

# MA544: Qual Problems

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# Course Notes

These notes roughly correspond to chapters 2 through 8 of Wheeden and Zygmund's *Measure and Integration* [1].

This first portion corresponds to material covered before Exam 1.

## 1.1 Preliminaries

Here is some precursor material to the Lebesgue theory of integration.

### Points and sets in $\mathbb{R}^n$

From this section, we need not say much only a few results and definitions are important.

If  $\mathcal{F}$  is a *countable* collection of subsets of  $\mathbb{R}^n$ , it will be called a *sequence of sets* and denoted  $\{E_k\}$  for  $k \in \mathbb{N}$ . The corresponding *union* and *intersection* will be written  $\bigcup_k E_k$  and  $\bigcap_k E_k$ . A sequence  $\{E_k\}$  is said to *increase* to  $\bigcup_k E_k$  if  $E_k \subset E_{k+1}$  for all  $k$  and to *decrease* to  $\bigcap_k E_k$  if  $E_k \supset E_{k+1}$  for all  $k$ ; we use the notation  $E_k \nearrow \bigcup_k E_k$  and  $E_k \searrow \bigcap_k E_k$  to denote these two possibilities. If  $\{E_k\}$  is a sequence of sets, we define

$$\limsup E_k := \bigcap_{j=1}^{\infty} \left( \bigcup_{k=j}^{\infty} E_k \right), \quad \liminf E_k := \bigcup_{j=1}^{\infty} \left( \bigcap_{k=j}^{\infty} E_k \right), \quad (1)$$

noting that the subsets  $U_j := \bigcup_{k=j}^{\infty} E_k$  and  $V_j := \bigcap_{k=j}^{\infty} E_k$  satisfy  $U_j \searrow \limsup E_k$  and  $V_j \nearrow \liminf E_k$ .\*

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\*Carlos: Make note of this. It is often a good strategy to decompose a set  $E$  into the intersection or union of a sequence  $E_k$ . Making appropriate manipulations, we often get  $E_k \searrow E$  or  $E_k \nearrow E$  and make limiting arguments about properties of the set, i.e., measure or the integral of some function whose domain is in  $E$ , etc.

$\mathbb{R}^n$  as a metric space

A student who has taken 504 or 571 will know most of the material under this section. We include it here as a useful reference to some of the more useful results of the properties of  $\mathbb{R}^n$  as a metric space.

If  $\mathbf{x} \in \mathbb{R}^n$ , we say that a sequence  $\{\mathbf{x}_k\}$  *converges* to  $\mathbf{x}$ , or that  $\mathbf{x}$  is the *limit* of  $\{\mathbf{x}_k\}$ , if  $|\mathbf{x} - \mathbf{x}_k| \rightarrow 0$  as  $k \rightarrow \infty$ . We denote this by writing either  $\lim_{k \rightarrow \infty} \mathbf{x}_k = \mathbf{x}$  or  $\mathbf{x}_k \rightarrow \mathbf{x}$  as  $k \rightarrow \infty$ . A point  $\mathbf{x} \in \mathbb{R}^n$  is called a *limit point of a set*  $E$  if it is the limit of a sequence of distinct points of  $E$ . A point  $\mathbf{x} \in E$  is called an *isolated point* of  $E$  if it is not the limit point of any sequence in  $E$  (excluding the trivial sequence  $\{\mathbf{x}_k\}$  where  $\mathbf{x}_k = \mathbf{x}$  for all  $k$ ). It follows that a point  $\mathbf{x}$  is isolated if and only if there is a  $\delta > 0$  such that  $|\mathbf{x} - \mathbf{y}| > \delta$  for every  $\mathbf{y} \in E$ ,  $\mathbf{y} \neq \mathbf{x}$ .

For sequences  $\{x_k\}$  in  $\mathbb{R}$ , we will write  $\lim_{k \rightarrow \infty} x_k = \infty$ , or  $x_k \rightarrow \infty$  as  $k \rightarrow \infty$ , if given  $M > 0$  there is an integer  $N$  such that  $x_k \geq M$  whenever  $k \geq N$ . A similar definition holds for  $\lim_{k \rightarrow \infty} x_k = -\infty$ .<sup>†</sup>

A sequence  $\{\mathbf{x}_k\}$  in  $\mathbb{R}^n$  is called a *Cauchy sequence* if given  $\varepsilon > 0$  there is an integer  $N$  such that  $|\mathbf{x}_k - \mathbf{x}_\ell| < \varepsilon$  for all  $k, \ell \geq N$ . We say that a metric space  $(X, |\cdot|)$  is *complete with respect to the metric*  $|\cdot|$  if every Cauchy sequence in  $X$  converges.

A set  $E_0 \subset E$  is said to be *dense* in  $E$  if for every  $\mathbf{y} \in E$  and  $\varepsilon > 0$ , there is a point  $\mathbf{x} \neq \mathbf{y}$  in  $E_0$  such that  $0 < |\mathbf{y} - \mathbf{x}| < \varepsilon$ . Thus,  $E_0$  is dense in  $E$  if every point of  $E$  is a limit point of  $E_0$ . If  $E_0 = E$ , we say  $E$  is *dense in itself*. As an example  $\mathbb{Q}^n \subset \mathbb{R}^n$  is dense in  $\mathbb{R}^n$ . Since this set is also countable, it follows that  $\mathbb{R}^n$  is *separable*, by which we mean that  $\mathbb{R}^n$  has a countable dense subset.

For a nonempty subset  $E$  of  $\mathbb{R}^n$ , we use the standard notation  $\sup E$  and  $\inf E$  for the *supremum* (least upper bound) and *infimum* (greatest lower bound) of  $E$ . In case  $\sup E$  is in  $E$ , it will be called  $\max E$ ; similarly, if  $\inf E \in E$ ,  $\inf E$  will be called  $\min E$ .

If  $\{a_k\}$  is a sequence of points in  $\mathbb{R}$ , let  $b_j := \sup_{k \geq j} a_k$  and  $c_j := \inf_{k \geq j} a_k$ ,  $j \in \mathbb{N}$ . Then  $-\infty \leq c_j \leq b_j \leq \infty$ , and  $\{b_j\}$  and  $\{c_j\}$  are monotone decreasing and increasing, respectively; i.e.,  $b_j \geq b_{j+1}$  and  $c_j \leq c_{j+1}$ . Define  $\limsup_{k \rightarrow \infty} a_k$  and  $\liminf_{k \rightarrow \infty} a_k$  by

$$\begin{aligned} \limsup_{k \rightarrow \infty} a_k &:= \lim_{j \rightarrow \infty} b_j = \lim_{j \rightarrow \infty} \{ \sup_{k \geq j} a_k \}, \\ \liminf_{k \rightarrow \infty} a_k &:= \lim_{j \rightarrow \infty} c_j = \lim_{j \rightarrow \infty} \{ \inf_{k \geq j} a_k \}. \end{aligned} \tag{2}$$

**Theorem 1** (1.4).

- (a)  $L = \limsup_{k \rightarrow \infty} a_k$  if and only if (i), there is a subsequence  $\{a_{k_j}\}$  of  $\{a_k\}$  that converges to  $L$  and (ii) if  $L' > L$ , there is an integer  $N$  such that  $a_k < L'$  for all  $k \geq N$ .
- (b)  $\ell = \liminf_{k \rightarrow \infty} a_k$  if and only if (i), there is a subsequence  $\{a_{k_j}\}$  of  $\{a_k\}$  that converges to  $\ell$  and (ii) if  $\ell' < \ell$ , there is an integer  $N$  such that  $a_k > \ell'$  for all  $k \geq N$ .

When they are finite,  $\limsup a_k$  and  $\liminf a_k$  are the largest and smallest limit points of  $\{a_k\}$ , respectively. It's not too difficult to show that  $\{a_k\}$  converges to  $a$ ,  $-\infty \leq a \leq \infty$ , if and only if  $\limsup a_k = \liminf a_k$ .

We can also use the metric on  $\mathbb{R}^n$  to define the *diameter* of a set  $E$  by letting

$$\text{diam}(E) := \sup \{ |\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in E \} \tag{3}$$

<sup>†</sup>Carlos: In fact, we can define it by saying that  $\lim_{k \rightarrow \infty} x_k = -\infty$  if  $\lim_{k \rightarrow \infty} -x_k = \infty$ .

If the diameter of  $E$  is finite,  $E$  is said to be *bounded*. Equivalently,  $E$  is bounded if there is a finite constant  $M$  such that  $|\mathbf{x}| \leq M$  for all  $\mathbf{x} \in E$ . If  $E_1$  and  $E_2$  are two sets, the *distance between  $E_1$  and  $E_2$*  is defined by

$$d(E_1, E_2) := \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{x} \in E_1, \mathbf{y} \in E_2\}. \quad (4)$$

### Open and closed sets in $\mathbb{R}^n$ , and special sets

For  $\mathbf{x} \in \mathbb{R}^n$  and  $\varepsilon > 0$ , the set

$$B_\varepsilon(\mathbf{x}) := B(\mathbf{x}, \varepsilon) := \{\mathbf{y} : |\mathbf{x} - \mathbf{y}| < \varepsilon\}. \quad (5)$$

A point  $\mathbf{x} \in E$  is called an *interior point* of  $E$  if there exists  $\varepsilon > 0$  such that  $B_\varepsilon(\mathbf{x}) \subset E$ . The collection of all interior points of  $E$  is called the *interior of  $E$*  and denoted  $E^\circ$ . A set is said to be *open* if  $E = E^\circ$ . The empty set  $\emptyset$  is open by convention. The whole space  $\mathbb{R}^n$  is clearly open and it is easy to see that  $B_\varepsilon(\mathbf{x})$  is open for any  $\varepsilon > 0$ . We shall generally denote open sets by the letter  $G$ .

A set  $E$  is *closed* if  $\mathbb{R}^n \setminus E$  is open. Thus,  $\emptyset$  and  $\mathbb{R}^n$  are closed (being the complements of each other). Closed sets will generally be denoted by the letter  $F$ . The union of the set  $E$  and all of its limit points is called the *closure* of  $E$  and written  $\bar{E}$ . By the *boundary* of  $E$ , we mean the set  $\partial E := \bar{E} \setminus E^\circ$ .

Now, consider a collection of sets  $\mathcal{A} = \{A\}$ . A set is said to be of *type  $A_\delta$*  if it can be written as a countable intersection of sets in  $\mathcal{A}$  and to be of *type  $A_\sigma$*  if it can be written as a countable union of sets in  $\mathcal{A}$ . The most common usage of this notation is  $G_\delta$  and  $F_\sigma$  sets where  $\mathcal{G} = \{G\}$  denotes the open sets in  $\mathbb{R}^n$  and  $\mathcal{F} = \{F\}$  the closed sets. Hence,  $E$  is of type  $G_\delta$  if

$$E = \bigcap_k G_k, \quad G_k \text{ open}, \quad (6)$$

and of type  $F_\sigma$  if

$$E = \bigcup_k F_k, \quad F_k \text{ is closed}. \quad (7)$$

The complement of a  $G_\delta$  set is an  $F_\sigma$  set and vice-versa.

Another type of special set we will have the occasion to use is a *perfect set*, by which we mean a closed set  $C$  each of whose points is a limit point of  $C$ . Thus, a perfect set is a closed set that is dense in itself.

**Theorem 2 (1.9).** *A perfect set is uncountable.*

An  $n$ -dimensional interval  $I$  is a subset of  $\mathbb{R}^n$  of the form

$$I = \{(x_1, \dots, x_n) : a_k \leq x_k \leq b_k, \text{ for } k = 1, \dots, n\}. \quad (8)$$

An  $n$ -interval is closed and has edges parallel to the coordinate axes. If the edge lengths  $b_k - a_k$  are all equal,  $I$  will be called an  *$n$ -dimensional cube* or simply an  *$n$ -cube*. Cubes will usually be denoted by the letter  $Q$ . Two intervals  $I_1$  and  $I_2$  are said to be *nonoverlapping* if their interiors are disjoint, i.e., if the most they have in common is some part of their boundary. A set equal to an  $n$ -interval minus some part of its boundary is called a *partly open interval*. By definition, the *volume*  $\text{Vol } I$  of the interval  $I = \{(x_1, \dots, x_n)\}$  is

$$\text{Vol } I = (b_1 - a_1) \cdots (b_n - a_n). \quad (9)$$

Somewhat more generally, if  $\{\mathbf{e}_k\}_{k=1}^n$  is any given set of  $n$  vectors emanating from a point in  $\mathbb{R}^n$ , we will consider the closed *parallelepiped*

$$P := \left\{ \mathbf{x} : \sum_{k=1}^n t_k \mathbf{e}_k, 0 \leq t_k \leq 1 \right\} \quad (10)$$

Note that the edges of  $P$  are parallel translates of the  $\mathbf{e}_k$ . Thus,  $P$  is an interval if the  $\mathbf{e}_k$  are parallel to the coordinate axes. The *volume*  $\text{Vol } P$  of  $P$  is by definition the absolute value of the  $n \times n$  determinant having  $\mathbf{e}_1, \dots, \mathbf{e}_n$  as rows. A linear transformation  $T$  of  $\mathbb{R}^n$  transforms a parallelepiped  $P$  into a parallelepiped  $P'$  with volume  $\text{Vol } P' = |\det T| \text{Vol } P$ . In particular, a rotation of axes in  $\mathbb{R}^n$  (which is an orthogonal linear transformation) does not change the volume of the parallelepiped. We will assume basic facts about the volume: for example, if  $N$  is finite and  $P$  is parallelepiped with  $P \subset \bigcup_{k=1}^N I_k$  then  $\text{Vol } P \leq \sum_{k=1}^N \text{Vol}(I_k)$ , and if  $\{I_k\}_{k=1}^N$  are nonoverlapping intervals contained in a parallelepiped  $P$ , then  $\sum_{k=1}^N \text{Vol } I_k \leq \text{Vol } P$ .

Now we shall use the notion of interval to obtain a very useful decomposition of open sets in  $\mathbb{R}^n$ . This will be the foundation of many of our results later on.

**Theorem 3** (1.10.). *Every open set in  $\mathbb{R}$  can be written as a countable union of disjoint open intervals.*

This construction, however, fails in  $\mathbb{R}^n$  for  $n > 1$  since the union of (overlapping) intervals is not generally an interval. However, the following weaker, but sufficient, theorem does hold.

**Theorem 4.** *Every open set in  $\mathbb{R}^n$ ,  $n \geq 1$ , can be written as a countable union of nonoverlapping (closed) cubes. It can also be written as a countable union of disjoint partly open subsets.*

You can find a proof of the preceding theorems in [1]

The collection  $\{Q : Q \in K_j, j = 1, 2, \dots\}$  which is constructed in the proof of the previous theorem is called a family of *dyadic cubes*.

### Compact Sets and the Heine–Borel Theorem

By an *cover* of a set  $E$ , we mean a family  $\mathcal{F}$  of sets  $A$  such that  $E \subset \bigcap_{A \in \mathcal{F}} A$ . A *subcover*  $\mathcal{F}_0$  of a cover  $\mathcal{F}$  is a cover with the property that  $A_0 \in \mathcal{F}$  whenever  $A_0 \in \mathcal{F}_0$ . A cover  $\mathcal{F}$  is called an *open cover* if each set in  $\mathcal{F}$  is open. We say  $E$  is *compact* if every open cover of  $E$  has a finite subcover.

**Theorem 5** (1.12).

- (a) *The Heine–Borel theorem: A set  $E \subset \mathbb{R}^n$  is compact if and only if it is closed and bounded.*
- (b) *A set  $E \subset \mathbb{R}^n$  is compact if and only if every sequence of points of  $E$  has a subsequence that converges to a point of  $E$ .*

## 1.2 Functions of bounded variation and the Riemann–Stieltjes integral

In this section, we introduce functions of bounded variation as well as the definition of the Riemann integral. We conclude with a proof that the

### Functions of bounded variation

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a real-valued function defined for all  $a \leq x \leq b$  and finite; let  $\Gamma = \{x_0, \dots, x_m\}$  be a *partition* of  $[a, b]$ , i.e., a collection of points  $x_i$ ,  $i = 0, \dots, m$ , satisfying  $x_0 = a$  and  $x_m = b$ , and  $x_{i-1} < x_i$  for  $i = 1, \dots, m$ . To each partition  $\Gamma$ , we associated a sum

$$S_\Gamma := S_\Gamma[f; a, b] := \sum_{i=1}^m |f(x_i) - f(x_{i-1})|. \quad (11)$$

The *variation* (or *total variation*) of  $f$  over  $[a, b]$  is defined as

$$V := V[f; a, b] := \sup_{\Gamma} S_\Gamma, \quad (12)$$

where the supremum is taken over all partitions  $\Gamma$  of  $[a, b]$ . If  $V < \infty$ ,  $f$  is said to be of *bounded variation on  $[a, b]$* ; if  $V = \infty$ ,  $f$  is of *unbounded variation on  $[a, b]$* .

Before going on to prove important properties about (12), let us look at some common examples (and nonexamples) of functions  $f$  of bounded variation.

**Examples 1.** Suppose  $f$  is *monotone* in  $[a, b]$ . Then, clearly, each  $S_\Gamma$  is equal to  $|f(a) - f(b)|$  for every partition  $\Gamma^\dagger$ , and therefore  $V = |f(b) - f(a)|$ .

**Examples 2.** Suppose the graph of  $f$  can be split into a finite number of monotone arcs, i.e., suppose  $[a, b] = \bigcup_{i=1}^k [a_{i-1}, a_i]$  and  $f$  is monotone in each  $[a_{i-1}, a_i]$ . Then  $V = \sum_{i=1}^k |f(a_i) - f(a_{i-1})|$ . To see this, we use the result of Example 1 above and the fact, yet to be proven, that  $V = V[a, b] = \sum_{i=1}^k V[a_i, a_{i-1}]$ .

If  $\Gamma = \{x_0, \dots, x_m\}$  is a partition of  $[a, b]$ , let  $|\Gamma|$ , called the *norm* of  $\Gamma$ , be defined as the length of the longest subinterval of  $\Gamma$

$$|\Gamma| := \max_i (x_i - x_{i-1}). \quad (13)$$

If  $f$  is continuous on  $[a, b]$  and  $\{\Gamma_j\}$  is a sequence of partitions of  $[a, b]$  with  $|\Gamma_j| \rightarrow 0$ , we shall see that  $V = \lim_{j \rightarrow \infty} S_{\Gamma_j}$ .

**Examples 3.** Let  $f$  be the *Dirichlet function*, defined by  $f(x) = 1$  for  $x$  rational and  $f(x) = 0$  for  $x$  irrational. Then, clearly,  $V[a, b] = \infty$  for any interval of  $[a, b]$ .<sup>§</sup>

**Examples 4.** A function that is continuous on an interval, however, need not be of bounded variation on that interval. Take for example the following construction: let  $\{a_j\}$  and  $\{d_j\}$ ,  $j = 1, 2, \dots$ , be monotone decreasing sequences in  $(0, 1]$  with  $a_1 = \lim_{j \rightarrow \infty} a_j = \lim_{j \rightarrow \infty} d_j = 0$  and  $\sum d_j = \infty$ . Construct a continuous function  $f$  as follows. On each subinterval  $[a_{j+1}, a_j]$ , the graph of  $f$  consists of the sides of the isosceles triangle with base  $[a_{j+1}, a_j]$  and height  $d_j$ . Thus,  $f(a_j) = 0$ , and if  $m_j$  denotes the midpoint of  $[a_{j+1}, a_j]$ , then  $f(m_j) = d_j$ . If we define  $f(0) = 0$ , then  $f$  is continuous on  $[0, 1]$ . Taking  $\Gamma_k$  to be the partition defined by the points  $0, \{a_j\}_{j=1}^{k+1}$  and  $\{m_j\}_{j=1}^k$ , we see that  $S_\Gamma = 2 \sum_{j=1}^k d_j$ . Hence,  $V[f; 0, 1] = \infty$ .

<sup>†</sup>Carlos: This may not be clear at a first glance, but, upon closer inspection, this is true by monotonicity. If  $a < x < b$ , we have  $|f(b) - f(a)| = |f(b) - f(x)| + |f(x) - f(a)|$ . This holds for an arbitrary partitions  $\Gamma$ .

<sup>§</sup>Carlos: By the density of  $\mathbb{Q}$  in  $\mathbb{R}$  (and by restriction,  $[a, b]$ , since  $[a, b]$  is path-connected), for any positive integer  $N$ , we may choose a partition  $\Gamma$  of  $[a, b]$  containing  $N + 1$  rational numbers so  $S_\Gamma = N + 1 > N$ .

### 1.3 The Lebesgue integral

This portion corresponds to material covered before the second exam.

## 1.4 Differentiation

This portion of the notes corresponds to material covered before the final.

This section deals with questions of differentiability and culminates with a couple of results tying together the Lebesgue integral with the derivative à la the familiar fundamental theorem of calculus for Riemann integrals.

### The indefinite integral

If  $f$  is a Riemann integrable function on an interval  $[a, b]$  of  $\mathbb{R}$ , then the familiar definition for its *indefinite integral* is

$$F(x) = \int_a^x f(y) dy, \quad a \leq x \leq b. \quad (14)$$

The *fundamental theorem of calculus* then asserts that  $F' = f$  if  $f$  is continuous. In this section, we study the analogue of this result for Lebesgue integrable functions.

Since we want to generalize our results to  $\mathbb{R}^n$ , first we must find a suitable notion of indefinite integral for multivariable functions. In two dimensions we might, for instance, define the indefinite integral  $F$  of  $f$  to be

$$F(x_1, x_2) := \int_{a_1}^{x_1} \int_{a_2}^{x_2} f(y_1, y_2) dy_2 dy_1. \quad (15)$$

As it turns out, it is better to abandon the notion that the indefinite integral be a function of a point and instead let it be a function of a set. Therefore, given a function  $f$ , integrable on some measurable subset  $A$  of  $\mathbb{R}^n$ , we define the *indefinite integral of  $f$*  to be the function

$$F(E) := \int_E f, \quad (16)$$

where  $E$  is a measurable subset of  $A$ .

The function  $F$  is an example of a *set function*, by which we mean any real-valued function  $F$  defined on a  $\sigma$ -algebra  $\Sigma$  of measurable sets such that

- (i)  $F(E)$  is finite for every  $E \in \Sigma$ .
- (ii)  $F$  is *countably additive*; i.e., if  $E$  is the union of disjoint sets  $E_k \in \Sigma$ ,  $k = 1, 2, \dots$ , then

$$F(E) = \sum_{k \in \mathbb{N}} F(E_k). \quad (17)$$

## 1.5 $L^p$ Classes

Let's take a small detour to ch. 5 of [?] to talk about  $L^p$  spaces.

### The relation between the Riemann–Stieltjes integral and the Lebesgue integral, and the $L^p$ spaces, $0 < p < \infty$

As it turns out, there is a remarkably simple and useful representation of the Lebesgue integral (over measurable subsets of  $\mathbb{R}^n$ ) in terms of the Riemann–Stieltjes integrals (over measurable subset of  $\mathbb{R}$ ). In order to establish this relationship, we will need to study the function

$$\omega(\alpha) := \omega_{f,E}(\alpha) := |\{ \mathbf{x} \in E : f(\mathbf{x}) > \alpha \}|, \quad (18)$$



where  $f$  is a measurable function on  $E$  and  $-\infty < \alpha < \infty$ . We call  $\omega_{f,E}$  (or simply  $\omega$ ) the *distribution function of  $f$  on  $E$* .

The function  $\omega$  is clearly not affected by changing  $f$  in a set of measure zero, and is decreasing. As  $\alpha \nearrow \infty$ , we have

$$\{\mathbf{x} \in E : f(\mathbf{x}) > \alpha\} \searrow \{\mathbf{x} \in E : f(\mathbf{x}) = \infty\}.$$

hence, assuming that  $f$  is finite a.e. in  $E$ , by Theorem 3.62(ii),  $\lim_{\alpha \rightarrow \infty} \omega = 0$ , unless  $\omega(\alpha) \equiv \infty$ . Similarly, we have  $\lim_{\alpha \rightarrow -\infty} \omega = |E|$ . For now, let us assume that the measure of  $E$  is finite; this will ensure that  $\omega$  is bounded.

In the following results, we assume that  $f$  is a measurable function that is finite a.e. in  $E$ ,  $|E| < \infty$ , and write

$$\omega(\alpha) = \omega_{f,E}(\alpha), \quad \{f > \alpha\} = \{\mathbf{x} \in E : f(\mathbf{x}) > \alpha\},$$

etc.

**Lemma 6** (5.38). *If  $\alpha < \beta$ , then  $|\{\alpha \leq f \leq \beta\}| = \omega(\alpha) - \omega(\beta)$ .*

*Proof.* For  $\alpha < \beta$ , we have  $\{f > \beta\} \subset \{f > \alpha\}$  and  $\{\gamma < f \leq \beta\} = \{f > \alpha\} \setminus \{f > \beta\}$ . Since  $|\{f > \beta\}| < \infty$ , the lemma follows from Corollary 3.25.  $\blacksquare$

Given  $\alpha$ , let

$$\omega(\alpha+) := \lim_{\varepsilon \searrow 0} \omega(\alpha + \varepsilon) \quad \omega(\alpha-) := \lim_{\varepsilon \searrow 0} \omega(\alpha - \varepsilon).$$

denote the limits of  $\omega$  from the right and left at  $\alpha$ .

**Lemma 7** (5.39).

- (a)  $\omega(\alpha+) = \omega(\alpha)$ ; i.e.,  $\omega$  is continuous from the right.
- (b)  $\omega(\alpha-) = |\{f \geq \alpha\}|$ .

**Corollary 8** (5.40).

- (a)  $\omega(\alpha-) - \omega(\alpha) = |\{f = \alpha\}|$ ; in particular,  $\omega$  is continuous at  $\alpha$  if and only if  $|\{f = \alpha\}| = 0$ .
- (b)  $\omega$  is constant in an open interval  $(\alpha, \beta)$  if and only if  $|\{\alpha < f < \beta\}| = 0$ , that is, if and only if  $f$  takes almost no values between  $\alpha$  and  $\beta$ .

The rest of this section establishes the relations between the Lebesgue and Riemann–Stieltjes integrals. As always, we assume  $f$  is measurable and finite a.e. in  $E$ ,  $|E| < \infty$  and  $\omega = \omega_{E,f}$ .

**Theorem 9** (5.41). *If  $a \leq f(\mathbf{x}) \leq b$  ( $a$  and  $b$  are finite) for all  $\mathbf{x} \in E$ , then*

$$\int_E f = - \int_a^b \alpha d\omega(\alpha).$$

*Proof.* The Lebesgue integral on the left-hand side exists since  $f$  is bounded and  $|E| < \infty$ . The Riemann–Stieltjes integral on the right-hand side exists by Theorem 2.24. To show that they are equal, let us partition the interval the interval  $[a, b]$  by  $a = \alpha_0 < \alpha_1 < \cdots < \alpha_k = b$  and let

$E_j = \{\alpha_{j-1} < f \leq \alpha_j\}$ . The  $E_j$  are disjoint and  $E = \bigcup_{j=1}^k E_j$ . Hence,  $\int_E f = \sum_{j=1}^k \int_{E_j} f$  and, therefore

$$\sum_{j=1}^{\infty} \alpha_{j-1} |E_j| \leq \int_E f \leq \sum_{j=1}^k \alpha_j |E_j|.$$

By Lemma 5.38,  $|E_j| = \omega(\alpha_j) - \omega(\alpha_{j-1})$ . Hence, the sums are Riemann–Stieltjes sums for  $-\int_a^b \alpha d\omega(\alpha)$ . Since the sums must converge to  $-\int_a^b \alpha d\omega(\alpha)$  as the norm of the partition tends to zero, the conclusion follows. ■

We can extend the conclusion of Theorem 5.41 to the case when  $f$  is not bounded as follows.

**Theorem 10** (5.42). *Let  $f$  be any measurable function on  $E$ , and let  $E_{ab} := \{\mathbf{x} \in E : a < f(\mathbf{x}) < b\}$  ( $a$  and  $b$  finite). Then,*

$$\int_{E_{ab}} f = - \int_a^b \alpha d\omega(\alpha).$$

*Sketch of proof.* Take  $\omega_{ab}(\alpha) := |\{\mathbf{x} \in E_{ab} : f(\mathbf{x}) > \alpha\}|$ . By Theorem 5.41, we have

$$\int_{E_{ab}} f = - \int_a^b \alpha d\omega_{ab}(\alpha).$$

Taking the limit of Riemann–Stieltjes sums that approximate the integrals, it suffices to show that  $\omega_{ab}(\alpha) - \omega_{ab}(\beta) = \omega(\alpha) - \omega(\beta)$ . Then The expression on the right-hand side of the equation above, is seen to be  $\int_a^b \alpha d\omega(\alpha)$ . ■

**Theorem 11** (5.43). *If either  $\int_E f$  or  $\int_{-\infty}^{\infty} \alpha d\omega(\alpha)$  exist and is finite, then the other exists and is finite, and*

$$\int_E f = - \int_{-\infty}^{\infty} \alpha d\omega(\alpha).$$

Two measurable functions  $f$  and  $g$  are said to be *equimeasurable*, or *equidistributed*, if

$$\omega_{f,E}(\alpha) = \omega_{g,E}(\alpha)$$

for all  $\alpha$ .

We may intuitively think of equimeasurable functions as being *rearrangements* of each other. For such functions, we have

$$|\{a < f \leq b\}| = |\{a < g \leq b\}| \quad |\{f = a\}| = |\{g = a\}|,$$

etc. We also gave the following immediate corollary of Theorem 5.43.

**Corollary 12** (5.44). *If  $f$  and  $g$  are equimeasurable on  $E$  and  $f \in L(E)$ , then  $g \in L(E)$  and*

$$\int_E f = \int_E g.$$

The method used to derive Theorem 5.41 through 5.43 illustrates a basic difference between the Lebesgue and the Riemann integral. The Riemann integral is defined by a limiting process whose initial step involves partitioning the domain of  $f$ . On the other hand, the Lebesgue integral can be obtained from a process that partitions the *range* of  $f$ . In order to define the process more clearly, let  $f$  be a nonnegative measurable function that is finite a.e. in  $E$ ,  $|E| < \infty$ . Let  $\Gamma = \{0 = \alpha_0 < \alpha_1 < \dots\}$  be a partition of the positive ordinate axis by a countable number of points  $\alpha_k \rightarrow \infty$ , and let  $|\Gamma| = \sup_k(\alpha_{k+1} - \alpha_k)$ . Set  $E_k := \{\alpha_k \leq f < \alpha_{k+1}\}$  and  $Z := \{f = \infty\}$ . Then the  $E_k$  are measurable and disjoint,  $|Z| = 0$  and  $E = (\bigcup E_k) \cup Z$ , so that  $|E| = \sum_k |E_k|$ . Let

$$S_\Gamma := \sum_{k \in \mathbb{N}} \alpha_k |E_k|, \quad S_\Gamma := \sum_{k \in \mathbb{N}} \alpha_{k+1} |E_k|.$$

## 1.6 $L^p$ Classes

Let's talk about  $L^p$  classes now and some important results about  $L^p$  spaces.

### Definition of $L^p$

If  $E$  is a measurable subset of  $\mathbb{R}^n$  and satisfies  $0 < p < \infty$ , then  $L^p(E)$  denotes the collection of measurable  $f$  for which  $\int_E |f|^p$  is finite, i.e.,

$$L^p(E) := \left\{ f : \int_E |f|^p < \infty \right\} \quad (19)$$

for  $0 < p < \infty$ . Here,  $f$  may be complex-valued, in which case, if  $f = f_1 + if_2$  for measurable real-valued  $f_1$  and  $f_2$ , we have  $|f|^2 = f_1^2 + f_2^2$ , so that

$$|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|.$$

It follows that  $f \in L^p(E)$  if and only if both  $f_1, f_2 \in L^p(E)$ .

We shall write

$$\|f\|_{p,E} := \left( \int_E |f|^p \right)^{1/p},$$

for  $0 < p < \infty$ . Thus,  $L^p(E)$  is the set of measurable  $f$  for which  $\|f\|_{p,E}$  is finite. Whenever it is clear from context, we will omit  $E$  in  $L^p(E)$  and  $\|f\|_{p,E}$ , and instead write  $L^p$  and  $\|f\|_p$ . Also note that  $L = L^1$ .

In order to define  $L^\infty(E)$ , let  $f$  be real-valued and measurable on a set  $E$  of positive measure. Define the *essential supremum* of  $f$  on  $E$  to be

$$\operatorname{ess\,sup}_E f := \inf \{ \alpha : |\{ \mathbf{x} \in E : f(\mathbf{x}) > \alpha \}| = 0 \}. \quad (20)$$

In words, this the essential supremum of  $f$  is the least upper bound of  $f$  outside of a set of measure zero. It can be restated as such:  $\operatorname{ess\,sup} f$  is the smallest number  $M$ ,  $-\infty \leq M \leq \infty$ , such that  $f(\mathbf{x}) \leq M$  almost everywhere in  $E$ .

In the definition of  $\operatorname{ess\,sup} f$ , we have made the explicit assumption that the measure of  $E$  is nonzero. Otherwise,  $\operatorname{ess\,sup} f = -\infty$  which can result in awkward or incorrect statements of results involving  $L^p$  spaces. Therefore, we shall adopt the convention that  $\operatorname{ess\,sup} f = 0$  if  $|E| = 0$ .

A real or complex-valued measurable  $f$  is said to be *essentially bounded*, or simply *bounded* almost everywhere on  $E$  if  $\text{ess sup } |f|$  is finite. The class of all functions that are essentially bounded on  $E$  is denoted by  $L^\infty(E)$ . Clearly,  $f \in L^\infty(E)$  if and only if its real and imaginary parts belong to  $L^\infty(E)$ . We shall use the notation  $\|f\|_\infty$  synonymously with  $\text{ess sup } f$ .

The following theorem gives some good motivation for the use the notation  $\|f\|_\infty$ , at least in the case  $|E| < \infty$ .

**Theorem 13** (8.1). *If  $|E| < \infty$ , then  $\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p$ .*

*Sketch of proof.* We may assume that  $|E| > 0$ , for otherwise we have a trivial statement, i.e.,  $\|f\|_p = 0$  for all  $p$  and by convention  $\|f\|_\infty = 0$  so clearly  $\|f\|_p \rightarrow \|f\|_\infty$  as  $p \rightarrow \infty$ . Set  $M := \|f\|_\infty$ . If  $M' < M$ , ■

# MA 544 (Spring 2016)

## 2.1 Exam 1 Prep

**Problem 2.1.** Let  $E \subset \mathbb{R}^n$  be a measurable set,  $r \in \mathbb{R}$  and define the set  $rE = \{r\mathbf{x} : \mathbf{x} \in E\}$ . Prove that  $rE$  is measurable, and that  $|rE| = |r|^n|E|$ .

*Proof.* Define a linear map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $\mathbf{x} \mapsto r\mathbf{x}$ . Using the standard basis for  $\mathbb{R}^n$ , this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \quad (21)$$

which has determinant  $\det T = r^n$ . By 3.35, we have  $|E| = |T(E)| = r^n|E| = |rE|$ . ■

**Problem 2.2.** Let  $\{E_k\}$ ,  $k \in \mathbb{N}$  be a collection of measurable sets. Define the set

$$\liminf_{k \rightarrow \infty} E_k = \bigcup_{k=1}^{\infty} \left( \bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|.$$

*Proof.* If the  $\liminf_{k \rightarrow \infty} |E_k| = \infty$  the inequality holds trivially. Hence, we may, without loss of generality, assume that  $\liminf_{k \rightarrow \infty} |E_k| < \infty$ . By 3.20, the set  $\liminf_{k \rightarrow \infty} E_k$  is measurable and we have

$$\left| \liminf_{k \rightarrow \infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \quad (22)$$

where  $F_k = \bigcap_{n=k}^{\infty} E_n$ . Now, note that the collection of sets  $F'_k = \bigcup_{\ell=1}^k F_\ell$  forms an increasing sequence of measurable sets  $F'_k \nearrow F'$ , where  $F' = \bigcup_{k=1}^{\infty} F_k = \liminf E_k$ . Then, by 3.26 (i), we have

$$\lim_{k \rightarrow \infty} |F'_k| = |F'| = \left| \liminf_{k \rightarrow \infty} E_k \right|. \quad (23)$$

Hence, it suffices to show that  $|F'_k| \leq |E_k|$  for all  $k$ , but this follows by monotonicity of the outer measure, 3.3, since  $F'_k \subset E_k$ . Thus, we have the desired inequality

$$\left| \liminf_{k \rightarrow \infty} E_k \right| \leq \liminf_{k \rightarrow \infty} |E_k|. \quad (24)$$

■

**Problem 2.3.** Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here  $B(\mathbf{0}, r) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r\}$ . Prove that  $F$  is monotonic increasing and continuous.

*Proof.* That  $F$  is increasing is immediate from the monotonicity of the outer measure since for  $x < x'$  we have  $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$  so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \leq |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that  $F$  is continuous, we will prove the following lemma

**Lemma 14.** For any  $x > 0$ ,  $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$ .

*Proof of lemma.* If  $\mathbf{y} \in xB(\mathbf{0}, 1)$  then  $\mathbf{y} = x\mathbf{y}'$  for  $\mathbf{y}' \in B(\mathbf{0}, 1)$ . Thus,  $|\mathbf{y}'| = |\mathbf{y}|/x < 1$  so  $|\mathbf{y}| < x$  implies that  $\mathbf{y} \in B(\mathbf{0}, x)$ . Hence, we have the containment  $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$ .

On the other hand, if  $\mathbf{y} \in B(\mathbf{0}, x)$  then  $|\mathbf{y}| < x$  so  $|\mathbf{y}/x| < 1$ . Hence,  $\mathbf{y}/x \in B(\mathbf{0}, 1)$  so  $x(\mathbf{y}/x) = \mathbf{y} \in xB(\mathbf{0}, 1)$ . Thus,  $B(\mathbf{0}, x) \subset xB(\mathbf{0}, 1)$  and equality holds. ♣

In light of Lemma 14 and 3.35, for  $x > 0$ , we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n |B(\mathbf{0}, 1)|. \quad (25)$$

It is clear that  $F$  is continuous on the interval  $[0, \infty)$  since  $F$  is a polynomial in  $x$ . ■

**Problem 2.4.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $C$  be the set of all points at which  $f$  is continuous. Show that  $C$  is a set of type  $G_\delta$ .

*Proof.* From the topological definition of continuity,  $f$  is continuous at  $x \in C$  if and only if for every neighborhood  $U$  of  $f(x)$ , the preimage  $f^{-1}(U)$  is a neighborhood of  $x$ . Now, ■

Let  $x \in C$ . Then, by the definition of continuity, for every natural number  $n > 0$  there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies

$$|f(x) - f(x')| < \frac{1}{2n}. \quad (26)$$

Let  $x'', x' \in B(x, \delta)$ . Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x'')| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \quad (27)$$

In view of these estimates, define the set

$$A_n = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \quad (28)$$

Good Lord, that was a long definition! We claim that  $C = \bigcap_{n=1}^{\infty} A_n$  and that  $A_n$  is open for all  $n$ .

First, let us show that  $C = \bigcap_{n=1}^{\infty} A_n$ . Let  $x \in C$ . Then for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . Thus,  $x \in A_n$  for all  $n$  so  $x \in \bigcap A_n$ . On the other hand, if  $x \in \bigcap A_n$  for every  $n > 0$ , there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . Fix  $\varepsilon > 0$ . By the Archimedean principle, there exists  $N > 0$  such that  $\varepsilon > 1/N$ . Then, since  $x \in A_N$  it follows that for some  $\delta' > 0$ ,  $|x - x'| < \delta'$  implies  $|f(x) - f(x')| < 1/N < \varepsilon$ . Thus,  $x \in C$  and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$ .

Lastly, we show that  $A_n$  is open. Let  $x \in A_n$ . Then there exists  $\delta > 0$  such that  $|x - x'| < \delta$  implies  $|f(x) - f(x')| < 1/n$ . In particular, this means that  $B(x, \delta) \subset A_n$  for any  $x' \in B(x, \delta)$  satisfies  $|f(x) - f(x')| < 1/n$ . Thus,  $A_n$  is open and we conclude that  $C = \bigcap_{n=1}^{\infty} A_n$  is a  $G_\delta$  set.

**Problem 2.5.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Is it true that if the sets  $\{f = r\}$  are measurable for all  $r \in \mathbb{R}$ , then  $f$  is measurable?

*Proof.* No. Recall that, by definition, or 4.1,  $f$  is measurable if and only if  $\{f > a\}$  for all  $a \in \mathbb{R}$ . ■

**Problem 2.6.** Let  $\{f_k\}_{k=1}^{\infty}$  be a sequence of measurable functions on  $\mathbb{R}$ . Prove that the set  $\{x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists}\}$  is measurable.

*Proof.* The idea here should be to rewrite

$$E = \left\{ x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists} \right\} \quad (29)$$

as a countable union/intersection of measurable sets. Let  $x \in E$ . By the Cauchy criterion, for every  $N > 0$  there exists a positive integer  $M$  such that  $m, n \geq M$  implies  $|f_n(x) - f_m(x)| < 1/N$ . With this in mind, define

$$E_N = \left\{ x : \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \quad (30)$$

Then, like for Problem 1.4, it is not too hard to see that the  $E_n$ 's are open and that  $E = \bigcap_{n=1}^{\infty} E_n$ . Thus,  $E$  is a  $G_\delta$  set and therefore measurable. ■

**Problem 2.7.** A real valued function  $f$  on an interval  $[a, b]$  is said to be *absolutely continuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Show that an absolutely continuous function on  $[a, b]$  is of bounded variation on  $[a, b]$ .

*Proof.* Suppose  $f: [a, b] \rightarrow \mathbb{R}$  is absolutely continuous. Then for fixed  $\varepsilon = 1$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , we have  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Let  $\Gamma = \{x_k\}_{k=1}^N$  be a partition of  $[a, b]$  into closed intervals such that  $x_{k+1} - x_k < \delta$ , then by absolute continuity we have

$$V[f; \Gamma] = \sum_{k=1}^N |f(x_{k+1}) - f(x_k)| < 1. \quad (31)$$

Thus,  $f$  is b.v. on  $[a, b]$ . ■

**Problem 2.8.** Let  $f$  be a continuous function from  $[a, b]$  into  $\mathbb{R}$ . Let  $\chi_{\{c\}}$  be the characteristic function of a singleton  $\{c\}$ , i.e.,  $\chi_{\{c\}}(x) = 0$  if  $x \neq c$  and  $\chi_{\{c\}}(c) = 1$ . Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}.$$

*Proof.* ■



## 2.2 Exam 1

### 2.3 Exam 2 Prep

**Problem 2.9.** Define for  $\mathbf{x} \in \mathbb{R}^n$ ,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that  $f$  is integrable outside any ball  $B_\varepsilon(\mathbf{0})$ , and that there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x}) d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

*Proof.* Recall that a real-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is (Lebesgue) integrable over a subset  $E$  of  $\mathbb{R}^n$  (or, alternatively,  $f$  belongs to  $L(E)$ ) if

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty.$$

Put  $E = \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$ . Then, to show that  $f$  belongs to  $L(E)$  it suffices to prove the inequality

$$\int_E f(\mathbf{x}) d\mathbf{x} < \frac{C}{\varepsilon} \quad (32)$$

for some appropriate constant  $C$ . We proceed by directly computing the Lebesgue integral of  $f$  and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x}) d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{(x_1^2 + \cdots + x_n^2)^{(n+1)/2}} \end{aligned}$$

let  $E_i$  denote the projection of  $E$  onto its  $i$ -th coordinate and make the trigonometric substitution  $x_1 = \sqrt{x_2^2 + \cdots + x_n^2} \tan \theta$ ,  $dx_1 = \sqrt{x_2^2 + \cdots + x_n^2} \sec^2 \theta d\theta$  with  $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$  giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[ \frac{\cos^{n-1} \theta}{(x_2^2 + \cdots + x_n^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{(x_2^2 + \cdots + x_n^2)^{n/2}} \left[ \int_{E_\theta} \cos^{n-1} \theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1} \theta d\theta < \infty. \quad (33)$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned} \int \cdots \int_E f(\mathbf{x}) d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{x_n^2} \\ &= 2C' \int_\varepsilon^\infty \frac{dx_n}{x_n^2} \\ &= \frac{C}{\varepsilon} \end{aligned} \tag{34}$$

as desired. ■

**Problem 2.10.** Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $\mathbb{R}^n$ , and assume that  $f_k$  converges pointwise almost everywhere to a function  $f$ . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets  $E$  of  $\mathbb{R}^n$ . Moreover, show that this is not necessarily true if  $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$ .

*Proof.* This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit  $f$  of  $\{f_k\}$  must be nonnegative a.e. in  $\mathbb{R}^n$ . For assume otherwise. Then there exists a collection of points  $\mathbf{x}$  in  $\mathbb{R}^n$  of nonzero  $\mathbb{R}^n$ -Lebesgue measure such that  $f(\mathbf{x}) < 0$ . But  $f_k(\mathbf{x}) \geq 0$  for all  $k \in \mathbb{N}$ . Set  $0 < \varepsilon < |f(\mathbf{x})|$  then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{35}$$

for all  $k$  which contradicts our assumption that  $f_k \rightarrow f$  a.e. on  $\mathbb{R}^n$ . Therefore, the set of points  $\mathbf{x} \in \mathbb{R}^n$  where  $f(\mathbf{x}) < 0$  must have measure zero.

Now, based on pointwise convergence a.e. to  $f$ , given  $\varepsilon > 0$  for a.e.  $\mathbf{x} \in \mathbb{R}^n$  we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{36}$$

for sufficiently large  $k$ ; say  $k$  greater than or equal to some index  $N \in \mathbb{N}$ . Moreover, we are given convergence in  $L(\mathbb{R}^n)$  of  $f_k$  to  $f$

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f < \infty. \tag{37}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{38}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{39}$$

for all  $k \in \mathbb{N}$ . By Theorem 5.5(ii),  $f$  and the  $f_k$ 's are finite a.e. in  $\mathbb{R}^n$  so for some sufficiently large real number  $M$ ,  $|f|, |f_k| \leq M$  for a.e.  $\mathbf{x} \in \mathbb{R}^n$ . In particular, for any measurable subset  $E$  of  $\mathbb{R}^n$ ,  $|f|, |f_k| \leq M$  for a.e.  $\mathbf{x} \in E$  so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \rightarrow \int_E f < \infty. \quad (40)$$

However, if  $f$  does not belong to  $L(\mathbb{R}^n)$ , i.e., its integral over  $\mathbb{R}^n$  is infinity, there is no guarantee that  $f$  will be finite a.e. in  $\mathbb{R}^n$ . This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset  $E$  of  $\mathbb{R}^n$ . Let us demonstrate this with an example. Consider the sequence of functions ■

**Problem 2.11.** Assume that  $E$  is a measurable set of  $\mathbb{R}^n$ , with  $|E| < \infty$ . Prove that a nonnegative function  $f$  defined on  $E$  is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}| < \infty.$$

*Proof.* If  $f$  is integrable over a measurable subset  $E$  of  $\mathbb{R}^n$ , then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \quad (41)$$

Set  $E_k = \{\mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k\}$  and  $F_k = \{\mathbf{x} \in E : f(\mathbf{x}) \geq k\}$ . Note the following properties about the sets we have just defined: first, the  $E_k$ 's are pairwise disjoint and the  $F_k$ 's are nested in the following way  $F_{k+1} \subset F_k$ ; second,  $E = \bigcup_{k=1}^{\infty} E_k$  and  $E_k = F_k \setminus F_{k+1}$ . By Theorem 3.23, since the  $E_k$ 's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \quad (42)$$

Now, since  $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$  on  $E_k$ , we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \quad (43)$$

Then we have the following upper and lower estimates on the integral of  $f$  over  $E$

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \quad (44)$$

But note that  $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$  by Corollary 3.25 since the measures of  $E_k$ ,  $F_k$ , and  $F_{k+1}$  are all finite. Hence, (44) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \quad (45)$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned} \tag{47}$$

Thus, from (46) and (47)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \tag{48}$$

so the integral  $\int_E f$  converges if and only if the sum  $\sum_{k=0}^{\infty} |F_k|$  converges. ■

**Problem 2.12.** Suppose that  $E$  is a measurable subset of  $\mathbb{R}^n$ , with  $|E| < \infty$ . If  $f$  and  $g$  are measurable functions on  $E$ , define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$  if and only if  $f_k$  converges to  $f$  as  $k \rightarrow \infty$ .

*Proof.*  $\implies$  : First note that  $\rho$  is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$ . Then, given  $\varepsilon > 0$  there exist an

sufficiently large index  $N$  such that for every  $k \geq N$  we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \quad (49)$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in  $E$  which happens if  $|f_k - f| = 0$  a.e. in  $E$ .

$\Leftarrow$  : Suppose that  $f_k \rightarrow f$  as  $k \rightarrow \infty$ .

I don't know how to solve this. This is the intended solution:

$\Rightarrow$  : Given  $\varepsilon > 0$ ,  $\rho(f_k, f) \rightarrow 0$  implies that

$$\int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \rightarrow 0.$$

Observe that the function  $\Phi: \mathbb{R}^+ \rightarrow \mathbb{R}$  given by  $\Phi(x) = x/(1+x)$  is increasing on  $\mathbb{R}^+$  and  $0 < \Phi(x) < 1$ , hence

$$\begin{aligned} \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1 + \varepsilon} |\{x \in E : |f_k(x) - f(x)| > \varepsilon\}|. \end{aligned}$$

Therefore,

$$|\{x \in E : |f_k(x) - f(x)| > \varepsilon\}| \leq \frac{1 + \varepsilon}{\varepsilon} \int_{\{x \in E : |f_k(x) - f(x)| > \varepsilon\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \rightarrow 0$$

as  $k \rightarrow \infty$ .

$\Leftarrow$  : Conversely, given  $\delta > 0$ , we have

$$\begin{aligned} \rho(f_k, f) &= \int_{\{x \in E : |f_k(x) - f(x)| > \delta\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\quad + \int_{\{x \in E : |f_k(x) - f(x)| \leq \delta\}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\leq |\{x \in E : |f_k(x) - f(x)| > \delta\}| + \frac{\delta}{1 + \delta} |E|. \end{aligned}$$

Since  $|E| < \infty$  and  $\delta/(1+\delta) \searrow 0$ , then for any  $\varepsilon > 0$ , there exists  $\delta' > 0$  such that

$$\frac{\delta'}{1 + \delta'} |E| < \frac{\varepsilon}{2}.$$

If  $f_k \rightarrow f$  as  $k \rightarrow \infty$  in measure, then for the above  $\delta'$  there is an index  $N > 0$  such that  $k \geq N$  implies

$$|\{x \in E : |f_k(x) - f(x)| > \delta'\}| < \frac{\varepsilon}{2}.$$

Therefore,  $f_k \rightarrow f$  in measure implies  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$ . ■

**Problem 2.13.** Define the *gamma function*  $\Gamma: \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function*  $\beta: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function  $u \mapsto e^{-u} u^{y-1}$  is in  $L(\mathbb{R}^+)$  for all  $y \in \mathbb{R}^+$ .

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

*Proof.* (a) Fix  $y \in \mathbb{R}^+$ . Then we must show that  $\Gamma(y) < \infty$ . First, since  $(0, 1)$  and  $[1, \infty)$  are disjoint measurable subsets of  $\mathbb{R}$ , by Theorem 5.7 we can split the integral  $\Gamma(y)$  into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u} u^{y-1} du}_{I_1} + \underbrace{\int_1^\infty e^{-u} u^{y-1} du}_{I_2}. \quad (50)$$

We will show, separately, that  $I_1$  and  $I_2$  are finite.

To see that  $I_1$  is finite, note that

$$\begin{aligned} e^{-u} u^{y-1} &= e^{-u} e^{(y-1) \log u} \\ &= e^{-u+(y-1) \log u} \\ &\leq e^{(y-1) \log u} \\ &= u^{y-1} \end{aligned} \quad (51)$$

since  $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u} u^{y-1} du \\ &\leq \int_0^1 u^{y-1} du \\ &= \left[ \frac{u^y}{y} \right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \quad (52)$$

To see that  $I_2$  is finite, note that

$$e^{-u} u^{y-1} \leq e^{-u} \quad (53)$$

**Intended solution:**

(b)

■

**Problem 2.14.** Let  $f \in L(\mathbb{R}^n)$  and for  $\mathbf{h} \in \mathbb{R}^n$  define  $f_{\mathbf{h}}: \mathbb{R}^n \rightarrow \mathbb{R}$  be  $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$ . Prove that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

*Proof.* Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{54}$$

■

**Problem 2.15.** (a) If  $f_k, g_k, f, g \in L(\mathbb{R}^n)$ ,  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ ,  $|f_k| \leq g_k$  and

$$\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if  $f_k, f \in L(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e. in  $\mathbb{R}^n$ , then

$$\int_{\mathbb{R}^n} |f_k - f| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \rightarrow \int_{\mathbb{R}^n} |f| \quad \text{as} \quad k \rightarrow \infty.$$

*Proof.* (a) Since  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. and  $|f_k| \leq g_k$ , then by Fatou's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} (g - f) &= \int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} g_k - f_k \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_k - f_k, \\ \int_{\mathbb{R}^n} g + f &\int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} g_k + f_k \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} g_k + f_k. \end{aligned}$$

Since  $f_k, g_k, f, g \in L(\mathbb{R}^n)$  and  $\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g$ , then using the similar argument as problem 2, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f &\geq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k, \\ \int_{\mathbb{R}^n} f &\leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k. \end{aligned}$$

Therefore,  $\int_{\mathbb{R}^n} f_k \rightarrow \int_{\mathbb{R}^n} f$ .

(b)  $\implies$  : This direction is obvious by the inequality

$$\left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \leq \int_{\mathbb{R}^n} ||f_k| - |f|| \leq \int_{\mathbb{R}^n} |f_k - f|.$$



$\Leftarrow$  : Let  $g_k = |f_k| + |f|$  and  $g = 2|f|$ . Since  $f_k, f \in L(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e., then  $g_k, g \in L(\mathbb{R}^n)$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ . By the assumption,  $\int_{\mathbb{R}^n} g_k \rightarrow \int_{\mathbb{R}^n} g$ .

Let  $\tilde{f}_k = |f_k - f|$ . Then  $\tilde{f}_k \rightarrow 0$  a.e. in  $\mathbb{R}^n$  and  $\tilde{f}_k \leq g_k$ . Applying part (a) to  $\tilde{f}_k$  we have

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

■

## 2.4 Midterm 2

**Problem 2.16.** Assume that  $f \in L(\mathbb{R}^n)$ . Show that for every  $\varepsilon > 0$  there exists a ball  $B$ , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

*Proof.* Recall that  $f \in L(\mathbb{R}^n)$  if and only if  $|f| \in L(\mathbb{R}^n)$ . Let  $B_k = B(\mathbf{0}, k)$  for  $k \in \mathbb{N}$  and  $\chi_{B_k}$  be the indicator function associated with  $B_k$ . Then, the sequence of maps  $\{|f_k|\}$  defined  $f_k = f\chi_{B_k}$  converge pointwise to  $|f|$ . Since  $|f| \in L(\mathbb{R}^n)$ , by the monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbb{R}^n} |f|. \quad (55)$$

But this means, exactly, that for every  $\varepsilon > 0$  there exists sufficiently large  $N \in \mathbb{N}$  such that

$$\begin{aligned} \varepsilon &> \left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \\ &= - \int_{\mathbb{R}^n} |f_k| + \int_{\mathbb{R}^n} |f| \\ &= - \int_{\mathbb{R}^n} |f| + \int_{\mathbb{R}^n} |f| \\ &= - \int_{B_k} |f| + \int_{\mathbb{R}^n} |f| \\ &= \int_{\mathbb{R}^n \setminus B_k} |f| \end{aligned} \quad (56)$$

as desired. ■

**Problem 2.17.** Let  $f \in L(E)$ , and let  $\{E_j\}$  be a countable collection of pairwise disjoint measurable subsets of  $E$ , such that  $E = \bigcup_{j=1}^{\infty} E_j$ . Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

*Proof.* First, since the  $E_j$ 's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \quad (57)$$

Let  $\chi_{E_j}$  be the characteristic function of the subset  $E_j$  of  $E$  and define  $f_j = f\chi_{E_j}$  for  $j \in \mathbb{N}$ . Note that, since both  $f$  and  $\chi_{E_j}$  are measurable on  $E$ ,  $f_j$  is measurable on  $E$  and  $\sum_{j=1}^{\infty} f_j = f$ . Moreover, since  $E_j \subset E$ , by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \setminus E_j} f = \int_E f_j + \int_{E \setminus E_j} f. \quad (58)$$

Hence, because the  $E_j$ 's are disjoint  $(E \setminus E_k) \setminus E_\ell = (E \setminus E_\ell) \setminus E_k$  so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \quad (59)$$

as desired. ■

**Problem 2.18.** Let  $\{f_k\}$  be a family in  $L(E)$  satisfying the following property: For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|A| < \delta$  implies

$$\int_A |f_k| < \varepsilon$$

for all  $k \in \mathbb{N}$ . Assume  $|E| < \infty$ , and  $f_k(x) \rightarrow f(x)$  as  $k \rightarrow \infty$  for a.e.  $x \in E$ . Show that

$$\lim_{k \rightarrow \infty} \int_E f_k = \int_E f.$$

(Hint: Use Egorov's theorem.)

*Proof.* Let  $\varepsilon > 0$  be given. Then, by the hypothesis, there exists  $\delta > 0$  such that  $|A| < \delta$  implies

$$\int_A |f_k| < \varepsilon \quad (60)$$

for all  $k \in \mathbb{N}$ . By Egorov's theorem, there exists a closed subset  $F$  of  $E$  such that  $|E \setminus F| < \delta$  and  $f_k \rightarrow f$  uniformly on  $F$ . Then, by the uniform convergence theorem,

$$\int_F f_k \rightarrow \int_F f \quad (61)$$

as  $k \rightarrow \infty$ . But by hypothesis, we have

$$\int_{E \setminus F} |f_k| < \varepsilon. \quad (62)$$

Letting  $\varepsilon \rightarrow 0$ , we achieved the desired convergence. ■

**Problem 2.19.** Let  $I = [0, 1]$ ,  $f \in L(I)$ , and define  $g(x) = \int_x^1 t^{-1} f(t) dt$  for  $x \in I$ . Prove that  $g \in L(I)$  and

$$\int_I g = \int_I f.$$

*Proof.* By Lusin's theorem, there exists a closed subset  $F$  of  $I$  with  $|I \setminus F| < \varepsilon$  such that the restriction of  $f$  to  $F = I \setminus E$  is continuous. Now, since  $F$  is closed in  $I$  and  $I$  is compact, it follows that  $F$  is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials  $\{p_k\}$  such that  $p_k \rightarrow f$  uniformly on  $F$ . Then, by the uniform convergence theorem, we have

$$\int_F p_k \rightarrow \int_F f \quad (63)$$

so

$$\begin{aligned}\int_F \left[ \int_x^1 t^{-1} p_k(t) dt \right] dx &= \int_F \left[ \int_x^1 a t^{-1} + q_k(t) dt \right] dx \\ &= \int_F q'_k(x) - a \log(x) dx \\ &< \infty\end{aligned}\tag{64}$$

for all  $k$  and converges uniformly to  $g$  so  $g \in L(I)$ . I don't know how to show that in fact  $\int_I g = \int_I f$ . Perhaps you show that the places where they differ is a set of measure zero. ■

## 2.5 Final Practice

**Problem 2.20.** Suppose  $f \in L^1(\mathbb{R})$  and that  $x$  is a point in the Lebesgue set of  $f$ . For  $r > 0$ , let

$$A(r) := \frac{1}{r} \int_{B(0,r)} |f(x-y) - f(x)| dy.$$

Show that:

- (a)  $A(r)$  is a continuous function of  $r$ , and  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ ;
- (b) there exists a constant  $M > 0$  such that  $A(r) \leq M$  for all  $r > 0$ .

*Proof.* ■

**Problem 2.21.** Let  $E \subset \mathbb{R}^n$  be a measurable set,  $1 \leq n < \infty$ . Assume  $\{f_k\}$  is a sequence in  $L^p(E)$  converging pointwise a.e. on  $E$  to a function  $f \in L^p(E)$ . Prove that

$$\|f_k - f\|_p \rightarrow 0$$

if and only if

$$\|f_k\|_p \rightarrow \|f\|_p$$

as  $k \rightarrow \infty$ .

*Proof.* ■

**Problem 2.22.** Let  $1 < p < \infty$ ,  $f \in L^p(E)$ ,  $g \in L^{p'}(E)$ .

- (a) Prove that  $f * g \in C(\mathbb{R}^n)$ .
- (b) Does this conclusion continue to be valid when  $p = 1$  and  $p = \infty$ ?

*Proof.* ■

**Problem 2.23.** Let  $f \in L(\mathbb{R})$ , and let  $F(t) := \int_{\mathbb{R}} f(x) \cos(tx) dx$ .

- (a) Prove that  $F(t)$  is continuous for  $t \in \mathbb{R}$ .
- (b) Prove the following *Riemann-Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

*Proof.* ■

**Problem 2.24.** Let  $f$  be of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular. Show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh$$

for all functions  $\varphi$  continuous on  $[a, b]$ .

*Proof.* ■

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