MA 54400-MIDTERM 1 PRACTICE PROBLEMS

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1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$, and define the set $rE = \{rx | x \in E\}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Consider the map $T : \mathbb{R}^n \to \mathbb{R}^n$, $T(x) = rx$. Then T is a linear transformation of $\mathbb{R}^n$, and $|\det T| = |r|^n$.

$E \subset \mathbb{R}^n$ is a measurable subset, then by theorem (3.33), $rE = T(E)$ is also measurable. Moreover, $|rE| = |T(E)| = |\det T||E| = |r|^n|E|$ by theorem (3.35). □

2. Let $\{E_k\}$, $k \in \mathbb{N}$, be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \cup_{k=1}^{\infty}(\cap_{n=k}^{\infty} E_n).$$

Show that

$$|\liminf_{k\to\infty} E_k| \le \liminf_{k\to\infty} |E_k|.$$

State and prove an analogous result for $\limsup_{k\to\infty} E_k$.

Proof. Because

$$\cap_{n=k}^{\infty} E_n \subset E_k, \text{for } k = 1, 2, \dots, \tag{1}$$

we have

$$|\cap_{n=k}^{\infty} E_n| \le |E_k|, \text{for } k = 1, 2, \dots.$$

Observe that

$$\cap_{n=k}^{\infty} E_n \nearrow \cup_{k=1}^{\infty}(\cap_{n=k}^{\infty} E_n)(= \liminf_{k\to\infty} E_k)$$

Hence by theorem (3.26(i)) and (**??**),

$$|\liminf_{k\to\infty} E_k| = \lim_{k\to\infty} |\cap_{n=k}^{\infty} E_n| \le \liminf_{k\to\infty} |E_k|.$$

Analogous result for $\limsup_{k\to\infty} E_k$:

Let $\{E_k\}$, $k \in \mathbb{N}$, be a collection of measurable sets. Define the set

$$\limsup_{k\to\infty} E_k = \cap_{k=1}^{\infty}(\cup_{n=k}^{\infty} E_n).$$

If $|\cup_{n=k_0}^{\infty} E_n| < +\infty$, for some $k_0 \in \mathbb{N}$, then

$$|\limsup_{k\to\infty} E_k| \ge \limsup_{k\to\infty} |E_k|.$$

proof of this result:

$$\cup_{n=k}^{\infty} E_n \supseteq E_k, \text{for } k = 1, 2, \dots,$$

implies:

$$|\cup_{n=k}^{\infty} E_n| \ge |E_k|, \text{for } k = 1, 2, \dots.$$

Since
$$\cup_{n=k}^{\infty} E_n \searrow \cap_{k=1}^{\infty} (\cup_{n=k}^{\infty} E_n) (= \limsup_{k\to\infty} E_k),$$
and
$$|\cup_{n=k_0}^{\infty} E_n| < +\infty, \text{for some } k_0 \in \mathbb{N},$$
we can apply theorem (3.26(ii)) and get the conclusion.

□

3. Let $E \subset \mathbb{R}^n$ be a measurable set, with $|E| = \infty$. Show that for any $C > 0$ there exists a measurable set $f \subset E$ such that $C < |F| < \infty$.

Proof. Let $E_n = E \cap B(0, n)$ for $n = 1, 2, \ldots$, where $B(0, n)$ is the ball in $\mathbb{R}^n$ centered at 0 with radius n. Then each E_n is measurable, $|E_n| < \infty$ and $E_n \nearrow E$.

By theorem (3.26(i)), $\lim_{n\to\infty} |E_n| = |E| = \infty$. Hence for any $C > 0$, there exists $n_0 \in \mathbb{N}$, such that $C < |E_{n_0}| < \infty$.

Let $F = E_{n_0}$.

□

4. Consider the function:
$$F(x) = \begin{cases} |B(0,x)| & x > 0 \\ 0 & x = 0 \end{cases}$$

Here $B(0, x) = \{x \in \mathbb{R}^n \| x| < r\}$.

Prove that F is monotone increasing and continuous.

Proof. First we claim: For any $r > 0$, $rB(0, 1) = B(0, r)$, where $rB(0, 1) = \{rx : x \in B(0, 1)\}$.

Proof of the claim: Given $y \in B(0, r)$, $y = r\frac{y}{r}$ with $|\frac{y}{r}| = \frac{|y|}{r} < 1$ (Here $|\cdot|$ denotes the Euclidean distance from a point in $\mathbb{R}^n$ to the origin). Hence $y \in rB(0, 1)$. On the other side, given $rx \in rB(0, 1)$ with $x \in B(0, 1)$, $|rx| = r|x| < r$, hence $rx \in B(0, r)$.

Apply the result from problem 1, $|B(0, r)| = r^n|B(0, 1)|$. (Here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$.) Hence, for any $x > 0$, $F(x) = |B(0, x)| = x^n|B(0, 1)| = Cx^n$, where $C = |B(0, 1)|$ is a constant. Since $F(0) = 0$ by definition, we have $F(x) = Cx^n$ on $[0, \infty)$. x^n for $n \in \mathbb{N}$ is monotone increasing and continuous on $[0, \infty)$, hence the same holds for F. □

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. Given $x_0 \in \mathbb{R}$, let $m_{x_0}(\delta) = \sup_{x,y\in(x_0-\delta,x_0+\delta)} |f(x) - f(y)|$ for $\delta \in (0, 1)$, and $w(x_0) = \lim_{\delta\to 0^+} m_{x_0}(\delta)$, which makes sense because $m_{x_0}(\delta)$ is nondecreasing on $(0, 1)$.

Claim: $E_t = \{x \in \mathbb{R} | w(x) < t\}$ is an open set in $\mathbb{R}$ for any $t > 0$.

Proof of the claim: For any $x \in E_t$, by the definition of the oscillation function w, there exists $\delta_x > 0$ such that for any $0 < \delta \le \delta_x$, $m_x(\delta) < t$. If we can show $(x - \delta_x, x + \delta_x) \subset E_t$, then we are done. In fact, for any

$x' \in (x - \delta_x, x + \delta_x)$, when δ' is small enough, say $0 < \delta' < \min\{|x + \delta_x - x'|, |x' - x - \delta_x|, 1\}$, we have $(x' - \delta', x' + \delta') \subset (x - \delta_x, x + \delta_x)$. Hence $m_{x'}(\delta') \leq m_x(\delta_x) < t$ by the definition of $m_{x'}$. Hence $w(x') = \lim_{\delta' \to 0^+} m_{x'}(\delta') < t$, which means $x' \in E_t$.

Notice that f is continuous at $x \iff x \in \cap_{n=1}^{\infty} E_{\frac{1}{n}}$. Hence $C = \{x \in \mathbb{R} | f \text{ is continuous at } x\} = \cap_{n=1}^{\infty} E_{\frac{1}{n}}$ is a set of type G_δ. □

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{f = r\}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

It is not true.

Proof. Take the nonmeasurable set $E \subset (0, 1)$ by corollary (3.39). Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & x \in (\mathbb{R} \setminus (0,1)) \cup E \\ x + 10 & x \in (0,1) \setminus E \end{cases}$$

Since for all $r \in \mathbb{R}$, $\{f = r\}$ contains at most two real numbers, we have $|\{f = r\}| = 0$, which is measurable. But f is not measurable, because it's not hard to see that $\{0 < f < 1\} = E$, which is not measurable. □

7. Let $\{f_k\}$ be a sequence of measurable function on $\mathbb{R}$. Prove that the set $\{x | \exists \lim_{k \to \infty} f_k(x)\}$ is measurable.

Proof. $\{f_k\}$ be a sequence of measurable function on $\mathbb{R}$, then by theorem (4.12) $\limsup_{k \to \infty} f_k$ and $\liminf_{k \to \infty} f_k$ are also measurable. Hence by theorem (4.7), $\{\limsup f_k > \liminf f_k\}$ is measurable. Since

$$\{x | \exists \lim_{k \to \infty} f_k(x)\} = \mathbb{R} \setminus \{\limsup f_k > \liminf f_k\},$$

then it's also measurable. (Notice, here we did not exclude the case when $\lim f_k = +\infty$ or $\lim f_k = -\infty$, and we use the fact that $+\infty > -\infty$.) □