MA52300 FALL 2016

Homework Assignment 7

Due Wed, Nov 2, 2016

1. Solve the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2 \\ u = x_1 & \text{on } |x| = 1 \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

   Hint: Use Laurent series.

2. Let Ω be a bounded domain with a C^1 boundary, $g \in C(\partial\Omega)$ and $f \in C(\overline{\Omega})$. Consider then the so-called *Neumann problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{$*$}$$

   where ν is the outer normal on $\partial\Omega$. Show that the solution of $(*)$ in $C^2(\Omega) \cap C^1(\overline{\Omega})$ is unique up to a constant; i.e., if u_1 and u_2 are both solutions of $(*)$, then $u_2 = u_1 + \mathit{const}$ in Ω.

   Hint: Look at the proof of the uniqueness for the Dirichlet problem by energy methods, [E, 2.2.5a].

3. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ \qquad\qquad\quad u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

   where $c \in \mathbb{R}$.

   Hint: Rewrite the problem in terms of $v(x,t) := e^{ct} u(x,t)$.