MA544 Exam 1 Study Guide

Review the following topics:
1) Define the Riemann integral of a function.
2) What is the oscillation of a function at a point?
3) Characterize the functions that are Riemann integrable.
5) Review the construction of the Cantor sets of measure zero and measure not zero.
6) Define a σ-algebra and a measure.
7) Define a measurable function.
8) Recall the citeria for a function to be measurable.
9) What are Borel sets?
10) Recall the construction of the Lebesgue integral.
11) Recall the proof of the two main convergence theorems and Fatou's Lemma.
12) Recall the definition of an outer measure of a set in $\mathbb{R}^n$.
13) Recall the definition of a Lebesgue measurable set .
14) Recall the proof that Riemann integrable functions are also Lebesgue itegrable
15) Recall the definition of L^p spaces and their main properties.
16) Review the homework exercises.

Review Exercises: These are some exercises that review the important topics we have seen. **You can turn these this on Monday, April 8th for an extra 30 points on the exam.**

Let $(X, \mathcal{M}, \mu)$ be a measure space. We also use $(\mathbb{R}^n, \mathcal{M}, \mu)$ for $\mathbb{R}^n$ equipped with the Lebesgue σ-algebra and measure.

1) If $f : X \longrightarrow \mathbb{R}$ is such that $f^{-1}((\lambda, \infty]) \in \mathcal{M}$ for every $\lambda \in \mathbb{Q}$. Is f measurable?
2) If If $f : X \longrightarrow \mathbb{R}$ is such that $\int_A f \, d\mu = 0$ for all $A \in \mathcal{M}$. Show that $f = 0$ a.e.
2) Let $f_n : X \longrightarrow \mathbb{R}$ be measurable, $n \in \mathbb{N}$. Let $A = \{x : \lim_{n\to\infty} f(x) \text{ exists } \}$. Show that A is measurable.
3) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$
a) Show that if is differentiable, then $f'(x)$ is Lebesgue measurable.
b) Show that f is Lebesgue measurable if and only if there exists a Borel measurable function g such that $f = g$ a.e.
c) If f is Lebesgue measurable and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and for every $U \subset \mathbb{R}$ with $\mu(U) = 0$, $\phi^{-1}(N)$ has measure zero. Show that $f \circ \phi$ is Lebesgue measurable.
4) Show that

$$\sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2}} (1 - (\sin x)^r)^n \ \cos x \ \ dx, \quad r < 1,$$

converges, and find its value.
5) Suppose $\mu(X) < \infty$, and let $f : X \longrightarrow [0, \infty)$. Prove that $\lim_{n\to\infty} \int_X f^n(x) \, d\mu$ exists and is finite if and only if $\mu(f^{-1}(1, \infty)) = 0$.

6) Let $f_n : X \longrightarrow [0,\infty]$ be measurable, $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} f_n(x) = 0$ a.e. and that $\lim_{n\to\infty} \int_X f_n \, d\mu = 0$. Is it true that $\lim_{n\to\infty} \int_E f_n \, d\mu = 0$, for all $E \in \mathcal{M}$?
7) Let $\phi_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, $j \in \mathbb{N}$. Suppose that $||\phi_j||_{L^2} = 1$ and that $\int_{\mathbb{R}} \phi_j(x)\phi_k(x) \, d\mu = 0$ if $j \neq k$. Let$s_N(x) = \sum_{j=1}^{N} C_j \phi_j(x)$, and assume that $\sum_{j=1}^{\infty} C_j^2 < \infty$. Show that s_N converges in $L^2(\mathbb{R}^n)$.
8) Examine the proof of Hölder's inequality and determine when equality holds.
9) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is such that $f \in L^2(\mathbb{R}^n)$. Show that if

$$\int_K x^j f(x) \;\; d\nu = 0, \quad j = 0, 1, 2, \dots$$

for every compact subset $K \subset \mathbb{R}^n$, then $f = 0$ a.e. Remark: Use Weierstrass theorem: The space of polynomials is dense in the set of continuos functions in K, with the uniform convergence topology.
10) Show that if $\mu(X) < \infty$ then $L^q(X) \subset L^p(X)$, for $1 < p < q \leq \infty$, but this is not true in general. Moreover, show that $\cup_{p>2} L^p([0,1]) \neq L^2([0,1])$.
11) Let $f \in L^p(\mathbb{R}^n)$. Compute

$$\lim_{h\to 0} \int_{\mathbb{R}^n} |f(x) - f(x+h)|^p \, d\mu,$$

12) Let $0 < p < r < q < \infty$, and $f \in L^p(X) \cap L^q(X)$. Show that $||f||_r \leq ||f||_p^{1-t} ||f||_q^t$ with $t \in (0,1)$ such that $\frac{1}{r} = \frac{1-t}{p} + \frac{t}{q}$.
13) Suppose that $\mu(X) < \infty$. Let $f : X \longrightarrow [0,\infty)$, be such that $0 < ||f||_\infty < \infty$, and let

$$\phi(p) = \int_X f^p \, d\mu = ||f||_p^p.$$

a) Prove that $\log \phi(p)$ is a convex function.
b) Prove that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.
14) Let $f \in L^1(\mathbb{R}^n)$ and let

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x) \;\; d\mu, \quad \langle x,\xi\rangle = x_1\xi_1 + \dots x_n\xi_n.$$

a) Show that $f \in C^\infty(\mathbb{R}^n)$.
b) Suppose that f is continuous and use that $e^{i\pi} = -1$ to show that

$$\widehat{f}(\xi) = -\int_{\mathbb{R}^n} e^{-i\langle \xi, x - \frac{\pi\xi}{|\xi|^2}\rangle} f(x) \;\; d\mu = -\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f\left(x + \frac{\pi\xi}{|\xi|^2}\right) \, dx.$$

Then

$$2\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \left[f(x) - f\left(x + \frac{\pi\xi}{|\xi|^2}\right)\right] \, dx.$$

Show that $\lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0$.
c) Prove that this is also true if $f \in L^1(\mathbb{R}^n)$.

d) Let $A \subset \mathbb{R}$ be measurable and $\mu(A) < \infty$. Show that

$$\lim_{n\to\infty} \int_A e^{inx}\, dx = 0.$$