MA166: Recitation 10

Carlos Salinas

March 30, 2016

1 Homework

Here is this week's WebAssign homework and outlines of the solutions to the problems.

I have three sections the first one beginning at **1:30 PM** and the last one at 3:30 pm in **REC 315**. Students should only ask questions about homework # 25, 26, and 27.

성준아, here is the material you need to know to give this recitation tomorrow. The subject for tomorrow's recitation are the different convergence tests for series found in sections §11.6 to 11.8 of Stewart's book. Here is a rough summary of the material on those sections:

Stewart §11.6–§11.8 Summary

§11.6: Absolute Convergence and the Ratio and Root Tests

Definition 1. A series $\sum a_n$ is called *absolutely convergent* if the series of absolute values $\sum |a_n|$ is convergent.

Definition 2. A series $\sum a_n$ is called *conditionally convergent* if it is convergent but not absolutely convergent.

Theorem 1. *If a series $\sum a_n$ is absolutely convergent, then it is conditionally convergent.*

Theorem 2 (The Ratio Test)**.** (i) *If* $\lim |a_{n+1}/a_n| = L < 1$*, then the series* $\sum a_n$ *is absolutely convergent (and therefore convergent).*

(ii) *If* $\lim |a_{n+1}/a_n| = L > 1$ *or* ∞*, then the series* $\sum a_n$ *is divergent.*

(iii) *If* $\lim |a_{n+1}/a_n| = L = 1$*, the* ratio test *is inconclusive; that is, conclusion can be drawn about the convergence or divergence of* $suma_n$.

Theorem 3 (The Root Test)**.** (i) *If* $\lim \sqrt[n]{|a_n|} = L < 1$*, the series* $\sum a_n$ *is is absolutely convergent (and therefore convergent).*

(ii) *If* $\lim \sqrt[n]{|a_n|} = L > 1$ *or* ∞*, then the series* $\sum a_n$ *is divergent.*

(iii) *If* $\lim \sqrt[n]{|a_n|} = L = 1$*, the* root test *is inconclusive.*

§11.7: Strategy for Testing Series

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.

2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$.

3. If the series has a form similar to a p-series or geometric series, then one of the comparison tests should be considered.

4. If you can see $\lim a_n \neq 0$, then the test for divergence should be used.

5. If the series has the form $\sum (-1)^{n+1} b_n$ or $\sum (-1)^n b_n$, the alternating test is an obvious choice.

6. Series that involve factorials and other products are handled conveniently with the ratio test.

7. If a_n has the form ${b_n}^n$, then the root test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,\mathrm{d}x$ is easily evaluated, the integral test is effective.

§11.8: Power Series

Definition 3. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

where x is a variable and the c_n's are constants called *coefficients* of the series.

Definition 4. More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2 + (x-a)^2 + \cdots$$

is called a *power series in* $(x-a)$ or a *power series centered at* a or a *power series series about* a.

Theorem 4. *For a given power series* $\sum c_n(x-a)^n$ *there are only three possibilities:*

(i) *The series converges only when* $x = a$.
(ii) *The series converges for all* x.
(iii) *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

The number R in (iii) is called the *radius of convergence* of the power series. By convention, $R = 0$ in case (i) and ∞ in case (ii). The *interval of convergence* of a power series is the interval that consists of all values of x for which the series converges.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum n! x^n$	$r = 0$	$\{0\}$
Example 2	$\sum \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

Now

Homework 25

Problem 1 (WebAssign HW 25, # 1)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3-9}}.$$

Problem 2 (WebAssign HW 25, # 2)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{\sin 7n}{5^n}.$$

Problem 3 (WebAssign HW 25, # 3)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{12^n}{(n+1)5^{2n+1}}.$$

Problem 4 (WebAssign HW 25, # 4)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln 6n}.$$

Problem 5 (WebAssign HW 25, # 5)**.** The terms of a series are defined recursively by the equations

$$a_1 = 4 \qquad a_{n+1} = \frac{7n+1}{3n+9} \cdot a_n.$$

Determine whether $\sum a_n$ is absolutely convergent, conditionally convergent, or divergent.

Homework 26

Problem 6 (WebAssign HW 26, # 1)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \left(\frac{n^2+4}{5n^2+2}\right)^n.$$

Problem 7 (WebAssign HW 26, # 2)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{1}{n+6^n}$$

Problem 8 (WebAssign HW 26, # 3)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{(4n+1)^n}{n^{5n}}.$$

Problem 9 (WebAssign HW 26, # 4)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty}(-1)^n \frac{5n}{n+3}.$$

Problem 10 (WebAssign HW 26, # 5)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty}\frac{n^2 8^{n-1}}{(-9)^n}.$$

Problem 11 (WebAssign HW 26, # 6)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty}\frac{1}{8n+5}.$$

Problem 12 (WebAssign HW 26, # 7)**.** Test the series for convergence or divergence.

$$\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln 7n}}.$$

Problem 13 (WebAssign HW 26, # 8)**.** Test the series for convergence or divergence.

$$\sum_{k=1}^{\infty}\frac{6^k k!}{(k+2)!}.$$

Problem 14 (WebAssign HW 26, # 9)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty}\frac{6n!}{e^{n^2}}.$$

Homework 27

Problem 15 (WebAssign HW 27, # 1)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=2}^{\infty}\frac{x^{n+1}}{2n!}.$$

Find the interval, I, of convergence of the series.

Problem 16 (WebAssign HW 27, # 2)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=1}^{\infty}\frac{8^n x^n}{n^3}.$$

Find the interval, I, of convergence of the series.

Problem 17 (WebAssign HW 27, # 3)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n\sqrt{n}} x^n.$$

Find the interval, I, of convergence of the series.

Problem 18 (WebAssign HW 27, # 4)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{12^n \ln n}.$$

Find the interval, I, of convergence of the series.

Problem 19 (WebAssign HW 27, # 5)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=0}^{\infty} \frac{(x-4)^n}{n^6+1}.$$

Find the interval, I, of convergence of the series.

Problem 20 (WebAssign HW 27, # 6)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=1}^{\infty} \frac{6^n (x+4)^n}{\sqrt{n}}.$$

Find the interval, I, of convergence of the series.

Problem 21 (WebAssign HW 27, # 7)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n}.$$

Find the interval, I, of convergence of the series.

Problem 22 (WebAssign HW 27, # 8)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=1}^{\infty} \frac{n}{b^n} (x-a)^n, \qquad b > 0.$$

Find the interval, I, of convergence of the series.

Problem 23 (WebAssign HW 27, # 9)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=1}^{\infty} n!(3x-1)^n.$$

Find the interval, I, of convergence of the series.

Problem 24 (WebAssign HW 27, # 10)**.** Find the radius of convergence, R, of the series.

$$\sum_{n=2}^{\infty} \frac{x^{6n}}{n(\ln n)^8}.$$

Find the interval, I, of convergence of the series.

2 Relevant Exam Problems

If you run out of things to talk about within the first fifteen to twenty minutes, talk about these

Problem 25 (Exam 3, Spring 2015, # 2)**.** The series

$$\sum_{n=1}^{\infty} \frac{1}{(n^{3\alpha}+9)^{1/8}}$$

if and only if α is?

Solution. Note that

$$(n^{3\alpha}+9)^{1/8} > n^{3\alpha/8}$$

so that

$$\frac{1}{(n^{3\alpha}+9)^{1/8}} < \frac{1}{n^{3\alpha/8}}$$

the left of which forms a p-series with $p = 3\alpha/8$. For this p-series to converge $3\alpha/8 > 1$ so $\alpha > 8/3$. This narrows down the answer to exactly one of the choices, in the exam namely C. ☺

Problem 26 (Exam 3, Spring 2015, # 3)**.** Test the following series for convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \arctan(\pi/2n)$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+9}{(n^3+4)\sqrt{n}}$.

Solution. In the case of (a), you've seen time and time again that the limit of the sequence $a_n := n\sin(1/n)$ goes to 1 as $n \to \infty$ (if you are not convinced, use l'Hôpital's rule on the limit $x\sin(1/x)$ as $x \to \infty$). In particular, since the limit is nonzero, this series cannot converge.

In the case of (b), note that $a_n := \arctan(\pi/2n)$ fails to satisfy the second condition for the alternating series test, namely, since $\pi/2n \to \infty$ as $n \to \infty$, we have $\arctan(\pi/2n) \to \pi/2 \neq 0$, so the series must diverge.

Lastly, in the case of (c), we have

$$(n^3+4)\sqrt{n} > n^3\sqrt{n} = n^{7/2}$$

so

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^2+9}{(n^3+4)\sqrt{n}} &< \sum_{n=1}^{\infty} \frac{n^2+9}{n^{7/2}} \\ &= \sum_{n=1}^{\infty} \frac{n^2}{n^{7/2}} + 9\sum_{n=1}^{\infty} \frac{1}{n^{7/2}}\end{aligned}$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}}_{S_1} + 9 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{7/2}}}_{S_2}$$

Both of which are p-series with $p_1 = 5/2 > 1$ and $p_2 = 7/2 > 1$, respectively. This implies that S_1 and S_2 converge so by the (as Stewart calls it) direct comparison test, the original series converges. Therefore, the letter answer to this problem is D.

Correction!!! It seems the answer is actually C and the difference here is that the test writer is claiming (b) converges. The only explanation can see for this is that he meant to write $\arctan(\pi/(2n))$. In this case the conditions of the alternating series tests are satisfied since (i) $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(\pi/(2x))) = -\pi x^{-2}/(2 + \pi^2 x^{-2}/2) < 0$ on $1 \leq x \leq \infty$ so the terms of the series are decreasing and (ii) $\lim \arctan(\pi/(2n)) = 0$ since $\pi/(2n) \to 0$ as $n \to 0$ and $\arctan(0) = 0$. ☺

Problem 27 (Exam 3, Spring 2015, # 7)**.** Suppose that the powers

$$\sum_{n=0}^{\infty} c_n (x-5)^n$$

converges when $x = 2$ and diverges when $x = 10$.

From the above information, which of the following statements can we conclude to be true?

I. The radius of convergence R satisfies $3 \leq R \leq 5$.

II. We *cannot* determine the interval of convergence from the above information only.

III. The derivative of the power series is $\sum_{n=1}^{\infty} n c_n (x-5)^{n-1}$ which converges when $x = 3$.

Solution. For case I just recall what it means to be the radius of convergence. It's a real number $R \geq 0$ such that $|x - 5| < R$ implies that the series converges. In this case, we are told that for $x = 2$ and for $x = 10$, which tells us that the series converges for $|x-5| \leq 3$ and diverges $|x-5| \geq 5$ and so we know that the radius of convergence cannot exceed 5 and is at least as big as 3. Hence, I holds.

For case II, the information cannot be used to determine the interval of convergence since, given that $3 \leq R \leq 5$ is true, the interval of convergence may be any one of the intervals with $(2, 8)$ or $(0, 10)$ as endpoints (possibly including the endpoints).

For case III, given that we know $|2 - 5| \leq 3$ implies $\sum c_n (2-5)^n$ converges, by the limit comparison test (as Stewart calls it), since $c_n/(nc_n) \to 0$ as $n \to \infty$, this implies that $\sum n c_n (2-5)^n$ converges. Therefore, the letter answer is D. (I and II are true). ☺

고마워, 성준!