MA544: Qual Problems

Carlos Salinas

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0.1 Final Exam Review

Material covered since exam 2.

If f is a Riemann integrable function on an interval $[a, b]$ in $\mathbb{R}$, then the familiar definition of its indefinite integral is

$$F(x) = \int_a^x f(y)dy, \qquad a \leq x \leq b.$$

The fundamental theorem of calculus asserts that $F' = f$ if f is continuous. We will study an analogue of this result for Lebesgue integrable f and higher dimensions.

We must first find an appropriate definition of the indefinite integral. In two dimensions, for example, we might choose

$$F(x_1, x_2) = \int_{a_1}^{x_1} \int_{a_2}^{x_2} f(y_1, y_2)dy_1dy_2.$$

It turns out, however, to be better to abandon the notion that the indefinite integral be a function of point and adopt the idea that it be a function of set. Thus, given $f \in L(A)$, where A is a measurable subset of $\mathbb{R}^n$, we define the *indefinite integral of* f to be the function

$$F(E) = \int_E f,$$

where E is any measurable subset of A.

F is an example of a *set function*, by which we mean any real-valued function F defined on a σ-algebra Σ of measurable sets such that

(i) $F(E)$ is finite for every $E \in \Sigma$.

(ii) F is *countably additive*; that is, if $E = \bigcup_k E_k$ is a union of disjoint $E_k \in \Sigma$, then

$$F(E) = \sum_k F(E_k).$$

By Theorem 5.5 and 5.24, the indefinite integral of $f \in L(A)$ satisfies (i) and (ii) for the σ-algebra of measurable subsets of A.

Recall that the diameter of a set E is the value

$$\sup\{\, \|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in E \,\}.$$

A set function $F(E)$ is called *continuous* if $F(E)$ tends to zero as the diameter of E tends to zero; i.e., $F(E)$ is continuous if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|F(E)| < \varepsilon$ whenever the diameter of E is less than δ. An example of a function that is *not* continuous can be obtained by setting $F(E) = 1$ for any measurable set that contains the origin, and $F(E) = 0$ otherwise.[1]

A set function F is called *absolutely continuous with respect to the Lebesgue measure*, or simply *absolutely continuous* if $F(E)$ tends to zero as the measure of E tends to zero. Thus, F is absolutely

[1] Why is this function not continuous. Consider the following argument: Let $\varepsilon = 1/2$ and let $B_k = B(\mathbf{0}, 1/k)$. Then as the diameter of B_k goes to zero, $F(B_k) = 1$ for all k so $F(B_k) \to 1 > 1/2$.

continuous if given a $\varepsilon > 0$ there exists $\delta > 0$ such that $|F(E)| < \varepsilon$ whenever the measure of E is less than δ.

A set function that is absolutely continuous is clearly continuous[2], however, the converse is false, as shown in the following example. Let A be the unit square in $\mathbb{R}^2$, let D be the diagonal of A, and consider the σ-algebra of measurable subsets E of A for which $E \cap D$ is linearly measurable. For such E, let $F(E)$ be the linear measure of $E \cap D$. Then F is a continuous set function. However, it is not absolutely continuous since the sets E containing a fixed segment of D whose $\mathbb{R}^2$-measures are arbitrarily small.

Theorem 1 (7.1)**.** *If $f \in L(A)$, its definite integral is absolutely continuous.*

Proof. We may assume that $f \geq 0$ by considering f^+ and f^-. Fix k and write $f = g+h$, where $g = f$ whenever $f \leq k$ and $g = k$ otherwise. Given $\varepsilon > 0$, we may choose k so large that $0 \leq \int_A h < \varepsilon/2$ and, *a fortiori*, $0 \leq \int_E f < \varepsilon/2$. Since

$$\int |f - C| \leq \int |f - f_{k_0}| + \int |f_{k_0} - C| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

we see that f has property $\mathcal{A}$.

To prove the lemma, let $f \in L(\mathbb{R})$. Writing $f = f^+ - f^-$, we may assume that $f \geq 0$. Then

$$\int |\chi_G - \chi_E| = |G \setminus E| < \varepsilon.$$

so we may assume that $f = \chi_G$ for open set G of finite measure. Using Theorem 1.11, write G as the union of (partly open) disjoint intervals $G = \bigcup I_k$. If we let f_N be the characteristic function of $\bigcup_{k=1}^N I_k$, we obtain

$$\int |f - f_N| = \sum_{k=N+1}^{\infty} |I_k| \to 0$$

since $\sum_k |I_k| = |G| < \infty$, i.e., the series converges. By (2), it is enough to show that each f_N has property $\mathcal{A}$. But f_N is the sum χ_{I_k}, $k = 1, \ldots, N$, so it suffices by (1) to show that the characteristic function of any partly open interval I has property $\mathcal{A}$. This is practically self-evident: if I^* denotes an interval that contains the closure of I in its interior and that satisfies $|I^* \setminus I| < \varepsilon$, then for any continuous C, $0 \leq C \leq 1$, which is 1 in I and 0 outside I^*, we have

$$\int |\chi_I - C| \leq |I^* - I| < \varepsilon.$$

∎

Theorem 2 (Simple Vitali lemma)**.** *Let E be a subset of $\mathbb{R}^n$ with $|E|_e < \infty$, and let K be a collection of cubes Q covering E, then there exists a positive constant β, depending only on n, and a finite number of disjoint cubes, $Q_1, \ldots, Q_N$ in K such that*

$$\sum_{j=1}^{N} |Q_j| \geq \beta |E|_e$$

[2]Suppose F is absolutely continuous. Then, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|F(E)| < \varepsilon$ whenever $|E| < \delta$.

As an application of Vitali's covering lemma, we will prove some basic result concerning the differentiability of monotone functions on $\mathbb{R}$. If $f(x)$ is a real-valued function defined and finite in a neighborhood of x_0, consider the four *Dini numbers* (or *derivatives*),

$$D_1 f(x_0) \coloneqq \limsup_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h},$$
$$D_2 f(x_0) \coloneqq \liminf_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h},$$
$$D_3 f(x_0) \coloneqq \limsup_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h},$$
$$D_4 f(x_0) \coloneqq \liminf_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h}.$$

Clearly, $D_2 f \le D_1 f$ and $D_4 f \le D_3 f$. If all four Dini numbers are equal, that is, if $\lim_{h\to 0}[f(x_0+h)-f(x_0)]/h$ exist, finite or infinite, we say that f has a *derivative at* x_0 and call the value the *derivative* $f'(x_0)$ at x_0. Thus, $-\infty \le f'(x_0) \le \infty$ if $f'(x_0)$ exists.

Theorem 3 (7.21)**.** *Let f be monotone increasing function and finite on an open interval $(a,b)\subset\mathbb{R}$. Then f has a measurable, nonnegative, finite derivative f' a.e. in (a,b). Moreover,*

$$0 \le \int_a^b f' \le f(b-) - f(a+).$$

Corollary 4 (7.23)**.** *If f is of bounded variation on $[a,b]$, then f' exists a.e. in $[a,b]$, and $f' \in L[a,b]$.*

Theorem 5 (7.24)**.** *If f is of bounded variation on $[a,b]$ and $V(x)$ is the variation of f on $[a,b]$, $a\le x\le b$, then*

$$V'(x) = |f'(x)|$$

for a.e. $x\in[a,b]$.

Lemma 6 (7.25, Fubini)**.** *Let $\{f_k\}$ be a sequence of monotone increasing functions on $[a,b]$. If the series $s(x)=\sum f_k(x)$ converges on $[a,b]$, (equivalently, if $s(a)$ and $s(b)$ are finite), then*

$$s'(x) = \sum f_k'(x)$$

a.e. in $[a,b]$. In particular, $f_k' \to 0$ a.e. in $[a,b]$.

The Cantor–Lebesgue function is an example of an increasing function f whose derivative is integrable on $[0,1]$, but for which $\int_0^1 f' \ne f(1)-f(0)$.

A function on a finite interval $[a,b]$ is said to be *absolutely continuous* on $[a,b]$ if given $\varepsilon>0$, there exists $\delta>0$ such that for any collection $\{[a_i,b_i]\}$ (finite or not) of nonoverlapping subintervals of $[a,b]$,

$$\sum |f(b_i)-f(a_i)| < \varepsilon$$

if $\sum(b_i-a_i)<\delta$.

E.g., if f is integrable on $[a,b]$ and $f(x)=\int_a^x g(x)$ and $a\le x\le b$, then

$$\sum |f(b_i)-f(a_i)| \le \int_{\bigcup[a_i,b_i]} |g|$$

for any overlapping $[a_i, b_i]$. By Theorem 7.1, $\int_E |g|$ is an absolutely continuous *set function* let

Theorem 7 (7.27)**.** *If f is absolutely continuous on $[a,b]$, then it is of bounded variation on $[a,b]$.*

Proof. Choose δ so that $\sum |f(b_i) - f(a_i)| \leq 1$ for any collection of nonoverlapping intervals with $\sum(b_i - a_i) \leq \delta$. Then the variation of f over any subinterval of $[a,b]$ with length less than δ is at most 1. ■

Skip to chapter 8, what we're doing right now.

If E is a measurable subset of $\mathbb{R}^n$ and p satisfies $0 < p < \infty$, then $L^p(E)$ denotes the collection of measurable f for which $\int_E |f|^p$ is finite, that is,

$$L^p(E) := \left\{ f : \int_E |f|^p < \infty \right\}, \quad 0 < p < \infty.$$

Here, f is may be complex-valued. In this case, if $f = f_1 + i f_2$ for measurable real-valued f_1 and f_2, we have $|f|^2 = |f_1|^2 + |f_2|^2$

We shall write

$$\|f\|_{p,E} := \left(\int_E |f|^p \right)^{1/p}, \quad 0 < p < \infty;$$

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1.1 Exam 1 Prep

Problem 1.1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{ r\mathbf{x} : \mathbf{x} \in E \}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Define a linear map $T \colon \mathbb{R}^n \to \mathbb{R}^n$ by $\mathbf{x} \mapsto r\mathbf{x}$. Using the standard basis for $\mathbb{R}^n$, this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \tag{1}$$

which has determinant $\det T = r^n$. By 3.35, we have $|E| = |T(E)| = r^n |E| = |rE|$. ∎

Problem 1.2. Let $\{E_k\}$, $k \in \mathbb{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k\to\infty} E_k \right| \le \liminf_{k\to\infty} |E_k|.$$

Proof. If the $\liminf |E_k| = \infty$ the inequality holds trivially. Hence, we may, without loss of generality, assume that $\liminf |E_k| < \infty$. By 3.20, the set $\liminf E_k$ is measurable and we have

$$\left| \liminf_{k\to\infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \tag{2}$$

where $F_k = \bigcap_{n=k}^{\infty} E_n$. Now, note that the collection of sets $F'_k = \bigcup_{\ell=1}^{k} F_\ell$ forms an increasing sequence of measurable sets $F'_k \nearrow F'$, where $F' = \bigcup_{k=1}^{\infty} F_k = \liminf E_k$. Then, by 3.26 (i), we have

$$\lim_{k\to\infty} |F'_k| = |F'| = \left| \liminf_{k\to\infty} E_k \right|. \tag{3}$$

Hence, it suffices to show that $|F'_k| \le |E_k|$ for all k, but this follows by monotonicity of the outer measure, 3.3, since $F'_k \subset E_k$. Thus, we have the desired inequality

$$\left| \liminf_{k\to\infty} E_k \right| \le \liminf_{k\to\infty} |E_k|. \tag{4}$$

∎

Problem 1.3. Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) = \{ \mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r \}$. Prove that F is monotonic increasing and continuous.

Proof. That F is increasing is immediate from the monotonicity of the outer measure since for $x < x'$ we have $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$ so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \leq |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that F is continuous, we will prove the following lemma

Lemma 8. *For any $x > 0$, $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$.*

Proof of lemma. If $\mathbf{y} \in xB(\mathbf{0}, 1)$ then $\mathbf{y} = x\mathbf{y'}$ for $\mathbf{y'} \in B(\mathbf{0}, 1)$. Thus, $|\mathbf{y'}| = |\mathbf{y}|/x < 1$ so $|\mathbf{y}| < x$ implies that $\mathbf{y} \in B(\mathbf{0}, x)$. Hence, we have the containment $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$.

On the other hand, if $\mathbf{y} \in B(\mathbf{0}, x)$ then $|\mathbf{y}| < x$ so $|\mathbf{y}/x| < 1$. Hence, $\mathbf{y}/x \in B(\mathbf{0}, 1)$ so $x(\mathbf{y}/x) = \mathbf{y} \in B(\mathbf{0}, x)$. Thus, $B(\mathbf{0}, x) \subset xB(\mathbf{0},)$ and equality holds. ♣

In light of Lemma 8 and 3.35, for $x > 0$, we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n|B(\mathbf{0}, 1)|. \tag{5}$$

It is clear that F is continuous on the interval $[0, \infty)$ since F is a polynomial in x. ■

Problem 1.4. Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. From the topological definition of continuity, f is continuous at $x \in C$ if and only if for every neighborhood U of $f(x)$, the preimage $f^{-1}(U)$ is a neighborhood of x. Now, ■

Let $x \in C$. Then, by the definition of continuity, for every natural number $n > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f(x) - f(x')| < \frac{1}{2n}. \tag{6}$$

Let $x'', x' \in B(x, \delta)$. Then, by the triangle inequality, we have

$$\begin{aligned}
|f(x') - f(x)''| &= |f(x') - f(x) - (f(x'') - f(x))| \\
&\leq |f(x') - f(x)| + |f(x'') - f(x)| \\
&< \frac{1}{2n} + \frac{1}{2n} \\
&= \frac{1}{n}.
\end{aligned} \tag{7}$$

In view of these estimates, define the set

$$A_n = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \tag{8}$$

Good Lord, that was a long definition! We claim that $C = \bigcap_{n=1}^{\infty} A_n$ and that A_n is open for all n.

First, let us show that $C = \bigcap_{n=1}^{\infty} A_n$. Let $x \in C$. Then for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Thus, $x \in A_n$ for all n so $x \in \bigcap A_n$. On the other hand, if $x \in \bigcap A_n$ for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$.

Fix $\varepsilon > 0$. By the Archimedean principle, there exists $N > 0$ such that $\varepsilon > 1/N$. Then, since $x \in A_N$ it follows that for some $\delta' > 0$, $|x - x'| < \delta'$ implies $|f(x) - f(x')| < 1/N < \varepsilon$. Thus, $x \in C$ and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$.

Lastly, we show that A_n is open. Let $x \in A_n$. Then there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. In particular, this means that $B(x, \delta) \subset A_n$ for any $x' \in B(x, \delta)$ satisfies $|f(x) - f(x')| < 1/n$. Thus, A_n is open and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$ is a G_δ set.

Problem 1.5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Proof. No. Recall that, by definition, or 4.1, f is measurable if and only if $\{ f > a \}$ for all $a \in \mathbb{R}$. ∎

Problem 1.6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set $\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \}$ is measurable.

Proof. The idea here should be to rewrite

$$E = \left\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \right\} \tag{9}$$

as a countable union/intersection of measurable sets. Let $x \in E$. By the Cauchy criterion, for every $N > 0$ there exists a positive integer M such that $m, n \geq M$ implies $|f_n(x) - f_m(x)| < 1/N$. With this in mind, define

$$E_N = \left\{ x : \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \tag{10}$$

Then, like for Problem 1.4, it is not too hard to see that the E_n's are open and that $E = \bigcap_{n=1}^{\infty} E_n$. Thus, E is a G_δ set and therefore measurable. ∎

Problem 1.7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, one has $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. Suppose $f\colon [a, b] \to \mathbb{R}$ is absolutely continuous. Then for fixed $\varepsilon = 1$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, we have $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Let $\Gamma = \{x_k\}_{k=1}^{N}$ be a partition of $[a, b]$ into closed intervals such that $x_{k+1} - x_k < \delta$, then by absolute continuity we have

$$\begin{aligned} V[f; \Gamma] &= \sum_{k=1}^{N} |f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \tag{11}$$

Thus, f is b.v. on $[a, b]$. ∎

Problem 1.8. Let f be a continuous function from $[a, b]$ into $\mathbb{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, i.e., $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(a) & \text{if } c = b \end{cases} .$$

Proof. ■

2 Exam 1

2.1 Exam 2 Prep

Problem 2.1. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B_\varepsilon(\mathbf{0})$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x}) d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f \colon \mathbb{R}^n \to \mathbb{R}$ is (Lebesgue) integrable over a subset E of $\mathbb{R}^n$ (or, alternatively, f belongs to $L(E)$) if

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty.$$

Put $E = \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$. Then, to show that f belongs to $L(E)$ it suffices to prove the inequality

$$\int_E f(\mathbf{x}) d\mathbf{x} < \frac{C}{\varepsilon} \tag{12}$$

for some appropriate constant C. We proceed by directly computing the Lebesgue integral of f and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x}) d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{({x_1}^2 + \cdots + {x_n}^2)^{(n+1)/2}} \end{aligned}$$

let E_i denote the projection of E onto its i-th coordinate and make the trigonometric substitution $x_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \tan\theta$, $dx_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \sec^2\theta d\theta$ with $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$ giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[\frac{\cos^{n-1}\theta}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1}\theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1}\theta d\theta < \infty. \tag{13}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned}\int \cdots \int_E f(\mathbf{x}) d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{{x_n}^2} \\ &= 2C' \int_\varepsilon^\infty \frac{dx_n}{{x_n}^2} \\ &= \frac{C}{\varepsilon}\end{aligned} \tag{14}$$

as desired. ■

Problem 2.2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit f of $\{f_k\}$ must be nonnegative a.e. in $\mathbb{R}^n$. For assume otherwise. Then there exists a collection of points $\mathbf{x}$ in $\mathbb{R}^n$ of nonzero $\mathbb{R}^n$-Lebesgue measure such that $f(\mathbf{x}) < 0$. But $f_k(\mathbf{x}) \geq 0$ for all $k \in \mathbb{N}$. Set $0 < \varepsilon < |f(\mathbf{x})|$ then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{15}$$

for all k which contradicts our assumption that $f_k \to f$ a.e. on $\mathbb{R}^n$. Therefore, the set of points $\mathbf{x} \in \mathbb{R}^n$ where $f(\mathbf{x}) < 0$ must have measure zero.

Now, based on pointwise convergence a.e. to f, given $\varepsilon > 0$ for a.e. $\mathbf{x} \in \mathbb{R}^n$ we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{16}$$

for sufficiently large k; say k greater than or equal to some index $N \in \mathbb{N}$. Moreover, we are given convergence in $L(\mathbb{R}^n)$ of f_k to f

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f < \infty. \tag{17}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{18}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{19}$$

for all $k \in \mathbb{N}$. By Theorem 5.5(ii), f and the f_k's are finite a.e. in $\mathbb{R}^n$ so for some sufficiently large real number M, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in \mathbb{R}^n$. In particular, for any measurable subset E of $\mathbb{R}^n$, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in E$ so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \to \int_E f < \infty. \tag{20}$$

However, if f does not belong to $L(\mathbb{R}^n)$, i.e., its integral over $\mathbb{R}^n$ is infinity, there is no guarantee that f will be finite a.e. in $\mathbb{R}^n$. This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset E of $\mathbb{R}^n$. Let us demonstrate this with an example. Consider the sequence of functions ■

Problem 2.3. Assume that E is a measurable set of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of $\mathbb{R}^n$, then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \tag{21}$$

Set $E_k = \{\, \mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k \,\}$ and $F_k = \{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{22}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \tag{23}$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \tag{24}$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (24) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \tag{25}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (26) \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \qquad (27) \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned}$$

Thus, from (26) and (27)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \qquad (28)$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ∎

Problem 2.4. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. $\implies$: First note that ρ is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that $\rho(f_k, f) \to 0$ as $k \to \infty$. Then, given $\varepsilon > 0$ there exist an sufficiently large index N such that for every $k \geq N$ we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \tag{29}$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in E which happens if $|f_k - f| = 0$ a.e. in E.

$\Longleftarrow$: Suppose that $f_k \to f$ as $k \to \infty$.

I don't know how to solve this. This is the intended solution:

$\Longrightarrow$: Given $\varepsilon > 0$, $\rho(f_k, f) \to 0$ implies that

$$\int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f} dx \longrightarrow 0.$$

Observe that the function $\Phi \colon \mathbb{R}^+ \to \mathbb{R}$ given by $\Phi(x) = x/(1+x)$ is increasing on $\mathbb{R}^+$ and $0 < \Psi(x) < 1$, hence

$$\begin{aligned}\int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1 + \varepsilon} |\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}|.\end{aligned}$$

Therefore,

$$|\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}| \leq \frac{1 + \varepsilon}{\varepsilon} \int_{\{ x \in E : |f_k(x) - f(x)| > \varepsilon \}} \frac{|f_k - f|}{1 + |f_k - f|} dx \longrightarrow 0$$

as $k \to \infty$.

$\Longleftarrow$: Conversely, given $\delta > 0$, we have

$$\begin{aligned}\rho(f_k, f) = &\int_{\{ x \in E : |f_k(x) - f(x)| > \delta \}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &+ \int_{\{ x \in E : |f_k(x) - f(x)| \leq \delta \}} \frac{|f_k - f|}{1 + |f_k - f|} dx \\ &\leq |\{ x \in E : |f_k(x) - f(x)| > \delta \}| + \frac{\delta}{1 + \delta} |E|.\end{aligned}$$

Since $|E| < \infty$ and $\delta/(1 + \delta) \searrow 0$, then for any $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\frac{\delta'}{1 + \delta'} |E| < \frac{\varepsilon}{2}.$$

If $f_k \to f$ as $k \to \infty$ in measure, then for the above δ' there is an index $N > 0$ such that $k \geq N$ implies

$$|\{ x \in E : |f_k(x) - f(x)| > \delta' \}| < \frac{\varepsilon}{2}.$$

Therefore, $f_k \to f$ in measure implies $\rho(f_k, f) \to 0$ as $k \to \infty$. ∎

Problem 2.5. Define the *gamma function* $\Gamma\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function* $\beta\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. (a) Fix $y \in \mathbb{R}^+$. Then we must show that $\Gamma(y) < \infty$. First, since $(0,1)$ and $[1,\infty)$ are disjoint measurable subsets of $\mathbb{R}$, by Theorem 5.7 we can split the integral $\Gamma(y)$ into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u}u^{y-1}du}_{I_1} + \underbrace{\int_1^\infty e^{-u}u^{y-1}du}_{I_2}. \tag{30}$$

We will show, separately, that I_1 and I_2 are finite.

To see that I_1 is finite, note that

$$\begin{aligned} e^{-u}u^{y-1} &= e^{-u}e^{(y-1)\log u} \\ &= e^{-u+(y-1)\log u} \\ &\leq e^{(y-1)\log u} \\ &= u^{y-1} \end{aligned} \tag{31}$$

since $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u}u^{y-1}du \\ &\leq \int_0^1 u^{y-1}du \\ &= \left[\frac{u^y}{y}\right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \tag{32}$$

To see that I_2 is finite, note that

$$e \tag{33}$$

Intended solution:

(b) ∎

Problem 2.6. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}} \colon \mathbb{R}^n \to \mathbb{R}$ be $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \to \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{34}$$

■

Problem 2.7. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \leq g_k$ and

$$\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \to 0 \qquad \text{as} \qquad k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \to \int_{\mathbb{R}^n} |f| \qquad \text{as} \qquad k \to \infty.$$

Proof. (a) Since $f_k \to f$ and $g_k \to g$ a.e. and $|f_k| \leq g_k$, then by Fatou's theorem,

$$\int_{\mathbb{R}^n} (g - f) = \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k - f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k - f_k,$$
$$\int_{\mathbb{R}^n} g + f \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k + f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k + f_k.$$

Since $f_k, g_k, f, g \in L(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$, then using the similar argument as problem 2, we have

$$\int_{\mathbb{R}^n} f \geq \limsup_{k \to \infty} \int_{\mathbb{R}^n} f_k,$$
$$\int_{\mathbb{R}^n} f \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} f_k.$$

Therefore, $\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f$.

(b) $\Longrightarrow$: This direction is obvious by the inequality

$$\left| \int_{\mathbb{R}^n} |f_k| - |f| \right| \leq \int_{\mathbb{R}^n} ||f_k| - |f|| \leq \int_{\mathbb{R}^n} |f_k - f|.$$

$\Longleftarrow$: Let $g_k = |f_k| + |f|$ and $g = 2|f|$. Since $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e., then $g_k, g \in L(\mathbb{R}^n)$ and $g_k \to g$ a.e. in $\mathbb{R}^n$. By the assumption, $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$.

Let $\tilde{f}_k = |f_k - f|$. Then $\tilde{f}_k \to 0$ a.e. in $\mathbb{R}^n$ and $\tilde{f}_k \leq g_k$. Applying part (a) to $\tilde{f}_k$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

$\blacksquare$

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Problem 2.8. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Proof. Recall that $f \in L(\mathbb{R}^n)$ if and only if $|f| \in L(\mathbb{R}^n)$. Let $B_k = B(\mathbf{0}, k)$ for $k \in \mathbb{N}$ and χ_{B_k} be the indicator function associated with B_k. Then, the sequence of maps $\{|f_k|\}$ defined $f_k = f\chi_{B_k}$ converge pointwise to $|f_k|$. Since $|f| \in L(\mathbb{R}^n)$, by the monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbb{R}^n} |f|. \tag{35}$$

But this means, exactly, that for every $\varepsilon > 0$ there exists sufficiently large $N \in \mathbb{N}$ such that

$$\begin{aligned}
\varepsilon &> \left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \\
&= -\int_{\mathbb{R}^n} |f_k| + \int_{\mathbb{R}^n} |f| \\
&= -\int_{\mathbb{R}^n} |f| + \int_{\mathbb{R}^n} |f| \\
&= -\int_{B_k} |f| + \int_{\mathbb{R}^n} |f| \\
&= \int_{\mathbb{R}^n \setminus B_k} |f|
\end{aligned} \tag{36}$$

as desired. ■

Problem 2.9. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. First, since the E_j's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \tag{37}$$

Let χ_{E_j} be the characteristic function of the subset E_j of E and define $f_j = f\chi_{E_j}$ for $j \in \mathbb{N}$. Note that, since both f and χ_{E_j} are measurable on E, f_j is measurable on E and $\sum_{j=1}^{\infty} f_j = f$. Moreover, since $E_j \subset E$, by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \setminus E_j} f = \int_E f_j + \int_{E \setminus E_j} f. \tag{38}$$

Hence, because the E_j's are disjoint $(E \setminus E_k) \setminus E_\ell = (E \setminus E_\ell) \setminus E_k$ so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \tag{39}$$

as desired. ■

Problem 2.10. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbb{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Proof. Let $\varepsilon > 0$ be given. Then, by the hypothesis, there exists $\delta > 0$ such that such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon \tag{40}$$

for all $k \in \mathbb{N}$. By Egorov's theorem, there exists a closed subset F of E such that $|E \setminus F| < \delta$ and $f_k \to f$ uniformly on F. Then, by the uniform convergence theorem,

$$\int_F f_k \longrightarrow \int_F f \tag{41}$$

as $k \to \infty$. But by hypothesis, we have

$$\int_{E\setminus F} |f_k| < \varepsilon. \tag{42}$$

Letting $\varepsilon \to 0$, we achieved the desired convergence. ■

Problem 2.11. Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)dt$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Proof. By Lusin's theorem, there exists a closed subset F of I with $|I \setminus F| < \varepsilon$ such that the restriction of f to $F = I \setminus E$ is continuous. Now, since F is closed in I and I is compact, it follows that I is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials $\{p_k\}$ such that $p_k \to f$ uniformly on F. Then, by the uniform convergence theorem, we have

$$\int_F p_k \longrightarrow \int_F f \tag{43}$$

so

$$
\begin{aligned}
\int_F \left[\int_x^1 t^{-1} p_k(t) dt \right] dx &= \int_F \left[\int_x^1 a t^{-1} + q_k(t) dt \right] dx \\
&= \int_F q_k'(x) - a \log(x) dx \\
&< \infty
\end{aligned}
\tag{44}
$$

for all k and converges uniformly to g so $g \in L(I)$. I don't know how to show that in fact $\int_I g = \int_I f$. Perhaps you show that the places where they differ is a set of measure zero. ■