MA557 Homework 6

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November 2, 2015

Problem 6.1

Let R be a Noetherian ring and I, J R-ideals. Write $I^{\langle J\rangle} = \bigcup_{n\geq 1}(I : J^n)$, which is called the *saturation of I with respect to J*. Show:

(a) If $I = \bigcap_{i=1}^m \mathfrak{q}_i$ with $\mathfrak{q}_i$ $\mathfrak{p}_i$-primary, then $I^{\langle J\rangle} = \bigcap_{J\not\subset\mathfrak{p}_i} \mathfrak{q}_i$.

(b) $I^{\langle J\rangle}$ is the unique largest R-ideal that coincides with I locally on the open set $\operatorname{Spec}(R) \smallsetminus V(J)$.

Proof. (a) We shall demonstrate double inclusion: Let $\bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal decomposition of I into primary ideals where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. $\implies$ Suppose $x \in I^{\langle J\rangle}$ then $xJ^n \subset I$ for some $n \geq 1$. Given i such that $\mathfrak{p}_i \not\supset J^*$ take $y \in J \smallsetminus \mathfrak{p}_i$. Then $xy^n \in \mathfrak{q}_i$ so $x \in \mathfrak{q}_i$ since $\mathfrak{q}_i$ is primary and $y \notin \mathfrak{p}_i$. Hence, $I^{\langle J\rangle} \subset \bigcap_{J\not\subset\mathfrak{p}_i} \mathfrak{q}_i$. $\impliedby$ Conversely, suppose that $x \in \bigcap_{J\not\subset\mathfrak{p}_i} \mathfrak{q}_i$ then $x \in \mathfrak{q}_i$ for all $\mathfrak{q}_i \not\supset J$. Take any $\mathfrak{p}_j$ containing J. Then $\mathfrak{p}_j = \operatorname{nil}(R/\mathfrak{q}_j)^c$ (this is easily seen from the fact that $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, i.e., $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary and the correspondence theorem for ideals) so there exists n_j with $xJ^{n_j} \subset \mathfrak{q}_j$ (since, in the quotient, $\bar{J}$ is nilpotent). Let n be the maximum of all such n_j then $xJ^n\mathfrak{q}_i$ for all i, i.e, $x \in (I : J^n) = \bigcap_i^m(\mathfrak{q}_i : J^n)$. Thus, $x \in I^{\langle J\rangle}$.

(b) We will prove that $I^{\langle J\rangle}$ is precisely the set of all $x \in R$ such that $x/1$ vanishes in $R_\mathfrak{p}$ for all $\mathfrak{p} \not\supset J$. $\implies$ Given $x \in I^{\langle J\rangle}$, $xJ^n \subset I$ for some $n \geq 1$. Let $\mathfrak{p}$ be a prime ideal not containing J and let $y \in J \smallsetminus \mathfrak{p}$. Then $xy^n \in I$ and $y^n \notin \mathfrak{p}$ so $x/1 = 0$ in $R_\mathfrak{p}$. $\impliedby$ Conversely, suppose that $x/1$ vanishes in $R_\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset R$. Then $xy = 0$ for some $y \in R \smallsetminus \mathfrak{p}$. Since $\mathfrak{p} = \sqrt{\mathfrak{q}_i}$ for some i, $y^n \in \mathfrak{q}_i$ for some $n \geq 1$. Let $\bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal decomposition of 0 (one exists since R is Noetherian) where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. By part (a), it suffices to show that ■

*Why does such an ideal exist? Well, suppose that $\mathfrak{p}_i \supset J$ for all $1 \leq i \leq m$. Then $J \subset \bigcap_{i=1}^m \mathfrak{p}_i = \bigcap_{i=1}^m \sqrt{\mathfrak{q}_i} = \sqrt{\bigcap_{i=1}^m \mathfrak{q}_i} = \sqrt{I}$ so that

Problem 6.2

Let R be a Noetherian ring. Show that R is reduced if and only if $\mathrm{Quot}(R)$ is a finite direct product of fields.

Proof. $\implies$ Suppose that R is reduced then the nilradical of R is precisely the 0 ideal. By 5.1(b), we know that the nilradical is equivalent to the union $\bigcup_{\mathfrak{p}\in \operatorname{Ass} R} \mathfrak{p}$. Moreover, by 5.4, we know that the set of associated primes of R is finite, say $\operatorname{Ass} R = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$. Therefore, $\mathrm{Quot}(R) = S^{-1}R$ where $S = R \smallsetminus \bigcup_{i=1}^n \mathfrak{p}_i$. Now, observe that by 5.3 the prime ideals of $\mathrm{Quot}(R)$ are precisely the ideals $S^{-1}\mathfrak{p}_i$ ■

Problem 6.3

Let R be a Noetherian ring and $x \in R$ an R-regular element. Show that $\operatorname{Ass}_R(R/(x^n)) = \operatorname{Ass}_R(R/(x))$ for every $n \geq 1$.

Proof. ■

Problem 6.4

Let $\varphi\colon R \to T$ be a homomorphism of rings where T is Noetherian, let ${}^a\varphi$ be the induced map on the spectra, and let N be a T-module. Show:

(a) $\operatorname{Ass}_R(N) = {}^a\varphi(\operatorname{Ass}_T(N))$.

(b) If N is finitely generated as a T-module then $\operatorname{Ass}_R(N)$ is finite.

Proof. ■

Problem 6.5

Let K be a field that is a finitely generated $\mathbb{Z}$-algebra. Show that K is a finite field.

Proof. ■

Problem 6.6

Let k be a Noetherian ring, R a finitely generated k-algebra, and $\mathrm{Aut}_k(R)$ the group of k-algebra automorphisms of R. For a subgroup G of $\mathrm{Aut}_k(R)$ write $R^G = \{\, x \in R \mid \sigma(x) = x \text{ for every } \sigma \in G \,\}$, which is called the *ring of invariants* of G. Show that if G is finite then R^G is a finitely generated k-algebra (and hence a Noetherian ring).

Proof. ■