

## MA 544: Homework 12

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**PROBLEM 12.1 (WHEEDEN & ZYGMUND §8, EX. 2)**

Prove the converse of Hölder's inequality for  $p = 1$  and  $\infty$ . Show also that for  $1 \leq p \leq \infty$ , a real-valued measurable  $f$  belongs to  $L^p(E)$  if  $fg \in L^1(E)$  for every  $g \in L^{p'}(E)$ ,  $1/p + 1/p' = 1$ . The negation is also of interest: if  $f \in L^p(E)$  then there exists  $g \in L^{p'}(E)$  such that  $fg \notin L^1(E)$ . (To verify the negation, construct  $g$  of the form  $\sum a_k g_k$  satisfying  $\int_E fg_k \rightarrow \infty$ .)

*Proof.* Recall the statement of Hölder's inequality: If  $1 \leq p \leq \infty$  and  $1/p + 1/p' = 1$ , then  $\|fg\|_1 \leq \|f\|_p \|g\|_{p'}$ .

To prove the converse of Hölder's inequality, we must show that if  $\|fg\|_1 \leq \|f\|_p \|g\|_{p'}$  for all  $f, g$  ■

**PROBLEM 12.2 (WHEEDEN & ZYGMUND §8, EX. 3)**

Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when  $p < 1$ .

*Proof.*



**PROBLEM 12.3 (WHEEDEN & ZYGMUND §8, EX. 4)**

Let  $f$  and  $g$  be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let  $1 < p < \infty$ . Prove that equality holds in the inequality  $|\int fg| \leq \|f\|_p \|g\|_{p'}$  if and only if  $fg$  has constant sign a.e. and  $|f|^p$  is a multiple of  $|g|^{p'}$  a.e.

If  $\|f + g\|_p = \|f\|_p + \|g\|_p$  and  $g \neq 0$  in Minkowski's inequality, show that  $f$  is a multiple of  $g$ .

Find analogues of these results for the spaces  $\ell^p$ .

*Proof.*

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**PROBLEM 12.4 (WHEEDEN & ZYGMUND §8, EX. 5)**

For  $0 < p \leq \infty$  and  $0 < |E| < \infty$ , define

$$N_p[f] := \left( \frac{1}{|E|} \int_E |f|^p \right)^{1/p},$$

where  $N_\infty[f]$  means  $\|f\|_\infty$ . Prove that if  $p_1 < p_2$ , then  $N_{p_1}[f] \leq N_{p_2}[f]$ . Prove also that if  $1 \leq p \leq \infty$ , then  $N_p[f + g] \leq N_p[f] + N_p[g]$ ,  $(1/|E|) \int_E |fg| \leq N_p[f]N_{p'}[g]$ ,  $1/p + 1/p' = 1$ , and  $\lim_{p \rightarrow \infty} N_p[f] = \|f\|_\infty$ . Thus,  $N_p$  behaves like  $\|\cdot\|_p$  but has the advantage of being monotone in  $p$ . Recall Exercise 28 of Chapter 5.

*Proof.*

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**PROBLEM 12.5 (WHEEDEN & ZYGMUND §8, EX. 6)**

- (a) Let  $1 \leq p_i, r \leq \infty$  and  $\sum_{i=1}^k 1/p_i = 1/r$ . Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

- (b) Let  $1 \leq p < r < q \leq \infty$  and define  $\theta \in (0, 1)$  by  $1/r = \theta/p + (1 - \theta)/q$ . Prove the interpolation estimate

$$\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}.$$

In particular, if  $A := \max\{\|f\|_p, \|f\|_q\}$ , then  $\|f\|_r \leq A$ .

*Proof.*

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**PROBLEM 12.6 (WHEEDEN & ZYGMUND §8, EX. 9)**

If  $f$  is real-valued and measurable on  $E$ ,  $|E| > 0$ , define its essential infimum on  $E$  by

$$f := \sup\{\alpha : |\{x \in E : f(x) < \alpha\}| = 0\}.$$

If  $f \geq 0$ , show that  ${}_E f = (1/f)^{-1}$ .

*Proof.*

■



**PROBLEM 12.7 (WHEEDEN & ZYGMUND §8, EX. 11)**

If  $f_k \rightarrow f$  in  $L^p$ ,  $1 \leq p < \infty$ ,  $g_k \rightarrow g$  pointwise, and  $\|g_k\|_\infty < M$  for all  $k$ , prove that  $f_k g_k \rightarrow fg$  in  $L^p$ .

*Proof.*

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