MA 519: Homework, Midterms and Practice Problems Solutions

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Contents

1 Homework Solutions

1.1 Homework 1

EXERCISE 1.1 (Handout 1, # 5). A closet contains five pairs of shoes. If four shoes are selected at random, what is the probability that there is at least one complete pair among the four?

> SOLUTION. Let A denote the event that "at least 1 complete pair is among the 4 selected shoes." We compute the probability of $\neg A$ the event that there is no complete pair among the 4 selected shoes.
>
> We can choose 4 shoes from the 5 pairs in $\binom{10}{4} = 210$ ways (this is the order of the sample space), in order that we do not select a pair, we choose one of 2 a left or a right from 4 pairs of shoes which can be chosen in $\binom{5}{4}$ ways.
>
> Thus,
>
> $$\begin{aligned} P(A) &= 1 - P(\neg A) \\ &= 1 - \frac{\binom{5}{4}2^4}{\binom{10}{4}} \\ &= 1 - \frac{8}{21} \\ &\approx 0.619. \end{aligned}$$

∎

EXERCISE 1.2 (Handout 1, # 7). A gene consists of 10 subunits, each of which is normal or mutant. For a particular cell, there are 3 mutant and 7 normal subunits. Before the cell divides into 2 daughter cells, the gene duplicates. The corresponding gene of cell 1 consists of 10 subunits chosen from the 6 mutant and 14 normal units. Cell 2 gets the rest. What is the probability that one of the cells consists of all normal subunits.

> SOLUTION. Let A denote the event that at least one of the cells consists of all normal subunits. Assuming that these cell units can be arranged in any order, each new cell can get 10 of the 20 subunits in $\binom{20}{10} = 184756$ ways. Suppose cell 1 receives all 14 of the normal subunits. There are $\binom{14}{10} = 1001$ ways to do this. Since we can do this for cell 2 as well, we have
>
> $$P(A) = \frac{2\binom{14}{10}}{\binom{20}{10}} \approx 0.011.$$

∎

EXERCISE 1.3 (Handout 1, # 9). From a sample of size n, r elements are sampled at random. Find the probability that none of the N prespecified elements are included in the sample, if sampling is

(a) with replacement;

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(b) without replacement.

Compute it for $r = N = 10$, $n = 100$.

SOLUTION. For part (a) there are a total of n^r possible draws (this is the order of the sample space). There are $n - N$ elements that have not been prespecified and $(n - N)^r$ ways to draw them. Therefore, the probability of the event A that none of the N prespecified elements are included in the sample with replacement is

$$P(A) = \frac{(n - N)^r}{n^r}.$$

For part (b) there are a total of $\binom{n}{r}$ possible draws (this is the order of the sample space). Again, there are $n - N$ elements that have not been prespecified and $\binom{n-N}{r}$ ways to draw them. Thus, the probability of the event A that none of the N prespecified elements are included in the sample without replacement is

$$P(A) = \frac{\binom{n-N}{r}}{\binom{n}{r}}.$$

∎

EXERCISE 1.4 (Handout 1, # 11). Let E, F, and G be three events. Find expressions for the following events:

(a) only E occurs;
(b) both E and G occur, but not F;
(c) all three occur;
(d) at least one of the events occurs;
(e) at most two of them occur.

SOLUTION. The solution to this problem reduces to naive set theory.

For part (a), the event that only E occurs is the event

$$E \cap \neg F \cap \neg G.$$

For part (b), the event that both E and G occur, but not F is

$$E \cap G \cap \neg F.$$

For part (c), the event that all three of E, F, and G occur is

$$E \cap F \cap G.$$

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For part (d), the event that at least one of E, F, or G occur is

$$E \cup F \cup G.$$

For part (e), the event that at most two ocur is the event that not all three occur, this is

$$\neg(E \cap F \cap G) = \neg E \cup \neg F \cup \neg G.$$

∎

EXERCISE 1.5 (Handout 1, # 12). Which is more likely:

(a) Obtaining at least one six in six rolls of a fair die, or
(b) Obtaining at least one double six in six rolls of a pair of fair dice.

SOLUTION. For part (a), we compute the probability of the complement of the event, i.e., the event that we do not roll a 6 in 6 rolls of a fair die. This is

$$P(A) = 1 - P(\neg A) = 1 - \left(\frac{5}{6}\right)^6 \approx 0.665.$$

For part (b), we, again, compute the probability of the complement of the event, i.e., the event that no double 6 is rolled. This is

$$P(A) = 1 - P(\neg A) = 1 - \left(\frac{35}{36}\right)^6 \approx 0.156.$$

Therefore, it is more likely that you roll a 6 in 6 rolls of a fair die than that you roll a double 6 in 6 rolls of a pair of fair dice. ∎

EXERCISE 1.6 (Handout 1, # 13). There are n people are lined up at random for a photograph. What is the probability that a specified set of r people happen to be next to each other?

SOLUTION. The r specified people can stand as a group starting at positions $1, 2, \ldots, n - r + 1$. They can be permuted among themselves in $r!$ ways. The remaining $n - r$ people can be permuted among themselves in $(n - r)!$ ways. Thus, we have

$$P = \frac{(n - r + 1)r!(n - r)!}{n!} = \frac{(n - r + 1)!r!}{n!}.$$

∎

EXERCISE 1.7 (Handout 1, # 16). Consider a particular player, say North, in a Bridge game. Let X be the number of aces in his hand. Find the distribution of X.

SOLUTION. ■

EXERCISE 1.8 (Handout 1, # 20). If 100 balls are distributed completely at random into 100 cells, find the expected value of the number of empty cells.
Replace 100 by n and derive the general expression. Now approximate it as n tends to ∞.

SOLUTION. ■

1.2 Homework 2

EXERCISE 1.9 (Handout 2, # 5). Four men throw their watches into the sea, and the sea brings each man one watch back at random. What is the probability that at least one man gets his own watch back?

SOLUTION. ∎

EXERCISE 1.10 (Handout 2, # 7). Calculate the probability that in Bridge, the hand of at least one player is void in a particular suit.

SOLUTION. ∎

EXERCISE 1.11 (Handout 2, # 12). If n balls are placed at random into n cells, find the probability that exactly one cell remains empty.

SOLUTION. ∎

EXERCISE 1.12 (Handout 2, # 13). *Spread of rumors.* In a town of $n+1$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, etc. At each step the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor told r times without:

(a) returning to the originator,
(b) being repeated to any person.

Do the same problem when at each step the rumor told by one person to a gathering of N randomly chosen people. (The first question is the special case $N = 1$).

SOLUTION. ∎

EXERCISE 1.13 (Handout 2, # 14). What is the probability that

(a) the birthdays of twelve people will fall in twelve different calendar months (assume equal probabilities for the twelve months),
(b) the birthdays of six people will fall in exactly two calendar months?

SOLUTION. ∎

EXERCISE 1.14 (Handout 2, # 15). A car is parked among N cars in a row, not at either end. On his return the owner finds exactly r of the N places still occupied. What is the probability that both neighboring places are empty?

SOLUTION. ∎

EXERCISE 1.15 (Handout 2, # 16). Find the probability that in a random arrangement of 52 bridge card no two aces are adjacent.

Solution. ■

EXERCISE 1.16 (Handout 2, # 17). Suppose $P(A) = 3/4$, and $P(B) = 1/3$.
Prove that $P(A \cap B) \geq 1/12$. Can it be equal to $1/12$?

Solution. ■

EXERCISE 1.17 (Handout 2, # 18). Suppose you have infinitely many events $A_1, A_2, \ldots$, and each one is sure to occur, i.e., $P(A_i) = 1$ for any i.
Prove that $P\left(\bigcap_{i=1}^{n} A_i\right) = 1$.

Solution. ■

EXERCISE 1.18 (Handout 2, # 19). There are n blue, n green, n red, and n white balls in an urn. Four balls are drawn from the urn with replacement. Find the probability that there are balls of at least three different colors among the four drawn.

Solution. ■

1.3 Homework 3

EXERCISE 1.19 (Handout 3, # 3). n sticks are broken into one short and one long part. The $2n$ parts are then randomly paired up to form n new sticks. Find the probability that

(a) the parts are joined in their original order, i.e., the new sticks are the same as the old sticks;
(b) each long part is paired up with a short part.

SOLUTION. ∎

EXERCISE 1.20 (Handout 3, # 5). In a town, there are 3 plumbers. On a certain day, 4 residents need a plumber and they each call one plumber at random.

(a) What is the probability that all the calls go to one plumber (not necessarily a specific one)?
(b) What is the expected value of the number of plumbers who get a call?

SOLUTION. ∎

EXERCISE 1.21 (Handout 4, # 7). (*Polygraphs*). Polygraphs are routinely administered to job applicants for sensitive government positions. Suppose someone actually lying fails the polygraph 90% of the time. But someone telling the truth also fails the polygraph 15% of the time. If a polygraph indicates that an applicant is lying, what is the probability that he is in fact telling the truth? Assume a general prior probability p that the person is telling the truth.

SOLUTION. ∎

EXERCISE 1.22 (Handout 4, # 8). In a bolt factory machines A, B, C manufacture, respectively, 25, 35, and 40 per cent of the total. Of their output 5, 4, and 2 per cent are defective bolts. A bolt is drawn at random from the produce and is found defective. What are the probabilities that it was manufactured by machines A, B, C?

SOLUTION. ∎

EXERCISE 1.23 (Handout 4, # 9). Suppose that 5 men out of 100 and 25 women out of 10 000 are colorblind. A colorblind person is chosen at random. What is the probability of his being male? (Assume males and females to be in equal numbers.)

SOLUTION. ∎

EXERCISE 1.24 (Handout 4, # 10). (*Bridge.*) In a Bridge party West has no ace. What probability should he attribute to the event of his partner having

(a) no ace,

(b) two or more aces?

Verify the result by a direct argument.

SOLUTION. ∎

EXERCISE 1.25 (Handout 4, # 12). A true-false question will be posed to a couple on a game show. The husband and the wife each has a probability p of picking the correct answer. Should they decide to let one of the answer the question, or decide that they will give the common answer if they agree and toss a coin to pick the answer if they disagree?

SOLUTION. ∎

EXERCISE 1.26 (Handout 4, # 13). An urn containing 5 balls has been filled up by taking 5 balls at random from a second urn which originally had 5 black and 5 white balls. A ball is chosen at random from the first urn and is found to be black. What is the probability of drawing a white ball if a second ball is chosen from among the remaining 4 balls in the first urn?

SOLUTION. ∎

EXERCISE 1.27 (Handout 4, # 15). Events A, B, C have probabilities p_1, p_2, p_3. Given that exactly two of the three events occured, the probability that C occured is greater than $1/2$ if and only if ... (write down the necessary and sufficient condition).

SOLUTION. ∎

EXERCISE 1.28 (Handout 5, # 1). There are five coins on a desk: 2 are double-headed, 2 are double-tailed, and 1 is a normal coin.
One of the coins is selected at random and tossed. It shows heads.
What is the probability that the other side of this coin is a tail?

SOLUTION. ∎

EXERCISE 1.29 (Handout 5, # 2). (*Genetic testing*). There is a 50-50 chance that the Queen carries the gene for hemophilia. If she does, then each Prince has a 50-50 chance of carrying it. Three Princesses were recently tested and found to be non-carriers. Find the following probabilities:

  (a) that the Queen is a carrier;
  (b) that the fourth Princess is a carrier.

SOLUTION. ∎

EXERCISE 1.30 (Handout 5, # 4). (*Is Johnny in Jail*). Johnny and you are roommates. You are a terrific student and spend Friday evenings drowned in books. Johnny always goes out on Friday evenings. 40% of the times, he goes out with his girlfriend, and 60% of the times he goes to a bar. If he goes out with his girlfriend, 30% of the times he is just too lazy to come back and spends the night at hers. If he goes to a bar, 40% of the times he gets mad at the person sitting on his right, beats him up, and goes to jail.
On one Saturday morning, you wake up to see Johnny is missing. Where is Johnny?

SOLUTION. ∎

2 Homework 4

EXERCISE 2.1 (Handout 5, # 2). In an urn, there are 12 balls. 4 of these are white. Three players: A, B, and C, take turns drawing a ball from the urn, in the alphabetical order. The first player to draw a white ball is the winner. Find the respective winning probabilities: assume that at each trial, the ball drawn in the trial before is put back into the urn (i.e., selection *with replacement*).

  SOLUTION.   ■

EXERCISE 2.2 (Handout 5, # 8). Consider n families with 4 children each. How large must n be to have a 90% probability that at least 3 of the n families are all girl families?

  SOLUTION.   ■

EXERCISE 2.3 (Handout 5, # 10). (*Yahtzee*). In Yahtzee, five fair dice are rolled. Find the probability of getting a Full House, which is three rolls of one number and two rolls of another, in Yahtzee.

  SOLUTION.   ■

EXERCISE 2.4 (Handout 5, # 12). The probability that a coin will show all heads or all tails when tossed four times is 0.25. What is the probability that it will show two heads and two tails?

  SOLUTION.   ■

EXERCISE 2.5 (Handout 5, # 13). Let the events $A_1, A_2, \ldots, A_n$ be independent and $P(A_k) = p_k$. Find the probability p that none of the events occurs.

  SOLUTION.   ■

EXERCISE 2.6 (Handout 6, # 5). Suppose a fair die is rolled twice and suppose X is the absolute value of the difference of the two rolls. Find the PMF and the CDF of X and plot the CDF. Find a median of X; is the median unique?

  SOLUTION.   ■

EXERCISE 2.7 (Handout 6, # 7). Find a discrete random variable X such that $E(X) = E(X^3) = 0$; $E(X^2) = E(X^4) = 1$.

  SOLUTION.   ■

EXERCISE 2.8 (Handout 6, # 9). (*Runs*). Suppose a fair die is rolled n times. By using the indicator variable method, find the expected number of times that a six is followed by at least two other sixes. Now compute the value when $n = 100$.

SOLUTION. ∎

EXERCISE 2.9 (Handout 6, # 10). (*Birthdays*). For a group of n people find the expected number of days of the year which are birthdays of exactly k people. (Assume 365 days and that all arrangements are equally probable.)

SOLUTION. ∎

EXERCISE 2.10 (Handout 6, # 11). (*Continuation*). Find the expected number of multiple birthdays. How large should n be to make this expectation exceed 1?

SOLUTION. ∎

EXERCISE 2.11 (Handout 6, # 12). (*The blood-testing problem*). A large number, N, of people are subject to a blood test. This can be administered in two ways, (i) Each person can be tested separately. In this case N tests are required, (ii) The blood samples of k people can be pooled and analyzed together. If the test is negative, this one test suffices for the k people. If the test is positive, each of the k persons must be tested separately, and in all $k + 1$ tests are required for the k people. Assume the probability p that the test is positive is the same for all people and that people are stochastically independent.

(b) What is the expected value of the number, X, of tests necessary under plan (ii)?
(c) Find an equation for the value of k which will minimize the expected number of tests under the second plan. (Do not try numerical solutions.)

SOLUTION. ∎

EXERCISE 2.12 (Handout 6, # 13). (*Sample structure*). A population consists of r (classes whose sizes are in the proportion $p_1 : p_2 : \cdots : p_r$. A random sample of size n is taken with replacement. Find the expected number of classes not represented in the sample.

SOLUTION. ∎

3 Homework 5

EXERCISE 3.1 (Handout 7, # 6(d, f)). Find the variance of the following random variables

(d) X = # of tosses of a fair coin necessary to obtain a head for the first time.
(f) X = # matches observed in random sitting of 4 husbands and their wives in opposite sides of a linear table.
This is an example of the *matching problem*.

SOLUTION. ∎

EXERCISE 3.2 (Handout 7, # 8). (*Nonexistence of variance*).

(a) Show that for a suitable positive constant c, the function $p(x) = c/x^3$, $x = 1, \ldots$, is a valid probability mass function (PMF).
(b) Show that in this case, the expectation of the underlying random variable exists, but the variance does not!

SOLUTION. ∎

EXERCISE 3.3 (Handout 7, # 9). In a box, there are 2 black and 4 white balls. These are drawn out one by one at random (without replacement).

(a) Let X be the draw at which the first black ball comes out. Find the mean the variance of X.
(b) Let X be the draw at which the second black ball comes out. Find the meman the variance of X.

SOLUTION. ∎

EXERCISE 3.4 (Handout 7, # 10). Suppose X has a *discrete uniform distribution* on the set $\{1, \ldots, N\}$.
Find formulas for the mean and the variance of X.

SOLUTION. ∎

EXERCISE 3.5 (Handout 7, # 11). (*Be Original*) Give an example of a random variable with mean 1 and variance 100.

SOLUTION. ∎

EXERCISE 3.6 (Handout 7, # 13). (*Be Original*). Suppose a random variable X has the property that its second and fourth moment are both 1.
What can you say about the nature of X?

SOLUTION. ∎

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EXERCISE 3.7 (Handout 7, # 14). (*Be Original*). One of the following inequalities is true in general for all nonnegative random variables. Identify which one!

$$E(X)E(X^4) \geq E(X^2)E(X^3);$$
$$E(X)E(X^4) \leq E(X^2)E(X^2).$$

SOLUTION. ■

EXERCISE 3.8 (Handout 7, # 15). Suppose X is the number of heads obtained in 4 tosses of a fair coin.

Find the expected value of the weird function

$$\log\big(2 + \sin(\tfrac{\pi}{4}x)\big).$$

SOLUTION. ■

EXERCISE 3.9 (Handout 7, # 16). In a sequence of Bernoulli trials let X be the length of the run (of either successes or failures) started by the first trial.

(a) Find the distribution of X, $E(X)$, $\text{Var}(X)$.

SOLUTION. ■

EXERCISE 3.10 (Handout 7, # 17). A man with n keys wants to open his door and tries the keys independently and at random. Find the mean and variance of the number of trials

(a) if unsuccessful keys are not eliminated from further selections;
(b) if they are.

(Assume that only one key fits the door. The exact distributions are given in II, 7, but are not required for the present problem.)

SOLUTION. ■

4 Midterms, Exams, and Qualifying Exams

4.1 Qualifying Exams, August '99

EXERCISE 4.1. The number of fish that Anirban catches on any given day has a Poisson distribution with mean 20. Due to the legendary softness of his heart, he sets free, on average, 3 out of the 4 fish he catches. Find the mean and the variance of the number of fish Anirban takes home on a given day.

SOLUTION. ∎

EXERCISE 4.2. A fair die is rolled and at the same time a fair coin is tossed. This is done repeatedly. Find the probability that head occurs (strictly) before six occurs.

SOLUTION. ∎

EXERCISE 4.3. X, Y are independent random variables with a common density $f(x) = e^{-|x|}/2$, $x \in (-\infty, \infty)$. Find the density function of $X + Y$.

SOLUTION. ∎

EXERCISE 4.4. Let X_n denote the distance between two points chosen independently at random from the unit cube in $\mathbb{R}^n$. Evaluate
$$\lim_{n \to \infty} \frac{E(X_n)}{\sqrt{n}}.$$

SOLUTION. ∎

EXERCISE 4.5. Let X be distributed as Uniform$[0, 1]$. What is the probability that the digit 5 does not occur in the decimal expansion of X?

SOLUTION. ∎

4.2 Qualifying Exam, January '06

EXERCISE 4.6. The birthdays of 5 people are known to fall in exactly 3 calendar months. What is the probability that exactly two of the 5 were born in January?

SOLUTION. ∎

EXERCISE 4.7. Coupons are drawn, independently, with replacement, one at a time, from a set of 10 coupons. Find, explicitly, the expected number of draws

  (a) until the first draw coupon is drawn again;
  (b) until a duplicate occurs.

SOLUTION. ∎

EXERCISE 4.8. Let N be a positive integer. Choose an integer at random from $\{1, \ldots, N\}$. Let E be the event that your chosen random number is divisible by 3, and divisible by at least one of 4 and 6, but not divisible by 5. Find, explicitly, $\lim_{N \to \infty} P(E)$.

SOLUTION. ∎

EXERCISE 4.9. Anirban is driving his Dodge on a highway with 4 lanes each way. He is wired to change lanes every minute on the minute. He changes with equal probability to either adjacent lane if there are two adjacent lanes, and the successive changes are mutually independent. Find, explicitly, the probability that after 4 minutes, Anirban is back to the lane he started from

  (a) if he started at an outside lane;
  (b) if he started at an inside lane.

SOLUTION. ∎

EXERCISE 4.10. Burgess is going to Moose Pass, Alaska. He is driving his Dodge. He puts his car on cruise control at 70 mph. Gas stations are located every 30 miles, starting from his home. His car runs out of gas at a time distributed as an exponential with mean 4 hours. When that happens, he gets out, takes his bik out of his trunk, and bikes to the next gas station say M, at 10 mph. Let the time elapsed between when Burgess starts his trip and when he arrives at the gas station M be T. Find $E(T)$.

SOLUTION. ∎

EXERCISE 4.11. A fair coin is tossed n times. Suppose X heads are obtained. Given $X = x$, let Y be generated according to the Poisson distribution with mean x. Find the unconditional variance of Y, and then find the limit of the probability $P\big(|Y - n/2| > n^{3/4}\big)$, as $n \to \infty$.

SOLUTION. ∎

EXERCISE 4.12. Anirban plays a game repeatedly. On each play he wins an amount uniformly distributed in $(0,1)$ dollars, and then he tips the lady in charge of the game the square of the amount he has won. Then he plays again, tips again, and so on. Approximately calculate the probability that if he plays and tips six hundred times, his total winnings minus his total tips will exceed \$105.

SOLUTION. ∎

EXERCISE 4.13. Anirban's dog got mad at him and broke his walking cane, first uniformly into two peices, and then the long piece again uniformly into two pieces. Find the probability that Anirban can make a triangle out of the three pieces of his cane.

SOLUTION. ∎

EXERCISE 4.14. Suppose X, Y, Z are identically independently distributed Exp(1) random variables. Find the joint density of (X, XY, XYZ).

SOLUTION. ∎

EXERCISE 4.15. Let X be the number of Kings and Y the number of Hearts in a Bridge hand. Find the correlation between X and Y.

SOLUTION. ∎

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4.3 Qualifying Exam, August '14

EXERCISE 4.16.

(a) 3 balls are distributed one by one and at random in 3 boxes. What is the probability that exactly one box remains empty?
(b) n balls are distributed one by one and at random in n boxes. Find the probability that exactly one box remains empty.
(c) n balls are distributed one by one and at random in n boxes Find the probability that exactly two boxes remain empty.

SOLUTION. ■

EXERCISE 4.17. n players each roll a fair die. For any pair of players i, j, $i < j$, who roll the same number, the group is awarded one point.

(a) Find the mean of the total points of the group.
(b) Find the variance of the total points of the group.

SOLUTION. ■

EXERCISE 4.18. Suppose $X_1, X_2, \ldots$, is an infinite sequence of independently identically distributed Uniform$[0, 1]$ random variables. Find the limit

$$\lim_{n \to \infty} P\left[\frac{\left(\prod_{i=1}^{n} X_i\right)^{1/n}}{\left(\sum_{i=1}^{n} X_i\right)/n} > \frac{3}{4}\right].$$

SOLUTION. ■

EXERCISE 4.19. Suppose X is an exponential random variable with density $e^{-x/\sigma_1}/\sigma_1$ and Y is another exponential random variable with density $e^{-y/\sigma_2}/\sigma_2$, and that X, Y are independent.

(a) Find the CDF of $X/(X + Y)$.
(b) In the case $\sigma_1 = 2$, $\sigma_2 = 1$, find the mean of $X/(X + Y)$.

SOLUTION. ■

EXERCISE 4.20. Ten independently picked Uniform$[0, 100]$ numbers are each rounded to the nearest integer. Use the central limit theorem to approximate the probability that the sum of the ten rounded numbers equals the rounded value of the sum of the ten original numbers.

SOLUTION. ■

EXERCISE 4.21. Suppose for some given $m \geq 2$, we choose m independently identically distributed Uniform$[0,1]$ random variables $X_1, \ldots, X_m$. Let $X_{\min}$ denote their minimum and $X_{\max}$ denote their maximum. Now continue sampling $X_{m+1}, \ldots,$ from the Uniform$[0,1]$ density. Let N be the first index k such that X_{m+k} falls outside the interval $[X_{\min}, X_{\max}]$.

(a) Find a formula for $P(N > n)$ for a general n.
(b) Hence, explicitly find $E(N)$.

SOLUTION. ∎

EXERCISE 4.22. A $G_{n,p}$ graph on n vartices is obtained by adding each of the $\binom{n}{2}$ possible edges into the graph mutually independently with probability p. If vertex subsets A, B both have k vertices, and each vertex A shares an edge with each vertex in B, but there are no edges among the vertices within A or within B, then A, B generate a complete bipartate subgraph of order k denoted as $K_{k,k}$.

(a) For a given n and p, find an expression for the expected number of complete bipartate subgraphs $K_{3,3}$ of order $k = 3$ in a $G_5 n, p$ graph.
(b) Let p_n denote the value of p for which the expected value in part (a) equals one. Identify constants α, β such that $\lim_{n \to \infty} n^\alpha p_n = \beta$.

SOLUTION. ∎