operatorname{ann} x^{n+1} = \cdots$. We claim that x is a non-zerodivisor in $R/\operatorname{ann} x^n$. Suppose $\bar{x}\bar{y} = 0$ in $R/\operatorname{ann} x^n$. Then $xy \in \operatorname{ann} x^n$ so $(xy)x^n = x^{n+1}y = 0$. Hence, $y \in \operatorname{ann} x^{n+1}$. But $\operatorname{ann} x^{n+1} = \operatorname{ann} x^n$ so $y \in \operatorname{ann} x^n$, i.e., $\bar{y} = 0$. Thus, x is a non-zerodivisor in $R/\operatorname{ann} x^n$. ∎

PROBLEM 3.5

Let R be an Artinian ring. Show that $R \cong R_1 \times \cdots \times R_n$ with R_i Artinian local rings.

Proof. Since R is Artinian, it is semilocal (this was shown in the proof of 3.16, but not given as a proposition). Let us prove this:

Proposition. *Let R be an Artinian ring. Then R is semilocal.*

Proof of proposition. Let Γ be the set of all maximal ideals of R (which is nonempty by 1.4). Suppose, towards a contradiction, that $|\Gamma| = \infty$ and consider the descending chain of ideals $\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \cdots$ where $\mathfrak{m}_i \in \Gamma$. By the dcc on R, this chain stabilizes for some n, i.e., $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = (\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n) \cap \mathfrak{m}_{n+1}$. Since $\mathfrak{m}_1 \cdots \mathfrak{m}_n \subset \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ then $\mathfrak{m}_1 \cdots \mathfrak{m}_n \subset \mathfrak{m}_{n+1}$. Then since $\mathfrak{m}_{n+1}$ is maximal, it is prime so, by 1.8, $\mathfrak{m}_i \subset \mathfrak{m}_{n+1}$ for some $1 \leq i \leq n$. This is a contradiction. Thus, $|\Gamma| < \infty$, i.e, R is semilocal. ♣

Then $\operatorname{rad} R = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$. So, by the Chinese remainder theorem, $\operatorname{rad} R = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. By 3.15, $\operatorname{rad} R$ is nilpotent so $(\operatorname{rad} R)^k = (\mathfrak{m}_1 \cdots \mathfrak{m}_n)^k = \mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = 0$ for some positive integer k. Then the $\mathfrak{m}_i^k$'s are pairwise comaximal since the $\mathfrak{m}_i$'s are pairwise comaximal (a consequence of the following lemma)

Lemma. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals. Then $\sqrt{\mathfrak{a}^n + \mathfrak{b}^m} \supset \mathfrak{a} + \mathfrak{b}$.*

Proof of lemma. Without loss of generality, suppose $n > m$. Let $x \in \mathfrak{a} + \mathfrak{b}$. Then $x = a + b$ for some $a \in \mathfrak{a}$, $b \in \mathfrak{b}$. Then by the binomial expansion theorem

$$x^{m+n-1} = (a+b)^{m+n-1} = \sum_{i+j=m+n-1} \binom{n+m-1}{j} x^i y^j \in \sqrt{\mathfrak{a}^n + \mathfrak{b}^m}$$

since (ignoring coefficients) the product $x^i y^j$ is in $\mathfrak{a}^n + \mathfrak{b}^m$ since if $i \leq n-1$ and $j \leq m-1$ then $i + j \leq m + n - 2$. Thus, $x \in \sqrt{\mathfrak{a}^n + \mathfrak{b}^m}$ and the containment $\sqrt{\mathfrak{a}^n + \mathfrak{b}^m} \supset \mathfrak{a} + \mathfrak{b}$ holds. ♣

It follows immediately from the lemma that $\sqrt{\mathfrak{m}_i^k + \mathfrak{m}_j^k} \supset \mathfrak{m}_i + \mathfrak{m}_j = R$ so $\mathfrak{m}_i^k + \mathfrak{m}_j^k = R$. Thus, by the Chinese remainder theorem, there is an isomorphism $R \cong R/(\operatorname{rad} R)^k \cong R/\mathfrak{m}_1^k \times \cdots \times R/\mathfrak{m}_n^k$. We claim that each $R/\mathfrak{m}_i^k$ is local.

By 1.2, there is a one-one correspondence between the ideals of $R/\mathfrak{m}_i^k$ and the ideals of $\mathfrak{m}$ of R containing $\mathfrak{m}_i^k$. In particular, if $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{m}_i^k$, then $\mathfrak{m}_i \subset \mathfrak{m}$ (since $\mathfrak{m}$ is prime). Thus, $\mathfrak{m} = \mathfrak{m}_i$ by the maximality of $\mathfrak{m}_i$. Passing to the quotient, we see that $\mathfrak{m}_i/\mathfrak{m}_i^k$ is the unique maximal ideal of $R/\mathfrak{m}_i^k$. Hence, $R/\mathfrak{m}_i^k$ is a local (and by 3.7) Artinian. ∎

Problem 3.6

Let R be an Artinian ring all of whose maximal ideals are principal. Show that every ideal in R is principal.

Proof. Since R is Artinian, by 3.17, it is Noetherian. Therefore, every ideal $\mathfrak{a}$ of R is finitely generated. Without loss of generality, suppose $\mathfrak{a} = a_1R + a_2R$. Then, by 1.4, there exists a maximal ideal $\mathfrak{m} \supset \mathfrak{a}$. By assumption, $\mathfrak{a} = mR$ so we have $a_1R + a_2R \subset mR$. ■

Problem 3.7

Prove 2.11 (the snake lemma).

Proof. ■