# MA571 Problem Set 7

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## Problem 7.1 (Munkres §26, Ex. 8)

**Theorem.** *Let $f\colon X \to Y$; let $Y$ be compact Hausdorff. Then $f$ is continuous if and only if the* graph *of $f$,*

$$G_f = \{\,(x, f(x)) \mid x \in X\,\},$$

*is closed in $X \times Y$.*

[*Hint:* If $G_f$ is closed and $V$ is a neighborhood of $f(x_0)$, then the intersection of $G_f$ and $X \times (Y - V)$ is closed. Apply Exercise 7.]

*Proof.* As we demonstrated in Problem 2.7 (Munkres §18, Ex. 17) $Y$ is Hausdorff if and only if the diagonal, $\Delta_Y = \{\,(y, y) \mid y \in Y\,\}$, is a closed subset of $Y \times Y$. Consider the map $F\colon X \times Y \to Y \times Y$ defined by $(x, y) \mapsto (f(x), y)$. This map is continuous by Theorem 18.4 as $f$ is, by assumption, continuous and $\mathrm{id}_Y$ is continuous by 18.2(b) (since it is the inclusion $Y \hookrightarrow Y$). Then

$$\begin{aligned}
F^{-1}(\Delta_Y) &= \{\,(x, y) \mid F(x, y) \in \Delta_Y,\, x \in X,\, y \in Y\,\}\\
&= \{\,(x, y) \mid (f(x), y) \in \Delta_Y,\, x \in X,\, y \in Y\,\}\\
&= \{\,(x, y) \mid f(x) = y,\, x \in X,\, y \in Y\,\}\\
&= \{\,(x, f(x)) \mid x \in X,\, y \in Y\,\}\\
&= G_f
\end{aligned}$$

is closed by Theorem 18.1(3).

Conversely, suppose $G_f$ is closed in $X \times Y$. Fix a point $x_0 \in X$ and let $V \subset Y$ be an arbitrary neighborhood of $f(x_0)$. Then $Y - V$ is a closed subset of $Y$ so, by Problem 2.1 (Munkres §17, Ex. 3), the product $X \times (Y - V)$ is closed in $Y \times Y$. In particular, by Theorem 17.1(2), the intersection $B = G_f \cap X \times (Y - V)$ is closed in $X \times Y$. Thus, by Problem 6.5 (Munkres §26, Ex. 7), since $Y$ is a compact Hausdorff space, the projection $\pi_1(B)$ onto $X$ is a closed subset of $X$. But the intersection

$$\begin{aligned}
B &= \{\,(x, y) \mid (x, y) \in G_f \text{ and } (x, y) \in X \times (Y - V)\,\}\\
&= \{\,(x, y) \mid y = f(x) \text{ and } (x, y) \in X \times (Y - V)\}\\
&= \{\,(x, f(x)) \mid f(x) \in Y - V\,\}
\end{aligned}$$

so we have that $\pi_1(B) = f^{-1}(Y - V)$. One containment is easy to see, namely "$\subset$": if $x \in B$ then $x = \pi_1(x, f(x))$ for at least one $f(x) \in Y - V$. To see the reverse inclusion, take $x \in f^{-1}(Y - V)$, then $f(x) \in Y - V$ so $(x, f(x)) \in U$, thus $x \in \pi_1(U)$. Thus, by Theorem 18.1(3), it follows that $f$ is continuous. ■

## Problem 7.2 (Munkres §26, Ex. 9)

Generalize the tube lemma as follows:

**Theorem.** *Let $A$ and $B$ be subspaces of $X$ and $Y$, respectively; let $N$ be an open set in $X \times Y$ containing $A \times B$. If $A$ and $B$ are compact, then there exist open sets $U$ and $V$ in $X$ and $Y$, respectively, such that*

$$A \times B \subset U \times V \subset N.$$

*Proof.* Keeping the same notation as the prompt of the theorem: Since $A$ and $B$ are compact, by Theorem 26.7, $A \times B$ are compact. ■

## Problem 7.3 (Munkres §26, Ex. 12)

**Theorem.** *Let $X$ be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of $X$ that is simply ordered by proper inclusion. Then*

$$Y = \bigcap_{A \in \mathcal{A}} A.$$

*Proof.* ■

## Problem 7.4 (Munkres §27, Ex. 2(b,d))

Let $X$ be a metric space with metric $d$; let $A \subset X$ be nonempty.

(b) Show that if $A$ is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(d) Assume that $A$ is compact; let $U$ be an open set containing $A$. Show that some $\varepsilon$-neighborhood of $A$ is contained in $U$.

*Proof.* ∎

## Problem 7.5 (Munkres §27, Ex. 5)

Let $X$ be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of $X$. Show that if each set $A_n$ has empty interior in $X$, then the union $\bigcup A_n$ has empty interior in $X$. [*Hint:* Imitate the proof of Theorem 27.7.]

This is a special case of the *Baire category theorem*, which we shall study in Chapter 8.

*Proof.* ■

## Problem 7.6 (Munkres §28, Ex. 2(a))

Let $\{X_\alpha\}$ be a nindexed family of nonempty spaces.

(a) Show that if $\prod X_\alpha$ is locally compact, then each $X_\alpha$ is locally compact and $X_\alpha$ is compact for all but finitely many values of $\alpha$.

*Proof.* ■

## Problem 7.7 (Munkres §28, Ex. 10)

Show that if $X$ is a Hausdorff space that is locally compact at the point $x$, then for each neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $V$ is compact and $\overline{V} \subset U$.

*Proof.* ■

## Problem 7.8

*Proof.* ■

## Problem 7.9 (A)

Let $S^1$ denote the circle
$$S^1 = \{ (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1 \}$$
and let $B^2$ denote the closed disk
$$B^2 = \{ (x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1 \}.$$
Prove that the quotient space $(S^1 \times [0, 1])/(S^1 \times 0)$ (see HW #4 for the notation) is homeomorphic to $B^2$.

*Proof.* ■