

## MA 544: Homework 9

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March 7, 2016



**PROBLEM 9.1 (WHEEDEN & ZYGMUND §6, EX. 1)**

- (a) Let  $E$  be a measurable subset of  $\mathbf{R}^2$  such that for almost every  $x \in \mathbf{R}^1$ ,  $\{y \mid (x, y) \in E\}$  has  $\mathbf{R}^1$ -measure zero. Show that  $E$  has measure zero and that for almost every  $y \in \mathbf{R}^1$ ,  $\{x \mid (x, y) \in E\}$  has measure zero.
- (b) Let  $f(x, y)$  be nonnegative and measurable in  $\mathbf{R}^2$ . Suppose that for almost every  $y \in \mathbf{R}^1$ ,  $f(x, y)$  is finite for almost every  $x$ . Show that for almost  $y \in \mathbf{R}^1$ ,  $f(x, y)$  is finite for almost every  $x$ .

*Proof.*

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**PROBLEM 9.2 (WHEEDEN & ZYGMUND §6, EX. 3)**

Let  $f$  be measurable and finite a.e. on  $[0, 1]$ . If  $f(x) - f(y)$  is integrable over the square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , show that  $f \in L[0, 1]$ .

*Proof.*

■

**PROBLEM 9.3 (WHEEDEN & ZYGMUND §6, EX. 4)**

Let  $f$  be measurable and periodic with period 1:  $f(t+1) = f(t)$ . Suppose there is a finite  $c$  such that

$$\int_0^1 |f(a+t) - f(b+t)| dt \leq c$$

for all  $a$  and  $b$ . Show that  $f \in L[0, 1]$ . (Set  $a = x$ ,  $b = -x$ , integrate with respect to  $x$ , and make the change of variables  $\chi = x + t$ ,  $\eta = -x + t$ .)

*Proof.*

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**PROBLEM 9.4 (WHEEDEN & ZYGMUND §6, EX. 6)**

For  $f \in L(\mathbf{R}^1)$ , define the *Fourier transform*  $\hat{f}$  of  $f$  by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-ixt} dt$$

for  $x \in \mathbf{R}^1$ . (For complex-valued function  $F = F_0 + iF_1$  whose real and imaginary parts  $F_0$  and  $F_1$  are integrable, we define  $\int F = \int F_0 + i \int F_1$ .) Show that if  $f$  and  $g$  belong to  $L(\mathbf{R}^1)$ , then

$$(\widehat{f * g})(x) = 2\pi \hat{f}(x)\hat{g}(x).$$

*Proof.*

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**PROBLEM 9.5 (WHEEDEN & ZYGMUND §6, EX. 7)**

Let  $F$  be a closed subset of  $\mathbf{R}^1$  and let  $\delta(x) = \delta(x, F)$  be the corresponding distance function. If  $\lambda > 0$  and  $f$  is nonnegative and integrable over the complement of  $F$ , prove that the function

$$\int_{\mathbf{R}^1} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}} dt$$

is integrable over  $F$  and so is finite a.e. in  $F$ . (In case  $f = \chi_{(a,b)}$ , this reduces to Theorem 6.17.)

*Proof.*

■

**PROBLEM 9.6 (WHEEDEN & ZYGMUND §6, EX. 9)**

- (a) Show that  $M_\lambda(x; F) = +\infty$  if  $x \notin F$ ,  $\lambda > 0$ .
- (b) Let  $F = [c, d]$  be a closed subinterval of a bounded open interval  $(a, b) \subset \mathbf{R}^1$ , and let  $M_\alpha$  be the corresponding Marcinkiewicz integral,  $\lambda > 0$ . Show that  $M_\lambda$  is finite for every  $x \in (c, d)$  and that  $M_\lambda(c) = M_\lambda(d) = \infty$ . Show also that  $\int M_\lambda \leq \lambda^{-1}|G|$ , where  $G = (a, b) - [c, d]$ .

*Proof.*

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