MA 523: Homework 1

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Problem 1.1 (Taylor's formula)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to \mathbf{0}$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) \coloneqq f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. ▸ Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) \coloneqq f(tx)$. We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Proof of claim. We shall proceed by induction on m. The case $m = 1$ follows easily from the chain rule:

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\ &= D^{(1,0,\ldots,0)} f(tx) x_1 + \cdots + D^{(0,\ldots,0,1)} f(tx) x_n \\ &= \big(D^{(1,0,\ldots,0)} x_1 + \cdots + D^{(0,\ldots,0,1)} x_n\big) f(tx) \end{aligned}$$

which we can write compactly as

$$= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

More generally, applying the equation above recursively, we have

$$\frac{d^m}{dt^m} g(t) = \big(D^{(1,0,\ldots,0)} x_1 + \cdots + D^{(0,\ldots,0,1)} x_n\big)^m f(tx)$$

by the multinomial theorem

$$\begin{aligned} &= \sum_{|\alpha|=m} \binom{|\alpha|}{\alpha} D^\alpha x^\alpha f(tx) \\ &= \sum_{|\alpha|=m} \binom{|\alpha|}{\alpha} D^\alpha f(tx) x^\alpha \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha \end{aligned}$$

as desired. □

Now, applying Taylor's formula in 1 variable to $g(t)$

$$\begin{aligned} g(t) &= \sum_{i=0}^{k} \frac{g^{(i)}(0)}{i!} t^i + \mathrm{R}_k(g) \\ &= \sum_{i=0}^{k} \frac{1}{i!} \sum_{|\alpha|=i} \frac{i!}{\alpha!} D^\alpha f(tx) x^\alpha + \mathrm{R}_k(g) \\ &= \sum_{i=0}^{k} \sum_{|\alpha|=i} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha t^i + \mathrm{R}_k(g) \\ &= \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha t^i + \mathrm{R}_k(g) \end{aligned}$$

and evaluating at $t = 1$ we have

$$\begin{aligned} g(1) &= \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha t^i \\ &= \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{R}_k(g) \\ &= \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{R}_k(g) \end{aligned}$$

where the remainder is given by

$$\mathrm{R}_k(g) = \frac{1}{k!} \int_0^1 (1-\tau) \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} D^\alpha f(0) x^\alpha \sim \mathrm{O}\left(|x|^{k+1}\right)$$

so

$$g(1) = f(x) = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}_k\left(|x|^{k+1}\right)$$

as desired ◂

Problem 1.2

Write down the characteristic equation for the p.d.e.

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ▸ For reference, formula (5) in §2.1.2 of [E] is

$$u(x,t) = g(x - tb) + \int_0^1 f\big(x + (s - t)b, s\big)\, ds \qquad (x \in \mathbb{R}^n, t > 0)$$

Let $x \in \mathbb{R}^n$ ◂

Problem 1.3

Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ►

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Problem 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\left(u_{x_1}^2 + u_{x_2}^2\right)$$

find a solution with $u(x_1, 0) = \left(1 - x_1^2\right)/2$.

Solution. ►

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