MA544: Qual Problems

Carlos Salinas

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1 MA 544 Spring 2016

1.1 Exam 1 Prep

Problem 1.1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{ r\mathbf{x} \mid \mathbf{x} \in E \}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Define a linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by $\mathbf{x} \mapsto r\mathbf{x}$. Using the standard basis for $\mathbb{R}^n$, this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \tag{1}$$

which has determinant $\det T = r^n$. By 3.35, we have $|E| = |T(E)| = r^n|E| = |rE|$. ∎

Problem 1.2. Let $\{E_k\}$, $k \in \mathbb{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k\to\infty} E_k \right| \leq \liminf_{k\to\infty} |E_k|.$$

Proof. ∎

Problem 1.3. Let $E \subset \mathbb{R}^n$ be a measurable set, with $|E| = \infty$. Show that for any $C > 0$ there exists a measurable set $F \subset E$ such that $C < |F| < \infty$.

Proof. ∎

Problem 1.4. Consider the function

$$F(\mathbf{x}) := \begin{cases} |B(x, 0)| & \mathbf{x} > 0 \\ 0 & \mathbf{x} = 0 \end{cases}.$$

Here $B(r, 0) := \{ \mathbf{y} \in \mathbb{R}^n \mid |\mathbf{y}| < r \}$. Prove that F is monotonic increasing and continuous.

Proof. ∎

Problem 1.5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. ∎

Problem 1.6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Proof. ∎

Problem 1.7. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set $\{\, x \mid \lim_{k\to\infty} f_k(x) \text{ exists} \,\}$ is measurable.

Proof. ∎

Problem 1.8. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, one has $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. ∎

Problem 1.9. Let f be a continuous function from $[a, b]$ into $\mathbb{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, i.e., $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f \,\mathrm{d}\,\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(a) & \text{if } c = b \end{cases} .$$

Proof. ∎