MA 519: Homework 1

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Problem 1.1 (Handout 1, # 5 [Feller Vol. 1])

A closet contains five pairs of shoes. If four shoes are selected at random, what is the probability that there is at least one complete pair among the four?

Solution. ▸ First, since the closet contains 5 pairs of shoes, it contains, in total, 10 shoes. Now, let us count the number of points in the sample space: since we are selecting 4 shoes out of 10 and the order does not matter, the cardinality of the sample space Ω is

$$\#\Omega = \binom{10}{4} = \frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{4 \cdot 3 \cdot 2 \cdot 6!} = 10 \cdot 3 \cdot 7 = 210 \tag{1.1}$$

total sample points. Now, which of these points are actually ones we care about? Let A denote the event that at least 1 complete pair of shoes is among the 4. Then we can split A (as a set) as the union of

$$A_1 := \{\text{exactly 1 pair is among the 4}\}$$

and

$$A_1 := \{\text{exactly 2 pairs are among the 4}\}.$$

We can count the number of points in A_2 easily enough these are

$$\#A_2 = \binom{5}{2} = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3!}{2 \cdot 3!} = 5 \cdot 2 = 10, \tag{1.2}$$

since we are not taking into consideration the order in which we select the pair.

To count the points in A_1 we observe that there are 5 pairs to choose from and for the remaining two shoes we must choose one shoe (either a right or a left) from the remaining 4 pairs which leaves $7 - 1 = 6$ other shoes to choose from; *i.e.* there are exactly

$$5 \cdot 4 \cdot 6 = 120 \tag{1.3}$$

ways to choose one pair. Taking (1.2) and (1.3), the probability that there is at least one complete pair among the four is

$$\#A_1 = p(A) = p(A_1) + p(A_2) = \frac{120}{210} + \frac{10}{210} = \frac{130}{210} \approx 0.6190.$$

◂

Problem 1.2 (Handout 1, # 7 [Feller Vol. 1])

A gene consists of 10 subunits, each of which is normal or mutant. For a particular cell, there are 3 mutant and 7 normal subunits. Before the cell divides into 2 daughter cells, the gene duplicates. The corresponding gene of cell 1 consists of 10 subunits chosen from the 6 mutant and 14 normal units. Cell 2 gets the rest. What is the probability that one of the cells consists of all normal subunits.

Solution. ► First, let A denote the event that one of the cells contains all normal units. Like Problem 1.1, we can reduce the problem of finding the probability of A to finding the probability of

$$A_1 := \left\{ \text{cell 1 consists of all normal subunits} \right\}$$

and

$$A_2 := \left\{ \text{cell 1 contains 6 mutant cells} \right\}$$

and taking their sum.

Now, let us count the number of points in our sample space Ω. Assuming the configuration of the subunits in a gene does not matter, we have

$$\#\Omega = \binom{20}{10} = 184756 \tag{1.4}$$

sample points.

Now we count the number of points in A_1 and A_2 these are: for A_1 we choose 10 subunits from among the 14 normal subunits giving us

$$\#A_1 = \binom{14}{10} = 1001 \tag{1.5}$$

sample points. For A_2, we must choose all 6 mutant subunits leaving 4 choices from among the 14 normal subunits giving us

$$\#A_1 = \binom{14}{4} = 1001. \tag{1.6}$$

Thus, we have

$$p(A) = p(A_1) + p(A_2) = \frac{1001}{184756} + \frac{1001}{184756} \approx 0.01083.$$

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Problem 1.3 (Handout 1, # 9 [Feller Vol. 1])

From a sample of size n, r elements are sampled at random. Find the probability that none of the N prespecified elements are included in the sample, if sampling is

(a) with replacement;

(b) without replacement.

Compute it for $r = N = 10$, $n = 100$.

Solution. ▸ Assuming there is no restriction on the prespecified elements, the order in which we choose r elements does not matter. Moreover, for this question to make any sense we must have $N \leq r$.

For part (a), with replacement, the number of points in the sample space Ω_a is given by the expression

$$\#\Omega_a = \frac{n^r}{r!}. \tag{1.7}$$

Now, let N be a sequence of prespecified elements.

For part (b), without replacement, the number of points in the sample space $\Omega_b$is given by the expression

$$\#\Omega_b = n!. \tag{1.8}$$

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Problem 1.4 (Handout 1, # 11 [Text 1.3])

A telephone number consists of ten digits, of which the first digit is one of $1, 2, \ldots, 9$ and the others can be $0, 1, 2, \ldots, 9$. What is the probability that 0 appears at most once in a telephone number, if all the digits are chosen completely at random?

Solution. ► First, since order matters, the number of points in the sample space Ω is

$$\#\Omega = 9 \cdot 10^9. \tag{1.9}$$

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Problem 1.5 (Handout 1, # 12 [Text 1.6])

Events A, B and C are defined in a sample space Ω. Find expressions for the following probabilities in terms of $P(A)$, $P(B)$, $P(C)$, $P(AB)$, $P(AC)$, $P(BC)$ and $P(ABC)$; here AB means $A \cap B$, etc.:

(a) the probability that exactly two of A, B, C occur;

(b) the probability that exactly one of these events occur;

(c) the probability that none of these events occur.

Solution. ►

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Problem 1.6 (Handout 1, # 13 [Text 1.8])

Mrs. Jones predicts that if it rains tomorrow it is bound to rain the day after tomorrow. She also thinks that the chance of rain tomorrow is $1/2$ and that the chance of rain the day after tomorrow is $1/3$. Are these subjective probabilities consistent with the axioms and theorems of probability?

Solution. ►

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Problem 1.7 (Handout 1, # 16)

Consider a particular player, say North, in a Bridge game. Let X be the number of aces in his hand. find the distribution of be the number of aces in his hand. find the distribution of X.

Solution. ▸

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Problem 1.8 (Handout 1, # 20)

If 100 balls are distributed completely at random into 100 cells, find the expected value of the number of empty cells.

Replace 100 by n and derive the general expression. Now approximate it as n tends to ∞.

Solution. ► ◄