Math 535 - General Topology
Fall 2012
Homework 6 Solutions

Problem 1. Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers. Let $n \geq 1$ and denote by $\mathbb{F}[x_1, x_2, \ldots, x_n]$ the set of all polynomials in n variables with coefficients in $\mathbb{F}$.

A subset $C \subseteq \mathbb{F}^n$ of n-dimensional space will be called **Zariski closed** if it is the zero locus of some polynomials:

$$C = V(S) := \{x \in \mathbb{F}^n \mid f(x) = 0 \text{ for all } f \in S\}$$

for some $S \subseteq \mathbb{F}[x_1, \ldots, x_n]$.

Note: The zero locus $V(S)$ is sometimes called the *algebraic variety* associated to S, hence the letter V.

For example, in $\mathbb{R}^2$, the subset $V(x_1^2 + x_2^2 - 9) \subset \mathbb{R}^2$ is the circle of radius 3 centered at the origin, which is therefore a Zariski closed subset.

By convention, let's say S is not allowed to be empty, though you will show in part (a) that it doesn't matter.

a. Show that the notion of "Zariski closed" subset does define a topology on $\mathbb{F}^n$, sometimes called the **Zariski topology**.

Solution. The entire space $\mathbb{F}^n$ is Zariski-closed:

$$\mathbb{F}^n = V(0).$$

Note: This is why we might as well allow S to be empty: $V(\emptyset) = \mathbb{F}^n = V(0)$.

The empty subset $\emptyset \subset \mathbb{F}^n$ is Zariski-closed:

$$\emptyset = V(1).$$

Since Zariski-closed subsets are defined as (arbitrary) intersections

$$V(S) = \bigcap_{f \in S} V(f)$$

of basic closed set $V(f)$, it suffices to check that a finite union of basic closed sets is an intersection of basic closed sets. For any polynomials f and g, we have

$$\begin{aligned} V(f) \cup V(g) &= \{x \in \mathbb{F}^n \mid f(x) = 0 \text{ or } g(x) = 0\} \\ &= \{x \in \mathbb{F}^n \mid f(x)g(x) = 0\} \\ &= V(fg). \quad \square \end{aligned}$$

b. Show that the Zariski topology is *strictly* coarser (i.e. smaller) and the usual metric topology on $\mathbb{F}^n$.

Solution. Any polynomial function $f\colon \mathbb{F}^n \to \mathbb{F}$ is metrically continuous, therefore its zero set $V(f) = f^{-1}(\{0\})$ is metrically closed. This prove $\mathcal{T}_{\text{Zar}} \leq \mathcal{T}_{\text{met}}$.

To show that the inequality is strict, consider the subset

$$C := \{x \in \mathbb{F}^n \mid x_n \geq 0\}$$

(or the real part $\operatorname{Re}(x_n) \geq 0$ in case $\mathbb{F} = \mathbb{C}$). Then C is metrically closed. However, C is not Zariski-closed. To prove this, let $V(f)$ be a Zariski-closed subset containing C, so that f vanishes on C.

For any fixed $a_1, \ldots a_{n-1} \in \mathbb{F}$, the polynomial

$$f(a_1, \ldots, a_{n-1}, x_n)$$

in one variable x_n vanishes for infinitely many values of x_n, thus is the zero polynomial. Since the a_i were arbitrary, this implies $f = 0$ and thus $V(f) = \mathbb{F}^n$. Therefore the Zariski closure of C is $\overline{C} = \mathbb{F}^n \neq C$. $\square$

c. Show that the Zariski topology on $\mathbb{F}^n$ is T_1.

Solution. For any $a \in \mathbb{F}^n$, the singleton $\{a\}$ is the Zariski-closed set

$$\{a\} = V(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n). \quad \square$$

d. Show that the Zariski topology on $\mathbb{F}^n$ is not T_2, i.e. not Hausdorff.

Solution. It suffices to show that any two basic open subsets have a non-empty intersection. Equivalently, any two basic closed subsets have a union which is not all of $\mathbb{F}^n$.

For any (non-zero) polynomials f and g, we have $V(f) \cup V(g) = V(fg)$. Since fg is not the zero polynomial, there is a point $a \in \mathbb{F}^n$ satisfying $(fg)(a) \neq 0$, so that $a \notin V(fg)$. $\square$

e. In the one-dimensional case $n = 1$, show that the Zariski topology on $\mathbb{F}$ is the cofinite topology.

Solution. ($\mathcal{T}_{\text{cof}} \leq \mathcal{T}_{\text{Zar}}$) Since the Zariski topology is T_1, every finite subset of $\mathbb{F}^n$ is Zariski-closed.

($\mathcal{T}_{\text{Zar}} \leq \mathcal{T}_{\text{cof}}$) Let $C \subset \mathbb{F}$ be Zariski-closed. If C is not all of $\mathbb{F}$, then $C \subseteq V(f)$ for non-zero polynomial f. But a non-zero polynomial $f(x_1)$ in one variable has (at most) finitely many zeroes, so that $V(f)$ is finite, as is C. $\square$

Problem 2. Two points x and y in a topological space X are **topologically distinguishable** if there is an open subset $U \subset X$ that contains one of the points but not the other. A space X is T_0 if any distinct points are topologically distinguishable.

Two points x and y are **topologically indistinguisable** if they are not topologically distinguishable, which amounts to x and y having exactly the same neighborhoods. One readily checks that topological indistinguishability is an equivalence relation on X, which we will denote by $x \sim y$.

The **Kolmogorov quotient** of X is the quotient $KQ(X) := X/{\sim}$, where topologically indistinguisable points become identified. In particular, X is T_0 if and only if X is homeomorphic to its Kolmogorov quotient.

a. Show that the Kolmogorov quotient satisfies the following universal property.

1. The natural map $\pi \colon X \twoheadrightarrow KQ(X)$ is continuous.
2. $KQ(X)$ is a T_0 space.
3. For any T_0 space Y and continuous map $f \colon X \to Y$, there is a unique continuous map $\overline{f} \colon KQ(X) \to Y$ satisfying $f = \overline{f} \circ \pi$, i.e. making the diagram

$$\begin{tikzcd} X \arrow[r, two heads, "\pi"] \arrow[dr, "f"'] & KQ(X) \arrow[d, dashed, "\exists!\,\overline{f}"] \\ & Y \end{tikzcd}$$

   commute.

In other words, $KQ(X)$ is the "closest T_0 space which X maps into".

Solution.

1. $\pi \colon X \twoheadrightarrow KQ(X)$ is continuous, since it is the canonical projection to the quotient space.
2. Let $a, b \in KQ(X)$ be distinct points. Pick representatives $x, y \in X$ of a and b respectively. Since a and b are distinct, x and y are topologically distinguishable. Let $U \subset X$ be an open subset that distinguishes x and y, say WLOG $x \in U$ and $y \notin U$.

   By definition of $\sim$, U is a union of equivalence classes (i.e. $z \in U$ implies that any $z' \sim z$ is also in U), which is saying $\pi^{-1}\pi(U) = U$. Therefore $\pi(U)$ is open in $KQ(X)$ and contains $\pi(x) = a$. However $\pi(U)$ does not contain b, since every $u \in U$ is distinguishable from y, so that $\pi(u) \neq \pi(y) = b$.
3. By the universal property of the quotient topology, if suffices to show that such a map $f \colon X \to Y$ is constant on equivalence classes, i.e. $x \sim x'$ implies $f(x) = f(x')$.

   Assuming $f(x) \neq f(x')$, there is an open $U \subset Y$ that distinguishes $f(x)$ and $f(x')$ since Y is T_0. Let's say WLOG $f(x) \in U$ and $f(x') \notin U$. Then $f^{-1}(U) \subset X$ is an open that distinguishes x and x', as $x \in f^{-1}(U)$ and $x' \notin f^{-1}(U)$. We conclude $x \not\sim x'$. $\square$

b. Show that X is regular if and only if its Kolmogorov quotient $KQ(X)$ is T_3.

Solution.

Lemma. *T_0 and regular implies T_3.*

Proof. Let us show that the space X is T_2. Let $x, y \in X$ be distinct points. Since X is T_0, there is an open $U \subset X$ distinguishing x and y, say WLOG $x \in U$ and $y \notin U$, i.e. $y \in U^c$. Since U^c is closed and disjoint from x, by regularity there are open sets U_x and V satisfying $x \in U_x$, $U^c \subseteq V$ and $U_x \cap V = \emptyset$. We have $y \in U^c \subseteq V$, so that U_x and V are neighborhoods that separate x and y. □

($\Rightarrow$) Note that $KQ(X)$ is automatically T_0. By the lemma, it suffices to show that $KQ(X)$ is regular. Let $C \subset KQ(X)$ be closed, and $a \notin C$. Then the preimages $\pi^{-1}(a)$ and $\pi^{-1}(C)$ are disjoint subsets of X, and $\pi^{-1}(C)$ is closed. Pick any representative $x \in \pi^{-1}(a)$. Since X is regular, there are open subsets $U, V \subset X$ satisfying $x \in U$, $\pi^{-1}(C) \subseteq V$, and $U \cap V = \emptyset$.

By the argument in part (a), $\pi(U)$ and $\pi(V)$ are open in $KQ(X)$ and contain a and C respectively (since π is surjective). Moreover, $\pi(U)$ and $\pi(V)$ are disjoint. Indeed, U and V are disjoint and open, so that any points $u \in U$ and $v \in V$ are distinguishable, i.e. $\pi(u) \neq \pi(v)$.

($\Leftarrow$) Let $C \subset X$ be closed and $x \notin C$. Because open subsets of X are unions of equivalence classes, the same holds for closed subsets. Therefore $\pi(C) \subset KQ(X)$ is closed and disjoint from the point $\pi(x)$.

Since $KQ(X)$ is regular, there are open subsets $U, V \subset KQ(X)$ satisfying $\pi(x) \in U$, $\pi(C) \subseteq V$, and $U \cap V = \emptyset$. Therefore $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open in X, contain x and C respectively, and are disjoint. □

Problem 3. Let (X, d) be a metric space.

a. Let $S \subseteq X$ be a non-empty subset and consider the function $f_S \colon X \to \mathbb{R}$ defined by

$$f_S(x) = d(x, S).$$

Show that f_S is Lipschitz continuous with Lipschitz constant 1, i.e.

$$|f_S(x) - f_S(y)| \leq d(x, y) \text{ for all } x, y \in X.$$

In particular, f_S is continuous.

Solution. For every $x, y \in X$ and $s \in S$, we have

$$d(x, s) \leq d(x, y) + d(y, s)$$

and taking the infimum over $s \in S$ yields

$$d(x, S) \leq d(x, y) + d(y, S)$$

which can be rewritten as

$$d(x, S) - d(y, S) \leq d(x, y).$$

Interchanging the role of x and y, we also obtain

$$d(y, S) - d(x, S) \leq d(x, y)$$

and therefore

$$|f_S(x) - f_S(y)| = |d(x, S) - d(y, S)| \leq d(x, y). \quad \square$$

b. Show that closed subsets of X can be **precisely** separated by functions, i.e. for any $A, B \subset X$ disjoint closed subsets of X, there is a continuous function $f \colon X \to [0, 1]$ satisfying

$$\begin{cases} f(a) = 0 \text{ for all } a \in A \\ f(b) = 1 \text{ for all } b \in B \\ f(x) \in (0, 1) \text{ for all } x \notin A \cup B. \end{cases}$$

First assume A and B are non-empty. Then treat the case $B = \emptyset$ separately.

Hint: Recall the equivalence $d(x, S) = 0$ if and only if $x \in \overline{S}$.

Solution. **When A and B are non-empty.** Then the functions $f_A, f_B \colon X \to \mathbb{R}$ are well-defined and continuous. Consider the function $f \colon X \to \mathbb{R}$ defined by

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}.$$

This function satisfies the desired properties.

- f is well-defined since the denominator is strictly positive on X:

$$\begin{aligned} f_A(x) + f_B(x) = 0 &\Leftrightarrow f_A(x) = 0 \text{ and } f_B(x) = 0 \\ &\Leftrightarrow x \in \overline{A} = A \text{ and } x \in \overline{B} = B \\ &\Leftrightarrow x \in A \cap B = \emptyset. \end{aligned}$$

- f is continuous, since the sum $f_A + f_B$ is continuous, so that the quotient $f = \frac{f_A}{f_A + f_B}$ is continuous on X.
- f takes values in $[0,1]$, by the inequalities $0 \leq f_A(x) \leq f_A(x) + f_B(x)$ for all $x \in X$.
- f satisfies:

$$\begin{aligned} f(x) = 0 &\Leftrightarrow \frac{f_A(x)}{f_A(x) + f_B(x)} = 0 \\ &\Leftrightarrow f_A(x) = 0 \\ &\Leftrightarrow x \in \overline{A} = A. \end{aligned}$$

- f satisfies:

$$\begin{aligned} f(x) = 1 &\Leftrightarrow \frac{f_A(x)}{f_A(x) + f_B(x)} = 1 \\ &\Leftrightarrow f_A(x) = f_A(x) + f_B(x) \\ &\Leftrightarrow f_B(x) = 0 \\ &\Leftrightarrow x \in \overline{B} = B. \end{aligned}$$

When $B = \emptyset$ is empty. Then replace f_B by the constant function 1:

$$f(x) = \frac{f_A(x)}{1 + f_A(x)}.$$

As in the previous case, the denominator $1 + f_A(x)$ is strictly positive on X, f is continuous, takes values in $[0,1]$, and vanishes precisely on A. It remains it check:

$$\begin{aligned} f(x) = 1 &\Leftrightarrow \frac{f_A(x)}{1 + f_A(x)} = 1 \\ &\Leftrightarrow f_A(x) = 1 + f_A(x) \\ &\Leftrightarrow x \in \emptyset = B. \quad \square \end{aligned}$$

Remark. A space is called **perfectly normal** if its closed subsets can be precisely separated by functions. A space is called T_6 if it is T_1 and perfectly normal. We have just shown that every metric space is T_6.

Problem 4. In this problem, we will show that a countable product of metrizable spaces is metrizable.

$\boxed{\textbf{a.}}$ Let (X, d) be a metric space. Consider the function $\rho\colon X \times X \to \mathbb{R}$ defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that ρ is a metric on X.

Solution. Write $\rho = h \circ d$ where $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the rescaling function defined by

$$h(t) = \frac{t}{1 + t}.$$

We check the three properties of a metric.

1. Positivity:
$$\rho(x, y) = h\left(d(x, y)\right) \geq 0 \quad \text{for all } x, y \in X$$
since $h(t) \geq 0$ for all $t \geq 0$.
$$\begin{aligned} \rho(x, y) = 0 &\Leftrightarrow h\left(d(x, y)\right) = 0 \\ &\Leftrightarrow d(x, y) = 0 \\ &\Leftrightarrow x = y \end{aligned}$$
where we used the property $h(t) = 0 \Leftrightarrow t = 0$.

2. Symmetry:
$$\begin{aligned} \rho(y, x) &= h\left(d(y, x)\right) \\ &= h\left(d(x, y)\right) \\ &= \rho(x, y). \end{aligned}$$

3. Triangle inequality: First note that h satisfies the “sublinearity” condition
$$h(s + t) \leq h(s) + h(t)$$

for all $s, t \geq 0$. Indeed, we have:

$$\begin{aligned}
h(s) + h(t) &= \frac{s}{1+s} + \frac{t}{1+t} \\
&= \frac{s(1+t) + t(1+s)}{(1+s)(1+t)} \\
&= \frac{s+t+2st}{1+s+t+st} \\
&\geq \frac{s+t+st}{1+s+t+st} \\
&\geq \frac{s+t+\frac{s+t}{1+s+t}st}{1+s+t+st} \\
&= \frac{(s+t)(1+s+t)+(s+t)st}{(1+s+t+st)(1+s+t)} \\
&= \frac{(s+t)(1+s+t+st)}{(1+s+t+st)(1+s+t)} \\
&= \frac{s+t}{1+s+t} \\
&= h(s+t).
\end{aligned}$$

Thus the triangle inequality for d

$$d(x, y) \leq d(x, z) + d(z, y)$$

along with the fact that h is non-decreasing implies

$$\begin{aligned}
\rho(x, y) &= h\left(d(x, y)\right) \\
&\leq h\left(d(x, z) + d(z, y)\right) \\
&\leq h\left(d(x, z)\right) + h\left(d(z, y)\right) \\
&= \rho(x, z) + \rho(z, y). \quad \square
\end{aligned}$$

$\boxed{\textbf{b.}}$ Show that the metric ρ from part (a) induces the same topology on X as the original metric d.

Solution. Given $h(0) = 0$, continuity of h at 0 means that for any $\epsilon > 0$, there is a $\delta > 0$ guaranteeing $h(t) < \epsilon$ whenever $t < \delta$. Substituting $t = d(x, y)$, we obtain $\rho(x, y) < \epsilon$ whenever $d(x, y) < \delta$, i.e.

$$B^d_\delta(x) \subseteq B^\rho_\epsilon(x)$$

where the superscript denotes which metric is being used. This proves that every ρ-open is d-open.

Since h is continuous and strictly increasing, it is a homeomorphism of a (small) neighborhood of 0 onto a (small) neighborhood of $h(0) = 0$. The local inverse h^{-1} satisfies $h^{-1}(0) = 0$ and is continuous at 0, therefore the argument above applies again.

For any $\epsilon > 0$, there is a $\delta > 0$ guaranteeing $h^{-1}(s) < \epsilon$ whenever $s < \delta$. Substituting $s = \rho(x, y)$, we obtain $d(x, y) < \epsilon$ whenever $\rho(x, y) < \delta$, i.e.

$$B^\rho_\delta(x) \subseteq B^d_\epsilon(x)$$

so that every d-open is ρ-open. $\square$

Remark. We could also have used the formula $\rho(x, y) = \min\{d(x, y), 1\}$. The goal was just to find a metric ρ which is topologically equivalent to d and is bounded.

$\boxed{\textbf{c.}}$ Let $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ be a countable family of metric spaces, where each metric d_i is bounded by 1, i.e.

$$d_i(x_i, y_i) \leq 1 \text{ for all } x_i, y_i \in X_i.$$

Write $X := \prod_{i\in\mathbb{N}} X_i$ and consider the function $d\colon X \times X \to \mathbb{R}$ defined by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i(x_i, y_i).$$

Show that d is a metric on X. (First check that d is a well-defined function.)

Solution. For any $x, y \in X$, the series defining $d(x, y)$ has only non-negative terms and is bounded:

$$\sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i(x_i, y_i) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

hence the series converges, so that $d(x, y)$ is well-defined.

We check the three properties of a metric.

1. Positivity:

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i(x_i, y_i) \geq \sum_{i=1}^{\infty} \frac{1}{2^i}\,(0) = 0$$

$$\begin{aligned} d(x, y) &= \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i(x_i, y_i) = 0 \\ &\Leftrightarrow \frac{1}{2^i}\, d_i(x_i, y_i) = 0 \text{ for all } i \in \mathbb{N} \\ &\Leftrightarrow d_i(x_i, y_i) = 0 \text{ for all } i \in \mathbb{N} \\ &\Leftrightarrow x_i = y_i \text{ for all } i \in \mathbb{N} \\ &\Leftrightarrow x = y. \end{aligned}$$

2. Symmetry:

$$\begin{aligned} d(y, x) &= \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i(y_i, x_i) \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i(x_i, y_i) \\ &= d(x, y). \end{aligned}$$

3. Triangle inequality: For any $x, y, z \in X$, we have

$$\begin{aligned}
d(x,y) &= \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i(x_i, y_i) \\
&\leq \sum_{i=1}^{\infty} \frac{1}{2^i}\, \left(d_i(x_i, z_i) + d_i(z_i, y_i)\right) \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i, z_i) + \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i(z_i, y_i) \\
&= d(x,z) + d(z,y). \quad \square
\end{aligned}$$

$\boxed{\textbf{d.}}$ Show that the metric d from part (c) induces the product topology on $X = \prod_{i \in \mathbb{N}} X_i$.

Solution. ($\mathcal{T}_{\text{met}} \leq \mathcal{T}_{\text{prod}}$) Let $\epsilon > 0$ and consider the open ball $B_\epsilon(x)$ centered at a point $x \in X$. We want to find an open "large box" $U = \prod_{i \in \mathbb{N}} U_i$ inside $B_\epsilon(x)$.

Let $N \in \mathbb{N}$ be large enough to guarantee the inequality

$$\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2}.$$

For the indices $1 \leq i \leq N$, take the radii $\epsilon_i := \frac{\epsilon}{2}$ and consider the open "large box" $U = \prod_{i \in \mathbb{N}} U_i$ defined by

$$U_i := \begin{cases} B_{\epsilon_i}(x_i) & \text{if } i \leq N \\ X_i & \text{if } i > N. \end{cases}$$

Note that U is indeed open in the product topology, since $U_i \subseteq X_i$ is open for all $i \in \mathbb{N}$ and $U_i \neq X_i$ for finitely many indices i.

For any point $y \in U$, its distance to x is

$$\begin{aligned} d(x, y) &= \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i) \\ &= \sum_{i=1}^{N} \frac{1}{2^i} d_i(x_i, y_i) + \sum_{i=N+1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i) \\ &\leq \sum_{i=1}^{N} \frac{1}{2^i} d_i(x_i, y_i) + \sum_{i=N+1}^{\infty} \frac{1}{2^i} (1) \\ &< \sum_{i=1}^{N} \frac{1}{2^i} d_i(x_i, y_i) + \frac{\epsilon}{2} \\ &< \sum_{i=1}^{N} \frac{1}{2^i} \epsilon_i + \frac{\epsilon}{2} \\ &= \sum_{i=1}^{N} \frac{1}{2^i} \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2} \sum_{i=1}^{N} \frac{1}{2^i} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

which proves the inclusion $U \subseteq B_\epsilon(x)$.

($\mathcal{T}_{\text{prod}} \leq \mathcal{T}_{\text{met}}$) Let $U = \prod_{i \in \mathbb{N}} U_i$ be an open "large box" and $x \in U$. We want to find an open ball around x satisfying $B_\epsilon(x) \subseteq U$.

By definition of large box, there is an index $N \in \mathbb{N}$ satisfying $U_i = X_i$ for all $i > N$. For the indices $1 \leq i \leq N$, we have $x_i \in U_i$ and $U_i \subseteq X_i$ is open, so that there is a radius $\epsilon_i > 0$ satisfying $x_i \in B_{\epsilon_i}(x_i) \subseteq U_i$.

Take the radius $\epsilon = \frac{1}{2^N} \min\{\epsilon_1, \ldots, \epsilon_N\}$ and consider the open ball $B_\epsilon(x)$. We claim $B_\epsilon(x) \subseteq U$.

Let $y \in B_\epsilon(x)$. For indices $1 \leq i \leq N$, the point y satisfies

$$\begin{aligned} \frac{1}{2^i}\, d_i(x_i, y_i) &\leq \sum_{j=1}^{\infty} \frac{1}{2^j}\, d_j(x_j, y_j) \\ &= d(x, y) \\ &< \epsilon \\ &\leq \frac{1}{2^N}\, \epsilon_i \\ &\leq \frac{1}{2^i}\, \epsilon_i \end{aligned}$$

or equivalently $d_i(x_i, y_i) < \epsilon_i$, which implies $y_i \in B_{\epsilon_i}(x_i) \subseteq U_i$.

For the remaining indices $i > N$, there is no constraint on y_i, namely $y_i \in U_i = X_i$ automatically. This proves $y \in \prod_{i \in \mathbb{N}} U_i = U$ and therefore the inclusion $B_\epsilon(x) \subseteq U$. $\square$