MA 544: Homework 5

Carlos Salinas

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Problem 5.1 (Wheeden & Zygmund §3, Ex. 14)

Show that the conclusion of part (ii) of Exercise 13 (Problem) is false if $|E|_e = +\infty$.

Proof. Let $V \subset [0,1]$ denote the Vitali set defined in 3.38 and consider the union $E \coloneqq V \cup (2, \infty)$. It is clear that the inner and outer measure of E is ∞. However, E itself is unmeasurable since otherwise $E \cap [0,1] = V \cap [0,1] = V$ would be measurable. ■

Problem 5.2 (Wheeden & Zygmund §3, Ex. 16)

Prove (3.34).

Proof.

Lemma. $|P| = v(P)$.

The result is trivial by 3.36, but then again, it is used to prove 3.36.

Let $\{\mathbf{e}_k\}_{k=1}^n$ be a set of orthogonal vectors emenating from a point in $\mathbb{R}^n$. The closed parallelapiped corresponding to $\{\mathbf{e}_k\}_{k=1}^n$ is the set

$$P = \left\{ \mathbf{x} \;\middle|\; \mathbf{x} = \sum_{k=1}^n t_k \mathbf{e}_k,\, 0 \le t \le 1 \right\}. \tag{1}$$

Let $\{I_k\}_{k=0}^\infty$ be a cover of P by n-intervals. ■

Problem 5.3 (Wheeden & Zygmund §3, Ex. 18)

Prove that outer measure is *translation invariant*; that is, if $E_{\mathbf{h}} \coloneqq \{ \mathbf{x}+\mathbf{h} \mid \mathbf{x} \in E \}$ is the translate of E by $\mathbf{h}$, $\mathbf{h} \in \mathbb{R}^n$, show that $|E_{\mathbf{h}}|_e = |E|_e$. If E is measurable, show that $E_{\mathbf{h}}$ is also measurable. [This fact was used in proving (3.37).]

Proof. By 3.6, given $\varepsilon > 0$, there exists an open set $G \supset E$ with $|G|_e \leq |E|_e + \varepsilon$. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ denote the linear transformation $\mathbf{x} \mapsto \mathbf{x}+\mathbf{h}$, $\mathbf{h} \in \mathbb{R}^n$. By 3.35 we have $|G|_e = |G| = |T(G)| = |T(G)|_e$ and $T(G)$ is an open set containing $E_{\mathbf{h}}$. Hence, we have an upper bound on the outer measure of $E_{\mathbf{h}}$ given by the inequality

$$|E_{\mathbf{h}}|_e \leq |T(G)|_e = |G|_e \leq |E|_e + \varepsilon. \tag{2}$$

On the other hand, by 3.6 there exists an open set $H \supset E_{\mathbf{h}}$ with $|H|_e \leq |E_{\mathbf{h}}| + \varepsilon$. Then by 3.35, we get the inequality

$$|E|_e \leq \left|T^{-1}(H)\right|_e = |H|_e \leq |E_{\mathbf{h}}|_e + \varepsilon. \tag{3}$$

Putting (2) and (3) we have

$$|E|_e - \varepsilon \leq |E_{\mathbf{h}}|_e \leq |E|_e + \varepsilon.$$

Letting $\varepsilon \to 0$, we have $|E|_e = |E_{\mathbf{h}}|_e$. It then follows that if E is measurable then $E_{\mathbf{h}}$ is measurable since $E_{\mathbf{h}} = T(E)$ and T is a Lipschitz transformation and $|E| = |E_{\mathbf{h}}|$. ■

Problem 5.4 (Wheeden & Zygmund §4, Ex. 1)

Prove corollary (4.2) and theorem (4.8)

Proof.

Corollary (Wheeden & Zygmund, 4.2)**.** *If f is measurable, then $\{ f > -\infty \}$, $\{ f < +\infty \}$, $\{ f = +\infty \}$, $\{ a \leq f \leq b \}$, $\{ f = a \}$, etc., are all measurable. Moreover f is measurable if and only if $\{ a < f < +\infty \}$ is measurable for every finite a.*

Suppose that f is measurable. By 4.1, we have $\{ f \geq a \}$ and $\{ f \leq a \}$ are measurable so

$$\{ f = a \} = \{ f \geq a \} \cap \{ f \leq a \} \tag{4}$$

is measurable and for $b > a$

$$\{ a \leq f \leq b \} = \{ f \geq a \} \cap \{ f \leq b \}. \tag{5}$$

Proof of corollary 4.2. Now, consider the sequence of measurable sets $\{E_k\}_{k=0}^{\infty}$ where $E_k := \{ f < a + k \}$. Then $\{ f < \infty \} = \bigcup_{k=0}^{\infty} E_k$ and since $E_k \nearrow \{ f < \infty \}$ (take $\mathbf{x} \in E_k$ then $f(\mathbf{x}) < a + k$ so $f(\mathbf{x}) < a + k + 1 \implies \mathbf{x} \in E_{k+1}$), by 3.26, we have $\{ f < \infty \}$ is measurable.

Similarly for $\{ f > -\infty \}$ we may consider the family $\{E_k\}_{k=0}^{\infty}$ where $E_k := \{ f > a - k \}$ (take $\mathbf{x} \in E_k$ then $f(\mathbf{x}) > a - k$ so $f(\mathbf{x}) > a - k - 1 \implies \mathbf{x} \in E_{k+1}$) and taking the limit as $k \to \infty$ we have $\{ f > -\infty \}$ is measurable.

Last but not least, since $\{ f < \infty \}$ is measurable, $\{ f = \infty \} = \{ f < \infty \}^{\mathsf{C}}$ is measurable.

Now, $\implies$ suppose f is measurable. Then $\{ a < f < b \} = \{ a \leq f \leq b \} \cap \{ f = a \}^{\mathsf{C}} \cap \{ f = b \}^{\mathsf{C}}$ is measurable for all finite $a < b$. Moreover, the family $\{E_k\}_{k=0}^{\infty}$ of sets $\{E_k\}_{k=0}^{\infty}$ where $E_k := \{ a \leq f < b + k \}$ is measurable for all k so, by 3.26, $\{ a \leq f < \infty \}$ is measurable since $E_k \nearrow \{ a \leq f < \infty \}$.

$\impliedby$ On the other hand, suppose that $\{ a \leq f < \infty \}$ is measurable for every finite a. Then, for fixed $a \in \mathbb{R}$ the family $\{E_k\}_{k=0}^{\infty}$ where $E_k := \{a - k \leq f < \infty\}$ is measurable. By 3.26, $\{ f < \infty \}$ is measurable so $\{ f = \infty \} = \{ f < \infty \}^{\mathsf{C}}$ is measurable. Thus,

$$\{ f > a \} = \{ a < f < \infty \} \cup \{ f = \infty \}$$

is measurable so f is measurable. ♣

Theorem (Wheeden & Zygmund, 4.8)**.** *If f is measurable and λ is any real number, then $f + \lambda$ and λf are measurable.*

Proof of theorem 4.8. If f is measurable, then $\{ f > a \}$ is measurable for all a so $\{ f > a - \lambda \} = \{ f + \lambda > a \}$ is measurable for all a. Hence, $f + \lambda$ is measurable.

If $\lambda \neq 0$, then $\{ f > a/\lambda \}$ is measurable for all a so λf is measurable. If $\lambda = 0$ then $\lambda f = 0$ is clearly measurable since $\{ 0 > a \} = (a, 0)$ is open for all a (possibly empty if $a \geq 0$, but still an open set).

Thus, $f + \lambda$ and λf are measurable. ♣

■

Problem 5.5 (Wheeden & Zygmund §4, Ex. 2)

Let f be a simple function, taking its distinct values on disjoint sets $E_1, ..., E_N$. Show that f is measurable if and only if $E_1, ..., E_N$ are measurable.

Proof. $\Longrightarrow$ Suppose f is a simple function taking distinct values on disjoint sets $E_1, ..., E_N$. Then $f = \sum_{k=1}^{N} a_k \chi_{E_k}$. If f is measurable, $\{ f > a \}$ is measurable for all finite a. In particular, $\{ f > a_k \} = E_k$ is measurable.

$\Longleftarrow$ On the other hand, suppose that E_k is measurable for all $1 \leq k \leq N$. Then χ_{E_k} is measurable and by Problem 5.4, the sum $f = \sum_{k=1}^{N} a_k \chi_{E_k}$ is measurable. ■