MA544: Qual Problems

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Contents

CHAPTER 1

Course Notes

These notes roughly correspond to chapters 2 through 8 of Wheeden and Zygmund's *Measure and Integration* [1].

This first portion corresponds to material covered before Exam 1.

1.1 Preliminaries

Here is some precursor material to the Lebesgue theory of integration.

Points and sets in $\mathbb{R}^n$

From this section, we need not say much only a few results and definitions are important.

If $\mathcal{F}$ is a *countable* collection of subsets of $\mathbb{R}^n$, it will be called a *sequence of sets* and denoted $\{E_k\}$ for $k \in \mathbb{N}$. The corresponding *union* and *intersection* will be written $\bigcup_k E_k$ and $\bigcap_k E_k$. A sequence $\{E_k\}$ is said to *increase* to $\bigcup_k E_k$ if $E_k \subset E_{k+1}$ for all k and to *decrease* to $\bigcap_k E_k$ if $E_k \supset E_{k+1}$ for all k; we use the notation $E_k \nearrow \bigcup_k E_k$ and $E_k \searrow \bigcap_k E_k$ to denote these two possibilities. If $\{E_k\}$ is a sequence of sets, we define

$$\limsup E_k \coloneqq \bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} E_k\right), \quad \liminf E_k \coloneqq \bigcup_{j=1}^{\infty}\left(\bigcap_{k=j}^{\infty} E_k\right), \tag{1.1}$$

noting that the subsets $U_j \coloneqq \bigcup_{k=j}^{\infty} E_k$ and $V_j \coloneqq \bigcap_{k=j}^{\infty} E_k$ satisfy $U_j \searrow \limsup E_k$ and $V_j \nearrow \liminf E_k$.*

*Carlos: Make note of this. It is often a good strategy to decompose a set E into the intersection or union of a sequence E_k. Making appropriate manipulations, we often get $E_k \searrow E$ or $E_k \nearrow E$ and make limiting arguments about properties of the set, i.e., measure or the integral of some function whose domain is in E, etc.

$\mathbb{R}^n$ as a metric space

A student who has taken 504 or 571 will know most of the material under this section. We include it here as a useful reference to some of the more useful results of the properties of $\mathbb{R}^n$ as a metric space.

If $\mathbf{x} \in \mathbb{R}^n$, we say that a sequence $\{\mathbf{x}_k\}$ *converges* to $\mathbf{x}$, or that $\mathbf{x}$ is the *limit* of $\{\mathbf{x}_k\}$, if $|\mathbf{x} - \mathbf{x}_k| \to 0$ as $k \to \infty$. We denote this by writing either $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$ or $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$. A point $\mathbf{x} \in \mathbb{R}^n$ is called a *limit point of a set* E if it is the limit of a sequence of distinct points of E. A point $\mathbf{x} \in E$ is called an *isolated point* of E if it is not the limit point of any sequence in E (excluding the trivial sequence $\{\mathbf{x}_k\}$ where $\mathbf{x}_k = \mathbf{x}$ for all k). It follows that a point $\mathbf{x}$ is isolated if and only if there is a $\delta > 0$ such that $|\mathbf{x} - \mathbf{y}| > \delta$ for every $\mathbf{y} \in E$, $\mathbf{y} \neq \mathbf{x}$.

For sequences $\{x_k\}$ in $\mathbb{R}$, we will write $\lim_{k\to\infty} x_k = \infty$, or $x_k \to \infty$ as $k \to \infty$, if given $M > 0$ there is an integer N such that $x_k \geq M$ whenever $k \geq N$. A similar definition holds for $\lim_{k\to\infty} x_k = -\infty$.†

A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is called a *Cauchy sequence* if given $\varepsilon > 0$ there is an integer N such that $|\mathbf{x}_k - \mathbf{x}_\ell| < \varepsilon$ for all $k, \ell \geq N$. We say that a metric space $(X, |\cdot|)$ is *complete with respect to the metric* $|\cdot|$ if every Cauchy sequence in X converges.

A set $E_0 \subset E$ is said to be *dense* in E if for every $\mathbf{y} \in E$ and $\varepsilon > 0$, there is a point $\mathbf{x} \neq \mathbf{y}$ in E_0 such that $0 < |\mathbf{y} - \mathbf{x}| < \varepsilon$. Thus, E_0 is dense in E if every point of E is a limit point of E_0. If $E_0 = E$, we say E is *dense in itself.* As an example $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense in $\mathbb{R}^n$. Since this set is also countable, it follows that $\mathbb{R}^n$ is *separable*, by which we mean that $\mathbb{R}^n$ has a countable dense subset.

For a nonempty subset E of $\mathbb{R}^n$, we use the standard notation $\sup E$ and $\inf E$ for the *supremum* (*least upper bound*) and $\inf E$ for the *infimum* (*greatest lower bound*) of E. In case $\sup E$ is in E, it will be called $\max E$; similarly, if $\inf E \in E$, ∞E will be called $\min E$.

If $\{a_k\}$ is a sequence of points in $\mathbb{R}$, let $b_j := \sup_{k\geq j} a_k$ and $c_j := \inf_{k\geq j} a_k$, $j \in \mathbb{N}$. Then $-\infty \leq c_j \leq b_j \leq \infty$, and $\{b_j\}$ and $\{c_j\}$ are monotone decreasing and increasing, respectively; i.e., $b_j \geq b_{j+1}$ and $c_j \leq c_j + 1$. Define $\limsup_{k\to\infty} a_k$ and $\liminf_{k\to\infty} a_k$ by

$$\begin{aligned} \limsup_{k\to\infty} a_k &:= \lim_{j\to\infty} b_j = \lim_{j\to\infty} \{\, \sup\nolimits_{k\geq j} a_k \},\\ \liminf_{k\to\infty} a_k &:= \lim_{j\to\infty} c_j = \lim_{j\to\infty} \{\, \inf\nolimits_{k\geq j} a_k \}. \end{aligned} \tag{1.2}$$

Theorem 1 1.4.

(a) *$L = \limsup_{k\to\infty} a_k$ if and only if* (i)*, there is a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ that converges to L and* (ii) *if $L' > L$, there is an integer N such that $a_k < L'$ for all $k \geq N$.*

(b) *$\ell = \limsup_{k\to\infty} a_k$ if and only if* (i)*, there is a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ that converges to ℓ and* (ii) *if $\ell' < \ell$, there is an integer N such that $a_k > \ell'$ for all $k \geq N$.*

When they are finite, $\limsup a_k$ and $\liminf a_k$ are the largest and smallest limit points of $\{a_k\}$, respectively. It's not too difficult to show that $\{a_k\}$ converges to a, $-\infty \leq a \leq \infty$, if and only if $\limsup a_k = \liminf a_k$.

We can also use the metric on $\mathbb{R}^n$ to define the *diameter of a set* E by letting

$$\operatorname{diam}(E) := \sup\{\, |\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in E \,\} \tag{1.3}$$

†Carlos: In fact, we can define it by saying that $\lim_{k\to\infty} x_k = -\infty$ if $\lim_{k\to\infty} -x_k = \infty$.

If the diameter of E is finite, E is said to be *bounded*. Equivalently, E is bounded if there is a finite constant M such that $|\mathbf{x}| \le M$ for all $\mathbf{x} \in E$. If E_1 and E_2 are two sets, the *distance between* E_1 *and* E_2 is defined by

$$d(E_1, E_2) \coloneqq \inf\{\, |\mathbf{x} - \mathbf{y}| : \mathbf{x} \in E_1,\, \mathbf{y} \in E_2 \,\}. \tag{1.4}$$

Open and closed sets in $\mathbb{R}^n$, and special sets

For $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon > 0$, the set

$$B_\varepsilon(\mathbf{x}) \coloneqq B(\mathbf{x}, \varepsilon) \coloneqq \{\, \mathbf{y} : |\mathbf{x} - \mathbf{y}| < \varepsilon \,\}. \tag{1.5}$$

A point $\mathbf{x} \in E$ is called an *interior point* of E if there exists $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset E$. The collection of all interior points of E is called the *interior of* E and denoted E°. A set is said to be *open* if $E = E^\circ$. The empty set $\emptyset$ is open by convention. The whole space $\mathbb{R}^n$ is clearly open and it is easy to see that $B_\varepsilon(\mathbf{x})$ is open for any $\varepsilon > 0$. We shall generally denote open sets by the letter G.

A set E is *closed* if $\mathbb{R}^n \setminus E$ is open. Thus, $\emptyset$ and $\mathbb{R}^n$ are closed (being the complements of each other). Closed sets will generally be denoted by the letter F. The union of the set E and all of its limit points is called the *closure* of E and written $\bar{E}$. By the *boundary* of E, we mean the set $\partial E \coloneqq \bar{E} \setminus E^\circ$.

Now, consider a collection of sets $\mathcal{A} = \{A\}$. A set is said to be of *type* A_δ if it can be written as a countable intersection of sets in A and to be of *type* A_σ if it can be written as a countable union of sets in A. The most common usage of this notation is G_δ and F_σ sets where $\mathcal{G} = \{G\}$ denotes the open sets in $\mathbb{R}^n$ and $\mathcal{F} = \{F\}$ the closed sets. Hence, E is of type G_δ if

$$E = \bigcap_k G_k,\ G_k \text{ open}, \tag{1.6}$$

and of type F_σ if

$$E = \bigcup_k F_k,\ F_k \text{ is closed}. \tag{1.7}$$

The complement of a G_δ set is an F_σ set and vice-versa.

Another type of special set we will have the occasion to use is a *perfect set*, by which we mean a closed set C each of whose points is a limit point of C. Thus, a perfect set is a closed set that is dense in itself.

Theorem 2 1.9. *A perfect set is uncountable.*

An n-dimensional interval I is a subset of $\mathbb{R}^n$ of the form

$$I = \{\, (x_1, \dots, x_n) : a_k \le x_k \le b_k, \text{ for } k = 1, \dots, n \,\}. \tag{1.8}$$

An n-interval is closed and has edges parallel to the coordinate axes. If the edge lengths $b_k - a_k$ are all equal, I will be called an *n-dimensional cube* or simply an *n-cube*. Cubes will usually be denoted by the letter Q. Two intervals I_1 and I_2 are said to be *nonoverlapping* if their interiors are disjoint, i.e., if the most they have in common is some part of their boundary. A set equal to an n-interval minus some part of its boundary is called a *partly open interval*. By definition, the *volume* $\operatorname{Vol} I$ of the interval $I = \{\, (x_1, \dots, x_n) \,\}$ is

$$\operatorname{Vol} I = (b_1 - a_1) \cdots (b_n - a_n). \tag{1.9}$$

Somewhat more generally, if $\{\mathbf{e}_k\}_{k=1}^n$ is any given set of n vectors emanating from a point in $\mathbb{R}^n$, we will consider the closed *parallelepiped*

$$P := \left\{ \mathbf{x} : \sum\nolimits_{k=1}^{n} t_k \mathbf{e}_k,\ 0 \le t_k \le 1 \right\} \tag{1.10}$$

Note that the edges of P are parallel translates of the $\mathbf{e}_k$. Thus, P is an interval if the $\mathbf{e}_k$ are parallel to the coordinate axes. The *volume* $\operatorname{Vol} P$ of P is by definition the absolute value of the $n \times n$ determinant having $\mathbf{e}_1, \dots, \mathbf{e}_n$ as rows. A linear transformation T of $\mathbb{R}^n$ transforms a parallelepiped P into a parallelepiped P' with volume $\operatorname{Vol} P' = |\det T| \operatorname{Vol} P$. In particular, a rotation of axes in $\mathbb{R}^n$ (which is an orthogonal linear transformation) does not change the volume of the parallelepiped. We will assume basic facts about the volume: for example, if N is finite and P is parallelepiped with $P \subset \bigcup_{k=1}^N I_k$ then $\operatorname{Vol} P \le \sum_{k=1}^N \operatorname{Vol}(I_k)$, and if $\{I_k\}_{k=1}^N$ are nonoverlapping intervals contained in a parallelepiped P, then $\sum_{k=1}^N \operatorname{Vol} I_k \le \operatorname{Vol} P$.

Now we shall use the notion of interval to obtain a very useful decomposition of open sets in $\mathbb{R}^n$. This will be the foundation of many of our results later on.

Theorem 3 1.10.. *Every open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals.*

This construction, however, fails in $\mathbb{R}^n$ for $n > 1$ since the union of (overlapping) intervals is not generally an interval. However, the following weaker, but sufficient, theorem does hold.

Theorem 4. *Every open set in $\mathbb{R}^n$, $n \ge 1$, can be written as a countable union of nonoverlapping (closed) cubes. It can also be written as a countable union of disjoint partly open subsets.*

You can find a proof of the preceding theorems in [1]

The collection $\{ Q : Q \in K_j,\ j = 1, 2, \dots \}$ which is constructed in the proof of the previous theorem is called a family of *dyadic cubes*.

Compact Sets and the Heine–Borel Theorem

By an *cover* of a set E, we mean a family $\mathcal{F}$ of sets A such that $E \subset \bigcap_{A \in \mathcal{F}} A$. A *subcover* $\mathcal{F}_0$ of a cover $\mathcal{F}$ is a cover with the property that $A_0 \in \mathcal{F}$ whenever $A_0 \in \mathcal{F}_0$. A cover $\mathcal{F}$ is called an *open cover* if each set in $\mathcal{F}$ is open. We say E is *compact* if every open cover of E has a finite subcover.

Theorem 5 1.12**.**

(a) *The Heine–Borel theorem: A set $E \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*
(b) *A set $E \subset \mathbb{R}^n$ is compact if and only if every sequence of points of E has a subsequence that converges to a point of E.*

Functions

By a function $f = f(\mathbf{x})$ defined for $\mathbf{x}$ in a subset E of $\mathbb{R}^n$, we will always mean a *real-valued* function, unless otherwise specified. By *real-valued*, we generally mean *extended real-valued*, i.e., f may take the values $\pm\infty$. If $|f(\mathbf{x})| < \infty$ for all $\mathbf{x} \in E$, f is *finite* on E. A finite function f is said to be *bounded* if there is a finite number M such that $|f(\mathbf{x})| \le M$ for all $\mathbf{x} \in E$; i.e., f is bounded on E if $\sup_{\mathbf{x} \in E} |f(\mathbf{x})|$ is finite. A sequence $\{f_k\}$ of functions is said to be *uniformly bounded* on E if there is a finite M such that $|f_k(\mathbf{x})| \le M$ for all $\mathbf{x} \in E$ and all k.

By the *support* of f, we mean the closure of the set where f is not zero. Thus, the support of a function is always closed. It follows that a function defined in $\mathbb{R}^n$ has *compact support* if and only if it vanishes outside some bounded set.

A function f defined on an interval I in $\mathbb{R}$ is called *monotone increasing (decreasing)* if $f(x) \leq f(y)$ (or $f(x) \geq f(y)$) whenever $x < y$, $x, y \in I$. By *strictly increasing (decreasing)* if $f(x) < f(y)$ (or $f(x) > f(y)$) whenever $x < y$, $x, y \in I$.

Let f be defined on $E \subset \mathbb{R}^n$ and let $\mathbf{x}_0$ be a limit point of E. Let $B'_\delta(\mathbf{x}_0) = B_\delta(\mathbf{x}_0) \setminus \{\mathbf{x}_0\}$ denote the puncture ball with center $\mathbf{x}_0$ and radius δ, and let

$$M_\delta(\mathbf{x}) = \sup_{\mathbf{x} \in B'_\delta(\mathbf{x}_0) \cap E} f(\mathbf{x}), \quad m_\delta(\mathbf{x}) = \inf_{\mathbf{x} \in B'_\delta(\mathbf{x}_0) \cap E} f(\mathbf{x}). \tag{1.11}$$

As $\delta \searrow 0$, $M_\delta(\mathbf{x}_0)$ decreases and $m_\delta(\mathbf{x}_0)$ increases, and we define

$$\begin{aligned} \limsup_{\substack{\mathbf{x} \to \mathbf{x}_0 \\ \mathbf{x} \in E}} f(\mathbf{x}) &= \limsup_{\delta \to 0} M_\delta(\mathbf{x}_0) \\ \liminf_{\substack{\mathbf{x} \to \mathbf{x}_0 \\ \mathbf{x} \in E}} f(\mathbf{x}) &= \lim_{\delta \to 0} m_\delta(\mathbf{x}_0). \end{aligned} \tag{1.12}$$

The following charaterizations about Equations (1.11) and (1.12) are valid.

Theorem 6 1.14**.**

(a) $M = \limsup_{\mathbf{x} \to \mathbf{x}_0, \mathbf{x} \in E} f(\mathbf{x})$ *if and only if* (i) *there exists* $\{\mathbf{x}_k\}$ *in* $E \setminus \{\mathbf{x}_0\}$ *such that* $\mathbf{x}_k \to \mathbf{x}_0$ *and* $f(\mathbf{x}_K) \to M$ *and* (ii) $M' > M$*, there exist* $\delta > 0$ *such that* $f(\mathbf{x}) < M'$ *for* $\mathbf{x} \in B'_\delta(\mathbf{x}_0)$ *for* $\mathbf{x} \in B'_\delta(\mathbf{x}_0) \cap E$.

(b) $m = \liminf_{\mathbf{x} \to \mathbf{x}_0, \mathbf{x} \in E} f(\mathbf{x})$ *if and only if* (i) *there exists* $\{\mathbf{x}_k\}$ *in* $E \setminus \{\mathbf{x}_0\}$ *such that* $\mathbf{x}_k \to \mathbf{x}_0$ *and* $f(\mathbf{x}_K) \to m$ *and* (ii) $m' < m$*, there exist* $\delta > 0$ *such that* $f(\mathbf{x}) > m'$ *for* $\mathbf{x} \in B'_\delta(\mathbf{x}_0)$ *for* $\mathbf{x} \in B'_\delta(\mathbf{x}_0) \cap E$.

1.2 Functions of bounded variation and the Riemann–Stieltjes integral

In this section, we introduce functions of bounded variation as well as the definition of the Riemann integral. We conclude with a proof that the

Functions of bounded variation

Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined for all $a \leq x \leq b$ and finite; let $\Gamma = \{x_0, \ldots, x_m\}$ be a *partition* of $[a, b]$, i.e., a collection of points x_i, $i = 0, \ldots, m$, satisfying $x_0 = a$ and $x_m = b$, and $x_{i-1} < x_i$ for $i = 1, \ldots, m$. To each partition Γ, we associated a sum

$$S_\Gamma \coloneqq S_\Gamma[f; a, b] \coloneqq \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|. \tag{1.13}$$

The *variation* (or *total variation*) *of* f *over* $[a, b]$ is defined as

$$V \coloneqq V[f; a, b] \coloneqq \sup_\Gamma S_\Gamma, \tag{1.14}$$

where the supremum is taken over all partitions Γ of $[a,b]$. If $V<\infty$, f is said to be of *bounded variation on* $[a,b]$; if $V=\infty$, f is of *unbounded variation on* $[a,b]$.

Before going on to prove important properties about (1.14), let us look at some common examples (and nonexamples) of functions f of bounded variation.

Examples 1. Suppose f is *monotone* in $[a,b]$. Then, clearly, each S_Γ is equal to $|f(a)-f(b)|$ for every partition $\Gamma^\ddagger$, and therefore $V=|f(b)-f(a)|$.

Examples 2. Suppose the graph of f can be split into a finite number of monotone arcs, i.e., suppose $[a,b]=\bigcup_{i=1}^{k}[a_{i-1},a_i]$ and f is monotone in each $[a_{i-1},a_i]$. Then $V=\sum_{i-1}^{k}|f(a_i)-f(a_{i-1})|$. To see this, we use the result of Example 1 above and the fact, yet to be proven, than $V=V[a,b]=\sum_{i=1}^{k}V[a_i,a_{i-1}]$.

If $\Gamma=\{x_0,\dots,x_m\}$ is a partition of $[a,b]$, let $|\Gamma|$, called the *norm of* Γ, be defined as the length of the longest subinterval of Γ

$$|\Gamma|\coloneqq\max_i(x_i-x_{i-1}). \tag{1.15}$$

If f is continuous on $[a,b]$ and $\{\Gamma_j\}$ is a sequence of partitions of $[a,b]$ with $|\Gamma_j|\to 0$, we shall see that $V=\lim_{j\to\infty}S_{\Gamma_j}$.

Examples 3. Let f be the *Dirichlet function*, defined by $f(x)=1$ for x rational and $f(x)=0$ for x irrational. Then, clearly, $V[a,b]=\infty$ for any interval of $[a,b]$.§

Examples 4. A function that is continuous on an interval, however, need not be of bounded variation on that interval. Take for example the following construction: let $\{a_j\}$ and $\{d_j\}$, $j=1,2,\dots$, be monotone decreasing sequences in $(0,1]$ with $a_1=\lim_{j\to\infty}a_j=\lim_{j\to\infty}d_j=0$ and $\sum d_j=\infty$. Construct a continuous function f as follows. On each subinterval $[a_{j+1},a_j]$, the graph of f consists of the sides of the isosceles triangle with base $[a_{j+1},a_j]$ and height d_j. Thus, $f(a_j)=0$, and if m_j denotes the midpoint of $[a_{j+1},a_j]$, then $f(m_j)=d_j$. If we define $f(0)=0$, then f is continuous on $[0,1]$. Taking Γ_k to be the partition defined by the points 0, $\{a_j\}_{j=1}^{k+1}$ and $\{m_j\}_{j=1}^{k}$, we see that $S_\Gamma=2\sum_{j=1}^{k}d_j$. Hence, $V[f;0,1]=\infty$.

‡Carlos: This may not be clear at a first glance, but, upon closer inspection, this is true by monotoncity. If $a<x<b$, we have $|f(b)-f(a)|=|f(b)-f(x)|+|f(x)-f(a)|$. This holds for an arbitrary partitions Γ.

§Carlos: By the density of $\mathbb{Q}$ in $\mathbb{R}$ (and by restriction, $[a,b]$, since $[a,b]$ is path-connected), for any positive integer N, we may choose a partition Γ of $[a,b]$ containing $N+1$ rational numbers so $S_\Gamma=N+1>N$.

1.3 The Lebesgue integral

This portion corresponds to material covered before the second exam.

1.4 Differentiation

This portion of the notes corresponds to material covered before the final.

This section deals with questions of differentiability and culminates with a couple of results tying together the Lebesgue integral with the derivative à la the familiar fundamental theorem of calculus for Riemann integrals.

The indefinite integral

If f is a Riemann integrable function on an interval $[a,b]$ of $\mathbb{R}$, then the familiar definition for its *indefinite integral* is

$$F(x) = \int_a^x f(y)dy, \quad a \leq x \leq b. \tag{1.16}$$

The *fundamental theorem of calculus* then asserts that $F' = f$ if f is continuous. In this section, we study the analogue of this result for Lebesgue integrable functions.

Since we want to generalize our results to $\mathbb{R}^n$, first we must find a suitable notion of indefinite integral for multivariable functions. In two dimensions we might, for instance, define the indefinite integral F of f to be

$$F(x_1, x_2) := \int_{a_1}^{x_1} \int_{a_2}^{x_2} f(y_1, y_2)\, dy_2\, dy_1. \tag{1.17}$$

As it turns out, it is better to abandon the notion that the indefinite integral be a function of a point an instead let it be a function of a set. Therefore, given a function f, integrable on some measurable subset A of $\mathbb{R}^n$, we define the *indefinite integral of* f to be the function

$$F(E) := \int_E f, \tag{1.18}$$

where E is a measurable subset of A.

The function F is an example of a *set function*, by which we mean any real-valued function F defined on a σ-algebra Σ of measurable sets such that

(i) $F(E)$ is finite for every $E \in \Sigma$.

(ii) F is *countably additive*; i.e., if E is the union of disjoint sets $E_k \in \Sigma$, $k = 1, 2, \ldots$, then

$$F(E) \sum_{k \in \mathbb{N}} F(E_k). \tag{1.19}$$

1.5 L^p Classes

Let's take a small detour to ch. 5 of [**?**] to talk about L^p spaces.

The relation between the Riemann–Stieltjes integral and the Lebesgue integral, and the L^p spaces, $0 < p < \infty$

As it turns out, there is a remarkably simple and useful representation of the Lebesgue integral (over measurable subsets of $\mathbb{R}^n$) in terms of the Riemann–Stiltjes integrals (over measurable subset of $\mathbb{R}$). In order to establish this relationship, we will need to study the function

$$\omega(\alpha) := \omega_{f,E}(\alpha) := |\{\, \mathbf{x} \in E : f(\mathbf{x}) > \alpha \,\}|, \tag{1.20}$$

where f is a measurable function on E and $-\infty < \alpha < \infty$. We call $\omega_{f,E}$ (or simply ω) the *distribution function of f on E*.

The function ω is clearly not affected by changing f in a set of measure zero, and is decreasing. As $\alpha \nearrow \infty$, we have

$$\{\, \mathbf{x} \in E : f(\mathbf{x}) > \alpha \,\} \searrow \{\, \mathbf{x} \in E : f(\mathbf{x}) = \infty \,\}.$$

hence, assuming that f is finite a.e. in E, by Theorem 3.62(ii), $\lim_{\alpha\to\infty} \omega = 0$, unless $\omega(\alpha) \equiv \infty$. Similarly, we have $\lim_{\alpha\to-\infty} \omega = |E|$. For now, let us assume that the measure of E is finite; this will ensure that ω is bounded.

In the following results, we assume that f is a measurable function that is finite a.e. in E, $|E| < \infty$, and write

$$\omega(\alpha) = \omega_{f,E}(\alpha), \quad \{\, f > a \,\} = \{\, \mathbf{x} \in E : f(\mathbf{x}) > \alpha \,\},$$

etc.

Lemma 7 5.38**.** *If $\alpha < \beta$, then $|\{\, \alpha \le f \le \beta \,\}| = \omega(\alpha) - \omega(\beta)$.*

Proof. For $\alpha < \beta$, we have $\{\, f > \beta \,\} \subset \{\, f > \alpha \,\}$ and $\{\, \gamma < f \le \beta \,\} = \{\, f > \alpha \,\} \setminus \{\, f > \beta \,\}$. Since $|\{\, f > \beta \,\}| < \infty$, the lemma follows from Corollary 3.25. ∎

Given α, let

$$\omega(\alpha+) \coloneqq \lim_{\varepsilon\searrow 0} \omega(\alpha+\varepsilon) \qquad \omega(\alpha-) \coloneqq \lim_{\varepsilon\searrow 0} \omega(\alpha-\varepsilon).$$

denote the limits of ω from the right and left at α.

Lemma 8 5.39**.**

(a) *$\omega(\alpha+) = \omega(\alpha)$; i.e., ω is continuous from the right.*
(b) *$\omega(\alpha-) = |\{\, f \ge \alpha \,\}|$.*

Corollary 9 5.40**.**

(a) *$\omega(\alpha-) - \omega(\alpha) = |\{\, f = \alpha \,\}|$; in particular, ω is continuous at α if and only if $|\{\, f = \alpha \,\}| = 0$.*
(b) *ω is constant in an open interval (α, β) if and only if $|\{\, \alpha < f < \beta \,\}| = 0$, that is, if and only if f takes almost no values between α and β.*

The rest of this section establishes the relations between the Lebesgue and Riemann–Stieltjes integrals. As always, we assume f is measurable and finite a.e. in E, $|E| < \infty$ and $\omega = \omega_{E,f}$.

Theorem 10 5.41**.** *If $a \le f(\mathbf{x}) \le b$ (a and b are finite) for all $\mathbf{x} \in E$, then*

$$\int_E f = -\int_a^b \alpha \, d\omega(\alpha).$$

Proof. The Lebesgue integral on the left-hand side exists since f is bounded and $|E| < \infty$. The Riemann–Stieltjes integral on the right-hand side exists by Theorem 2.24. To show that they are equal, let us partition the interval the interval $[a, b]$ by $a = \alpha_0 < \alpha_1 < \cdots < \alpha_k = b$ and let

$E_j = \{, \alpha_{j-1} < f \leq \alpha_j \}$. The E_j are disjoint and $E = \bigcup_{j=1}^k E_j$. Hence, $\int_E f = \sum_{j=1}^k \int_{E_j} f$ and, therefore

$$\sum_{j=1}^{\infty} \alpha_{j-1}|E_j| \leq \int_E f \leq \sum_{j=1}^{k} \alpha_j |E_j|.$$

By Lemma 5.38, $|E_j| = \omega(\alpha_j)\omega(\alpha_j) = -[\omega(\alpha_j) - \omega(\alpha_{j-1})]$. Hence, the sums are Riemann–Stieltjes sums for $-\int_a^b \alpha\, d\omega(\alpha)$. Since the sums must converge to $-\int_a^b \alpha\, d\omega(\alpha)$ as the norm of the partition tends to zero, the conclusion follows. ∎

We can extend the conclusion of Theorem 5.41 to the case when f is not bounded as follows.

Theorem 11 5.42**.** *Let f be any measurable function on E, and let $E_{ab} \coloneqq \{ \mathbf{x} \in E : a < f(\mathbf{x}) < b \}$ (a and b finite). Then,*

$$\int_{E_{ab}} f = -\int_a^b \alpha\, d\omega(\alpha).$$

Sketch of proof. Take $\omega_{ab}(\alpha) \coloneqq |\{ \mathbf{x} \in E_{ab} : f(\mathbf{x}) > \alpha \}|$. By Theorem 5.41, we have

$$\int_{E_{ab}} f = -\int_a^b \alpha\, d\omega_{ab}(\alpha).$$

Taking the limit of Riemann–Stieltjes sums that approximate the integrals, it suffices to show that $\omega_{ab}(\alpha) - \omega_{ab}(\beta) = \omega(\alpha) - \omega(\beta)$. Then The expression on the right-hand side of the equation above, is seen to be $\int_a^b \alpha\, d\omega(\alpha)$. ∎

Theorem 12 5.43**.** *If either $\int_E f$ or $\int_{-\infty}^{\infty} \alpha\, d\omega(\alpha)$ exist and is finite, then the other exists and is finite, and*

$$\int_E f = -\int_{-\infty}^{\infty} \alpha\, d\omega(\alpha).$$

Two measurable functions f and g are said to be *equimeasurable*, or *equidistributed*, if

$$\omega_{f,E}(\alpha) = \omega_{g,E}(\alpha)$$

for all α.

We may intuitively think of equimeasurable functions as being *rearrangements* of each other. For such functions, we have

$$|\{ a < f \leq b \}| = |\{ a < g \leq b \}| \quad |\{ f = a \}| = |\{ g = a \}|,$$

etc. We also gave the following immediate corollary of Theorem 5.43.

Corollary 13 5.44**.** *If f and g are equimeasurable on E and $f \in L(E)$, then $g \in L(E)$ and*

$$\int_E f = \int_E g.$$

The method used to derive Theorem 5.41 through 5.43 illustrates a basic difference between the Lebesgue and the Riemann integral. The Riemann integral is defined by a limiting process whose initial step involves partitioning the domain of f. On the other hand, the Lebesgue integral can be obtained from a process that partitions the *range* of f. In order to define the process more clearly, let f be a nonnegative measurable function that is finite a.e. in E, $|E| < \infty$. Let $\Gamma = \{ 0 = \alpha_0 < \alpha_1 < \cdots \}$ be a partition of the positive ordinate axis by a countable number of points $\alpha_k \to \infty$, and let $|\Gamma| = \sup_k(\alpha_{k+1} - \alpha_k)$. Set $E_k \coloneqq \{ \alpha_k \le f < \alpha_{k+1} \}$ and $Z \coloneqq \{ f = \infty \}$. Then the E_k are measurable and disjoint, $|Z| = 0$ and $E = (\bigcup E_k) \cup Z$, so that $|E| = \sum_k |E_k|$. Let

$$S_\Gamma \coloneqq \sum_{k \in \mathbb{N}} \alpha_k |E_k|, \quad S_\Gamma \coloneqq \sum_{k \in \mathbb{N}} \alpha_{k+1} |E_k|.$$

1.6 L^p Classes

Let's talk about L^p classes now and some important results about L^p spaces.

Definition of L^p

If E is a measurable subset of $\mathbb{R}^n$ and satisfies $0 < p < \infty$, then $L^p(E)$ denotes the collection of measurable f for which $\int_E |f|^p$ is finite, i.e.,

$$L^p(E) \coloneqq \left\{ f : \int_E |f|^p < \infty \right\} \tag{1.21}$$

for $0 < p < \infty$. Here, f may be complex-valued, in which case, if $f = f_1 + i f_2$ for measurable real-valued f_1 and f_2, we have $|f|^2 = {f_1}^2 + {f_2}^2$, so that

$$|f_1|, |f_2| \le |f| \le |f_1| + |f_2|.$$

It follows that $f \in L^p(E)$ if and only if both $f_1, f_2 \in L^p(E)$.

We shall write

$$\|f\|_{p,E} \coloneqq \left(\int_E |f|^p \right)^{1/p},$$

for $0 < p < \infty$. Thus, $L^p(E)$ is the set of measurable f for which $\|f\|_{p,E}$ is finite. Whenever it is clear from context, we will omit E in $L^p(E)$ and $\|f\|_{p,E}$, and instead write L^p and $\|f\|_p$. Also note that $L = L^1$.

In order to define $L^\infty(E)$, let f be real-valued and measurable on a set E of positive measure. Define the *essential supremum* of f on E to be

$$\operatorname*{ess\,sup}_E f \coloneqq \inf\{ \alpha : |\{ \mathbf{x} \in E : f(\mathbf{x}) > \alpha \}| = 0 \}. \tag{1.22}$$

In words, this the essential supremum of f is the least upper bound of f outside of a set of measure zero. It can be restated as such: $\operatorname{ess\,sup} f$ is the smallest number M, $-\infty \le M \le \infty$, such that $f(\mathbf{x}) \le M$ almost everywhere in E.

In the definition of $\operatorname{ess\,sup} f$, we have made the explicit assumption that the measure of E is nonzero. Otherwise, $\operatorname{ess\,sup} f = -\infty$ which can result in awkward or incorrect statements of results involving L^p spaces. Therefore, we shall adopt the convention that $\operatorname{ess\,sup} f = 0$ if $|E| = 0$.

A real or complex-valued measurable f is said to be *essentially bounded*, or simply *bounded* almost everywhere on E if $\operatorname{ess\,sup}|f|$ is finite. The class of all functions that are essentially bounded on E is denoted by $L^\infty(E)$. Clearly, $f \in L^\infty(E)$ if and only if its real and imaginary parts belong to $L^\infty(E)$. We shall use the notation $\|f\|_\infty$ synonymously with $\operatorname{ess\,sup} f$.

The following theorem gives some good motivation for the use the notation $\|f\|_\infty$, at least in the case $|E| < \infty$.

Theorem 14 8.1. *If $|E| < \infty$, then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.*

Sketch of proof. We may assume that $|E| > 0$, for otherwise we have a trivial statement, i.e., $\|f\|_p = 0$ for all p and by convention $\|f\|_\infty = 0$ so clearly $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$. Set $M \coloneqq \|f\|_\infty$. If $M' < M$, ■

CHAPTER 2

MA 544 (Spring 2016)

2.1 Exam 1 Prep

Problem 2.1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{ r\mathbf{x} : \mathbf{x} \in E \}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

Proof. Define a linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by $\mathbf{x} \mapsto r\mathbf{x}$. Using the standard basis for $\mathbb{R}^n$, this map has the matrix presentation

$$T\mathbf{x} = \begin{bmatrix} r & & \\ & \ddots & \\ & & r \end{bmatrix} \mathbf{x} \tag{2.1}$$

which has determinant $\det T = r^n$. By 3.35, we have $|E| = |T(E)| = r^n|E| = |rE|$. ∎

Problem 2.2. Let $\{E_k\}$, $k \in \mathbb{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k\to\infty} E_k \right| \le \liminf_{k\to\infty} |E_k|.$$

Proof. If the $\liminf |E_k| = \infty$ the inequality holds trivially. Hence, we may, without loss of generality, assume that $\liminf |E_k| < \infty$. By 3.20, the set $\liminf E_k$ is measurable and we have

$$\left| \liminf_{k\to\infty} E_k \right| = \left| \bigcup_{k=1}^{\infty} F_k \right|, \tag{2.2}$$

where $F_k = \bigcap_{n=k}^{\infty} E_n$. Now, note that the collection of sets $F'_k = \bigcup_{\ell=1}^{k} F_\ell$ forms an increasing sequence of measurable sets $F'_k \nearrow F'$, where $F' = \bigcup_{k=1}^{\infty} F_k = \liminf E_k$. Then, by 3.26 (i), we have

$$\lim_{k\to\infty} |F'_k| = |F'| = \left|\liminf_{k\to\infty} E_k\right|. \tag{2.3}$$

Hence, it suffices to show that $|F'_k| \le |E_k|$ for all k, but this follows by monotonicity of the outer measure, 3.3, since $F'_k \subset E_k$. Thus, we have the desired inequality

$$\left|\liminf_{k\to\infty} E_k\right| \le \liminf_{k\to\infty} |E_k|. \tag{2.4}$$

■

Problem 2.3. Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) = \{\, \mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r \,\}$. Prove that F is monotonic increasing and continuous.

Proof. That F is increasing is immediate from the monotonicity of the outer measure since for $x < x'$ we have $B(\mathbf{0}, x) \subset B(\mathbf{0}, x')$ so, by 3.2, we have

$$F(x)|B(\mathbf{0}, x)| \le |B(\mathbf{0}, x')| = F(x')$$

as desired.

To see that F is continuous, we will prove the following lemma

Lemma 15. *For any $x > 0$, $xB(\mathbf{0}, 1) = B(\mathbf{0}, x)$.*

Proof of lemma. If $\mathbf{y} \in xB(\mathbf{0}, 1)$ then $\mathbf{y} = x\mathbf{y}'$ for $\mathbf{y}' \in B(\mathbf{0}, 1)$. Thus, $|\mathbf{y}'| = |\mathbf{y}|/x < 1$ so $|\mathbf{y}| < x$ implies that $\mathbf{y} \in B(\mathbf{0}, x)$. Hence, we have the containment $xB(\mathbf{0}, 1) \subset B(\mathbf{0}, x)$.

On the other hand, if $\mathbf{y} \in B(\mathbf{0}, x)$ then $|\mathbf{y}| < x$ so $|\mathbf{y}/x| < 1$. Hence, $\mathbf{y}/x \in B(\mathbf{0}, 1)$ so $x(\mathbf{y}/x) = \mathbf{y} \in B(\mathbf{0}, x)$. Thus, $B(\mathbf{0}, x) \subset xB(\mathbf{0},)$ and equality holds. ♣

In light of Lemma 15 and 3.35, for $x > 0$, we have

$$F(x) = |B(\mathbf{0}, x)| = |xB(\mathbf{0}, 1)| = x^n |B(\mathbf{0}, 1)|. \tag{2.5}$$

It is clear that F is continuous on the interval $[0, \infty)$ since F is a polynomial in x. ■

Problem 2.4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. From the topological definition of continuity, f is continuous at $x \in C$ if and only if for every neighborhood U of $f(x)$, the preimage $f^{-1}(U)$ is a neighborhood of x. Now, ■

Let $x \in C$. Then, by the definition of continuity, for every natural number $n > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies

$$|f(x) - f(x')| < \frac{1}{2n}. \tag{2.6}$$

Let $x'', x' \in B(x, \delta)$. Then, by the triangle inequality, we have

$$\begin{aligned} |f(x') - f(x)''| &= |f(x') - f(x) - (f(x'') - f(x))| \\ &\leq |f(x') - f(x)| + |f(x'') - f(x)| \\ &< \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n}. \end{aligned} \tag{2.7}$$

In view of these estimates, define the set

$$A_n = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } x', x'' \in B(x, \delta) \text{ implies } |f(x') - f(x'')| < \frac{1}{n} \right\}. \tag{2.8}$$

Good Lord, that was a long definition! We claim that $C = \bigcap_{n=1}^{\infty} A_n$ and that A_n is open for all n.

First, let us show that $C = \bigcap_{n=1}^{\infty} A_n$. Let $x \in C$. Then for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Thus, $x \in A_n$ for all n so $x \in \bigcap A_n$. On the other hand, if $x \in \bigcap A_n$ for every $n > 0$, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. Fix $\varepsilon > 0$. By the Archimedean principle, there exists $N > 0$ such that $\varepsilon > 1/N$. Then, since $x \in A_N$ it follows that for some $\delta' > 0$, $|x - x'| < \delta'$ implies $|f(x) - f(x')| < 1/N < \varepsilon$. Thus, $x \in C$ and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$.

Lastly, we show that A_n is open. Let $x \in A_n$. Then there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < 1/n$. In particular, this means that $B(x, \delta) \subset A_n$ for any $x' \in B(x, \delta)$ satisfies $|f(x) - f(x')| < 1/n$. Thus, A_n is open and we conclude that $C = \bigcap_{n=1}^{\infty} A_n$ is a G_δ set.

Problem 2.5. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Proof. No. Recall that, by definition, or 4.1, f is measurable if and only if $\{ f > a \}$ for all $a \in \mathbb{R}$. ■

Problem 2.6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set $\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \}$ is measurable.

Proof. The idea here should be to rewrite

$$E = \left\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \right\} \tag{2.9}$$

as a countable union/intersection of measurable sets. Let $x \in E$. By the Cauchy criterion, for every $N > 0$ there exists a positive integer M such that $m, n \geq M$ implies $|f_n(x) - f_m(x)| < 1/N$. With this in mind, define

$$E_N = \left\{ x : \text{there exists } M \text{ such that } m, n \geq M \text{ implies } |f_n(x) - f_m(x)| < \frac{1}{N} \right\}. \tag{2.10}$$

Then, like for Problem 1.4, it is not too hard to see that the E_n's are open and that $E = \bigcap_{n=1}^{\infty} E_n$. Thus, E is a G_δ set and therefore measurable. ■

Problem 2.7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Proof. Suppose $f\colon [a, b] \to \mathbb{R}$ is absolutely continuous. Then for fixed $\varepsilon = 1$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, we have $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$. Let $\Gamma = \{x_k\}_{k=1}^N$ be a partition of $[a, b]$ into closed intervals such that $x_{k+1} - x_k < \delta$, then by absolute continuity we have

$$\begin{aligned} V[f;\Gamma] &= \sum_{k=1}^{N} |f(x_{k+1}) - f(x_k)| \\ &< 1. \end{aligned} \tag{2.11}$$

Thus, f is b.v. on $[a, b]$. ■

Problem 2.8. Let f be a continuous function from $[a, b]$ into $\mathbb{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, i.e., $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b) \\ -f(a) & \text{if } c = a \\ f(a) & \text{if } c = b \end{cases}.$$

Proof. ■

2.2 Exam 1

2.3 Exam 2 Prep

Problem 2.9. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B_\varepsilon(\mathbf{0})$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \smallsetminus B_\varepsilon(\mathbf{0})} f(\mathbf{x})\, d\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f\colon \mathbb{R}^n \to \mathbb{R}$ is (Lebesgue) integrable over a subset E of $\mathbb{R}^n$ (or, alternatively, f belongs to $L(E)$) if

$$\int_E f(\mathbf{x})\, d\mathbf{x} < \infty.$$

Put $E = \mathbb{R}^n \smallsetminus B_\varepsilon(\mathbf{0})$. Then, to show that f belongs to $L(E)$ it suffices to prove the inequality

$$\int_E f(\mathbf{x})\, d\mathbf{x} < \frac{C}{\varepsilon} \tag{2.12}$$

for some appropriate constant C. We proceed by directly computing the Lebesgue integral of f and employing Tonelli's theorem:

$$\begin{aligned} \int_E f(\mathbf{x})\, d\mathbf{x} &= \int_E \frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}} \\ &= \int \cdots \int_E \frac{dx_1 \cdots dx_n}{({x_1}^2 + \cdots + {x_n}^2)^{(n+1)/2}} \end{aligned}$$

let E_i denote the projection of E onto its i-th coordinate and make the trigonometric substitution $x_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \tan\theta$, $dx_1 = \sqrt{{x_2}^2 + \cdots + {x_n}^2} \sec^2\theta d\theta$ with $\theta \in (-\pi/2, -\tan^{-1}(\varepsilon)) \cup (\tan^{-1}(\varepsilon), \pi/2)$ giving us the integral

$$= \int_{E_n} \cdots \int_{E_2} \left[\frac{\cos^{n-1}\theta}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} d\theta \right] dx_2 \cdots dx_n$$

which, by Tonelli's theorem, is

$$= \int_{E_n} \cdots \int_{E_2} \frac{dx_2 \cdots dx_n}{({x_2}^2 + \cdots + {x_n}^2)^{n/2}} \left[\int_{E_\theta} \cos^{n-1}\theta d\theta \right]$$

where the integral

$$\int_{E_\theta} \cos^{n-1}\theta d\theta < \infty. \tag{2.13}$$

Proceeding in this manner, we eventually achieve the inequality

$$\begin{aligned}\int \cdots \int_E f(\mathbf{x}) d\mathbf{x} &< C' \int_{E_n} \frac{dx_n}{{x_n}^2} \\ &= 2C' \int_\varepsilon^\infty \frac{dx_n}{{x_n}^2} \\ &= \frac{C}{\varepsilon}\end{aligned} \tag{2.14}$$

as desired. ■

Problem 2.10. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. This is probably some theorem I can't remember right now. But anyway, first we shall establish that the limit f of $\{f_k\}$ must be nonnegative a.e. in $\mathbb{R}^n$. For assume otherwise. Then there exists a collection of points $\mathbf{x}$ in $\mathbb{R}^n$ of nonzero $\mathbb{R}^n$-Lebesgue measure such that $f(\mathbf{x}) < 0$. But $f_k(\mathbf{x}) \geq 0$ for all $k \in \mathbb{N}$. Set $0 < \varepsilon < |f(\mathbf{x})|$ then we have

$$|f(\mathbf{x}) - f_k(\mathbf{x})| > |f(\mathbf{x})| > \varepsilon \tag{2.15}$$

for all k which contradicts our assumption that $f_k \to f$ a.e. on $\mathbb{R}^n$. Therefore, the set of points $\mathbf{x} \in \mathbb{R}^n$ where $f(\mathbf{x}) < 0$ must have measure zero.

Now, based on pointwise convergence a.e. to f, given $\varepsilon > 0$ for a.e. $\mathbf{x} \in \mathbb{R}^n$ we have the following estimate

$$|f(\mathbf{x}) - f_k(\mathbf{x})| < \varepsilon \tag{2.16}$$

for sufficiently large k; say k greater than or equal to some index $N \in \mathbb{N}$. Moreover, we are given convergence in $L(\mathbb{R}^n)$ of f_k to f

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f < \infty. \tag{2.17}$$

By monotonicity of the Lebesgue integral (Theorem 5.5(iii)), this implies that

$$\int_E f \leq \int_{\mathbb{R}^n} f < \infty \tag{2.18}$$

and

$$\int_E f_k \leq \int_{\mathbb{R}^n} f_k < \infty \tag{2.19}$$

for all $k \in \mathbb{N}$. By Theorem 5.5(ii), f and the f_k's are finite a.e. in $\mathbb{R}^n$ so for some sufficiently large real number M, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in \mathbb{R}^n$. In particular, for any measurable subset E of $\mathbb{R}^n$, $|f|, |f_k| \leq M$ for a.e. $\mathbf{x} \in E$ so, by the bounded convergence theorem, we have the desired convergence

$$\int_E f_k \to \int_E f < \infty. \tag{2.20}$$

However, if f does not belong to $L(\mathbb{R}^n)$, i.e., its integral over $\mathbb{R}^n$ is infinity, there is no guarantee that f will be finite a.e. in $\mathbb{R}^n$. This means that the bounded convergence theorem will fail to ensure convergence in integral for any measurable subset E of $\mathbb{R}^n$. Let us demonstrate this with an example. Consider the sequence of functions ■

Problem 2.11. Assume that E is a measurable set of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of $\mathbb{R}^n$, then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \tag{2.21}$$

Set $E_k = \{\, \mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k \,\}$ and $F_k = \{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{2.22}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \tag{2.23}$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \tag{2.24}$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (2.24) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \tag{2.25}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (2.26) \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|) &= \sum_{k=0}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty} (k+1)|F_k| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty} (k+2)|F_{k+1}| - \sum_{k=0}^{\infty} (k+1)|F_{k+1}| \qquad (2.27) \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned}$$

Thus, from (2.26) and (2.27)

$$\sum_{k=1}^{\infty} |F_k| \le \int_E f(\mathbf{x}) d\mathbf{x} \le \sum_{k=0}^{\infty} |F_k| \qquad (2.28)$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ■

Problem 2.12. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f,g) = \int_E \frac{|f-g|}{1+|f-g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. $\Longrightarrow$: First note that ρ is strictly greater than or equal to zero since it is the integral of a nonnegative function. Suppose that $\rho(f_k, f) \to 0$ as $k \to \infty$. Then, given $\varepsilon > 0$ there exist an sufficiently large index N such that for every $k \geq N$ we have

$$\rho(f_k, g) = \int_E \frac{|f_k - f|}{1 + |f_k - f|} < \varepsilon. \tag{2.29}$$

By Theorem 5.11, this means that the map

$$\frac{|f_k - f|}{1 + |f_k - f|}$$

is zero a.e. in E which happens if $|f_k - f| = 0$ a.e. in E.

$\Longleftarrow$: Suppose that $f_k \to f$ as $k \to \infty$.

I don't know how to solve this. This is the intended solution:

$\Longrightarrow$: Given $\varepsilon > 0$, $\rho(f_k, f) \to 0$ implies that

$$\int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{|f_k - f|}{1 + |f_k - f} dx \longrightarrow 0.$$

Observe that the function $\Phi \colon \mathbb{R}^+ \to \mathbb{R}$ given by $\Phi(x) = x/(1+x)$ is increasing on $\mathbb{R}^+$ and $0 < \Psi(x) < 1$, hence

$$\begin{aligned}\int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{|f_k - f|}{1 + |f_k - f|} dx &\geq \int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{\varepsilon}{1 + \varepsilon} dx \\ &= \frac{\varepsilon}{1 + \varepsilon} |\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}|.\end{aligned}$$

Therefore,

$$|\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}| \leq \frac{1 + \varepsilon}{\varepsilon} \int_{\{\, x \in E : |f_k(x) - f(x)| > \varepsilon \,\}} \frac{|f_k - f|}{1 + |f_k - f|}\, dx \longrightarrow 0$$

as $k \to \infty$.

$\Longleftarrow$: Conversely, given $\delta > 0$, we have

$$\begin{aligned}\rho(f_k, f) = &\int_{\{\, x \in E : |f_k(x) - f(x)| > \delta \,\}} \frac{|f_k - f|}{1 + |f_k - f|}\, dx \\ &+ \int_{\{\, x \in E : |f_k(x) - f(x)| \leq \delta \,\}} \frac{|f_k - f|}{1 + |f_k - f|}\, dx \\ &\leq |\{\, x \in E : |f_k(x) - f(x)| > \delta \,\}| + \frac{\delta}{1 + \delta} |E|.\end{aligned}$$

Since $|E| < \infty$ and $\delta/(1 + \delta) \searrow 0$, then for any $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\frac{\delta'}{1 + \delta'} |E| < \frac{\varepsilon}{2}.$$

If $f_k \to f$ as $k \to \infty$ in measure, then for the above δ' there is an index $N > 0$ such that $k \geq N$ implies

$$|\{ x \in E : |f_k(x) - f(x)| > \delta' \}| < \frac{\varepsilon}{2}.$$

Therefore, $f_k \to f$ in measure implies $\rho(f_k, f) \to 0$ as $k \to \infty$. ■

Problem 2.13. Define the *gamma function* $\Gamma \colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1}\, du,$$

and the *beta function* $\beta \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. (a) Fix $y \in \mathbb{R}^+$. Then we must show that $\Gamma(y) < \infty$. First, since $(0,1)$ and $[1, \infty)$ are disjoint measurable subsets of $\mathbb{R}$, by Theorem 5.7 we can split the integral $\Gamma(y)$ into

$$\Gamma(y) = \underbrace{\int_0^1 e^{-u}u^{y-1}\, du}_{I_1} + \underbrace{\int_1^\infty e^{-u}u^{y-1}\, du}_{I_2}. \tag{2.30}$$

We will show, separately, that I_1 and I_2 are finite.

To see that I_1 is finite, note that

$$\begin{aligned} e^{-u}u^{y-1} &= e^{-u}e^{(y-1)\log u} \\ &= e^{-u+(y-1)\log u} \\ &\leq e^{(y-1)\log u} \\ &= u^{y-1} \end{aligned} \tag{2.31}$$

since $0 < u < 1$

$$\begin{aligned} I_1 &= \int_0^1 e^{-u}u^{y-1}\, du \\ &\leq \int_0^1 u^{y-1}\, du \\ &= \left[\frac{u^y}{y}\right]_0^1 \\ &= \frac{1}{y} \\ &< \infty. \end{aligned} \tag{2.32}$$

To see that I_2 is finite, note that

$$e \tag{2.33}$$

Intended solution:

(b) ■

Problem 2.14. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}} \colon \mathbb{R}^n \to \mathbb{R}$ be $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \to \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. Note that by the triangle inequality, we have the following estimate on the integral

$$\int_{\mathbb{R}^n} |f_{\mathbf{h}}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \leq \tag{2.34}$$

■

Problem 2.15. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \leq g_k$ and

$$\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \to 0 \qquad \text{as} \qquad k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \to \int_{\mathbb{R}^n} |f| \qquad \text{as} \qquad k \to \infty.$$

Proof. (a) Since $f_k \to f$ and $g_k \to g$ a.e. and $|f_k| \leq g_k$, then by Fatou's theorem,

$$\int_{\mathbb{R}^n} (g - f) = \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k - f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k - f_k,$$
$$\int_{\mathbb{R}^n} g + f \int_{\mathbb{R}^n} \liminf_{k \to \infty} g_k + f_k \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} g_k + f_k.$$

Since $f_k, g_k, f, g \in L(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$, then using the similar argument as problem 2, we have

$$\int_{\mathbb{R}^n} f \geq \limsup_{k \to \infty} \int_{\mathbb{R}^n} f_k,$$
$$\int_{\mathbb{R}^n} f \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} f_k.$$

Therefore, $\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f$.

(b) $\implies$: This direction is obvious by the inequality

$$\left|\int_{\mathbb{R}^n} |f_k| - |f|\right| \le \int_{\mathbb{R}^n} ||f_k| - |f|| \le \int_{\mathbb{R}^n} |f_k - f|.$$

$\impliedby$: Let $g_k = |f_k| + |f|$ and $g = 2|f|$. Since $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e., then $g_k, g \in L(\mathbb{R}^n)$ and $g_k \to g$ a.e. in $\mathbb{R}^n$. By the assumption, $\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g$.

Let $\tilde{f}_k = |f_k - f|$. Then $\tilde{f}_k \to 0$ a.e. in $\mathbb{R}^n$ and $\tilde{f}_k \le g_k$. Applying part (a) to $\tilde{f}_k$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \tilde{f}_k = \lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k - f| = 0.$$

■

2.4 Midterm 2

Problem 2.16. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \smallsetminus B} |f| < \varepsilon.$$

Proof. Recall that $f \in L(\mathbb{R}^n)$ if and only if $|f| \in L(\mathbb{R}^n)$. Let $B_k = B(\mathbf{0}, k)$ for $k \in \mathbb{N}$ and χ_{B_k} be the indicator function associated with B_k. Then, the sequence of maps $\{|f_k|\}$ defined $f_k = f\chi_{B_k}$ converge pointwise to $|f_k|$. Since $|f| \in L(\mathbb{R}^n)$, by the monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} |f_k| = \int_{B_k} |f| \longrightarrow \int_{\mathbb{R}^n} |f|. \tag{2.35}$$

But this means, exactly, that for every $\varepsilon > 0$ there exists sufficiently large $N \in \mathbb{N}$ such that

$$\begin{aligned} \varepsilon &> \left| \int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f| \right| \\ &= -\int_{\mathbb{R}^n} |f_k| + \int_{\mathbb{R}^n} |f| \\ &= -\int_{\mathbb{R}^n} |f| + \int_{\mathbb{R}^n} |f| \\ &= -\int_{B_k} |f| + \int_{\mathbb{R}^n} |f| \\ &= \int_{\mathbb{R}^n \smallsetminus B_k} |f| \end{aligned} \tag{2.36}$$

as desired. ∎

Problem 2.17. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. First, since the E_j's are pairwise disjoint, by Theorem 3.23, we have

$$|E| = \sum_{j=1}^{\infty} |E_j|. \tag{2.37}$$

Let χ_{E_j} be the characteristic function of the subset E_j of E and define $f_j = f\chi_{E_j}$ for $j \in \mathbb{N}$. Note that, since both f and χ_{E_j} are measurable on E, f_j is measurable on E and $\sum_{j=1}^{\infty} f_j = f$. Moreover, since $E_j \subset E$, by monotonicity of the integral we have

$$\int_E f = \int_{E_j} f + \int_{E \smallsetminus E_j} f = \int_E f_j + \int_{E \smallsetminus E_j} f. \tag{2.38}$$

Hence, because the E_j's are disjoint $(E \smallsetminus E_k) \smallsetminus E_\ell = (E \smallsetminus E_\ell) \smallsetminus E_k$ so

$$\int_E f = \sum_{j=1}^{\infty} \int_E f_j = \sum_{j=1}^{\infty} \int_{E_j} f \tag{2.39}$$

as desired. ∎

Problem 2.18. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbb{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Proof. Let $\varepsilon > 0$ be given. Then, by the hypothesis, there exists $\delta > 0$ such that such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon \tag{2.40}$$

for all $k \in \mathbb{N}$. By Egorov's theorem, there exists a closed subset F of E such that $|E \smallsetminus F| < \delta$ and $f_k \to f$ uniformly on F. Then, by the uniform convergence theorem,

$$\int_F f_k \longrightarrow \int_F f \tag{2.41}$$

as $k \to \infty$. But by hypothesis, we have

$$\int_{E\smallsetminus F} |f_k| < \varepsilon. \tag{2.42}$$

Letting $\varepsilon \to 0$, we achieved the desired convergence. ∎

Problem 2.19. Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)\, dt$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Proof. By Lusin's theorem, there exists a closed subset F of I with $|I \smallsetminus F| < \varepsilon$ such that the restriction of f to $F = I \smallsetminus E$ is continuous. Now, since F is closed in I and I is compact, it follows that I is compact. Hence, by the Stone–Weierstraß approximation theorem, there exist a sequence of polynomials $\{p_k\}$ such that $p_k \to f$ uniformly on F. Then, by the uniform convergence theorem, we have

$$\int_F p_k \longrightarrow \int_F f \tag{2.43}$$

so

$$\begin{aligned}\int_F \left[\int_x^1 t^{-1} p_k(t)dt\right] dx &= \int_F \left[\int_x^1 at^{-1} + q_k(t)dt\right] dx \\ &= \int_F q_k'(x) - a\log(x)\, dx \\ &< \infty\end{aligned} \tag{2.44}$$

for all k and converges uniformly to g so $g \in L(I)$. I don't know how to show that in fact $\int_I g = \int_I f$. Perhaps you show that the places where they differ is a set of measure zero. ■

2.5 Final Practice

Problem 2.20. Suppose $f \in L^1(\mathbb{R})$ and that x is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) \coloneqq \frac{1}{r} \int_{B(0,r)} |f(x-y) - f(x)| \, dy.$$

Show that:

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$;
(b) there exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

Proof. ∎

Problem 2.21. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \leq n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0$$

if and only if

$$\|f_k\|_p \longrightarrow \|f\|_p$$

as $k \to \infty$.

Proof. ∎

Problem 2.22. Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

Proof. ∎

Problem 2.23. Let $f \in L(\mathbb{R})$, and let $F(t) \coloneqq \int_{\mathbb{R}} f(x) \cos(tx) dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t \to \infty} F(t) = 0.$$

Proof. ∎

Problem 2.24. Let f be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular. Show that

$$\int_a^b \varphi \, df = \int_a^b \varphi f' dx + \int_a^b \varphi \, dh$$

for all functions φ continuous on $[a, b]$.

Proof. ∎

2.6 Bañuelos Quals

August 2000

Problem 2.25. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions with the property that for all $n \geq 1$,

$$\mu(\{ x \in X : |f_n(x)| \geq \lambda \}) \leq Ce^{-\lambda^2/n}$$

for all $\lambda > 0$. (Here C is a constant independent of n.) Let $n_k = 2^k$. Prove that

$$\limsup_{k\to\infty} \frac{|f_{n_k}}{\sqrt{n_k \ln(ln(n_k))}} \leq 1$$

a.e.

Proof. ∎

Problem 2.26. let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let f_n be a sequence of measurable functions with $f_1 \in L^1(\mu)$ and with the property that

$$\mu(\{ x \in X : |f_n(x)| > \lambda \}) \leq \mu(\{ x \in X : |f_1(x)| > \lambda \})$$

for all n and all $\lambda > 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_X \left[\max_{1\leq j\leq n} |f_j| \right] d\mu = 0.$$

(*Hint*: You may use the fact that $\|f\|_1 = \int_0^\infty \{ |f(x)| > \lambda \} \, d\lambda$.)

Proof. ∎

Problem 2.27. (i) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions. Prove that $f_n \to f$ is measurable if and only if every subsequence $\{f_{n_k}\}$ contains a further subsequence $\{f_{n_{k_j}}\}$ that converges a.e. to f.

(ii) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $F \colon \mathbb{R} \to \mathbb{R}$ be continuous and $f_n \to f$ in measure. Prove that $F(f_n) \to F(f)$ in measure. (You may assume, of course, that f_n, F, $F(f_n)$, and $F(f)$ are all measurable.)

Proof. ∎

Problem 2.28. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and suppose $f \in L^1(\mu)$ is nonnegative. Suppose $1 < q < \infty$ and let $1 \leq \leq \infty$ be its conjugate exponent, i.e., $1/p + 1/q = 1$. Suppose f has the property that

$$\int_E f \, d\mu \leq \mu(E)^{1/q}$$

for all measurable sets E. Prove that $f \in L^r(\mu)$ for any $1 \leq r < p$.
(*Hint*: Consider)

Proof. ∎

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