MA544: Qual Preparation

Carlos Salinas

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Contents

1 Danielli

1.1 Danielli: Practice Exams Spring 2016

1.1.1 Exam 1 Practice

Problem 1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{\, r\mathbf{x} : \mathbf{x} \in E \,\}$. Prove that rE is measurable, and that $|rE| = |r|^n|E|$.

Solution. ► Define a map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by $T\mathbf{x} = r\mathbf{x}$. Note that T is *Lipschitz continuous* since for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the equality

$$|T\mathbf{x} - T\mathbf{y}| = |r\mathbf{x} - r\mathbf{y}| = |r||\mathbf{x} - \mathbf{y}| \tag{1}$$

is satisfied. By Theorem 3.33 from [5, Ch. 3, p.55], the image of E under T is measurable. Moreover, by Theorem 3.35 [5, Ch. 3, p. 56], since T is linear, it follows that $|T(E)| = |\det T||E|$ where $\det T = |r|^n$. Lastly, we note that the image of E under T is precisely the set rE so that $|T(E)| = |rE| = |r|^n|E|$, as was to be shown. ◄

Problem 2. Let $\{E_k\}$, $k \in \mathbf{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty} E_n\right).$$

Show that

$$\left|\liminf_{k\to\infty} E_k\right| \le \liminf_{k\to\infty}|E_k|.$$

Solution. ► Following the style of [5, Ch. 1, p. 2], particularly, the sets defined after the introduction of equation (1.1), set

$$V_k = \bigcap_{\ell=k}^{\infty} E_\ell. \tag{2}$$

Note that the collection of sets $\{V_k\}$ forms an increasing sequence, that is, if $\mathbf{x} \in V_k$ then, by (2), $\mathbf{x}$ is in the intersection $E_k \cap \left(\bigcap_{\ell=k+1} E_\ell\right)$, but, by (2), $\bigcap_{\ell=k+1} E_\ell = V_{k+1}$ thus, $\mathbf{x}$ is in V_{k+1} so $V_{k+1} \supset V_k$. Hence, we have $V_k \nearrow \liminf E_k$.

Now, consider the sequence $\{|V_k|\}$ formed by the Lebesgue measure of the V_k. By Theorem 3.26 from [5, Ch. 3, p. 51], since $V_k \nearrow \liminf E_k$,

$$\lim_{k\to\infty} |V_k| = \lim_{k\to\infty}\left|\bigcap_{\ell=k}^{\infty} E_\ell\right| = \left|\liminf_{k\to\infty} E_k\right|. \tag{3}$$

But note that, by the monotonicity of the Lebesgue measure, we have

$$\left|\bigcap_{\ell=k}^{\infty} E_\ell\right| \leq |E_k|, \tag{4}$$

so, by properties of the lim inf, in particular, by Theorem 19(v) from [2, Ch. 1, p. 23], we have

$$\limsup_{k\to\infty} |V_k| \leq \liminf_{k\to\infty} |E_k|. \tag{5}$$

Hence, by (3) and Proposition 19 (iv), since the sequence $\{|V_k|\}$ converges and is equal to the measure of $\liminf E_k$, by (5), we have

$$\left|\liminf_{k\to\infty} E_k\right| \leq \liminf_{k\to\infty} |E_k| \tag{6}$$

as was to be shown. ◀

Problem 3. Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) = \{\, \mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r \,\}$. Prove that F is monotonic increasing and continuous.

Solution. ► Define the linear map $T\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ by $T(r)\mathbf{x} = r\mathbf{x}$. We claim that $B(\mathbf{0}, r) = T(r, B(\mathbf{0}, 1))$. To reduce notation, set $B_1 = B(\mathbf{0}, 1)$ and $B_r = B(\mathbf{0}, r)$.

Proof of claim. ► $\subset$: Let $\mathbf{x} \in B_r$. Then $|\mathbf{x}| < r$ so $|\mathbf{x}|/r < 1$. Thus, $|\mathbf{x}|/r \in B_1$ so it is in the image of B_1 under the map $T(r, \cdot)$.

$\supset$: On the other hand, suppose $\mathbf{x} \in T(r, B_1)$. Then $\mathbf{x} = r\mathbf{y}$ for some $\mathbf{y} \in B_1$. Then, since $|\mathbf{y}| < 1$, $|\mathbf{x}| = r|\mathbf{y}| < r$ so $\mathbf{x} \in B_r$. ◀

From the claim, we see that $F(x) = |T(x, B(\mathbf{0}, 1))|$ which, by Problem 1, is nothing more that the polynomial $|B_1|x^n$. It is clear, from this equivalence, that F is monotonically increasing: Take $x, y \in [0,\infty)$ such that $x < y$, then $x^n < y^n$ so

$$F(x) = |B_1|x^n < |B_1|y^n = F(y). \tag{7}$$

Thus, F is monotonically increasing.

In the argument above, since $F(x) = |B_1|x^n$ is a polynomial in $[0,\infty)$ (and polynomials are continuous on $\mathbb{R}$) F is continuous on $[0,\infty)$. ◀

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Solution. ► (Without much motivation) let us consider the collection of sets $\{E_k\}$ defined by

$$E_k = \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } y, z \in B(x, \delta) \text{ implies } |f(y) - f(z)| < \frac{1}{k} \right\}. \tag{8}$$

We claim that $C = \bigcap_{k=1}^{\infty} E_k$ and that each E_k is open.

Proof of claim. ► First, we demonstrate equality. $\subset$: Suppose $x \in C$. Then, by the definition of continuity, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $y \in B(x, \delta)$ implies $|f(x) - f(y)| < \delta$. In particular, for every k, there exists $\delta > 0$ such that for $y \in B(x, \delta)$ the inequality $|f(x) - f(y)| < 1/k$ holds. Thus, x is in $\bigcap_{k=1}^{\infty} E_k$.

$\supset$: On the other hand, suppose that $x \in \bigcap_{k=1}^{\infty} E_k$. Then, given $\varepsilon > 0$, by the Archimedean property, there exists a positive integer N such that $1/N < \varepsilon$. Then, since $x \in \bigcap_{k=1}^{\infty} E_k$, $x \in E_N$ so

$$|f(x) - f(y)| < \frac{1}{N} < \varepsilon. \tag{9}$$

Thus, x is in C and $C = \bigcap_{k=1}^{\infty} E_k$.

All that remains to be shown is that the E_k are open. But this is clear by the way we defined E_k in (8): Let $x \in E_k$, then there exists $\delta > 0$ such that for any $y, z \in B(x, \delta)$, $|f(y) - f(z)| < 1/k$; Let $x' \in B(x, \delta)$ and set $\delta' = \min\{|(x + \delta) - x'|, |(x - \delta) - x|\}$. Then, since $B(x', \delta') \subset B(x, \delta)$, for every $y, z \in B(x', \delta')$, we have $|f(y) - f(z)| < 1/k$. Hence, $x' \in E_k$ for any $x' \in B(x, \delta)$ so $B(x, \delta) \subset E_k$. ◄

Since C can be expressed as the countable intersection of open sets E_k, it follows that C is a G_δ set. ◄

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Solution. ► If $\{ f = r \}$ are measurable for all $r \in \mathbb{R}$, it is not necessarily the case that f is measurable. Consider the following construction: Let $E \subset (0, 1)$ be an unmeasurable set.* Define a map $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{R} \setminus ((0, 1) \setminus E), \\ x + 1 & \text{if } x \in (0, 1) \setminus E. \end{cases} \tag{10}$$

*It's construction does not concern us. The interested reader such direct their refer to Theorem 3.38 from [5, Ch. 3, p. 57-58] or Theorem 17 from [2, Ch. 2§7, p. 48].

By the definition, it is clear that $\{ f = r \}$ is measurable and $|\{ f = r \}| = 0$ since $\{ f = r \}$ contains at most two elements. However, the set $\{ 0 < f < 1 \} = E$ is not measurable. Thus, f is not measurable. ◄

Problem 6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set $\{ x : \lim_{k\to\infty} f_k(x) \text{ exists} \}$ is measurable.

Solution. ► By Theorem 4.12 from [5, Ch. 4, p. 67], $\liminf_{k\to\infty} f_k$ and $\limsup_{k\to\infty} f_k$ are measurable. By Theorem 4.7 from [5, Ch. 4, p. 66]

$$\left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\} \tag{11}$$

is measurable. Since

$$\left\{ \lim_{k\to\infty} f_k \text{ exists} \right\} = \left\{ \limsup_{k\to\infty} f_k = \liminf_{k\to\infty} f_k \right\} = \mathbb{R} \setminus \left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\}, \tag{12}$$

by Theorem 3.17 from [5, Ch. 3, p. 48], the set $\{ \lim_{k\to\infty} f_k \text{ exists} \}$ is measurable. ◄

Problem 7. A real valued function f on an interval $[a, b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, one has $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$.

Solution. ► Suppose f is absolutely continuous on $[a, b]$. Let $\varepsilon = 1$. Then, there exists $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{N}$ of open intervals in (a, b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, one has $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < 1$. Let $N = \lceil (b - a)/\delta \rceil$, that is, N is the smallest integer greater than $(b - a)/\delta$, and consider the partition $\Gamma = \{x_k\}$ where $x_k = a + k(b - a)/N$, for $k = 0, \ldots, N$. Then $x_k - x_{k-1} < (b - a)/N < \delta$ so, by Theorem 2.2(i) from [5, Ch. 2, p. 19], we have $V[f; x_{k-1}, x_k] < 1$ for $k = 0, \ldots, N$. In follows by Theorem 2.2(ii) that

$$V[f; a, b] = \sum_{k=1}^{N} V[f; x_{k-1}, x_k] < N. \tag{13}$$

Thus, f is b.v. on $[a, b]$. ◄

Problem 8. Let f be a continuous function from $[a,b]$ into $\mathbb{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, that is, $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a,b), \\ -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

Solution. ► The result follows quite easily from more sophisticated measure theoretic arguments. At this point, however, such language has not been discussed so we shall prove this using nothing but the definition of the Riemann–Stieltjes integral and properties thereof.

Let us consider each case $c \in (a,b)$, $c = a$, and $c = b$ separately.

Recall that the given a partition $\Gamma = \{x_0, \ldots, x_m\}$ of $[a,b]$, the Riemann–Stieltjes sum of f with respect to φ is

$$R_\Gamma = \sum_{k=1}^{m} f(\xi_k)[\varphi(x_k) - \varphi(x_{k-1})]. \tag{14}$$

The Riemann–Stieltjes integral is defined as the limit

$$\int_a^b f \, \mathrm{d}\varphi = \lim_{|\Gamma| \to 0} R_\Gamma \tag{15}$$

if it exists.

Suppose $c \in (a,b)$. Then, for any partition Γ of $[a,b]$, either $c \in \Gamma$ or $c \notin \Gamma$. In the latter case, $R_\Gamma = 0$. In the former case c is one of the x_k, say $c = x_\ell$ for $0 < \ell < m$. Then

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^{m} f(\xi_k)[\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= 0 + \cdots + 0 + f(\xi_{\ell-1}) - f(\xi_\ell) + 0 + \cdots + 0 \\ &= f(\xi_{\ell-1}) - f(\xi_\ell). \end{aligned} \tag{16}$$

Since f is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi_\ell - \xi_{\ell-1}| < \delta$ implies $|f(\xi_\ell) - f(\xi_{\ell-1})| < \varepsilon$. It follows that the quantity in (16) approaches 0 as $|\Gamma|$ approaches 0. Therefore, $\int_a^b f \, \mathrm{d}\chi_{\{c\}} = 0$.

Suppose $c = a$. Then, since any partition Γ of $[a,b]$ must contain the point a, we have

$$
\begin{aligned}
R_\Gamma &= \sum_{k=1}^{m} f(\chi_k)[\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\
&= f(\xi_1)[\chi_{\{c\}}(x_1) - \chi_{\{c\}}(x_0)] + f(\xi_2)[\chi_{\{c\}}(x_2) - \chi_{\{c\}}(x_1)] \\
&\qquad\qquad\qquad + \cdots + f(\xi_m)[\chi_{\{c\}}(x_m) - \chi_{\{c\}}(x_{m-1})] \\
&= -f(\xi_1) + 0 + \cdots + 0 \\
&= -f(\xi_1)
\end{aligned} \tag{17}
$$

Taking the limit as $|\Gamma| \to 0$, $\xi_1 \to a$ so, by continuity of f, $f(\xi_1) \to f(a)$. Thus, $\int_a^b f \,\mathrm{d}\chi_{\{c\}} = -f(a)$.

A similar argument to the one above shows that, if $c = b$, the Riemann–Stieltjes integral $\int_a^b f \,\mathrm{d}\chi_{\{c\}} = f(b)$. ◀

1.1.2 Exam 1

Problem 1.

Solution. ► ◄

Problem 2.

Solution. ► ◄

Problem 3.

(i) Show that if $B_r = \{ \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < r \}$, then there exists a constant C such that $|B_r| = Cr^n$.

(*Hint*: Think of B_r as $\{ r\mathbf{x} : \mathbf{x} \in B_1 \}$.)

(ii) Let $E \subset \mathbb{R}^n$ be a measurable set and let $\varphi_E \colon \mathbb{R}^n \to \mathbb{R}$ be defined $\varphi_E(\mathbf{x}) = \left|E \cap B_{|\mathbf{x}|}\right|$. Use part (i) to prove that φ_E is continuous.

Solution. ► (i) To prove this result, we use the map constructed in Problem 1 of the review sheet for Exam 1, the map $T \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Set $T_r \colon \mathbb{R}^n \to \mathbb{R}^n$ to be $T_r = T(r)$. Then, we claim $B_r = T_r(B_1)$ and $|B_r| = |T_r(B_1)|$, which, as we saw in Problem 1 of the review sheet, has measure $|B_1||r|^n$. Setting $C = |B_1|$, we have $|B_r| = C|r|^n$ as desired.

(ii) To prove that φ_E is continuous, we provide an (ε, δ)-argument. Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that $\mathbf{y} \in B(\mathbf{x}, \delta)$ implies

$$|\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| < \varepsilon. \tag{1}$$

First, note that since $\mathbf{x} \mapsto |\mathbf{x}|$ is continuous and polynomials $p \colon \mathbb{R}^n \to \mathbb{R}^n$ are continuous, then the composition $\mathbf{x} \mapsto |\mathbf{x}|^n$ is continuous. Therefore, there exists $\delta > 0$ such that $\mathbf{y} \in B(\mathbf{x}, \delta)$ implies

$$||\mathbf{x}|^n - |\mathbf{y}|^n| < \frac{\varepsilon}{C}, \tag{2}$$

where $C = |B_1|$.

Now, let $x \in \mathbb{R}^n$ and $\mathbf{y} \in B(\mathbf{x}, \delta)$ as above. Then, by (2) we have

$$\begin{aligned} |\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| &= \left| |E \cap B_{|\mathbf{x}|}| - |E \cap B_{|\mathbf{y}|}| \right| \\ &\leq \left| |B_{|\mathbf{x}|}| - |B_{|\mathbf{y}|}| \right| \\ &= C \left| |\mathbf{x}|^n - |\mathbf{y}|^n \right| \\ &\leq C \left[\frac{\varepsilon}{C} \right] \\ &= \varepsilon. \end{aligned} \tag{3}$$

It follows that φ_E is continuous. ◄

Problem 4. Assume that $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$. Prove that f is measurable.

Solution. ► By Jordan's theorem (Corollary 2.7 from [5, Ch. 2, p. 21]), the function f is of bounded variation on $[a, b]$ if and only if it can be written as the difference $f_1 - f_2$ of two bounded functions f_1 and f_2 that are monotone increasing on $[a, b]$. Then, f_1 and f_2 are continuous a.e. on $[a, b]$ and hence, are measurable. ◄

1.1.3 Exam 2 Practice Problems

Problem 1. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B(\mathbf{0}, \varepsilon)$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \setminus B(\mathbf{0}, \varepsilon)} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Solution. ► Recall that a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgueintegrable on a subset E of $\mathbb{R}^n$ if

$$\int_E f(\mathbf{x})\,\mathrm{d}\mathbf{x} < \infty. \tag{1}$$

Let f be as given in the statement of the problem and set $B_\varepsilon = B(\mathbf{0}, \varepsilon)$. Consider the change of variables to *hyperspherical coordinates* $(x_1, \ldots, x_n) \mapsto (r, \Theta)$ where $\Theta = (\theta_1, \ldots, \theta_{n-1})$.† By Theorem 7.26(iii) from [4, Ch. 7, p. 123], we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\varepsilon} f(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^n \setminus B_\varepsilon} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n \setminus B_\varepsilon} \frac{1}{|\mathbf{x}|^{n+1}}\,\mathrm{d}\mathbf{x}. \\ &= \int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}}\,\mathrm{d}r dV, \end{aligned} \tag{2}$$

where S_r^{n-1} is the $(n-1)$-sphere centered at $\mathbf{0}$ with radius r, that is, the subset $\{\, \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1 \,\}$ of $\mathbb{R}^n$ and dV is the *volume element* of S_r^{n-1}. Since $1/|r|^{n+1}$ is nonnegative, by Tonelli's theorem the the iterated integrals in (2) may be exchange, that is,

$$\int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}}\,\mathrm{d}r dV = \int_\varepsilon^\infty \left(\int_{S_r^{n-1}} 1\,\mathrm{d}V \right) \frac{1}{|r|^{n+1}}\,\mathrm{d}r. \tag{3}$$

Now, note that from Problem 1 of the review sheet for Exam 1, we have

$$\int_{S_r^{n-1}} 1\,\mathrm{d}V = |S_r^{n-1}|_{\mathbb{R}^{n-1}} = |S^{n-1}|_{\mathbb{R}^{n-1}} |r|^{n-1}. \tag{4}$$

†The explicit construction of the map $(x_1, \ldots, x_n) \mapsto (r, \Theta)$ is of no concern to us for now. What is important is that it exists.

Set $C = |S^{n-1}|_{\mathbb{R}^{n-1}}$. Putting equations (2), (3), and (4) together, we have

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_\varepsilon} f(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_\varepsilon^\infty C|r|^{n-1} \frac{1}{|r|^{n+1}}\,\mathrm{d}r \\
&= \int_\varepsilon^\infty \frac{C}{|r|^2}\,\mathrm{d}r \\
&= \lim_{x\to\infty}\left[-\frac{C}{x} - \left(-\frac{C}{\varepsilon}\right)\right] \\
&= \frac{C}{\varepsilon},
\end{aligned} \tag{5}$$

as was to be shown. ◀

Problem 2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty} \int_E f_k$$

for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Solution. ▶ Let $E \subset \mathbb{R}^n$ be a measurable subset of $\mathbb{R}^n$. Then, since $f_k \to f$ pointwise a.e. on $\mathbb{R}^n$, then $f_k \to f$ pointwise a.e. on E and $\mathbb{R}^n \setminus E$. To prove that the limit of the sequence of integrals $\left\{\int_E f_k\right\}$ exist and is equal to $\int_E f$, it suffices to prove that

$$\int_E f \le \liminf_{k\to\infty} \int_E f_k \le \limsup_{k\to\infty} \int_E f_k \le \int_E f. \tag{6}$$

The lower bound in (6) follows from an application of Fatou's lemma:

$$\int_E f = \int_E \liminf_{k\to\infty} f \le \liminf_{k\to\infty} \int_E f_k. \tag{7}$$

Also by Fatou's lemma, we have

$$\int_{\mathbb{R}^n\setminus E} f = \int_{\mathbb{R}^n\setminus E} \liminf_{k\to\infty} f \le \liminf_{k\to\infty} \int_{\mathbb{R}^n\setminus E} f_k. \tag{8}$$

Now, since $f \in L^1(\mathbb{R}^n)$, by equation (8) and properties of the lim inf and lim sup[‡] we have

$$\begin{aligned}\int_E f = \int_{\mathbb{R}^n} f - \int_{\mathbb{R}^n\setminus E} f &\geq \limsup_{k\to\infty} \int_{\mathbb{R}^n} f - \liminf_{k\to\infty} \int_{\mathbb{R}^n\setminus E} f_k \\ &\geq \limsup_{k\to\infty} \int_{\mathbb{R}^n} f_k - \limsup_{k\to\infty} \int_{\mathbb{R}^n\setminus E} f_k \\ &= \limsup_{k\to\infty} \left[\int_{\mathbb{R}^n} f_k - \int_{\mathbb{R}^n\setminus E} f_k \right] \\ &= \limsup_{k\to\infty} \int_E f_k.\end{aligned} \tag{9}$$

By equations (7) and (9) it follows that $\lim_{k\to\infty} \int_E f_k$ exists and is equal to $\int_E f$.

To see that the result need not be true if $\int_E f = \infty$, consider the following example: Let $f_k : \mathbb{R} \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} k^2/2 & \text{if } x \in (-1/k, 1/k), \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

and $f = 1$.

It is easy to see that $f_k \to f$ a.e. in $\mathbb{R}$ and that both $\int_{\mathbb{R}} f = \infty$ and $\lim_{k\to\infty} f_k = \infty$. However, if $E = (-1, 1)$ then $\int_E f = 1$, but $\lim_{k\to\infty} \int_E f = \infty$. ◀

Problem 3. Assume that E is a measurable set of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{ \mathbf{x} \in E : f(\mathbf{x}) \geq k \}| < \infty.$$

Solution. ► If f is integrable over a measurable subset E of $\mathbb{R}^n$, then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \tag{11}$$

Set $E_k = \{ \mathbf{x} \in E : k + 1 > f(\mathbf{x}) \geq k \}$ and $F_k = \{ \mathbf{x} \in E : f(\mathbf{x}) \geq k \}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise

[‡]Namely, for any sequence of positive real numbers $\{a_k\}$ the inequality $\liminf a_k \leq \limsup a_k$ holds

disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{12}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \tag{13}$$

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)|E_k|. \tag{14}$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (14) becomes

$$\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) \leq \int_E f(\mathbf{x}) d\mathbf{x} \leq \sum_{k=0}^{\infty} (k+1)(|F_k| - |F_{k+1}|). \tag{15}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k(|F_k| - |F_{k+1}|) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty} (k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (16) \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty}(k+1)(|F_k|-|F_{k+1}|) &= \sum_{k=0}^{\infty}(k+1)|F_k| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty}(k+1)|F_k| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty}(k+2)|F_{k+1}| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \qquad (17) \\
&= |F_0| + \sum_{k=0}^{\infty}|F_{k+1}| \\
&= \sum_{k=0}^{\infty}|F_k|.
\end{aligned}$$

Thus, from (16) and (17)

$$\sum_{k=1}^{\infty}|F_k| \leq \int_E f(\mathbf{x})d\mathbf{x} \leq \sum_{k=0}^{\infty}|F_k| \qquad (18)$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty}|F_k|$ converges. ◀

Problem 4. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f,g) = \int_E \frac{|f-g|}{1+|f-g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Solution. ► ◀

Problem 5. Define the *gamma function* $\Gamma\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^{\infty} e^{-u}u^{y-1}\,\mathrm{d}u,$$

and the *beta function* $\beta\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.

(b) Show that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Solution. ► ◄

Problem 6. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}} : \mathbb{R}^n \to \mathbb{R}$ be $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h}\to 0} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Solution. ► ◄

Problem 7. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \le g_k$ and

$$\int_{\mathbb{R}^n} g_k \longrightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \qquad \text{as } k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \longrightarrow \int_{\mathbb{R}^n} |f| \qquad \text{as } k \to \infty.$$

Solution. ► (a) $\Longrightarrow$ (b): Assume part (a) then $\Longrightarrow$ if

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \tag{19}$$

as $k \to \infty$, we have

(b): ◄

1.1.4 Exam 2 (2010)

Problem 1. Suppose $f \in L^1(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Hint: Use the monotone convergence theorem.

Solution. ► ◄

Problem 2. (a) Prove the following generalization of *Chebyshev's inequality*: Let $0 < p < \infty$ and $E \subset \mathbb{R}^n$ be measurable. assume that $|f|^p \in L^1(E)$. Then

$$|\{\, x \in E : f(\mathbf{x}) > \alpha \,\}| \leq \frac{1}{\alpha^p} \int_{\{\, f > \alpha \,\}} f^p,$$

for $\alpha > 0$.

(b) Let p, E, and f be as in part (a). In addition, assume that $\{f_k\}$ is a sequence such that $\int_E |f_k - f|^p \to 0$ as $k \to \infty$. Show that $f_k \to f$ in measure on E.

Recall that $f_k \to f$ in measure on E if and only if for every $\varepsilon > 0$

$$\lim_{k \to \infty} |\{\, \mathbf{x} \in E : |f_k(\mathbf{x}) - f(\mathbf{x})| > \varepsilon \,\}| = 0.$$

Solution. ► ◄

Problem 3. Let $f \in L^1(\mathbb{R})$, and define

$$F(\xi) = \int_{\mathbb{R}} f(x) \cos(2\pi x \xi) \, \mathrm{d}x.$$

Prove that F is continuous and bounded on $\mathbb{R}$.

Solution. ► ◄

Problem 4. Use repeated integration techniques to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2} \, \mathrm{d}\mathbf{x} = \pi^{n/2}.$$

Hint: Start from the case $n = 1$ by using the polar coordinates in

$$\left[\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x\right]^2 = \left[\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x\right]\left[\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}y\right]$$

Solution. ► ◄

Problem 5.

Solution. ► ◄

1.1.5 Exam 2

Problem 1. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Solution. ► ◄

Problem 2. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Solution. ► ◄

Problem 3. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbf{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Solution. ► ◄

Problem 4. Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)\,\mathrm{d}t$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Solution. ► ◄

1.1.6 Final Exam Practice Problems

Problem 1. Suppose $f \in L^1(\mathbb{R}^n)$ and that x is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) = \frac{1}{|r|^n} \int_{B(0,r)} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{y}.$$

Show that:

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$;
(b) there exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

Solution. ► (a) Without loss of generality, we may assume $r < s$. Then, we want to show that as $r \to s$, the quantity

$$|A(s) - A(r)| \longrightarrow 0.$$

Set $F(\mathbf{y}) = |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})|$ and consider said quantity

$$\begin{aligned}
|A(s) - A(r)| &= \left| \frac{1}{|s|^n} \int_{B_s} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} - \frac{1}{|r|^n} \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} + \frac{1}{|s|^n} \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} - \frac{1}{|r|^n} \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} + \left(\frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right| \\
&\leq \underbrace{\frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y}}_{I_1} + \underbrace{\left(\frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y}}_{I_2}.
\end{aligned}$$

Hence, we must show that the quantities $I_1, I_2 \to 0$ as $r \to s$.

To see that $A(r) \to 0$ as $r \to 0$, note that x is a point of the Lebesgue set of f and that

$$0 = \lim_{B_r \searrow \mathbf{x}} \frac{1}{|B_1||r|^n} \int_{B_r} |f(\mathbf{y}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{y} = \frac{1}{|B_1|} \lim_{B_r \searrow \mathbf{x}} \frac{1}{|r|^n} \int_{B_r} |f(\mathbf{t}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{t} = \lim_{r \to 0} A(r).$$

by making the change of variables $\mathbf{t} = \mathbf{x} - \mathbf{y}$.

(b) ◄

Problem 2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \leq n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0$$

if and only if

$$\|f_k\|_p \longrightarrow \|f\|_p$$

as $k \to \infty$.

Solution. ► ◄

Problem 3. Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

Solution. ► ◄

Problem 4. Let $f \in L(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x)\cos(tx)dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Solution. ► ◄

Problem 5. Let f be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular. Show that

$$\int_a^b \varphi \, \mathrm{d}f = \int_a^b \varphi f' dx + \int_a^b \varphi \, \mathrm{d}h$$

for all functions φ continuous on $[a, b]$.

Solution. ► ◄

1.1.7 Final Exam 2010

Problem 1. Suppose that $f \in L^1(\mathbb{R}^n)$, and that $\mathbf{x}$ is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) = \frac{1}{r^n} \int_{B_r} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{y},$$

where $B_r = B(\mathbf{0}, r)$.

Show that

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$.
(b) There exists a constant $M > 0$ such that $A(r) \leq M$ for all $r > 0$.

Solution. ► (a)

(b) ◄

Problem 2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \leq p < \infty$. assume that $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0 \iff \|f_k\|_p \longrightarrow \|f\|_p$$

Hint: To prove one of the implications, you can use the following fact without proving it:

$$\left| \frac{a-b}{2} \right| \leq \frac{|a|^p + |b|^p}{2}$$

for all $a, b \in \mathbb{R}$.

Solution. ► ◄

Problem 3. Let $0 < p < q < r \leq \infty$, $E \subset \mathbb{R}^n$ be a measurable set. Show that each $f \in L^q(E)$ is the sum of a function $g \in L^p(E)$ and a function $h \in L^r(E)$.

Solution. ► ◄

Problem 4. Prove that $f : [a, b] \to \mathbb{R}$ is Lipschitz continuous if and only if f is absolutely continuous and there exists a constant $M > 0$ such that $|f'| < M$ a.e. on $[a, b]$.

Solution. ► ◄

Problem 5. Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ or $p = \infty$?.

Solution. ► ◄

1.1.8 Final Exam

1.2 Danielli: Winter 2012

Problem 1. Let $f(x,y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $\partial f(x,y)/\partial x$ is a bounded function of (x,y). Prove that $\partial f(x,y)/\partial x$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\,\mathrm{d}y = \int_0^1 \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y.$$

Solution. ► ◄

Problem 2. Let f be a function of bounded variation on $[a,b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular, show that

$$\int_a^b \varphi\,\mathrm{d}f = \int_a^b \varphi f'\,\mathrm{d}x + \int_a^b \varphi\,\mathrm{d}h.$$

Hint: A function h is said to be singular if $h' = 0$.

Solution. ► ◄

Problem 3. Let $E \subset \mathbb{R}$ be a measurable set, and let K be a measurable function on $E \times E$. Assume that there exists a positive constant C such that

$$\int_E K(x,y)\,\mathrm{d}x \le C \tag{1}$$

for a.e. $y \in E$, and

$$\int_E K(x,y)\,\mathrm{d}y \le C \tag{2}$$

for a.e. $x \in E$.

Let $1 < p < \infty$, $f \in L^p(E)$, and define

$$T_f(x) = \int_E K(x,y) f(y)\,\mathrm{d}y.$$

(a) Prove that $T_f \in L^p(E)$ and

$$\|T_f\|_p \le C\|f\|_p. \tag{3}$$

(b) Is (3) still valid if $p = 1$ or ∞? If so, are assumptions (1) and (2) needed?

Solution. ► ◄

Problem 4. Let f be a nonnegative measurable function on $[0, 1]$ satisfying

$$|\{ x \in [0,1] : f(x) > \alpha \}| < \frac{1}{1+\alpha^2} \tag{4}$$

for $\alpha > 0$.

(a) Determine values of $p \in [1, \infty)$ for which $f \in L^p[0, 1]$.
(b) If p_0 is the minimum value of p for which p may fail to be in L^p, give an example of a function which satisfies (4), but which is not in $L^{p_0}[0, 1]$.

Solution. ► ◄

1.3 Danielli: Summer 2011

Problem 1. Let $f \in L^1(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x)\cos(tx)\,\mathrm{d}x$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Hint: Start by proving the statement for $f = \chi_{[a,b]}$.

Solution. ►

◄

Problem 2. (a) Suppose that $f_k, f \in L^2(E)$, with E a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E fg \tag{1}$$

as $k \to \infty$ for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \to \|f\|_2$ show that f_k converges to f in L^2, i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as $k \to \infty$.

(b) Provide an example of a sequence f_k in L^2 and a function f in L^2 satisfying (1), but such that f_k does *not* converge to f in L^2.

Solution. ► ◄

Problem 3. A bounded function f is said to be of bounded variation on $\mathbb{R}$ if it is of bounded variation on any finite subinterval $[a,b]$, and moreover $A = \sup_{a,b} V[a,b;f] < \infty$. Here, $V[a,b;f]$ denotes the total variation of f over the interval $[a,b]$. Show that:

(a) $\displaystyle\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x \le A|h|$ for all $h \in \mathbb{R}$.

Hint: For $h > 0$, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,\mathrm{d}x.$$

(b) $\left|\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x\right| \leq A$, where φ is any function of class C^1, of bounded variation, compactly supported, with $\sup_{x\in\mathbb{R}} |\varphi(x)| \leq 1$.

Solution. ► ◄

Problem 4. (a) Prove the *generalized Hölder's inequality*: Assume $1 \leq p \leq \infty$, $j = 1, \dots, n$, with $\sum_{j=1}^{\infty} 1/p_j = 1/r \leq 1$. If E is a measurable set and $f_j \in L^{p_j}(E)$ for $j = 1, \dots, n$, then $\prod_{j=1}^{n} f_j \in L^r(E)$ and

$$\|f_1 \cdots f_n\|_r \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if $1 \leq p, q, r \leq \infty$, with $1/p + 1/q = 1/r + 1$, $f \in L^p(\mathbb{R})$, and $g \in L^p(\mathbb{R})$, then

$$|(f * g)(x)| \leq \|f\|_p{}^{r-p}\|g\|_q{}^{r-q} \int |f(y)|^p |g(x-y)|^q \,\mathrm{d}y.$$

(Recall that $(f * g)(x) = \int f(y)g(x-y)\,\mathrm{d}y$.)

(c) Prove *Young's convolution theorem*: Assume that p, q, r, f, and g are as in part (b). Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Solution. ► ◄

2 Bañuelos

2.1 Bañuelos: Summer 2000

Problem 1. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions with the property that for all $n \geq 1$

$$\mu(\{ x \in X : |f_n(x)| \geq \lambda \}) \leq C \exp(-\lambda^2/n)$$

for all $\lambda > 0$. (Here C is a constant independent of n.) Let $n_k = 2^k$. Prove that

$$\limsup_{k\to\infty} \frac{|f_{n_k}|}{\sqrt{n_k \log(\log(n_k))}} \leq 1 \quad \text{a.e.}$$

Solution. ► Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions such that

$$\mu(\{ x \in X : |f_n(x)| \geq \lambda \}) \leq C \exp(-\lambda^2/n) \tag{1}$$

for all λ. Now, consider the subsequence $\{f_{2^k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$. We aim to show that

$$\limsup_{k\to\infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} \leq 1$$

almost everywhere. To that end, it suffices to show that the set

$$E = \left\{ x \in X : \limsup_{k\to\infty} \frac{|f_{2^k}|}{\sqrt{2^k \log(\log(2^k))}} > 1 \right\}$$

has measure zero. Let $x \in E$ then

$$\limsup_{k\to\infty} \frac{|f_{2^k}(x)|}{\sqrt{2^k \log(\log(2^k))}} > 1.$$

This means that there exists some subsequence $\{k_m\}_{m=1}^{\infty} \subset \{k\}_{n=1}^{\infty}$ such that

$$\lim_{m\to\infty} \frac{|f_{2^{k_m}}(x)|}{\sqrt{2^{k_m} \log(\log(2^{k_m}))}} > 1.$$

This means that, for sufficiently large N

$$|f_{2^{k_n}}(x)| > \sqrt{2^{k_n} \log(\log(2^{k_n}))}$$

for all $n \geq N$. But by Equation (1) we have

$$\begin{aligned}
\mu\left(\left\{ x \in X : \frac{|f_{2^{k_n}}(x)|}{\sqrt{2^{k_n}\log(\log(2^{k_n}))}} \geq 1 \right\}\right) &\leq C\exp\left(-\left(\sqrt{2^{k_n}\log(\log(2^{k_n}))}\right)^2 \Big/ 2^{k_n}\right) \\
&= C\exp\left(-2^{k_n}\log(\log(2^{k_n}))\Big/2^{k_n}\right) \\
&= C\exp\left(-\log(\log(2^{k_n}))\right) \\
&= C\exp\left(\log(1/\log(2^{k_n}))\right) \\
&= \frac{C}{\log(2^{k_n})}.
\end{aligned} \tag{2}$$

Letting $n \to \infty$, we see that the measure of the set on the left-hand side of Equation (2) must go to 0 so $\mu(E) = 0$. ◄

Problem 2. Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let f_n be a sequence of measurable functions with $f_1 \in L^1(\mu)$ and with the property that

$$\mu(\{ x \in X : |f_n(x)| > \lambda \}) \leq \mu(\{ x \in X : |f_1(x)| > \lambda \})$$

for all n and all $\lambda > 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{n}\int_X \left[\max_{1\leq j\leq n} |f_j|\right] \mathrm{d}\mu = 0.$$

[*Hint*: You may use the fact that $\|f\|_1 = \int_0^\infty \mu(\{ |f(x)| > \lambda \})\,\mathrm{d}\lambda$.]

Solution. ► Define $g_n, h_n : \mathcal{F} \to [0,\infty]$ for $n \in \mathbb{N}$ by

$$g_n(\lambda) = \mu(\{ x \in X : |f_n(x)| > \lambda \}), \quad h_n(\lambda) = \mu\left(\left\{ x \in X : \max_{1\leq i\leq n} |f_i(x)| > \lambda \right\}\right).$$

Now, note that, by the monotonicity of μ, we have

$$h_n(\lambda) \leq \sum_{i=1}^{n} g_n(\lambda) \leq n g_1(\lambda).$$

Thus,

$$\frac{h_n(\lambda)}{n} \leq g_1(\lambda).$$

Since $\|f_1\|_1 = \int_0^\infty g_1(\lambda)\,\mathrm{d}\lambda$, by Lebesgue's dominated convergence theorem, we have

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \int_X \left[\max_{1\leq j\leq n} |f_j|\right] \mathrm{d}\mu &= \lim_{n\to\infty} \int_X \frac{h_n(x)}{n}\,\mathrm{d}\mu \\
&= \int_X \lim_{n\to\infty} \frac{h_n(x)}{n}\,\mathrm{d}\mu \\
&\leq \int_X \lim_{n\to\infty} \frac{\mu(X)}{n} \\
&= 0
\end{aligned}
$$

as we wanted to show. ◄

Problem 3.

(i) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions. Prove that $f_n \to f$ is measurable if and only if every subsequence $\{f_{n_k}\}$ contains a further subsequence $\{f_{n_{k_j}}\}$ that converges a.e. to f.

(ii) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $F\colon \mathbb{R} \to \mathbb{R}$ be continuous and $f_n \to f$ in measure. Prove that $F(f_n) \to F(f)$ in measure. (You may assume, of course, that f_n, F, $F(f_n)$, and $F(f)$ are all measurable.)

Solution. ► Recall that a sequence of measurable functions $\{f_n\}$ converge in measure to a limit f if for every $\varepsilon > 0$ the limit

$$\lim_{n\to\infty} \mu(\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\}) = 0.$$

For part (i) $\implies$ suppose that $f_n \to f$ in measure. Then given $\varepsilon > 0$ and $\delta > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\mu(\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\}) < \delta.$$

In particular, given $\varepsilon = k^{-1}$ and $\delta = 2^{-k}$, consider the countable collection of measurable sets $\{E_k\}_{k=1}^\infty$ given by

$$E_k = \left\{ x \in X : |f(x) - f_{n_k}(x)| \geq \frac{1}{k} \right\},$$

where $n_k \geq N(k)$ (which depends on our choice of k) such that

$$\mu(E_k) < \frac{1}{2^k}.$$

Now, by the Borel–Cantelli lemma, since

$$\sum_{k=1}^{\infty} \mu(E_k) < \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty,$$

for almost every $x \in X$, there exists $N_x \in \mathbb{N}$ such that $x \notin E_k$ for $k \geq N_x$. This means that for $k \geq N_x$, we have

$$|f(x) - f_{n_k}(x)| < \frac{1}{k}.$$

Let $\{f_{n_{k+1}}\}$ be the subsequence of $\{f_{n_k}\}$. Then

$$\lim_{k\to\infty} f_{n_{k+1}} = f$$

as desired.

$\Longleftarrow$ On the other hand, suppose that every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ contains a subsequence $\{f_{n_{k_j}}\}$ that converges to f. Seeking a contradiction, suppose that given $\varepsilon > 0$ there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$M = \mu\big(\{\, x \in X : |f(x) - f_{n_k}(x)| \geq \varepsilon \,\}\big) > 0.$$

But by assumption there exists a subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ that converges almost everywhere to f. We claim that this implies that $f_{n_{k_j}} \to f$ in measure.

Proof of claim. This is adapted from a proof in Royden, Proposition 3, Ch. 5.

First note that f is measurable since it is the pointwise limit almost everywhere of a sequence of measurable functions. Let $\varepsilon, \delta > 0$ be given. Here is where the assumption that $\mu(X) < \infty$ is essential! By Egorov's theorem, there is a measurable subset $E \subset X$ with $\mu(X \setminus E) < \delta$ such that $f_n \to f$ uniformly on E. Thus, there is an index N such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. Thus, for $n \geq N$,

$$\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\} \subset X \setminus E$$

so

$$\mu(\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\}) < \varepsilon.$$

Thus, we have

$$\lim_{n\to\infty} \mu(\{\, x \in X : |f(x) - f_n(x)| \geq \varepsilon \,\}) = 0,$$

i.e., $f_n \to f$ in measure. ∎

Hence, since $f_{n_{k_j}} \to f$ in measure, but $M > 0$ we have a contradiction.

For (ii) since F is continuous given $\varepsilon > 0$ there exist $\delta > 0$ such that $|x - x'| < \delta$ implies $|F(x) - F(x')| < \varepsilon$. By part (i), $f_n \to f$ in measure if and only if every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ contains a subsequence $\{f_{n_{k_j}}\}$ that converges to f almost everywhere, i.e., given $\delta > 0$ there exists an index N such that $n_{k_j} \geq N$ implies

$$|f(x) - f_{n_{k_j}}(x)| < \delta$$

for almost every $x \in X$. Thus,

$$\left|F(f(x)) - F(f_{n_{k_j}}(x))\right| < \varepsilon$$

and we see that for every subsequence $\{F \circ f_{n_k}\}$ of $\{F \circ f_n\}$ we can find a subsequence $\{F \circ f_{n_{k_j}}\}$ that converges almost everywhere to $F \circ f$. ◀

Problem 4. Let $(X, \mathscr{F}, \mu)$ be a finite measure space and suppose $f \in L^1(\mu)$ is nonnegative. Suppose $1 < p < \infty$ and let $1 < q < \infty$ be its conjugate exponent, i.e., $1/p + 1/q = 1$. Suppose f has the property that

$$\int_E f \, \mathrm{d}\mu \leq \mu(E)^{1/q}$$

for all measurable sets E. Prove that $f \in L^r(\mu)$ for any $1 \leq r < p$.

[*Hint*: Consider $\left\{ x \in X : 2^n \leq f(x) < 2^{n+1} \right\}$, if you like.]

Solution. ▶ By previous problems, we know that if $\mu(X) < \infty$ and $f \in L^p(X)$, then $f \in L^r(X)$ for $1 \leq r < p$, so it suffices to show that $\|f\|_p < \infty$.

Instead of following the hint, consider the set

$$E_t = \{ x \in X : f(x) \geq t \}$$

and let

$$\omega(t) = \mu(E_t),$$

i.e., the distribution function of f. Then, we have

$$\int_0^\infty \omega(t) \, \mathrm{d}t = \int_X f \, \mathrm{d}\mu.$$

In particular, if we make the substitution $\alpha = t^{1/p}$, $\mathrm{d}\alpha = t^{1/q}/p \, \mathrm{d}t = \alpha^{p/q}/p \, \mathrm{d}t$, we have

$$\int_X f^r \, \mathrm{d}\mu = \int_0^\infty p\alpha^{-p/q}\omega(\alpha) \, \mathrm{d}\alpha.$$

Now, by Chebyshev's inequality, we have

$$t\omega(t) \leq \int_{E_t} f \,\mathrm{d}\mu \leq \omega(t)^{1/q}$$

so

$$\omega(t) \leq t^{-p}.$$

Thus,

$$\int_X f^r \,\mathrm{d}\mu = \int_0^\infty p\alpha^{-p/q}\omega(\alpha)\,\mathrm{d}\alpha \leq \int_0^\infty p\alpha^{-p-p/q}\,\mathrm{d}\alpha.$$

Since $p + p/q > 1$, the integral above is finite. Thus, $f \in L^p(X)$ and we have $f \in L^r(X)$ for all $1 \leq r < p$. ◀

Problem 5. Let f be a continuous function on $[-1, 1]$. Find

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|)\,\mathrm{d}x.$$

Solution. ► To find the limit of the integral

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|)\,\mathrm{d}x$$

we first make the following substitutions: Let $y = nx$, $\mathrm{d}y = n\,\mathrm{d}x$. Then

$$\int_{-1/n}^{1/n} f(x)(1 - n|x|)\,\mathrm{d}x = \frac{1}{n}\int_{-1}^{1} f(y/n)(1 - |y|)\,\mathrm{d}y.$$

By the extreme value theorem, since f is continuous and $[-1, 1]$ is compact f is bounded on $[-1, 1]$ by, say M. Let $g(x) = M$. Then $g \in L^1(X)$ since $\|g\|_1 = 2M$. Thus, by the Lebesgue dominated convergence theorem, since

$$|f(y/n)(1 - |y|)| \leq M$$

on $[-1, 1]$ and $g \in L^1([-1, 1])$ it follows that

$$\begin{aligned}
\lim_{n\to\infty} \int_{-1/n}^{1/n} f(x)(1 - n|x|)\,\mathrm{d}x &= \lim_{n\to\infty} \frac{1}{n}\int_{-1}^{1} f(y/n)(1 - |y|)\,\mathrm{d}y \\
&= \int_{-1}^{1} \lim_{n\to\infty}\left[\frac{f(y/n)(1 - |y|)}{n}\right]\mathrm{d}y \\
&= \int_{-1}^{1} \lim_{n\to\infty}\left[\frac{f(y/n)}{n} - \frac{|y|}{n}\right]\mathrm{d}y \\
&= 0.
\end{aligned}$$

◀

Problem 6. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $f \in L^p(\mu)$, $1 \le p < \infty$. Suppose E_n is a sequence of measurable sets satisfying $\mu(E_n) = 1/n$ for all n. Prove that

$$\lim_{n\to\infty}\left[n^{(p-1)/p}\int_{E_n}|f|\,\mathrm{d}\mu\right] = 0.$$

Solution. ► The result follows immediately by Hölder's inequality. Let $C = \|f\|_p$. Since $f \in L^p(X)$, then $f \in L^p(E_n)$ for all $n \in \mathbb{N}$. Thus, by Hölder's inequality

$$\begin{aligned}\|f\|_{L^1(E_n)} &\le \|f\|_{L^p(E_n)}\mu(E)^{1/q}\\ &\le C\mu(E)^{1/q}\\ &= C\mu(E)^{p/(p-1)}\\ &= Cn^{-p/(p-1)}\\ &= Cn^{p/(1-p)}.\end{aligned}$$

Hence, the integral is bounded above by

$$\begin{aligned}0 \le n^{(p-1)/p}\int_{E_n}|f|\,\mathrm{d}\mu &\le Cn^{(p-1)/p+p/(1-p)}\\ &= Cn^{(2p-1)/(p(1-p))}.\end{aligned}$$

Since $p > 1$, $1-p < 0$ and $2p-1 > 0$ so the exponent $(2p-1)/(p(1-p)) < 0$. Thus, as $n \to \infty$

$$Cn^{(2p-1)/(p(1-p))} \longrightarrow 0.$$

It follows that

$$\lim_{n\to\infty}\left[n^{(p-1)/p}\int_{E_n}|f|\,\mathrm{d}\mu\right] = 0.$$

◄

Problem 7. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{g_n\}$ be a sequence of nonnegative measurable functions with the property that $g_n \in L^1(\mu)$ for every n and $g_n \to g$ in $L^1(\mu)$. Let $\{f_n\}$ be another sequence of nonnegative measurable functions on $(X, \mathcal{F}, \mu)$.

(i) If $f_n \le g_n$ almost everywhere for every n, prove that

$$\limsup_{n\to\infty}\int_X f_n\,\mathrm{d}\mu \le \int_X \limsup_{n\to\infty} f_n\,\mathrm{d}\mu.$$

[*Hint*: Start by considering a subsequence $\{f_{n_k}\}$ such that

$$\lim_{n_k \to \infty} \int_X f_{n_k} \, \mathrm{d}\mu = \limsup_{n \to \infty} \int_X f_n \, \mathrm{d}\mu$$

and let $\{g_{n_{k_j}}\}$ be a subsequence of $\{g_{n_k}\}$ such that $g_{n_{k_j}} \to g$ almost everywhere.]

(ii) If $f_n \to f$ almost everywhere and if $f_n \leq g_n$ almost everywhere for all n, then $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Solution. ► Part (i) is a generalization of what is colloquially known as the reverse Fatou's lemma. Consider the sequence of measurable functions $\{h_n\}$ where $g_n - f_n$. Note that $g_n - f_n \geq 0$ for all $x \in X$ since $g_n \geq f$ for all $x \in X$. Thus, by Fatou's lemma, we have

$$\begin{aligned}
\liminf_{n \to \infty} \int_X g_n - f_n \, \mathrm{d}\mu &\leq \int_X \liminf_{n \to \infty} \int_X g_n - f_n \, \mathrm{d}\mu \\
\liminf_{n \to \infty} \left[\int_X g_n \, \mathrm{d}\mu - \int_X f_n \, \mathrm{d}\mu \right] &\leq \int_X \left[\liminf_{n \to \infty} g_n + \liminf_{n \to \infty} -f_n \right] \mathrm{d}\mu \\
\liminf_{n \to \infty} \int_X g_n \, \mathrm{d}\mu + \liminf \left[- \int_X f_n \, \mathrm{d}\mu \right] &\leq \int_X \liminf_{n \to \infty} g_n + \int_X \liminf_{n \to \infty} (-f_n) \, \mathrm{d}\mu \\
\liminf_{n \to \infty} \int_X g_n \, \mathrm{d}\mu - \limsup \int_X f_n \, \mathrm{d}\mu &\leq \int_X \liminf_{n \to \infty} g_n - \int_X \limsup_{n \to \infty} f_n \, \mathrm{d}\mu \\
\int_X g \, \mathrm{d}\mu - \limsup \int_X f_n \, \mathrm{d}\mu &\leq \int_X \liminf_{n \to \infty} g_n - \int_X \limsup_{n \to \infty} f_n \, \mathrm{d}\mu.
\end{aligned}$$

Now, let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ such that

$$\int_X f_{n_k} \, \mathrm{d}\mu \longrightarrow \limsup_{n \to \infty} \int_X f_n \, \mathrm{d}\mu.$$

Since $g_n \to g$ in $L^1(X)$, for every subsequence $\{g_{n_k}\}$ there exists a subsequence $\{g_{n_{k_j}}\}$ that converges to g. ◄

Problem 8. Let $f \in L^1(\mathbb{R})$. Consider the function

$$F(x) = \int_{\mathbb{R}} \exp(\mathrm{i}xt) f(t) \, \mathrm{d}t.$$

(i) Show that $F \in L^\infty(\mathbb{R})$ and that F is continuous at every $x \in \mathbb{R}$. Moreover, if $|t|^k f(t) \in L^\infty(\mathbb{R})$ for all $k \geq 1$, show that F is infinitely differentiable, i.e., $F \in C^\infty(\mathbb{R})$.

(ii) Suppose f is continuous as well as in $L^1(\mathbb{R})$. Show that $\lim_{|x|\to\infty} F(x) = 0$.

[*Hint*: Using $\exp(-\mathrm{i}\pi) = -1$, write $F(x) = \left(\int_{\mathbb{R}}(\exp(\mathrm{i}xt) - \exp(\mathrm{i}xt - \mathrm{i}\pi))\right)/2$.]

Solution. ► For part (a), we claim that

$$|F(x)| \leq \|f\|_1.$$

Since $|-|\colon \mathbb{C} \to \mathbb{R}$ is convex, by Jensen's inequality, we have

$$\begin{aligned}
|F(x)| &= \left|\int_X \exp(\mathrm{i}xt) f(t)\,\mathrm{d}t\right| \\
&\leq \int_X |\exp(\mathrm{i}xt) f(t)|\,\mathrm{d}t \\
&\leq \int_X |f(t)|\,\mathrm{d}t \\
&= \|f\|_1.
\end{aligned}$$

Thus, $\operatorname{ess\,sup}_{x\in\mathbb{R}} F \leq \|f_1\|$ so $F \in L^\infty(\mathbb{R})$. To see that F is continuous, take let $\varepsilon > 0$ be given. Consider the sequence of functions $f_n = f\chi_{\{|t|\leq n\}}$. Then $f_n \to f$ and, by Lebesgue's dominated convergence theorem, there exists an index N such that for every $n \geq N$ we have

$$\int_{\mathbb{R}} |f|\,\mathrm{d}t - \int_{\mathbb{R}} |f_n|\,\mathrm{d}t = \int_{\{|t|>n\}} |f|\,\mathrm{d}t < \frac{\varepsilon}{4}.$$

Let $\delta < \varepsilon/$ ◄

2.2 Bañuelos: Summer 2000

Problem 1. For any two subsets A and B of $\mathbb{R}$ define $A+B = \{ a+b : a \in A, b \in B \}$.

(i) Suppose A is closed and B is compact. Prove that $A+B$ is closed.
(ii) Give an example that shows that (i) may be false if we only assume that A and B are closed.

Solution. ► ◄

Problem 2. Suppose $f : [0,1] \to \mathbb{R}$ is differentiable at every $x \in [0,1]$ where by differentiability at 0 and 1 we mean right and left differentiability, respectively. Prove that f' is continuous if and only if f is uniformly differentiable. That is, if and only if for all $\varepsilon > 0$ there is an $h_0 > 0$ such that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

whenever $0 \leq x, x+h \leq 1$, $0 < |h| < h_0$.

Solution. ► ◄

Problem 3. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$ and let $F_1, \ldots, F_{17}$ be seventeen measurable subsets of X with $\mu(F_j) = 1/4$ for every j.

(i) Prove that five of these subsets must have an intersection of positive measure. That is, if $E_1, \ldots, E_k$ denotes the collection of all nonempty intersections of the F_j taken five at a time ($k \leq 6188$), show that at least one of these sets must have positive measure.
(ii) Is the conclusion in (i) true if we take sixteen sets instead of seventeen?

Solution. ► ◄

Problem 4. Let $f_n : X \to [0, \infty)$ be a sequence of measurable functions on the measure space $(X, \mathcal{F}, \mu)$. Suppose there is a positive constant M such that the functions $g_n(x) = f(x)\chi_{\{ f_n \leq M \}}(x)$ satisfy $\|g_n\|_1 \leq A/n^{4/3}$ and for which $\mu(\{ x \in X : f_n(x) > M \}) \leq B/n^{5/4}$, where A and B are positive constants independent of n. Prove that

$$\sum_{n=1}^{\infty} f_n < \infty$$

almost everywhere.

Solution. ► ◄

Problem 5. Let $\{g_n\}$ be a bounded sequence of functions on $[0,1]$ which is uniformly Lipschitz. That isthere is a constant M (independent of n) such that for all n, $|g_n(x) - g_n(y)| \le M|x-y|$ for all $x, y \in [0,1]$ and $|g_n(x)| \le M$ for all $x \in [0,1]$.

(i) Prove that for any $0 \le a \le b \le 1$,

$$\lim_{n\to\infty} \int_a^b g_n(x) \sin(2n\pi x)\,\mathrm{d}x = 0.$$

(ii) Prove that for any $f \in L^1[0,1]$,

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x) \sin(2n\pi x)\,\mathrm{d}x = 0.$$

Solution. ► ◄

Problem 6. Let $\{f_n\}$ be a sequence of nonnegative functions in $L^1[0,1]$ with the property that $\int_0^1 f_n(t)\,\mathrm{d}t = 1$ and $\int_{1/n}^1 f_n(t)\,\mathrm{d}t \le 1/n$ for all n. Define $h(x) = \sup_n f_n(x)$. Prove that $h \notin L^1[0,1]$.

Solution. ► ◄

2.3 Bañuelos: Winter 2007

Problem 1. Let $f\colon [0,1] \to \mathbb{R}$.

(i) Define what it means for f to be absolutely continuous.
(ii) Define what it means for f to be of bounded variation.
(iii) Let $V(f;0,x)$ be the total variation of f on $[0,x]$. Prove that if f is absolutely continuous on $[0,1]$ so is $V(f;0,x)$.

Solution. ► ◄

Problem 2.

(i) Suppose that $f\colon [0,1] \to \mathbb{R}$ is nondecreasing with $f(0) = 0$ and $f(1) = 1$. For $a > 0$, let A be set of all $x \in (0,1)$ for which

$$\limsup_{h\to 0} \frac{f(x+h) - f(x)}{h} > a.$$

Prove that $m^*(A) < 1/a$, where m^* denotes the Lebesgue outer measure.
(ii) Prove that there is no Lebesgue measurable set A in $[0,1]$ with the property that $m(A \cap I) = m(I)/4$ for every interval I.

[*Hint*: Consider the function $f(x) = \chi_A(x)$.]

Solution. ► ◄

Problem 3. Let $\{E_j\}_{j=1}^\infty$ be Lebesgue measurable sets in $[0,1]$ and let $E = \bigcup_{j=1}^\infty E_j$ and suppose there is an $\varepsilon > 0$ such that $\sum_{j=1}^\infty m(E_j) \le m(E) + \varepsilon$.

(i) Show that for all measurable sets $A \subset [0,1]$

$$\sum_{j=1}^{\infty} m(A \cap E_j) \le m(A \cap E) + \varepsilon.$$

(ii) Let A be the set of all $x \in [0,1]$ which are in at least two of E_j'. Prove that $m(A) \le \varepsilon$.

Solution. ► ◄

Problem 4. Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n\colon X \to [0,\infty)$ be a sequence measurable functions and suppose that $\|f_n\|_p \le 1$, $1 < p < \infty$, and that $f_n \to f$ almost everywhere. Prove

(i) $f \in L^p(\mu)$.
(ii) $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Solution. ►

◄

Problem 5.

Solution. ► ◄

Problem 6.

Solution. ► ◄

2.4 Bañuelos: Winter 2013

Problem 1.

(a)

(i) Define almost uniform convergence on the measure space $(X, \mathcal{F}, \mu)$.
(ii) Let f_n be a sequence of nonnegative measurable functions converging almost uniformly to the nonnegative function f. Prove that $\sqrt{f_n}$ converges almost uniformly to $\sqrt{f}$.

(b)

(i) Suppose f_n has the property that $\int_X |f_n| \, \mathrm{d}\mu \to 0$.
(ii) Does it follow that $f_n \to 0$ almost everywhere? Justify your answer.
(iii) Does it follow that $f_n \to 0$ almost uniformly? Justify your answer.

Solution. ► ◄

Problem 2. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1 \leq p \leq \infty$ and q be its conjugate exponent. Suppose $f_n \to f$ in L^p and $g_n \to g$ in L^q. Prove that $f_n g_n \to fg$ in L^1.

Solution. ► ◄

Problem 3. Let $\{a_k\}$ be a sequence of positive numbers converging to infinity. Prove that the following limit exists

$$\lim_{k \to \infty} \int_0^\infty \frac{\exp(-x) \cos x}{a_k x^2 + (1/a_k)} \, \mathrm{d}x$$

and find it. Make sure to justify all steps.

Solution. ► ◄

Problem 4. Let $(X, \mathcal{F}, \mu)$ be σ-finite and f be measurable such that for all $\lambda > 0$

$$\mu(\{ x \in X : |f(x)| > \lambda \}) \leq \frac{20}{\lambda^p}$$

where $1 < p < \infty$. Let q be the conjugate exponent of p. Prove that there is a constant C depending only on p such that

$$\int_E |f(x)|\,\mathrm{d}\mu \le Cm(E)^{1/q},$$

for all measurable sets E with $0 < \mu(E) < \infty$. (The inequality holds trivially when $\mu(E) = 0$ or $\mu(E) = \infty$.)

[*Hint*: Recall $\int_E |f(x)|\,\mathrm{d}\mu = \int_0^\infty ?\,\mathrm{d}\lambda$ and "break it" at the right place!]

Solution. ► ◄

Problem 5. Suppose $f\colon [0,1] \to \mathbb{R}$ is of bounded variation with $V(f;0,1) = \alpha$. For any $\beta > \alpha$, set

$$A = \left\{ x \in (0,1) : \limsup_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|} > \beta \right\}.$$

Prove that for any $0 < p < 1$, $m(A) \le (\alpha/\beta)^p$, where m denotes the Lebesgue measure.

Solution. ► ◄

Problem 6. Let $f \in L^1(0,1)$ and for $x \in (0,1)$, define

$$h(x) = \int_x^1 \frac{f(t)}{t}\,\mathrm{d}t.$$

(i) Prove that h is continuous on $(0,1)$.
(ii) Show that

$$\int_0^1 h(t)\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t.$$

Solution. ► ◄

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