

MA571 Homework 10

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PROBLEM 10.1 (MUNKRES §52, EX. 2)

Let α be a path in X from x_0 to x_1 ; let β be a path in X from x_1 to x_2 . Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. By Theorem 52.1, the paths α and β induce a group homomorphism $\hat{\alpha}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ and $\hat{\beta}: \pi_1(X, x_1) \rightarrow \pi_1(X, x_2)$, respectively. We want to show therefore that the induced homomorphism $\hat{\gamma} = \widehat{\alpha * \beta}$ is in fact equivalent to the composition $\hat{\beta} \circ \hat{\alpha}$. Let $[f]$ be a loop based at x_0 then

$$\begin{aligned}\hat{\gamma}([f]) &= \widehat{\alpha * \beta}([f]) \\ &= [\overline{\alpha * \beta}] * [f] * [\alpha * \beta] \\ &= [\bar{\beta} * \bar{\alpha}] * [f] * [\alpha] * [\beta]\end{aligned}$$

by the well-definedness of the path product operation, we have

$$= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta]$$

by associativity of the path product,

$$\begin{aligned}&= [\bar{\beta}] * ([\bar{\alpha}] * [f] * [\alpha]) * [\beta] \\ &= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta]\end{aligned}$$

where $\alpha([f])$ is a loop based at x_1 so

$$\begin{aligned}&= \hat{\beta}(\hat{\alpha}([f])) \\ &= (\hat{\beta} \circ \hat{\alpha})([f]).\end{aligned}$$

Thus, the following diagram commutes

$$\begin{array}{ccc}\pi_1(X, x_0) & \xrightarrow{\hat{\alpha}} & \pi_1(X, x_1) \\ & \searrow \hat{\gamma} = \widehat{\alpha * \beta} & \downarrow \hat{\beta} \\ & & \pi_1(X, x_2).\end{array}$$

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PROBLEM 10.2 (MUNKRES §52, EX. 3)

Let x_0 and x_1 be points of the path-connected space X . Show that $\pi_1(X, x_0)$ is Abelian if and only if for every pair α and β of paths from x_0 to x_1 , we have $\hat{\alpha} = \hat{\beta}$.

Proof. \implies Suppose that $\pi_1(X, x_0)$ is Abelian. Then for any class of loops about x_0 , say $[f]$ and $[g]$, the product $[f] * [g] = [g] * [f]$. Let α and β be paths from x_0 to x_1 . Then the induced map on fundamental groups $\hat{\alpha}$ and $\hat{\beta}$ yield isomorphism by Theorem 52.1 so that the map $\hat{\beta} \circ \hat{\alpha}$ is an automorphism of $\pi_1(X, x_0)$. Moreover, we have

$$\begin{aligned}\hat{\beta} \circ \hat{\alpha}([f]) &= \hat{\beta}(\hat{\alpha}([f])) \\ &= \hat{\beta}([\bar{\alpha}] * [f] * [\alpha]) \\ &= [\beta] * ([\bar{\alpha}] * [f] * [\alpha]) * [\bar{\beta}]\end{aligned}$$

by associativity of the path product, we may rewrite the above expression as

$$= ([\beta] * [\bar{\alpha}]) * [f] * ([\alpha] * [\bar{\beta}])$$

noting that $[\beta] * [\bar{\alpha}]$ and $[\alpha] * [\bar{\beta}]$ are loops based at x_0 , since $\pi_1(X, x_0)$ is Abelian, we have

$$\begin{aligned}&= ([\beta] * [\bar{\alpha}]) * ([\alpha] * [\bar{\beta}]) * [f] \\ &= [e_{x_0}] * [f] \\ &= [f].\end{aligned}$$

Thus, $\hat{\beta} \circ \hat{\alpha} = \text{id}_{\pi_1(X, x_0)}$, i.e., $\hat{\alpha} = \hat{\beta}$.

\Leftarrow Let f and g be loops about x_0 . Then, since X is path connected, we claim that f and g are homotopic to the path product $\alpha_1 * \bar{\beta}_1$ and $\alpha_2 * \bar{\beta}_2$ where α_i, β_i are paths from x_0 to x_1 . More precisely, split f into the paths $f_1 = f(t/2)$ and $f_2 = f((t+1)/2)$; it is clear that $f = f_1 * f_2$. Let $x_2 := f_1(1)$ then there exists a path α from x_2 to x_1 since X is path connected. Now we claim that the following

$$H(x, t) := f_1(x) * \alpha(tx) * \bar{\alpha}((1-t)x) * f_2(x)$$

is a homotopy from $f = f_1 * f_2$ to the extended loop $\tilde{f} = f_1 * \alpha * \bar{\alpha} * f_2$.

Proof of claim. It is clear that H is continuous since it is a path products and multiplication on the unit interval I is continuous so tx is continuous. Lastly, $H(x, 0) = f_1(x) * \alpha(0) * \bar{\alpha}(0) * f_2(x)$ and $H(x, 1) = f_1(x) * \alpha(x) * \bar{\alpha}(x) * f_2(x)$. ♣

Now, let $f \simeq_p \alpha_1 * \bar{\beta}_1$ and $g \simeq_p \alpha_2 * \bar{\beta}_2$ where α_i, β_i are paths from x_0 to x_1 . Then we have

$$\begin{aligned}[f] * [g] * [\bar{f}] * [\bar{g}] &= [\alpha_1 * \bar{\beta}_1] * [\alpha_2 * \bar{\beta}_2] * [\overline{\alpha_1 * \bar{\beta}_1}] * [\overline{\alpha_2 * \bar{\beta}_2}] \\ &= [\alpha_1 * \bar{\beta}_1] * [\alpha_2 * \bar{\beta}_2] * [\beta_1 * \bar{\alpha}_1] * [\beta_2 * \bar{\alpha}_2] \\ &= [\alpha_1] * [\bar{\beta}_1] * [\alpha_2] * [\bar{\beta}_2] * [\beta_1] * [\bar{\alpha}_1] * [\beta_2] * [\bar{\alpha}_2] \\ &= \hat{\alpha}_1([\bar{\beta}_1] * [\alpha_2] * [\bar{\beta}_2] * [\beta_1]) * [\beta_2] * [\alpha_2] \\ &= \hat{\alpha}_1(\hat{\beta}_2([\alpha_2] * [\bar{\beta}_2])) * [\beta_2] * [\bar{\alpha}_2] \\ &= [\alpha_2] * [\bar{\beta}_2] * [\beta_2] * [\bar{\alpha}_2]\end{aligned}$$

$$\begin{aligned} &= [\alpha_2] * [e_{x_0}] * [\bar{\alpha}_2] \\ &= [\alpha_2] * [\bar{\alpha}_2] \\ &= [e_{x_0}]. \end{aligned}$$

Thus, $\pi_1(X, x_0)$ is Abelian. ■

PROBLEM 10.3 (MUNKRES §52, EX. 4)

Let $A \subset X$; suppose $r: X \rightarrow A$ is continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A .) If $a_0 \in A$, show that

$$r_*: \pi_1(X, x_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

Proof. Suppose f is a loop in A based at a . Then, extending the codomain of f to X , f is a loop in X based at a . Then, since $r(a) = a$ for all a and $f(I) \subset A$, $r_*([f]) = [p(f)] = [f]$ so r_* is surjective. ■

PROBLEM 10.4 (MUNKRES §53, EX. 6)

Show that if X is path connected, the homomorphism induced by a continuous map is independent of the base point, up to isomorphisms of the groups involved. More precisely, let $h: X \rightarrow Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1 , and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps “commutes”

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \hat{\alpha} \downarrow & & \downarrow \hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1). \end{array}$$

Proof. Unpacking the expression on the left, we have the following sequence of equalities: Let f be a loop in X based at x_0 then

$$\begin{aligned} (\hat{\beta} \circ (h_{x_0})_*)([f]) &= \hat{\beta}((h_{x_0})_*([f])) \\ &= \hat{\beta}([h(f)]) \\ &= [\bar{\beta}] * [h(f)] * [\beta] \\ &= [\overline{h \circ \alpha}] * [h(f)] * [h \circ \alpha] \\ &= [h \circ \bar{\alpha}] * [h(f)] * [h \circ \alpha] \end{aligned}$$

but since $(h_{x_1})_*$ is a homomorphism

$$\begin{aligned} &= (h_{x_0})_*(\bar{\alpha} * f * \alpha) \\ &= (h_{x_1})_*(\hat{\alpha}([f])) \\ &= ((h_{x_1})_* \circ \hat{\alpha})([f]). \end{aligned}$$

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PROBLEM 10.5 (MUNKRES §55, EX. 1)

Show that if A is a retract of B^2 , then every continuous map $f: A \rightarrow A$ has a fixed point.

Proof. Suppose that A is a retract of B^2 . Let $r: B^2 \rightarrow A$ be one such retraction. If $f: A \rightarrow A$ is a continuous map, then $f \circ r$ is a continuous map, by Theorem 18.2(c), from B^2 to A . Expanding the codomain of f to B^2 , i.e., composing with the canonical injection $\iota: A \hookrightarrow B^2$, we have a continuous mapping $\tilde{f}: B^2 \rightarrow B^2$ that coincides with f in A . Then, by Theorem 55.6, \tilde{f} has a fixed point, i.e., $\tilde{f}(x) = x$ for some $x \in B^2$. By the Brouwer fixed-point theorem for the disc, there exists a point $x \in B^2$ such that $\tilde{f}(x) = x$, but $\text{im } \tilde{f} = \text{im } f \subset A$ so $x \in A$. It follows that f has a fixed point. ■

PROBLEM 10.6 (MUNKRES §55, EX. 2)

Show that if $h: S^1 \rightarrow S^1$ is nullhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof.

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PROBLEM 10.7 ((A))

Prove that every m -manifold is locally path-connected.

Proof. Suppose M is an m -manifold. Let $x \in M$ and U' be an arbitrary neighborhood of x . Then, since M is a manifold, there exists an open neighborhood U of x that is homeomorphic to an open subset, say V , of \mathbf{R}^m . Let $h: U \rightarrow V$ be a homeomorphism. Then $h(U \cap U')$ is open in U so by Theorem 16.2, $h(U \cap U')$ is open in \mathbf{R}^m . Therefore, for sufficiently small values of $\delta > 0$, we have the inclusion $B(h(x), \delta) \subset h(U \cap U')$. We claim that $W := h^{-1}(B(h(x), \delta))$ is a path-connected neighborhood of x contained in U' .

Containment is clear for h is a bijection and we have that $W \subset U \cap U' \subset U'$. By Example 3 in Munkres §24 we know that open balls in \mathbf{R}^m are path-connected therefore given $y_0 = h(x_0), y_1 = h(x_1) \in h(W)$, there exists a path $p: I \rightarrow h(W)$ with $p(0) = y_0$ and $p(1) = y_1$. Then $q := ((h^{-1})|_{h(W)} \circ p): I \rightarrow W$ is a path in W from x_0 to x_1 . It is clear that q is continuous by Theorem 18.2(c) since it is a composition of continuous functions (where $(h^{-1})|_{h(W)}$ is continuous by Theorem 18.2(d) since it is the restriction of a continuous function). Lastly, $q(0) = x_0$ and $q(1) = x_1$. Since x_0 and x_1 were arbitrary, it follows that W is path-connected. Therefore, M is locally path-connected. ■

PROBLEM 10.8 ((B))

Prove that every m -manifold is regular.

Proof. Let $x \in M$ and U' be an arbitrary neighborhood of x . Then, since M is a manifold, there exists an open neighborhood U of x that is homeomorphic to an open subset, say V , of \mathbf{R}^m . Let $h: U \rightarrow V$ be a homeomorphism. Then $h(U \cap U')$ is open in U so by Theorem 16.2, $h(U \cap U')$ is open in \mathbf{R}^m . Since \mathbf{R}^m is regular, there exist an neighborhood W of $h(x)$ such that $\overline{W} \subset h(U \cap U')$. We claim that $h^{-1}(W) \subset U'$ is a neighborhood of x such that $\overline{h^{-1}(W)} \subset U'$. ■

PROBLEM 10.9 ((C))

Prove that there is no 1-1 continuous function $\iota: S^1 \rightarrow \mathbf{R}$. You may assume any fact about trigonometric functions. (Note: this shows in particular that there is no $\iota: S^1 \rightarrow \mathbf{R}$ with $p \circ \iota$ equal to the identity map, where p is the map in the note on the Fundamental Group of the Circle.)

Proof.

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PROBLEM 10.10 ((D))

Prove Proposition C from the note on the Fundamental Group of the Circle.

Proof.

