MA557 Homework 9

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Problem 9.1

Let R be a Noetherian ring, $R \subset S$ an extension of rings, and $x \in S$. Show that x is integral over R if and only if for every minimal prime $\mathfrak{q}$ of S, the image of x in $S/\mathfrak{q}$ is integral over $R/\mathfrak{q} \cap R$.

Proof. $\Longrightarrow$ Suppose that x is integral over R. Then x satisfies a monic polynomial of degree n, say $f(X) = X^n + a_1 X^{n-1} + \cdots + a_n$. Let $\mathfrak{q}$ be a minimal prime of S and consider the quotient ring $S/\mathfrak{q}$. If $x \in \mathfrak{q}$ there is nothing to show as $\bar{x} = \bar{0}$ hence satisfies the polynomial X over $R/\mathfrak{q} \cap S$. Suppose $x \notin \mathfrak{q}$. Then

$$\bar{0} = \overline{x^n + a_1 x^{n-1} + \cdots + a_n} = \bar{x}^n + \bar{a}_1 \bar{x}^{n-1} + \cdots + \bar{a}_n$$

so $\bar{x}$ satisfies the polynomial $\bar{f}(X)$. Hence, $\bar{x}$ is integral over $R/\mathfrak{q} \cap S$.

$\Longleftarrow$ Conversely, suppose that for $x \in S$ the image of x in $S/\mathfrak{q}$ is integral over $R/\mathfrak{q} \cap S$. Then we shall show that x is integral over R. For this, it suffices to show that $R[x]$ is a finite R-module.

Since I've not been successful at showing my assertion let us make an extra assumption on S. In particular, we shall assume that S is Noetherian. Since S is Noetherian, S contains finitely many minimal primes $\mathfrak{q}_1, ..., \mathfrak{q}_n$. Let $f_i(X) \in R[X]$ be the minimal polynomial of x in $S/\mathfrak{q}_i$, i.e., $f_i(x)\mathfrak{q}_i$. Then

$$f(x) = f_1(x) \cdots f_n(x) \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = \operatorname{nil} S.$$

Since $\operatorname{nil} S$ is nilpotent, $f(x)^m = 0$ for some positive integer m. Thus, x is integral over R.

More generally, that is, not assuming S is Noetherian, consider the set

$$W = \left\{ x^n + a_1 x^{n-1} + \cdots + a_n \,\middle|\, a_i \in R \right\} \cup \{1\},$$

i.e., the set of all monic polynomials in x union 1. Note that this set is multiplicatively closed since for any two polynomials $p(x)$ and $q(x)$ in W their product $p(x)q(x)$ is another monic polynomial in x. Now, it suffices to show that $0 \in W$. We shall proceed by contradiction. Suppose $0 \notin W$. Then $(0) \cap W = \emptyset$. Therefore, there exists a minimal prime $\mathfrak{q}$ over (0) such that $\mathfrak{q} \cap W = \emptyset$. But $\bar{x}$ is integral over $R/\mathfrak{q} \cap R$, i.e., $f(x) \in \mathfrak{q}$ for some monic polynomial $f \in R[X]$. Thus, $f(x) \in \mathfrak{q} \cap W$ this is a contradiction. Therefore, $0 \in W$ and it follows that x is integral over R. ■

Problem 9.2

Let d be a square-free integer and R the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{d})$. Show that

$$R = \begin{cases} \mathbf{Z}[\sqrt{d}] & \text{if } d \not\equiv 1 \mod 4 \\ \mathbf{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \mod 4 \end{cases}.$$

Proof. Courtesy of Dummit & Foote: Since d satisfies the polynomial $X^2 - d$, respectively $X^2 - X + (1-d)/4$ for $d \equiv 2$ or $3 \mod 4$, it follows that $\sqrt{d}$ is integral in $\mathbf{Q}(\sqrt{d})$ so $\mathbf{Z}[\sqrt{d}]$ is contained in the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{d})$. To see the reverse containment let $\alpha = a + b\sqrt{d}$ with $a, b \in \mathbf{Q}$ and suppose that α is integral. If $b = 0$, then $\alpha \in \mathbf{Q}$ so $a \in \mathbf{Z}$. So suppose $b \neq 0$. Then the minimal polynomial of α is $X^2 - 2aX + (a^2 - b^2 d)$ and $2a, a^2 - b^2 d \in \mathbf{Z}$. Thus,

$$4(a^2 - b^2 d) = (2a)^2 - (2b)^2 d$$

so $4b^2 d \in \mathbf{Z}$. Since d is square-free it follows that $2b$ is an integer, $x^2 - y^2 d \equiv 0 \mod 4$. Since 0 and 1 are the only squares mod 4 and d is not divisible by 4, it we claim that (i) $d \equiv 2$ or $3 \mod 4$ and x, y are both even, or (2) $d \equiv 1 \mod 4$ and x, y are both odd. In the first case, $a, b \in \mathbf{Z}$ and $\alpha \in \mathbf{Z}[\sqrt{d}]$. In the latter case, $a + b\sqrt{d} = r + s\sqrt{d}$ where $r = (x-y)/2$ and $s = y$ are both integers, so again $\alpha \in \mathbf{Z}[\sqrt{d}]$. ∎

Problem 9.3

Let $R \subset S$ be an integral extension of rings and I and R-ideal. Show that

(a) $\operatorname{ht} IS \leq \operatorname{ht} I$

(b) $\operatorname{ht} IS = \operatorname{ht} I$ if S is a domain and R is normal.

Proof. (a) Let $s = \operatorname{ht} I$ and let $\mathfrak{q} \supset I$ be a prime ideal in R with height s, i.e., there exists a proper chain of ideals

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_s = \mathfrak{q}.$$

Then by lying over there exists a prime ideal $\mathfrak{p}_0 \subset S$ which contracts to $\mathfrak{q}_0$ so that by going up we get the chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_s = \mathfrak{p} \tag{1}$$

where $\mathfrak{p} \cap R = \mathfrak{q}$. We claim that $\operatorname{ht} \mathfrak{q} = s$. It is clear that $\operatorname{ht} \mathfrak{q} \geq s$ by (1). To see that $\operatorname{ht} \mathfrak{q} \leq s$ suppose that we have the refinement

$$\mathfrak{p}_0' \subsetneq \mathfrak{p}_1' \subsetneq \cdots \subsetneq \mathfrak{p}_r' = \mathfrak{p}.$$

Write $\mathfrak{q}_i' = (\mathfrak{p}_i') \cap R$. Then the contracted chain

$$\mathfrak{q}_0' \subsetneq \mathfrak{q}_1' \subsetneq \cdots \subsetneq \mathfrak{q}_r' = \mathfrak{q}$$

is a refinement of $\mathfrak{q}$. Hence, $r \leq s$. It follows that $\operatorname{ht} p = s$. Thus, $\operatorname{ht} IS \leq \operatorname{ht} I$.

(b) By part (a) it suffices to show that $\operatorname{ht} IS \geq \operatorname{ht} I$. Without loss of generality, assume $\operatorname{ht} IS < \infty$, say $\operatorname{ht} IS = s$. Suppose $\mathfrak{p} \supset IS$ with $\operatorname{ht} \mathfrak{p} = \operatorname{ht} IS = s$. Then

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_s = \mathfrak{p}.$$

Then, setting $\mathfrak{q}_i := \mathfrak{p}_i \cap R$ and $\mathfrak{q} := \mathfrak{p} \cap R$, we get a chain

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_s = \mathfrak{q}$$

where, by 7.12(b), this chain cannot be made smaller. Now, suppose we have another chain of prime ideals leading up to $\mathfrak{q}_s$, say

$$\mathfrak{q}_0' \subsetneq \mathfrak{q}_1' \subsetneq \cdots \subsetneq \mathfrak{q}_t' = \mathfrak{q}_s.$$

Then, by going up since R is normal, this lifts to a chain

$$\mathfrak{q}_0' S \subsetneq \mathfrak{q}_1' S \subsetneq \cdots \subsetneq \mathfrak{q}_t' S = \mathfrak{p}.$$

But $\operatorname{ht} \mathfrak{p} = s$ so $t \leq s$. Thus, $\operatorname{ht} I \leq \operatorname{ht} \mathfrak{q} \leq \operatorname{ht} IS$ so in particular we have $\operatorname{ht} I = \operatorname{ht} IS$. ∎