MA 523: Homework 6

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Problem 6.1

For $n = 2$ find Green's function for the quadrant $U := \{ x_1, x_2 > 0 \}$ by repeated reflection.

Solution. Taking the hit, set $x' := (x_1, -x_2)$, $x'' := (-x_1, x_2)$, and define

$$\varphi^x(y) := \Phi(x - y) + \Phi(x' - y) - \Phi(x'' - y). \tag{6.1}$$

We claim that φ^x, as defined above, solves

$$\begin{cases} \Delta\varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(x - y) & \text{on } \partial U. \end{cases}$$

It is clear that $\Delta\varphi^x = 0$ since it is built up from the fundamental solutions on $\mathbb{R}^n$ (this follows from the linearity of the Laplace operator). To see that $\varphi^x(y) = \Phi(x - y)$ on ∂U, we do a case by case analysis.

Note that on $\{0\} \times \{x_2 > 0\} \subset \partial U$, ■

Problem 6.2

(Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B(0,r) = \{\, x \in \mathbb{R}^n : |x| < r \,\}$.

Solution. ■

Problem 6.3

Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$\begin{cases} P_k(\lambda x) = \lambda^k P_k(x), \quad P_m(\lambda x) = \lambda^m P_m(x) & \text{for every } x \in \mathbb{R}^n,\ \lambda > 0, \\ \quad \Delta P_k = 0, \qquad\qquad\quad \Delta P_m = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

(a) Show that

$$\left\{ \frac{\partial P_k}{\partial \nu} = k P_k(x), \quad \frac{\partial P_m}{\partial \nu} = m P_m(x) \quad \text{on } \partial B(0,1), \right.$$

where $B(0,1) = \{\, x \in \mathbb{R}^n : |x| < 1 \,\}$ and ν is the outward normal on $\partial B(0,1)$.

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x)\, d\sigma = 0, \qquad \text{if } k \neq m.$$

Solution. ■