

MA 598 (Algebraic Geometry): Homework 1

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Problem 1.1 (5-lemma)

Given a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h & & \\
 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0,
 \end{array}$$

suppose f and h are isomorphisms. Prove that g is an isomorphism.

Solution.

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Problem 1.2 (Snake lemma)

Given the diagram with exact rows

$$\begin{array}{ccccccc}
 & \text{Ker } f & \longrightarrow & \text{Ker } g & \longrightarrow & \text{Ker } h & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Coker } f & \longrightarrow & \text{Coker } g & \longrightarrow & \text{Coker } h,
 \end{array}$$

show that the sequence

$$0 \longrightarrow \text{Ker } f \longrightarrow \text{Ker } g \longrightarrow \text{Ker } h \longrightarrow \text{Coker } f \longrightarrow \text{Coker } g \longrightarrow \text{Coker } h \longrightarrow 0$$

is exact.

Solution.

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Problem 1.3

Given an exact sequence of modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

fix any $M \in R\text{-Mod}$.

(a) Show that

$$0 \longrightarrow \text{Hom}(M, A) \longrightarrow \text{Hom}(M, B) \longrightarrow \text{Hom}(M, C)$$

is exact.

(b) Show that

$$0 \longrightarrow \text{Hom}(C, M) \longrightarrow \text{Hom}(B, M) \longrightarrow \text{Hom}(A, M)$$

is exact.

Solution.

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Problem 1.4

A short exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{p} 0$$

splits if there is a homomorphism $s : C \rightarrow B$ called the *splitting* such that $p \circ s = \text{id}_C$. In which case, we can put

$$0 \longrightarrow \text{Hom}(C, M) \longrightarrow \text{Hom}(B, M) \longrightarrow \text{Hom}(A, M) \longrightarrow 0$$

and

$$0 \longrightarrow \text{Hom}(M, A) \longrightarrow \text{Hom}(M, B) \longrightarrow \text{Hom}(M, C) \longrightarrow 0.$$

Solution.

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Problem 1.5

Find an example which shows that $\text{Hom}(-, M)$ is *not* exact.

Solution. Consider the following example: let $A = B = M = \mathbb{Z}$ and $C = \mathbb{Z}/2\mathbb{Z}$ then the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f=2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact, however Hom

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