Instructor: Tatsunari Watanabe **Name:**____________________.
TA: Carlos Salinas

MA 26500-215 Quiz 4

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1. (12 points) List all the properties a set V must satisfy in order to be a vector space.

   (*Hint:* there are eight of them.)

   Solution: The axioms that a *real* vector space (i.e., a vector space with scalar in $\mathbb{R}$, because you can have *complex* vector spaces with scalars in the complex or *imaginary* numbers $\mathbb{C}$) must satisfy are

   1. associativity of addition: $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$;
   2. commutativity of addition: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{u}$;
   3. additivie identity: there exists some element (called a *zero*) $\mathbf{0} \in V$ such that for any $\mathbf{v} \in V$, $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$.
   4. additive inverse: for every $\mathbf{v} \in V$ there exist an inverse element $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. **Note** the vector $\mathbf{w}$ is not always $-\mathbf{v}$, it just so happens that in the linear algebra we cover it will almost always be $-\mathbf{v}$;
   5. compatibility of scalar multiplication: if $a, b \in \mathbb{R}$, $a(b\mathbf{v}) = (ab)\mathbf{v}$ for all $\mathbf{v} \in V$;
   6. multiplicative identity: there exist element $1 \in \mathbb{R}$ such that $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$;
   7. distributivity with respect to vector addition: if $a \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in V$ then $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$;
   8. distributivity with respect to scalar addition: if $a, b \in \mathbb{R}$, $\mathbf{v} \in V$ then $(a + b)\mathbf{v} = a\mathbf{v} + a\mathbf{w}$.

2. Which of the following subsets W of $\mathbb{R}^3$ are subspaces?
   (a) (2 points) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$.
   (b) (2 points) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}$.

(c) (2 points) $W = \left\{ \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] : x^2 + y^2 + z^2 = 1 \right\}$.

(d) (2 points) $W = \left\{ \left[\begin{smallmatrix} x+2y+3z \\ z \\ 0 \end{smallmatrix}\right] : x, y, z \in \mathbb{R} \right\}$.

(By now you should have a feel of what a vector spaces is so you do not need to check all of the conditions; but for those that are not subspaces, give me a reason, e.g., the set is not closed under addition, multiplication by scalars, etc.)

Solution: The set in (a) is not subspace and this is easy to see because $[-1, 0, 1]$ is in the set, but $-1[-1, 0, 1] = [1, 0, -1]$ is not since $1 \nleq 0 \nleq -1$.

The set in (b) is a vector space because, first, it is nonempty since it contains $[0, 0, 0]$ and if $c \in \mathbb{R}$, $[x_1, y_1, z_1], [x_2, y_2, z_2] \in W$ then

$$c \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

and both satisfy

$$\begin{aligned} cx_1 + cy_1 + cz_1 &= c(x_1 + y_1 + z_1) \\ &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 + y_1 + y_2 + z_1 + z_2 &= (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) \\ &= 0. \end{aligned}$$

The set in (c) is the sphere. It is clearly not a subspace since $[1, 0, 0], [0, 1, 0] \in W$, but $[1, 0, 0] + [0, 1, 0] = [1, 1, 0]$ has norm $1^2 + 1^2 + 0^2 = 2 \neq 0$ hence $[1, 1, 0] \notin W$.

Lastly, the set in (d) is a subspace because, first, it is nonempty since $[0, 0, 0] \in W$ and if $c \in \mathbb{R}$, $[x_1 + 2y_1 + 3z_1, z_1, 0], [x_2 + 2y_2 + 3z_2, z_2, 0] \in W$ then

$$\begin{aligned} c \begin{bmatrix} x_1 + 2y_1 + 3z_1 \\ z_1 \\ 0 \end{bmatrix} &= \begin{bmatrix} c(x_1 + 2y_1 + 3z_1) \\ cz_1 \\ c \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} cx_1 + 2cy_1 + 3cz_1 \\ cz_1 \\ 0 \end{bmatrix} && (\clubsuit) \\ \begin{bmatrix} x_1 + 2y_1 + 3z_1 \\ z_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 + 2y_2 + 3z_2 \\ z_2 \\ 0 \end{bmatrix} &= \begin{bmatrix} x_1 + 2y_1 + 3z_1 + x_2 + 2y_2 + 3z_2 \\ z_1 + z_2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (x_1 + x_2) + 2(y_1 + y_2) + 3(z_1 + z_2) \\ (z_1 + z_2) \\ 0 \end{bmatrix} && (\spadesuit) \end{aligned}$$

Putting $x = cx_1, y = cy_1, z = cz_1$ for the entries in (♣), we see that the vector in (♣) indeed belongs to W since it has the shape $[x + 2y + 3z, z, 0]$. Putting $x = x_1 + x_2, y = y_1 + y_2, z = z_1 + z_2$ we see that the vector in (♠) also belongs to W. Thus, W is a subspace.