MA 572: Homework 4

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Problem 4.1 (Hatcher §2.1, Ex. 20)

Show that $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Proof. First note that the reduced suspension of X, ΣX, which is homotopy equivalent to SX, can be realized as the quotient space CX/X. Given the imbedding $X \hookrightarrow CX$, by 2.16 and excision (or 2.22) we have the long exact sequence in

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_n(X) \arrow[r] & \widetilde{H}_n(CX) \arrow[r] & \widetilde{H}_n(CX,X) \arrow[dll, out=-5, in=175] \\ & \widetilde{H}_{n-1}(X) \arrow[r] & \widetilde{H}_{n-1}(CX) \arrow[r] & \widetilde{H}_{n-1}(CX,X) \arrow[r] & \cdots, \end{tikzcd} \tag{1}$$

where $\widetilde{H}_n(CX,X) \cong \widetilde{H}_n(CX/X) \cong \widetilde{H}_n(SX)$. Since CX is contractible, we have $H_n(CX) = 0$ for all n and the long exact sequence (1) yields an isomorphism

$$\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X).$$

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PROBLEM 4.2 (HATCHER §2.1, EX. 22)

Prove by induction on the dimension the following facts about the homology of a finite dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:

(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.

(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.

(c) If X is has k n-cells, then $H_n(X)$ is generated by at most k elements.

Proof. To make Hatcher's comment more precise, if $X = \bigcup X^k$ is a finite-dimensional CW-complex, by excision the relative homology

$$H_n\big(X^k, X^{k-1}\big) \cong \widetilde{H}_n\big(X^k/X^{k-1}\big) \cong \widetilde{H}_n\big(\textstyle\bigvee_{\alpha_k} S^k\big) = \displaystyle\bigoplus_{\alpha_k} \widetilde{H}_n\big(S^k\big) = \begin{cases} \bigoplus_{\alpha_k} \mathbf{Z} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where α_k is the index over the k-dimensional cells of X.

(a) By induction on d, if $d = 0$ then X^0 is a collection of points.

(b)

(c) ∎

Problem 4.3 (Hatcher §2.2, Ex. 2)

Given a map $f\colon S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbf{R}\mathrm{P}^{2n} \to \mathbf{R}\mathrm{P}^{2n}$ has a fixed point. Construct maps $\mathbf{R}\mathrm{P}^{2n-1} \to \mathbf{R}\mathrm{P}^{2n-1}$ without fixed points from linear transformations $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ without eigenvectors.

Proof. ∎