MA571 Homework 12

Carlos Salinas

November 18, 2015

Problem 12.1 (Munkres §58, Ex. 7(c))

Let A be a subspace of X; let $j\colon A \hookrightarrow X$ be the inclusion map, and let $f\colon X \to A$ be a continuous map. Suppose there is a homotopy $H\colon X \times I \to X$ between the map $j \circ f$ and the identity map of X.

(c) Give an example in which j_* is not an isomorphism.

Example. ♠

Problem 12.2 (Munkres §58, Ex.9(a,b,c))

We define the *degree* of a continuous map $h\colon S^1 \to S^1$ as follows:

Let b_0 be the point $(0,1)$ of S^1; choose a generator γ for the infinite cyclic group $\pi_1(S^1, b_0)$. If x_0 is any point of S^1, choose a path α in S^1 from b_0 to x_0 and define $\gamma(x_0) = \hat{\alpha}(\gamma)$. Then $\gamma(x_0)$ generates $\pi_1(S^1, x_0)$. The element $\gamma(x_0)$ is independent of the choice of the path α, since the fundamental group of S^1 is Abelian.

Now given $h\colon S^1 \to S^1$, choose $x_0 \in S^1$ and let $h(x_0) = x_1$. consider the homomorphism

$$h_*\colon \pi_1(S^1, x_0) \longrightarrow \pi_1(S^1, x_1).$$

Since both groups are infinite cyclic, we have

$$h_*(\gamma(x_0)) = d \cdot \gamma(x_1) \tag{*}$$

for some integer d, if the group is written additively. The integer d is called the *degree* of h is denoted by $\deg h$.

The degree of h is independent of the choice of the generator γ; choosing the other generator woul merely change the sign of both sides of (*).

(a) Show that d is independent of the choice of x_0.

(b) Show that if $h, k\colon S^1 \to S^1$ are homotopic, they have the same degre.

(c) Show that $\deg(h \circ k) = (\deg h) \cdot (\deg k)$.

Proof. ∎

Problem 12.3 (Munkres §60, Ex. 2)

Let X be the quotient space obtained from B^2 by identifying each point x of S^1 with its antipode $-x$. Show that X is homeomorphic to the projective plane P^2.

Proof. ■

For the problems to come we need the following definitions:

Definition. Let M be an m-manifold.

(i) A *linear* path in $\mathbf{R}^n$ is a path $f\colon [a,b] \to \mathbf{R}^n$ with $f(s) = \frac{1}{b-a}[(b-s)z_1 + (s-a)z_2]$ for two points z_1 and z_2.

(ii) A *quasi-linear* path in M is a path $g\colon [a,b] \to M$ for which there is an open set U containing $g([a,b])$ and a homeomorphism h from U to an open set $\mathbf{R}^m$ such that $h \circ g$ is linear.

(iii) A *piecewise quasi-linear* path in M is a path $g\colon [a,b] \to M$ for which there is a partition of $[a,b]$ into subintervals such that the restriction of g to each subinterval of the partition is quasi-linear.

Problem 12.4 (A)

(i) Let M be an m-manifold, let U an open set in M which is homeomorphic to an open *ball* in $\mathbf{R}^m$, and let g be a path in U. Prove that g is a path-homotopic to a quasi-linear path. (Hint: straight-line homotopy.)

(ii) Prove that every path in an m-manifold is path-homotopic to a piecewise quasi-linear path. (Hint: Theorem 51.3, Lebesgue Lemma and part (i)).

Proof. ■

Problem 12.5 (B)

Prove piecewise quasi-linear path in an m-manifold with $m > 1$ cannot be onto. (Hint: Use Problem A from HW 2; you may *assume*, without proving it, that the image of a linear path does not contain an open set of $\mathbf{R}^m$ if $m > 1$.)

Proof. ∎

Problem 12.6 (C)

(i) S^m is an m-manifold for all m (you don't have to prove this, it follows easily from the solution of HW 8 #3). Prove that S^m is simply connected for $m \geq 2$. Do not use Section 59. (Hint: Use Problems A and B from assignment and Problem C from HW 11.)

(ii) Prove that $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^2$ for $n \neq 2$. (Hint: You may use Theorem A from the note on the Fundamental Group of the Circle.)

Proof. ■