MA544: Qual Preparation

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Prof. Bañuelos, este es otro problema con el que no he podido avanzar.

Problem 1. Let $f_n : X \to [0, \infty)$ be a sequence of measurable functions on the measure space $(X, \mathscr{F}, \mu)$. Suppose there is a positive constant M such that the functions $g_n(x) = f_n(x)\chi_{\{ f_n \le M \}}(x)$ satisfy $\|g_n\|_1 \le An^{-4/3}$ and for which $\mu \{ x \in X : f_n(x) > M \} \le Bn^{-5/4}$, where A and B are positive constants independent of n. Prove that

$$\sum_{n=1}^{\infty} f_n < \infty$$

almost everywhere.

Solution. ► Let

$$E = \left\{ x \in X : \sum_{n \in \mathbb{N}} f_n(x) = \infty \right\}.$$

We must show that for every $\varepsilon > 0$, $\mu(E) < \varepsilon$, i.e., E is a set of measure zero.

Seeking a contradiction, suppose that $\mu(E) > 0$. We know that

$$\mu \{ f_n > M \} \le \frac{B}{n^{5/4}}$$

and that

$$\|g_n\|_1 = \int_{\{ f_n \le M \}} f_n(x) \, \mathrm{d}x \le \frac{A}{n^{4/3}}.$$

Take $\mathrm{re}(z)$, $\mathrm{im}(z)$ ◄