MA166: Exam 2 Prep

Carlos Salinas

March 8, 2016

As I promised, here are somewhat detailed solutions to two of the last Exam 2's for MA 166.

1 MA 166 Exam 2, Spring 2014

Problem 1.1. Evaluate the following integral

$$\int_0^\pi \sin^2 x \cos^2 x \, dx$$

Solution. The following trig identities are useful

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{1}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}. \tag{2}$$

With that in mind we compute the integral

$$\begin{aligned}
\int_0^\pi \sin^2 x \cos^2 x \, dx &= \int_0^\pi (\sin x \cos x)^2 \, dx \\
&= \int_0^\pi \left(\frac{1}{2}\sin 2x\right)^2 \, dx \\
&= \frac{1}{4}\int_0^\pi \sin^2 2x \, dx \\
&= \frac{1}{4}\int_0^\pi \frac{1-\cos 4x}{2} \, dx \\
&= \frac{1}{8}\int_0^\pi 1-\cos 4x \, dx \\
&= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^\pi \\
&= \frac{1}{8}[\pi - 0 - (0-0)] \\
&= \boxed{\frac{\pi}{8}.}
\end{aligned}$$

Answer: B. ☺

Problem 1.2. Evaluate the following integral

$$\int_0^{\pi/4} \sec^4 x \tan x \, dx.$$

Solution. The following identities are useful

$$\sec^2\theta - \tan^2\theta = 1. \tag{3}$$

Substitute $u = \tan x$, $du = \sec^2 x\ dx$

$$\begin{aligned}
\int_0^{\pi/4} \sec^4 x \tan x\ dx &= \int_0^{\pi/4} \sec^2 x \sec^2 x \tan x\ dx \\
&= \int_0^{\pi/4} \sec^2 x(1+\tan^2 x)\tan x\ dx \\
&= \int_0^{1} (1+u^2)u\ dx \\
&= \int_0^{1} u+u^3\ dx \\
&= \left[\frac{1}{2}u^2 + \frac{1}{4}u^4\right]_0^1 \\
&= \frac{1}{2}+\frac{1}{4}-0-0 \\
&= \boxed{\frac{3}{4}.}
\end{aligned}$$

Answer: A. ☺

Problem 1.3. After the appropriate trigonometric substitution, the integral

$$\int_2^5 \frac{dx}{\sqrt{x^2-4x+13}}$$

becomes

Solution. The general strategy for these types of problems is to complete the square in the denominator and make some sort of trig substitution

$$x^2-4x+13 = \left(x^2-4x+4\right)+9 = (x-2)^2+9.$$

Make the u-substitution $u = (x-2)/3$, $du = dx/3$

$$\begin{aligned}
\int_2^5 \frac{dx}{\sqrt{x^2-4x+13}} &= \int_2^5 \frac{dx}{3\sqrt{(x-2)^2/9+1}} \\
&= \frac{1}{3}\int_2^5 \frac{dx}{\sqrt{\left(\frac{x-2}{3}\right)^2+1}} \\
&= \int_0^1 \frac{du}{\sqrt{u^2+1}}
\end{aligned}$$

follow it up with the trig substitution $\tan\theta = u$, $\sec^2\theta\ d\theta = du$, $0 \le \theta \le \pi/4$

$$= \int_0^{\pi/4} \sec^2\theta \cos\theta\ d\theta$$

$$
\begin{aligned}
&= \int_0^{\pi/4} \sec\theta \, d\theta \\
&= [\ln|\sec\theta + \tan\theta|]_0^{\pi/4} \\
&= \ln\left|\frac{\sec\pi/4 + \tan\pi/4}{\sec 0 + \tan 0}\right| \\
&= \ln\left|\frac{\sqrt{2}+1}{1+0}\right| \\
&= \boxed{\ln\left|\sqrt{2}+1\right|}
\end{aligned}
$$

Answer: A. ☺

Problem 1.4. Compute

$$\int_3^4 \frac{3}{x^2 - x - 2} \, dx.$$

Solution. Factor

$$x^2 - x - 2 = (x-2)(x+1)$$

and use partial fractions

$$
\begin{aligned}
\frac{3}{(x-2)(x+1)} &= \frac{A}{x-2} + \frac{B}{x+1} \\
3 &= A(x+1) + B(x-2) \\
0x + 3 &= (A+B)x + A - 2B
\end{aligned}
$$

gives you $A - 2B = 3$, $A + B = 0$ so $A = -B$, $-B - 2B = 3$, $B = -1$ and $A = 1$. Now we can compute the integral

$$
\begin{aligned}
\int_3^4 \frac{3}{x^2 - x - 2} \, dx &= \int_3^4 \left[\frac{1}{x-2} - \frac{1}{x+1}\right] dx \\
&= [\ln|x-2| - \ln|x+1|]_3^4 \\
&= \left[\ln\left|\frac{x-2}{x+1}\right|\right]_3^4 \\
&= \ln\left|\frac{2}{5}\right| - \ln\left|\frac{1}{4}\right| \\
&= \ln\left|\frac{2/5}{1/4}\right| \\
&= \boxed{\ln\left|\frac{8}{5}\right|.}
\end{aligned}
$$

Remember your log properties!

Answer: B. ☺

Problem 1.5. It is known that

$$\int \frac{2x-3}{x(x^2+1)}dx = a\ln x + b\ln\left(x^2+1\right) + c\tan x + C$$

for some constants a, b, c and C. What is b?

Solution. There is a typo in the original problem; instead of $c\tan x$ it should be a $c\tan^{-1} x$. One thing you can do is use the fundamental theorem of calculus

$$f(x) = \frac{d}{dt}\int_a^x f(t)\,dt. \tag{4}$$

Applying the fundamental theorem on our function, we get

$$\begin{aligned}
\frac{2x-3}{x(x^2+1)} &= \frac{a}{x} + \frac{2bx}{x^2+1} + \frac{c}{x^2+1} \\
&= \frac{a}{x} + \frac{2bx+c}{x^2+1} \\
&= \frac{a\left(x^2+1\right) + (2bx+c)x}{x(x^2+1)} \\
&= \frac{(a+2b)x^2 + cx + a}{x(x^2+1)}.
\end{aligned}$$

Now we solve for the values in the numerator by noting that $a+2b=0$, $c=2$ and $a=-3$, so $b=3/2$.

Answer: E. ☺

Problem 1.6. Evaluate the integral

$$\int \frac{x^2+5x+1}{\left(x^2+1\right)^2}dx.$$

Solution. Rewrite the function

$$\frac{x^2+5x+1}{\left(x^2+1\right)^2} = \frac{\left(x^2+1\right)+5x}{\left(x^2+1\right)^2} = \underbrace{\frac{1}{x^2+1}}_{f_1} + \underbrace{\frac{5x}{\left(x^2+1\right)^2}}_{f_2}.$$

Let's compute these separately. If you recognize the integral of $1/\left(x^2+1\right)$ as being $\tan^{-1}(x)$, good for you; if not we can use the trig substitution $\tan\theta = x$, $\sec^2\theta\,d\theta = dx$

$$\begin{aligned}
I_1 &= \int \frac{dx}{x^2+1} \\
&= \int \sec^2\theta\cos^2\theta\,d\theta \\
&= \int 1\,d\theta
\end{aligned}$$

$$= \theta$$

substitute back $\theta = \tan^{-1}(x)$ and we have

$$= \tan^{-1}(x).$$

Now we compute I_2 by using the substitution $u = x^2$, $du = 2x\ dx$

$$\begin{aligned} I_2 &= \int \frac{5x}{\left(x^2+1\right)^2} \\ &= \frac{5}{2}\int \frac{1}{(u+1)^2} du \\ &= -\frac{5}{2}(u+1) \\ &= -\frac{5}{2}(x^2+1) \end{aligned}$$

so the integral is

$$I_1 + I_2 = \boxed{\tan^{-1}(x) - \frac{5}{2}\left(x^2+1\right)^{-1} + C.}$$

Answer: C. ☺

Problem 1.7. Approximate $\int_{-1}^{3} x^4\ dx$ using Simpson's rule with $n = 4$ subintervals.

Solution. Remember Simpson's rule? Neither do I, so here it is

$$\int_{x_0}^{x_1} f(x)\ dx = \int_{x_0}^{x_0+2\Delta x} \approx \frac{\Delta x}{3}(f(x_0) + 4f(x_0+\Delta x) + f(x_0+2\Delta x)). \tag{5}$$

Now, let's partition the interval $-1 \le x \le 3$ into 4 subintervals, $\Delta x = (3-(-1))/4 = 1$ so $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$. We have

$$\begin{aligned} \int_{-1}^{3} x^4\ dx &= \int_{-1}^{1} x^4\ dx + \int_{1}^{3} x^4\ dx \\ &\approx \frac{1}{3}\left(\left({x_0}^4 + 4{x_1}^4 + {x_2}^4\right) + \left({x_2}^4 + 4{x_3}^4 + {x_4}^4\right)\right) \\ &= \frac{1}{3}(1+0+1+1+64+81) \\ &= \boxed{\frac{148}{3}.} \end{aligned}$$

Answer: E. ☺

Problem 1.8. Which of the following improper integrals converge?

I. $\int_0^\infty xe^{-x^2}\,dx$

II. $\int_{-\infty}^\infty \frac{dx}{x}$

III. $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$

Solution. First, let's compute the integrals I, II and III. Here's I

$$\begin{aligned} I_1 &= \int_0^\infty xe^{-x^2}\,dx \\ &= \frac{1}{2}\int_0^\infty e^{-u}\,du \\ &= \left[-\frac{1}{2}e^{-u}\right]_0^\infty \\ &= \frac{1}{2}. \end{aligned}$$

Here's II

$$\begin{aligned} I_2 &= \int_{-\infty}^\infty \frac{dx}{x} \\ &= \int_{-\infty}^0 \frac{dx}{x} + \int_0^\infty \frac{dx}{x} \\ &= -\int_\infty^0 \frac{du}{u} + \int_0^\infty \frac{dx}{x} \\ &= \int_0^\infty \frac{du}{u} + \int_0^\infty \frac{dx}{x} \\ &= [\ln u]_0^\infty + [\ln x]_0^\infty \end{aligned}$$

this clearly divereges since $\ln x \to \infty$ as $x \to 0$ and $lnx \to \infty$ as $x \to \infty$. The same goes for $\ln u$. You can't win. Here's III

$$\begin{aligned} I_3 &= \int_{-1}^1 \frac{dx}{\sqrt[3]{x}} \\ &= \int_{-1}^1 x^{1/3}\,dx \\ &= \frac{3}{4}\left[x^{4/3}\right]_{-1}^1 \\ &= 0. \end{aligned}$$

Hence, I and III converge, but III does not. ☺

Problem 1.9. Find the exact length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/3$.

Solution. First find the derivative with respect to x

$$\frac{dy}{dx} = \tan x.$$

Then

$$\begin{aligned} S(0, \pi/3) &= \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx \\ &= \int_0^{\pi/3} \sec x\, dx \\ &= [\ln|\sec x + \tan x|]_0^{\pi/3} \\ &= \ln\left(2 + \sqrt{3}\right) - \ln(1 - 0) \\ &= \boxed{\ln\left(2 + \sqrt{3}\right).} \end{aligned}$$

☺

Problem 1.10. The curve $y = 2 - x^2$, $0 \leq x \leq 1$, is rotated around the y-axis to generate the surface S. Which is of the following formulas represents the area of the surface S?

Solution. First we find the derivative

$$\frac{dy}{dy} = -2x.$$

Now the arc will be

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \sqrt{1 + (-2x)^2} dx \\ &= \sqrt{1 + 4x^2} dx \end{aligned}$$

Then

$$\int_0^2 2\pi x\, ds = \int_0^2 2\pi x \sqrt{1 + 4x^2} dx$$

substitute $u = 1 + 4x^2$, $du = 8x\, dx$

$$\begin{aligned} &= \frac{\pi}{4} \int_0^{17} \sqrt{u}\, du \\ &= \frac{\pi}{4} \left[\frac{2}{3} x^{3/2}\right]_0^{17} \\ &= \frac{\pi}{6} \left[x^{3/2}\right]_0^{17} \end{aligned}$$

$$= \frac{17\sqrt{17}\pi}{6}.$$

Answer: D. ☺

Problem 1.11. Let $(\bar{x}, \bar{y})$ denote the coordinates of the center of mass of a region bounded by the curves $y = x^4$, $y = 0$ and $x = 1$, with density $\rho = 1$. What is $\bar{x}$?

Solution. Remember the formulas for the moments? Here they are

$$M_x = \rho \int_a^b \frac{f(x)^2 - g(x)^2}{2} dx \qquad M_y = \rho \int_a^b x(f(x) - g(x))\, dx. \tag{6}$$

Now, compute M_x and M_y

$$\begin{aligned} M_x &= \frac{1}{2}\int_0^1 x^8\, dx \\ &= \frac{1}{18}\left[x^9\right]_0^1 \\ &= \frac{1}{18} \end{aligned} \qquad \begin{aligned} M_y &= \int_0^1 x^5\, dx \\ &= \frac{1}{6}\left[x^6\right]_0^1 \\ &= \frac{1}{6}. \end{aligned}$$

And the area under the curve is

$$\begin{aligned} A &= \int_0^1 x^4\, dx \\ &= \frac{1}{5}\left[x^5\right]_0^1 \\ &= \frac{1}{5}. \end{aligned}$$

So

$$\bar{x} = \frac{M_y}{A} = \frac{1/6}{1/5} = \frac{5}{6}.$$

Answer: D. ☺

Problem 1.12. Determine whether the following sequences are convergent or divergent.

(1) $\left\{ a_n = \dfrac{2n}{3n+1} \right\}$

(2) $\{ a_n = \cos n\pi \}$

(3) $\left\{ a_n = n \sin\left(\dfrac{1}{n}\right) \right\}$.

Solution. (2) Recall that $a_n \cos n\pi = (-1)^n$. This sequence clearly does not converge because for its value goes from -1 to 1 and the distance between any two member $|a_n - a_{n-1}| = 2$, i.e., never gets any smaller.

(1) This converges. Set $n = x$ and by L'Hôpital's rule we have

$$\lim_{x\to\infty} \frac{2x}{3x+1} = \lim_{x\to\infty} \frac{2}{3} = 1.$$

So the sequence converges to 1.

(3) Lastly, we can show this sequence converges by making the substitution as $m = 1/n$ and now $1/n \to 0$ as $n \to \infty$ so we want

$$\lim_{m\to 0} \frac{1}{m} \sin(m).$$

This is a well known limit and you can use some geometry to show that

$$\cos m \leq \frac{\sin m}{m} \leq 1$$

so by the squeeze theorem, $\lim_{m\to 0} \sin(m)/m = 1$.

Answer: C ☺

2 Formula Sheet

Here is some useful stuff you should know before you take the exam

Section 2.1: Trigonometric identities

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{7}$$
$$\sec^2\theta - \tan^2\theta = 1 \tag{8}$$
$$\csc^2\theta - \cot^2\theta = 1. \tag{9}$$

Square and double-angle formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{10}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \tag{11}$$
$$\sin 2\theta = 2\sin\theta\cos\theta \tag{12}$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{13}$$

Section 2.2: Approximate Integrals

For the integral

$$\int_a^b f(x)\, dx$$

set $\Delta x = (b-a)/n$ where n is the number of desired steps and $\bar{x}_i = (x_{i-1}+x_i)/2$, i.e., the midpoint and $x_i = x_{i-1} + \Delta x$. Then

$$M_n = \Delta[f(\bar{x}_1) + \cdots + f(\bar{x}_n)] \tag{14}$$

$$T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \tag{15}$$

$$S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]. \tag{16}$$

Where M_n is the midpoint rule, T_n is the trapezoidal rule and S_n is Simpson's rule with n steps.

Section 2.3: Arc Length and Surface Area

Let $f(x)$ be an integrable function (yes, there are functions you cannot integrate) of x and $a \leq x \leq b$ then

Arc-length

The arc-length of f is

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx. \tag{17}$$

Surface area

The surface area of f is

$$\int 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{18}$$

about the y-axis and

$$\int 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \tag{19}$$

If $f(y)$ is a function of x,

$$\int 2\pi f(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dx \tag{20}$$

about the y-axis and

$$\int 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dx. \tag{21}$$

Section 2.4: Moments and centroids

Let ρ be the density and f, g be the curves for $a \leq x \leq b$. Then the area is

$$A = \int_a^b f(x) - g(x)\ dx, \tag{22}$$

the mass is

$$m = \rho A, \tag{23}$$

the moments are

$$M_x = \rho \int_a^b \frac{f(x)^2 - g(x)^2}{2} dx \qquad M_y = \rho \int_a^b x(f(x) - g(x))\ dx \tag{24}$$

and the centroids are

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}. \tag{25}$$