MA544: Qual Problems

Carlos Salinas

April 23, 2016

Contents

CHAPTER 1

Course Notes

These notes roughly correspond to chapters 2 through 8 of Wheeden and Zygmund's *Measure and Integration* [1].

This first portion corresponds to material covered before Exam 1.

1.1 Functions of bounded variation and the Riemann–Stieltjes integral

In this section, we introduce functions of bounded variation as well as the definition of the Riemann integral. We conclude with a proof that the

Functions of bounded variation

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined for all $a \leq x \leq b$ and finite; let $\Gamma = \{x_0, \dots, x_m\}$ be a *partition* of $[a,b]$, i.e., a collection of points x_i, $i = 0, \dots, m$, satisfying $x_0 = a$ and $x_m = b$, and $x_{i-1} < x_i$ for $i = 1, \dots, m$. To each partition Γ, we associated a sum

$$S_\Gamma \coloneqq S_\Gamma[f; a, b] \coloneqq \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|. \tag{1}$$

The *variation* (or *total variation*) *of f over* $[a,b]$ is defined as

$$V \coloneqq V[f; a, b] \coloneqq \sup_\Gamma S_\Gamma, \tag{2}$$

where the supremum is taken over all partitions Γ of $[a,b]$. If $V < \infty$, f is said to be of *bounded variation on* $[a,b]$; if $V = \infty$, f is of *unbounded variation on* $[a,b]$.

Before going on to prove important properties about (2), let us look at some common examples (and nonexamples) of functions f of bounded variation.

Examples 1. Suppose f is *monotone* in $[a,b]$. Then, clearly, each S_Γ is equal to $|f(a) - f(b)|$ for every partition Γ*, and therefore $V = |f(b) - f(a)|$.

*Carlos: This may not be clear at a first glance, but, upon closer inspection, this is true by monotoncity. If $a < x < b$, we have $|f(b) - f(a)| = |f(b) - f(x)| + |f(x) - f(a)|$. This holds for an arbitrary partitions Γ.

Examples 2. Suppose the graph of f can be split into a finite number of monotone arcs, i.e., suppose $[a,b] = \bigcup_{i=1}^{k} [a_{i-1}, a_i]$ and f is monotone in each $[a_{i-1}, a_i]$. Then $V = \sum_{i-1}^{k} |f(a_i) - f(a_{i-1})|$. To see this, we use the result of Example 1 above and the fact, yet to be proven, than $V = V[a,b] = \sum_{i=1}^{k} V[a_i, a_{i-1}]$.

If $\Gamma = \{x_0, \ldots, x_m\}$ is a partition of $[a,b]$, let $|\Gamma|$, called the *norm of* Γ, be defined as the length of the longest subinterval of Γ

$$|\Gamma| \coloneqq \max_i (x_i - x_{i-1}). \tag{3}$$

If f is continuous on $[a,b]$ and $\{\Gamma_j\}$ is a sequence of partitions of $[a,b]$ with $|\Gamma_j| \to 0$, we shall see that $V = \lim_{j\to\infty} S_{\Gamma_j}$.

Examples 3. Let f be the *Dirichlet function*, defined by $f(x) = 1$ for x rational and $f(x) = 0$ for x irrational. Then, clearly, $V[a,b] = \infty$ for any interval of $[a,b]$.†

Examples 4. A function that is continuous on an interval, however, need not be of bounded variation on that interval. Take for example the following construction: let $\{a_j\}$ and $\{d_j\}$, $j = 1, 2, \ldots,$ be monotone decreasing sequences in $(0,1]$ with $a_1 = \lim_{j\to\infty} a_j = \lim_{j\to\infty} d_j = 0$ and $\sum d_j = \infty$. Construct a continuous function f as follows. On each subinterval $[a_{j+1}, a_j]$, the graph of f consists of the sides of the isosceles triangle with base $[a_{j+1}, a_j]$ and height d_j. Thus, $f(a_j) = 0$, and if m_j denotes the midpoint of $[a_{j+1}, a_j]$, then $f(m_j) = d_j$. If we define $f(0) = 0$, then f is continuous on $[0,1]$. Taking Γ_k to be the partition defined by the points 0, $\{a_j\}_{j=1}^{k+1}$ and $\{m_j\}_{j=1}^{k}$, we see that $S_\Gamma = 2\sum_{j=1}^{k} d_j$. Hence, $V[f;0,1] = \infty$.

†Carlos: By the density of $\mathbb{Q}$ in $\mathbb{R}$ (and by restriction, $[a,b]$, since $[a,b]$ is path-connected), for any positive integer N, we may choose a partition Γ of $[a,b]$ containing $N+1$ rational numbers so $S_\Gamma = N+1 > N$.

1.2 The Lebesgue integral

This portion corresponds to material covered before the second exam.

1.3 Differentiation

This portion of the notes corresponds to material covered before the final.

This section deals with questions of differentiability and culminates with a couple of results tying together the Lebesgue integral with the derivative à la the familiar fundamental theorem of calculus for Riemann integrals.

The indefinite integral

If f is a Riemann integrable function on an interval $[a, b]$ of $\mathbb{R}$, then the familiar definition for its *indefinite integral* is

$$F(x) = \int_a^x f(y)dy, \quad a \leq x \leq b. \tag{4}$$

The *fundamental theorem of calculus* then asserts that $F' = f$ if f is continuous. In this section, we study the analogue of this result for Lebesgue integrable functions.

Since we want to generalize our results to $\mathbb{R}^n$, first we must find a suitable notion of indefinite integral for multivariable functions. In two dimensions we might, for instance, define the indefinite integral F of f to be

$$F(x_1, x_2) := \int_{a_1}^{x_1} \int_{a_2}^{x_2} f(y_1, y_2)\, dy_2\, dy_1. \tag{5}$$

As it turns out, it is better to abandon the notion that the indefinite integral be a function of a point an instead let it be a function of a set. Therefore, given a function f, integrable on some measurable subset A of $\mathbb{R}^n$, we define the *indefinite integral of* f to be the function

$$F(E) := \int_E f, \tag{6}$$

where E is a measurable subset of A.

The function F is an example of a *set function*, by which we mean any real-valued function F defined on a σ-algebra Σ of measurable sets such that

(i) $F(E)$ is finite for every $E \in \Sigma$.
(ii) F is *countably additive*; i.e., if E is the union of disjoint sets $E_k \in \Sigma$, $k = 1, 2, \ldots$, then

$$F(E) \sum_{k \in \mathbb{N}} F(E_k). \tag{7}$$

1.4 L^p Classes

Let's take a small detour to ch. 5 of [1] to talk about L^p spaces.

The relation between the Riemann–Stieltjes integral and the Lebesgue integral, and the L^p spaces, $0 < p < \infty$

As it turns out, there is a remarkably simple and useful representation of the Lebesgue integral (over measurable subsets of $\mathbb{R}^n$) in terms of the Riemann–Stiltjes integrals (over measurable subset of $\mathbb{R}$). In order to establish this relationship, we will need to study the function

$$\omega(\alpha) := \omega_{f,E}(\alpha) := |\{\, \mathbf{x} \in E : f(\mathbf{x}) > \alpha \,\}|, \tag{8}$$

where f is a measurable function on E and $-\infty < \alpha < \infty$. We call $\omega_{f,E}$ (or simply ω) the *distribution function of f on E*.

The function ω is clearly not affected by changing f in a set of measure zero, and is decreasing. As $\alpha \nearrow \infty$, we have

$$\{\, \mathbf{x} \in E : f(\mathbf{x}) > \alpha \,\} \searrow \{\, \mathbf{x} \in E : f(\mathbf{x}) = \infty \,\}.$$

hence, assuming that f is finite a.e. in E, by Theorem 3.62(ii), $\lim_{\alpha\to\infty} \omega = 0$, unless $\omega(\alpha) \equiv \infty$. Similarly, we have $\lim_{\alpha\to-\infty} \omega = |E|$. For now, let us assume that the measure of E is finite; this will ensure that ω is bounded.

In the following results, we assume that f is a measurable function that is finite a.e. in E, $|E| < \infty$, and write

$$\omega(\alpha) = \omega_{f,E}(\alpha), \quad \{\, f > a \,\} = \{\, \mathbf{x} \in E : f(\mathbf{x}) > \alpha \,\},$$

etc.

Lemma 1 (5.38)**.** *If $\alpha < \beta$, then $|\{\, \alpha \le f \le \beta \,\}| = \omega(\alpha) - \omega(\beta)$.*

Proof. For $\alpha < \beta$, we have $\{\, f > \beta \,\} \subset \{\, f > \alpha \,\}$ and $\{\, \gamma < f \le \beta \,\} = \{\, f > \alpha \,\} \setminus \{\, f > \beta \,\}$. Since $|\{\, f > \beta \,\}| < \infty$, the lemma follows from Corollary 3.25. ■

Given α, let

$$\omega(\alpha+) \coloneqq \lim_{\varepsilon\searrow 0} \omega(\alpha+\varepsilon) \qquad \omega(\alpha-) \coloneqq \lim_{\varepsilon\searrow 0} \omega(\alpha-\varepsilon).$$

denote the limits of ω from the right and left at α.

Lemma 2 (5.39)**.**

(a) $\omega(\alpha+) = \omega(\alpha)$*; i.e., ω is continuous from the right.*
(b) $\omega(\alpha-) = |\{\, f \ge \alpha \,\}|$.

Corollary 3 (5.40)**.**

(a) $\omega(\alpha-) - \omega(\alpha) = |\{\, f = \alpha \,\}|$*; in particular, ω is continuous at α if and only if* $|\{\, f = \alpha \,\}| = 0$.
(b) *ω is constant in an open interval (α, β) if and only if $|\{\, \alpha < f < \beta \,\}| = 0$, that is, if and only if f takes almost no values between α and β.*

The rest of this section establishes the relations between the Lebesgue and Riemann–Stieltjes integrals. As always, we assume f is measurable and finite a.e. in E, $|E| < \infty$ and $\omega = \omega_{E,f}$.

Theorem 4 (5.41)**.** *If $a \le f(\mathbf{x}) \le b$ (a and b are finite) for all $\mathbf{x} \in E$, then*

$$\int_E f = -\int_a^b \alpha \, d\omega(\alpha).$$

Proof. The Lebesgue integral on the left-hand side exists since f is bounded and $|E| < \infty$. The Riemann–Stieltjes integral on the right-hand side exists by Theorem 2.24. To show that they are equal, let us partition the interval the interval $[a, b]$ by $a = \alpha_0 < \alpha_1 < \cdots < \alpha_k = b$ and let

$E_j = \{, \alpha_{j-1} < f \leq \alpha_j \}$. The E_j are disjoint and $E = \bigcup_{j=1}^{k} E_j$. Hence, $\int_E f = \sum_{j=1}^{k} \int_{E_j} f$ and, therefore

$$\sum_{j=1}^{\infty} \alpha_{j-1}|E_j| \leq \int_E f \leq \sum_{j=1}^{k} \alpha_j |E_j|.$$

By Lemma 5.38, $|E_j| = \omega(\alpha_j)\omega(\alpha_j) = -[\omega(\alpha_j) - \omega(\alpha_{j-1})]$. Hence, the sums are Riemann–Stieltjes sums for $-\int_a^b \alpha \, d\omega(\alpha)$. Since the sums must converge to $-\int_a^b \alpha \, d\omega(\alpha)$ as the norm of the partition tends to zero, the conclusion follows. ■

We can extend the conclusion of Theorem 5.41 to the case when f is not bounded as follows.

Theorem 5 (5.42)**.** *Let f be any measurable function on E, and let $E_{ab} \coloneqq \{ \mathbf{x} \in E : a < f(\mathbf{x}) < b \}$ (a and b finite). Then,*

$$\int_{E_{ab}} f = -\int_a^b \alpha \, d\omega(\alpha).$$

Sketch of proof. Take $\omega_{ab}(\alpha) \coloneqq |\{ \mathbf{x} \in E_{ab} : f(\mathbf{x}) > \alpha \}|$. By Theorem 5.41, we have

$$\int_{E_{ab}} f = -\int_a^b \alpha \, d\omega_{ab}(\alpha).$$

Taking the limit of Riemann–Stieltjes sums that approximate the integrals, it suffices to show that $\omega_{ab}(\alpha) - \omega_{ab}(\beta) = \omega(\alpha) - \omega(\beta)$. Then The expression on the right-hand side of the equation above, is seen to be $\int_a^b \alpha \, d\omega(\alpha)$. ■

Theorem 6 (5.43)**.** *If either $\int_E f$ or $\int_{-\infty}^{\infty} \alpha \, d\omega(\alpha)$ exist and is finite, then the other exists and is finite, and*

$$\int_E f = -\int_{-\infty}^{\infty} \alpha \, d\omega(\alpha).$$

Two measurable functions f and g are said to be *equimeasurable*, or *equidistributed*, if

$$\omega_{f,E}(\alpha) = \omega_{g,E}(\alpha)$$

for all α.

We may intuitively think of equimeasurable functions as being *rearrangements* of each other. For such functions, we have

$$|\{ a < f \leq b \}| = |\{ a < g \leq b \}| \quad |\{ f = a \}| = |\{ g = a \}|,$$

etc. We also gave the following immediate corollary of Theorem 5.43.

Corollary 7 (5.44)**.** *If f and g are equimeasurable on E and $f \in L(E)$, then $g \in L(E)$ and*

$$\int_E f = \int_E g.$$

The method used to derive Theorem 5.41 through 5.43 illustrates a basic difference between the Lebesgue and the Riemann integral. The Riemann integral is defined by a limiting process whose initial step involves partitioning the domain of f. On the other hand, the Lebesgue integral can be obtained from a process that partitions the *range* of f. In order to define the process more clearly, let f be a nonnegative measurable function that is finite a.e. in E, $|E| < \infty$. Let $\Gamma = \{ 0 = \alpha_0 < \alpha_1 < \cdots \}$ be a partition of the positive ordinate axis by a countable number of points $\alpha_k \to \infty$, and let $|\Gamma| = \sup_k (\alpha_{k+1} - \alpha_k)$. Set $E_k := \{ \alpha_k \leq f < \alpha_{k+1} \}$ and $Z := \{ f = \infty \}$. Then the E_k are measurable and disjoint, $|Z| = 0$ and $E = (\bigcup E_k) \cup Z$, so that $|E| = \sum_k |E_k|$. Let

$$S_\Gamma := \sum_{k \in \mathbb{N}} \alpha_k |E_k|, \quad S_\Gamma := \sum_{k \in \mathbb{N}} \alpha_{k+1} |E_k|.$$

CHAPTER 2

MA 544 (Spring 2016)

2.1 Exam 1 Prep

Problem 2.1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{ r\mathbf{x} : \mathbf{x} \in E \}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Define a linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by $\mathbf{x} \mapsto r\mathbf{x}$. Using the standard basis for $\mathbb{R}^n$, this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \tag{9}$$

which has determinant $\det T = r^n$. By 3.35, we have $|E| = |T(E)| = r^n |E| = |rE|$. ■

Problem 2.2. Let $\{E_k\}$, $k \in \mathbb{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k\to\infty} E_k \right| \leq \liminf_{k\to\infty} |E_k|.$$

Proof. If the $\liminf |E_k| = \infty$ the inequality holds trivially. Hence, we may, without loss of generality, assume that $\liminf |E_k| < \infty$. By 3.20, the set $\liminf E_k$ is measurable and we have

$$\left| \liminf_{k\to\infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \tag{10}$$

where $F_k = \bigcap_{n=k}^{\infty} E_n$. Now, note that the collection of sets $F'_k = \bigcup_{\ell=1}^{k} F_\ell$ forms an increasing sequence of measurable sets $F'_k \nearrow F'$, where $F' = \bigcup_{k=1}^{\infty} F_k = \liminf E_k$. Then, by 3.26 (i), we have

$$\lim_{k\to\infty} |F'_k| = |F'| = \left|\liminf_{k\to\infty} E_k\right|. \tag{11}$$

Hence, it suffices to show that $|F'_k| \le |E_k|$ for all k, but this follows by monotonicity of the outer measure, 3.3, since $F'_k \subset E_k$. Thus, we have the desired inequality

$$\left|\liminf_{k\to\infty} E_k\right| \le \liminf_{k\to\infty} |E_k|. \tag{12}$$

■

Problem 2.3. Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) = \{\, \mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r \,\}$. Prove that F is monotonic increasing and continuous.

Proof. That F is increasing is immediate from the monotonicity of the outer measure since for $x < x'$ we have $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$ so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \le |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that F is continuous, we will prove the following lemma

Lemma 8. *For any $x > 0$, $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$.*

Proof of lemma. If $\mathbf{y} \in xB(\mathbf{0}, 1)$ then $\mathbf{y} = x\mathbf{y}'$ for $\mathbf{y}' \in B(\mathbf{0}, 1)$. Thus, $|\mathbf{y}'| = |\mathbf{y}|/x < 1$ so $|\mathbf{y}| < x$ implies that $\mathbf{y} \in B(\mathbf{0}, x)$. Hence, we have the containment $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$.

On the other hand, if $\mathbf{y} \in B(\mathbf{0}, x)$ then $|\mathbf{y}| < x$ so $|\mathbf{y}/x| < 1$. Hence, $\mathbf{y}/x \in B(\mathbf{0}, 1)$ so $x(\mathbf{y}/x) = \mathbf{y} \in B(\mathbf{0}, x)$. Thus, $B(\mathbf{0}, x) \subset xB(\mathbf{0},)$ and equality holds. ♣

In light of Lemma 8 and 3.35, for $x > 0$, we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n |B(\mathbf{0}, 1)|. \tag{13}$$

It is clear that F is continuous on the interval $[0, \infty)$ since F is a polynomial in x. ■

Problem 2.4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. From the topological definition of continuity, f is continuous at $x \in C$ if and only if for every neighborhood U of $f(x)$, the preimage $f^{-1}(U)$ is a neighborhood of x. Now, ■

Let $x \in C$. Then, by the definition of continuity, for every natural number $n > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f(x) - f(x')| < \frac{1}{2n}. \tag{14}$$

Let $x'', x' \in B(x, \delta)$. Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x)''| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \tag{15}$$

In view of these estimates, define the set

$$A_n = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \tag{16}$$

Good Lord, that was a long definition! We claim that $C = \bigcap_{n=1}^{\infty} A_n$ and that A_n is open for all n.

First, let us show that $C = \bigcap_{n=1}^{\infty} A_n$. Let $x \in C$. Then for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Thus, $x \in A_n$ for all n so $x \in \bigcap A_n$. On the other hand, if $x \in \bigcap A_n$ for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Fix $\varepsilon > 0$. By the Archimedean principle, there exists $N > 0$ such that $\varepsilon > 1/N$. Then, since $x \in A_N$ it follows that for some $\delta' > 0$, $|x - x'| < \delta'$ implies $|f(x) - f(x')| < 1/N < \varepsilon$. Thus, $x \in C$ and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$.

Lastly, we show that A_n is open. Let $x \in A_n$. Then there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. In particular, this means that $B(x, \delta) \subset A_n$ for any $x' \in B(x, \delta)$ satisfies $|f(x) - f(x')| < 1/n$. Thus, A_n is open and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$ is a G_δ set.

Problem 2.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Proof. No. Recall that, by definition, or 4.1, f is measurable if and only if $\{ f > a \}$ for all $a \in \mathbb{R}$. ■

Problem 2.6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set $\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \}$ is measurable.

Proof. The idea here should be to rewrite

$$E = \left\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \right\} \tag{17}$$

as a countable union/intersection of measurable sets. Let $x \in E$. By the Cauchy criterion, for every $N > 0$ there exists a positive integer M such that $m, n \geq M$ implies $|f_n(x) - f_m(x)| < 1/N$. With this in mind, define

$$E_N = \left\{ x : \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \tag{18}$$

Then, like for Problem 1.4, it is not too hard to see that the E_n's are open and that $E = \bigcap_{n=1}^{\infty} E_n$. Thus, E is a G_δ set and therefore measurable. ■

Problem 2.7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. Suppose $f\colon [a, b] \to \mathbb{R}$ is absolutely continuous. Then for fixed $\varepsilon = 1$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, we have $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Let $\Gamma = \{x_k\}_{k=1}^N$ be a partition of $[a, b]$ into closed intervals such that $x_{k+1} - x_k < \delta$, then by absolute continuity we have

$$\begin{aligned} V[f; \Gamma] &= \sum_{k=1}^N |f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \tag{19}$$

Thus, f is b.v. on $[a, b]$. ■

Problem 2.8. Let f be a continuous function from $[a, b]$ into $\mathbb{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, i.e., $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(a) & \text{if } c = b \end{cases}.$$

Proof. ■

2.2 Exam 1

2.3 Exam 2 Prep

Problem 2.9. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B_\varepsilon(\mathbf{0})$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x})\, d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f \colon \mathbb{R}^n \to \mathbb{R}$ is (Lebesgue) integrable over a subset E of $\mathbb{R}^n$ (or, alternatively, f belongs to $L(E)$) if

$$\int_E f(\mathbf{x})\, d\mathbf{x} < \infty.$$

Put $E = \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$. Then, to show that f belongs to $L(E)$ it suffices to prove the inequality

$$\int_E f(\mathbf{x})\, d\mathbf{x} < \frac{C}{\varepsilon} \tag{20}$$

for some appropriate constant C. We proceed by directly computing the Lebesgue integral of f and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x})\, d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{({x_1}^2 + \cdots + {x_n}^2)^{(n+1)/2}} \end{aligned}$$

let E_i denote the projection of E onto its i-th coordinate and make the trigonometric substitution $x_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \tan\theta$, $dx_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \sec^2\theta d\theta$ with $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$ giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[\frac{\cos^{n-1}\theta}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1}\theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1}\theta d\theta < \infty. \tag{21}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned}\int \cdots \int_E f(\mathbf{x})d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{{x_n}^2} \\ &= 2C' \int_{\varepsilon}^{\infty} \frac{dx_n}{{x_n}^2} \\ &= \frac{C}{\varepsilon}\end{aligned} \tag{22}$$

as desired. ■

Problem 2.10. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit f of $\{f_k\}$ must be nonnegative a.e. in $\mathbb{R}^n$. For assume otherwise. Then there exists a collection of points $\mathbf{x}$ in $\mathbb{R}^n$ of nonzero $\mathbb{R}^n$-Lebesgue measure such that $f(\mathbf{x}) < 0$. But $f_k(\mathbf{x}) \geq 0$ for all $k \in \mathbb{N}$. Set $0 < \varepsilon < |f(\mathbf{x})|$ then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{23}$$

for all k which contradicts our assumption that $f_k \to f$ a.e. on $\mathbb{R}^n$. Therefore, the set of points $\mathbf{x} \in \mathbb{R}^n$ where $f(\mathbf{x}) < 0$ must have measure zero.

Now, based on pointwise convergence a.e. to f, given $\varepsilon > 0$ for a.e. $\mathbf{x} \in \mathbb{R}^n$ we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{24}$$

for sufficiently large k; say k greater than or equal to some index $N \in \mathbb{N}$. Moreover, we are given convergence in $L(\mathbb{R}^n)$ of f_k to f

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f < \infty. \tag{25}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{26}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{27}$$

for all $k \in \mathbb{N}$. By Theorem 5.5(ii), f and the f_k's are finite a.e. in $\mathbb{R}^n$ so for some sufficiently large real number M, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in \mathbb{R}^n$. In particular, for any measurable subset E of $\mathbb{R}^n$, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in E$ so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \to \int_E f < \infty. \tag{28}$$

However, if f does not belong to $L(\mathbb{R}^n)$, i.e., its integral over $\mathbb{R}^n$ is infinity, there is no guarantee that f will be finite a.e. in $\mathbb{R}^n$. This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset E of $\mathbb{R}^n$. Let us demonstrate this with an example. Consider the sequence of functions ■

Problem 2.11. Assume that E is a measurable set of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of $\mathbb{R}^n$, then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \tag{29}$$

Set $E_k = \{\, \mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k \,\}$ and $F_k = \{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{30}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \tag{31}$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \tag{32}$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (32) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \tag{33}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (34) \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \qquad (35) \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned}$$

Thus, from (34) and (35)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \qquad (36)$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ■

Problem 2.12. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. $\Longrightarrow$: First note that ρ is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that $\rho(f_k, f) \to 0$ as $k \to \infty$. Then, given $\varepsilon > 0$ there exist an sufficiently large index N such that for every $k \geq N$ we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \tag{37}$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in E which happens if $|f_k - f| = 0$ a.e. in E.

$\Longleftarrow$: Suppose that $f_k \to f$ as $k \to \infty$.

I don't know how to solve this. This is the intended solution:

$\Longrightarrow$: Given $\varepsilon > 0$, $\rho(f_k, f) \to 0$ implies that

$$\int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f} dx \longrightarrow 0.$$

Observe that the function $\Phi \colon \mathbb{R}^+ \to \mathbb{R}$ given by $\Phi(x) = x/(1+x)$ is increasing on $\mathbb{R}^+$ and $0 < \Psi(x) < 1$, hence

$$\begin{aligned} \int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1 + \varepsilon} |\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}|. \end{aligned}$$

Therefore,

$$|\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}| \leq \frac{1 + \varepsilon}{\varepsilon} \int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f|} \, dx \longrightarrow 0$$

as $k \to \infty$.

$\Longleftarrow$: Conversely, given $\delta > 0$, we have

$$\begin{aligned} \rho(f_k, f) = &\int_{\{ x \in E : |f_k(x) - f(x)| > \delta \}} \frac{|f_k - f|}{1 + |f_k - f|} \, dx \\ &+ \int_{\{ x \in E : |f_k(x) - f(x)| \leq \delta \}} \frac{|f_k - f|}{1 + |f_k - f|} \, dx \\ &\leq |\{ x \in E : |f_k(x) - f(x)| > \delta \}| + \frac{\delta}{1 + \delta} |E|. \end{aligned}$$

Since $|E| < \infty$ and $\delta/(1+\delta) \searrow 0$, then for any $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\frac{\delta'}{1 + \delta'} |E| < \frac{\varepsilon}{2}.$$

If $f_k \to f$ as $k \to \infty$ in measure, then for the above δ' there is an index $N > 0$ such that $k \geq N$ implies

$$|\{ x \in E : |f_k(x) - f(x)| > \delta' \}| < \frac{\varepsilon}{2}.$$

Therefore, $f_k \to f$ in measure implies $\rho(f_k, f) \to 0$ as $k \to \infty$. ∎

Problem 2.13. Define the *gamma function* $\Gamma\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1}\, du,$$

and the *beta function* $\beta\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. (a) Fix $y \in \mathbb{R}^+$. Then we must show that $\Gamma(y) < \infty$. First, since $(0, 1)$ and $[1, \infty)$ are disjoint measurable subsets of $\mathbb{R}$, by Theorem 5.7 we can split the integral $\Gamma(y)$ into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u}u^{y-1}\, du}_{I_1} + \underbrace{\int_1^\infty e^{-u}u^{y-1}\, du}_{I_2}. \tag{38}$$

We will show, separately, that I_1 and I_2 are finite.

To see that I_1 is finite, note that

$$\begin{aligned} e^{-u}u^{y-1} &= e^{-u}e^{(y-1)\log u} \\ &= e^{-u+(y-1)\log u} \\ &\leq e^{(y-1)\log u} \\ &= u^{y-1} \end{aligned} \tag{39}$$

since $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u}u^{y-1}\, du \\ &\leq \int_0^1 u^{y-1}\, du \\ &= \left[\frac{u^y}{y}\right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \tag{40}$$

To see that I_2 is finite, note that

$$e \tag{41}$$

Intended solution:

(b) ∎

Problem 2.14. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}} \colon \mathbb{R}^n \to \mathbb{R}$ be $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \to \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{42}$$

■

Problem 2.15. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \leq g_k$ and

$$\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \to 0 \qquad \text{as} \qquad k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \to \int_{\mathbb{R}^n} |f| \qquad \text{as} \qquad k \to \infty.$$

Proof. (a) Since $f_k \to f$ and $g_k \to g$ a.e. and $|f_k| \leq g_k$, then by Fatou's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} (g - f) = \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k - f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k - f_k, \\ \int_{\mathbb{R}^n} g + f \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k + f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k + f_k. \end{aligned}$$

Since $f_k, g_k, f, g \in L(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$, then using the similar argument as problem 2, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f \geq \limsup_{k \to \infty} \int_{\mathbb{R}^n} f_k, \\ \int_{\mathbb{R}^n} f \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} f_k. \end{aligned}$$

Therefore, $\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f$.

(b) $\implies$: This direction is obvious by the inequality

$$\left| \int_{\mathbb{R}^n} |f_k| - |f| \right| \leq \int_{\mathbb{R}^n} ||f_k| - |f|| \leq \int_{\mathbb{R}^n} |f_k - f|.$$

$\Longleftarrow$: Let $g_k = |f_k| + |f|$ and $g = 2|f|$. Since $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e., then $g_k, g \in L(\mathbb{R}^n)$ and $g_k \to g$ a.e. in $\mathbb{R}^n$. By the assumption, $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$.

Let $\tilde{f}_k = |f_k - f|$. Then $\tilde{f}_k \to 0$ a.e. in $\mathbb{R}^n$ and $\tilde{f}_k \le g_k$. Applying part (a) to $\tilde{f}_k$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

■

2.4 Midterm 2

Problem 2.16. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Proof. Recall that $f \in L(\mathbb{R}^n)$ if and only if $|f| \in L(\mathbb{R}^n)$. Let $B_k = B(\mathbf{0}, k)$ for $k \in \mathbb{N}$ and χ_{B_k} be the indicator function associated with B_k. Then, the sequence of maps $\{|f_k|\}$ defined $f_k = f\chi_{B_k}$ converge pointwise to $|f_k|$. Since $|f| \in L(\mathbb{R}^n)$, by the monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbb{R}^n} |f|. \tag{43}$$

But this means, exactly, that for every $\varepsilon > 0$ there exists sufficiently large $N \in \mathbb{N}$ such that

$$\begin{aligned} \varepsilon &> \left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \\ &= -\int_{\mathbb{R}^n} |f_k| + \int_{\mathbb{R}^n} |f| \\ &= -\int_{\mathbb{R}^n} |f| + \int_{\mathbb{R}^n} |f| \\ &= -\int_{B_k} |f| + \int_{\mathbb{R}^n} |f| \\ &= \int_{\mathbb{R}^n \setminus B_k} |f| \end{aligned} \tag{44}$$

as desired. ■

Problem 2.17. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. First, since the E_j's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \tag{45}$$

Let χ_{E_j} be the characteristic function of the subset E_j of E and define $f_j = f\chi_{E_j}$ for $j \in \mathbb{N}$. Note that, since both f and χ_{E_j} are measurable on E, f_j is measurable on E and $\sum_{j=1}^{\infty} f_j = f$. Moreover, since $E_j \subset E$, by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \setminus E_j} f = \int_E f_j + \int_{E \setminus E_j} f. \tag{46}$$

Hence, because the E_j's are disjoint $(E \setminus E_k) \setminus E_\ell = (E \setminus E_\ell) \setminus E_k$ so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \tag{47}$$

as desired. ■

Problem 2.18. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbb{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Proof. Let $\varepsilon > 0$ be given. Then, by the hypothesis, there exists $\delta > 0$ such that such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon \tag{48}$$

for all $k \in \mathbb{N}$. By Egorov's theorem, there exists a closed subset F of E such that $|E \setminus F| < \delta$ and $f_k \to f$ uniformly on F. Then, by the uniform convergence theorem,

$$\int_F f_k \longrightarrow \int_F f \tag{49}$$

as $k \to \infty$. But by hypothesis, we have

$$\int_{E\setminus F} |f_k| < \varepsilon. \tag{50}$$

Letting $\varepsilon \to 0$, we achieved the desired convergence. ■

Problem 2.19. Let $I = [0,1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)\,dt$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Proof. By Lusin's theorem, there exists a closed subset F of I with $|I \setminus F| < \varepsilon$ such that the restriction of f to $F = I \setminus E$ is continuous. Now, since F is closed in I and I is compact, it follows that I is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials $\{p_k\}$ such that $p_k \to f$ uniformly on F. Then, by the uniform convergence theorem, we have

$$\int_F p_k \longrightarrow \int_F f \tag{51}$$

so

$$\begin{aligned}
\int_F \left[\int_x^1 t^{-1} p_k(t)dt\right] dx &= \int_F \left[\int_x^1 at^{-1} + q_k(t)dt\right] dx \\
&= \int_F q_k'(x) - a\log(x)\, dx \\
&< \infty
\end{aligned} \tag{52}$$

for all k and converges uniformly to g so $g \in L(I)$. I don't know how to show that in fact $\int_I g = \int_I f$. Perhaps you show that the places where they differ is a set of measure zero. ■

2.5 Final Practice

Problem 2.20. Suppose $f \in L^1(\mathbb{R})$ and that x is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) \coloneqq \frac{1}{r} \int_{B(0,r)} |f(x-y) - f(x)| \, dy.$$

Show that:

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$;
(b) there exists a constant $M > 0$ such that $A(r) \le M$ for all $r > 0$.

Proof. ∎

Problem 2.21. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \le n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0$$

if and only if

$$\|f_k\|_p \longrightarrow \|f\|_p$$

as $k \to \infty$.

Proof. ∎

Problem 2.22. Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

Proof. ∎

Problem 2.23. Let $f \in L(\mathbb{R})$, and let $F(t) \coloneqq \int_{\mathbb{R}} f(x) \cos(tx) dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t \to \infty} F(t) = 0.$$

Proof. ∎

Problem 2.24. Let f be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular. Show that

$$\int_a^b \varphi \, df = \int_a^b \varphi f' dx + \int_a^b \varphi \, dh$$

for all functions φ continuous on $[a, b]$.

Proof. ∎

Bibliography

[1] R. Wheeden, R.L. Wheeden, and A. Zygmund. *Measure and Integral: An Introduction to Real Analysis*. Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1977. ISBN 9780824764999. URL `https://books.google.com/books?id=YDkDmQ_hdmcC`.