MA 544: Homework 2

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Problem 2.1

Show that the boundary of any interval has outer measure zero.

Proof. Let $I = \bigtimes_{i=1}^{n} I_i$ be an n-dimensional interval where $I_n = [a_i, b_i]$ is an interval in $\mathbf{R}$. Consider the boundary

$$\partial I = \bigcup_{i=1}^{2n} I_1 \times \cdots \times \{c_i\} \times \cdots \times I_n$$

where $c_i := a_i$ for $1 \leq i \leq n$ is odd and $c_i := b_i$ for $n < i \leq 2n$. Thus, by theorem 3.4, it suffices to show that each $J_i := I_1 \times \cdots \times \{c_i\} \times \cdots \times I_n$ has measure zero. Let $\varepsilon > 0$ be given. Put $M := \prod_{j \neq i} I_j$. Then the family of intervals

$$I_k := I_1 \times \cdots \times \left[c_i - \frac{\varepsilon}{M 2^{k+1}}, c_i + \frac{\varepsilon}{M 2^{k+1}}\right] \times \cdots \times I_n,$$

is a countable collection of n-dimensional intervals containing J_i. Hence, by theorem 3.4 (subadditivity), we have

$$|J_i|_e \leq \sum_{k=0}^{\infty} M\left(\frac{\varepsilon}{M 2^k}\right) = \sum_{k=0}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon. \tag{1}$$

Letting $\varepsilon \to 0$, we have $|J_i|_e = 0$. Again, by theorem 3.4, we have

$$|\partial I|_e \leq \sum_{i=1}^{2n} |J_i|_e$$

where, by (1), the right is zero so $|\partial I|_e = 0$. ■

Problem 2.2

Show that a set consisting of a single point has outer measure zero.

Proof. Let x be a point in $\mathbf{R}^n$. Let $\varepsilon > 0$ be given. Consider the family of intervals

$$I_k := \bigtimes_{i=1}^{n} \left[x_i - \sqrt[n]{\frac{\varepsilon}{2^{k+2}}}, x_i + \sqrt[n]{\frac{\varepsilon}{2^{k+2}}} \right]$$

where by x_i we mean the ith coordinate of x. It is clear that each I_k contains $\{x\}$ since in the projection, the projection of each interval $\pi_i(I_k)$ contains an x_i. Then, by theorem 3.4, we have

$$\begin{aligned}
|\{x\}|_e &\leq \sum_{k=0}^{\infty} |I_k|_e \\
&= \sum_{k=0}^{\infty} \left[\prod_{i=1}^{n} \left(x_i + \sqrt[n]{\frac{\varepsilon}{2^{k+2}}} - \left(x_i - \sqrt[n]{\frac{\varepsilon}{2^{k+2}}} \right) \right) \right] \\
&= \sum_{k=0}^{\infty} \left(\prod_{i=1}^{n} \sqrt[n]{\frac{\varepsilon}{2^k}} \right) \\
&= \sum_{k=0}^{\infty} \frac{\varepsilon}{2^k} \\
&= \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we see that $|\{x\}|_e = 0$. ■