

# MA571 Homework 10

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**PROBLEM 10.1 (MUNKRES §52, EX. 2)**

Let  $\alpha$  be a path in  $X$  from  $x_0$  to  $x_1$ ; let  $\beta$  be a path in  $X$  from  $x_1$  to  $x_2$ . Show that if  $\gamma = \alpha * \beta$ , then  $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$ .

*Proof.* By Theorem 52.1, the paths  $\alpha$  and  $\beta$  induce a group homomorphism  $\hat{\alpha}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$  and  $\hat{\beta}: \pi_1(X, x_1) \rightarrow \pi_1(X, x_2)$ , respectively. We want to show therefore that the induced homomorphism  $\hat{\gamma} = \widehat{\alpha * \beta}$  is in fact equivalent to the composition  $\hat{\beta} \circ \hat{\alpha}$ . Let  $[f]$  be a loop based at  $x_0$  then

$$\begin{aligned}\hat{\gamma}([f]) &= \widehat{\alpha * \beta}([f]) \\ &= [\overline{\alpha * \beta}] * [f] * [\alpha * \beta] \\ &= [\bar{\beta} * \bar{\alpha}] * [f] * [\alpha] * [\beta]\end{aligned}$$

by the well-definedness of the path product operation, we have

$$= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta]$$

by associativity of the path product,

$$\begin{aligned}&= [\bar{\beta}] * ([\bar{\alpha}] * [f] * [\alpha]) * [\beta] \\ &= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta]\end{aligned}$$

where  $\alpha([f])$  is a loop based at  $x_1$  so

$$\begin{aligned}&= \hat{\beta}(\hat{\alpha}([f])) \\ &= (\hat{\beta} \circ \hat{\alpha})([f]).\end{aligned}$$

Thus, the following diagram commutes

$$\begin{array}{ccc}\pi_1(X, x_0) & \xrightarrow{\hat{\alpha}} & \pi_1(X, x_1) \\ & \searrow \hat{\gamma} = \widehat{\alpha * \beta} & \downarrow \hat{\beta} \\ & & \pi_1(X, x_2).\end{array}$$

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**PROBLEM 10.2 (MUNKRES §52, EX. 3)**

Let  $x_0$  and  $x_1$  be points of the path-connected space  $X$ . Show that  $\pi_1(X, x_0)$  is Abelian if and only if for every pair  $\alpha$  and  $\beta$  of paths from  $x_0$  to  $x_1$ , we have  $\hat{\alpha} = \hat{\beta}$ .

*Proof.*  $\implies$  Suppose that  $\pi_1(X, x_0)$  is Abelian. Then for any class of loops about  $x_0$ , say  $[f]$  and  $[g]$ , the product  $[f] * [g] = [g] * [f]$ . Let  $\alpha$  and  $\beta$  be paths from  $x_0$  to  $x_1$ . Then the induced map on fundamental groups  $\hat{\alpha}$  and  $\hat{\beta}$  yield isomorphism by Theorem 52.1 so that the map  $\hat{\beta} \circ \hat{\alpha}$  is an automorphism of  $\pi_1(X, x_0)$ . Moreover, we have

$$\begin{aligned}\hat{\beta} \circ \hat{\alpha}([f]) &= \hat{\beta}(\hat{\alpha}([f])) \\ &= \hat{\beta}([\bar{\alpha}] * [f] * [\alpha]) \\ &= [\beta] * ([\bar{\alpha}] * [f] * [\alpha]) * [\bar{\beta}]\end{aligned}$$

by associativity of the path product, we may rewrite the above expression as

$$= ([\beta] * [\bar{\alpha}]) * [f] * ([\alpha] * [\bar{\beta}])$$

noting that  $[\beta] * [\bar{\alpha}]$  and  $[\alpha] * [\bar{\beta}]$  are loops based at  $x_0$ , since  $\pi_1(X, x_0)$  is Abelian, we have

$$\begin{aligned}&= ([\beta] * [\bar{\alpha}]) * ([\alpha] * [\bar{\beta}]) * [f] \\ &= [e_{x_0}] * [f] \\ &= [f].\end{aligned}$$

Thus,  $\hat{\beta} \circ \hat{\alpha} = \text{id}_{\pi_1(X, x_0)}$ , i.e.,  $\hat{\alpha} = \hat{\beta}$ .

$\Leftarrow$  Let  $f$  and  $g$  be loops about  $x_0$ . Then, since  $X$  is path connected, we claim that  $f$  and  $g$  are homotopic to the path product  $\alpha_1 * \bar{\beta}_1$  and  $\alpha_2 * \bar{\beta}_2$  where  $\alpha_i, \beta_i$  are paths from  $x_0$  to  $x_1$ . More precisely, split  $f$  into the paths  $f_1 = f(t/2)$  and  $f_2 = f((t+1)/2)$ ; it is clear that  $f = f_1 * f_2$ . Let  $x_2 := f_1(1)$  then there exists a path  $\alpha$  from  $x_2$  to  $x_1$  since  $X$  is path connected. Now we claim that the following

$$H(x, t) := f_1(x) * \alpha(tx) * \bar{\alpha}((1-t)x) * f_2(x)$$

is a homotopy from  $f = f_1 * f_2$  to the extended loop  $\tilde{f} = f_1 * \alpha * \bar{\alpha} * f_2$ . It is clear that  $H$  is continuous since it is a path products and multiplication on the interval  $I$ ,  $tx$ , is continuous. Lastly,  $H(x, 0) = f_1(x) * \alpha(0) * \bar{\alpha}(0) * f_2(x)$  and  $H(x, 1) = f_1(x) * \alpha(x) * \bar{\alpha}(x) * f_2(x)$ . Now  $\blacksquare$

**PROBLEM 10.3 (MUNKRES §52, EX. 4)**

Let  $A \subset X$ ; suppose  $r: X \rightarrow A$  is continuous map such that  $r(a) = a$  for each  $a \in A$ . (The map  $r$  is called a *retraction* of  $X$  onto  $A$ .) If  $a_0 \in A$ , show that

$$r_*: \pi_1(X, x_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

*Proof.*



**PROBLEM 10.4 (MUNKRES §53, EX. 6)**

Show that if  $X$  is path connected, the homomorphism induced by a continuous map is independent of the base point, up to isomorphisms of the groups involved. More precisely, let  $h: X \rightarrow Y$  be continuous, with  $h(x_0) = y_0$  and  $h(x_1) = y_1$ . Let  $\alpha$  be a path in  $X$  from  $x_0$  to  $x_1$ , and let  $\beta = h \circ \alpha$ . Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps “commutes”

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \hat{\alpha} \downarrow & & \downarrow \hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1). \end{array}$$

*Proof.*

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**PROBLEM 10.5 (MUNKRES §55, EX. 1)**

Show that if  $A$  is a retract of  $B^2$ , then every continuous map  $f: A \rightarrow A$  has a fixed point.

*Proof.*

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**PROBLEM 10.6 (MUNKRES §55, EX. 2)**

Show that if  $h: S^1 \rightarrow S^1$  is nullhomotopic, then  $h$  has a fixed point and  $h$  maps some point  $x$  to its antipode  $-x$ .

*Proof.*

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**PROBLEM 10.7 ((A))**

Prove that every  $m$ -manifold is locally path-connected.

*Proof.*

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**PROBLEM 10.8 ((B))**

Prove that every  $m$ -manifold is regular.

*Proof.*

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**PROBLEM 10.9 ((C))**

Prove that there is no 1-1 continuous function  $\iota: S^1 \rightarrow \mathbf{R}$ . You may assume any fact about trigonometric functions. (Note: this shows in particular that there is no  $\iota: S^1 \rightarrow \mathbf{R}$  with  $p \circ \iota$  equal to the identity map, where  $p$  is the map in the note on the Fundamental Group of the Circle.)

*Proof.*

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**PROBLEM 10.10 ((D))**

Prove Proposition C from the note on the Fundamental Group of the Circle.

*Proof.*

