

# MA553: Qual Preparation

Carlos Salinas

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## 1 Ulrich

### 1.1 Ulrich: Winter 2002

**Problem 1.** Let  $G$  be a group and  $H$  a subgroup of finite index. Show that there exists a normal subgroup  $N$  of  $G$  of finite index with  $N \subset H$ .

**Solution.** ►

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**Problem 2.** Show that every group of order 992 ( $= 2^5 \cdot 31$ ) is solvable.

**Solution.** ►

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**Problem 3.** Let  $G$  be a group of order 56 with a normal 2-Sylow subgroup  $Q$ , and let  $P$  be a 7-Sylow subgroup of  $G$ . Show that either  $G \simeq P \times Q$  or  $Q \simeq \mathbb{Z}/(2) \times \mathbb{Z}/(2) \times \mathbb{Z}/(2)$ .

[Hint:  $P$  acts on  $Q \setminus \{e\}$  via conjugation. Show that this action is either trivial or transitive.]

**Solution.** ►

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**Problem 4.** Let  $R$  be a commutative ring and  $\text{Rad}(R)$  the intersection of all maximal ideals of  $R$ .

- (a) Let  $a \in R$ . Show that  $a \in \text{Rad}(R)$  if and only if  $1 + ab$  is a unit for every  $b \in R$ .
- (b) Let  $R$  be a domain and  $R[X]$  the polynomial ring over  $R$ . Deduce that  $\text{Rad}(R[X]) = 0$ .

**Solution.** ►

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**Problem 5.** Let  $R$  be a unique factorization domain and  $P$  a prime ideal of  $R[X]$  with  $P \cap R = 0$ .

- (a) Let  $n$  be the smallest possible degree of a nonzero polynomial in  $P$ . Show that  $P$  contains a primitive polynomial  $f$  of degree  $n$ .
- (b) Show that  $P$  is the principal ideal generated by  $f$ .

**Solution.** ►

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**Problem 6.** Let  $k$  be a field of characteristic zero. assume that every polynomial in  $k[X]$  of odd degree and every polynomial in  $k[X]$  of degree two has a root in  $k$ . Show that  $k$  is algebraically closed.

**Solution.** ►

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**Problem 7.** Let  $k \subset K$  be a finite Galois extension with Galois group  $\text{Gal}(K/k)$ , let  $L$  be a field with  $k \subset L \subset K$ , and set  $H = \{ \sigma \in \text{Gal}(K/k) : \sigma(L) = L \}$ .

- (a) Show that  $H$  is the normalizer of  $\text{Gal}(K/L)$  in  $\text{Gal}(K/k)$ .
- (b) Describe the group  $H/\text{Gal}(K/L)$  as an automorphism group.

**Solution.** ►

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