MA 544: Homework 11

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Problem 11.1 (Wheeden & Zygmund §7, Ex. 11)

Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt.$$

(Consider the case when f is the characteristic function of an interval, an open set, etc.)

Proof. Recall that, by Theorem 5.21, f is integrable (or in L^1) on $[\alpha, \beta]$ if and only if $|f|$ is integrable on $[\alpha, \beta]$. Therefore, it suffices to prove the result for the case $f \geq 0$. We split the proof of the result into a series of claims and then proceed to show the more general result.

Claim 1. *Let g be as above and G be an open subset of $[\alpha, \beta]$. Then*

$$|g(G)| = \int_G g'(t)dt.$$

Proof of claim 1. Let G be an open subset of (a, b) then, by Theorem 1.10, G can be written as the countable union of disjoint open intervals $\{I_k\}$. By Theorem 5.7, since g' is nonnegative and measurable and $\int_G g'$ is finite (in particular, it is bounded above by $\int_a^b g'$), we have

$$\int_G g'(t)dt = \sum_k \int_{I_k} g'(t)dt. \tag{11.1}$$

But by Theorem 7.27, since g is absolutely continuous on $[\alpha, \beta]$, g is b.v. on $[\alpha, \beta]$ so by Theorem 7.30

$$|g(I_k)| = g(\beta_k) - g(\alpha_k) = V[g; \alpha_k, \beta_k] = \int_{\alpha_k}^{\beta_k} g'(t)dt$$

where α_k is the left-most endpoint of I_k and β_k the right-most. By Equation (11.1), on the right-hand side, we have

$$\int_{I_k} g'(t)dt = |g(I_k)|$$

so, by Theorem 3.23, we have

$$\int_G g'(t)dt = \sum_k |g(I_k)| = |g(\textstyle\bigcup_k I_k)| = |g(G)| \tag{11.2}$$

as desired. ♣

Claim 2. *Let g be as above and E be a G_δ-subset of $[\alpha, \beta]$. Then*

$$|g(E)| = \int_E g'(t)dt.$$

Proof of claim 2. Suppose E is a G_δ-set, then E is the countable intersection of open subsets $\{G_k\}$ of $[\alpha, \beta]$. We may choose G_k's such that $G_k \searrow E$ (for example, taking our original collection of open subsets $\{G_k\}$ and taking the finite intersection $\bigcap_{j=1}^k G_j$). Hence, we have $\chi_{G_k} \searrow \chi_E$ and consequently $\chi_{G_k} g' \searrow \chi_E g'$. Thus, we have

$$\lim_{k\to\infty} \int_E \chi_{G_k} g'(t)dt = \lim_{k\to\infty} |g(G_k)| = |g(E)| \tag{11.3}$$

by Claim 1 and Theorem 3.10. Thus, by the monotone convergence theorem together with Equation (11.3), we have

$$|g(E)| = \lim_{k\to\infty} \int_E \chi_{G_k} g'(t)dt = \int_E \chi_{G_k} g'(t)dt \tag{11.4}$$

as desired. ♣

Claim 3.

■

Problem 11.2 (Wheeden & Zygmund §7, Ex. 15)

Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\varphi(x) = \int_a^x f(t)dt + \varphi(a)$ in (a, b) and f is monotone increasing, then φ is convex in (a, b). (Use Exercise 14.)

Proof. We will assume the result in Exercise 14. First we check that φ is continuous. Since f is monotone increasing, f is b.v. on $[a, b]$ so f is bounded a.e. on (a, b) by a previous exercise. Thus, $f \in L(a, b)$ so by Theorem 7.1, φ is absolutely continuous and hence, continuous.

Now, let $x_1, x_2 \in (a, b)$ and, without loss of generality, assume $x_1 < x_2$. Then, we have

$$\begin{aligned}\varphi\left(\frac{x_1 + x_2}{2}\right) &= \int_a^{(x_1+x_2)/2} f(t)dt + \varphi(a)\\ &= \int_a^{x_1} f(t)dt + \int_{x_1}^{(x_1+x_2)/2} f(t)dt + \varphi(a)\end{aligned}$$

since f is monotone increasing, we have $\int_{x_1}^{(x_1+x_2)/2} f(t)dt \leq \int_{(x_1+x_2)/2}^{x_2} f(t)dt$ so

$$\begin{aligned}&= \int_a^{x_1} f(t)dt + \frac{1}{2}\left[2\int_{x_1}^{(x_1+x_2)/2} f(t)dt\right] + \varphi(a)\\ &\leq \int_a^{x_1} f(t)dt + \frac{1}{2}\left[\int_{x_1}^{(x_1+x_2)/2} f(t)dt + \int_{(x_1+x_2)/2}^{x_2} f(t)dt\right] + \varphi(a)\\ &= \frac{1}{2}\left[\int_a^{x_1} f(t)dt + \varphi(a)\right] + \frac{1}{2}\left[\int_a^{x_1} f(t)dt + \int_{x_1}^{(x_1+x_2)/2} f(t)dt + \int_{(x_1+x_2)/2}^{x_2} f(t)dt + \varphi(a)\right]\\ &= \frac{1}{2}\left[\int_a^{x_1} f(t)dt + \varphi(a)\right] + \frac{1}{2}\left[\int_a^{x_2} f(t)dt + \varphi(a)\right]\\ &= \frac{\varphi(x_1) + \varphi(x_2)}{2}.\end{aligned}$$

Thus, by Exercise 14, φ is convex. ∎

Problem 11.3 (Wheeden & Zygmund §5, Ex. 8)

Prove (5.49).

Proof. Recall the content of equation 5.49: For f measurable, we have

$$\omega(\alpha) \leq \frac{1}{\alpha^p} \int\limits_{\{ f>\alpha \}} f^p, \quad \alpha > 0. \tag{11.5}$$

Consider the L^p-norm of f raised to the p-th power

$$\|f\|_p{}^p = \int |f(x)|^p dx$$

since f is measurable, f is measurable so $\{ f > \alpha \}$ is measurable hence, by the monotonicity of the Lebesgue integral, we have

$$\begin{aligned} &\geq \int_{\{ f>\alpha \}} f^p dx \\ &\geq \int_{\{ f>\alpha \}} \alpha^p dx \\ &= \alpha^p |\{ f > \alpha \}| \\ &= \alpha^p \omega(\alpha). \end{aligned}$$

Thus, we have

$$\omega(\alpha) \leq \frac{1}{\alpha^p} \int_{\{ f>\alpha \}} f^p$$

as desired. ■

Problem 11.4 (Wheeden & Zygmund §5, Ex. 11)

For which p does $1/x \in L^p(0,1)$? $L^p(1,\infty)$? $L^p(0,\infty)$?

Proof. For the case $1/x \in L^p(0,1)$, this happens if and only if $\int_0^1 x^{-p}dx < \infty$ if and only if $p < 1$.

In the second case $1/x \in L^p(1,\infty)$ if and only if $p > 1$.

Lastly, we have $1/x \in L^p(0,\infty)$ if and only if $1/x \in L^p(0,1)$ and $1/x \in L^p(1,\infty)$. By our previous arguments, this is impossible. Thus, $1/x \notin L^p(0,\infty)$. ■

Problem 11.5 (Wheeden & Zygmund §5, Ex. 12)

Give an example of a bounded continuous f on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

Proof. An example, given in class, is the following: Set

$$f(x) \coloneqq \begin{cases} 1 & x \leq e \\ 1/\ln x & x \geq e. \end{cases} \tag{11.6}$$

This function is bounded (above by 1), continuous ($\lim_{x+\to e} f(x) = 1 = \lim_{x-\to e} f(x)$) and $\lim_{x\to\infty} f(x) = 0$ Now, observe that, for every $p > 0$, we have $\ln(x) \leq x^{1/p}$ for x larger than some number K depending on p. Thus,

$$\int_K^\infty \frac{dx}{\ln x} \geq \int_K^\infty \frac{dx}{x} = \infty.$$

so f cannot be in $L^p(0, \infty)$ for any $p > 0$. ■

Problem 11.6 (Wheeden & Zygmund §5, Ex. 17)

If $f \geq 0$ and $\omega(\alpha) \leq c(1+\alpha)^p$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.

Proof. Assuming the results of Exercise 16, it suffices to show that

$$\int_0^\infty \alpha^{r-1}\omega(\alpha)d\alpha \leq c\int_0^\infty \frac{\alpha^{r-1}}{(1+\alpha)^p}d\alpha < \infty \tag{11.7}$$

for all $r \in (0,p)$. The integral is improper only near ∞, and convergence there follows from the fact that

$$\frac{\alpha^{r-1}}{(1+\alpha)^p} < \frac{\alpha^{r-1}}{\alpha^p} = \frac{1}{\alpha^{p-(r-1)}}$$

L for sufficiently large α. Since $r < p$, we have $p-(r-1) > 1$, hence

$$\int_{K_p}^\infty \frac{d\alpha}{\alpha^{p-(r-1)}}$$

converges. ■

Problem 11.7 (Wheeden & Zygmund §8, Thm. 8.3)

If $f, g \in L^p(E)$, $p > 0$, then $f + g \in L^p(E)$ and $cf \in L^p(E)$ for any constant c.

Proof. Suppose $f, gi \in L^p(E)$ and c is any constant, then, by Minkowski's inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p < \infty$$

and

$$\|cf\|_p = \left(\int_E |cf|^p\right)^{1/p} = \left(\int_E |c|^p|f|^p\right)^{1/p} = |c|\left(\int_E |c|^p|f|^p\right)^{1/p} < \infty.$$

Thus, $f + g, cf \in L^p(E)$ ■