MA557 Problem Set 5

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Problem 5.1

For I an R-ideal consider the multiplicatively closed set $S = 1 + I$. Show that

(a) $\tilde{S} = R \smallsetminus \bigcup \mathfrak{m}$, where the union is taken over all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R) \cap V(I)$.
(b) $\mathfrak{m}\text{-Spec}(S^{-1}R)$ and $\mathfrak{m}\text{-Spec}(R/I)$ are homeomorphic.

Proof. (a) By 4.19, we have

$$\tilde{S} = R \smallsetminus \bigcup_{\substack{\mathfrak{p}\in\mathrm{Spec}(R)\\ \mathfrak{p}\cap S=\emptyset}} \mathfrak{p}.$$

But $\mathfrak{p} \cap S = \mathfrak{p} \cap (1 + I) = \emptyset$ if and only if $\mathfrak{p} + I \neq R$ if and only if there is some maximal ideal $\mathfrak{m} \supset \mathfrak{p} + I$.

For the former equivalence: $\implies$ Suppose that $\mathfrak{p} \cap S = \mathfrak{p} \cap (1 + I) = \emptyset$, then if $\mathfrak{p} + I = R$ for some $x \in \mathfrak{p}$, $y \in I$ we have $x + y = 1$. But then $x = 1 - y \in \mathfrak{p} \cap S$; this is a contradiction. $\impliedby$ Conversely, if $\mathfrak{p} \cap S \neq \emptyset$, $x = 1 + y \in \mathfrak{p}$ for some $y \in I$ so $x - y = (1 + y) - y = 1 \in \mathfrak{p} + I$ implies $\mathfrak{p} + I = R$.

For the latter equivalence: $\implies$ Suppose $\mathfrak{p} + I \neq R$, then $\mathfrak{p} + I$ is a proper ideal of R so, by 1.5, is contained in a maximal ideal $\mathfrak{m}$. $\impliedby$ Conversely, if $\mathfrak{m} \subsetneq R$ is a maximal ideal containing $\mathfrak{p} + I$ then $\mathfrak{p} + I \neq R$ for otherwise $\mathfrak{m} = R$. Then it suffices to take the union over all maximal ideals $\mathfrak{m} \supset I$.

(b) We will show that $\mathfrak{m}\text{-Spec}(S^{-1}R) \approx \mathfrak{m}\text{-Spec}(R) \cap V(I)$ and $\mathfrak{m}\text{-Spec}(R/I) \approx \mathfrak{m}\text{-Spec}(R) \cap V(I)$ so that, by the transitivity of homeomorphism, we have $\mathfrak{m}\text{-Spec}(S^{-1}R) \approx \mathfrak{m}\text{-Spec}(R/I)$. By 4.21(a), $\mathrm{Spec}(R/I) \approx V(I)$ so the restriction $\mathfrak{m}\text{-Spec}(R/I) \approx \mathfrak{m}\text{-Spec}(R/I) \cap V(I)$. To see that $\mathfrak{m}\text{-Spec}(S^{-1}R) \approx \mathfrak{m}\text{-Spec}(R) \cap V(I)$, let $\varphi \colon R \to S^{-1}R$ be the canonical homomorphism sending $x \mapsto x/1$, then φ induces a continuous closed map ${}^a\varphi \colon \mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ taking $\bar{\mathfrak{p}} \mapsto \mathfrak{p}$, i.e., ideal extension. Thus, by 4.13(d), there is a one-to-one correspondence between $\bar{\mathfrak{p}} \in \mathrm{Spec}(S^{-1}M)$ and its extension $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathfrak{p} \cap S = \emptyset$ so that it suffices to show that ${}^a\varphi(\mathfrak{m}\text{-Spec}(S^{-1}R)) = \mathfrak{m}\text{-Spec}(R) \cap V(I)$. But this is easy: If $\bar{\mathfrak{m}} \in \mathfrak{m}\text{-Spec}(S^{-1}R)$ then its contraction is a maximal ideal $\mathfrak{m} \supset I$ by part (a), hence is in $\mathfrak{m}\text{-Spec}(R) \cap V(I)$. Conversely, if $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R) \cap V(I)$, again, by part (a), $\mathfrak{m}$ is a maximal ideal not meeting S so that by 4.13(d), there exist some maximal ideal $\bar{\mathfrak{m}}$ contracting to $\mathfrak{m}$. It follows that $\mathfrak{m}\text{-Spec}(S^{-1}R) \approx \mathfrak{m}\text{-Spec}(R/I)$. ■

Problem 5.2

Show that the following are equivalent for a ring R:

(a) there exist rings $R_1 \neq 0$ and $R_2 \neq 0$ so that $R \cong R_1 \times R_2$;
(b) there exist an idempotent $e \in R$ with $e \neq 0$ and $e \neq 1$;
(c) $\operatorname{Spec}(R)$ is disconnected.

Proof. (a) $\iff$ (b) is immediate for suppose $R \cong R_1 \times R_2$ by $\varphi\colon R \to R_1 \times R_2$. Then, since φ is a bijection, there exist an $r \in R$ that maps to the idempotent element $(1,0) \in R_1 \times R_2$.

Conversely, suppose $e \in R$ is idempotent. Then $e' = 1 - e$ is also idempotent since

$$(e')^2 = (1-e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e.$$

Moreover

$$ee' = e(1-e) = e - e^2 = e - e = 0.$$

Let R_1 and R_2 be the subrings of R generated by e and e', respectively. Then we claim that $R \cong R_1 \times R_2$ via $\varphi(r) = (re, re')$. It is clear that φ is a ring homomorphism: take $r_1, r_2 \in R$ then

$$\begin{aligned} \varphi(r_1 + r_2) &= ((r_1+r_2)e, (r_1+r_2)e') \\ &= (r_1 e + r_2 e, r_1 e' + r_2 e') \\ &= (r_1 e, r_1 e') + (r_2 e, r_2 e') \\ &= \varphi(r_1) + \varphi(r_2) \end{aligned} \qquad \begin{aligned} \varphi(r_1 r_2) &= (r_1 r_2 e, r_1 r_2 e') \\ &= \big(r_1 r_2 e^2, r_1 r_2 (e')^2\big) \\ &= (r_1 e, r_1 e')(r_2 e, r_2 e') \\ &= \varphi(r_1)\varphi(r_2). \end{aligned}$$

To prove surjective take $(r,s) \in R_1 \times R_2$ then, $r = r_1 e$ and $s = r_2 e'$ for $r_1, r_2 \in R$ then

$$\begin{aligned} \varphi(r_1 e + r_2 e') &= \varphi(r_1 e) + \varphi(r_2 e') \\ &= (r_1 e, r_1 e e') + (r_2 e' e, r_2 e e') \\ &= (r_1 e, 0) + (0, r_2 e') \\ &= (r_1 e, r_2 e') \\ &= (r, s). \end{aligned}$$

To prove injectivity take $r \in \ker \varphi$. Then $\varphi(r) = (re, re') = (0,0)$. Then $re - re' = r(e - e') = r \cdot 1 = 0$ so $r = 0$.

(a) $\implies$ (c) Recall that a topological space X is disconnected if there exist disjoint open sets A, B with $X = A \cup B$. Suppose $R \cong R_1 \times R_2$. Then $\operatorname{Spec}(R) \approx \operatorname{Spec}(R_1 \times R_2)$: Keeping the notation as before, φ is a set bijection so it induces a bijection, call it φ^*, on $\operatorname{Spec}(R) \to \operatorname{Spec}(R_1 \times R_2)$ by sending $\operatorname{Spec}(I) \mapsto \operatorname{Spec}(\varphi(I))$; Now let $I \subset R$ be an ideal, then

$$\varphi^*(V(I)) = \varphi^*(V(eI + e'I)) = V(\varphi(eI) + \varphi(e'I)) = V(eI \times e'I)$$

is closed. Thus, φ^* is a homeomorphism. Now, we claim that the sets $A = V(R_1 \times 0)$ and $B = V(0 \times R_2)$ constitute a separation of R. First note by 4.20(2) that

$$A \cup B = V(R_1 \times 0) \cup V(0 \times R_2) = V((R_1 \times 0) \cap (0 \times R_2)) = V(0) = \operatorname{Spec}(R).$$

Moreover

$$A \cap B = V(R_1 \times 0) \cap V(0 \times R_2) = V(R_1 \times 0 + 0 \times R_2) = V(R) = \emptyset.$$

Since both A and B are closed, it follows that A, B forms a separation of $\operatorname{Spec}(R)$.

(c) $\implies$ (b) As suggested by Matsumura: If $\operatorname{Spec}(R) = V(I_1) \cup V(I_2)$ with $V(I_1) \cap V(I_2) \cap = \emptyset$, then $I_1 + I_2 = R$ and $I_1 I_2 \subset \operatorname{nil}(R)$. So $1 = e_1 + e_2$ and $(e_1 e_2)^n = 0$ for some $e_1 \in I_1$, $e_2 \in I_2$. So we have that

$$1 = (e_1 + e_2)^{2n} = e_1^n x_1 + e_2^n x_2$$

with $x_1, x_2 \in R$. So $e = e_1^n x_1$ satisfies $e(1-e) = 0$, i.e., e is idempotent. ■

PROBLEM 5.3

A topological space is called *Noetherian* if the set of closed sets satisfies the dcc. Show that if a ring R is Noetherian then so is $\operatorname{Spec}(R)$, but that the converse does not hold.

Proof. We will first prove the following useful results:

Lemma. *Let R be a commutative ring with identity. Then*

(i) $V(I) = V(\sqrt{I})$.
(ii) $I \subset J$ *implies* $V(I) \supset V(J)$.
(iii) $V(I) \supset V(J)$ *implies* $\sqrt{I} \subset \sqrt{J}$.

Proof of lemma. (i) It is clear that for every prime ideal $\mathfrak{p} \supset \sqrt{I}$ we have $\mathfrak{p} \supset I$ so it suffice to prove that if $\mathfrak{p} \supset I$ then $\mathfrak{p} \supset \sqrt{I}$. But this is clear since if $x \in \sqrt{I}$ then $x^k \in I$ for some positive integer k so $x^k \in \mathfrak{p}$ and since $\mathfrak{p}$ is prime $x \in \mathfrak{p}$. Thus, $V(I) = V(\sqrt{I})$.

(ii) Suppose $I \subset J$. Then every prime ideal $\mathfrak{p} \supset J$ must also contain I. Thus, $V(I) \supset V(J)$.

(iii) Suppose $V(I) \supset V(J)$. Then, for every prime ideal $\mathfrak{p} \supset J$, $\mathfrak{p} \supset I$ so

$$\sqrt{J} = \bigcap_{\mathfrak{p} \supset J} \mathfrak{p} \supset \bigcap_{\mathfrak{p} \supset J} \mathfrak{p} \cap \bigcap_{\substack{\mathfrak{q} \supset I \\ \mathfrak{q} \not\supset J}} \mathfrak{q} = \sqrt{I}.$$ ♣

It suffices to reduce to the case of varieties of ideals in R since varieties generate the Zariski topology on $\operatorname{Spec}(R)$. Suppose

$$V(I_1) \supset V(I_2) \supset \cdots$$

is a descending chain of varieties in $\operatorname{Spec}(R)$. Then, by the (iii) of the lemma and the nullstellensatz, the latter chain is in one-to-one correspondence with the ascending chain of radical ideals

$$\sqrt{I_1} \subset \sqrt{I_2} \subset \cdots$$

which must stabilize since R is Noetherian. It follows that the chain $V(I_1) \supset V(I_2) \supset \cdots$ stabilizes so $\operatorname{Spec}(R)$ is Noetherian. ■

Problem 5.4

A nonempty closed subset V of a topological space is called *irreducible* if $V = V_1 \cup V_2$, V_1 and V_2 closed subset, implies $V_1 = V$ or $V_2 = V$.

(a) Show that in a Noetherian topological space every nonempty closed subset is a finite union of irreducible closed subsets.
(b) Show that $V(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec}(R)$, are exactly the irreducible closed subsets of $\operatorname{Spec}(R)$.

Proof. (a) Let X be a Noetherian topological space. Let

$$\Lambda = \{\, V \subset X \mid V \text{ is closed and not a finite union of irreducible closed subsets}\}.$$

Then, by the dcc, Λ contains a minimal element, say W. Then W is not irreducible so we can write $W = W_1 \cup W_2$ where $W_1 \neq W$ and $W_2 \neq W$. By minimality of W, W_1 and W_2 are finite unions of irreducible closed subsets so $W_1 = \bigcup_{i=1}^k W_1^{(i)}$ and $W_2 = \bigcup_{i=1}^\ell W_2^{(i)}$ so

$$W = W_1 \cup W_2 = \left(\bigcup_{i=1}^{\ell} W_1^{(i)}\right) \cup \left(\bigcup_{i=1}^{k} W_2^{(i)}\right)$$

a contradiction. Thus, every closed subset V can be expressed as the finite union of irreducible closed subsets.

(b) We prove the contrapositive. Suppose that $I \subset R$ is not prime. Then we can find $x, y \in R$ with $xy \in I$, but $x \notin I$, $y \notin I$. Thus,

$$V((I,x)) \cup V((I,y)) = V((I,x) \cap (I,y)) = V(I),$$

but neither $V((I,x)) \neq V(I)$ or $V((I,y)) \neq V(I)$ so $V(I)$ is not irreducible. ■

Problem 5.5

Show that a Noetherian ring has only finitely many minimal prime ideals.

Proof. Since R is Noetherian, by Problem 5.3, $\operatorname{Spec}(R)$ is Notherian, that is, it satisfies the dcc. Thus, by Problem 5.4, $\operatorname{Spec}(R)$ is the union of finitely many irreducible subsets $V(0) = \bigcup_{i=1}^{n} V(\mathfrak{p}_i)$ where $\mathfrak{p}_i$ is prime. Now, suppose $\mathfrak{q}$ is a minimal prime (we are guaranteed one if $R \neq 0$ by the next problem). Then $\mathfrak{q} \in V(0)$ so $\mathfrak{q} \in V(\mathfrak{p}_i)$ for some $1 \leq i \leq n$. Then $\mathfrak{q} \supset \mathfrak{p}_i$, but by minimality $\mathfrak{q} = \mathfrak{p}_i$. It follows that R contains $\leq n$ minimal prime ideals, in particular, finitely many. ■

Problem 5.6

Show that a nonzero ring has at least one minimal prime ideal.

Proof. We will proceed by Zorn's lemma. Let Λ be the set of all prime ideals of R. This set is nonempty by 1.4 since every nonzero ring contains a maximal ideal, in particular, a prime ideal. Λ is ordered by reverse-inclusion. Let $L \subset \Lambda$ be a totally ordered subset of Λ. We claim that $\mathfrak{q} = \bigcap_{\mathfrak{p}\in L} \mathfrak{p}$ is a prime ideal: It is clear that $\mathfrak{q}$ is an ideal. Suppose $xy \in \mathfrak{q}$. Then $\mathfrak{x}\mathfrak{y} \in \mathfrak{p}$ for every $\mathfrak{p} \in L$. Let K be the collection of all $\mathfrak{p} \ni x$. Then either $\bigcap_{\mathfrak{p}\in K} \mathfrak{p} = \mathfrak{q}$ (in which case we are done) or $\bigcap_{\mathfrak{p}\in K} \mathfrak{p} \supset \mathfrak{q}$ where $x \notin \mathfrak{q}$. But then, since every $\mathfrak{p} \in L$ is prime, they must contain x so $x \in \bigcap_{\mathfrak{p}\in K} \mathfrak{p}$. Thus, $\mathfrak{q}$ is prime. By Zorn's lemma, it follows that Λ contains a minimal element, in this case, a smallest (by reverse-inclusion) prime ideal. ∎