MA 544: Homework 4

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February 7, 2016

Problem 4.1 (Wheeden & Zygmund §3, Ex. 12)

If E_1 and E_2 are measurable sets in $\mathbb{R}^1$, show $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1||E_2|$. (Interpret $0 \cdot \infty$ as 0.) [Hint: Use a characterization of measurability.]

Proof. By (3.28) (i) we may write E_1 and E_2 as the unions $H_1 \cup Z_1$ and $H_2 \cup Z_2$, respectively, where H_1 and H_2 are F_σ and Z_1 and Z_2 are measure zero. Now, by elementary set theory, the Cartesian product $E_1 \times E_2$ can then be written as

$$E_1 \times E_2 = (H_1 \cup Z_1) \times (H_2 \cup Z_2) = \underbrace{H_1 \times H_2}_{H} \cup \underbrace{H_1 \times Z_2 \cup Z_1 \times H_2 \cup Z_1 \times Z_2}_{Z}. \tag{1}$$

Hence, we win by (3.28) (i) if we can show that the Cartesian of two F_σ sets is an F_σ set and if the Cartesian product of a measurable set with a set of measure zero is measure zero.

First, we prove the former, since the argument to be made is little more than elementary set theory. Let F_1 and F_2 be F_σ. Write $F_1 = \bigcup F_k'$ and $F_2 = \bigcup F_k''$ where the F_k''s and the F_k'''s are closed subsets of $\mathbb{R}$. Then, $F_k' \times F_\ell''$ are closed subsets of $\mathbb{R}^2$ by elementary topology. Moreover, $F_k' \times F_\ell'' \subset F_1 \times F_2$ hence, $\bigcup_{k,\ell} F_k' \times F_\ell'' \subset F_1 \times F_2$. Thus, it suffices to show that $\bigcup_{k,\ell} F_k' \times F_\ell'' \supset F_1 \times F_2$. Let $(x, y) \in F_1 \times F_2$. Then $x \in F_1$ and $y \in F_2$. But since $F_1 = \bigcup F_k'$ and $F_2 = \bigcup F_k''$ then $x \in F_k'$ and $x \in F_\ell''$ for some k, ℓ. In other words, $(x, y) \in F_k' \times F_\ell''$ so (x, y) is in the union $\bigcup_{k,\ell} F_k' \times F_\ell''$. Hence, we have $F_1 \times F_2 = \bigcup_{k,\ell} F_k' \times F_\ell''$. We conclude that if F_1 and F_2 are F_σ, then so is their Cartesian product $F_1 \times F_2$.

Let E be a measurable set with $|E| < \infty$ and Z a set of measure zero. Then, for every $\varepsilon > 0$ there exists a countable collection of intervals $\{I_k\}$ containing Z such that $\sum \mathrm{vol}(I_k) < \varepsilon$. Similarly, we can find a collection $\{I_k'\}$ of intervals containing E such that $\sum \mathrm{vol}(I_k') < |E| + \varepsilon$. Then, $\{I_k' \times I_\ell\}$ is a countable collection of 2-intervals containing $E \times Z$ with

$$\begin{aligned}
\sum_{k,\ell} \mathrm{vol}(I_k' \times I_\ell) &= \sum_{k,\ell} \mathrm{vol}(I_k')\,\mathrm{vol}(I_\ell) \\
&= \sum_k \sum_\ell \mathrm{vol}(I_k')\,\mathrm{vol}(I_\ell) \\
&= \left(\sum_k \mathrm{vol}(I_k')\right)\left(\sum_\ell \mathrm{vol}(I_\ell)\right) \\
&= (|E| + \varepsilon)\varepsilon
\end{aligned}$$

Letting $\varepsilon \to 0$, we have $E \times Z$ is measure zero. If $|E| = \infty$, partition E into disjoint finite measure subsets of $\mathbb{R}$ by taking the following intersection

$$E_k = E \cap (B(0, k) \setminus B(0, k-1))$$

for $k \in \mathbb{N}$.[1] By our previous argument, $E_k \times Z$ is measure zero so $\{E_k \times Z\}$ is a cover of $E \times Z$

[1] In fact, it might be quicker from now on to quote the fact that $\mathbb{R}^n$ is σ-finite.

hence, by (3.24), we have

$$\begin{aligned}
|E \times Z| &= \left|\left(\bigcup_k E_k\right) \times Z\right| \\
&= \left|\bigcup_k E_k \times Z\right| \\
&= \sum_k |E_k \times Z| \\
&= 0.
\end{aligned}$$

Thus, $E \times Z$ is measure zero.

Hence, $E_1 \times E_2$ is measurable with $|E_1 \times E_2| = |H_1 \times H_2|$. It's left to show is that $|H_1 \times H_2| = |H_1||H_2|$.

Let H_1 and H_2 be F_σ sets of finite measure. Then, for every $\varepsilon > 0$, there exists a collection of intervals $\{I_k\}$ and $\{I_k'\}$ covering H_1 and H_2 respectively such that

$$\sum_k \operatorname{vol}(I_k) < |H_1| + \varepsilon \qquad \sum_k \operatorname{vol}(I_k') < |H_2| + \varepsilon.$$

Then the collection $\{I_k \times I_\ell'\}$ is a cover of $H_1 \times H_2$ by 2-intervals and we have ■

Problem 4.2 (Wheeden & Zygmund §3, Ex. 13)

Motivated by (3.7), define the *inner measure* of E by $|E|_i = \sup|F|$, where the supremum is taken over all closed subsets F of E. Show that

(i) $|E|_i \leq |E|_e$, and

(ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$.

[Use (3.22).]

Proof. (i) If the outer measure of E is infinite, the inequality holds trivially. Suppose $|E|_e < \infty$. Since closed sets are measurable and their outer measure is equal to their Lebesgue measure, then we may replace $|F|$ by $|F|_e$ to mirror the definition of the outer-measure and, by the monotonicity of the outer measure, we have

$$|F| = |F|_e \leq |E|_e. \tag{2}$$

Taking the supremum on both sides of (2), we obtain the desired inequality

$$|E|_i \leq |E|_e. \tag{3}$$

(ii) $\Longrightarrow$ Suppose E is measurable with $|E| < \infty$. Then for every $\varepsilon > 0$ there exists an open set $G \supset E$ such that $|G|_e < |E|_e + \varepsilon$.

$\Longleftarrow$ Suppose that $|E|_i = |E|_e$. Then ■

Problem 4.3 (Wheeden & Zygmund §3, Ex. 14)

Show that the conclusion of part (ii) of Exercise 13 is false if $|E|_e = +\infty$.

Proof. ■

Problem 4.4 (Wheeden & Zygmund §3, Ex. 15)

If E is measurable and A is any subset of E, show that $|E| = |A|_i + |E - A|_e$. (See Exercise 13 for the definition of $|A|_i$.)

Proof. ■