MRC 2016 Report: Tropicalization of Character Varieties

Shams Alyusof[1], Corry Bedwell[4], Ellie Dannenberg[2], Dmitry Gekhtman[5], Charlie Katerba[1], Jack Love[1], Chris Manon[1], Giuseppe Martone[3] and Carlos Salinas[6]

[1]George Mason
[2]University of Chicago
[3]University of South California
[4]University of Maryland–College Park
[5]California Institute of Technology
[6]Purdue University

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1 Overview

Tropical geometry is a new and exciting area of mathematics that is best described as piece-wise linear algebraic geometry. In tropical geometetry the sum of two real numbers $x \oplus y$ is their maximum and the product $x \odot y$ their sum. This, together with a minimum element $-\infty$, gives us the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$. In the tropical setting, polynomials become picewise linear functions and algebraic varieties give way to tropical varieties—which are in some sense "skeletons" of the original variety.

During the introductory talks by Manon, we decided to try our hand at "tropicalizing" some of the $\mathrm{SL}(2, \mathbb{C})$-character varieties that were presented Lawton's talk and, further, tropicalize $\mathrm{PSL}(2, \mathbb{C})$-, $\mathrm{SL}(3, \mathbb{C})$-, an $\mathrm{Sp}(4)$-character varieties. We broke out—loosely—into two groups: one concerned with the tropicalization of $\mathrm{PSL}(2, \mathbb{C})$-character varieties and the other with visualization of the Newton polytopes that came about from tropicalization.

Here is a summary of the observations made by the group.

2 Tropicalization of $\mathfrak{X}(F_3, \mathrm{SL}(2, \mathbb{C}))$

From Lawton [], we deduced that the character variety $\mathfrak{X}(F_3, \mathrm{SL}(2, \mathbb{C}))$ is cut out by the polynomial in 7 variables

$$\begin{aligned} f = X_1X_2X_3X_7 - {X_4}^2 + {X_5}^2 + {X_1}^2 \\ + {X_2}^2 + {X_3}^2 + {X_7}^2 \\ + X_1X_6X_7 + X_2X_5X_7 + X_3X_4X_7 \\ + X_1X_2X_4 + X_1X_3X_5 + X_2X_3X_6 - 4. \end{aligned} \tag{1}$$

With the help of `gfan`—a software package for computing Gröbner fans and tropical varieties [1]—and `Mathematica` we were were able to find $\mathrm{Trop}(\mathfrak{X}(F_3, \mathrm{SL}(2, \mathbb{C})))$. Its tropicalization is the codimension 1-cones of the dual fan to the Newton polytope $\mathrm{Newt}(f)$. Figure 1 is a picture made we made using the Ti*k*Z graphics language to draw the edge graph of $\mathrm{Newt}(f)$

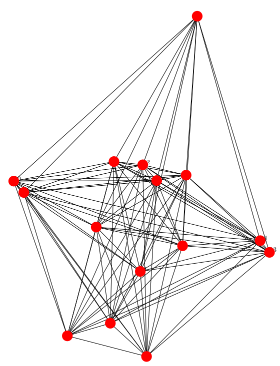

Figure 1: Edge graph of the Newton polytope of $\mathfrak{X}(F_3, \mathrm{SL}(2, \mathbb{C}))$.

3 Tropicalization of $\mathfrak{X}(F_2, \mathrm{SL}_3\mathbb{C}))$

The character variety $\mathfrak{X}(F_2, \mathrm{SL}_3\mathbb{C}))$ is cut out by the huge polynomial in 9 variables

$$
\begin{aligned}
f = 9 + 3X_9 - 6X_1X_2 - 6X_3X_4 - 6X_5X_6 - 6X_7X_8 + {X_9}^2 - X_1X_2X_9 \\
&- X_3X_4X_9 - X_5X_6X_9 \ - X_7X_8X_9 + {X_1}^3 \\
&+ {X_3}^3 + {X_5}^3 + {X_7}^3 + {X_2}^3 \\
&+ {X_4}^3 + {X_6}^3 + {X_8}^3 - 3X_2X_8X_6 \\
&- 3X_1X_5X_7 - 3X_3X_5X_8 - 3X_4X_6X_7 + 3X_1X_4X_8 \\
&+ 3X_2X_3X_7 + 3X_1X_3X_6 + 3X_2X_4X_5 - X_1X_2X_3X_4X_9 \\
&+ X_1X_3X_6X_9 + X_2X_4X_5X_9 + X_1X_4X_8X_9 + X_2X_3X_7X_9 \\
&+ X_1X_2X_3X_4 + X_3X_4X_5X_6 + X_1X_2X_7X_8 + X_3X_4X_7X_8 \\
&+ X_1X_2X_5X_6 + X_5X_6X_7X_8 + X_4{X_8}^2X_6 + X_3X_5{X_7}^2 \\
&+ {X_2}^2X_4X_8 + {X_1}^2X_3X_7 + X_1{X_4}^2X_6 + X_2{X_3}^2X_5 \\
&+ {X_1}^2X_8X_6 + {X_2}^2X_5X_7 + X_3X_8{X_6}^2 + X_4{X_5}^2X_7 \\
&+ {X_2}^2X_3X_6 + {X_1}^2X_5X_4 + X_1{X_3}^2X_8 + X_2{X_4}^2X_7 \\
&+ {X_4}^2X_5X_8 + {X_3}^2X_6X_7 + X_1X_5{X_8}^2 + X_2X_6{X_7}^2 \\
&+ X_2{X_5}^2X_8 + X_1{X_6}^2X_7 - 2X_2X_4{X_6}^2 - 2X_1X_3{X_5}^2 \\
&- 2X_2X_3{X_8}^2 - 2X_1X_4{X_7}^2 + {X_2}^2{X_4}^2X_6 + {X_1}^2{X_3}^2X_5 \\
&+ {X_2}^2{X_3}^2X_8 + {X_1}^2{X_4}^2X_7 - X_1X_3{X_4}^2X_8 - X_2{X_3}^2X_4X_7 \\
&- {X_1}^2X_2X_3X_6 - X_1{X_2}^2X_4X_5 - X_1{X_3}^2X_4X_6 - X_2X_3{X_4}^2X_5 \\
&- {X_1}^2X_2X_4X_8 - X_1{X_2}^2X_3X_7 - X_1X_2{X_4}^3 - X_1X_2{X_3}^3 \\
&- {X_2}^3X_3X_4 - {X_1}^3X_3X_4 - X_2X_3X_4X_8X_6 - X_1X_3X_4X_5X_7 \\
&- X_1X_2X_3X_5X_8 - X_1X_2X_4X_6X_7 + {X_1}^2{X_2}^2X_3X_4 + X_1X_2{X_3}^2{X_4}^2
\end{aligned}
$$

It's the edge graph of its Newton polytope is shown in Figure 2.

4 Tropicalization of $\mathrm{PSL}(2, \mathbb{C})$-character varieties

Let $F_3 = \langle A, B, C \rangle$; that is, the free group on 3 letters. We claim that $\mathbb{C}[\mathfrak{X}(F_3, \mathrm{PSL}_2\mathbb{C})] = \mathbb{C}[\mathfrak{X}(F_3, \mathrm{SL}(2, \mathbb{C}))]^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ is generated the following family of trace polynomials

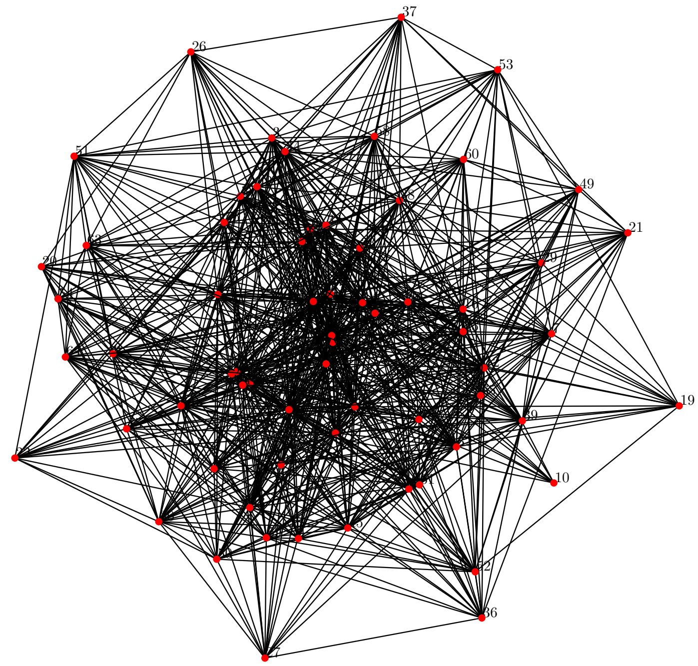

Figure 2: Edge graph of the Newton polytope of $\mathfrak{X}(F_3, \mathrm{SL}(3,\mathbb{C}))$.

⋆ Type χ:

$$\chi_A = (\operatorname{tr} A)^2 \qquad \chi_B = (\operatorname{tr} B)^2 \qquad \chi_C = (\operatorname{tr} C)^2$$
$$\chi_{AB} = (\operatorname{tr} AB)^2 \qquad \chi_{AC} = (\operatorname{tr} AC)^2$$
$$\chi_{BC} = (\operatorname{tr} BC)^2 \qquad \chi_{ABC} = (\operatorname{tr} ABC)^2$$

⋆ Type τ:

$$\tau_{AB} = \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AB \quad \tau_{AC} = \operatorname{tr} A \operatorname{tr} C \operatorname{tr} AC \quad \tau_{BC} = \operatorname{tr} B \operatorname{tr} C \operatorname{tr} BC$$

⋆ Type Λ:

$$\Lambda_A = \operatorname{tr} B \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} AC \qquad \Lambda_B = \operatorname{tr} A \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} BC$$
$$\Lambda_C = \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AC \operatorname{tr} BC$$

⋆ Lonely Δ:

$$\Delta = \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C \operatorname{tr} ABC$$

⋆ Equally lonely Σ:

$$\Sigma = \operatorname{tr} AB \operatorname{tr} AC \operatorname{tr} BC$$

⋆ Type Θ:

$$\begin{aligned} \Theta_A &= \operatorname{tr} A \operatorname{tr} BC \operatorname{tr} ABC & \Theta_B &= \operatorname{tr} B \operatorname{tr} AC \operatorname{tr} ABC \\ \Theta_C &= \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} ABC \end{aligned}$$

4.1 Relations

Explicit example:

$$\Sigma^2 = (\operatorname{tr} AB \operatorname{tr} AC \operatorname{tr} BC)^2 = \chi_{AB}\chi_{AC}\chi_{BC}$$

4.2 (Binomial) Relations

$$\begin{aligned} \tau_{AB}^2 &= \chi_A\chi_B\chi_{AB} & \tau_{AC}^2 &= \chi_A\chi_C\chi_{AC} \\ \tau_{BC}^2 &= \chi_B\chi_C\chi_{BC} \end{aligned}$$

$$\begin{aligned} \Lambda_A^2 &= \chi_B\chi_C\chi_{AB}\chi_{AC} & \Lambda_B^2 &= \chi_A\chi_C\chi_{AB}\chi_{BC} \\ \Lambda_C^2 &= \chi_A\chi_B\chi_{AC}\chi_{BC} \end{aligned}$$

$$\begin{aligned} \Theta_A^2 &= \chi_A\chi_{BC}\chi_{ABC} & \Theta_B^2 &= \chi_B\chi_{AC}\chi_{ABC} \\ \Theta_C^2 &= \chi_C\chi_{AB}\chi_{ABC} \end{aligned}$$

$$\Sigma^2 = \chi_{AB}\chi_{AC}\chi_{BC} \qquad \Delta^2 = \chi_A\chi_B\chi_C\chi_{ABC}.$$

and finally the relation coming from $\mathfrak{X}(F_3, \mathrm{SL}(2,\mathbb{C}))$ can be written as

$$\begin{aligned} \chi_A + \chi_B + \chi_C + \chi_{AB} + \chi_{AC} &= \tau_{AB} + \tau_{AC} + \tau_{BC} \\ +\chi_{BC} + \chi_{ABC} + \Sigma + \Delta \quad &+ \Theta_A + \Theta_B + \Theta_C + 4. \end{aligned}$$

References

[1] Jensen, A. N. Gfan, a software system for Gröbner fans and tropical varieties. Available at `http://home.imf.au.dk/jensen/software/gfan/gfan.html`.