MA 598 (Algebraic Geometry): Homework 1

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PROBLEM 1.1 (5-LEMMA)

Given a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0,
\end{array}
$$

suppose f and h are isomorphisms. Prove that g is an isomorphism.

SOLUTION.   ∎

Problem 1.2 (Snake lemma)

Given the diagram with exact rows

$$
\begin{array}{ccccc}
\operatorname{Ker} f & \longrightarrow & \operatorname{Ker} g & \longrightarrow & \operatorname{Ker} h \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Coker} f & \longrightarrow & \operatorname{Coker} g & \longrightarrow & \operatorname{Coker} h,
\end{array}
$$

show that the sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} g \longrightarrow \operatorname{Coker} h \longrightarrow 0$$

is exact.

Solution.   ∎

PROBLEM 1.3

Given an exact sequence of modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

fix any $M \in R\text{-}\mathbf{Mod}$.

(a) Show that
$$0 \longrightarrow \operatorname{Hom}(M, A) \longrightarrow \operatorname{Hom}(M, B) \longrightarrow \operatorname{Hom}(M, C)$$
is exact.

(b) Show that
$$0 \longrightarrow \operatorname{Hom}(C, M) \longrightarrow \operatorname{Hom}(B, M) \longrightarrow \operatorname{Hom}(A, M)$$
is exact.

SOLUTION.   ∎

PROBLEM 1.4

A short exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{p} 0$$

splits if there is a homomorphism $s \colon C \to B$ called the *splitting* such that $p \circ s = \mathrm{id}_C$. In which case, we can put

$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C) \longrightarrow 0.$$

SOLUTION.   ■

Problem 1.5

Find an example which shows that $\mathrm{Hom}(-, M)$ is *not* exact.

Solution. Consider the following example: let $A = B = M = \mathbf{Z}$ and $C = \mathbf{Z}/2\mathbf{Z}$ then the sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\ f=2\ } \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0$$

is exact, however Hom   ∎