Math 527 - Homotopy Theory
Spring 2013
Homework 5 Solutions

Problem 1. Find a space X such that *no* choice of basepoint will make it well-pointed.

Solution. Consider the space $\mathbb{Q}$ of rational numbers, with its standard metric topology. For any point $q \in \mathbb{Q}$, there is a homeomorphism $\mathbb{Q} \xrightarrow{\cong} \mathbb{Q}$ sending q to 0, e.g. translation by $-q$. Therefore it suffices to show that the inclusion of some basepoint, say, $i\colon \{7\} \hookrightarrow \mathbb{Q}$ is not a cofibration.

Consider the mapping cyclinder $M(i) \cong \mathbb{Q} \times \{0\} \cup \{7\} \times I \subset \mathbb{Q} \times I$, with the natural maps $f\colon \mathbb{Q} \to M(i)$ and $H\colon \{7\} \times I \to M(i)$. Recall that $\mathbb{Q}$ is totally disconnected (i.e. connected components are all singletons), and the same argument shows that singletons $\{(q,0)\} \subset M(i)$ with $q \neq 7$ are connected components of $M(i)$. In particular, any path in $M(i)$ starting at $(q,0)$ must be constant. Therefore, any homotopy G from f to a map $g\colon \mathbb{Q} \to M(i)$ must satisfy

$$G(q,t) = G(q,0) = f(q) = (q,0)$$

for all $t \in I$ and all $q \neq 7$. At time $t = 1$, the resulting map $g\colon \mathbb{Q} \to M(i)$ satisfies $g(q) = (q,0)$ for all $q \neq 7$. By continuity, it also satisfies

$$g(7) = (7,0) \neq (7,1) = H(7,1).$$

Therefore the homotopy $H\colon \{7\} \times I \to M(i)$ cannot be extended to a homotopy $\widetilde{H}\colon \mathbb{Q} \times I \to M(i)$. $\square$

Problem 2. (May § 9.4 Lemma) Show that for all $n \geq 0$, the functor $\pi_n \colon \mathbf{Top}_* \to \mathbf{Set}_*$ preserves products. In other words, for all pointed spaces X and Y, there is a natural isomorphism

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y).$$

Solution. Recall that the Cartesian product is the product in $\mathbf{Top}_*$

$$(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0))$$

as well as in the homotopy category $\mathrm{Ho}(\mathbf{Top}_*)$. Therefore the natural map

$$\pi_n(X \times Y) = [S^n, X \times Y]_* \xrightarrow{\cong} [S^n, X]_* \times [S^n, Y]_* = \pi_n(X) \times \pi_n(Y)$$

is an isomorphism (in $\mathbf{Set}_*$). $\square$

Remark. This argument works for arbitrary products, not just finite:

$$\varphi \colon \pi_n\left(\prod_\alpha X_\alpha\right) \xrightarrow{\cong} \prod_\alpha \pi_n(X_\alpha).$$

Moreover, the isomorphism φ is an isomorphism of groups, since each of its coordinates

$$\pi_n\left(\prod_\alpha X_\alpha\right) \xrightarrow{\varphi} \prod_\alpha \pi_n(X_\alpha) \twoheadrightarrow \pi_n(X_\beta)$$

is a group homomorphism, namely $\pi_n(p_\beta)$ induced by the projection $p_\beta \colon \prod_\alpha X_\alpha \twoheadrightarrow X_\beta$.

Problem 3. (May § 9.6 Problem 1) Let X and Y be pointed spaces, and $n \geq 2$.

a. Show that the map $j_* \colon \pi_n(X \times Y) \to \pi_n(X \times Y, X \vee Y)$ is zero.

Solution. Consider the natural isomorphisms

$$\pi_n(X \times Y) \xrightarrow[\cong]{(p_{X*}, p_{Y*})} \pi_n(X) \times \pi_n(Y) \xleftarrow[\cong]{} \pi_n(X) \oplus \pi_n(Y)$$

where the last step comes from the fact that $\pi_n(X)$ and $\pi_n(Y)$ are abelian groups ($n \geq 2$). One readily checks that the inverse isomorphism is

$$\pi_n(X) \oplus \pi_n(Y) \xrightarrow[\cong]{(\iota_{X*}, \iota_{Y*})} \pi_n(X \times Y)$$

where $\iota_X \colon X \to X \times Y$ is the "slice inclusion" $\iota_X(x) = (x, y_0)$ and likewise for ι_Y. Therefore, any element $\theta \in \pi_n(X \times Y)$ can be (uniquely) written as a sum $\theta = \theta_X + \theta_Y$ with $\theta_X \in \operatorname{im} \iota_{X*}$ and $\theta_Y \in \operatorname{im} \iota_{Y*}$.

Any element $\theta_X \in \operatorname{im} \iota_{X*}$ is represented by a map $D^n \to X \times Y$ whose image is contained in $X \times \{y_0\} \subseteq X \vee Y$, which implies $j_*(\theta_X) = 0 \in \pi_n(X \times Y, X \vee Y)$, and likewise $j_*(\theta_Y) = 0$. We conclude

$$\begin{aligned} j_*(\theta) &= j_*(\theta_X + \theta_Y) \\ &= j_*(\theta_X) + j_*(\theta_Y) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

using the fact that $\pi_n(X \times Y, X \vee Y)$ is a group (as $n \geq 2$) and j_* is a group homomorphism. $\square$

b. Show that there is an isomorphism

$$\pi_n(X \vee Y) \simeq \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$

Solution. Note that the slice inclusions ι_X and ι_Y factor through the subspace $X \vee Y \subseteq X \times Y$:

$$\begin{tikzcd} X \arrow[r, hook, "\iota_X"] \arrow[rr, bend right, "\iota_X"'] & X \vee Y \arrow[r, hook, "i"] & X \times Y. \end{tikzcd}$$

This provides a section of the map i_* as follows:

$$\begin{tikzcd} \pi_n(X \vee Y) \arrow[r, "i_*"] & \pi_n(X \times Y) \\ & \pi_n(X) \oplus \pi_n(Y). \arrow[ul, "(\iota_{X*}, \iota_{Y*})"] \arrow[u, "(\iota_{X*}, \iota_{Y*})"', "\cong"] \end{tikzcd}$$

Hence the long exact sequence

$$\begin{tikzcd} \pi_{n+1}(X \times Y) \arrow[r, "j_*", "0"'] & \pi_{n+1}(X \times Y, X \vee Y) \arrow[r, "\partial"] & \pi_n(X \vee Y) \arrow[r, "i_*"] & \pi_n(X \times Y) \arrow[r, "j_*", "0"'] \arrow[l, dashed, bend left] & \pi_n(X \times Y, X \vee Y) \end{tikzcd}$$

breaks down to a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_{n+1}(X \times Y, X \vee Y) \arrow[r, "\partial"] & \pi_n(X \vee Y) \arrow[r, "i_*"] & \pi_n(X \times Y) \arrow[r] \arrow[l, dashed, bend left] & 0 \end{tikzcd}$$

which yields the isomorphism

$$\begin{aligned} \pi_n(X \vee Y) &\cong \pi_n(X \times Y) \oplus \pi_{n+1}(X \times Y, X \vee Y) \\ &\cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y). \quad \square \end{aligned}$$

Problem 4. (May § 9.4 Lemma) Let $n \geq 2$ and consider the n-dimensional real projective space $\mathbb{R}P^n$. Show that the following holds: $\pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}/2$ and $\pi_k(\mathbb{R}P^n) \simeq \pi_k(S^n)$ for all $k \geq 2$.

Solution. Recall that $\mathbb{R}P^n$ is obtained as the quotient

$$\mathbb{R}P^n = S^n/O(1)$$

of the sphere by the free action of the group $O(1) = \{-1, 1\} \subset \mathbb{R}^\times$. The quotient map $p\colon S^n \twoheadrightarrow \mathbb{R}P^n$ is thus a two-sheeted covering, and is in fact the universal cover of $\mathbb{R}P^n$, since S^n is simply-connected for $n \geq 2$. Therefore $\pi_1(\mathbb{R}P^n)$ is isomorphic to $O(1) \simeq \mathbb{Z}/2$.

For $k \geq 2$, the covering map p induces an isomorphism $p_*\colon \pi_k(S^n) \xrightarrow{\simeq} \pi_k(\mathbb{R}P^n)$. □

Problem 5. (May § 9.6 Problem 2) Let $n \geq 3$.

a. Compute the group $\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1})$.

Solution. The standard inclusion $i\colon \mathbb{R}P^{n-1} \to \mathbb{R}P^n$ lifts to universal covers to the standard inclusion $\widetilde{i}\colon S^{n-1} \to S^n$, as illustrated in the commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "\widetilde{i}"] \arrow[d, "p"'] & S^n \arrow[d, "p"] \\ \mathbb{R}P^{n-1} \arrow[r, "i"] & \mathbb{R}P^n. \end{tikzcd}$$

Note that $\widetilde{i}\colon S^{n-1} \to S^n$ is null-homotopic, as $\pi_{n-1}(S^n) = 0$. For any $k \geq 2$, applying π_k to this diagram yields

$$\begin{tikzcd} \pi_k(S^{n-1}) \arrow[r, "\widetilde{i}_* = 0"] \arrow[d, "p_*"', "\simeq"] & \pi_k(S^n) \arrow[d, "p_*", "\simeq"'] \\ \pi_k(\mathbb{R}P^{n-1}) \arrow[r, "i_*"] & \pi_k(\mathbb{R}P^n) \end{tikzcd}$$

from which we conclude $i_* = 0$. Hence the long exact sequence

$$\pi_n(\mathbb{R}P^{n-1}) \xrightarrow[0]{i_*} \pi_n(\mathbb{R}P^n) \xrightarrow{j_*} \pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \xrightarrow{\partial} \pi_{n-1}(\mathbb{R}P^{n-1}) \xrightarrow[0]{i_*} \pi_{n-1}(\mathbb{R}P^n)$$

breaks down to a short exact sequence

$$0 \longrightarrow \pi_n(\mathbb{R}P^n) \xrightarrow{j_*} \pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \xrightarrow{\partial} \pi_{n-1}(\mathbb{R}P^{n-1}) \longrightarrow 0$$

which is automatically split, since $\pi_{n-1}(\mathbb{R}P^{n-1}) \simeq \mathbb{Z}$ is a projective $\mathbb{Z}$-module. This yields the isomorphism

$$\begin{aligned} \pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) &\simeq \pi_n(\mathbb{R}P^n) \oplus \pi_{n-1}(\mathbb{R}P^{n-1}) \\ &\cong \pi_n(S^n) \oplus \pi_{n-1}(S^{n-1}) \\ &\cong \mathbb{Z} \oplus \mathbb{Z}. \quad \square \end{aligned}$$

b. Deduce that the quotient map of pairs

$$q\colon (\mathbb{R}P^n, \mathbb{R}P^{n-1}) \to (\mathbb{R}P^n/\mathbb{R}P^{n-1}, *)$$

does not induce an isomorphism on homotopy groups.

Solution. Recall that the standard CW-structure on $\mathbb{R}P^n$ has one cell in each dimension $0, 1, \ldots, n$ and $\mathbb{R}P^{n-1}$ is the $(n-1)$-skeleton of $\mathbb{R}P^n$. Therefore we have a homeomorphism

$$\mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n.$$

The quotient map q cannot induce an isomorphism on the n^{th} relative homotopy groups, as the two groups are non-isomorphic:

$$\begin{aligned}
&\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \simeq \mathbb{Z} \oplus \mathbb{Z} \\
&\pi_n(\mathbb{R}P^n/\mathbb{R}P^{n-1}, *) \cong \pi_n(S^n, *) \cong \mathbb{Z} \not\cong \mathbb{Z} \oplus \mathbb{Z}. \quad \square
\end{aligned}$$