MA571 Homework 8

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Problem 8.1 (Munkres §46, Ex. 6)

Show that the compact-open topology, $\mathcal{C}(X,Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular. [*Hint:* If $\overline{U} \subset V$, then $\overline{S(C,U)} \subset S(C,V)$.]

Proof. Suppose that Y is Hausdorff. Let f and g be distinct continuous functions from X to Y. Then there exists a point $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Since Y is Hausdorff there exists disjoint neighborhoods U and V of $f(x_0)$ and $g(x_0)$, respectively. Now, we claim that

Claim. *If $C \subset X$ is finite, C is compact.*

Proof. Write $C = \{x_1, ..., x_n\}$. Let $\mathcal{A}$ be an open cover of C. Then since $C \subset \bigcup_{U_\alpha \in \mathcal{A}} U_\alpha$ we can choose A_i containing x_i for every $1 \leq i \leq n$. Thus, the subcollection $\{U_i\}_{i=1}^n$ covers C. ♣

Let $U' = S(\{x_0\}, U)$ and $V' = S(\{x_0\}, V)$. Note that U' and V' are nonempty since $f \in U'$ and $g \in V'$. Moreover, their intersection is empty for suppose $h \in U' \cap V'$, then $h(x_0) \in U \cap V$, but $U \cap V = \emptyset$. Then, since U' and V' are subbasis elements for the compact-open topology on $\mathcal{C}(X,Y)$ and they "separate" f and g, it follows that $\mathcal{C}(X,Y)$ is Hausdorff.

Now, suppose that Y is regular. We shall proceed by the hint and Lemma 31.1(b). Consider the subbasis element $S(C,U)$. Since Y is regular, there exists a neighborhood $V \supset U$ such that $V \supset \overline{U}$. Let $f \in \overline{S(C,U)}$. Then, we claim that $f \in S(C,V)$. For suppose not, then there exists an element $x_0 \in C$ such that $f(x_0) \notin V$. Then, since $\overline{U} \subset V$, by hypothesis, $f(x_0) \notin \overline{U}$. Consider the subbasic neighborhood $S(\{x_0\}, Y - \overline{U})$ of f. Then, $S(\{x_0\}, Y - \overline{U}) \cap S(C,U)$ is nonempty. Let g be in the aforementioned intersection. Then $g(x_0) \in g(C) \subset U$, but $g(x_0) \in Y - \overline{U}$. This is a contradiction. It follows by Lemma 31.1(b) that $\mathcal{C}(X,Y)$ is regular. ∎

Problem 8.2 (Munkres §46, Ex. 7)

Show that if Y is locally compact Hausdorff, then composition of maps

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$$

is continuous, provided the compact-open topology is used throughout. [*Hint:* If $g \circ f \in S(C, U)$, find V such that $f(C) \subset V$ and $g(\overline{V}) \subset U$.]

Proof.   ∎

PROBLEM 8.3 (MUNKRES §46, EX. 8)

Let $\mathcal{C}'(X, Y)$ denote the set $\mathcal{C}(X, Y)$ in some topology $\mathcal{T}$. Show that if the evaluation map

$$e \colon X \times \mathcal{C}'(X, Y) \longrightarrow Y$$

is continuous, then $\mathcal{T}$ contains the compact-open topology. [*Hint:* The induced map $E \colon \mathcal{C}'(X, Y) \to \mathcal{C}(X, Y)$ is continuous.]

Proof.   ∎

PROBLEM 8.4 ((A))

Definition 1. Definition. If X is a locally compact Hausdorff space then the space Y given by Theorem 29.1 is called the *one-point compactification* of X.

Let X be a compact Hausdorff space and let W be an open subset of X (so W is locally compact by Corollary 29.3) with $W \neq X$. Prove that the one-point compactification of W is homeomorphic to the quotient space $X/(X - W)$.

Proof.   ∎

PROBLEM 8.5 ((B))

Let X be a compact Hausdorff space, let Y be a topological space, and let $p\colon X \to Y$ be a closed surjective continuous map. Prove that Y is Hausdorff. [*Hint:* one ingredient in the proof is p. 171 # 5.]

Note: combining this with HW 4 Problem E and HW 6 Problem A gives a necessary and sufficient condition for a quotient of a compact Hausdorff space to be Hausdorff.

Proof.   ■

Problem 8.6 ((C))

Let $S^2 \subset \mathbf{R}^3$ be the subspace
$$\left\{\, (x, y, z) \mid x^2 + y^2 + z^2 = 1 \,\right\}.$$

Prove that S^2 is a 2-manifold. (The definition of m-manifold, where m is a positive whole number, is given at the top of page 225.)

Proof.   ∎

Problem 8.7 ((D))

Prove that the union of the x and y-axes in $\mathbf{R}^2$ is not a 1-manifold.

Proof.   ∎