Math 527 - Homotopy Theory
Spring 2013
Homework 11 Solutions

Problem 1. Show that a path-connected space is weakly equivalent to a product of Eilenberg-MacLane spaces if and only if it admits a Postnikov tower of principal fibrations with trivial k-invariants (all of them).

Note. Here, we follow Hatcher's convention that the k-invariants are used to build the Postnikov tower of X starting from $P_1 X$ and not $P_0 X$. In other words, by "Postnikov tower of principal fibrations", we mean that the maps $P_n X \to P_{n-1} X$ are principal fibrations for all $n \geq 2$. Using $n \geq 1$ instead would force $\pi_1 X$ to be abelian.

Solution. Some preliminary observations.

1. A path-connected space X is weakly equivalent to a product of Eilenberg-MacLane spaces if and only if it is weakly equivalent to $\prod_{i \geq 1} K(\pi_i X, i)$.

2. A projection $\pi_B \colon B \times F \twoheadrightarrow B$ is always a fibration, so that the sequence

$$F \xrightarrow{\iota} B \times F \xrightarrow{\pi_B} B$$

is a fiber sequence. Here $\iota = (c_{b_0}, \mathrm{id}_F) \colon F \to B \times F$ denotes the "slice inclusion" $\iota(f) = (b_0, f)$.

3. The homotopy fiber of the constant map $c \colon X \to Y$ is

$$\begin{aligned} F(c) &= \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = c(x), \gamma(1) = y_0\} \\ &= \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = \gamma(1) = y_0\} \\ &= X \times \Omega Y. \end{aligned}$$

Iterating the homotopy fiber once more yields the fiber sequence

$$\Omega Y \xrightarrow{\iota} X \times \Omega Y \xrightarrow{\pi_X} X \xrightarrow{c} Y.$$

In particular, the fibration $\pi_B \colon B \times F \twoheadrightarrow B$ can be extended to the right by the constant map if and only if F admits a delooping.

Now onto the proof of the statement.

($\Rightarrow$) Assume given a (zigzag of, but WLOG a single) weak equivalence $\varphi \colon X \xrightarrow{\sim} \prod_{i \geq 1} K(\pi_i X, i)$.

Then the successive projections

$$
\begin{tikzcd}
 & & \vdots \arrow[d] \\
 & & \prod_{i=1}^{n} K(\pi_i X, i) = P_n X \arrow[d] \\
 & & \prod_{i=1}^{n-1} K(\pi_i X, i) = P_{n-1} X \arrow[d] \\
 & & \vdots \arrow[d] \\
 & & K(\pi_1 X, 1) \times K(\pi_2 X, 2) = P_2 X \arrow[d] \\
X \arrow[r, "\varphi", "\sim"'] & \prod_{i \geq 1} K(\pi_i X, i) \arrow[uuuur] \arrow[uuur] \arrow[ur] \arrow[r] \arrow[dr] & K(\pi_1 X, 1) = P_1 X \arrow[d] \\
 & & * = P_0 X
\end{tikzcd}
$$

form a Postnikov tower for X.

The truncation map between successive stages $P_n X \to P_{n-1} X$ is a projection with fiber $K(\pi_n X, n)$, which admits a delooping $K(\pi_n X, n+1)$ since $\pi_n X$ is abelian (as $n \geq 2$).

By observation (3), $P_n X \to P_{n-1} X$ is a principal fibration which can be extended to the right by the constant map

$$P_n X \to P_{n-1} X \xrightarrow{*} K(\pi_n X, n+1)$$

so that the k-invariant $k_{n-1} \in H^{n+1}(P_{n-1} X; \pi_n X)$ is trivial.

($\Leftarrow$) Assume all k-invariants of X are trivial, i.e. for all $n \geq 2$ we have fiber sequences

$$P_n X \to P_{n-1} X \xrightarrow{*} K(\pi_n X, n+1).$$

By observation (3), this implies the equivalence

$$\begin{aligned} P_n X &\simeq P_{n-1} X \times \Omega K(\pi_n X, n+1) \\ &\simeq P_{n-1} X \times K(\pi_n X, n). \end{aligned}$$

Repeating this equivalence inductively, we conclude that for all $n \geq 1$, the Postnikov stages of X are products

$$P_n X \simeq \prod_{i=1}^{n} K(\pi_i X, i).$$

Since these truncation maps $P_n X \to P_{n-1} X$ are projections, in particular fibrations, the homotopy limit of the tower is (equivalent to) its strict limit. We conclude:

$$\begin{aligned} X &\xrightarrow{\sim} \operatorname*{holim}_n P_n X \\ &\simeq \operatorname*{holim}_n \prod_{i=1}^{n} K(\pi_i X, i) \\ &\simeq \lim_n \prod_{i=1}^{n} K(\pi_i X, i) \\ &\cong \prod_{i=1}^{\infty} K(\pi_i X, i) \end{aligned}$$

and thus X is weakly equivalent to a product of Eilenberg-MacLane spaces. $\square$

Problem 2. Let X be a path-connected CW complex and G a group. Show that the map

$$\pi_1 \colon [X, K(G,1)]_* \to \mathrm{Hom}_{\mathbf{Gp}} \left(\pi_1(X), G \right)$$

is a bijection.

Solution. WLOG X has a single 0-cell. Indeed, X is pointed homotopy equivalent to such a CW complex, and the functors on both sides $[-, K(G,1)]_*$ and $\mathrm{Hom}_{\mathbf{Gp}}(\pi_1(-), G)$ are invariant under pointed homotopy equivalence.

WLOG X is 2-dimensional. Indeed, the skeletal inclusion $\iota_2 \colon X_2 \hookrightarrow X$ induces an isomorphism $\iota_{2*} \colon \pi_1(X_2) \xrightarrow{\cong} \pi_1(X)$ and a bijection

$$\iota_2^* \colon [X, K(G,1)]_* \xrightarrow{\cong} [X_2, K(G,1)]_*$$

as shown in the notes from 5/29.

True for wedges of circles. When X is a wedge of circles $X \simeq \bigvee_{j \in J} S^1$, then π_1 does induce a bijection, as shown by the commutative diagram:

$$\begin{tikzcd}
{[\bigvee_j S^1, K(G,1)]_*} \arrow[r, "\pi_1"] \arrow[dd, "\cong"'] & \mathrm{Hom}_{\mathbf{Gp}} \left(\pi_1(\bigvee_j S^1), G \right) \arrow[d, "\cong"] \\
 & \mathrm{Hom}_{\mathbf{Gp}} \left(*_j \pi_1(S^1), G \right) \arrow[d, "\cong"] \\
\prod_j [S^1, K(G,1)]_* \arrow[r, "\prod_j \pi_1"] \arrow[d, equal] & \prod_j \mathrm{Hom}_{\mathbf{Gp}} \left(\pi_1(S^1), G \right) \arrow[d, "\cong"] \\
\prod_j \pi_1 K(G,1) \arrow[r, "\prod_j \psi", "\simeq"'] & \prod_j G
\end{tikzcd}$$

where $\psi \colon \pi_1 K(G,1) \xrightarrow{\cong} G$ is some fixed identification.

True in general. Let X be a 2-dimensional CW complex with a single 0-cell. WLOG all attaching maps of the 2-cells are pointed, so that $X = X_2$ sits in a cofiber sequence

$$\bigvee S^1 \to X_1 \to X_2 \to \bigvee S^2. \tag{1}$$

By the theorem on the fundamental group of CW complexes, applying π_1 to this specific cofiber sequence (1) yields a right exact sequence of groups

$$\pi_1(\bigvee S^1) \to \pi_1(X_1) \twoheadrightarrow \pi_1(X_2) \to 0.$$

Applying $\mathrm{Hom}_{\mathbf{Gp}}(-, G)$ then yields a left exact sequence of pointed sets, which is the bottom row in the diagram below.

Applying $[-, K(G,1)]_*$ to the cofiber sequence (1) yields an exact sequence of pointed sets. The natural transformation π_1 yields a map of exact sequences:

$$\begin{tikzcd}
{[\bigvee S^2, K(G,1)]_* = 0} \arrow[r] \arrow[d] & {[X_2, K(G,1)]_*} \arrow[r] \arrow[d, "\pi_1"'] & {[X_1, K(G,1)]_*} \arrow[r] \arrow[d, "\pi_1"', "\cong"] & {[\bigvee S^1, K(G,1)]_*} \arrow[d, "\pi_1"', "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathbf{Gp}}\left(\pi_1(X_2), G\right) \arrow[r] & \mathrm{Hom}_{\mathbf{Gp}}\left(\pi_1(X_1), G\right) \arrow[r] & \mathrm{Hom}_{\mathbf{Gp}}\left(\pi_1(\bigvee S^1), G\right).
\end{tikzcd}$$

Because $X_1 \simeq \bigvee S^1$ is also a wedge of circles, the two downward maps to the right are bijections, and hence so is the downward map

$$\pi_1 \colon [X_2, K(G,1)]_* \to \mathrm{Hom}_{\mathbf{Gp}}\left(\pi_1(X_2), G\right). \quad \square$$

Problem 3. Let X be a CW complex, with n-skeleton X_n, and let Y be a path-connected simple space. Let $n \geq 2$, and let $f_n, g_n \colon X_n \to Y$ be two maps which agree on X_{n-1}, i.e.

$$f_n|_{X_{n-1}} = g_n|_{X_{n-1}}.$$

Let $d(f_n, g_n) \in C^n(X; \pi_n Y)$ denote their difference cochain.

Show that $f_n \simeq g_n$ rel X_{n-2} holds if and only if $[d(f_n, g_n)] = 0 \in H^n(X; \pi_n Y)$ holds, i.e. $d(f_n, g_n)$ is a coboundary.

Solution. Since Y is path-connected and simple, we can safely ignore basepoints and work with unpointed maps.

WLOG $X = X_n$.

Consider the map

$$S \colon (X_n \times \partial I) \cup (X_{n-1} \times I) \to Y$$

defined by

$$\begin{aligned} S|_{X_n \times \{0\}} &= f_n \\ S|_{X_n \times \{1\}} &= g_n \\ S|_{X_{n-1} \times \{t\}} &= f_{n-1} \text{ for all } t \in I. \end{aligned}$$

(The letter S was chosen for "Stationary".)

The condition $f_n \simeq g_n$ rel X_{n-2} can be stated as being able to extend the restriction

$$S_{n-1} := S|_{X_n \times \partial I \cup X_{n-2} \times I}$$

to all of $X_n \times I$. In other words, $S = S_n$ is defined on the relative n-skeleton of the relative CW complex

$$(X_n \times I, X_n \times \partial I)$$

and we want to extend its restriction S_{n-1} from the relative $(n-1)$-skeleton to the relative $(n+1)$-skeleton $X_n \times I$. **There exists such an extension if and only if the obstruction class of S_n**

$$c(S_n) \in C^{n+1}(X_n \times I, X_n \times \partial I; \pi_n Y)$$

is a coboundary.

The short exact sequence of cellular chain complexes

$$0 \to C_*(X \times \partial I) \to C_*(X \times I) \to C_*(X \times I, X \times \partial I) \to 0$$

yields a short exact sequence of cellular cochain complexes

$$0 \to C^*(X \times I, X \times \partial I; \pi_n Y) \to C^*(X \times I; \pi_n Y) \to C^*(X \times \partial I; \pi_n Y) \to 0.$$

Using the fact that $C_*(I)$ is finitely generated and free in each degree, we obtain the isomorphism

$$C^{n+1}(X_n \times I, X_n \times \partial I; \pi_n Y) \cong C^n(X_n; \pi_n Y) \otimes_{\mathbb{Z}} C^1(I) \tag{2}$$

and moreover, the coboundary operator in the relative cellular cochain complex $C^*(X_n \times I, X_n \times \partial I; \pi_n Y)$ corresponds to the coboundary in $C^*(X_n; \pi_n Y)$. In other words, the diagram

$$\begin{tikzcd}
C^{n+1}(X_n \times I, X_n \times \partial I; \pi_n Y) & C^n(X_n; \pi_n Y) \otimes_{\mathbb{Z}} C^1(I) \arrow[l, "\cong"'] \arrow[r, "\cong"] & C^n(X_n; \pi_n Y) \\
C^n(X_n \times I, X_n \times \partial I; \pi_n Y) \arrow[u, "\delta"] & C^{n-1}(X_n; \pi_n Y) \otimes_{\mathbb{Z}} C^1(I) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[u, "\delta\otimes\mathrm{id}"] & C^{n-1}(X_n; \pi_n Y) \arrow[u, "\delta"]
\end{tikzcd}$$

commutes.

Therefore, the obstruction class $c(S_n) \in C^{n+1}(X_n \times I, X_n \times \partial I; \pi_n Y)$ is a coboundary if and only if the corresponding cochain in $C^n(X_n; \pi_n Y)$ is a coboundary.

Relative $(n+1)$-cells of $(X_n \times I, X_n \times \partial I)$ are of the form $e^n_\alpha \times e^1$ for some n-cell e^n_α of X_n with attaching map $\varphi_\alpha \colon S^{n-1} \to X_{n-1}$ and characteristic map

$$\Phi_\alpha \colon (D^n, S^{n-1}) \to (X_n, X_{n-1}).$$

Here e^1 denotes the unique 1-cell of the interval I.

The value of the cochain $c(S_n)$ on the relative $(n+1)$-cell $e^n_\alpha \times e^1$ is the composite

$$\begin{tikzcd}
\partial(D^n \times D^1) \arrow[r] \arrow[d, equal] & (X_n \times I)_n \arrow[r, "S"] \arrow[d, equal] & Y. \\
\partial D^n \times D^1 \cup D^n \times \partial D^1 \arrow[r, "\varphi_\alpha \times \mathrm{id}_I \cup \Phi_\alpha \times \mathrm{id}_{\partial I}"'] & X_{n-1} \times I \cup X_n \times \partial I \arrow[ur] &
\end{tikzcd}$$

By definition of S, that composite is homotopic to the map $d(f_n, g_n)(e^n_\alpha \times e^1) \in \pi_n Y$ (or minus it, depending on our sign convention in the definition of the difference construction). This proves the equality

$$c(S_n) = \pm d(f_n, g_n)$$

via the isomorphism (2).

Therefore the obstruction class $c(S_n)$ is a coboundary if and only if the difference cochain $d(f_n, g_n)$ is a coboundary. $\square$