Spring 2016 Notes

Carlos Salinas

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Chapter 1

Notes from Wheeden and Zygmund

1.1 Preliminaries

Here are some results if think were useful to look over.

If $\mathcal{F}$ is a countable collection of sets, it will be called a *sequence of sets* and denoted $\mathcal{F} = \{E_k : k = 1, ...\}$. The corresponding union and intersection will be written $\bigcup_k E_k$ and $\bigcap_k E_k$. A sequence of $\{E_k\}$ of sets is said to *increase* to $\bigcup_k E_k$ if $E_k \subset E_{k+1}$ and to *decrease* to $\bigcap_k E_k$ to denote these two possibilities. If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets, we define

$$\overline{\lim}\, E_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k, \qquad \underline{\lim}\, E_k = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k, \tag{1.1}$$

noting that the sets $U_j = \bigcup_{k=j}^{\infty} E_k$

Chapter 2

Hatcher Algebraic Topology Notes

2.1 The Fundamental Group

The van Kampen Theorem

Suppose a space X is decomposed as the union of a collection of pathconnected open subsets A_α, each of which contains the basepoint $x_0 \in X$. By the remarks of the preceding paragraph, the homomorphism $j_\alpha\colon \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \hookrightarrow X$ extend to a homomorphism $\Phi\colon *_\alpha\, \pi_1(A_\alpha) \to \pi_1(X)$. The van Kapmen theorem will say that Φ is very often surjective, but we can expect Φ to have a nontrivial kernel in general. For if $i_{\alpha\beta}\colon \pi_1(A_\alpha \cap B_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$, then $j_\alpha \circ i_{\alpha\beta} = j_\beta \circ i_{\alpha\beta}$, both these compositions being of the form $i_{\alpha\beta}(\omega) i_{\alpha\beta}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$. Van Kampen's theorem asserts that under fairly broad hypotheses this gives a full description of Φ

Theorem 1 (Hatcher, 1.20, p. 43)**.** *If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi\colon *_\alpha\, \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$.*

Example 1 (1.21: Wedge Sums)**.** In chapter 0 of Hatcher, we defined the wedge sum $\bigvee_\alpha X_\alpha$ of a collection of spaces X_α with basepoints $x_\alpha \in X_\alpha$ to be the quotient space of the disjoint union $\bigsqcup_\alpha X_\alpha$ in which all the basepoints x_α are identified to a single point. If each x_α is a deformation retract of an open neighborhood U_α in X_α, then X_α is a deformation retract of its open neighborhood $A_\alpha = X_\alpha \vee \bigvee_{\beta \neq \alpha} U_\beta$. The intersection of two or more distinct A_α's is $\bigvee_\alpha U_\alpha$, which deformation retracts to a point. Van Kampen's theorem then implies that $\Phi\colon *_\alpha\, \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$ is an isomorphism.

Thus for a wedge sum $\bigvee_\alpha S^1_\alpha$ of circles, $\pi_1\big(\bigvee_\alpha S^1_\alpha\big)$ is a free group, the free product of copies of $\mathbf{Z}$, one for each circle S^1_α. In particular, $\pi_1\big(S^1 \vee S^1\big)$ is the free group $\mathbf{Z} * \mathbf{Z}$, as in the example at the beginning of this section.

2.2 Homology

Simplicial and Singular Homology

Δ-complexes

The idea of the Δ-complex generalizes the construction of a topological space via the quotient of some triangularization of a polygon in $\mathbf{R}^n$. The n-dimensional analogue of the triangle is called the *n-simplex.* This is the smallest convex set in a Euclidean space $\mathbf{R}^m$ containing $n+1$ points $v_0, ..., v_n$ that do not lie in a less than n dimensional hyperplane. An equivalent condition is that the difference vectors $v_1 - v_0, ..., v_n - v_0$ are linearly independent. The points v_i are the *vertices* of the simplex, and the simplex itself is denoted $[v_0, ..., v_n]$. For example, there is a standard n-simplex

$$\Delta^n = \left\{ (t_0, ..., t_n) \in \mathbf{R}^n \;\middle|\; \sum_i t_i \text{ and } t_i \geq 0 \text{ for all } i \right\}$$

whose vertices are the unit vectors along the coordinate axes. For the purposes of homology, it is important that we keep track of the ordering on the vertices v_i, so an 'n-simplex' will always mean an 'n-simplex with an ordering on its vertices.' As a consequence, there is a natural ordering on the edges $[v_i, v_j]$ according to increasing subscripts.[1] Specifying the ordering of the vertices also determines a canonical linear homeomorphism from the standard n-simplex Δ^n onto any other n-simplex $[v_0, ..., v_n]$, preserving the order of the vertices, namely, $(t_0, ..., t_n) \mapsto \sum_i t_i v_i$. The coefficients t_i are *barycentric coordinates* of the point $\sum_i t_i v_i$ in $[v_0, ..., v_i]$.

If we delete one of the $n+1$ vertices of the n-simplex $[v_0, ..., v_n]$, then the remaining n-vertices span an $(n-1)$-simplex called a *face* of $[v_0, ..., v_n]$. We adopt the following convention

> The vertices of a face, or of any complex spanned by a subset of the vertices, will always be ordered according to their order in the larger simplex.

The union of the faces of Δ^n is the *boundary* of Δ^n, written
partialΔ^n. The *open simplex* $\Delta^{n\,\circ}$ is $\Delta^n \smallsetminus \partial\Delta^n$.

A Δ-complex structure on a space X is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$ with n-depending on the index of α such that:

(i) The restriction $\sigma_\alpha \restriction \Delta^{n\,\circ}$ is injective, and each point of X is in the image of exactly one such restriction.

(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

(iii) A set $A \subset X$is open iff ${\sigma_\alpha}^{-1}(A)$ is open in Δ^n for each σ_α.

Among other things, this last condition rules out trivialities like regarding all of the points in X as individual vertices.

A consequence of (iii) is that X can be built as a quotient space of a collection of disjoint simplices Δ^n_α, one for each $\sigma_\alpha \colon \Delta^n \to X$, the quotient space obtained by identifying each face of

[1] I'm not sure what Hatcher means here, unless he is choosing the natural ordering on the indices $I \subset \mathbf{N}$, i.e., the ordering $1 < 2 < \cdots$

Δ_α^n with the Δ_β^{n-1} corresponding to the restriction σ_β of σ_α to the face in question, as in (ii). One can think of building the quotient space inductively: starting with a discrete set of vertices, then attaching edges to these to produce a graph, the attaching 2-simplices to the graph, and so on. From this viewpoint we see that the data specifying a Δ-complex can be described in a purely combinatorial way as collections of n-simplices Δ_α^n for each n together with functions associating to each face of each n-simplex Δ_α^n an $(n-1)$-simplex Δ_β^{n-1}.

More generally, Δ-complexes can be built from collections of disjoint simplices by identifying varies subsimplices spanned by a subsets of the vertices, where the identifications are performed using the canonical linear homeomorphism that preserve the ordering of the vertices.

Thinking of a Δ-complex X as the quotient space of a collection of disjoint simplices, it's not hard to see that X must be a Hausdorff space. Indeed, if $x, y \in X$ they lie in the image of the same simplex $\operatorname{Im} \sigma_\alpha$ we may take their preimage ${\sigma_\alpha}^{-1}(x)$ and ${\sigma_\alpha}^{-1}(y)$ and find disjoint neighborhoods U_x and U_y containing these subsets of Δ^k. Then $\sigma_\alpha(U_x)$ and $\sigma_\alpha(U_y)$ are disjoint and contain x and y.

2.3 Cohomology