Math 527 - Homotopy Theory
Spring 2013
Homework 14 Solutions

Problem 1. Let $n \geq 2$. For any $f \in \pi_{2n-1}(S^n)$, denote its Hopf invariant by $H(f) \in \mathbb{Z}$.

a. Show that if n is odd, then $H(f) = 0$ holds for all $f \in \pi_{2n-1}(S^n)$.

Solution. Let $\alpha_f \in H^n(C(f)) \simeq \mathbb{Z}$ and $\beta_f \in H^{2n}(C(f)) \simeq \mathbb{Z}$ denote the standard generators. By graded commutativity of the cup product, we have

$$\begin{aligned} \alpha_f^2 &= (-1)^{|\alpha_f||\alpha_f|}\alpha_f^2 \\ &= (-1)^{n^2}\alpha_f^2 \\ &= -\alpha_f^2 \end{aligned}$$

so that $2\alpha_f^2 = 0$ holds, which implies $\alpha_f^2 = 0$ since $H^{2n}(C(f))$ is torsionfree. $\square$

b. Show that if n is not a power of 2, then there is no $f \in \pi_{2n-1}(S^n)$ satisfying $H(f) = 1$.

Solution. Consider reduction of coefficients mod 2

$$\varphi\colon H^*(C(f);\mathbb{Z}) \to H^*(C(f);\mathbb{Z}/2)$$

induced by the map of rings $\mathbb{Z} \to \mathbb{Z}/2$, and denote the reduced cohomology class by $\varphi(x) = \overline{x}$. By the universal coefficient theorem (or cellular cohomology), the mod 2 cohomology of $C(f)$ is given by

$$H^k(C(f);\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } k = 0, n, 2n \\ 0 & \text{otherwise} \end{cases}$$

and φ is just the usual reduction $\mathbb{Z} \to \mathbb{Z}/2$ mod 2 in each degree. Hence, to prove the result, it suffices to show that $\overline{\alpha_f^2} = 0 \in H^{2n}(C(f);\mathbb{Z}/2)$ holds for all f. Now we have

$$\begin{aligned} \overline{\alpha_f^2} &= \overline{\alpha_f}^2 \text{ since } \varphi \text{ is a ring map} \\ &= \mathrm{Sq}^n\,\overline{\alpha_f} \end{aligned}$$

where Sq^i denotes Steenrod squares. Since n is not a power of 2, Sq^n is decomposable, i.e. it can be written as a sum

$$\mathrm{Sq}^n = \sum \mathrm{Sq}^{I^l}$$

where each $I^l = (i_1^l, i_2^l, \ldots, i_{m_l}^l)$ is a multi-index where all indices satisfy $i_j^l < n$. Since $H^k(C(f);\mathbb{Z}/2) = 0$ holds for $n < k < 2n$, we have

$$\begin{aligned} \mathrm{Sq}^{I^l}\,\overline{\alpha_f} &= \mathrm{Sq}^{i_1^l}\,\mathrm{Sq}^{i_2^l} \ldots \mathrm{Sq}^{i_{m_l}^l}\,\overline{\alpha_f} \\ &= \mathrm{Sq}^{i_1^l}\,\mathrm{Sq}^{i_2^l} \ldots \mathrm{Sq}^{i_{m_l-1}^l}(0) \\ &= 0 \end{aligned}$$

for all such multi-indices I^l, and therefore:

$$\begin{aligned} \mathrm{Sq}^n\,\overline{\alpha_f} &= \sum \mathrm{Sq}^{I^l}\,\overline{\alpha_f} \\ &= \sum 0 \\ &= 0. \quad \square \end{aligned}$$

c. Let $\eta\colon S^3 \to S^2$ and $\nu\colon S^7 \to S^4$ denote the Hopf bundles. Show that these two maps satisfy $H(\eta) = \pm 1$ and $H(\nu) = \pm 1$.

Solution. The cofiber of η is complex projective space $C(\eta) \cong \mathbb{C}P^2$, whose cohomology is

$$H^*(\mathbb{C}P^2) \cong \mathbb{Z}[\alpha]/\alpha^3$$

where $\alpha = \alpha_\eta$ is a generator of $H^2(\mathbb{C}P^2) \simeq \mathbb{Z}$. Thus we have $\beta_\eta = \pm\alpha^2$.

Likewise, the cofiber of η is quaternionic projective space $C(\nu) \cong \mathbb{H}P^2$, whose cohomology is

$$H^*(\mathbb{H}P^2) \cong \mathbb{Z}[\alpha]/\alpha^3$$

where $\alpha = \alpha_\nu$ is a generator of $H^4(\mathbb{H}P^2) \simeq \mathbb{Z}$. Thus we have $\beta_\nu = \pm\alpha^2$.

Problem 2. Let n be a positive *even* integer. Let $\iota \in \pi_n(S^n)$ denote the class of the identity map, and consider the Whitehead product $[\iota, \iota] \in \pi_{2n-1}(S^n)$. Show that its Hopf invariant $H([\iota,\iota])$ is equal to 2.

Solution. Consider the map of cofiber sequences

$$\begin{tikzcd} S^{2n-1} \arrow[r, "W"] \arrow[d, "\mathrm{id}"'] & S^n \vee S^n \arrow[r] \arrow[d, "\nabla"'] & S^n \times S^n \arrow[r] \arrow[d, "\varphi"'] & S^{2n} \arrow[d, "\mathrm{id}"'] \\ S^{2n-1} \arrow[r, "{[\iota,\iota]}"'] & S^n \arrow[r] & C([\iota,\iota]) \arrow[r] & S^{2n} \end{tikzcd}$$

where ∇ denotes the fold map. The horizontal maps in the square

$$\begin{tikzcd} S^n \vee S^n \arrow[r] \arrow[d, "\nabla"'] & S^n \times S^n \arrow[d, "\varphi"'] \\ S^n \arrow[r] & C([\iota,\iota]) \end{tikzcd}$$

induce isomorphisms on H^n, and the fold map induces the diagonal map on H^n

$$\nabla^* \colon H^n(S^n) \to H^n(S^n \vee S^n) \cong H^n(S^n) \oplus H^n(S^n)$$
$$\alpha \mapsto \alpha_1 + \alpha_2$$

where $\alpha \in H^n(S^n) \simeq \mathbb{Z}$ denotes the standard generator and $\alpha_1, \alpha_2 \in H^n(S^n \vee S^n) \simeq \mathbb{Z} \oplus \mathbb{Z}$ denote the two generators.

By abuse of notation, we use the same notation for the corresponding classes in $H^n\left(C([\iota,\iota])\right) \cong H^n(S^n)$ and $H^n(S^n \times S^n) \cong H^n(S^n \vee S^n)$. Thus the map induced by φ on H^n satisfies $\varphi^*(\alpha) = \alpha_1 + \alpha_2$.

The horizontal maps in the square

$$\begin{tikzcd} S^n \times S^n \arrow[r] \arrow[d, "\varphi"'] & S^{2n} \arrow[d, "\mathrm{id}"'] \\ C([\iota,\iota]) \arrow[r] & S^{2n} \end{tikzcd}$$

induce isomorphisms on H^{2n}, and so does id, and hence φ does as well. Denote by $\beta \in H^{2n}(S^{2n}) \simeq \mathbb{Z}$ the standard generator and, by abuse of notation, the corresponding class in $H^{2n}\left(C([\iota,\iota])\right) \cong H^{2n}(S^{2n})$. Then the square implies the relation $\varphi^*(\beta) = \beta \in H^{2n}(S^{2n} \times S^{2n})$.

The class $\alpha^2 \in H^{2n}\left(C([\iota,\iota])\right)$ pulls back to $H^{2n}(S^n \times S^n)$ via φ to the class:

$$\begin{aligned}
\varphi^*(\alpha^2) &= \varphi^*(\alpha)^2 \\
&= (\alpha_1 + \alpha_2)^2 \\
&= \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2\alpha_1 + \alpha_2^2 \\
&= \alpha_1\alpha_2 + \alpha_2\alpha_1 \ \text{ by the Kunneth formula } H^*(S^n \times S^n) \cong H^*(S^n) \otimes_{\mathbb{Z}} H^*(S^n) \\
&= \alpha_1\alpha_2 + (-1)^{|\alpha_1||\alpha_2|}\alpha_1\alpha_2 \\
&= \alpha_1\alpha_2 + (-1)^{n^2}\alpha_1\alpha_2 \\
&= 2\alpha_1\alpha_2 \\
&= 2\beta \\
&= 2\varphi^*(\beta) \\
&= \varphi^*(2\beta).
\end{aligned}$$

Since $\varphi^* \colon H^{2n}\left(C([\iota,\iota])\right) \xrightarrow{\cong} H^{2n}(S^n \times S^n)$ is an isomorphism, the equation

$$\alpha^2 = 2\beta$$

holds in $H^{2n}\left(C([\iota,\iota])\right)$, which proves $H([\iota,\iota]) = 2$. $\square$

Problem 3. Compute the following rational cohomology algebras.

a. $H^*(K(\mathbb{Z},3);\mathbb{Q})$

Solution. First note that by Hurewicz, given $n \geq 2$, we have

$$H_k(K(\mathbb{Z},n);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } 0 < k < n \\ \mathbb{Z} & \text{if } k = n. \end{cases}$$

By the universal coefficient theorem, we have

$$H^k(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 0 \\ 0 & \text{if } 0 < k < n \\ \mathbb{Q} & \text{if } k = n. \end{cases}$$

Recall that $H^*(K(\mathbb{Z},2);\mathbb{Q}) \cong \mathbb{Q}[\iota_2]$ is a polynomial algebra on a class $\iota_2 \in H^2(K(\mathbb{Z},2);\mathbb{Q})$ which is the image in $\mathbb{Q}$-coefficients of the fundamental class in $H^2(K(\mathbb{Z},2);\mathbb{Z})$.

Consider the path loop fibration

$$K(\mathbb{Z},2) \to PK(\mathbb{Z},3) \to K(\mathbb{Z},3)$$

where the base space $K(\mathbb{Z},3)$ is simply-connected. Consider its cohomology Serre spectral sequence with rational coefficients

$$E_2^{p,q} = H^p(K(\mathbb{Z},3);H^q(K(\mathbb{Z},2);\mathbb{Q})) \Rightarrow H^{p+q}(\mathrm{pt};\mathbb{Q}).$$

Observe that the E_2-term is concentrated in even rows, and therefore all differentials d_r with r even are zero (since they go down $r-1$ steps).

Also note that the columns $E_2^{1,*}$ and $E_2^{2,*}$ are zero. In particular, the entry $E_2^{4,0} = H^4(K(\mathbb{Z},3);\mathbb{Q})$ could only be hit by the differential $d_4 \equiv 0$, and thus $E_2^{4,0}$ is never hit. We deduce:

$$H^4(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{4,0} = E_\infty^{4,0} = 0$$

and therefore the column $E_2^{4,*}$ is also zero. The E_2-term looks like this:

6	$\mathbb{Q}\iota_2^3$			$\mathbb{Q}\iota_3\iota_2^3$		?
5						
4	$\mathbb{Q}\iota_2^2$			$\mathbb{Q}\iota_3\iota_2^2$		?
3						
2	$\mathbb{Q}\iota_2$			$\mathbb{Q}\iota_3\iota_2$		?
1						
0	$\mathbb{Q}1$			$\mathbb{Q}\iota_3$		?
	0	1	2	3	4	5

where elements denote generators (as $\mathbb{Q}$-vector spaces), and blank entries are zero.

Since $d_2 \equiv 0$ is zero, we have $E_3 = E_2$. The class $\iota_3 \in E_3^{3,0}$ can only be hit by d_3, and therefore

$$d_3 \colon \mathbb{Q} \simeq E_3^{0,2} \to E_3^{3,0} \simeq \mathbb{Q}$$

is an isomorphism, or equivalently:

$$d_3(\iota_2) = c\iota_3$$

for some scalar $c \neq 0$. By the (graded) Leibniz rule and graded commutativity, we have

$$\begin{aligned} d_3(\iota_2^k) &= d_3(\iota_2)\iota_2^{k-1} + (-1)^{|\iota_2|}\iota_2 d_3(\iota_2)\iota_2^{k-2} + \ldots + (-1)^{(k-1)|\iota_2|}\iota_2^{k-1} d_3(\iota_2) \\ &= k d_3(\iota_2)\iota_2^{k-1} \\ &= kc\, \iota_3 \iota_2^{k-1}. \end{aligned}$$

Therefore the differential

$$d_3 \colon \mathbb{Q} \simeq E_3^{0,2k} \to E_3^{3,2k-2} \simeq \mathbb{Q}$$

is an isomorphism for all $k \geq 1$. It follows that d_3 from the column 3 to the column 6 is zero:

$$d_3 \colon E_3^{3,q} \xrightarrow{0} E_3^{6,q-2}.$$

The E_3-term looks like this:

6	$\mathbb{Q}\iota_2^3$			$\mathbb{Q}\iota_3\iota_2^3$		?	?
5							
4	$\mathbb{Q}\iota_2^2$			$\mathbb{Q}\iota_3\iota_2^2$		?	?
3							
2	$\mathbb{Q}\iota_2$			$\mathbb{Q}\iota_3\iota_2$		?	?
1							
0	$\mathbb{Q}1$			$\mathbb{Q}\iota_3$		?	?
	0	1	2	3	4	5	6

where the drawn differentials d_3 are isomorphisms. Therefore the E_4 and E_5 terms look like this:

6						?	?
5							
4						?	?
3							
2						?	?
1							
0	$\mathbb{Q}1$					?	?
	0	1	2	3	4	5	6

Note that $E_2^{5,0} = H^5(K(\mathbb{Z},3);\mathbb{Q})$ could only be hit by the differential $d_2 \equiv 0$ and by d_5, but now we learn from the E_5-term that $E_5^{5,0} = E_2^{5,0}$ is not hit by d_5, because of the equality $E_5^{0,4} = 0$. We deduce:

$$H^5(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{5,0} = E_\infty^{5,0} = 0$$

and therefore the column $E_2^{5,*}$ was zero to begin with.

Likewise, $E_2^{6,0} = H^6(K(\mathbb{Z},3);\mathbb{Q})$ could only be hit by the differential d_3, which we showed does not happen, and by $d_6 \equiv 0$. We deduce:

$$H^6(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{6,0} = E_\infty^{6,0} = 0$$

and therefore the column $E_2^{6,*}$ was zero to begin with.

But now $E_2^{7,0} = H^7(K(\mathbb{Z},3);\mathbb{Q})$ could not have been hit by d_3, because of the equality $E_3^{4,2} = 0$, nor can it be hit by further differentials, because of the zeroes in the E_4-term. We deduce:

$$H^7(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{7,0} = E_\infty^{7,0} = 0$$

and therefore the column $E_2^{7,*}$ was zero to begin with. Repeating this argument inductively, we deduce:

$$H^p(K(\mathbb{Z},3);\mathbb{Q}) = E_2^{p,0} = E_\infty^{p,0} = 0$$

for all $p \geq 7$ and therefore the E_2-term was concentrated in the columns 0 and 3. We conclude

$$H^p(K(\mathbb{Z},3);\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } p = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

and the multiplicative structure is determined by the relation $\iota_3^2 = 0$ given by graded commutativity.

In summary, the cohomology algebra

$$H^*(K(\mathbb{Z},3);\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}[\iota_3]$$

is an exterior algebra over $\mathbb{Q}$ on the generator $\iota_3 \in H^3(K(\mathbb{Z},3);\mathbb{Q})$. $\square$

b. $H^*(K(\mathbb{Z},4);\mathbb{Q})$

Solution. Consider the path loop fibration

$$K(\mathbb{Z},3) \to PK(\mathbb{Z},4) \to K(\mathbb{Z},4)$$

where the base space $K(\mathbb{Z},4)$ is simply-connected. Consider its cohomology Serre spectral sequence with rational coefficients

$$E_2^{p,q} = H^p\left(K(\mathbb{Z},4); H^q\left(K(\mathbb{Z},3);\mathbb{Q}\right)\right) \Rightarrow H^{p+q}(\mathrm{pt};\mathbb{Q}).$$

By part (a), the E_2-term is concentrated in two rows $q = 0, 3$ and looks like this:

4							
3	$\mathbb{Q}\iota_3$				$\mathbb{Q}\iota_4\iota_3$	?	?
2							
1							
0	$\mathbb{Q}1$				$\mathbb{Q}\iota_4$	?	?
	0	1	2	3	4	5	6

There is only room for one non-trivial differential, namely d_4, and it must destroy everything, except the $\mathbb{Q}$ in the corner $(0,0)$. It follows that

$$d_4 \colon \mathbb{Q} \simeq E_4^{0,3} \to E_4^{4,0} \simeq \mathbb{Q}$$

is an isomorphism, or equivalently:

$$d_4(\iota_3) = x_4 = c\iota_4$$

for some scalar $c \neq 0$. The E_4-term looks like this:

4									
3	$\mathbb{Q}\iota_3$				$\mathbb{Q}x_4\iota_3$				$\mathbb{Q}x_4^2\iota_3$
2									
1									
0	$\mathbb{Q}1$				$\mathbb{Q}x_4$				$\mathbb{Q}x_4^2$
	0	1	2	3	4	5	6	7	8

Indeed, we have $E_4^{4,3} \simeq \mathbb{Q}x_4\iota_3$ and the differential

$$d_4 \colon E_4^{4,3} \xrightarrow{\simeq} E_4^{8,0}$$

is an isomorphism, sending the generator $x_4\iota_3$ to a generator

$$\begin{aligned} d_4(x_4\iota_3) &= d_4(x_4)\iota_3 + (-1)^{|x_4|}x_4 d_4(\iota_3) \\ &= 0 + x_4x_4 \\ &= x_4^2. \end{aligned}$$

Repeating this argument inductively, we obtain for all $k \geq 0$ the isomorphism $E_4^{4k,3} \simeq \mathbb{Q}x_4^k\iota_3$ and the differential

$$d_4 \colon E_4^{4k,3} \xrightarrow{\simeq} E_4^{4(k+1),0}$$

is an isomorphism, sending the generator $x_4^k\iota_3$ to a generator

$$\begin{aligned} d_4(x_4^k\iota_3) &= d_4(x_4^k)\iota_3 + (-1)^{|x_4^k|}x_4^k d_4(\iota_3) \\ &= x_4^{k+1} \end{aligned}$$

and those are the only non-trivial differentials. We conclude

$$H^p(K(\mathbb{Z},4);\mathbb{Q}) = \begin{cases} \mathbb{Q}x_4^k \cong \mathbb{Q}\iota_4^k & \text{if } p = 4k \\ 0 & \text{otherwise} \end{cases}$$

so that the cohomology algebra

$$H^*(K(\mathbb{Z},4);\mathbb{Q}) \cong \mathbb{Q}[\iota_4]$$

is a polynomial algebra over $\mathbb{Q}$ on the generator $\iota_4 \in H^4(K(\mathbb{Z},4);\mathbb{Q})$. $\square$