MA 54400 - Final Exam Practice Problems
Spring 2016
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1. Suppose that $f \in L^1(\mathbb{R}^n)$, and that x is a point in the Lebesgue set of f. For $r > 0$, let
$$A(r) = \frac{1}{r^n} \int_{B(0,r)} |f(x-y) - f(x)| \; dy.$$
Show that:
(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$;
(b) There exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \leq p < \infty$. Assume that $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that
$$\|f_k - f\|_p \to 0 \Leftrightarrow \|f_k\|_p \to \|f\|_p \text{ as } k \to \infty.$$

3. Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$.
(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ or $p = \infty$?

4. Let $f \in L^1(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x) \cos(tx) \; dx$.
(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemann-Lebesgue Lemma*:
$$\lim_{t \to \infty} F(t) = 0.$$

5. Let f be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular, show that
$$\int_a^b \phi \; df = \int_a^b \phi f' \; dx \; + \; \int_a^b \phi \; dh$$
for all functions ϕ continuous on $[a, b]$.