MA571 Homework 11

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Problem 11.1 (Munkres §53, Ex. 7(abcd))

Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]

(d) Show that $\pi_1(G, x_0)$ is Abelian.

Proof. For part (a) we need to show that the operation (0) $\otimes$ is associative, (1) $\Omega(G, x_0)$ is closed under $\otimes$, (2) $\Omega(G, x_0)$ contains an identity element e and (3) for every $f \in \Omega(G, x_0)$ there exists an element $\bar{f} \in \Omega(G, x_0)$ such that $f \otimes \bar{f} = \bar{f} \otimes f = e$. We shall proceed in order: (0) Let $f, g, h \in \Omega(G, x_0)$. Then $(f \otimes g) \otimes h = f \otimes (g \otimes f)$ since the multiplication $\cdot$ is associative in G, i.e., since given $t \in I$ we have $(f(t) \cdot g(t)) \cdot h(t) = f(t) \cdot (g(t) \cdot h(t))$, in particular this holds for all $\in I$. (1) Let f and g be loops at x_0 then $f \otimes g = f(s) \cdot g(s)$ ■

Problem 11.2 ((A))

Prove Proposition F from the note on the Fundamental Group of the Circle.

Proof. ■

Problem 11.3 ((B))

Prove Lemma G from the note on the Fundamental Group of the Circle. (Hint: one way to do this is to use the fact, which you don't have to prove, that if $\sim$ is the equivalence relation on $[a, a+1]$ which identifies a and $a+1$ then the restriction of p induces a homeomorphism $[a, a+1]/\sim \to S^1$.)

Proof. ∎

Problem 11.4 ((C))

Show that for every point $x \in S^n$ the space $S^n - x$ is homeomorphic to $\mathbf{R}^n$. You may use the fact, shown in Step 1 of the proof of Theorem 59.3, that S^n with the *north pole* removed is homeomorphic to $\mathbf{R}^n$. (Hint: linear algebra.)

Proof. ■

Problem 11.5 ((D))

Show that every loop in S^n which is not onto is path-homotopic to a constant path. (Hint: use Problem C).

Proof. ∎

PROBLEM 11.6 ((E))

Let X be a topological space and let $A \subset X$ be a deformation retract. In the space X/A, the set A is a point (because it is an equivalence class). Show that this point is a deformation retract of X/A. (Hint: use p. 289 # 9.)

Proof. ∎