Wavelets and Approximation Theory

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These notes are incomplet and inkorrect (as an old computer documentation joke goes). Nonetheless, I've decided to distribute them in case they prove useful to someone.

My goal is to present certain results that can be proved in a (relatively) straightforward way. So, for some problems we present complete proofs in L_2 and partial proofs (or none at all) for L_p, $p \neq 2$. To simplify arguments even more, we consider only wavelet spaces of piecewise constant functions on uniform grids of size $2^k \times 2^k$ on the unit square. As far as I know, all results can be extended to approximations of higher order. We don't consider at all the area of "construction of wavelets," which is a large topic in itself. I've presented some of these results in graduate "topics" courses at Purdue University.

The notes have certain themes: generalized wavelets (results can be generalized to box splines, for example), nonlinearity (nonlinear approximations, nonlinear wavelet decompositions, avoiding arguments that use linear functionals, etc.), non-Hilbert spaces (L_p and ℓ_p for $p \neq 2$), nonconvexity (L_p and ℓ_p for $0 < p < 1$).

My approach has been inspired by and is derived from the approach of Ron DeVore, who has made tremendous contributions to this field and with whom I collaborated for about a decade.

Some parts of these notes are influenced directly by the survey paper "Wavelets," by Ronald A. DeVore and Bradley J. Lucier, *Acta Numerica* **1** (1992), 1–56. Specific examples of nonlinear, piecewise constant, wavelet approximations appeared in "Image compression through wavelet transform coding," by Ronald A. DeVore, Bjorn Jawerth, and Bradley J. Lucier, *IEEE Transactions on Information Theory*, **38** (1992), 719–746. And the approach in that paper was in turn influenced by the proofs in "Interpolation of Besov spaces," by Ronald A. DeVore and Vasil A. Popov, *Transactions of the American Mathematical Society* **305** (1988), 397–414.

But my goal is not to document everything that influenced these notes, for two reasons: (1) I can't remember where many ideas came from and (2) nearly all these results are now over 20 years old, so can appear, perhaps, in these informal notes without strict attribution.

Please e-mail me at `lucier@math.purdue.edu` to get the latest version of these notes, or to point out errors or gaps in arguments.

I thank Jeffrey Gaither, who helped correct and complete some aspects of these notes.

Background

We use the following terminology.

A *norm* on a vector space X, $\|\cdot\|\colon X \to [0,\infty) \subset \mathbb{R}$, satisfies (1) $(\forall a \in \mathbb{R})$ $(\forall x \in X)$ $\|ax\| = |a|\,\|x\|$, (2) $(\forall x, y \in X)$ $\|x+y\| \le \|x\| + \|y\|$, and (3) $(\forall x \in X \mid \|x\| = 0)$ $x = 0$.

A *semi-norm* satisfies (1) and (2), but not necessarily (3).

A *quasi-norm* satisfies (1), (3), and (2′): $(\exists C > 1)$ $(\forall x, y \in X)$ $\|x+y\| \le C(\|x\|+\|y\|)$.

And a *semi-quasi-norm* (or quasi-semi-norm) satisfies (1) and (2′).

Elements of the *sequence space*

$$\ell_p = \left\{ x = (x_1, x_2, \dots) \;\middle|\; x_k \in \mathbb{R},\ \|x\|_{\ell_p}^p := \sum_{k\ge 0} |x_k|^p < \infty \right\}$$

for $0 < p < \infty$ (with the usual change when $p = \infty$) are denoted by (x_k) or $\{x_k\}$.

The notation $A(f) \asymp B(f)$, $A, B\colon X \to [0,\infty) \subset \mathbb{R}$, means that there exist positive constants $\overline{c}$, $\underline{c}$ such that for all $f \in X$,

$$\underline{c}A(f) \le B(f) \le \overline{c}A(f).$$

We say that A is *equivalent* to B on X. It is a standard result that all pairs of (quasi-)norms on a finite-dimensional space are equivalent.

We use the following two inequalities extensively.

For any bounded domain $\Omega \subset \mathbb{R}^2$, let $|\Omega|$ be the volume of Ω. If $0 < q < p < \infty$, we can use Hölder's inequality on Ω with

$$s = \frac{p}{q} \ge 1 \text{ and } \frac{1}{r} + \frac{1}{s} = 1$$

to see that for any

$$f \in L_p(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \|f\|_{L_p(\Omega)}^p := \int_\Omega |f|^p < \infty \right\}$$

we have

$$\frac{1}{|\Omega|}\int_\Omega (|f|^q \cdot 1) \le \left(\frac{1}{|\Omega|}\int_\Omega (|f|^q)^s\right)^{1/s} \left(\frac{1}{|\Omega|}\int_\Omega 1^r\right)^{1/r} = \left(\frac{1}{|\Omega|}\int_\Omega (|f|^q)^{p/q}\right)^{q/p}.$$

Taking qth roots shows that

$$\left(\frac{1}{|\Omega|}\int_\Omega |f|^q\right)^{1/q} \le \left(\frac{1}{|\Omega|}\int_\Omega |f|^p\right)^{1/p}. \tag{1}$$

On the other hand, for $0 < q \le p \le \infty$ and sequence norms we have

$$\|(x_k)\|_{\ell_p} \le \|(x_k)\|_{\ell_q}; \tag{2}$$

when $p = \infty$ this is obvious, while for other values of p this inequality will be proved later.

Nested finite-dimensional linear spaces

Let I be the unit interval $[0,1]^2$; for $k \ge 0$ and

$$j \in \mathbb{Z}_2^k := \{j = (j_1, j_2) \in \mathbb{Z}^2 \mid 0 \le j_1, j_2 < 2^k\}$$

let $I_{j,k}$ be the square with opposite corners $j/2^k$ and $(j + (1,1))/2^k$; in other words

$$I_{j,k} = 2^{-k}(I + j).$$

The corners of the squares $I_{j,k}$ and the squares themselves are called *dyadic*, as the corners are pairs of rational numbers whose denominators are powers of 2.

We define the characteristic function of $I_{j,k}$ to be

$$\chi_{j,k}(x) := \begin{cases} 1, & x \in I_{j,k}, \\ 0, & \text{otherwise.} \end{cases}$$

We want to project a function f defined on I onto functions that are constant on each $I_{j,k}$. We'll call the space of such functions

$$S^k = \{ f : I \to \mathbb{R} \mid f\big|_{I_{j,k}} \text{ is a constant} \}.$$

These spaces have the property that $S^k \subset S^{k+1}$, i.e., this is an expanding sequences of finite-dimensional spaces.

Projections as near-best approximations

For now, let's take

$$P_k f\big|_{I_{j,k}} = \frac{1}{|I_{j,k}|} \int_{I_{j,k}} f,$$

i.e., $P_k f$ in $I_{j,k}$ is the *average* of f on $I_{j,k}$. $P_k f$ is the best $L_2(I)$ approximation to f from S^k, as calculus shows.

We make a series of claims about P_k. First, note that $P_k f$ is bounded on $L_p(I)$ for $1 \le p \le \infty$, because, for each $I_{j,k}$, by (1),

$$|P_k f| = \left| \frac{1}{|I_{j,k}|} \int_{I_{j,k}} f \right| \le \left(\frac{1}{|I_{j,k}|} \int_{I_{j,k}} |f|^p \right)^{1/p}.$$

So,

$$\int_{I_{j,k}} |P_k f|^p \le \int_{I_{j,k}} \frac{1}{|I_{j,k}|} \int_{I_{j,k}} |f|^p = \int_{I_{j,k}} |f|^p;$$

summing this inequality over j and taking pth roots gives $\|P_k f\|_{L_p(I)} \le \|f\|_{L_p(I)}$.

Second, if $S \in S^k$, then $P_k S = S$ (which is obvious). Third, $P_k f$ is additive with respect to elements of S^k: for any S in S^k,

$$P_k(f + S) = P_k f + P_k S = P_k f + S, \quad S \in S^k.$$

P_k is in fact *linear*, but we only need additivity.

These three properties imply that $P_k f$ is a *near-best* approximation to f in S^k, because for *any* $S \in S^k$, we have

$$\begin{aligned} \|P_k f - f\|_{L_p(I)} &= \|(P_k f - P_k S) + (S - f)\|_{L_p(I)} \le \|P_k f - P_k S\|_{L_p(I)} + \|f - S\|_{L_p(I)} \\ &\le \|f - S\|_{L_p(I)} + \|f - S\|_{L_p(I)}. \end{aligned}$$

So for all $f \in L_p(I)$,

$$\|P_k f - f\|_{L_p(I)} \le 2 \inf_{S \in S^k} \|f - S\|_{L_p(I)}. \tag{3}$$

The same would be true of any (possibly nonlinear) projector of $L_p(I)$ onto S^k that satisfies these three properties.

Moduli of smoothness

We want to compare how close f is to $P_k f$ by considering the *smoothness* of f. So for any $x \in I$, nonnegative integer r, and $h \in \mathbb{R}^2$ for which this makes sense, we define the differences

$$\Delta_h^0 f(x) := f(x), \quad \Delta_h^{r+1} f(x) := \Delta_h^r f(x + h) - \Delta_h^r f(x), \quad r \ge 0.$$

Thus

$$\Delta_h^1 f(x) = f(x + h) - f(x),\ \Delta_h^2 f(x) = f(x + 2h) - 2f(x + h) + f(x),\ \text{etc.};$$

by induction,

$$\Delta_h^r f(x) = \sum_{s=0}^{r} \binom{r}{s} (-1)^{r+s} f(x + sh).$$

We have $x + sh \in I$ for $s = 0, \dots, r$, only for x in

$$I_{rh} := \{x \in I \mid x + rh \in I\},$$

so for $f(x)$ defined for $x \in I$, $\Delta_h^r f(x)$ is defined for $x \in I_{rh}$.

For any $0 < p \le \infty$, the rth modulus of smoothness in $L_p(I)$ is defined by

$$\omega_r(f,t)_p := \sup_{|h|<t} \|\Delta_h^r f\|_{L_p(I_{rh})}.$$

Bounding approximation error by modulus of smoothness

We claim that for $1 \le p \le \infty$, we have

$$\|f - P_k f\|_{L_p(I)} \le (2\pi)^{1/p}\omega_1(f, 2^{-k}\sqrt{2})_p.$$

Why? Let's consider $1 \le p < \infty$. On each $I_{j,k}$, we have

$$\int_{I_{j,k}} |f(x) - P_k f(x)|^p dx = \int_{I_{j,k}} \left| f(x) - \frac{1}{|I_{j,k}|} \int_{I_{j,k}} f(y)\, dy \right|^p dx.$$

But now, because $f(x)$ does not depend on y we can pull $f(x)$ into the integral in y; since $1 \le p$, we can use (1) to show the latter quantity is bounded by

$$\int_{I_{j,k}} \left| \frac{1}{|I_{j,k}|} \int_{I_{j,k}} f(x) - f(y)\, dy \right|^p dx \le \int_{I_{j,k}} \frac{1}{|I_{j,k}|} \int_{I_{j,k}} |f(x) - f(y)|^p\, dy\, dx.$$

Note that for a given $x \in I_{j,k}$, the values of y are also restricted to $I_{j,k}$; so, in fact, if we define $h = y - x$, we have $|h| \le 2^{-k}\sqrt{2}$ and we have the further bound

$$\int_{I_{j,k}} \frac{1}{|I_{j,k}|} \int_{\substack{|h|\le\sqrt{2}\,2^{-k} \\ x\in I_{j,k},\ x+h\in I}} |f(x+h) - f(x)|^p\, dh\, dx.$$

We now sum over all j to find

$$\int_I |f(x) - P_k f(x)|^p\, dx \le \frac{1}{|I_{j,k}|} \int_I \int_{\substack{|h|\le\sqrt{2}\,2^{-k} \\ x\in I_h}} |f(x+h) - f(x)|^p\, dh\, dx.$$

Change the order of integration, take a supremum over all $|h| \le 2^{-k}\sqrt{2}$, and get

$$\begin{aligned}\int_I |f(x) - P_k f(x)|^p\, dx &\le \frac{|\{h \mid |h| \le 2^{-k}\sqrt{2}\}|}{|I_{j,k}|} \sup_{|h|\le 2^{-k}\sqrt{2}} \int_{x\in I_h} |f(x+h) - f(x)|^p\, dx \\ &= 2\pi\omega_1(f, 2^{-k}\sqrt{2})_p^p.\end{aligned}$$

Thus

$$\|f - P_k f\|_{L_p(I)} \le (2\pi)^{1/p}\omega_1(f, 2^{-k}\sqrt{2})_p.$$

A different argument, which we leave to the reader, is needed when $p = \infty$.

At this point I'm supposed to present various properties of $\omega_r(f,t)_p$, $1 \le p \le \infty$, like

(a) $\omega_r(f,t)_p$ is nondecreasing,
(b) $\omega_r(f,t)_p \to 0$ as $t \to 0$ if $1 \le p < \infty$ and $f \in L_p(I)$ or if $p = \infty$ and f is uniformly continuous on I,
(c) $\omega_r(f,nt)_p \le n^r \omega_r(f,t)_p$ and $\omega_r(f,\lambda t)_p \le (\lambda+1)^r \omega_r(f,t)_p$ for integer $n > 0$ and real $\lambda > 0$,
(d) $\omega_r(f+g,t)_p \le \omega_r(f,t)_p + \omega_r(g,t)_p$,
(e) $\omega_r(f,t)_p = 0$ for all $t > 0$ iff f is a polynomial of total degree $< r$ on I,
(f) $\omega_r(f,t)_p \le 2^r \|f\|_{L_p(I)}$.

See Chapter 2 of DeVore and Lorentz, *Constructive Approximation Theory*, for the one-dimensional theory. Note that property (d) implies that $\omega_r(\,\cdot\,,t)_p$ is a semi-norm on $L_p(I)$. The "only if" part of (e) is nontrivial. These properties hold for more general domains than our square I, for example they hold for functions defined on open, convex Ω.

Because of (c) we have, in fact,

$$\|f - P_k f\|_{L_p(I)} \le C\omega_1(f, 2^{-k})_p. \tag{4}$$

And because of (4) and (b), we have that $P_k f \to f$ in $L_p(I)$ as $k \to \infty$. Thus, if we define $P_{-1} f = 0$ we can write

$$P_m f = (P_m f - P_{m-1} f) + \cdots + (P_1 f - P_0 f) = \sum_{k=0}^{m} (P_k f - P_{k-1} f).$$

Let $m \to \infty$ to see that

$$f = \lim_{m\to\infty} P_m f = \sum_{k=0}^{\infty} (P_k f - P_{k-1} f),$$

where the sum converges in $L_p(I)$.

Since $P_{k-1} f \in S^{k-1} \subset S^k$ and $P_k f \in S^k$, $P_k f - P_{k-1} f \in S^k$.

So we can write any f in $L_p(I)$ as a convergent sum of elements of S^k, $k \ge 0$.

$0 < p < 1$: Not as strange as it might seem

If we use

$$P_k f(x) = \frac{1}{|I_{j,k}|} \int_{I_{j,k}} f \quad \text{for } x \in I_{j,k}$$

as our projector, then the above stuff works only for $p \ge 1$. For certain applications, we will need $p < 1$ and it will be nice to get it to work there, too.

The basic inequality for $0 < p \le 1$ is

$$\left|\sum_i |t_i|\right|^p \le \sum_i |t_i|^p. \tag{5}$$

We can prove this as follows.

Without loss of generality we assume that $0 \le b \le a$ and

$$|a+b|^p = |a|^p|1+(b/a)|^p = a^p(1+x)^p, \quad 0 \le x = b/a \le 1.$$

We now note that for $x \ge 0$,

$$F(x) := \frac{(1+x)^p}{1+x^p}$$

satisfies $F(0) = 1$, and for $0 < x < 1$,

$$F'(x) = \frac{(1+x^p)p(1+x)^{p-1} - (1+x)^p p x^{p-1}}{(1+x^p)^2}.$$

The numerator is

$$p(1+x)^{p-1}[1+x^p-(1+x)x^{p-1}] = p(1+x)^{p-1}[1+x^p-x^{p-1}-x^p] = p(1+x)^{p-1}[1-x^{p-1}].$$

Since $p-1 \le 0$ and $0 < x < 1$, this quantity is negative. Thus, $F(x)$ is nonincreasing on $[0,1]$ and

$$\frac{(1+x)^p}{1+x^p} \le F(0) = 1, \text{ i.e., } (1+x)^p \le 1+x^p.$$

Backing up gives

$$|a+b|^p = a^p(1+(b/a))^p \le a^p(1+(b/a)^p) = a^p + b^p,$$

which in turn gives (5).

From (5) we can derive other useful inequalities. For example, if $0 < s \le r < \infty$ and $t_i = |x_i|^r$, then set $p = s/r \le 1$ to see that

$$\left(\sum |x_i|^r\right)^{s/r} \le \sum |x_i|^{r\cdot s/r},$$

or, on taking sth roots of each side,

$$\left(\sum |x_i|^r\right)^{1/r} \le \left(\sum |x_i|^s\right)^{1/s},$$

as claimed earlier.

Formula (5) immediately gives

$$\int_I |f(x)+g(x)|^p \, dx \le \int_I \bigl(|f(x)|^p + |g(x)|^p\bigr)\, dx,$$

i.e.,

$$\|f+g\|^p_{L_p(I)} \le \|f\|^p_{L_p(I)} + \|g\|^p_{L_p(I)}. \tag{6}$$

The function $F(\xi) = \xi^p$ is concave on $[0,\infty)$ for $0 < p \le 1$, so for $0 \le t \le 1$ and any nonnegative a and b

$$tF(a) + (1-t)F(b) \le F\bigl(ta + (1-t)b\bigr).$$

Setting $t = 1/2$, $a = \|f\|_{L_p(I)}$, and $b = \|g\|_{L_p(I)}$, we get with some algebra

$$\|f\|^p_{L_p(I)} + \|g\|^p_{L_p(I)} \le 2^{1-p}\bigl(\|f\|_{L_p(I)} + \|g\|_{L_p(I)}\bigr)^p.$$

Combining this with (6) gives

$$\|f+g\|_{L_p(I)} \le 2^{\frac{1}{p}-1}\bigl(\|f\|_{L_p(I)} + \|g\|_{L_p(I)}\bigr),$$

i.e., $\|\cdot\|_{L_p(I)}$ is a *quasi-norm*. The completeness of $L_p(I)$ for $p < 1$ is proved in the same way as for $1 \le p < \infty$.

Projections and moduli of smoothness in $L_p(I)$, $0 < p \le \infty$

For any nontrivial measurable set Ω we can define a (possibly non-unique) *median* of f on Ω to be any number m for which

$$|\{f(x) \ge m\}| \ge \frac{1}{2}|\Omega| \text{ and } |\{f(x) \le m\}| \ge \frac{1}{2}|\Omega|. \tag{7}$$

A median is a best $L_1(\Omega)$ constant approximation to f on Ω. If $k < m$, for example, then because $|\{f \ge m\}| \ge \frac{1}{2}|\Omega|$

$$\begin{aligned}
\int |f-k| &= \int_{f\ge k}(f-k) + \int_{f<k}(k-f) \\
&= \int_{f\ge m}(f-m) + \int_{f\ge m}(m-k) + \int_{k\le f<m}(f-k) \\
&\quad + \int_{f<m}(m-f) + \int_{f<m}(k-m) - \int_{k\le f<m}(k-f) \\
&= \int |f-m| + (m-k)(|\{f\ge m\}| - |\{f<m\}|) + 2\int_{k\le f<m}(f-k) \\
&\ge \int |f-m|.
\end{aligned}$$

We can try a new projector

$$f_{j,k} := P_k f\Big|_{I_{j,k}} := \text{a median of } f \text{ on } I_{j,k}.$$

Note that this isn't necessarily unique, so just pick one.

We note that

$$\int_{\{x\in I_{j,k} | |f(x)|\geq |f_{j,k}|\}} |f_{j,k}|^p \, dx \leq \int_{I_{j,k}} |f(x)|^p \, dx$$

No matter the choice of median, the measure of the set of all x such that $|f(x)| \geq |f_{j,k}|$ on $I_{j,k}$ is at least $\frac{1}{2}|I_{j,k}|$, so the left hand side is

$$\begin{aligned}
|\{x \in I_{j,k} \mid |f(x)| \geq |f_{j,k}|\}| \, |f_{j,k}|^p &= \frac{|\{x \in I_{j,k} \mid |f(x)| \geq |f_{j,k}|\}|}{|I_{j,k}|} |I_{j,k}| \, |f_{j,k}|^p \\
&\geq \frac{1}{2} \int_{I_{j,k}} |f_{j,k}|^p,
\end{aligned}$$

so

$$\int_{I_{j,k}} |P_k f|^p \leq 2 \int_{I_{j,k}} |f|^p.$$

Summing over all j gives

$$\|P_k f\|_{L_p(I)}^p \leq 2\|f\|_{L_p(I)}^p.$$

or

$$\|P_k f\|_{L_p(I)} \leq 2^{1/p}\|f\|_{L_p(I)}.$$

Note that this argument works for any $0 < p < \infty$, and the case $p = \infty$ is trivial.

Although P_k is no longer linear, one sees immediately that $P_k S = S$ for $S \in S^k$ and P_k is still additive with respect to elements of S^k, i.e., one can choose medians so that

$$P_k(f + S) = P_k f + P_k S = P_k f + S.$$

So, for $1 \leq p$ and P_k the median operator, we have that $\|P_k f\|_{L_p(I)} \leq 2\|f\|_{L_p(I)}$ and the same argument as for the averaging operator shows that

$$\|f - P_k f\|_{L_p(I)} \leq 3 \inf_{S\in S^k} \|f - S\|_{L_p(I)}.$$

For $0 < p < 1$, use (6) to see that for any $S \in S^k$,

$$\begin{aligned}
\|f - P_k f\|_{L_p(I)}^p &= \|f - S + S - P_k f\|_{L_p(I)}^p \\
&\leq \|f - S\|_{L_p(I)}^p + \|S - P_k f\|_{L_p(I)}^p \\
&= \|f - S\|_{L_p(I)}^p + \|P_k(S - f)\|_{L_p(I)}^p \\
&\leq 3\|f - S\|_{L_p(I)}^p
\end{aligned}$$

so

$$\|f - P_k\|_{L_p(I)} \le 3^{1/p} \inf_{S\in S^k} \|f - S\|_{L_p(I)},$$

i.e., $P_k f$ is a near-best approximation to f in S^k with a constant that depends on p.

If we now distinguish $P_k^{\text{average}} f$ from P_k^{median} we see that for $p \ge 1$ and $k \ge 0$,

$$\|f - P_k^{\text{median}} f\|_{L_p(I)} \le C \inf_{S\in S^k} \|f - S\|_{L_p(I)} \le C\|f - P_k^{\text{average}} f\|_{L_p(I)} \le C\omega_1(f, 2^{-k})_p.$$

A different argument works for *all* $0 < p < \infty$. For fixed $x \in I_{j,k}$, we note that for points y that cover at least half the measure of $I_{j,k}$, we have

$$|f(x) - P_k f| \le |f(x) - f(y)|,$$

since $f(y)$ and $f(x)$ will lie on "opposite" sides of the median. So for $x \in I_{j,k}$ and $p > 0$,

$$\begin{aligned}
|f(x) - P_k f|^p \frac{1}{2}|I_{j,k}| &\le |f(x) - P_k f|^p \big|\{y \in I_{j,k} \mid |f(x) - f(y)| \ge |f(x) - P_k f|\}\big| \\
&= \int_{\{y\in I_{j,k} \mid |f(x)-f(y)|\ge|f(x)-P_k f|\}} |f(x) - P_k f|^p \, dy \\
&\le \int_{\{y\in I_{j,k} \mid |f(x)-f(y)|\ge|f(x)-P_k f|\}} |f(x) - f(y)|^p \, dy \\
&\le \int_{I_{j,k}} |f(x) - f(y)|^p \, dy.
\end{aligned}$$

So

$$\frac{1}{2}|I_{j,k}| \int_{I_{j,k}} |f(x) - P_k f|^p \, dx \le \int_{I_{j,k}} \int_{I_{j,k}} |f(x) - f(y)|^p \, dy \, dx$$

or

$$\int_{I_{j,k}} |f(x) - P_k f|^p \, dx \le \frac{2}{|I_{j,k}|} \int_{I_{j,k}} \int_{I_{j,k}} |f(x) - f(y)|^p \, dy \, dx$$

and we proceed as with the average projector to show that

$$\|f - P_k f\|_{L_p(I)} \le C(p)\omega_1(f, 2^{-k})_p.$$

What are the properties of the modulus of smoothness when $0 < p \le 1$? When $p \le 1$, property (d) for the modulus of smoothness is modified to

$$\omega_r(f + g, t)_p^p \le \omega_r(f, t)_p^p + \omega_r(g, t)_p^p,$$

or, for all $0 < p \le \infty$,

$$\omega_r(f+g,t)_p^{\min(p,1)} \le \omega_r(f,t)_p^{\min(p,1)} + \omega_r(g,t)_p^{\min(p,1)},$$

Similarly, because

$$\begin{aligned}
\int_{I_{rh}} \left|\Delta_h^r f(x)\right|^p dx &= \int_{I_{rh}} \left|\sum_{s=0}^{r}(-1)^{r+s}\binom{r}{s} f(x+sh)\right|^p dx \\
&\le \int_{I_{rh}} \sum_{s=0}^{r} \binom{r}{s}^p |f(x+sh)|^p\, dx \\
&\le \int_{I_{rh}} \sum_{s=0}^{r} \binom{r}{s} |f(x+sh)|^p\, dx \\
&\le 2^r \|f\|_{L_p(I)}^p
\end{aligned}$$

for $0 < p \le 1$, (f) is now

$$\omega_r(f,t)_p^{\min(p,1)} \le 2^r \|f\|_{L_p(I)}^{\min(p,1)} \quad \text{for } 0 < p \le \infty.$$

Finally, for integer $n > 0$ and real $\lambda > 0$, (c) is now

$$\omega_r(f,nt)_p^{\min(p,1)} \le n^r \omega_r(f,t)_p^{\min(p,1)} \text{ and } \omega_r(f,\lambda t)_p^{\min(p,1)} \le (\lambda+1)^r \omega(f,t)_p^{\min(p,1)}.$$

We can choose many other nonlinear projectors. One that I find interesting is

$$P_k f = \operatorname{round}\left(\frac{1}{|I_{j,k}|}\int_{I_{j,k}} f\right)$$

where round(x) is the integer closest to x (breaking ties arbitrarily). It is shown in DeVore, Jawerth, Lucier "Image compression through wavelet transform coding" that if f takes on integer values (as it does from a digital camera, for example), then this $P_k f$ is stable for all $0 < p$. This leads to so-called "integer-to-integer" wavelet transforms (which necessarily are nonlinear) and wavelet transforms for binary images.

One can even calculate the averages using fixed-point arithmetic with enough precision and still have a stable transform. This nonlinear theory can accomodate many computational "crimes".

Besov smoothness spaces

We measure the smoothness of f by considering how fast the modulus of smoothness decays as $t \to 0$. One natural measure may be: $(\exists C, \alpha > 0)$ $(\forall t > 0)$,

$$\omega_r(f,t)_p \le Ct^\alpha.$$

This is equivalent to

$$t^{-\alpha}\omega_r(f,t)_p \le C$$

and, in fact, we could take

$$\sup_t t^{-\alpha}\omega_r(f,t)_p$$

as a *semi-norm* if $p \ge 1$ (and a semi-quasi-norm if $0 < p < 1$).

Unfortunately, this is not quite general enough for our purposes. We need to make a more subtle distinction in the decay of $\omega_r(f,t)_p$, so we add another parameter q and define for $0 < p, q \le \infty$ and $r - 1 \le \alpha < r$ the (quasi-)semi-norms for the Besov spaces $B^\alpha_q(L_p(I))$ by

$$|f|_{B^\alpha_q(L_p(I))} := \left(\int_0^1 [t^{-\alpha}\omega_r(f,t)_p]^q \,\frac{dt}{t}\right)^{1/q}, \quad 0 < q < \infty,$$

and

$$|f|_{B^\alpha_\infty(L_p(I))} := \sup_{0<t<1} t^{-\alpha}\omega_r(f,t)_p.$$

We also define the (quasi-)norm

$$\|f\|_{B^\alpha_q(L_p(I))} := |f|_{B^\alpha_q(L_p(I))} + \|f\|_{L_p(I)}.$$

(And after that bit of pedantic distinction between (quasi-)(semi-)norms and (semi-)norms, we shall in the future call quasi-norms "norms.")

Now

$$\int_0^1 [t^{-\alpha}\omega_r(f,t)_p]^q \,\frac{dt}{t} = \sum_{k\ge 0}\int_{2^{-k-1}}^{2^{-k}} [t^{-\alpha}\omega_r(f,t)_p]^q \,\frac{dt}{t}.$$

and we have for $2^{-k-1} \le t \le 2^{-k}$

$$[2^{\alpha k}\omega_r(f,2^{-(k+1)})_p]^q 2^k \le [t^{-\alpha}\omega_r(f,t)_p]^q \frac{1}{t} \le [2^{\alpha(k+1)}\omega_r(f,2^{-k})_p]^q 2^{k+1}$$

We integrate over $[2^{-(k+1)}, 2^{-k}]$ (which has length $2^{-(k+1)}$), sum over k, and use property (c) for moduli of smoothness (arguing separately for $p \ge 1$ and $0 < p < 1$) to see that there are positive constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c}\left(\sum_{k\ge 0}[2^{\alpha k}\omega_r(f,2^{-k})_p]^q\right)^{1/q} \le |f|_{B^\alpha_q(L_p(I))} \le \overline{c}\left(\sum_{k\ge 0}[2^{\alpha k}\omega_r(f,2^{-k})_p]^q\right)^{1/q},$$

i.e.,

$$|f|_{B_q^\alpha(L_p(I))} \asymp \|\{2^{\alpha k}\omega_r(f,2^{-k})_p\}\|_{\ell_q}.$$

So we can take as the norm of $B_q^\alpha(L_p(I))$ the quantity

$$\|f\|_{B_q^\alpha(L_p(I))} := \|f\|_{L_p(I)} + \|\{2^{\alpha k}\omega_r(f,2^{-k})_p\}\|_{\ell_q}.$$

Sequence norms of wavelet coefficients: Part I, the direct inequality

We have decomposed $f \in L_p(I)$ by

$$f = \sum_{k\geq 0}(P_k f - P_{k-1} f), \quad P_{-1}f = 0,$$

where we can take P_k to be the average projector onto S^k when $p \geq 1$ and we take P_k to be the median projector onto S^k for any $p > 0$.

Because

$$P_{k-1}f \in S^{k-1} \subset S^k,$$

we have $P_k f - P_{k-1} f \in S^k$ so we define $d_{j,k}$ by

$$P_k f - P_{k-1} f = \sum_{j\in\mathbb{Z}_k^2} d_{j,k}\chi_{j,k};$$

note that because the interiors of the various $I_{j,k}$s for $j \in \mathbb{Z}_k^2$ don't overlap, we have

$$\|P_k f - P_{k-1} f\|_{L_p(I)} = \left(\sum_j \|d_{j,k}\chi_{j,k}\|_{L_p(I)}^p\right)^{1/p} = \|\{\|d_{j,k}\chi_{j,k}\|_{L_p(I)}\}\|_{\ell_p(j\in\mathbb{Z}_k^2)}.$$

Now for $0 < p \leq \infty$ and $k \geq 1$ we have

$$\begin{aligned}
\|P_k f - P_{k-1} f\|_{L_p(I)} &= \|P_k f - f + f - P_{k-1} f\|_{L_p(I)} \\
&\leq C_p(\|P_k f - f\|_{L_p(I)} + \|P_{k-1} f - f\|_{L_p(I)}) \\
&\leq C_p(\omega_1(f,2^{-k})_p + \omega_1(f,2^{-(k+1)})_p) \\
&\leq C_{p,r}\omega_1(f,2^{-k})_p,
\end{aligned}$$

while for $k = 0$,

$$\|P_0 f - P_{-1} f\|_{L_p(I)} = \|P_0 f\|_{L_p(I)} = |d_{0,0}| \leq C\|f\|_{L_p(I)}.$$

Combining these bounds, we see that for any $0 < q \le \infty$ we have

$$\begin{aligned}\|\{2^{\alpha k}\|P_k f - P_{k-1}f\|_{L_p(I)}\}\|_{\ell_q(k\ge 0)} &= \|\{2^{\alpha k}\|\{\|d_{j,k}\chi_{j,k}\|_{L_p(I)}\}\|_{\ell_p(j\in\mathbb{Z}_k^2)}\}\|_{\ell_q(k\ge 0)} \\ &\le C(\|\{2^{\alpha k}\omega_1(f,2^{-k})_p\}\|_{\ell_q(k\ge 0)} + \|f\|_{L_p(I)}) \\ &= C\|f\|_{B_q^\alpha(L_p(I))}.\end{aligned}$$

We tend to keep $\|\chi_{j,k}\|_{L_p(I)}$ in there as a *weight*; we can also write the left hand-side of the inequality as

$$\|\{2^{\alpha k}\|\chi_{j,k}\|_{L_p(I)}\|\{d_{j,k}\}\|_{\ell_p(j\in\mathbb{Z}_k^2)}\}\|_{\ell_q(k\ge 0)}.$$

Since we have the explicit value $\|\chi_{j,k}\|_{L_p(I)} = 2^{-2k/p}$ we can also write the left-hand-side of the inequality as

$$\|\{2^{(\alpha-2/p)k}\|\{d_{j,k}\}\|_{\ell_p(j\in\mathbb{Z}_k^2)}\}\|_{\ell_q(k\ge 0)}.$$

If we use a different normalization for $\chi_{j,k}$ then the weight $2^{(\alpha-2/p)k}$ will differ; normalizing $\chi_{j,k}$ to be have $\|\chi_{j,k}\|_{L_p(I)} = 1$ rather than just being the characteristic function of $I_{j,k}$, for example, will lead to a weight of $2^{\alpha k}$.

The *direct inequality*

$$\|\{2^{\alpha k}\|\chi_{j,k}\|_{L_p(I)}\|\{d_{j,k}\}\|_{\ell_p(j\in\mathbb{Z}_k^2)}\}\|_{\ell_q(k\ge 0)} \le C\|f\|_{B_q^\alpha(L_p(I))}, \tag{8}$$

which followed from the *Jackson inequality*

$$\|f - P_k f\|_{L_p(I)} \le C\omega_1(f,2^{-k})_p,$$

is the easy part.

Sequence norms of wavelet coefficients: Part II, the inverse inequality

The harder part is to show that

$$\|f\|_{B_q^\alpha(L_p(I))} \le C\|\{2^{\alpha k}\|P_k f - P_{k-1}f\|_{L_p(I)}\}\|_{\ell_q(k\ge 0)},$$

which is called the *inverse inequality.* The inverse inequality will follow by a bound on the modulus of smoothness of any element of S^k; specifically for $S \in S^k$ we have the *Bernstein inequality*

$$\omega_1(S,t)_p \le C\begin{cases} \|S\|_{L_p(I)}, & 2^{-k} \le t, \\ 2^{k/p}t^{1/p}\|S\|_{L_p(I)}, & 0 < t \le 2^{-k}. \end{cases} \tag{9}$$

The first inequality is a special case of

$$\omega_r(S,t)_p^{\min(1,p)} \le 2^r \|S\|_{L_p(I)}^{\min(1,p)}.$$

To prove the second, we start with $h=(h_1,h_2)$, $|h|\le t$, and

$$f(x+h)-f(x) = f(x+(h_1,h_2)) - f(x+(h_1,0)) + f(x+(h_1,0)) - f(x),$$

so

$$\|\Delta_h^1 f\|_{L_p(I_h)} \le C_p(\|\Delta_{(h_1,0)}^1 f\|_{L_p(I_{(h_1,0)})} + \|\Delta_{(0,h_2)}^1 f\|_{L_p(I_{(0,h_2)})}) \tag{10}$$

and we need to consider only offsets parallel to the coordinate axes with $|h_1|,|h_2|\le t$.

Now we denote

$$S = \sum_j s_{j,k}\chi_{j,k}.$$

When $x\in I_{j,k}$ and $0\le h\le t\le 2^{-k}$, we have

$$S(x+(h,0)) - S(x) = \begin{cases} s_{j,k}-s_{j,k}=0, & x+(h,0)\in I_{j,k}, \\ s_{j+(1,0),k}-s_{j,k}, & x+(h,0)\in I_{j+(1,0),k}. \end{cases}$$

The area where $x\in I_{j,k}$ and $x+(h,0)\in I_{j+(1,0),k}$ is $h\cdot 2^{-k}$; therefore

$$\begin{aligned}\int_{I_{(h,0)}} |S(x+(h,0))-S(x)|^p\,dx &= \sum_j \int_{I_{j,k}\cap I_{(h,0)}} |S(x+(h,0))-S(x)|^p\,dx \\ &= \sum_{j_1+1<2^k} |s_{j+(1,0),k}-s_{j,k}|^p h 2^{-k} \\ &= 2^k h \sum_{j_1+1<2^k} |s_{j+(1,0),k}-s_{j,k}|^p 2^{-2k} \\ &= 2^k h \|\Delta_{(2^{-k},0)}^1 S\|_{L_p(I_{(2^{-k},0)})}^p \\ &\le C 2^k h \|S\|_{L_p(I)}^p.\end{aligned}$$

So

$$\sup_{0\le h\le t\le 2^k} \|S(\cdot+(h,0))-S\|_{L_p(I_{(h,0)})} \le C 2^{k/p} t^{1/p} \|S\|_{L_p(I)};$$

combining this with (10) implies the second part of (9).

If we write $S_k = P_k f - P_{k-1} f \in S^k$, we have

$$f = \sum_{k\ge 0} S_k.$$

If $0 < p \le 1$ we have

$$\int_{I_h} |\Delta_h^1 f(x)|^p = \int_{I_h} \Big|\sum_{k\ge 0} \Delta_h^1 S_k\Big|^p \le \sum_{k\ge 0} \int_{I_h} |\Delta_h^1 S_k|^p.$$

If we now take $|h| \le 2^{-m}$ and use the two parts of (9), we find

$$\begin{aligned}\int_{I_h} |\Delta_h^1 f|^p &\le C\Big(\sum_{0\le k<m} \int_{I_h} |\Delta_h^1 S_k|^p + \sum_{m\le k} \int_{I_h} |\Delta_h^1 S_k|^p\Big)\\ &\le C\Big(\sum_{0\le k<m} (2^k 2^{-m})\|S_k\|_{L_p(I)}^p + \sum_{m\le k} \|S_k\|_{L_p(I)}^p\Big).\end{aligned}$$

Taking suprema over $|h| \le 2^{-m}$ gives

$$\omega_1(f, 2^{-m})_p^p \le C\Big(\sum_{0\le k<m} 2^{k-m}\|S_k\|_{L_p(I)}^p + \sum_{m\le k} \|S_k\|_{L_p(I)}^p\Big), \quad m \ge 0,$$

and multiplying by $2^{\alpha mp}$ gives

$$2^{\alpha mp}\omega_1(f, 2^{-m})_p^p \le C\Big(\sum_{0\le k<m} 2^{k+(\alpha p-1)m}\|S_k\|_{L_p(I)}^p + \sum_{m\le k} 2^{\alpha mp}\|S_k\|_{L_p(I)}^p\Big).$$

Now we see that we want to have $2^{\alpha kp}\|S_k\|_{L_p(I)}^p$ on the right hand side, so we multiply and divide each term on the right by $2^{\alpha kp}$:

$$2^{\alpha mp}\omega_1(f, 2^{-m})_p^p \le C\Big(\sum_{0\le k<m} 2^{(\alpha p-1)(m-k)}2^{\alpha kp}\|S_k\|_{L_p(I)}^p + \sum_{m\le k} 2^{\alpha(m-k)p}2^{\alpha kp}\|S_k\|_{L_p(I)}^p\Big)$$

This set of inequalities can be written in vector form. We set

$$x_m := 2^{\alpha mp}\omega_1(f, 2^{-m})_p^p \text{ and } y_k := 2^{\alpha kp}\|S_k\|_{L_p(I)}^p.$$

Then, componentwise in m,

$$(x_m) \le C\begin{pmatrix} 1 & 2^{-\alpha p} & 2^{-2\alpha p} & 2^{-3\alpha p} & \dots \\ 2^{(\alpha p-1)} & 1 & 2^{-\alpha p} & 2^{-2\alpha p} & \dots \\ 2^{2(\alpha p-1)} & 2^{(\alpha p-1)} & 1 & 2^{-\alpha p} & \dots \\ 2^{3(\alpha p-1)} & 2^{2(\alpha p-1)} & 2^{(\alpha p-1)} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}(y_k).$$

If $p \geq 1$ we proceed similarly, but now we use the triangle inequality and end up with

$$2^{\alpha m}\omega_1(f,2^{-m})_p \leq C\Bigg(\sum_{0\leq k<m} 2^{(\alpha-1/p)(m-k)}2^{\alpha k}\|S_k\|_{L_p(I)} + \sum_{m\leq k} 2^{\alpha(m-k)}2^{\alpha k}\|S_k\|_{L_p(I)}\Bigg).$$

Now we set

$$x_m := 2^{\alpha m}\omega_1(f,2^{-m})_p \text{ and } y_k := 2^{\alpha k}\|S_k\|_{L_p(I)}$$

and get the vector inequality

$$(x_m) \leq C\begin{pmatrix} 1 & 2^{-\alpha} & 2^{-2\alpha} & 2^{-3\alpha} & \dots \\ 2^{(\alpha-1/p)} & 1 & 2^{-\alpha} & 2^{-2\alpha} & \dots \\ 2^{2(\alpha-1/p)} & 2^{(\alpha-1/p)} & 1 & 2^{-\alpha} & \dots \\ 2^{3(\alpha-1/p)} & 2^{2(\alpha-1/p)} & 2^{(\alpha-1/p)} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}(y_k).$$

These two vector inequalities, which are the crux of the matter, were derived by Larry Brown at Purdue.

In either case for $\ell \geq 0$ we set

$$T_\ell(y_0,y_1,y_2,\dots) := (y_\ell,y_{\ell+1},\dots) \text{ and } T_{-\ell}(y_0,y_1,y_2,\dots) := (\underbrace{0,\dots,0}_{\ell\text{ times}},y_0,y_1,\dots).$$

For any $0<r\leq\infty$, we obviously have $\|T_\ell(y_k)\|_r \leq \|(y_k)\|_r$.

For $0<p\leq 1$ our vector inequality can be written

$$(x_m) \leq C\Bigg(\sum_{0\leq\ell} 2^{-\alpha\ell p}T_\ell(y_k) + \sum_{1\leq\ell} 2^{\ell(\alpha p-1)}T_{-\ell}(y_k)\Bigg).$$

We set $r=q/p$; if $0<r\leq 1$, we have

$$\|(x_m)\|_r^r \leq C\Bigg(\sum_{0\leq\ell} 2^{-\alpha\ell pr}\|T_\ell(y_k)\|_r^r + \sum_{1\leq\ell} 2^{\ell(\alpha p-1)r}\|T_{-\ell}(y_k)\|_r^r\Bigg).$$

Both these sums are finite if $\alpha p-1<0$, i.e., $\alpha<1/p$, and we get

$$\|(x_m)\|_r^r = \sum_{m\geq 0}[2^{\alpha mp}\omega_1(f,2^{-m})_p^p]^{q/p} \leq C\|(y_k)\|_r^r = C\sum_{k\geq 0}[2^{\alpha kp}\|S_k\|_{L_p(I)}^p]^{q/p}$$

which is what we want upon taking qth roots. If $1\leq r$, then we have

$$\|(x_m)\|_r \leq C\Bigg(\sum_{0\leq\ell} 2^{-\alpha\ell p}\|T_\ell(y_k)\|_r + \sum_{1\leq\ell} 2^{\ell(\alpha p-1)}\|T_{-\ell}(y_k)\|_r\Bigg);$$

again, both these sums are finite if $\alpha < 1/p$, and

$$\|(x_m)\|_r \le C\|(y_k)\|_r,$$

which again gives what we want upon taking pth roots.

For $1 \le p$ our vector inequality becomes

$$(x_m) \le C\biggl(\sum_{0\le\ell} 2^{-\alpha\ell}T_\ell(y_k) + \sum_{1\le\ell} 2^{\ell(\alpha-1/p)}T_{-\ell}(y_k)\biggr).$$

Now we take $r = q$; again, whether $0 < r \le 1$ or $1 \le r$, we have for $\alpha < 1/p$

$$\sum_{m\ge 0}[2^{\alpha m}\omega_1(f, 2^{-m})_p]^q \le C\sum_{k\ge 0}[2^{\alpha k}\|S_k\|_{L_p(I)}]^q.$$

In a similar way we can show

$$\|f\|_{L_p(I)} \le C\|\{2^{\alpha k}\|S_k\|_{L_p(I)}\}\|_{\ell_q(0\le k)}.\tag{11}$$ [1]

So we have

$$\|f\|_{B_q^\alpha(L_p(I))} \le C\biggl(\sum_{k\ge 0}[2^{\alpha k}\|P_k f - P_{k-1}f\|_{L_p(I)}]^q\biggr)^{1/q},\tag{12}$$

and from (8) and (12) we have proved the following theorem:

[1] Indeed, if $p \ge 1$ then $\|f\|_{L_p(I)} \le \sum_{k\ge 0}\|S_k\|_{L_p(I)}$; if $q \le 1$ we have

$$\sum_{k\ge 0}\|S_k\|_{L_p(I)} \le \biggl(\sum_{k\ge 0}\|S_k\|_{L_p(I)}^q\biggr)^{1/q} \le \biggl(\sum_{k\ge 0}2^{\alpha kq}\|S_k\|_{L_p(I)}^q\biggr)^{1/q}$$

and if $q \ge 1$ we have (with $1/q + 1/q' = 1$)

$$\sum_{k\ge 0}\|S_k\|_{L_p(I)} = \sum_{k\ge 0}\|S_k\|_{L_p(I)}2^{\alpha k}2^{-\alpha k} \le \biggl(\sum_{k\ge 0}\|S_k\|_{L_p(I)}^q 2^{\alpha kq}\biggr)^{1/q}\biggl(\sum_{k\ge 0}2^{-\alpha kq'}\biggr)^{1/q'},$$

so we've proved (11) for $p \ge 1$. If $p \le 1$ then $\|f\|_{L_p(I)}^p \le \sum_{k\ge 0}\|S_k\|_{L_p(I)}^p$; if $r = q/p \le 1$ then

$$\sum_{k\ge 0}\|S_k\|_{L_p(I)}^p \le \biggl(\sum_{k\ge 0}\|S_k\|_{L_p(I)}^{pr}\biggr)^{1/r} = \biggl(\sum_{k\ge 0}\|S_k\|_{L_p(I)}^q\biggr)^{p/q} \le \biggl(\sum_{k\ge 0}2^{\alpha kq}\|S_k\|_{L_p(I)}^q\biggr)^{p/q},$$

which implies (11), and if $r \ge 1$ we have (again with $1/r + 1/r' = 1$)

$$\sum_{k\ge 0}\|S_k\|_{L_p(I)}^p = \sum_{k\ge 0}\|S_k\|_{L_p(I)}^p 2^{\alpha kp}2^{-\alpha kp} \le \biggl(\sum_{k\ge 0}\|S_k\|_{L_p(I)}^{pr}2^{\alpha kpr}\biggr)^{1/r}\biggl(\sum_{k\ge 0}2^{-\alpha kpr'}\biggr)^{1/r'},$$

which, since $pr = q$, again implies (11).

THEOREM 1. *Assume* $0 < p \le \infty$, $0 < \alpha < \min(1, 1/p)$, $0 < q \le \infty$. *For* $x \in I_{j,k}$ *we let*

$$P_k f(x) = \begin{cases} \dfrac{1}{|I_{j,k}|} \displaystyle\int_{I_{j,k}} f, & 1 \le p, \\ \text{a median of } f \text{ on } I_{j,k}, & 0 < p. \end{cases}$$

Then

$$\|f\|_{B_q^\alpha(L_p(I))} \asymp \|\{2^{\alpha k} \|P_k f - P_{k-1} f\|_{L_p(I)}\}\|_{\ell_q(0 \le k)}. \tag{13}$$

Embeddings of Besov spaces

One can derive a number of properties of Besov spaces from Theorem 1.

Because $L_p(I) \subset L_{p'}(I)$ when $p' \le p$, we have in that case $B_q^\alpha(L_p(I)) \subset B_q^\alpha(L_{p'}(I))$.

Because $\ell_q \subset \ell_{q'}$ if $0 < q < q' \le \infty$, we have $B_q^\alpha(L_p(I)) \subset B_{q'}^\alpha(L_p(I))$ for $0 < q < q' \le \infty$. In particular, $B_q^\alpha(L_p(I)) \subset B_\infty^\alpha(L_p(I))$ for all $0 < q < \infty$.

If $\alpha' > \alpha$, however, then $B_{q'}^{\alpha'}(L_p(I)) \subset B_q^\alpha(L_p(I))$ for any q and q'. One sees this by noting that $B_{q'}^{\alpha'}(L_p(I)) \subset B_\infty^{\alpha'}(L_p(I))$ and $f \in B_\infty^{\alpha'}(L_p(I))$ implies there is a C such that

$$2^{\alpha' k} \|P_k f - P_{k-1} f\|_{L_p(I)} \le C.$$

This means that for all k

$$2^{\alpha k} \|P_k f - P_{k-1} f\|_{L_p(I)} \le C 2^{-(\alpha' - \alpha)k},$$

and the right-hand side, being a decreasing geometric sequence, is in ℓ_q for any q.

Thus α is the main determiner of smoothness for fixed p, and the second parameter q allows us to make finer distinctions.

We now fix $\delta \in \mathbb{R}$ and consider the pairs α, q ($q > 0$, $0 < \alpha < \min(1, 1/q)$) that satisfy

$$\frac{1}{q} - \frac{\alpha}{2} = \delta. \tag{14}$$

For any pair satisfying (14) we have with $P_k f - P_{k-1} f = \sum_{j \in \mathbb{Z}_k^2} d_{j,k} \chi_{j,k}$,

$$\begin{aligned} \|\{2^{\alpha k} \|P_k f - P_{k-1} f\|_{L_q(I)}\}\|_{\ell_q(0 \le k)}^q &= \sum_{k \ge 0} 2^{\alpha k q} \sum_{j \in \mathbb{Z}_k^2} |d_{j,k}|^q \|\chi_{j,k}\|_{L_q(I)}^q \\ &= \sum_{k \ge 0} 2^{\alpha k q} \sum_{j \in \mathbb{Z}_k^2} |d_{j,k}|^q 2^{-2k} \\ &= \sum_{k \ge 0} \sum_{j \in \mathbb{Z}_k^2} |2^{(\alpha - 2/q)k} d_{j,k}|^q \\ &= \sum_{k \ge 0} \sum_{j \in \mathbb{Z}_k^2} |2^{-2k\delta} d_{j,k}|^q. \end{aligned}$$

So

$$\|f\|_{B^\alpha_q(L_q(I))} \asymp \|\{2^{\alpha k}\|P_kf - P_{k-1}f\|_{L_q(I)}\}\|_{\ell_q(0\le k)} = \|\{2^{-2k\delta}d_{j,k}\}\|_{\ell_q(k\ge 0,\ j\in\mathbb{Z}_k^2)}. \tag{15}$$

If the pair α', q' also satisfies (14) with $\alpha' > \alpha$, then $q' < q$; because $\ell_{q'} \subset \ell_q$ in this case, we have that $B^{\alpha'}_{q'}(L_{q'}(I)) \subset B^\alpha_q(L_q(I))$.

All these inclusions come with norm inequalities, so they are in fact *embeddings*.

To summarize: for $\alpha, \alpha' > 0$, $0 < p, p', q, q' \le \infty$,

$$\begin{aligned}
B^\alpha_q(L_{p'}(I)) &\hookrightarrow B^\alpha_q(L_p(I)) \quad \text{when } p' > p;\\
B^\alpha_{q'}(L_p(I)) &\hookrightarrow B^\alpha_q(L_p(I)) \quad \text{when } q' < q;\\
B^{\alpha'}_{q'}(L_p(I)) &\hookrightarrow B^\alpha_q(L_p(I)) \quad \text{when } \alpha' > \alpha \text{ for any } q, q';\\
B^{\alpha'}_{q'}(L_{q'}(I)) &\hookrightarrow B^\alpha_q(L_q(I)) \quad \text{when } \frac{1}{q} - \frac{\alpha}{2} = \frac{1}{q'} - \frac{\alpha'}{2} \text{ and } \alpha' > \alpha.
\end{aligned}$$

In fact, we have for $0 < \alpha$, $0 < p < \infty$, and

$$\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{p} \quad \Big(\text{so } \delta = \frac{1}{p}\Big),$$

that $B^\alpha_q(L_q(I)) \hookrightarrow L_p(I)$. This can be shown simply when $p = 2$, so

$$\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{2} \quad \Big(\text{so } \delta = \frac{1}{2}\Big),$$

P_kf is the best $L_2(I)$ approximation to f on S^k, and $0 < \alpha < 1$. For then $P_kf - P_{k-1}f$ is orthogonal to all $S \in S^\ell$ for $\ell \le k-1$, and for fixed k the $\chi_{j,k}$, $j \in \mathbb{Z}_k^2$, are orthogonal because they have essentially disjoint support.

Thus we have

$$\begin{aligned}
\|f\|^2_{L_2(I)} &= \sum_{k\ge 0} \|P_kf - P_{k-1}f\|^2_{L_2(I)} = \sum_{k\ge 0}\sum_{j\in\mathbb{Z}_k^2} \|d_{j,k}\chi_{j,k}\|^2_{L_2(I)}\\
&= \sum_{k\ge 0}\sum_{j\in\mathbb{Z}_k^2} |2^{-k}d_{j,k}|^2 = \sum_{k\ge 0}\sum_{j\in\mathbb{Z}_k^2} |2^{-2k\delta}d_{j,k}|^2. \qquad (16)\\
&= \|\{2^{-2k\delta}d_{j,k}\}\|_{\ell_2(k\ge 0,\ j\in\mathbb{Z}_k^2)}.
\end{aligned}$$

Again, because $q < 2$, $\ell_q \hookrightarrow \ell_2$. Thus from (16) and (15) we have, as claimed,

$$B^\alpha_q(L_q(I)) \hookrightarrow L_2(I).$$

We summarize these results in the important Figure 1. A function space with smoothness α in $L_p(I)$ is graphed at the point $(1/p, \alpha)$. This concept is not so precise, so the Sobolev space $W^{1,1}(I)$, the Besov space $B^1_q(L_1(I))$ for any $0 < q \le \infty$, and the space of functions of bounded variation $\mathrm{BV}(I)$ would all be represented by the point $(1, 1)$. Similarly, all Besov spaces $B^\alpha_q(L_p(I))$ for any q would be graphed at the point $(1/p, \alpha)$, and the space $L_p(I)$ would be represented by the point $(1/p, 0)$ (since functions in $L_p(I)$ have zero smoothness).

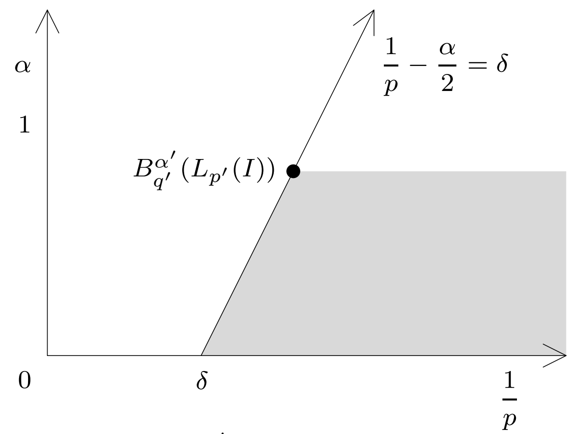

FIGURE 1. The Besov space $B^{\alpha'}_{q'}(L_{p'}(I))$ is embedded in every Besov space $B^{\alpha}_{q}(L_p(I))$ with $(1/p, \alpha)$ in at least the (open) shaded area.

Two-dimensional Haar transform

This may not look much like "regular" wavelets, so we specialize a bit further and see how this applies to two-dimensional Haar wavelets.

For each $k > 0$, each interval

$$I_{j,k-1} = I_{2j,k} \cup I_{2j+(1,0),k} \cup I_{2j+(0,1),k} \cup I_{2j+(1,1),k}\,,$$

so naturally we can write on $I_{j,k-1}$

$$\begin{aligned} P_k f - P_{k-1} f &= d_{2j,k}\chi_{2j,k} + d_{2j+(1,0),k}\chi_{2j+(1,0),k} \\ &\qquad + d_{2j+(0,1),k}\chi_{2j+(0,1),k} + d_{2j+(1,1),k}\chi_{2j+(1,1),k} \\ &= \boxed{\begin{matrix} d_{2j+(0,1),k} & d_{2j+(1,1),k} \\ d_{2j,k} & d_{2j+(1,0),k} \end{matrix}}\,, \end{aligned}$$

where we have boxed the values of the difference of projections to suggest the values in each quarter of the interval $I_{j,k-1}$

We choose a different basis for this four-dimensional space: For any numbers a, b, c, and d, we can write

$$\boxed{\begin{matrix} a & b \\ c & d \end{matrix}} = \alpha \boxed{\begin{matrix} +1 & -1 \\ +1 & -1 \end{matrix}} + \beta \boxed{\begin{matrix} -1 & -1 \\ +1 & +1 \end{matrix}} + \gamma \boxed{\begin{matrix} -1 & +1 \\ +1 & -1 \end{matrix}} + \delta \boxed{\begin{matrix} +1 & +1 \\ +1 & +1 \end{matrix}}. \tag{17}$$

Therefore, for *any* projector P_k we can write $P_k f - P_{k-1} f$ as a linear combination

$$P_k f - P_{k-1} f = \sum_{j \in \mathbb{Z}^2_{k-1}} \sum_{\psi \in \Psi} c_{j,k-1,\psi} \psi_{j,k-1}$$

where

$$\Psi = \{\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \psi^{(4)}\}$$

and each ψ is defined on I by

$$\begin{aligned}
\psi^{(1)}(x_1,x_2) &= \begin{cases} 1, & 0 \le x_1 < 1/2, \\ -1, & \text{otherwise,} \end{cases} \\
\psi^{(2)}(x_1,x_2)) &= \begin{cases} 1, & 0 \le x_2 < 1/2, \\ -1, & \text{otherwise,} \end{cases} \\
\psi^{(3)}(x_1,x_2) &= \begin{cases} 1 & 0 \le x_1, x_2 < 1/2 \text{ or } 1/2 \le x_1, x_2 \le 1, \\ -1, & \text{otherwise,} \end{cases} \\
\psi^{(4)}(x_1,x_2) &= 1 \quad \text{for all } (x_1,x_2) \in I.
\end{aligned}$$

So we can write

$$f = \sum_{k \ge 0} \sum_{j \in \mathbb{Z}_k^2} \sum_{\psi \in \Psi} c_{j,k,\psi} \psi_{j,k}. \tag{18}$$

On each interval $I_{j,k-1}$, we have for any $0 < p < \infty$

$$\int_{I_{j,k-1}} \Big(\sum_{\ell} d_{\ell,k} \chi_{\ell,k} \Big)^p \asymp \sum_{\psi} \int_{I_{j,k-1}} |c_{j,k-1,\psi} \psi_{j,k-1}|^p, \tag{19}$$

with constants that depend only on p because all (quasi-)norms on a four-dimensional space are equivalent. Thus, from (13) and (19) we have the norm equivalence

$$\|f\|_{B_q^\alpha(L_p(I))} \asymp \left(\sum_{k \ge 0} \left[2^{\alpha k} \Big(\sum_{j,\psi} \|c_{j,k,\psi} \psi_{j,k}\|_{L_p(I)}^p \Big)^{1/p} \right]^q \right)^{1/q}.$$

When P_k is the *average projector*,

$$P_k f(x) = \frac{1}{|I_{j,k}|} \int_{I_{j,k}} f = f_{j,k} \quad \text{for } x \in I_{j,k},$$

we can specialize things a bit more. We have on $I_{j,k-1}$

$$f_{j,k-1} = \frac{1}{4}(f_{2j,k} + f_{2j+(1,0),k} + f_{2j+(0,1),k} + f_{2j+(1,1),k}),$$

i.e., $f_{j,k-1}$ is the average of the four k-level "pixel values" on $I_{j,k-1}$, and the coefficient δ in (17) is *zero*. (When $k-1=0$, δ is the average value of f on I.)

We then write $\Psi_0 = \Psi$ and

$$\Psi_k = \{\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\}, \quad k > 0,$$

and normalize $\psi_{j,k}$ in $L_2(I)$:

$$\psi_{j,k}(x) = 2^k \psi(2^k x - j), \quad k \geq 0,\ \psi \in \Psi_k, j \in \mathbb{Z}_k^2.$$

The set $\{\psi_{j,k} \mid k \geq 0,\ \psi \in \Psi_k,\ j \in \mathbb{Z}_k^2\}$ is *orthonormal* and, because of (18), forms an orthonormal basis for $L_2(I)$. In particular, if

$$f = \sum_{k\geq 0} \sum_{\psi\in\Psi_k} \sum_{j\in\mathbb{Z}_k^2} c_{j,k,\psi}\psi_{j,k} \quad \text{(the } \textit{Haar} \text{ transform)},$$

then

$$\|f\|_{L_2(I)} = \|(c_{j,k,\psi})\|_{\ell_2(0\leq k, j\in\mathbb{Z}_k^2, \psi\in\Psi)}. \tag{20}$$

Furthermore, since

$$\|\psi_{j,k}\|_{L_p(I)} = \left(\int_{I_{j,k}} [2^k]^p\right)^{1/p} = (2^{-2k}2^{kp})^{1/p} = 2^{k(1-2/p)} \quad \text{for all } \psi \in \Psi_k \text{ and } j \in \mathbb{Z}_k^2,$$

we can write for $0 < \alpha < \frac{1}{p}$ and $p \geq 1$

$$\|f\|_{B_q^\alpha(L_p(I))} \asymp \left(\sum_{k\geq 0}\left[2^{\alpha k}2^{k(1-2/p)}\left(\sum_{\psi\in\Psi_k}\sum_{j\in\mathbb{Z}_k^2}|c_{j,k,\psi}|^p\right)^{1/p}\right]^q\right)^{1/q}$$

because the average projector P_k is bounded on $L_p(I)$.

In particular, if $0 < \alpha < 1$ and $p = q$ satisfies

$$\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{2},$$

so $1 < q < 2$, then the exponent of 2 in the previous formula satisfies

$$\left(\alpha + 1 - \frac{2}{p}\right)k = 0,$$

and

$$\|f\|_{B_q^\alpha(L_q(I))} \asymp \|(c_{j,k,\psi})\|_{\ell_q(0\leq k, j\in\mathbb{Z}_k^2, \psi\in\Psi)}. \tag{21}$$

Since $\ell_q \subset \ell_2$, we see immediately from (20) and (21) that $B_q^\alpha(L_q(I))$ is embedded in $L_2(I)$.

In the following we accept implicitly for the Haar transform that $\psi^{(4)}$ will be used only when $k = 0$.

The big picture, part I: Linear approximation

In the next two sections we discuss two big ideas in the context of wavelets:

(1) Nonlinear approximation is better than linear approximation.
(2) Approximation is equivalent to smoothness.

We ask a number of questions:

(1) What do we mean by linear and nonlinear approximation by wavelets?
(2) What does it mean to approximate a function well by wavelets?
(3) Can one characterize the set of functions that are approximated well by wavelets?

We note that the average projector $P_k f$ is the *best* approximation in $L_2(I)$ to f on S^k, and using the Haar wavelets we developed in the previous section we can write

$$P_k f = \sum_{0\le\ell<k,\ j\in\mathbb{Z}^2_\ell,\ \psi\in\Psi} c_{j,\ell,\psi}\psi_{j,\ell}, \quad c_{j,\ell,\psi} = \langle f, \psi_{j,\ell}\rangle. \tag{22}$$

There are 2^{2k} terms in the sum in (22), and the dimension of S^k is 2^{2k}. So for each k we have chosen, a priori, a set of 2^{2k} wavelet terms

$$\{\psi_{j,\ell} \mid 0 \le \ell < k,\ j \in \mathbb{Z}^2_\ell,\ \psi \in \Psi\}$$

before even looking at f. We also have that

$$P_k(\alpha f + \beta g) = \alpha P_k f + \beta P_k g,$$

i.e., this *approximation process* is *linear*.

So if we define for $1 \le p \le \infty$

$$E_N(f)_p = \inf_{S\in S^k} \|f - S\|_{L_p(I)}, \quad N = 2^{2k} \text{ (the dimension of } S^k),$$

the error of best approximation of f in $L_p(I)$, we have by (3) and (4),

$$E_{2^{2k}}(f)_p \le \|f - P_k f\|_{L_p(I)} \le 2E_{2^{2k}}(f)_p \le C\omega_1(f, 2^{-k})_p.$$

So, for any $1 \le p < \infty$, $0 < \alpha < 1/p$, and $0 < q \le \infty$,

$$\|\{2^{\alpha k} E_{2^{2k}}(f)_p\}\|_{\ell_q(k\ge0)} \le C\|\{2^{\alpha k}\omega_1(f, 2^{-k})_p\}\|_{\ell_q(k\ge0)} \asymp C\|f\|_{B^\alpha_q(L_p(I))}.$$

Conversely, from Theorem 1,

$$\begin{aligned}\|f\|_{B^\alpha_q(L_p(I))} &\asymp \|\{2^{\alpha k}\|P_kf - P_{k-1}f\|_{L_p(I)}\}\|_{\ell_q(k\ge 0)} \\ &\le \|\{2^{\alpha k}(\|P_kf - f\|_{L_p(I)} + \|f - P_{k-1}f\|_{L_p(I)})\}\|_{\ell_q(k\ge 0)} \\ &\le C_q\|\{2^{\alpha k}\|P_kf - f\|_{L_p(I)}\}\|_{\ell_q(k\ge 0)} \\ &\le C\|\{2^{\alpha k}E_{2^{2k}}(f)_p\}\|_{\ell_q(k\ge 0)}.\end{aligned}$$

Combining the previous two inequality gives

$$\|\{2^{\alpha k}E_{2^{2k}}(f)_p\}\|_{\ell_q(k\ge 0)} \asymp \|f\|_{B^\alpha_q(L_p(I))},$$

i.e., *f can be approximated in S^k at a certain rate if and only if f is in a specific Besov smoothness space.*

This is the first example of the dictum:

Approximation is equivalent to smoothness.

The big picture, part II: Compression of wavelet coefficients

Note: This section is not yet written in a way that fits in with the previous material.

We choose an *error space*; for the moment, we choose $L_2(I)$. We also work with orthogonal Haar wavelets $\{\psi_{j,k}\}$.

We note that if we want to approximate

$$f = \sum_{j,k,\psi} c_{j,k,\psi}\psi_{j,k}$$

by a sum

$$\tilde f = \sum_{c_{j,k,\psi}\psi_{j,k}\in\Lambda} c_{j,k,\psi}\psi_{j,k}$$

with no more than N terms in Λ and we want to minimize

$$\|f-\tilde f\|_{L_2(I)} = \left(\sum_{c_{j,k,\psi}\psi_{j,k}\notin\Lambda} \|c_{j,k,\psi}\psi_{j,k}\|^2_{L_2(I)}\right)^{1/2}$$

then we should put into Λ the N terms $c_{j,k,\psi}\psi_{j,k}$ with the largest values of $\|c_{j,k,\psi}\psi_{j,k}\|$; we call that approximation f_N. (Break ties in an arbitrary manner.)

If we sort $\{c_{j,k,\psi}\psi_{j,k}\}$ in nonincreasing order of $\|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)}$, and call the resulting sequence

$$\{c_i\} = \{\|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)}\}, \quad c_i \ge c_{i+1},$$

then

$$\|f - f_N\|_{L_2(I)} = \left(\sum_{N<i} c_i^2\right)^{1/2}.$$

We now want to find an equivalence between the rate of decay of $\|f - f_N\|_{L_2(I)}$ as $N \to \infty$ and the *smoothness* of f in $B_q^\alpha(L_q(I))$, if in fact

$$\|f\|_{B_q^\alpha(L_q(I))} \asymp \left(\sum_{j,k,\psi} \|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)}^q\right)^{1/q} = \left(\sum_i c_i^q\right)^{1/q} < \infty.$$

This is done in Appendix A, but a simple bound on $\|f - f_N\|_{L_2(I)}$ when $f \in B_q^\alpha(L_q(I))$, $1/q = \alpha/2 + 1/2$, can be obtained as follows.

We choose an $\epsilon > 0$ and ask "How many coefficients can satisfy $|c_{j,k,\psi}| > \epsilon$?" If we denote this number by N, then we must have

$$(N\epsilon^q)^{1/q} \le \left(\sum_{j,k,\psi} \|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)}^q\right)^{1/q} \le C\|f\|_{B_q^\alpha(L_q(I))},$$

so

$$N \le C\epsilon^{-q}\|f\|_{B_q^\alpha(L_q(I))}^q \text{ and } \epsilon < CN^{-1/q}\|f\|_{B_q^\alpha(L_q(I))}.$$

We put these N coefficients into Λ, the resulting error is

$$\begin{aligned}
\|f - f_N\|_{L_2(I)} = \left(\sum_{c_i\le\epsilon} c_i^2\right)^{1/2} &\le \sup_{c_i\le\epsilon} c_i^{(2-q)/2}\left(\sum_{c_i\le\epsilon} c_i^q\right)^{1/2} \\
&\le \epsilon^{(2-q)/2}\left(\sum c_i^q\right)^{1/2} \\
&\le C(N^{-1/q}\|f\|_{B_q^\alpha(L_q(I))})^{(2-q)/2}\|f\|_{B_q^\alpha(L_q(I))}^{q/2} \\
&= CN^{-(2-q)/(2q)}\|f\|_{B_q^\alpha(L_q(I))} \\
&= CN^{-\alpha/2}\|f\|_{B_q^\alpha(L_q(I))}
\end{aligned}$$

since

$$\frac{1}{q} - \frac{1}{2} = \frac{\alpha}{2}.$$

We can rewrite this as

$$N^{\alpha/2}\|f - f_N\|_{L_2(I)} \le C\|f\|_{B_q^\alpha(L_q(I))},$$

or

$$\|\{N^{\alpha/2}\|f - f_N\|_{L_2(I)}\}\|_\infty \le C\|f\|_{B_q^\alpha(L_q(I))}. \tag{23}$$

In fact, we have the more subtle estimate

$$\begin{aligned}\Big(\sum_{N\ge1}[N^{\alpha/2}\|f - f_{N-1}\|_{L_2(I)}]^q\frac{1}{N}\Big)^{1/q} &= \Big(\sum_{N\ge1}\Big[N^{\alpha/2}\Big(\sum_{i=N}^\infty c_i^2\Big)^{1/2}\Big]^q\frac{1}{N}\Big)^{1/q}\\ &\asymp \Big(\sum_{i=1}^\infty c_i^q\Big)^{1/q} \asymp \|f\|_{B_q^\alpha(L_q(I))},\end{aligned} \tag{24}$$

which we'll prove in Appendix A.

Quantization. In practice, the coefficients of a compressed image, etc., are calculated by a practice called quantization. There are many different methods of quantizing coefficients: scalar quantization, vector quantization, zero-tree coding, etc.

Nonetheless, all quantization strategies of which I'm aware take a function f with wavelet coefficients $c_{j,k,\psi}$, and derive a *compressed* function $\bar f$ with *compressed coefficients* $\bar c_{j,k,\psi}$ satisfying the following properties.

These compressed coefficients satisfy for a parameter $\epsilon > 0$ and two constants C_1 and C_2:

(1) $\|(c_{j,k,\psi} - \bar c_{j,k,\psi})\psi_{j,k}\|_{L_p(I)} \le C_1\epsilon$.
(2) $\|c_{j,k,\psi}\psi_{j,k}\|_{L_p(I)} < C_2\epsilon \implies \bar c_{j,k,\psi} = 0$.

Why the C_1 and C_2? Ideally, we could pick $C_1 = C_2 = 1$ and prove something, but in practice methods implement a so-called "dead zone" around zero by choosing $C_2 > 1$, and recreating $\bar c_{j,k,\psi}$ often introduces a small extra error that causes $C_1 > 1$. So we'll keep these two constants in this analysis.

For our proof, we assume that we measure error in $L_2(I)$, so $p = 2$; later we'll consider the special case of $p = 1$. A more general theory is available (see DeVore, Jawerth, and Popov, Compression of Wavelet Coefficients) but the proofs are beyond the level of this class.

We assume that $f \in B_q^\alpha(L_q(I))$ with $\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{2}$, so $B_q^\alpha(L_q(I)) \hookrightarrow L_2(I)$.

Again, we ask how many nonzero $\bar c_{j,k,\psi}$ (denoted by N) can we have? If $\bar c_{j,k,\psi} \ne 0$, then $\|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)} \ge C_2\epsilon$, so for all those $\bar c_{j,k,\psi}$ we have

$$N(C_2\epsilon)^q \le \sum \|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)}^q = \|f\|_{B_q^\alpha(L_q(I))}^q,$$

so $N \le C_2^{-q}\epsilon^{-q}\|f\|_{B_q^\alpha(L_q(I))}^q$ and $\epsilon \le N^{-1/q}C_2^{-1}\|f\|_{B_q^\alpha(L_q(I))}$.

If we write $\gamma_{j,k,\psi} = c_{j,k,\psi} - \bar{c}_{j,k,\psi}$, then we have from the two properties of $\bar{c}_{j,k,\psi}$

$$\|\gamma_{j,k,\psi}\psi_{j,k}\|_{L_2(I)} \leq \begin{cases} \|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)}, & \|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)} < C_2\epsilon, \\ C_1\epsilon \leq C_1/C_2\|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)}, & \|c_{j,k,\psi}\psi_{j,k}\|_{L_2(I)} \geq C_2\epsilon, \end{cases}$$

and in all cases $\|\gamma_{j,k,\psi}\psi_{j,k}\|_{L_2(I)} \leq \max(C_1, C_2)\epsilon$.

So if we denote $\sum \bar{c}_{j,k,\psi}\psi_{j,k}$ by $\bar{f}_N$, we use these two inequalities to find

$$\begin{aligned} \|f - \bar{f}_N\|^2_{L_2(I)} &= \sum \|\gamma_{j,k,\psi}\psi_{j,k}\|^2_{L_2(I)} \\ &= \sum \|\gamma_{j,k,\psi}\psi_{j,k}\|^q_{L_2(I)} \, \|\gamma_{j,k,\psi}\psi_{j,k}\|^{2-q}_{L_2(I)} \\ &\leq \big(\max(C_1, C_2)\epsilon\big)^{2-q} \max(1, C_1/C_2)^q \sum \|c_{j,k,\psi}\psi_{j,k}\|^q_{L_2(I)} \\ &= \max(C_1, C_2)^2 C_2^{-q}\epsilon^{2-q}\|f\|^q_{B^\alpha_q(L_q(I))}. \end{aligned}$$

Because $\epsilon \leq N^{-1/q}C_2^{-1}\|f\|_{B^\alpha_q(L_q(I))}$, we have

$$\epsilon^{2-q} \leq N^{-(2/q-1)}C_2^{-(2-q)}\|f\|^{2-q}_{B^\alpha_q(L_q(I))}$$

and

$$\begin{aligned} \|f - \bar{f}_N\|^2_{L_2(I)} &\leq \max(C_1, C_2)^2 C_2^{-q} N^{-(2/q-1)} C_2^{-(2-q)} \|f\|^{2-q}_{B^\alpha_q(L_q(I))} \|f\|^q_{B^\alpha_q(L_q(I))} \\ &= \max(C_1, C_2)^2 C_2^{-2} N^{-\alpha} \|f\|^2_{B^\alpha_q(L_q(I))}, \end{aligned}$$

because $2/q - 1 = \alpha$. So

$$\|f - \bar{f}_N\|_{L_2(I)} \leq \max(C_1/C_2, 1)N^{-\alpha/2}\|f\|_{B^\alpha_q(L_q(I))}.$$

So now I'm glad that I carried through the dependence on C_1 and C_2, as it's clear that the error bound really does depend only on the ratio C_1/C_2, because the number of nonzero coefficients N depends only on $C_2\epsilon$, so we could define a new $\epsilon' = C_2\epsilon$ and in terms of this ϵ' we'd have $C_2' = 1$ and $C_1' = C_1/C_2$.

Scalar quantization. In general, when measuring error in $L_p(I)$ we will want to ensure that

(1) $\|(c_{j,k,\psi} - \bar{c}_{j,k,\psi})\psi_{j,k}\|_{L_p(I)} \leq C_1\epsilon$.
(2) $\|c_{j,k,\psi}\psi_{j,k}\|_{L_p(I)} < C_2\epsilon \implies \bar{c}_{j,k,\psi} = 0$.

We won't prove this except in special cases of p, but it's true in general.

So let's set C_1 to 1 for the moment and design a quantization strategy.

We will calculate an integer *code*, $\text{code}_{j,k,\psi}$, for each coefficient $c_{j,k,\psi}$ as follows. We start with

$$\|(c_{j,k,\psi}-\bar c_{j,k,\psi})\psi_{j,k}\|_{L_p(I)}\le\epsilon$$

and divide by 2ϵ:

$$\Bigl|\frac{c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}}{2\epsilon}-\frac{\bar c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}}{2\epsilon}\Bigr|\le\frac12.$$

If we define $\text{round}(x)$ to be the nearest integer to a real number x (we don't care what happens when x is precisely between two integers), then we set

$$\text{code}_{j,k,\psi}=\text{round}\Bigl(\frac{c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}}{2\epsilon}\Bigr),$$

so that

$$\Bigl|\frac{c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}}{2\epsilon}-\text{code}_{j,k,\psi}\Bigr|\le\frac12.$$

We could set $\bar c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}=2\epsilon\,\text{code}_{j,k,\psi}$ and obtain

$$\|(c_{j,k,\psi}-\bar c_{j,k,\psi})\psi_{j,k}\|_{L_p(I)}\le\epsilon.$$

Furthermore, $\text{code}_{j,k,\psi}=\bar c_{j,k,\psi}=0$ if

$$\Bigl|\frac{c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}}{2\epsilon}\Bigr|<\frac12$$

or

$$\|c_{j,k,\psi}\psi_{j,k}\|_{L_p(I)}<\epsilon,$$

so $C_2=1$ with this strategy.

When it comes time to reconstruct the nonzero $\bar c_{j,k,\psi}$ we can run into practical problems. It could happen that $\frac{c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}}{2\epsilon}$ are statistically distributed so that nearly all the $\text{code}_{j,k,\psi}$ are zero, and that if, e.g.,

$$\text{code}_{j,k,\psi}=\text{round}\Bigl(\frac{c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}}{2\epsilon}\Bigr)=1,$$

then $\Bigl(\frac{c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}}{2\epsilon}\Bigr)$ is much more likely to be near the left end of the interval $[1/2,3/2]$ than the right, so that the reconstructed $\bar c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}=2\epsilon\,\text{code}_{j,k,\psi}$ could be nearly twice the size of $c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}$.

And we'll see that in the image—the reconstructed $\bar{c}_{j,k,\psi}\psi_{j,k}$ will be isolated at a given scale (because there will be few nonzero reconstructed wavelet terms at that scale) and be about twice the size of the original coefficient.

If $\text{code}_{j,k,\psi} = 2$, then $\left(\frac{c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}}{2\epsilon}\right)$ was inside $[3/2, 5/2]$, so using the naive reconstruction formula for $\bar{c}_{j,k,\psi}$ could result in a reconstructed coefficient being 4/3 the size of the original, which is less noticeable.

Some people try to get rid of the possibility of the reconstructed coefficients are significantly larger than the original coefficients by creating a "dead zone"—they set $\bar{c}_{j,k,\psi} = 0$ if $|\text{code}_{j,k,\psi}| \le 1$ or even $|\text{code}_{j,k,\psi}| \le 2$. This makes $C_2 > 1$ in our quantization rules.

In our code we use a different strategy. We assume that for each k and ψ the coefficient $c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}/(2\epsilon)$ is a random variable (**Note:** the first time the term "random variable" has come up in this course) with a probability distribution function

$$2a\, e^{-a|x|} \quad \text{for some } a > 0.$$

We estimate a from the observed fraction of $\text{code}_{j,k,\psi}$ that are zero (i.e., so the original $c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}/(2\epsilon)$ are in the interval $[-1/2, 1/2]$). Using this estimated p.d.f., we set $\bar{c}_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}/(2\epsilon)$ to be the expected value of $c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}/(2\epsilon)$ in $[\text{code}_{j,k,\psi} - 1/2, \text{code}_{j,k,\psi} + 1/2]$.

So, when the observed fraction of the $\text{code}_{j,k,\psi}$ being zero is very high, near 1, in the rare case when $c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}/(2\epsilon)$ is close to, but just under, 3/2, the reconstructed $\bar{c}_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}/(2\epsilon)$ will be just above 1/2. So the error $\|(c_{j,k,\psi} - \bar{c}_{j,k,\psi})\psi_{j,k}\|_{L_p(I)}$ will be just about twice what it would otherwise be, i.e., we'll have $C_1 = 2$. So in this rare case the error will be larger than usual, while in the much more common case when $c_{j,k,\psi}\|\psi_{j,k}\|_{L_p(I)}/(2\epsilon)$ is barely above 1/2, the error will be much smaller.

This probability model was chosen because the offset depends only on the current estimate of a, and not on the value of $\text{code}_{j,k,\psi}$. Note also that because the estimated parameter a depends only on the observed number of zero codes $\text{code}_{j,k,\psi}$, the decoder can apply the same calculation to compute the same a.

It is the sequence of integers $\{\text{code}_{j,k,\psi} \mid k \ge 0, j \in \mathbb{Z}_k^2, \psi \in \Psi\}$ that is coded into a bit stream, while the decoder reverses the process and reconstructs $\bar{f}_N$.

Image Compression "Crimes." In practice, we aren't given the exact light vector field that is hitting the focal plane of the camera (or, more generally, receptor). The perturbations caused by fixed pixel resolution, grey-scale resolution, point-spread-function of the lens system, etc., all introduce changes (sometimes nonlinear) that affect the image that is handed to our image compression routine. We shall call these perturbations that make our simple theory incomplete or incorrect "crimes" in parallel to the term "variational crimes" used in the finite element method literature.

“What is a wavelet?”

In the course of presenting these notes I was asked “But what *is* a wavelet?”.

There are many people who can give a better answer than I can, but I’ll give my perspective.

I prefer the question “*What is a wavelet transform*?” The reader should realize that the rest of this section is meta-mathematics, not mathematics.

Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be a *refinable function*, that is, there are finitely many coefficients a_j, $j \in \mathbb{Z}^2$, for which

$$\phi(x) = \sum_j a_j \phi(2x - j).$$

For each $k \ge 0$, let S^k be the span of the functions

$$\phi_{jk}(x) = \phi(2^k x - j), \quad j \in \mathbb{Z}^2,$$

whose support intersects with the unit interval I nontrivially.

Assume that there are (possibly nonlinear) projectors $P_k\colon L_p(I) \to S^k$ for some range of p and a positive constant C such that for some $r > 0$ and all $k \ge 0$ and all $f \in L_p(I)$,

$$\|P_k f - f\|_{L_p(I)} \le C\omega_r(f, 2^{-k})_p. \tag{25}$$

Assume also that there is some $\beta > 0$ and $C > 0$ such that for all $S \in S^k$,

$$\omega_r(S,t)_p \le C \begin{cases} \|S\|_{L_p(I)}, & 2^{-k} \le t, \\ 2^{k\beta/p} t^{\beta/p} \|S\|_{L_p(I)}, & 0 < t \le 2^{-k}. \end{cases} \tag{26}$$

In other words, Jackson (25) and Bernstein (26) inequalities hold for the spaces S^k.

In this case, the series

$$f = \sum_{k \ge 0} (P_k f - P_{k-1} f)$$

converges in $L_p(I)$ and Theorem 1 will hold for some range of α and p.

Conclusion: This combination of a family of subspaces S^k, generated by the dyadic dilates and translates of a function ϕ, together with specific projectors P_k such that Jackson (25) and Bernstein (26) inequalities hold is a *wavelet transform*.

In most cases people construct linear projectors P_k such that that S^k is the direct sum of S^{k-1} and the range of $P_k - P_{k-1}$, which we’ll denote by W^{k-1}. So

$$S^k = S^{k-1} \oplus W^{k-1}.$$

Because ϕ is refinable, the space W^{k-1} is generated by $2^d - 1$ functions $\psi \in \Psi$, where we're working in $\mathbb{R}^d$ ($d = 2$ up until now). If the functions $\phi(\cdot - j)$, $j \in \mathbb{Z}^2$, are mutually orthogonal, then the functions $\psi(\cdot - j)$, $j \in \mathbb{Z}^2$, $\psi \in \Psi$ can be taken to be mutually orthogonal. This is the situation for the Haar transform, for which $\phi = \chi_I$, the characteristic function of I, and P_k is the $L_2(I)$ projection onto S^k.

If P_k is the median transform, however, then it's not hard to see that the range of $P_k - P_{k-1}$ is S^k itself. Nonetheless, with this specific way of calculating coefficients, Jackson and Bernstein inequalities hold, and we consider this a wavelet transform.

This definition is at the very least imprecise and most likely incorrect in important ways. But it is general enough to allow the notion of nonlinear wavelet transforms, binary wavelet transforms (apply median projectors to functions f whose ranges take only the values 0 and 1), integer-to-integer wavelet transforms (see Devore-Jawerth-Lucier, "Image compression through wavelet transform coding"), etc.

Appendix A

In this Appendix we give a short proof of the equivalence between nonlinear rates of approximation in $L_2(I)$ using orthogonal wavelets.

which we prove using a lemma from DeVore and Temlyakov, Advances in Computational Math, Vol 5, 1996, 173–197. I'm responsible for any errors in translation. Note that (24) implies (23) since $\sum_N 1/N$ diverges. Let $\{a_k\}$ be a non-negative, non-increasing sequence,

$$\sigma_m^2 = \sum_{k=m}^{\infty} a_k^2,$$

and

$$\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{2}.$$

Then we have

$$a_{2m} \le a_{2m-1} \le \left(\frac{1}{m} \sum_{k=m}^{2m-1} a_k^2 \right)^{1/2} \le \frac{1}{m^{1/2}} \sigma_m.$$

So

$$\sum_{m=1}^{\infty} a_m^q \le 2 \sum_{m=1}^{\infty} \frac{1}{m^{q/2}} \sigma_m^q = 2 \sum_{m=1}^{\infty} m^{\alpha q/2} \sigma_m^q \frac{1}{m}$$

or

$$\left(\sum_{m=1}^{\infty} a_m^q \right)^{1/q} \le 2^{1/q} \left(\sum_{m=1}^{\infty} [m^{\alpha/2} \sigma_m]^q \frac{1}{m} \right)^{1/q}.$$

In the other direction,

$$
\begin{aligned}
\sigma_{2^m} &= \Big(\sum_{k=2^m}^{\infty} a_k^2\Big)^{1/2} \\
&\le \Big(\sum_{k=m}^{\infty} 2^k a_{2^k}^2\Big)^{1/2} \qquad \text{since } a_k \le a_{2^m} \text{ for the } 2^m \text{ terms } k = 2^m, \dots, 2^{m+1}-1 \\
&\le \Big(\sum_{k=m}^{\infty} 2^{kq/2} a_{2^k}^q\Big)^{1/q}
\end{aligned}
$$

since $q < 2$.

Thus,

$$
\begin{aligned}
\sum_{m=0}^{\infty} 2^{m\alpha q/2}\sigma_{2^m}^q &\le \sum_{m=0}^{\infty} 2^{m\alpha q/2} \sum_{k=m}^{\infty} 2^{kq/2} a_{2^k}^q \\
&= \sum_{k=0}^{\infty} 2^{kq/2} \sum_{m=0}^{k} 2^{m\alpha q/2} a_{2^k}^q \quad \text{(change order of summation)} \\
&= \sum_{k=0}^{\infty} 2^{kq/2} \frac{2^{(k+1)\alpha q/2}-1}{2^{\alpha q/2}-1} a_{2^k}^q \quad \text{(geometric series)} \\
&\le \frac{2^{\alpha q/2}}{2^{\alpha q/2}-1} \sum_{k=0}^{\infty} 2^{kq/2} 2^{\alpha k q/2} a_{2^k}^q \\
&= \frac{2^{\alpha q/2}}{2^{\alpha q/2}-1} \sum_{k=0}^{\infty} 2^k a_{2^k}^q, \quad \text{since } q/2 + \alpha q/2 = 1, \\
&\le \frac{2^{\alpha q/2+1}}{2^{\alpha q/2}-1} \sum_{j=1}^{\infty} a_j^q
\end{aligned}
$$

since $a_{2^k} \le a_j$ for the 2^{k-1} terms with $j = 2^{k-1}+1, \dots, 2^k$. We also have

$$
\sigma_{2^m} \ge \sigma_j \ge \sigma_{2^{m+1}} \text{ and } 2^{m\alpha q/2} \le j^{\alpha q/2} < 2^{(m+1)\alpha q/2}
$$

for the 2^m terms with $2^m \le j \le 2^{m+1}-1$. So

$$
\sum_{j=2^m}^{2^{m+1}-1} j^{\alpha q/2}\sigma_j^q \frac{1}{j} \le 2^{(m+1)\alpha q}\sigma_{2^m}^q \frac{1}{2^m} 2^m \le 2^{\alpha q/2} 2^{m\alpha q/2}\sigma_{2^m}^q.
$$

Combining these inequalities gives us

$$
\sum_{j=1}^{\infty} [j^{\alpha/2}\sigma_j]^q \frac{1}{j} \le 2^{\alpha q/2} \sum_{m=0}^{\infty} 2^{m\alpha q/2}\sigma_{2^m}^q \le \frac{2^{\alpha q+1}}{2^{\alpha q/2}-1} \sum_{j=1}^{\infty} a_j^q,
$$

which is what we needed.

Appendix B

The following seems to be a start of a proof that ended up not going anywhere, but I'm keeping it here for the record. We first note that

$$[N^{\alpha/2}\|f - f_{N-1}\|_{L_2(I)}]^q \frac{1}{N} = \left(\frac{\sum_{N\le i} c_i^2}{N}\right)^{q/2},$$

so we want to prove that

$$\left(\sum_N \left(\frac{\sum_{N\le i} c_i^2}{N}\right)^{q/2}\right)^{1/q} \asymp \left(\sum_i c_i^q\right)^{1/q} = \left(\sum_i (c_i^2)^{q/2}\right)^{1/q}$$

or

$$\left(\sum_N \left(\frac{\sum_{N\le i} c_i^2}{N}\right)^{q/2}\right)^{2/q} \asymp \left(\sum_i (c_i^2)^{q/2}\right)^{2/q}.$$

We now pull out of our hat Theorem 344 of *Inequalities* by Hardy-Littlewood-Polya, which states that when $\sigma < 1$ and $a_n \ge 0$,

$$\sum \left(\frac{a_n + a_{n+1} + \cdots}{n}\right)^{\sigma} > \sigma^{\sigma} \sum a_n^{\sigma}.$$

When we set $\sigma = q/2 < 1$ and $a_n = c_n^2$, this gives

$$\left(\sum_N \left(\frac{\sum_{N\le i} c_i^2}{N}\right)^{q/2}\right)^{2/q} > \left(\frac{q}{2}\right)\left(\sum_i (c_i^2)^{q/2}\right)^{2/q}.$$

The inequality in the other direction is a bit trickier, and we must use the fact that the c_i are *non-increasing*.

We set $a_i = c_i^2$ and note that, trivially,

$$\begin{aligned}
a_1 &= (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + (a_4 - a_5) + \cdots \\
a_2 &= \qquad\qquad\quad\; (a_2 - a_3) + (a_3 - a_4) + (a_4 - a_5) + \cdots \\
a_3 &= \qquad\qquad\qquad\qquad\qquad\;\; (a_3 - a_4) + (a_4 - a_5) + \cdots \\
a_4 &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; (a_4 - a_5) + \cdots .
\end{aligned}$$

So we have the vector equality

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix} = (a_1 - a_2) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + (a_2 - a_3) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + (a_3 - a_4) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} + (a_4 - a_5) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} + \cdots$$

$$=: b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + b_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + b_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} + b_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} + \cdots =: \sum b_\ell v_\ell .$$

with $b_i \geq 0$ for all i.

We now set

$$T_k x = T_k \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ x_k \\ x_{k+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_k \\ x_{k+1} \\ \vdots \end{pmatrix} .$$

Since $\sigma = q/2 < 1$ and

$$\sum_{k \leq i} c_i^2 = \sum_{k \leq i} a_i = \|T_k a\|_1 ,$$

we need to bound

$$\|T_1 a\|_1^\sigma + \frac{1}{2^\sigma} \|T_2 a\|_1^\sigma + \frac{1}{3^\sigma} \|T_3 a\|_1^\sigma + \cdots .$$

But

$$\|T_k a\|_1^\sigma = \| \sum_\ell b_\ell T_k v_\ell \|_1^\sigma \leq \sum_\ell b_\ell^\sigma \|T_k v_\ell\|_1^\sigma$$

So the left-hand side of (?) is bounded by

$$\sum_\ell b_\ell^\sigma \|T_1 v_\ell\|_1^\sigma + \frac{1}{2^\sigma} \sum_\ell b_\ell^\sigma \|T_2 v_\ell\|_1^\sigma + \frac{1}{3^\sigma} \sum_\ell b_\ell^\sigma \|T_3 v_\ell\|_1^\sigma + \cdots$$
$$= \sum_\ell b_\ell^\sigma \Big(\|T_1 v_\ell\|_1^\sigma + \frac{1}{2^\sigma} \|T_2 v_\ell\|_1^\sigma + \frac{1}{3^\sigma} \|T_3 v_\ell\|_1^\sigma + \cdots \Big).$$

Since

$$\|T_k v_\ell\|_1^\sigma = \left(\sum_{j=k}^{\ell} 1 \right)^\sigma = (\ell - k + 1)^\sigma ,$$

we have

$$\begin{aligned}\|T_1 v_\ell\|_1^\sigma + \frac{1}{2^\sigma}\|T_2 v_\ell\|_1^\sigma + \frac{1}{3^\sigma}\|T_3 v_\ell\|_1^\sigma + \cdots \\ = \ell^\sigma + \frac{1}{2^\sigma}(\ell-1)^\sigma + \frac{1}{3^\sigma}(\ell-2)^\sigma + \cdots + \frac{1^\sigma}{\ell^\sigma} = \sum_{j=1}^{\ell} \frac{(\ell-j+1)^\sigma}{j^\sigma}.\end{aligned}$$

End of proof going nowhere.