MA 544: Homework 10

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Problem 10.1 (Wheeden & Zygmund §7, Ex. 1)

Let f be measurable in $\mathbf{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(\mathbf{x}) \geq c\|\mathbf{x}\|^{-n}$ for $\|\mathbf{x}\| \geq 1$.

Proof. Suppose that f is measurable and nonzero on a subset E of $\mathbf{R}^n$ with positive measure. Assume E is bounded. Since f is measurable $|f|$ is measurable so the set $E_a \coloneqq \{\, \mathbf{x} \in E : |f|(\mathbf{x}) > a \,\}$, for a finite, is a measurable bounded subset of $\mathbf{R}^n$. Let χ_a denote the characteristic function of E_a. Then, by Chebyshev's inequality, we have

$$\begin{aligned} {\chi_a}^*(\mathbf{x}) &= \sup_Q \frac{1}{|Q|} \int_Q |\chi_a(\mathbf{y})| d\mathbf{y} \\ &\leq \sup_Q \frac{1}{|Q|} \left[\frac{1}{a} \int_Q |f(\mathbf{y})| d\mathbf{y} \right] \\ &= \frac{1}{a} f^*(\mathbf{x}). \end{aligned} \tag{10.1}$$

By the commentary on p. 138, there exists constants c_1 and c_2 such that

$$c_1 \frac{|E_a|}{\|\mathbf{x}\|^n} \leq {\chi_a}^*(\mathbf{x}) \leq c_2 \frac{|E_a|}{\|\mathbf{x}\|^n} \tag{10.2}$$

for all large $\|\mathbf{x}\|$. Putting (10.1) and (10.2) together, we obtain

$$ac_1 \frac{|E_a|}{\|\mathbf{x}\|^n} \leq f^*(\mathbf{x}). \tag{10.3}$$

Setting $c \coloneqq ac_1|E|$, we have the desired lower bound $c\|\mathbf{x}\|^{-n} \leq f^*(\mathbf{x})$ (assuming $\|\mathbf{x}\|$ is large). ∎

Problem 10.2 (Wheeden & Zygmund §7, Ex. 2)

Let $\varphi(\mathbf{x}), \mathbf{x} \in \mathbf{R}^n$, be a bounded measurable function such that $\varphi(\mathbf{x}) = 0$ for $\|\mathbf{x}\| \geq 1$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(\mathbf{x}) = \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon)$. ($\varphi_\varepsilon$ is called an *approximation to the identity.*) If $f \in L(\mathbf{R}^n)$, show that

$$\lim_{\varepsilon \to 0}(f * \varphi_\varepsilon)(\mathbf{x}) = f(\mathbf{x})$$

in the Lebesgue set of f. (Note that $\int \varphi_\varepsilon = 1$, $\varepsilon > 0$, so that

$$(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x}) = \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{y}) d\mathbf{y}.$$

Use Theorem 7.16.)

Proof. First note that, making the change of variables $\mathbf{u} = \mathbf{x}/\varepsilon$ (with Jacobian $\mathbf{J}(\mathbf{x}, \mathbf{u}) = \varepsilon^n$), we have

$$\begin{aligned}
\int \varphi_\varepsilon(\mathbf{x}) d\mathbf{x} &= \int \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon) d\mathbf{x} \\
&= \int_{B(\mathbf{0},\varepsilon)} \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon) d\mathbf{x} \\
&= \int_{B(\mathbf{0},1)} \varphi(\mathbf{u}) d\mathbf{u} \\
&= \int \varphi(\mathbf{x}) d\mathbf{u} \\
&= 1.
\end{aligned} \tag{10.4}$$

Hence, by the hint and the definition of the convolution, we have

$$\begin{aligned}
|(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x})| &= \left| \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{x}) d\mathbf{x} \right| \\
&= \left| \int_{B(\mathbf{0},\varepsilon)} [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{x}) d\mathbf{x} \right| \\
&\leq \int_{B(\mathbf{0},\varepsilon)} |[f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{x})| d\mathbf{x}.
\end{aligned} \tag{10.5}$$

Now, since φ is bounded, say by M, we have

$$\varphi_\varepsilon(\mathbf{y}) = \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon) \leq M. \tag{10.6}$$

Then, we have an estimate on (10.5)

$$\begin{aligned}
\int_{B(\mathbf{0},\varepsilon)} |[f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\varphi_\varepsilon(\mathbf{x})| d\mathbf{x} &\leq \frac{M}{\varepsilon^n} \int_{B\mathbf{0},\varepsilon} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| \\
&\leq \frac{M}{\varepsilon^n} \int_{B(\mathbf{0},\varepsilon)} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})|.
\end{aligned} \tag{10.7}$$

Now, let Q_ε be the larges cube centered at $\mathbf{x}$ contained in $B(\mathbf{0}, \mathbf{x})$. Then, as we have previously shown, the volume of Q_ε is $C\varepsilon^n$ for some positive real number C. Making a change of variables $\mathbf{v} = \mathbf{x} - \mathbf{y}$ gives us

$$|(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x})| \leq C\frac{M}{|Q_\varepsilon|}\int_{Q_\varepsilon + bfx} |f(\mathbf{v}) - f(\mathbf{x})|d, \tag{10.8}$$

which goes to 0 an $\varepsilon \to 0$ by Theorem 7.16 since $\mathbf{x}$ is a point in the Lebesgue set of f. ■

Problem 10.3 (Wheeden & Zygmund §7, Ex. 6)

Show that if $\alpha > 0$, then x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Proof. ■

Problem 10.4 (Wheeden & Zygmund §7, Ex. 8)

Prove the following converse of Theorem 7.31: If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Proof. ∎

Problem 10.5 (Wheeden & Zygmund §7, Ex. 9)

If f is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then f is absolutely continuous on $[a, b]$. (For the second part, use Theorems 2.2(ii) and 7.24 to show that $V(x)$ is absolutely continuous and then use the result of Exercise 8).

Proof. ∎

Problem 10.6 (Wheeden & Zygmund §7, Ex. 12)

Use Jensen's inequality to prove that if $a, b \geq 0$, $p, q > 1$, $(1/p) + (1/q) = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}.$$

More generally, show that

$$a_1 \cdots a_N = \sum_{j=1}^{N} \frac{{a_j}^{p_j}}{p_j},$$

where $a_j \geq 0$, $p_j > 1$, $\sum_{j=1}^{N}(1/p_j) = 1$. (Write $a_j = e^{x_j/p_j}$ and use the convexity of e^x).

Proof. ■

Problem 10.7 (Wheeden & Zygmund §7, Ex. 13)

Prove Theorem 7.36.

Proof. Recall the statement of Theorem 7.36

Theorem. (i) *If φ_1 and φ_2 are convex in (a, b), then $\varphi_1 + \varphi_2$ is convex in (a, b).*

(ii) *If φ is convex in (a, b) and c is a positive constant, then $c\varphi$ is convex in (a, b).*

(iii) *If φ_k, $k = 1, 2, \ldots$, are convex in (a, b) and $\varphi_k \to \varphi$ in (a, b), then φ is convex in (a, b).*

■