MA 544: Homework 8

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Problem 8.1 (Wheeden & Zygmund §5, Ex. 2)

Show that the conclusion of (5.32) are not true without the assumption that $\varphi \in L(E)$. [In part (ii), for example, take $f_k = \chi_{(k,\infty)}$.]

Proof. (ii) Following the hint, consider the family of decreasing functions $f_k = \chi_{(k,\infty)}$ on $\mathbf{R}$ which converge pointwise to 0. Since $\int_{\mathbf{R}} f_k = \int_{\mathbf{R}} \chi_{(k,\infty)} = |(k,\infty)| = \infty$ for all k, the sequence of integrals $\int_{\mathbf{R}} f_k \to \infty$, but $\int_{\mathbf{R}} 0 = 0$.

For part (i) we may again consider the indicator function $\chi_{(k,\infty)}$ and define $f_k := -\chi_{(k,\infty)}$. Then $f_k \nearrow 0$, but $\int_{\mathbf{R}} f_k = -\int_{\mathbf{R}} \chi_{(k,\infty)} = -\infty$ for all k. ■

Problem 8.2 (Wheeden & Zygmund §5, Ex. 4)

If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and $\int_0^1 x^k f(x)\,dx \to 0$.

Proof. Since x^k is a polynomial and therefore continuous, x^k is measurable as a consequence of Theorem 4.3. Moreover, since $\left|x^k\right| \leq 1$ for all k, by Theorem 5.30, $x^k f \in L(0,1)$.

Now, by the Stone–Weistraß approximation theorem, given $\varepsilon > 0$ there exists a polynomial p such that $|p - f| < \varepsilon$ for every $x \in [0,1]$. Hence, we have

$$\begin{aligned}\left|x^k f\right| &= \left|x^k f + x^k p - x^k p\right| \\ &= \left|x^k (f-p)\right| + \left|x^k p\right| \\ &= \left|x^k\right| (\varepsilon + |p|)\end{aligned}$$

which goes to 0 as $k \to \infty$ a.e. except at $x = 1$. Thus, by Lebesgue's dominated convergence theorem, since $x^k f \to 0$ a.e. on $[0,1]$ and $\left|x^k f\right| \leq |f|$ a.e. in $[0,1]$ for all k, we have $\int_0^1 x^k f \to 0$ as desired. ■

Problem 8.3 (Wheeden & Zygmund §5, Ex. 6)

Let $f(x,y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $\partial f(x,y)/\partial x$ is a bounded function of (x,y). Show that $\partial f(x,y)/\partial x$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy.$$

Proof. First we will show that $\partial f/\partial x$ is measurable as a function of y. Since $\int_0^1 f(x,y)\,dy$ exists for every x, by Theorem 5.1 $f(x,y)$ is measurable as a function of y so $\partial f/\partial x$ is measurable by Theorem 4.12 since it is limit of the sequence

$$f_n(x,y) \coloneqq \frac{f(x+(1/n),y) - f(x,y)}{1/n} \tag{8.1}$$

which is a sum of measurable functions of y.

To prove the second half we will show that the sequence

$$g_n \coloneqq \int_0^1 f_n(x)\,dy \longrightarrow \frac{d}{dx}\int_0^1 f(x,y)\,dy. \tag{8.2}$$

Since $\partial f/\partial x$ is bounded, there exists M such that $|\partial f/\partial x| < M$ so convergence of f_n to $\partial f/\partial x$ means that for every $\varepsilon > 0$, there exists N such that $n \ge N$ implies $|f_n| < M + \varepsilon$. Then by the bounded convergence theorem, $\int_0^1 f_n \to \int_0^1 \to \partial f/\partial x$. But by definition the limit as $n \to \infty$ of

$$\begin{aligned}\int_0^1 f_n(x)\,dy &= \int_0^1 \frac{f(x+1/n,y) - f(x,y)}{1/n}\,dy \\ &= \frac{\int_0^1 f(x+1/n,y)\,dy - \int_0^1 f(x,y)\,dy}{1/n}\end{aligned}$$

is

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy.$$

Hence, by the uniqueness of limit, we have

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy$$

a.e. on $[0,1]$. ∎

Problem 8.4 (Wheeden & Zygmund §5, Ex. 7)

Give an example of an f that is not integrable, but whose improper Riemann integral exists and is finite.

Proof. The following is a standard example of a function f that is improperly Riemann integrable, but not Lebesgue integrable. Set $f \coloneqq \sin(x)/x$. Then the Riemann $\int_{-\infty}^{\infty} f\, dx = 2\pi$ may be computed fairly easily by contour integration noting that $f = \Im\big(e^{ix}/x\big)$ and applying Jordan's lemma.

However, by Theorem 5.21, f is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable however, we show that $\int_{\mathbf{R}} |f|\, dx = \infty$. To see this note that

$$\int_{\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx$$

make the substitution $x = t + k\pi$, then

$$\begin{aligned} &= \sum_{k=1}^{n} \int_0^{\pi} \left|\frac{\sin(t+k\pi)}{t+k\pi}\right| dt \\ &= \sum_{k=1}^{n} \int_0^{\pi} \left|\frac{\sin(t+k\pi)}{t+k\pi}\right| dt \end{aligned}$$

and since $t + k\pi \leq \pi + k\pi$ for $0 \leq t \leq \pi$ we have

$$\begin{aligned} &\geq \sum_{k=1}^{n} \int_0^{\pi} \left|\frac{\sin(t+k\pi)}{\pi+k\pi}\right| dt \\ &\geq \sum_{k=1}^{n} \frac{1}{\pi(k+1)} \int_0^{\pi} \sin(t+k\pi)\, dt \\ &\geq \sum_{k=1}^{n} \frac{2}{\pi(k+1)} \end{aligned}$$

which clearly diverges as $n \to \infty$ since the lower bound above is the scaled harmonic series starting at 2. Thus, $|f|$ is not integrable over $\mathbf{R}$ so f is not integrable over $\mathbf{R}$. ∎

Problem 8.5 (Wheeden & Zygmund §5, Ex. 21)

If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

Proof. Since E is measurable, $\int_E f = 0$ so f is measurable. Write E as the union

$$E = \{ f > 0 \} \cup \{f = 0\} \cup \{ f < 0 \}. \tag{8.3}$$

Then

$$\begin{aligned} \int_{\{ f>a \}} f \, dx &= \int_E f^+ \, dx \\ &= 0 \end{aligned} \qquad \begin{aligned} \int_{\{ f<0 \}} f \, dx &= -\int_E f^- \, dx \\ &= 0. \end{aligned}$$

Hence, by Theorem 5.11, $f^+ = 0$ and $f^- = 0$ a.e. on E. Thus, $f = f^+ - f^- = 0$ a.e. on E ■

Problem 8.6 (Wheeden & Zygmund §6, Ex. 10)

Let V_n be the volume of the unit ball in $\mathbf{R}^n$. Show by using Fubini's theorem that

$$V_n = 2V_{n-1}\int_0^1 \left(1-t^2\right)^{(n-1)/2}\,dt.$$

(We also observe that by setting $w = t^2$, the integral is a multiple of a classical β-function and so can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$, $s > 0$.)

Proof. We prove this by induction. Since $V_0 = 0$ (unless we set $V_0 := 1$ we cannot start induction at $n = 1$), we start induction at $n = 2$. For $n = 2$ we have

$$V_2 = 2V_1\int_0^1 \sqrt{1-t^2}\,dt = 4\int_0^1 \sqrt{1-t^2} = \pi. \tag{8.4}$$

Assume by induction that for $k < n$

$$V_k = 2V_{k-1}\int_0^1 \left(1-t^2\right)^{(k-1)/2}\,dt \tag{8.5}$$

holds. Then, since $B_1(\mathbf{0})$ is the set of all points $\mathbf{x} \in \mathbf{R}^n$ such that ${x_1}^2 + \cdots + {x_n}^2 \leq 1$ we have

$$\begin{aligned} V_n &= \int\cdots\int_{{x_1}^2+\cdots+{x_n}^2\leq 1} dx_1\cdots dx_n \\ &= \int_{-1}^1\int\cdots\int_{{x_2}^2+\cdots+{x_n}^2\leq 1-{x_1}^2} dx_1\cdots dx_n. \end{aligned}$$

now, make the change of variables $u_i = x_i/\sqrt{1-{x_1}^2}$, $du_i = dx_i/\sqrt{1-{x_1}^2}$ and we have

$$= \int_{-1}^1\int\cdots\int_{{y_2}^2+\cdots+{y_n}^2\leq 1}\left(1-{x_1}^2\right)^{(n-1)/2}\,dx_1dy_2\cdots dy_n$$

then by Fubini's theorem

$$\begin{aligned} &= \int_{-1}^1\left[\int\cdots\int_{{y_2}^2+\cdots+{y_n}^2\leq 1} dy_2\cdots dy_n\right]\left(1-{x_1}^2\right)^{(n-1)/2}\,dx_1 \\ &= V_{n-1}\int_{-1}^1\left(1-{x_1}^2\right)^{(n-1)/2}\,dx_1 \\ &= 2V_{n-1}\int_0^1\left(1-{x_1}^2\right)^{(n-1)/2}\,dx_1 \end{aligned}$$

as desired. ∎

Problem 8.7 (Wheeden & Zygmund §6, Ex. 11)

Use Fubini's theorem to prove that

$$\int_{\mathbf{R}^n} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = \pi^{n/2}.$$

(For $n = 1$, write $\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dxdy$ and use polar. For $n > 1$, use the formula $e^{-|\mathbf{x}|^2} = e^{-x_1{}^2}\cdots e^{-x_n{}^2}$ and Fubini's theorem to reduce the case $n = 1$.)

Proof. Using the hint, by induction for $n = 1$ we have

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dxdy. \tag{8.6}$$

Now we make a change of $x = r\cos\theta$, $y = r\sin\theta$, $0 \le r \le \infty$, $0 \le \theta \le \pi$ and the determinant of the Jacobian is r so we have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dxdy &= \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\, drd\theta \\ &= 2\pi\int_0^{\infty} re^{-r^2}\, dr\end{aligned}$$

by making the substitution, $u = r^2$

$$\begin{aligned}&= \pi\int_0^{\infty} e^{-u}\, du \\ &= \pi\end{aligned}$$

so $\int_{\mathbf{R}} e^{-x^2} = \sqrt{\pi}$.

Assume by induction that $\int_{\mathbf{R}^k} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = \pi^{k/2}$ for all $k < n$. Then writing

$$\int_{\mathbf{R}^n} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = e^{-|\mathbf{x}|^2} = \int\cdots\int_{\mathbf{R}^n} e^{-x_1{}^2}\cdots e^{-x_n{}^2}\, dx_1\cdots dx_n \tag{8.7}$$

Then by Fubini's theorem

$$\begin{aligned}\int\cdots\int_{\mathbf{R}^n} e^{-x_1{}^2}\cdots e^{-x_n{}^2} dx_1\cdots dx_n &= \int_{\mathbf{R}}\left[\int\cdots\int_{\mathbf{R}^{n-1}} e^{-x_1{}^2}\cdots e^{-x_n{}^2}\right] dx_1 \\ &= \int_{\mathbf{R}}\left[\int\cdots\int_{\mathbf{R}^{n-1}} e^{-x_1{}^2}\cdots e^{-x_n{}^2}\right] dx_1 \\ &= \int_{\mathbf{R}}\left[\int\cdots\int_{\mathbf{R}^{n-1}} e^{-x_2{}^2}\cdots e^{-x_n{}^2}\right] e^{-x_1{}^2}\, dx_1 \\ &= \pi^{(n-1)/2}\int_{\mathbf{R}} e^{-x_1{}^2}\, dx_1 \\ &= \pi^{(n-1)/2}\pi^{1/2} \\ &= \pi^{n/2}\end{aligned}$$

as desired. ∎