MA 523: Homework 2

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Problem 2.1

Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the noncharacteristic boundary condition when Γ is flat near x^0 since $\nu(x^0) = (0, \dots, 0, -1)$ giving us

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \dots, 0, -1) \\ &= -F_{p_n}(p^0, z^0, x^0) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

To show (2.1), we will straighten the boundary near x^0 and apply the noncharacteristic boundary conditions. Let $\Phi, \Psi \colon \mathbb{R}^n \to \mathbb{R}^n$ be smooth maps such that $\Psi = \Phi^{-1}$ and Φ straightens out ∂U near x^0. Then, setting $y^0 := (y_1, \dots, y_{n-1}, 0) = \Phi(x^0)$ and $v(y) = u(\Psi(y))$, our PDE becomes

$$0 = F\big(Dv(y) D\Phi(\Psi(y)), v(y), \Psi(y)\big).$$

From here we follow the proof of Lemma 1 in [E, §3.2.3]. Let $G \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the map given by

$$\begin{cases} G^i(p, y) = p_i - g_{x_i}(y), \\ G^n(p, y) = F(p, g(y), y). \end{cases}$$

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Problem 2.2

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. ►

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Problem 2.3

Show that the function $u(x,t)$ defined by $t \geq 0$ by

$$u(x,t) = \begin{cases} -\dfrac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burger's equation*).

Solution. ► ◄