MA 523: Homework 2

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Problem 2.1

Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 because in such case we have $\nu(x^0) = (0, \ldots, 0, 1)$ so

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \ldots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall verify the noncharacteristic condition (2.1) by first flattening the boundary near x^0 and then applying the noncharacteristic boundary conditions to the flattened region. Assuming some regularity near x^0, Γ is a hypersurface of dimension $n - 1$. Assuming some degree of regularity near x^0, e.g., that the boundary be smooth, we may express Γ near x^0 as the graph of a smooth function $f \colon \mathbb{R}^{n-1} \to \mathbb{R}$, i.e., $x = (y, f(y))$ on Γ and $x_n \geq f(y)$ after reorienting the coordinate axes. Then we flatten out Γ via the map $\Phi(x) \colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\begin{cases} y_1 = x_1 = \Phi^1(x), \\ \quad \vdots \\ y_{n-1} = x_{n-1} = \Phi^{n-1}(x), \\ y_n = x_n - f(x_1, \ldots, x_{n-1}) = \Phi^n(x) \end{cases}$$

and write $y = \Phi(x)$. ◄

Problem 2.2

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. ►

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Problem 2.3

Show that the function $u(x,t)$ defined by $t \geq 0$ by

$$u(x,t) = \begin{cases} -\dfrac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burger's equation*).

Solution. ► ◄