MA 544: Homework 10

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Problem 10.1 (Wheeden & Zygmund §7, Ex. 1)

Let f be measurable in $\mathbf{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(\mathbf{x}) \geq c\|\mathbf{x}\|^{-n}$ for $\|\mathbf{x}\| \geq 1$.

Proof. Suppose that f is measurable and nonzero on a subset E of $\mathbf{R}^n$ with positive measure. Assume E is bounded. Since f is measurable $|f|$ is measurable so the set $E_a \coloneqq \{\, \mathbf{x} \in E : |f|(\mathbf{x}) > a \,\}$, for a finite, is a measurable bounded subset of $\mathbf{R}^n$. Let χ_a denote the characteristic function of E_a. Then, by Chebyshev's inequality, we have

$$\begin{aligned}
{\chi_a}^*(\mathbf{x}) &= \sup_Q \frac{1}{|Q|} \int_Q |\chi_a(\mathbf{y})| d\mathbf{y} \\
&\leq \sup_Q \frac{1}{|Q|} \left[\frac{1}{a} \int_Q |f(\mathbf{y})| d\mathbf{y} \right] \\
&= \frac{1}{a} f^*(\mathbf{x}).
\end{aligned} \tag{10.1}$$

By the commentary on p. 138, there exists constants c_1 and c_2 such that

$$c_1 \frac{|E_a|}{\|\mathbf{x}\|^n} \leq {\chi_a}^*(\mathbf{x}) \leq c_2 \frac{|E_a|}{\|\mathbf{x}\|^n} \tag{10.2}$$

for all large $\|\mathbf{x}\|$. Putting (10.1) and (10.2) together, we obtain

$$a c_1 \frac{|E_a|}{\|\mathbf{x}\|^n} \leq f^*(\mathbf{x}). \tag{10.3}$$

Setting $c \coloneqq a c_1 |E|$, we have the desired lower bound $c\|\mathbf{x}\|^{-n} \leq f^*(\mathbf{x})$ (assuming $\|\mathbf{x}\|$ is large). ∎

Problem 10.2 (Wheeden & Zygmund §7, Ex. 2)

Let $\varphi(\mathbf{x}), \mathbf{x} \in \mathbf{R}^n$, be a bounded measurable function such that $\varphi(\mathbf{x}) = 0$ for $\|\mathbf{x}\| \geq 1$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(\mathbf{x}) = \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon)$. ($\varphi_\varepsilon$ is called an *approximation to the identity.*) If $f \in L(\mathbf{R}^n)$, show that

$$\lim_{\varepsilon \to 0}(f * \varphi_\varepsilon)(\mathbf{x}) = f(\mathbf{x})$$

in the Lebesgue set of f. (Note that $\int \varphi_\varepsilon = 1$, $\varepsilon > 0$, so that

$$(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x}) = \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{y})d\mathbf{y}.$$

Use Theorem 7.16.)

Proof. First note that, making the change of variables $\mathbf{u} = \mathbf{x}/\varepsilon$ (with Jacobian $\mathbf{J}(\mathbf{x}, \mathbf{u}) = \varepsilon^n$), we have

$$\begin{aligned} \int \varphi_\varepsilon(\mathbf{x})d\mathbf{x} &= \int \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon)d\mathbf{x} \\ &= \int_{B(\mathbf{0},\varepsilon)} \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon)d\mathbf{x} \\ &= \int_{B(\mathbf{0},1)} \varphi(\mathbf{u})d\mathbf{u} \\ &= \int \varphi(\mathbf{x})d\mathbf{u} \\ &= 1. \end{aligned} \tag{10.4}$$

Hence, by the hint and the definition of the convolution, we have

$$\begin{aligned} |(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x})| &= \left| \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{x})d\mathbf{x} \right| \\ &= \left| \int_{B(\mathbf{0},\varepsilon)} [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{x})d\mathbf{x} \right| \\ &\leq \int_{B(\mathbf{0},\varepsilon)} |[f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{x})|d\mathbf{x}. \end{aligned} \tag{10.5}$$

Now, since φ is bounded, say by M, we have

$$\varphi_\varepsilon(\mathbf{y}) = \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon) \leq M. \tag{10.6}$$

Then, we have an estimate on (10.5)

$$\begin{aligned} \int_{B(\mathbf{0},\varepsilon)} |[f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{x})|d\mathbf{x} &\leq \frac{M}{\varepsilon^n} \int_{B\mathbf{0},\varepsilon} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| \\ &\leq \frac{M}{\varepsilon^n} \int_{B(\mathbf{0},\varepsilon)} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})|. \end{aligned} \tag{10.7}$$

Now, let Q_ε be the larges cube centered at $\mathbf{x}$ contained in $B(\mathbf{0}, \mathbf{x})$. Then, as we have previously shown, the volume of Q_ε is $C\varepsilon^n$ for some positive real number C. Making a change of variables $\mathbf{v} = \mathbf{x} - \mathbf{y}$ gives us

$$|(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x})| \leq C\frac{M}{|Q_\varepsilon|}\int_{Q_\varepsilon + bfx} |f(\mathbf{v}) - f(\mathbf{x})| d\mathbf{v}, \tag{10.8}$$

which goes to 0 an $\varepsilon \to 0$ by Theorem 7.16 since $\mathbf{x}$ is a point in the Lebesgue set of f. ∎

Problem 10.3 (Wheeden & Zygmund §7, Ex. 6)

Show that if $\alpha > 0$, then x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Proof. Recall that a function $f\colon [a,b] \to \mathbf{R}$ is absolutely continuous on $[a,b]$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that for any collection $\{[a_j, b_j]\}$ of nonoverlapping subintervals of $[a,b]$, $\sum |b_i - a_i| < \delta$ implies $\sum |f(b_i) - f(a_i)| < \varepsilon$.

Now, by Theorem 7.29 on the bounded interval $[a,b] \subset [0,\infty)$ we may write $F(x) = x^\alpha$ as the integral

$$F(x) = \alpha \int_a^x x^{\alpha-1} + c \tag{10.9}$$

for some finite constant c. Since constants and indefinite integrals are absolutely continuous, F is absolutely continuous if we can show $x^{\alpha-1}$ is integrable on $[a,b]$. This is true unless $a = 0$ and $\alpha < 1$. If $a \neq 0$ and $\alpha \geq 1$, $x^{\alpha-1}$ is Riemann integrable and positive, so it is Lebesgue integrable and its Riemann integral equals its Lebesgue integral. ■

Problem 10.4 (Wheeden & Zygmund §7, Ex. 8)

Prove the following converse of Theorem 7.31: If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Proof. ■

Problem 10.5 (Wheeden & Zygmund §7, Ex. 9)

If f is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then f is absolutely continuous on $[a, b]$. (For the second part, use Theorems 2.2(ii) and 7.24 to show that $V(x)$ is absolutely continuous and then use the result of Exercise 8).

Proof. (Sorry, I skipped this and the last one since I was at a conference and its easier to show things about concavity...) ■

Problem 10.6 (Wheeden & Zygmund §7, Ex. 12)

Use Jensen's inequality to prove that if $a, b \geq 0$, $p, q > 1$, $(1/p) + (1/q) = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

More generally, show that

$$a_1 \cdots a_N = \sum_{j=1}^{N} \frac{{a_j}^{p_j}}{p_j},$$

where $a_j \geq 0$, $p_j > 1$, $\sum_{j=1}^{N}(1/p_j) = 1$. (Write $a_j = e^{x_j/p_j}$ and use the convexity of e^x).

Proof. Suppose that a, b, p and q satisfy the conditions given in the statement of the problem. The claim is true if $a = 0$ and $b = 0$, so assume $a, b > 0$. Set $t := 1/p$ and $(1-t) = 1/q$. Then, since the logarithm ln is strictly concave, we have

$$\begin{aligned} \ln(ta^p + (1-t)b^q) &\geq t\ln(a^p) + (1-t)\ln(b^q) \\ &= \ln(a) + \ln(b) \\ &= \ln(ab). \end{aligned} \tag{10.10}$$

Lastly, exponentiating the left and right inequalities in (10.10), we have

$$\begin{aligned} ab &\leq ta^p + (1-t)b^q \\ &= \frac{a^p}{p} + \frac{b^q}{q} \end{aligned} \tag{10.11}$$

as desired. The generalization follows from Jensen's inequality. ■

Problem 10.7 (Wheeden & Zygmund §7, Ex. 13)

Prove Theorem 7.36.

Proof. Recall the statement of Theorem 7.36

Theorem. (i) *If φ_1 and φ_2 are convex in (a,b), then $\varphi_1+\varphi_2$ is convex in (a,b).*

(ii) *If φ is convex in (a,b) and c is a positive constant, then $c\varphi$ is convex in (a,b).*

(iii) *If φ_k, $k=1,2,\dots,$ are convex in (a,b) and $\varphi_k\to\varphi$ in (a,b), then φ is convex in (a,b).*

(i) Let $x,y\in(a,b)$ and φ_1 and φ_2 be convex. Then for every $t\in[0,1]$, we have

$$\begin{aligned}\varphi_1(tx+(1-t)y)+\varphi_2(tx+(1-t)y) &\le t\varphi_1(x)+(1-t)\varphi_1(y)+t\varphi_2(x)+(1-t)\varphi_2(y)\\ &= t(\varphi_1(x)+\varphi_2(x))+(1-t)(\varphi_1(y)+\varphi_2(y)).\end{aligned} \tag{10.12}$$

Hence, $\varphi_1+\varphi_2$ is convex.

(ii) Let $c>0$. Then for all $t\in[0,1]$, we have

$$\begin{aligned}c\varphi(tx+(1-t)y) &\le ct\varphi(x)+c(1-t)\varphi(y)\\ &= c(t\varphi(x)+(1-t)\varphi(y))\end{aligned} \tag{10.13}$$

So $c\varphi$ is convex.

(iii) For $t\in[0,1]$ and all $k\ge 1$, we have

$$\varphi_k(tx+(1-t)y)\le t\varphi_k(x)+(1-t)\varphi_k(y). \tag{10.14}$$

Letting $k\to\infty$, since $\varphi_k\to\varphi$ pointwise on (a,b), we have

$$\begin{aligned}\varphi(tx+(1-t)y) &= \lim_{k\to\infty}\varphi_k(tx+(1-t)y)\\ &\le \lim_{k\to\infty}[t\varphi_k(x)+(1-t)\varphi_k(y)]\\ &= t\varphi(x)+(1-t)\varphi(y).\end{aligned} \tag{10.15}$$

Hence φ is convex on (a,b). ■