Each problem is worth 14 points

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

1. Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,
$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

2. Let X be a topological space and let A be a dense subset of X. Let Y be a Hausdorff space, and let $g, h : X \to Y$ be continuous functions which agree on A. Prove that $g = h$.

3. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let G_f (called the *graph* of f) be the subspace $\{\, (x, f(x)) \mid x \in X \,\}$ of $X \times Y$. Prove that if Y is Hausdorff then G_f is closed.

4. Let X be a topological space and let $f, g : X \to \mathbb{R}$ be continuous. Define $h : X \to \mathbb{R}$ by
$$h(x) = \min\{f(x), g(x)\}$$
Use the pasting lemma to prove that h is continuous. (You will not get full credit for any other method.)

5. Let X and Y be topological spaces and let $f : X \to Y$ be a function with the property that
$$f(\overline{A}) \subset \overline{f(A)}$$
for all subsets A of X.

   Prove that f is continuous.

6. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. **Prove** that
$$f(\overline{A}) \subset \overline{f(A)}$$
for all subsets A of X.

7. Let X be any topological space and let Y be a Hausdorff space. Let $f, g : X \to Y$ be continuous functions.

   Prove that the set $\{x \in X \mid f(x) = g(x)\}$ is closed.

8. Let X be a topological space and A a subset of X. Suppose that
$$A \subset \overline{X - \overline{A}}.$$

   Prove that $\overline{A}$ does not contain any nonempty open set.

9. Let X be a topological space with a countable basis. **Prove** that every open cover of X has a countable subcover.

10. Let X_α be an infinite family of topological spaces.

    (a) (6 points) Define the product topology on
    $$\prod_\alpha X_\alpha$$

    (b) (8 points) For each α, let A_α be a subspace of X_α. Prove that
    $$\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}.$$

11. Suppose that we are given an indexing set A, and for each $\alpha \in A$ a topological space X_α.

    Suppose also that for each $\alpha \in A$ we are given a point $b_\alpha \in X_\alpha$.

    Let $Y = \prod_\alpha X_\alpha$, with the product topology. Let $\pi_\alpha : Y \to X_\alpha$ be the projection.

    Prove that the set
    $$S = \{y \in Y \mid \pi_\alpha y = b_\alpha \text{ except for finitely many } \alpha\}$$
    is dense in Y (that is, its closure is Y).

12. Let X be the Cartesian product
    $$\prod_{i=1}^{\infty} \mathbb{R}$$
    with the **box topology** (recall that a basis for this topology consists of all sets of the form $\prod_{i=1}^{\infty} U_i$, where each U_i is open in $\mathbb{R}$).

    Let
    $$f : \mathbb{R} \to X$$
    be the function which takes t to $(t, t, t, \ldots)$

    Prove that f is not continuous.

13. **Prove** that the countable product
    $$\prod_{n=1}^{\infty} \mathbb{R}$$
    (with the product topology) has the following property: there is a countable family $\mathfrak{F}$ of neighborhoods of the point
    $$\mathbf{0} = (0, 0, \ldots)$$
    such that for every neighborhood V of $\mathbf{0}$ there is a $U \in \mathfrak{F}$ with $U \subset V$.

    Note: the book proves that $\prod_{n=1}^{\infty} \mathbb{R}$ is a metric space, but you may not use this in your proof. Use the definition of the product topology.

14. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's.

    For each $n \geq 1$, let $y_n \in Y$ be the element $(1, 1, \ldots, 1, 0, 0, \ldots)$, with n 1's at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1's. **Prove** that the set $\{y_n\}_{n \geq 1} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

15. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's.

    Let A be the subset of Y consisting of sequences with only a finite number of 1's. Is A closed? **Prove or disprove.**

16. Let Y be a topological space.

    Let X be a set and let $f : X \to Y$ be a function. Give X the topology in which the open sets are the sets $f^{-1}(V)$ with V open in Y (you do **not** have to verify that this is a topology).

    Let $a \in X$ and let B be a closed set in X not containing a.

    Prove that $f(a)$ is not in the closure of $f(B)$.

17. Let $f : X \to Y$ be a function that takes closed sets to closed sets. Let $y \in Y$ and let U be an open set containing $f^{-1}(y)$.

    Prove that there is an open set V containing y such that $f^{-1}(V)$ is contained in U.

18. Let X be a topological space with an equivalence relation $\sim$. Suppose that the quotient space $X/\sim$ is Hausdorff.

    Prove that the set
    $$S = \{(x, y) \in X \times X \mid x \sim y\}$$
    is a closed subset of $X \times X$.

19. Let $p : X \to Y$ be a quotient map.

    Let us say that a subset S of X is *saturated* if it has the form $p^{-1}(T)$ for some subset T of Y.

    Suppose that for every $y \in Y$ and every open neighborhood U of $p^{-1}(y)$ there is a saturated open set V with $p^{-1}(y) \subset V \subset U$.

    Prove that p takes closed sets to closed sets.

20. Let X be a topological space, let D be a connected subset of X, and let $\{E_\alpha\}$ be a collection of connected subsets of X.

    Prove that if $D \cap E_\alpha \neq \emptyset$ for all α, then $D \cup (\bigcup_\alpha E_\alpha)$ is connected.

21. Let X and Y be connected. Prove that $X \times Y$ is connected.

22. For any space X, let us say that two points are "inseparable" if there is no separation $X = U \cup V$ into disjoint open sets such that $x \in U$ and $y \in V$.

    Write $x \sim y$ if x and y are inseparable. Then $\sim$ is an equivalence relation (you don't have to prove this).

    Now suppose that X is locally connected (this means that for every point x and every open neighborhood U of x, there is a connected open neighborhood V of x contained in U).

    Prove that each equivalence class of the relation $\sim$ is connected.

23. Let X be a topological space.

    Let $A \subset X$ be connected.

    Prove $\bar{A}$ is connected.

24. Let $X_1, X_2, \ldots$ be topological spaces.

    Suppose that $\prod_{n=1}^{\infty} X_n$ is locally connected.

    Prove that all but finitely many X_n are connected.

25. Let X be a connected space and let $f : X \to Y$ be a function which is continuous and onto. **Prove** that Y is connected. (This is a theorem in Munkres—prove it from the definitions).

26. Given:

    - $p : X \to Y$ is a quotient map.
    - Y is connected.
    - For every $y \in Y$, the set $p^{-1}(\{y\})$ is connected.

    Prove that X is connected.

27. Let A be a subset of $\mathbb{R}^2$ which is homeomorphic to the open unit interval $(0, 1)$.

    Prove that A does not contain a nonempty set which is open in $\mathbb{R}^2$.

28. Let X be a connected space. Let $\mathcal{U}$ be an open covering of X and let U be a nonempty set in $\mathcal{U}$. Say that a set V in $\mathcal{U}$ is *reachable from* U if there is a sequence

    $$U = U_1, U_2, \ldots, U_n = V$$

    of sets in $\mathcal{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for each i from 1 to $n-1$.

    Prove that every nonempty V in $\mathcal{U}$ is reachable from U.

29. Suppose that X is connected and every point of X has a path-connected open neighborhood. **Prove** that X is path-connected.

30. Let X be a topological space and let $f, g : X \to [0, 1]$ be continuous functions.

    Suppose that X is connected and f is onto.

    Prove that there must be a point $x \in X$ with $f(x) = g(x)$.