MA 523: Homework 8

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November 9, 2016

CARLOS SALINAS PROBLEM 8.1

Problem 8.1

Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t)$$

where

$$\Phi(x,t) = \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}}$$

is positive for |x| < 1, t > 0.

(*Hint*: Show that u satisfies $u_t = u_{xx}$ for t > 0,

$$\begin{cases} u = 0 & \text{on } \{ |x| = 1 \} \times \{ t \ge 0 \}, \\ u = \delta_0 & \text{on } \{ |x| = 1 \} \times \{ t = 0 \}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{|x| \le 1\} \times \{\varepsilon \le t \le T\}$ for small $\varepsilon > 0$ and large T > 0 pass to the limit as $\varepsilon \to 0^+$ and $T \to \infty$.)

SOLUTION. Taking the hint, let us verify that $u_t = u_{xx}$, for t > 0. By direct computation, we have

$$\Phi_{x}(x,t) = \frac{\partial}{\partial x} \left(\frac{e^{-x^{2}/(4t)}}{\sqrt{4\pi t}} \right) \qquad \qquad \Phi_{xx}(x,t) = \frac{\partial}{\partial x} \left(-\frac{xe^{-x^{2}/(4t)}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \right) \\
= -\frac{xe^{-x^{2}/(4t)}}{2\sqrt{4\pi}t^{\frac{3}{2}}}, \qquad \qquad \qquad = \frac{x^{2}e^{-x^{2}/(4t)}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{e^{-x^{2}/(4t)}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\
= \frac{(x^{2} - 2t)e^{-x^{2}/(4t)}}{4\sqrt{4\pi}t^{\frac{5}{2}}},$$

and

$$\Phi_t(x,t) = \frac{\partial}{\partial t} \left(\frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}} \right)$$

$$= \frac{x^2 e^{-x^2/(4t)}}{4\sqrt{4\pi t}^{\frac{5}{2}}} - \frac{e^{-x^2/(4t)}}{2\sqrt{4\pi t}^{\frac{3}{2}}}$$

$$= \frac{(x^2 - 2t)e^{-x^2/(4t)}}{4\sqrt{4\pi t}^{\frac{5}{2}}}.$$

Since $\Phi_t = \Phi_{xx}$ it follows that $u_t = u_{xx}$ (for t > 0).

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Next we show that u=0 on $\{|x|=1\} \times \{t \geq 0\}$ and $u=\delta_0$ on $\{|x|=1\} \times \{t=0\}$. To show u=0 fix a $t\geq 0$ and, after relabeling if necessary, assume that x=1 which gives us

$$u(1,t) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-k^2/(4t)}}{\sqrt{4\pi t}}$$
$$= \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} (-1)^k (e^{-1/(4t)})^{k^2}$$
$$= \frac{2}{\sqrt{4\pi t}} \sum_{k=0}^{\infty} (-1)^k (e^{-1/(4t)})^{k^2}$$

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PROBLEM 8.2 (TIKHONOV'S EXAMPLE)

Let

$$g(t) := \begin{cases} e^{-t^2} & t > 0, \\ 0 & t \le 0. \end{cases}$$

Then $g \in C^{\infty}(\mathbb{R})$ and we define

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that u(x,t) solves the heat equation in $\mathbb{R} \times (0,\infty)$ with the initial condition u(x,0) = 0, $x \in \mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

Solution. Let u be as above. Then

$$u_t(x,t) = \frac{\partial}{\partial t} \left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right)$$
$$= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k}$$
$$= \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2},$$

and

$$u_{x}(x,t) = \frac{\partial}{\partial x} \left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right) \qquad u_{xx}(x,t) = \frac{\partial}{\partial x} \left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1} \right)$$

$$= \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2kx^{2k-1} \qquad \qquad = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} (2k-1)x^{2k-2} + \frac{\partial}{\partial x} g^{(0)}(t)$$

$$= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}, \qquad \qquad = \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}.$$

Thus, $u_t - \Delta u = 0$; i.e., u solves the heat equation.

CARLOS SALINAS PROBLEM 8.3

Problem 8.3

Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax) e^{-x^2} dx, \qquad (a > 0).$$

(*Hint:* Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.)

SOLUTION. Write

$$\cos(ax) = \frac{1}{2} \left(e^{iax} + e^{-iax} \right).$$

Then,

$$\cos(ax)e^{-x^2} = \frac{1}{2}(e^{iax-x^2} + e^{-iax-x^2})$$

so

$$\int_{-\infty}^{\infty} \cos(ax) e^{-x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \left(e^{iax - x^2} + e^{-iax - x^2} \right) dx$$

$$=$$