MA52300 FALL 2016

Homework Assignment 9 – *Solutions*

1. (a) Show that for $n = 3$ the general solution of the wave equation $u_{tt} - \Delta u = 0$ with spherical symmetry about the origin has the form

$$u = \frac{F(r+t) + G(r-t)}{r}, \quad r = |x|$$

with suitable F, G.

(b) Show that the solution with initial data of the form

$$u = 0, \quad u_t = h(r)$$

($h =$ even function of r) is given by

$$u = \frac{1}{2r} \int_{r-t}^{r+t} \rho h(\rho)\, d\rho.$$

Solution. (a) Let $u(x,t) = \phi(|x|, t)$. Then obviously

$$U(r,t) := \fint_{B(0,r)} u(y,t) d\sigma_y = \fint_{B(0,r)} \phi(r,t) d\sigma_y = \phi(r,t).$$

Hence $u(x,t) = U(r,t)$ with $r = |x|$. On the other hand, the spherical means $U(r,t)$ satisfy the Euler-Poisson-Darboux equation

$$U_{tt} - U_{rr} - \frac{2}{r} U_r = 0.$$

Arguing as in the derivation of Kirchhoff's formula, set

$$\tilde{U} = rU.$$

Then

$$\tilde{U}_{tt} - \tilde{U}_{rr} = 0$$

is a solution of the one-dimensional wave equation. Hence

$$\tilde{U}(r,t) = F(r+t) + G(r-t), \quad \text{for } r, t > 0$$

with suitable F and G. Then

$$u(x,t) = U(r,t) = \frac{\tilde{U}(r,t)}{r} = \frac{F(r+t) + G(r-t)}{r}, \quad r = |x|$$

(b) Assume now

$$u(x,0) = 0, \quad u_t(x,0) = h(r).$$

Then will have

$$\tilde{U}(r,0) = 0, \quad \tilde{U}_t(r,0) = rh(r) \quad \text{for } r > 0.$$

Besides

$$\tilde{U}(0,t) = 0, \quad \text{for } t \geq 0.$$

Thus, by the reflection principle, $\tilde{U}$ can be obtained by solving the initial-value problem

$$\tilde{U}(r,0) = 0, \quad \tilde{U}_t(r,0) = rh(r) \quad \text{for } -\infty < r < \infty.$$

(Observe that $rh(r)$ is an odd function of r, since we assume h is even.) By d'Alembert's formula

$$\tilde{U}(r,t) = \frac{1}{2}\int_{r-t}^{r+t} \rho h(\rho) d\rho,$$

which implies

$$u(x,t) = \frac{1}{2r}\int_{r-t}^{r+t} \rho h(\rho) d\rho.$$

□

2. Show that the solution $w(x_1,t)$ of the initial-value problem for the *Klein-Gordon equation*

$$w_{tt} = w_{x_1 x_1} - \lambda^2 w \tag{1}$$

$$w(x_1,0) = 0, \quad w_t(x_1,0) = h(x_1) \tag{2}$$

is given by

$$w(x_1,t) = \frac{1}{2}\int_{x_1-t}^{x_1+t} J_0(\lambda s)\, h(y_1)\, dy_1.$$

Here

$$s^2 = t^2 - (x_1 - y_1)^2,$$

while J_0 denotes the Bessel function defined by

$$J_0(z) = \frac{2}{\pi}\int_0^{\pi/2} \cos(z \sin\theta)\, d\theta.$$

Hint: "Descend" to (1) from the two-dimensional wave equation satisfied by

$$u(x_1,x_2,t) = \cos(\lambda x_2) w(x_1,t).$$

Solution. Following the hint, let

$$u(x_1,x_2,t) = \cos(\lambda x_2) w(x_1,t).$$

Then

$$u_{tt} = \cos(\lambda x_2) w_{tt}, \quad u_{x_1 x_1} = \cos(\lambda x_2) w_{x_1 x_1}, \quad u_{x_2 x_2} = -\lambda^2 \cos(\lambda x_2) w$$

which implies that

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2}.$$

By the Poisson's formula for the solutions of the two-dimensional wave equation, we will have

$$w(x_1,t) = u(x_1,0,t) = \frac{1}{2\pi}\iint_{(x_1-y_1)^2+y_2^2<t^2} \frac{\cos(\lambda y_2)h(y_1)}{\sqrt{t^2-(x_1-y_1)^2-y_2^2}}\,dy_1dy_2.$$

Let now $s = \sqrt{t^2-(x_1-y_1)^2}$. Then

$$\begin{aligned} w(x_1,t) &= \frac{1}{2\pi}\int_{x_1-t}^{x_1+t}\int_{-s}^{s} \frac{\cos(\lambda y_2)h(y_1)}{\sqrt{s^2-y_2^2}}\,dy_2dy_1 \\ &= \frac{1}{2\pi}\int_{x_1-t}^{x_1+t} K(\lambda,s)h(y_1)\,dy_1, \end{aligned}$$

where

$$K(\lambda,s) = \int_{-s}^{s}\frac{\cos(\lambda y_2)}{\sqrt{s^2-y_2^2}}\,dy_2.$$

Finally, by substituting $y_2 = s\sin\theta$ we realize that

$$K(\lambda,s) = 2\int_0^{\pi/2}\cos(\lambda s\sin\theta)\,d\theta = \pi J_0(\lambda s).$$

This implies

$$w(x_1,t) = \frac{1}{2}\int_{x_1-t}^{x_1+t} J_0(\lambda s)\,h(y_1)\,dy_1$$

as required. □

3. Let u solve

$$\begin{cases} u_{tt}-\Delta u = 0 & \text{in } \mathbb{R}^3\times(0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^3\times\{t=0\} \end{cases}$$

where g, h, are smooth and have compact support. Show there exists a constant C such that

$$|u(x,t)| \le C/t \quad (x\in\mathbb{R}^3, t>0).$$

Solution. Let $\operatorname{supp} g$, $\operatorname{supp} h \subset B_a$. By Kirchhoff's formula

$$u(x,t) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)}[th(y)+g(y)+Dg(y)\cdot(y-x)]d\sigma_y$$

Now observe that we can replace the integral over $\partial B(x,t)$ by an integral over $\partial B(x,t)\cap B(0,a)$. Moreover,

$$\text{Area}(\partial B(x,t)\cap B(0,a)) \le \begin{cases} C, & t\ge a \\ 4\pi t^2, & 0<t<a \end{cases},$$

where $C = C(a)$ is independent of t, as it follows from simple geometric considerations. Hence,

$$\begin{aligned} |u(x,t)| &\leq \frac{1}{4\pi t^2} \int_{\partial B(x,t)\cap B(0,a)} (Mt + M + Mt) d\sigma_y \\ &\leq \frac{C(t+1)}{4\pi t^2} \operatorname{Area}(\partial B(x,t) \cap B(0,a)) \\ &\leq \frac{C}{t}, \end{aligned}$$

where in the last step we have to consider the cases $t \geq a$ and $0 < t < a$ separately. $\square$