MA571 Midterm 1: Practice Problems

Carlos Salinas

September 30, 2015

Problem Mid-1.1

Let $A \subset X$ and $B \subset Y$. Show that the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

Proof. Before we proceed, we need to prove the following nontrivial facts:

Claim 1 (Munkres §17, Ex. 3)**.** *If A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.*

Proof of claim. We will show that the complement of $A \times B$ is open in $X \times Y$. Let $(x, y) \in (X \times Y) \smallsetminus (A \times B)$. Then $x \notin A$ and $y \notin B$. Since A and B are closed in X and Y, respectively, there exist neighborhoods U and V of x and y, respectively, such that $U \subset X \smallsetminus A$ and $V \subset Y \smallsetminus B$. Then $U \times V \subset (X \times Y) \smallsetminus (A \times B)$ is a neighborhood of (x, y) so, by Lemma C, $(X \times Y) \smallsetminus (A \times B)$ is open. Thus, $A \times B$ is closed. ♣

Claim 2 (Munkres §17, Ex. 6(a))**.** *If $A \subset B$, then $\overline{A} \subset \overline{B}$.*

Proof of claim. By Theorem 17.5(a), $x \in \overline{A}$ if and only if for every neighborhood $U \ni x$, $U \cap A \neq \emptyset$. In particular, since $A \subset B$, $U \cap B \neq \emptyset$ for every neighborhood $U \ni x$ for every $x \in \overline{A}$. Thus, $x \in \overline{B}$ and we have the following containment $\overline{A} \subset \overline{B}$. ♣

Since $A \subset \overline{A}$ and $B \subset \overline{B}$ then $A \times B \subset \overline{A} \times \overline{B}$. Then by Claim 2 $\overline{A \times B} \subset \overline{\overline{A} \times \overline{B}}$, but by Claim 1 $\overline{\overline{A} \times \overline{B}} = \overline{A} \times \overline{B}$ so $\overline{A \times B} \subset \overline{A} \times \overline{B}$. To see the reverse containment, take an element $(x, y) \in \overline{A} \times \overline{B}$ then for $x \in \overline{A}$ and $y \in \overline{B}$. Thus, by Theorem 17.5(a) for every neighborhood $U \ni x$ and $V \ni y$, we have $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. Thus, $U \times V \cap A \times B \neq \emptyset$ so by Theorem 17.5(b), since $U \times V$ is a basis element for the topology on $X \times Y$, $(x, y) \in \overline{A \times B}$. Thus, $\overline{A \times B} \supset \overline{A} \times \overline{B}$ and the equality $\overline{A \times B} = \overline{A} \times \overline{B}$ holds. ■

Problem Mid-1.2

Let X be a topological space and let A be a dense subset of X. Let Y be a Hausdorff space and let $g, h \colon X \to Y$ be continuous functions which agree on A. Prove that $g = h$.

Proof. Suppose, towards a contradiction, that $g \neq h$. Then $g(x) \neq h(x)$ for some $x \in X \smallsetminus A$. Since Y is Hausdorff, there exists neighborhoods $U \ni g(x)$ and $V \ni h(x)$ with $U \cap V = \emptyset$. Since g and h are continuous, $g^{-1}(U)$ and $h^{-1}(V)$ are neighborhoods of x. In particular, $g^{-1}(U) \cap h^{-1}(U)$ is a nonempty neighborhood of x. Since $\overline{A} = X$, by Theorem 17.5(a), $\big(g^{-1}(U) \cap h^{-1}(V)\big) \cap A \neq \emptyset$. Let $y \in \big(g^{-1}(U) \cap h^{-1}(V)\big) \cap A$. Then $g(y) = h(y) \in U \cap V$. This contradicts the fact that U and V were chosen to be disjoint. ■

Problem Mid-1.3

Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Let G_f (called the *graph* of f) be the subspace $\{\, x \times f(x) \mid x \in X \,\}$ of $X \times Y$. Prove that if Y is Hausdorff then G_f is closed.

Proof. ■

Problem Mid-1.4

Let X be a topological space and let $f, g\colon X \to \mathbf{R}$ be continuous. Define $h\colon X \to \mathbf{R}$ by

$$h(x) = \min\{(f(x), g(x)\}.$$

Use the pasting lemma to prove that h is continuous. (You will not get full credit for any other method.)

Proof. Ngger ■

Problem Mid-1.5

Let X and Y be topological spaces and let $f\colon X \to Y$ be a function with the property that

$$f(\overline{A}) \subset \overline{f(A)}$$

for all subsets A of X. Prove that f is continuous.

Proof. ■

Problem Mid-1.6

Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Prove that

$$f(\overline{A}) \subset \overline{f(A)}$$

for all subsets A of X.

Proof. ■

Problem Mid-1.7

Let X be any topological space and let Y be a Hausdorff space. Let $f, g\colon X \to Y$ be continuous functions. Prove that the set $\{\, x \in X \mid f(x) = g(x) \,\}$ is closed.

Proof. ■

Problem Mid-1.8

Let X be a topological space and A a subset of X. Suppose that

$$A \subset \overline{X \setminus \overline{A}}.$$

Prove that $\overline{A}$ does not contain any nonempty open set.

Proof. ■

Problem Mid-1.9

Let X be a topological space with a countable basis. Prove that every open cover of X has a countable subcover.

Proof. ■

Problem Mid-1.10

Let X_α be an infinite family of topological spaces.

(a) Define the product topology on $\prod X_\alpha$.
(b) For each α, let A_α be a subspace of X_α. Prove that $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$.

Proof. ∎

Problem Mid-1.11

Suppose that we are given an indexing set A, and for each $\alpha \in A$ a topological space X_α. Suppose also that for each $\alpha \in A$ we are given a point $b_\alpha \in X_\alpha$. Let $Y = \prod X_\alpha$ with the product topology. Let $\pi_\alpha : Y \to X_\alpha$ be the projection. Prove that the set

$$S = \{\, y \in Y \mid \pi_\alpha(y) = b_\alpha \text{ except for finitely many } \alpha \,\}$$

is dense in Y (that is, its closure is Y).

Proof. ∎

Problem Mid-1.12

Let X be the Cartesian product $\mathbf{R}^\omega = \prod_{i=1}^\infty \mathbf{R}$ with the box topology (recall that a basis for this topology consists of all sets of the form $\prod_{i=1}^\infty U_i$, where each U_i is open in $\mathbf{R}$). Let $f : \mathbf{R} \to X$ be the function which takes t to $(t, t, t, ...)$. Prove that f is not continuous.

Proof. ∎

Problem Mid-1.13

Prove that the countable product $\mathbf{R}^\omega$ (with the product topology) has the following property: there is a countable family $\mathcal{F}$ of neighborhoods of the point $\mathbf{0} = (0, 0, 0, ...)$ such that for every neighborhood V of $\mathbf{0}$ there is a $U \in \mathcal{F}$ with $U \subset V$.

Note: the book proves that $\mathbf{R}^\omega$ is a metric space, but you may not use this in your proof. Use the definition of the product topology.

Proof. ∎

Problem Mid-1.14

Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X, thus an element of Y is a sequence of 0's and 1's. For each $n \geq 1$, let $y_0 \in Y$ be the element $(1, 1, 1, ..., 1, 0, 0, 0, ..)$, with n 1's at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1s. Prove that the set $\{y_n\}_{n \geq 1} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

Proof. ∎

Problem Mid-1.15

Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's. Let A be the subset of Y consisting of sequences with only a finite number of 1's. Is A closed? Prove or disprove.

Proof. ∎

Problem Mid-1.16

Let Y be a topological space.Let X be a set and let $f: X \to Y$ be a function. Give X the topology in which the open sets are the sets $f^{-1}(V)$ with V open in Y (you do not have to verify that this is a topology). Let $a \in X$ and let B be a closed set in X not containing a. Prove that $f(a)$ is not in the closure of $f(B)$.

Proof. ■

Problem Mid-1.17

Let $f: X \to Y$ be a function that takes closed sets to closed sets. Let $y \in Y$ and let U be an open set containing $f^{-1}(y)$. Prove that there is an open set V containing y such that $f^{-1}(V)$ is contained in U.

Proof. ■

Problem Mid-1.18

Let X be a topological space with an equivalence relation $\sim$. Suppose that the quotient space $X/\sim$ is Hausdorff. Prove that the set $S = \{ x \times y \in X \times X \mid x \sim y \}$ is a closed subset of $X \times X$.

Proof. ■

Problem Mid-1.19

Let $p: X \to Y$ be a quotient map. Let us say that a subset S of X is *saturated* if it has the form $p^{-1}(T)$ for some subset T of Y. Suppose that for every $y \in Y$ and every open neighborhood U of $p^{-1}(y)$ there is a saturated open set V with $p^{-1}(y) \subset V \subset U$. Prove that p takes closed sets to closed sets.

Proof. ■

Problem Mid-1.20

Let X be a topological space, let D be a connected subset of X, and let $\{E_\alpha\}$ be a collection of connected subsets of X.

Proof. ■

Problem Mid-1.21

Let X and Y be connected. Prove that $X \times Y$ is connected.

Proof. ■

Problem Mid-1.22

For any space X, let us say that two points are "inseparable" if there is no separation $X = U \cup V$ into disjoint open sets such that $x \in U$ and $y \in V$. Write $x \sim y$ if x and y are inseparable. Then $\sim$ is an equivalence relation (you don't have to prove this). Now suppose that X is locally connected (this means that for every point x and every open neighborhood U of x, there is a connected open neighborhood V of x contained in U). Prove that ecah equivalence class of the relation $\sim$ is connected.

Proof. ■

Problem Mid-1.23

Let X be a topological space. Let $A \subset X$ be connected. Prove $\overline{A}$ is connected.

Proof. ■

Problem Mid-1.24

Let $X_1, X_2, \ldots$ be topological spaces. Suppose $\prod_{n=1}^{\infty} X_n$ is locally connected. Prove that al lbut finitely many X_n are connected.

Proof. ■

Problem Mid-1.25

LEt X be a connected space and let $f\colon X \to Y$ be a function which is continuous and onto. Prove that Y is connected. (This is a theorem in Munkres—prove it from the definitions).

Proof. ■

Problem Mid-1.26

Given:

(i) $p\colon X \to Y$ is a quotient map.
(ii) Y is connected.
(iii) For every $y \in Y$, the set $p^{-1}(y)$ is connected.

Prove that X is connected.

Proof. ■

Problem Mid-1.27

Let A be a subset of $\mathbf{R}^2$ which is homeomorphic to the open unit interval $(0,1)$. Prove that A does not contain a nonempty set which is open in $\mathbf{R}^2$.

Proof. ■

Problem Mid-1.28

Let X be a connected space. Let $\mathcal{U}$ be an open covering of X and let U be a nonempty set in $\mathcal{U}$. Say that a set V in $\mathcal{U}$ is *reachable from* U if there is a sequence $U = U_1, U_2, ..., U_n = V$ of sets in $\mathcal{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for each i from 1 to $n - 1$. Prove that every nonempty V in $\mathcal{U}$ is reachable from U.

Proof. ■

Problem Mid-1.29

Suppose that X is connected and every point of X has a path-connected open neighborhood. Prove that X is path-connected.

Proof. ■

Problem Mid-1.30

Let X be a topological space and let $f, g\colon X \to [0, 1]$ be continuous functions. Suppose that X is connected and f is onto. Prove that there must be a point $x \in X$ with $f(x) = g(x)$.

Proof. ■