

## MA 26500-215 Quiz 10

July 29, 2016

1. Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear map that sends

$$T(1, 0, 0) = (3, 2, 4)$$

$$T(0, 1, 0) = (2, 0, 2)$$

$$T(0, 0, 1) = (4, 2, 3).$$

- (a) (4 points) Find the value of  $T(2, 1, -1)$ .

**Solution:** Since  $T$  is a linear map, we know that  $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$  and  $T(c\mathbf{v}) = cT(\mathbf{v})$  so

$$\begin{aligned} T(2, 1, -1) &= T((2, 0, 0) + (0, 1, 0) + (0, 0, -1)) \\ &= T(2, 0, 0) + T(0, 1, 0) + T(0, 0, -1) \\ &= 2T(1, 0, 0) + T(0, 1, 0) - T(0, 0, 1) \\ &= 2(3, 2, 4) + (2, 0, 2) - (4, 2, 3) \\ &= (6, 4, 8) + (2, 0, 2) + (-4, -2, -3) \\ &= (6 + 2 - 4, 4 + 0 - 2, 8 + 2 - 3) \\ &= (4, 2, 7). \end{aligned}$$

- (b) (6 points) Find the matrix representation of  $T$  with respect to the standard basis on  $\mathbb{R}^3$ .

**Solution:** Using the standard basis on  $\mathbb{R}^3$  which, by the way, is the set  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ , for a general vector

$$\mathbf{x} = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1),$$

we have

$$T(\mathbf{x}) = (3x_1 + 2x_2 + 4x_3, 2x_1 + 0x_2 + 2x_3, 4x_1 + 2x_2 + 3x_3).$$

Thus,

$$\begin{aligned} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{bmatrix} \\ &= \begin{bmatrix} 3x_1 + 2x_2 + 4x_3 \\ 2x_1 + 0x_2 + 2x_3 \\ 4x_1 + 2x_2 + 3x_3 \end{bmatrix}. \end{aligned}$$

This tells us that the matrix must be

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

- (c) (10 points) Using the matrix representation of  $T$ , find the characteristic polynomial. **You do not have to simplify it.**

**Solution:** To find the minimal polynomial of  $T$ , we find

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3 - \lambda & 2 & 4 \\ 2 & 0 - \lambda & 2 \\ 4 & 2 & 3 - \lambda \end{bmatrix} \\ &= (3 - \lambda) \det \begin{bmatrix} 0 - \lambda & 2 \\ 2 & 3 - \lambda \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 2 \\ 4 & 3 - \lambda \end{bmatrix} \\ &\quad + 4 \det \begin{bmatrix} 2 & 0 - \lambda \\ 4 & 2 \end{bmatrix} \\ &= (3 - \lambda)(\lambda^2 - 3\lambda - 4) - 2(6 - 2\lambda - 8) + 4(4 + 4\lambda) \\ &= (3\lambda^2 - 9\lambda - 12 - \lambda^3 + 3\lambda^2 + 4\lambda) \\ &\quad + (-12 + 4\lambda + 16) \\ &\quad + (16 + 16\lambda) \\ &= -\lambda^3 + 6\lambda^2 + 15\lambda + 8. \end{aligned}$$

Using this equation, we can find the eigenvalues of  $T$ . One thing you can do is first try to find a root of  $\lambda^3 - 6\lambda^2 - 15\lambda - 8$ . As it turns out,  $\lambda = -1$  is a root since

$$(-1)^3 - 6(-1)^2 - 15(-1) - 8 = -1 - 6 + 15 - 8 = -15 + 15 = 0.$$

So, using long division, we can factor  $(\lambda + 1)$  from  $\lambda^3 - 6\lambda^2 - 15\lambda - 8$  and so

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda + 1)(\lambda^2 - 7\lambda - 8).$$

And you can do it again since  $(-1)^2 - 7(-1) - 8 = 1 + 7 - 8 = 0$  so

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda + 1)^2(\lambda - 8).$$

Thus, the eigenvalues are  $-1$  and  $8$ .