# MA 523: Homework 5

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October 4, 2016

## Problem 5.1

Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if $O$ is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$, $x \in \mathbb{R}^n$, then $\Delta v = 0$.

*Solution.* Let

$$O = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be an orthogonal $n \times n$ matrix. We will show that $\Delta v = 0$, where $v(x) = u(Ox)$.

First, let us compute the gradient of $v$:

$$\begin{aligned} Dv(x) &= Du(Ox) \\ &= Du\big(a_{11}x_1 + \cdots + a_{1n}x_n, \dots, a_{n1}x_1 + \cdots + a_{nn}x_n\big) \\ &= \Big(\sum\nolimits_{j=1}^n a_{j1}u_{x_j}, \dots, \sum\nolimits_{j=1}^n a_{jn}u_{x_j}\Big) \\ &= O^{\mathrm{T}} Du(x). \end{aligned}$$

Lastly, we compute the divergence of $Dv$:

$$\begin{aligned} \Delta v(x) &= \operatorname{div} Dv(x) \\ &= \operatorname{div}\Big(\sum\nolimits_{j=1}^n a_{j1}u_{x_j}, \dots, \sum\nolimits_{j=1}^n a_{jn}u_{x_j}\Big). \end{aligned}$$

Here the partial derivatives become unwieldy so we will first examine the partial $\frac{\partial}{\partial x_1}$ of the first term and proceed from there. In this case,

$$\begin{aligned} \frac{\partial}{\partial x_1}\sum_{j=1}^n a_{j1}u_{x_j} &= a_{11}\tfrac{\partial}{\partial x_1}u_{x_1} + a_{12}\tfrac{\partial}{\partial x_1}u_{x_2} + \cdots + a_{1n}\tfrac{\partial}{\partial x_1}u_{x_n} \\ &= a_{11}\big(a_{11}u_{x_1x_1} + a_{21}u_{x_1x_2} + \cdots + a_{n1}u_{x_1x_n}\big) \\ &\quad + \cdots + a_{1n}\big(a_{11}u_{x_1x_n} + a_{21}u_{x_2x_n} + \cdots + a_{n1}u_{x_nx_n}\big). \end{aligned}$$

Similarly, taking the $k^{\text{th}}$ partial of the $k^{\text{th}}$ entry of $Dv$, we have

$$\begin{aligned} \frac{\partial}{\partial x_k}\sum_{j=1}^n a_{jk}u_{x_j} &= a_{k1}\big(a_{1k}u_{x_1x_1} + \cdots + a_{nk}u_{x_1x_n}\big) \\ &\quad + \cdots + a_{kn}\big(a_{1k}u_{x_1x_n} + \cdots + a_{nk}u_{x_nx_n}\big). \end{aligned} \tag{5.1}$$

Now, since $O$ is orthogonal, we have

$$\begin{aligned} O^{\mathrm{T}}O &= \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}^2 + \cdots + a_{nn}^2 & & & \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \end{aligned}$$

$\blacksquare$

## Problem 5.2

Let $n = 2$ and $U$ be the halfplane $\{\, x_2 > 0 \,\}$. Prove that

$$\sup_U u = \sup_{\partial U} u$$

for $u \in C^2(U) \cap C(\bar{U})$ which are harmonic in $U$ under the additional assumption that $u$ is bounded from above in $\bar{U}$. (The additional assumption is needed to exclude examples like $u = x_2$.)
[*Hint:* Take for $\varepsilon > 0$ the harmonic function

$$u(x_1, x_2) + \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region $\left\{\, x_1^2 + (x_2+1)^2 < a_2, x_2 > 0 \,\right\}$ with large $a$. Let $\varepsilon \to 0$.]

*Solution.* ∎

## Problem 5.3

Let $U \subset \mathbb{R}^n$ be an open set. We say $v \in C^2(U)$ is subharmonic if

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be smooth and convex. Assume $u^1, \ldots, u^m$ are harmonic in $U$ and

$$v := \varphi(u_1, \ldots, u_m).$$

Prove $v$ is sub harmonic.

[*Hint:* Convexity for a smooth function $\varphi(z)$ is equivalent to $\sum_{j,k=1}^m \varphi_{z_j,z_k}(z)\xi_j\xi_j \geq 0$ for any $\xi \in \mathbb{R}^m$.]

(b) Prove $v := |Du|^2$ is subharmonic, whenever $u$ is harmonic. (Assume that harmonic functions are $C^\infty$.)

*Solution.* ■