

# MA 544: Homework 5

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**PROBLEM 5.1 (WHEEDEN & ZYGMUND §3, EX. 14)**

Show that the conclusion of part (ii) of Exercise 13 (Problem) is false if  $|E|_e = +\infty$ .

*Proof.* Let  $V \subset [0, 1]$  denote the Vitali set defined in 3.38 and consider the union  $E := V \cup (2, \infty)$ . It is clear that the inner and outer measure of  $E$  is  $\infty$ . However,  $E$  itself is unmeasurable since otherwise  $E \cap [0, 1] = V \cap [0, 1] = V$  would be measurable. ■

**PROBLEM 5.2 (WHEEDEN & ZYGMUND §3, EX. 16)**

Prove (3.34).

*Proof.*

**Lemma.**  $|P| = v(P)$ .

Let  $\{\mathbf{e}_k\}_{k=1}^n$  be a set of orthogonal vectors emanating from a point in  $\mathbb{R}^n$ . The closed parallelepiped corresponding to  $\{\mathbf{e}_k\}_{k=1}^n$  is the set

$$P = \left\{ \mathbf{x} \mid \mathbf{x} = \sum_{k=1}^n t_k \mathbf{e}_k, 0 \leq t_k \leq 1 \right\}. \quad (1)$$

■

**PROBLEM 5.3 (WHEEDEN & ZYGMUND §3, EX. 18)**

Prove that outer measure is *translation invariant*; that is, if  $E_{\mathbf{h}} := \{\mathbf{x} + \mathbf{h} \mid \mathbf{x} \in E\}$  is the translate of  $E$  by  $\mathbf{h}$ ,  $\mathbf{h} \in \mathbb{R}^n$ , show that  $|E_{\mathbf{h}}|_e = |E|_e$ . If  $E$  is measurable, show that  $E_{\mathbf{h}}$  is also measurable. [This fact was used in proving (3.37).]

*Proof.* By 3.6, given  $\varepsilon > 0$ , there exists an open set  $G \supset E$  with  $|G|_e \leq |E|_e + \varepsilon$ . Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote the linear transformation  $\mathbf{x} \mapsto \mathbf{x} + \mathbf{h}$ ,  $\mathbf{h} \in \mathbb{R}^n$ . By 3.35 we have  $|G|_e = |G| = |T(G)| = |T(G)|_e$  and  $T(G)$  is an open set containing  $E_{\mathbf{h}}$ . Hence, we have an upper bound on the outer measure of  $E_{\mathbf{h}}$  given by the inequality

$$|E_{\mathbf{h}}|_e \leq |T(G)|_e = |G|_e \leq |E|_e + \varepsilon. \quad (2)$$

On the other hand, by 3.6 there exists an open set  $H \supset E_{\mathbf{h}}$  with  $|H|_e \leq |E_{\mathbf{h}}|_e + \varepsilon$ . Then by 3.35, we get the inequality

$$|E|_e \leq |T^{-1}(H)|_e = |H|_e \leq |E_{\mathbf{h}}|_e + \varepsilon. \quad (3)$$

Putting (2) and (3) we have

$$|E|_e - \varepsilon \leq |E_{\mathbf{h}}|_e \leq |E|_e + \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$ , we have  $|E|_e = |E_{\mathbf{h}}|_e$ . It then follows that if  $E$  is measurable then  $E_{\mathbf{h}}$  is measurable since  $E_{\mathbf{h}} = T(E)$  and  $T$  is a Lipschitz transformation and  $|E| = |E_{\mathbf{h}}|$ . ■

**PROBLEM 5.4 (WHEEDEN & ZYGMUND §4, EX. 1)**

Prove corollary (4.2) and theorem (4.8)

*Proof.*

**Corollary** (Wheeden & Zygmund, 4.2). *If  $f$  is measurable, then  $\{f > -\infty\}$ ,  $\{f < +\infty\}$ ,  $\{f = +\infty\}$ ,  $\{a \leq f \leq b\}$ ,  $\{f = a\}$ , etc., are all measurable. Moreover  $f$  is measurable if and only if  $\{a < f < +\infty\}$  is measurable for every finite  $a$ .*

Suppose that  $f$  is measurable. By 4.1, we have  $\{f \geq a\}$  and  $\{f \leq a\}$  are measurable so

$$\{f = a\} = \{f \geq a\} \cap \{f \leq a\} \quad (4)$$

is measurable and for  $b > a$

$$\{a \leq f \leq b\} = \{f \geq a\} \cap \{f \leq b\}. \quad (5)$$

*Proof of corollary 4.2.* Now, consider the sequence of measurable sets  $\{E_k\}_{k=0}^{\infty}$  where  $E_k := \{f < a + k\}$ . Then  $\{f < \infty\} = \bigcup_{k=0}^{\infty} E_k$  and since  $E_k \nearrow \{f < \infty\}$  (take  $\mathbf{x} \in E_k$  then  $f(\mathbf{x}) < a + k$  so  $f(\mathbf{x}) < a + k + 1 \implies \mathbf{x} \in E_{k+1}$ ), by 3.26, we have  $\{f < \infty\}$  is measurable.

Similarly for  $\{f > -\infty\}$  we may consider the family  $\{E_k\}_{k=0}^{\infty}$  where  $E_k := \{f > a - k\}$  (take  $\mathbf{x} \in E_k$  then  $f(\mathbf{x}) > a - k$  so  $f(\mathbf{x}) > a - k - 1 \implies \mathbf{x} \in E_{k+1}$ ) and taking the limit as  $k \rightarrow \infty$  we have  $\{f > -\infty\}$  is measurable.

Last but not least, since  $\{f < \infty\}$  is measurable,  $\{f = \infty\} = \{f < \infty\}^c$  is measurable.

Now,  $\implies$  suppose  $f$  is measurable. Then  $\{a < f < b\} = \{a \leq f \leq b\} \cap \{f = a\}^c \cap \{f = b\}^c$  is measurable for all finite  $a < b$ . Moreover, the family  $\{E_k\}_{k=0}^{\infty}$  of sets  $\{E_k\}_{k=0}^{\infty}$  where  $E_k := \{a \leq f < b + k\}$  is measurable for all  $k$  so, by 3.26,  $\{a \leq f < \infty\}$  is measurable since  $E_k \nearrow \{a \leq f < \infty\}$ .

$\Leftarrow$  On the other hand, suppose that  $\{a \leq f < \infty\}$  is measurable for every finite  $a$ . Then, for fixed  $a \in \mathbb{R}$  the family  $\{E_k\}_{k=0}^{\infty}$  where  $E_k := \{a - k \leq f < \infty\}$  is measurable. By 3.26,  $\{f < \infty\}$  is measurable so  $\{f = \infty\} = \{f < \infty\}^c$  is measurable. Thus,

$$\{f > a\} = \{a < f < \infty\} \cup \{f = \infty\}$$

is measurable so  $f$  is measurable. ♣

**Theorem** (Wheeden & Zygmund, 4.8). *If  $f$  is measurable and  $\lambda$  is any real number, then  $f + \lambda$  and  $\lambda f$  are measurable.*

*Proof of theorem 4.8.* If  $f$  is measurable, then  $\{f > a\}$  is measurable for all  $a$  so  $\{f > a - \lambda\} = \{f + \lambda > a\}$  is measurable for all  $a$ . Hence,  $f + \lambda$  is measurable.

If  $\lambda \neq 0$ , then  $\{f > a/\lambda\}$  is measurable for all  $a$  so  $\lambda f$  is measurable. If  $\lambda = 0$  then  $\lambda f = 0$  is clearly measurable since  $\{0 > a\} = (a, 0)$  is open for all  $a$  (possibly empty if  $a \geq 0$ , but still an open set).

Thus,  $f + \lambda$  and  $\lambda f$  are measurable. ♣

■

**PROBLEM 5.5 (WHEEDEN & ZYGMUND §4, EX. 2)**

Let  $f$  be a simple function, taking its distinct values on disjoint sets  $E_1, \dots, E_N$ . Show that  $f$  is measurable if and only if  $E_1, \dots, E_N$  are measurable.

*Proof.*  $\implies$  Suppose  $f$  is a simple function taking distinct values on disjoint sets  $E_1, \dots, E_N$ . Then  $f = \sum_{k=1}^N a_k \chi_{E_k}$ . If  $f$  is measurable,  $\{f > a\}$  is measurable for all finite  $a$ . In particular,  $\{f > a_k\} = E_k$  is measurable.

$\impliedby$  On the other hand, suppose that  $E_k$  is measurable for all  $1 \leq k \leq N$ . Then  $\chi_{E_k}$  is measurable and by Problem 5.4, the sum  $f = \sum_{k=1}^N a_k \chi_{E_k}$  is measurable. ■