MA 519: Homework 7

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Problem 7.1 (Handout 10, # 4)

(*Poisson Approximation.*) One hundred people will each toss a fair coin 200 times. Approximate the probability that at least 10 of the 100 people would each have obtained exactly 100 heads and 100 tails.

Solution. Let X denote the number of people who obtain exactly 100 heads and (consequently) 100 tails. First, we compute the probability that any one given person obtains exactly 100 heads. There are 2^{200} possible outcomes for 200 tosses of a fair coin, and $\binom{200}{100}$ possible ways of obtaining exactly 100 heads. Thus, the probability that any one person obtains exactly 100 head in 200 tosses of a fair coin is

$$p = \frac{\binom{200}{100}}{2^{200}} \approx 0.056.$$

Now, assuming $X \sim \text{Poisson}(5.635)$, the probability that at least 10 of the 100 people have each obtained exactly 100 heads and 100 tails is

$$\begin{aligned}
P(X \geq 10) &= 1 - P(X < 10) \\
&= 1 - \sum_{i=1}^{9} P(X = i) \\
&= 1 - e^{-5.635} \sum_{i=0}^{9} \frac{5.635^i}{i!} \\
&= 1 - 0.883 \\
&\approx 0.117.
\end{aligned}$$

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Problem 7.2 (Handout 10, # 5)

(*A Pretty Question.*) Suppose X is a Poisson distributed random variable. Can three different values of X have an equal probability?

Solution. No. Let $X \sim \text{Poisson}(\lambda)$. First, we show that given any two values $k_1, k_2 \in \mathbb{Z}_{\geq 0}$, there exists λ such that $p(k_1) = p(k_2)$.

Observe that for $p(k_1) = p(k_2)$ we must have

$$\begin{aligned} e^{-\lambda}\frac{\lambda^{k_1}}{k_1!} &= e^{-\lambda}\frac{\lambda^{k_2}}{k_2!} \\ \lambda^{k_1-k_2} &= \frac{k_1!}{k_2!} \end{aligned}$$

this implies that, given k_1 and k_2,

$$\begin{aligned} (k_1 - k_2)\ln\lambda &= \ln(k_1!/k_2!) \\ \lambda(k_1, k_2) &= e^{\ln(k_1!/k_2!)/(k_1-k_2)}. \end{aligned}$$

For example, $\lambda(3, 5) \approx 4.472$ and

$$p(3) \approx 0.170 \approx p(5).$$

We now prove our original claim. Assume for a moment that X is continuous. We will show that the PMF p of X has at most one critical point. Write the PMF of p as its continuous analogue

$$p(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{\Gamma(x)}.$$

Then, taking the derivative of p, we have

$$\begin{aligned} p'(x) &= e^{-\lambda}\frac{\Gamma(x)\lambda^x\ln\lambda - \lambda^x\Gamma(x)}{\Gamma(x)^2} \\ &= \frac{e^{-\lambda}\lambda^x}{\Gamma(x)}\left[\ln\lambda - \frac{\Gamma'(x)}{\Gamma(x)}\right] \\ &= p(x)\left[\ln\lambda - \frac{\Gamma'(x)}{\Gamma(x)}\right]. \end{aligned}$$

Since $\Gamma, \Gamma' > 0$ for all $x \in \mathbb{R}_{\geq 0}$, $p'(x) = 0$ if and only if

$$\ln\lambda = \frac{\Gamma'(x)}{\Gamma(x)}$$

which happens at most once since the quotient. ■

Problem 7.3 (Handout 10, # 6)

(*Poisson Approximation.*) There are 20 couples seated at a rectangular table, husbands on one side and the wives on the other, in a random order. Using a Poisson approximation, find the probability that exactly two husbands are seated directly across from their wives; at least three are; at most three are.

Solution. Let X count the number of husbands that have been seated directly across from their respective wives. We approximate the probabilities that (i) exactly two husbands are seated directly across from their wives, (ii) at least three are, and (iii) at most three are, by assuming that $X \sim \text{Poisson}(\lambda)$. But first, we compute the mean of X.

Fixing the men, the probability that any given man is sitting directly across from his wife is $1/20$. Since the distribution of X is binomial, its mean is $20(1/20) = 1$. Thus, $\lambda = 1$.

For (i), the probability that exactly two husbands are seated directly from their wives is

$$p(2; 1) = e^{-1}\frac{1}{2!} \approx 0.184.$$

For (iii), the probability is

$$p(x \geq 3; 1) = 1 - e^{-1}\left[1 + 1 + \frac{1}{2}\right] \approx 0.080.$$

For (iii), the probability is

$$p(x \leq 3; 1) = e^{-1}\left[1 + 1 + \frac{1}{2} + \frac{1}{6}\right] \approx 0.981.$$

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Problem 7.4 (Handout 10, # 7)

(*Poisson Approximation.*) There are 5 coins on a desk, with probabilities 0.05, 0.1, 0.05, 0.01, and 0.04 for heads. By using a Poisson approximation, find the probability of obtaining at least one head when the five coins are each tossed once.
Is the number of heads obtained binomially distributed in this problem?

Solution. First, let us find the probability p that the toss of a coin randomly selected from the 5 comes up heads. This is given by

$$p = \frac{0.05 + 0.1 + 0.05 + 0.01 + 0.04}{5} = 0.05.$$

Now, $\lambda = 0.25$ so the probability of obtaining at least one head is

$$e^{-0.25} \sum_{i=1}^{5} \frac{0.25^i}{i!} \approx 0.221.$$

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Problem 7.5 (Handout 10, # 8)

A book of 500 pages contains 500 misprints. Estimate the chances that a given page contains at least three misprints.

Solution. We approximate the distribution of X the number of misprints on a given page using a Poisson distribution with parameter $\lambda = 1$ the density of misprints throughout the book. Under these assumptions, the probability that a given page contains at least three misprints is given by

$$1 - e^{-1}\frac{1}{1!} - e^{-1}\frac{1}{1!} - e^{-1}\frac{1}{2!} \approx 0.080.$$

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Problem 7.6 (Handout 10, # 9)

Estimate the number of raisins which a cookie should contain on the average if it is desired that not more than one cookie out of a hundred should be without raisin.

Solution. The number of raisins in a cookie is given by a poisson distribution: Let X_λ be the number of raisins in a cookie that came out of a batch where the average number of raisins per cookie was λ. We want $P(X_\lambda = 0) \leq 0.01$. That is, we want

$$\frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} \leq 0.01$$

which occurs for $\lambda \geq -\ln(0.01) = \ln(100) \approx 4.605$.

That is, we want there to be about 4.6 raisins per cookie in order to guarantee that no more than one out of one hundred cookies is raisinless. ∎

Problem 7.7 (Handout 10, # 10)

The terms $p(k;\lambda)$ of the Poisson distribution reach their maximum when k is the largest integer not exceeding λ.

Solution. First, set $\lfloor\lambda\rfloor$ equal to the largest integer not exceeding λ. Then we show $p(k;\lambda)$ is increasing on $[0,\lfloor\lambda\rfloor]$ and decreasing on $[\lfloor\lambda\rfloor,\infty)$; this suffices to show that the maximum occurs at $\lfloor\lambda\rfloor$.

Now, note that for all k,

$$\begin{aligned} p(k;\lambda)-p(k+1;\lambda) &= \frac{\lambda^k e^{-\lambda}}{k!}-\frac{\lambda^{k+1}e^{-\lambda}}{(k+1)!}\\ &= \frac{e^{-\lambda}\lambda^k}{k!}\left[1-\frac{\lambda}{k+1}\right]\\ &= \frac{e^{-\lambda}\lambda^k}{k!}\left[1-\frac{\lambda}{k+1}\right] \end{aligned}$$

which is positive exactly when $k<\lambda+1$ and negative exactly when $k\geq lambda+1$. That is, $p(k+1;\lambda)$ is larger than $p(k;\lambda)$ exactly when $k<\lambda+1$; that is, $p(k;\lambda)$ increases until the largest integer not exceeding λ; $p(k;\lambda)$ is increasing on $[0,\lfloor\lambda\rfloor$ and decreasing on $[\lfloor\lambda\rfloor,\infty)$, as desired. ∎

Problem 7.8 (Handout 10, # 11)

Prove

$$p(0,\lambda)+\cdots+p(n,\lambda)=\frac{1}{n!}\int_\lambda^\infty e^{-x}x^n\,dx.$$

Solution. By integration by parts, we have

$$\begin{aligned}\frac{1}{n!}\int_\lambda^\infty e^{-x}x^n\,dx &= \frac{1}{n!}e^{-x}x^n\Big|_\lambda^\infty-\frac{1}{n!}\int_\lambda^\infty -ne^{-x}x^{n-1}\,dx\\ &=\frac{e^{-\lambda}\lambda^n}{n!}+\frac{1}{(n-1)!}\int_\lambda^\infty e^{-x}x^{n-1}\,dx\\ &\vdots\\ &=\frac{e^{-\lambda}\lambda^n}{n!}+\frac{e^{-\lambda}\lambda^{n-1}}{(n-1)!}+\cdots+\frac{e^{-\lambda}\lambda^0}{0!}\\ &=p(n;\lambda)+\cdots+p(0;\lambda),\end{aligned}$$

as was to be shown. ■

Problem 7.9 (Handout 10, # 12)

There is a random number N of coins in your pocket, where N has a Poisson distribution with mean μ. Each one is tossed once.
Let X be the number of times a head shows.
Find the distribution of X.

Solution. First, note that $P(N = n) = e^{-\mu}\mu^n/n!$. Given that there are n coins in your pocket, let X_n denote the number of heads shown. Then $P(X_n = m) = \binom{n}{m}(0.5)^n$. So, the distribution of X is given by

$$\begin{aligned}
P(X = m) &= \sum_{n=0}^{\infty} P(X_n = m)P(N = n) \\
&= \sum_{n=0}^{\infty} \binom{n}{m}(0.5)^n \frac{e^{-\mu}\mu^n}{n!} \\
&= e^{-\mu} \sum_{n=0}^{\infty} \frac{n!}{(n-m)!m!n!}(0.5\mu)^n \\
&= \frac{e^{-\mu}}{m!} \sum_{n=0}^{\infty} \frac{(0.5\mu)^n}{(n-m)!}
\end{aligned}$$

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Problem 7.10 (Handout 10, # 14)

Find the MGF of a general Poisson distribution, and hence prove that the mean and the variance of an arbitrary Poisson distribution are equal.

Solution. Let X be a poisson distribution with mean λ. Then

$$\begin{aligned} M_X(t) &= \sum_{n=0}^{\infty} e^{tn} P(X = n) \\ &= \sum_{n=0}^{\infty} e^{tn} \frac{e^{-\lambda}\lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(e^t\lambda)^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^t} \end{aligned}$$

Note that $M'_X(t) = e^{-\lambda}\left(\lambda e^{\lambda e^t + t}\right)$ and $M''_X(t) = e^{-\lambda}\left(\lambda e^{\lambda e^t + t}\right)(\lambda e^t + 1)$.

Evaluating at 0, we get that $E(X) = M'_X(0) = \lambda$ and $E(X^2) = M''_X(0)/2 = \lambda$. ■

Problem 7.11 (Handout 10, # 17 (a))

(*Poisson approximations.*) 20 couples are seated in a rectangular table, husbands on one side and the wives on the other. First, find the expected number of husbands that sit directly across from their wives. Then, using a Poisson approximation, find the probability that two do; three do; at most five do.

Solution. Like in Problem 7.3, $\lambda = 1$.

Using a Poisson approximation, the probability that two husbands are seated directly across from their wives is

$$p(2;1) = e^{-1}\frac{1}{2} \approx 0.184.$$

The probability that three do is

$$p(3;1) = e^{-1}\frac{1}{6} \approx 0.061.$$

The probability that at most five do

$$p(x \leq 5;1) = e^{-1}\left[1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}\right] \approx 0.999.$$

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