MA 523: Homework 6

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Problem 6.1

For $n = 2$ find Green's function for the quadrant $U = \{ x_1, x_2 > 0 \}$ by repeated reflection.

Solution. Taking the hit, set $x' = (x_1, -x_2)$ and $x'' = (-x_1, x_2)$ and define

$$\varphi^x(y) := \Phi(x - y) + \Phi(x' - y) - \Phi(x'' - y). \tag{6.1}$$

We claim that φ^x, as defined above, solves

$$\begin{cases} \Delta\varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(x - y) & \text{on } \partial U. \end{cases}$$

■

Problem 6.2

(Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B(0,r) = \{ x \in \mathbb{R}^n : |x| < r \}$.

Solution. ■

Problem 6.3

Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$\begin{aligned} P_k(\lambda x) &= \lambda^k P_k(x), & P_m(\lambda x) &= \lambda^m P_m(x) & &\text{for every } x \in \mathbb{R}^n,\ \lambda > 0, \\ \Delta P_k &= 0, & \Delta P_m &= 0 & &\text{in } \mathbb{R}^n. \end{aligned}$$

(a) Show that

$$\frac{\partial P_k}{\partial \nu} = k P_k(x), \qquad \frac{\partial P_m}{\partial \nu} = m P_m(x) \qquad \text{on } \partial B(0,1)$$

where $B(0,1) = \{ x \in \mathbb{R}^n : |x| < 1 \}$ and ν is the outward normal on $\partial B(0,1)$.

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x)\, d\sigma = 0, \qquad \text{if } k \neq m.$$

Solution. ■