

# MA 523: Homework 8

Carlos Salinas

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## PROBLEM 8.1

Show that the function

$$u(x, t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x - 2k, t)$$

where

$$\Phi(x, t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

is positive for  $|x| < 1$ ,  $t > 0$ .

(Hint: Show that  $u$  satisfies  $u_t = u_{xx}$  for  $t > 0$ ,

$$\begin{cases} u = 0 & \text{on } \{|x| = 1\} \times \{t \geq 0\}, \\ u = \delta_0 & \text{on } \{|x| = 1\} \times \{t = 0\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain  $\{|x| \leq 1\} \times \{\varepsilon \leq t \leq T\}$  for small  $\varepsilon > 0$  and large  $T > 0$  pass to the limit as  $\varepsilon \rightarrow 0^+$  and  $T \rightarrow \infty$ .)

*SOLUTION.* Taking the hint, let us verify that  $u_t = u_{xx}$ , for  $t > 0$ . By direct computation, we have

$$\begin{aligned} \Phi_x(x, t) &= \frac{\partial}{\partial x} \left( \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \right) & \Phi_{xx}(x, t) &= \frac{\partial}{\partial x} \left( -\frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}} \right) \\ &= -\frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}}, & &= \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}} \\ & & &= \frac{(x^2 - 2t) e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}}, \end{aligned}$$

and

$$\begin{aligned} \Phi_t(x, t) &= \frac{\partial}{\partial t} \left( \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \right) \\ &= \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}} \\ &= \frac{(x^2 - 2t) e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}}. \end{aligned}$$

Since  $\Phi_t = \Phi_{xx}$  it follows that  $u_t = u_{xx}$  (for  $t > 0$ ).

Next we show that  $u = 0$  on  $\{|x| = 1\} \times \{t \geq 0\}$  and  $u = \delta_0$  on  $\{|x| = 1\} \times \{t = 0\}$ . To show  $u = 0$  fix a  $t \geq 0$  and after relabeling, we may assume that  $x = 1$  which gives us

$$\begin{aligned} u(1, t) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-\frac{k^2}{4t}}}{\sqrt{4\pi t}} \\ &= \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} (-1)^k (e^{-\frac{1}{4t}})^{k^2} \\ &= \frac{2}{\sqrt{4\pi t}} \sum_{k=0}^{\infty} (-1)^k (e^{-\frac{1}{4t}})^{k^2} \\ &= \end{aligned}$$

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## PROBLEM 8.2 (TIKHONOV'S EXAMPLE)

Let

$$g(t) := \begin{cases} e^{-t^2} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then  $g \in C^\infty(\mathbb{R})$  and we define

$$u(x, t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that  $u(x, t)$  solves the heat equation in  $\mathbb{R} \times (0, \infty)$  with the initial condition  $u(x, 0) = 0$ ,  $x \in \mathbb{R}$ . Why doesn't this contradict the uniqueness theorem for the initial value problem?

*SOLUTION.* Let  $u$  be as above. Then

$$\begin{aligned} u_t(x, t) &= \frac{\partial}{\partial t} \left( \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}, \end{aligned}$$

and

$$\begin{aligned} u_x(x, t) &= \frac{\partial}{\partial x} \left( \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2k x^{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}, \\ u_{xx}(x, t) &= \frac{\partial}{\partial x} \left( \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1} \right) \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} (2k-1) x^{2k-2} + \frac{\partial}{\partial x} g^{(0)}(t) \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}. \end{aligned}$$

Thus,  $u_t - \Delta u = 0$ ; i.e.,  $u$  solves the heat equation. ■

## PROBLEM 8.3

Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax) e^{-x^2} dx, \quad (a > 0).$$

(*Hint:* Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.)

*SOLUTION.*

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