MA 544: Homework 7

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Problem 7.1 (Wheeden & Zygmund §4, Ex. 9)

(a) Show that the limit of a decreasing (increasing) sequence of functions usc (lsc) at $\mathbf{x}_0$ is usc (lsc) at $\mathbf{x}_0$. In particular, the limit of a decreasing (increasing) sequence of functions continuous at $\mathbf{x}_0$ is usc (lsc) at $\mathbf{x}_0$.

(b) Let f be usc and less than ∞ on $[a, b]$. Show that there exists continuous f_k on $[a, b]$ such that $f_k \searrow f$.

Proof. (a) Suppose $\{f_k \colon E \to \mathbf{R}\}_{k=1}^{\infty}$ is a sequence of decreasing functions which are usc at $\mathbf{x}_0 \in E$ and put $f := \lim f_k$. Then, since the f_k's are decreasing, we have

$$f(\mathbf{x}) \leq f_k(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in E. \tag{7.1}$$

Moreover, f_k being usc at $\mathbf{x}_0$ means that for any sequence $\mathbf{x}_k \to \mathbf{x}_0$, $\overline{\lim}_{\mathbf{x}\to\mathbf{x}_0} f_k(\mathbf{x}) \leq f_k(\mathbf{x}_0)$. Thus, by (7.1) we have

$$\overline{\lim_{\mathbf{x}\to\mathbf{x}_0}} f(\mathbf{x}) \leq \overline{\lim_{\mathbf{x}\to\mathbf{x}_0}} f_k(\mathbf{x}) \leq f_k(\mathbf{x}_0). \tag{7.2}$$

Now, let $k \to \infty$ and we have $\overline{\lim}_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) \leq f(\mathbf{x}_0)$. Thus, f is usc at $\mathbf{x}_0$.

(b) Mimicking the construction of the Cantor–Lebesgue function, divide the interval into $2n$ closed intervals $I_1 := [y_0, y_1], ..., I_{2n} := [y_{2n-1}, y_{2n}]$ where $y_0 = a$, $y_{2n} = b$. Define

$$f_n(y_k) := \max_{I_k \cup I_{k+1}} f(x) \tag{7.3}$$

and $f_n(x)$ is linear on I_n, i.e., is a line connecting $f(y_{n-1})$ to $f(y_n)$. Then the function f_n is clearly continuous on $[a, b]$ since it is piece-wise linear and it is decreasing since f is usc (on smaller intervals, the jumps are smaller). We claim that $f_n \to f$. Fix $x \in [a, b]$. Given $\varepsilon > 0$, for sufficiently large index $N \in \mathbf{N}$ for every $n \geq N$ we have

$$|f(x) - f_n(x)| < |f(x) - f_n(y_{k-1})|$$

for $x \in I_{k-1} \cup I_k$

$$= \left|f(x) - \max_{y \in I_{k-1}\cup I_k} f(y)\right|$$
$$< \varepsilon$$

for sufficiently small I_k's (I'm sure). ∎

Problem 7.2 (Wheeden & Zygmund §4, Ex. 11)

Let f be defined on $\mathbf{R}^n$ and let $B(\mathbf{x})$ denote the open ball $\{\, \mathbf{y} \mid |\mathbf{x}-\mathbf{y}| < r \,\}$ with center $\mathbf{x}$ and fixed radius r. Show that the function $g(\mathbf{x}) = \sup\{\, f(\mathbf{y}) \mid \mathbf{y} \in B(\mathbf{x}) \,\}$ is lsc and the function $h(\mathbf{x}) = \inf\{\, f(\mathbf{y}) \mid \mathbf{y} \in B(\mathbf{x}) \,\}$ is usc on $\mathbf{R}^n$. Is the same true for the closed ball $\{\, \mathbf{y} \mid |\mathbf{x}-\mathbf{y}| \le r \,\}$?

Proof. Fix $\mathbf{x}_0 \in \mathbf{R}^n$. By 4.14(ii), given a real number m such that $f(\mathbf{x}_0) > m$, it suffices to show that there exists $\delta > 0$ such that for every $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ we have $g(\mathbf{x}) > m$. Since $g(\mathbf{x}_0)$ is the supremum of $f(\mathbf{x})$ for all $\mathbf{x} \in B(\mathbf{x}_0)$, given $\varepsilon > 0$ such that $g(\mathbf{x}_0) - \varepsilon > m$ there exists $\mathbf{y} \in B(\mathbf{x}_0)$ such that $f(\mathbf{y}) > g(\mathbf{x}_0) - \varepsilon$. Let $\delta \coloneqq \frac{1}{2}(r - |\mathbf{y} - \mathbf{x}_0|)$ (δ is positive since $\mathbf{y} \in B(\mathbf{x}_0)$ hence $|\mathbf{y}-\mathbf{x}_0| < r$). Then for any $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ we have

$$\begin{aligned}
|\mathbf{x}-\mathbf{y}| &= |\mathbf{x}-\mathbf{x}_0 - (\mathbf{y}-\mathbf{x}_0)| \\
&\le |\mathbf{x}-\mathbf{x}_0| + |\mathbf{y}-\mathbf{x}_0| \\
&= \tfrac{1}{2}(r - |\mathbf{y}-\mathbf{x}_0|) + |\mathbf{y}-\mathbf{x}_0| \\
&= \tfrac{1}{2}r + \tfrac{1}{2}|\mathbf{y}-\mathbf{x}_0| \\
&< r
\end{aligned} \tag{7.4}$$

so $\mathbf{y} \in B(\mathbf{x})$. Hence, $g(\mathbf{x}) \ge f(\mathbf{y}) > g(\mathbf{x}_0) - \varepsilon > m$, i.e., $B_\delta(\mathbf{x}_0) \subset \{\, \mathbf{x} \mid f(\mathbf{x}) > m \,\}$. Thus, g is lsc on $\mathbf{R}^n$.

The proof for h is similar to g. In fact, note that for any set $E \subset \mathbf{R}$ we have $\inf E = -\sup(-E)$ so that if we set $f' \coloneqq -f$ and define $g'(\mathbf{x}) \coloneqq \sup\{\, f'(\mathbf{y}) \mid \mathbf{y} \in B(\mathbf{x}) \,\}$. By the above, g' is lsc in $\mathbf{R}^n$ so $h = -g'$ is usc in $\mathbf{R}^n$.

For the last part consider the map

$$f(x) \coloneqq \begin{cases} 1 & \text{if } x < x_0 \\ 0 & \text{if } x \ge x_0 \end{cases} \tag{7.5}$$

defined earlier in the section. Let $h(x) \coloneqq \inf\big\{\, f(y) \;\big|\; y \in \overline{B(x)} \,\big\}$. Let x be a distance of r to the right of x_0. Then $1/2 > h(x) = 0$. However, for any $\delta > 0$, $y \in B_\delta(x)$ we have $x' \in \overline{B(y)}$ such that $x' < x_0$. Hence, $h(y) = 1 > 1/2$ so h is not usc. ∎

Problem 7.3 (Wheeden & Zygmund §4, Ex. 15)

Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. If $|f_k(M)| \leq M < \infty$ for all k for each $\mathbf{x} \in E$, show that given $\varepsilon > 0$, there is closed $F \subset E$ and finite M such that $|E \smallsetminus F| < \varepsilon$ and $|f_k(\mathbf{x})| \leq M$ for all $\mathbf{x} \in F$.

Proof. Define $f := \sup |f_k|$. Note that, since $|f_k| = f^+ + f^-$ and f^+ and f^- are measurable, $|f_k|$ is measurable hence, by 4.11, f is measurable. Now, given $\varepsilon > 0$ by Lusin's theorem f has the $\mathcal{C}$-property on E, i.e., there exists a closed subset F of E such that $|E \smallsetminus F| < \varepsilon/2$ and f is continuous when restricted to F. Take the $\delta > 0$ such that $\left|E \smallsetminus \overline{B_\delta(\mathbf{0})}\right| < \varepsilon/2$. Then $F \cap \overline{B_\delta(\mathbf{0})}$ is closed and compact and we have

$$\begin{aligned}
\left|E \smallsetminus (F \cap \overline{B_\delta(\mathbf{0})})\right| &= \left|E \smallsetminus F \cup \left(E \smallsetminus \overline{B_\delta(\mathbf{0})}\right)\right| \\
&\leq |E \smallsetminus F| + \left|E \smallsetminus \overline{B_\delta(\mathbf{0})}\right| \\
&< \varepsilon.
\end{aligned}$$

By Problem 6.2 (W&Z, 4.7) f achieves its maximum M on $F \cap \overline{B_\delta(\mathbf{0})}$. Thus, $|f_k| \leq M$ for all $\mathbf{x} \in F \cap \overline{B_\delta(\mathbf{0})}$. ∎

Problem 7.4 (Wheeden & Zygmund §4, Ex. 18)

If f is measurable on E, define $\omega_f(a) = |\{ f > a \}|$ for $-\infty < a < \infty$. If $f_k \nearrow f$, show that $\omega_{f_k} \nearrow \omega_f$. If $f_k \to f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f. [For the second part, show that if $f_k \to f$, then $\overline{\lim}_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$ and $\underline{\lim}_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$ for every $\varepsilon > 0$.]

Proof. For fixed a define $E_k \coloneqq \{ f_k > a \}$. Then we have $E_1 \subset E_2 \subset \cdots$ so $E_k \nearrow \bigcup E_k$. Now, it is clear that $\{ f > a \} \supset \bigcup E_k$ since $\mathbf{x} \in \bigcup E_k$ implies that $\mathbf{x} \in E_k$ for some k so $f_k(\mathbf{x}) > a$ for all $K \geq k$. In particular, $f(\mathbf{x}) > a$. Thus, $\mathbf{x} \in \{ f > a \}$. On the other hand, if $\mathbf{x} \in \{ f > a \}$ then $f(\mathbf{x}) > a$ so $f_k \nearrow f$ implies that for sufficiently large N we have $f_N(\mathbf{x}) > a$. Thus, $\mathbf{x} \in E_N$ so $\mathbf{x} \in \bigcup E_k$ and we have

$$\{ f > a \} = \bigcup E_k. \tag{7.6}$$

It follows by 3.26(i) that $\omega_{f_k} \nearrow \omega_f$ point-wise. ■

Problem 7.5 (Wheeden & Zygmund §5, Ex. 1)

If f is a simple measurable function (not necessarily positive) taking values a_j on E_j, $j = 1, ..., N$, show that $\int_E f = \sum_{j=1}^N a_j|E_j|$. [Use (5.24)].

Proof. Since f is a simple measurable function $E_k \cap E_\ell = \emptyset$ for $k \neq \ell$. Since $E \coloneqq \bigcup_{j=1}^N E_j$ is countable, by 5.24 we have

$$\int_E f = \sum_{j=1}^N \int_{E_j} f = \sum_{j=1}^N \int a_j \chi_{E_j} = \sum_{j=1}^N a_j |E_j|.$$

■

Problem 7.6 (Wheeden & Zygmund §5, Ex. 3)

Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \le f$ a.e. on E, show that $\int_E f_k \to \int_E f$.

Proof. By Fatou's lemma we have

$$\int_E f = \int_E \varliminf_{k\to\infty} f_k \le \varliminf_{k\to\infty} \int_E f_k. \tag{7.7}$$

Now, by 5.10 since $f_k \le f$ a.e. on E we have $\int_E f_k \le \int_E f$ so

$$\varlimsup_{k\to\infty} \int f_k \le \int_E f. \tag{7.8}$$

But $\varliminf \int_E f_k \le \varlimsup \int_E f_k$ so we must have $\lim \int_E f_k \to \int_E f$ as $k \to \infty$. ∎