# Introduction to Character Varieties, Part I

Or how I learned to stop worrying and love examples.

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# Outline

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   - Ex: $\mathrm{SL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{R})$ or $\mathrm{SU}(n)$ or $\mathrm{SO}(n)$

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- However, this does show that

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- Since $G$ is admits a faithful linear representation $G \hookrightarrow \mathrm{GL}(V)$, $\mathrm{Hom}(\Gamma, G) \subset \mathrm{Hom}(\Gamma, \mathrm{GL}(V))$ is subspace of traditional representations.

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- If $G$ is complex reductive, then the character variety is the GIT quotient (hence an algebraic set; a union of varieties)

# The General Framework

Given a concrete category $\mathcal{C}$, suppose you would like to understand the objects in $\mathcal{C}$, denoted $\mathrm{Obj}(\mathcal{C})$. Suppose there is an equivalence relation $\sim$ on $\mathrm{Obj}(\mathcal{C})$ given by morphisms in $\mathcal{C}$.

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Form the category $\mathcal{C}$ with objects the polystable $G$-representations of $\Gamma$, and morphisms given by post-composing with $\mathrm{Aut}(G)$; and equivalence classes given by the restricted action of $\mathrm{Inn}(G)$. Then $\mathrm{Iso}(\mathcal{C}) = \mathfrak{X}_\Gamma(G)$.

# Just one matrix.

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- $\mathrm{Hom}(\mathbb{Z}, \mathsf{SL}(2, \mathbb{C})) \cong \mathsf{SL}(2, \mathbb{C})$ since a homomorphism is determined by where 1 maps to, and every choice is possible.

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- Thus, $\mathfrak{X}_{\mathbb{Z}}(\mathrm{SL}(2, \mathbb{C})) = \mathbb{C}$.

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- As with the first example, the closed orbits correspond to diagonalizable matrices over $\mathbb{C}$.
- If a matrix in $\mathsf{SL}(2, \mathbb{R})$ is diagonalizable over $\mathbb{R}$, then it corresponds to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some $\lambda \in \mathbb{R}^*$.

- Since $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ are conjugate in $\mathsf{SL}(2, \mathbb{R})$, we can suppose $|\lambda| > |\lambda^{-1}|$ (if $\lambda \neq \pm 1$).

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- Thus the elements of $\mathfrak{X}_{\mathbb{Z}}(\mathsf{SL}(2,\mathbb{R}))$ corresponding to these kind of matrices are parametrized by the space

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- Similarly the space of matrices in $\mathrm{SL}(2,\mathbb{R})$ diagonalizable over $\mathbb{C} \setminus \mathbb{R}$ is parametrized by

$$\mathcal{D}_{\mathbb{C}} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^*, \ z + z^{-1} \in \mathbb{R} \right\}.$$

- The condition $z + z^{-1} \in \mathbb{R}$ is equivalent to $|z| = 1$, so these matrices are in fact in $\mathrm{SO}(2, \mathbb{C})$.

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- Hence the corresponding ones in $\mathrm{SL}(2, \mathbb{R})$ belong to $\mathrm{SO}(2)$, and are of the form

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with $0 \leq \theta < 2\pi$.

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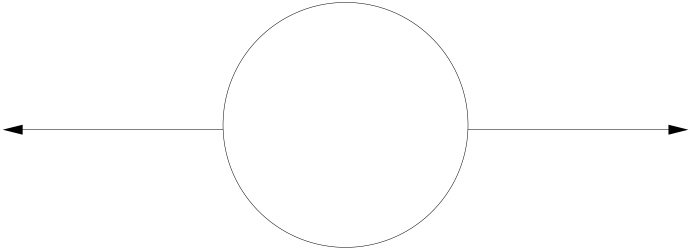

Figure : $\mathfrak{X}_{\mathbb{Z}}(\mathrm{SL}(2,\mathbb{R})) \cong \mathbb{R} \setminus (-1,1) \cup \{z \in \mathbb{C} \setminus \mathbb{R} \mid |z| = 1\}$

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## Punctured Torus $T$: fundamental group is free of rank 2.

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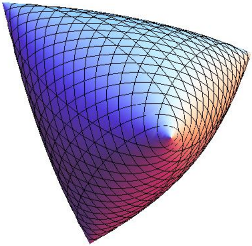

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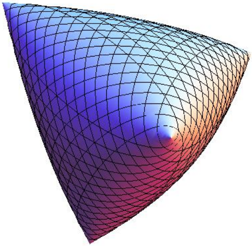

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3. Thus, $\mathfrak{X}_{\mathbb{Z}^2}(\mathsf{SU}(2)) \cong \partial\mathfrak{X}_{\pi_1(T)}(\mathsf{SU}(2)) \cong S^2$.

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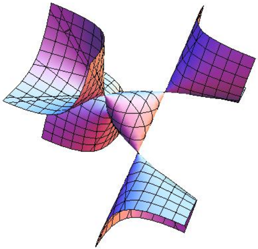
Here is what $\mathfrak{X}_{\pi_1(T)}(\mathrm{SU}(2)) \cup W$ looks like:



5. In 2006, L- showed

$$\begin{tikzcd} \mathfrak{X}_{\pi_1(T)}(\mathsf{SL}(3,\mathbb{C})) \arrow[r, hook] \arrow[d] & \mathbb{C}^9 \\ \mathbb{C}^8 & \end{tikzcd}$$

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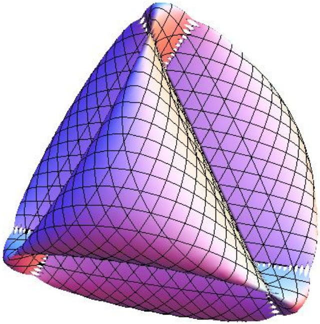
6. Which led to $\mathfrak{X}_{\pi_1(T)}(\mathrm{SU}(3)) \cong S^8$ (Florentino & L-, 2008).

# An example with relations in Γ.

7. $\mathfrak{X}_{\mathbb{Z}^3}(\mathrm{SU}(2))$ is a 3 dimensional orbifold with 8 singularities; each locally $\mathcal{C}_{\mathbb{R}}(\mathbb{R}P^2)$



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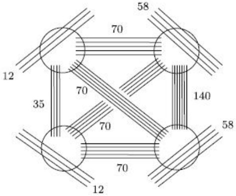


Figure : $m = 42$, $n = 30$; Source: Topology Appl. 156 (2009), no. 14

# If you Google any of the following key words,

you will find that the study of character varieties *at least* touches their corresponding theories:

flat $G$-bundles, $G$-Higgs bundles, holomorphic vector bundles, $(G, X)$-structures, Mirror symmetry, String vacua, Yang-Mills connections, knot invariants, Geometric Langlands, Quantization, Spin Networks, $A$-polynomial, hyperbolic manifolds

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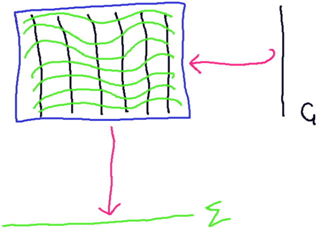

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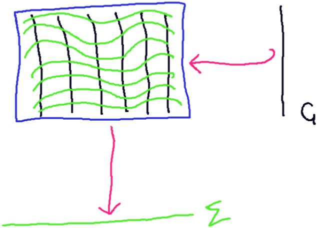
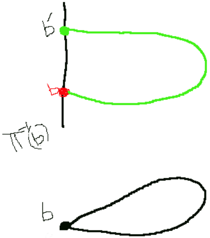



- $\pi_1(\Sigma, b)$ acts (properly and freely) on the trivial bundle by deck transformations on the universal cover $\widetilde{\Sigma}$, and by $\rho$ on $G$.

- Conversely, picking a base point $b$ in the fiber, there must be a $g \in G$ so $g \cdot b = b'$. This defines the *holonomy* homomorphism $\pi_1(\Sigma, b) \longrightarrow G$.




- Changing the base point of the holonomy results in its conjugation.

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Summary and general theme: $\Gamma$ encodes topology and $G$ encodes geometry, and $\mathfrak{X}_\Gamma(G)$ is the space of geometries (of fixed type) that can be imposed on a fixed topology.

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- When $G$ is the isometry group of a model geometry $X$, and there are transverse sections to the flat bundles, the sections define charts from (compact) $\Sigma$ to $X$, and the holonomy defines chart transitions in $G$.

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- This construction includes all (compact) geometric manifolds (with possible non-empty boundary) and orbifolds (Euclidean, Hyperbolic, Projective, Affine, etc).
- Therefore, the moduli spaces of $(G, X)$-structures on $\Sigma$ (when they exists) are analytic subspaces of $\mathfrak{X}_\Gamma(G)$.

## Mathematical Physics

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# They are even getting famous now.

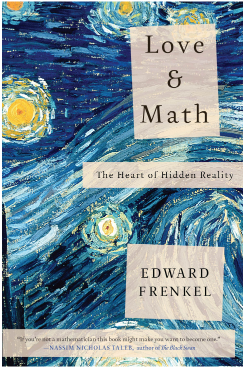


Figure : Character varieties are featured in the recent love story about the Geometric Langlands Program.

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3. And last but not least, recent work from Kapovich & Millson, and also Rapinchuk, have shown that just about all algebraic varieties arise as character varieties (up to a point). So their general study has bearing on the structure of algebraic varieties at large.

# Thank you!

- Part II: Working with Character Varieties, Thursday 10/22/2015
- I gratefully acknowledge support from:

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