

## MA 523: Homework 2

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**Problem 2.1**

Verify assertion (36) in [E, §3.2.3], that when  $\Gamma$  is not flat near  $x^0$  the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here  $\nu(x^0)$  denotes the normal to the hypersurface  $\Gamma$  at  $x^0$ ).

**Solution.** ► First, note that the condition

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the noncharacteristic boundary condition when  $\Gamma$  is flat near  $x^0$  since  $\nu(x^0) = (0, \dots, 0, -1)$  giving us

$$\begin{aligned} 0 &\neq D_p F(p^0, z^0, x^0) \cdot (0, \dots, 0, -1) \\ &= -F_{p_n}(p^0, z^0, x^0) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

To show (2.1), we will straighten the boundary near  $x^0$  and apply the noncharacteristic boundary conditions. Let  $\Phi, \Psi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be smooth maps such that  $\Psi = \Phi^{-1}$  and  $\Phi$  straightens out  $\partial U$  near  $x^0$ . Then, setting  $y^0 := (y_1, \dots, y_{n-1}, 0) = \Phi(x^0)$  and  $v(y) = u(\Psi(y))$ , our PDE becomes

$$0 = F(Dv(y)D\Phi(\Psi(y)), v(y), \Psi(y)).$$

From here we follow the proof of Lemma 1 in [E, §3.2.3]. Let  $G: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the map given by

$$\begin{cases} G^i(p, y) = p_i - g_{x_i}(y), \\ G^n(p, y) = F(p, g(y), y). \end{cases}$$

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**Problem 2.2**

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions  $u(x, 0) = g(x)$  is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some  $t > 0$ , unless  $a(g(x))$  is a nondecreasing function of  $x$ .

**Solution.** ►

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**Problem 2.3**

Show that the function  $u(x, t)$  defined by  $t \geq 0$  by

$$u(x, t) = \begin{cases} -\frac{2}{3} \left( t + \sqrt{3x + t^2} \right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law  $u_t + (u^2/2)_x = 0$  (*inviscid Burger's equation*).

**Solution.** ►

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