Rank 1 Character Varieties-Part II Generators

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MRC - Snowbird

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3. Therefore $\mathrm{Hom}(\mathsf{F}_r, G)$ is an affine variety:

$$\left\{\left(v_{11}^1, v_{12}^1, v_{21}^1, v_{22}^1, ..., v_{11}^r, v_{12}^r, v_{21}^r, v_{22}^r\right) \in \mathbb{C}^{4r} \mid v_{11}^k v_{22}^k - v_{12}^k v_{21}^k = 1,\ 1 \leq k \leq r\right\}.$$

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4. Therefore it has a coordinate ring $\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, \mathrm{SL}(2,\mathbb{C}))] \cong$

$$\mathbb{C}[x_{ij}^k \mid 1 \le i,j \le 2,\ 1 \le k \le r]/\langle x_{11}^k x_{22}^k - x_{12}^k x_{21}^k - 1 \mid 1 \le k \le r \rangle.$$

5. G acts on $\mathrm{Hom}(\mathsf{F}_r, G)$ by $g \cdot \rho = g\rho g^{-1}$; or equivalently on $G^{\times r}$ by $g \cdot (g_1, ..., g_r) = (gg_1g^{-1}, ..., gg_rg^{-1})$.

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7. Since G is reductive, the ring of invariants $\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, G)]^G := \{f \in \mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, G)] \mid g \cdot f = f\}$ is finitely generated.

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8. Therefore, $\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, G)]^G \cong \mathbb{C}[t_1, ..., t_N]/\mathfrak{I}$ for some ideal $\mathfrak{I}$.

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10. Hilbert's Nullstellensatz says that the maximal ideals in $\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, G)]^G$ are exactly of the form $\langle t_1 - a_1, \ldots, t_N - a_N\rangle + \mathfrak{I}$ where $(a_1, \ldots, a_N) \in \mathbb{C}^N$ is in the simultaneous zero set of the generators of $\mathfrak{I}$.

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11. This correspondence, $\langle t_1 - a_1, ..., t_N - a_N\rangle + \mathfrak{I} \mapsto (a_1, ..., a_N)$, determines an algebraic set.
12. This space is the character variety

$$\mathfrak{X}_r(\mathrm{SL}(2, \mathbb{C})).$$

Generators and the Non-Commutative Picture

- Recall that $\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, \mathsf{SL}(2,\mathbb{C}))] = \mathbb{C}[x_{ij}^k]/\Delta$ where Δ is the ideal generated by the r irreducible polynomials

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- We note that
$$\begin{pmatrix} x_{11}^k & x_{12}^k \\ x_{21}^k & x_{22}^k \end{pmatrix}$$
are simply called *generic matrices*.

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- Closely related to $\mathbb{C}[\mathfrak{X}_r] := \mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, \mathsf{SL}(2,\mathbb{C}))]^{\mathsf{SL}(2,\mathbb{C})}$ is the ring of invariants

$$\mathbb{C}[\mathfrak{Y}_r] := \mathbb{C}[\mathfrak{gl}(2,\mathbb{C})^{\times r}]^{\mathsf{SL}(2,\mathbb{C})} = \mathbb{C}[x_{ij}^k]^{\mathsf{SL}(2,\mathbb{C})}.$$

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- In fact $\mathbb{C}[\mathfrak{Y}_r]/\Delta \approx \mathbb{C}[\mathfrak{X}_r]$.
- Otherwise stated,

$$\mathbb{C}[x_{ij}^k]^{\mathrm{SL}(2,\mathbb{C})}/\Delta \approx \left(\mathbb{C}[x_{ij}^k]/\Delta\right)^{\mathrm{SL}(2,\mathbb{C})};$$

which is true because $\mathrm{SL}(2,\mathbb{C})$ is *linearly* reductive and the generators of Δ are invariants.

First Fundamental Theorem of Matrix Invariants

In 1976 Procesi proved

Theorem (Procesi)

$\mathbb{C}[\mathfrak{Y}_r]$ *is generated by the invariants* $\mathrm{tr}(\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_k})$*, where* $\mathbf{X}_j$ *are generic matrices.*

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- As an immediate corollary, we have that $\mathbb{C}[\mathfrak{X}_r]$ is generated by the invariants $\mathrm{tr}(\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_k})$, where $\mathbf{X}_j$ are generic *unimodular* matrices.

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- Executive Comment: As the determinant measures volume, one should think of the trace as measuring length. So Procesi's coordinates are "length coordinates."

Introduction to "word play"

- The Cayley-Hamilton equation gives

$$\mathbf{X}^2 - \mathrm{tr}(\mathbf{X})\mathbf{X} + \det(\mathbf{X})\mathbf{I} = \mathbf{0}.$$

And if we assume $\det(\mathbf{X}) = 1$, as is the case in $\mathbb{C}[\mathfrak{X}_r]$, we easily derive $\mathrm{tr}(\mathbf{X}^{-1}) = \mathrm{tr}(\mathbf{X})$ and $\mathrm{tr}(\mathbf{X}^2) = \mathrm{tr}(\mathbf{X})^2 - 2$.

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Example: $r = 1$

- We also get from the characteristic equation (multiplying by $\mathbf{X}^{n-2}$): $\mathbf{X}^n - \mathrm{tr}(\mathbf{X})\mathbf{X}^{n-1} + \mathbf{X}^{n-2} = \mathbf{0}$, which in turn gives $\mathrm{tr}(\mathbf{X}^n) = \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{X}^{n-1}) - \mathrm{tr}(\mathbf{X}^{n-2})$.

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- Precisely, the words are $\{\mathbf{X}_1, ..., \mathbf{X}_r\}$.
- Since the dimension of $\mathfrak{X}_1$ is 1, we have proved: $\mathbb{C}[\mathfrak{X}_1] \cong \mathbb{C}[t]$ where t corresponds to the invariant function $\mathrm{tr}(\mathbf{X})$.

Example: $r = 2$

- More generally, note that the dimension of $\mathfrak{X}_r$ is equal to $3r - 3$ for $r \geq 2$.
- Multiplying the Cayley-Hamilton equation on both sides by words $\mathbf{U}$ and $\mathbf{V}$ allows us to freely eliminate the generators of type: $\mathrm{tr}(\mathbf{U}\mathbf{X}^n\mathbf{V})$ as long as $n \geq 2$ and at least one of $\mathbf{U}$ or $\mathbf{V}$ is not the identity.

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- So for the case, $\mathbb{C}[\mathfrak{X}_2]$ we are left with the generators $\mathrm{tr}(\mathbf{X}_1), \mathrm{tr}(\mathbf{X}_2), \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2)$ since any other expression in two letters would result in a sub-expression with an exponent greater than one, which we just showed was impossible.

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- Consequently, there are exactly $\binom{r}{2}$ generators of type $\mathrm{tr}(\mathbf{XY})$ in $\mathbb{C}[\mathfrak{X}_r]$.
- This also gives a direct (and short) proof of the Fricke-Vogt theorem: $\mathfrak{X}_2 \cong \mathbb{C}^3$ (equivalently $\mathbb{C}[\mathfrak{X}_2] \cong \mathbb{C}[x, y, z]$).

A general remark

The only connected rank 1 complex Lie groups are $\mathsf{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$. They are related by the quotient map $\mathsf{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ with fibre $\{\pm I\}$. Since the fibre is central (i.e. commutes with the conjugation action) there is a natural map $\mathfrak{X}_\Gamma(\mathsf{SL}(2,\mathbb{C})) \to \mathfrak{X}_\Gamma(\mathrm{PSL}(2,\mathbb{C}))$ with fibre $\mathfrak{X}_\Gamma(\mathbb{Z}/2\mathbb{Z})$. Using this map, one can determine the relations and generators of $\mathfrak{X}_\Gamma(\mathrm{PSL}(2,\mathbb{C}))$ from thos of $\mathfrak{X}_\Gamma(\mathsf{SL}(2,\mathbb{C}))$.
See *G-Character varieties for $G = \mathsf{SO}(n,\mathbb{C})$ and other not simply connected groups* by Adam S. Sikora.

First step to fundamental relation: Polarization

- Replacing $\mathbf{X}$ with $\mathbf{X}+\mathbf{Y}$ in the Cayley-Hamilton equation gives

$$(\mathbf{X}+\mathbf{Y})^2 - \mathrm{tr}(\mathbf{X}+\mathbf{Y})(\mathbf{X}+\mathbf{Y}) + \det(\mathbf{X}+\mathbf{Y})\mathbf{I} = \mathbf{0}.$$

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- Simplifying this expression yields

$$\mathbf{XY}+\mathbf{YX}=\mathrm{tr}(\mathbf{X})\mathbf{Y}+\mathrm{tr}(\mathbf{Y})\mathbf{X}-\mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\mathbf{I}+\mathrm{tr}(\mathbf{XY})\mathbf{I}.$$

Second step, but an important step...

Multiplying on the right by $\mathbf{Z}$ we get the expression

$$\begin{aligned}\mathrm{tr}(\mathbf{XYZ}) + \mathrm{tr}(\mathbf{YXZ}) =& \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{YZ}) + \mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{XZ}) \\ &- \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{Z}) + \mathrm{tr}(\mathbf{XY})\mathrm{tr}(\mathbf{Z}).\end{aligned}$$

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At this point, we see that we only need $\binom{r}{3}$ generators of the form $\mathrm{tr}(\mathbf{XYZ})$, and no others of length 3 or more in three letters Remember we already have shown we never need exponents beyond 1 in any letter.

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- Now taking this relation and substituting $\mathbf{Z} \mapsto \mathbf{ZW}$ gives a relation for $\mathrm{tr}(\mathbf{XYZW}) + \mathrm{tr}(\mathbf{YXZW})$.

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- Sending $\mathbf{X} \mapsto \mathbf{XW}$ gives $\mathrm{tr}(\mathbf{XWYZ}) + \mathrm{tr}(\mathbf{YXWZ})$; and $\mathbf{Z} \mapsto \mathbf{WZ}$ gives $\mathrm{tr}(\mathbf{XYWZ}) + \mathrm{tr}(\mathbf{YXWZ})$.

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- Subtracting, adding, and subtracting these four relations gives and expression for $\mathrm{tr}(\mathbf{XYZW}) - \mathrm{tr}(\mathbf{XYWZ})$.

Fundamental Relation

- However, sending $\mathbf{X} \mapsto \mathbf{W} \mapsto \mathbf{Y} \mapsto \mathbf{Z} \mapsto \mathbf{X}$ in the first expression gives $\mathrm{tr}(\mathbf{XYZW}) + \mathrm{tr}(\mathbf{XYWZ})$.

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- So length 4 words are not need to generate the ring.

- Since **W** can be any word in the generic matrices, we have proved that $\mathbb{C}[\mathfrak{X}_r]$ is generated by at most $\binom{r}{1} + \binom{r}{2} + \binom{r}{3} = \frac{r(r^2+5)}{6}$ generators (so the ring is finitely generated!)

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- In particular, here are the generators:
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- We will see in a minute that this is a minimal generating set (we can't get rid of any either!)

Geometrically, this says that the minimal (trace) embedding of $\mathfrak{X}_r$ into $\mathbb{C}^N$ is when $N = \frac{r(r^2+5)}{6}$ and the mapping is exactly

$$[\rho] \mapsto \Big(\mathrm{tr}(\rho(\gamma_1)), \ldots, \mathrm{tr}(\rho(\gamma_r)), \mathrm{tr}(\rho(\gamma_1\gamma_2)), \ldots, \mathrm{tr}(\rho(\gamma_{r-1}\gamma_r)),$$
$$\mathrm{tr}(\rho(\gamma_1\gamma_2\gamma_3)), \ldots, \mathrm{tr}(\rho(\gamma_{r-2}\gamma_{r-1}\gamma_r)) \Big).$$

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- There are then only 7 generators.
- If $\mathrm{tr}(\mathbf{XYZ})$ was allowed to be eliminated, we would conclude that $\mathfrak{X}_3$ was affine $\mathbb{C}^6$.
- However, it is not hard to show there exists two representations which agree on the six generators of word length two or less but differ at $\mathrm{tr}(\mathbf{XYZ})$.

For instance, $\mathbf{X} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,$\mathbf{Y} = \begin{pmatrix} 0 & 2 \\ -1/2 & 0 \end{pmatrix}$,and $\mathbf{Z} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ or $\mathbf{Z} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ gives two such representations.

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Remark

One can further show there exists a product relation for $\mathrm{tr}(\mathbf{XYZ})\mathrm{tr}(\mathbf{YXZ})$. Together with the sum relation, we conclude that $\mathfrak{X}_3$ is a hypersurface.

Deriving the Product Relation

$$\begin{aligned}
\mathrm{tr}(\mathbf{XYZ})\mathrm{tr}(\mathbf{XZY}) &= \mathrm{tr}(\mathbf{X})^2 + \mathrm{tr}(\mathbf{Y})^2 + \mathrm{tr}(\mathbf{Z})^2 \\
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&- \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{XY}) - \mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{Z})\mathrm{tr}(\mathbf{YZ}) \\
&- \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Z})\mathrm{tr}(\mathbf{XZ}) \\
&+ \mathrm{tr}(\mathbf{XY})\mathrm{tr}(\mathbf{YZ})\mathrm{tr}(\mathbf{XZ}) - 4
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- Using the characteristic equation, we derive that $(*)\ \mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - \mathrm{tr}(\mathbf{A}^{-1}\mathbf{B})$.

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- Again using $(*)$, we can simplify $\mathrm{tr}(\mathbf{B}^{-1}\mathbf{ABA})$, $\mathrm{tr}(\mathbf{ACA}^{-1}\mathbf{C}^{-1})$, and $\mathrm{tr}(\mathbf{ABAC})$.
- Together, these formulae give the product relation.

Exercises

1. By hand fill in the details for the derivations for the product formula $\mathrm{tr}(\mathbf{ABC})\mathrm{tr}(\mathbf{ACB})$.
2. By hand fill in the details for the derivation for the $\mathrm{tr}(\mathbf{XYZW})$.
3. Verify that the example representations I gave are the same on $\mathrm{tr}(\mathbf{A}), \mathrm{tr}(\mathbf{B}), \mathrm{tr}(\mathbf{C}), \mathrm{tr}(\mathbf{AB}), \mathrm{tr}(\mathbf{AC}), \mathrm{tr}(\mathbf{BC})$ but differ on $\mathrm{tr}(\mathbf{ABC})$ and $\mathrm{tr}(\mathbf{ACB})$.
4. Write algorithms by hand that turn $\mathrm{tr}(\mathbf{W})$ into a trace expression with every letter of every represented word having exponent 1 (non-negative and no-multiplicity) and no word having length greater than 3.
5. Together implement the above algorithm in *Mathematica*.