Snowbird Workshop – Character Varieties

Carlos Salinas

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Contents

Chapter 1

Workshop Notes

I will write a summary of the work done at Snowbird over the week. Here is some preparation for the workshop.

1.1 Knot Theory

1.1.1 Definitions and First Examples

Definition 1 (Provisional)**.** A *knot* is a closed loop of string in $\mathbb{R}^3$; two knots are equivalent if one can be wiggled around, stretched, tangled and untangled until it coincides with the other. Cutting and rejoining is *not* permitted.

All this means is that if K_1 and K_2 are two knots in $\mathbb{R}^3$, then we say $K_1 \simeq K_2$ if and only if K_1 can be deformed *continuously* into K_2.

Remark 1. Can one produce a table of the simplest knot types (a *knot type* means an equvialence class of knots, in other words a *topological* as opposed to a *geometrical* knot: often, we will simply call it a knot). Simplest is clearly something we will need to define: How should one measure the complexity of knots?

Definition 2. A *link* is simply a collection of (finitely-many) disjoint closed loops of string in $\mathbb{R}^3$; each loop is called a *component* of the link. Equivalence is defined in the obvious way. A knot is therefore just a one-component link.

Definition 3. The *crossing number* $c(K)$ of a knot K is the minimal number of crossings in any diagram of that knot. (This is a natural measure of complexity.) A *minimal diagram* of K is one with $c(K)$ crossings.

1.1.2 Operations on Knots

Definition 4. The *mirror-image* $\bar{K}$ of a knot K is obtained by reflecting it in a plane in $\mathbb{R}^3$. It may also be defined given by the diagram D of K: One simply exchanges all crossings of D. This is evident if one considers reflecting in the plane of the page.

Definition 5. A knot is called *amphichiral* if it is equivalent to its own mirror-image. How might one detect amphichirality? The trefoil is in fact not amphichiral, whilst the figure-eight is.

Definition 6. An *oriented* knot is one with a chosen direction or arrow of circulation along the string. Under equivalence this direction is carried along as well, so one may talk about equivalence (meaning *orientation-preserving equivalence*) of oriented knots.

Definition 7. The *reverse* rK of an oriented knot K is simply the same knot with the opposite orientation. One may also define the *inverse* $r\bar{K}$ as the composition of the reversal and the mirror-image. By analogy with amphichirality, we have a notion of a knot being *reversible* or *invertible* if it is equivalent to its reverse or inverse. Reversibility is very difficult to detect; the knot 8_{17} is the first non-reversible one (discovered by Trotter in the 60s).

Definition 8. If K_1, K_2 are oriented knots, one may form their *connected sum* $K_1 \# K_2$ by removing a little arc from each and splicing up the ends to get a single component making sure that the orientations glue to get a consistent orientation on the result.

This operation behaves rather like multiplication on the positive integers. It is commutative with the unknot as identity. A natural question is whether there is an inverse; could one somehow cancel out the knottedness of a knot K by connect-summing with another knot? This seems implausible, and we will prove it false. Thus, knots form a *semigroup* under connected-sum. In this semigroup, just as in the positive integers under multiplication, there is a notion of *prime factorisation*, which we shall study later.

1.1.3 Alternating Diagrams

Definition 9. An *alternating diagram* D of a knot K is a diagram such that it passes alternatively over and under crossings, when circling completely around the diagram from some arbitrary starting point. An *alternating knot* K is one which possesses *some* alternating diagram.

1.1.4 Unknotting number

Definition 10. The *unknotting number* $u(K)$ of a knot K is the minimum, over all diagrams D of K, of the minimal number of crossing changes required to turn D into a diagram of the unknot.

1.2 Formal Definitions and Reidemeister Moves

1.2.1 Knots and equivalence

How should we formulate the definition of a knot?

The most obvious way is to consider parametric curves in $\mathbb{R}^3$. Let $I := [0, 1]$ be the closed unit interval in $\mathbb{R}$. A continuous vector-valued function $\mathbf{x}(s)$ with domain I defines such a curve; the continuity requirement makes sure that it is unbroken. If we also impose the condition $\mathbf{x}(0) = \mathbf{x}(1)$ then the initial and the final point are made to coincide, so we have a parametric representation of a *closed loop*, rather than just an arc. If we require that the map $s \mapsto \mathbf{x}(s)$ is injective on the interval $[0, 1)$, then we enforce that the curve *does not intersect itself.* These three conditions constitute a reasonable definition of a knot, which we now proceed to study further.

Next question: How should we formulate the notion of deformation of a knot?

Deformation is best visualized as a time-dependent process. Imagine starting at time $t = 0$ with a knot K_0, and deforming it through a family of intermediate knots K_t to a final one K_1. We need to make sure this process of deformation is continuous in t. So we could consider vector-valued functions of two variables $\mathbf{x}(s,t)$ for $(s,t) \in I \times I$, with the requirement thatfor each fixed value t, the function $s \mapsto \mathbf{x}(s,t)$ obeys the three conditions making it a knot K_t.

Thus, we can define two knots K and K' to be equivalent if there exists a deformation running from $K_0 = K$ to $K_1 = K'$.

Unfortunately, this definition is not correct as it allows for pathological knots with infinitely much knotting. Instead, we can define two knots to be equivalent if they are *ambient isotopic*, meaning that there exists an (orientation-preserving) homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ carrying one to the other. This definition turns out to fix the second problem—now, not all knots are equivalent to one another—but it does not rule out pathological knots.

The easiest way to do that is to require that each knot should be representable as a *knotted polygon* in $\mathbb{R}^3$, that is a subset made up of *finitely many* straight-line arc segments. All *tame* knots can be approximated arbitrarily closely by polygonal subsets—we just have to use a very large number of tiny edges. On the other hand, pathological knots (ones that are infinitely knotted) cannot be represented using a knotted polygon.

Definition 11. If K is a subset of $\mathbb{R}^3$ which can be written as the union of arc segments

$$K = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{n-2}, a_{n-1}] \cup [a_{n-1}, a_0]$$

such that the segments are disjoint from one another except when consecutive (in which case they intersect at a single point, i.e., $[a_{k-2}, a_{k-1}] \cap [a_{k-1}, a_k] = \{a_k\}$) then we will say K is a *knot*.

Definition 12. Suppose K is a knot in $\mathbb{R}^3$ having $[a_i, a_{i+1}]$ as one of its edges, and suppose $T \coloneqq [a_i, x, a_{i+1}]$ is a closed solid triangle in $\mathbb{R}^3$ which intersects K only along the edge $[a_i, a_{i+1}]$. Then we may slide the edge across the triangle without hitting anything: Replace the edge $[a_i, a_{i+1}]$ of K by two new edges $[a_i, x] \cup [x, a_{i+1}]$ so as to form a new knot K' with one more edge in total. Such a move is called a Δ-move.

A Δ-move is clearly an absolutely basic kind of deformation of knots which should be regarded as an equivalence. We will in fact define our equivalence relation to be the one "generated by Δ-moves" as follows:

Definition 13. Two knots K, J are *equivalent* (or *isotopic*) if there is a sequence of knots $K = K_0, K_1, K_2, \ldots, K_n = J$ of knots such that each pair K_i, K_{i+1} is related by a Δ-move or the reverse of a Δ-move.

This clearly defines an equivalence relation on knots.

We will often confuse knots in $\mathbb{R}^3$ with their equivalence classes, which are the things we are really interested in topologically.

1.2.2 Projections and diagrams

Definition 14. If K is a knot in $\mathbb{R}^3$, its *projection* is $\pi(K) \subset \mathbb{R}^2$, where π is the projection along the z-axis onto the xy-plane. The projection is said to be *regular* if the preimage of a point of $\pi(K)$ conists of either one or two points of K, in the latter case neither being a vertex of K. Clearly a knot has an *irregular* projection if it has any edges parallel to the z-axis, if it has three or more points lying above each otehr, or any vertex lying above or below another point of K; on the other hand, a regular projection of a knot consists of a polygonal circle drawn in the plane with only "transverse double points" as self-intersections.

Definition 15. If K has a regular projection then we can define the corresponding *knot diagram* D by redrawing it with a broken arc near each *crossing* to incorporate the over/under information. If K had an irregular projection then we would not be able to easily reconstruct it from this sort of picture so it is important that we can find regular projections of knots easily.

Definition 16. Define an *ε-perturbation* of a knot K in $\mathbb{R}^3$ to be any knot K' obtained by moving each of the vertices of K a distance less than ε, and reconnecting them with straight edges in the same fashion as K.

1.3 Algebraic Geometry

Generally speaking, in order to do geometry we need

(1) A topological space.

(2) A notion of locally standard objects. (For example, in the case of real manifolds, a ball in $\mathbb{R}^n$. In the case of complex manifolds, a ball in $\mathbb{C}^n$. In the case of algebraic varieties, a system of polynomials that are locally 0.)

(3) Some set of functions on the space (perhaps locally defined). For example, in the real case, C^k-functions, or smooth functions, or analytic functions In the complex case, holomorphic functions, or analytic functions. In the complex case, holomorphic functions.

(4) Maps between the objects given by (1), (2), (3).

Another theme in algebraic geometry is that of classifying a space (or moduli space). Assume that we have some geometric algebraic object X.This object X is at least a topological space.

Question 2. Given X, with some topological structure, classify all algebraic structures it carries, compatible with the underlying topological structure.

Examples 1. Consider the elliptic curve of equation

$$y^2 = ax^3 + bx + c, \qquad (a, b, c \in \mathbb{C}),$$

where the right hand side has distinct roots. Geometrically, this is a genus 1 complex surface with one point missing. If we compactify, we obtain the usual torus.

Here are two things that we can ask ourveles.

What are the algebraic structures carried by the torus?

Given X an algebraic variety, classify all subobjects of X.

This problem can only be handled if we fix some discrete invariants. Then it might be possible to classify the subobjects, and the classifying space might also be an algebraic variety.

Consider the special case where $\Bbbk = \mathbb{C}$ and $X = \mathbb{A}^n$. We would like to classify all the subvarieties of $\mathbb{A}^n$ traced by polynomials $\mathbf{f}_1, \dots, \mathbf{f}_m \in \mathbb{C}[X_1, \dots, X_n]$. Let us consider the easier problem which is to classify the linear subvarieties of $\mathbb{A}^n$. Using translation, we may assume without loss of generality that they pass through the origin. The invariant is the dimension d, where $0 \le d \le n$. The cases $d = 0$, n are trivial. Let $G(n, d)$ denote the space of all linear transformations of dimension d in $\mathbb{A}^n$ through $\mathbf{0}$.

Observe that there is an isomorphism

$$G(n, d) \simeq G(n, n - d)$$

given by duality. We will treat the case $d = 1$, since it is simpler. We need to classify all lines through the origin $\mathbf{0}$ in $\mathbb{A}^n$. Let Σ be the unit sphere in $\mathbb{A}^n$, that is, the set

$$\Sigma := \left\{ \mathbf{z} : \sum |z_i|^2 = 1, \mathbf{z} = (z_1, \dots, z_n) \right\}$$