MA 523: Homework 9

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Problem 9.1

(a) Show that for $n = 3$ the general solution to the wave equation $u_{tt} - \Delta u = 0$ with spherical symmetry about the origin has the form

$$u = \tfrac{1}{r}F(r+t) + \tfrac{1}{r}G(r-t), \quad r = |x|,$$

with suitable F and G.

(b) Show that the solution with initial data of the form

$$u(r,0) = 0, \quad u_t(r,0) = h(r)$$

(h is an even function of r) is given by

$$u = \frac{1}{2r}\int_{r-t}^{r+t} \rho h(\rho)\, d\rho.$$

Solution. For part (a): We show that

$$u = \tfrac{1}{r}F(r+t) + \tfrac{1}{r}G(r-t)$$

is in fact a solution to the wave equation in $\mathbb{R}^3 \times (0,\infty)$. By direct calculation, we have

$$\begin{aligned}
u_{tt} &= \tfrac{1}{r}F''(r+t) + \tfrac{1}{r}G''(r-t),\\
u_r &= \tfrac{1}{r}F'(r+t) + \tfrac{1}{r}G'(r-t) - \tfrac{1}{r^2}F(r+t) - \tfrac{1}{r^2}G(r-t),\\
u_{rr} &= \tfrac{1}{r}F''(r+t) + \tfrac{1}{r}G''(r-t)\\
&\qquad - \tfrac{2}{r^2}F'(r+t) - \tfrac{2}{r^2}G'(r-t)\\
&\qquad + \tfrac{2}{r^3}F(r+t) + \tfrac{2}{r^3}G(r-t),\\
\Delta_{S^2} u &= 0,
\end{aligned}$$

and lastly,

$$\begin{aligned}
\Delta u &= \tfrac{\partial}{\partial r^2}u + \tfrac{2}{r}\tfrac{\partial}{\partial r}u + \tfrac{1}{r^2}\Delta_{S^2}u\\
&= \tfrac{1}{r}F''(r+t) + \tfrac{1}{r}G''(r-t)\\
&\qquad - \tfrac{2}{r^2}F'(r+t) - \tfrac{2}{r^2}G'(r-t) + \tfrac{2}{r^3}F(r+t) + \tfrac{2}{r^3}G(r-t)\\
&\qquad + \tfrac{2}{r^2}F'(r+t) + \tfrac{2}{r^2}G'(r-t) - \tfrac{2}{r^3}F(r+t) - \tfrac{2}{r^3}G(r-t)\\
&\quad + 0\\
&= \tfrac{1}{r}F''(r+t) + \tfrac{1}{r}G''(r-t).
\end{aligned}$$

Therefore, looking at the equation for u_{tt}, we see that

$$u_{tt} - \Delta u = 0;$$

i.e., $u(r,t) := \frac{1}{r}F(r+t) + \frac{1}{r}G(r-t)$ is a solution to the wave equation with F and G at least twice differentiable and such that they satisfy the initial conditions of the (nonhomogeneous) wave equation.

We still need to show that if u is a solution to the wave equation with spherical symmetry it has the form prescribed above. We trust this can be done for now and return to this problem as time permits.

For part (b): We show by a direct computation that

$$u = \frac{1}{2r}\int_{r-t}^{r+t} \rho h(\rho)\, d\rho$$

solve the initial-value problem for the wave equation. First, we compute the necessary partial derivatives of u,

$$\begin{aligned}
u_t(r,t) &= \tfrac{1}{2r}[(r+t)h(r+t) - (r-t)h(r-t)],\\
u_{tt}(r,t) &= \tfrac{1}{2r}[(r+t)h'(r+t) + h(r+t) + (r-t)h'(r-t) + h(r-t)],\\
u_r(r,t) &= \tfrac{1}{2r}[(r+t)h(r+t) - (r-t)h(r-t)] - \frac{1}{2r^2}\int_{r-t}^{r+t} \rho h(\rho)\, d\rho\\
&= \tfrac{1}{2r}[(r+t)h(r+t) - (r-t)h(r-t)] - \tfrac{1}{r}u(r,t),\\
u_{rr}(r,t) &= h'(r+t) - h'(r-t) - \tfrac{1}{r}u_r(r,t) + \tfrac{1}{r^2}u(r,t),
\end{aligned}$$

■

Problem 9.2

Show that the solution $w(x_1, t)$ of the initial-value problem for the *Klein–Gordon equation*

$$\begin{cases} w_{tt} = w_{x_1x_1} - \lambda^2 w, \\ \qquad w(x_1, 0) = 0, \quad w_t(x_1, 0) = h(x_1) \end{cases} \tag{9.1}$$

is given by

$$w(x_1, t) = \frac{1}{2}\int_{x_1-t}^{x_1+t} J_0(\lambda s)h(y_1)\, dy_1.$$

Here $s^2 = t^2 - (x_1 - y_1)^2$, while J_0 denotes the Bessel function defined by

$$J_0(z) := \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(z\sin\theta)\, d\theta.$$

(*Hint:* Descend to (9.1) from the two-dimensional wave equation satisfied by

$$u(x_1, x_2, t) = \cos(\lambda x_2)w(x_1, t).)$$

Solution. Taking the hint, the equation $u(x_1, x_2, t) := \cos(\lambda x_2)w(x_1, t)$ satisfies the two-dimensional wave equation as we will shortly see. First, let us compute find the necessary partial derivatives of u,

$$\begin{aligned} u_{tt} &= \cos(\lambda x_2)w_{tt}(x_1, t), \\ u_{x_1} &= \cos(\lambda x_2)w_{x_1}(x_1, t), \\ u_{x_1x_1} &= \cos(\lambda x_2)w_{x_1x_1}(x_1, t), \\ u_{x_2} &= -\lambda\sin(\lambda x_2)w(x_1, t), \\ u_{x_2x_2} &= -\lambda^2\cos(\lambda x_2)w(x_1, t). \end{aligned}$$

Then, by (9.1) together with the equations above we have

$$\begin{cases} u_{tt} - \Delta u = \cos(\lambda x_2)(w_{tt} - w_{x_1x_1} + \lambda^2 w) = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \qquad u = 0, \quad u_t = \cos(\lambda x_2)h(x_1) & \text{on } \mathbb{R}^3 \times \{\, t = 0\,\}. \end{cases}$$

By Kirchhoff's formula,

$$u(x_1, x_2, t) = \fint_{\partial B(x_1, x_2, t)} t\cos(\lambda y_2)h(y_1)\, dS(y_1, y_2)$$

solves the above initial-value problem. Thus, $u(x_1, 0, t) = w(x_1, t)$ solves the Klein–Gordon equation (9.1). Making the substitution $y_2 \mapsto s = \sqrt{t^2 - (x_1 - y_1)^2}$ we can rewrite $u(x_1, 0, t)$ as

$$\begin{aligned} w(x_1, t) &= u(x_1, 0, t) \\ &= \frac{1}{4\pi t^2}\int_{\partial B(x_1, 0, t)} t\cos(\lambda s)h(y_1)\, dS(y_1, s) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi} \int_{\partial B(0,0,1)} \cos(\lambda t z_2) h(x_1 + t z_1)\, dS(z_1, z_2) \\
&= \frac{1}{4\pi} \int_{-1}^{1} \left[\int_{z_2 \in S^1} \cos(\lambda t z_2)\, dz_2 \right] h(x_1 + t z_1)\, dz_1 \\
&= \frac{1}{4\pi} \int_{x_1 - t}^{x_1 + t} \left[4 \int_0^{\frac{\pi}{2}} \cos(\lambda s \sin\theta)\, d\theta \right] h(y_1)\, dy_1 \\
&= \frac{1}{2} \int_{x_1 - t}^{x_1 + t} J_0(s) h(y_1)\, dy_1.
\end{aligned}
$$

as desired. ■

Problem 9.3

Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u = g, \quad u_t = h & \text{on } \mathbb{R}^3 \times \{ t = 0 \} \end{cases}$$

where g and h are smooth and have compact support. Show there exists a constant C such that

$$|u(x,t)| \leq \frac{C}{t} \quad (x \in \mathbb{R}^3,\, t > 0).$$

Solution. Since h and g are compactly supported, we can find sufficiently large balls about their supports $B(x, R)$ independent of the size of $B(x,t)$. Now, by Kirchhoff's formula we have

$$\begin{aligned} |u(x,t)| &= \left| \fint_{\partial B(x,t)} th(y) + g(y) + Dg(y) \cdot (y - x)\, dS(y) \right| \\ &= \frac{1}{4\pi t^2} \left| \int_{\partial B(x,t)} th(y) + g(y) + Dg(y) \cdot (y - x)\, dS(y) \right| \\ &\leq \frac{1}{4\pi t^2} \int_{\partial B(x,R)} t \sup\{|h|, |g|, |Dg|\}\, dS(y) \\ &= \frac{M |\partial B(x,R)|}{4\pi t} \\ &= \frac{C}{t}, \end{aligned}$$

where $M < \infty$ since h and g are smooth and compactly supported (and hence h, g, and $|Dg|$ achieve their supremum in $B(x,R)$). ■