Math 527 - Homotopy Theory
Spring 2013
Homework 6 Solutions

Problem 1. (The Hopf fibration) Let $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ be the unit sphere. Stereographic projection provides a homeomorphism $S^2 \cong \mathbb{C}P^1$, where the "North pole" corresponds to $[0:1] \in \mathbb{C}P^1$. The composite

$$S^3 \hookrightarrow \mathbb{C}^2 \setminus \{0\} \twoheadrightarrow \mathbb{C}P^1$$

where the second map is the natural quotient map, is called the **Hopf map** and is usually denoted by $\eta \colon S^3 \to S^2$.

a. Show that $\eta \colon S^3 \to S^2$ is a fiber bundle with fiber S^1.

Solution. Let us show more generally that the quotient map $q \colon S^{2n+1} \twoheadrightarrow \mathbb{C}P^n$ is a fiber bundle with fiber S^1. Recall that the affine patches

$$U_i := \{[z_0 : z_1 : \ldots : z_n] \in \mathbb{C}P^n \mid z_i \neq 0\} \cong \mathbb{C}^n$$

with $0 \leq i \leq n$ form an open cover of $\mathbb{C}P^n$. We will show that each U_i is a trivializing neighborhood for q. Consider the preimage

$$q^{-1}(U_i) = \{(z_0, z_1, \ldots, z_n) \in S^{2n+1} \mid z_i \neq 0\}$$

and consider the (continuous) map $\varphi \colon q^{-1}(U_i) \to U_i \times S^1$ defined by

$$\varphi(z) = \left(q(z), \frac{z_i}{|z_i|} \right).$$

Now consider the map $\psi \colon U_i \times S^1 \to p^{-1}(U_i)$ defined by

$$\psi\left([z_0 : \ldots : z_n], e^{i\theta}\right) = e^{i\theta} \left(\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i} \right) \frac{|z_i|}{\|z\|}.$$

One readily checks that ψ is well defined, continuous, and inverse to φ. □

b. Show that for all $n \geq 3$, the Hopf map induces an isomorphism $\eta_* \colon \pi_n(S^3) \xrightarrow{\simeq} \pi_n(S^2)$. Deduce in particular the isomorphism $\pi_3(S^2) \simeq \mathbb{Z}$, where the class $[\eta] \in \pi_3(S^2)$ is a generator.

Solution. Consider the long exact sequence of the fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$. For all $n \geq 3$, the relevant part of the exact sequence is

$$0 = \pi_n(S^1) \longrightarrow \pi_n(S^3) \xrightarrow{\eta_*} \pi_n(S^2) \longrightarrow \pi_{n-1}(S^1) = 0$$

which shows that $\pi_n(S^3) \xrightarrow{\eta_*} \pi_n(S^2)$ is an isomorphism.

In particular, for $n = 3$, we obtain the isomorphism $\pi_3(S^3) \xrightarrow{\eta_*} \pi_3(S^2)$. Given $\pi_3(S^3) \cong \mathbb{Z}$ with generator $[\mathrm{id}_{S^3}]$, we conclude $\pi_3(S^2) \simeq \mathbb{Z}$ with generator $\eta_*[\mathrm{id}_{S^3}] = [\eta \circ \mathrm{id}_{S^3}] = [\eta]$. □

Problem 2. Let (E, e_0) and (B, b_0) be pointed spaces and $p\colon E \to B$ a pointed map. Denote by $F := p^{-1}(b_0)$ the strict fiber of p, and $F(p)$ the homotopy fiber of p, defined as

$$F(p) = \{(e, \gamma) \in E \times B^I \mid \gamma(0) = p(e), \gamma(1) = b_0\}.$$

There is a canonical "inclusion of the strict fiber into the homotopy fiber" $\varphi\colon F \to F(p)$ defined by

$$\varphi(e) = (e, c_{b_0})$$

where $c_{b_0}\colon I \to B$ is the constant path at b_0.

Show that if $p\colon E \to B$ is a fibration, then the map $\varphi\colon F \to F(p)$ is a homotopy equivalence.

Solution. Denote by

$$P(p) = \{(e, \gamma) \in E \times B^I \mid \gamma(0) = p(e)\} = E \times_B B^I$$

the path space construction on p. Since p is a fibration, the canonical map $(\mathrm{ev}_0, p_*)\colon E^I \to P(p)$ admits a section $s\colon P(p) \to E^I$. Note that the restriction of s to $F(p) \subseteq P(p)$ takes values in paths that end in F. In other words, for $(e, \gamma) \in F(p)$, we have

$$\begin{aligned} p\left(s(e,\gamma)(1)\right) &= \left(p_* s(e,\gamma)\right)(1) \\ &= \gamma(1) \\ &= b_0 \end{aligned}$$

which means $s(e,\gamma)(1) \in p^{-1}(b_0) = F$. Now consider the composite

$$\begin{tikzcd} F(p) \arrow[r, "s"] \arrow[d, hook] \arrow[rr, bend left, "\psi"] & \mathrm{ev}_1^{-1}(F) \arrow[r, "\mathrm{ev}_1"] \arrow[d, hook] & F \arrow[d, hook] \\ P(p) \arrow[r, "s"] & E^I \arrow[r, "\mathrm{ev}_1"] & E. \end{tikzcd}$$

We claim that $\psi\colon F(p) \to F$ is homotopy inverse to $\varphi\colon F \to F(p)$.

a) $\psi\varphi \simeq \mathrm{id}_F$. Note that s restricted to $F \subseteq F(p)$ takes values in paths that live entirely in F. Indeed, the path $s(\varphi(e))$ projects down to the constant path $p\left(s(\varphi(e))\right) = c_{b_0}$ in B. Consider the commutative diagram

$$\begin{tikzcd} F \arrow[r, "s"] \arrow[d, hook, "\varphi"'] & F^I \arrow[r, "\mathrm{ev}_1"] \arrow[d, hook] & F \arrow[d, equal] \\ F(p) \arrow[r, "s"] \arrow[d, hook] \arrow[rr, bend left, "\psi"] & \mathrm{ev}_1^{-1}(F) \arrow[r, "\mathrm{ev}_1"] \arrow[d, hook] & F \arrow[d, hook] \\ P(p) \arrow[r, "s"] & E^I \arrow[r, "\mathrm{ev}_1"] & E \end{tikzcd}$$

and note that the formula $ev_t \circ s\colon F \to F$ for $t \in I$ provides a homotopy from $\mathrm{ev}_0 \circ s = \mathrm{id}_F$ to $\mathrm{ev}_1 \circ s = \psi\varphi$.

b) $\varphi\psi \simeq \mathrm{id}_{F(p)}$. The family of maps parametrized by $t \in I$

$$F(p) \to F(p)$$
$$(e, \gamma) \mapsto \big(s(e, \gamma)(t), \gamma_{[t,1]}\big)$$

provides a homotopy from the map

$$(e, \gamma) \mapsto \big(s(e, \gamma)(0), \gamma_{[0,1]}\big) = (e, \gamma)$$

to the map

$$(e, \gamma) \mapsto \big(s(e, \gamma)(1), \gamma_{[1,1]}\big) = (\psi(e, \gamma), c_{b_0}) = \varphi\psi(e, \gamma)$$

which proves the relation $\varphi\psi \simeq \mathrm{id}_{F(p)}$. $\square$

Problem 3. Let X be a CW-complex with skeletal filtration $X_0 \subseteq X_1 \subseteq \ldots \subseteq X = \operatorname{colim}_n X_n$. Show that for any $k \geq 0$, there is a natural isomorphism

$$\pi_k(X) \cong \operatorname*{colim}_n \pi_k(X_n).$$

Solution. Let us show that the natural map

$$\varphi\colon \operatorname*{colim}_n \pi_k(X_n) \to \pi_k(\operatorname*{colim}_n X_n) = \pi_k(X)$$

is an isomorphism.

a) φ is surjective. Let $[\alpha] \in \pi_k(X)$ be represented by a pointed map $\alpha\colon S^k \to X$. Since S^k is compact, the image $\alpha(S^k) \subseteq X$ is compact, and thus meets only finitely many cells of X. Therefore α factors through some finite skeleton $X_N \subseteq X$, as illustrated in the diagram:

$$\begin{tikzcd} S^k \arrow[r] \arrow[rr, bend left, "\alpha"] & X_N \arrow[r, hook] & X. \end{tikzcd}$$

Therefore $[\alpha]$ is in the image of the composite

$$\pi_k(X_N) \to \operatorname*{colim}_n \pi_k(X_n) \xrightarrow{\varphi} \pi_k(X)$$

where the first map is the "summand inclusion" in the colimiting cocone.

b) φ is injective. Let $\beta' \in \ker \varphi \subseteq \operatorname{colim}_n \pi_k(X_n)$. The condition $\varphi(\beta') = 0 \in \pi_k(X)$ means that there is a (pointed) null-homotopy

$$H\colon S^k \wedge I_+ \to X$$

of $\varphi(\beta')$. Again, $S^k \wedge I_+$ is compact, so that the image $H(S^k \wedge I_+) \subseteq X$ meets only finitely many cells of X. Therefore H factors through some finite skeleton $X_N \subseteq X$, as illustrated in the diagram:

$$\begin{tikzcd} S^k \wedge I_+ \arrow[r] \arrow[rr, bend left, "H"] & X_N \arrow[r, hook] & X. \end{tikzcd}$$

Therefore β' is the image via the "summand inclusion"

$$\pi_k(X_N) \to \operatorname*{colim}_n \pi_k(X_n)$$

of an element which was already 0 in $\pi_k(X_N)$, which proves $\beta' = 0$. $\square$

Remark. The same statement holds for homology $H_k(X) \cong \operatorname{colim}_n H_k(X_n)$, by compactness of the simplex Δ^k.

Alternate solution. By cellular approximation, we have $\pi_k(X) \cong \pi_k(X_n)$ for all $n \geq k+1$. More precisely, the sequence

$$\pi_k(X_k) \twoheadrightarrow \pi_k(X_{k+1}) \xrightarrow{\cong} \pi_k(X_{k+2}) \xrightarrow{\cong} \pi_k(X_{k+3}) \xrightarrow{\cong} \dots$$

stabilizes, and skeletal inclusions $X_n \subseteq X$ induce isomorphisms $\pi_k(X_n) \xrightarrow{\cong} \pi_k(X)$ for all $n \geq k+1$. Therefore the natural map $\operatorname{colim}_n \pi_k(X_n) \to \pi_k(X)$ is an isomorphism. $\square$

Remark. It may seem that the stronger result obtained in the alternate solution makes the first solution useless. However, the argument used in the first solution is much more general, and applies for example to cell complexes

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X = \operatorname*{colim}_n X_n$$

where cells of arbitrary dimensions are added at each stage. More generally, it applies to any sequence of closed embeddings

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X = \operatorname*{colim}_n X_n$$

assuming the spaces X_n are compactly generated weakly Hausdorff. See May-Ponto, Proposition 2.5.4 and Corollary 2.5.6.

Problem 4. In this problem, feel free to refer to Homework 5 Problem 4. Consider infinite-dimensional real projective space $\mathbb{R}P^\infty = \operatorname{colim}_n \mathbb{R}P^n$.

a. Compute all homotopy groups of $\mathbb{R}P^\infty$.

Solution. Using the standard cell structure on $\mathbb{R}P^\infty$, note that $\mathbb{R}P^\infty$ is path-connected: $\pi_0(\mathbb{R}P^\infty) = \pi_0(\mathbb{R}P^1) = *$. Moreover we have

$$\pi_1(\mathbb{R}P^\infty) \cong \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$$

and for all $k \geq 2$ we have

$$\pi_k(\mathbb{R}P^\infty) \cong \pi_k(\mathbb{R}P^{k+1}) \cong \pi_k(S^{k+1}) = 0. \quad \square$$

b. Show that $\mathbb{R}P^2$ and $S^2 \times \mathbb{R}P^\infty$ are not homotopy equivalent.

Solution. The two spaces have very different homology. For instance, we have

$$H_k(\mathbb{R}P^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } 0 \leq k \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

whereas for all $k \geq 2$, the Künneth theorem yields

$$\begin{aligned} H_k(S^2 \times \mathbb{R}P^\infty; \mathbb{Z}/2) &\cong H_0(S^2; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H_k(\mathbb{R}P^\infty; \mathbb{Z}/2) \oplus H_2(S^2; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H_{k-2}(\mathbb{R}P^\infty; \mathbb{Z}/2) \\ &\cong \mathbb{Z}/2 \otimes_{\mathbb{Z}/2} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \otimes_{\mathbb{Z}/2} \mathbb{Z}/2 \\ &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\ &\not\cong H_k(\mathbb{R}P^2; \mathbb{Z}/2). \quad \square \end{aligned}$$

Alternate solution. The two spaces have a different π_1-action on π_2.

By naturality of the π_1-action, the action of

$$\pi_1(S^2 \times \mathbb{R}P^\infty) \cong \pi_1(S^2) \times \pi_1(\mathbb{R}P^\infty) \cong \pi_1(\mathbb{R}P^\infty)$$

on

$$\pi_2(S^2 \times \mathbb{R}P^\infty) \cong \pi_2(S^2) \times \pi_2(\mathbb{R}P^\infty) \cong \pi_2(S^2)$$

is trivial.

On the other hand, the action of $\pi_1(\mathbb{R}P^2) \cong O(1) = \{-1, 1\}$ on $\pi_2(\mathbb{R}P^2) \cong \pi_2(S^2)$ is induced by deck transformations on the universal cover $S^2 \twoheadrightarrow \mathbb{R}P^2$. One can show, say, using the Hurewicz theorem, that the action of the antipodal map (scalar multiplication by -1)

$$\tau \colon S^2 \to S^2$$

on $\pi_2(S^2) \cong \mathbb{Z}$ is multiplication by the degree

$$\deg(\tau) = (-1)^{2+1} = -1$$

which is a non-trivial action. $\square$