MA571 Problem Set 4

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Problem 4.1 (Munkres §20, Ex. 4(a))

Consider the product, uniform, and box topologies on $\mathbf{R}^\omega$.

(a) In which topologies are the following functions from $\mathbf{R}$ to $\mathbf{R}^\omega$ continuous?

$$\begin{aligned} f(t) &= (t, 2t, 3t, ...) \\ g(t) &= (t, t, t, ...) \\ h(t) &= (t, \tfrac{1}{2}t, \tfrac{1}{3}t, ...). \end{aligned}$$

Proof. The maps f, g and h are, evidently, continuous by Theorem 19.6 and the following lemmas (they may be useful in the future so we prove them here):

Lemma 8 (Munkres §18, Ex. 1)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f\colon X \to Y$ is continuous in ε-δ sense. Then f is continuous in the open set sense.*

Proof. Suppose f is continuous in the ε-δ sense, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x_0, x) < \delta$ implies $d_Y(f(x_0), f(x)) < \varepsilon$. Now, let U be an open set in $\mathbf{R}$ and let $x_0 \in f^{-1}(U)$. Since U is open, there exists a real number $\varepsilon > 0$ such that $B_{d_Y}(f(x_0), \varepsilon) \subset U$. Since f is ε-δ continuous, there exists $\delta > 0$ such that $x \in B_{d_X}(x_0, \delta)$ implies $f(x) \in B_{d_Y}(f(x_0), \varepsilon)$ so $B_{d_X}(x_0, \delta) \subset f^{-1}(U)$ (this is because if $x \in B_{d_X}(x_0, \delta)$, then $f(x) \in B_{d_Y}(f(x_0), \varepsilon) \subset U$ so $f(x) \in U$ and in particular $x \in f^{-1}(U)$). Since x_0 was arbitrary, we conclude that $f^{-1}(U)$ is open. ♣

Lemma 9. *Suppose $f, g\colon \mathbf{R} \to \mathbf{R}$ are continuous. Then the following hold*

(i) The sum $(f + g)(x) = f(x) + g(x)$ is continuous.
(ii) The product $fg(x) = f(x)g(x)$ is continuous.

Proof. By Lemma 8, it suffices to show that $f + g$ and fg are continuous in the ε-δ sense: Let $x_0 \in \mathbf{R}$ and let $\varepsilon > 0$ be given.

(i) Since f and g are continuous in the ε-δ sense there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x_0 - x| < \delta_1$ implies $|f(x_0) - f(x)| < \varepsilon/2$ and $|x_0 - x| < \delta_2$ implies $|g(x_0) - g(x)| < \varepsilon/2$ respectively. Take $\delta = \min\{\delta_1, \delta_2\}$. Then, by the triangle inequality (cf. Munkres §20 the definition of a metric in p. 119) we have

$$\begin{aligned} |(f + g)(x_0) - (f + g)(x)| &= |f(x_0) + g(x_0) - f(x) - g(x)| \\ &= |f(x_0) - f(x) + g(x_0) - g(x)| \\ &\le |f(x_0) - f(x)| + |g(x_0) - g(x)| \\ &\le \varepsilon \end{aligned}$$

(ii) Since f and g are continuous in the ε-δ sense, by the triangle inequality we have

$$\begin{aligned}|fg(x_0) - fg(x)| &= |f(x_0)g(x_0) - f(x)g(x)| \\ &= |f(x_0)g(x_0) - f(x_0)g(x) + f(x_0)g(x) - f(x)g(x)| \\ &= |f(x_0)g(x_0) - f(x_0)g(x)| + |f(x_0)g(x) - f(x)g(x)| \\ &= |f(x_0)||g(x_0) - g(x)| + |f(x_0) - f(x)||g(x)|.\end{aligned}$$

To bound this expression, consider the following: Let $\delta_1 > 0$ such that $|f(x_0) - f(x)| < \varepsilon/2$. Since g is continuous, choose $\delta_2 > 0$ such that $|g(x_0) - g(x)| < 1$. Then $g(x) < g(x_0) + 1$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Finally, if choose $\delta_3 > 0$ such that $|g(x_0) - g(x)| < \varepsilon/2f(x_0)$. Then $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ gives a bound to the expression

$$|f(x_0)||g(x_0) - g(x)| + |f(x_0) - f(x)||g(x)| < \varepsilon.$$

Note that if $f(x_0) = 0$, we discard δ_3 and we obtain a stricter bound on our estimates. In any case, fg is continuous. ♣

Corollary. *Polynomials from* $\mathbf{R}$ *to* $\mathbf{R}$ *are continuous.*

Proof of Corollary. It is immediate from Lemma 9(i,ii) and Theorem 18.2(a,b) from Munkres. Here is a sketch: By Theorem 18.2(a) constant functions are continuous, therefore $x \mapsto a_0$ for $a_0 \in \mathbf{R}$ is continuous. By Theorem 18.2(b), the map $x \mapsto x$ is continuous so by Lemma 9(ii), $x \mapsto x^2$ is continuous. By induction on n, $x \mapsto x^n$ is continuous. Similarly, we have that $x \mapsto a_n x^n$ is continuous. Thus, by Lemma 9(i), the map

$$x \longmapsto a_n x^n + \cdots + a_1 x + a_0$$

is continuous. ♣

Now, for the box topology, consider our favorite neighborhood of $\mathbf{0}$ (as seen in Munkres §19, p. 117) given by

$$U = \prod_{n \in \mathbf{Z}_+} \left(-\frac{1}{n}, \frac{1}{n}\right).$$

The set U is clearly open since it is a basis element, by Theorem 19.2. However, the preimage

$$h^{-1}(U) = \bigcap_{n \in \mathbf{Z}_+} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open in $\mathbf{R}$ so h is not open in $\mathbf{R}^\omega$ with the box topology.

Finally, we will show that h is continuous in the ε-δ sense: Given $\varepsilon > 0$ and $x_0 \in \mathbf{R}$, let $\delta = \varepsilon$, then for any $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ we have

$$d_{\bar{\rho}}(h(x_0), h(x)) = |x_0 - x| < \varepsilon.$$

Thus, since h is continuous in the ε-δ sense, by Lemma 8, we have that h is continuous in the open set sense. ■

Problem 4.2 (Munkres §20, Ex. 4(b))

Consider the product, uniform, and box topologies on $\mathbf{R}^\omega$.

(b) In which topologies do the following sequences converge?

$$
\begin{aligned}
\mathbf{w}_1 &= (1,1,1,1,...), & \mathbf{x}_1 &= (1,1,1,1,...),\\
\mathbf{w}_2 &= (0,2,2,2,...), & \mathbf{x}_2 &= (0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},...),\\
\mathbf{w}_3 &= (0,0,3,3,...), & \mathbf{x}_3 &= (0,0,\tfrac{1}{3},\tfrac{1}{3},...),\\
&\vdots & &\vdots\\
\mathbf{y}_1 &= (1,0,0,0,...) & \mathbf{z}_1 &= (1,1,0,0,...),\\
\mathbf{y}_2 &= (\tfrac{1}{2},\tfrac{1}{2},0,0,...) & \mathbf{z}_2 &= (\tfrac{1}{2},\tfrac{1}{2},0,0,...),\\
\mathbf{y}_3 &= (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,...) & \mathbf{z}_3 &= (\tfrac{1}{3},\tfrac{1}{3},0,0,...),\\
&\vdots & &\vdots
\end{aligned}
$$

Proof. By Lemma D (from Prof. McClure's notes) if $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$ and $\{\mathbf{z}_n\}$ converge in the box topology, they converge to $\mathbf{0}$ since they converge to $\mathbf{0}$ in the product topology (and this can be readily seen by applying Problem 3.5 [Munkres §19, Ex. 6]).

However, for the sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ we see that the neighborhood of $\mathbf{0}$ given by

$$U = \prod_{n\in\mathbf{Z}_+}\left(-\frac{1}{n},\frac{1}{n}\right)$$

does not contain any term of either sequence since for any $k \in \mathbf{Z}_+$, the term

$$\mathbf{x}_k = (0,0,...,1/k,1/k,...) \notin (-1,1)\times\cdots\times(-1/k,1/k)\times(-1/(k-1),1/(k-1))\times\cdots.$$

Similarly, we can see that $\mathbf{y}_k$ will not be in U for any k so the sequence $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ will not converge in the box topology.

Although $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ do not converge in the box topology we claim that the sequence $\{\mathbf{z}_n\}$ does converge. To see this it is enough to consider basic open neighborhoods of $\mathbf{0}$. Let $U = \prod(a_n, b_n)$ be a basis element containing $\mathbf{0}$. Then we must show that for N sufficiently big, $\mathbf{x}_n \in U$ for all $n \geq N$. Let $b = \min\{b_1, b_2\}$. Since $b > 0$, by the Archimedean property (Munkres Theorem 4.2), there exists $N \in \mathbf{Z}_+$ such that $1/N < b$. Thus, $\mathbf{z}_n \in U$ for all $n \geq N$ so $\mathbf{z}_n \to \mathbf{0}$ in the box topology. ■

Problem 4.3 (Munkres §20, Ex. 6(b))

Let $\bar{\rho}$ be the uniform metric on $\mathbf{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, x_3, ...) \in \mathbf{R}^\omega$ and given $0 < \varepsilon < 1$, let

$$U(\mathbf{x}, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \times \cdots.$$

(b) Show that $U(\mathbf{x}, \varepsilon)$ is not even open in the uniform topology.

Proof of (b). It is sufficient to find a point $\mathbf{x}_0 \in U(\mathbf{x}, \varepsilon)$ such that $B_{\bar{\rho}}(\mathbf{x}_0, \delta) \not\subset U(\mathbf{x}, \varepsilon)$ for any $\delta > 0$. Let $\mathbf{x}_0$ be the point

$$\mathbf{x}_0 = \prod_{n \in \mathbf{Z}_+} \left(x_n + \left(\frac{n-1}{n} \right) \varepsilon \right).$$

Now consider the open ball $B_{\bar{\rho}}(\mathbf{x}_0, \delta)$ for $\delta > 0$. Now, pick a point $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}_0, \delta)$ given by

$$\mathbf{y} = \prod_{n \in \mathbf{Z}_+} \left(x_n + \left(\frac{n-1}{n} \right) \varepsilon + \frac{\delta}{2} \right).$$

Clearly $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}_0, \delta)$ since

$$\bar{\rho}(\mathbf{x}_0, \mathbf{y}) = \sup_{n \in \mathbf{Z}_+} \{\min\{|x_n - y_n|, 1\}\} = \min\{\delta/2, 1\} \leq \delta/2.$$

However, by the Archimedean property, there exists $k \in \mathbf{Z}_+$ such that $\delta/2 > 1/k$ so $n \geq k$ implies

$$y_n = x_n + \left(\frac{n-1}{n} \right) \varepsilon + \frac{\delta}{2} > x_n + \varepsilon$$

so $\mathbf{y}$ is in $B_{\bar{\rho}}(\mathbf{x}_0, \delta)$ but not in $U(\mathbf{x}, \varepsilon)$. Since δ was arbitrary, we conclude that $U(\mathbf{x}, \varepsilon)$ is not open. ∎

Problem 4.4 (A)

Prove Theorem Q.2 from the notes on Quotient Spaces.

Proof. Recall the statement of the theorem:

Theorem (Theorem Q.2)**.** *A function $f\colon X/{\sim} \to Y$ is continuous if and only if the composite*

$$X \xrightarrow{q} X/{\sim} \xrightarrow{f} Y$$

is continuous.

The direction $\implies$ follows from Theorem 18.2(c) in Munkres.
$\impliedby$ Suppose that the composite

$$X \xrightarrow{q} X/{\sim} \xrightarrow{f} Y$$

is continuous. Then for every open set $U \subset Y$, the preimage $(f \circ q)^{-1}(U)$ is open in X. But the preimage

$$(f \circ q)^{1}(U) = q^{-1}(f^{-1}(U))$$

and since q is a quotient map by definition (cf. Munkres §22, p. 137) $f^{-1}(U)$ is open in $X/{\sim}$ if and only if $q^{-1}(f^{-1}(U))$ is open in X. Thus, the map $f\colon X/{\sim} \to Y$ is continuous. ■

Problem 4.5 (B)

Prove Proposition Q.5 from the notes on Quotient Spaces.

Proof. Recall the definition and the proposition:

Definition. Let X and Y be topological spaces. A map $p\colon X \to Y$ is a *Munkres quotient map* if $\bar{p}\colon X/{\sim_p} \to Y$ is a homeomorphism.

Proposition (Proposition Q.5)**.** *A map $p\colon X \to Y$ satisfies Definition Q.4 if and only if it satisfies the definition at the top of page 137 in Munkres.*

and Munkres's definition:

Definition (Munkres §22, p. 137)**.** Let X and Y be topological spaces; let $p\colon X \to Y$ be a surjective map. The map p is said to be a *quotient map* provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

$\implies$ Now, suppose that $\bar{p}\colon X/{\sim_p} \to Y$ is a homeomorphism. Then $\bar{p}$ is continuous with a continuous inverse $\bar{p}^{-1}\colon Y \to X/{\sim_p}$. Let $q\colon X \to X/{\sim_p}$ bet he map which takes x in X to its equivalence class $[x]$ in $X/{\sim_p}$. Then by Problem 4.5(A), the composite

$$X \xrightarrow{q} X/{\sim_p} \xrightarrow{\bar{p}} Y$$

is continuous if and only if $\bar{p}$ is continuous. Moreover, since $\bar{p}$ is bijective, it is surjective and q is clearly surjective so the map $p = \bar{p} \circ q$ is surjective. Let us prove this claim:

Lemma 10. *Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are surjective maps. Then the composite map $g \circ f\colon X \to Z$ is surjective.*

Proof. Since $g\colon Y \to Z$ is surjective, for every $z \in Z$ there exists a $y \in Y$ such that $g(y) = z$. Similarly, for every $y' \in Y$ there exists a $x' \in X$ such that $f(x') = y'$, in particular there exists a $x \in X$ such that $f(x) = y$. Thus, $g(f(x)) = g \circ f(x) = z$. Since z was arbitrary, we conclude that the composition of surjective maps is again surjective. ♣

Now suppose U is open in Y. Then the preimage $p^{-1}(U) = (\bar{p} \circ q)^{-1}(U) = q^{-1}(\bar{p}^{-1}(U))$

$\impliedby$ Now suppose that $p\colon X \to Y$ is a Munkres quotient map. That is, the map $p\colon X \to Y$ is surjective ∎

Problem 4.6 (C)

Prove Proposition Q.6 from the notes on Quotient Spaces.

Proof. Recall the statement of the proposition:

Proposition (Proposition Q.6)**.** *Let $p\colon X \to Y$ be a Munkres quotient map. A function $f\colon Y \to Z$ is continuous if and only if the composite*

$$X \xrightarrow{p} Y \xrightarrow{f} Z$$

is continuous.

■

Problem 4.7 (D)

(Do not use Problem E to do this problem). Let $\sim$ be the equivalence relation on the interval $[-1, 1]$ defined by $x \sim y$ if and only if $x = y$ or $x = -y$ with $y \in (-1, 1)$ (you do not have to prove that this is an equivalence relation). Prove that $[-1, 1]/\sim$ is not Hausdorff.

Proof. ■

Problem 4.8 (E)

Let X be a topological space with an equivalence relation $\sim$. Suppose that the quotient space $X/\sim$ is Hausdorff.

Prove that the set
$$S = \{\, x \times y \in X \times X \mid x \sim y \,\}$$
is a closed subset of $X \times X$.

Proof. ∎

Problem 4.9 (F)

For problem F you need the following definition: if Y is a topological space and S is a subset of Y, we write Y/S for the quotient space $Y/{\sim}$, where $\sim$ is defined by $x \sim y$ if and only if $x = y$ or $\{x, y\} \subset S$. (Intuitively, Y/S is obtained from Y by collapsing S to a point.)

Let X be a topological space. Let U be an open set in X, and let A be a subset of U. Give U the subspace topology. Let $\iota \colon U/A \to X/A$ be the map which takes $[x]$ to $[x]$ (you do not have to prove that this is well-defined).

(i) Prove that ι is continuous.
(ii) Prove that ι is an open map.

Proof. (i)

(ii) ∎

Problem 4.10 (G)

Let X be a topological space satisfying the first countability axiom (see the bottom of page 130 and the top of page 131). Let $A \subset X$ and let $x \in \overline{A}$. Prove that there is a sequence in A which converges to x (see the top of page 131 for a hint).

Proof. ■