Profinite Groups and Group Cohomology

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Motivation

Profinite groups and group cohomology arise in many areas of mathematics. I will provide some broad motivation for each topic. The reason for this motivation is that in treating these topics rigorously and in some generality, we will see that there is a fair amount of language, notation, .etc needed for these topics. I cannot do both topics the time they separately deserve and have chosen profinite groups as the main topic.

There are many views one can take in exploring the topic of profinite groups. In Galois theory, one sees that profinite groups arise naturally in taking closures or limits in extension theory. Profinite groups arise in covering space theory of manifolds and varieties as limits of all of the finite deck/Galois groups of the covers. The profinite completion of a finitely generated group is a universal object for the finite representation theory of the group. All of these examples illustrate why profinite groups of centrally important in mathematics. They serve as completions/limits and as universal objects.

Group cohomology from my personal perspective is a tool/view that every mathematician should be casual familiar with. Cohomology theory, in the broadest sense, is amazingly diverse. It could measure the failure of satisfying a certain condition/equation. As such, it can be viewed as an obstruction. It could also parameterizes objects or encoding some infinitesimal deformation theory. For me, cohomology theory comes to life when it is used in a specific setting. For instance, if Γ is a group, there is an associated topological space $K(\Gamma,1)$ that has the same cohomology theory as Γ. The best example of such an association is the circle $S^1 = \mathbf{R}/\mathbf{Z}$ with the integers $\mathbf{Z}$. This gives a concrete picture to the abstract treatment I will give here. Time permitting, I may give a lecture at the end of the course on Galois cohomology in a concrete setting like Brauer groups. It would serve solely as a highlight for cohomology theory and lack in details.

The class notes will follow Wilson's book [2]. Another suggested reference for the class is Ribes–Zalesskii [1]. Additional references will be added as the lecture notes develop.

Chapter 1

Preliminaries

In this preliminary module, we will quickly review some point-set topology and topological groups that we will use in the sequel. Important terms will be listed in **bold**. Blue text indicates that there is a link to a webpage with additional information. Often the webpages will be wikipedia pages.

1.1 Lecture 1. Topological Spaces

By a **topological space** we mean a pair $(X, \mathscr{T})$, where X is a set and $\mathscr{T}$ is a set of subsets of X satisfying:

(i) $\emptyset, X \in \mathscr{T}$.

(ii) If $U_1, U_2 \in \mathscr{T}$, then $U_1 \cap U_2 \in \mathscr{T}$.

(iii) For any subset $\mathscr{S} \subset \mathscr{T}$, $\bigcup_{U \in S} U \in \mathscr{T}$.

The sets $U \in \mathscr{T}$ are referred to as **$\mathscr{T}$–open sets** or simply **open** when the topology $\mathscr{T}$ is understood. A subset $C \subset X$ is **$\mathscr{T}$–closed** if $X - C \in \mathscr{T}$. Given a subset $Y \subset X$, we define the **closure** of Y in X to be the intersection of all the closed sets $C \subset X$ such that $Y \subset C$. We denote the closure of Y by $\overline{Y}$. We say $Y \subset X$ is **dense** if $\overline{Y} = X$.

For each $x \in X$, we say $U \in \mathscr{T}$ is an **open neighborhood of x** or simply a **neighborhood of x** if $x \in U$. A **base** for a topology $\mathscr{T}$ is any subset $\mathscr{B}$ of $\mathscr{T}$ such that every $U \in \mathscr{T}$ can be expressed as a union of open sets in $\mathscr{B}$. A **neighborhood base** at x is any collection $\mathscr{B}_x$ of neighborhoods of x such that every neighborhood of x can be expressed as a union of sets in $\mathscr{B}_x$.

Example: Discrete Topology

If $\mathscr{T}$ is the power set $\mathscr{P}(X)$ of X, then $\mathscr{T}$ is a topology. This topology is called the **discrete topology**. Every subset of X is both open and closed.

Given a topological space $(X,\mathscr{T})$ and a subset $Y \subset X$, we define the **subspace topology** $\mathscr{T}_{X,Y}$ on Y by

$$\mathscr{T}_{X,Y} = \{U \cap Y \;:\; U \in \mathscr{T}\}.$$

We will refer to $(Y,\mathscr{T}_{X,Y})$ as a subspace of $(X,\mathscr{T})$.

We say that $(X,\mathscr{T})$ is **compact** if given any subset $\mathscr{C} \subset \mathscr{T}$ such that $X = \bigcup_{U\in\mathscr{C}} U$, there exists a finite subset $\mathscr{C}_0 \subset S$ such that $X = \bigcup_{U\in\mathscr{S}_0} U$. We will say that a subset $Y \subset (X,\mathscr{T})$ is compact if $(Y,\mathscr{T}_{X,Y})$ is a compact space. The following lemma is immediate from the definitions of closed and compact.

Lemma 1.1. *Let $(X,\mathscr{T})$ be a compact space. If $\mathscr{S}$ is a collection of closed sets of X such that for any finite subset $\mathscr{S}_0 \subset \mathscr{S}$, we have $\bigcap_{C\in\mathscr{S}_0} C \neq \emptyset$, then $\bigcap_{C\in\mathscr{S}} C \neq \emptyset$.*

We say a space $(X,\mathscr{T})$ is **Hausdorff** if given distinct $x_1,x_2 \in X$, there exists disjoint open sets $U_1,U_2 \in \mathscr{T}$ such that $x_i \in U_i$ for $i = 1,2$. If X is a Hausdorff space, then $\{x\}$ is closed for all $x \in X$. We say a space $(X,\mathscr{T})$ is **connected** if X cannot be expressed as the union of two disjoint closed sets. We say a space $(X,\mathscr{T})$ is **totally disconnected** if every connected subspace has at most one element.

Lemma 1.2. *Let X be a compact Hausdorff space.*

(a) If C_1,C_2 are disjoint closed subsets of X, then there exist disjoint open subsets U_1,U_2 of X such that $C_i \subset U_i$ for $i = 1,2$.

(b) If $x \in X$ and A_x is the intersection of all sets U containing x that are both open and closed, then A_x is connected.

(c) If X is also totally disconnected, then every open set is a union of sets that are both open and closed.

Proof: We start with (a). First, we assert that for each $x \in C_1$, there exist disjoint open sets U_x,V_x such that $x \in U_x$ and $C_2 \subset V_x$. For each $y \in C_2$, there exist disjoint open sets

$U_{x,y}, V_{x,y}$ such that $x \in U_{x,y}$ and $y \in V_{x,y}$. The set of open sets $\mathscr{C}_x = \{X - C_2\} \cup \{V_{x,y}\}_{y \in C_2}$ is an open cover of X. Since X is compact, there exists a finite subset $\{y_1, \dots, y_n\}$ of C_2 such that X is a union of $X - C_2$ and the sets V_{x,y_i}. Taking

$$U_x = \bigcap_{i=1}^{n} U_{x,y_i}, \quad V_x = \bigcup_{i=1}^{n} V_{x,y_i}$$

verifies our first assertion. Varying U_x over $x \in C_1$, we obtain disjoint open sets U_x, V_x such that $x \in U_x$ and $C_2 \subset V_x$. As before, the sets $X - C_1$ and U_x are an open cover of X. Since X is compact, there exists a finite set $\{x_1, \dots, x_m\}$ of C_1 such that X is a union of $X - C_1$ and the sets U_{x_i}. Finally, we define

$$U_1 = \bigcup_{i=1}^{m} U_{x_i}, \quad U_2 = \bigcap_{i=1}^{m} V_{x_i}.$$

Next we prove (b). Given $x \in X$, recall that A_x is the intersection of all subsets of X that contain x and are both open and closed. Assume that $A_x = C_1 \cup C_2$, where C_1, C_2 are open in A_x and disjoint. Since A_x is the intersection of closed sets, A_x is closed in X. Since $C_{i+1} = A - C_i$, both C_1, C_2 are closed in A_x. In tandem, we see that C_1, C_2 are also closed in X. By (a), there exists disjoint open sets U_1, U_2 in X such that $C_i \subset U_i$. Set $C = X - (U_1 \cup U_2)$ and let $\mathscr{S}$ be the collection of closed sets comprised of C and every set in X that contains x and is both open and closed. By construction, $\mathscr{S}$ is a collection of closed sets such that $\bigcap_{C' \in \mathscr{S}} C' = \emptyset$; for the latter, simply note that C is disjoint from A_x. Since X is compact, there exists a finite collection of subsets $V_1, \dots, V_n$ of X that contain x, are both open and closed, and satisfy

$$C \cap \left(\bigcap_{i=1}^{n} V_i \right) = \emptyset.$$

Set $V = \bigcap_{i=1}^{n} V_i$ and note that $V \subset U_1 \cup U_2$. In particular, V is a disjoint union of $W_1 = V \cap U_1$ and $W_2 = V \cap U_2$, and so W_1, W_2 are both open and closed in V. Since V is also both open and closed in X, the subsets W_1, W_2 are both open and closed in X. Since $x \in V$, it follows that $x \in W_1$ or $x \in W_2$. If $x \in W_1$, then $A_x \subset W_1 \subset U_1$ and so $C_2 \subset A_x \cap U_2 \subset U_1 \cap U_2 = \emptyset$. Similarly, if $x \in W_2$, then $A_x \subset W_2 \subset U_2$ and so $C_1 \subset A_x \cap U_1 \subset U_1 \cap U_2 = \emptyset$. Hence, we see that A_x is connected.

Finally, we prove (c). Let U be an open set in X and let $x \in U$. For each $y \in X$ distinct from x, there exists a set F_y in X that is both open and closed and satisfies $x \in F_y, y \notin F_y$.

Setting $U_y = X - F_y$, we see that X is the union of U with the sets U_y. Since X is compact, there exists a finite subset $\{y_1, \dots, y_n\}$ of $X - \{x\}$ such that

$$X = U \cup \left(\bigcup_{i=1}^{n} (X - F_{y_i}) \right) = U \cup \left(X - \left(\bigcap_{i=1}^{n} F_{y_i} \right) \right).$$

Therefore,

$$\bigcap_{i=1}^{n} F_{y_i} \subset U.$$

Since this open and closed set contains x and x is arbitrary, the result follows. ♠

Given a pair of topological spaces X, Y, we say a function $f\colon X \to Y$ is **continuous** if for each open subset $V \subset Y$, the subset $f^{-1}(V) \subset X$ is open. Alternatively, $f^{-1}(C)$ is closed in X for every closed subset $C \subset Y$. We say a function $f\colon X \to Y$ is a **homeomorphism** if f is bijective and f, f^{-1} are both continuous.

Lemma 1.3.

(a) *Every closed subset of a compact space is compact.*

(b) *Every compact subset of a Hausdorff space is closed.*

(c) *If $f\colon X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

(d) *If $f\colon X \to Y$ is continuous and bijective, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

(e) *If $f, g\colon X \to Y$ are continuous and Y is Hausdorff, then the set*

$$E(f,g) = \{x \in X \ : \ f(x) = g(x)\}$$

is a closed subset of X.

Proof: To prove (a), let X be a compact space and C is a closed subset of X. If $\mathscr{C}$ is an open covering of C, then for each $V \in \mathscr{C}$, there is an open subset U_V of X such that $V = C \cap U_V$. The open sets $X - C$ and the open sets $\{U_V\}_{V \in \mathscr{C}}$ form an open covering of X. Since X is compact, there is a finite subset $\mathscr{C}_0$ of $\mathscr{C}$ such that $X - C$ and $\{U_V\}_{V \in \mathscr{C}_0}$ cover X. It is clear that $\mathscr{C}_0$ is a cover of C.

To prove (b), let X be a Hausdorff space and $C \subset X$ a compact subspace. To prove that C is closed, we will prove that $X - C$ is open. For each $x \in X - C$ and $y \in C$, there exist disjoint open subsets $U_{x,y}, V_{x,y}$ in X such that $x \in U_{x,y}$ and $y \in V_{x,y}$. The collection $\mathscr{C} = \left\{V_{x,y}\right\}_{y \in C}$ is an open cover of C and since C is compact, there exists a finite subset $\{y_1, \dots, y_n\}$ of C such that $\left\{V_{x,y_i}\right\}_{i=1}^{n}$ is a cover of C. The open set $U = \bigcap_{i=1}^{n} U_{x,y_i}$ is an open subset of X that contains x and is contained in $X - C$. Since $x \in X$ was arbitrary, we see that $X - C$ is open.

To prove (c), giving any open cover $\mathscr{C}$ of $f(X)$, we have an associated open cover $f^{-1}(\mathscr{C}) = \left\{f^{1}(U)\right\}_{U \in \mathscr{C}}$. Since X is compact, there is a finite subset $\mathscr{C}_0$ of $\mathscr{C}$ such that $\left\{f^{-1}(U)\right\}_{U \in \mathscr{C}_0}$ is a cover. It is straightforward to see that $\mathscr{C}_0$ is the desired finite subcover needed to prove $f(X)$ is compact.

To prove (d), it is enough to prove that f is a closed mapping which follows from (a)-(c).

To prove (e), we show that $U_{f,g} = X - E_{f,g}$ is open. Given $x \in U_{f,g}$, by definition $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open sets V_f, V_g of Y such that $\alpha(x) \in V_\alpha$ for $\alpha = f, g$. Since f, g are continuous, $f^{-1}(V_f) \cap f^{-1}(V_g)$ is an open subset of $U_{f,g}$ containing x, and so $U_{f,g}$ is open since x was arbitrary. ♠

Lemma 1.4. *If X is totally disconnected, then $\{x\}$ is closed in X for every $x \in X$.*

Proof: Let C_x denote the closure of $\{x\}$ in X and assume that $C_x = U_1 \cup U_2$ where U_1, U_2 are disjoint union of open sets U_j in X. The sets $C_j = X - U_j$ are closed subsets and so one of them must contain x. Without loss of generality, we assume $x \in C_1$. Since C_x is the closure of $\{x\}$, we see that $C_x \subset U_1$ and so $U_2 = \emptyset$. ♠

Given an equivalence relationship $\sim$ on a space X, the quotient space of equivalence classes will be denoted by $X/\sim$. The set $X/\sim$ can be given a topological structure by taking the weakest topology on $X/\sim$ such that $\pi_\sim \colon X \to X/\sim$ is continuous. Specifically, $V \subset X/\sim$ is open if and only if $\pi_\sim^{-1}(V)$ is open in X. This topology is called the **quotient topology** and we refer to $X/\sim$ with this topology as the **quotient space**. This topology satisfies the following universal mapping property: If $f \colon X \to Y$ is a continuous function such that $f(x) = f(y)$ whenever $x \sim y$, then there exists a unique function $\widetilde{f} \colon (X/\sim) \to Y$ such that $f = \widetilde{f} \circ \pi_\sim$. We will refer to $\pi_\sim$ as the **quotient map** and will often simply denote it by π.

1.2 Lecture 2. Filters and Product Spaces

Given a family $\{X_\lambda\}_{\lambda\in\Lambda}$ of topological spaces, the product $\prod_{\lambda\in\Lambda} X_\lambda$ is the set of functions $x\colon \Lambda \to \cup_\lambda X_\lambda$ such that $x(\lambda) \in X_\lambda$. It is sometime convenient to view these functions as lists or vector $x = (x_\lambda)$ where $x_\lambda = x(\lambda)$. For each $\lambda_0 \in \Lambda$, we have a projection map $\pi_{\lambda_0}\colon \prod_{\lambda\in\Lambda} X_\lambda \to X_{\lambda_0}$ given by $\pi_{\lambda_0}(x) = x_{\lambda_0}$. We endow $\prod_{\lambda\in\Lambda} X_\lambda$ with a topology generated by basic open sets of the form $\bigcap_{i=1}^n \pi_{\lambda_i}^{-1}(U_i)$ as we vary $n \in \mathbf{N}$, the indices $\lambda_1,\dots,\lambda_n \in \Lambda$, and open subsets $U_i \subset X_{\lambda_i}$. This topology is called the **product topology**. We refer to $\prod_{\lambda\in\Lambda} X_\lambda$ with the product topology as the **product space**. The following is left as an exercise.

Lemma 1.5. *Let $\{X_\lambda\}$ be a family of topological spaces with product space $\prod_{\lambda\in\Lambda} X_\lambda$. Given a topological space Z, there exists continuous functions $f_\lambda : Z \to X_\lambda$ for each $\lambda \in \Lambda$ if and only if there exists a continuous function $f\colon Z \to \prod_{\lambda\in\Lambda} X_\lambda$ such that $f_\lambda = \pi_\lambda \circ f$ for all $\lambda \in \Lambda$.*

For $a = (a_\lambda) \in \prod_{\lambda\in\Lambda} X_\lambda$ and an open neighborhood N_a of a in $\prod_{\lambda\in\Lambda} X_\lambda$, by definition of the product topology, there exists $n \in \mathbf{N}$, $\lambda_1,\dots,\lambda_n \in \Lambda$, and open sets $U_i \subset X_{\lambda_i}$ such that

$$\bigcap_{i=1}^n \pi_{\lambda_i}^{-1}(U_i) \subset N_a.$$

In particular, the $a_{\lambda_i} \in U_i$ for $i = 1,\dots,n$ and

$$\left\{ x \in \prod_{\lambda\in\Lambda} X_\lambda \ : \ \pi_{\lambda_i}(x) = a_{\lambda_i} \text{ for } i = 1,\dots,n \right\} \subset N_a.$$

Theorem 1.6. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of topological spaces with product space $\prod_{\lambda\in\Lambda} X_\lambda$.*

(a) If each X_λ is Hausdorff, then $\prod_{\lambda\in\Lambda} X_\lambda$ is Hausdorff.

(b) If each X_λ is totally disconnected, then $\prod_{\lambda\in\Lambda} X_\lambda$ is totally disconnected.

(c) If each X_λ is compact, then $\prod_{\lambda\in\Lambda} X_\lambda$ is compact.

Proof of (a) and (b): To prove (a), given distinct points $x, y \in \prod_{\lambda\in\Lambda} X_\lambda$, we have $x = (x_\lambda)$, $y = (y_\lambda)$. As $x \neq y$, there exists $\lambda_0 \in \Lambda$ such that $x_{\lambda_0} \neq y_{\lambda_0}$. Since X_{λ_0} is Hausdorff, there exists disjoint opens sets $U_{\lambda_0}, V_{\lambda_0} \subset X_{\lambda_0}$ such that $x \in U_{\lambda_0}$, $y \in V_{\lambda_0}$. It follows that $U_x = \pi_{\lambda_0}^{-1}(U_{\lambda_0})$ and $V_x = \pi_{\lambda_0}^{-1}(V_{\lambda_0})$ separate x, y in $\prod_\lambda X_\lambda$.

To prove (b), if $C \subset \prod_{\lambda\in\Lambda} X_\lambda$ is a non-empty connected subset, then $\pi_\lambda(C) = C_\lambda$ is connected for each λ since π_λ is continuous. As X_λ is totally disconnected and C is not empty, C_λ is a singleton set $\{c_\lambda\}$ for each λ and hence $C = \{c\}$ where $c = (c_\lambda)$. ♠

To prove Theorem 1.6 (c), which is known as **Tychonoff's Theorem**, we will use ultrafilters and Zorn's lemma.

Recall in a non-empty partially ordered set S with relation $\leq$, a subset $C \subset S$ is a **chain** if for any pair of elements $c_1, c_2 \in C$, we have either $c_1 \leq c_2$ or $c_2 \leq c_1$. An element $m \in S$ is **maximal** if $m \leq s$ implies $s = m$.

Lemma 1.7 (Zorn's Lemma)**.** *If S is a non-empty partially ordered set such that for each chain $C \subset S$, there exists an element $s_C \in S$ such that $c \leq s_C$ for each $c \in C$, then S contains a maximal element.*

Given a set X and a family of subsets $\mathscr{L}$ of X that is closed under finite intersections and finite unions, we say a family of sets $\mathscr{F} \subset \mathscr{L}$ is a **filter** if

(i) If $F_1, F_2 \in \mathscr{F}$, then $F_1 \cap F_2 \in \mathscr{F}$.

(ii) If $F \in \mathscr{F}$ and $F \subset F'$ for some $F' \in \mathscr{F}$, then $F' \in \mathscr{F}$.

(iii) $\emptyset \notin \mathscr{F}$.

In discussing filters, we will often omit which family of sets $\mathscr{L}$ our filters reside in. The two primary examples for $\mathscr{L}$ will be the set of all closed sets and the power set on X. Unless specified otherwise, the reader can take $\mathscr{L} = \mathscr{P}(X)$.

The simplest examples of filters are principal filters given as follows. For any set $S \subset X$, we define $\mathscr{F}_S$ to be the set of subsets of X that contain S. It is a simple matter to see that (i)–(iii) is satisfied by $\mathscr{F}_S$. The filter $\mathscr{F}_S$ is called a **principal filter** associated to S. If X is a topological space and S is a closed subset, we take our filter $\mathscr{F}_S$ to be comprised of all supersets of S that are closed in X.

Another important example of a filter on X is the **cofinite filter** or **Fréchet filter**. This filter $\mathscr{F}_{cf}(X)$ is defined as

$$\mathscr{F}_{cf}(X) = \{Y \subset X \ : \ X - Y \text{ is finite}\}.$$

This is a filter provided X is infinite since $\emptyset \in \mathscr{F}_{cf}(X)$ when X is finite.

The set of all filters in X that are contained in $\mathscr{L}$ is a partially ordered set via set inclusion. A filter in $\mathscr{L}$ that is maximal with respect to this partial ordering is called a **ultrafilter** in $\mathscr{L}$. Note that since the union of a chain of filters in $\mathscr{L}$ is easily seen to be a filter, every filter $\mathscr{F}$ in $\mathscr{L}$ is contained in an ultrafilter in $\mathscr{L}$ by Zorn's Lemma. The following lemma contains some useful results on filters and ultrafilters.

Lemma 1.8.

(a) Every filter $\mathscr{F}$ is contained in an ultrafilter.

(b) A filter $\mathscr{F}$ in X is an ultrafilter if and only if for each $Y \subset X$, either $Y \in \mathscr{F}$ or $X - Y \in \mathscr{F}$.

(c) For any $x \in X$, the principal filter $\mathscr{F}_{\{x\}}$ is an ultrafilter.

(d) If X is finite, every ultrafilter $\mathscr{F}$ in X is principal.

(e) If X is infinite and $\mathscr{F}$ is a non-principal ultrafilter, then $\mathscr{F}$ contains $\mathscr{F}_{cf}$.

The proofs of the assertions are fairly easy and left to the reader.

Exercise. Prove Lemma 1.8

We say that a set $\mathscr{S} \subset \mathscr{P}(X)$ satisfies the **finite intersection property** if for each finite subset $\mathscr{S}_0 \subset \mathscr{S}$, we have $\bigcap_{S \in \mathscr{S}_0} S \neq \emptyset$. By definition, a filter $\mathscr{F}$ satisfies the finite intersection property. Moreover, any collection of sets with the finite intersection property is contained in a filter.

Lemma 1.9. *If $\mathscr{S}$ is a family of subsets in X with the finite intersection property, then there exists a filter $\mathscr{F}$ that contains $\mathscr{S}$. If X is a topological space and $\mathscr{S}$ consists of closed subsets of X, the filter $\mathscr{F}$ can be taken to contain only closed subsets of X.*

Proof: The filter $\mathscr{F}$ can be built in two stages. First, we add all the finite intersections of the members of $\mathscr{S}$. Second, we take all of the supersets of the members of this new family to get the filter. If X is a topological space and $\mathscr{S}$ consists of closed subsets of X, then in the first stage, the new members will be closed. In the second stage, we simply add only the supersets of S that are closed in X. ♠

We saw in Lemma 1.1 that a topological space X is compact if any only if for each family of closed sets $\mathscr{S}$ of X with the finite intersection property must satisfy $\bigcap_{S \in \mathscr{S}} S \neq \emptyset$.

Lemma 1.10. *A topological space X is compact if and only if for each ultrafilter $\mathscr{F}$ in X of closed sets, we have $\bigcap_{F\in\mathscr{F}} F \neq \emptyset$.*

Proof: For the direct implication, simply note that since X is compact, by Lemma 1.1, we must have $\bigcap_{F\in\mathscr{F}} F \neq \emptyset$ for any filter $\mathscr{F}$ in X consisting of closed sets. For the converse direction, by Lemma 1.1, we must show that for any collection of closed sets $\mathscr{S}$ with the finite intersection property, that we have $\bigcap_{S\in\mathscr{S}} S \neq \emptyset$. By Lemma 1.9, $\mathscr{S}$ is contained in a filter $\mathscr{F}$ of closed sets and hence by hypothesis $\emptyset \neq \bigcap_{F\in\mathscr{F}} F \subset \bigcap_{S\in\mathscr{S}} S$. ♠

We are now ready to prove Theorem 1.6 (c).

Proof of Theorem 1.6 (c): By Lemma 1.10, it suffices to prove that for any ultrafilter $\mathscr{F}$ of closed subsets of $\prod_{\lambda\in\Lambda} X_\lambda$ must satisfy $\bigcap_{F\in\mathscr{F}} F \neq \emptyset$. For each $\lambda \in \Lambda$, we have the filter $\mathscr{F}_\lambda$ in X_λ given by taking all of the closed sets $C \subset X_\lambda$ such that $\pi_\lambda(F) \subset C$ for some $F \in \mathscr{F}$. Since X_λ is compact, we know that $\bigcap_{C\in\mathscr{F}_\lambda} C \neq \emptyset$ and we select $y_\lambda \in X_\lambda$ such that $y_\lambda \in C$ for all $C \in \mathscr{F}_\lambda$. We assert that $y = (y_\lambda)$ is contained in each $F \in \mathscr{F}$. If not, since $\prod_{\lambda\in\Lambda} X_\lambda - F$ is open, there exist open sets $U_1,\dots,U_n$ in $X_{\lambda_1},\dots,X_{\lambda_n}$ such that $y \in \pi_{\lambda_i}^{-1}(U_i)$ for each i and F is disjoint from $\bigcap_{i=1}^n \pi_{\lambda_i}^{-1}(U_i)$. In particular,

$$F \subset \bigcup_{i=1}^n \left(\prod_{\lambda\in\Lambda} X_\lambda - \pi_{\lambda_i}^{-1}(U_i) \right).$$

Since $\mathscr{F}$ is an ultrafilter, by Lemma 1.8 (b), $\prod_{\lambda\in\Lambda} X_\lambda - \pi_{\lambda_{i_0}}^{-1}(U_{i_0}) \in \mathscr{F}$ for some i_0. However, we see that

$$\pi_{\lambda_{i_0}}(\prod_{\lambda\in\Lambda} X_\lambda - \pi_{\lambda_{i_0}}^{-1}(U_{i_0})) \subset X_{\lambda_{i_0}} - U_{i_0},$$

and so $X_{\lambda_{i_0}} - U_{i_0} \in \mathscr{F}_{\lambda_{i_0}}$. Since $y \in \pi_{\lambda_{i_0}}^{-1}(U_{i_0})$, we see that $y_{\lambda_{i_0}} \notin X_{\lambda_{i_0}} - U_{i_0}$, contradicting the fact that $y_\lambda \in C$ for all $\lambda \in \Lambda$ and $C \in \mathscr{F}_\lambda$. Therefore, we conclude that $y \in F$ for all $F \in \mathscr{F}$ and so $\bigcap_{F\in\mathscr{F}} F \neq \emptyset$ as needed to verify that $\prod_{\lambda\in\Lambda} X_\lambda$ is compact. ♠

We now record some results on topological groups that we will need in our discussion on profinite groups. By a **topological group**, we mean a set G that is both a topological space and a group (the binary operation will be denoted multiplicatively) such that the function $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is continuous where the domain is equipped with the product topology. We will denote the identity element in G by 1_G and simply by 1 when G is understood. Given a topological group G, an element $g \in G$, and subsets

$U,V \subset G$, we define

$$\begin{aligned} gU &= \{gu \ : \ u \in U\} \\ Ug &= \{ug \ : \ u \in U\} \\ U^{-1} &= \left\{u^{-1} \ : \ u \in U\right\} \\ UV &= \{uv \ : \ u \in U, \ v \in V\}. \end{aligned}$$

We now state several lemmas.

Lemma 1.11. *Let G be a topological group.*

(a) The function $G \to G$ given by $g \mapsto g^{-1}$ is a homeomorphism. In particular, the function $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1 g_2$ is continuous.

(b) For each $g \in G$, the functions $L_g, R_g \colon G \to G$ given by $L_g(h) = gh$, $R_g(h) = hg$ are homeomorphisms.

Lemma 1.12. *Let G be a topological group.*

(a) If H is an open or closed subgroup of G, then gH, Hg are open or closed for all $g \in G$.

(b) Every open subgroup of G is closed and every closed subgroup of finite index is open. If G is compact, every open subgroup of G has finite index.

(c) If H is a subgroup of G, then H is a topological group with respect to the subspace topology.

(d) If K is a normal subgroup of G, then G/K is a topological group with respect to the quotient topology and the quotient map $G \to G/K$ is an open mapping.

Lemma 1.13. *Let G be a topological group.*

(a) G is Hausdorff if and only if $\{1\}$ is closed.

(b) If K is a normal subgroup, then G/K is Hausdorff if and only of K is closed.

(c) If G is totally disconnected, then G is Hausdorff.

(d) If G is compact and Hausdorff and C_1, C_2 are closed subsets of G, then C_1C_2 is closed.

We leave the proofs of Lemmas 1.11, 1.12, and 1.13 to the reader.

Lemma 1.14. *Let G be a compact topological group and $\{X_\lambda\}_{\lambda\in\Lambda}$ a family of closed subsets with the property that for all $\lambda_1, \lambda_2 \in \Lambda$, there exists $\mu \in \Lambda$ such that $X_\mu \subset X_{\lambda_1} \cap X_{\lambda_2}$. If $Y \subset G$ is a closed subset, then*

$$\left(\bigcap_{\lambda\in\Lambda} X_\lambda\right) Y = \bigcap_{\lambda\in\Lambda} (X_\lambda Y).$$

Proof: Visibly, $(\bigcap_\lambda X_\lambda) Y \subset \bigcap_\lambda (X_\lambda Y)$. For the reverse inclusion, we argue by contradiction and assume that there exists $x \in \bigcap_\lambda (X_\lambda Y)$ and $x \notin (\bigcap_\lambda X_\lambda) Y$. It follows that $xY^{-1} \cap (\bigcap_\lambda X_\lambda) = \emptyset$. Since G is compact and the subsets xY^{-1}, X_λ are closed, there exists $n \in \mathbf{N}$ and $i_1, \ldots, i_n \in \mathscr{I}$ such that $xY^{-1} \cap \bigcap_{i=1}^n X_{\lambda_i} = \emptyset$. By hypothesis, there exists $\mu \in \mathscr{I}$ such that $X_\mu \subset \bigcap_{i=1}^n X_{\lambda_i}$ and so $xY^{-1} \cap X_\mu = \emptyset$. However, that implies $x \notin X_\mu Y$, which is impossible since $x \in \bigcap_\lambda (X_\lambda Y)$. ♠

Lemma 1.15. *If G is a compact topological group and C is a subset of G that is both open and closed and contains 1, then C contains an open normal subgroup.*

Proof: To prove this lemma, we claim the following:

Claim: *There exists an open subset $U \subset C$ such that $1 \in U$ and $U = U^{-1}$.*

Assuming the claim, we prove the lemma. Set $U^n = UU^{n-1}$ where $U^1 = U$, and let $H = \bigcup_{n\in\mathbf{N}} U^n$ be the subgroup generated by U. Since H is open, by Lemma 1.12, H is also closed and finite index in G. As $U^n \subset C$ for all n, we see that H is contained in C. Since H is finite index, it has only finitely many conjugates in G and taking the intersection of all of the distinct G–conjugates of H, we obtain an open and closed normal subgroup of G inside of C. We leave the proof of the claim as an exercise. ♠

Exercise. Prove the above claim.

Proposition 1.16. *Let G be a compact, totally disconnected, topological group.*

(a) Every open set in G is a union of cosets of open normal subgroups.

(b) A subset of G is both open and closed if and only if it is a union of finitely many cosets of open normal subgroups.

(c) If $S \subset G$, then

$$\overline{S} = \bigcap_{\substack{N \lhd G \\ N \text{ open}}} NS.$$

In particular,

$$C = \bigcap_{\substack{N \lhd G \\ N \text{ open}}} NC$$

and so

$$\{1\} = \bigcap_{\substack{N \lhd G \\ N \text{ open}}} N.$$

Proof: (a): Let $U \subset G$ be an open subset and $x \in U$. The subset $x^{-1}U$ is open and contains 1, and hence by Lemma 1.15 contains an open normal subgroup. As x was arbitrary, we result follows.

(b): If U is both open and closed, by Part (a) we know that U is a union of cosets of open normal subgroups. Since U is closed and G is compact, U is also compact. Hence, U is a finite union of cosets of open normal subgroups. The converse is straightforward.

(c): This follows from Part (a) after taking complements. ♠

Finally, we note that if $\{G_\lambda\}_{\lambda \in \Lambda}$ is a family of topological groups, then $\prod_\lambda G_\lambda$ is a topological group with the product topology and binary operation given coordinate-wise. Note that if G_λ is compact, Hausdorff, and/or totally disconnected for each $\lambda \in \Lambda$, then $\prod_\lambda G_\lambda$ is compact, Hausdorff, and/or totally disconnected by Theorem 1.6.

Chapter 2

Profinite Groups and Completions

In this chapter, we introduce profinite groups and establish some basic results with regard to profinite groups.

2.1 Lecture 3. Inverse Limits

A **directed set** is a partially ordered set $(\mathscr{I}, \leq)$ such that for any $i_1, i_2 \in \mathscr{I}$ there exists $\ell \in \mathscr{I}$ with $i_1, i_2 \leq \ell$. An **inverse system of sets indexed by a directed set $\mathscr{I}$** is a pair (X_i, φ_{ij}) where $\{X_i\}_{i \in \mathscr{I}}$ is a family of sets and $\left\{\varphi_{ij} \colon X_j \to X_i\right\}_{\substack{i,j \in \mathscr{I} \\ i \leq j}}$ is a family of functions satisfying $\varphi_{ii} = \mathrm{Id}_{X_i}$ and $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ for all $i \leq j \leq k$. When each set X_i is a group, we require the functions φ_{ij} be homomorphisms. Similarly, if the sets X_i are equipped with topological or topological group structures, we require that the functions φ_{ij} be continuous functions or continuous homomorphisms. With those restrictions on the functions φ_{ij}, we will call such inverse systems, **inverse systems of groups**, **inverse systems of topological spaces**, or **inverse systems of topological groups**. For any inverse system of sets (X_i, φ_{ij}), there is an associated inverse system of topological spaces given by endowing the sets X_i with the discrete topology.

Example. Let $\mathscr{I} = \mathbf{N}$ and p be a fixed, integral prime in $\mathbf{N}$. Let $Q_i = \mathbf{Z}/p^i\mathbf{Z}$ with $\varphi_{ij} \colon \mathbf{Z}/p^j\mathbf{Z} \to \mathbf{Z}/p^i\mathbf{Z}$ given by $\varphi_{ij}(n + p^j\mathbf{Z}) = n + p^i\mathbf{Z}$. The pair $(Z/p^i\mathbf{Z}, q_{ij})$ is an inverse system of finite groups/rings. We will discuss this example in detail later in these notes.

Example. Let G be a group and $\mathscr{I}$ a collection of normal subgroups H of G such that if $H_1, H_2 \in \mathscr{I}$ then there is $H \in \mathscr{I}$ with $H \subset H_1 \cap H_2$. The set $\mathscr{I}$ is a directed set under reverse-inclusion; $H \leq H'$ if and only if $H' \subset H$. We can form an inverse system of groups $(Q_H, \varphi_{H,H'})$ where $Q_H = G/H$ and $\varphi_{H,H'} \colon G/H' \to G/H$ when $H \leq H'$; the map $\varphi_{H,H'}$ comes from the isomorphism theorems. We will discuss examples like this one in more detail later in these notes.

Given an inverse system of sets (X_i, φ_{ij}) and a set Y, we say that a family of functions $\psi_i \colon Y \to X_i$ is **compatible with** (X_i, φ_{ij}) provided the diagram

$$\begin{tikzcd} & Y \arrow[dl, "\psi_j"'] \arrow[dr, "\psi_i"] & \\ X_j \arrow[rr, "\varphi_{ij}"'] & & X_i \end{tikzcd}$$

commutes for all $i, j \in \mathscr{I}$ with $i \leq j$; that is, $\varphi_{ij} \circ \psi_j = \psi_i$. We will sometimes denote a compatible family of functions by (Y, ψ_i). If the inverse system is a system of groups, topological spaces, or topological groups, we insist that Y have the same structure and the functions ψ_i be homomorphisms, continuous, or continuous homomorphisms.

Definition 2.1 (Inverse Limit)**.** *Let (X_i, φ_{ij}) be an inverse system of set (groups, rings, topological spaces, topological groups, .etc) with indexing set $\mathscr{I}$. We say that (X, φ_i) is an* ***inverse limit*** *of (X_i, φ_{ij}) if X is a set (group, ring, topological space, topological group, .etc) with a compatible family of functions φ_i that satisfies the following universal mapping property: If Y is another set (group, ring, topological space, topological group, .etc) with a compatible family of functions ψ_i, there exists a unique function $\psi \colon Y \to X$ such that the diagram*

$$\begin{tikzcd} & Y \arrow[dl, "\psi"'] \arrow[dr, "\psi_i"] & \\ X \arrow[rr, "\varphi_i"'] & & X_i \end{tikzcd}$$

commutes for all $i \in \mathscr{I}$.

We now prove that inverse limits exists and are unique.

Theorem 2.2. *Let (X_i, φ_{ij}) be an inverse system of sets with indexing set $\mathscr{I}$.*

(a) If (X, φ_i) and (Y, ψ_i) are inverse limits of the inverse system (X_i, φ_{ij}), then there is a bijection (homeomorphism, isomorphism, .etc) $\overline{\varphi}\colon X \to Y$ such that $\psi_i \circ \overline{\varphi} = \varphi_i$ for all $i \in \mathscr{I}$.

(b) Define

$$X = \left\{ x \in \prod_i X_i \ : \ \varphi_{ij}(\pi_j(x)) = \pi_i(x) \textit{ for all } i, j \in \mathscr{I} \textit{ with } i \leq j \right\}$$

with functions $\varphi_i\colon X \to X_i$ given by $\varphi_i(x) = \pi_i(x)$. Then (X, φ_i) is an inverse limit of the inverse system (X_i, φ_{ij}).

(c) If (X_i, φ_{ij}) is an inverse system of topological groups and (X, φ_i) is an inverse limit, then X is a topological group and the functions φ_i are continuous homomorphisms.

Proof: (a): By the universal mapping property for inverse limits, we have functions $\varphi_X : X \to Y$ and $\varphi_Y : Y \to X$ such that

$$\varphi_i = \psi_i \circ \varphi_X, \quad \varphi_i \circ \varphi_Y = \psi_i.$$

Consequently,

$$\varphi_i \circ \varphi_Y \circ \varphi_X = \varphi_i,$$

and so $\varphi_Y \circ \varphi_X = \mathrm{Id}_X$ by the uniqueness of mapping in Definition 2.1. Similarly,

$$\psi_i \circ \varphi_X \circ \varphi_Y = \psi_i,$$

and so $\varphi_X \circ \varphi_Y = \mathrm{Id}_Y$. Hence φ_X is a bijection.

(b): We must prove that (X, φ_i) is an inverse limit on (X_i, φ_{ij}) where

$$X = \left\{ x \in \prod_i X_i \ : \ \varphi_{ij}(\pi_j(x)) = \pi_i(x), \text{ for all } i, j \in \mathscr{I} \text{ with } i \leq j \right\}$$

and the functions φ_i are the restriction of the projection maps $\pi_i\colon \prod_j X_j \to X_i$ to X. Given a set Y with a compatible family of functions $\psi_i\colon Y \to X_i$, we need a unique function $\psi\colon Y \to X$ such that $\varphi_i \circ \psi = \psi_i$ for all $i \in \mathscr{I}$. We define $\psi\colon Y \to \prod_i X_i$ by $\psi(y) = (\psi_i(y))$ and note that the compatibility of the family ψ_i implies the image of ψ is in X. We leave the uniqueness of the map ψ to the reader.

(c): If X_i are topological spaces, X is a topological space via the subspace topology coming from the product space $\prod_i X_i$. If the X_i are also groups, X is a topological group and the functions φ_i are continuous homomorphisms. ♠

Given an inverse system of topological groups (X_i, φ_{ij}), Theorem 2.2 shows that there is a unique inverse limit (X, φ_i) and it is a topological group. We denote the limit by $\varprojlim X_i$.

Proposition 2.3. *Let (X_i, φ_{ij}) be an inverse system of topological spaces with indexing set $\mathscr{I}$ and $X = \varprojlim X_i$.*

(a) If each X_i is Hausdorff, then X is Hausdorff and $X \subset \prod_i X_i$ is closed.

(b) If each X_i is totally disconnected, then X is totally disconnected.

(c) If each X_i is compact and Hausdorff, then X is compact and Hausdorff. Moreover, if each X_i is non-empty, X is non-empty.

Proof: (a): By Theorem 2.2 (a)-(b), we can view $X \subset \prod_i X_i$. Since X_i is Hausdorff for each $i \in \mathscr{I}$, by Theorem 1.6 (a), the product $\prod_i X_i$ is Hausdorff. As any subspace of a Hausdorff space is Hausdorff, X is Hausdorff. By Lemma 1.3 (e), we know that

$$E_{i,j} = \left\{ x \in \prod_i X_i \ : \ \varphi_{ij}(\pi_j(x)) = \pi_i(x) \right\}$$

is a closed subset and so $X = \bigcap_{i \leq j} E_{i,j}$ is closed.

(b): Any subspace of a totally disconnected space is totally disconnected.

(c): By Theorem 1.6 (c), $\prod_i X_i$ is compact and by Part (a), $X \subset \prod_i X_i$ is Hausdorff and closed. Hence, X is compact and Hausdorff by Lemma 1.3 (a). To prove that X is non-empty with each X_i is a non-empty, compact, Hausdorff space, one can check that $\bigcap_{i \leq j} E_{i,j} \neq \emptyset$. We leave that as an exercise. ♠

Exercise. Prove that $\bigcap_{i \leq j} E_{i,j} \neq \emptyset$.

We require the following technical result in the sequel.

Proposition 2.4. *Let (X_i, φ_{ij}) be an inverse system of non-empty, compact Hausdorff spaces with indexing set $\mathscr{I}$ and $(X, \varphi_i) = \varprojlim (X_i, \varphi_{ij})$.*

(a) $\varphi_i(X) = \bigcap_{j \geq i} \varphi_{ij}(X_j)$ for each $i \in \mathscr{I}$.

(b) The set of subsets $\varphi_i^{-1}(U)$, varying over $i \in \mathscr{I}$ and open subsets $U \subset X_i$, form a base for X.

(c) If $Y \subset X$ satsfies $\varphi_i(Y) = X_i$ for each $i \in \mathscr{I}$, then Y is dense in X.

(d) If $\psi\colon Y \to X$ is a function and Y is a topological space, then ψ is continuous if and only if $\varphi_i \circ \psi$ is continuous for each $i \in \mathscr{I}$.

(e) If $f\colon X \to A$ is a continuous function and A is a discrete space, then there exists $r \in \mathscr{I}$ and a continuous function $f_r\colon X_r \to A$ such that $f = f_r \circ \varphi_r$.

Proof: As before, we will take $X \subset \prod_i X_i$ as in Theorem 2.2 (b).

(a): By compatibility of mappings, we know that $\varphi_i(X) = \varphi_{ij}(\varphi_j(X)) \subset \varphi_{ij}(X_j)$ for all $j \geq i$. Consequently, $\varphi_i(X) \subset \bigcap_{j\geq i} \varphi_{ij}(X_j)$. For the reverse inclusion, given $x \in \bigcap_{j\geq i} \varphi_{ij}(X_j)$ and $j \geq i$, we define

$$Y_j = \left\{y \in X_j \;:\; \varphi_{ij}(y) = x\right\} = \varphi_{ij}^{-1}(\{x\}).$$

As $\{x\}$ is closed, φ_{ij} is continuous, and X_j is compact, Y_j is compact by Lemma 1.3 (a). If $i \leq j \leq k$ and $y \in Y_k$, we see that $\varphi_{jk}(y) \in Y_j$ since $\varphi_{ij}(\varphi_{jk}(y)) = \varphi_{ik}(y) = x$. In particular, the family (Y_j, φ_{jk}) is an inverse limit system of non-empty compact, Hausdorff spaces and so $Y = \varprojlim Y_j$ is a non-empty closed subset of $\prod_j Y_j$. For any $y \in Y$ with $y = (y_j)$, we have $\varphi_{jk}(y_k) = y_j$ and $y_i = x$. If $\ell \in \mathscr{I}$ and $i \nleq \ell$, there exists $j \in \mathscr{I}$ such that $i, \ell \leq j$. We define $y_\ell \in Y_\ell$ by $\varphi_{\ell j}(y_j)$ and note that this is independent of j by compatibility of mappings. This yields $y \in X$ such that $\varphi_i(y) = x$.

(b): Every open subset in X is of the form

$$U = X \cap \left(\bigcap_{j=1}^{n} \pi_{i_j}^{-1}(U_j)\right)$$

where $n \in \mathbf{N}$, $i_1, \ldots, i_n \in \mathscr{I}$, and $U_j \subset X_{i_j}$ is an open subset. For $u \in U$ with $u = (u_i)$, we select $k \geq i_1, \ldots, i_n$. The set $\varphi_{i_jk}^{-1}(U_j)$ is open for all $j \in \{1, \ldots, n\}$ and contains u_k. For $V_k = \bigcap_{j=1}^{n} \varphi_{i_jk}^{-1}(U_j)$, we see that $\varphi_k^{-1}(V_k) \subset U$ and contains u. Hence, U is expressible as a union of open subsets of the form $\varphi_k^{-1}(U)$, and so this collection of open subsets of X is a base for X.

(c): For each $i \in \mathscr{I}$ and non-empty open subset $U \subset X_i$, we have $\varphi_i(Y) \cap U \neq \emptyset$ and so $Y \cap \varphi_i^{-1}(U) \neq \emptyset$. Hence, Y is dense since the open subsets $\varphi_i^{-1}(U)$ form a base for X by Part (b); recall that Y is dense if and only if $Y \cap U \neq \emptyset$ for any non-empty open set U.

(d): The direct implication follows from the continuity of the composition of continuous functions. For the reverse implication, we know that $\psi^{-1}(\varphi_i^{-1}(U))$ is an open subset of Y for all $i \in \mathscr{I}$ and open subsets $U \subset X_i$. Since $\varphi_i^{-1}(U)$ form a base by Part (b), it follows that ψ is continuous.

(e): Setting $A_0 = f(X)$, we see that A_0 is finite since it is compact and discrete. For each $a \in A_0$, the set $f^{-1}(a)$ is compact and open. Therefore, $f^{-1}(a)$ is a finite union of open sets of the form $\varphi_i^{-1}(U)$. Since A_0 is finite, there exists $n \in \mathbf{N}$, $i_1, \ldots, i_n \in \mathscr{I}$, and open sets $U_j \subset X_{i_j}$ such that for each $a \in A_0$, the set $f^{-1}(a)$ is a finite union of subsets from $\left\{\varphi_{i_1}^{-1}(U_1), \ldots, \varphi_{i_n}^{-1}(U_n)\right\}$. Now, select $\ell \in \mathscr{I}$ such that $\ell \geq i_j$ for all $j = 1, \ldots, n$. By compatibility of mappings, we have $\varphi_{i_j}^{-1}(U_j) = \varphi_\ell^{-1}(\varphi_{i_j\ell}(U_j))$ for each j. Therefore, for each $a \in A_0$, we have $f^{-1}(a) = \varphi_\ell^{-1}(U_a)$ for some open subset $U_a \subset X_\ell$. Next, set $C = X_\ell - \bigcup_{a \in A_0} U_a$. Since for every $x \in X$, we know that $\varphi_\ell(x) \in U_a$ for $a = f(x)$, we see that $C \cap \varphi_\ell(X) = \emptyset$. By Part (a), we have

$$C \cap \left(\bigcap_{k \geq \ell} \varphi_{\ell k}(X_k) \right) = \emptyset.$$

Since X_k is compact, there exists $m \in \mathbf{N}$ and $r_1, \ldots, r_m \in \mathscr{I}$ such that

$$C \cap \left(\bigcap_{s=1}^{m} \varphi_{\ell r_s}(X_{r_s}) \right) = \emptyset.$$

Now take $r \geq r_1, \ldots, r_m$ and observe, by compatibility of mappings, that $C \cap \varphi_{\ell r}(X_r) = \emptyset$ and $\varphi_{\ell r}(X_r) \subset \bigcup_{a \in A_0} U_a$. For each $a \in A_0$, set $W_a = \varphi_{\ell r}^{-1}(U_a) \subset X_r$ and note that if $a \neq a'$, then $W_a \cap W_{a'} = \emptyset$. Moreover, for each $x \in X_r$, we know that $\varphi_{\ell r}(x) \in U_a$ for $f(x) = a$, and so $x \in W_a$. Hence, X_r is the disjoint union of the open and closed subsets W_a. The function $f_r \colon X_r \to A$ given by $f_r(x) = a$ for $x \in W_a$ is continuous and satisfies $f = f_r \circ \varphi_r$ ♠

Proposition 2.5. *Let X be a totally disconnected, compact, Hausdorff space. Then X is an inverse limit of its discrete quotient spaces.*

Proof: Let $\mathscr{I}$ be the set of all partitions of X into finitely many subsets that are both open and closed. For each $i \in \mathscr{I}$, we have the associated quotient space X_i given by the equivalence relation associated to the partition i and associated quotient map $q_i \colon X \to X_i$. Since the partition sets are both open and closed, X_i is a discrete space. As X is compact, these are precisely the discrete quotient spaces of X.

For $i, j \in \mathscr{I}$, we write $i \leq j$ if and only if there exists $q_{ij} \colon X_j \to X_i$ satisfying $q_{ij} \circ q_j = q_i$. As the maps q_j are surjective, if q_{ij} exists, it must be unique. Given $i, j \in \mathscr{I}$, we have associated partitions

$$i = \{U_1, \ldots, U_m\}, \quad j = \{V_1, \ldots, V_n\}.$$

For these partitions, we obtain a refinement

$$k = \{U_r \cap V_s\}_{r,s=1}^{m,n}$$

with $i,j \leq k$, and so $\mathscr{I}$ is a directed set. Hence, we have an inverse system (X_i, q_{ij}) and a compatible family of maps (X, q_i). Take $Y = \varprojlim(X_i, q_{ij})$ with projection maps $\widehat{q_i}\colon Y \to X_i$. By the universal mapping property for Y, there exists a unique continuous function $q\colon X \to Y$ such that $\widehat{q_i} \circ q = q_i$. We assert that q is a homeomorphism. Since q is continuous and X, Y are compact and Hausdorff, it suffices to show that q is bijective by Lemma 1.3 (d). Given $x_1, x_2 \in X$ with $q(x_1) = q(x_2)$, then $q_i(x_1) = q_i(x_2)$ for every $i \in \mathscr{I}$. In particular, no open and closed set of X can contain just one of x_1, x_2. Since X is totally disconnected, we must have $x_1 = x_2$. To see that q is onto, note that since $\widehat{q_i}(q(X)) = q_i(X) = X_i$, $q(X)$ is dense in Y. However, $q(X)$ is closed by Lemma 1.3 and so q is onto. ♠

2.1.1 Prelude: After hours thoughts

Inverse limits are central to the theory of profinite groups and a great deal of time in this section was spent on developing the basic theory of profinite groups. A profinite group is an inverse limit of an inverse system of finite groups equipped with the discrete topology. In particular, profinite groups are totally disconnected, compact, Hausdorff topological groups. The points of the inverse limit are described concretely via the embedding of the completion $\varprojlim X_i \subset \prod_i X_i$. Specifically, it is the maximal subset of $\prod_i X_i$ such that the restriction of the projection maps π_i to the subset yields a compatible family of functions. The points of the inverse limit can be thought of as functions $f\colon \mathscr{I} \to \bigcup_i X_i$ such that $f(i) \in X_i$ and f is compatible with the maps φ_{ij}. These functions are called nets on the disjoint union of the sets $\bigcup_i X_i$ indexed by $\mathscr{I}$, and X can be viewed as the set of such nets that are also compatible with the maps φ_{ij}.

Example. X_i is finite, $\mathscr{I} = \mathbf{N}$ and each φ_{ij} is surjective.

The inverse limit of (X_i, φ_{ij}) is the set of sequences gives as follows. For $i = 0$, we pick $x_0 \in X_0$. Next, we pick $x_1 \in X_1$ with $\varphi_{01}(x_1) = x_0$, and so forth to obtain a sequence $x_j \in X_j$. At each stage x_j, at the $(j+1)$–stage we can select any $x_{j+1} \in \varphi_{jj+1}^{-1}(x_j)$. In particular, the set of all such sequences can be thought of as a $|X_0|$–rooted forest with vertices X_j and roots vertices in X_0.

2.2 Lecture 4. Profinite groups

Chapter 3

Galois Groups

Chapter 4

Topologically Countable and Finitely Generated Profinite Groups

Chapter 5

Group Cohomology

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