

MA 523: Homework 1

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PROBLEM 1.1 (TAYLOR'S FORMULA)

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$$

as $x \rightarrow 0$ for each $k = 1, 2, \dots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m .

Solution. ► Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Proof of claim. We shall proceed by induction on m . The case $m = 1$ follows easily from the chain rule:

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\ &= D^{(1,0,\dots,0)} f(tx) x_1 + \dots + D^{(0,\dots,0,1)} f(tx) x_n \\ &= (D^{(1,0,\dots,0)} x_1 + \dots + D^{(0,\dots,0,1)} x_n) f(tx) \end{aligned}$$

which we can write compactly as

$$= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Now, assume the result for $n \leq m - 1$. Then

$$\begin{aligned} \frac{d^m}{dt^m} g(t) &= \frac{d}{dt} \left[\frac{d^{m-1}}{dt^{m-1}} g(t) \right] \\ &= \frac{d}{dt} \left[\sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} D^\alpha f(tx) x^\alpha \right] \end{aligned}$$

since the derivative is a linear operator, we have

$$\begin{aligned} &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \frac{d}{dt} [D^\alpha f(tx) x^\alpha] \\ &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \sum_{|\beta|=1} D^{\alpha+\beta} f(tx) x^{\alpha+\beta} \end{aligned}$$

but since f is smooth, the order in which we take derivatives does not matter and, hence the operators commute giving us

$$= \left[\sum_{|\beta|=1} (Dx)^\beta \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha \right] f(tx). \quad (1.1)$$

From here it suffices to do some combinatorics on the operators and reduce it to the desired expression. By the multinomial theorem, we have

$$\left(\sum_{|\alpha'|=1} (Dx)^{\alpha'} \right)^{m-1} = \sum_{|\alpha|=m-1} \binom{|\alpha|}{\alpha} (Dx)^\alpha = \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha.$$

Thus (1.1) becomes

$$\begin{aligned} \left[\sum_{|\beta|=1} (Dx)^\beta \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^\alpha \right] f(tx) &= \left[\sum_{|\beta|=1} (Dx)^\beta \left(\sum_{|\alpha'|=1} (Dx)^{\alpha'} \right)^{m-1} \right] f(tx) \\ &= \left[\left(\sum_{|\beta|=1} (Dx)^\beta \right)^m \right] f(tx) \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} (Dx)^\alpha f(tx) \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha, \end{aligned}$$

as desired. □

Now, applying Taylor's formula in 1 variable to $g(t)$ and evaluating at $t = 1$ we have

$$\begin{aligned} f(x) &= g(1) \\ &= \sum_{i=0}^k \frac{g^{(i)}(0)}{i!} 1^i + O(|x|^{k+1}) \\ &= \sum_{i=0}^k \frac{1}{i!} \sum_{|\alpha|=i} \frac{i!}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}) \\ &= \sum_{i=0}^k \sum_{|\alpha|=i} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}) \\ &= \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}) \end{aligned}$$

as desired. ◀

PROBLEM 1.2

Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \quad (*)$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve (*) subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ► For reference, formula (5) in §2.1.2 of [E] is the solution to the nonhomogeneous problem

$$u(\mathbf{x}, t) = g(\mathbf{x} - t\mathbf{b}) + \int_0^t f(\mathbf{x} + (s - t)\mathbf{b}, s) ds$$

where $\mathbf{x} \in \mathbb{R}^n$, $t > 0$.

To make the notation more bearable, we will use \mathbf{b} and \mathbf{x} to denote the original vectors in (*). First, we write (*) as the directional derivative along $(\mathbf{b}, 1)$, (note the abuse of notation)

$$\begin{aligned} f &= u_t + \mathbf{b} \cdot Du \\ &= (\mathbf{b}, 1) \cdot Du. \end{aligned}$$

Using the structure of characteristic ODE, we have the PDE

$$F(p, z, x) = (\mathbf{b}, 1) \cdot p$$

with characteristics

$$\dot{x} = b, \quad \dot{t} = 1, \quad \dot{z} = f.$$

Now, given a point $(\mathbf{x}, t) \in \mathbb{R}^n \times (0, \infty)$ we can solve the ODEs \dot{x} and \dot{t} easily as the lines $x(s) = \mathbf{x} - \mathbf{b}t + \mathbf{b}s$ and $t = s$. Substituting these solutions into \dot{z} , we have

$$\dot{z} = f(x(s), t(s)) = f(\mathbf{x} + \mathbf{b}(s - t), s)$$

so

$$\begin{aligned} \int_0^t f(x(s), t(s)) ds &= \int_0^t f(\mathbf{x} + \mathbf{b}(s - t), s) ds = \int_0^t \dot{z} ds \\ &= z(t) - z(0) \\ &= u(\mathbf{x}, t) - u(\mathbf{x} - \mathbf{b}t, 0). \end{aligned}$$

Thus,

$$\begin{aligned} u(\mathbf{x}, t) &= u(\mathbf{x} - \mathbf{b}t, 0) + \int_0^t f(x(s), t(s)) ds = f(\mathbf{x} + \mathbf{b}(s - t), s) ds \\ &= g(\mathbf{x} - \mathbf{b}t) + \int_0^t f(x(s), t(s)) ds = f(\mathbf{x} + \mathbf{b}(s - t), s) ds \end{aligned}$$

as desired. ◀

PROBLEM 1.3

Solve using the characteristics:

- (a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.
- (b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.
- (c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ► For part (a), employing the method of characteristics, we write

$$F(p, z, x) = (x_1^2, x_2^2) \cdot p = z^2.$$

From here, we have

$$\dot{x} = (x_1^2, x_2^2), \quad \dot{z} = z^2.$$

Now fix $\mathbf{x} = (2t, t)$ on the line $x_2 = 2x_1$ and solve for \dot{x} ,

$$\begin{aligned} \int_0^{2t} ds &= \int_0^{2t} \frac{1}{x_1(s)^2} dx_1(s) & \int_0^t ds &= \int_0^t \frac{1}{x_1(s)^2} dx_1(s) \\ 2t &= \frac{1}{x_1(2t)} - \frac{1}{x_1(0)} & t &= \frac{1}{x_2(s)} - \frac{1}{x_2(0)}. \end{aligned}$$

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PROBLEM 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with $u(x_1, 0) = (1 - x_1^2)/2$.

Solution. ►

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