MA571 Problem Set 4

Carlos Salinas

September 21, 2015

Problem 4.1 (Munkres §20, Ex. 4(a))

Consider the product, uniform, and box topologies on $\mathbf{R}^\omega$.

(a) In which topologies are the following functions from $\mathbf{R}$ to $\mathbf{R}^\omega$ continuous?

$$\begin{aligned} f(t) &= (t, 2t, 3t, \ldots) \\ g(t) &= (t, t, t, \ldots) \\ h(t) &= (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots). \end{aligned}$$

Proof. The maps f, g and h are, evidently, continuous by Theorem 19.6 and the following lemmas (they may be useful in the future so we prove them here):

Lemma 8 (Munkres §18, Ex. 1)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f\colon X \to Y$ is continuous in ε-δ sense. Then f is continuous in the open set sense.*

Proof. Suppose f is continuous in the ε-δ sense, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x_0, x) < \delta$ implies $d_Y(f(x_0), f(x)) < \varepsilon$. Now, let U be an open set in $\mathbf{R}$ and let $x_0 \in f^{-1}(U)$. Since U is open, there exists a real number $\varepsilon > 0$ such that $B_{d_Y}(f(x_0), \varepsilon) \subset U$. Since f is ε-δ continuous, there exists $\delta > 0$ such that $x \in B_{d_X}(x_0, \delta)$ implies $f(x) \in B_{d_Y}(f(x_0), \varepsilon)$ so $B_{d_X}(x_0, \delta) \subset f^{-1}(U)$ (this is because if $x \in B_{d_X}(x_0, \delta)$, then $f(x) \in B_{d_Y}(f(x_0), \varepsilon) \subset U$ so $f(x) \in U$ and in particular $x \in f^{-1}(U)$). Since x_0 was arbitrary, we conclude that $f^{-1}(U)$ is open. ♣

Lemma 9. *Suppose $f, g\colon \mathbf{R} \to \mathbf{R}$ are continuous. Then the following hold*

(i) The sum $(f+g)(x) = f(x) + g(x)$ is continuous.
(ii) The product $fg(x) = f(x)g(x)$ is continuous.

Proof. By Lemma 8, it suffices to show that $f + g$ and fg are continuous in the ε-δ sense: Let $x_0 \in \mathbf{R}$ and let $\varepsilon > 0$ be given.

(i) Since f and g are continuous in the ε-δ sense there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x_0 - x| < \delta_1$ implies $|f(x_0) - f(x)| < \varepsilon/2$ and $|x_0 - x| < \delta_2$ implies $|g(x_0) - g(x)| < \varepsilon/2$ respectively. Take $\delta = \min\{\delta_1, \delta_2\}$. Then, by the triangle inequality (cf. Munkres §20 the definition of a metric in p. 119) we have

$$\begin{aligned} |(f+g)(x_0) - (f+g)(x)| &= |f(x_0) + g(x_0) - f(x) - g(x)| \\ &= |f(x_0) - f(x) + g(x_0) - g(x)| \\ &\le |f(x_0) - f(x)| + |g(x_0) - g(x)| \\ &\le \varepsilon \end{aligned}$$

(ii) Since f and g are continuous in the ε-δ sense, by the triangle inequality we have

$$\begin{aligned}|fg(x_0) - fg(x)| &= |f(x_0)g(x_0) - f(x)g(x)| \\ &= |f(x_0)g(x_0) - f(x_0)g(x) + f(x_0)g(x) - f(x)g(x)| \\ &= |f(x_0)g(x_0) - f(x_0)g(x)| + |f(x_0)g(x) - f(x)g(x)| \\ &= |f(x_0)||g(x_0) - g(x)| + |f(x_0) - f(x)||g(x)|.\end{aligned}$$

To bound this expression, consider the following: Let $\delta_1 > 0$ such that $|f(x_0) - f(x)| < \varepsilon/2$. Since g is continuous, choose $\delta_2 > 0$ such that $|g(x_0) - g(x)| < 1$. Then $g(x) < g(x_0) + 1$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Finally, if choose $\delta_3 > 0$ such that $|g(x_0) - g(x)| < \varepsilon/2f(x_0)$. Then $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ gives a bound to the expression

$$|f(x_0)||g(x_0) - g(x)| + |f(x_0) - f(x)||g(x)| < \varepsilon.$$

Note that if $f(x_0) = 0$, we discard δ_3 and we obtain a stricter bound on our estimates. In any case, fg is continuous. ♣

Corollary. *Polynomials from* $\mathbf{R}$ *to* $\mathbf{R}$ *are continuous.*

Proof of Corollary. It is immediate from Lemma 9(i,ii) and Theorem 18.2(a,b) from Munkres. Here is a sketch: By Theorem 18.2(a) constant functions are continuous, therefore $x \mapsto a_0$ for $a_0 \in \mathbf{R}$ is continuous. By Theorem 18.2(b), the map $x \mapsto x$ is continuous so by Lemma 9(ii), $x \mapsto x^2$ is continuous. By induction on n, $x \mapsto x^n$ is continuous. Similarly, we have that $x \mapsto a_n x^n$ is continuous. Thus, by Lemma 9(i), the map

$$x \longmapsto a_n x^n + \cdots + a_1 x + a_0$$

is continuous. ♣

Now, for the box topology, consider our favorite neighborhood of $\mathbf{0}$ (as seen in Munkres §19, p. 117) given by

$$U = \prod_{n \in \mathbf{Z}_+} \left(-\frac{1}{n}, \frac{1}{n}\right).$$

The set U is clearly open since it is a basis element, by Theorem 19.2. However, the preimage

$$h^{-1}(U) = \bigcap_{n \in \mathbf{Z}_+} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open in $\mathbf{R}$ so h is not open in $\mathbf{R}^\omega$ with the box topology.

Finally, we will show that h is continuous in the ε-δ sense: Given $\varepsilon > 0$ and $x_0 \in \mathbf{R}$, let $\delta = \varepsilon$, then for any $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ we have

$$d_{\bar{\rho}}(h(x_0), h(x)) = |x_0 - x| < \varepsilon.$$

Thus, since h is continuous in the ε-δ sense, by Lemma 8, we have that h is continuous in the open set sense. ■

Problem 4.2 (Munkres §20, Ex. 4(b))

Consider the product, uniform, and box topologies on $\mathbf{R}^\omega$.

(b) In which topologies do the following sequences converge?

$$\begin{aligned}
\mathbf{w}_1 &= (1,1,1,1,...), & \mathbf{x}_1 &= (1,1,1,1,...),\\
\mathbf{w}_2 &= (0,2,2,2,...), & \mathbf{x}_2 &= (0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},...),\\
\mathbf{w}_3 &= (0,0,3,3,...), & \mathbf{x}_3 &= (0,0,\tfrac{1}{3},\tfrac{1}{3},...),\\
&\vdots & &\vdots\\
\mathbf{y}_1 &= (1,0,0,0,...) & \mathbf{z}_1 &= (1,1,0,0,...),\\
\mathbf{y}_2 &= (\tfrac{1}{2},\tfrac{1}{2},0,0,...) & \mathbf{z}_2 &= (\tfrac{1}{2},\tfrac{1}{2},0,0,...),\\
\mathbf{y}_3 &= (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,...) & \mathbf{z}_3 &= (\tfrac{1}{3},\tfrac{1}{3},0,0,...),\\
&\vdots & &\vdots
\end{aligned}$$

Proof. By Lemma D (from Prof. McClure's notes) if $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$ and $\{\mathbf{z}_n\}$ converge in the box topology, they converge to $\mathbf{0}$ since they converge to $\mathbf{0}$ in the product topology (and this can be readily seen by applying Problem 3.5 [Munkres §19, Ex. 6]).

However, for the sequences $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ we see that the neighborhood of $\mathbf{0}$ given by

$$U = \prod_{n\in\mathbf{Z}_+}\left(-\frac{1}{n},\frac{1}{n}\right)$$

does not contain any term of either sequence since for any $k \in \mathbf{Z}_+$, the term

$$\mathbf{x}_k = (0,0,...,1/k,1/k,...) \notin (-1,1)\times\cdots\times(-1/k,1/k)\times(-1/(k-1),1/(k-1))\times\cdots.$$

Similarly, we can see that $\mathbf{y}_k$ will not be in U for any k so the sequence $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ will not converge in the box topology.

Although $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ do not converge in the box topology we claim that the sequence $\{\mathbf{z}_n\}$ does converge. To see this it is enough to consider basic open neighborhoods of $\mathbf{0}$. Let $U = \prod(a_n, b_n)$ be a basis element containing $\mathbf{0}$. Then we must show that for N sufficiently big, $\mathbf{x}_n \in U$ for all $n \geq N$. Let $b = \min\{b_1, b_2\}$. Since $b > 0$, by the Archimedean property (Munkres Theorem 4.2), there exists $N \in \mathbf{Z}_+$ such that $1/N < b$. Thus, $\mathbf{z}_n \in U$ for all $n \geq N$ so $\mathbf{z}_n \to \mathbf{0}$ in the box topology. ∎

Problem 4.3 (Munkres §20, Ex. 6(b))

Let $\bar{\rho}$ be the uniform metric on $\mathbf{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, x_3, ...) \in \mathbf{R}^\omega$ and given $0 < \varepsilon < 1$, let

$$U(\mathbf{x}, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \times \cdots.$$

(b) Show that $U(\mathbf{x}, \varepsilon)$ is not even open in the uniform topology.

Proof of (b). It is sufficient to find a point $\mathbf{x}_0 \in U(\mathbf{x}, \varepsilon)$ such that $B_{\bar{\rho}}(\mathbf{x}_0, \delta) \not\subset U(\mathbf{x}, \varepsilon)$ for any $\delta > 0$. Let $\mathbf{x}_0$ be the point

$$\mathbf{x}_0 = \prod_{n \in \mathbf{Z}_+} \left(x_n + \left(\frac{n-1}{n} \right) \varepsilon \right).$$

Now consider the open ball $B_{\bar{\rho}}(\mathbf{x}_0, \delta)$ for $\delta > 0$. Now, pick a point $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}_0, \delta)$ given by

$$\mathbf{y} = \prod_{n \in \mathbf{Z}_+} \left(x_n + \left(\frac{n-1}{n} \right) \varepsilon + \frac{\delta}{2} \right).$$

Clearly $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}_0, \delta)$ since

$$\bar{\rho}(\mathbf{x}_0, \mathbf{y}) = \sup_{n \in \mathbf{Z}_+} \{ \min\{ |x_n - y_n|, 1 \} \} = \min\{\delta/2, 1\} \leq \delta/2.$$

However, by the Archimedean property, there exists $k \in \mathbf{Z}_+$ such that $\delta/2 > 1/k$ so $n \geq k$ implies

$$y_n = x_n + \left(\frac{n-1}{n} \right) \varepsilon + \frac{\delta}{2} > x_n + \varepsilon$$

so $\mathbf{y}$ is in $B_{\bar{\rho}}(\mathbf{x}_0, \delta)$ but not in $U(\mathbf{x}, \varepsilon)$. Since δ was arbitrary, we conclude that $U(\mathbf{x}, \varepsilon)$ is not open. ■

Problem 4.4 (A)

Prove Theorem Q.2 from the notes on Quotient Spaces.

Proof. Recall the statement of the theorem:

Theorem (Theorem Q.2)**.** *A function $f\colon X/\!\sim\, \to Y$ is continuous if and only if the composite*

$$X \xrightarrow{q} X/\!\sim\, \xrightarrow{f} Y$$

is continuous.

The direction $\implies$ follows from Theorem 18.2(c) in Munkres.
$\impliedby$ Suppose that the composite

$$X \xrightarrow{q} X/\!\sim\, \xrightarrow{f} Y$$

is continuous. Then for every open set $U \subset Y$, the preimage $(f \circ q)^{-1}(U)$ is open in X. But the preimage

$$(f \circ q)^1(U) = q^{-1}(f^{-1}(U))$$

and since q is a quotient map by definition (cf. Munkres §22, p. 137) $f^{-1}(U)$ is open in $X/\!\sim$ if and only if $q^{-1}(f^{-1}(U))$ is open in X. Thus, the map $f\colon X/\!\sim\, \to Y$ is continuous. ∎

Problem 4.5 (B)

Prove Proposition Q.5 from the notes on Quotient Spaces.

Proof. Recall the definition and the proposition:

Definition. Let X and Y be topological spaces. A map $p\colon X \to Y$ is a *Munkres quotient map* if $\bar{p}\colon X/{\sim_p} \to Y$ is a homeomorphism.

Proposition (Proposition Q.5)**.** *A map $p\colon X \to Y$ satisfies Definition Q.4 if and only if it satisfies the definition at the top of page 137 in Munkres.*

and Munkres's definition:

Definition (Munkres §22, p. 137)**.** Let X and Y be topological spaces; let $p\colon X \to Y$ be a surjective map. The map p is said to be a *quotient map* provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

$\implies$ Now, suppose that $\bar{p}\colon X/{\sim_p} \to Y$ is a homeomorphism. Then $\bar{p}$ is continuous with a continuous inverse $\bar{p}^{-1}\colon Y \to X/{\sim_p}$. Let $q\colon X \to X/{\sim_p}$ bet he map which takes x in X to its equivalence class $[x]$ in $X/{\sim_p}$. Then by Problem 4.5(A), the composite

$$X \xrightarrow{q} X/{\sim_p} \xrightarrow{\bar{p}} Y$$

is continuous if and only if $\bar{p}$ is continuous. Moreover, since $\bar{p}$ is bijective, it is surjective and q is clearly surjective so the map $p = \bar{p} \circ q$ is surjective. Let us prove this claim:

Lemma 10. *Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are surjective maps. Then the composite map $g \circ f\colon X \to Z$ is surjective.*

Proof. Since $g\colon Y \to Z$ is surjective, for every $z \in Z$ there exists a $y \in Y$ such that $g(y) = z$. Similarly, for every $y' \in Y$ there exists a $x' \in X$ such that $f(x') = y'$, in particular there exists a $x \in X$ such that $f(x) = y$. Thus, $g(f(x)) = g \circ f(x) = z$. Since z was arbitrary, we conclude that the composition of surjective maps is again surjective. ♣

Now suppose U is open in Y. Then the preimage

$$p^{-1}(U) = (\bar{p} \circ q)^{-1}(U) = q^{-1}(\bar{p}^{-1}(U))$$

is open since p is continuous. Conversely, suppose that the preimage $p^{-1}(U)$ is open in X for $U \subset Y$. Then we have that

$$p^{-1}(U) = (\bar{p} \circ q)^{-1}(U) = q^{-1}(\bar{p}^{-1}(U))$$

so $\bar{p}^{-1}(U)$ is open in $X/{\sim}$. Hence, we have that

$$\bar{p}(\bar{p}^{-1}(U)) = (\bar{p} \circ \bar{p}^{-1})(U) = \mathrm{id}_Y(U) = U$$

is open in Y since $\bar{p}$ is a homeomorphism.

$\impliedby$ Now suppose that $p\colon X \to Y$ is a Munkres quotient map. That is, the map $p\colon X \to Y$ is surjective with U open in Y if and only if $p^{-1}(U)$ is open in X. We claim that he map $\bar{p}\colon X/{\sim} \to Y$

is a homeomorphism with continuous inverse $\bar{p}^{-1}\colon Y \to X/{\sim_p}$ given by $y \mapsto [x]$ where x' is in the equivalence class $[x]$ if and only if $x' \in p^{-1}(y)$. First, it is clear that the map $\bar{p}$ is continuous by Problem 4.4 (A) (that is, Theorem Q.2 from the notes) since $p\colon X \to Y$ is continuous. Now we check that $\bar{p}^{-1}$ is indeed the inverse map of $\bar{p}$. Let $y \in Y$. Since p is surjective, there exists $x \in X$ such that $p(x) = y$; take this x to be the our representative of the equivalence class $[x]$ of points in X which map to y. Then

$$\begin{aligned} \bar{p} \circ \bar{p}^{-1}(y) &= \bar{p}(\bar{p}^{-1}(y)) \\ &= \bar{p}([x]) \\ &= y \\ &= \mathrm{id}_Y(y) \end{aligned} \qquad\qquad \begin{aligned} \bar{p}^{-1} \circ \bar{p}([x]) &= \bar{p}^{-1}(\bar{p}([x])) \\ &= \bar{p}^{-1}(y) \\ &= [x] \\ &= \mathrm{id}_{X/\sim_p}([x]). \end{aligned}$$

Lastly, we will show that $\bar{p}^{-1}$ is continuous. Let U be open in $X/{\sim_p}$. Then $(\bar{p}^{-1})^{-1}(U)$ is open in Y if and only if $p^{-1}(\bar{p}^{-1})^{-1}(U)$ is open in X, that is,

$$\begin{aligned} p^{-1}(\bar{p}^{-1})^{-1}(U) &= (\bar{p} \circ q)(\bar{p}^{-1})^{-1}(U) \\ &= q^{-1}\Big(\bar{p}^{-1}\Big((\bar{p}^{-1})^{-1}(U)\Big)\Big), \end{aligned}$$

but since $\bar{p}$ is bijective, in particular surjective, by Problem 1.1 (Munkres §2, Ex. 1(b)), we have

$$= q^{-1}(U)$$

which is by definition open in X. Thus, $(\bar{p}^{-1})^{-1}(U)$ is open in Y and we see that $\bar{p}^{-1}$ is continuous. We conclude that the map $\bar{p}\colon X/{\sim_p} \to Y$ is a homeomorphism. ■

Remarks. In retrospect it would have been easier to show that the map $\bar{p}\colon X/{\sim_p} \to Y$ is an open map (at least conceptually and the notation would have been easier to digest). Observe how much cleaner this is: Let U be an open set in $X/{\sim_p}$, the image $\bar{p}(U)$ is open in Y if and only if $p^{-1}(\bar{p}(U))$ is open in X, but as sets

$$p^{-1}(\bar{p}(U)) = q^{-1}(\bar{p}^{-1}(\bar{p}(U))) = q^{-1}(U)$$

where $\bar{p}^{-1}(\bar{p}(U))$ follows from the bijectivity of $\bar{p}$ which we previously demonstrated. It is clear then that $\bar{p}$ is a homeomorphism. Both proofs are correct, but we leave this here for pedantic purposes.

Problem 4.6 (C)

Prove Proposition Q.6 from the notes on Quotient Spaces.

Proof. Recall the statement of the proposition:

Proposition (Proposition Q.6)**.** *Let $p\colon X \to Y$ be a Munkres quotient map. A function $f\colon Y \to Z$ is continuous if and only if the composite*

$$X \xrightarrow{p} Y \xrightarrow{f} Z$$

is continuous.

Identify Y with the quotient $X/{\sim_p}$ via the homeomorphism $\bar{p}\colon X/{\sim_p} \to Y$ given above in Problem 4.5 (Proposition Q.5) then apply Problem 4.4 (Theorem Q.2).

More precisely, we have

$$X \xrightarrow{q} X/{\sim_p} \xrightarrow{\cong} Y \xrightarrow{f} Z$$

where $\bar{p} \circ q = p$ and we see that the map $f \circ \bar{p}$ is continuous if and only if the composition $(f \circ \bar{p}) \circ q$ is continuous. Then $\bar{p}$ is a quotient map (because it is an homeomorphism) since U is open in Y if and only if $\bar{p}^{-1}(U)$ is open in $X/{\sim_p}$. Thus, by Problem 4.4 (Theorem Q.2), the map f is continuous if and only if the composition $f \circ \bar{p}$ is continuous. Now we tie in the chain of implications: f is continuous $\implies$ $f \circ \bar{p}$ is continuous $\implies$ $(f \circ \bar{p}) \circ q = f \circ (\bar{p} \circ q) = f \circ p$ is continuous. Conversely $f \circ p = f \circ \bar{p} \circ q = (f \circ \bar{p}) \circ q$ is continuous $\implies$ $f \circ \bar{p}$ is continuous $\implies$ f is continuous. ∎

***Remarks**.* Here's an alternative way I thought about doing this before I realized $\bar{p}$ being a homeomorphism implies it is a quotient map and so we can apply Theorem Q.2 on it.

Here's the idea: Since $X/{\sim_p} \cong Y$, there exists a continuous inverse $\bar{p}^{-1}: Y \to X/{\sim_p}$ to $\bar{p}$. Then we can prove that f is continuous, which is the difficult direction, by invoking Theorem 18.2(c) on $f \circ \bar{p}$ and $\bar{p}^{-1}$ (since they are continuous by hypothesis) then $(f \circ \bar{p}) \circ \bar{p}^{-1} = f \circ (\bar{p} \circ \bar{p}^{-1}) = f$ is continuous.

Problem 4.7 (D)

(Do not use Problem E to do this problem). Let $\sim$ be the equivalence relation on the interval $[-1,1]$ defined by $x \sim y$ if and only if $x = y$ or $x = -y$ with $y \in (-1,1)$ (you do not have to prove that this is an equivalence relation). Prove that $[-1,1]/\sim$ is not Hausdorff.

Proof. We will show that for any open neighborhood U and V of $[1]$ and $[-1]$ respectively, the intersection $U \cap V \neq \emptyset$. Let U and V be as above, then by the definition of the quotient map $q^{-1}(U)$ and $q^{-1}(V)$ are open neighborhoods of 1 and -1 respectively. Then by the definition of the subspace topology, there exists $\varepsilon_1, \varepsilon_2 > 0$ such that

$$B(1,\varepsilon_1) \cap [-1,1] = (1-\varepsilon_1, 1] \subset q^{-1}(U) \quad \text{and} \quad B(-1,\varepsilon_2) \cap [-1,1] = [-1,-1+\varepsilon_2) \subset q^{-1}(V).$$

Then, by Problem 1.1 (Munkres §2, Ex. 2(b)) and the transitivity of the subset relation, $U_0 = q((1-\varepsilon_1, 1]) \subset U$ and $V_0 = q([-1,-1+\varepsilon_2)) \subset V$ so $U_0 \cap V_0 \subset U \cap V$. Let us prove this claim:

Lemma 11. *Suppose $A \subset C$ and $B \subset D$. Then $A \cap B \subset C \cap D$.*

Proof of lemma. Suppose $x \in A \cap B$ if and only if $x \in A$ and $x \in B$ which implies $x \in C$ and $x \in D$ since $A \subset C$ and $B \subset D$. But this is true if and only if $x \in C \cap D$. Thus, $A \cap B \subset C \cap D$. ♣

Now we will show that $U_0 \cap V_0 \neq \emptyset$. For if we take the preimage of U_0 and V_0 under q we have

$$\begin{aligned} q^{-1}(U_0) &= \{\, x \in [-1,1] \mid x \sim x' \text{ for every } x' \in (1-\varepsilon_1, 1] \,\} \\ &= (-1,-1+\varepsilon_1) \cup (-1-\varepsilon_1, 1] \\ q^{-1}(V_0) &= \{\, x \in [-1,1] \mid x \sim x' \text{ for every } x' \in [-1,-1+\varepsilon_2) \,\} \\ &= [-1,-1+\varepsilon_2) \cup (1-\varepsilon_2, 1) \end{aligned}$$

where one can see that the points $\pm \min\{\varepsilon_1, \varepsilon_2\}$ are in the intersection $q^{-1}(U_0) \cap q^{-1}(V_0)$. Thus, $q^{-1}(U_0) \cap q^{-1}(V_0) = q^{-1}(U_0 \cap V_0) \neq \emptyset$ so $U_0 \cap V_0 \neq \emptyset$. In particular $U \cap V \neq \emptyset$ so $[-1,1]/\sim$ is not Hausdorff. ■

Problem 4.8 (E)

Let X be a topological space with an equivalence relation $\sim$. Suppose that the quotient space $X/\sim$ is Hausdorff.

Prove that the set

$$S = \{\, x \times y \in X \times X \mid x \sim y \,\}$$

is a closed subset of $X \times X$.

Proof. We will show that $(X \times X) \setminus S$ is open in $X \times X$. Let $x \times y \in (X \times X) \setminus S$. Then $q(x) \neq q(y)$ in the quotient $X/\sim$ since $x \nsim y$. Hence, there exist open neighborhoods U and V of $q(x)$ and $q(y)$, respectively, such that $U \cap V = \emptyset$. Then $q^{-1}(U)$ and $q^{-1}(V)$ are open neighborhoods of x and y respectively with $q^{-1}(U) \cap q^{-1}(V) = q^{-1}(U \cap V) = \emptyset$. Then $q^{-1}(U) \times q^{-1}(V)$ is a basis element of $X \times X$ containing $x \times y$ with $q^{-1}(U) \times q^{-1}(V) \subset (X \times X) \setminus S$ (otherwise there is an $x' \times y' \in q^{-1}(U) \times q^{-1}(V)$ with $x' \sim y'$, but then $q(x') = q(y') \in U \cap V$ which contradicts our choice of U and V). Since $x \times y$ was chosen arbitrarily, we conclude that $(X \times X) \setminus S$ is open in $X \times X$ and therefore, its complement S is closed in $X \times X$. ■

Problem 4.9 (F)

For problem F you need the following definition: if Y is a topological space and S is a subset of Y, we write Y/S for the quotient space $Y/{\sim}$, where $\sim$ is defined by $x \sim y$ if and only if $x = y$ or $\{x, y\} \subset S$. (Intuitively, Y/S is obtained from Y by collapsing S to a point.)

Let X be a topological space. Let U be an open set in X, and let A be a subset of U. Give U the subspace topology. Let $\iota^*\colon U/A \to X/A$ be the map which takes $[x]$ to $[x]$ (you do not have to prove that this is well-defined).

(i) Prove that ι^* is continuous.
(ii) Prove that ι^* is an open map.

Proof. (i) Since the composition $p \circ \iota\colon U \to X/A$ in the diagram below is continuous by Theorem 18.2(b) and by the definition of the quotient map p

$$\begin{tikzcd} U \arrow[r, hook, "\iota"] \arrow[d, two heads, "q"] & X \arrow[d, two heads, "p"] \\ U/A \arrow[r, "\iota^*"] & X/A, \end{tikzcd}$$

it follows by Problem 4.4 (Theorem Q.2) that ι^* is continuous. Alternatively, we note that $\iota^*\colon U/A \to X/A$ is the inclusion map, and therefore, is continuous.

(ii) We prove the following stronger but simple (to prove) result:

Lemma 12. *Suppose $Y \subset X$ is open. The inclusion $\iota\colon Y \hookrightarrow X$ is an open map.*

Proof. Let U be an open in Y. Then, by Lemma 16.2, $\iota(U) = U$ is open in X. Thus ι is an open map. ♣

If we can show that U/A is open in X/A, it follows from Lemma 12 that ι^* is an open map. Looking at the diagram in part (i) above, the we have that

$$\iota\Big(q^{-1}(\iota^*)^{-1}(U/A)\Big) = \iota(q^{-1}(U/A)) = \iota(U) = U = p^{-1}(U/A) = p^{-1}(\iota^*(U/A))$$

is open in X, hence, by the definition of the quotient map, U/A is open in X/A. Thus, the map ι^* is open in X/A. ■

Problem 4.10 (G)

Let X be a topological space satisfying the first countability axiom (see the bottom of page 130 and the top of page 131). Let $A \subset X$ and let $x \in \overline{A}$. Prove that there is a sequence in A which converges to x (see the top of page 131 for a hint).

Proof. Suppose that X satisfies the first countability axiom (cf. Munkres, §21, pp. 130-131). Let $x \in \overline{A}$. We will construct a sequence $\{x_n\}$ which converges to x. Since x is in the closure of A, for every neighborhood U of x the intersection $U \cap A$ is nonempty. In particular, since X is first countable, there is a countable collection $\{U_n\}$ of neighborhoods of x with $U_n \cap A \neq \emptyset$ for all n. Now, define a nested sequence of sets $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$ where $V_n = \bigcap_{i=1}^n U_i$ and let $x_n \in V_n \cap A$. (Note that V_n is nonempty since it is a neighborhood of x so for some positive integer N the neighborhood $U_N \subset V_n$. Moreover $V_n \cap A$ is nonempty since V_n is a neighborhood of x which is in the closure of A.) We claim that the sequence we just created, $\{x_n\}$, converges to x. Let U be any neighborhood of x. Then $U_N \subset U$ for some positive integer N. Hence $x_n \in U$ for every $n \geq N$ (by construction). Thus, the sequence $x_n \to x$. ■