MA 519: Homework 2

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Problem 2.1 (Handout 2, # 5)

Four men throw their watches into the sea, and the sea brings each man one watch back at random. What is the probability that at least one man gets his own watch back?

Solution. ▶ Let Ω denote the sample space and let A denote the event that at least one man gets his own watch back. Since the order of distributing wallets to each man matters,

$$\#\Omega = 4! = 24.$$

Now, let A_i be the event that man i gets his wallet back, where $1 \leq i \leq 4$. Then $A = A_1 \cup A_2 \cup A_3 \cup A_4$ and by the inclusion-exclusion principle, we have

$$\begin{aligned} P(A) &= 4P(A_1) - 6P(A_1 \cap A_2) + 4P(A_1 \cap A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3 \cap A_4) \\ &= 4\frac{3!}{24} - 6\frac{2!}{24} + 4\frac{1}{24} - \frac{1}{24} \\ &= 1 - \frac{1}{2} + \frac{3}{24} \\ &= \frac{1}{2} + \frac{3}{24} \\ &= \frac{15}{24} \end{aligned}$$

where, because of the symmetry of the events, $P(A_i \cap A_j) = P(A_k \cap A_\ell)$, $P(A_i \cap A_j \cap A_k) = P(A_\ell \cap A_n \cap A_m)$ etc., thereby simplifying the above expression. ◀

Problem 2.2 (Handout 2, # 7)

Calculate the probability that in Bridge, the hand of at least one player is void in a particular suit.

Solution. ▶ We employ the same strategy as the last problem, i.e., we will use the inclusion-exclusion principle to solve this. We may, without loss of generality, assume that the particular suit is heart. Now, by Feller's solution to the occupancy problem, there are $\binom{13+4-1}{13} = 560$ ways to distribute 13 hearts to four people; this is the size of our sample space. Then, there are

$$\begin{aligned}\binom{13+3-1}{13} &= \binom{15}{13} = 15 \cdot 7 = 105\\ \binom{13+2-1}{13} &= \binom{14}{13} = 14\\ \binom{13+1-1}{13} &= \binom{13}{13} = 1\end{aligned}$$

ways to distribute 13 hearts between 3 (excluding N), 2 (excluding N and E), and 1 person (excluding N, E and S). By the inclusion-exclusion principle and symmetry of the events, we have

$$\begin{aligned}P(\text{at least one player is void in hearts}) &= \frac{4 \cdot 105 - 6 \cdot 14 + 4 \cdot 1}{560}\\ &= \frac{340}{560}\\ &\approx 0.6071\end{aligned}$$

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Problem 2.3 (Handout 2, # 12)

If n balls are placed at random into n cells, find the probability that exactly one cell remains empty.

Solution. ▶ Let n balls be placed at random into n cells. (As was pointed out to us in the last homework, we will assume that the n balls are distinct.) Thus, there are n^n ways to arrange the n balls into the n bins.

Now, there are

$$\binom{n}{1}\binom{n}{2}\binom{n-1}{1}(n-2)!$$

ways to put n balls into n slots with exactly one remaining empty (we pick 1 of our n slots to be empty, we pick 2 of our n balls to share a slot, we pick one of our remaining $n-1$ slots for those 2 balls to go into, we fill the rest with one ball each.)

Thus, the probability that exactly 1 cell remains empty is

$$\begin{aligned}\frac{\binom{n}{1}\binom{n}{2}\binom{n-1}{1}(n-2)!}{n^n} &= \frac{n\binom{n}{2}(n-1)(n-2)!}{n^n}\\ &= \frac{n!(n-1)/2}{n^n}\end{aligned}$$

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Problem 2.4 (Handout 2, # 13)

Spread of rumors. In a town of $n+1$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, etc. At each step the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor told r times without:

(a) returning to the originator,
(b) being repeated to any person.

Do the same problem when at each step the rumor told by one person to a gathering of N randomly chosen people. (The first question is the special case $N = 1$).

Solution. ▶ For part (a), the originator can chose from among n different people and at each step, each person can chose from among $n-1$ people (excluding the originator). This gives us a probability of

$$\frac{n(n-1)^r}{n^r} = \left(\frac{n-1}{n}\right)^{r-1}$$

of the rumor not returning to the originator.

For part (b), the originator can chose from among n different people and at each step i each person can chose from among $n-i+1$ people excluding the people who told him the rumor. This gives us a probability of

$$\frac{n(n-1)\cdots(n-r+1)}{n^r}$$

of the rumor not being repeated to anybody.

Lastly, there are $\binom{n}{N}^r$ ways to pick a group of N people to tell the rumor to r times; this is the size of the sample space. Now, the originator has $\binom{n}{N}$ choices he can make of group of N people to tell the rumor to and similarly at step i, the rumor spreader has $\binom{n-(i-1)N}{N}$ choices. Thus, the probability of the rumor not returning to the originator with this strategy is

$$\frac{\binom{n}{N}\binom{n-N}{N}\cdots\binom{n-(r-1)N}{N}}{\binom{n}{N}^r}.$$

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Problem 2.5 (Handout 2, # 14)

A family problem. In a certain family four girls take turns at washing dishes. Out of a total of four breakages, three were caused by the youngest girl, and she was thereafter called clumsy. Was she justified in attributing the frequency of breakages to chance? Discuss the connection with random placement of balls.

Solution. ▶ By assuming that every sample point in our sample space is equally likely, we can relate the problem at hand to the probability of the event A that out of four indistinguishable balls, three fall in a specific bin, say, the first one. First, $\#\Omega = 4^4$ since we are including the possibility that no sister is responsible for breaking the dish. Now, there are 3 ways to attribute dishes to the youngest sister and the last dish can go into one of 4 places, the three other sisters or none of them. This gives us a probability of

$$P(A) = \frac{4 \cdot 3}{4^4} = \frac{12}{265} \approx 0.0469.$$

Since this probability is rather small, the sisters are justified in calling her clumsy. ◀

Problem 2.6 (Handout 2, # 15)

A car is parked among N cars in a row, not at either end. On his return the owner finds exactly r of the N places still occupied. What is the probability that both neighboring places are empty?

Solution. ▸ There are $r-1$ cars not belonging to the owner with $N-1$ spots for them. Thus, there are $\binom{N-1}{r-1}$ possible arrangements. Out of those arrangements, we insist that that the spots to the left and to the right of the car's owner be empty, leaving $N-3$ spots for $r-1$ cars. This gives us a probability of

$$\frac{\binom{N-3}{r-1}}{\binom{N-1}{r-1}}.$$

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Problem 2.7 (Handout 2, # 16)

Find the probability that in a random arrangement of 52 bridge card no two aces are adjacent.

Solution. ▸ We can simplify this problem by using the sticks and stars analogy. There are 4 aces and, if we want to avoid placing the aces next to each other, we count the spaces between the remaining cards 48 cards, this is $52 - 4 + 1 = 49$ and place the 4 aces in between. This gives a total of $\binom{48}{4}$ arrangements (where the order in which we do it does not matter) out $\binom{52}{4}$ possible arrangements including the possibility of two aces being next to each other. Thus, the probability that no two aces are adjacent is

$$\frac{\binom{48}{4}}{\binom{52}{4}} \approx 0.7187.$$

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Problem 2.8 (Handout 2, # 17)

Suppose $P(A) = 3/4$, and $P(B) = 1/3$.

Prove that $P(A \cap B) \geq 1/12$. Can it be equal to $1/12$?

Solution. ▸ By the inclusion-exclusion principle, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

so

$$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

But, since $A \cup B \subseteq \Omega$ and $P(\Omega) = 1$, $P(A \cup B) \leq P(\Omega) = 1$ so $-P(A \cup B) \geq -1$. Thus,

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &\geq P(A) + P(B) - 1 \\ &= \frac{3}{4} + \frac{1}{3} - 1 \\ &= \frac{9}{12} + \frac{4}{12} - \frac{12}{12} \\ &= \frac{1}{12}. \end{aligned}$$

Lastly, we show that $P(A \cap B)$ can in fact be equal to $1/12$. Consider the closed unit interval $I = [0, 1]$ equipped with the Lebesgue measure. Set $A = (0, 3/4)$ and $B = (1 - 1/3, 1) = (2/3, 1)$. Then the intersection $A \cap B = (2/3, 3/4)$ has Lebesgue measure

$$m(A \cap B) = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}.$$

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Problem 2.9 (Handout 2, # 18)

Suppose you have infinitely many events $A_1, A_2, \ldots$, and each one is sure to occur, i.e., $P(A_i) = 1$ for any i.

Prove that $P\left(\bigcap_{i=1}^{n} A_i\right) = 1$.

Solution. ▸ Consider the sequence of probabilities $\{P_n : n \in \mathbb{N}\}$ where $P_n = P\left(\bigcap_{i=1}^{n} A_i\right)$. Note that $\bigcap_{i=1}^{n} A_i \downarrow \bigcap_{i=1}^{\infty} A_i$. First we show, by induction, that $P_n = 1$.

The case $n = 1$ is trivial. Now, assume the result holds for $n-1$ and consider $P_n = P\left(\bigcap_{i=1}^{n} A_i\right)$. Writing $A' = \bigcap_{i=1}^{n-1} A_i$, we have

$$P_n = P(A' \cap A_n).$$

By the inclusion-exclusion principle,

$$\begin{aligned} P_n &= P(A') + P(A_n) - P(A' \cup A_n) \\ &= 1 + 1 - P(A' \cup A_n) \end{aligned}$$

and since $P(A' \cup A_n) \geq P(A') = 1$ by the monotonicity of the probability measure, $P(A' \cup A_n) = 1$ since $P(A' \cup A_n) \leq P(\Omega) = 1$, thus

$$\begin{aligned} &= 1 + 1 - 1 \\ &= 1. \end{aligned}$$

It follows that $\{P_n : n \in \mathbb{N}\}$ is the constant sequence $\{1\}$.

By Theorem 1.1 from DasGupta (this theorem is proved in Wheeden and Zygmund in more generality, not this problem, but DasGupta's Theorem 1.1), we have

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P_n = 1.$$

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Problem 2.10 (Handout 2, # 19)

There are n blue, n green, n red, and n white balls in an urn. Four balls are drawn from the urn with replacement. Find the probability that there are balls of at least three different colors among the four drawn.

Solution. ▶ Each ball has an equal probability of being drawn on each draw. That is, each (ordered) set of 4 draws is equally likely to occur.

There are 4^4 ways to draw 4 balls.

There are 4! ways to draw one ball of each color.

There are $4 \cdot 3\binom{4}{2}2!$ ways to draw 4 balls, missing exactly one color (pick a color to be missed, pick a color to be drawn twice, pick two draws for that color to be drawn on, rearrange the other two colors into the other two draws.)

Thus, the probability of having at least three different colors being drawn among those four is

$$\begin{aligned}\frac{4! + 4 \cdot 3\binom{4}{2}2!}{4^4} &= \frac{3! + 6 \cdot 4!/(2!2!)}{4^3} \\ &= \frac{6+36}{4^3} \\ &\cong 0.65625\end{aligned}$$

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