MA571 Homework 8

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Problem 8.1 (Munkres §46, Ex. 6)

Show that the compact-open topology, $\mathcal{C}(X,Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular. [*Hint:* If $\overline{U} \subset V$, then $\overline{S(C,U)} \subset S(C,V)$.]

Proof. Suppose that Y is Hausdorff. Let f and g be distinct continuous functions from X to Y. Then there exists a point $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Since Y is Hausdorff there exists disjoint neighborhoods U and V of $f(x_0)$ and $g(x_0)$, respectively. Now, we claim that

Claim. *If $C \subset X$ is finite, C is compact.*

Proof. Write $C = \{x_1, ..., x_n\}$. Let $\mathcal{A}$ be an open cover of C. Then since $C \subset \bigcup_{U_\alpha \in \mathcal{A}} U_\alpha$ we can choose A_i containing x_i for every $1 \leq i \leq n$. Thus, the subcollection $\{U_i\}_{i=1}^n$ covers C. ♣

Let $U' = S(\{x_0\}, U)$ and $V' = S(\{x_0\}, V)$. Note that U' and V' are nonempty since $f \in U'$ and $g \in V'$. Moreover, their intersection is empty for suppose $h \in U' \cap V'$, then $h(x_0) \in U \cap V$, but $U \cap V = \emptyset$. Then, since U' and V' are subbasis elements for the compact-open topology on $\mathcal{C}(X,Y)$ and they "separate" f and g, it follows that $\mathcal{C}(X,Y)$ is Hausdorff.

Now, suppose that Y is regular. We shall proceed by the hint and Lemma 31.1(b). Consider the subbasis element $S(C,U)$. Since Y is regular, there exists a neighborhood $V \supset U$ such that $V \supset \overline{U}$. Let $f \in \overline{S(C,U)}$. Then, we claim that $f \in S(C,V)$. For suppose not, then there exists an element $x_0 \in C$ such that $f(x_0) \notin V$. Then, since $\overline{U} \subset V$, by hypothesis, $f(x_0) \notin \overline{U}$. Consider the subbasic neighborhood $S\big(\{x_0\}, Y - \overline{U}\big)$ of f. Then, $S\big(\{x_0\}, Y - \overline{U}\big) \cap S(C,U)$ is nonempty. Let g be in the aforementioned intersection. Then $g(x_0) \in g(C) \subset U$, but $g(x_0) \in Y - \overline{U}$. This is a contradiction. It follows by Lemma 31.1(b) that $\mathcal{C}(X,Y)$ is regular. ■

Problem 8.2 (Munkres §46, Ex. 7)

Show that if Y is locally compact Hausdorff, then composition of maps

$$\mathcal{C}(X,Y)\times\mathcal{C}(Y,Z)\longrightarrow\mathcal{C}(X,Z)$$

is continuous, provided the compact-open topology is used throughout. [*Hint:* If $g\circ f\in S(C,U)$, find V such that $f(C)\subset V$ and $g(\overline{V})\subset U$.]

Proof. Let $F\colon\mathcal{C}(X,Y)\times\mathcal{C}(Y,Z)\to\mathcal{C}(X,Z)$ given by $(f,g)\mapsto g\circ f$. Suppose $g\circ f\in S(C,U)$. Then $g(f(C))\subset U$ and since is continuous, we have that $g^{-1}(U)$ is an open set containing $f(C)$. Thus, by theorem 29.2, for every $x\in f(C)$ there exists an open neighborhood V_x of x such that $\overline{V_x}\subset g^{-1}(U)$ is compact. Then the collection of all such open neighborhoods, $\{V_x\}_{x\in f(C)}$, forms an open cover of $f(C)$. Since $f(C)$ is compact, by Theorem 26.5 since C is compact and f is continuous, then by Lemma 26.1 there exists a finite subcollection, say $\{V_i\}_{i=1}^n$, that covers C. Let $V=\bigcup_{i=1}^n V_i$. We claim that $\overline{V}\subset U$ and is compact. More generally, we have

Lemma 16 (Munkres §26, Ex. 3)**.** *A finite union of compact subspaces of X is compact.*

Proof of lemma. Suppose $C_1,...,C_n\subset X$ are compact and write $C=\bigcup_{i=1}^n C_i$. Let $\mathcal{A}=\{U_\alpha\}$ be an open cover of C. Then $C_i\subset\bigcup U_\alpha$ so, since C_i is compact, there exists a finite subcollection $\mathcal{A}_i=\{U_j^i\}_{j=1}^{n_i}$ that covers C_i. Then $\mathcal{B}=\bigcup_{i=1}^n\mathcal{A}_i$ is a finite subcollection of $\mathcal{A}$ that covers C, i.e., C is compact. ♣

By Lemma 16, $\overline{V}$ is compact since, by induction on Problem 2.2 (Munkres §17, Ex. 6(b)), it is the union of finitely many compact sets $\overline{V}=\bigcup_{i=1}^n\overline{V_i}$. Moreover, by Lemma 5 (from HW # 2[1]) we have that $f(C)\subset V\subset\overline{V}\subset g^{-1}(U)$. At last, tying these results together, we have

$$F\big(S(C,V)\times(\overline{V},U)\big)\subset S(C,U),$$

since $f'\in S(C,V)$ if $f'(C)\subset V$ and $g'\in S\big(\overline{V},U\big)$ if $g'\big(\overline{V}\big)\subset U$ so $g'(f'(C))\subset g'\big(\overline{V}\big)\subset U$ so $g'\circ f'\in S(C,U)$. It follows, by Theorem 18.1(4), that F is continuous. ■

[1] This states that if $A_\alpha\subset C$ then $\bigcup A_\alpha\subset C$.

Problem 8.3 (Munkres §46, Ex. 8)

Let $\mathcal{C}'(X,Y)$ denote the set $\mathcal{C}(X,Y)$ in some topology $\mathcal{T}$. Show that if the evaluation map

$$e\colon X \times \mathcal{C}'(X,Y) \longrightarrow Y$$

is continuous, then $\mathcal{T}$ contains the compact-open topology. [*Hint:* The induced map $E\colon \mathcal{C}'(X,Y) \to \mathcal{C}(X,Y)$ is continuous.]

Proof. Suppose that the evaluation map $e\colon X \times \mathcal{C}'(X,Y) \longrightarrow Y$ is continuous. Then, by Theorem 46.11 the induced map $E\colon \mathcal{C}'(X,Y) \to \mathcal{C}(X,Y)$ in

$$X \times \mathcal{C}'(X,Y) \xrightarrow{(\mathrm{id}_X, E)} X \times \mathcal{C}(X,Y) \xrightarrow{e'} Y$$

is continuous. In fact, it is easy to see that the induced map E is the identity map on $\mathcal{C}(X,Y)$ for $e(x,f) = f(x) = f'(x) = e'(f',x) = e'(E(f),x)$ for all x so $f = f'$. Now, let $S(C,U)$ be a subbasic open set in $\mathcal{C}(X,U)$. Then $E^{-1}(S(C,U)) = S(C,U)$ is open in $\mathcal{C}'(X,Y)$. Thus $\mathcal{T}$ is finer than the compact-open topology. ■

Problem 8.4 ((A))

Definition 1. Definition. If X is a locally compact Hausdorff space then the space Y given by Theorem 29.1 is called the *one-point compactification* of X.

Let X be a compact Hausdorff space and let W be an open subset of X (so W is locally compact by Corollary 29.3) with $W \neq X$. Prove that the one-point compactification of W is homeomorphic to the quotient space $X/(X - W)$.

Proof. Let W_∞ denote the one-point compactification of W and define the map $p\colon X \to W_\infty$ by

$$p(x) = \begin{cases} x, & x \in W \\ \infty, & x \in X - W. \end{cases}$$

We claim that p is continuous. It suffices to show that the preimage of a basic open set in W_∞ is open in X. Suppose U is a type 1 open subset of W_∞, that is, U does not contain the point at infinity. Then $U \subset W$ so is open in X by Theorem 16.2. Suppose that U is a type 2 open subset of W_∞. Then $C = W_\infty - U$ is a compact subset of W_∞. Moreover $C \subset W$ so C is a compact subset of X, that is to say, if $\{U_\alpha\}$ is an open cover of C in X, then $\{U_\alpha \cap W\}$ is an open cover of C in W and since C is compact in W, there exists a finite subcollection $\{U_i \cap W\}_{i=1}^n$ in Y that covers C hence, the collection $\{U_i\}_{i=1}^n$ is a finite subcollection in X that covers C. It follows by Theorem 26.3 that C is closed so $p^{-1}(U) = X - C$ is open in X. Thus, p is continuous. By Theorem Q.3, it follows that the induced map $\bar{p}\colon X/(X - W) \to W_\infty$ is continuous. Moreover, p preserves the equivalence relation: Suppose $x \sim y$ then either $x = y \in W$ or $x, y \in X - W$; in the former we have $p(x) = x = y = p(y)$; in the latter we have $p(x) = \infty = p(y)$.

By Theorem 26.6, since the quotient $X/(X-W)$ is compact and W_∞ is Hauusdorff, it suffices to show that $\bar{p}$ is bijective. It is clear that $\bar{p}$ is surjective since p is surjective ($p(X) = p(W \cup (X-W)) = p(W) \cup p(X - W) = W \cup \{\infty\} = W_\infty$). To see that $\bar{p}$ is injective suppose $p([x]) = p([y])$. Then $p([x]) = \infty$ or $p([x]) \neq \infty$. If $p([x]) \neq \infty$, then $p([x]) = x = y = p([x])$ or $p([x]) = \infty = p([y])$. In either case, $x \sim y$ so $[x] = [y]$. Thus, $\bar{p}$ is bijective. It follows that $\bar{p}$ is a homeomorphism. ∎

Problem 8.5 ((B))

Let X be a compact Hausdorff space, let Y be a topological space, and let $p\colon X \to Y$ be a closed surjective continuous map. Prove that Y is Hausdorff. [*Hint:* one ingredient in the proof is p. 171 # 5.]

Note: combining this with HW 4 Problem E and HW 6 Problem A gives a necessary and sufficient condition for a quotient of a compact Hausdorff space to be Hausdorff.

Proof. Let x and y be distinct points in Y. Since p is surjective, there exist x_0 and y_0 in X such that $p(x_0) = x$ and $p(y_0) = y$. Then, since X is Hausdorff, by Theorem 17.8, x_0 and y_0 are closed in X so x and y are closed in Y. Then $p^{-1}(x)$ and $p^{-1}(y)$ are closed since

$$X - p^{-1}(x) = p^{-1}(Y - x)$$

which is open in X since $Y - x$ is open in Y and p is continuous. Moreover, $p^{-1}(x)$ and $p^{-1}(y)$ are clearly disjoint for otherwise $p(z) = x = y$, but $x \neq y$. Now, by Theorem 32.3, X is normal since it is a compact Hausdorff space (alternatively we may appeal to Theorem 26.3 and Munkres §26, Ex. 5 as suggested in the hint) so there exist disjoint open sets U and V containing $p^{-1}(x)$ and $p^{-1}(y)$, respectively. Then $X - U$ and $X - V$ are closed so $p(X - U)$ and $p(X - V)$ are closed in Y. Then, we claim $U' = Y - p(X - U)$ and $V' = Y - p(X - V)$ are disjoint neighborhoods of x and y, respectively. It is clear that U' and V' are open, since their complements are closed. Moreover, $U' \ni x$ and $V' \ni y$ since $Y - U' = p(X - U)$ does not contain x and $Y - V' = p(X - V)$ does not contain y. Lastly, $U' \cap V' = \emptyset$ for otherwise there is $z \in U' \cap V'$ so $z \notin p(X - U)$ and $z \notin p(X - V)$ so $z \in Y - (p(X - U) \cup p(X - V))$, but $p(X - U) \cup p(X - V) \supset p((X - U) \cup p(X - V)) = p(X)$ so $z \in \emptyset$, this is a contradiction. Thus, Y is Hausdorff. ■

Problem 8.6 ((C))

Let $S^2 \subset \mathbf{R}^3$ be the subspace

$$\{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}.$$

Prove that S^2 is a 2-manifold. (The definition of m-manifold, where m is a positive whole number, is given at the top of page 225.)

Proof. ■

Problem 8.7 ((D))

Prove that the union of the x and y-axes in $\mathbf{R}^2$ is not a 1-manifold.

Proof. ∎