Exercises in Basic Mathematics

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Definition 1. Given any ideal $\mathfrak{a} \subset k[X_1, \dots, X_q]$, define $V_k(\mathfrak{a})$ by

$$V_k(\mathfrak{a}) := \{\, \mathbf{x} \in \mathbf{A}^q : \text{for every } f \in \mathfrak{a},\ f(\mathbf{x}) = 0 \,\}.$$

We call $V_k(\mathfrak{a})$ the *set of Ω-points of the affine k-variety determined by $\mathfrak{a}$.* With a slight abuse of language, we call $V_k(\mathfrak{a})$ the *affine k-variety determined by $\mathfrak{a}$.* Similarly, given by any ideal $\mathfrak{a} \subset \bar{k}[X_1, \dots, X_q]$, defined by $V_{\bar{k}}(\mathfrak{a})$ by

$$V_{\bar{k}}(\mathfrak{a}) := \{\, \mathbf{x} \in \mathbf{A}^q : \text{for every } f \in \mathfrak{a},\ f(\mathbf{x}) = 0 \,\}.$$

We call $V_{\bar{k}}(\mathfrak{a})$ the *set of Ω-points of the (geometric) affine $\bar{k}$-variety determined by $\mathfrak{a}$,* or for short, the *(geometric) affine variety determined by $\mathfrak{a}$.*

To ease the notation, we usually drop the subscript k or $\bar{k}$ and simply write V.

If A is a (commutative) ring (with unit 1), recall that the *radical*, $\sqrt{\mathfrak{b}}$, of an ideal, $\mathfrak{a} \subset A$, is defined by

$$\sqrt{\mathfrak{a}} := \{\, a \in A : \text{there exists } n \geq 1,\ a^n \in \mathfrak{a} \,\}.$$

A *radical ideal* is an ideal, $\mathfrak{a}$, such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

The following properties are easily verified. We state them for V_k, but they also hold for $V_{\bar{k}}$:

$$\begin{aligned}
&V(0) = \mathbf{A}^n,\ V(A) = \emptyset \\
&V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) \\
&\mathfrak{a} \subset \mathfrak{b} \text{ implies that } V(\mathfrak{b}) \subset V(\mathfrak{b}) \\
&V\Big(\textstyle\sum_\alpha \mathfrak{a}_\alpha\Big) = \textstyle\bigcap_\alpha V(\mathfrak{a}_\alpha) \\
&V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})
\end{aligned}$$

CHAPTER 4

Differential Geometry Exercises

4.1 The Matrix Exponential; Some Matrix Lie Groups

The Exponential Map