MA166: Solutions to Homework 1-10

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1 Solutions to Homework

These are solutions the assigned problems from Stewart's *Calculus, Early Transcendentals, 7th ed.* from the assignment sheet. I plan to keep updating the solutions sheet as the semester goes on.

Problem 1.1 (Stewart §12.1, Exercise 6). (a) What does the equation $x = 4$ represent in $\mathbb{R}^2$? What does it represent in $\mathbb{R}^3$? Illustrate with a sketches.

(b) What does the equation $y = 3$ represent in $\mathbb{R}^3$? What does $z = 5$ represent? What does the pair of equations $y = 3$, $z = 5$ represent? In other words, describe the set of points (x, y, z) such that $y = 3$ and $z = 5$. Illustrate with a sketch.

Solution. In such problems, your first instinct should be to ask yourself "What is a point on the equation I am given?."

(a) Let us begin by making a table of values of $x = 4$ in $\mathbb{R}^2$:

x	y
4	0
4	1
4	-1
4	1000

.

Recognize the pattern? In fact, for any value of y that we choose, x will always be 4 so the equation represents a vertical line which perpendicular to the x-axis, and touching the x-axis at the point $(0, 4)$.

We can use the same method to figure out what this equation, $x = 4$, represents in $\mathbb{R}^3$:

x	y	z
4	0	0
4	1	0
4	0	1
4	1	1

.

If you know some linear algebra, we have enough linearly independent vectors to span a plane. If you don't know any linear algebra, just note that if we choose any values for y and z the point $(4, y, z)$ will still be on the equation $x = 4$. Hence, the equation $x = 4$ represents a plane perpendicular to x-axis at the point $(4, 0, 0)$.

(b) Again, we may repeat the previous strategy to solve this problem $y = 3$,i.e., we make a table with points in the equation $y = 3$

x	y	z
0	3	0
1	3	0
1	3	-1

.

Again, we see that no matter what value of x and z we pick, the point $(x, 3, z)$ will always be in the equation $y = 3$. Hopefully, you can see why this equation represents a plane perpendicular to the y-axis at the point $(0, 3, 0)$.

Hopefully you are able to see the pattern now and can tell me all on your own what the equation $z = 5$ represents in $\mathbb{R}^3$. Right! It will be a plane perpendicular to the z-axis and touching the z-axis at the point $(0,0,5)$. In fact, we can make a categorical statement about what $x = a$ and $y = b$ or $z = c$ represent in $\mathbb{R}^3$:

Definition 1. Let a stand-in for one of the x, y, or z variables and let b be any real number. Then the equation

$$a = b \tag{1}$$

represents a plane perpendicular to the a-axis and touching the a-axis at the point $(b,0,0)$ (or $(0,b,0)$, or $(0,0,b)$ as it may be).

Now, it is not so clear what the pair of equations $y = 3$ and $z = 5$, but we can make a table and take a look at some of the values these equations take on

x	y	z
-1	3	5
0	3	5
-1	3	5
1000	3	5

So no matter what value of x we pick, the point $(x,3,5)$ will be in the pair of equations $y = 3$ and $z = 5$. Hence, we can see that this pair of equations represents a line extending in the same direction x-axis which touches the point $(0,3,5)$. That last bit is a bit arbitrary, we can say it touches the point $(1,3,5)$ or even $(10000,3,5)$, there is not much else we can say since the line never touches the any of the axes, but it does cross the yz-plane at the point $(0,3,5)$, which is questionably useful.

In fact, an easier way to say this is that the pair of equations $y = 3$ and $z = 5$ represents the intersection of the planes $y = 3$ and $z = 5$. Finding the line where these planes intersect is equivalent to finding the points that the pair of equations satisfy. ■

Remarks. In all this discussion, I failed to address one thing and that is, for example, why the point $(0,-3,5)$ is not on the pair of equations $y = 3$ and $z = 5$. Well, that is simply because we have a restriction on y and z, i.e., y must be 3 and z must be 5 or else we are looking at something all together different.

Problem 1.2 (Stewart §12.1, Exercise 16)**.** Show that the equation represents a sphere, and find its center and radius.

$$x^2 + y^2 + z^2 - 2x - 4y + 8z = 15.$$

Solution. You have no doubt seen the *standard equation* for a sphere of radius r centered at (x_0, y_0, z_0); in case you missed it here it is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \tag{2}$$

What is scary is that an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

is called a conic section and if we are given an equation like the one above, we cannot always factor it into something nice. Enough of that, our equation is clearly an equation of a sphere (you can tell this because it does not contain any mixed terms, i.e., terms of the form xy, xz, or yz) so we can use the method of completing the square to express the equation in standard form:

$$\begin{aligned} x^2 + y^2 + z^2 - 2x - 4y + 8z &= 15 \\ \left(x^2 - 2x\right) + \left(y^2 - 4y\right) + \left(z^2 + 8z\right) &= 15 \\ \left(x^2 - 2x + 1\right) - 1 + \left(y^2 - 4y + 2\right) - 2 + \left(z^2 + 8z + 16\right) - 16 &= 15 \\ \left(x^2 - 2x + 1\right) + \left(y^2 - 4y + 2\right) + \left(z^2 + 8z + 16\right) &= 15 + 1 + 2 + 16 \\ (x-1)^2 + (x-2)^2 + (x+4)^2 &= 34 \end{aligned}$$

Now that we've gotten the equation down the standard form of a sphere, we can read off the necessary values: So the equation

$$x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$$

represents a <u>sphere centered at $(1, 2, 4)$ with radius $\sqrt{34}$</u>. ■

Problem 1.3 (Stewart §12.1, Exercise 31)**.** Describe in words the region of $\mathbb{R}^3$ represented by the inequality

$$x^2 + y^2 + z^2 \leq 3.$$

Solution. Remember the equation of our good old friend the sphere? Well, it's not quite a sphere that we have but something like it. The equation we are given is indeed that of a sphere, but the radius is allowed to change from 0 to $\sqrt{3}$ so it is in fact a union of spheres of every radius between 0 and $\sqrt{3}$. In other words, it is the <u>solid sphere (also called a ball) of radius 3</u>. ■

Problem 1.4 (Stewart §12.1, Exercise 33)**.** Describe in words the region of $\mathbb{R}^3$ represented by the inequality

$$x^2 + z^2 \leq 9.$$

Solution. This looks something like the circle of radius 3 in the xz-plane. In fact, we again have all of the circles of radius 0 to 3 in the xz-plane and this gives us a <u>solid circle of radius (also called a disk) 3</u>. ■

Problem 1.5 (Stewart §12.1, Exercise 37)**.**

Solution. ■

Problem 1.6 (Stewart §12.1, Exercise 38)**.**

Solution. ■

Problem 1.7 (Stewart §12.2, Exercise 3)**.**

Solution. ■

Problem 1.8 (Stewart §12.2, Exercise 5)**.**

Solution. ■

2 Solutions to Homework 2

Problem 2.1 (Stewart §12.2, Exercise 11).

Solution. ■

Problem 2.2 (Stewart §12.2, Exercise 13).

Solution. ■

Problem 2.3 (Stewart §12.2, Exercise 17).

Solution. ■

Problem 2.4 (Stewart §12.2, Exercise 19).

Solution. ■

Problem 2.5 (Stewart §12.2, Exercise 25).

Solution. ■

Problem 2.6 (Stewart §12.2, Exercise 26).

Solution. ■

Problem 2.7 (Stewart §12.3, Exercise 11).

Solution. ■

Problem 2.8 (Stewart §12.3, Exercise 5).

Solution. ■

Problem 2.9 (Stewart §12.3, Exercise 8).

Solution. ■

Problem 2.10 (Stewart §12.3, Exercise 9).

Solution. ■

3 Solutions to Homework 3

4 Solutions to Homework 4

5 Solutions to Homework 5

6 Homework 6 Solutions

Sorry, no pictures. Unless I can get a hang of Inkscape's graphing syntax, I won't be able to generate all of these images with just Ti*k*Z; it is simply too much work. I am willing to plot integrals, cross-sectional areas, etc., but no shells and solids of revolution.

Problem 6.1 (Stewart §6.2, Exercise 2)**.** Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical washer.

$y = 1 - x^2$, $y = 0$; about the x-axis

Solution. We first need to figure out an equation for the cross-sectional area $A(y)$ for our solid. Since we are rotating about the x-axis, our problem is most easily approached by solving for A in terms of x. Since we are rotating, $A(y) = \pi y^2$ will be the area of a circle. Solving for $A(y)$ in terms of x, we have $A(x) = \pi(1 - x^2)^2$.

Next we need to find the points where the graph $y = 1 - x^2$ intersects the line $y = 0$, i.e., where $1 - x^2 = 0$. This is easy as, $-x^2 = -1$ so $x^2 = 1$ hence, $x = -1$ or 1.

Now we are ready to start calculating the volume of the solid of revolution: Since the graph of y is symmetric about the y-axis, it suffices to compute the following integral from 0 to 1 and multiply by 2. Hence, we have

$$\begin{aligned}
2\pi \int_0^1 \left(1 - x^2\right)^2 \mathrm{d}x &= 2\pi \int_0^1 1 - 2x^2 + x^4 \,\mathrm{d}x \\
&= 2\pi \left(x - \frac{2x^3}{3} + \frac{x^5}{5} \bigg|_0^1 \right) \\
&= 2\pi \left(1 - \frac{2}{3} + \frac{1}{5} - (0 - 0 + 0) \right) \\
&= 2\pi \left(\frac{6}{15} \right) \\
&= \boxed{\frac{12\pi}{15}.}
\end{aligned}$$

■

Problem 6.2 (Stewart §6.2, Exercise 3)**.** $y = \sqrt{x - 1}$, $y = 0$, $x = 5$; about the x-axis.

Solution. Same procedure, set $A(x) = \pi y^2 = \pi\left(\sqrt{x-1}\right)^2$. Find the point x where $y = \sqrt{x-1}$ intersects the lines $y = 0$ and $x = 5$. These values are $x = 1$ and $x = 5$. Now we are ready to compute the integral:

$$\begin{aligned}
\pi \int_1^5 x - 1 \,\mathrm{d}x &= \pi \left(\frac{x^2}{2} - x \bigg|_1^5 \right) \\
&= \pi \left(\frac{25}{2} - 5 - \frac{1}{2} + 1 \right) \\
&= \boxed{8\pi.}
\end{aligned}$$

$\blacksquare$

Problem 6.3 (Stewart §6.2, Exercise 8)**.** $y = \frac{1}{4}x^2$, $y = 5 - x^2$; about the x-axis.

Solution. First, let us find the points of intersection of the graphs $y = \frac{1}{4}x^2$, $y = 5 - x^2$ as such

$$\begin{aligned}
\frac{1}{4}x^2 &= 5 - x^2 \\
x^2 &= 20 - 4x^2 \\
5x^2 &= 20 \\
x^2 &= 4
\end{aligned}$$

so $x = \pm 2$. Now, since $y = 5 - x^2$ is above $y = \frac{1}{4}x^2$ for all $-2 \leq x \leq 2$, the cross-section $A(y) = \pi y^2$ varies as the difference $5 - x^2 - \frac{1}{4}x^2 = 5 - \frac{5}{4}x^2$ so

$$A(x) = \pi\left(5 - \frac{5}{4}x^2\right)^2$$

and we have

$$\begin{aligned}
\pi \int_{-2}^{2} \left(5 - \frac{5}{4}x^2\right)^2 \mathrm{d}x &= \pi \int_{-2}^{2} 25 - \frac{5}{2}x^2 + \frac{25}{16}x^4 \, \mathrm{d}x \\
&= 2\pi \int_{0}^{2} 25 - \frac{5}{2}x^2 + \frac{25}{16}x^4 \, \mathrm{d}x \\
&= 2\pi\left(25x - \frac{5}{6}x^3 + \frac{5}{16}x^5 \Big|_0^2\right) \\
&= 2\pi\left(25 \cdot 2 - \frac{5 \cdot 8}{6} + \frac{5 \cdot 2^5}{16}\right) \\
&= 2\pi\left(50 - \frac{20}{3} + 10\right) \\
&= 2\pi\left(\frac{150 - 20 + 30}{3}\right) \\
&= \boxed{\frac{320\pi}{3}}
\end{aligned}$$

$\blacksquare$

Problem 6.4 (Stewart §6.2, Exercise 9)**.** $y^2 = x$, $x = 2y$; about the y-axis.

Solution. Since we are rotating about the y-axis, we want to consider the cross-sectional area A perpendicular to the y-axis. Therefore, we want to solve for A in terms of x. Now, let us find the values of y where the equations $x = y^2$ and $x = 2y$ intersect. These points are

$$\begin{aligned}
y^2 &= 2y \\
y^2 - 2y &= 0 \\
y(y - 2) &= 0
\end{aligned}$$

so $y = 0$ or $y = 2$. Now that we have our bounds, let us express the radius of our cross-sectional area in terms of y, i.e., it will be the difference $y^2 - 2y$ so that $A(y) = \pi(y(y-2))^2$ and we are ready to integrate:

$$\begin{aligned}
\pi \int_0^2 y^2(y-2)^2 \, \mathrm{d}y &= \pi \int_0^2 y^2(y^2 - 2y + 4) \, \mathrm{d}y \\
&= \pi \int_0^2 y^4 - 2y^3 + 4y^2 \, \mathrm{d}y \\
&= \pi \left(\frac{y^5}{5} - \frac{y^4}{2} + \frac{4y^3}{3} \bigg|_0^2 \right) \\
&= \pi \left(\frac{2^5}{5} - \frac{2^4}{4} + \frac{4 \cdot 2^3}{3} \right) \\
&= \boxed{\frac{196\pi}{15}.}
\end{aligned}$$

■

Problem 6.5 (Stewart §6.2, Exercise 19)**.**

Solution. ■

Problem 6.6 (Stewart §6.2, Exercise 21)**.**

Solution. ■

Problem 6.7 (Stewart §6.2, Exercise 24)**.**

Solution. ■

Problem 6.8 (Stewart §6.2, Exercise 26)**.**

Solution. ■

Problem 6.9 (Stewart §6.2, Exercise 27)**.**

Solution. ■

7 Homework 7 Solutions

8 Homework 8 Solutions

Problem 8.1 (Stewart §6.4, Exercise 3)**.** A variable force of $5x^{-2}$ pounds moves an object along a straight line when it is x feet from the origin. Calculate the work done in moving the object from $x = 1$ft to $x = 10$ft.

Solution. ■

Problem 8.2 (Stewart §6.4, Exercise 5)**.** Shown is the graph of a force function (in newtons) that increases to its maximum value and then remains constant. How much work is done by the force in moving an object a distance of 8 m?

Solution. ■

Problem 8.3 (Stewart §6.4, Exercise 10)**.** If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stretch it 9 in beyond its natural length?

Solution. ■

Problem 8.4 (Stewart §6.4, Exercise 19)**.** An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is $1000\,\mathrm{kg}\cdot\mathrm{m}^{-3}$.)

Solution. First we find the mass of the water in the rectangular aquarium. ■

Problem 8.5 (Stewart §6.4, Exercise 21)**.**

Solution. ■

Problem 8.6 (Stewart §6.5, Exercise 11)**.**

Solution. ■

Problem 8.7 (Stewart §6.5, Exercise 14)**.**

Solution. ■

9 Homework 9 Solutions

Problem 9.1 (Stewart §7.1, Exercise 1)**.**

Solution. ∎

Problem 9.2 (Stewart §7.1, Exercise 3)**.**

Solution. ∎

Problem 9.3 (Stewart §7.1, Exercise 10)**.**

Solution. ∎

Problem 9.4 (Stewart §7.1, Exercise 17)**.**

Solution. ∎

Problem 9.5 (Stewart §7.1, Exercise 27)**.**

Solution. ∎

Problem 9.6 (Stewart §7.1, Exercise 37)**.**

Solution. ∎

Problem 9.7 (Stewart §7.1, Exercise 62)**.**

Solution. ∎

10 Homework 10 Solutions

Problem 10.1 (Stewart §7.2, Exercise 1).

Solution. ∎

Problem 10.2 (Stewart §7.2, Exercise 7).

Solution. ∎

Problem 10.3 (Stewart §7.2, Exercise 11).

Solution. ∎

Problem 10.4 (Stewart §7.2, Exercise 17).

Solution. ∎

Problem 10.5 (Stewart §7.2, Exercise 23).

Solution. ∎

Problem 10.6 (Stewart §7.2, Exercise 24).

Solution. ∎

Problem 10.7 (Stewart §7.2, Exercise 35).

Solution. ∎

Problem 10.8 (Stewart §7.2, Exercise 61).

Solution. ∎