

# MA 523: Homework, Midterms and Practice Problems Solutions

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# 1 Homework Solutions

## 1.1 Homework 1

PROBLEM 1.1.1 (Taylor's formula). Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be smooth,  $n \geq 2$ . Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$$

as  $x \rightarrow \mathbf{0}$  for each  $k = 1, 2, \dots$ , assuming that you know this formula for  $n = 1$ .

*Hint:* Fix  $x \in \mathbb{R}^n$  and consider the function of one variable  $g(t) := f(tx)$ . Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on  $m$ .

*SOLUTION.* Taking the hint, apply Taylor's formula to the function  $g(t) = f(tx)$ ,

$$g(t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} t^k$$

■

PROBLEM 1.1.2. Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{*}$$

on  $\mathbb{R}^n \times (0, \infty)$ , where  $b \in \mathbb{R}^n$ . Using the characteristic equation, solve (\*) subject to the initial condition

$$u = g$$

on  $\mathbb{R}^n \times \{t = 0\}$ . Make sure the answer agrees with formula (5) in §2.1.2 of [E].

*SOLUTION.*

■

PROBLEM 1.1.3. Solve using the characteristics:

(a)  $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$ ,  $u = 1$  on the line  $x_2 = 2x_1$ .

(b)  $u u_{x_1} + u_{x_2} = 1$ ,  $u(x_1, x_1) = \frac{x_1}{2}$ .

(c)  $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$ ,  $u(x_1, x_2, 0) = g(x_1, x_2)$ .

*SOLUTION.*

■

PROBLEM 1.1.4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with  $u(x_1, 0) = \frac{1-x_1^2}{2}$ .

*SOLUTION.*

■

## 1.2 Homework 2

PROBLEM 1.2.1. Verify assertion (36) in [E, §3.2.3], that when  $\Gamma$  is not flat near  $x^0$  the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here  $\nu(x^0)$  denotes the normal to the hypersurface  $\Gamma$  at  $x^0$ ).

SOLUTION. ■

PROBLEM 1.2.2. Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions  $u(x, 0) = g(x)$  is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some  $t > 0$ , unless  $a(g(x))$  is a nondecreasing function of  $x$ .

SOLUTION. ■

PROBLEM 1.2.3. Show that the function  $u(x, t)$  defined for  $t \geq 0$  by

$$u(x, t) = \begin{cases} -\frac{2}{3} \left( t + \sqrt{3x + t^2} \right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law  $u_t + \left(\frac{u^2}{2}\right)_x = 0$  (*inviscid Burgers' equation*).

SOLUTION. ■

### 1.3 Homework 3

PROBLEM 1.3.1. Consider the initial value problem

$$\begin{cases} u_t = \sin u_x, \\ u(x, 0) = \frac{\pi}{4}x. \end{cases}$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

*SOLUTION.* ■

PROBLEM 1.3.2. Consider the Cauchy problem for  $u(x, y)$

$$\begin{cases} u_y = a(x, y, u)u_x + b(x, y, u), \\ u(x, 0) = 0, \end{cases}$$

let  $a$  and  $b$  be analytic functions of their arguments. Assume that  $D^\alpha a(0, 0, 0) \geq 0$  and  $D^\alpha b(0, 0, 0) \geq 0$  for all  $\alpha$ . (Remember by definition, if  $\alpha = 0$  then  $D^\alpha f = f$ .)

- (a) Show that  $D^\beta u(0, 0) \geq 0$  for all  $|\beta| \leq 2$ .
- (b) Prove that  $D^\beta u(0, 0) \geq 0$  for all  $\beta = (\beta_1, \beta_2)$ . (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in  $\beta_2$ )

*SOLUTION.* ■

PROBLEM 1.3.3. (Kovalevskaya’s example) show that the line  $\{t = 0\}$  is characteristic for the heat equation  $u_t = u_{xx}$ . Show there does not exist an analytic solution  $u$  of the heat equation in  $\mathbb{R} \times \mathbb{R}$ , with  $u = \frac{1}{1+x^2}$  on  $\{t = 0\}$ . (*Hint:* assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of  $(0, 0)$ .)

*SOLUTION.* ■

## 1.4 Homework 4

PROBLEM 1.4.1 (Legendre transform). Let  $u(x_1, x_2)$  be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of  $\mathbb{R}^2$ , where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u(\mathbf{x}(p)),$$

where  $\mathbf{x} = (x^1, x^2)$ ,  $p = (p_1, p_2)$ . Show that  $v$  satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

(*Hint*: See [Evans, 4.4.3b], prove the identities (29)).

SOLUTION. ■

PROBLEM 1.4.2. Find the solution  $u(x, t)$  of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant  $x > 0, t > 0$  for which

$$\begin{cases} u(x, 0) = f(x), & u_t(x, 0) = g(x), & \text{for } x > 0, \\ u_t(0, t) = \alpha u_x(0, t), & & \text{for } t > 0, \end{cases}$$

where  $\alpha \neq -1$  is a given constant. Show that generally no solution exists when  $\alpha = -1$ . (*Hint*: Use a representation  $u(x, t) = F(x - t) + G(x + t)$  for the solution.)

SOLUTION. ■

PROBLEM 1.4.3. (a) Let  $u$  be a solution of the wave equation  $u_{tt} - c^2 u_{xx} = 0$  for  $0 < x < \pi, t > 0$  such that  $u(0, t) = u(\pi, t) = 0$ . Show that the *energy*

$$E(t) = \frac{1}{2} \int_0^\pi (u_t^2 + c^2 u_x^2) dx, \quad t > 0$$

is independent of  $t$ ; i.e.,  $\frac{d}{dt}E = 0$  for  $t > 0$ . Assume that  $u$  is  $C^2$  up to the boundary.

(b) Express the energy  $E$  of the Fourier series solution

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(nct) + b_n \sin(nct)) \sin(nx)$$

in terms of coefficients  $a_n, b_n$ .

SOLUTION. ■

## 1.5 Homework 5

PROBLEM 1.5.1. Prove that Laplace's equation  $\Delta u = 0$  is rotation invariant; that is, if  $O$  is an orthogonal  $n \times n$  matrix and we define  $v(x) := u(Ox)$ ,  $x \in \mathbb{R}^n$ , then  $\Delta v = 0$ .

SOLUTION. ■

PROBLEM 1.5.2. Let  $n = 2$  and  $U$  be the halfplane  $\{x_2 > 0\}$ . Prove that

$$\sup_U u = \sup_{\partial U} u$$

for  $u \in C^2(U) \cap C(\bar{U})$  which are harmonic in  $U$  under the additional assumption that  $u$  is bounded from above in  $\bar{U}$ . (The additional assumption is needed to exclude examples like  $u = x_2$ .)

[Hint: Take for  $\varepsilon > 0$  the harmonic function

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region  $\{x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0\}$  with large  $a$ . Let  $\varepsilon \rightarrow 0$ .]

SOLUTION. ■

PROBLEM 1.5.3. Let  $U \subset \mathbb{R}^n$  be an open set. We say  $v \in C^2(U)$  is subharmonic if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Let  $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}$  be smooth and convex. Assume  $u^1, \dots, u^m$  are harmonic in  $U$  and

$$v := \varphi(u_1, \dots, u_m).$$

Prove  $v$  is subharmonic.

[Hint: Convexity for a smooth function  $\varphi(z)$  is equivalent to  $\sum_{j,k=1}^m \varphi_{z_j, z_k}(z) \xi_j \xi_k \geq 0$  for any  $\xi \in \mathbb{R}^m$ .]

(b) Prove  $v := |Du|^2$  is subharmonic, whenever  $u$  is harmonic. (Assume that harmonic functions are  $C^\infty$ .)

SOLUTION. ■

## 1.6 Homework 6

PROBLEM 1.6.1. For  $n = 2$  find Green's function for the quadrant  $U := \{x_1, x_2 > 0\}$  by repeated reflection.

SOLUTION. Taking the hit, set  $x' := (x_1, -x_2)$ ,  $x'' := (-x_1, x_2)$ ,  $x''' := (-x_1, -x_2)$ , and define

$$\varphi^x(y) := \Phi(y - x') + \Phi(y - x'') - \Phi(y - x'''). \quad (1)$$

We claim that  $\varphi^x$ , as defined above, solves

$$\begin{cases} \Delta \varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(y - x) & \text{on } \partial U. \end{cases}$$

It is clear that  $\Delta \varphi^x = 0$  since it is built up from the fundamental solutions on  $\mathbb{R}^n$  (this follows from the linearity of the Laplace operator). To see that  $\varphi^x(y) = \Phi(y - x)$  on  $\partial U$ , we do a case by case analysis.

Note that on  $\{x_1 = 0\} \subset \partial U$ , we have

$$\varphi^x(y_1, 0) = \Phi(-x_1, y_2 + x_2) + \Phi(-x_1, y_2 - x_2) - \Phi(x_1, y_2 + x_2),$$

where, since the fundamental solution is radial, we have  $\Phi(-x_1, y_2 + x_2) = \Phi(x_1, y_2 + x_2)$ , and hence the above equals

$$\begin{aligned} &= \Phi(-x_1, y_2 - x_2) \\ &= \Phi(y - x) \end{aligned}$$

and on  $\{x_2 = 0\} \subset \partial U$ , we have

$$\varphi^x(0, y_2) = \Phi(y_1 - x_1, x_2) + \Phi(y_1 + x_1, -x_2) - \Phi(y_1 + x_1, x_2)$$

where, again because  $\Phi$  is radial,  $\Phi(y_1 + x_1, -x_2) = \Phi(y_1 + x_1, x_2)$ , thus the above equals

$$\begin{aligned} &= \Phi(y_1 - x_1, x_2) \\ &= \Phi(y - x). \end{aligned}$$

Thus,  $\varphi^x(y) = \Phi(y - x)$  on  $\partial U$ .

Therefore, Green's function on  $U$  is

$$G(x, y) = \Phi(y - x) - \varphi^x(y) = \Phi(y - x) - \Phi(y - x') - \Phi(y - x'') + \Phi(y - x'''). \quad \blacksquare$$

PROBLEM 1.6.2. (Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0)$$

whenever  $u$  is positive and harmonic in  $B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$ .

*SOLUTION.* Recall Poisson's formula for the ball

$$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y), \quad (2)$$

where  $x \in B(0, r)$  and  $u$  solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0, r), \\ u = g & \text{on } \partial B(0, r). \end{cases}$$

For fixed  $x \in B(0, r)$ , write

$$u(x) = r^{n-2}(r+|x|)(r-|x|) \left[ \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y) \right].$$

Now, since  $r+|x| \geq |x-y| \geq r-|x|$  for all  $y \in \partial B(0, r)$ , we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} \int_{\partial B(0,r)} g(y) dS(y) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} \int_{\partial B(0,r)} g(y) dS(y). \quad (3)$$

Since  $u = g$  on the boundary  $\partial B(0, r)$ , by applying the mean-value property on (??) we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} u(0),$$

as desired. ■

**PROBLEM 1.6.3.** Let  $P_k(x)$  and  $P_m(x)$  be homogeneous harmonic polynomials in  $\mathbb{R}^n$  of degrees  $k$  and  $m$  respectively; i.e.,

$$\begin{cases} P_k(\lambda x) = \lambda^k P_k(x), & P_m(\lambda x) = \lambda^m P_m(x) & \text{for every } x \in \mathbb{R}^n, \lambda > 0, \\ \Delta P_k = 0, & \Delta P_m = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

(a) Show that

$$\begin{cases} \frac{\partial P_k}{\partial \nu} = k P_k(x), & \frac{\partial P_m}{\partial \nu} = m P_m(x) & \text{on } \partial B(0, 1), \end{cases}$$

where  $B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$  and  $\nu$  is the outward normal on  $\partial B(0, 1)$ .

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x) d\sigma = 0, \quad \text{if } k \neq m.$$

*SOLUTION.* For part (a), let

$$P_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha.$$



Then, since  $\nu = (x_1, \dots, x_n)$ , the derivative along  $\nu$  is given by

$$\begin{aligned} \frac{\partial P_k(x)}{\partial \nu} &= \sum_{i=1}^n (P_k)_{x_i} x_i \\ &= \sum_{i=1}^n \left( \sum_{|\alpha|=k} a_\alpha x^\alpha \right)_{x_i} x_i \\ &= \sum_{i=1}^n \left( \sum_{j=1}^m a_\alpha x_1^{\alpha_1^j} \cdots x_i^{\alpha_i^j} \cdots x_n^{\alpha_n^j} \right)_{x_i} x_i \end{aligned}$$

where  $\sum_{i=1}^n \alpha_i^j = k$  and  $1 \leq j \leq \binom{n+k-1}{n} =: m$  (by the stars and bars theorem)

$$\begin{aligned} &= \sum_{i=1}^n \sum_{j=1}^m \left( \alpha_i^j a_\alpha x_1^{\alpha_1^j} \cdots x_i^{\alpha_i^j-1} \cdots x_n^{\alpha_n^j} \right) x_i \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i^j a_\alpha x_1^{\alpha_1^j} \cdots x_i^{\alpha_i^j} \cdots x_n^{\alpha_n^j} \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i^j a_\alpha x^\alpha \end{aligned}$$

switching the order of summation, we have

$$\begin{aligned} &= \sum_{j=1}^m \sum_{i=1}^n \alpha_i^j a_\alpha x^\alpha \\ &= \sum_{j=1}^m k a_\alpha x^\alpha \\ &= k \sum_{j=1}^m a_\alpha x^\alpha \\ &= k P_k(x). \end{aligned}$$

This argument, of course, applies to every  $k \in \mathbb{N}$ .

For part (b), by Green's theorem, we have

$$\begin{aligned} \int_{B(0,r)} P_k(x) \Delta P_m(x) - (\Delta P_k(x)) P_m(x) dx &= \int_{\partial B(0,r)} P_k(x) \frac{\partial}{\partial \nu} P_m(x) - \frac{\partial}{\partial \nu} P_k(x) P_m(x) dS(x) \\ &= \int_{\partial B(0,r)} (m-k) P_k(x) P_m(x) dS(x), \end{aligned}$$

where the left-hand side is equal to 0 since both  $\Delta P_k$  and  $\Delta P_m$  are 0. Since  $m \neq k$ , it must be the case that

$$\int_{\partial B(0,r)} P_k(x) P_m(x) dS(x) = 0.$$

■

## 2 Exams

### 2.1 Midterm Practice Problems

PROBLEM 2.1.1. Solve  $u_{x_1}^2 + x_2 u_{x_2} = u$  with initial conditions  $u(x_1, 1) = \frac{x_1^2}{4} + 1$ .

*SOLUTION.* By inspection, we may suspect that  $v(x_1, x_2) = \frac{x_1^2}{4} + x_2$  is a solution to the PDE. It certainly satisfies the boundary condition. A routine calculation shows that  $v$  is in fact a solution to the PDE. Lucky guess!

More formally, let us solve this problem using the method of characteristics. First, write

$$F(p, z, x) = (p^1(s))^2 + x^2(s)p^2(s) - z(s) = 0.$$

Then, the characteristic ODEs are

$$\left\{ \begin{array}{l} (\dot{p}^1(s), \dot{p}^2(s)) = -(0, p^2(s)) + (p^1(s), p^2(s)) \\ \quad = (p^1(s), 0), \\ \dot{z}(s) = (2p^1(s), x^2(s)) \cdot (p^1(s), p^2(s)) \\ \quad = 2p^1(s)^2 + x^2(s)p^2(s), \\ (\dot{x}^1(s), \dot{x}^2(s)) = (2p^1(s), x^2(s)). \end{array} \right.$$

Now, for  $(x^1(0), x^2(0)) = (x^0, 1)$ , integrating the characteristics, we get

$$\left\{ \begin{array}{l} (p^1(s), p^2(s)) = (p_0^1 e^s, p_0^2), \\ (x^1(s), x^2(s)) = (2p_0^1 e^s + x_0^1, x_0^2 e^s), \\ z(s) = \frac{(x^0)^2}{4} e^{2s} + p_0^2 e^s + z^0 \end{array} \right.$$

Using the initial condition and the PDE, we find that

$$\begin{aligned} p_0^1 &= \frac{x^0}{2}, \quad p_0^2 = \frac{(x^0)^2}{4} + 1 - \frac{(x^0)^2}{4} = 1, \\ x_0^1 &= 0, \quad x_0^2 = 1 \\ z^0 &= 0, \end{aligned}$$

and consequently

$$\left\{ \begin{array}{l} (x^1(s), x^2(s)) = (x^0 e^s, e^s), \\ z(s) = \frac{(x^0)^2}{4} e^{2s} + e^s \end{array} \right.$$

so, rewriting  $z$  in terms of  $(x^1, x^2)$ , we have

$$\begin{aligned} z(s) &= \frac{(x^0)^2}{4} e^{2s} + e^s \\ &= \frac{(x^1(s))^2}{4} + x^2(s), \end{aligned}$$

so the solution in terms of  $(x_1, x_2)$ , is

$$u(x_1, x_2) = \frac{x_1^2}{4} + x_2,$$

just as we suspected. ■

PROBLEM 2.1.2. Find the maximal  $t_0 > 0$  for which the (classical) solution of the Cauchy problem

$$\begin{cases} uu_x + u_t = 0, \\ u(x, 0) = e^{-\frac{x^2}{2}}, \end{cases}$$

exists in  $\mathbb{R} \times [0, t)$ ; i.e., the first time  $t = t_0$  when the shock develops.

*SOLUTION.* First, let us find a solution to the PDE using the method of characteristics. Write

$$F(p, z, x) = z(s)p^1(s) + p^2(s).$$

Then, the characteristic ODEs are

$$\begin{cases} (\dot{p}^1(s), \dot{p}^2(s)) = -(0, 0) - p^1(p^1(s), p^2(s)) \\ \quad = (-p^1(s)^2, -p^1(s)p^2(s)), \\ \dot{z}(s) = (z(s), 1) \cdot (p^1(s), p^2(s)) \\ \quad = z(s)p^1(s) + p^2(s) \\ \quad = 0, \\ (\dot{x}(s), \dot{t}(s)) = (z(s), 1). \end{cases}$$

Thus, integrating the characteristic ODEs from  $(x^0, 0)$ , we have

$$\begin{cases} \dot{z}(s) = z^0, \\ (x(s), t(s)) = (z^0 s + x^0, s); \end{cases}$$

since the PDE is quasilinear, we disregard  $(p^1, p^2)$ .

Applying the boundary conditions, we see that

$$z^0 = u(x^0, 0) = e^{-\frac{(x^0)^2}{2}}.$$

Here's where it gets tricky. After a little struggling, we see that there is really no way to solve for  $z$  in terms of  $(x(s), t(s))$ . However, we can solve for the projected characteristics:

$$(x(t, y), t) = (e^{-\frac{y^2}{2}}t + y, t);$$

and this is really all that matters for us to find the time  $t_0$  when the shock develops, i.e., the time when the projected characteristic fails to be injective.

A little calculation shows that this happens precisely when  $t = e^{-\frac{1}{2}}$ . ■

PROBLEM 2.1.3. If  $\rho_0$  denotes the maximum density of cars on a highway (i.e., under bumper-to-bumper conditions), then a reasonable model for traffic density  $\rho$  is given by

$$\begin{cases} \rho_t + (F(\rho))_x = 0, \\ F(\rho) = c\rho\left(1 - \frac{\rho}{\rho_0}\right), \end{cases}$$

where  $c > 0$  is a constant (free speed of highway). Suppose the initial density is

$$\rho(x, 0) = \begin{cases} \frac{1}{2}\rho_0 & \text{if } x < 0, \\ \rho_0 & \text{if } x > 0. \end{cases}$$

Find the shock curve and describe the weak solution. (Interpret your result for the traffic flow.)

*SOLUTION.* First, note that

$$\begin{aligned} (F(\rho))_x &= F'(\rho)\rho_x \\ &= \left[-c\frac{\rho}{\rho_0} + c\left(1 - \frac{\rho}{\rho_0}\right)\right]\rho_x \\ &= \left(c - \frac{2c\rho}{\rho_0}\right)\rho_x. \end{aligned}$$

Let us find a solution to the PDE using the method of characteristics. Write

$$G(p, z, x) = p^2(s) + F'(z(s))p^1(s).$$

Then, the characteristic ODEs are

$$\begin{cases} (\dot{p}^1(s), \dot{p}^2(s)) = (-F''(z(s))p^1(s), -F''(z(s))p^2(s)), \\ \dot{z}(s) = F'(z(s))p^1(s) + p^2(s) \\ \quad = 0, \\ (\dot{x}^1(s), \dot{x}^2(s)) = (F'(z(s)), 1). \end{cases}$$

Now, integrating the characteristics, we have

$$\begin{cases} z(s) = z^0, \\ (x^1(s), x^2(s)) = (F'(z^0)s + x^0, s). \end{cases}$$

We have two cases to consider,  $x^0 < 0$  or  $x^0 > 0$ . For  $x^0 < 0$ ,  $z^0 = \frac{\rho_0}{2}$  and the projected characteristics look like

$$\begin{aligned} (F'(\frac{\rho_0}{2})t + x^0, t) &= \left(\left[c - \frac{2c(\frac{\rho_0}{2})}{\rho_0}\right]t + x^0, t\right) \\ &= (0 \cdot t + x^0, t) \\ &= (x^0, t) \end{aligned}$$

(where we have replaced  $s$  with the more appropriate  $t$ ). Whereas for  $x^0 > 0$ , we have

$$\begin{aligned}(F'(\rho_0)t + x^0, t) &= \left( \left[ c - \frac{2c\rho_0}{\rho_0} \right] t + x^0, t \right) \\ &= (-ct + x^0, t).\end{aligned}$$

These characteristics intersect precisely when

$$t = \frac{x_1^0 - x_2^0}{c},$$

where  $x_1^0 > 0$ ,  $x_2^0 < 0$ . ■

PROBLEM 2.1.4. Find the characteristics of the second order equation

$$u_{xx} - (2 \cos x)u_{xy} - (3 + \sin^2 x)u_{yy} - yu_y = 0,$$

and transform it to the canonical form.

SOLUTION. First, writing the PDE in the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + 2Du_x + 2Eu_y + Fu = 0,$$

we see that  $A = 1$ ,  $B = -\cos x$ ,  $C = -3\sin^2 x$ , and  $E = -\frac{y}{2}$ . We solve for the characteristic curve by find a solution to the ODEs

$$\begin{aligned}\frac{dy}{dx} &= \frac{B \pm \sqrt{B^2 - AC}}{A} \\ &= -\cos x \pm \sqrt{\cos^2 x + 3 + \sin^2 x} \\ &= -\cos x \pm 2.\end{aligned}$$

The solutions give us the following ODEs

$$\begin{cases} y = -\sin x + 2x + \xi(x, y), \\ y = -\sin x - 2x + \eta(x, y). \end{cases}$$

Integrating these equations, we have

$$\begin{cases} \xi(x, y) = y + \sin x - 2x, \\ \eta(x, y) = y + \sin x + 2x. \end{cases}$$

These are the characteristic strips for the PDE.

To put this PDE in canonical form, we first compute the following partial derivatives

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x, \\ u_y &= u_\xi \xi_y + u_\eta \eta_y, \\ u_{xx} &= u_\xi \xi_{xx} + u_\eta \eta_{xx} + (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) \xi_x + (u_{\xi\eta} \xi_x + u_{\eta\eta} \eta_x) \eta_x \\ &= u_{\xi\xi} (\xi_x)^2 + u_{\eta\eta} (\eta_x)^2 + 2u_{\xi\eta} \xi_x \eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx},\end{aligned}$$

exploiting symmetry, we can find  $u_{yy}$  by replacing  $x$  with  $y$  above

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + u_{\eta\eta}(\eta_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\xi}\xi_{yy} + u_{\eta}\eta_{yy},$$

the last thing we need to figure out is the mixed partial

$$\begin{aligned} u_{xy} &= u_{\xi}\xi_{xy} + u_{\eta}\eta_{xy} + (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + (u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y)\eta_x \\ &= u_{\xi\xi}\xi_x\xi_y + u_{\eta\eta}\eta_x\eta_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\xi}\xi_{xy} + u_{\eta}\eta_{xy}. \end{aligned}$$

Now find the partials  $\xi_x, \eta_x, \xi_y, \eta_y, \xi_{xy}, \dots$ , etc.

$$\begin{aligned} \xi_x &= \cos x - 2, & \eta_x &= \cos x + 2, \\ \xi_{xx} &= -\sin x, & \eta_{xx} &= -\sin x, \\ \xi_{xy} &= 0, & \eta_{xy} &= 0, \\ \xi_y &= 1, & \eta_y &= 1, \\ \xi_{yy} &= 0, & \eta_{yy} &= 0. \end{aligned}$$

Thus,

$$\left\{ \begin{aligned} u_x &= (\cos x - 2)u_{\xi} + (\cos x + 2)u_{\eta}, \\ u_y &= u_{\xi} + u_{\eta}, \\ u_{xx} &= (\cos x - 2)^2 u_{\xi\xi} + (\cos x + 2)^2 u_{\eta\eta} \\ &\quad + 2(\cos x + 2)(\cos x - 2)u_{\xi\eta} - (\sin x)u_{\xi} - (\sin x)u_{\eta} \\ &= (\cos^2 x - 4\cos x + 4)u_{\xi\xi} + (\cos^2 x + 4\cos x + 4)u_{\eta\eta} \\ &\quad + 2(\cos^2 x - 4)u_{\xi\eta} - (\sin x)u_{\xi} - (\sin x)u_{\eta} \\ u_{yy} &= u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}, \\ u_{xy} &= (\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}, \end{aligned} \right.$$

so the canonical form is

$$\begin{aligned} 0 &= u_{xx} - (2\cos x)u_{xy} - (3\sin^2 x)u_{yy} - yu_y \\ &= \xi^2 u_{\xi\xi} + \eta^2 u_{\eta\eta} \\ &\quad + 2\xi\eta u_{\xi\eta} - (\sin x)u_{\xi} - (\sin x)u_{\eta} \\ &\quad - (2\cos x)((\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}) \\ &\quad - (3\sin^2 x)(u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}) \\ &\quad - y(u_{\xi} + u_{\eta}) \end{aligned}$$

Who cares. ■

PROBLEM 2.1.5. Let  $Lu := u_{xx} - 4u_{yy} + \sin(y + 2x)u_x = 0$ .

- Consider the level curve  $\Gamma = \{(x, y) : \varphi(x, y) = C\}$  where  $|D\varphi| \neq 0$  on  $\Gamma$ . Define what it means for  $\Gamma$  to be characteristic with respect to  $L$  at a point  $(x_0, y_0) \in \Gamma$ .
- Find the points at which the curve  $x^2 + y^2 = 5$  is characteristic.

- (c) Is it true that every smooth simple closed curve  $\Gamma$  in  $\mathbb{R}^2$  has at least one point at which it is characteristic with respect to  $L$ ?

*SOLUTION.* ■

PROBLEM 2.1.6. Consider the second order equation

$$Lu := u_{xx} - 2xu_{xy} + x^2u_{yy} - 2u_y = 0.$$

- (a) Find the characteristic curves of  $Lu = 0$ . What is the type of this equation?  
 (b) Find the points on the line  $\Gamma := \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$  at which  $\Gamma$  is characteristic with respect to  $Lu = 0$ .

*SOLUTION.* ■

PROBLEM 2.1.7. Solve the initial boundary value problem for the equation  $u_{tt} = u_{xx}$  in  $\{x > 0, t > 0\}$  satisfying

$$\begin{cases} u(x, 0) = \sin^2 x, & u_t(x, 0) = \sin x, \\ u(0, t) = 0. \end{cases}$$

*SOLUTION.* ■

PROBLEM 2.1.8. Consider the initial/boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } 0 < x < \pi, t > 0, \\ u(x, 0) = x, & u_t(x, 0) = 0 \quad \text{for } 0 < x < \pi, \\ u_x(0, t) = 0, & u_x(\pi, t) = 0 \quad \text{for } t > 0. \end{cases}$$

- (a) Find a weak solution of the problem.  
 (b) Is the solution unique? Continuous?  $C^1$ ?

*SOLUTION.* ■

PROBLEM 2.1.9. Let  $B_1^+$  denote the open half-ball  $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$ . Assume  $u \in C(\bar{B}_1^+)$  is harmonic in  $B_1^+$  with  $u = 0$  on  $\partial B_1^+ \cap \{x_n = 0\}$ . Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0, \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0, \end{cases}$$

for  $x \in B_1$ . Prove  $v$  is harmonic in  $B_1$ .

*Hint:* It will be enough to prove that  $\int_B \nabla v \nabla \eta \, dx = 0$  for any test function  $\eta \in C_0^\infty(B_1)$ . Split  $\int_{B_1} = \int_{B_1^+} + \int_{B_1^-}$  and apply the integration by parts formula to each of  $\int_{B_1^\pm}$ .

*SOLUTION.*

■

PROBLEM 2.1.10. Let  $u$  and  $v$  be harmonic functions in the unit ball  $B_1 \subset \mathbb{R}^n$ . What can you conclude about  $u$  and  $v$  if

- (a)  $D^\alpha u(0) = D^\alpha v(0)$  for every multiindex  $\alpha$ ?
- (b)  $u(x) \leq v(x)$  for every  $x \in B_1$  and  $u(0) = v(0)$ ?

Justify your answer in each case.

*SOLUTION.*

■

PROBLEM 2.1.11. Let  $\Phi$  be the fundamental solution of the Laplace equation in  $\mathbb{R}^n$  and  $f \in C_0^\infty(\mathbb{R}^n)$ . Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy$$

is a solution to the Poisson equation  $-\Delta u = f$  in  $\mathbb{R}^n$ . Show that if  $f$  is radial, i.e.,  $f(y) = f(|y|)$ , and supported in  $B_R := \{|x| < R\}$ , then

$$u(x) = c\Phi(x)$$

for any  $x \in \mathbb{R}^n \setminus B_R$ , where

$$c = \int_{\mathbb{R}^n} f(y) dy.$$

[*Hint:* Use polar (spherical) coordinates and apply the mean value property for harmonic functions.]

*SOLUTION.*

■



### 3 Qualifying Exams

#### 3.1 Qualifying Exam, August '04

PROBLEM 3.1.1. Consider the initial value problem

$$\begin{cases} a(x, y)u_x + b(x, y)u_y = -u, \\ u = f \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\},$$

where  $a$  and  $b$  satisfy

$$a(x, y) + b(x, y)y > 0$$

for any  $x, y \in \mathbb{R}^n \setminus \{(0, 0)\}$ .

- (a) Show that the initial value problem has a unique solution in a neighborhood of  $S^1$ . Assume that  $a$ ,  $b$ , and  $f$  are smooth.
- (b) Show that the solution of the initial value problem actually exists in  $\mathbb{R}^2 \setminus \{(0, 0)\}$ .

*SOLUTION.*

■

PROBLEM 3.1.2. Let  $u \in C^2(\mathbb{R} \times [0, \infty))$  be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where  $f$  and  $g$  have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dx,$$

and the potential energy by

$$P(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) dx.$$

Show that

- (a)  $K(t) + P(t)$  is constant in  $t$ ,
- (b)  $K(t) = P(t)$  for all large enough times  $t$ .

*SOLUTION.*

■

PROBLEM 3.1.3. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = e^{-t}g(x) & \text{for } x \in \mathbb{R}^3, t > 0, \\ u(x, 0) = u_t(x, 0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution  $u = e^{-t} + t - 1$  when  $g(x) = 1$ .

*SOLUTION.* ■

PROBLEM 3.1.4. Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and  $g \in C_0^\infty(\Omega)$ . Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega, t > 0, \\ u(x, 0) = g(x) & \text{for } x \in \Omega, \\ u(x, t) = 0 & \text{for } x \in \partial\Omega, t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n, t > 0, \\ v(x, 0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put  $g = 0$  outside  $\Omega$ .

(a) Show that

$$-v(x, t) \leq u(x, t) \leq v(x, t)$$

for any  $x \in \Omega, t > 0$ .

(b) Use (a) to conclude that

$$\lim_{t \rightarrow \infty} u(x, t) = 0,$$

for any  $x \in \Omega$ .

*SOLUTION.* ■

PROBLEM 3.1.5. Let  $P_k(x)$  and  $P_m(x)$  be homogeneous harmonic polynomials in  $\mathbb{R}^n$  of degrees  $k$  and  $m$  respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \quad P_m(\lambda x) = \lambda^m P_m(x),$$

for any  $x \in \mathbb{R}^n, \lambda > 0$ ,

$$\Delta P_k = 0, \quad \Delta P_m = 0$$

in  $\mathbb{R}^n$ .

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = k P_k(x), \quad \frac{\partial P_m(x)}{\partial \nu} = m P_m(x)$$

on  $\partial B_1$ , where  $B_1 = \{|x| < 1\}$  and  $\nu$  is the outward normal on  $\partial B_1$ .

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x) dS = 0,$$

if  $k \neq m$ .

*SOLUTION.* ■

## 3.2 Qualifying Exam, August '05

PROBLEM 3.2.1.

- (a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ u = 1 \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\}.$$

- (b) Is the solution unique in a neighborhood of the point  $(1, 0)$ ? Justify your answer.

*SOLUTION.* The solution to the first part is

$$u(x, y) = \frac{x^2 + y^2}{4} + \frac{3}{4}.$$

■

PROBLEM 3.2.2. Consider the second order PDE in  $\{x > 0, y > 0\} \subset \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

- (a) Classify the equation and reduce it to the canonical form.  
 (b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy} G\left(\frac{x}{y}\right).$$

*SOLUTION.*

■

PROBLEM 3.2.3. Let  $\Phi$  be the fundamental solution of the Laplace equation in  $\mathbb{R}^3$  and  $f \in C_0^\infty(\mathbb{R}^n)$ . Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy$$

is a solution of the Poisson equation  $-\Delta u = f$  in  $\mathbb{R}^n$ . Show that if  $f$  is radial (i.e.,  $f(y) = f(|y|)$ ) and supported in  $B_R = \{|x| < R\}$ , then

$$u(x) = c\Phi(x),$$

for any  $x \in \mathbb{R}^n \setminus B_R$ , where

$$c = \int_{\mathbb{R}^n} f(y) dy.$$

[Hint: Use spherical (polar) coordinates and the mean value property.]

*SOLUTION.*

■

PROBLEM 3.2.4. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where  $g, h \in C_0^\infty(\mathbb{R}^2)$  and  $\alpha \geq 0$  is a constant.

- (a) Show that for an appropriate choice of constant  $\lambda$  and  $\mu$  the function

$$v(x_1, x_2, x_3, t) := e^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation  $v_{tt} - \Delta v = 0$ .

- (b) Use (a) to prove the following domain of dependence result: for any point  $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$  the value  $u(x_0, t_0)$  is uniquely determined by values of  $g$  and  $h$  in  $\overline{B_{t_0}(x_0)} := \{|x - x_0| \leq t_0\}$ . (You may use the corresponding result for the wave equation without proof.)

*SOLUTION.* ■

PROBLEM 3.2.5. Let  $u(x, t)$  be a bounded solution of the heat equation  $u_t = u_{xx}$  in  $\mathbb{R} \times (0, \infty)$  with the initial condition

$$u(x, 0) = u_0(x)$$

for  $x \in \mathbb{R}$ , where  $u_0 \in C^\infty$  is  $2\pi$ -periodic, i.e.,  $u_0(x + 2\pi) = u_0(x)$ . Show that

$$\lim_{t \rightarrow \infty} u(x, t) = a_0,$$

uniformly in  $x \in \mathbb{R}$ , where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x) dx.$$

*SOLUTION.* ■

### 3.3 Qualifying Exam, January '14

PROBLEM 3.3.1. Consider the first order equation in  $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

- (a) Find the characteristic curves of the equation.
- (b) Consider the Cauchy problem for this equation prescribed on the line  $x_1 = 1$ :

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on  $f$  so that the problem is solvable in a neighborhood of the point  $(1, 0)$ .

*SOLUTION.* ■

PROBLEM 3.3.2. Let  $u$  be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{x_n > 0\}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where  $g$  is a continuous function with compact support in  $\mathbb{R}^{n-1}$ . Here  $n \geq 2$ . Prove that

$$u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty,$$

for  $x \in \{x_n > 0\}$ .

*SOLUTION.* ■

PROBLEM 3.3.3. Let  $u$  be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \quad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions  $u(x, 0) = g(x)$ , where  $g$  is a bounded continuous function on  $\mathbb{R}$  satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^\alpha, \quad x, y \in \mathbb{R}$$

with a constant  $\alpha \in (0, 1]$ . Show that

$$\begin{aligned} |u(x, t) - u(y, t)| &\leq M|x - y|^\alpha, & x, y \in \mathbb{R}, t > 0, \\ |u(x, t) - u(x, s)| &\leq C_\alpha M|t - s|^{\frac{\alpha}{2}}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of  $u(x, t)$  as a convolution with the heat kernel  $\Phi(y, t)$ , make a change of variables  $z = \frac{y}{\sqrt{t}}$  and use that  $|\sqrt{t} - \sqrt{s}| \leq \sqrt{|t - s|}$ .]

*SOLUTION.* ■

PROBLEM 3.3.4. Let  $u$  be a positive harmonic function in the unit ball  $B_1$  in  $\mathbb{R}^n$ . Show that

$$|D(\ln u)| \leq M \quad \text{in } B_{\frac{1}{2}}$$

for a constant  $M$  depending only on the dimension  $n$ .

[*Hint:* Use the interior derivative estimate  $|Du(x)| \leq (\frac{C_n}{r}) \sup_{B_r(x)} |u|$  for  $B_r(x) \subset B_1$  as well as the Harnack inequality for harmonic functions.]

SOLUTION. ■

PROBLEM 3.3.5. Let  $u$  be a  $C^2$  solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = 0, \quad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some  $k \geq 0$ . Prove that there exists a function  $\varphi(r)$  such that

$$u(x, t) = t^{k+2} \varphi\left(\frac{|x|}{t}\right).$$

[*Hint:* As one of the steps show that  $u$  is  $(k+2)$ -homogeneous in  $(x, t)$  variables, i.e.,  $u(\lambda x, \lambda t) = \lambda^{k+2} u(x, t)$  for any  $\lambda > 0$ .]

SOLUTION. ■