Fall 2016 Notes

Carlos Salinas

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Chapter 1

Some Category Theory

In an effort to help Vinh prepare for his talk, here are some notes I have compiled on category theory. Here are the books and notes that I used: the *CRing project* notes; *A First Course in Commutative Algebra* by Altman and Kleiman; and *Foundations of Algebraic Geometry* by Ravi Vakil.

1.1 Basics

Here are some of the basic ideas (and frankly, the most boring part of category theory).

Chapter 2

Probability

Some (mostly discrete) probability theory for MA 51900.

2.1 Basics

In this section we will talk about concepts related to discrete probability. Before we begin, we introduce the axioms we will be working under. First and foremost, to do probability we need a *sample space* Ω and a *probability* $p\colon \mathscr{M} \to [0,1]$ which assigns values between 0 and 1 to *special* subsets of Ω which we denote by $\mathscr{M}$ (more formally, this $\mathscr{M}$ is called a *σ-algebra* by analysts or a *algebra of events* by probabilists and p is called a *probability measure* and there are certain axioms it must satisfy for us to be able to assign consistent values to subsets of Ω with p). An element $\omega \in \Omega$ is called a *sample point* and a (special) collection of ω, $A \in \mathscr{M}$, is called an event. We call the triplet $(\Omega, \mathscr{M}, p)$ a probability space.

The algebra of events comes $\mathscr{M}$ with a natural multiplication and addition given, naturally, by union and intersection of events (*i.e.* $A + B := A \cup B$ and $AB := A \cap B$) and an additive as well as multiplicative identity $\emptyset$ and Ω, *etc.* If $AB = \emptyset$ we say that the events A and B are *mutually exclusive*.

Remark 2.1. We won't always use the notation $A + B$ and AB to mean $A \cup B$ and $A \cap B$, respectively (since I prefer the set-theoretic notation over the algebraic one), but Prof. DasGupta makes has a preference for the latter and Feller uses a mix of the two. Now, you may ask "Why introduce this notation at all if you are going to disregard it?" The reason is that I will be using examples from Feller and DasGupta's book and sometimes I will be too rushed to bother translating the notation and though I don't expect anybody but myself to read this, it may very well happen that I pass these notes on to somebody else.

In this section, we shall assume that our sample space Ω is discrete,*i.e.* $\#\Omega < \infty$ or at the very least $\aleph^0$. We additionally require that for each point ω in the space Ω its probability $p(\omega)$ in non-negative and

$$\sum_{\omega \in \Omega} p(\omega) = 1. \tag{2.1}$$

There are of course a whole number of beautiful relationships that p satisfies (those that any sane measure would satisfy like countable additivity, subadditivity, *etc.*), but we shall not talk about them here, instead let us get down to the crux of the matter (at least at this point in the class): counting. Since our sample spaces will be finite (at least for now), we need to be able to count sample points in Ω by way of combinatorics (this is in my opinion, a lot tougher than working with infinite sample spaces for which we must make certain assumptions about the sample points and the probability measure – it is less tedious to solve problems with sane assumptions than it is to count points).

Basic Combinatorics

It is often reasonable to assume that the probability of any particular sample point $\omega \in \Omega$ is just as likely as that of any other sample point. We say that in such a sample space each sample point is *equally likely* to happen. This means that the probability of $\omega \in \Omega$ happening is precisely

$$p(\omega) = \frac{1}{\#\Omega}.$$

Thus, to compute the probability of an event A happening, we need only count the number of points in A and divide by the cardinality of Ω, $\#\Omega$.

Here are a couple of results about counting that will be useful to us throughout this section:

Lemma 2.2. *With m elements $a_1, \ldots, a_m$ and n elements $b_1, \ldots, b_n$, it is possible to form mn pairs (a_i, b_j) containing one element from each group.*

Proof. Arrange the elements $a_1, \ldots, a_m$ in a column and $b_1, \ldots, b_n$ in a row and form a multiplication table where the ij-th element is $a_i b_j$. Then this assertion becomes obvious.

More formally, choose an element a from among $a_1, \ldots, a_m$ (there are m choices) then to form a pair $(a, -)$ we can choose from among n elements $b_1, \ldots, b_n$. Thus, the number of pairs that can be formed with a is n and there are m possible choices for a so there are mn possible pairs. ∎