

# MA 519: Homework 4

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## PROBLEM 4.1 (HANDOUT 5, # 2)

In an urn, there are 12 balls. 4 of these are white. Three players:  $A$ ,  $B$ , and  $C$ , take turns drawing a ball from the urn, in the alphabetical order. The first player to draw a white ball is the winner. Find the respective winning probabilities: assume that at each trial, the ball drawn in the trial before is put back into the urn (i.e., selection *with replacement*).

*SOLUTION.* Denote the events that player  $A$  wins, player  $B$  wins, and player  $C$  wins by  $A$ ,  $B$ , and  $C$  respectively.

Note that  $A$  wins if some multiple of 3 losses occurs, followed by a win. Also,  $B$  wins if some multiple of three losses occurs, followed by a loss, then a win. Last,  $C$  wins if some multiple of three losses occurs, followed by two losses then a win.

A win occurs with probability  $1/3$  each time, and a loss occurs with probability  $2/3$  each time. Thus, because each draw is independent,

$$P(A) = \sum_{i=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{3i}, \quad P(B) = \sum_{i=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{3i+1}, \quad P(C) = \sum_{i=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^{3i+2}.$$

That is,

$$P(A) = \sum_{i=0}^{\infty} \frac{9}{19} \approx 0.47, \quad P(B) = \sum_{i=0}^{\infty} \frac{6}{19} \approx 0.32, \quad P(C) = \sum_{i=0}^{\infty} \frac{4}{19} \approx 0.21.$$

■

## PROBLEM 4.2 (HANDOUT 5, # 8)

Consider  $n$  families with 4 children each. How large must  $n$  be to have a 90% probability that at least 3 of the  $n$  families are all girl families?

*SOLUTION.* The probability that a family has all girls is  $(0.5)^4$ .

In  $n$  families, the probability that at least 3 are all-girl is  $1 - P(0) - P(1) - P(2)$ , where  $P(m)$  is the probability that exactly  $m$  families are all-girl.

Note that

$$\begin{aligned} P(0) &= (1 - (0.5)^4)^n \\ P(1) &= \binom{n}{1} (0.5)^4 (1 - (0.5)^4)^{n-1} \\ P(2) &= \binom{n}{2} (0.5)^8 (1 - (0.5)^4)^{n-2} \end{aligned}$$

(we choose a number of families to be all-girl, and then find the probability that that family is all-girl, while all of the rest are not all-girl).

So that the probability that at least 3 are all girl is

$$1 - (1 - (0.5)^4)^n - n(0.5)^4 (1 - (0.5)^4)^{n-1} - n(n-1)(0.5)^8 (1 - (0.5)^4)^{n-2}.$$

■

## PROBLEM 4.3 (HANDOUT 5, # 10)

(*Yahtzee*). In Yahtzee, five fair dice are rolled. Find the probability of getting a Full House, which is three rolls of one number and two rolls of another, in Yahtzee.

*SOLUTION.* There are 30 different kinds of full house (6 different three of a kinds, and 5 different kinds of different two of a kind).

The probability of rolling a specific kind of full house is

$$\binom{5}{3} \left(\frac{1}{6}\right)^3 \left(\frac{1}{6}\right)^2$$

(Choose 3 dice to be the three of a kind, have them all rolled the specific number, have the other two dice rolled the other specific number.)

So the probability of rolling some kind of full house is

$$30 \binom{5}{3} \left(\frac{1}{6}\right)^3 \left(\frac{1}{6}\right)^2 = \frac{25}{648} \approx 0.039.$$

■

## PROBLEM 4.4 (HANDOUT 5, # 12)

The probability that a coin will show all heads or all tails when tossed four times is 0.25. What is the probability that it will show two heads and two tails?

*SOLUTION.* Let  $H$  denote the event that, on a given coin toss, that coin is heads, and let  $T$  denote the event that, on a given coin toss, that coin is tails. Then, because each coin toss is independent, this says that

$$P(H)^4 + P(T)^4 = \frac{1}{4}.$$

Moreover, because this is a coin,

$$P(H) + P(T) = 1.$$

So,

$$\begin{aligned} P(H)^4 + (1 - P(H))^4 &= \frac{1}{4} \\ 2P(H)^4 - 4P(H)^3 + 6P(H)^2 - 4P(H) + 1 &= \frac{1}{4} \end{aligned}$$

We can approximate a solution to the above by  $P(H) \approx 0.299$  or by  $P(H) \approx 0.701$ .

Now, the probability that two coin flips of four are heads is

$$\binom{4}{2} P(H)^2 P(T)^2 \approx 0.263.$$

■

## PROBLEM 4.5 (HANDOUT 5, # 13)

Let the events  $A_1, A_2, \dots, A_n$  be independent and  $P(A_k) = p_k$ . Find the probability  $p$  that none of the events occurs.

*SOLUTION.* Let the events  $A_1, A_2, \dots, A_n$  be independent. Then the events  $A_1^\sim, A_2^\sim, \dots, A_n^\sim$  are independent. (Where  $A$  denotes the event that  $A$  does *not* happen, i.e.,  $A^\sim = \Omega \setminus A$ ). The events  $A_i^\sim$  each have probability  $1 - p_i$  of occurring. Thus, the probability,  $p$ , that none of the events occur is

$$p = \prod_{k=1}^n (1 - p_k)$$

■

## PROBLEM 4.6 (HANDOUT 6, # 5)

Suppose a fair die is rolled twice and suppose  $X$  is the absolute value of the difference of the two rolls. Find the PMF and the CDF of  $X$  and plot the CDF. Find a median of  $X$ ; is the median unique?

*SOLUTION.* First, we compute the probability mass function. Note that  $X$  is an integer between 0 and 5, so computing the probabilities that  $X$  is each of 0 through 5 suffices to describe the PMF of  $X$ .

We calculate the probabilities, by counting.

$$\begin{aligned} P(0) &= \frac{6}{36} = \frac{1}{6} \\ P(1) &= \frac{10}{36} = \frac{5}{18} \\ P(2) &= \frac{8}{36} = \frac{2}{9} \\ P(3) &= \frac{6}{36} = \frac{1}{6} \\ P(4) &= \frac{4}{36} = \frac{1}{9} \\ P(5) &= \frac{2}{36} = \frac{1}{18} \end{aligned}$$

We calculate the CDF by summing.

$$\begin{aligned} \text{CDF}(0) &= \frac{6}{36} = \frac{1}{6} \\ \text{CDF}(1) &= \frac{16}{36} = \frac{4}{9} \\ \text{CDF}(2) &= \frac{24}{36} = \frac{2}{3} \\ \text{CDF}(3) &= \frac{30}{36} = \frac{5}{6} \\ \text{CDF}(4) &= \frac{34}{36} = \frac{17}{18} \\ \text{CDF}(5) &= \frac{36}{36} = 1 \end{aligned}$$

The median is 2. ■



## PROBLEM 4.7 (HANDOUT 6, # 7)

Find a discrete random variable  $X$  such that  $E(X) = E(X^3) = 0$ ;  $E(X^2) = E(X^4) = 1$ .

*SOLUTION.* Set  $\Omega = \{0, 1\}$  and define a random variable  $X: \Omega \rightarrow \mathbb{R}$  by  $X(0) = -1$ ,  $X(1) = 1$  as well as a probability  $P(0) = P(1) = 1/2$ . Then

$$E(X) = -1(1/2) + 1(1/2) = 0 = (-1)^3(1/2) + 1^3(1/2) = E(X^3),$$

whereas

$$E(X^2) = (-1)^2(1/2) + 1^2(1/2) = 1 = (-1)^4(1/2) + 1^4(1/2) = E(X^4),$$

as desired. ■

## PROBLEM 4.8 (HANDOUT 6, # 9)

(Runs). Suppose a fair die is rolled  $n$  times. By using the indicator variable method, find the expected number of times that a six is followed by at least two other sixes. Now compute the value when  $n = 100$ .

*SOLUTION.* Let  $\Omega$  denote the sample space and define a random variable  $X: \Omega \rightarrow \mathbb{R}$  by  $x \mapsto k$  the number of times a subsequence of three or more consecutive sixes shows up in  $x$ . To compute the mean value of  $X$ , we must first find the PMF of  $X$  to assign probabilities to the various possible outcomes.

Fix an integer  $k \leq n$ . We must count all possible sequences  $x \in \Omega$  which contain exactly  $k$  subsequences of three or more consecutive sixes. ■

## PROBLEM 4.9 (HANDOUT 6, # 10)

(*Birthdays*). For a group of  $n$  people find the expected number of days of the year which are birthdays of exactly  $k$  people. (Assume 365 days and that all arrangements are equally probable.)

*SOLUTION.* Let  $\Omega$  denote the sample space and  $X: \Omega \rightarrow \mathbf{R}$ . ■

## PROBLEM 4.10 (HANDOUT 6, # 11)

(*Continuation*). Find the expected number of multiple birthdays. How large should  $n$  be to make this expectation exceed 1?

SOLUTION. ■

## PROBLEM 4.11 (HANDOUT 6, # 12)

(*The blood-testing problem*). A large number,  $N$ , of people are subject to a blood test. This can be administered in two ways, (i) Each person can be tested separately. In this case  $N$  tests are required, (ii) The blood samples of  $k$  people can be pooled and analyzed together. If the test is negative, this one test suffices for the  $k$  people. If the test is positive, each of the  $k$  persons must be tested separately, and in all  $k + 1$  tests are required for the  $k$  people. Assume the probability  $p$  that the test is positive is the same for all people and that people are stochastically independent.

- (b) What is the expected value of the number,  $X$ , of tests necessary under plan (ii)?
- (c) Find an equation for the value of  $k$  which will minimize the expected number of tests under the second plan. (Do not try numerical solutions.)

SOLUTION. ■

## PROBLEM 4.12 (HANDOUT 6, # 13)

(*Sample structure*). A population consists of  $r$  (classes whose sizes are in the proportion  $p_1 : p_2 : \cdots : p_r$ ). A random sample of size  $n$  is taken with replacement. Find the expected number of classes not represented in the sample.

SOLUTION. ■