Introduction to Character Varieties, Part II

$\mathrm{SL}(2, \mathbb{C})$ as a case study.

Sean Lawton

George Mason University

Korean Institute for Advanced Study

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  4. In this case, it is exactly given by $\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, \mathrm{SL}(2,\mathbb{C}))] \cong$

$$\mathbb{C}[x_{ij}^k \mid 1 \leq i,j \leq 2,\ 1 \leq k \leq r]/\langle x_{11}^k x_{22}^k - x_{12}^k x_{21}^k - 1 \mid 1 \leq k \leq r \rangle.$$

5. G acts on $\mathrm{Hom}(\mathsf{F}, G)$ by $g \cdot \rho = g\rho g^{-1}$; or equivalently on $G^{\times r}$ by $g \cdot (g_1, ..., g_r) = (gg_1g^{-1}, ..., gg_rg^{-1})$.

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11. This correspondence, $\langle t_1 - a_1, ..., t_N - a_N\rangle + \mathfrak{I} \mapsto (a_1, ..., a_N)$, determines an algebraic set. This space is the character variety described before.

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- Then $\mathbb{C}[R_{\mathbb{Z}}(G)] = \mathbb{C}[\mathrm{SL}(2,\mathbb{C})] = \mathbb{C}[z,w,u,v]/(zw-uv-1)$.

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- Let $f \in \mathbb{C}[R_{\mathbb{Z}}(G)]^G$, then $f(\rho) = f(g_t \rho g_t^{-1}) = f(\epsilon_t)$.
- If $t = \pm 2$, then ρ is either conjugate to ϵ_t or is $\pm\mathbf{1}$.

- In the former case, ϵ_t is conjugate to either one of $\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$, and thus $f(\rho) = f(g_t \rho g_t^{-1}) = f(\epsilon_t)$.

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- So again we conclude that $\mathfrak{X}_{\mathbb{Z}}(\mathrm{SL}(2, \mathbb{C})) \cong \mathbb{C}$ since Hilbert's Nullstellensatz implies all maximal ideals in $\mathbb{C}[t]$ are of the form $(t - a)$ for $a \in \mathbb{C}$.

Generators and the Non-Commutative Picture

- Recall that $\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, \mathsf{SL}(2,\mathbb{C}))] = \mathbb{C}[x_{ij}^k]/\Delta$ where Δ is the ideal generated by the r irreducible polynomials

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- We note that
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- Otherwise stated,

$$\mathbb{C}[x_{ij}^k]^{\mathsf{SL}(2,\mathbb{C})}/\Delta \approx \left(\mathbb{C}[x_{ij}^k]/\Delta\right)^{\mathsf{SL}(2,\mathbb{C})};$$

which is true because $\mathsf{SL}(2,\mathbb{C})$ is *linearly* reductive and the generators of Δ are invariants.

First Fundamental Theorem of Matrix Invariants

In 1976 Procesi proved

Theorem (Procesi)

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- Evidently, this ring is multigraded.
- Finding minimal generators amounts to finding all linear relations among generators of the same multidegree in the vector space
$$\mathbb{C}[\mathfrak{Y}_r]^+ / \left(\mathbb{C}[\mathfrak{Y}_r]^+\right)^2$$
where $\mathbb{C}[\mathfrak{Y}]^+$ is the ideal of positive terms.

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And if we assume $\det(\mathbf{X}) = 1$, as is the case in $\mathbb{C}[\mathfrak{X}_r]$, we easily derive $\mathrm{tr}(\mathbf{X}^{-1}) = \mathrm{tr}(\mathbf{X})$ and $\mathrm{tr}(\mathbf{X}^2) = \mathrm{tr}(\mathbf{X})^2 - 2$.

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Remark

Minimal generators for $\mathbb{C}[\mathfrak{Y}_r]$ were first worked out by Sibirskii in 1968.

Example: $r = 1$ (again)

- We also get from the characteristic equation (multiplying by $\mathbf{X}^{n-2}$): $\mathbf{X}^n - \mathrm{tr}(\mathbf{X})\mathbf{X}^{n-1} + \mathbf{X}^{n-2} = \mathbf{0}$, which in turn gives $\mathrm{tr}(\mathbf{X}^n) = \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{X}^{n-1}) - \mathrm{tr}(\mathbf{X}^{n-2})$.

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- Consequently, there are exactly r generators of type $\mathrm{tr}(\mathbf{X})$ in $\mathbb{C}[\mathfrak{X}_r]$ and none of type $\mathrm{tr}(\mathbf{X}^n)$, $n \neq 1$; and this is minimal among these generators.

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- Precisely, the words are $\{\mathbf{X}_1, ..., \mathbf{X}_r\}$.
- Hence we recover the fact we proved earlier: $\mathbb{C}[\mathfrak{X}_1] \cong \mathbb{C}[t]$ where t corresponds to the invariant function $\mathrm{tr}(\mathbf{X})$.

Example: $r = 2$ (again)

- The dimension of $\mathfrak{X}_r$ is equal to $3r - 3$ for $r \geq 2$, and 1 for $r = 1$. The last example proves the $r = 1$ case by explicit computation, and we will prove the $r \geq 2$ case later.

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- Multiplying the Cayley-Hamilton equation on both sides by words $\mathbf{U}$ and $\mathbf{V}$ allows us to freely eliminate the generators of type: $\mathrm{tr}(\mathbf{U}\mathbf{X}^n\mathbf{V})$ as long as $n \geq 2$ and at least one of $\mathbf{U}$ or $\mathbf{V}$ is not the identity.

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- So for the case, $\mathbb{C}[\mathfrak{X}_2]$ we are left with the generators $\mathrm{tr}(\mathbf{X}_1), \mathrm{tr}(\mathbf{X}_2), \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)$ since any other expression in two letters would result in a sub-expression with an exponent greater than one, which we just showed was impossible.

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- This also gives a direct (and short) proof of the Fricke-Vogt theorem: $\mathfrak{X}_2 \cong \mathbb{C}^3$ (equivalently $\mathbb{C}[\mathfrak{X}_2] \cong \mathbb{C}[x, y, z]$).

First step to fundamental relation: Polarization

- Replacing $\mathbf{X}$ with $\mathbf{X} + \mathbf{Y}$ in the Cayley-Hamilton equation gives

$$(\mathbf{X} + \mathbf{Y})^2 - \mathrm{tr}(\mathbf{X} + \mathbf{Y})(\mathbf{X} + \mathbf{Y}) + \det(\mathbf{X} + \mathbf{Y})\mathbf{I} = \mathbf{0}.$$

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- Simplifying this expression yields

$$\mathbf{XY}+\mathbf{YX} = \mathrm{tr}(\mathbf{X})\mathbf{Y} + \mathrm{tr}(\mathbf{Y})\mathbf{X} - \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\mathbf{I} + \mathrm{tr}(\mathbf{XY})\mathbf{I}.$$

Second step, but an important step...

Multiplying on the right by $\mathbf{Z}$ we get the expression

$$\begin{aligned}\mathrm{tr}(\mathbf{XYZ}) + \mathrm{tr}(\mathbf{YXZ}) =& \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{YZ}) + \mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{XZ}) \\ & - \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{Z}) + \mathrm{tr}(\mathbf{XY})\mathrm{tr}(\mathbf{Z}).\end{aligned}$$

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- Subtracting, adding, and subtracting these four relations gives and expression for $\mathrm{tr}(\mathbf{XYZW}) - \mathrm{tr}(\mathbf{XYWZ})$.

Fundamental Relation

- However, sending $\mathbf{X} \mapsto \mathbf{W} \mapsto \mathbf{Y} \mapsto \mathbf{Z} \mapsto \mathbf{X}$ in the first expression gives $\mathrm{tr}(\mathbf{XYZW}) + \mathrm{tr}(\mathbf{XYWZ})$.

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- This adds to our sum to give:

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\end{aligned}$$

- So length 4 words are not need to generate the ring.

- Since $\mathbf{W}$ can be any word in the generic matrices, we have proved that $\mathbb{C}[\mathfrak{X}_r]$ is generated by at most $\binom{r}{1}+\binom{r}{2}+\binom{r}{3}=\frac{r(r^2+5)}{6}$ generators (so the ring is finitely generated!)

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- In particular, here are the generators:
$\mathcal{G}_1 = \{\mathrm{tr}(\mathbf{X}_1), ..., \mathrm{tr}(\mathbf{X}_r)\}$ of order r.
$\mathcal{G}_2 = \{\mathrm{tr}(\mathbf{X}_i\mathbf{X}_j) \mid 1 \leq i, j \leq r \text{ and } i \neq j\}$ of order $\frac{r(r-1)}{2}$.
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- We will see in a minute that this is a minimal generating set (we can't get rid of any either!)

Geometrically, this says that the minimal (trace) embedding of $\mathfrak{X}_r$ into $\mathbb{C}^N$ is when $N = \frac{r(r^2+5)}{6}$ and the mapping is exactly

$$[\rho] \mapsto \Big(\mathrm{tr}(\rho(\gamma_1)), \ldots, \mathrm{tr}(\rho(\gamma_r)), \mathrm{tr}(\rho(\gamma_1\gamma_2)), \ldots, \mathrm{tr}(\rho(\gamma_{r-1}\gamma_r)),$$

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It is clear that these functions are contained in the ring of invariants, but it is not obvious that these are all of them (this is what Procesi proved).

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It is clear that these functions are contained in the ring of invariants, but it is not obvious that these are all of them (this is what Procesi proved). Even more surprising is that the maximum word length is independent of the rank r (this is what we just showed!).

A further word about embeddings

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- We must show there is $[\rho_0]$ so $\dim T_{[\rho]}(\mathfrak{X}_r(\mathrm{SL}(n, \mathbb{C}))) = N_{\mathrm{tr}}$.
- Indeed, let ρ_0 be the trivial representation (all generators map to the identity). Then the tangent space at the identity representation is $T_0\left(\mathfrak{sl}(n, \mathbb{C})^r /\!\!/ \mathrm{SL}(n, \mathbb{C})\right)$.

- However, $\mathfrak{sl}(n, \mathbb{C})^r /\!\!/ \mathrm{SL}(n, \mathbb{C})$ is exactly $\mathfrak{gl}(n, \mathbb{C})^r /\!\!/ \mathrm{SL}(n, \mathbb{C})$ with the r traces $\mathrm{tr}(\mathbf{X}_i)$ specialized to 0 (which leaves only homogeneous generators of degree two or more).

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- Thus the dimension at 0 is N_{tr}, as required.

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- There are then only 7 generators.
- If $\mathrm{tr}(\mathbf{XYZ})$ was allowed to be eliminated, we would conclude that $\mathfrak{X}_3$ was affine $\mathbb{C}^6$.
- However, it is not hard to show there exists two representations which agree on the six generators of word length two or less but differ at $\mathrm{tr}(\mathbf{XYZ})$.

For instance, $\mathbf{X} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{Y} = \begin{pmatrix} 0 & 2 \\ -1/2 & 0 \end{pmatrix}$, and $\mathbf{Z} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ or $\mathbf{Z} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ gives two such representations.

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One can further show there exists a product relation for $\mathrm{tr}(\mathbf{XYZ})\mathrm{tr}(\mathbf{YXZ})$. *Together with the sum relation, we conclude that* $\mathfrak{X}_3$ *is a hypersurface and the generator of the ideal is an irreducible quadratic polynomial in* $\mathrm{tr}(\mathbf{XYZ})$ *over the other 6 generators.*

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Deriving the Product Relation

$$\begin{aligned}
\operatorname{tr}(\mathbf{XYZ})\operatorname{tr}(\mathbf{XZY}) = {} & \operatorname{tr}(\mathbf{X})^2 + \operatorname{tr}(\mathbf{Y})^2 + \operatorname{tr}(\mathbf{Z})^2 \\
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- Together, these formulae give the product relation.

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- Such a set cannot have any relations among themselves alone; that is, they are *algebraically independent*.

Since $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are affine, and $\mathfrak{X}_3$ is a hypersurface. One might expect

Theorem

The following subsets of minimal generators are together algebraically independent:
$\{\mathrm{tr}(\mathbf{X}_i) \mid 1 \leq i \leq r\}$ *of order* r
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There are $3r-3$ of these generators, and so this set is maximal.

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  (from $\mathbf{X}_1$) x^1_{11}
  (from $\mathbf{X}_2$) x^2_{11}, x^2_{22},
  (from $\mathbf{X}_k$) $x^k_{11}, x^k_{12}, x^k_{22}$
- Putting $\mathrm{tr}(\mathbf{X}_r), \mathrm{tr}(\mathbf{X}_1\mathbf{X}_r), \mathrm{tr}(\mathbf{X}_2\mathbf{X}_r)$ in the last 3 rows we get a block diagonal matrix.

Sketch of Proof

This is an adaptation of the proof Aslaksen, Tan, and Zhu used in 1994 to establish a similar result for $\mathbb{C}[\mathfrak{Y}_r]$. Teranishi in 1988 has a different proof in this case.

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- Putting $\mathrm{tr}(\mathbf{X}_r), \mathrm{tr}(\mathbf{X}_1\mathbf{X}_r), \mathrm{tr}(\mathbf{X}_2\mathbf{X}_r)$ in the last 3 rows we get a block diagonal matrix. By induction we must show these three traces are independent in the variables from $\mathbf{X}_r$.

We first show that the matrix entries we have chosen are independent:
Generically, we can assume the first matrix is diagonal
$\mathbf{X}_1 = \begin{pmatrix} x_{11}^1 & 0 \\ 0 & 1/x_{22}^1 \end{pmatrix}$.

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Since this leaves us with only $3r - 3$ elements, they must be independent since the dimension is $3r - 3$ also.

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Remark

This directly shows that these coordinates are local and their differentials generate cotangent spaces (generically).

The Magnus Trace Map

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- In the cases, $r = 1, 2, 3$ the natural map $\mathfrak{X}_r \to \mathbb{C}^{3r-3}$ is surjective and there is always a slice (a map back).
- Unfortunately, this is not always the case:

Theorem (Florentino, 2007)

$\mathfrak{X}_r \longrightarrow \mathbb{C}^{3r-3}$ is only surjective in the cases $r = 1, 2, 3$; but in general the image omits only a subset of a codimension 1 subspace.

"Simple" Relations

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- Recall,

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- Let $\mathfrak{J}$ be the ideal of relations for $\mathbb{C}[\mathfrak{Y}_r]$ and enumerate the minimal generators $t_1, \ldots, t_{N_r}$. Then $\mathbb{C}[\mathfrak{Y}_r] = \mathbb{C}[t_1, \ldots, t_{N_r}]/\mathfrak{J}$.
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- Then $\mathfrak{J}/\mathbb{C}[t_1, \ldots, t_{N_r}]^+\mathfrak{J}$ is a vector space. Its basis are the generators of $\mathfrak{J}$.
- We expect that the highest weight vectors of this vector space under the natural action of GL_r will give proof that the resulting relations are minimal. We also expect that it is a Gröbner basis.

Description of Ideal

In general,

$$\mathfrak{X}_r = \mathrm{Spec}_{max}\left(\mathbb{C}[t_1, ..., t_{\frac{r(r^2+5)}{6}}]/\mathfrak{I}_r\right)$$

where $\mathfrak{I}_r$ is an ideal generated by $\frac{1}{2}\left(\binom{r}{3}^2 + \binom{r}{3}\right) + r\binom{r}{4}$ polynomials.

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Here is the description:
Let $\mathbf{Z}_i = \mathbf{X}_i - \frac{1}{2}\mathrm{tr}(\mathbf{X}_i)\mathbf{I}$ (generic traceless matrix) and let $s_3(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) = \sum_{\sigma \in S_3} \mathrm{sign}(\sigma)\mathbf{A}_{\sigma(1)}\mathbf{A}_{\sigma(2)}\mathbf{A}_{\sigma(3)}$.

- Type 1 relations:

$$\mathrm{tr}(s_3(\mathbf{Z}_{i_1},\mathbf{Z}_{i_2},\mathbf{Z}_{i_3}))\mathrm{tr}(s_3(\mathbf{Z}_{j_1},\mathbf{Z}_{j_2},\mathbf{Z}_{j_3}))+18\det(\mathrm{tr}\,((\mathbf{Z}_{i_{row}}\mathbf{Z}_{j_{column}})) = 0,$$

for $1 \le i_1 < i_2 < i_3 \le r$, $1 \le j_1 < j_2 < j_3 \le r$.

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- Type 2:

$$\sum_{k=0}^{3}(-1)^k \mathrm{tr}(\mathbf{Z}_i\mathbf{Z}_{p_k})\mathrm{tr}(s_3(\mathbf{Z}_{p_0}, \ldots, \hat{\mathbf{Z}}_{p_k}, \ldots, \mathbf{Z}_{p_3})) = 0,$$

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- Note: this relation shows up only at the rank 4 case.

Sketch of Proof

Again,
$\mathbb{C}[\mathfrak{X}_r] \cong \mathbb{C}[\mathrm{SL}(2,\mathbb{C})^{\times r} /\!\!/ \mathrm{SL}(2,\mathbb{C})] \cong \mathbb{C}[\mathfrak{gl}(2,\mathbb{C})^{\times r} /\!\!/ \mathrm{SL}(2,\mathbb{C})]/\Delta.$

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However,

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But Weyl solved the relations problem for orthogonal invariants in 1946 in his book *The Classical Groups, Their Invariants and Representations*.
Rewriting those invariants in terms of traces then gives the result.

Relations in Γ

Also, for a finitely generated Γ, $\mathfrak{X}_\Gamma(G)$ is always cut out of $\mathfrak{X}_r(G)$ by using the relations in Γ; this can be made explicit in the $G = \mathrm{SL}(2,\mathbb{C})$ case without using an (elimination ideal) algorithm.

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Theorem

Let $\Gamma = \langle \gamma_1, ..., \gamma_r \mid R_i,\ \ i \in I \rangle$, *and denote* $\gamma_0 = 1$. *Then* $\mathfrak{X}_\Gamma(\mathrm{SL}(2, \mathbb{C}))$ *is given by*

$$\{[\rho] \in \mathfrak{X}_r(\mathrm{SL}(2, \mathbb{C})) \mid \mathrm{tr}(\rho(R_i\gamma_j)) - \mathrm{tr}(\rho(\gamma_j)) = 0, \forall i, j\}.$$

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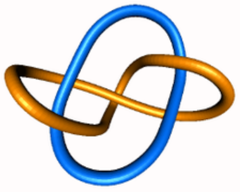
- Therefore, $\mathfrak{X}_{\Gamma_n}(\mathrm{SL}(2,\mathbb{C})) = \{t \in \mathbb{C} \mid P_n(t) = 0 = Q_n(t)\}$
- Experimentally, the solution sets have all orders (coming in pairs), and so we conjecture that all dimension 0 varieties arise this way (up to isomorphism).

Whitehead Link



- Recall, $\mathfrak{X}_2(\mathrm{SL}(2, \mathbb{C}) = \mathbb{C}^3$ and so for all $w \in F_2 = \langle a, b \rangle$, there is a unique $P_w \in \mathbb{C}[x, y, z]$ so

$$P_w(\mathrm{tr}(a), \mathrm{tr}(b), \mathrm{tr}(ab)) = \mathrm{tr}(w).$$

- The fundamental group of the complement in S^3 admits the presentation

$$\Gamma = \left\langle a, b \mid \overbrace{a^{-1}b^{-1}aba^{-1}bab^{-1}aba^{-1}b^{-1}ab^{-1}a^{-1}b}^{w} \right\rangle.$$

- So the character variety $\mathfrak{X}_\Gamma(\mathrm{SL}(2,\mathbb{C}))$ is given by $\{(x,y,z) \in \mathbb{C}^3 \mid P_w(x,y,z) - 2 = 0,\ P_{aw}(x,y,z) - x = 0,\ P_{bw}(x,y,z) - y = 0, P_{abw}(x,y,z) - z = 0\}$.

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- Using a Groebner Basis algorithm, we then get $\{(x,y,z)\in\mathbb{C}^3 \mid x^5y-2x^4y^2z-x^4z+x^3y^3z^2+2x^3y^3+4x^3yz^2-7x^3y-2x^2y^4z-3x^2y^2z^3+5x^2y^2z-2x^2z^3+6x^2z+xy^5+4xy^3z^2-7xy^3+3xyz^4-13xyz^2+12xy-y^4z-2y^2z^3+6y^2z-z^5+6z^3-8z=0\}$.

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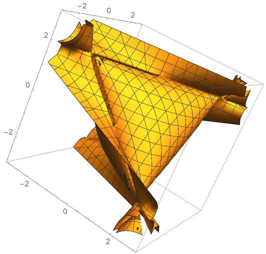

Rank 4 Case

- The fundamental group of the 5-holed sphere is a free group on four letters with the following presentation:

$$\pi = \langle a, b, c, d, e \mid abcde = 1 \rangle \cong \langle a, b, c, d \rangle,$$

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- The character variety is by definition $\mathrm{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))/\!\!/\mathsf{SL}(2,\mathbb{C}) \cong \mathsf{SL}(2,\mathbb{C})^{\times 4}/\!\!/\mathsf{SL}(2,\mathbb{C})$, given by

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- So this is the moduli space of (polystable) flat $\mathrm{SL}(2,\mathbb{C})$-bundles over the 5-holed sphere.

- The coordinate ring has the following presentation:

$$\mathbb{C}[\mathrm{SL}(2,\mathbb{C})^{\times 4}/\!\!/\mathrm{SL}(2,\mathbb{C})] = \mathbb{C}[r_1, \ldots, r_9][t_1, \ldots, t_5]/(f_1, \ldots, f_{14}),$$

where $\{r_1, \ldots, r_9\}$ is a minimal generating set for the rational function field, $\{r_1, \ldots, r_9, t_1, \ldots, t_5\}$ is a minimal generating set for the coordinate ring, and $\{f_1, \ldots, f_{14}\}$ is a minimal generating set for the ideal of relations in terms of the generators.

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- Consequences are that $\mathrm{SL}(2,\mathbb{C})^{\times 4}/\!\!/\mathrm{SL}(2,\mathbb{C})$ embeds in $\mathbb{C}^{14}$ and its dimension is 9. Since it has 14 relations it is very far from a complete intersection like the rank $1, 2, 3$ cases.

- The coordinate ring has the following presentation:

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- More still, at a generic smooth point $[\rho]$, $\{dr_1, ..., dr_9\}$ generates $T^*_{[\rho]}\left(\mathrm{SL}(2,\mathbb{C})^{\times 4} /\!\!/ \mathrm{SL}(2,\mathbb{C})\right) \cong \mathbb{C}^9$.

Here are the formulas for the generators:
$r_1 = \mathrm{tr}(\mathbf{A})$, $r_2 = \mathrm{tr}(\mathbf{B})$, $r_3 = \mathrm{tr}(\mathbf{C})$, $r_4 = \mathrm{tr}(\mathbf{D})$, $r_5 = \mathrm{tr}(\mathbf{AB})$, $r_6 = \mathrm{tr}(\mathbf{AC})$, $r_7 = \mathrm{tr}(\mathbf{AD})$, $r_8 = \mathrm{tr}(\mathbf{BC})$, $r_9 = \mathrm{tr}(\mathbf{BD})$
$t_1 = \mathrm{tr}(\mathbf{CD})$, $t_2 = \mathrm{tr}(\mathbf{ABC})$, $t_3 = \mathrm{tr}(\mathbf{ABD})$, $t_4 = \mathrm{tr}(\mathbf{ACD})$, $t_5 = \mathrm{tr}(\mathbf{BCD})$

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$t_1 = \mathrm{tr}(\mathbf{CD})$, $t_2 = \mathrm{tr}(\mathbf{ABC})$, $t_3 = \mathrm{tr}(\mathbf{ABD})$, $t_4 = \mathrm{tr}(\mathbf{ACD})$, $t_5 = \mathrm{tr}(\mathbf{BCD})$

There are two types of relations (degree 5 and degree 6 in the matrix entries) and the rest is combinatorics.

Here are the formulas for the ideal of relations (f_1 through f_4 are all of one type, f_5 through f_8 are the rank 3 relation for each set of 3, and f_9 through f_{14} are a generalized relation of the same type as f_5 through f_8).

$$
\begin{aligned}
f_1 = {} & 3\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})^2 - 3\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{A})^2 - \\
& 3\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{A})^2 + 3\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{A})^2 + \\
& 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A}) - 3\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{A}) + \\
& 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A}) + 3\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{A}) - \\
& 3\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{A}) - 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{A}) + \\
& 3\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{A}) - 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{A}) - \\
& 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{ABC}) + 6\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{ABD}) - 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{ACD}) - \\
& 12\mathrm{tr}(\mathbf{BCD}) + 6\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B}) + 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C}) + \\
& 6\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{C}) + 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D}) - 6\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})
\end{aligned}
$$

$$
\begin{aligned}
f_2 = {} & \\
& -3\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})^2 + 3\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2 + 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{B})^2 - \\
& 3\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{B})^2 - 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{B}) + \\
& 3\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{B}) - 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B}) - \\
& 3\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - 3\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{B}) + \\
& 3\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{B}) + 3\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{B}) + \\
& 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{B}) - 6\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ABC}) + 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ABD}) + \\
& 12\mathrm{tr}(\mathbf{ACD}) + 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BCD}) - 6\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A}) - 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C}) - \\
& 6\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{D}) - 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D}) + 6\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})
\end{aligned}
$$

$$
\begin{aligned}
f_3 = {} & 3\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})^2 - 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})^2 - \\
& 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{C})^2 + 3\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{C})^2 + \\
& 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{C}) - 3\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{C}) + \\
& 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{C}) - 3\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) + \\
& 3\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 3\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + \\
& 3\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{C}) - 3\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{C}) - \\
& 6\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABC}) - 12\mathrm{tr}(\mathbf{ABD}) - 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ACD}) + \\
& 6\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BCD}) + 6\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{A}) + 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A}) + \\
& 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B}) + 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{D}) - 6\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})
\end{aligned}
$$

$$f_4 = -3\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})^2 + 3\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})^2 + 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 - 3\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 - 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D}) + 3\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{D}) - 3\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{D}) - 3\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) + 3\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 3\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 3\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 3\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 12\mathrm{tr}(\mathbf{ABC}) + 6\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABD}) - 6\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ACD}) + 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BCD}) - 6\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A}) - 6\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{B}) - 6\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B}) - 6\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C}) + 6\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})$$

$$f_5 = 36\mathrm{tr}(\mathbf{AB})^2 + 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{AB}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AB}) - 36\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AB}) + 36\mathrm{tr}(\mathbf{AC})^2 + 36\mathrm{tr}(\mathbf{BC})^2 + 36\mathrm{tr}(\mathbf{ABC})^2 + 36\mathrm{tr}(\mathbf{A})^2 + 36\mathrm{tr}(\mathbf{B})^2 + 36\mathrm{tr}(\mathbf{C})^2 - 36\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A}) - 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{B}) - 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) - 36\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 36\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 144$$

$$f_6 = 36\mathrm{tr}(\mathbf{AC})^2 + 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{AC}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AC}) - 36\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AC}) + 36\mathrm{tr}(\mathbf{AD})^2 + 36\mathrm{tr}(\mathbf{CD})^2 + 36\mathrm{tr}(\mathbf{ACD})^2 + 36\mathrm{tr}(\mathbf{A})^2 + 36\mathrm{tr}(\mathbf{C})^2 + 36\mathrm{tr}(\mathbf{D})^2 - 36\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A}) - 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{C}) - 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 36\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) - 144$$

$$f_7 = 36\mathrm{tr}(\mathbf{BC})^2 + 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{BC}) - 36\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{BC}) - 36\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BC}) + 36\mathrm{tr}(\mathbf{BD})^2 + 36\mathrm{tr}(\mathbf{CD})^2 + 36\mathrm{tr}(\mathbf{BCD})^2 + 36\mathrm{tr}(\mathbf{B})^2 + 36\mathrm{tr}(\mathbf{C})^2 + 36\mathrm{tr}(\mathbf{D})^2 - 36\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{B}) - 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{C}) - 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 36\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) - 144$$

$$f_8 = 36\mathrm{tr}(\mathbf{AB})^2 + 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{AB}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AB}) - 36\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AB}) + 36\mathrm{tr}(\mathbf{AD})^2 + 36\mathrm{tr}(\mathbf{BD})^2 + 36\mathrm{tr}(\mathbf{ABD})^2 + 36\mathrm{tr}(\mathbf{A})^2 + 36\mathrm{tr}(\mathbf{B})^2 + 36\mathrm{tr}(\mathbf{D})^2 - 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A}) - 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{B}) - 36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 36\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 144$$

$$f_9 = 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{AC})^2 - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{AC}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{AC}) - 18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AC}) + 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AC}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AC}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AC}) - 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AC}) + 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AC}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AC}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AC}) + 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{A})^2 + 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})^2 + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C})^2 + 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{C})^2 - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})^2 - 36\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{AD}) - 72\mathrm{tr}(\mathbf{BD}) - 36\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{CD}) + 36\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{ACD}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A}) - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{C})\cdots$$

$$\cdots - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})^2\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})^2\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})^2\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 36\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})$$

$$f_{10} = -18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{AB})^2 + 18\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AB})^2 + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{AB}) + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{AB}) + 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AB}) - 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AB}) - 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AB}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AB}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})^2 - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})^2 + 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{AD}) + 36\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{BD}) + 72\mathrm{tr}(\mathbf{CD}) + 36\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{ABD}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A}) - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})$$

$$f_{11} = -18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})^2 + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BC})^2 + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{BC}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{BC}) - 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{BC}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{BC}) - 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BC}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BC}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})^2 - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C})^2 + 72\mathrm{tr}(\mathbf{AD}) + 36\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BD}) + 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{CD}) + 36\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{BCD}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})^2\mathrm{tr}(\mathbf{D}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})$$

$$
\begin{aligned}
f_{12} = &-18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{AD})^2 + 18\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AD})^2 + \\
&18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{AD}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{AD}) - \\
&18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{AD}) - 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{AD}) + \\
&18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AD}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{AD}) - \\
&18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})^2 - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D})^2 + 18\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 + \\
&36\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{AC}) + 72\mathrm{tr}(\mathbf{BC}) + 36\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{CD}) + \\
&36\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{ACD}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A}) - \\
&18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - \\
&18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) + 18\mathrm{tr}(\mathbf{A})^2\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 36\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - \\
&18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{D}) - \\
&18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) - \\
&18\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})
\end{aligned}
$$

$$f_{13} = 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})^2 - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{BD}) -$$
$$18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{BD}) - 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{BD}) +$$
$$18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{BD}) - 18\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{BD}) +$$
$$18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{BD}) + 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BD}) -$$
$$18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BD}) + 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BD}) -$$
$$18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{BD}) + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{B})^2 +$$
$$18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})^2 + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{D})^2 +$$
$$18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{D})^2 - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})^2 +$$
$$18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 -$$
$$18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})^2 - 72\mathrm{tr}(\mathbf{AC}) - 36\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BC}) -$$
$$36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{CD}) + 36\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{BCD}) -$$
$$18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{B}) +$$
$$18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{C}) + 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) +$$
$$18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{D})\cdots$$

$$\cdots - 18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{ABD})\mathrm{tr}(\mathbf{D}) - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{D}) +$$
$$18\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) +$$
$$18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D}) -$$
$$18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{B})^2\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) + 18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D}) +$$
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$$
\begin{aligned}
f_{14} = &-18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{CD})^2 + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{CD})^2 + \\
&18\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{CD}) + 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BD})\mathrm{tr}(\mathbf{CD}) - \\
&18\mathrm{tr}(\mathbf{BCD})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{CD}) - 18\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{CD}) + \\
&18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{CD}) - 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{CD}) - \\
&18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{C})^2 + 18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C})^2 - 18\mathrm{tr}(\mathbf{AB})\mathrm{tr}(\mathbf{D})^2 + \\
&18\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{D})^2 + 72\mathrm{tr}(\mathbf{AB}) + 36\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{BC}) + \\
&36\mathrm{tr}(\mathbf{AD})\mathrm{tr}(\mathbf{BD}) + 36\mathrm{tr}(\mathbf{ACD})\mathrm{tr}(\mathbf{BCD}) - 36\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - \\
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&18\mathrm{tr}(\mathbf{BC})\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{C}) - 18\mathrm{tr}(\mathbf{AC})\mathrm{tr}(\mathbf{B})\mathrm{tr}(\mathbf{C}) - \\
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\end{aligned}
$$

The above remarks can be generalized to any rank free group (any n-holed sphere). I used a *Mathematica* notebook to perform routine computations, but at no point was an (elimination ideal) algorithm used to generate relations.

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- From results of Florentino-Lawton (2012), we know that the singular locus is exactly the reducible representations (and so corresponds to the free Abelian character variety).

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- So we not only know the local structure but also know the Betti numbers of as well.

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- All of these statements generalize to arbitrary free groups explicitly.

Thank you!

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