

# MA 523: Homework, Midterms and Practice Problems Solutions

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## Contents

|          |  |           |
|----------|--|-----------|
| <b>1</b> | <b>Homework Solutions</b>              | <b>2</b>  |
| 1.1      | Homework 1 . . . . .                   | 2         |
| 1.2      | Homework 2 . . . . .                   | 7         |
| 1.3      | Homework 3 . . . . .                   | 10        |
| 1.4      | Homework 4 . . . . .                   | 12        |
| 1.5      | Homework 5 . . . . .                   | 14        |
| 1.6      | Homework 6 . . . . .                   | 15        |
| 1.7      | Homework 7 . . . . .                   | 18        |
| <b>2</b> | <b>Homework 8</b>                      | <b>21</b> |
| <b>3</b> | <b>Exams</b>                           | <b>26</b> |
| 3.1      | Midterm Practice Problems . . . . .    | 26        |
| 3.2      | Final Practice Problems . . . . .      | 33        |
| <b>4</b> | <b>Qualifying Exams</b>                | <b>37</b> |
| 4.1      | Qualifying Exam, August '04 . . . . .  | 37        |
| 4.2      | Qualifying Exam, August '05 . . . . .  | 39        |
| 4.3      | Qualifying Exam, January '14 . . . . . | 41        |

# 1 Homework Solutions

These are my (corrected) solutions to Petrosyan's Math 523 homework for the fall semester of 2016.

## 1.1 Homework 1

PROBLEM 1.1.1 (Taylor's formula). Let  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  be smooth,  $n \geq 2$ . Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$$

as  $x \rightarrow \mathbf{0}$  for each  $k = 1, 2, \dots$ , assuming that you know this formula for  $n = 1$ .

*Hint:* Fix  $x \in \mathbf{R}^n$  and consider the function of one variable  $g(t) := f(tx)$ . Prove that

$$\frac{d^k}{dt^k} g(t) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on  $m$ .

*SOLUTION.* Taking the hint, let us consider the function in one variable  $g(t) := f(tx)$  for  $x \in \mathbf{R}^n$  fixed. We show by induction on  $k$  that

$$\frac{d^k}{dt^k} g(t) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha f(tx) x^\alpha. \quad (1.1.1)$$

Once we have shown (1.1.1) holds, evaluating  $g$  at  $t = 1$  gives us the desired equality; i.e.,

$$f(x) = g(1)$$

which, by Taylor's formula in one variable, is

$$= \sum_{j=0}^k \frac{g^{(j)}(0)}{j!} 1^j + O(|x|^{k+1})$$

applying (1.1.1) here gives us

$$\begin{aligned} &= \sum_{j=0}^k \frac{1}{j!} \left[ \sum_{|\alpha|=j} \frac{j!}{\alpha!} D^\alpha f(tx) x^\alpha \right] + O(|x|^{k+1}) \\ &= \sum_{j=0}^k \left[ \sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha \right] + O(|x|^{k+1}) \\ &= \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}) \end{aligned}$$

as desired.

Let us now show that (1.1.1) holds. To prove this we consider the algebra on the differential operator  $d/dt$ . By the chain rule, we have

$$\frac{d}{dt}(\cdot) = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}(\cdot).$$

Since  $f$  is smooth by Schwartz's theorem the differential operators  $\partial/\partial x_j$  and  $\partial/\partial x_l$  commute for all  $1 \leq j, l \leq n$ . Therefore, by the multinomial theorem,

$$\frac{d^k}{dt^k}(\cdot) = \left( \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}(\cdot) \right)^k = \sum_{|\alpha|=k} \frac{m!}{\alpha!} x^\alpha D^\alpha(\cdot). \quad \blacksquare$$

PROBLEM 1.1.2. Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \quad (*)$$

on  $\mathbf{R}^n \times (0, \infty)$ , where  $b \in \mathbf{R}^n$ . Using the characteristic equation, solve  $(*)$  subject to the initial condition  $u = g$  on  $\mathbf{R}^n \times \{t = 0\}$ . Make sure the answer agrees with formula (5) in §2.1.2 of [E].

SOLUTION. Write

$$F(p, z, x, t) := (b, 1) \cdot p - f = 0.$$

Then the characteristic equations to the problem  $(*)$  with the initial value  $u(\cdot, 0) = g(\cdot)$  are given by

$$\begin{cases} \dot{p} = -D_{x,t}F - D_z F p = 0, \\ \dot{z} = D_p F \cdot p = (b, 1) \cdot p = f, \\ (\dot{x}, \dot{t}) = D_p F = (b, 1). \end{cases}$$

Now let us solve the characteristic equations above subject to the initial values  $(x(0), t(0)) = (x^0, 0) \in \mathbf{R}^n \times (0, \infty)$ ; these are

$$\begin{cases} x(s) = x^0 + bs, & t(s) = s, \\ z(s) = z(0) + \int_0^s f(x(\tau), t(\tau)) d\tau \\ \quad = g(x^0) + \int_0^s f(x^0 + b\tau, \tau) d\tau. \end{cases}$$

Solving back, we have  $t = s$ ,  $x^0 = x - bs = x - bt$ , and therefore

$$u(x, t) = z(s) = g(x - bt, t) + \int_0^t f(x - b\tau, \tau) d\tau$$

solves the transport equation  $(*)$  with initial value  $u(\cdot, 0) = g(\cdot)$ .

This verifies formula [5] from [E, §2.1.2]. ■

PROBLEM 1.1.3. Solve using the characteristics:

- (a)  $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$ ,  $u = 1$  on the line  $x_2 = 2x_1$ .  
(b)  $uu_{x_1} + u_{x_2} = 1$ ,  $u(x_1, x_1) = x_1/2$ .  
(c)  $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$ ,  $u(x_1, x_2, 0) = g(x_1, x_2)$ .

*SOLUTION.* For part (a): Write  $F := (x_1^2, x_2^2) \cdot p - z^2 = 0$ . We have the characteristic equations

$$\begin{cases} \dot{p} = -D_x F - D_z F p = 2((x_1 - z)p_1, (x_2 - z)p_2), \\ \dot{z} = D_p \cdot p = z^2, \\ \dot{x} = (x_1^2, x_2^2). \end{cases}$$

We can solve the characteristic equations with respect to the initial conditions  $x(0) = (x^0, 2x^0)$ ,  $z(0) = 1$  on the line  $x_1 = 2x_2$ ; these are

$$\begin{cases} x_1(s) = \frac{x^0}{1 + x^0 s}, & x_2(s) = \frac{2x^0}{1 + 2x^0 s}, \\ z(s) = \frac{1}{1 - s}. \end{cases}$$

Now we solve these in terms of the coordinates  $(x_1, x_2)$ . Assuming  $x^0 \neq 0$ , we have

$$s = \frac{1}{x^0} - \frac{1}{x_1} \quad \text{and} \quad s = \frac{1}{2x^0} - \frac{1}{x_2}.$$

Therefore,

$$\begin{aligned} s &= 2\left(\frac{1}{2x^0} - \frac{1}{x_2}\right) - \left(\frac{1}{x^0} - \frac{1}{x_1}\right) \\ &= \frac{1}{x_1} - \frac{2}{x_2}. \end{aligned}$$

Thus,

$$u(x_1, x_2) = \frac{1}{1 - \left(\frac{1}{x_1} - \frac{2}{x_2}\right)} = \frac{x_1 x_2}{x_1 x_2 - x_2 - 2x_1}$$

solves the PDE  $F$  for  $(x_1, x_2)$  on the line  $x_1 = 2x_2$  away from the origin.

For part (b): Write  $F = (z, 1) \cdot p - 1 = 0$ . Then we have the characteristic equations

$$\begin{cases} \dot{p} = -D_x F - D_z p = -(p_1, 0) \\ \dot{z} = D_p \cdot p = 1 \\ \dot{x} = D_p F = (z, 1) \end{cases}$$

Next we solve the characteristic equations subject to the initial conditions  $x(0) = (x^0, x^0)$ ,  $z(0) = x^0/2$  on the line  $x_1 = x_2$ ; these are

$$\begin{cases} z(s) = \frac{1}{2}x^0 + s, \\ x_1(s) = x^0 + \frac{1}{2}(x^0 s + s^2), & x_2(s) = x^0 + s. \end{cases}$$

Then, solving in terms of the coordinates  $(x_1, x_2)$ , we have

$$x^0 = 2(x_2 - z) \quad \text{and} \quad s = 2z - x_2.$$

Therefore,

$$\begin{aligned} x_1 &= 2(x_2 - z) + (x_2 - z)(2z - x_2) + \frac{1}{2}(2z - x_2)^2 \\ &= -\frac{1}{2}x_2(x_2 - 4) + (x_2 - 2)z. \end{aligned}$$

Hence,

$$u(x_1, x_2) = \frac{2x_1 + x_2^2 - 4x_2}{2(x_2 - 2)}$$

solves the PDE  $F$  subject to the condition  $u(x_1, x_1) = x_1/2$  provided  $x_2 \neq 2$ .

For part (c): Write  $F := (x_1, 2x_2, 1) \cdot p - 3z = 0$ . Then the characteristic equations are

$$\begin{cases} \dot{p} = -D_x F - D_z p = (2p_1, p_2, 3p_3) \\ \dot{z} = D_p F \cdot p = 3z \\ \dot{x} = D_p F = (x_1, 2x_2, 1) \end{cases}$$

Next we solve the characteristic equations subject to the initial conditions  $x(0) = (x_1^0, x_2^0, 0)$ ,  $z(s) = g(x_1^0, x_2^0)$ ; these are

$$\begin{cases} x_1(s) = x_1^0 e^s, & x_2(s) = x_2^0 e^{2s}, & x_3(s) = s, \\ z(s) = g(x_1^0, x_2^0) e^{3s}. \end{cases}$$

Then, solving for  $u$  in terms of the coordinates  $(x_1, x_2, x_3)$ , we have

$$s = x_3, \quad x_1^0 = x_1 e^{-s}, \quad \text{and} \quad x_2^0 = x_2 e^{-2s}.$$

Thus,

$$u(x_1, x_2, x_3) = g(x_1 e^{-x_3}, x_2 e^{-2x_3}) e^{3x_3}$$

solves the PDE  $F$  subject to the condition  $u(x_1, x_2, 0) = g(x_1, x_2)$ . ■

PROBLEM 1.1.4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with  $u(x_1, 0) = (1 - x_1^2)/2$ .

*SOLUTION.* The equation is nonlinear and therefore, we do not expect the method of characteristics to yield a unique solution to the PDE

$$F := x_1 p_1 + x_2 p_2 + \frac{1}{2}(p_1^2 + p_2^2) - z.$$

Let us find the characteristic equations for  $F$ ; these are

$$\begin{cases} \dot{p} = -D_x F - D_z F p = -(p_1, p_2) - (-1)(p_1, p_2) = 0, \\ \dot{z} = D_p F \cdot p = (x_1 + p_1, x_2 + p_2) \cdot (p_1, p_2) = (x_1 + p_1)p_1 + (x_2 + p_2)p_2, \\ \dot{x} = D_p F = (x_1 + p_1, x_2 + p_2), \end{cases}$$

Next we solve the characteristic equations subject to the initial values  $x(0) = (x^0, 0)$ ,  $z(0) = \frac{1}{2}(1 - (x^0)^2)$  and, after revisiting the equation  $F$ ,  $p_1(0) = -x^0$  and

$$p_2(0)^2 = 2 \left( -(x^0)^2 + \frac{1}{2}(x^0)^2 + \frac{1}{2}(1 - (x^0)^2) \right) = 1$$

so  $p_2(0) = \pm 1$ . Therefore, the solution to the characteristic equations subject to these initial values is

$$\begin{cases} p_1(s) = -x^0, & p_2(s) = \pm 1, \\ x_1(s) = x^0, & x_2(s) = \pm 1(e^s - 1), \\ z(s) = \frac{1}{2}(1 - (x^0)^2) + (e^s - 1). \end{cases}$$

Thus, solving for  $s$  and  $x^0$  in terms of the coordinates  $(x_1, x_2)$ , we have

$$u(x_1, x_2) = \frac{1}{2}(1 - x_1^2) \pm x_2. \quad \blacksquare$$

## 1.2 Homework 2

PROBLEM 1.2.1. Verify assertion (36) in [E, §3.2.3], that when  $\Gamma$  is not flat near  $x^0$  the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here  $\nu(x^0)$  denotes the normal to the hypersurface  $\Gamma$  at  $x^0$ ).

*SOLUTION.* Throughout this, let  $(p^0, z^0, x^0)$  denote an admissible triple to the PDE  $F$  at some point  $x^0$  in its domain. First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$$

reduces to the standard noncharacteristic boundary condition if  $\Gamma$  is flat near  $x^0$  since in that case the normal to the hypersurface at  $x^0$  will be  $(0, \dots, 0, 1)$ ; i.e.,

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall therefore proceed to flatten the hypersurface  $\Gamma$  near  $x^0$  and apply the standard noncharacteristic boundary condition.

Assuming  $\Gamma$  is reasonably tame, by the implicit function theorem, it can be written as the graph  $\{x_n = \varphi(x_1, \dots, x_{n-1})\}$  on some neighborhood  $U$  of  $x^0$  for  $\varphi$  smooth. Now consider the smooth mapping  $\Phi: U \rightarrow V$  given by

$$\begin{cases} y_j := \Phi^j(x) := x_j, & 1 \leq j \leq n-1, \\ y_n := \Phi^n(x) := x_n - \varphi(x_1, \dots, x_{n-1}), \end{cases}$$

where we use  $y$  to denote new coordinates on the image of  $\Phi$ . Note that  $\nu(x^0)$  is parallel to the gradient  $D_x \Phi^n = (-\varphi_{x_1}, \dots, -\varphi_{x_{n-1}}, 1)$  so the inner product of the latter with  $F_{p_n}(p^0, z^0, x^0)$  is nonzero if and only if the inner product of  $\nu(x^0)$  with  $F_{p_n}(p^0, z^0, x^0)$  is nonzero.

Set  $\Delta := \Phi(\Gamma)$  and define  $v(y) := u(\Phi^{-1}(y))$ . Then  $u(x) = v(\Phi(x))$ . Moreover, by the chain rule we have

$$D_{x_i} u = \sum_{j=1}^n D_{y_j} v D_{x_j} \Phi^j, \quad 1 \leq j \leq n;$$

i.e.,  $D_x u = D_y v D_x \Phi$ . Thus,  $v$  satisfies the PDE

$$G(D_y v, v, y) := F(D_y v D_x \Phi, v, \Phi^{-1}(y)) = 0$$

in  $\Delta$  and, since  $\Delta$  has been flattened near  $y^0 := \Phi(x^0)$ , applying the noncharacteristic condition, we have

$$\begin{aligned} D_{p_n} G &= (D_{p_1} F)(D_{x_1} \Phi^n) + \dots + (D_{p_n} F)(D_{x_n} \Phi) \\ &= D_p F \cdot D_x \Phi^n. \end{aligned}$$

Therefore, if  $(p^0, z^0, x^0)$  is a compatible triple for  $F$  and  $(q^0, z^0, y^0) = (p^0 D_x \Phi(x^0), z^0, \Phi(x^0))$  is the corresponding for  $G$ , then

$$D_{p_n} G(q^0, z^0, y^0) = D_p F(p^0, z^0, x^0) \cdot D_x \Phi^n(x^0);$$

i.e.,

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$$

where  $\nu(x^0)$  is the normal vector to  $\Gamma$  at  $x^0$ . ■

PROBLEM 1.2.2. Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions  $u(x, 0) = g(x)$  is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some  $t > 0$ , unless  $a(g(x))$  is a nondecreasing function of  $x$ .

*SOLUTION.* Write  $F(p, z, x, t) := (a(z), 1) \cdot p = 0$ . Using the method of characteristics, we have the following characteristic ODEs to solve

$$\begin{cases} \dot{p} = -D_{x,t}F - D_z F p = -a'(z)p_1(p_1, p_2), \\ \dot{z} = D_p F \cdot p = (a(z), 1) \cdot p = 0, \\ \dot{x} = D_{p_1} F = a(z), \quad \dot{t} = D_{p_2} F = 1. \end{cases}$$

Solving these subject to the initial conditions  $x(0) = x^0$ ,  $t(0) = 0$ , and  $z(0) = g(x^0)$ , we have

$$\begin{cases} x(s) = x^0 + a(g(x^0))s, & t(s) = s, \\ z(s) = g(x^0). \end{cases}$$

Thus, we see that  $u$  is constant on the projected characteristics

$$x = x^0 + a(g(x^0))t; \tag{1.2.1}$$

i.e.,  $u = g(x^0)$ .

Solving for  $u$  in terms of  $x$  and  $t$ , we have

$$x^0 = x - a(u)t$$

so

$$u = g(x - a(u)t).$$

Now, choose another starting point  $y^0 < x^0$ . Then we must have  $u = g(y^0)$  on the curve

$$x = y^0 + a(g(y^0))t. \tag{1.2.2}$$

Thus, if  $g(y^0) > g(x^0)$  the two characteristics (1.2.1) and (1.2.2) will cross at some  $t^0 > 0$  and we cannot have a continuous solution up to that point. If  $g(y^0) < g(x^0)$  the characteristics (1.2.1) and (1.2.2) will not cross and therefore,  $u$  solves the PDE on the upper halfplane  $\{t > 0\}$ . ■



PROBLEM 1.2.3. Show that the function  $u(x, t)$  defined for  $t \geq 0$  by

$$u(x, t) = \begin{cases} -\frac{2}{3} \left( t + \sqrt{3x + t^2} \right) & \text{for } 4x + t^2 > 0, \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law  $u_t + (u^2/2)_x = 0$  (*inviscid Burgers' equation*).

*SOLUTION.* First we show that  $u$  is in fact a weak solution of the inviscid Burgers' equation. That is, write  $u_l$  and  $u_r$  for  $u$  restricted to the domains  $\{4x + t^2 < 0\}$  and  $\{4x + t^2 > 0\}$ , respectively. Then a trivial calculation shows that  $u_l$  and  $u_r$  are solutions to the inviscid Burgers' equation: It is clear that  $u_l$  is a solution of the Burgers' equation since it is the trivial solution; for  $u_r$  we must work a little harder;

$$\begin{aligned} (u_r)_t &= -\frac{2}{3} \left( 1 + \frac{t}{\sqrt{3x + t^2}} \right), \\ (u_r)_x &= -\left( \frac{1}{\sqrt{3x + t^2}} \right) \end{aligned}$$

hence,

$$\begin{aligned} u_t + (u^2/2)_x &= u_t + u_x u \\ &= -\frac{2}{3} \left( 1 + \frac{t}{\sqrt{3x + t^2}} \right) + \frac{2}{3} \left( \frac{1}{\sqrt{3x + t^2}} \right) (t + \sqrt{3x + t^2}) \\ &= 0. \end{aligned}$$

Lastly, we need to verify the Rankine–Hugoniot condition on the curve  $\Gamma := \{4x + t^2 = 0\}$ . Write  $s(t) = x = -t^2/4$  (the natural parametrization of the curve  $\Gamma$ ), then

$$\begin{array}{lll} \sigma = \dot{s}(t), & \llbracket u \rrbracket = u_l - u_r, & \llbracket F \rrbracket = F(u_l) - F(u_r) \\ = -\frac{t}{2} & = 0 + \frac{2}{3} \left( t + \sqrt{-\frac{3}{4}t^2 + t^2} \right) & = 0 - \frac{\llbracket u_r \rrbracket^2}{2} \\ & = 0 + \frac{2}{3} \left( \frac{3}{2}t \right) & = 0 - \frac{t^2}{2} \\ & = t & = -\frac{t^2}{2}. \end{array}$$

Thus,

$$\llbracket F \rrbracket = -\frac{t^2}{2} = \left( -\frac{t}{2} \right) t = \sigma \llbracket u \rrbracket$$

so the PDE satisfies the Rankine–Hugoniot condition and hence, is an integral solution.

Finally, to check that  $u$  is an entropy solution we note that  $u^2/2$  is strictly convex and therefore, we must show that  $u_l > u_r$ . But this is obvious since  $u_l - u_r = t$  which is strictly greater than zero. ■

### 1.3 Homework 3

PROBLEM 1.3.1. Consider the initial value problem

$$\begin{cases} u_t = \sin u_x, \\ u(x, 0) = \frac{\pi}{4}x. \end{cases}$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

*SOLUTION.* The equation  $F(p, z, x, t) := \sin p_1 - p_2 = 0$  is a fully nonlinear first-order PDE. We first verify that the curve  $\Gamma$  is characteristic near the origin; i.e., we must show that  $F_p \cdot \nu \neq 0$  where  $\nu$  is the normal vector to  $\Gamma$  at the origin. In this case,  $\nu = (0, 1)$  and  $F_p = (\cos p_1, -1)$ ; hence,

$$F_p \cdot \nu = (\cos p_1, -1) \cdot (0, 1) = -1 \neq 0.$$

Moreover, the curve  $\Gamma$  is analytic (since it is cut out by the equation  $\frac{\pi}{4}x$ ) and the initial conditions are analytic. Therefore, the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and we can obtain an analytic solution

$$u(x, t) = \sum_{k,l} \frac{a_{k,l}}{k!l!} x^k t^l$$

about the origin.

First, we must compute the coefficients  $a_{k,l}$ . To this end, we must find the partial derivatives  $u_{k,l}$  and potentially, relations among them which will help us to find these coefficients. Naïvely listing the partials with respect to  $t$  and  $x$ , we have

$$\begin{array}{ll} u(0, 0) = 0 & u_x(0, 0) = \frac{\pi}{4} \\ u_t(0, 0) = \sin u_x(0, 0) = \frac{\sqrt{2}}{2} & u_{xx}(0, 0) = 0 \\ u_{tx}(0, 0) = 0 & u_{tt}(0, 0) = -\cos(u_x(0, 0))u_{xt}(0, 0) = 0 \\ u_{xxx}(0, 0) = 0 & u_{ttx}(0, 0) = 0 \\ \vdots & \vdots \end{array}$$

It is not difficult to see that higher derivatives of  $u$  will be zero. Thus,

$$u(x, t) = \frac{\pi}{4}x + \frac{\sqrt{2}}{2}t.$$

Plugging this equation into  $F$  we see that

$$u_t - \sin u_x = \frac{\sqrt{2}}{2} - \sin(\pi/4) = 0;$$

i.e.,  $u(x, t)$ , as defined above, is an analytic solution to the PDE  $F$ . ■

PROBLEM 1.3.2. Consider the Cauchy problem for  $u(x, y)$

$$\begin{cases} u_y = a(x, y, u)u_x + b(x, y, u), \\ u(x, 0) = 0 \end{cases}$$

let  $a$  and  $b$  be analytic functions of their arguments. Assume that  $D^\alpha a(0, 0, 0) \geq 0$  and  $D^\alpha b(0, 0, 0) \geq 0$  for all  $\alpha$ . (Remember by definition, if  $\alpha = 0$  then  $D^\alpha f = f$ .)

- (a) Show that  $D^\beta u(0, 0) \geq 0$  for all  $|\beta| \leq 2$ .
- (b) Prove that  $D^\beta u(0, 0) \geq 0$  for all  $\beta = (\beta_1, \beta_2)$ .

*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in  $\beta_2$ .

*SOLUTION.* For part (a): We compute all partial  $D^\beta u$  at  $(0, 0)$  for  $|\beta| \leq 2$  explicitly; these are

$$\begin{aligned} u(0, 0) &= u_x(0, 0) = u_{xx}(0, 0) = 0, \\ u_y(0, 0) &= a(0, 0, 0)u_x(0, 0) + b(0, 0, 0) = b \geq 0, \\ u_{xy}(0, 0) &= (a_x(0, 0, 0) + a_u(0, 0, 0)u_x(0, 0))u_y(0, 0) + a(0, 0, 0)u_{xx}(0, 0) + b_x(0, 0, 0) \\ &\quad + b_z(0, 0, 0)u_x(0, 0) \geq 0, \\ u_{yy}(0, 0) &= (a_y(0, 0, 0) + a_u(0, 0, 0)u_y(0, 0))u_x(0, 0) \\ &\quad + b_y(0, 0, 0) + b_u(0, 0, 0)u_y(0, 0) \geq 0. \end{aligned}$$

For part (b):

■

PROBLEM 1.3.3. (Kovalevskaya’s example) show that the line  $\{t = 0\}$  is characteristic for the heat equation  $u_t = u_{xx}$ . Show there does not exist an analytic solution  $u$  of the heat equation in  $\mathbf{R} \times \mathbf{R}$ , with  $u = 1/(1 + x^2)$  on  $\{t = 0\}$ .

*Hint:* Assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of  $(0, 0)$ .

*SOLUTION.*

■

## 1.4 Homework 4

PROBLEM 1.4.1 (Legendre transform). Let  $u(x_1, x_2)$  be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of  $\mathbf{R}^2$ , where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u(\mathbf{x}(p)),$$

where  $\mathbf{x} = (x^1, x^2)$ ,  $p = (p_1, p_2)$ . Show that  $v$  satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

*Hint:* See [Evans, 4.4.3b], prove the identities (29).

*SOLUTION.* ■

PROBLEM 1.4.2. Find the solution  $u(x, t)$  of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant  $x > 0, t > 0$  for which

$$\begin{cases} u(x, 0) = f(x), & u_t(x, 0) = g(x), & \text{for } x > 0, \\ u_t(0, t) = \alpha u_x(0, t), & & \text{for } t > 0, \end{cases}$$

where  $\alpha \neq -1$  is a given constant. Show that generally no solution exists when  $\alpha = -1$ .

*Hint:* Use a representation  $u(x, t) = F(x - t) + G(x + t)$  for the solution.

*SOLUTION.* ■

PROBLEM 1.4.3. (a) Let  $u$  be a solution of the wave equation  $u_{tt} - c^2 u_{xx} = 0$  for  $0 < x < \pi, t > 0$  such that  $u(0, t) = u(\pi, t) = 0$ . Show that the *energy*

$$E(t) = \frac{1}{2} \int_0^\pi (u_t^2 + c^2 u_x^2) dx, \quad t > 0$$

is independent of  $t$ ; i.e.,  $dE/dt = 0$  for  $t > 0$ . Assume that  $u$  is  $C^2$  up to the boundary.

(b) Express the energy  $E$  of the Fourier series solution

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(nct) + b_n \sin(nct)) \sin(nx)$$

in terms of coefficients  $a_n, b_n$ .

*SOLUTION.*

■

## 1.5 Homework 5

PROBLEM 1.5.1. Prove that Laplace's equation  $\Delta u = 0$  is rotation invariant; that is, if  $O$  is an orthogonal  $n \times n$  matrix and we define  $v(x) := u(Ox)$ ,  $x \in \mathbf{R}^n$ , then  $\Delta v = 0$ .

SOLUTION. ■

PROBLEM 1.5.2. Let  $n = 2$  and  $U$  be the halfplane  $\{x_2 > 0\}$ . Prove that

$$\sup_U u = \sup_{\partial U} u$$

for  $u \in C^2(U) \cap C(\bar{U})$  which are harmonic in  $U$  under the additional assumption that  $u$  is bounded from above in  $\bar{U}$ . (The additional assumption is needed to exclude examples like  $u = x_2$ .)

*Hint:* Take for  $\epsilon > 0$  the harmonic function

$$u(x_1, x_2) - \epsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region  $\{x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0\}$  with large  $a$ . Let  $\epsilon \rightarrow 0$ .

SOLUTION. ■

PROBLEM 1.5.3. Let  $U \subseteq \mathbf{R}^n$  be an open set. We say  $v \in C^2(U)$  is subharmonic if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Let  $\varphi: \mathbf{R}^m \rightarrow \mathbf{R}$  be smooth and convex. Assume  $u^1, \dots, u^m$  are harmonic in  $U$  and

$$v := \varphi(u_1, \dots, u_m).$$

Prove  $v$  is subharmonic.

*Hint:* Convexity for a smooth function  $\varphi(z)$  is equivalent to  $\sum_{j,k=1}^m \varphi_{z_j, z_k}(z) \xi_j \xi_k \geq 0$  for any  $\xi \in \mathbf{R}^m$ .

(b) Prove  $v := |Du|^2$  is subharmonic, whenever  $u$  is harmonic. (Assume that harmonic functions are  $C^\infty$ .)

SOLUTION. ■

## 1.6 Homework 6

PROBLEM 1.6.1. For  $n = 2$  find Green's function for the quadrant  $U := \{x_1, x_2 > 0\}$  by repeated reflection.

*SOLUTION.* Taking the hit, set  $x' := (x_1, -x_2)$ ,  $x'' := (-x_1, x_2)$ ,  $x''' := (-x_1, -x_2)$ , and define

$$\varphi^x(y) := \Phi(y - x') + \Phi(y - x'') - \Phi(y - x'''). \quad (1.6.1)$$

We claim that  $\varphi^x$ , as defined above, solves

$$\begin{cases} \Delta \varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(y - x) & \text{on } \partial U. \end{cases}$$

It is clear that  $\Delta \varphi^x = 0$  since it is built up from the fundamental solutions on  $\mathbf{R}^n$  (this follows from the linearity of the Laplace operator). To see that  $\varphi^x(y) = \Phi(y - x)$  on  $\partial U$ , we do a case by case analysis.

Note that on  $\{x_1 = 0\} \subseteq \partial U$ , we have

$$\varphi^x(y_1, 0) = \Phi(-x_1, y_2 + x_2) + \Phi(-x_1, y_2 - x_2) - \Phi(x_1, y_2 + x_2),$$

where, since the fundamental solution is radial, we have  $\Phi(-x_1, y_2 + x_2) = \Phi(x_1, y_2 + x_2)$ , and hence the above equals

$$\begin{aligned} &= \Phi(-x_1, y_2 - x_2) \\ &= \Phi(y - x) \end{aligned}$$

and on  $\{x_2 = 0\} \subseteq \partial U$ , we have

$$\varphi^x(0, y_2) = \Phi(y_1 - x_1, x_2) + \Phi(y_1 + x_1, -x_2) - \Phi(y_1 + x_1, x_2)$$

where, again because  $\Phi$  is radial,  $\Phi(y_1 + x_1, -x_2) = \Phi(y_1 + x_1, x_2)$ , thus the above equals

$$\begin{aligned} &= \Phi(y_1 - x_1, x_2) \\ &= \Phi(y - x). \end{aligned}$$

Thus,  $\varphi^x(y) = \Phi(y - x)$  on  $\partial U$ .

Therefore, Green's function on  $U$  is

$$G(x, y) = \Phi(y - x) - \varphi^x(y) = \Phi(y - x) - \Phi(y - x') - \Phi(y - x'') + \Phi(y - x'''). \quad \blacksquare$$

PROBLEM 1.6.2. (Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0)$$

whenever  $u$  is positive and harmonic in  $B(0, r) = \{x \in \mathbf{R}^n : |x| < r\}$ .

*SOLUTION.* Recall Poisson's formula for the ball

$$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y), \quad (1.6.2)$$

where  $x \in B(0, r)$  and  $u$  solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0, r), \\ u = g & \text{on } \partial B(0, r). \end{cases}$$

For fixed  $x \in B(0, r)$ , write

$$u(x) = r^{n-2}(r+|x|)(r-|x|) \left[ \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y) \right].$$

Now, since  $r+|x| \geq |x-y| \geq r-|x|$  for all  $y \in \partial B(0, r)$ , we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} \int_{\partial B(0,r)} g(y) dS(y) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} \int_{\partial B(0,r)} g(y) dS(y). \quad (1.6.3)$$

Since  $u = g$  on the boundary  $\partial B(0, r)$ , by applying the mean-value property on (1.6.3) we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} u(0),$$

as desired. ■

**PROBLEM 1.6.3.** Let  $P_k(x)$  and  $P_m(x)$  be homogeneous harmonic polynomials in  $\mathbf{R}^n$  of degrees  $k$  and  $m$  respectively; i.e.,

$$\begin{cases} P_k(\lambda x) = \lambda^k P_k(x), & P_m(\lambda x) = \lambda^m P_m(x) & \text{for every } x \in \mathbf{R}^n, \lambda > 0, \\ \Delta P_k = 0, & \Delta P_m = 0 & \text{in } \mathbf{R}^n. \end{cases}$$

(a) Show that

$$\begin{cases} \frac{\partial P_k}{\partial \nu} = k P_k(x), & \frac{\partial P_m}{\partial \nu} = m P_m(x) & \text{on } \partial B(0, 1), \end{cases}$$

where  $B(0, 1) = \{x \in \mathbf{R}^n : |x| < 1\}$  and  $\nu$  is the outward normal on  $\partial B(0, 1)$ .

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x) d\sigma = 0, \quad \text{if } k \neq m.$$

*SOLUTION.* For part (a), let

$$P_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha.$$



Then, since  $\nu = (x_1, \dots, x_n)$ , the derivative along  $\nu$  is given by

$$\begin{aligned} \frac{\partial P_k(x)}{\partial \nu} &= \sum_{j=1}^n (P_k)_{x_j} x_j \\ &= \sum_{j=1}^n \left( \sum_{|\alpha|=k} a_\alpha x^\alpha \right)_{x_j} x_j \\ &= \sum_{j=1}^n \left( \sum_{l=1}^m a_\alpha x_1^{\alpha_1^l} \dots x_j^{\alpha_j^l} \dots x_n^{\alpha_n^l} \right)_{x_j} x_j \end{aligned}$$

where  $\sum_{j=1}^n \alpha_j^l = k$  and  $1 \leq j \leq \binom{n+k-1}{n} =: m$  (by the stars and bars theorem)

$$\begin{aligned} &= \sum_{j=1}^n \sum_{l=1}^m \left( \alpha_j^l a_\alpha x_1^{\alpha_1^l} \dots x_j^{\alpha_j^l-1} \dots x_n^{\alpha_n^l} \right) x_j \\ &= \sum_{j=1}^n \sum_{l=1}^m \alpha_j^l a_\alpha x_1^{\alpha_1^l} \dots x_j^{\alpha_j^l} \dots x_n^{\alpha_n^l} \\ &= \sum_{j=1}^n \sum_{l=1}^m \alpha_j^l a_\alpha x^\alpha \end{aligned}$$

switching the order of summation, we have

$$\begin{aligned} &= \sum_{l=1}^m \sum_{j=1}^n \alpha_j^l a_\alpha x^\alpha \\ &= \sum_{l=1}^m k a_\alpha x^\alpha \\ &= k \sum_{l=1}^m a_\alpha x^\alpha \\ &= k P_k(x). \end{aligned}$$

This argument, of course, applies to every  $k \in \mathbb{N}$ .

For part (b), by Green's theorem, we have

$$\begin{aligned} \int_{B(0,r)} P_k(x) \Delta P_m(x) - (\Delta P_k(x)) P_m(x) dx &= \int_{\partial B(0,r)} P_k(x) \frac{\partial}{\partial \nu} P_m(x) - \frac{\partial}{\partial \nu} P_k(x) P_m(x) dS(x) \\ &= \int_{\partial B(0,r)} (m-k) P_k(x) P_m(x) dS(x), \end{aligned}$$

where the left-hand side is equal to zero since both  $\Delta P_k$  and  $\Delta P_m$  are zero. Since  $m \neq k$ , it must be the case that

$$\int_{\partial B(0,r)} P_k(x) P_m(x) dS(x) = 0.$$

■

## 1.7 Homework 7

PROBLEM 1.7.1. Solve the Dirichlet problem for the Laplace equation in  $\mathbf{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ u = x_1 & \text{on } |x| = 1, \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

*Hint:* Use Laurent series.

*SOLUTION.* First, let us make the change of variables  $(x_1, x_2) \mapsto re^{i\theta}$  to the Dirichlet problem in question:

$$\begin{cases} \Delta u = 0 & \text{in } 1 < r < 2, \\ u = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) & \text{on } r = 1, \\ u = 1 + \frac{1}{i}(e^{i2\theta} - e^{-i2\theta}) & \text{on } r = 2. \end{cases} \quad (1.7.1)$$

Now, suppose  $u$  is a solution, of the form

$$u(re^{i\theta}) = b \ln r + \sum_{n=-\infty}^{\infty} (a_n r^n + \overline{a_{-n}} r^{-n}) e^{in\theta},$$

to the problem (1.7.1). It is easy to see that  $u$  is in fact harmonic:

$$\begin{aligned} \Delta u &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \\ &= -br^{-2} + br^{-2} + \sum_{n=-\infty}^{\infty} [(n(n-1) + n - n^2) a_n r^n \\ &\quad + (n(n-1) + n - n^2) \overline{a_{-n}} r^{-n}] e^{in\theta} \\ &= 0. \end{aligned}$$

Next we use the boundary data to compute the coefficients  $a_n$ ,  $n \in \mathbf{Z}$ . Using the data (1.7.1), on  $r = 1$  we have

$$\frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \sum_{n=-\infty}^{\infty} (a_n + \overline{a_{-n}}) e^{in\theta},$$

and on  $r = 2$

$$1 + \frac{1}{i}(e^{i2\theta} - e^{-i2\theta}) = b \ln 2 + \sum_{n=-\infty}^{\infty} (2^n a_n + 2^{-n} \overline{a_{-n}}) e^{in\theta}.$$

These equations immediately tell us that  $b = 1/\ln 2$ . Moreover, the following relations on the coefficients hold

$$\begin{cases} \frac{1}{2} = a_1 + \overline{a_{-1}} & \frac{1}{2} = a_{-1} + \overline{a_1}, \\ \frac{1}{i} = 2^2 a_2 + 2^{-2} \overline{a_{-2}}, & -\frac{1}{i} = 2^2 a_{-2} + 2^{-2} \overline{a_2}, \\ 0 = a_n + \overline{a_{-n}} & \text{for } n \neq \pm 1, \\ 0 = 2^n a_n + 2^{-n} \overline{a_{-n}} & \text{for } n \neq \pm 2. \end{cases}$$

A little calculation shows that

$$\begin{cases} a_1 = -\frac{1}{6}, & a_{-1} = \frac{2}{3}, \\ a_2 = -\frac{4i}{15}, & a_{-2} = -\frac{4i}{15}, \\ a_n = 0 & \text{for } n \neq \pm 1, \pm 2. \end{cases}$$

Thus,

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{\ln 2} \ln r + \left(-\frac{4i}{15}r^{-2} + \frac{4i}{15}r^2\right)e^{-i2\theta} + \left(\frac{2}{3}r^{-1} - \frac{1}{6}r\right)e^{-i\theta} \\ &\quad + \left(-\frac{1}{6}r + \frac{2}{3}r^{-1}\right)e^{i\theta} + \left(-\frac{4i}{15}r^2 + \frac{4i}{15}r^{-2}\right)e^{i2\theta} \\ &= \frac{1}{\ln 2} \ln r - \frac{8}{15}r^{-4} \left(\frac{r^2 e^{i2\theta} - r^2 e^{-i2\theta}}{2i}\right) + \frac{8}{15} \left(\frac{r^2 e^{i2\theta} - r^2 e^{-i2\theta}}{2i}\right) \\ &\quad + \frac{4}{3}r^{-2} \left(\frac{r e^{i\theta} + r e^{-i\theta}}{2}\right) - \frac{1}{3} \left(\frac{r e^{i\theta} + r e^{-i\theta}}{2}\right). \end{aligned}$$

Substituting back, we have

$$u(x_1, x_2) = \frac{1}{\ln 2} \ln(x_1^2 + x_2^2) - \frac{16x_1x_2}{15(x_1^2 + x_2^2)^2} + \frac{16x_1x_2}{15} + \frac{4x_1}{3(x_1^2 + x_2^2)} - \frac{x_1}{3}. \quad (1.7.2)$$

It is easy to see that (1.7.2) satisfies the boundary data at  $|x| = 1$  and  $|x| = 2$ . ■

**PROBLEM 1.7.2.** Let  $\Omega$  be a bounded domain with a  $C^1$  boundary,  $g \in C^2(\partial\Omega)$  and  $f \in C(\bar{\Omega})$ . Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \quad (*)$$

where  $\nu$  is the outer normal on  $\partial\Omega$ . Show that the solution of  $(*)$  in  $C^2(\Omega) \cap C^1(\bar{\Omega})$  is unique up to a constant; i.e., if  $u_1$  and  $u_2$  are both solutions of  $(*)$ , then  $u_2 = u_1 + \text{const.}$  in  $\Omega$ .

*Hint:* Look at the proof of the uniqueness for the Dirichlet problem by energy methods [E, 2.2.5a].

*SOLUTION.* Suppose  $u_1$  and  $u_2$  are solutions to the Neumann problem (\*). Define  $v := u_1 - u_2$ . Then  $v$  is harmonic in  $\Omega$  and  $\partial v / \partial \nu = 0$  on  $\partial\Omega$ . Consider the energy functional

$$E[v] = \frac{1}{2} \int_{\Omega} |Dv|^2 dx.$$

By Green's formula version (ii),

$$\begin{aligned} E[v] &= \frac{1}{2} \int_{\Omega} |Dv|^2 dx \\ &= -\frac{1}{2} \int_{\Omega} v \Delta v dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} v dS(x) \\ &= 0. \end{aligned}$$

This implies that  $|Dv|^2 = Dv \cdot Dv = 0$  which, since the standard inner product on  $\mathbf{R}^n$  is positive-definite, implies that  $Dv \equiv 0$ . It follows that  $u_1 = u_2 + \text{const}$ , i.e., the solution  $u$  to (\*) is unique up to a constant. ■

PROBLEM 1.7.3. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta_x u + cu = f & \text{in } \mathbf{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbf{R}^n \times \{t = 0\}, \end{cases}$$

where  $c \in \mathbf{R}$ .

*Hint:* Rewrite the problem in terms of  $v(x, t) := e^{ct}u(x, t)$ .

*SOLUTION.* Taking the hint, let us rewrite the problem in terms of  $v(x, t) = e^{ct}u(x, t)$ :

$$\begin{cases} v_t - \Delta_x v = e^{ct}f & \text{in } \mathbf{R}^n \times (0, \infty), \\ v = g & \text{on } \mathbf{R}^n \times \{t = 0\}. \end{cases} \quad (1.7.3)$$

By Duhamel's principle, the problem (1.7.3) is solved by

$$v(x, t) = \int_{\mathbf{R}^n} \Phi(x - y, t) g(y) dy + \int_0^t \int_{\mathbf{R}^n} \Phi(x - y, t - s) e^{cs} f(y, s) dy ds,$$

where  $\Phi$  is the fundamental solution to the heat equation. Thus, the formula

$$u(x, t) = e^{-ct}v(x, t) = e^{-ct} \int_{\mathbf{R}^n} \Phi(x - y, t) g(y) dy + e^{-ct} \int_0^t \int_{\mathbf{R}^n} \Phi(x - y, t - s) e^{cs} f(y, s) dy ds$$

solves the original problem. ■

## 2 Homework 8

PROBLEM 2.0.1. Show that the function

$$u(x, t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x - 2k, t)$$

where

$$\Phi(x, t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

is positive for  $|x| < 1$ ,  $t > 0$ .

*Hint:* Show that  $u$  satisfies  $u_t = u_{xx}$  for  $t > 0$ ,

$$\begin{cases} u = 0 & \text{on } \{|x| = 1\} \times \{t \geq 0\}, \\ u = \delta_0 & \text{on } \{|x| \leq 1\} \times \{t = 0\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain  $\{|x| \leq 1\} \times \{\epsilon \leq t \leq T\}$  for small  $\epsilon > 0$  and large  $T > 0$  pass to the limit as  $\epsilon \rightarrow 0+$  and  $T \rightarrow \infty$ .

*SOLUTION.* Taking the hint, let us verify that  $u_t = u_{xx}$ , for  $t > 0$ . By direct computation, we have

$$\begin{aligned} \Phi_x(x, t) &= \frac{\partial}{\partial x} \left( \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \right) & \Phi_{xx}(x, t) &= \frac{\partial}{\partial x} \left( -\frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}} \right) \\ &= -\frac{x e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}}, & &= \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}} \\ & & &= \frac{(x^2 - 2t) e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}}, \end{aligned}$$

and

$$\begin{aligned} \Phi_t(x, t) &= \frac{\partial}{\partial t} \left( \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \right) \\ &= \frac{x^2 e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi t}^{\frac{3}{2}}} \\ &= \frac{(x^2 - 2t) e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi t}^{\frac{5}{2}}}. \end{aligned}$$

Since  $\Phi_t = \Phi_{xx}$  it follows that  $u_t = u_{xx}$  (assuming uniform convergence of  $u$ ).

Next we show that  $u = 0$  on  $\{|x| = 1\} \times \{t \geq 0\}$  and  $u = \delta_0$  on  $\{|x| = 1\} \times \{t = 0\}$ . To show  $u = 0$  fix a  $t \geq 0$  and, after relabeling if necessary, assume that  $x = 1$  which gives us

$$\begin{aligned} u(1, t) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-\frac{(1-2k)^2}{4t}}}{\sqrt{4\pi t}} \\ &= \frac{1}{\sqrt{4\pi t}} \left( \dots - e^{-\frac{9}{4t}} + e^{-\frac{1}{4t}} - e^{-\frac{1}{4t}} + e^{-\frac{9}{4t}} + \dots \right) \\ &= 0. \end{aligned}$$

Similarly for  $u(-1, t) = 0$ .

For  $u(|x| \leq 1, 0)$ , we have a

$$\begin{aligned} u(|x| \leq 1, 0) &= \sum_{k=-\infty}^{\infty} (-1)^k \lim_{t \rightarrow 0^+} \left[ e^{-\frac{(x-2k)^2}{4t}} / \sqrt{4\pi t} \right] \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \delta_0(x - 2k) \\ &= \delta_0(x) \end{aligned}$$

since  $|x| \leq 1$  and values  $\delta_0$  is zero for values  $x - 2k$  outside of the interval  $[-1, 1]$ .

At last we show that  $u$  is positive for  $|x| < 1$ ,  $t > 0$ . Seeking a contradiction, suppose  $u$  is negative on some point  $(x_0, t_0)$  in  $\{|x| < 1\} \times \{\epsilon \leq t \leq T\}$ . Then by the minimum principle,  $u$  achieves its minimum somewhere on the bottom boundary  $\{|x| = 1\} \times \{t = \epsilon\}$ . Therefore, there exists a sequence  $(x_n, t_n) \rightarrow (x, 0)$ , where  $|x_n|, |x| < 1$ , such that  $u(x, 0) < 0$ . However, we have shown above that  $u(x, 0) = \delta_0(x)$  for  $|x| < 1$ ; i.e., either  $u(x, 0) = 0$  or  $u(x, 0) = +\infty$ . This is a contradiction. Therefore, it must be the case that  $u \geq 0$  for  $|x| < 1$ ,  $t > 0$ . ■

PROBLEM 2.0.2 (Tikhonov's example). Let

$$g(t) := \begin{cases} e^{-t^{-2}} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then  $g \in C^\infty(\mathbf{R})$  and we define

$$u(x, t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that  $u(x, t)$  solves the heat equation in  $\mathbf{R} \times (0, \infty)$  with the initial condition  $u(x, 0) = 0$ ,  $x \in \mathbf{R}$ . Why doesn't this contradict the uniqueness theorem for the initial value problem?

*SOLUTION.* Let  $u$  be as above. Then

$$\begin{aligned} u_t(x, t) &= \frac{\partial}{\partial t} \left( \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} \\ &= \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}, \end{aligned}$$

and

$$\begin{aligned} u_x(x, t) &= \frac{\partial}{\partial x} \left( \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right) & u_{xx}(x, t) &= \frac{\partial}{\partial x} \left( \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1} \right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2k x^{2k-1} & &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} (2k-1) x^{2k-2} + \frac{\partial}{\partial x} g^{(0)}(t) \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}, & &= \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}. \end{aligned}$$

Thus,  $u_t - \Delta u = 0$ ; i.e.,  $u$  solves the heat equation. As this example shows, unless some assumptions on  $u$  such as subexponential (cf. [E §2.3], Theorem 7) growth is assumed. ■

PROBLEM 2.0.3. Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax) e^{-x^2} dx, \quad (a > 0).$$

*Hint:* Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.

*SOLUTION.* By separation of variables,

$$u(x, t) = \cos(ax) e^{-a^2 t}$$

is a solution to the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbf{R} \times (0, \infty), \\ u = \cos(ax) & \text{on } \mathbf{R} \times \{t = 0\}. \end{cases}$$

However, the convolution

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \cos(ay) e^{-\frac{|x-y|^2}{4t}} dy$$

is also a solution to the Cauchy problem. Now note that

$$\begin{aligned}\int_{-\infty}^{\infty} \cos(ay) e^{-y^2} dy &= \sqrt{\pi} \cdot u(0, \tfrac{1}{4}) \\ &= \sqrt{\pi} e^{-\frac{a^2}{4}}.\end{aligned}$$

■



PROBLEM 2.0.4. (a) Show that for  $n = 3$  the general solution to the wave equation  $u_{tt} - \Delta u = 0$  with spherical symmetry about the origin has the form

$$u = \frac{1}{r}F(r+t) + \frac{1}{r}G(r-t), \quad r = |x|,$$

with suitable  $F$  and  $G$ .

(b) Show that the solution with initial data of the form

$$u(r, 0) = 0, \quad u_t(r, 0) = h(r)$$

( $h$  is an even function of  $r$ ) is given by

$$u = \frac{1}{2r} \int_{r-t}^{r+t} \rho h(\rho) d\rho.$$

SOLUTION. ■

PROBLEM 2.0.5. Show that the solution  $w(x_1, t)$  of the initial-value problem for the *Klein-Gordon equation*

$$\begin{cases} w_{tt} = w_{x_1 x_1} - \lambda^2 w, \\ w(x_1, 0) = 0, \end{cases} \quad w_t(x_1, 0) = h(x_1) \quad (2.0.1)$$

is given by

$$w(x_1, t) = \frac{1}{2} \int_{x_1-t}^{x_1+t} J_0(\lambda s) h(y_1) dy_1.$$

Here  $s^2 = t^2 - (x_1 - y_1)^2$ , while  $J_0$  denotes the Bessel function defined by

$$J_0(z) := \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(z \sin \theta) d\theta.$$

*Hint:* Descend to (2.0.1) from the two-dimensional wave equation satisfied by

$$u(x_1, x_2, t) = \cos(\lambda x_2) w(x_1, t).$$

SOLUTION. ■

PROBLEM 2.0.6. Let  $u$  solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbf{R}^3 \times (0, \infty), \\ u = g, \quad u_t = h & \text{on } \mathbf{R}^3 \times \{t = 0\} \end{cases}$$

where  $g$  and  $h$  are smooth and have compact support. Show there exists a constant  $C$  such that

$$|u(x, t)| \leq Ct^{-1} \quad (x \in \mathbf{R}^3, t > 0).$$

SOLUTION. ■

### 3 Exams

#### 3.1 Midterm Practice Problems

PROBLEM 3.1.1. Solve  $u_{x_1}^2 + x_2 u_{x_2} = u$  with initial conditions  $u(x_1, 1) = x_1^2/4 + 1$ .

*SOLUTION.* By inspection, we may suspect that  $v(x_1, x_2) = x_1^2/4 + x_2$  is a solution to the PDE. It certainly satisfies the boundary condition. A routine calculation shows that  $v$  is in fact a solution to the PDE. Lucky guess!

More formally, let us solve this problem using the method of characteristics. First, write

$$F(p, z, x) = (p^1(s))^2 + x^2(s)p^2(s) - z(s) = 0.$$

Then, the characteristic ODEs are

$$\left\{ \begin{array}{l} (\dot{p}^1(s), \dot{p}^2(s)) = -(0, p^2(s)) + (p^1(s), p^2(s)) \\ \quad = (p^1(s), 0), \\ \dot{z}(s) = (2p^1(s), x^2(s)) \cdot (p^1(s), p^2(s)) \\ \quad = 2p^1(s)^2 + x^2(s)p^2(s), \\ (\dot{x}^1(s), \dot{x}^2(s)) = (2p^1(s), x^2(s)). \end{array} \right.$$

Now, for  $(x^1(0), x^2(0)) = (x^0, 1)$ , integrating the characteristics, we get

$$\left\{ \begin{array}{l} (p^1(s), p^2(s)) = (p_0^1 e^s, p_0^2), \\ (x^1(s), x^2(s)) = (2p_0^1 e^s + x_0^1, x_0^2 e^s), \\ z(s) = \frac{(x^0)^2}{4} e^{2s} + p_0^2 e^s + z^0 \end{array} \right.$$

Using the initial condition and the PDE, we find that

$$\begin{aligned} p_0^1 &= \frac{x^0}{2}, \quad p_0^2 = \frac{(x^0)^2}{4} + 1 - \frac{(x^0)^2}{4} = 1, \\ x_0^1 &= 0, \quad x_0^2 = 1 \\ z^0 &= 0, \end{aligned}$$

and consequently

$$\left\{ \begin{array}{l} (x^1(s), x^2(s)) = (x^0 e^s, e^s), \\ z(s) = \frac{(x^0)^2}{4} e^{2s} + e^s \end{array} \right.$$

so, rewriting  $z$  in terms of  $(x^1, x^2)$ , we have

$$\begin{aligned} z(s) &= \frac{(x^0)^2}{4} e^{2s} + e^s \\ &= \frac{(x^1(s))^2}{4} + x^2(s), \end{aligned}$$

so the solution in terms of  $(x_1, x_2)$ , is

$$u(x_1, x_2) = \frac{x_1^2}{4} + x_2,$$

just as we suspected. ■

PROBLEM 3.1.2. Find the maximal  $t_0 > 0$  for which the (classical) solution of the Cauchy problem

$$\begin{cases} uu_x + u_t = 0, \\ u(x, 0) = e^{-\frac{x^2}{2}}, \end{cases}$$

exists in  $\mathbf{R} \times [0, t)$ ; i.e., the first time  $t = t_0$  when the shock develops.

*SOLUTION.* First, let us find a solution to the PDE using the method of characteristics. Write

$$F(p, z, x) = z(s)p^1(s) + p^2(s).$$

Then, the characteristic ODEs are

$$\begin{cases} (\dot{p}^1(s), \dot{p}^2(s)) = -(0, 0) - p^1(p^1(s), p^2(s)) \\ \quad = (-p^1(s)^2, -p^1(s)p^2(s)), \\ \dot{z}(s) = (z(s), 1) \cdot (p^1(s), p^2(s)) \\ \quad = z(s)p^1(s) + p^2(s) \\ \quad = 0, \\ (\dot{x}(s), \dot{t}(s)) = (z(s), 1). \end{cases}$$

Thus, integrating the characteristic ODEs from  $(x^0, 0)$ , we have

$$\begin{cases} \dot{z}(s) = z^0, \\ (x(s), t(s)) = (z^0 s + x^0, s); \end{cases}$$

since the PDE is quasilinear, we disregard  $(p^1, p^2)$ .

Applying the boundary conditions, we see that

$$z^0 = u(x^0, 0) = e^{-(x^0)^2/2}.$$

Here's where it gets tricky. After a little struggling, we see that there is really no way to solve for  $z$  in terms of  $(x(s), t(s))$ . However, we can solve for the projected characteristics:

$$(x(t, y), t) = (e^{-y^2/2}t + y, t);$$

and this is really all that matters for us to find the time  $t_0$  when the shock develops, i.e., the time when the projected characteristic fails to be injective.

A little calculation shows that this happens precisely when  $t = e^{-1/2}$ . ■

PROBLEM 3.1.3. If  $\rho_0$  denotes the maximum density of cars on a highway (i.e., under bumper-to-bumper conditions), then a reasonable model for traffic density  $\rho$  is given by

$$\begin{cases} \rho_t + (F(\rho))_x = 0, \\ F(\rho) = c\rho\left(1 - \frac{\rho}{\rho_0}\right), \end{cases}$$

where  $c > 0$  is a constant (free speed of highway). Suppose the initial density is

$$\rho(x, 0) = \begin{cases} \frac{1}{2}\rho_0 & \text{if } x < 0, \\ \rho_0 & \text{if } x > 0. \end{cases}$$

Find the shock curve and describe the weak solution. (Interpret your result for the traffic flow.)

*SOLUTION.* First, note that

$$\begin{aligned} (F(\rho))_x &= F'(\rho)\rho_x \\ &= \left[-c\frac{\rho}{\rho_0} + c\left(1 - \frac{\rho}{\rho_0}\right)\right]\rho_x \\ &= \left(c - \frac{2c\rho}{\rho_0}\right)\rho_x. \end{aligned}$$

Let us find a solution to the PDE using the method of characteristics. Write

$$G(p, z, x) = p^2(s) + F'(z(s))p^1(s).$$

Then, the characteristic ODEs are

$$\begin{cases} (\dot{p}^1(s), \dot{p}^2(s)) = (-F''(z(s))p^1(s), -F''(z(s))p^2(s)), \\ \dot{z}(s) = F'(z(s))p^1(s) + p^2(s) \\ \quad = 0, \\ (\dot{x}^1(s), \dot{x}^2(s)) = (F'(z(s)), 1). \end{cases}$$

Now, integrating the characteristics, we have

$$\begin{cases} z(s) = z^0, \\ (x^1(s), x^2(s)) = (F'(z^0)s + x^0, s). \end{cases}$$

We have two cases to consider,  $x^0 < 0$  or  $x^0 > 0$ . For  $x^0 < 0$ ,  $z^0 = \frac{\rho_0}{2}$  and the projected characteristics look like

$$\begin{aligned} \left(F'\left(\frac{\rho_0}{2}\right)t + x^0, t\right) &= \left(\left[c - \frac{2c(\frac{\rho_0}{2})}{\rho_0}\right]t + x^0, t\right) \\ &= (0 \cdot t + x^0, t) \\ &= (x^0, t) \end{aligned}$$

(where we have replaced  $s$  with the more appropriate  $t$ ). Whereas for  $x^0 > 0$ , we have

$$\begin{aligned}(F'(\rho_0)t + x^0, t) &= \left( \left[ c - \frac{2c\rho_0}{\rho_0} \right] t + x^0, t \right) \\ &= (-ct + x^0, t).\end{aligned}$$

These characteristics intersect precisely when

$$t = \frac{x_1^0 - x_2^0}{c},$$

where  $x_1^0 > 0$ ,  $x_2^0 < 0$ . ■

PROBLEM 3.1.4. Find the characteristics of the second order equation

$$u_{xx} - (2 \cos x)u_{xy} - (3 + \sin^2 x)u_{yy} - yu_y = 0,$$

and transform it to the canonical form.

SOLUTION. First, writing the PDE in the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + 2Du_x + 2Eu_y + Fu = 0,$$

we see that  $A = 1$ ,  $B = -\cos x$ ,  $C = -3 \sin^2 x$ , and  $E = -\frac{y}{2}$ . We solve for the characteristic curve by find a solution to the ODEs

$$\begin{aligned}\frac{dy}{dx} &= \frac{B \pm \sqrt{B^2 - AC}}{A} \\ &= -\cos x \pm \sqrt{\cos^2 x + 3 + \sin^2 x} \\ &= -\cos x \pm 2.\end{aligned}$$

The solutions give us the following ODEs

$$\begin{cases} y = -\sin x + 2x + \xi(x, y), \\ y = -\sin x - 2x + \eta(x, y). \end{cases}$$

Integrating these equations, we have

$$\begin{cases} \xi(x, y) = y + \sin x - 2x, \\ \eta(x, y) = y + \sin x + 2x. \end{cases}$$

These are the characteristic strips for the PDE.

To put this PDE in canonical form, we first compute the following partial derivatives

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x, \\ u_y &= u_\xi \xi_y + u_\eta \eta_y, \\ u_{xx} &= u_\xi \xi_{xx} + u_\eta \eta_{xx} + (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) \xi_x + (u_{\xi\eta} \xi_x + u_{\eta\eta} \eta_x) \eta_x \\ &= u_{\xi\xi} (\xi_x)^2 + u_{\eta\eta} (\eta_x)^2 + 2u_{\xi\eta} \xi_x \eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx},\end{aligned}$$

exploiting symmetry, we can find  $u_{yy}$  by replacing  $x$  with  $y$  above

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + u_{\eta\eta}(\eta_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\xi}\xi_{yy} + u_{\eta}\eta_{yy},$$

the last thing we need to figure out is the mixed partial

$$\begin{aligned} u_{xy} &= u_{\xi}\xi_{xy} + u_{\eta}\eta_{xy} + (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + (u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y)\eta_x \\ &= u_{\xi\xi}\xi_x\xi_y + u_{\eta\eta}\eta_x\eta_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\xi}\xi_{xy} + u_{\eta}\eta_{xy}. \end{aligned}$$

Now find the partials  $\xi_x, \eta_x, \xi_y, \eta_y, \xi_{xy}, \dots$ , etc.

$$\begin{array}{ll} \xi_x = \cos x - 2, & \eta_x = \cos x + 2, \\ \xi_{xx} = -\sin x, & \eta_{xx} = -\sin x, \\ \xi_{xy} = 0, & \eta_{xy} = 0, \\ \xi_y = 1, & \eta_y = 1, \\ \xi_{yy} = 0, & \eta_{yy} = 0. \end{array}$$

Thus,

$$\left\{ \begin{array}{l} u_x = (\cos x - 2)u_{\xi} + (\cos x + 2)u_{\eta}, \\ u_y = u_{\xi} + u_{\eta}, \\ u_{xx} = (\cos x - 2)^2 u_{\xi\xi} + (\cos x + 2)^2 u_{\eta\eta} \\ \quad + 2(\cos x + 2)(\cos x - 2)u_{\xi\eta} - (\sin x)u_{\xi} - (\sin x)u_{\eta} \\ \quad = (\cos^2 x - 4\cos x + 4)u_{\xi\xi} + (\cos^2 x + 4\cos x + 4)u_{\eta\eta} \\ \quad + 2(\cos^2 x - 4)u_{\xi\eta} - (\sin x)u_{\xi} - (\sin x)u_{\eta} \\ u_{yy} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}, \\ u_{xy} = (\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}, \end{array} \right.$$

so the canonical form is

$$\begin{aligned} 0 &= u_{xx} - (2\cos x)u_{xy} - (3\sin^2 x)u_{yy} - yu_y \\ &= \xi^2 u_{\xi\xi} + \eta^2 u_{\eta\eta} \\ &\quad + 2\xi\eta u_{\xi\eta} - (\sin x)u_{\xi} - (\sin x)u_{\eta} \\ &\quad - (2\cos x)((\cos x - 2)u_{\xi\xi} + (\cos x + 2)u_{\eta\eta} + 2(\cos x)u_{\xi\eta}) \\ &\quad - (3\sin^2 x)(u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}) \\ &\quad - y(u_{\xi} + u_{\eta}) \end{aligned}$$

Who cares. ■

PROBLEM 3.1.5. Let  $Lu := u_{xx} - 4u_{yy} + \sin(y + 2x)u_x = 0$ .

- Consider the level curve  $\Gamma = \{(x, y) : \varphi(x, y) = C\}$  where  $|D\varphi| \neq 0$  on  $\Gamma$ . Define what it means for  $\Gamma$  to be characteristic with respect to  $L$  at a point  $(x_0, y_0) \in \Gamma$ .
- Find the points at which the curve  $x^2 + y^2 = 5$  is characteristic.

- (c) Is it true that every smooth simple closed curve  $\Gamma$  in  $\mathbf{R}^2$  has at least one point at which it is characteristic with respect to  $L$ ?

*SOLUTION.* ■

PROBLEM 3.1.6. Consider the second order equation

$$Lu := u_{xx} - 2xu_{xy} + x^2u_{yy} - 2u_y = 0.$$

- (a) Find the characteristic curves of  $Lu = 0$ . What is the type of this equation?  
 (b) Find the points on the line  $\Gamma := \{(x, y) \in \mathbf{R}^2 : x + y = 1\}$  at which  $\Gamma$  is characteristic with respect to  $Lu = 0$ .

*SOLUTION.* ■

PROBLEM 3.1.7. Solve the initial boundary value problem for the equation  $u_{tt} = u_{xx}$  in  $\{x > 0, t > 0\}$  satisfying

$$\begin{cases} u(x, 0) = \sin^2 x, & u_t(x, 0) = \sin x, \\ u(0, t) = 0. \end{cases}$$

*SOLUTION.* ■

PROBLEM 3.1.8. Consider the initial/boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } 0 < x < \pi, t > 0, \\ u(x, 0) = x, & u_t(x, 0) = 0 \quad \text{for } 0 < x < \pi, \\ u_x(0, t) = 0, & u_x(\pi, t) = 0 \quad \text{for } t > 0. \end{cases}$$

- (a) Find a weak solution of the problem.  
 (b) Is the solution unique? Continuous?  $C^1$ ?

*SOLUTION.* ■

PROBLEM 3.1.9. Let  $B_1^+$  denote the open half-ball  $\{x \in \mathbf{R}^n : |x| < 1, x_n > 0\}$ . Assume  $u \in C(\bar{B}_1^+)$  is harmonic in  $B_1^+$  with  $u = 0$  on  $\partial B_1^+ \cap \{x_n = 0\}$ . Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0, \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0, \end{cases}$$

for  $x \in B_1$ . Prove  $v$  is harmonic in  $B_1$ .

*Hint:* It will be enough to prove that  $\int_B \nabla v \nabla \eta \, dx = 0$  for any test function  $\eta \in C_0^\infty(B_1)$ . Split  $\int_{B_1} = \int_{B_1^+} + \int_{B_1^-}$  and apply the integration by parts formula to each of  $\int_{B_1^\pm}$ .

*SOLUTION.* ■

PROBLEM 3.1.10. Let  $u$  and  $v$  be harmonic functions in the unit ball  $B_1 \subseteq \mathbf{R}^n$ . What can you conclude about  $u$  and  $v$  if

- (a)  $D^\alpha u(0) = D^\alpha v(0)$  for every multiindex  $\alpha$ ?
- (b)  $u(x) \leq v(x)$  for every  $x \in B_1$  and  $u(0) = v(0)$ ?

Justify your answer in each case.

*SOLUTION.* ■

PROBLEM 3.1.11. Let  $\Phi$  be the fundamental solution of the Laplace equation in  $\mathbf{R}^n$  and  $f \in C_0^\infty(\mathbf{R}^n)$ . Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbf{R}^n} \Phi(x - y) f(y) dy$$

is a solution to the Poisson equation  $-\Delta u = f$  in  $\mathbf{R}^n$ . Show that if  $f$  is radial, i.e.,  $f(y) = f(|y|)$ , and supported in  $B_R := \{|x| < R\}$ , then

$$u(x) = c\Phi(x)$$

for any  $x \in \mathbf{R}^n \setminus B_R$ , where

$$c = \int_{\mathbf{R}^n} f(y) dy.$$

*Hint:* Use polar (spherical) coordinates and apply the mean value property for harmonic functions.

*SOLUTION.* ■



### 3.2 Final Practice Problems

PROBLEM 3.2.1. Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$  with smooth boundary. Show that the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u + \alpha \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

has at most one solution in  $C^2(\Omega) \cap C(\bar{\Omega})$  if  $\alpha > 0$ . Here  $\nu$  is the outward normal on  $\partial\Omega$  and  $f, g$  are assumed to be smooth.

*SOLUTION.* Let us assume that  $\Omega$  is also a connected subset of  $\mathbf{R}^n$ . We will use energy methods to show that there is only one solution to the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u + \alpha \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases} \quad (3.2.1)$$

Suppose  $u_1$  and  $u_2$  are two distinct solutions to the problem (3.2.1). Define  $v := u_1 - u_2$ . Then  $v$  solves the problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ v + \alpha \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the energy

$$E[v] = \frac{1}{2} \int_{\Omega} |Dv|^2 dx.$$

By Green's formula, we may recast the expression above as the sum

$$\begin{aligned} E[v] &= -\frac{1}{2} \left[ \int_{\Omega} v \Delta v dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} v dS(x) \right] \\ &= -\frac{\alpha}{2} \int_{\partial\Omega} v^2 dS(x) \\ &\geq 0. \end{aligned}$$

However, since  $\alpha > 0$  and  $v^2$  is strictly positive, it must be the case that  $v \equiv 0$  on  $\partial\Omega$ . The maximum principle then implies that  $v \equiv 0$  in  $\Omega$ . It follows that  $u_1 = u_2$ ; i.e., the solution is unique. ■

PROBLEM 3.2.2. Let  $g$  be a continuous function with compact support in  $\mathbf{R}^n$ . Write the formula for the bounded solution of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbf{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbf{R}^n \times \{t = 0\}. \end{cases}$$

Prove that  $\lim_{t \rightarrow \infty} u(x, t) = 0$ , where the convergence is uniform in  $x \in \mathbf{R}^n$ .

*SOLUTION.* From previous work on the heat equation, we know that the convolution

$$u(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy$$

the initial-value problem above. A crude estimate on the magnitude of  $u$  gives us

$$\begin{aligned} |u(x, t)| &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left| \int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy \right| \\ &\leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbf{R}^n} \left| e^{-\frac{|x-y|^2}{4t}} g(y) \right| dy \\ &< M t^{-\frac{n}{2}}; \end{aligned}$$

where  $M < \infty$  is chosen such that

$$\frac{1}{(4\pi)^{\frac{n}{2}}} \int_{\text{Supp } g} \left| e^{-\frac{|x-y|^2}{4t}} g(y) \right| dy < M.$$

Having established this estimate, we have  $\lim_{t \rightarrow \infty} u(x, t) = 0$  uniformly. ■

PROBLEM 3.2.3. Find an explicit solution to the problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{on } \mathbf{R} \times (0, \infty), \\ u = e^{3x} & \text{in } \mathbf{R} \times \{t = 0\}. \end{cases}$$

*SOLUTION.* By separation of variables, suppose we can write  $u(x, t)$  as the product  $X(x)T(t)$ . Then

$$\begin{cases} X(x)T'(t) - X''(x)T(t) = 0 & \text{in } \mathbf{R} \times (0, \infty), \\ X(x)T(0) = e^{3x} & \text{on } \mathbf{R} \times \{t = 0\}. \end{cases}$$

Then, it suffices to solve the system of ODEs

$$\begin{cases} X''(x) - 9X(x) = 0, \\ T'(t) - 9T(t) = 0 \end{cases}$$

which satisfy  $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = 9$ . The solution to these ODEs are

$$\begin{cases} X(x) = C_1 e^{3x}, \\ T(t) = C_2 e^{9t}. \end{cases}$$

Then,

$$u(x, t) = X(x)T(t) = C e^{3x+9t}.$$

Analyzing the initial conditions, we see  $C = 1$ .

Another way to solve this problem is by computing the convolution

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-\frac{|x-y|^2}{4t}} e^{3y} dy.$$

Putting this through WolframAlpha gives

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \left[ \sqrt{4\pi t} e^{9t+3x} \right] = e^{9t+3x}$$

which agrees with our ‘separation of variables’ solution. ■

PROBLEM 3.2.4. Find a formula for the solution of

$$\begin{cases} u_{tt} - u_{xx} + u = 0 & \text{in } \mathbf{R} \times (0, \infty), \\ u = f, \quad u_t = g & \text{on } \mathbf{R} \times \{t = 0\} \end{cases}$$

where  $f, g \in C_0^\infty(\mathbf{R})$ .

*Hint:* Method I: Use Hadamard’s method of descent. Namely, find  $h(y)$  such that  $v(x, y, t) := h(y)u(x, t)$  solves

$$v_{tt} - (v_{xx} + v_{yy}) = 0.$$

Method II: Use the Fourier transform.

*SOLUTION.* ■

PROBLEM 3.2.5. Let  $u \in C^2(\mathbf{R}^n \times [0, \infty))$  satisfy

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbf{R}^n \times (0, \infty), \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x). \end{cases}$$

Show that if both  $g$  and  $h$  are radial, then so is  $u(\cdot, t)$  for any  $t > 0$ . (Recall that the function  $f$  is called radial if  $f(x) = f(|x|)$ .)

*SOLUTION.* ■

PROBLEM 3.2.6. Find the value of the solution  $u$  of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{for } x \in \mathbf{R}^3, t > 0, \\ u(x, 0) = 0, u_t(x, 0) = \varphi(x), \end{cases}$$

where

$$\varphi(x) := \begin{cases} 1 & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a \end{cases}$$

at a point  $(x, t)$  such that  $|x| + t < a$ .

*SOLUTION.* ■

PROBLEM 3.2.7. Let  $u$  be a nonzero harmonic function in  $B(0, R) := \{x \in \mathbf{R}^n : |x| < R\}$ . Define

$$E(r) := \int_{\partial B(0, r)} u^2(y) d\sigma_y.$$

Show that  $\ln E(r)$  is a convex function of  $\ln r$ ; i.e.,

$$E(\sqrt{ab})^2 \leq E(a)E(b), \quad \text{for } a, b > 0,$$

for any  $0 < a \leq c < R$ .

(Hint: Use the representation of  $u$  as a uniformly convergent series

$$u(x) = \sum_{k=0}^{\infty} p_k(x), \quad |x| < R,$$

where  $p_k(x)$  is a homogeneous harmonic polynomial of order  $k$ .)

SOLUTION. ■

PROBLEM 3.2.8. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution to the following Cauchy problem,

$$\begin{cases} u_{tt} - \Delta u = e^{-t} f(x) & x \in \mathbf{R}^3, t > 0, \\ u(x, 0) = u_t(x, 0) = 0, & x \in \mathbf{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution  $u = e^{-t} + t - 1$  when  $f(x) = 1$ .

SOLUTION. ■

PROBLEM 3.2.9. Let  $f(x) = e^{-|x|^2}$ ,  $x \in \mathbf{R}^n$ . Find  $f * f$ .

(Hint: Use either the heat equation or the Fourier transform.)

SOLUTION. ■

PROBLEM 3.2.10. Recall that a solution to the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbf{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbf{R}^n \times \{t = 0\} \end{cases}$$

is given by

$$u(x, t) = \int_{\mathbf{R}^n} \Phi(x - y, t) g(y) dy,$$

where, for  $t > 0$ ,

$$\Phi(z, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|z|^2}{4t}}.$$

Assume that  $g$  is continuous and compactly supported. Show that there exists a  $C > 0$  such that

$$|Du(x, t)| \leq C t^{-\frac{1}{2}} \|g\|_{L^\infty}.$$

SOLUTION. ■

## 4 Qualifying Exams

### 4.1 Qualifying Exam, August '04

PROBLEM 4.1.1. Consider the initial value problem

$$\begin{cases} a(x, y)u_x + b(x, y)u_y = -u, \\ u = f \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\},$$

where  $a$  and  $b$  satisfy

$$a(x, y) + b(x, y)y > 0$$

for any  $x, y \in \mathbf{R}^n \setminus \{(0, 0)\}$ .

- (a) Show that the initial value problem has a unique solution in a neighborhood of  $S^1$ . Assume that  $a$ ,  $b$ , and  $f$  are smooth.
- (b) Show that the solution of the initial value problem actually exists in  $\mathbf{R}^2 \setminus \{(0, 0)\}$ .

*SOLUTION.* ■

PROBLEM 4.1.2. Let  $u \in C^2(\mathbf{R} \times [0, \infty))$  be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbf{R} \times (0, \infty), \\ u = f, \quad u_t = g & \text{in } \mathbf{R} \times 0, \end{cases}$$

where  $f$  and  $g$  have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dx,$$

and the potential energy by

$$P(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) dx.$$

Show that

- (a)  $K(t) + P(t)$  is constant in  $t$ ,
- (b)  $K(t) = P(t)$  for all large enough times  $t$ .

*SOLUTION.* ■

PROBLEM 4.1.3. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = e^{-t}g(x) & \text{for } x \in \mathbf{R}^3, t > 0, \\ u(x, 0) = u_t(x, 0) = 0 & \text{for } x \in \mathbf{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution  $u = e^{-t} + t - 1$  when  $g(x) = 1$ .

*SOLUTION.* ■

PROBLEM 4.1.4. Let  $\Omega$  be a bounded open set in  $\mathbf{R}^n$  and  $g \in C_0^\infty(\Omega)$ . Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega, t > 0, \\ u(x, 0) = g(x) & \text{for } x \in \Omega, \\ u(x, t) = 0 & \text{for } x \in \partial\Omega, t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbf{R}^n, t > 0, \\ v(x, 0) = |g(x)| & \text{for } x \in \mathbf{R}^n, \end{cases}$$

where we put  $g = 0$  outside  $\Omega$ .

(a) Show that

$$-v(x, t) \leq u(x, t) \leq v(x, t)$$

for any  $x \in \Omega, t > 0$ .

(b) Use (a) to conclude that

$$\lim_{t \rightarrow \infty} u(x, t) = 0,$$

for any  $x \in \Omega$ .

*SOLUTION.* ■

PROBLEM 4.1.5. Let  $P_k(x)$  and  $P_m(x)$  be homogeneous harmonic polynomials in  $\mathbf{R}^n$  of degrees  $k$  and  $m$  respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \quad P_m(\lambda x) = \lambda^m P_m(x),$$

for any  $x \in \mathbf{R}^n, \lambda > 0$ ,

$$\Delta P_k = 0, \quad \Delta P_m = 0$$

in  $\mathbf{R}^n$ .

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = k P_k(x), \quad \frac{\partial P_m(x)}{\partial \nu} = m P_m(x)$$

on  $\partial B_1$ , where  $B_1 = \{ |x| < 1 \}$  and  $\nu$  is the outward normal on  $\partial B_1$ .

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x) dS = 0,$$

if  $k \neq m$ .

*SOLUTION.* ■

## 4.2 Qualifying Exam, August '05

PROBLEM 4.2.1.

- (a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ u = 1 \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\}.$$

- (b) Is the solution unique in a neighborhood of the point  $(1, 0)$ ? Justify your answer.

*SOLUTION.* The solution to the first part is

$$u(x, y) = \frac{x^2 + y^2}{4} + \frac{3}{4}.$$

■

PROBLEM 4.2.2. Consider the second order PDE in  $\{x > 0, y > 0\} \subseteq \mathbf{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

- (a) Classify the equation and reduce it to the canonical form.  
 (b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy}G(x/y).$$

*SOLUTION.*

■

PROBLEM 4.2.3. Let  $\Phi$  be the fundamental solution of the Laplace equation in  $\mathbf{R}^3$  and  $f \in C_0^\infty(\mathbf{R}^n)$ . Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbf{R}^n} \Phi(x - y) f(y) dy$$

is a solution of the Poisson equation  $-\Delta u = f$  in  $\mathbf{R}^n$ . Show that if  $f$  is radial (i.e.,  $f(y) = f(|y|)$ ) and supported in  $B_R = \{|x| < R\}$ , then

$$u(x) = c\Phi(x),$$

for any  $x \in \mathbf{R}^n \setminus B_R$ , where

$$c = \int_{\mathbf{R}^n} f(y) dy.$$

[Hint: Use spherical (polar) coordinates and the mean value property.]

*SOLUTION.*

■

PROBLEM 4.2.4. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbf{R}^2 \times (0, \infty), \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{for } x \in \mathbf{R}^2, \end{cases}$$

where  $g, h \in C_0^\infty(\mathbf{R}^2)$  and  $\alpha \geq 0$  is a constant.

- (a) Show that for an appropriate choice of constant  $\lambda$  and  $\mu$  the function

$$v(x_1, x_2, x_3, t) := e^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation  $v_{tt} - \Delta v = 0$ .

- (b) Use (a) to prove the following domain of dependence result: for any point  $(x_0, t_0) \in \mathbf{R}^2 \times (0, \infty)$  the value  $u(x_0, t_0)$  is uniquely determined by values of  $g$  and  $h$  in  $\overline{B_{t_0}}(x_0) := \{|x - x_0| \leq t_0\}$ . (You may use the corresponding result for the wave equation without proof.)

*SOLUTION.* ■

PROBLEM 4.2.5. Let  $u(x, t)$  be a bounded solution of the heat equation  $u_t = u_{xx}$  in  $\mathbf{R} \times (0, \infty)$  with the initial condition

$$u(x, 0) = u_0(x)$$

for  $x \in \mathbf{R}$ , where  $u_0 \in C^\infty$  is  $2\pi$ -periodic, i.e.,  $u_0(x + 2\pi) = u_0(x)$ . Show that

$$\lim_{t \rightarrow \infty} u(x, t) = a_0,$$

uniformly in  $x \in \mathbf{R}$ , where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x) dx.$$

*SOLUTION.* ■



### 4.3 Qualifying Exam, January '14

PROBLEM 4.3.1. Consider the first order equation in  $\mathbf{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

- (a) Find the characteristic curves of the equation.
- (b) Consider the Cauchy problem for this equation prescribed on the line  $x_1 = 1$ :

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on  $f$  so that the problem is solvable in a neighborhood of the point  $(1, 0)$ .

*SOLUTION.* ■

PROBLEM 4.3.2. Let  $u$  be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{x_n > 0\}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbf{R}^{n-1}, \end{cases}$$

where  $g$  is a continuous function with compact support in  $\mathbf{R}^{n-1}$ . Here  $n \geq 2$ . Prove that

$$u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty,$$

for  $x \in \{x_n > 0\}$ .

*SOLUTION.* ■

PROBLEM 4.3.3. Let  $u$  be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \quad \text{in } \mathbf{R} \times (0, \infty),$$

with the initial conditions  $u(x, 0) = g(x)$ , where  $g$  is a bounded continuous function on  $\mathbf{R}$  satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^\alpha, \quad x, y \in \mathbf{R}$$

with a constant  $\alpha \in (0, 1]$ . Show that

$$\begin{aligned} |u(x, t) - u(y, t)| &\leq M|x - y|^\alpha, & x, y \in \mathbf{R}, t > 0, \\ |u(x, t) - u(x, s)| &\leq C_\alpha M|t - s|^{\alpha/2}, & x \in \mathbf{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of  $u(x, t)$  as a convolution with the heat kernel  $\Phi(y, t)$ , make a change of variables  $z = y/\sqrt{t}$  and use that  $|\sqrt{t} - \sqrt{s}| \leq \sqrt{|t - s|}$ .]

*SOLUTION.* ■

PROBLEM 4.3.4. Let  $u$  be a positive harmonic function in the unit ball  $B_1$  in  $\mathbf{R}^n$ . Show that

$$|D(\ln u)| \leq M \quad \text{in } B_{1/2}$$

for a constant  $M$  depending only on the dimension  $n$ .

[*Hint:* Use the interior derivative estimate  $|Du(x)| \leq (C_n/r) \sup_{B_r(x)} |u|$  for  $B_r(x) \subseteq B_1$  as well as the Harnack inequality for harmonic functions.]

*SOLUTION.* ■

PROBLEM 4.3.5. Let  $u$  be a  $C^2$  solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbf{R}^n \times (0, \infty), \\ u = 0, \quad u_t = 0 & \text{on } \mathbf{R}^n \times \{0\}. \end{cases}$$

for some  $k \geq 0$ . Prove that there exists a function  $\varphi(r)$  such that

$$u(x, t) = t^{k+2} \varphi(|x|/t).$$

[*Hint:* As one of the steps show that  $u$  is  $(k+2)$ -homogeneous in  $(x, t)$  variables, i.e.,  $u(\lambda x, \lambda t) = \lambda^{k+2} u(x, t)$  for any  $\lambda > 0$ .]

*SOLUTION.* ■