MRC 2016: Character Varieties

Tropicalization of Character Varieties

Tropical Geometry Group

Snowbird, 2016

The main task of this group was to explore the tropical geometry arising from the tropicalization of character varieties of some finitely generated groups Γ into $\mathrm{SL}_2\,\mathbb{C}$ and $\mathrm{PSL}_2\,\mathbb{C}$.

The main task of this group was to explore the tropical geometry arising from the tropicalization of character varieties of some finitely generated groups Γ into $\mathrm{SL}_2\,\mathbb{C}$ and $\mathrm{PSL}_2\,\mathbb{C}$.
Using results we got from `Mathematica` and `GFan`, we conjectured that, at least in the case of free groups, the
$\mathrm{Trop}(\mathfrak{X}(F_n, \mathrm{SL}_2\,\mathbb{C})) = \mathrm{Trop}(\mathfrak{X}(F_n, \mathrm{PSL}_2\,\mathbb{C})$.
Additionally, Charlie Katerba

$S_{2,g}$ - a recursive approach

Definition

$S_{2,g}$ is the number of words over an alphabet of size g such that no letter appears (considering absolute multiplicities) more than twice.

$S_{2,g}$ - a recursive approach

Definition

$S_{2,g}$ is the number of words over an alphabet of size g such that no letter appears (considering absolute multiplicities) more than twice.

Definition

$S_{2,g}^{d,l,u}$ is the number of words counted by $S_{2,g}$ which have exactly d exponents of ± 2, have *reduced* length l (exponents are ignored), and use exactly u distinct letters.

$S_{2,g}$ - a recursive approach

Definition

$S_{2,g}$ is the number of words over an alphabet of size g such that no letter appears (considering absolute multiplicities) more than twice.

Definition

$S_{2,g}^{d,l,u}$ is the number of words counted by $S_{2,g}$ which have exactly d exponents of ± 2, have *reduced* length l (exponents are ignored), and use exactly u distinct letters.

$$S_{2,g} = \sum_{d=0}^{g} \sum_{l=0}^{2g} \sum_{u=0}^{g} S_{2,g}^{d,l,u},$$

and

$$S_{2,0}^{0,0,0} = 1.$$

The recursion

$$
\begin{aligned}
S_{2,g}^{d,l,u} =& S_{2,g-1}^{d,l,u} \\
&+ \sum_{j=0}^{u-1} \binom{g-1}{j} \cdot j! \cdot 2^j \cdot (l-2j) \cdot \left(S_{2,g-1-j}^{d,l-1-2j,u-j-1} + S_{2,g-1-j}^{d-1,l-1-2j,u-j-1} \right) \cdot 2 \\
&+ \sum_{j=0}^{u-1} \sum_{k=0}^{u-1-j} \binom{g-1}{j+k} \cdot (j+k)! \cdot 2^{j+k} \cdot \frac{(l-2(j+k)-1)\cdot(l-2(j+k)-2)}{2} \cdot S_{2,g-1-j-k}^{d,l-2(j+k)-2,u-1-j-k} \cdot 4 \\
&+ \sum_{j=0}^{0} \sum_{k=1}^{u-1-j} \binom{g-1}{j+k} \cdot (j+k)! \cdot 2^{j+k} \cdot \frac{(l-2(j+k)-1)\cdot 2}{2} \cdot S_{2,g-1-j-k}^{d,l-2(j+k)-2,u-1-j-k} \cdot 4 \\
&+ \sum_{j=1}^{u-1} \sum_{k=0}^{0} \binom{g-1}{j+k} \cdot (j+k)! \cdot 2^{j+k} \cdot \frac{(l-2(j+k)-1)\cdot 2}{2} \cdot S_{2,g-1-j-k}^{d,l-2(j+k)-2,u-1-j-k} \cdot 4 \\
&- \sum_{j=1}^{u-1} \sum_{k=1}^{u-1-j} \binom{g-1}{j+k} \cdot (j+k)! \cdot 2^{j+k} \cdot \frac{(l-2(j+k)-1)\cdot 2}{2} \cdot S_{2,g-1-j-k}^{d,l-2(j+k)-2,u-1-j-k} \cdot 4 \\
&+ \sum_{j=1}^{u-1} \sum_{k=0}^{u-1-j} \binom{g-1}{j+k} \cdot (j+k)! \cdot 2^{j+k} \cdot \frac{(l-2(j+k)-1)\cdot 2(j-1)}{2} \cdot S_{2,g-1-j-k}^{d,l-2(j+k)-2,u-1-j-k} \cdot 4 \\
&+ \sum_{j=1}^{u-1} \sum_{k=1}^{u-1-j} \binom{g-1}{j+k} \cdot (j+k)! \cdot 2^{j+k} \cdot \frac{(l-2(j+k)-1)\cdot 2)}{2} \cdot S_{2,g-1-j-k}^{d,l-2(j+k)-2,u-1-j-k} \cdot 4
\end{aligned}
$$

Some calculations

This has been implemented, with the following results

$$
\begin{aligned}
S_{2,0} &= 1 \\
S_{2,1} &= 5 \\
S_{2,2} &= 105 \\
S_{2,3} &= 6061 \\
S_{2,4} &= 668753 \\
&\vdots
\end{aligned}
$$

Conclusion: it is impractical to consider nave generators when examining representations F_n to $\mathrm{SL}_2\,\mathbb{C}$.

References I

[]

Melissa L. Macasieb, Kathleen L. Petersen, and Ronald M. van Luijk.
On character varieties of two-bridge knot groups.
2009.

Qingchun Ren, Steven V Sam, and Bernd Sturmfels.
Tropicalization of classical moduli spaces.
2013.

Qingchun Ren, Kristin Shaw, and Bernd Sturmfels.
Tropicalization of del pezzo surfaces, 2014.