MA 166: Quiz 3

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You have **15 minutes** to complete this quiz. You may work in groups, but you are not allowed to use any other resources.

Problem 1. Let $\mathbf{u} = \langle 6, 3, 1 \rangle$, $\mathbf{v} = \langle 0, 1, 2 \rangle$, and $\mathbf{w} = \langle 4, -2, 5 \rangle$.

(i) Find the scalar projection $\operatorname{comp}_{\mathbf{v}} \mathbf{w}$.

(ii) Find the projection $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$.

(iii) Find the cross product $\mathbf{v} \times \mathbf{w}$.

(iv) What is a vector orthogonal to $\mathbf{v}$ and $\mathbf{w}$?

(v) Find the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

(vi) Are the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ coplanar?

Problem 2. Find the area enclosed by the regions

(i) $y = x^3$, and $y = |x|$.

(ii) $y = e^x$, $y = e^2 x$, and $x = \ln 2$.

Solutions

Solution to Problem 1. (i) Recall the formula for the scalar projection of the vector $\mathbf{b}$ onto $\mathbf{a}$

$$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}. \tag{1}$$

All we need to do for this problem is to substitute $\mathbf{v}$ for $\mathbf{a}$ and $\mathbf{w}$ for $\mathbf{b}$ in equation (1) and we have

$$\begin{aligned}
\operatorname{comp}_{\mathbf{v}} \mathbf{w} &= \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|} \\
&= \frac{\langle 0, 1, 2\rangle \cdot \langle 4, -2, 5\rangle}{|\langle 0, 1, 2\rangle|} \\
&= \frac{0 \cdot 4 + 1(-2) + 2 \cdot 5}{\sqrt{0^2 + 1^2 + 2^2}} \\
&= \boxed{\frac{8}{\sqrt{5}}.}
\end{aligned}$$

(ii) The equation for the projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = (\operatorname{comp}_{\mathbf{a}} \mathbf{b}) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a}. \tag{2}$$

Then substituting $\mathbf{u}$ for $\mathbf{a}$ and $\mathbf{v}$ for $\mathbf{b}$, by equation 2, we have

$$\begin{aligned}
\operatorname{proj}_{\mathbf{u}} \mathbf{v} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \right) \mathbf{u} \\
&= \left(\frac{\langle 6, 3, 1\rangle \cdot \langle 0, 1, 2\rangle}{|\langle 6, 3, 1\rangle|^2} \right) \langle 6, 3, 1\rangle \\
&= \frac{6 \cdot 0 + 3 \cdot 1 + 1 \cdot 2}{(6^2 + 3^2 + 1)^2} \langle 6, 3, 1\rangle \\
&= \frac{3 + 2}{36 + 9 + 1} \langle 6, 3, 1\rangle \\
&= \boxed{\frac{5}{46} \langle 6, 3, 1\rangle.}
\end{aligned}$$

(iii) The cross product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle. \tag{3}$$

hen substituting $\mathbf{v}$ for $\mathbf{a}$ and $\mathbf{w}$ for $\mathbf{b}$, by equation 3, we have

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \langle 1 \cdot 5 - 2(-2), 2 \cdot 4 - 0 \cdot 5, 0(-2) - 1 \cdot 4 \rangle \\ &= \boxed{\langle 9, 8, -4 \rangle.} \end{aligned}$$

Of course, I would never try to memorize that horrible formula, but instead write the vectors as the rows of a matrix like so

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix}$$

and taking the determinant like so

$$\begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \mathbf{i} + \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix} \mathbf{j} + \begin{bmatrix} 0 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{k},$$

and again

$$(1 \cdot 5 - 2(-2))\mathbf{i} + (2 \cdot 4 - 0 \cdot 5)\mathbf{j} + (0(-2) - 1 \cdot 4)\mathbf{k} = \boxed{9\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}.}$$

(iv) Remember that the cross product of $\mathbf{a}$ and $\mathbf{b}$ has the property that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. In the case of $\mathbf{v}$ and $\mathbf{w}$ we can demonstrate that the the cross product $\mathbf{v} \times \mathbf{w}$ is in fact orthogonal to $\mathbf{v}$ and $\mathbf{w}$:

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= \langle 0, 1, 2 \rangle \cdot \langle 9, 8, -4 \rangle \\ &= 0 \cdot 9 + 1 \cdot 8 + 2(-4) \\ &= 8 - 8 \\ &= 0 \end{aligned} \qquad \begin{aligned} \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) &= \langle 4, -2, 5 \rangle \cdot \langle 9, 8, -4 \rangle \\ &= 4 \cdot 9 + (-2)8 + 5(-4) \\ &= 36 - 16 - 20 \\ &= 0. \end{aligned}$$

(v) Let's take our value for the cross product of $\mathbf{v}$ with $\mathbf{w}$ from part (iii) and dot it with $\mathbf{u}$ this gives us

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \langle 6, 3, 1\rangle \cdot \langle 9, 8, -4\rangle \\ &= 6 \cdot 9 + 3 \cdot 8 + 1(-4) \\ &= 74.\end{aligned}$$

(vi) Remember, three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar if they all lie on a plane. This means that if we can find a vector which is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ then it will be orthogonal to $\mathbf{c}$. For the case of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, from the previous problem we have $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 72 \neq 0$ so $\mathbf{u}$ is not orthogonal to $\mathbf{v} \times \mathbf{w}$ so it cannot lie in the same plane as $\mathbf{v}$ and $\mathbf{w}$ ∎

Solution to Problem 2. ((i)) A great way to begin this problem is by plotting the curves $y = |x|$ and $y = x^3$. As you can see from Figure 1 the curves $y = |x|$ and $y = x^3$ intersect at the points $x = 0$ and $x = 1$. Moreover, we see that on this interval, $0 \leq x \leq 1$, $y = x^3$ is always smaller than $y = |x|$ so evaluating the integral

$$\int_0^1 \left||x| - x^3\right| \, \mathrm{d}x$$

comes down to computing the integral

$$\int_0^1 |x| - x^3 \, \mathrm{d}x.$$

Now, remember the definition of the absolute value: If $f(x)$ then its absolute value is the piecewise defined function

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0. \end{cases} \tag{4}$$

Since x is ≥ 0 on $0 \leq x \leq 1$, $y = |x| = x$ on the interval $0 \leq x \leq 1$ so

$$\begin{aligned}
\int_0^1 |x| - x^3 \,\mathrm{d}x &= \int_0^1 x - x^3 \,\mathrm{d}x \\
&= \frac{x^2}{2} - \frac{x^4}{4}\bigg|_0^1 \\
&= \frac{1^4}{2} - \frac{1^4}{4} - \left(\frac{0^2}{2} - \frac{0^4}{4}\right) \\
&= \boxed{\frac{1}{4}.}
\end{aligned}$$

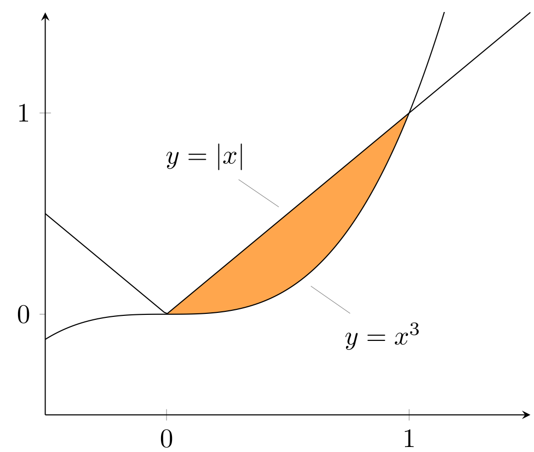

Figure 1: The area enclosed by the curves $y = |x|$ and $y = x^3$ at their points of intersection.

((ii)) As before, it helps to graph these curves (see Figure 2). From the figure we see that $y = e^{2x}$ and e^x intersect at $x = 0$ where $e^{2\cdot 0} = 1 = e^0$ and they both intersect the vertical line $x = \ln 2$ at, well, obviously at $x = \ln 2$. So our integral will be

$$\int_0^{\ln 2} \left|e^{2x} - e^x\right| \mathrm{d}x.$$

Like before, note that $e^{2x} > e^x$ for any $0 \leq x \leq \ln 2$ so we can forget about the absolute value and evaluate the integral

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\begin{aligned}
\int_0^{\ln 2} e^{2x} - e^x \,\mathrm{d}x &= \frac{e^{2x}}{2} - e^x \Big|_0^{\ln 2} \\
&= \frac{e^{2\ln 2}}{2} - e^{\ln 2} - \left(\frac{e^{2\cdot 0}}{2} - e^0\right) \\
&= \frac{e^{\ln 2^2}}{2} - 2 - \left(\frac{1}{2} - 1\right) \\
&= \frac{2^2}{2} - 2 + \frac{1}{2} \\
&= 2 - 2 + \frac{1}{2} \\
&= \boxed{\frac{1}{2}.}
\end{aligned}
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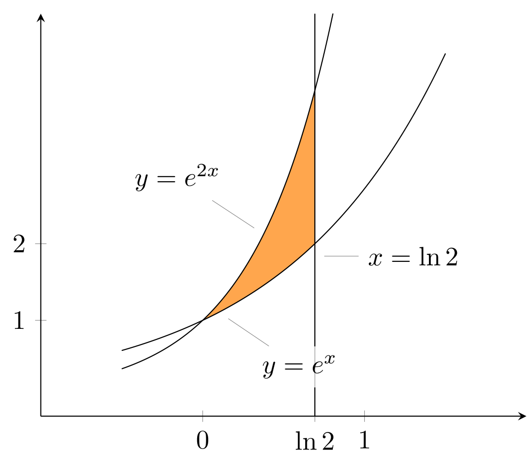

Figure 2: The area enclosed by the curves $y = e^x$, $y = e^{2x}$, and $x = \ln 2$ at their points of intersection.