MA571 Problem Set 2

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Problem 2.1 (Munkres §17, p. 100, Exercise 3)

Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Proof. Before proceeding with our main result we will prove the following useful set theoretic results which we have taken (and modified) from Munkres §1, p. 14, Exercises 2(n) and 2(o):

Lemma 4. *For sets A, B, C and D we the following equalities hold:*

(a) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(b) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.
(c) $(A \setminus C) \times B = (A \times B) \setminus (C \times B)$.

that is, the Cartesian product distributes over taking complements.

Proof of Lemma 4. (a) The equality follows (rather straightforwardly) from the definition of the Cartesian product and the complement of a set for $x \times y \in (A \times B) \cap (C \times D)$ if and only if $x \times y \in A \times B$ and $x \times y \in C \times D$ if and only if $x \in A$ and $x \in C$ and $y \in B$ and $y \in D$ if and only if $x \in A \cap C$ and $y \in B \cap D$ if and only if $x \times y \in (A \cap C) \times (B \cap D)$.

(b) The point $x \times y \in A \times (B \setminus C)$ if and only if $x \in A$ and $y \in B \setminus C$ if and only if $x \in A$ and $y \in B$ and $y \notin C$ if and only if $x \times y \in A \times B$ and $x \times y \notin A \times C$ if and only if $x \times y \in (A \times B) \setminus (A \times C)$.

(c) The very same argument as part (b) can be used, taking B to be a subset of A and replacing (where appropriate) A by $A \setminus B$ and $B \setminus C$ by C, to prove that

$$(A \setminus B) \times C = (A \times C) \setminus (B \times C). \qquad \clubsuit$$

Now let's turn our attention back to the problem at hand. Since A is closed in X and B is closed in Y, their complements, $X \setminus A$ and $Y \setminus B$, are open in X and Y, respectively (this is by definition cf. Munkres §17, p. 93). Hence, the sets

$$(X \setminus A) \times Y \quad \text{and} \quad X \times (Y \setminus B)$$

are open in $X \times Y$ since they are basis elements of the product topology on $X \times Y$ (cf. definition of the product topology on Munkres §15, p. 86). Hence, their complements are closed. By Lemma 4(b) and 4(c), we may rewrite the complements of $(X \setminus A) \times Y$ and $X \times (Y \setminus B)$ as

$$(X \times Y) \setminus ((X \setminus A) \times Y) = A \times Y \quad \text{and} \quad (X \times Y) \setminus (X \times (Y \setminus B)) = X \times B,$$

respectively. Then, by Theorem 17.(b), the intersection

$$(A \times Y) \cap (X \times B)$$

is closed since $A \times Y$ and $X \times B$ are closed. At last, by Lemma 4(a), we may rewrite the former intersection as

$$(A \times Y) \cap (X \times B) = (A \cap X) \times (Y \cap B) = A \times B.$$

Thus $A \times B$ is closed in $X \times Y$. ■

Problem 2.2 (Munkres §17 p. 101, Exercise 6(b))

Let A, B and A_α denote subsets of a space X. Prove the following:

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. The containment $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ is immediate from the definition of the closure of a set (cf. Munkres §17, p. 95) since $\overline{A} \cup \overline{B}$ is a closed set (by Theorem 17.1(a)) which contains $A \cup B$, hence must contain the closure of $A \cup B$. To see the reverse containment note we will make use of the following lemma (which I was not able to immediately find in Munkres):

Lemma 5. *If $A \subset C$ and $B \subset C$ then $A \cup B \subset C$.*

Proof of Lemma 5. By the definition of subset and union (cf. Munkres §1, pp. 4-5) if $x \in A \cup B$ then $x \in A$ or $x \in B$. Since $A \subset C$ and $B \subset C$, in either case we have that $x \in C$. Thus $A \cup B \subset C$. ♣

Armed with Lemma 5, note that $A \subset \overline{A \cup B}$ and $B \subset \overline{A \cup B}$ so $\overline{A \cup B}$ contains the closure of A and B so it must contain the union of their respective closures, i.e., $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

Naturally, this result may be extended, by induction, to show that the closure of a finite union of sets is the union of the closure of said sets. ∎

Problem 2.3 (Munkres §17 p. 101, Exercise 6(c))

Let A, B and A_α denote subsets of a space X. Prove the following:

(b) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$; give an example where equality fails.

Proof. The containment $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$ follows immediately from the definition of closure since $\overline{\bigcup A_\alpha}$ is a closed set containing A_α so must contain $\overline{A_\alpha}$ for each α.

The reverse containment is not true in general (in fact, as Theorem 17.1(3) suggests, an arbitrary union of closed sets is not even necessarily closed). As a counter example, consider the family of subsets $A_q = \{q\}$, for $q \in \mathbf{Q}$, of $\mathbf{R}$. Since $\mathbf{R}$ is Hausdorff, by Theorem 17.8, the closure of A_q is itself. Hence, we see that the union

$$\bigcup_{q \in \mathbf{Q}} \overline{A_q} = \mathbf{Q},$$

but, (by Munkres §17, Example 6) $\overline{\mathbf{Q}} = \mathbf{R}$. ∎

Problem 2.4 (Munkres §17 p. 101, Exercise 7)

Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \overline{A_\alpha}$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so x must belong to the closure of some A_α. Therefore, $x \in \bigcup \overline{A_\alpha}$.

Critique. The claim is false in general as the counterexample in the preceding problem demonstrates. The main problem with this proof lies in the assertion U intersecting some A_α implies "x must belong to the closure of some A_α." But a different neighborhood of x may intersect a different A_α in the union. Recall, by Theorem 17.5(a), if x is in the closure of A_α, then $U \cap A_\alpha \neq \emptyset$ for every neighborhood U of x. That is, the proof is claiming that for every neighborhood U of x there exists some A_α in the union $\bigcup A_\alpha$ such that $U \cap A_\alpha \neq \emptyset$, i.e., $x \in \overline{A_\alpha}$. But for x to be in $\bigcup \overline{A_\alpha}$ we need that for some A_α for every neighborhood U of x, $U \cap A_\alpha \neq \emptyset$. These are not equivalent statements. ♥

Problem 2.5 (Munkres §17, p. 101, 9)

Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

Proof. By Problem 2.1, $\overline{A} \times \overline{B}$ is a closed set which contains $A \times B$ so it must contain the closure of $A \times B$, i.e., $\overline{A \times B} \subset \overline{A} \times \overline{B}$. To see the reverse containment, take a point $x \times y \in \overline{A} \times \overline{B}$. Then, by Theorem 17.5(a), for every neighborhood U of x and every neighborhood V of y, the intersections $U \cap A$ and $V \cap B$ are nonempty. Thus, by Lemma 4(a), the set

$$(V \times U) \cap (A \times B) = (V \cap A) \times (U \cap B)$$

is nonempty. Then, since $U \times V$ is an arbitrary basis element containing $x \times y$, by Theorem 17.5(b) $x \times y \in \overline{A \times B}$. Thus, $\overline{A \times B} = \overline{A} \times \overline{B}$. ∎

Problem 2.6 (Munkres §17, p. 101, 10)

Show that every order topology is Hausdorff.

Proof. Let $(X, <)$ denote a nonempty set equipped with a simple order relation. Then by the definition on Munkres §14, p. 84, a basis for the order topology on X are sets of the following types:

(1) All open intervals (a, b) in X.
(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

Let a and b be two distinct points in X; we may assume, without loss of generality, that $a < b$. Then, we must show that there exists neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$.

If X set with finite cardinality the order topology on X will coincide with the discrete topology so that we may take $\{a\}$ and $\{b\}$ to be neighborhoods of a and b. Then, $\{a\} \cap \{b\} = \emptyset$ so X is Hausdorff.

Now, suppose X is not of finite cardinality. Define the sets

$$C = (a, b), \quad A = \{ x \in X \mid x < a \} \quad \text{and} \quad B = \{ x \in X \mid x > b \}.$$

Then at least one of A, B or C is nonempty and has infinite cardinality.

Suppose A is nonempty, but B and C are empty. Take any element $x \in A$, then (x, b) is a neighborhood of a and b must be a largest element so $(a, b_0] = C \cup \{b\} = \{b\}$ is a neighborhood of b satisfying $(x, b) \cap \{b\} = \emptyset$. Similarly, if B is nonempty, but A and C are empty, $\{a\}$ and (a, x) for some $x \in B$ are neighborhoods of a and b, respectively, with $\{a\} \cap (a, x) = \emptyset$.

If C is nonempty but A and B are empty, a must be a smallest element and b must be a largest element. Then, since X is not finite, there exist at least two distinct elements x and y in C with $x < y$ so $[a, x)$ and $(y, b]$ are neighborhoods of a and b, respectively, with $[a, x) \cap (y, b] = \emptyset$.

Now, suppose at least two of A, B and C are nonempty. If C is empty, but A and B are nonempty. Then the intervals $(x, b) = (x, a]$ and $(a, y) = [b, y)$ are neighborhoods of a and b respectively with $(x, b) \cap (a, y) = \emptyset$. If A is empty, but B and C are nonempty, then a is a smallest element. Then there exists at least two distinct elements x and y with $x < y$ in C so that $[a, x)$ and (y, b) are neighborhoods of a and b, respectively, with $[a, x) \cap (y, b) = \emptyset$. Similarly, if B is empty, but A and C are nonempty, for any $x < y$ in C, (a, x) and $(y, b]$ are neighborhoods of a and b, respectively, with $(a, x) \cap (y, b]$.

Lastly, if A, B and C are nonempty we win! Then, for any $x \in A$, $y \in B$ and $z, w \in C$ with $z < w$ the intervals (x, z) and (w, y) are neighborhoods of a and b, respectively, with $(x, z) \cap (w, y) = \emptyset$.

In every case, X satisfies the Hausdorff property. ■

***Remarks**.* Perhaps there is a better way to approach this problem. The demonstration is thorough and covers every case, but we still desire a more elegant proof.

Problem 2.7 (Munkres §17, p. 101, 13)

Show that X is Hausdorff if and only if the *diagonal* $\Delta = \{ x \times x \mid x \in X \}$ is closed in $X \times X$.

Proof. $\implies$ Suppose X is Hausdorff. The diagonal Δ is closed, by definition, if and only if its complement, $(X \times X) \setminus \Delta$, is open in $X \times X$. Let $x \times y \in (X \times X) \setminus \Delta$. Since X is Hausdorff, there exists neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$. Thus, $U \times V$ is a neighborhood of $x \times y$ contained in $(X \times X) \setminus \Delta$. By the definition (cf. Munkres §13 p. 78), since for every point $x \times y \in (X \times X) \setminus \Delta$ we may find a basis element $U \times V \subset (X \times X) \setminus \Delta$ containing $x \times y$, it follows that $(X \times X) \setminus \Delta$ is open. Thus, Δ is closed.

$\impliedby$ Suppose Δ is closed. Then the complement of Δ is open in $X \times X$. Thus, for every $x \times y$ in the complement of Δ, we may find a basis element $U \times V \subset (X \times X) \setminus \Delta$ containing $x \times y$. Thus, U and V are neighborhoods of x and y, respectively, such that $U \cap V = \emptyset$ (for otherwise $z \times z \in U \times V$ but $U \times V$ is in the complement of Δ). Thus, X is Hausdorff. ■

Problem 2.8 (Munkres §18, p. 111, 4)

Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f\colon X \to X \times Y$ and $g\colon Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

Proof. Let $Z = \operatorname{im} f$. To show that $f\colon X \to X \times Y$ is an imbedding, we will show that the map $f'\colon X \to Z$, which is obtained by restricting the codomain of f is a continuous injection with a continuous inverse g. First we shall show injectivity. To see that f is continuous we note that f can be written as the tuple $f'(x) = (f_1, f_2)$ where $f_1 = \mathrm{id}_X$ and f_2 is the constant map $x \mapsto y_0$ for all $x \in X$. The maps f_1 and f_2 are continuous (by Theorem 18.2(a) and (b)) so, by Theorem 18.4, f is continuous. To prove that f is bijective it suffices to exhibit an inverse. We claim that the map $F = \pi_X | Z$ is an inverse (continuity follows of F from Theorem 18.2(d) and the fact that projections are continuous as discussed on §15 pp. 87-88). But this claim is clear since

$$\begin{aligned} F \circ f(x) &= F(f(x)) \\ &= F(x \times y_0) \\ &= x \\ &= \mathrm{id}_X(x) \end{aligned} \qquad \begin{aligned} f \circ F(x \times y_0) &= f(F(x \times y_0)) \\ &= f(x) \\ &= x \times y_0 \\ &= \mathrm{id}_Z(x \times y_0). \end{aligned}$$

Thus, f is an imbedding.

The proof that g is an imbedding is analogous (it is sufficient to replace f by g, F by G, $x \times y_0$ by $x_0 \times y$, $x \mapsto y_0$ by $y \mapsto x_0$, π_X by π_Y, and id_X by id_Y in the argument above). So as not to be penalized for not providing the proof for g we copy and paste, making the appropriate replacements, here:

Let $Z = \operatorname{im} g$. To show that $g\colon Y \to X \times Y$ is an imbedding, we will show that the map $g'\colon Y \to Z$, which is obtained by restricting the codomain of g is a continuous injection with a continuous inverse g. First we shall show injectivity. To see that g is continuous we note that g can be written as the tuple $g'(x) = (g_1, g_2)$ where $g_1 = \mathrm{id}_Y$ and g_2 is the constant map $y \mapsto x_0$ for all $y \in Y$. The maps g_1 and g_2 are continuous g is continuous. To prove that g is bijective it suffices to exhibit an inverse. We claim that the map $G = \pi_Y | Z$ is an inverse (the continuity of G follows from he fact that it is the restriction of a projection). But this claim is clear since

$$\begin{aligned} G \circ g(x) &= G(g(x)) \\ &= G(x_0 \times y) \\ &= y \\ &= \mathrm{id}_Y(y) \end{aligned} \qquad \begin{aligned} g \circ G(x_0 \times y) &= g(G(x_0 \times y)) \\ &= g(y) \\ &= x_0 \times y \\ &= \mathrm{id}_Z(x_0 \times y). \end{aligned}$$

Thus, g is an imbedding. ∎

Problem 2.9 (Munkres §18, p. 111-112, 8(a,b))

Let Y be an ordered set in the order topology. Let $f, g\colon X \to Y$ be continuous.

(a) Show that the set $\{\, x \mid f(x) \leq g(x) \,\}$ is closed in X.
(b) Let $h\colon X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [*Hint:* Use the pasting lemma.]

Proof. (a) Let $A = \{\, x \mid f(x) \leq g(x) \,\}$. To prove that A is closed, we will demonstrate that its complement,

$$X \setminus A = \{\, x \mid f(x) > g(x) \,\},$$

is open. Let $x \in X \setminus A$. Then $f(x) \neq g(x)$. By Problem 2.6, Y is Hausdorff so there exist neighborhoods U and V of $f(x)$ and $g(x)$, respectively, such that $U \cap V = \emptyset$. Without loss of generality, we may assume U and V are basis elements, i.e., $U = (x_3, x_4)$ and $V = (x_1, x_2)$. Then, since f and g are continuous (cf. Munkres §18, p. 102), the intersection $f^{-1}(U) \cap g^{-1}(V)$ in a neighborhood of x contained entirely in $X \setminus A$ (for otherwise there exists a $y \in (f^{-1}(U) \cap g^{-1}(V)) \cap A$ which simultaneously satisfies $x_1 < g(y) < x_2 < x_3 < f(y) < x_4$ and $f(y) \leq g(y)$, but this is absurd).

(b) Define the sets

$$A = \{\, x \mid f(x) \leq g(x) \,\} \quad \text{and} \quad B = \{\, x \mid f(x) \geq g(x) \,\}.$$

By part (a), A and B are closed in X. Lastly, define $f' = f|A$ and $g' = g|B$ (by Theorem 18.2(d) f' and g' are continuous). Since $f' = g'$ on $A \cap B$ (by construction), by the pasting lemma, we have that

$$h(x) = \min\{f(x), g(x)\} = \begin{cases} f'(x) & \text{if } x \in A, \\ g'(x) & \text{if } x \in B \end{cases}$$

is continuous. ∎

Problem 2.10

Given: X is a topological space with open sets $U_1, ..., U_n$ such that $\overline{U}_i = X$ for all i. Prove that the closure of $U_1 \cap \cdots \cap U_n$ is X.

Proof. ****Opening remarks**:** This property of U, that $\overline{U} = X$, is called *density* (and is not defined until Munkres §30, p. 190), but should be recognizable to anyone who has taken a course in real analysis so I don't feel any qualms about using said adjective here. At any rate, we shall proceed by induction on n the number of sets in the intersection.

Consider the base case $n = 2$: Suppose U_1 and U_2 are dense open subsets of X. Let $x \in \overline{U_1} = X$. Then, by Theorem 17.5(a), for any neighborhood U of x, $U \cap U_1 \neq \emptyset$. In particular, note that $U \cap U_1$ is open since it is a finite intersection of open sets (cf. Munkres §13 definition of topology). Let $y \in U \cap U_1$. Then, since $y \in \overline{U_2}$ and $U \cap U_1$ is a neighborhood of y, we have that

$$(U \cap U_1) \cap U_2 = U \cap (U_1 \cap U_2) \neq \emptyset.$$

Hence, x is in the closure of $U_1 \cap U_2$ for any $x \in X$ so $\overline{U_1 \cap U_2} = \emptyset$.

Suppose the property holds for he intersection of $n - 1$ such open dense sets. Suppose $U_1, ..., U_n$ are open dense subsets in X. Let $U' = \bigcap_{i=1}^{n-1} U_i$. Then, by the induction hypothesis, U' is an open set with $\overline{U'} = X$. Again, as in the base case, we have $U' \cap U$ is the intersection of open dense subsets of X so

$$\overline{U' \cap U} = X = \overline{U_1 \cap \cdots \cap U_{n-1} \cap U_n}.$$

■