MA557 Problem Set 1

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Problem 1.1

Show that $\operatorname{rad}(R[x]) = \operatorname{nil}(R[x])$.

Proof. Suppose R is a commutative ring with identity and $R[x]$ is the polynomial ring over R in the indeterminate x. Then, it is clear that $\operatorname{rad}(R[x]) \supset \operatorname{nil}(R[x])$ since $\operatorname{nil}(R[x])$ is the intersection of all prime ideals of $R[x]$ and every maximal ideal is a prime ideal. To show the reverse containment, we will first prove the following results found in Dummit and Foote, §7.3, p. 33:

Lemma 1. *Let $f = a_n x^n + \cdots + a_1 x + a_0 \in R[x]$. Then*

(a) f is a unit in $R[x]$ if and only if a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent in R;
(b) f is nilpotent in $R[x]$ if and only if $a_0, a_1, \ldots, a_n$ are nilpotent elements of R.

Proof of lemma. (a) Suppose f is a unit. Then there exists $g = b_m x^m + \cdots + b_0$ in $R[x]$ such that $fg = 1$. In particular,

$$\begin{aligned} fg - 1 &= (a_n x^n + \cdots + a_0)(b_m x^m + \cdots + b_0) - 1 \\ &= \sum_{\substack{k_0 + \cdots + k_n = m \\ \ell_0 + \cdots + \ell_m = m}} a_0^{k_0} \cdots a_n^{k_n} b_0^{\ell_0} \cdots b_m^{\ell_m} x^{mn + \sum i k_i + \sum j \ell_j} - 1 \\ &= 0, \end{aligned}$$

is true if and only if

(b) ♦

■

Problem 1.2

Let I and J be R-ideals. Show that

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

Proof. ■

Problem 1.3

Let S be a subset of a ring R. Show that the following are equivalent:

(i) $R \setminus S$ is a union of prime ideals.
(ii) $1 \in S$, and for any elements x, y of R, $x \in S$ and $y \in S$ if and only if $xy \in S$.

Proof. ∎

Problem 1.4

Show that the set of all zero divisors in a ring is a union of prime ideals.

Proof. ■

Problem 1.5

Let $\varphi\colon R \to S$ be a surjective homomorphism of rings.

(a) Show that $\varphi(\operatorname{rad}(R)) \subset \operatorname{rad}(S)$, but that equality does not hold in general.
(b) Show that $\varphi(\operatorname{rad}(R)) = \operatorname{rad}(S)$ if R is semilocal.

Proof. ∎

Problem 1.6

An element $e \in R$is called *idempotent* if $e^2 = e$. Show that in a local ring, 0 and 1 are the only idempotents.

Proof. ∎

Problem 1.7

Let I be an R-ideal. Show that I is finitely generated and $I^2 = I$ if and only if $I = Re$ with e idempotent.

Proof. ■