MA557 Problem Set 6

Carlos Salinas

October 20, 2015

Problem 6.1

For an n by n matrix φ with entries in R write $I_t(\varphi)$ for the R-ideal generated by all the t by t minors of φ (set $I_t(\varphi) = R$ for $t \leq 0$ and $I_t(\varphi) = 0$ for $t > \min\{m, n\}$). Thinking of φ as an R-linear map $\varphi\colon R^m \to R^n$ set $M = \operatorname{coker}(\varphi)$ and define $F_i(M) = \operatorname{Fitt}_i(M) = I_{n-i}(\varphi)$. This ideal is called the *ith Fitting ideal of M*. Show:

(a) $F_i(M)$ only depends on i and M (but not on m, n, φ).

(b) $(\operatorname{ann}(M))^n \subset F_0(M) \subset \operatorname{ann}(M)$.

(c) In case R is local, $F_i(M) = R$ if and only if $\mu(M) \leq i$.

(d) $V(F_i(M)) = \left\{ \mathfrak{p} \in \operatorname{Spec}(R) \;\middle|\; \mu_{R_\mathfrak{p}}(M_\mathfrak{p}) > i \right\}$.

Proof. ∎

Problem 6.2

Let I be an ideal in a Noetherian ring. Show that either I contains an R-regular element or else $aI = 0$ for some $0 \neq a \in R$.

Proof. Suppose R is a Noetherian ring and $I \subset R$ an ideal. Then, by 3.2, I is finitely generated, say $I = (a_1, ..., a_n)$, for $a_1, ..., a_n \in R$. Then, either I contains an R-regular element or it does not. If I does not contain an R-regular element, then for every a_i there exists $x_i \in R$ such that $a_i x_i = 0$. Thus, $I \subset \bigcup_{i=1}^{n} \operatorname{ann}(x_i)$, but each $\operatorname{ann}(x_i) \subset \mathfrak{p}_i$ for some $\mathfrak{p} \in \operatorname{Ass}(R)$ so $I \bigcup_{i=1}^{m} \mathfrak{p}_i$ for $m \leq n$. By the prime avoidance lemma, 1.7, it follows that $I \subset \mathfrak{p}_i = \operatorname{ann}(y_i)$ for some $1 \leq i \leq m$. Thus, $y_i I = 0$. ■

Problem 6.3

Let $I \subset J$ be ideals in a Noetherian ring. Show that if $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ for every associated prime $\mathfrak{p}$ of I, then $I = J$.

Proof. ■

Problem 6.4

Let R be a Noetherian ring and M a finite R-module. Show that $\ell(M) < \infty$ if and only if $\operatorname{Supp}(M) \subset \mathfrak{m}\text{-}\operatorname{Spec}(R)$.

Proof. ∎

Problem 6.5

Let R be a Noetherian ring, $M \neq 0$ a finite R-module, and

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

a chain of submodules with $M_i/M_{i-1} \cong R/\mathfrak{p}_i$, $\mathfrak{p}_i \in \operatorname{Spec}(R)$.

(a) Show that $\operatorname{Ass}(M) \subset \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ and that the minimal elements of the two sets coincide (hence only depend on M).

(b) Let $\mathfrak{p}$ be minimal in $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$. Show that in any chain as above, the multiplicity with witch the factor $R/\mathfrak{p}$ appears is $\ell_{R_\mathfrak{p}}(M_\mathfrak{p})$ (hence only depends on M).

Proof. ∎

Problem 6.6

Let $R = k[X, Y]$ be a polynomial ring over a field and $I = (X^2, XY) \subset R$. Find two distinct shortest primary decompositions of I.

Proof. ■