MA 544: Homework 3

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Problem 3.1 (Wheeden & Zygmund §3, Ex. 5)

Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage each interval removed has length $\delta 3^{-k}$, $0<\delta<1$. Show that the resulting set is perfect, has measure $1-\delta$, and contains no interval.

Proof. Put $C_0 \coloneqq [0,1]$. We begin constructing our desired set by removing the open set $\left(\frac{\delta}{3}, 1-\frac{\delta}{3}\right)$ from the closed interval $[0,1]$. This separates $[0,1]$ into the union of two disjoint closed (and bounded therefore, compact) intervals $\left[0, \frac{\delta}{3}\right]$ and $\left[1-\frac{\delta}{3}, 1\right]$ which we shall call C_1. Next, we remove the open interval $\left(\frac{\delta}{9}, \frac{\delta}{3}-\frac{\delta}{9}\right)$ from $\left[0, \frac{\delta}{3}\right]$ and the open interval $\left(1-\frac{\delta}{3}+\frac{\delta}{9}, 1-\frac{\delta}{9}\right)$ and end up with the union C_2 of four disjoint closed intervals. Continue in this fashion ad infinitum. Note that each $C_{k+1}\subset C_k$ and each C_k is a finite union of closed subsets thus, by theorem 1.7 and the Cantor's intersection theorem, the set $C_\delta \coloneqq \bigcap_{i=1}^\infty C_i$ is closed, compact, and nonempty.

Now we show that the set we have constructed, C_δ, is perfect. Since C_δ is closed, it remains to show that C_δ contains no isolated points. Note that, in the construction of C_δ, we never removed the endpoints the intervals which union to C_k. Thus, the endpoints of the intervals which union to C_k are in C_δ. Before continuing, we need to figure out what the length of each interval at the kth stage in the construction is and in the process, we shall prove that $|C_\delta| = 1-\delta$.

At each stage in the construction (except for $k=0$) we removed 2^{k-1} open intervals of length $\delta 3^{-k}$. Thus, at the kth stage of the construction, the measure of C_k will be

$$|C_k| = 1-\sum_{i=1}^k \frac{2^{i-1}\delta}{3^i} = 1-\frac{\delta}{3}\sum_{i=1}^k\left(\frac{2}{3}\right)^{i-1}.$$

We immediately recognize the right-hand side as a geometric sum so letting $k\to\infty$, by theorem 3.26(ii), we have

$$|C_\delta| = \lim_{k\to\infty} 1-\frac{\delta}{3}\sum_{i=1}^k\left(\frac{2}{3}\right)^{i-1} = 1-\lim_{k\to\infty}\frac{\delta}{3}\sum_{i=1}^k\left(\frac{2}{3}\right)^{i-1} = 1-\frac{\delta}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1-\delta.$$

Now, let $\varepsilon>0$ be given. By the Archimedean principle, we may choose a sufficiently large natural number N so that $|C_N|2^{-N} < 2^{-N} < \varepsilon$. Let x be a point in C_δ, then $x\in C_N$ since $x\in C_k$ for all k. In particular, x is in one of he 2^N disjoint closed intervals that union to C_k, call it I. Let x' be the closest endpoint of I to x (if x is itself an endpoint, choose x' to be the opposite endpoint). Then, by the triangle inequality, we have $|x-x'|\le |C_N|2^{-N}<\varepsilon$. Hence, the open neighborhood $B(x,\varepsilon)\smallsetminus\{x\}\neq\emptyset$ for any ε. Thus, C_δ is perfect.

Last but not least, we show that C_δ contains no interval. Suppose that (a,b) is an interval contained in C_δ. Hence, $(a,b)\subset C_k$ for all k. (I don't know how to finish the proof without using a fact about connected 1-manifolds). Then, since (a,b) is connected, it must be contained in an connected component C of C_δ. However, the connected components of C_k, i.e., the closed intervals, have measure less than 2^{-k} so $b-a\le 2^{-k}$. Letting $k\to\infty$, we have $b-a\le 0$ which leads to a contradiction since the measure of an interval is strictly greater than 0. ■

Problem 3.2 (Wheeden & Zygmund §3, Ex. 7)

Prove (3.15).

Proof. Recall the statement of 3.15:

Lemma (Wheeden & Zygmund (3.15)). *If $\{I_k\}_k^N$ is a finite collection of nonoverlapping intervals, then $\bigcup I_k$ is measurable and $|\bigcup I_k| = \sum|I_k|$.*

Note that the proof follows exactly as corollary 3.24. By theorem 3.14, since $\bigcup I_k$ is a finite union of closed sets, $\bigcup I_k$ is measurable. To see that $|\bigcup I_k| = \sum|I_k|$. By subadditivity, we have

$$\left|\bigcup I_k\right| \leq \sum|I_k|.$$

On the other hand, since $|{I_k}^\circ| = |{I_k}^\circ| + |\partial I_k| = |{I_k}^\circ \cup \partial I| = |I_k|$, we have

$$\sum|I_k| = \sum|{I_k}^\circ| \leq \left|\bigcup I_k\right|.$$

Hence, $|\bigcup I_k| = \sum|I_k|$. ■

Problem 3.3 (Wheeden & Zygmund §3, Ex. 8)

Show that the Borel algebra $\mathcal{B}$ in $\mathbf{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbf{R}^n$.

Proof. We defined the Borel algebra $\mathcal{B}$ in $\mathbf{R}^n$ to be the smallest σ-algebra containing the open sets in $\mathbf{R}^n$. Since $\mathcal{B}$ contains all the closed subsets of $\mathbf{R}^n$, theorem 3.17, it suffices to show that $\mathcal{B}$ is the smallest such σ-algebra. Suppose $\mathcal{B}'$ is another σ-algebra containing all of the closed sets in $\mathbf{R}^n$. Then, by theorem 3.17, $\mathcal{B}'$ contains all of the open sets of $\mathbf{R}^n$ and, since $\mathcal{B}$ is the smallest σ-algebra containing the open subsets of $\mathbf{R}^n$, we have $\mathcal{B} \subset \mathcal{B}'$, as desired. ∎

Problem 3.4 (Wheeden & Zygmund §3, Ex. 9)

If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ (and also $\liminf E_k$) has measure zero.

Proof. Define $E \coloneqq \limsup E_k$ and $E'_\ell \coloneqq \bigcup_{k=\ell}^{\infty} E_k$. Then E'_ℓ is a decreasing (with respect to inclusion) sequence of sets with $\lim_\ell E'_\ell = E$. Then E is contained in the intersection $\bigcap_{\ell=n}^{\infty} E'_\ell$ for all n, so by the monotonicity of the outer measure we have

$$|E|_e \leq |E'_\ell|_e.$$

On the other hand, we also have

$$|E'_n|_e \leq \sum_{k=n}^{\infty} |E_k|_e$$

for all n. Since, by assumption, the sum $\sum |E_k|_e$ converges, we have, for every $\varepsilon > 0$, there exists N sufficiently large such that the sum $\sum_{k=n}^{\infty} |E_k|_e < \varepsilon$ for every $n \geq N$. Thus, $|E|_e \leq \varepsilon$ for every $\varepsilon > 0$. Let $\varepsilon \to 0$ and we have $|E|_e = 0$ as desired. Lastly, we note that for any sequence $\{a_k\} \subset \mathbf{R}$ we have $\liminf a_n \leq \limsup a_n$ so, naturally, $\liminf E_k = 0$. ■

Problem 3.5 (Wheeden & Zygmund §3, Ex. 10)

If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

Proof. Without loss of generality, we may assume that $|E_1|, |E_2| < \infty$, for otherwise the result holds trivially.

By Carathéodory, we have that

$$|E_1| = |E_1 \cap E_2| + |E_1 \smallsetminus E_2| \quad \text{and} \quad |E_2| = |E_1 \cap E_2| + |E_2 \smallsetminus E_1|. \tag{1}$$

Moreover, by elementary set theory, we have $(E_1 \cup E_2) \smallsetminus E_2 = E_1 \smallsetminus (E_1 \cap E_2)$ and $E_1 \subset E_1 \cup E_2$, and $E_1 \cap E_2 \subset E_1$ so by rearranging (1) we have

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|,$$

as desired. ■