MA557 Problem Set 2

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Problem 2.1

Let $\mathfrak{a}$ be an R-ideal and M a finite R-module. Show that

$$\sqrt{\operatorname{ann}(M/\mathfrak{a}M)} = \sqrt{\operatorname{ann}(M) + \mathfrak{a}}.$$

Proof. One inclusion is immediate, namely,

$$\sqrt{\operatorname{ann}(M/\mathfrak{a}M)} \subset \sqrt{\operatorname{ann}(M) + \mathfrak{a}}$$

since $x \in \sqrt{\operatorname{ann}(M/\mathfrak{a}M)}$ if $x^n \in \operatorname{ann}(M/\mathfrak{a}M)$ if $x^n M \subset \mathfrak{a}M$, i.e., $x^n m = \sum y_i m_i$ for $y_i \in \mathfrak{a}$, $m_i \in M$. But $x' \in \sqrt{\operatorname{ann}(M) + \mathfrak{a}}$ if $x'^n = n+y$ for $n \in \operatorname{ann}(M)$, $y \in \mathfrak{a}$, or $x'^n m = (n+y)m = nm+ ym = ym$, in particular, $x'^n m \in \mathfrak{a}M$ so $x' \in \sqrt{\operatorname{ann}(M/\mathfrak{a}M)}$. To see the reverse inclusion note that [cf. Matsumura, Theorem 2.1] if $x^n \in \operatorname{ann}(M/\mathfrak{a}M)$ then there exists a $y \in \mathfrak{a}$ such that $(x^n+y)M = 0$ or $x^n M = -yM \subset \mathfrak{a}M$ so $x \in \sqrt{\operatorname{ann}(M/\mathfrak{a}M)}$. Thus, $\sqrt{\operatorname{ann}(M) + \mathfrak{a}} \subset \sqrt{\operatorname{ann}(M/\mathfrak{a}M}$ and we have equality. ∎

Problem 2.2

Let R be a local ring and M, N finite R-modules. Show that $M \otimes_R N = 0$ if and only if $M = 0$ or $N = 0$.

Proof. $\Longleftarrow$: If $M = 0$ or $N = 0$, it is immediate that $M \otimes_R N = 0$.

$\Longrightarrow$: Let $\mathfrak{m}$ be a maximal ideal of R. Since $M \otimes_R N = 0$, by Theorem 2.7, we have

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Problem 2.3

Show that $R^n \cong R^m$ if and only if $n = m$.

Proof. $\blacksquare$

Problem 2.4

Prove 2.7.

Proof. Recall the statement of Theorem 2.7:

Theorem. *(a)* $M \otimes_R N \cong N \otimes_R M$ *via* $x \otimes y \mapsto y \otimes x$.
(b) $(M \otimes_R N) \otimes_R P \cong M \otimes_R N \otimes_R P \cong M \otimes_R (N \otimes_R P)$ *via* $(x \otimes y) \otimes z \mapsto x \otimes y \otimes z \mapsto x \otimes (y \otimes z)$.
(c) $(M \oplus N) \otimes_R P \cong (M \otimes_R P) \oplus (N \otimes_R P)$ *via* $(x + y) \otimes z \mapsto x \otimes z + y \otimes z$.
(d) $R \otimes_R M \cong M$ *via* $r \otimes x \mapsto rx$.

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Problem 2.5

Prove 2.8.

Proof. Recall the statement of Proposition 2.8:

Proposition. *Let M be an R-module, N an R-S-bimodule and P an S-module. Then:*

(a) $M \otimes_R N$ *is an* R-S-*bimodule via* $(\sum m_i \otimes n_i)s = \sum m_i \otimes (sn_i)$.
(b) *The free module* $(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P)$ *as* R-S-*bimodules via* $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$.

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Problem 2.6

Prove 2.9.

Proof. Recall the statement of Theorem 2.9:

Theorem. *Let $\psi\colon R \to S$ be a ring map and M and R module. Then $S \otimes_R M$ is an S-module (by Proposition 2.8) and $\mu\colon M \to S \otimes_R M$ with $\mu(m) = 1 \otimes m$ is an R-linear map. Moreover, for every R-linear map $\varphi\colon M \to N$, where N is any S-module, there exists a unique S-linear map f so that $\varphi = f \circ \mu$, i.e, the diagram commmutes*

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Problem 2.7

Prove 2.10.

Proof. Recall the statement of Proposition 2.10:

Proposition. *Let S and T be R-algebras. Then there is an R-algebra structure on $S \otimes_R T$ with $(s_1 \otimes t_1)(s_2 \otimes t_2) = (s_1 s_2) \otimes (t_1 t_2)$.*

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