MA544: Qual Preparation

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Contents

Contents **1**

1 MA 544 Spring 2016 **2**
- 1.1 Homework . . . 2
- 1.2 Exam 1 Prep . . . 5
- 1.3 Exam 1 . . . 10
- 1.4 Exam 2 Prep . . . 11
- 1.5 Midterm 2 . . . 19
- 1.6 Final Practice . . . 22

2 MA 544 Past Quals **23**
- 2.1 Danielli: Winter 2012 . . . 23
- 2.2 Danielli: Summer 2011 . . . 25

Bibliography **27**

CHAPTER 1

MA 544 Spring 2016

This is material from the course MA 544 as taught in the spring of 2016.

1.1 Homework

These exercises were assigned from Wheeden and Zygmund's *Measure and Integral*. Therefore, most of the theorems I reference will be from [4]. Other resources include [1] and [2]. For more elementary results, I cite [3].

Homework 1

Problem 1 (Wheeden & Zygmund Ch. 2, Ex. 1)**.** Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0,1]$, but that $V[f;0,1] = +\infty$.

Proof. It is clear that the function $f(x) = x\sin(1/x)$ is *bounded* on $[0,1]$ since $|\sin(1/x)| \leq 1$ and $|x| \leq 1$ on $[0,1]$. Moreover, by properties of *continuous functions* on $\mathbb{R}$, it is obvious that f is continuous on $(0,1)$.* What is not obvious is continuity at 0. To show that f is continuous at 0, by Theorem 4.6 from [3, Ch. 4, p. 86], it suffices to show that $\lim_{x\to 0} f(x) = 0$. Consider the sequence $\{1/k\}$. This sequence converges to 0. Moreover, given $\varepsilon > 0$, by the Archimedean principle, for sufficiently large K, the inequality $1/K < \varepsilon$ holds so for every $k \geq K$ we have

$$|(1/k)\sin(k) - 0| \leq |1/k| < \varepsilon. \tag{1}$$

Thus, $\lim_{k\to\infty} f(1/k) = 0$. Thus, f is continuous on all of $[0,1]$.

*You can, for example, take a look at Theorem 4.9 from [3, Ch. 4, p. 87].

Nevertheless, f is not of *bounded variation* on $[0,1]$. By Corollary 2.10 from [4, Ch. 2, p. 23], the *total variation* V of f on $[0,1]$ is given by

$$\begin{aligned}
V &= \int_0^1 |f'|\,dx \\
&= \int_0^1 |\sin(1/x) - (1/x)\cos(1/x)|\,dx \\
&= \int_1^\infty \frac{1}{u^2}|\sin u - u\cos u|\,dx \\
&\geq \int_M^\infty \frac{1}{2u}\,du \\
&= \infty,
\end{aligned} \tag{2}$$

where, for sufficiently large M, for $u \geq M$ we have $|\sin u - u\cos u| > u/2$. Thus, f is not of bounded variation. ■

Problem 2 (Wheeden & Zygmund Ch. 2, Ex. 2)**.** Prove theorem (2.1).

Proof. Recall the statement of theorem (2.1):

(a) If f is of bounded variation on $[a,b]$, then f is bounded on $[a,b]$.

(b) Let f and g be of bounded variation on $[a,b]$. Then cf (for any real constant c), $f+g$, and fg are of bounded variation on $[a,b]$. Moreover, f/g is of bounded variation on $[a,b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a,b]$.

(a) Recall that f is of b.v. on $[a,b]$ if the total variation V of f on $[a,b]$ is finite, where V is defined to be the supremum of the sum $\sum_{i=1}^m |f(x_i) - f(x_{i-1})|$ over all *partitions* $\Gamma = \{x_0, \ldots, x_m\}$ of $[a,b]$ of the sum.

(b) ■

Problem 3 (Wheeden & Zygmund Ch. 2, Ex. 3)**.** If $[a',b']$ is a subinterval of $[a,b]$ show that $P[a',b'] \leq P[a,b]$ and $N[a',b'] \leq N[a,b]$.

Proof. ■

Problem 4 (Wheeden & Zygmund Ch. 2, Ex. 11)**.** Show that $\int_a^b f\,d\varphi$ exists if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|, |\Gamma'| < \delta$.

Proof. ■

Problem 5 (Wheeden & Zygmund Ch. 2, Ex. 13)**.** Prove theorem (2.16).

Proof. Recall the statement of Theorem 2.16:

(i) If $\int_a^b f\,d\varphi$ exists, then so do $\int_a^b cf\,d\varphi$ and $\int_a^b f\,d(c\varphi)$ for any constant c, and

$$\int_a^b cf\,d\varphi = \int_a^b f\,d(c\varphi) = c\int_a^b f\,d\varphi.$$

(ii) If $\int_a^b f_1 \, d\varphi$ and $\int_a^b f_2 \, d\varphi$ both exist, so does $\int_a^b (f_1 + f_2) \, d\varphi$, and

$$\int_a^b (f_1 + f_2) \, d\varphi = \int_a^b f_1 \, d\varphi + \int_a^b f_2 \, d\varphi.$$

(iii) If $\int_a^b f \, d\varphi_1$ and $\int_a^b f \, d\varphi_2$ both exist, so does $\int_a^b f \, d(\varphi_1 + \varphi_2)$, and

$$\int_a^b f \, d(\varphi_1 + \varphi_2) = \int_a^b f \, d\varphi_1 + \int_a^b f \, d\varphi_2.$$

■

1.2 Exam 1 Prep

Problem 1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{\, r\mathbf{x} : \mathbf{x} \in E \,\}$. Prove that rE is measurable, and that $|rE| = |r|^n|E|$.

Proof. Define a map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by $T\mathbf{x} := r\mathbf{x}$. Note that T is *Lipschitz continuous* since for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the equality

$$|T\mathbf{x} - T\mathbf{y}| = |r\mathbf{x} - r\mathbf{y}| = |r||\mathbf{x} - \mathbf{y}| \tag{1}$$

is satisfied. By Theorem 3.33 from [4, Ch. 3, p.55], the image of E under T is measurable. Moreover, by Theorem 3.35 [4, Ch. 3, p. 56], since T is linear, it follows that $|T(E)| = |\det T||E|$ where $\det T = |r|^n$. Lastly, we note that the image of E under T is precisely the set rE so that $|T(E)| = |rE| = |r|^n|E|$, as was to be shown. ∎

Problem 2. Let $\{E_k\}$, $k \in \mathbb{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty} E_n\right).$$

Show that

$$\left|\liminf_{k\to\infty} E_k\right| \leq \liminf_{k\to\infty}|E_k|.$$

Proof. Following the style of [4, Ch. 1, p. 2], particularly, the sets defined after the introduction of equation (1.1), set

$$V_k := \bigcap_{\ell=k}^{\infty} E_\ell. \tag{2}$$

Note that the collection of sets $\{V_k\}$ forms an increasing sequence, that is, if $\mathbf{x} \in V_k$ then, by (2), $\mathbf{x}$ is in the intersection $E_k \cap \left(\bigcap_{\ell=k+1} E_\ell\right)$, but, by (2), $\bigcap_{\ell=k+1} E_\ell = V_{k+1}$ thus, $\mathbf{x}$ is in V_{k+1} so $V_{k+1} \supset V_k$. Hence, we have $V_k \nearrow \liminf E_k$.

Now, consider the sequence $\{|V_k|\}$ formed by the Lebesgue measure of the V_k. By Theorem 3.26 from [4, Ch. 3, p. 51], since $V_k \nearrow \liminf E_k$,

$$\lim_{k\to\infty}|V_k| = \lim_{k\to\infty}\left|\bigcap_{\ell=k}^{\infty} E_\ell\right| = \left|\liminf_{k\to\infty} E_k\right|. \tag{3}$$

But note that, by the monotonicity of the Lebesgue measure, we have

$$\left|\bigcap_{\ell=k}^{\infty} E_\ell\right| \leq |E_k|, \tag{4}$$

so, by properties of the $\liminf$, in particular, by Theorem 19(v) from [2, Ch. 1, p. 23], we have

$$\limsup_{k\to\infty}|V_k| \leq \liminf_{k\to\infty}|E_k|. \tag{5}$$

Hence, by (3) and Proposition 19 (iv), since the sequence $\{|V_k|\}$ converges and is equal to the measure of $\liminf E_k$, by (5), we have

$$\left|\liminf_{k\to\infty} E_k\right| \leq \liminf_{k\to\infty} |E_k| \tag{6}$$

as was to be shown. ■

Problem 3. Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) = \{\, \mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r \,\}$. Prove that F is monotonic increasing and continuous.

Proof. Define the linear map $T\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ by $T(r)\mathbf{x} \coloneqq r\mathbf{x}$. We claim that $B(\mathbf{0}, r) = T(r, B(\mathbf{0}, 1))$. To reduce notation, set $B_1 \coloneqq B(\mathbf{0}, 1)$ and $B_r \coloneqq B(\mathbf{0}, r)$.

Proof of claim. $\subset$: Let $\mathbf{x} \in B_r$. Then $|\mathbf{x}| < r$ so $|\mathbf{x}|/r < 1$. Thus, $|\mathbf{x}|/r \in B_1$ so it is in the image of B_1 under the map $T(r, -)$.

$\supset$: On the other hand, suppose $\mathbf{x} \in T(r, B_1)$. Then $\mathbf{x} = r\mathbf{y}$ for some $\mathbf{y} \in B_1$. Then, since $|\mathbf{y}| < 1$, $|\mathbf{x}| = r|\mathbf{y}| < r$ so $\mathbf{x} \in B_r$. ♣

From the claim, we see that $F(x) = |T(x, B(\mathbf{0}, 1))|$ which, by Problem 1, is nothing more that the polynomial $|B_1|x^n$. It is clear, from this equivalence, that F is monotonically increasing: Take $x, y \in [0, \infty)$ such that $x < y$, then $x^n < y^n$ so

$$F(x) = |B_1|x^n < |B_1|y^n = F(y). \tag{7}$$

Thus, F is monotonically increasing.

In the argument above, since $F(x) = |B_1|x^n$ is a polynomial in $[0, \infty)$ (and polynomials are continuous on $\mathbb{R}$) F is continuous on $[0, \infty)$. ■

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. (Without much motivation) let us consider the collection of sets $\{E_k\}$ defined by

$$E_k \coloneqq \left\{\, x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } y, z \in B(x, \delta) \text{ implies } |f(y) - f(z)| < \frac{1}{k} \,\right\}. \tag{8}$$

We claim that $C = \bigcap_{k=1}^{\infty} E_k$ and that each E_k is open.

Proof of claim. First, we demonstrate equality. $\subset$: Suppose $x \in C$. Then, by the definition of continuity, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $y \in B(x, \delta)$ implies $|f(x) - f(y)| < \delta$. In particular, for every k, there exists $\delta > 0$ such that for $y \in B(x, \delta)$ the inequality $|f(x) - f(y)| < 1/k$ holds. Thus, x is in $\bigcap_{k=1}^{\infty} E_k$.

$\supset$: On the other hand, suppose that $x \in \bigcap_{k=1}^{\infty} E_k$. Then, given $\varepsilon > 0$, by the Archimedean property, there exists a positive integer N such that $1/N < \varepsilon$. Then, since $x \in \bigcap_{k=1}^{\infty} E_k$, $x \in E_N$ so

$$|f(x) - f(y)| < \frac{1}{N} < \varepsilon. \tag{9}$$

Thus, x is in C and $C = \bigcap_{k=1}^{\infty} E_k$.

All that remains to be shown is that the E_k are open. But this is clear by the way we defined E_k in (8): Let $x \in E_k$, then there exists $\delta > 0$ such that for any $y, z \in B(x, \delta)$, $|f(y) - f(z)| < 1/k$; Let $x' \in B(x, \delta)$ and set $\delta' := \min\{|(x+\delta) - x'|, |(x-\delta) - x|\}$. Then, since $B(x', \delta') \subset B(x, \delta)$, for every $y, z \in B(x', \delta')$, we have $|f(y) - f(z)| < 1/k$. Hence, $x' \in E_k$ for any $x' \in B(x, \delta)$ so $B(x, \delta) \subset E_k$. ♣

Since C can be expressed as the countable intersection of open sets E_k, it follows that C is a G_δ set. ■

Problem 5. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Proof. If $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, it is not necessarily the case that f is measurable. Consider the following construction: Let $E \subset (0,1)$ be an unmeasurable set.[†] Define a map $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{R} \setminus ((0,1) \setminus E), \\ x+1 & \text{if } x \in (0,1) \setminus E. \end{cases} \tag{10}$$

By the definition, it is clear that $\{ f = r \}$ is measurable and $|\{ f = r \}| = 0$ since $\{ f = r \}$ contains at most two elements. However, the set $\{ 0 < f < 1 \} = E$ is not measurable. Thus, f is not measurable. ■

Problem 6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set $\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \}$ is measurable.

Proof. By Theorem 4.12 from [4, Ch. 4, p. 67], $\liminf_{k\to\infty} f_k$ and $\limsup_{k\to\infty} f_k$ are measurable. By Theorem 4.7 from [4, Ch. 4, p. 66]

$$\left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\} \tag{11}$$

is measurable. Since

$$\left\{ \lim_{k\to\infty} f_k \text{ exists} \right\} = \left\{ \limsup_{k\to\infty} f_k = \liminf_{k\to\infty} f_k \right\} = \mathbb{R} \setminus \left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\}, \tag{12}$$

by Theorem 3.17 from [4, Ch. 3, p. 48], the set $\{ \lim_{k\to\infty} f_k \text{ exists} \}$ is measurable. ■

Problem 7. A real valued function f on an interval $[a,b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{N}$ of open intervals in (a,b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, one has $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a,b]$ is of bounded variation on $[a,b]$.

[†] It's construction does not concern us. The interested reader such direct their refer to Theorem 3.38 from [4, Ch. 3, p. 57-58] or Theorem 17 from [2, Ch. 2§7, p. 48].

Proof. Suppose f is absolutely continuous on $[a,b]$. Let $\varepsilon := 1$. Then, there exists $\delta > 0$ such that for every finite disjoint collection $\{(a_k,b_k)\}_{k=1}^N$ of open intervals in (a,b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < 1$. Let $N := \lceil (b-a)/\delta \rceil$, that is, N is the smallest integer greater than $(b-a)/\delta$, and consider the partition $\Gamma = \{x_k\}$ where $x_k := a + k(b-a)/N$, for $k = 0, \ldots, N$. Then $x_k - x_{k-1} < (b-a)/N < \delta$ so, by Theorem 2.2(i) from [4, Ch. 2, p. 19], we have $V[f; x_{k-1}, x_k] < 1$ for $k = 0, \ldots, N$. In follows by Theorem 2.2(ii) that

$$V[f;a,b] = \sum_{k=1}^N V[f; x_{k-1}, x_k] < N. \tag{13}$$

Thus, f is b.v. on $[a,b]$. ∎

Problem 8. Let f be a continuous function from $[a,b]$ into $\mathbb{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, that is, $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a,b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}$$

Proof. The result follows quite easily from more sophisticated measure theoretic arguments. At this point, however, such language has not been discussed so we shall prove this using nothing but the definition of the Riemann–Stieltjes integral and properties thereof.

Let us consider each case $c \in (a,b)$, $c = a$, and $c = b$ separately.

Recall that the given a partition $\Gamma = \{x_0, \ldots, x_m\}$ of $[a,b]$, the Riemann–Stieltjes sum of f with respect to φ is

$$R_\Gamma := \sum_{k=1}^m f(\xi_k)[\varphi(x_k) - \varphi(x_{k-1})]. \tag{14}$$

The Riemann–Stieltjes integral is defined as the limit

$$\int_a^b f\, d\varphi := \lim_{|\Gamma| \to 0} R_\Gamma \tag{15}$$

if it exists.

Suppose $c \in (a,b)$. Then, for any partition Γ of $[a,b]$, either $c \in \Gamma$ or $c \notin \Gamma$. In the latter case, $R_\Gamma = 0$. In the former case c is one of the x_k, say $c = x_\ell$ for $0 < \ell < m$. Then

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^m f(\xi_k)[\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= 0 + \cdots + 0 + f(\xi_{\ell-1}) - f(\xi_\ell) + 0 + \cdots + 0 \\ &= f(\xi_{\ell-1}) - f(\xi_\ell). \end{aligned} \tag{16}$$

Since f is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi_\ell - \xi_{\ell-1}| < \delta$ implies $|f(\xi_\ell) - f(\xi_{\ell-1})| < \varepsilon$. It follows that the quantity in (16) approaches 0 as $|\Gamma|$ approaches 0. Therefore, $\int_a^b f\, d\chi_{\{c\}} = 0$.

Suppose $c = a$. Then, since any partition Γ of $[a, b]$ must contain the point a, we have

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^{m} f(\chi_k)[\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= f(\xi_1)[\chi_{\{c\}}(x_1) - \chi_{\{c\}}(x_0)] + f(\xi_2)[\chi_{\{c\}}(x_2) - \chi_{\{c\}}(x_1)] \\ &\qquad + \cdots + f(\xi_m)[\chi_{\{c\}}(x_m) - \chi_{\{c\}}(x_{m-1})] \\ &= -f(\xi_1) + 0 + \cdots + 0 \\ &= -f(\xi_1) \end{aligned} \tag{17}$$

Taking the limit as $|\Gamma| \to 0$, $\xi_1 \to a$ so, by continuity of f, $f(\xi_1) \to f(a)$. Thus, $\int_a^b f\, d\chi_{\{c\}} = -f(a)$.

A similar argument to the one above shows that, if $c = b$, the Riemann–Stieltjes integral $\int_a^b f\, d\chi_{\{c\}} = f(b)$. ■

1.3 Exam 1

Problem 1.

Proof. ■

Problem 2.

Proof. ■

Problem 3.

(i) Show that if $B_r \coloneqq \{\, \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < r \,\}$, then there exists a constant C such that $|B_r| = Cr^n$.

(*Hint*: Think of B_r as $\{\, r\mathbf{x} : \mathbf{x} \in B_1 \,\}$.)

(ii) Let $E \subset \mathbb{R}^n$ be a measurable set and let $\varphi_E \colon \mathbb{R}^n \to \mathbb{R}$ be defined $\varphi_E(\mathbf{x}) \coloneqq \big|E \cap B_{|\mathbf{x}|}\big|$. Use part (i) to prove that φ_E is continuous.

Proof. (i) To prove this result, we use the map constructed in Problem 1 of the review sheet for Exam 1, the map $T \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Set $T_r \colon \mathbb{R}^n \to \mathbb{R}^n$ to be $T_r \coloneqq T(r)$. Then, we claim $B_r = T_r(B_1)$ and $|B_r| = |T_r(B_1)|$, which, as we saw in Problem 1 of the review sheet, has measure $|B_1||r|^n$. Setting $C \coloneqq |B_1|$, we have $|B_r| = C|r|^n$ as desired.

(ii) To prove that φ_E is continuous, we provide an (ε, δ)-argument. Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that $\mathbf{y} \in B(\mathbf{x}, \delta)$ implies

$$|\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| < \varepsilon. \tag{1}$$

First, note that since $\mathbf{x} \mapsto |\mathbf{x}|$ is continuous and polynomials $p \colon \mathbb{R}^n \to \mathbb{R}^n$ are continuous, then the composition $\mathbf{x} \mapsto |\mathbf{x}|^n$ is continuous. Therefore, there exists $\delta > 0$ such that $\mathbf{y} \in B(\mathbf{x}, \delta)$ implies

$$||\mathbf{x}|^n - |\mathbf{y}|^n| < \frac{\varepsilon}{C}, \tag{2}$$

where $C \coloneqq |B_1|$.

Now, let $x \in \mathbb{R}^n$ and $\mathbf{y} \in B(\mathbf{x}, \delta)$ as above. Then, by (2) we have

$$\begin{aligned} |\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| &= \big|\big|E \cap B_{|\mathbf{x}|}\big| - \big|E \cap B_{|\mathbf{y}|}\big|\big| \\ &\le \big||B_{|\mathbf{x}|}| - |B_{|\mathbf{y}|}|\big| \\ &= C||\mathbf{x}|^n - |\mathbf{y}|^n| \\ &\le C\Big[\frac{\varepsilon}{C}\Big] \\ &= \varepsilon. \end{aligned} \tag{3}$$

It follows that φ_E is continuous. ■

Problem 4. Assume that $f \colon [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$. Prove that f is measurable.

Proof. By Jordan's theorem (Corollary 2.7 from [4, Ch. 2, p. 21]), the function f is of bounded variation on $[a, b]$ if and only if it can be written as the difference $f_1 - f_2$ of two bounded functions f_1 and f_2 that are monotone increasing on $[a, b]$. Then, f_1 and f_2 are continuous a.e. on $[a, b]$ and hence, are measurable. ■

1.4 Exam 2 Prep

Problem 1. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B_\varepsilon(\mathbf{0})$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})} f(\mathbf{x})\, d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f\colon \mathbb{R}^n \to \mathbb{R}$ is (Lebesgue) integrable over a subset E of $\mathbb{R}^n$ (or, alternatively, f belongs to $L(E)$) if

$$\int_E f(\mathbf{x})\, d\mathbf{x} < \infty.$$

Put $E = \mathbb{R}^n \setminus B_\varepsilon(\mathbf{0})$. Then, to show that f belongs to $L(E)$ it suffices to prove the inequality

$$\int_E f(\mathbf{x})\, d\mathbf{x} < \frac{C}{\varepsilon} \tag{1}$$

for some appropriate constant C. We proceed by directly computing the Lebesgue integral of f and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x})\, d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{({x_1}^2 + \cdots + {x_n}^2)^{(n+1)/2}} \end{aligned}$$

let E_i denote the projection of E onto its i-th coordinate and make the trigonometric substitution $x_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \tan\theta$, $dx_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \sec^2\theta d\theta$ with $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$ giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[\frac{\cos^{n-1}\theta}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1}\theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1}\theta d\theta < \infty. \tag{2}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned}\int \cdots \int_E f(\mathbf{x})d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{{x_n}^2} \\ &= 2C' \int_\varepsilon^\infty \frac{dx_n}{{x_n}^2} \\ &= \frac{C}{\varepsilon}\end{aligned} \tag{3}$$

as desired. ■

Problem 2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit f of $\{f_k\}$ must be nonnegative a.e. in $\mathbb{R}^n$. For assume otherwise. Then there exists a collection of points $\mathbf{x}$ in $\mathbb{R}^n$ of nonzero $\mathbb{R}^n$-Lebesgue measure such that $f(\mathbf{x}) < 0$. But $f_k(\mathbf{x}) \geq 0$ for all $k \in \mathbb{N}$. Set $0 < \varepsilon < |f(\mathbf{x})|$ then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{4}$$

for all k which contradicts our assumption that $f_k \to f$ a.e. on $\mathbb{R}^n$. Therefore, the set of points $\mathbf{x} \in \mathbb{R}^n$ where $f(\mathbf{x}) < 0$ must have measure zero.

Now, based on pointwise convergence a.e. to f, given $\varepsilon > 0$ for a.e. $\mathbf{x} \in \mathbb{R}^n$ we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{5}$$

for sufficiently large k; say k greater than or equal to some index $N \in \mathbb{N}$. Moreover, we are given convergence in $L(\mathbb{R}^n)$ of f_k to f

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f < \infty. \tag{6}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{7}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{8}$$

for all $k \in \mathbb{N}$. By Theorem 5.5(ii), f and the f_k's are finite a.e. in $\mathbb{R}^n$ so for some sufficiently large real number M, $|f|, |f_k| \le M$ for a.e. $\mathbf{x} \in \mathbb{R}^n$. In particular, for any measurable subset E of $\mathbb{R}^n$, $|f|, |f_k| \le M$ for a.e. $\mathbf{x} \in E$ so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \to \int_E f < \infty. \tag{9}$$

However, if f does not belong to $L(\mathbb{R}^n)$, i.e., its integral over $\mathbb{R}^n$ is infinity, there is no guarantee that f will be finite a.e. in $\mathbb{R}^n$. This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset E of $\mathbb{R}^n$. Let us demonstrate this with an example. Consider the sequence of functions ■

Problem 3. Assume that E is a measurable set of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, \mathbf{x} \in E : f(\mathbf{x}) \ge k \,\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of $\mathbb{R}^n$, then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \tag{10}$$

Set $E_k = \{\, \mathbf{x} \in E : k+1 > f(\mathbf{x}) \ge k \,\}$ and $F_k = \{\, \mathbf{x} \in E : f(\mathbf{x}) \ge k \,\}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{11}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \le f(\mathbf{x}) \le (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \le \int_{E_k} f(\mathbf{x}) d\mathbf{x} \le (k+1)|E_k|. \tag{12}$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \le \int_E f(\mathbf{x}) d\mathbf{x} \le \sum_{k=0}^{\infty} (k+1)|E_k|. \tag{13}$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (13) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \le \int_E f(\mathbf{x}) d\mathbf{x} \le \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \tag{14}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\ &= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\ &= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (15) \\ &= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\ &= \sum_{k=1}^{\infty} |F_{k+1}|\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\ &= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\ &= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \qquad (16) \\ &= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\ &= \sum_{k=0}^{\infty} |F_k|.\end{aligned}$$

Thus, from (15) and (16)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \qquad (17)$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ■

Problem 4. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. $\Longrightarrow$: First note that ρ is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that $\rho(f_k, f) \to 0$ as $k \to \infty$. Then, given $\varepsilon > 0$ there exist an sufficiently large index N such that for every $k \geq N$ we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \tag{18}$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in E which happens if $|f_k - f| = 0$ a.e. in E.

$\Longleftarrow$: Suppose that $f_k \to f$ as $k \to \infty$.

I don't know how to solve this. This is the intended solution:

$\Longrightarrow$: Given $\varepsilon > 0$, $\rho(f_k, f) \to 0$ implies that

$$\int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{|f_k - f|}{1 + |f_k - f} dx \longrightarrow 0.$$

Observe that the function $\Phi \colon \mathbb{R}^+ \to \mathbb{R}$ given by $\Phi(x) = x/(1+x)$ is increasing on $\mathbb{R}^+$ and $0 < \Psi(x) < 1$, hence

$$\begin{aligned} \int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1 + \varepsilon} |\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}|. \end{aligned}$$

Therefore,

$$|\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}| \leq \frac{1 + \varepsilon}{\varepsilon} \int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{|f_k - f|}{1 + |f_k - f|} \, dx \longrightarrow 0$$

as $k \to \infty$.

$\Longleftarrow$: Conversely, given $\delta > 0$, we have

$$\begin{aligned} \rho(f_k, f) = &\int_{\{\, x \in E : |f_k(x) - f(x)| > \delta \,\}} \frac{|f_k - f|}{1 + |f_k - f|} \, dx \\ &+ \int_{\{\, x \in E : |f_k(x) - f(x)| \leq \delta \,\}} \frac{|f_k - f|}{1 + |f_k - f|} \, dx \\ &\leq |\{\, x \in E : |f_k(x) - f(x)| > \delta \,\}| + \frac{\delta}{1 + \delta} |E|. \end{aligned}$$

Since $|E| < \infty$ and $\delta/(1+\delta) \searrow 0$, then for any $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\frac{\delta'}{1 + \delta'} |E| < \frac{\varepsilon}{2}.$$

If $f_k \to f$ as $k \to \infty$ in measure, then for the above δ' there is an index $N > 0$ such that $k \geq N$ implies

$$|\{\, x \in E : |f_k(x) - f(x)| > \delta' \,\}| < \frac{\varepsilon}{2}.$$

Therefore, $f_k \to f$ in measure implies $\rho(f_k, f) \to 0$ as $k \to \infty$. ■

Problem 5. Define the *gamma function* $\Gamma\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1}\, du,$$

and the *beta function* $\beta\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. (a) Fix $y \in \mathbb{R}^+$. Then we must show that $\Gamma(y) < \infty$. First, since $(0, 1)$ and $[1, \infty)$ are disjoint measurable subsets of $\mathbb{R}$, by Theorem 5.7 we can split the integral $\Gamma(y)$ into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u}u^{y-1}\, du}_{I_1} + \underbrace{\int_1^\infty e^{-u}u^{y-1}\, du}_{I_2}. \tag{19}$$

We will show, separately, that I_1 and I_2 are finite.

To see that I_1 is finite, note that

$$\begin{aligned} e^{-u}u^{y-1} &= e^{-u}e^{(y-1)\log u} \\ &= e^{-u+(y-1)\log u} \\ &\le e^{(y-1)\log u} \\ &= u^{y-1} \end{aligned} \tag{20}$$

since $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u}u^{y-1}\, du \\ &\le \int_0^1 u^{y-1}\, du \\ &= \left[\frac{u^y}{y}\right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \tag{21}$$

To see that I_2 is finite, note that

$$e \tag{22}$$

Intended solution:

(b) ■

Problem 6. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}} : \mathbb{R}^n \to \mathbb{R}$ be $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \to \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{23}$$

■

Problem 7. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \leq g_k$ and

$$\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \to 0 \qquad \text{as} \qquad k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \to \int_{\mathbb{R}^n} |f| \qquad \text{as} \qquad k \to \infty.$$

Proof. (a) Since $f_k \to f$ and $g_k \to g$ a.e. and $|f_k| \leq g_k$, then by Fatou's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} (g - f) = \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k - f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k - f_k, \\ \int_{\mathbb{R}^n} g + f \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k + f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k + f_k. \end{aligned}$$

Since $f_k, g_k, f, g \in L(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$, then using the similar argument as problem 2, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f \geq \limsup_{k \to \infty} \int_{\mathbb{R}^n} f_k, \\ \int_{\mathbb{R}^n} f \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} f_k. \end{aligned}$$

Therefore, $\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f$.

(b) $\Longrightarrow$: This direction is obvious by the inequality

$$\left| \int_{\mathbb{R}^n} |f_k| - |f| \right| \leq \int_{\mathbb{R}^n} ||f_k| - |f|| \leq \int_{\mathbb{R}^n} |f_k - f|.$$

$\Longleftarrow$: Let $g_k = |f_k| + |f|$ and $g = 2|f|$. Since $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e., then $g_k, g \in L(\mathbb{R}^n)$ and $g_k \to g$ a.e. in $\mathbb{R}^n$. By the assumption, $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$.

Let $\tilde{f}_k = |f_k - f|$. Then $\tilde{f}_k \to 0$ a.e. in $\mathbb{R}^n$ and $\tilde{f}_k \leq g_k$. Applying part (a) to $\tilde{f}_k$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

■

1.5 Midterm 2

Problem 1. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Proof. ∎

Problem 2. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. ∎

Problem 3. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbb{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k \to \infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Proof. ∎

Problem 4. Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)\, dt$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Proof. ∎

1.6 Final Practice

Problem 1. Suppose $f \in L^1(\mathbb{R})$ and that x is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) := \frac{1}{r} \int_{B(0,r)} |f(x-y) - f(x)| \, dy.$$

Show that:

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$;
(b) there exists a constant $M > 0$ such that $A(r) \le M$ for all $r > 0$.

Proof. ■

Problem 2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \le n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0$$

if and only if

$$\|f_k\|_p \longrightarrow \|f\|_p$$

as $k \to \infty$.

Proof. ■

Problem 3. Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

Proof. ■

Problem 4. Let $f \in L(\mathbb{R})$, and let $F(t) := \int_{\mathbb{R}} f(x) \cos(tx) dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t \to \infty} F(t) = 0.$$

Proof. ■

Problem 5. Let f be of bounded variation on $[a,b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular. Show that

$$\int_a^b \varphi \, df = \int_a^b \varphi f' \, dx + \int_a^b \varphi \, dh$$

for all functions φ continuous on $[a,b]$.

Proof. ■

CHAPTER 2

MA 544 Past Quals

2.1 Danielli: Winter 2012

Problem 1. Let $f(x,y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $\partial f(x,y)/\partial x$ is a bounded function of (x,y). Prove that $\partial f(x,y)/\partial x$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial f(x,y)}{\partial x}\,dy.$$

Proof. ■

Problem 2. Let f be a function of bounded variation on $[a,b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular, show that

$$\int_a^b \varphi\,df = \int_a^b \varphi f'\,dx + \int_a^b \varphi\,dh.$$

Hint: A function h is said to be singular if $h' = 0$.

Proof. ■

Problem 3. Let $E \subset \mathbb{R}$ be a measurable set, and let K be a measurable function on $E \times E$. Assume that there exists a positive constant C such that

$$\int_E K(x,y)\,dx \le C \tag{1}$$

for a.e. $y \in E$, and

$$\int_E K(x,y)\,dy \le C \tag{2}$$

for a.e. $x \in E$.

Let $1 < p < \infty$, $f \in L^p(E)$, and define

$$T_f(x) := \int_E K(x, y) f(y)\, dy.$$

(a) Prove that $T_f \in L^p(E)$ and

$$\|T_f\|_p \leq C\|f\|_p. \tag{3}$$

(b) Is (3) still valid if $p = 1$ or ∞? If so, are assumptions (1) and (2) needed?

Proof. ■

Problem 4. Let f be a nonnegative measurable function on $[0, 1]$ satisfying

$$|\{\, x \in [0, 1] : f(x) > \alpha \,\}| < \frac{1}{1 + \alpha^2} \tag{4}$$

for $\alpha > 0$.

(a) Determine values of $p \in [1, \infty)$ for which $f \in L^p[0, 1]$.

(b) If p_0 is the minimum value of p for which p may fail to be in L^p, give an example of a function which satisfies (4), but which is not in $L^{p_0}[0, 1]$.

Proof. ■

2.2 Danielli: Summer 2011

Problem 1. Let $f \in L^1(\mathbb{R})$, and let $F(t) := \int_{\mathbb{R}} f(x)\cos(tx)\,dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.

(b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Hint: Start by proving the statement for $f = \chi_{[a,b]}$.

Proof. ■

Problem 2. (a) Suppose that $f_k, f \in L^2(E)$, with E a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E fg \tag{1}$$

as $k \to \infty$ for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \to \|f\|_2$ show that f_k converges to f in L^2, i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as $k \to \infty$.

(b) Provide an example of a sequence f_k in L^2 and a function f in L^2 satisfying (1), but such that f_k does *not* converge to f in L^2.

Proof. ■

Problem 3. A bounded function f is said to be of bounded variation on $\mathbb{R}$ if it is of bounded variation on any finite subinterval $[a,b]$, and moreover $A := \sup_{a,b} V[a,b;f] < \infty$. Here, $V[a,b;f]$ denotes the total variation of f over the interval $[a,b]$. Show that:

(a) $\displaystyle\int_{\mathbb{R}} |f(x+h) - f(x)|\,dx \le A|h|$ for all $h \in \mathbb{R}$.

Hint: For $h > 0$, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,dx.$$

(b) $\displaystyle\left|\int_{\mathbb{R}} f(x)\varphi'(x)\,dx\right| \le A$, where φ is any function of class C^1, of bounded variation, compactly supported, with $\sup_{x\in\mathbb{R}} |\varphi(x)| \le 1$.

Proof. ■

Problem 4. (a) Prove the *generalized Hölder's inequality*: Assume $1 \le p \le \infty$, $j = 1, \dots, n$, with $\sum_{j=1}^{\infty} 1/p_j = 1/r \le 1$. If E is a measurable set and $f_j \in L^{p_j}(E)$ for $j = 1, \dots, n$, then $\prod_{j=1}^{n} f_j \in L^r(E)$ and

$$\|f_1 \cdots f_n\|_r \le \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if $1 \le p, q, r \le \infty$, with $1/p + 1/q = 1/r + 1$, $f \in L^p(\mathbb{R})$, and $g \in L^p(\mathbb{R})$, then

$$|(f * g)(x)| \le \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \int |f(y)|^p |g(x-y)|^q \, dy.$$

(Recall that $(f * g)(x) \coloneqq \int f(y) g(x-y) \, dy$.)

(c) Prove *Young's convolution theorem*: Assume that p, q, r, f, and g are as in part (b). Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Proof. ∎

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