MA571 Problem Set 7

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Problem 7.1 (Munkres §26, Ex. 8)

Theorem. *Let $f\colon X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the* graph *of f,*
$$G_f = \{ (x, f(x)) \mid x \in X \},$$
is closed in $X \times Y$.

[*Hint:* If G_f is closed and V is a neighborhood of $f(x_0)$, then the intersection of G_f and $X \times (Y - V)$ is closed. Apply Exercise 7.]

Proof. As we demonstrated in Problem 2.7 (Munkres §18, Ex. 17) Y is Hausdorff if and only if the diagonal, $\Delta_Y = \{ (y, y) \mid y \in Y \}$, is a closed subset of $Y \times Y$. Consider the map $F\colon X \times Y \to Y \times Y$ defined by $(x, y) \mapsto (f(x), y)$. This map is continuous by Theorem 18.4 as f is, by assumption, continuous and id_Y is continuous by 18.2(b) (since it is the inclusion $Y \hookrightarrow Y$). Then
$$\begin{aligned} F^{-1}(\Delta_Y) &= \{ (x, y) \mid F(x, y) \in \Delta_Y,\, x \in X,\, y \in Y \} \\ &= \{ (x, y) \mid (f(x), y) \in \Delta_Y,\, x \in X,\, y \in Y \} \\ &= \{ (x, y) \mid f(x) = y,\, x \in X,\, y \in Y \} \\ &= \{ (x, f(x)) \mid x \in X,\, y \in Y \} \\ &= G_f \end{aligned}$$
is closed by Theorem 18.1(3).

Conversely, suppose G_f is closed in $X \times Y$. Fix a point $x_0 \in X$ and let $V \subset Y$ be an arbitrary neighborhood of $f(x_0)$. Then $Y - V$ is a closed subset of Y so, by Problem 2.1 (Munkres §17, Ex. 3), the product $X \times (Y - V)$ is closed in $Y \times Y$. In particular, by Theorem 17.1(2), the intersection $B = G_f \cap X \times (Y - V)$ is closed in $X \times Y$. Thus, by Problem 6.5 (Munkres §26, Ex. 7), since Y is a compact Hausdorff space, the projection $\pi_1(B)$ onto X is a closed subset of X. But
$$\begin{aligned} B &= \{ (x, y) \mid (x, y) \in G_f \text{ and } (x, y) \in X \times (Y - V) \} \\ &= \{ (x, y) \mid y = f(x) \text{ and } (x, y) \in X \times (Y - V)\} \\ &= \{ (x, f(x)) \mid f(x) \in Y - V \} \end{aligned}$$
so we have that $\pi_1(B) = f^{-1}(Y - V) = X - f^{-1}(V)$. One containment is easy to see, namely "$\subset$": if $x \in B$ then $x = \pi_1(x, f(x))$ for at least one $f(x) \in Y - V$. To see the reverse inclusion, take $x \in f^{-1}(Y - V)$, then $f(x) \in Y - V$ so $(x, f(x)) \in B$, hence $x \in \pi_1(B)$. Thus, $X - \pi_1(B) = f^{-1}(V)$ is open so f is continuous. ∎

Problem 7.2 (Munkres §26, Ex. 9)

Generalize the tube lemma as follows:

Theorem. *Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that*

$$A \times B \subset U \times V \subset N.$$

Proof. We first prove the theorem for the case in which $A = \{a\}$. Since $a \times B$ is canonically homeomorphic to B, by Theorem 26.5, $a \times B$ is compact. Therefore, for every covering by open subsets $\{U_\alpha\}$ of $a \times B$, $U_\alpha \subset X \times Y$, there exists a finite subcollection, say $\{U_i\}_{i=1}^n$ covering $a \times B$. ∎

Problem 7.3 (Munkres §26, Ex. 12)

Let $p\colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(y)$ is compact, for each $y \in Y$. (Such a map is called a *perfect map.*) Show that if Y is compact, then X is compact.

[*Hint:* If U is an open set containing $p^{-1}(y)$, there is a neighborhood W of y such that $p^{-1}(W)$ is contained in U.]

Proof. ■

Problem 7.4 (Munkres §27, Ex. 2(b,d))

Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ε-neighborhood of A is contained in U.

Proof. ■

Problem 7.5 (Munkres §27, Ex. 5)

Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X. [*Hint:* Imitate the proof of Theorem 27.7.]

This is a special case of the *Baire category theorem*, which we shall study in Chapter 8.

Proof. ∎

Problem 7.6 (Munkres §28, Ex. 2(a))

Let $\{X_\alpha\}$ be a nindexed family of nonempty spaces.

(a) Show that if $\prod X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α.

Proof. ∎

Problem 7.7 (Munkres §28, Ex. 10)

Show that if X is a Hausdorff space that is locally compact at the point x, then for each neighborhood U of x, there is a neighborhood V of x such that V is compact and $\overline{V} \subset U$.

Proof. ∎

Problem 7.8

Proof. ■

Problem 7.9 (A)

Let S^1 denote the circle

$$S^1 = \{\, (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1 \,\}$$

and let B^2 denote the closed disk

$$B^2 = \{\, (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1 \,\}.$$

Prove that the quotient space $(S^1 \times [0, 1])/(S^1 \times 0)$ (see HW #4 for the notation) is homeomorphic to B^2.

Proof. ■