MA 519: Homework 11

Max Jeter, Carlos Salinas November 10, 2016

PROBLEM 11.1 (DASGUPTA 7.2 (A), (B), (C), (D), (E))

- (a) Suppose $E|X_n-c|^{\alpha}\to 0$, where $0<\alpha<1$. Does X_n necessarily converge in probability to c?
- (b) Suppose $a_n(X_n \theta) \xrightarrow{\mathcal{L}} N(0, 1)$. Under what condition on a_n can we conclude that $X_n \xrightarrow{\mathcal{P}} \theta$?
- (c) $o_p(1) + O_p(1) = ?$
- (d) $o_p(1)O_p(1) = ?$
- (e) $o_p(1) + o_p(1)O_p(1) = ?$

SOLUTION. For part (a) we show that indeed $E(|X_n - c|^{\alpha}) \to 0$ implies $X_n \xrightarrow{\mathcal{P}} c$. Let $\varepsilon > 0$ be given. By Markov's inequality, we have

$$P(|X_n - c| > \varepsilon) = P(|X_n - c|^{\alpha} > \varepsilon^{\alpha}) \le \frac{E(|X_n - c|^{\alpha})}{\varepsilon^{\alpha}}.$$

Since $E(|X_n - c|^{\alpha}) \to 0$ as $n \to \infty$, it follows that $P(|X_n - c| > \varepsilon)$ as $n \to \infty$; i.e., X_n converges to c in probability.

For part (b), suppose $a_n(X_n - \theta) \xrightarrow{\mathcal{L}} N(0, 1)$; i.e., $P(|a_n(X_n - \theta)| \leq x) \to \Phi(x)$ as $n \to \infty$. In words $X_n \xrightarrow{\mathcal{P}} \theta$ means that for every $\varepsilon > 0$ and every $\eta > 0$ there exists a positive integer N depending on ε and η such that $n \geq N$ implies

$$P(|X_n - \theta| \ge \varepsilon) < \eta.$$

First, let us find the PDF of the sequence $a_n(X_n - \theta)$. Let f_n denote the PDF of X_n , then the CDF of $a_n(X_n + \theta)$ is

$$P(|a_n(X_n - \theta)| \le x) = P(-x \le a_n(X_n - \theta) \le x)$$

$$= P\left(-\frac{x}{a_n} + \theta \le X_n \le \frac{x}{a_n} + \theta\right)$$

$$= \int_{-x/a_n + \theta}^{x/a_n + \theta} f(y) \, dy$$

$$= f(x/a_n + \theta) - f(-x/a_n + \theta),$$

therefore its PDF is

$$\frac{dP}{dx}(|a_n(X_n - \theta)| \le x) = \frac{d}{dx} \left[f\left(\frac{x}{a_n} + \theta\right) - f\left(-\frac{x}{a_n} + \theta\right) \right]$$
$$= \frac{1}{a_n} \left(f(x/a_n + \theta) + f(-x/a_n + \theta) \right)$$

For part (c), suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n = o_p(1)$ and $b_n = O_p(1)$, then for the sequence $\{c_n := a_n + b_n\}$ the most we can expect is $c_n = O_p(1)$. Indeed, we know that if a sequence is $o_p(1)$ then it is also $O_p(1)$ therefore there exists K_1 and K_2 such that $|a_n| \leq K_1$, $|b_n| \leq K_2$ for all $n \geq 1$. Therefore, $|c_n| \leq K_1 + K_2$ for all $n \geq 1$.

For part (d), suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n = o_p(1)$ and $b_n = O_p(1)$, then for the sequence $\{c_n := a_n b_n\}$ the most we can expect is $c_n = O_p(1)$. Again, since $\{a_n\}$ is $o_p(1)$ it

is $O_p(1)$ so there exists a constant $K_1 \ge 0$ such that $|a_n| \le K_1$ for all $n \ge 1$ and similarly for $\{b_n\}$ there exists a constant K_2 such that $|b_n| \le K_2$ for all $n \ge 1$. Therefore, $|c_n| \le K_1 K_2$ for all $n \ge 1$ so $c_n = O_p(1)$.

For part (e), suppose $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences such that $a_n, b_n = o_p(1)$ and $c_n = O_p(1)$, then for the sequence $\{d_n := a_n + b_n c_n\}$ the most we can expect is $d_n = O_p(1)$ since there exists contstants K_1 , K_2 , and K_3 such that $|a_n| \le K_1$, $|b_n| \le K_2$, and $|c_n| \le K_3$ for all $n \ge 1$. This implies that $|d_n| \le K_1 + K_2 K_3$ for all $n \ge 1$. Thus, $d_n = O_p(1)$.

Problem 11.2 (DasGupta 7.3 [Monte Carlo])

Consider the purely mathematical problem of finding a definite integral f(x) dx for some (possibly complicated) function f(x). Show that the SLLN provides a method for approximately finding the value of the integral by using appropriate averages $\frac{1}{n}\sum_{k=1}^{n} f(X_k)$. Numerical analysts call this Monte Carlo integration.

SOLUTION. Let X_k , for $1 \leq k \leq n$, be independent and identically distributed U[a,b] random variables and let $f:[a,b]\to\mathbb{R}$ be integrable on [a,b]. Moreover, let us denote the integral of f on [a, b] by

$$I := \int_{a}^{b} f \, dx$$

and the average of n random sample points from [a, b] by

$$I_n := \frac{1}{n} \sum_{k=1}^n f(X_k).$$

By the strong law of large numbers, we immediately have

$$I_n \longrightarrow E(f(X_1)) = \int_{-\infty}^{\infty} f(x) \chi_{[a,b]}(x) dx = \int_a^b f(x) dx,$$

as desired.

PROBLEM 11.3 (DASGUPTA 7.4 (A), (B))

Suppose X_1, \ldots , are i.i.d. and that $E(X_1) = \mu \neq 0$, $Var(X_1) = \sigma^2 < \infty$. Let $S_{m,p} = \sum_{k=1}^m X_k^p$, $m \geq 1, p = 1, 2$.

- (a) Identify with proof the almost sure limit of $S_{m,1}/S_{n,1}$ for fixed m, and $n \to \infty$.
- (b) Identify with proof the almost sure limit of $S_{n-m,1}/S_{n,1}$ for fixed m, and $n \to \infty$.

SOLUTION. For part (a), by the strong law of large numbers the average $\bar{X}_n = S_{n,1}/n \xrightarrow{\text{a.s.}} \mu$ as $n \to \infty$, so $S_{n,1} \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$. Therefore, since $S_{m,1}$ is a fixed, $S_{m,1}/S_{n,1} \xrightarrow{\text{a.s.}} 0$. For part (b), we have

$$\frac{S_{n-m,1}}{S_{n,1}} = \frac{S_{n,1} - S_{m,1}}{S_{n,1}}$$
$$= 1 - \frac{S_{m,1}}{S_{n,1}}$$

which converges a.s. to 1 since $S_{m,1}/S_{n,1} \xrightarrow{\text{a.s.}} 0$.

PROBLEM 11.4 (DASGUPTA 7.5 (A))

Let A_n , $n \ge 1$, A be events with respect to a common sample space Ω .

(a) Prove that $I_{A_n} \xrightarrow{\mathcal{L}} I_A$ if and only if $P(A_n) \to P(A)$.

Solution. One direction of this is obvious; namely, since I_{A_n} and I_A are indicator random variables $E(I_{A_n}) = P(A_n)$ and $E(I_A) = P(A)$ so $E(I_{A_n}) = P(A_n) \to P(A) = E(I_A)$ implies $I_{A_n} \xrightarrow{\mathcal{L}} I_A$. On the other hand, if $I_{A_n} \xrightarrow{\mathcal{L}} I_A$, then $P(I_{A_n} \leq x) \to P(I_{A_n} \leq x)$ so letting $x \to \infty$, $P(A_n) = P(I_{A_n} \leq \infty) \to P(I_A \leq \infty) = P(A)$.

PROBLEM 11.5 (DASGUPTA 7.11 [SAMPLE MAXIMUM])

Let X_k , $k \ge 1$, be an i.i.d. sequence, and $X_{(n)}$ the maximum of X_1, \ldots, X_n . Let $\xi(F) = \sup\{x : F(x) < 1\}$, where F is the common CDF of the X_k . Prove that $X_{(n)} \xrightarrow{\text{a.s.}} \xi(F)$.

SOLUTION. We point out that this is an immediate extension of Example 7.7 in DasGupta's book. Let $\varepsilon > 0$. Set $\xi = \xi(F)$. Then

$$\begin{split} P(|\forall n \geq m, \xi - X_{(n)}| \leq \varepsilon|) &= P(\forall n \geq m, \xi - X_{(n)} \leq \varepsilon) \\ &= P(\forall n \geq m, X_{(n)} \geq \xi - \varepsilon) \\ &= P(X_{(m)} \geq \xi - \varepsilon) \\ &= 1 - P(X_{(m)} < \xi - \varepsilon) \\ &= 1 - P(X_i < \xi - \varepsilon)^m \\ &\to 1 \end{split}$$

with convergence above being as $m \to \infty$.

That is, by definition, $X_{(n)} \to \xi$ almost surely.

PROBLEM 11.6 (DASGUPTA 7.14 (A))

Suppose X_k are i.i.d. standard Cauchy. Show that

(a)
$$P(|X_n| > n \text{ infinitely often}) = 1.$$

SOLUTION. Recall that the sum of two independent Cauchy variables is again Cauchy. Therefore, $\bar{X}_n = \sum_{k=1}^n X_n$ is Cauchy. Now by the weak law of large numbers, we have

$$P(\limsup |\bar{X}_n| = \infty) = 1.$$

This says that the average for any n, the average \bar{X}_n eventually exceeds n with probability 1. Therefore, $X_n > n$ infinitely often.

PROBLEM 11.7 (DASGUPTA 7.16 [COUPON COLLECTION])

Cereal boxes contain independently and with equal probability exactly one of n different celebrity pictures. Someone having the entire set of n pictures can cash them in for money. Let W_n be the minimum number of cereal boxes one would need to purchase to own a complete set of the pictures. Find a sequence a_n such that $W_n/a_n \xrightarrow{\mathcal{P}} 1$. (*Hint:* Approximate the mean of W_n .)

SOLUTION. We can model the scenario by $X_k \sim \text{Geom}(\frac{n-k+1}{n})$ where initially (k=1) we have a 100% chance of finding obtaining new picture. Therefore, the minimum number of boxes required to obtain a complete set of pictures W_n is the sum $\sum_{k=1}^n X_k$. Define $a_n := \sum_{k=1}^n n(n-k+1)^{-1} = n/(n^2 - nk + n)$. Then by the weak law of large numbers

Problem 11.8 (DasGupta 7.17)

Let $X_n \sim \text{Bin}(n, p)$. Show that $(X_n/n)^2$ and $X_n(X_n-1)/(n(n-1))$ both converge in probability to p^2 . Do they converge almost surely?

SOLUTION. First note that $(X_n/n)^2 \sim X_n(X_n-1)/(n(n-1))$ so it suffices to show that $(X_n/n)^2 \rightarrow p^2$. We show this explicitly. Let ε be given then we show that

$$P\left(\left|\left(\frac{X_n}{n}\right)^2 - p^2\right| \ge \varepsilon\right) \longrightarrow 0.$$

That is, given $\eta > 0$ there exists N such that $n \geq N$ implies

$$P\left(\left|\left(\frac{X_n}{n}\right)^2 - p^2\right| \ge \varepsilon\right) = P\left(\left(\frac{X_n}{n}\right)^2 - p^2 \ge \varepsilon\right) + P\left(\left(\frac{X_n}{n}\right)^2 - p^2 \le -\varepsilon\right)$$
< η .

From the calculations above, we have

$$P\left(\left(\frac{X_n}{n}\right)^2 - p^2 \ge \varepsilon\right) = P\left(X_n \ge n\sqrt{\varepsilon + p^2}\right)$$

$$= 1 - P\left(X_n < n\sqrt{\varepsilon + p^2}\right)$$

$$\approx \frac{1}{\sqrt{2\pi np(1-p)}} \int_{-\infty}^{n\sqrt{\varepsilon + p^2}} e^{-(x-np)^2/(2np(1-p))} dx$$

$$\sim Ce^{-C''n^2} \int_{-\infty}^{C'n} e^{-C'''x^2} dx$$

since both sequences in n above are convergent and the limit of the product of convergent sequences is the product of the limits, then the limit above equals 0.

PROBLEM 11.9 (DASGUPTA 7.21)

Let X_1, X_2, \ldots , be i.i.d. U[0, 1]. Let

$$G_n = (X_1 \cdots X_n)^{1/n}.$$

Find c such that $G_n \xrightarrow{\mathcal{P}} c$.

SOLUTION. Note that

$$\ln(G_n) = \frac{1}{n} \ln(X_1 X_2 \dots X_n)$$

$$= \frac{1}{n} \sum_{i=1}^n \ln(X_i)$$

$$\xrightarrow{\mathcal{P}} \int_0^1 \ln(x) dx$$

$$= -1$$

So that $\ln(G_n) \xrightarrow{\mathcal{P}} -1$; that is, $G_n \xrightarrow{\mathcal{P}} e^{-1}$.

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PROBLEM 11.10 (DASGUPTA 7.30 [CONCEPTUAL])

Suppose $X_n \xrightarrow{\mathcal{L}} X$, and also $Y_n \xrightarrow{\mathcal{L}} X$. Does this mean that $X_n - Y_n$ converge in distribution to (the point mass at) zero?

SOLUTION. No. Pick $X_n = U(\{-1,1\})$, and $Y_n = -X_n$. Then $X_n - Y_n = 2$ for all $n \in \mathbb{N}$, but X_n and Y_n are both uniformly distributed on $\{-1,1\}$, so they both converge (in distribution) to $U(\{-1,1\})$.

PROBLEM 11.11 (DASGUPTA 7.31 (A))

(a) Suppose $a_n(X_n - \theta) \to N(0, \tau^2)$; what can be said about the limiting distribution of $|X_n|$, when $\theta \neq 0$, $\theta = 0$?

SOLUTION.