# MA 519: Homework 11

Max Jeter, Carlos Salinas

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## Problem 11.1 (DasGupta 7.2 (a), (b), (c), (d), (e))

(a) Suppose $E|X_n - c|^\alpha \to 0$, where $0 < \alpha < 1$. Does $X_n$ necessarily converge in probability to $c$?
(b) Suppose $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0,1)$. Under what condition on $a_n$ can we conclude that $X_n \xrightarrow{\mathcal{P}} \theta$?
(c) $\mathrm{o}_p(1) + \mathrm{O}_p(1) = ?$
(d) $\mathrm{o}_p(1)\mathrm{O}_p(1) = ?$
(e) $\mathrm{o}_p(1) + \mathrm{o}_p(1)\mathrm{O}_p(1) = ?$

*Solution.* For part (a) we show that indeed $E(|X_n - c|^\alpha) \to 0$ implies $X_n \xrightarrow{\mathcal{P}} c$. Let $\varepsilon > 0$ be given. By Markov's inequality, we have

$$P(|X_n - c| > \varepsilon) = P(|X_n - c|^\alpha > \varepsilon^\alpha) \le \frac{E(|X_n - c|^\alpha)}{\varepsilon^\alpha}.$$

Since $E(|X_n - c|^\alpha) \to 0$ as $n \to \infty$, it follows that $P(|X_n - c| > \varepsilon)$ as $n \to \infty$; i.e., $X_n$ converges to $c$ in probability.

For part (b), suppose $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0,1)$; i.e., $P(|a_n(X_n - \theta)| \le x) \to \Phi(x)$ as $n \to \infty$. In words $X_n \xrightarrow{\mathcal{P}} \theta$ means that for every $\varepsilon > 0$ and every $\eta > 0$ there exists a positive integer $N$ depending on $\varepsilon$ and $\eta$ such that $n \ge N$ implies

$$P(|X_n - \theta| \ge \varepsilon) < \eta.$$

First, let us find the PDF of the sequence $a_n(X_n - \theta)$. Let $f_n$ denote the PDF of $X_n$, then the CDF of $a_n(X_n + \theta)$ is

$$\begin{aligned}
P(|a_n(X_n - \theta)| \le x) &= P(-x \le a_n(X_n - \theta) \le x) \\
&= P\left(-\frac{x}{a_n} + \theta \le X_n \le \frac{x}{a_n} + \theta\right) \\
&= \int_{-x/a_n+\theta}^{x/a_n+\theta} f(y)\, dy \\
&= f(x/a_n + \theta) - f(-x/a_n + \theta),
\end{aligned}$$

therefore its PDF is

$$\begin{aligned}
\frac{dP}{dx}(|a_n(X_n - \theta)| \le x) &= \frac{d}{dx}\left[f\left(\frac{x}{a_n} + \theta\right) - f\left(-\frac{x}{a_n} + \theta\right)\right] \\
&= \frac{1}{a_n}\big(f(x/a_n + \theta) + f(-x/a_n + \theta)\big)
\end{aligned}$$

For part (c), suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n = \mathrm{o}_p(1)$ and $b_n = \mathrm{O}_p(1)$, then for the sequence $\{c_n := a_n + b_n\}$ the most we can expect is $c_n = \mathrm{O}_p(1)$. Indeed, we know that if a sequence is $\mathrm{o}_p(1)$ then it is also $\mathrm{O}_p(1)$ therefore there exists $K_1$ and $K_2$ such that $|a_n| \le K_1$, $|b_n| \le K_2$ for all $n \ge 1$. Therefore, $|c_n| \le K_1 + K_2$ for all $n \ge 1$.

For part (d), suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n = \mathrm{o}_p(1)$ and $b_n = \mathrm{O}_p(1)$, then for the sequence $\{c_n := a_n b_n\}$ the most we can expect is $c_n = \mathrm{O}_p(1)$. Again, since $\{a_n\}$ is $\mathrm{o}_p(1)$ it is $\mathrm{O}_p(1)$ so there exists a constant $K_1 \geq 0$ such that $|a_n| \leq K_1$ for all $n \geq 1$ and similarly for $\{b_n\}$ there exists a constant $K_2$ such that $|b_n| \leq K_2$ for all $n \geq 1$. Therefore, $|c_n| \leq K_1 K_2$ for all $n \geq 1$ so $c_n = \mathrm{O}_p(1)$.

For part (e), suppose $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences such that $a_n, b_n = \mathrm{o}_p(1)$ and $c_n = \mathrm{O}_p(1)$, then for the sequence $\{d_n := a_n + b_n c_n\}$ the most we can expect is $d_n = \mathrm{O}_p(1)$ since there exists contstants $K_1$, $K_2$, and $K_3$ such that $|a_n| \leq K_1$, $|b_n| \leq K_2$, and $|c_n| \leq K_3$ for all $n \geq 1$. This implies that $|d_n| \leq K_1 + K_2 K_3$ for all $n \geq 1$. Thus, $d_n = \mathrm{O}_p(1)$. ∎

## Problem 11.2 (DasGupta 7.3 [Monte Carlo])

Consider the purely mathematical problem of finding a definite integral $f(x)\,dx$ for some (possibly complicated) function $f(x)$. Show that the SLLN provides a method for approximately finding the value of the integral by using appropriate averages $\frac{1}{n}\sum_{k=1}^{n} f(X_k)$.

Numerical analysts call this Monte Carlo integration.

*Solution.* Let $X_k$, for $1 \leq k \leq n$, be independent and identically distributed $U[a,b]$ random variables and let $f\colon [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Moreover, let us denote the integral of $f$ on $[a,b]$ by

$$I := \int_a^b f\,dx$$

and the average of $n$ random sample points from $[a,b]$ by

$$I_n := \frac{1}{n}\sum_{k=1}^{n} f(X_k).$$

We aim to show that the strong law of large numbers implies $I_n \to I$ almost surely as $n \to \infty$. ■

## Problem 11.3 (DasGupta 7.4 (a), (b))

Suppose $X_1, \dots,$ are i.i.d. and that $E(X_1) = \mu \neq 0$, $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Let $S_{m,p} = \sum_{k=1}^{m} X_k^p$, $m \geq 1$, $p = 1, 2$.

(a) Identify with proof the almost sure limit of $S_{m,1}/S_{n,1}$ for fixed $m$, and $n \to \infty$.
(b) Identify with proof the almost sure limit of $S_{n-m,1}/S_{n,1}$ for fixed $m$, and $n \to \infty$.

*Solution.* ■

## Problem 11.4 (DasGupta 7.5 (a))

Let $A_n$, $n \geq 1$, $A$ be events with respect to a common sample space $\Omega$.

(a) Prove that $I_{A_n} \xrightarrow{\mathcal{L}} I_A$ if and only if $P(A_n) \to P(A)$.

*Solution.* ■

## Problem 11.5 (DasGupta 7.11 [Sample Maximum])

Let $X_k$, $k \geq 1$, be an i.i.d. sequence, and $X_{(n)}$ the maximum of $X_1, \ldots, X_n$. Let $\xi(F) = \sup\{\, x : F(x) < 1 \,\}$, where $F$ is the common CDF of the $X_k$. Prove that $X_{(n)} \xrightarrow{\text{a.s.}} \xi(F)$.

*Solution.* ∎

## Problem 11.6 (DasGupta 7.14 (a))

Suppose $X_k$ are i.i.d. standard Cauchy. Show that

(a) $P(|X_n| > n \text{ infinitely often}) = 1$.

*Solution.* ■

## Problem 11.7 (DasGupta 7.16 [Coupon Collection])

Cereal boxes contain independently and with equal probability exactly one of $n$ different celebrity pictures. Someone having the entire set of $n$ pictures can cash them in for money. Let $W_n$ be the minimum number of cereal boxes one would need to purchase to own a complete set of the pictures. Find a sequence $a_n$ such that $W_n/a_n \xrightarrow{\mathcal{P}} 1$.
(*Hint:* Approximate the mean of $W_n$.)

*Solution.* Let $X_n \sim \text{Geom}(\frac{n-k}{n})$ ∎

## Problem 11.8 (DasGupta 7.17)

Let $X \sim \text{Bin}(n, p)$. Show that $(X_n/n)^2$ and $X_n(X_n - 1)/(n(n-1))$ both converging in probability to $p^2$. Do they converge almost surely?

*Solution.* ■

## Problem 11.9 (DasGupta 7.21)

Let $X_1, X_2, \dots,$ be i.i.d. $U[0,1]$. Let

$$G_n = (X_1 \cdots X_n)^{1/n}.$$

Find $c$ such that $G_n \xrightarrow{\mathcal{P}} c$.

*Solution.* ∎

## Problem 11.10 (DasGupta 7.30 [Conceptual])

Suppose $X_n \xrightarrow{\mathcal{L}} X$, and also $Y_n \xrightarrow{\mathcal{L}} X$. Does this mean that $X_n - Y_n$ converge in distribution to (the point mass at) zero?

*Solution.* ∎

## Problem 11.11 (DasGupta 7.31 (a))

(a) Suppose $a_n(X_n - \theta) \to N(0, \tau^2)$; what can be said about the limiting distribution of $|X_n|$, when $\theta \neq 0$, $\theta = 0$?

*Solution.* ■