MA166: Exam 9 Solutions

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1 Solutions to this week's homework

1.1 Homework # 22

Problem 1 (WebAssign HW # 22, # 1)**.** Determine whether the series is convergent or divergent.

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots .$$

Solution. The series is convergent. It is in fact this is a very important p-series whose sum is equal to $\pi^2/6$. ☺

Problem 2 (WebAssign HW # 22, # 2)**.** Determine whether the series is convergent or divergent.

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots .$$

Solution. This is again convergent because it is a p-series with form

$$\sum_{n=0}^{\infty} \frac{1}{n^{3/2}}$$

with $p = 3/2 > 1$, so the series converges. ☺

Problem 3 (WebAssign HW # 22, # 3)**.** Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+5}{n^2}.$$

Solution. Expand the series

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sqrt{n}+5}{n^2} &= \sum_{n=1}^{\infty} \left[\frac{\sqrt{n}}{n^2} + \frac{5}{n^2}\right] \\
&= \sum_{n=1}^{\infty} \left[\frac{1}{n^{3/2}} + \frac{5}{n^2}\right] \\
&= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}}_{S_1} + \underbrace{\sum_{n=1}^{\infty} \frac{5}{n^2}}_{S_2}.
\end{aligned}$$

The series S_1 is finite by the last problem since it is a p-series with $p = 3/2 > 1$ and the same goes for S_2. Thus, the series we were given is convergent. ☺

Problem 4 (WebAssign HW # 22, # 4)**.** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be convergent.

(a) If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?

(b) If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?

Solution. ☺

Problem 5 (WebAssign HW # 22, # 5)**.** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be divergent.

(a) If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?

(b) If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?

Solution. ☺

Problem 6 (WebAssign HW # 22, # 6)**.** Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{5n^3+1}.$$

Solution. Use the comparison test to determine whether the series converges

$$\begin{aligned}\frac{n}{5n^3+1} &< \frac{n}{5n^3} \\ &= \frac{1}{5n^2} \\ &< \frac{1}{n^2}.\end{aligned}$$

Hence, we have

$$\sum_{n=1}^{\infty} \frac{n}{5n^3+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

the right of which converges since it is a p-series with $p = 2 > 1$. ☺

Problem 7 (WebAssign HW # 22, # 7)**.** Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^4}{2n^5-1}.$$

Solution. By the integral test we have

$$\int_1^{\infty} \frac{x^4}{2x^5-1}\,\mathrm{d}x = \frac{1}{2}\int_1^{\infty} \frac{x^4}{x^5-1/2}\,\mathrm{d}x$$

make the u-substitution $u = x^5 - 1/2$, $\mathrm{d}u = 5x^4\,\mathrm{d}x$ so

$$\begin{aligned}&= \frac{1}{10}\int_{1/2}^{\infty} \frac{\mathrm{d}u}{u} \\ &= \left[\frac{1}{10}\ln|u|\right]_{1/2}^{\infty} \\ &= \infty.\end{aligned}$$

☺

Problem 8 (WebAssign HW # 22, # 8)**.** Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n+6}{n\sqrt{n}}.$$

Solution. Expand the series and simplify

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n+6}{n\sqrt{n}} &= \sum_{n=1}^{\infty} \frac{n+6}{n^{3/2}} \\ &= \sum_{n=1}^{\infty} \left[\frac{n}{n^{3/2}} + \frac{6}{n^{3/2}}\right] \\ &= \sum_{n=1}^{\infty} \left[\frac{1}{n^{1/2}} + \frac{6}{n^{3/2}}\right] \\ &= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}}_{S_1} + 6 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}}_{S_2}.\end{aligned}$$

The series S_2 is a convergent p-series with $p = 2 > 1$, but S_1 is not convergent since $p = 1/2 < 1$. Hence, the sum of S_1 and S_2 is divergent. ☺

Problem 9 (WebAssign HW # 22, # 9)**.** Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{7^n}{1+11^n}.$$

Solution. Apply the comparison test

$$\frac{7^n}{1+11^n} < \frac{7^n}{11^n} = \left(\frac{7}{11}\right)^n.$$

Then

$$\sum_{n=1}^{\infty} \frac{7^n}{1+11^n} < \sum_{n=1}^{\infty} \left(\frac{7}{11}\right)^n$$

which is a geometric series with radius $r = |7/11| < 1$, hence convergent so the original series is convergent. ☺

1.2 Homework # 23

Problem 10 (WebAssign HW # 23, # 1)**.** Determine whether the series converges or diverges.

$$\sum_{n=9}^{\infty} \frac{3\sqrt{n}}{n-8}.$$

Solution. By the comparison test

$$\begin{aligned}\frac{3\sqrt{n}}{n-8} &> \frac{3\sqrt{n}}{n}\\ &= \frac{3}{\sqrt{n}}\end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{n-8} > \sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$$

the right of which diverges since it is a p-series with $p = 1/2 < 1$. Hence, the original series diverges. ☺

Problem 11 (WebAssign HW # 23, # 2)**.** Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{9}{\sqrt{n^2+2}}.$$

Solution. By the comparison test

$$\begin{aligned}\frac{9}{\sqrt{n^2+2}} &> \frac{9}{\sqrt{n^2}}\\ &= \frac{9}{n}\end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{9}{\sqrt{n^2+2}} > \sum_{n=1}^{\infty} \frac{9}{n}$$

where the series on the right diverges, so the original series diverges. ☺

Problem 12 (WebAssign HW # 23, # 3)**.** Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5+8^n}{5+7^n}.$$

Solution. By the comparison test we have

$$\begin{aligned}\frac{5+8^n}{5+7^n} &> \frac{5+8^n}{7^n}\\ &= \frac{5}{7^n} + \left(\frac{8}{7}\right)^n.\end{aligned}$$

Then we have

$$\sum_{n=1}^{\infty} \frac{5+8^n}{5+7^n} > \sum_{n=1}^{\infty} \left[\frac{5}{7^n} + \left(\frac{8}{7}\right)^n\right]$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{5}{7^n}}_{S_1} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{8}{7}\right)^n}_{S_2}$$

where the series on the right is a geometric series with radius $r = |8/7| > 1$ hence diverges. Thus, the sum $S_1 + S_2$ diverges. ☺

Problem 13 (WebAssign HW # 23, # 4)**.** Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n + 2^n}{n + 8^n}.$$

Solution. By the comparison test we have

$$\begin{aligned} \frac{n + 2^n}{n + 8^n} &< \frac{n + 2^n}{8^n} \\ &= \frac{n}{8^n} + \left(\frac{2}{8}\right)^n. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{n + 2^n}{n + 8^n} &< \sum_{i=1}^{\infty} \left[\frac{n}{8^n} + \left(\frac{2}{8}\right)^n\right] \\ &= \underbrace{\sum_{i=1}^{\infty} \frac{n}{8^n}}_{S_1} + \underbrace{\sum_{i=1}^{\infty} \left(\frac{2}{8}\right)^n}_{S_2}. \end{aligned}$$

S_2 converges because it is a geometric series with radius $r = |2/8| = 1/4 < 1$. S_1 converges by the integral test

$$\int_1^{\infty} \frac{x}{8^x}\,\mathrm{d}x = \int_1^{\infty} \frac{x}{e^{x \ln 8}}\,\mathrm{d}x$$

by integration by parts

$$\begin{aligned} &= \left[-xe^{-x\ln 8}/\ln 8 - e^{-x \ln 8}/(\ln 8)^2\right]_1^{\infty} \\ &= 0 - (-8/\ln 8 - 8/(\ln 8)^2) \\ &< \infty \end{aligned}$$

(We don't care what the value is, just that it is not infinite.) Thus, $S_1 + S_2$ converges.at ☺

Problem 14 (WebAssign HW # 23, # 5)**.** Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} 5 \sin\left(\frac{3}{n}\right).$$

Solution. Set $b_n := 15/n$. By the limit comparison test, we have

$$\lim_{n\to\infty} \frac{5\sin(3/n)}{15/n} = \lim_{n\to\infty} \frac{\sin(3/n)}{n/3} = \lim_{k\to 0} \frac{\sin k}{k} = 1.$$

The above limit is well known. The reason we make a change of variables here is because we have $3/n \to 0$ as $n \to \infty$ so we may as well define $k = 3/n$ and look at the sequence of k as they go to 0. Since the series

$$\sum_{i=1}^{\infty} \frac{15}{n}$$

is harmonic, it diverges so the original series diverges. ☺

1.3 Homework # 24

Problem 15 (WebAssign HW # 24, # 1)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{8}} - \cdots .$$

Solution. First we find b_n. Looking at the pattern we quickly see that

$$b_n = \frac{1}{\sqrt{n+3}}.$$

Then $b_n > 0$ is a positive sequence with limit 0 so by the alternating series test, the series converges. ☺

Problem 16 (WebAssign HW # 24, # 2)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5n+1}.$$

Solution. The sequence

$$b_n = \frac{1}{5n+1}$$

is positive (always greater than or equal to 0) and converges to 0 so by the alternating series test, the series converges. ☺

Problem 17 (WebAssign HW # 24, # 3)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{5n+1}.$$

Identify b_n and evaluate the limit $\lim_{n\to\infty} b_n$.

Solution. Identifying b_n is the easiest part. It is the part of the series which does not have the alternating power $(-1)^n$, i.e., $b_n = (3n-1)/(5n+1)$. Now to compute the limit you can use l'Hôpital's rule on the functions $3x - 1/5x + 1$ to get $3/5$.

Since $\lim_{n\to\infty} b_n \neq 0$ and $b_{n+1} \geq b_n$ for all n, the series diverges. ☺

Problem 18 (WebAssign HW # 24, # 4)**.** Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n^5}.$$

Solution. The series converges since it satisfies

$$b_n = \frac{1}{3n^5} \geq \frac{1}{3(n+1)^5}$$

and

$$\lim_{n\to\infty} b_n = 0.$$

☺

Problem 19 (WebAssign HW # 24, #)**.** Approximate the sum of the series correct to four decimal places.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{7^n}.$$

Solution. ☺

Problem 20 (WebAssign HW # 24, #)**.**

Solution. ☺

2 Exam problems