MA52300 Fall 2016

Homework Assignment 4

Due Wed, Sep 28, 2016

1. (Legendre transform) Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2,x_2} = 0$$

is some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u(\mathbf{x}(p)),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies a *linear* equation

$$a^{22}(p)v_{p_1p_1} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_2p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29))

2. Find the solution $u(x, t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for } x > 0$$
$$u_t(0,t) = \alpha u_x(0,t), \quad \text{for } t > 0,$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$.

(*Hint:* use a representation $u(x, t) = F(x - t) + G(x + t)$ for the solution)

3. (a) Let u be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0, t) = u(\pi, t) = 0$. Show that the "energy"

$$E(t) = \frac{1}{2}\int_0^\pi (u_t^2 + c^2 u_x^2)dx, \quad t > 0$$

is independent of t; i.e., $\frac{d}{dt}E = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(nct) + b_n \sin(nct)) \sin nx$$

in terms of coefficients a_n, b_n.