MA571 Problem Set 7

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Problem 7.1 (Munkres §26, Ex. 8)

Theorem. *Let $f: X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the* graph *of f,*

$$G_f = \{ (x, f(x)) \mid x \in X \},$$

is closed in $X \times Y$.

[*Hint:* If G_f is closed and V is a neighborhood of $f(x_0)$, then the intersection of G_f and $X \times (Y - V)$ is closed. Apply Exercise 7.]

Proof. As we demonstrated in Problem 2.7 (Munkres §18, Ex. 17) Y is Hausdorff if and only if the diagonal, $\Delta_Y = \{ (y, y) \mid y \in Y \}$, is a closed subset of $Y \times Y$. Consider the map $F: X \times Y \to Y \times Y$ defined by $(x, y) \mapsto (f(x), y)$. This map is continuous by Theorem 18.4 as f is, by assumption, continuous and id_Y is continuous by 18.2(b) (since it is the inclusion $Y \hookrightarrow Y$). Then

$$\begin{aligned} F^{-1}(\Delta_Y) &= \{ (x, y) \mid F(x, y) \in \Delta_Y,\ x \in X,\ y \in Y \} \\ &= \{ (x, y) \mid (f(x), y) \in \Delta_Y,\ x \in X,\ y \in Y \} \\ &= \{ (x, y) \mid f(x) = y,\ x \in X,\ y \in Y \} \\ &= \{ (x, f(x)) \mid x \in X,\ y \in Y \} \\ &= G_f \end{aligned}$$

is closed by Theorem 18.1(3).

Conversely, suppose G_f is closed in $X \times Y$. Fix a point $x_0 \in X$ and let $V \subset Y$ be an arbitrary neighborhood of $f(x_0)$. Then $Y - V$ is a closed subset of Y so, by Problem 2.1 (Munkres §17, Ex. 3), the product $X \times (Y - V)$ is closed in $Y \times Y$. In particular, by Theorem 17.1(2), the intersection $B = G_f \cap X \times (Y - V)$ is closed in $X \times Y$. Thus, by Problem 6.5 (Munkres §26, Ex. 7), since Y is a compact Hausdorff space, the projection $\pi_1(B)$ onto X is a closed subset of X. But

$$\begin{aligned} B &= \{ (x, y) \mid (x, y) \in G_f \text{ and } (x, y) \in X \times (Y - V) \} \\ &= \{ (x, y) \mid y = f(x) \text{ and } (x, y) \in X \times (Y - V) \} \\ &= \{ (x, f(x)) \mid f(x) \in Y - V \} \end{aligned}$$

so we have that $\pi_1(B) = f^{-1}(Y - V) = X - f^{-1}(V)$. One containment is easy to see, namely "$\subset$": if $x \in B$ then $x = \pi_1(x, f(x))$ for at least one $f(x) \in Y - V$. To see the reverse inclusion, take $x \in f^{-1}(Y - V)$, then $f(x) \in Y - V$ so $(x, f(x)) \in B$, hence $x \in \pi_1(B)$. Thus, $X - \pi_1(B) = f^{-1}(V)$ is open so f is continuous. ∎

Problem 7.2 (Munkres §26, Ex. 9)

Generalize the tube lemma as follows:

Theorem. *Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that*

$$A \times B \subset U \times V \subset N.$$

Proof. The idea is to construct an appropriate covering of $A \times B$ using both compactness of A and compactness of B that will give us the open sets that we want. Fix an $a \in A$. Then, for every $b \in B$ there exists neighborhoods $U_b \subset X$ and $V_b \subset Y$ of a and b, respectively, such that $U_b \times V_b \subset N$ (by the definition of the product topology and since N is open). Then, since B is compact, by Lemma 26.1, there exists a finite subcollection, say $\{V_i\}_{i=1}^{n_a}$, that covers B. Let $U_a = \bigcup_{i=1}^{n_a} U_i$ and $V_a = \bigcup_{i=1}^{n_a} V_i$. Varying this over every $a \in A$, we obtain an open cover $\{U_a \times V_a\}_{a \in A}$; let's verify this: Let $(a, b) \in A \times B$, then $a \in U_a = \bigcup_{i=1}^{n_a} U_i$ (since each U_i is in fact a neighborhood of a) and $b \in V_a = \bigcup_{i=1}^{n_a} V_i$ so $b \in V_i$ for some $1 \leq q \leq n_a$. Thus, by Theorem 26.7, there exists a finite subcollection $\{U_i \times V_i\}_{i=1}^{n}$ covering $A \times B$. Take $U = \bigcup_{i=1}^{n} U_i$ and $V = \bigcap_{i=1}^{n} V_i$. Then, we claim that $A \times B \subset U \times V \subset N$.

It is clear, by construction of U and V, that $U \times V \subset N$ (and this follows from Lemma 5 proved on Homework 2, i.e., if $A, B \subset C$ then $A \cup B, A \cap B \subset C$). To see that $A \times B \subset U \times V$ take $(a, b) \in A \times B$. Then $a \in U_i$ for some $1 \leq i \leq n$ and $b \in V_i$ for all i (since $V_i \supset B$ for all $1 \leq i \leq n$)so $(a, b) \in U \times V$. Thus, we have

$$A \times B \subset U \times V \subset N$$

as desired. ∎

Problem 7.3 (Munkres §26, Ex. 12)

Let $p\colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(y)$ is compact, for each $y \in Y$. (Such a map is called a *perfect map.*) Show that if Y is compact, then X is compact.

[*Hint:* If U is an open set containing $p^{-1}(y)$, there is a neighborhood W of y such that $p^{-1}(W)$ is contained in U.]

Proof. First we shall prove Munkres's hint:

Claim. *Let $p\colon X \to Y$ be a closed map. If U is an open subset containing $p^{-1}(y)$ for some $y \in Y$, there exists a neighborhood W of y such that $p^{-1}(W) \subset U$.*

Proof of claim. Let $y \in Y$. Suppose that U is an open subset containing $p^{-1}(y)$. Then, $X - U$ is closed so $p(X-U)$ is closed. In particular, $y \notin p(X-U)$ (for if it were, we would have $p^{-1}(y) \subset X-U$, but $U \supset p^{-1}(y)$). Thus $Y - p(X-U)$ is a neighborhood of y so

$$p^{-1}(Y - p(X-U)) = p^{-1}(Y) - p^{-1}(p(X-U)) = X - p^{-1}(p(X-U)) \subset U$$

since, by Problem 1.1(a) (Munkres §2, Ex. 1(a)), we have that $p^{-1}(p(X-U)) \supset X - U$. ♣

Now let $\{U_\alpha\}$ be an open cover of X. Then, since $p^{-1}(y) \subset X = \bigcup U_\alpha$ is compact, by Lemma 26.1, there exists a finite subcollection, say $\{U_i\}_{i=1}^{n_y}$, that covers $p^{-1}(y)$. Let $U_y = \bigcup_{i=1}^{n_y} U_i$. Then, by the claim, there exists W_y neighborhood of y such that $p^{-1}(W_y) \subset \bigcup_{i=1}^{n_y} U_i$. We can do this for every $y \in Y$. In particular, we see that the collection $\{W_y\}_{y\in Y}$ is an open cover of Y so, since Y is compact, there exists a finite subcollection, say $\{W_{y_i}\}_{i=1}^{n}$, that covers Y. Then $p^{-1}(W_{y_i}) \subset U_{y_i}$ and

$$X = p^{-1}(Y) = \bigcup_{i=1}^{n} p^{-1}(W_{y_i}) \subset \bigcup_{i=1}^{n} U_{y_i}.$$

Thus, X is compact. ■

Problem 7.4 (Munkres §27, Ex. 2(b,d))

Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ε-neighborhood of A is contained in U.

Proof. (b) Fix $x \in X$ and consider the map $d_x : A \to \mathbf{R}$ given by $a \mapsto d(x, a)$. We claim that d_x is continuous so, assuming this has been proven, by the extreme value theorem there exists points $a, b \in A$ such that $d_x(a) \leq d_x(y) \leq d_x(b)$ for every $y \in A$. In particular, we have that $d(x, A) = \inf_{y \in A} d(x, y) = d(x, a) = d_x(a)$ ((i) $d_x(a) \leq d_x(y)$ for all y; (ii) if $d_x(a') \leq d_x(y)$ for all $y \in A$ then $d_x(a) = d_x(a')$ since $d_x(a) \leq d_x(y)$ for all $y \in A$).

That d_x is continuous follows from the following lemma (which we shall prove if we have time):

Lemma (Munkres §18, Ex;11)**.** *Let $f : X \times Y \to Z$. We say that F is* continuous in each variable separately *if for each y_0 in Y, the map $h : X \to Z$ defined by*

(d) For every $a \in A$, $r > 0$ such that $B_d(a, 2r) \subset U$ (we are guaranteed these exist since U is open in the metric topology) consider the collection $\{B_d(a, 2r)\}$. This collection is an open cover of A, so, by Lemma 26.1, there exists a finite subcollection, say $\{B_d(a_i, 2r_i)\}_{i=1}^n$ that covers A. Let $r = \min\{r_1, ..., r_n\}$ and a be the corresponding basepoint for the open ball. Then for every $b \in B_d(a_i, r_i)$, we have that

$$B_d(b, r) \subset B_d(a_i, r_i + \varepsilon) \subset B_d(a_i, 2r_i) \subset U$$

so, by part (c), $U(A, \varepsilon) = \bigcup_{b \in A} B_d(b, r) \subset U$ ■

Problem 7.5 (Munkres §27, Ex. 5)

Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X. [*Hint:* Imitate the proof of Theorem 27.7.]

This is a special case of the *Baire category theorem*, which we shall study in Chapter 8.

Proof. Mimicking the proof of Theorem 27.7, suppose $A \subset X$ is closed and $U \subset X$ is a nonempty open subset such that $U \not\subset X$. Then, since $U - A \neq \emptyset$ and X is a compact Hausdorff space, by Theorem 26.2, the union $A \cup (X - U)$ is compact so, by Theorem 26.4, there exist disjoint neighborhoods W and V about $A \cup (X - U)$ and x, respectively, such that

$$\overline{V} \subset X - (A \cup (X - U)) = (X - A) \cap U = U - A.$$

Now we show that any nonempty open set, U_0, has a point that is not in the union $\bigcup A_n$. For A_i, $i \geq 1$, U_{i-1} is a nonempty open subset such that $U_{i-1} \not\subset A_i$, hence, there is a nonempty open set $U_i \subset X$ such that $\overline{U}_i \subset U_{i-1} - A_i$. We thus have a nested sequence of nonempty closed subsets

$$\overline{U_1} \subset \overline{U_2} \subset \cdots$$

and their intersection is nonempty since X is compact, such that any point $x \in \bigcap \overline{U_i}$ belongs to U_0, but not to $\bigcup A_n$. ∎

Problem 7.6 (Munkres §29, Ex. 2(a))

Let $\{X_\alpha\}$ be an indexed family of nonempty spaces.

(a) Show that if $\prod X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α.

Proof of (a). Suppose $X = \prod X_\alpha$ is locally compact. Then, for every $\mathbf{x} \in X$, there exist a compact set C containing an open neighborhood U of $\mathbf{x}$. We may, without loss of generality, assume $U = \prod U_\alpha$ where $U_\alpha = X_\alpha$ for all but finitely many X_α. Suppose $U_\beta = X_\beta$. Then $\pi_\beta(C) = X_\beta$ is compact by Theorem 26.5. It follows that each X_α is compact for all but finitely man α. To see that each X_α is also locally compact we prove the following stronger result:

Lemma 15 (Munkres §20, Ex. 3)**.** *If $f\colon X \to Y$ is a continuous and open, then $f(X)$ is locally compact.*

Proof of lemma. Since X is locally compact, then for every $x \in X$ there exists a compact set C containing a neighborhood U of x. Then, $f(U) \subset f(C)$ is a compact set, by Theorem 26.5, containing a neighborhood, namely $f(U)$, of $f(x)$. Thus, $f(X)$ is locally compact. ♣

Now, since π_α is an open map (generalization of Munkres §16, Ex. 4), it follows that $\pi_\alpha(X) = X_\alpha$ is locally compact by Lemma 15. ■

Problem 7.7 (Munkres §29, Ex. 10)

Show that if X is a Hausdorff space that is locally compact at the point x, then for each neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Proof. Since x is locally compact, there exists a compact set C containing a neighborhood, say W, of x. Let U be an arbitrary neighborhood of x. Then $C - U \cap W$ is a closed subset of C, hence compact in the subspace topology on C so, by Lemma 26.1, it is compact in X. Moreover, $x \notin C - U \cap W$ so by Lemma 26.4, since X is Hausdorff, there exists disjoint neighborhoods V_1 and V_2 of x and $C - U \cap W$, respectively. Now, note that $\overline{V_1} \cap V_2 = \emptyset$ for otherwise for any point $x \in \overline{V_1} \cap V_2$, V_2 is a neighborhood of x so $V_1 \cap V_2 \neq \emptyset$, this is a contradiction. Let $V = V_1 \cap U \cap W$. Then $V \subset U$ and $V \subset C$ and, by Lemma B, $\overline{V} \subset C$, by Theorem 26.2, $\overline{V}$ is compact as desired. ∎

Problem 7.8 (A)

Let S^1 denote the circle
$$S^1 = \{ (x,y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1 \}$$
and let B^2 denote the closed disk
$$B^2 = \{ (x,y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1 \}.$$
Prove that the quotient space $(S^1 \times [0,1])/(S^1 \times 0)$ (see HW #4 for the notation) is homeomorphic to B^2.

Proof. Note that, by Theorem 26.6, it suffices to find a bijective continuous function $\bar{\varphi}$, since B^2 is Hausdorff and CS^1 is compact, by Theorem 26.5. Consider the map $\varphi\colon S^1 \times [0,1] \to B^2$ given by $(x,y,z) \mapsto (zx, zy)$. We will show that the induced map on the quotient space $\bar{\varphi}$ is a continuous bijection.

To see that $\bar{\varphi}$ is continuous, let $\Phi\colon S^1 \times [0,1] \to \mathbf{R}^2$ be φ with whose codomain has been extended. Then, note that $\pi_1 \circ \Phi(x,y) = zx$ and $\pi_2 \circ \Phi(x,y) = zy$ are continuous by Theorem 21.4, so by Theorem 18.4 and 18.2(e), φ is continuous. Moreover, if $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$ then $(x_1,y_1,z_1) = (x_2,y_2,z_2)$, in which case $\varphi(x_1,y_1,z_1) = \varphi(x_2,y_2,z_2)$, or $(x_1,y_1,0) = (x_2,y_2,0)$ for any $(x_1,y_1,0) \in S^1 \times 0$, in which case $\varphi(x_1,y_1,0) = 0 = \varphi(x_2,y_2,0)$, φ preserves the equivalence relation so by Theorem Q.3, $\bar{\varphi}$ is continuous.

Now we show that $\bar{\varphi}$ is bijective. Surjectivity of $\bar{\varphi}$ follows from surjectivity of φ. Let $(x,y) \in B^2$ and put $z_0 = \sqrt{x^2+y^2}$, $y_0 = y/z_0$ and $x_0 = x/x_0$. Note that $x_0^2 + y_0^2 = x^2/(x^2+y^2) + y^2/(x^2+y^2) = 1$ and $\sqrt{x^2+y^2} \leq 1$ for all x, y so $z \leq 1$ so $(x_0,y_0,z_0) \in S^1 \times [0,1]$. Thus, $\varphi(x_0,y_0,z_0) = (x,y)$ so φ is surjective. This implies that $\bar{\varphi}$ is surjective.

Finally, to see that $\bar{\varphi}$ is injective suppose $\bar{\varphi}([x_1,y_1,z_1]) = \bar{\varphi}([x_2,y_2,z_2])$ then
$$(z_1x_1, z_1y_1) = z_1(x_1,y_1) = z_2(x_2,y_2) = (z_2x_2, z_2y_2) \tag{(*)}$$
so, if $z_1 = 0$ we must have $z_2 = 0$ since $(0,0) \notin S^1$, hence $(x_1,y_1,z_1) = (x_1,y_1,0)$ and $(x_2,y_2,z_2) = (x_2,y_2,0)$ so $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$, if $z_1 \neq 0$ then we can divide by z_1 on both sides and we must have $z_2 = z_1$ since $\sqrt{(z_2x_2/z_1)^2 + (z_2y/z_1)^2} = |z_2/z_1|\sqrt{x_2^2+y_2^2} = 1$ and $z_1, z_2 \geq 0$ so, in fact, we have $(x_1,y_1,z_1) = (x_2,y_2,z_2)$ so in particular $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$. Thus, $\bar{\varphi}$ is injective.

We conclude, by Theorem 26.6, that φ is a homeomorphism and $CS^1 \cong B^2$. ∎