MA 523: Homework 7

Carlos Salinas

October 31, 2016

Problem 7.1

Solve the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ u = x_1 & \text{on } |x| = 1, \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

(*Hint:* Use Laurent series.)

Solution. Suppose

$$u_\ell(x_1, x_2) = \sum_{\alpha,\beta\in\mathbb{Z}} a_{\alpha\beta} x_1^\alpha x_2^\beta \tag{7.1}$$

is a Laurent series solution to the Dirichlet problem above. Then harmonicity implies that

$$\begin{aligned} 0 &= \Delta u_\ell(x_1, x_2) \\ &= \sum_{\alpha,\beta\in\mathbb{Z}} \alpha(\alpha-1) a_{\alpha\beta} x_1^{\alpha-2} x_2^\beta + \sum_{\alpha,\beta\in\mathbb{Z}} \beta(\beta-1) a_{\alpha\beta} x_1^\alpha, x_2^{\beta-2} \end{aligned}$$

where, after shifting indices on both series, we have the single series

$$= \sum_{\alpha,\beta\in\mathbb{Z}} \big((\alpha+2)(\alpha+1) a_{\alpha+2,\beta} + (\beta+2)(\beta+1) a_{\alpha,\beta+2}\big) x_1^\alpha x_2^\beta.$$

Thus, the coefficients must satisfy

$$(\alpha+2)(\alpha+1) a_{\alpha+2,\beta} + (\beta+2)(\beta+1) a_{\alpha,\beta+2} = 0. \tag{7.2}$$

In particular, if $\alpha = \beta$

$$a_{\alpha+2,\beta} = -a_{\alpha,\beta+2}.$$

■

Problem 7.2

Let Ω be a bounded domain with a C^1 boundary, $g \in C^2(\partial\Omega)$ and $f \in C(\bar{\Omega})$. Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where ν is the outer normal on $\partial\Omega$. Show that the solution of $(*)$ in $C^2(\Omega) \cap C^1(\bar{\Omega})$ is unique up to a constant; i.e., if u_1 and u_2 are both solutions of $(*)$, then $u_2 = u_1 + \text{const.}$ in Ω.
(*Hint:* Look at the proof of the uniqueness for the Dirichlet problem by energy methods [E, 2.2.5a].)

Solution. Suppose u_1 and u_2 are solutions to the Neumann problem $(*)$. Define $v := u_1 - u_2$. Then v is harmonic in Ω and $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$. Consider the energy functional

$$E[v] = \frac{1}{2} \int_\Omega |Dv|^2 \, dx.$$

By Green's formula version (ii),

$$\begin{aligned} E[v] &= \frac{1}{2} \int_\Omega |Dv|^2 \, dx \\ &= -\frac{1}{2} \int_\Omega v \Delta v \, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} v \, dS(x) \\ &= 0. \end{aligned}$$

This implies that $|Dv|^2 = Dv \cdot Dv = 0$ which, since the standard inner product on $\mathbb{R}^n$ is positive-definite, implies that $Dw \equiv 0$. It follows that $u_1 = u_2 + \text{const}$, i.e., the solution u to $(*)$ is unique up to a constant. ■

Problem 7.3

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\}, \end{cases}$$

where $c \in \mathbb{R}$.
(*Hint:* Rewrite the problem in terms of $v(x,t) := \mathrm{e}^{ct} u(x,t)$.)

Solution. ■