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IRREDUCIBLE OUTER AUTOMORPHISMS OF A FREE GROUP

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ABSTRACT. We give sufficient conditions for all positive powers of an outer automorphism of a finitely generated free group to be irreducible, in the sense of Bestvina and Handel. We prove a conjecture of Stallings (1982), that a PV automorphism in rank ≥ 3 has no nontrivial fixed points.

1. INTRODUCTION

A map of finite graphs $f: G \to G$ will be required to take vertices to vertices and map each edge to a path in G (thus $f^{-1}(V)$ is not required to be V, where V is the vertex set of G). If G is connected, and if f is a homotopy equivalence with $f|E$ an immersion for each edge E of G, then f is said to be a *topological realization* of the outer automorphism $\mathscr{O}(f)$ of $\pi_1(G, x)$ (for $x \in V$) determined by $\alpha \mapsto p \circ f(\alpha) \circ p^{-1}$; here p is a path from x to $f(x)$ and α is a loop based at x. It is readily checked that $\mathscr{O}(f)$ is independent of the choices of x and p.

An outer automorphism $\mathscr{O}$ of a finitely generated free group is said to be *reducible* (in the sense of Bestvina–Handel [BH]) if $\mathscr{O} = \mathscr{O}(f)$ where $f: G \to G$ is a topological realization such that (a) G has no vertices of valence one, (b) no f-invariant forest of G contains an edge, and (c) there exists an f-invariant subgraph G_0 of G (with G_0 not necessarily connected) so that $G_0 \neq G$ and so that $\pi_1(G_0, x) \neq \{1\}$ for some vertex x of G_0. If $\mathscr{O}$ is not reducible it is called irreducible. It is an easy matter to check whether $\mathscr{O}$ is reducible; one need only produce one graph G as in the definition. However it is by no means an easy matter to check irreducibility. We shall describe some classes of irreducible automorphisms and apply thereto a beautiful result [BH, Theorem 4.1]:

Theorem. *Suppose $\mathscr{O}$ is an outer automorphism of a finitely generated free group F such that every power $\mathscr{O}^k$, $k \geq 1$, is irreducible. If there is a nontrivial cyclic word $s \in F$ such that either $\mathscr{O}(s) = s$ or $\mathscr{O}(s) = s^{-1}$, then $\mathscr{O}$ is geometric.*

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That $\mathscr{O}$ is *geometric* means that it is the homomorphism induced on fundamental group by an autohomeomorphism of a surface. In the conclusion of the above theorem, the surface involved has exactly one boundary component. (The statement in [BH] carries the additional hypothesis that the Frobenius–Peron eigenvalue λ associated to a train track topological representative for $\mathscr{O}$ should satisfy $\lambda > 1$; however if $\lambda = 1$, the train track map is of finite order, and so if the rank of F is greater than one, $\mathscr{O}^k$ must be reducible for some $k \geq 1$.)

An outer automorphism $\mathscr{O}$ of a free group F_n of rank n determines an automorphism $\mathscr{O}_{ab}$ of the abelianization $(F_n)_{ab} \cong \mathbf{Z}^n$; if we choose a basis for $\mathbf{Z}^n$ we obtain a matrix representation for $\mathscr{O}_{ab}$. Let $\operatorname{char}\mathscr{O}$ denote the characteristic polynomial of $\mathscr{O}_{ab}$.

Recall that $p(x) \in \mathbf{Z}[x]$ is called a PV-polynomial if it is monic and if it has precisely one root λ (counted with multiplicity) with $|\lambda| > 1$, whereas for all other roots μ, $|\mu| < 1$. (The large root λ is a Pisot–Vijayaraghavan number.)

In what follows, a polynomial is said to be *irreducible* if it is irreducible over the rationals. We shall show:

Corollary 2.6. *If* $\operatorname{char}\mathscr{O}$ *is a PV-polynomial, then $\mathscr{O}^k$, $k \geq 1$, is irreducible.*

If M is a square matrix of real numbers, say M is *primitive* if each entry of M is nonnegative and if $\exists N \geq 1$ such that every entry of M^N is positive.

Corollary 2.7. *If* $\operatorname{char}\mathscr{O}$ *is irreducible and if some matrix representation for $\mathscr{O}_{ab}$ is primitive, then, for all $k \geq 1$, $\mathscr{O}^k$ is irreducible.*

Given an automorphism $\phi : F \to F$, we define the fixed subgroup, $\operatorname{Fix}(\phi) = \{w \in F | \phi(w) = w\}$. Finally, we settle a conjecture of [St]:

Corollary 2.8. *If ϕ is an automorphism of a finitely generated free group of rank ≥ 3, such that* $\operatorname{char}\phi$ *is a PV-polynomial, then* $\operatorname{Fix}(\phi) = \{1\}$.

2. The details

Let $p(x) \in \mathbf{Q}[x]$ be a monic polynomial of degree n with rational coefficients, and with complex roots $\lambda_1, \ldots, \lambda_n$. Thus, $p(x) = \prod_1^n (x - \lambda_i)$. We say that $p(x)$ is *super irreducible*, when, for every integer $k \geq 1$, the polynomial $p_k(x) = \prod_1^n (x - \lambda_i^k)$ is irreducible in $\mathbf{Q}[x]$. We say that an $n \times n$ matrix M over $\mathbf{Q}$ is *super irreducible*, when its characteristic polynomial $\operatorname{char} M$ is super irreducible; equivalently, as a linear transformation $M : \mathbf{Q}^n \to \mathbf{Q}^n$, no positive power M^k maps a proper subspace of $\mathbf{Q}^n$ into itself.

2.1. **Theorem.** *A monic irreducible polynomial $p(x) \in \mathbf{Q}[x]$ with roots $\lambda_1, \ldots, \lambda_n$ is super irreducible if and only if, for all i, j with $1 \leq i < j \leq n$, the quotient $\lambda_i \cdot \lambda_j^{-1}$ is not a root of unity.*

Proof. Let Γ denote the Galois group of the polynomial $p(x)$ over $\mathbf{Q}$. Since $p(x)$ is irreducible, Γ acts transitively on the roots $\{\lambda_i\}$. Therefore Γ acts transitively on the kth powers of these roots, which are the roots of $p_k(x)$. Thus, $p_k(x)$ is irreducible, if and only if, whenever $i \neq j$, then $\lambda_i^k \neq \lambda_j^k$; and this occurs if and only if $\lambda_i \cdot \lambda_j^{-1}$ is not a kth root of unity.

2.2. **Theorem.** *If the monic polynomial $p(x) \in \mathbf{Q}[x]$ with roots $\{\lambda_i\}$ is irreducible, and if, for all $i \neq 1$, $|\lambda_1| > |\lambda_i|$, then $p(x)$ is super irreducible.*

Proof. As before, the Galois group Γ acts transitively on the roots, so that for any $i \neq j$, there is $\sigma \in \Gamma$ such that $\sigma(\lambda_i) = \lambda_1$, and $\sigma(\lambda_j) = \lambda_r$ for $r \neq 1$. Then

$$|\sigma(\lambda_i \cdot \lambda_j^{-1})| = |\lambda_1 \cdot \lambda_r^{-1}| > 1,$$

so that $\sigma(\lambda_i \cdot \lambda_j^{-1})$ cannot be a root of unity, and therefore $\lambda_i \cdot \lambda_j^{-1}$ cannot be a root of unity. Thus Theorem 2.1 applies.

2.3. **Corollary.** *If $p(x)$ is a PV-polynomial, then $p(x)$ is super irreducible.*

Proof. $p(x)$ is a monic integer polynomial having a unique root of absolute value greater than 1, the other roots having absolute value less than 1. Since any monic integer polynomial has at least one root of absolute value at least 1, the polynomial $p(x)$ must be irreducible. Theorem 2.2 immediately applies to show that $p(x)$ is super irreducible.

2.4. **Corollary.** *If M is a primitive matrix of rational numbers with* $\operatorname{char} M$ *irreducible, then* $\operatorname{char} M$ *is super irreducible.*

Proof. From Perron's theorem [Ga, p. 53] one knows that M possesses a positive real eigenvalue λ of multiplicity one such that all other eigenvalues μ of M satisfy $|\mu| < \lambda$. With the assumption that $\operatorname{char} M$ is irreducible, Theorem 2.2 then applies to show that $\operatorname{char} M$ is super irreducible.

Recall that if $\mathcal{O}$ is an outer automorphism of a finitely generated free group F, then $\operatorname{char}\mathcal{O}$ is the characteristic polynomial of the abelianization of $\mathcal{O}$.

2.5. **Theorem.** *If* $\operatorname{char}\mathcal{O}$ *is super irreducible, then for every integer $k \geq 1$, the power $\mathcal{O}^k$ is an irreducible outer automorphism (in the sense of Bestvina and Handel).*

Proof. By assumption, $\operatorname{char}\mathcal{O}^k$ is irreducible for all $k \geq 1$. Suppose that $\mathcal{O}$ were reducible in the sense of Bestvina and Handel. Then there exists a topological representative $f : G \to G$ for $\mathcal{O} = \mathcal{O}(f)$, with G a connected graph having no vertex of valence 1, having only subsets of vertices as f-invariant forests, and possessing a subgraph $G_0 \neq G$ with $f(G_0) \subseteq G_0$ and $\pi_1(G_0, x) \neq \{1\}$ for some vertex x. If G_0' is the connected component of G_0 containing x, there exist $l, k > 0$ such that $f^{k+l}(G_0') \subseteq f^l(G_0')$; in fact, we can choose k and l so that f^k fixes a vertex $f^l(x)$ of G_0. Then $\pi_1(f^l(G_0'), f^l(x))$ is a proper free factor of $\pi_1(G, f^l(x))$ which is invariant under f_*^k. It follows that the matrix for $(f_*^k)_{ab}$ can be written in block form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

In particular char$\mathcal{O}^k$ would be reducible; this contradicts the assumption that char$\mathcal{O}$ is super irreducible. We can apply the same argument to $\mathcal{O}^r$ for any $r \geq 1$ to show that $\mathcal{O}^r$ is irreducible.

2.6. **Corollary.** *If* char$\mathcal{O}$ *is a PV-polynomial, then* $\mathcal{O}^k$, *for all* $k \geq 1$, *is irreducible.*

Proof. This follows from Corollary 2.3 and Theorem 2.5.

2.7. **Corollary.** *If* char$\mathcal{O}$ *is irreducible and if some matrix representation for* $\mathcal{O}_{ab}$ *is primitive, then, for all* $k \geq 1$, $\mathcal{O}^k$ *is irreducible.*

Proof. Corollary 2.4 plus Theorem 2.5.

2.8. **Corollary.** *If* ϕ *is an automorphism of a finitely generated free group* F_n *of rank* $n \geq 3$, *such that* charϕ *is a PV- polynomial, then* $\mathrm{Fix}(\phi) = \{1\}$.

Proof. By Corollary 2.6, if $\mathcal{O}$ is the outer automorphism represented by ϕ, then all positive powers $\mathcal{O}^k$ are irreducible in the sense of Bestvina and Handel. If $\mathrm{Fix}(\phi)$ were nontrivial, there would exist $s \in F_n$ with $s \neq 1$ such that $\phi(s) = s$. By Theorem 4.1 of [BH], quoted in the introduction, $\mathcal{O}$ would be geometric, induced by a homeomorphism of a compact surface with one boundary component. However, in [St] it is proved that if the rank of F_n is at least 3, then a PV outer automorphism is not induced by a homeomorphism of any bounded compact surface, regardless of the number of components of the boundary. This contradiction shows that $\mathrm{Fix}(\phi) = \{1\}$.

2.9. **Corollary.** *If* ϕ *is a PV-automorphism of a finitely generated free group of rank* ≥ 3, *then* ϕ *has no nontrivial recurrent words.*

Proof. The set of recurrent words is by definition $\bigcup_{k\geq 1} \mathrm{Fix}(\phi^k)$. But $\mathrm{Fix}(\phi^k) = \{1\}$ since ϕ^k is PV.

3. Remarks

3.1. **Example.** In Theorem 2.5 we need the hypothesis that for all $k \geq 1$ the polynomial char$\mathcal{O}^k$ is irreducible. The hypothesis that char$\mathcal{O}$ alone is irreducible does not imply that $\mathcal{O}$ is irreducible, as the following example shows. Let G be the graph with two vertices P and Q, and three edges A, B, and C; suppose that C joins P to Q, that A joins P to P, and that B joins Q to Q; thus G is a "pince-nez" graph, with two circles A and B and a midbar C. Let $f: G \to G$ be given by $P \mapsto Q$, $Q \mapsto P$, $A \mapsto B$, $B \mapsto \overline{A}$, $C \mapsto B\overline{C}A$.

Observe that there are no vertices of valence one and that $\{P, Q\}$ is the only f-invariant forest. However $G_0 = \{A, B, P, Q\}$ is an f-invariant subgraph. Thus $\mathscr{O}(f)$ is reducible. We can calculate $\operatorname{char}\mathscr{O}(f)$ with respect to the homology basis $\alpha = [A]$, $\beta = [B]$ to get $\operatorname{char}\mathscr{O}(f) = x^2 + 1$, which is irreducible.

3.2. **Example.** The characteristic polynomial for the matrix M given by

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

is PV, [Ge, p. 87]. It follows that if $\mathscr{O}$ is an outer automorphism such that $\mathscr{O}_{ab}$ has M as a matrix representation, then $\mathscr{O}^k$ is irreducible for all $k \geq 1$.

3.3. **Example.** Consider the 4×4 integral matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & b_1 & b_2 & b_3 \end{pmatrix}.$$

This matrix is primitive if all $b_i > 0$. The characteristic polynomial is $\operatorname{char} M = x^4 - b_3x^3 - b_2x^2 - b_1x - 1$. We shall determine when $\operatorname{char} M$ is irreducible and hence when Corollary 2.7 applies to an outer automorphism $\mathscr{O}$ with $\operatorname{char}\mathscr{O} = \operatorname{char} M$. The only possible linear factor of $\operatorname{char} M$ is $x + 1$ (since $+1$ is not a root) and this leads to the equation $b_2 = b_1 + b_3$. A factorization with quadratic factors is of the form

$$\operatorname{char} M = (x^2 + \alpha x + \beta)(x^2 + \gamma x + \delta).$$

This leads to equations

$$\begin{aligned} \alpha + \gamma &= -b_3, \\ \alpha\gamma + \beta + \delta &= -b_2, \\ \alpha\delta + \beta\gamma &= -b_1, \\ \beta\delta &= -1. \end{aligned}$$

Since $\beta + \delta = 0$ these equations simplify and we get $b_1^2 = b_3^2 + 4b_2$. Thus $\operatorname{char} M$ is irreducible iff both $b_2 \neq b_1 + b_3$ and $b_1^2 \neq b_3^2 + 4b_2$.

In the easier case of the matrix

$$M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & b_1 & b_2 \end{pmatrix}$$

where b_1 and b_2 are positive integers, $\operatorname{char} M_1$ is irreducible iff $b_1 \neq b_2 + 2$. In this case, either $\operatorname{char} M_1$ or $\operatorname{char} M_1^{-1}$ is PV.

3.4. **Question.** Finally we would like to raise a question about the split extension $E = F \times_{\mathscr{O}} \mathbf{Z}$, where F is a finitely generated free group and $\mathscr{O}$ is an outer automorphism of F. If for some $k \geq 1$, $\mathscr{O}^k$ fixed a nontrivial cyclic word, then E would possess a free abelian subgroup of rank 2 and hence E would not be word hyperbolic in the sense of Gromov [Gr]. A bold conjecture would be: If for all $k \geq 1$, $\mathscr{O}^k$ is irreducible and fixes no nontrivial cyclic word, then the split extension E is word hyperbolic.

Perhaps one should first try to see if this is true when $\mathscr{O}$ is PV.

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