MA 523: Homework 5

Carlos Salinas

October 4, 2016

Problem 5.1

Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$, $x \in \mathbb{R}^n$, then $\Delta v = 0$.

Solution. Let

$$O = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be an orthogonal $n \times n$ matrix. We will show that $\Delta v = 0$, where $v(x) = u(Ox)$.

First, let us compute the gradient of v,

$$\begin{aligned} Dv(x) &= Du(Ox) \\ &= Du\big(a_{11}x_1 + \cdots + a_{1n}x_n, \ldots, a_{n1}x_1 + \cdots + a_{nn}x_n\big) \\ &= \Big(\sum\nolimits_{j=1}^n a_{j1}u_{x_j}, \ldots, \sum\nolimits_{j=1}^n a_{jn}u_{x_j}\Big) \\ &= O^{\mathrm{T}} Du(x). \end{aligned}$$

Lastly, we compute the divergence of Dv,

$$\begin{aligned} \Delta v(x) &= \operatorname{div} Dv(x) \\ &= \operatorname{div}\Big(\sum\nolimits_{j=1}^n a_{j1}u_{x_j}, \ldots, \sum\nolimits_{j=1}^n a_{jn}u_{x_j}\Big). \end{aligned}$$

Here the partial derivatives become unwieldy so we will first examine the partial $\frac{\partial}{\partial x_1}$ of the first term and proceed from there. In this case,

$$\begin{aligned} \frac{\partial}{\partial x_1}\sum_{j=1}^n a_{j1}u_{x_j} &= a_{11}(u_{x_1})_{x_1} + a_{21}(u_{x_2})_{x_1} + \cdots + a_{n1}(u_{x_n})_{x_1} \\ &= a_{11}(a_{11}u_{x_1x_1} + a_{21}u_{x_1x_2} + \cdots + a_{n1}u_{x_1x_n}) \\ &\quad + \cdots + a_{n1}(a_{11}u_{x_1x_n} + a_{21}u_{x_2x_n} + \cdots + a_{n1}u_{x_nx_n}) \\ &= a_{11}^2u_{x_1x_1} + 2a_{11}a_{21}u_{x_1x_2} + 2a_{11}a_{31}u_{x_1x_3} + \cdots + a_{21}^2u_{x_2x_2} \\ &\quad + \cdots + a_{k1}^2u_{x_kx_k} + \cdots + a_{n1}^2u_{x_nx_n}. \end{aligned}$$

Similarly, taking the k^{th} partial of the k^{th} entry of Dv, we have

$$\begin{aligned} \frac{\partial}{\partial x_k}\sum_{j=1}^n a_{jk}u_{x_j} &= a_{1k}(a_{1k}u_{x_1x_1} + \cdots + a_{nk}u_{x_1x_n}) \\ &\quad + \cdots + a_{nk}(a_{1k}u_{x_1x_n} + \cdots + a_{nk}u_{x_nx_n}) \\ &= a_{1k}^2u_{x_1x_1} + a_{2k}^2u_{x_2x_2} + \cdots + a_{kk}^2u_{x_kx_k} \\ &\quad + \cdots + a_{nk}^2u_{x_nx_n} + \{\text{mixed terms}\}. \end{aligned} \tag{5.1}$$

Now, since O is orthogonal, we have

$$\begin{aligned}
OO^{\mathrm{T}} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}^2 + \cdots + a_{1n}^2 & a_{11}a_{21} + \cdots + a_{1n}a_{2n} & \cdots & a_{11}a_{n1} + \cdots + a_{1n}a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}a_{11} + \cdots + a_{nn}a_{1n} & a_{n1}a_{21} + \cdots + a_{nn}a_{2n} & \cdots & a_{n1}^2 + \cdots + a_{nn}^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}.
\end{aligned}$$

We can sum up the results of our calculation as

$$\begin{cases} \text{(a)} \ \sum_{j=1}^n a_{kj}a_{\ell j} = \sum_{j=1}^n a_{kj}^2 = 1 & \text{if } k = \ell, \\ \text{(b)} \ \sum_{j=1}^n a_{kj}a_{\ell j} = 0 & \text{if } k \neq \ell. \end{cases} \tag{5.2}$$

for $1 \leq k, \ell \leq n$.

Now, going back to (5.1), we have

$$\begin{aligned}
\operatorname{div} Dv &= \sum_{k=1}^n \left[\frac{\partial}{\partial x_k} \sum_{j=1}^n a_{jk} u_{x_j} \right] \\
&= (a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2) u_{x_1 x_1} + (a_{12}^2 + a_{22}^2 + \cdots + a_{2n}^2) u_{x_2 x_2} \\
&\quad + \cdots + (a_{1n}^2 + \cdots + a_{nn}^2) u_{x_n x_n} + \{\text{mixed terms}\} \\
&= u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} \\
&= 0,
\end{aligned} \tag{5.3}$$

as desired.

All that is left to show as that the mixed terms in the expression above actually have coefficients of the form in (5.2) (b). And in fact one can see, expanding (5.3), that the mixed terms have the form

$$\sum_{j=1}^n a_{kj}a_{\ell j} = 0.$$

For example, the first member in the mixed terms sequence is

$$2(a_{11}a_{21} + a_{12}a_{22} + \cdots + a_{1n}a_{2n}) u_{x_1 x_2} = 0.$$

(Time permits, I will provide a better argument than simply expanding (5.3); but a little routine calculation shows that these terms in fact have the form we have described.) ■

Problem 5.2

Let $n = 2$ and U be the halfplane $\{ x_2 > 0 \}$. Prove that

$$\sup_U u = \sup_{\partial U} u$$

for $u \in C^2(U) \cap C(\bar{U})$ which are harmonic in U under the additional assumption that u is bounded from above in $\bar{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.)
[*Hint:* Take for $\varepsilon > 0$ the harmonic function

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region $\left\{ x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0 \right\}$ with large a. Let $\varepsilon \to 0$.]

Solution. Consider the harmonic function

$$u_\varepsilon(x_1, x_2) := u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Set $U_a := \left\{ x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0 \right\}$.

First, we note that $u_\varepsilon \uparrow u$ as $\varepsilon \to 0$ pointwise, i.e., given $\eta > 0$, for

$$0 < \varepsilon(x_1, x_2) < \eta / \ln \sqrt{x_1^2 + (x_2 + 1)^2},$$

we have

$$\begin{aligned} |u_\varepsilon(x_1, x_2) - u(x_1, x_2)| &= \left| u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} - u(x_1, x_2) \right| \\ &= \left| \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} \right| \\ &= \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2} \\ &< \eta, \end{aligned}$$

for any $(x_1, x_2) \in U_a$.

Moreover, a simple calculation shows that u_ε is in fact harmonic. By the linearity the Laplacian, it suffices to show that the Laplacian of

$$v_\varepsilon(x) := \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}$$

is 0. First, we calculate he partial derivatives $\frac{\partial^2}{\partial x_1 \partial x_1}$ and $\frac{\partial^2}{\partial x_2 \partial x_2}$

$$\begin{aligned} \frac{(v_\varepsilon)_{x_1}}{\varepsilon} &= -\frac{x_1}{x_1^2 + (x_2 + 1)^2} & \frac{(v_\varepsilon)_{x_2}}{\varepsilon} &= -\frac{(x_2 + 1)}{x_1^2 + (x_2 + 1)^2} \\ \frac{(v_\varepsilon)_{x_1 x_1}}{\varepsilon} &= -\frac{-x_1^2 + (x_2 + 1)^2}{\left(x_1^2 + (x_2 + 1)^2\right)^2} & \frac{(v_\varepsilon)_{x_2 x_2}}{\varepsilon} &= -\frac{x_1^2 - (x_2 + 1)^2}{\left(x_1^2 + (x_2 + 1)^2\right)^2}. \end{aligned}$$

Thus,

$$\Delta u_\varepsilon = \Delta u + \Delta v_\varepsilon = \Delta v_\varepsilon = \varepsilon\left(-\frac{-x_1^2+(x_2+1)^2}{\left(x_1^2+(x_2+1)^2\right)^2} - \frac{x_1^2-(x_2+1)^2}{\left(x_1^2+(x_2+1)^2\right)^2}\right) = 0.$$

Now, observe that, for any a, by the maximum principle, we have

$$\max_{\bar{U}_a} u_\varepsilon = \max_{\partial U_a} u_\varepsilon$$

for any a. Choose a large enough so

$$\sup_{\partial U_a} u_\varepsilon \leq \sup_{\partial U} u.$$

Then,

$$\sup_{\bar{U}_a} u_\varepsilon \leq \sup_{\partial U} u$$

so, taking $a \to \infty$, we have

$$\sup_{\bar{U}} u_\varepsilon \leq \sup_{\partial U} u.$$

Thus, for every $x_1, x_2 \in U$,

$$u(x_1,x_2) - \varepsilon \ln\sqrt{x_1^2+(x_2+1)^2} < \sup_{\partial U} u.$$

Letting $\varepsilon \to 0$, we achieve the desired inequality, i.e.,

$$\sup_{\bar{U}} u \leq \sup_{\partial U} u.$$

The last inequality is obvious and stems from the fact that $\partial U \subset \bar{U}$, i.e., the inequality

$$\sup_{\partial U} u \leq \sup_{\bar{U}} u.$$

We conclude that

$$\sup_{\partial U} u = \sup_{\bar{U}} u,$$

as was to be shown. ■

Problem 5.3

Let $U \subset \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \ldots, u^m$ are harmonic in U and

$$v := \varphi(u_1, \ldots, u_m).$$

Prove v is subharmonic.

[*Hint:* Convexity for a smooth function $\varphi(z)$ is equivalent to $\sum_{j,k=1}^m \varphi_{z_j,z_k}(z)\xi_j\xi_k \geq 0$ for any $\xi \in \mathbb{R}^m$.]

(b) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. (Assume that harmonic functions are C^∞.)

Solution. For part (a), by the chain rule, we have

$$v_{x_i} = \varphi_{u_1} u^1_{x_i} + \cdots + \varphi_{u_m} u^m_{x_i}.$$

Taking another partial, we have

$$\begin{aligned} v_{x_i x_i} &= (v_{x_i})_{x_i} \\ &= \frac{\partial}{\partial x_i}\big(\varphi_{u_1} u^1_{x_i} + \cdots + \varphi_{u_m} u^m_{x_i}\big) \\ &= \varphi_{u_1} u^1_{x_i x_i} + \cdots + \varphi_{u_m} u^m_{x_i x_i} \\ &\quad + \big(\varphi_{u_1 u_1} u^1_{x_i} + \cdots + \varphi_{u_1 u_m} u^m_{x_i}\big) u^1_{x_i} \\ &\quad\quad + \cdots + \big(\varphi_{u_1 u_m} u^1_{x_i} + \cdots + \varphi_{u_m u_m} u^m_{x_i}\big) u^m_{x_i}. \end{aligned} \tag{5.4}$$

Now, taking the sum

$$\begin{aligned} \sum_{i=1}^n v_{x_i x_i} &= \sum_{i=1}^n \sum_{j=1}^m \varphi_{u_j} u^j_{x_i x_i} \\ &= \sum_{j=1}^m \sum_{i=1}^n \varphi_{u_j} u^j_{x_i x_i} \\ &= \sum_{j=1}^m \big(\varphi_{u_j} u^j_{x_1 x_1} + \cdots + \varphi_{u_j} u^j_{x_n x_n}\big) \\ &= \sum_{j=1}^m \varphi_{u_j}\big(u^j_{x_1 x_1} + \cdots + u^j_{x_n x_n}\big) \\ &= 0, \end{aligned}$$

since $\Delta u^j = 0$ for all j.

What about the remaining terms in (5.4)? These terms can be written in the form

$$\sum_{j,k=1}^{m} \varphi_{u_j u_k}(u)\xi_j\xi_k,$$

where $\xi_i = \big(u^1_{x_i}, \ldots, u^m_{x_i}\big)(x_1, \ldots, x_n) \in \mathbb{R}^m$ for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Since φ is convex, by assumption, this quantity is greater than or equal to 0.

Thus, $\Delta v \geq 0$ so v is subharmonic.

For part (b), we have

$$v = |Du|^2 = u^2_{x_1} + \cdots + u^2_{x_n}.$$

Taking the partial derivative with respect to x_i, we have

$$\begin{aligned} v_{x_i} &= \frac{\partial}{\partial x_i}\big(u^2_{x_1} + \cdots + u^2_{x_n}\big) \\ &= 2u_{x_1}u_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_n}, \end{aligned}$$

and again

$$\begin{aligned} v_{x_ix_i} &= (v_{x_i})_{x_i} \\ &= \frac{\partial}{\partial x_i}\big(2u_{x_1}u_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_n}\big) \\ &= 2u_{x_1}u_{x_1x_ix_i} + 2u^2_{x_1x_i} + \cdots + 2u_{x_n}u_{x_ix_ix_n} + 2u^2_{x_ix_n} \\ &= 2\sum_{j=1}^{n}\big(u_{x_j}u_{x_jx_ix_i} + u^2_{x_jx_i}\big). \end{aligned}$$

Then

$$\begin{aligned} \frac{\Delta v}{2} &= \sum_{i,j=1}^{n}\big(u_{x_j}u_{x_jx_ix_i} + u^2_{x_jx_i}\big) \\ &= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + \sum_{i,j=1}^{n} u^2_{x_jx_i}, \end{aligned}$$

splitting the second term into the sum

$$\begin{aligned} &= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + \sum_{1\leq j<i\leq n} u^2_{x_jx_i} \\ &\quad + \sum_{1\leq i<j\leq n} u^2_{x_jx_i} + \sum_{1\leq i=j\leq n} u^2_{x_ix_i}, \end{aligned}$$

where the last term is 0 since u is harmonic, giving us

$$\begin{aligned} &= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + \sum_{1\leq j<i\leq n} u^2_{x_jx_i} + \sum_{1\leq i<j\leq n} u^2_{x_jx_i} \\ &= \sum_{i,j=1}^{n} u_{x_j}u_{x_jx_ix_i} + 2\sum_{1\leq j<i\leq n} u^2_{x_jx_i}, \end{aligned}$$

here $\sum_{j=1}^n u_{x_i x_j x_j} = \Delta u_{x_i} = 0$ since the derivatives of harmonic functions are harmonic, so

$$\begin{aligned}
&= \sum_{j=1}^n u_{x_j}(\Delta u_{x_j}) + 2 \sum_{1 \le j < i \le n} u_{x_j x_i}^2 \\
&= 2 \sum_{1 \le j < i \le n} u_{x_j x_i}^2 \\
&\ge 0,
\end{aligned}$$

as desired. That is, $\Delta v \ge 0$ so v is subharmonic. ■