

MA557 Homework 12

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December 9, 2015

PROBLEM 12.1

Let R be a Noetherian domain. Show that the following are equivalent:

- (i) R is a unique factorization domain
- (ii) every prime ideal of R of height one is principal
- (iii) R is normal with $\text{Cl}(R) = 0$.

Proof. (i) \implies (ii) Suppose R is a Noetherian domain. Let \mathfrak{p} be a height one prime. Then there exists at least one nonzero element $x \in \mathfrak{p}$. Let $x = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the factorization of x into irreducible (prime) elements of R . Set $p := p_i$ for any prime in the factorization of x . Then the ideal generated by p is a prime ideal contained in \mathfrak{p} , i.e., $\langle p \rangle \subset \mathfrak{p}$. But $\text{ht}(\mathfrak{p}) = 1$. Thus, $\langle p \rangle = \mathfrak{p}$.

For the following implication we need the theorem:

Theorem (Krull's Principal Ideal Theorem). *In a Noetherian ring, every minimal prime ideal of a principal ideal has height at most 1.*

(ii) \implies (i) Suppose that every height one prime ideal in R is principal. To show that R is a UFD, it suffices to show that every irreducible element p is a prime element, that is, $\langle p \rangle$ is a prime ideal. Let \mathfrak{p} be the minimal prime containing p . By Krull's principal ideal theorem, $\text{ht } \mathfrak{p} \leq 1$. Since \mathfrak{p} is principal, $\mathfrak{p} = \langle x \rangle$ for some $x \in \mathfrak{p}$. Thus, $p = xy$ for some $y \in R$. But p is prime hence, irreducible so either x or y is a unit. If x is a unit, then $\mathfrak{p} = R$, which is a contradiction. Thus, y must be a unit and we see that $\langle p \rangle = \langle xy \rangle = \mathfrak{p}$ is prime.

Now, for the following implications we need to know a couple of definitions: Let $D(R)$ denote the set of divisional fractional R -ideals and $F(R)$ denote the set of all principal fractional ideals. Then the *divisor class group* of R is the quotient $\text{Cl}(R) := D(R)/F(R)$.

We show (i) \iff (iii). That is, we show that a fractional ideal I is invertible if and only if it is principal and nonzero. Suppose I is principal. Then $I = xR$ so $(xR)((1/x)R) = R$ so I is invertible. Conversely, if I is nonzero and invertible with inverse J , then $1 = \sum x_i y_i$ for $x_i \in I$, $y_i \in J$. Fix a nonzero $x \in I$. Then $x = \sum x_i y_i x$. But $y_i x \in R$ since $x \in I$ and $y_i \in J$. Since R is a UFD, we have a division algorithm, so set $d := \gcd\{x_i x\}$ and $x' := \sum (y_i x/d) x_i$. Then $x = dx'$, so I is principal. ■

PROBLEM 12.2

Let R be a ring with total ring of quotients K , M an R -module, and

$$\mathrm{Tor}(M) = \{x \in M \mid ax = 0 \text{ for some non zero-divisor } a \text{ of } R\}.$$

The submodule $\mathrm{Tor}(M)$ is called the *torsion of M* , and M is called *torsion free* if $\mathrm{Tor}(M) = 0$. Show

- (a) $\mathrm{Tor}(M) = \ker(M \rightarrow K \otimes_R M)$
- (b) $M/\mathrm{Tor}(M)$ is torsion free.

Proof. (a) Let S denote the set of all regular elements of R and let $\varphi: R \rightarrow K$, where $K := S^{-1}R$, be the canonical localization map $a \mapsto a/1$. We show, by way of double inclusion, that $\mathrm{Tor}(M) = \ker \Phi$, where $\Phi: M \rightarrow K \otimes_R M$ is the canonical map $x \mapsto 1 \otimes x$. Note that this map, Φ , is well defined by the UMP of the tensor product (HW 2). Now let us show the containment $\mathrm{Tor}(M) \subset \ker \Phi$: Let $x \in \mathrm{Tor}(M)$, then x is a non-zero divisor of R such that $ax = 0$. Since a is a non-zero divisor, $a \in S$ so $a/1 = 0/1$ in K . Thus, we have

$$\Phi(xm) = 1 \otimes x = a/1 \otimes x = 0 \otimes x = 0,$$

so $x \in \ker \Phi$. Conversely, suppose that $x \in \ker(\Phi)$. By some theorem from the localization section¹ we have $K \otimes_R M \cong S^{-1}M$. Thus $1 \otimes x = 0$ implies that $x = 0$ in the localization $S^{-1}M$. This is true if and only if $ax = 0$ for some non-zero divisor a of R . Thus, $x \in \ker \Phi$ and equality holds.

(b) We prove the statement elementwise. Let $x := x' + \mathrm{Tor}(M)$ be in $M/\mathrm{Tor}(M)$. Then $ax = 0$ for some non zero-divisor $a \in R$. This implies that $ax' + \mathrm{Tor}(M) = 0 + \mathrm{Tor}(M)$ or $ax' \in \mathrm{Tor}(M)$. Then $b(ax') = 0$ for some non zero-divisor $b \in R$. Since both a and b are non-zero divisors, and $(ba)x' = 0$ then $x' \in \mathrm{Tor}(M)$. Thus, $\mathrm{Tor}(M) = 0$. ■

¹Sorry! I misplaced my notebook and I've been taking notes on sheets of computer paper so I hate going through the mess.

PROBLEM 12.3

Let R be a Dedekind domain and M a finitely generated R -module of rank r . Show that:

- (a) If M is torsion free then M is projective (hint: induct on r).
- (b) $M \cong \text{Tor}(M) \oplus P$ with P projective.
- (c) If $M \neq 0$ is projective then $M \cong R^{r-1} \oplus I$ with $I \neq 0$ an ideal.
- (d) If M is torsion (i.e., $M = \text{Tor}(M)$) then

$$M \cong R/I_1 \oplus \cdots \oplus R/I_n \quad \text{with} \quad I_1 \supset \cdots \supset I_n \neq 0$$

ideals (hint: for p_1, \dots, p_s the minimal primes of $\text{ann}(M)$ and $S = R \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s)$, show that $S^{-1}R$ is a PID).

Proof. (a) First, we shall prove the following useful lemma:

Lemma. *Let R be an integral domain and M an R -module. Then M is torsionfree if and only if $M_{\mathfrak{m}}$ is a torsionfree $R_{\mathfrak{m}}$ -module for every $\mathfrak{m} \in \mathfrak{m}\text{-Spec } R$.*

Proof of lemma. \implies Suppose that $(r/s)(x/t) = 0$ for some $s, t \in R \setminus \mathfrak{m}$, $r \in R$. Then there exists some $u \in R$ such that $urx = 0$. If $x \neq 0$, then $ur = 0$. But, since R is an integral domain it follows that $r = 0$. Thus, $\text{Tor}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = 0$.

\Leftarrow Conversely, if M is not torsionfree, there exists a nonzero $x \in M$ with $\text{ann}_R x \neq 0$. Let $\mathfrak{m} \in \mathfrak{m}\text{-Spec } R$ contain $\text{ann}_R x$. Then, localizing at \mathfrak{m} , we have $\bar{a}\bar{x} = 0$ in $M_{\mathfrak{m}}$. ♣

Now, by induction, let M , generated by x , be a torsionfree R module. Let $\mathfrak{m} \in \mathfrak{m}\text{-Spec } R$. Then $M_{\mathfrak{m}}$ is a torsionfree $R_{\mathfrak{m}}$ -module.

Scratch that, let us proceed from the following: Suppose M has rank $\text{rk} = 1$. Then $M \otimes_R K \cong K$ via the R -linear isomorphism φ . Hence, M embeds into K , i.e., $M' := \varphi(M \otimes_R 1)$ is a finitely generated R -submodule of K . Then we claim that M' is a fractional ideal.

Proof of claim. ♣

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