

MA 523: Homework 2

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Problem 2.1

Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 because in such case we have $\nu(x^0) = (0, \dots, 0, 1)$ so

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall verify the noncharacteristic condition (2.1) by first flattening the boundary near x^0 and then applying the noncharacteristic boundary conditions to the flattened region. Assuming some degree of regularity near x^0 , e.g., that the boundary of U be smooth, we may express Γ near x^0 as the graph of a smooth function $f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, i.e., $x = (x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}))$ on Γ and $x_n \geq f(y)$ after reorienting the coordinate axes. Then we flatten out Γ via the map $\Phi(x): \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$\left\{ \begin{array}{l} y_1 = x_1 = \Phi^1(x), \\ \vdots \\ y_{n-1} = x_{n-1} = \Phi^{n-1}(x), \\ y_n = x_n - f(x_1, \dots, x_{n-1}) = \Phi^n(x) \end{array} \right.$$

and write $y = \Phi(x)$. Let $\Psi = \Phi^{-1}$ and rewrite our PDE F in terms of y as follows,

$$0 = F(Du(\Psi(y)), u(\Psi(y)), \Psi(y)). \tag{2.2}$$

Since $\Delta = \Phi(\Gamma)$ is flat near $y^0 = \Phi(x^0) = (y_1^0, \dots, y_{n-1}^0, 0)$, we may apply the standard noncharacteristic condition on (2.2) and get

$$0 \neq F_{u_{y_n}}(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)).$$

Before we move on to finding an expression for this derivative, let us consider the gradient $Du(\Psi(y))$. By the chain rule, we have

$$\begin{aligned} u_{y_i}(\Psi(y)) &= \sum_{j=1}^n u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_i} \\ &= u_{x_i}(\Psi(y)) + u_{x_n}(\Psi(y)) f_{y_i}(y_1, \dots, y_{n-1}), \\ u_{y_n}(\Psi(y)) &= \sum_{j=1}^n u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_n} \\ &= u_{x_n}(\Psi(y)), \end{aligned}$$

Then, substituting u_{y_n} for u_{x_n} , we have

$$u_{y_i}(\Psi(y)) = u_{x_i}(\Psi(y)) + u_{y_n}(\Psi(y))f_{y_i}(y_1, \dots, y_{n-1}),$$

Now, by the chain rule on (2.2), we have

$$\begin{aligned} 0 &\neq F_{u_{y_n}}(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)) \\ &= F_{u_{y_n}}(u_{x_1} + u_{y_n}f_{y_1}, \dots, u_{x_{n-1}} + u_{y_n}f_{y_{n-1}}, u_{y_n}, z^0, x^0) \\ &= F_{u_{x_1}}f_{y_1} + \dots + F_{u_{x_{n-1}}}f_{y_{n-1}} + F_{u_{x_n}} \\ &= D_p F(p^0, z^0, x^0) \cdot (Df(x^0), 1) \\ &= D_p F(p^0, z^0, x^0) \cdot \nu(x^0), \end{aligned}$$

as we set out to show. ◀

Problem 2.2

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x .

Solution. ► The characteristic ODEs of this PDE are

$$\dot{t} = 1, \quad \dot{x} = a(z), \quad \dot{z} = 0. \quad (2.3)$$

with initial conditions $t_0 = 0$, $x_0 = x(0)$ and $z(x_0, 0) = g(x_0)$ with $(x_0, 0) \in \mathbb{R} \times (0, \infty)$. Hence, we have

$$t(s) = s, \quad x(s) = a(g(x_0))s + x_0, \quad z(s) = g(x_0).$$

Thus, solving for x_0 and s in terms of t , x and z , we have

$$\begin{aligned} x &= a(g(x_0))s + x_0 \\ &= a(z)t + x_0, \end{aligned}$$

so, moving x_0 to the left-hand side

$$x_0 = x - a(z)t$$

hence,

$$z = g(x - a(z)t),$$

i.e.,

$$u = g(x - a(u)t),$$

as desired.

For the latter half of the problem, write

$$u(x + a(g(x))t, t) = g(x).$$

Suppose that $a(g(x))$ is not a nondecreasing function of x . Then, there exists $0 < x_1 < x_2$ such that $a(g(x_1)) > a(g(x_2))$. Define

$$y = -\frac{x_1 - x_2}{a(g(x_1)) - a(g(x_2))} > 0. \quad (2.4)$$

Then, we have

$$t_0 = x_1 + a(g(x_1))y = x_2 + a(g(x_2))y.$$

Thus,

$$\begin{aligned} u(x, t_0) &= g(x_1) \\ &= u(x_1 + a(g(x_1))t_0, t_0) \\ &= g(x_2) \\ &= u(x_2 + a(g(x_2))t_0, t_0). \end{aligned}$$

However, $g(x_1) \neq g(x_2)$ since $a(g(x_1)) > a(g(x_2))$. ◀

Problem 2.3

Show that the function $u(x, t)$ defined for $t \geq 0$ by

$$u(x, t) = \begin{cases} -\frac{2}{3} \left(t + \sqrt{3x + t^2} \right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burgers' equation*).

Solution. ► The shock occurs along the curve C given by $s(t) = -t^2/4$. First, we verify that the equation given by u above is in fact a solution to the inviscid Burgers' equation to the right and to the left of C : to the left of C , $4x + t^2 < 0$, the equation is trivially satisfied whereas to the right, $4x + t^2 > 0$, we have,

$$-\frac{2}{3} \left(1 + \frac{t}{\sqrt{3x + t^2}} \right) + \frac{2}{9} \left(3 + \frac{3t}{\sqrt{3x + t^2}} \right) = 0.$$

So u is indeed a solution to the inviscid Burgers' equation.

Now we examine the behavior of u along the curve C . First, we have

$$\begin{aligned} \sigma &= \dot{s}(t) \\ &= -\frac{t}{2}, \\ \llbracket u \rrbracket &= u_\ell - u_r \\ &= 0 + \frac{2}{3} \left(t + \sqrt{-\frac{3}{4}t^2 + t^2} \right) \\ &= 0 + \frac{2}{3} \left(\frac{3}{2}t \right) \\ &= t, \\ \llbracket F \rrbracket &= F(u_\ell) - F(u_r) \\ &= 0 - \frac{\llbracket u_r \rrbracket^2}{2} \\ &= 0 - \frac{t^2}{2}. \end{aligned}$$

Thus,

$$\llbracket F \rrbracket = -\frac{t^2}{2} = \left(-\frac{t}{2} \right) t = \sigma \llbracket u \rrbracket$$

satisfies the Rankine–Hugoniot condition and hence, is an integral solution.

Lastly, we verify that u satisfies the entropy condition, i.e., Eq. (17) from [E, §3.4.b], that $F'(u_\ell) > \sigma > F(u_r)$. Fix a point $x_0 \in \mathbb{R}$ and let $y > 0$. Then, if $x_0 > -t^2/4$, we have

$$\begin{aligned} u(x_0 + y, t) - u(x_0, t) &= \sup_{x > -t^2/4} \{u_x(x, t)\}y \\ &= \sup_{x > -t^2/4} \left\{ \frac{1}{\sqrt{3x + t^2}} \right\}y \\ &= \frac{2}{t}y \end{aligned}$$

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