#1 (a) Wlog assume $r > s$

$|A(r) - A(s)| =$

$$\left| \frac{1}{r^n} \int_{B_s} |f(x-y) - f(x)| + \frac{1}{r^n} \int_{B_r \setminus B_s} |f(x-y) - f(x)| - \frac{1}{s^n} \int_{B_s} |f(x-y) - f(x)| \right|$$

$$\leq \left(\frac{1}{s^n} - \frac{1}{r^n} \right) \int_{B_s} |f(x-y) - f(x)| \, dy + \frac{1}{r^n} \int_{B_r \setminus B_s} |f(x-y) - f(x)| \, dy$$

Since $|f(x-y) - f(x)| \in L^1_{loc}(dy)$, both terms go to 0 as $r \to s$ (use absolute continuity of integral in the second one).

$A(r) \to 0$ as $r \to 0$ because x is a Lebesgue pt. (change variable $x - y = z$)

(b) Part (a) $\Rightarrow \exists\, c$ s.t.

$A(r) \leq c \quad \forall\, r \in (0, 1]$

If $r \geq 1$

$$A(r) \leq \int_{\mathbb{R}^n} |f(x-y) - f(x)| \, dy$$

$$\leq 2 \|f\|_1 .$$

#2. $\Rightarrow$)

$$\|f_k\|_p \le \|f_k - f\|_p + \|f\|_p$$

$$\Rightarrow \|f_k\|_p - \|f\|_p \le \|f_k - f\|_p$$

Similarly $$\|f\|_p - \|f_k\|_p \le \|f_k - f\|_p$$

$$\Rightarrow 0 \le \big|\|f_k\|_p - \|f\|_p\big| \le \|f_k - f\|_p \to 0$$

$\Leftarrow$) Show first that

$$|a-b|^p \le 2^{p-1}(|a|^p + |b|^p) \quad \forall a, b \in \mathbb{R}.$$

Apply the inequality to $|f_k - f|^p$ and then apply (after proving it) Problem #23 on p. 110.

#3. Assume first $1 \le p < \infty$

$$|(f * g)(x+h) - (f*g)(x)|$$

$$\le \int_{\mathbb{R}^n} |f(x+h-y) - f(x-y)|\, |g(y)|$$

$$\le \|f(\cdot + h) - f\|_p \, \|g\|_{p'} \xrightarrow[h \to 0]{} 0$$

by continuity in L^p.

If $p = \infty$, then $p' = 1$ and the roles of p & p' can be reversed in the above argument.

#4. (a) $f(x)\cos(tx) \to f(x)\cos(t_0 x)$ as $t \to t_0$

$$|f(x)\cos(tx)| \le |f(x)|, \quad f \in L^1$$

Apply Lebesgue dominated convergence.

(b) Assume $f = \chi_{[a,b]}$

$$\left|\int_a^b \cos tx\right| = \left|\frac{1}{t} \sin tx \Big|_a^b\right|$$

$$= \left|\frac{\sin bx - \sin ax}{t}\right| \le \frac{2}{t} \xrightarrow[t\to\infty]{} 0$$

Let $f \in L^1(\mathbb{R}) \Rightarrow \exists\, s \in L^1(\mathbb{R})$,

s simple function s.t. $\|f-s\|_1 < \varepsilon/2$

$$\left|\int f(x)\cos(tx)\,dx\right| \le \int |f(x)-s(x)||\cos tx| + \left|\int s(x)\cos(tx)\right|$$

$$\le \|f-s\|_1 + \left|\int s(x)\cos(tx)\right|$$

$$< \varepsilon/2 + \varepsilon/2 \quad \text{if } t \ge T.$$

#5. $\int \phi \, df$ exists b/c $\phi \in C([a,b])$ and $f \in BV([a,b]) \Rightarrow \exists \int f \, d\phi$ and

$$\int \phi \, df = \phi(b) f(b) - \phi(a) f(a) - \int f \, d\phi$$

Now, (*) $\int f \, d\phi = \int g \, d\phi + \int h \, d\phi$

We have that $\int g \, d\phi$ exists b/c g is abs. cont. and ϕ is cont., and

$$-\int g \, d\phi = +\int \phi \, dg - \phi(b) g(b) + \phi(a) g(a)$$

$$= \int \phi g' - \phi(b) g(b) + \phi(a) g(a).$$

Moreover, we know from (*) that $\int h \, d\phi$ exists $\Rightarrow \int \phi \, dh$ exists and

$$\int \phi \, dh = \phi(b) h(b) - \phi(a) h(a) - \int h \, d\phi$$

Combining we get

$$\int \phi \, df = \phi(b) f(b) - \phi(a) f(a) + \int \phi g' - \phi(b) g(b) + \phi(a) g(a) + \int \phi \, dh - \phi(b) h(b) + \phi(a) h(a)$$

$$= \int \phi g' + \int \phi \, dh$$