

# Fall 2015 Notes

Carlos Salinas

September 24, 2015

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## 1 McClure's 571 Problems

### 1.1 Midterm I (Fall 2015)

**Problem 1.1.1.** Let  $A \subset X$  and  $B \subset Y$ . Show that the space  $X \times Y$ ,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

*Proof.* ■

**Problem 1.1.2.** Let  $X$  be a topological space and let  $A$  be a dense subset of  $X$ . Let  $Y$  be a Hausdorff space and let  $g, h: X \rightarrow Y$  be continuous functions which agree on  $A$ . Prove that  $g = h$ .

*Proof.* ■

**Problem 1.1.3.** Let  $X$  and  $Y$  be topological spaces and let  $f: X \rightarrow Y$  be a continuous function. Let  $G_f$  (called the *graph* of  $f$ ) be the subspace  $\{x \times f(x) \mid x \in X\}$  of  $X \times Y$ . Prove that if  $Y$  is Hausdorff then  $G_f$  is closed.

*Proof.* ■

**Problem 1.1.4.** Let  $X$  be a topological space and let  $f, g: X \rightarrow \mathbf{R}$  be continuous. Define  $h: X \rightarrow \mathbf{R}$  by

$$h(x) = \min\{f(x), g(x)\}.$$

Use the pasting lemma to prove that  $h$  is continuous. (You will not get full credit for any other method.)

*Proof.* ■

**Problem 1.1.5.** Let  $X$  and  $Y$  be topological spaces and let  $f: X \rightarrow Y$  be a function with the property that

$$f(\overline{A}) \subset \overline{f(A)}$$

for all subsets  $A$  of  $X$ . Prove that  $f$  is continuous.

*Proof.* ■

**Problem 1.1.6.** Let  $X$  and  $Y$  be topological spaces and let  $f: X \rightarrow Y$  be a continuous function. Prove that

$$f(\overline{A}) \subset \overline{f(A)}$$

for all subsets  $A$  of  $X$ .

*Proof.* ■

**Problem 1.1.7.** Let  $X$  be any topological space and let  $Y$  be a Hausdorff space. Let  $f, g: X \rightarrow Y$  be continuous functions. Prove that the set  $\{x \in X \mid f(x) = g(x)\}$  is closed.

*Proof.* ■

**Problem 1.1.8.** Let  $X$  be a topological space and  $A$  a subset of  $X$ . Suppose that

$$A \subset \overline{X \setminus \overline{A}}.$$

Prove that  $\overline{A}$  does not contain any nonempty open set.

*Proof.* ■

**Problem 1.1.9.** Let  $X$  be a topological space with a countable basis. Prove that every open cover of  $X$  has a countable subcover.

*Proof.* ■

**Problem 1.1.10.** Let  $X_\alpha$  be an infinite family of topological spaces.

- (a) Define the product topology on  $\prod X_\alpha$ .
- (b) For each  $\alpha$ , let  $A_\alpha$  be a subspace of  $X_\alpha$ . Prove that  $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$ .

*Proof.* ■

**Problem 1.1.11.** Suppose that we are given an indexing set  $A$ , and for each  $\alpha \in A$  a topological space  $X_\alpha$ . Suppose also that for each  $\alpha \in A$  we are given a point  $b_\alpha \in X_\alpha$ . Let  $Y = \prod X_\alpha$  with the product topology. Let  $\pi_\alpha: Y \rightarrow X_\alpha$  be the projection. Prove that the set

$$S = \{ y \in Y \mid \pi_\alpha(y) = b_\alpha \text{ except for finitely many } \alpha \}$$

is dense in  $Y$  (that is, its closure is  $Y$ ).

*Proof.* ■

**Problem 1.1.12.** Let  $X$  be the Cartesian product  $\mathbf{R}^\omega = \prod_{i=1}^\infty \mathbf{R}$  with the box topology (recall that a basis for this topology consists of all sets of the form  $\prod_{i=1}^\infty U_i$ , where each  $U_i$  is open in  $\mathbf{R}$ ). Let  $f: \mathbf{R} \rightarrow X$  be the function which takes  $t$  to  $(t, t, t, \dots)$ . Prove that  $f$  is not continuous.

*Proof.* ■

**Problem 1.1.13.** Prove that the countable product  $\mathbf{R}^\omega$  (with the product topology) has the following property: there is a countable family  $\mathcal{F}$  of neighborhoods of the point  $\mathbf{0} = (0, 0, 0, \dots)$  such that for every neighborhood  $V$  of  $\mathbf{0}$  there is a  $U \in \mathcal{F}$  with  $U \subset V$ .

Note: the book proves that  $\mathbf{R}^\omega$  is a metric space, but you may not use this in your proof. Use the definition of the product topology.

*Proof.* ■

**Problem 1.1.14.** Let  $X$  be the two-point set  $\{0, 1\}$  with the discrete topology. Let  $Y$  be a countable product of copies of  $X$ , thus an element of  $Y$  is a sequence of 0's and 1's. For each  $n \geq 1$ , let  $y_0 \in Y$  be the element  $(1, 1, 1, \dots, 1, 0, 0, 0, \dots)$ , with  $n$  1's at the beginning and all other entries 0. Let  $y \in Y$  be the element with all 1s. Prove that the set  $\{y_n\}_{n \geq 1} \cup \{y\}$  is closed. Give a clear explanation. Do not use a metric.

*Proof.* ■

**Problem 1.1.15.** Let  $X$  be the two-point set  $\{0, 1\}$  with the discrete topology. Let  $Y$  be a countable product of copies of  $X$ ; thus an element of  $Y$  is a sequence of 0's and 1's. Let  $A$  be the subset of  $Y$  consisting of sequences with only a finite number of 1's. Is  $A$  closed? Prove or disprove.

*Proof.* ■

**Problem 1.1.16.** Let  $Y$  be a topological space. Let  $X$  be a set and let  $f: X \rightarrow Y$  be a function. Give  $X$  the topology in which the open sets are the sets  $f^{-1}(V)$  with  $V$  open in  $Y$  (you do not have to verify that this is a topology). Let  $a \in X$  and let  $B$  be a closed set in  $X$  not containing  $a$ . Prove that  $f(a)$  is not in the closure of  $f(B)$ .

*Proof.* ■

**Problem 1.1.17.** Let  $f: X \rightarrow Y$  be a function that takes closed sets to closed sets. Let  $y \in Y$  and let  $U$  be an open set containing  $f^{-1}(y)$ . Prove that there is an open set  $V$  containing  $y$  such that  $f^{-1}(V)$  is contained in  $U$ .

*Proof.* ■

**Problem 1.1.18.** Let  $X$  be a topological space with an equivalence relation  $\sim$ . Suppose that the quotient space  $X/\sim$  is Hausdorff. Prove that the set  $S = \{x \times y \in X \times X \mid x \sim y\}$  is a closed subset of  $X \times X$ .

*Proof.* ■

**Problem 1.1.19.** Let  $p: X \rightarrow Y$  be a quotient map. Let us say that a subset  $S$  of  $X$  is *saturated* if it has the form  $p^{-1}(T)$  for some subset  $T$  of  $Y$ . Suppose that for every  $y \in Y$  and every open neighborhood  $U$  of  $p^{-1}(y)$  there is a saturated open set  $V$  with  $p^{-1}(y) \subset V \subset U$ . Prove that  $p$  takes closed sets to closed sets.

*Proof.* ■

**Problem 1.1.20.** Let  $X$  be a topological space, let  $D$  be a connected subset of  $X$ , and let  $\{E_\alpha\}$  be a collection of connected subsets of  $X$ .

*Proof.* ■

**Problem 1.1.21.** Let  $X$  and  $Y$  be connected. Prove that  $X \times Y$  is connected.

*Proof.* ■

**Problem 1.1.22.** For any space  $X$ , let us say that two points are “inseparable” if there is no separation  $X = U \cup V$  into disjoint open sets such that  $x \in U$  and  $y \in V$ . Write  $x \sim y$  if  $x$  and  $y$  are inseparable. Then  $\sim$  is an equivalence relation (you don't have to prove this). Now suppose that  $X$  is locally connected (this means that for every point  $x$  and every open neighborhood  $U$  of  $x$ , there is a connected open neighborhood  $V$  of  $x$  contained in  $U$ ). Prove that each equivalence class of the relation  $\sim$  is connected.

*Proof.* ■

**Problem 1.1.23.** Let  $X$  be a topological space. Let  $A \subset X$  be connected. Prove  $\overline{A}$  is connected.

*Proof.* ■

**Problem 1.1.24.** Let  $X_1, X_2, \dots$  be topological spaces. Suppose  $\prod_{n=1}^{\infty} X_n$  is locally connected. Prove that at most finitely many  $X_n$  are connected.

*Proof.* ■

**Problem 1.1.25.** Let  $X$  be a connected space and let  $f: X \rightarrow Y$  be a function which is continuous and onto. Prove that  $Y$  is connected. (This is a theorem in Munkres—prove it from the definitions).

*Proof.* ■

**Problem 1.1.26.** Give:

- (i)  $p: X \rightarrow Y$  is a quotient map.
- (ii)  $Y$  is connected.
- (iii) For every  $y \in Y$ , the set  $p^{-1}(y)$  is connected.

Prove that  $X$  is connected.

*Proof.* ■

**Problem 1.1.27.** Let  $A$  be a subset of  $\mathbf{R}^2$  which is homeomorphic to the open unit interval  $(0, 1)$ . Prove that  $A$  does not contain a nonempty set which is open in  $\mathbf{R}^2$ .

*Proof.* ■

**Problem 1.1.28.** Let  $X$  be a connected space. Let  $\mathcal{U}$  be an open covering of  $X$  and let  $U$  be a nonempty set in  $\mathcal{U}$ . Say that a set  $V$  in  $\mathcal{U}$  is *reachable from*  $U$  if there is a sequence  $U = U_1, U_2, \dots, U_n = V$  of sets in  $\mathcal{U}$  such that  $U_i \cap U_{i+1} \neq \emptyset$  for each  $i$  from 1 to  $n - 1$ . Prove that every nonempty  $V$  in  $\mathcal{U}$  is reachable from  $U$ .

*Proof.* ■

**Problem 1.1.29.** Suppose that  $X$  is connected and every point of  $X$  has a path-connected open neighborhood. Prove that  $X$  is path-connected.

*Proof.* ■

**Problem 1.1.30.** Let  $X$  be a topological space and let  $f, g: X \rightarrow [0, 1]$  be continuous functions. Suppose that  $X$  is connected and  $f$  is onto. Prove that there must be a point  $x \in X$  with  $f(x) = g(x)$ .

*Proof.* ■

## 1.2 Midterm II (Fall 2015)

## **2 Kaufmann's 571 Problems**

### **2.1 Midterm (Fall 2014)**

### **2.2 Final (Fall 2014)**

### **3 MA553 Qual Problems**

#### **3.1 Goins**

#### **3.2 Goldberg**

#### **3.3 Ulrich**