

# MA 544: Homework 9

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**PROBLEM 9.1 (WHEEDEN & ZYGMUND §6, EX. 1)**

- (a) Let  $E$  be a measurable subset of  $\mathbb{R}^2$  such that for almost every  $x \in \mathbb{R}^1$ ,  $\{y : (x, y) \in E\}$  has  $\mathbb{R}^1$ -measure zero. Show that  $E$  has measure zero and that for almost every  $y \in \mathbb{R}^1$ ,  $\{x : (x, y) \in E\}$  has measure zero.
- (b) Let  $f(x, y)$  be nonnegative and measurable in  $\mathbb{R}^2$ . Suppose that for almost every  $x \in \mathbb{R}^1$ ,  $f(x, y)$  is finite for almost every  $y$ . Show that for almost every  $y \in \mathbb{R}^1$ ,  $f(x, y)$  is finite for almost every  $x$ .

*Proof.* (a) That  $E$  has measure zero is a consequence of Fubini's theorem. Set  $E_x := \{y : (x, y) \in E\}$  and  $E_y := \{x : (x, y) \in E\}$  then, by Theorem 6.8, we have

$$|E| = \iint_{\mathbb{R}^2} \chi_E \, dx \, dy = \int_{\mathbb{R}} \left[ \int_{E_x} 1 \, dy \right] dx = \int_{\mathbb{R}} \left[ \int_{E_y} 1 \, dx \right] dy = 0. \quad (9.1)$$

Hence,  $E$  has measure zero. Moreover, we see that  $\int_{\mathbb{R}} \left[ \int_{E_y} 1 \, dx \right] dy = 0$  which means that for a.e.  $y \in \mathbb{R}$ ,  $E_y$  has  $\mathbb{R}^1$ -measure zero.

(b) Let  $E$  be the set of all pairs  $(x, y) \in \mathbb{R}^2$  such that  $f(x, y)$  is not finite. By hypothesis, the set  $E_x$  has  $\mathbb{R}^1$ -measure zero for a.e.  $x$ . Therefore, by part (a) the set  $E_y$  has measure zero. Hence, for a.e.  $y$ ,  $f(x, y)$  is finite for a.e.  $x$ . ■

**PROBLEM 9.2 (WHEEDEN & ZYGMUND §6, EX. 3)**

Let  $f$  be measurable and finite a.e. on  $[0, 1]$ . If  $f(x) - f(y)$  is integrable over the square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , show that  $f \in L[0, 1]$ .

*Proof.* Set  $I := [0, 1]$ . Suppose that  $f(x) - f(y) \in L(I \times I)$ . Then by Fubini's theorem we have

$$\iint_{I \times I} f(x) - f(y) \, dx \, dy = \int_I \left[ \int_I f(x) - f(y) \, dx \right] dy = \int_I \left[ \int_I f(x) - f(y) \, dy \right] dx < \infty. \quad (9.2)$$

Hence, for a.e.  $y \in \mathbb{R}$ ,  $f(x) - f(y)$  is integrable so  $f(x)$  is integrable. ■

**PROBLEM 9.3 (WHEEDEN & ZYGMUND §6, EX. 4)**

Let  $f$  be measurable and periodic with period 1:  $f(t+1) = f(t)$ . Suppose there is a finite  $c$  such that

$$\int_0^1 |f(a+t) - f(b+t)| dt \leq c$$

for all  $a$  and  $b$ . Show that  $f \in L[0, 1]$ . (Set  $a = x$ ,  $b = -x$ , integrate with respect to  $x$ , and make the change of variables  $\xi = x + t$ ,  $\eta = -x + t$ .)

*Proof.* Following the hint, write

$$c \geq \int_0^1 \int_0^1 |f(x+t) - f(-x+t)| dx dt$$

making the change of variables  $\xi = x + t$ ,  $\eta = -x + t$  and appropriate modification to the bounds of integration, i.e.,  $0 \leq \xi \leq 2$ ,  $-1 \leq \eta \leq 1$  we have

$$= \int_{-1}^1 \int_0^2 |f(\xi) - f(\eta)| (\det \mathbf{J}(\xi, \eta)) d\xi d\eta$$

by Fubini's theorem

$$= \int_0^2 \int_{-1}^1 |f(\xi) - f(\eta)| (\det \mathbf{J}(\xi, \eta)) d\eta d\xi$$

where  $\mathbf{J}(\xi, \eta) = \begin{bmatrix} \partial x / \partial \xi & \partial x / \partial \eta \\ \partial t / \partial \xi & \partial t / \partial \eta \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$  is the Jacobian of the linear transformation which sends the pair  $(\xi, \eta)$  to  $(1/2(\xi - \eta), 1/2(\xi + \eta))$ , hence we have

$$\begin{aligned} &= \frac{1}{2} \int_0^2 \int_{-1}^1 |f(\xi) - f(\eta)| d\xi d\eta \\ &= \frac{1}{2} \int_0^2 \int_{-1}^0 |f(\xi) - f(\eta)| d\xi d\eta + \frac{1}{2} \int_0^2 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta \end{aligned}$$

Here we use Theorem 3.35 to note that the translation  $\eta \mapsto \eta + 1$  and the fact that  $f$  is periodic with period 1 gives us

$$= \int_0^2 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta$$

similarly, we have

$$= 2 \int_0^1 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta.$$

Hence, the inequality

$$\int_0^1 \int_0^1 |f(\xi) - f(\eta)| d\xi d\eta \leq \frac{c}{2} \tag{9.3}$$

holds so by Problem 9.2 (§6, Ex. 3),  $|f| \in L[0, 1]$  hence,  $f \in L[0, 1]$ . ■

**PROBLEM 9.4 (WHEEDEN & ZYGMUND §6, EX. 6)**

For  $f \in L(\mathbb{R}^1)$ , define the *Fourier transform*  $\hat{f}$  of  $f$  by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt$$

for  $x \in \mathbb{R}^1$ . (For complex-valued function  $F = F_0 + iF_1$  whose real and imaginary parts  $F_0$  and  $F_1$  are integrable, we define  $\int F = \int F_0 + i \int F_1$ .) Show that if  $f$  and  $g$  belong to  $L(\mathbb{R}^1)$ , then

$$(\widehat{f * g})(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

*Proof.* By direct computation we have

$$(\widehat{f * g})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(s-t) g(t) dt \right] e^{-ixs} ds$$

now do this

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s-t) g(t) e^{-ixs} dt ds$$

make the substitution  $u = s - t$ , then the above becomes

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) g(t) e^{-ix(u+t)} dt du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-ixu} g(t) e^{-ixt} dt du \end{aligned}$$

by Fubini's theorem, this is just

$$\begin{aligned} &= 2\pi \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-ixu} du \right) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) e^{-ixt} dt \right) \\ &= 2\pi \hat{f}(x) \hat{g}(x) \end{aligned}$$

as desired. ■

**PROBLEM 9.5 (WHEEDEN & ZYGMUND §6, EX. 7)**

Let  $F$  be a closed subset of  $\mathbb{R}^1$  and let  $\delta(x) = \delta(x, F)$  be the corresponding distance function. If  $\lambda > 0$  and  $f$  is nonnegative and integrable over the complement of  $F$ , prove that the function

$$\int_{\mathbb{R}^1} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}} dt$$

is integrable over  $F$  and so is finite a.e. in  $F$ . (In case  $f = \chi_{(a,b)}$ , this reduces to Theorem 6.17.)

*Proof.* Set  $G := \mathbb{R} \setminus F$ . By assumption, we have

$$\int_{\mathbb{R} \setminus F} f(x) dx < \infty. \tag{9.4}$$

■

**PROBLEM 9.6 (WHEEDEN & ZYGMUND §6, EX. 9)**

- (a) Show that  $M_\lambda(x; F) = +\infty$  if  $x \notin F$ ,  $\lambda > 0$ .
- (b) Let  $F = [c, d]$  be a closed subinterval of a bounded open interval  $(a, b) \subset \mathbb{R}^1$ , and let  $M_\alpha$  be the corresponding Marcinkiewicz integral,  $\lambda > 0$ . Show that  $M_\lambda$  is finite for every  $x \in (c, d)$  and that  $M_\lambda(c) = M_\lambda(d) = \infty$ . Show also that  $\int M_\lambda \leq \lambda^{-1}|G|$ , where  $G = (a, b) - [c, d]$ .

*Proof.*

■