

# MA544: Qual Preparation

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# 1 MA 544 Spring 2016

This is material from the course MA 544 as it was taught in the spring of 2016.

## 1.1 Homework

These exercises were assigned from Wheeden and Zygmund's *Measure and Integral*, therefore, most of the theorems I reference will be from [4]. Other resources include [1] and [2]. For more elementary results, I cite [3]. Unless otherwise stated, whenever we quote a result, e.g., Theorem 1.1, it is understood to come from Wheeden and Zygmund's *Measure and Integral*.

Throughout these notes

$\mathbb{R}$	is the set of real numbers
$\mathbb{R}^+$	is the set of positive real numbers, that is, $x \in \mathbb{R}$ with $x \geq 0$
$\mathbb{C}$	is the set of complex numbers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{Z}$	is the set of the integers
$\mathbb{Z}^+$	is the set of positive integers, that is, $x \in \mathbb{Z}$ with $x \geq 0$
$\mathbb{N}$	is the set of the natural numbers $1, 2, \dots$
$A \setminus B$	is the set difference of $A$ and $B$ , that is, the complement of $A \cap B$ in $A$
$m^*(E)$	the outer measure of $E$
$m_*(E)$	the inner measure of $E$
$m(E)$	the Lebesgue measure of $E$
$\ -\ $	the standard Euclidean norm on $\mathbb{R}^n$
$f \asymp g$	means $f$ is asymptotically equivalent to $g$ , that is, $\lim_{x \rightarrow \infty} g(x)/f(x) = 1$

### 1.1.1 Homework 1

**Problem 1** (Wheeden & Zygmund Ch. 2, Ex. 1). Let  $f(x) = x \sin(1/x)$  for  $0 < x \leq 1$  and  $f(0) = 0$ . Show that  $f$  is bounded and continuous on  $[0, 1]$ , but that  $V[f; 0, 1] = \infty$ .

**Solution.** ► Let  $f$  equal  $x \sin(1/x)$ . We will show that  $f$  is bounded and continuous on  $[0, 1]$ , but that it is not of bounded variation on  $[0, 1]$ .

First we will show that  $f$  is bounded. Note that both  $|x|$  and  $|\sin(1/x)|$  are bounded by 1 on the interval  $[0, 1]$ . Since  $|f| = |x| |\sin(1/x)|$ , it follows that  $|f| \leq 1$  on  $[0, 1]$ . Thus,  $f$  is bounded on  $[0, 1]$ .

Next we show that  $f$  is continuous. It is easy to show that  $f$  is continuous on the subinterval  $(0, 1]$  since both  $|x|$  and  $\sin(1/x)$  are continuous on that interval and we know that the product of continuous functions is continuous. To see that  $f$  is continuous at 0 we must show that  $f(x^+) = f(0)$ ; that is, the limit of  $f$  as  $x$  approaches 0 from the right is  $f(0)$  which by definition is 0. To this end, it suffices to take a (monotonically decreasing) sequence  $x_n \downarrow 0$  and show that the limit of the sequence  $\{f(x_n)\}_{n=1}^\infty$  is 0. Let  $\varepsilon > 0$  be given then, since  $x_n$  converges to 0 there exists an index  $N$  such that  $|0 - x_n| < \varepsilon$  whenever  $n \geq N$ . Since  $|f(x_n)| \leq |x_n|$  on  $[0, 1]$ , the following inequality holds

$$\begin{aligned} |0 - f(x_n)| &= |0 - x_n \sin(1/x_n)| \\ &\leq |x_n| \\ &< \varepsilon. \end{aligned}$$

Thus,  $f$  is continuous at 0 and it converges to 0.

Despite the nice properties that  $f$  seemingly possesses,  $f$  is not b.v. on  $[0, 1]$ . To show that  $f$  is not b.v. on  $[0, 1]$  we must show that for any positive real number  $M$  there exists some partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of  $[0, 1]$  such that the sum associated to  $\Gamma$

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| > M.$$

Let  $N$  be the smallest integer greater than  $M$  and let  $n$  be the smallest integer greater than or equal to  $N/2$ . Then the partition  $\Gamma = \{x_0 = 1 < x_1 < \cdots < x_{n+1} = 1\}$  where  $x_i = 2/((3 + (n - i))\pi)$  for  $1 \leq i \leq N$ . Then we have the inequality

$$\begin{aligned} S_\Gamma &= \sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=2}^n |f(x_i) - f(x_{i-1})| + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &= N + |f(x_{n+1}) - f(x_n)| + |f(x_0) - f(x_1)| \\ &> M. \end{aligned}$$

Thus,  $f$  is not b.v. on  $[0, 1]$ . ◀

**Problem 2** (Wheeden & Zygmund Ch. 2, Ex. 2). Prove theorem (2.1).

**Solution.** ► Recall the statement of Theorem 2.1:

- (a) If  $f$  is of bounded variation on  $[a, b]$ , then  $f$  is bounded on  $[a, b]$ .
- (b) Let  $f$  and  $g$  be of bounded variation on  $[a, b]$ . Then  $cf$  (for any real constant  $c$ ),  $f + g$ , and  $fg$  are of bounded variation on  $[a, b]$ . Moreover,  $f/g$  is of bounded variation on  $[a, b]$  if there exists an  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for  $x \in [a, b]$ .

We shall prove these in alphabetical order:

For part (a) we shall proceed by contradiction. First, without loss of generality, we may assume that  $f(a) = 0$  since the function the variation of  $g(x) = f(x) - f(a)$  is equal to the variation of  $f$  and  $g(a) = 0$ . Suppose that  $f$  is b.v. on  $[a, b]$  with variation  $V = V[f; a, b]$ , but that  $f$  is unbounded on  $[a, b]$ ; that is, given a positive real number  $M$  there exists a point  $x$  in  $[a, b]$  such that  $|f(x)| > M$ . In particular, there exists  $x \in [a, b]$  such that  $|f(x)| > V$ . Hence, for any  $x \in [a, b]$  by the triangle inequality we have

$$\begin{aligned}
 V &< |f(x)| \\
 &= |f(x) - f(a) + f(a)| \\
 &\leq |f(x) - f(a)| + |f(a)| \\
 &\leq V.
 \end{aligned}$$

This is a contradiction. Therefore, it must be the case that if  $f$  is b.v. on  $[a, b]$  then  $f$  is bounded on  $[a, b]$ .

We break part (b) into three sections. Suppose  $f$  and  $g$  are b.v. on  $[a, b]$  with variation  $V$  and  $V'$ , respectively. We will show that (i)  $cf$ ; (ii)  $f + g$ ; and (iii)  $fg$  are b.v. on  $[a, b]$ . Moreover, we show that (iv)  $f/g$  is b.v. on  $[a, b]$  if there exists  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for all  $x \in [a, b]$ .

For part (i) above let  $c$  be a real number. Given a partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of  $[a, b]$ , we have

$$\begin{aligned}
 S_\Gamma &= \sum_{i=1}^n |cf(x_i) - cf(x_{i-1})| \\
 &= \sum_{i=1}^n |c||f(x_i) - f(x_{i-1})| \\
 &= |c| \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \\
 &\leq |c|V
 \end{aligned}$$

since  $V$  is the supremum of the sums of the form  $\sum_{i=1}^m |f(x_i) - f(x_{i-1})|$  over all partitions of  $[a, b]$ . Thus,  $V[cf; a, b] \leq |c|V$  so  $cf$  is b.v. on  $[a, b]$ .

For part (ii) given a partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of the interval  $[a, b]$ , by the triangle

inequality we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^n |(f(x_i) + g(x_i)) - (f(x_{i-1}) + g(x_{i-1}))| \\
&= \sum_{i=1}^n |(f(x_i) - f(x_{i-1})) + (g(x_i) - g(x_{i-1}))| \\
&\leq \sum_{i=1}^n |f(x_i) - f(x_{i-1})| + \sum_{i=1}^n |g(x_i) - g(x_{i-1})| \\
&\leq V + V'.
\end{aligned}$$

Thus,  $f + g$  is b.v. on  $[a, b]$

For part (iii) since  $f$  and  $g$  are b.v. on  $[a, b]$  by part (a)  $f$  and  $g$  are bounded on  $[a, b]$  by, say,  $M$  and  $N$ , respectively. Now, given a partition  $\Gamma = \{x_0 < x_1 < \dots < x_n\}$  of  $[a, b]$ , by the triangle inequality we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1}) \\
&\quad + f(x_i)g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^n |(f(x_i)g(x_i) - f(x_i)g(x_{i-1})) \\
&\quad - (f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1}))| \\
&\leq \sum_{i=1}^n |f(x_i)g(x_i) - f(x_i)g(x_{i-1})| \\
&\quad + \sum_{i=1}^n |f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^n |f(x_i)||g(x_i) - g(x_{i-1})| + \sum_{i=1}^n |g(x_{i-1})||f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^n M|g(x_i) - g(x_{i-1})| + \sum_{i=1}^n N|f(x_i) - f(x_{i-1})| \\
&\leq MV' + NV.
\end{aligned}$$

Thus,  $fg$  is b.v. on  $[a, b]$ .

Finally, for part (iv) suppose there exists  $\varepsilon > 0$  such that  $|g(x)| \geq \varepsilon$  for all  $x \in [a, b]$ . Then, given

a partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of  $[a, b]$ , largely by the triangle inequality, we have

$$\begin{aligned}
S_\Gamma &= \sum_{i=1}^n |f(x_i)/g(x_i) - f(x_{i-1})/g(x_{i-1})| \\
&= \sum_{i=1}^n \left| \frac{f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)}{g(x_i)g(x_{i-1})} \right| \\
&\leq \frac{1}{\varepsilon^2} \sum_{i=1}^n |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^n |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1}) \\
&\quad - (f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1}))| \\
&\leq \frac{1}{\varepsilon^2} \sum_{i=1}^n |g(x_{i-1})||f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} \sum_{i=1}^n |f(x_{i-1})||g(x_i) - g(x_{i-1})| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^n M_g |f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} \sum_{i=1}^n M_f |g(x_i) - g(x_{i-1})| \\
&= \frac{1}{\varepsilon^2} M_g \sum_{i=1}^n |f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} M_f \sum_{i=1}^n |g(x_i) - g(x_{i-1})| \\
&\leq \frac{1}{\varepsilon^2} (NV + MV')
\end{aligned}$$

where, as above,  $f$  is bounded by  $M$  and  $g$  is bounded by  $N$ . Thus,  $f/g$  is b.v. on  $[a, b]$ .

This concludes the proof of Theorem 2.1. ◀

**Problem 3** (Wheeden & Zygmund Ch. 2, Ex. 3). If  $[a', b']$  is a subinterval of  $[a, b]$  show that  $P[a', b'] \leq P[a, b]$  and  $N[a', b'] \leq N[a, b]$ .

**Solution.** ▶ We will prove this by digging in to the definition of  $N$  and  $P$ . Recall that given a partition  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  of the interval  $[a, b]$ ,  $P$  and  $N$  are defined to be the supremum over the sum of the positive and, respectively, the sum negative terms of  $S_\Gamma$ ; that is,  $P$  and  $N$  are the supremum over every partition  $\Gamma$  of  $[a, b]$  of

$$P_\Gamma = \sum_{i=1}^n [f(x_i) - f(x_{i-1})]^+ \quad \text{and} \quad N_\Gamma = \sum_{i=1}^n [f(x_i) - f(x_{i-1})]^-.$$

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation on  $[a, b]$  and let  $[a', b']$  be a subinterval of  $[a, b]$ . Without loss of generality, we may assume that  $[a', b']$  is strictly contained in  $[a, b]$ ; that is,  $a' \neq a$  and  $b' \neq b$ . We aim to show that  $P[a', b'] \leq P[a, b]$  and  $N[a', b'] \leq N[a, b]$ . Since the argument for  $N$  is similar to that of  $P$ , we will omit it here for the sake of brevity. Now, consider the closure of the complement of  $[a', b']$  in  $[a, b]$ ,  $[a, b] \setminus [a', b'] = [a, a'] \cup [b', b]$ . Since  $[a, a']$ ,  $[a', b']$

and  $[b', b]$  are close intervals we may take partitions

$$\begin{aligned}\Gamma_a &= \{x_0 < x_1 < \cdots < x_\ell\}, \\ \Gamma_{ab} &= \{x_\ell < x_{\ell+1} < \cdots < x_m\}\end{aligned}$$

and

$$\Gamma_b = \{x_m < x_{m+1} < \cdots < x_n\}$$

of  $[a, a']$ ,  $[a', b']$  and  $[b', b]$ , respectively and extend this to a partition

$$\Gamma = \{x_0 < x_1 < \cdots < x_\ell < x_{\ell+1} < \cdots < x_m < x_{m+1} < \cdots < x_n\}$$

of  $[a, b]$ . Then, by the definition of  $N$  we have the string of inequalities

$$\begin{aligned}P_\Gamma &= \sum_{i=1}^n [f(x_i) - f(x_{i-1})]^+ \\ &= \sum_{i=1}^{\ell} [f(x_i) - f(x_{i-1})]^+ \\ &\quad + \sum_{i=\ell+1}^m [f(x_i) - f(x_{i-1})]^+ \\ &\quad + \sum_{i=m+1}^n [f(x_i) - f(x_{i-1})]^+ \\ &= P_{\Gamma_{ab}} + P_{\Gamma_a} + P_{\Gamma_b} \\ &\leq P[a, b].\end{aligned}$$

Taking the supremum on the left, we have

$$P[a, a'] + P[a', b'] + P[b', b] \leq P[a, b].$$

Since  $P$  is strictly positive, it must be the case that  $P[a', b'] \leq P[a, b]$ . ◀

**Problem 4** (Wheeden & Zygmund Ch. 2, Ex. 11). Show that  $\int_a^b f d\varphi$  exists if and only if given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|R_\Gamma - R_{\Gamma'}| < \varepsilon$  if  $|\Gamma|, |\Gamma'| < \delta$ .

**Solution.** ▶ One direction is straightforward. Namely  $\Leftarrow$  : suppose that given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|R_\Gamma - R_{\Gamma'}| < \varepsilon$  whenever  $|\Gamma|$  and  $|\Gamma'|$  are less than  $\delta$ . Let  $\{\Gamma_n\}_{n=1}^\infty$  be a decreasing sequence of partitions (by which we mean  $\Gamma_n \subseteq \Gamma_{n+1}$  of  $[a, b]$  such that  $|\Gamma_n| \rightarrow 0$ ). Then, by convergence, there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies  $|\Gamma_n| < \delta$ . Then, for  $n, m \geq N$ , we have

$$|R_{\Gamma_n} - R_{\Gamma_m}| < \varepsilon.$$

Thus, by the Cauchy criterion for convergence, the sequence  $\{R_{\Gamma_n}\}_{n=0}^\infty$  converges and its limit is by definition the Riemann–Stieltjes integral  $\int_a^b f d\varphi$ .

On the other hand  $\implies$  : suppose that  $I = \int_a^b f \, d\varphi$  exists. Then given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|I - R_\Gamma| < \varepsilon/2$  whenever  $|\Gamma| < \delta$ . Let  $\Gamma$  and  $\Gamma'$  be two partitions of  $[a, b]$  with norm  $|\Gamma|, |\Gamma'| < \delta$ . Then we have

$$\begin{aligned} |R_\Gamma - R_{\Gamma'}| &= |R_\Gamma - I - (R_{\Gamma'} - I)| \\ &\leq |R_\Gamma - I| + |R_{\Gamma'} - I| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus,  $I$  satisfies the Cauchy condition. ◀

**Problem 5** (Wheeden & Zygmund Ch. 2, Ex. 13). Prove theorem (2.16).

**Solution.** ▶ Recall the statement of Theorem 2.16:

- (i) If  $\int_a^b f \, d\varphi$  exists, then so do  $\int_a^b cf \, d\varphi$  and  $\int_a^b f \, d(c\varphi)$  for any constant  $c$ , and

$$\int_a^b cf \, d\varphi = \int_a^b f \, d(c\varphi) = c \int_a^b f \, d\varphi.$$

- (ii) If  $\int_a^b f_1 \, d\varphi$  and  $\int_a^b f_2 \, d\varphi$  both exist, so does  $\int_a^b (f_1 + f_2) \, d\varphi$ , and

$$\int_a^b (f_1 + f_2) \, d\varphi = \int_a^b f_1 \, d\varphi + \int_a^b f_2 \, d\varphi.$$

- (iii) If  $\int_a^b f \, d\varphi_1$  and  $\int_a^b f \, d\varphi_2$  both exist, so does  $\int_a^b f \, d(\varphi_1 + \varphi_2)$ , and

$$\int_a^b f \, d(\varphi_1 + \varphi_2) = \int_a^b f \, d\varphi_1 + \int_a^b f \, d\varphi_2.$$

We prove this in (Roman) numerical order.

For (i) suppose that  $I = \int_a^b f \, d\varphi$  exists. Then, given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|I - R_\Gamma| < \varepsilon/|c|$  whenever  $\Gamma$  is a partition of  $[a, b]$  with  $|\Gamma| < \delta$ . We claim that  $\int_a^b cf \, d\varphi = |c|I$ . Let  $\Gamma = \{x_0 < x_1 < \dots < x_n\}$  be a partition  $[a, b]$  with  $|\Gamma| < \delta$ . Then the Riemann–Stieltjes sums  $R'_\Gamma$  of the pair  $(cf, \varphi)$  associated to  $\Gamma$  give us the chain of inequalities

$$\begin{aligned} ||c|I - R'_\Gamma| &= \left| |c|I - \sum_{i=1}^n cf(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c| \left| \sum_{i=1}^n f(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |c| |I - R_\Gamma| \\ &< |c| \frac{\varepsilon}{|c|} \\ &= \varepsilon. \end{aligned}$$



Thus,  $\int_a^b cf \, d\varphi$  is Riemann–Stieltjes integrable and its integral is equal to  $|c|I$ . A similar argument shows that  $\int_a^b f \, d(c\varphi)$  is Riemann–Stieltjes integrable with integral  $|c|I$ .

For (ii) let  $I_1 = \int_a^b f_1 \, d\varphi$  and  $I_2 = \int_a^b f_2 \, d\varphi$ . Then, we claim that  $I = \int_a^b (f_1 + f_2) \, d\varphi$  exists and that  $I = I_1 + I_2$ . Since both  $I_1$  and  $I_2$  exist, given  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$|I_1 - R_\Gamma^1| < \frac{\varepsilon}{2} \quad \text{and} \quad |I_2 - R_\Gamma^2| < \frac{\varepsilon}{2}$$

whenever  $|\Gamma| < \delta$ . Let  $\Gamma = \{x_0 < x_1 < \cdots < x_n\}$  be a partition of  $[a, b]$  with  $|\Gamma| < \delta$ . Then the Riemann–Stieltjes sums  $R_\Gamma$  of the pair  $(f_1 + f_2, \varphi)$  associated to  $\Gamma$  give is the following chain of inequalities

$$\begin{aligned} |(I_1 + I_2) - R_\Gamma| &= \left| (I_1 + I_2) - \sum_{i=1}^n (f_1(\xi_i) + f_2(\xi_i))[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= \left| I_1 - \sum_{i=1}^n f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right. \\ &\quad \left. + I_2 - \sum_{i=1}^n f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &\leq \left| I_1 - \sum_{i=1}^n f_1(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &\quad + \left| I_2 - \sum_{i=1}^n f_2(\xi_i)[\varphi(x_i) - \varphi(x_{i-1})] \right| \\ &= |I_1 - R_\Gamma^1| + |I_2 - R_\Gamma^2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus,  $I$  exists and it is equal to the sum  $I_1 + I_2$ .

Part (iii) is similar to part (ii) in the above equation except that instead of splitting the sum at  $f_1 + f_2$  part, we split it at  $\varphi_1 + \varphi_2$  part. ◀

### 1.1.2 Homework 2

**Problem 1.** Show that the boundary of any interval has outer measure zero.

**Solution.** ► Let  $I = \prod_{i=1}^n I_i$  be a closed interval in  $\mathbb{R}^n$  and let  $J$  be the boundary of  $I$ . We must show that given  $\varepsilon > 0$  there exists a countable collection of intervals  $\{I_n\}_{n \in J}$  covering  $J$  such that

$$\sum_{n \in J} \text{vol}(I_n) < \varepsilon.$$

First, note that we can write  $J$  as the union  $\bigcup_{i=1}^n J_i$  where

$$J_i = [a_1, b_1] \times \cdots \times \{a_i\} \times \cdots \times [a_n, b_n] \cup [a_1, b_1] \times \cdots \times \{b_i\} \times \cdots \times [a_n, b_n].$$

Since the countable union of null sets has measure zero, it suffices to show that the set

$$[a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$$

has measure zero. Consider the collection  $\{I_\varepsilon\}$  consisting of the single interval

$$I_\varepsilon = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \times \left[ a_n - \frac{\varepsilon}{2 \prod_{i=1}^{n-1} (b_i - a_i)}, a_n + \frac{\varepsilon}{2 \prod_{i=1}^{n-1} (b_i - a_i)} \right].$$

It is clear that  $I_\varepsilon \supseteq J$ . Now, computing the volume of this interval, we have

$$\begin{aligned} \text{vol}(I_\varepsilon) &= \prod_{i=1}^{n-1} (b_i - a_i) \left[ a_n + \frac{\varepsilon}{2 \prod_{i=1}^{n-1} (b_i - a_i)} - \left( a_n - \frac{\varepsilon}{2 \prod_{i=1}^{n-1} (b_i - a_i)} \right) \right] \\ &= \left[ \prod_{i=1}^{n-1} (b_i - a_i) \right] \frac{\varepsilon}{\prod_{i=1}^{n-1} (b_i - a_i)} \\ &= \varepsilon. \end{aligned}$$

Thus,  $J$  has measure zero. ◀

**Problem 2.** Show that a set consisting of a single point has outer measure zero.

**Solution.** ► Let  $\{a\}$  be the set consisting of a single point  $a \in \mathbb{R}$ . Then we must show that given  $\varepsilon > 0$  there exists a countable collection of intervals  $\{I_n\}$  such that

$$\sum_{n \in J} m(I_n) < \varepsilon.$$

Consider the collection  $\{I_\varepsilon\}$  consisting of the single interval

$$I_\varepsilon = \left[ a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2} \right].$$

It is clear that  $\{a\} \subseteq I_\varepsilon$ . Moreover,

$$\begin{aligned} \text{vol}(I_\varepsilon) &= a + \frac{\varepsilon}{2} - \left( a - \frac{1}{\varepsilon} \right) \\ &= \varepsilon. \end{aligned}$$

Thus,  $\{a\}$  has measure zero. ◀

### 1.1.3 Homework 3

**Problem 1** (Wheeden & Zygmund Ch. 3, Ex. 5). Construct a subset of  $[0, 1]$  in the same manner as the Cantor set, except that at the  $k$ th stage each interval removed has length  $\delta 3^{-k}$ ,  $0 < \delta < 1$ . Show that the resulting set is perfect, has measure  $1 - \delta$ , and contains no interval.

**Solution.** ► We construct the prescribed subset as follows: take the open interval  $(1/2 - \delta/6, 1/2 + \delta/6)$  and remove it from the closed interval  $[0, 1]$  the result is a union of two disjoint closed intervals

$$E_{1,1} = \left[0, \frac{1}{2} - \frac{1}{6}\delta\right], \quad E_{1,2} = \left[\frac{1}{2} + \frac{1}{6}\delta, 1\right],$$

whose union we call  $E_1$ ; this marks the first step in the construction of this Cantor-like set. Next, we remove the set

$$\left(\frac{1}{4} - \frac{5}{36}\delta, \frac{1}{4} + \frac{1}{36}\delta\right) \cup \left(\frac{3}{4} + \frac{\delta}{36}, \frac{3}{4} + \frac{5}{36}\delta\right)$$

from the set  $E_1$  which yields  $E_2$  the union of the four closed intervals

$$\begin{aligned} E_{2,1} &= \left[0, \frac{1}{4} - \frac{5}{36}\delta\right], & E_{2,2} &= \left[\frac{1}{4} + \frac{1}{36}\delta, \frac{1}{2} - \frac{1}{6}\delta\right], \\ E_{2,3} &= \left[\frac{1}{2} + \frac{1}{6}\delta, \frac{3}{4} + \frac{\delta}{36}\right], & E_{2,4} &= \left[\frac{3}{4} + \frac{5}{36}\delta, 1\right]. \end{aligned}$$

In the  $n$ th step of the construction, we remove an open interval of length  $3^{-n}\delta$  from the center of each interval  $E_{n-1,i}$  yielding  $E_n$  which is the union of  $2^n$  intervals  $E_{n,i}$  of length  $2^{-n} - \delta 2^{-n} \sum_{i=1}^n 2^{i-1} 3^{-i}$ . Let  $E$  be the intersection  $\bigcap_{i=1}^{\infty} E_i$ . This concludes our construction.

Next we show that  $E$  is perfect, has measure  $1 - \delta$  and contains no interval.

To see that  $E$  is perfect, we must show that  $E$  is closed and that and dense in itself. The set  $E$  is closed because it is the (arbitrary) intersection of closed intervals. To see that  $E$  is dense in itself, we must show that for every  $\varepsilon > 0$ , for every  $x \in E$ , the intersection  $(B(x, \varepsilon) \cap E) \setminus \{x\}$  is nonempty. Let  $\varepsilon > 0$  and  $x \in E$  be given. Then, since  $x \in E$ ,  $x \in E_n$  for every  $n$ . Thus,  $x$  is in some closed interval  $E_{n,i} \subseteq E_n$ . Let  $N$  be the smallest integer such that the length of  $E_{N,i} = [a, b]$  is less than  $\varepsilon$ . Then,  $a, b \in E$  and  $a, b \in B(x, \varepsilon)$  and  $x$  is not equal to both  $a$  and  $b$ . Thus,  $(E \cap B(x, \varepsilon)) \setminus \{x\} \neq \emptyset$ . It follows that  $E$  is a perfect set.

To see that the measure of  $E$  is  $1 - \delta$  by Theorem 3.26 (ii) since  $m(E_1) = 1 - \delta/3 < \infty$  and

$E_n \searrow E$  we have

$$\begin{aligned}
m(E) &= m\left(\bigcap_{i=1}^{\infty} E_i\right) \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n m(E_i) \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^{2^n} \left[ \frac{1}{2^n} - \frac{\delta}{2^n} \sum_{i=1}^n \frac{2^{i-1}}{3^i} \right] \\
&= \lim_{n \rightarrow \infty} \left[ 1 - \delta \sum_{i=1}^n \frac{2^{i-1}}{3^i} \right] \\
&= \lim_{n \rightarrow \infty} \left[ 1 - \frac{\delta}{3} \sum_{i=1}^n \left(\frac{2}{3}\right)^{i-1} \right]
\end{aligned}$$

letting  $j = i - 1$ , we can rewrite the series above as the geometric series

$$\begin{aligned}
&= 1 - \frac{\delta}{3} \lim_{n \rightarrow \infty} \sum_{j=0}^n \left(\frac{2}{3}\right)^j \\
&= 1 - \delta,
\end{aligned}$$

as desired.

Lastly, we must show that  $E$  contains no interval. Seeking a contradiction, suppose that  $E$  contains an interval  $I = [a, b]$  of length  $b - a$ . Then, since  $I \subseteq E$ ,  $I \subseteq E_n$  for all  $n$  so, since  $I$  is connected, it must be contained in one of the  $E_{n,i}$  for all  $n$ . Let  $N$  be the smallest integer such that  $m(E_{N,i}) < b - a$  and  $E_{N,i} = [c, d]$  contains  $I$ . Then, since  $I \subseteq E_{N,i}$ , both  $a$  and  $b$  are points in  $I$ ,  $|b - a| \leq |d - c| = m(E_{N,i})$ . This is a contradiction. Thus, it must be the case that  $E$  contains no interval.  $\blacktriangleleft$

**Problem 2** (Wheeden & Zygmund Ch. 3, Ex. 7). Prove (3.15).

**Solution.**  $\blacktriangleright$  Here is the statement of the lemma:

*If  $\{I_k\}_{k=1}^N$  is a finite collection of nonoverlapping intervals, then  $\bigcup_{k=1}^N I_k$  is measurable and  $m\left(\bigcup_{k=1}^N I_k\right) = \sum_{k=1}^N m(I_k)$ .*

By Theorem 3.12, the union  $\bigcup_{n=1}^N I_n$  is measurable. Hence, it remains to show that  $m\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n)$ .

We take the approach of extending the argument provided in Theorem 3.2. As in Theorem 3.2, we note that, since  $\{I_n\}_{n=1}^N$  covers the union  $\bigcup_{n=1}^N I_n$ , then

$$m\left(\bigcup_{n=1}^N I_n\right) \leq \sigma\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n).$$

On the other hand, note that  $I_n$  is the union  $I_n^\circ \cup \partial I_n$  of its interior and its boundary. In the previous homework, we showed that the boundary of an interval has measure zero. Hence, we have

$$m(I_n^\circ) \leq m(I_n) \leq m(I_n^\circ) + m(\partial I_n) = m(I_n^\circ)$$

so  $m(I_n) = m(I_n^\circ)$ . Now, note that

$$m\left(\bigcup_{n=1}^N I_n^\circ\right) = \sum_{n=1}^N m(I_n^\circ) = \sum_{n=1}^N m(I_n).$$

Hence, we have

$$\begin{aligned} \sum_{n=1}^N m(I_n) &= m\left(\bigcup_{n=1}^N I_n^\circ\right) \\ &\leq m\left(\bigcup_{n=1}^N I_n\right) \\ &\leq \sum_{n=1}^N m(I_n). \end{aligned}$$

Thus, equality  $m\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N m(I_n)$  holds. ◀

**Problem 3** (Wheeden & Zygmund Ch. 3, Ex. 8). Show that the Borel algebra  $\mathcal{B}$  in  $\mathbb{R}^n$  is the smallest  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ .

**Solution.** ▶ Since  $\mathcal{B}$  is the smallest  $\sigma$ -algebra containing all of the open sets of  $\mathbb{R}^n$ , it contains all of the closed sets of  $\mathbb{R}^n$ . Now, suppose that  $\mathcal{B}'$  is another  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ . Then,  $\mathcal{B}' \subseteq \mathcal{B}$  since  $\mathcal{B}$  contains all of the closed sets in  $\mathbb{R}^n$ . However, since  $\mathcal{B}'$  is a  $\sigma$ -algebra, it contains all of the open sets in  $\mathbb{R}^n$ , so  $\mathcal{B}' \subseteq \mathcal{B}$  since  $\mathcal{B}$  is the smallest  $\sigma$ -algebra containing the open sets in  $\mathbb{R}^n$ . Thus,  $\mathcal{B}' = \mathcal{B}$ . ◀

**Problem 4** (Wheeden & Zygmund Ch. 3, Ex. 9). If  $\{E_k\}_{k=1}^\infty$  is a sequence of sets with  $\sum m^*(E_k) < \infty$ , show that  $\limsup E_k$  (and also  $\liminf E_k$ ) has measure zero.

**Solution.** ▶ First, since  $\{E_n\}_{n=1}^\infty$  is a sequence of sets with

$$\sum_{i=1}^{\infty} m^*(E_i) < \infty$$

for every  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies

$$\sum_{i=n}^{\infty} m^*(E_i) < \varepsilon.$$

Let's put this aside for now.

Define  $E = \limsup_{n \rightarrow \infty} E_n$  and  $E'_n = \bigcup_{i=n}^{\infty} E_i$ . It is easy to see that  $\{E'_n\}_{n=1}^{\infty}$  is a decreasing sequence of sets whose intersection  $\bigcap_{n=1}^{\infty} E'_n$  is the limit supremum  $E$ . By the monotonicity of the outer measure, we have

$$m^*(E) \leq m^*(E'_n)$$

for all  $n \in \mathbb{N}$ . On the other hand,

$$m^*(E'_n) \leq \sum_{i=n}^{\infty} m^*(E_i) < \varepsilon$$

for every  $\varepsilon$ . Letting  $\varepsilon$  go to 0 we have  $m^*(E) = 0$ .

Lastly, we note that  $E' = \liminf_{n \rightarrow \infty} E_n$  is a subset of  $\limsup_{n \rightarrow \infty} E_n$ , so that  $m^*(E') = 0$ . ◀

**Problem 5** (Wheeden & Zygmund Ch. 3, Ex. 10). If  $E_1$  and  $E_2$  are measurable, show that  $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$ .

**Solution.** ▶ We may, without loss of generality, assume that  $m(E_1), m(E_2) < \infty$  for otherwise there is nothing to show as equality holds trivially.

Now, by Carathéodory's theorem we have the following characterization of measurability: a set  $E$  is measurable if and only if for every set  $A$  we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E).$$

Therefore, the following equalities hold

$$\begin{aligned} m(E_1) &= m(E_1 \cap E_2) + m(E_1 \setminus E_2) \\ m(E_2) &= m(E_1 \cap E_2) + m(E_2 \setminus E_1). \end{aligned}$$

Moreover, from elementary set theory we have

$$(E_1 \cup E_2) \setminus E_2 = E_1 \setminus (E_1 \cap E_2),$$

$E_1 \subseteq E_1 \cup E_2$  and  $E_1 \cap E_2 \subseteq E_1$  so

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

as desired. ◀

### 1.1.4 Homework 4

**Problem 1** (Wheeden & Zygmund Ch. 3, Ex. 12). If  $E_1$  and  $E_2$  are measurable sets in  $\mathbb{R}^1$ , show  $E_1 \times E_2$  is a measurable subset of  $\mathbb{R}^2$  and  $m(E_1 \times E_2) = m(E_1)m(E_2)$ . (Interpret  $0 \cdot \infty$  as 0.) [Hint: Use a characterization of measurability.]

**Solution.** ► The proof of this result is rather long and we shall omit it for now as I gain nothing from retracing my steps on this one. ◀

**Problem 2** (Wheeden & Zygmund Ch. 3, Ex. 13). Motivated by (3.7), define the *inner measure* of  $E$  by  $m_*(E) = \sup m(F)$ , where the supremum is taken over all closed subsets  $F$  of  $E$ . Show that

- (i)  $m_*(E) \leq m^*(E)$ , and
- (ii) if  $m^*(E) < \infty$ , then  $E$  is measurable if and only if  $m_*(E) = m^*(E)$ .

[Use (3.22).]

**Solution.** ► First we show part (i). If  $m^*(E) = \infty$ , the inequality holds trivially. Suppose that  $m^*(E) < \infty$ . Then, since  $F$  is closed, it is measurable and  $m(F) = m^*(F)$ . Moreover,  $F \subseteq E$  so by the monotonicity of the outer measure,

$$m(F) = m^*(F) < m^*(E).$$

Taking the supremum over all  $F$  on the left, we have

$$m_*(E) = \sup_{F \subseteq E} m(F) < m^*(E)$$

as we set out to show.

Next we show part (ii). Let  $E \subseteq \mathbb{R}^n$  with  $m^*(E) < \infty$ .  $\implies$  Suppose that  $E$  is measurable. Then, by Lemma 3.22, there exists a closed set  $F \subseteq E$  such that  $m^*(E \setminus F) < \varepsilon$ . Since closed sets are measurable, by Corollary 3.31, we have

$$m^*(E \setminus F) = m(E) - m(F) < \varepsilon$$

so

$$m(E) < m(F) + \varepsilon.$$

Letting  $\varepsilon$  go to 0, we have

$$m(E) \leq m(F);$$

and taking the supremum on the right

$$m(E) \leq m_*(E).$$

But, by part (i),  $m_*(E) \leq m^*(E) = m(E)$ . Thus,  $m_*(E) = m^*(E)$  as was to be shown.

$\Leftarrow$  On the other hand, suppose that  $m_*(E) = m^*(E)$ . Then, given  $\varepsilon > 0$  there exists an open set  $G$  containing  $E$  and a closed set  $F$  contained in  $E$  such that

$$\begin{aligned} m(G) - m^*(E) &< \frac{\varepsilon}{2} \\ m_*(E) - m(F) &< \frac{\varepsilon}{2}. \end{aligned}$$

Then

$$\begin{aligned}
m^*(E \setminus F) &< m^*(G \setminus F) \\
&= m^*(G) - m^*(G \cap F) \\
&= m^*(G) - m^*(F) \\
&< \frac{\varepsilon}{2} + m^*(E) - \left(m^*(E) - \frac{\varepsilon}{2}\right) \\
&= \varepsilon.
\end{aligned}$$

Thus, by Lemma 3.22,  $E$  is measurable. ◀

**Problem 3** (Wheeden & Zygmund Ch. 3, Ex. 15). If  $E$  is measurable and  $A$  is any subset of  $E$ , show that  $m(E) = m_*(A) + m^*(E \setminus A)$ . (See Exercise 13 for the definition of  $m_*(A)$ .)

**Solution.** ▶ Suppose  $A \subseteq E$ . If  $A$  is measurable, by Problem 2, the outer and inner measure of  $A$  agree; symbolically, we have  $m(A) = m^*(A) = m_*(A)$ . Thus, we have

$$m^*(E \setminus A) = m^*(E) - m^*(A) = m^*(E) - m_*(A).$$

If  $A$  is not measurable and  $m(E) < \infty$ , then we must have  $m^*(A), m^*(E \setminus A) < \infty$  by the monotonicity of the outer measure; since both  $A$  and  $E \setminus A$  are subsets of  $E$ . Hence, we may, without any ambiguity, subtract the quantity  $m^*(E \setminus A)$  from  $m(E)$  and we have

$$\begin{aligned}
m(E) - m^*(E \setminus A) &= m(E) - \inf\{m(G) : E \setminus A \subseteq G \text{ and } G \text{ is open}\} \\
&= m(E) - \inf\{m(G) : E \setminus A \subseteq G \subseteq E \text{ and } G \text{ is open}\} \\
&=
\end{aligned}$$
◀



### 1.1.5 Homework 5

**Problem 1** (Wheeden & Zygmund Ch. 3, Ex. 14). Show that the conclusion of part (ii) of Exercise 13 is false if  $m^*(E) = \infty$ .

**Solution.** ► Part (ii) of Exercise 13 is part (ii) of Problem 2 from the last section (Homework 4). In that problem we showed that if the outer measure of  $E$  is finite, then  $E$  is measurable if and only if its outer and inner measure agree. Here we construct a counter example to this when the outer measure of  $E$  is  $\infty$ ; that is, we show that there exists a set  $E$  with  $m^*(E) = \infty$  such that  $m^*(E) \neq m_*(E)$ . So, which set shall it be? Since we are unoriginal, we will pull an example from Wheeden and Zygmund itself.

Let  $V \subseteq [0, 1]$  be Vitali's unmeasurable (Theorem 3.38) and consider the union  $E = V \cup (2, \infty)$ . It is clear that the inner and outer measure of  $E$  are both  $\infty$ . However,  $E$  itself must be unmeasurable for otherwise  $E \cap [0, 1] = V$  is measurable. ◀

**Problem 2** (Wheeden & Zygmund Ch. 3, Ex. 16). Prove (3.34).

**Solution.** ► We must prove Equation 3.34; that is, if  $P$  is a parallelepiped

$$m(P) = \text{vol}(P).$$

We may, without loss of generality, assume that one of the vertices of  $P$  is  $\mathbf{0}$ . Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be a set of vectors such that

$$P = \left\{ x \in \mathbb{R}^n : x = \sum_{k=1}^n t_k \mathbf{e}_k, 0 \leq t_k \leq 1 \right\}.$$

By definition, the measure of  $P$  is

$$m(P) = \inf_{\mathcal{S}} \left[ \sum_{I_n \in \mathcal{S}} \text{vol}(I_n) \right]$$

where  $\mathcal{S}$  is a cover of  $P$  by intervals. Take the set of

*Remarks.* Literally nobody cares about this problem. I don't remember how to do it, but it must have been painful if I can't figure it out now, even. ◀

**Problem 3** (Wheeden & Zygmund Ch. 3, Ex. 18). Prove that outer measure is *translation invariant*; that is, if  $E_h = \{x + h : x \in E\}$  is the translate of  $E$  by  $h$ ,  $h \in \mathbb{R}^n$ , show that  $m^*(E_h) = m^*(E)$ . If  $E$  is measurable, show that  $E_h$  is also measurable. [This fact was used in proving (3.37).]

**Solution.** ► Let  $E \subseteq \mathbb{R}^n$  and  $h \in \mathbb{R}^n$  and define the set  $E_h$  to be the set  $E_h = \{x + h : x \in E\}$ . We will show that the outer measure of  $E$  is preserved under such translations. But first, let us point out that  $E_h$  is nothing more than the image of  $E$  under the linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$

given by  $x \mapsto x + h$ . By Theorem 3.35, such a map preserves measurability of sets and for any measurable set  $E' \subseteq \mathbb{R}^n$ ,  $m(T(E')) = (\det T)m(E') = m(E')$  (since  $\det T = 1$ ). Now, by Theorem 3.6, for every  $\varepsilon > 0$ , there exist an open set  $G \supseteq E$  such that  $m^*(G) \leq m^*(E) + \varepsilon$ . Consider the image of  $G$  under  $T$ ,  $T(G)$  is an open set containing  $E_h$  so  $m^*(G) \geq m^*(E)$  and

$$m^*(T(G)) = m^*(G) < m^*(E) + \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$ , we achieve the inequality

$$m^*(E_h) \leq m^*(E).$$

To get the other inequality, take the map  $T^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  which takes  $x \mapsto x - h$ ; this sends  $E_h$  to  $E$  and the same argument shows that

$$m^*(E) \leq m^*(E_h).$$

Thus, we have  $m^*(E) = m^*(E_h)$ , as was to be shown. ◀

**Problem 4** (Wheeden & Zygmund Ch. 4, Ex. 1). Prove corollary (4.2) and theorem (4.8)

**Solution.** ▶ The corollary and theorem in question are:

*If  $f$  is measurable, then  $\{f > -\infty\}$ ,  $\{f < +\infty\}$ ,  $\{f = +\infty\}$ ,  $\{a \leq f \leq b\}$ ,  $\{f = a\}$ , etc., are all measurable. Moreover  $f$  is measurable if and only if  $\{a < f < +\infty\}$  is measurable for every finite  $a$ .*

and

*If  $f$  is measurable and  $\lambda$  is any real number, then  $f + \lambda$  and  $\lambda f$  are measurable.*

Their proofs are quite simple. For the corollary: Suppose  $f: E \rightarrow \mathbb{R}$  is a measurable function. By Theorem 4.1,  $f$  is measurable if and only if for every finite  $\alpha \in \mathbb{R}$ , the sets

$$\begin{aligned} \{x \in E : f(x) \geq \alpha\} \\ \{x \in E : f(x) < \alpha\} \\ \{x \in E : f(x) \leq \alpha\} \end{aligned}$$

are measurable. Since measurable sets form a  $\sigma$ -algebra on  $\mathbb{R}^n$ , we know that the countable union and intersection of measurable sets is measurable. Thus,

$$\begin{aligned} \{x \in E : f(x) > -\infty\} &= \bigcup_{\alpha \in \mathbb{Z}} \{x \in E : f(x) > \alpha\} \\ \{x \in E : f(x) = \infty\} &= \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\} \\ \{x \in E : f(x) < \infty\} &= \bigcup_{\alpha \in \mathbb{Z}} \{x \in E : f(x) < \alpha\} \end{aligned}$$

are easily seen to be measurable.

Showing that  $\{x \in E : f(x) = \alpha\}$  and  $\{x \in E : \alpha < f(x) < \beta\}$  are measurable requires some clever (but not too clever) intersection/union of the sets we get from Theorem 4.1.

For the theorem: Suppose  $f$  is measurable and  $\lambda$  is a constant. By Theorem 4.1, for any finite  $\alpha \in \mathbb{R}$  we have

$$\{x \in E : f(x) > \alpha - \lambda\}$$

so

$$\{x \in E : f(x) + \lambda > \alpha\}$$

is measurable. Thus,  $f + \lambda$  is measurable. Similarly, for  $\lambda \neq 0$ , taking the set

$$\{x \in E : f(x) > \alpha/\lambda\} = \{x \in E : \lambda f(x) > \alpha\}$$

shows that  $\lambda f$  is measurable; otherwise, if  $\lambda = 0$ ,  $\lambda f = 0$  is constant and hence is continuous which in turn implies that it is measurable.  $\blacktriangleleft$

**Problem 5** (Wheeden & Zygmund Ch. 4, Ex. 2). Let  $f$  be a simple function, taking its distinct values on disjoint sets  $E_1, \dots, E_N$ . Show that  $f$  is measurable if and only if  $E_1, \dots, E_N$  are measurable.

**Solution.**  $\blacktriangleright \implies$  Suppose that  $f$  is measurable. Then, by Corollary 4.2, the sets of the form  $\{f = \alpha_n\} = E_n$  are measurable. So the sets  $E_n$  are measurable.

$\Leftarrow$  On the other hand, suppose that the sets  $E_n$  are measurable. Then,  $\chi_{E_n}$  is measurable so by Theorem 4.8,  $f$  is measurable since it is the sum

$$f = \sum_{n=1}^N \alpha_{E_n}.$$

$\blacktriangleleft$

### 1.1.6 Homework 6

**Problem 1** (Wheeden & Zygmund Ch. 4, Ex. 4). Let  $f$  be defined and measurable in  $\mathbb{R}^n$ . If  $T$  is a nonsingular linear transformation of  $\mathbb{R}^n$ , show that  $f(T(x))$  is measurable. [If  $E_1 = \{x : f(x) > a\}$  and  $E_2 = \{x : f(T(x)) > a\}$ , show  $E_2 = T^{-1}(E_1)$ .]

**Solution.** ► Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a measurable function and  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation. Then, we show that the composition  $f \circ T$  is measurable. Fix a finite  $\alpha \in \mathbb{R}$  and let

$$E_1 = \{x : f(x) > \alpha\}$$

$$E_2 = \{x : f(T(x)) > \alpha\}.$$

Then, by Theorem 3.35, it suffices to show that  $E_2 = T^{-1}(E_1)$  since  $T^{-1}$  is a nonsingular linear transformation so it sends measurable sets to measurable sets. But this equality is obvious: Suppose  $x \in E_2$ ; then  $f(T(x)) > \alpha$  so, because  $T$  is nonsingular and therefore bijective, clearly  $x \in T^{-1}(E_1)$  so  $E_2 \subseteq T^{-1}(E_1)$ . On the other hand, if  $x \in T^{-1}(E_1)$  then  $x$  is a point in  $E$  such that  $f(T(x)) > \alpha$  so  $x \in E_2$ . Thus,  $E_2 = T^{-1}(E_1)$  and consequently,  $f \circ T$  is a measurable function. ◀

**Problem 2** (Wheeden & Zygmund Ch. 4, Ex. 7). Let  $f$  be u.s.c. and less than  $\infty$  on a compact set  $E$ . Show that  $f$  is bounded above on  $E$ . Show also that  $f$  assumes its maximum on  $E$ , i.e., that there exists  $x_0 \in E$  such that  $f(x_0) \geq f(x)$  for all  $x \in E$ .

**Solution.** ► First we show that  $f$  is bounded. Suppose that  $f$  is u.s.c. on  $E$ . Then, by Theorem 4.14 (i), sets of the form  $\{x \in E : f(x) < \alpha\}$  are relatively open. Let  $\mathcal{G} = \{G_\alpha\}_{\alpha \in \mathbb{Z}}$  where  $G_\alpha = \{x \in E : f(x) < \alpha\}$ . Then  $\mathcal{G}$  forms an open cover of  $E$  and since  $E$  is compact there exists a finite subset  $\{G_{\alpha_n}\}_{n=1}^N$  for some finite subset  $\{\alpha_1, \dots, \alpha_N\}$  of  $\mathbb{Z}$ . Let  $\alpha = \max\{\alpha_1, \dots, \alpha_N\}$ . Then,  $f(x) < \alpha$  for all  $x \in E$  so  $f$  is bounded above by  $\alpha$ .

Next, we show that  $f$  in fact assumes its maximum (locally) on  $E$  by using only topological properties of  $f$ . Since sets of the form  $\{x \in E : f(x) \geq \alpha\}$  are relatively closed, by Theorem 4.14 (i), for fixed  $x \in E$  the sets  $F_x = \{y \in E : f(y) \geq f(x)\}$  are relatively closed. Consider the collection  $\{F_x\}_{x \in E}$  of closed subsets of  $E$ . First, note that each of these sets is nonempty since  $f(x) \geq f(x)$  so  $x \in F_x$  for every  $x \in E$ . Now, let  $\{x_n\}_{n=1}^N \subseteq E$  and consider the collection  $\{F_{x_n}\}_{n=1}^N$ . Then  $\bigcap_{n=1}^N F_{x_n} \neq \emptyset$  since for  $x$  the point in  $\{x_1, \dots, x_N\}$  such that  $f(x) = \min\{f(x_1), \dots, f(x_N)\}$ ,  $x \in F_{x_n}$  for all  $1 \leq n \leq N$ . Thus, by the finite intersection property, the intersection  $F = \bigcap_{x \in E} F_x$  is nonempty. Let  $y \in \bigcap_{x \in E} F_x$ , then  $f(y) \geq f(x)$  for all  $x \in E$  so  $f$  achieves its maximum (locally) on  $E$ . ◀

**Problem 3** (Wheeden & Zygmund Ch. 4, Ex. 8).

- Let  $f$  and  $g$  be two functions which are u.s.c. at  $x_0$ . Show that  $f + g$  is u.s.c. at  $x_0$ . Is  $f - g$  u.s.c. at  $x_0$ ? When is  $fg$  u.s.c. at  $x_0$ ?
- If  $\{f_k\}$  is a sequence of functions are u.s.c. at  $x_0$ , show that  $\inf f_k(x)$  is u.s.c. at  $x_0$ .
- If  $\{f_k\}$  is a sequence of functions which are u.s.c. at  $x_0$  and which converge uniformly near  $x_0$ , show that  $\lim f_k$  is u.s.c. at  $x_0$ .

**Solution.** ► We prove these in alphabetical order (a)  $\rightarrow$  (b)  $\rightarrow$  (c).

For (a), suppose that  $f$  and  $g$  are u.s.c. at  $x_0$ . Then given  $M > f(x_0), g(x_0)$  there exists  $\delta_1, \delta_2 > 0$  such that  $f(x), g(x) < M/2$  for all  $|x_1 - x_0| < \delta_1, |x_2 - x_0| < \delta_2$ , respectively. Let  $\delta$  be the minimum of  $\{\delta_1, \delta_2\}$ . Then for any  $x$  such that  $|x - x_0| < \delta$ , we have

$$\begin{aligned} |f(x) + g(x) - (f(x_0) + g(x_0))| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \frac{M}{2} + \frac{M}{2} \\ &= M. \end{aligned}$$

Thus,  $f + g$  is u.s.c.

For that second little part of (a), the one that asks “Is  $f - g$  u.s.c. at  $x_0$ ?” we provide a counterexample. In fact, the following is enough of a counterexample: Take  $f = 0$  (which is continuous everywhere) and  $g$  any function that is u.s.c., but not continuous, at  $x_0$  then  $f - g = -g$  is l.s.c. at  $x_0$ . Another counterexample is provided by the equations  $u_1$  and  $u_2$  from Ch. 4 of Wheeden and Zygmund: Fix an  $x_0 \in \mathbb{R}$  and define

$$u_1(x) = \begin{cases} 0 & \text{if } x < x_0, \\ 1 & \text{if } x \geq x_0, \end{cases} \quad u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

Then

$$u_1(x) - u_2(x) = \begin{cases} 0 & \text{if } x \leq x_0, \\ 1 & \text{if } x > x_0. \end{cases}$$

is not u.s.c. at  $x_0$  since being u.s.c. at  $x_0$  implies that for  $1/2 > f(x_0) = 0$  there exists  $\delta > 0$  such that  $f(x) < 1/2$  for all  $x \in (x_0 - \delta, x_0 + \delta)$ . But for any  $x' > x_0$  in  $(x_0 - \delta, x_0 + \delta)$ ,  $u(x') = 1 > 1/2$  which contradicts the assumption that  $u$  is u.s.c. at  $x_0$ .

For (b), suppose  $\{f_n\}_{n=1}^\infty$  is a sequence of functions that are u.s.c. at  $x_0$ . Then

$$\limsup_{\substack{x \rightarrow x_0 \\ x \in E}} f_n(x) \leq f_n(x_0)$$

for all  $n \in \mathbb{N}$ . We must show that

$$\limsup_{\substack{x \rightarrow x_0 \\ x \in E}} [\inf f_n(x)] \leq \inf f_n(x_0).$$

◀

### 1.1.7 Homework 7

**Problem 1** (Wheeden & Zygmund Ch. 4, Ex. 9).

- (a) Show that the limit of a decreasing (increasing) sequence of functions u.s.c. (l.s.c.) at  $x_0$  is u.s.c. (l.s.c.) at  $x_0$ . In particular, the limit of a decreasing (increasing) sequence of functions continuous at  $x_0$  is u.s.c. (l.s.c.) at  $x_0$ .
- (b) Let  $f$  be u.s.c. and less than  $\infty$  on  $[a, b]$ . Show that there exists continuous  $f_k$  on  $[a, b]$  such that  $f_k \downarrow f$ .

**Solution.** ► For part (a) we may as well assume that  $f \geq 0$  for all  $x$ . Let  $\{f_n\}$ ,  $n \in \mathbb{N}$ , be a sequence of decreasing functions with limit  $f$  which are u.s.c. at  $x_0$ . Then, for every  $n \in \mathbb{N}$ , for every sequence  $x \rightarrow x_0$ ,

$$\limsup_{x \rightarrow x_0} f_n(x) \leq f_n(x_0).$$

Now, we claim that  $f(x) \leq f_n(x)$  for every  $x$  and every  $n \in \mathbb{N}$ .

*Proof of claim.* Suppose  $f(x) > f_{N_1}(x)$  for some  $x$ ,  $N_1 \in \mathbb{N}$ . Then there exists a real number  $\varepsilon > 0$  such that  $0 < \varepsilon < |f(x) - f_{N_1}(x)|$  (we may, for example, take  $\varepsilon$  to be in  $\mathbb{Q}$  which is dense in  $\mathbb{R}$ ). Then, since  $f_n \downarrow f$ , there exists an index  $N_1 \in \mathbb{N}$  such that

$$|f(x) - f_n(x)| < \varepsilon.$$

However, since the sequence  $f_n$  decreases to  $f$ , for  $n \geq \max\{N_1, N_2\}$ ,  $f_n(x) \leq f_{N_1}(x)$  so

$$|f(x) - f_n(x)| > |f(x) - f_{N_1}(x)| > \varepsilon.$$

This is a contradiction. ■

Having established this, for every sequence  $x \rightarrow x_0$ , we have

$$\limsup_{x \rightarrow x_0} f(x) \leq \limsup_{x \rightarrow x_0} f_n(x) \leq f_n(x_0).$$

Letting  $n \rightarrow \infty$ ,

$$\limsup_{x \rightarrow x_0} f(x) \leq \lim_{n \rightarrow \infty} f_n(x_0) = f(x_0).$$

For part (b) suppose  $f: [a, b] \rightarrow \mathbb{R}$  is u.s.c. on  $[a, b]$  and  $f(x) < \infty$  for all  $x \in [a, b]$ . For a fixed  $x \in [a, b]$ ,  $f$  is u.s.c. at  $x$  if for every  $\varepsilon > 0$ , there exists a neighborhood  $B(x, \delta)$  such that  $f(y) < f(x) + \varepsilon$ . Now, let  $\varepsilon = 1/n$ . Then, for each  $x \in [a, b]$ , there exists a neighborhood  $B(x, \delta_x)$  such that  $f(y) < f(x) + \varepsilon$  for  $y \in B(x, \delta_x)$ .

The following post on the Mathematics [StackExchange](#) contains a solution to part (b) of this problem.

First, we claim that  $f(x) \neq \infty$  for any  $x \in [a, b]$ , it must be bounded.

*Proof of claim.* By Theorem 4.14 (a), sets of the form  $\{x \in [a, b] : f(x) < a\}$  is relatively open for all finite  $a$ . Define

$$E_n = \{x \in [a, b] : f(x) < n\}.$$

Then, the collection  $\mathcal{E} = \{E_n\}$ ,  $n \in \mathbb{N}$ , is an open cover of  $[a, b]$ . Since  $[a, b]$  is compact, there exists a finite subcover  $\{E_{n_1}, \dots, E_{n_m}\}$  of  $\mathcal{E}$ . Letting  $M = \max\{n_1, \dots, n_m\}$ , we have  $f < M$  for all  $x \in [a, b]$ . Thus,  $f$  is bounded on  $[a, b]$ . ■

Now that we have established that  $f$  is bounded on  $[a, b]$  by, say,  $M$  then  $\sup_{x \in [a, b]} f \leq M$ . Define

$$f_n(x) = \sup_{y \in [a, b]} [f(y) - n|x - y|].$$

We claim that this family of functions  $\{f_n\}$ ,  $n \in \mathbb{N}$ , is continuous and that  $f_n \rightarrow f$ . To see that  $f$  is continuous, we observe that this family of functions is in fact Lipschitz continuous

$$\begin{aligned} |f_n(x) - f_n(y)| &= \left| \sup_{z \in [a, b]} [f(z) - n|x - z|] - \sup_{z \in [a, b]} [f(z) - n|y - z|] \right| \\ &\leq \left| \sup_{z \in [a, b]} [f(z) - n|x - z| - f(z) - n|y - z|] \right| \\ &= \left| \sup_{z \in [a, b]} [-n|x - z| - n|y - z|] \right| \\ &= \left| \sup_{z \in [a, b]} [-n|x - y + (y - z)| - n|y - z|] \right| \\ &\leq \left| \sup_{z \in [a, b]} [-n|x - y| - 2n|y - z|] \right| \\ &= n|x - y|. \end{aligned}$$

Thus,  $f_n$  is Lipschitz and in particular, it is continuous.

To see that  $f_n \rightarrow f$  pointwise, let  $\varepsilon > 0$  be given then we must show that there exists some index  $N$  such that  $n \geq N$  implies

$$|f(x) - f_n(x)| < \varepsilon.$$

Expanding the equation above, we see that

$$|f(x) - f_n(x)| = \left| f(x) - \sup_{y \in [a, b]} [f(y) - n|x - y|] \right|$$

◀

**Problem 2** (Wheeden & Zygmund Ch. 4, Ex. 11). Let  $f$  be defined on  $\mathbb{R}^n$  and let  $B(x)$  denote the open ball  $\{y : |x - y| < r\}$  with center  $x$  and fixed radius  $r$ . Show that the function  $g(x) = \sup\{f(y) : y \in B(x)\}$  is l.s.c. and the function  $h(x) = \inf\{f(y) : y \in B(x)\}$  is u.s.c. on  $\mathbb{R}^n$ . Is the same true for the closed ball  $\{y : |x - y| \leq r\}$ ?

**Solution.** ▶ Note that, by properties of the infimum/supremum for any set of real numbers  $S \subset \mathbb{R}$ ,

$$\sup S = -\inf(-S)$$

where  $-S = \{-s : s \in S\}$ . Thus,

$$\begin{aligned} g(x) &= -\inf\{-f(y) : y \in B(x, r)\} \\ &= \sup\{f(y) : y \in B(x, r)\}. \end{aligned}$$

Letting  $f' = -f$ , it suffices to show that  $g'(x) = \inf\{f'(y) : y \in B(x, r)\}$  is u.s.c. since for any u.s.c. function  $f$ ,  $-f$  is l.s.c. Therefore, we show that  $h$  is u.s.c.

To see that  $h$  is u.s.c., let  $M > h(x_0)$ . Then we must show that there exists a neighborhood  $B(x_0, \delta)$  such that  $M > h(x)$  for every  $x \in B(x_0, \delta)$ . Since  $h(x_0)$  is the infimum of  $f(x)$  over all  $x \in B(x_0, r)$ , given  $\varepsilon > 0$  there exists  $x \in B(x_0, r)$  such that  $f(x) < h(x_0) + \varepsilon < M$ . Define  $\delta = (r - |x - y|)/2$ . Then we claim that for any  $x \in B(x_0, \delta)$ ,

$$g(x) < M.$$

*Proof of claim.* Let  $x \in B(x_0, \delta)$ . Then  $y \in B(x_0, \delta)$  since

$$\begin{aligned} |x - y| &= |x - x_0 - (y - x_0)| \\ &\leq |x - x_0| + |y - x_0| \\ &= (r - |y - x_0|)/2 + |y - x_0| \\ &= r/2 + |y - x_0|/2 \\ &< r. \end{aligned}$$

Thus,

$$g(x) \leq f(y) < g(x_0) + \varepsilon < M.$$

■

It follows that  $g$  is u.s.c. ◀

**Problem 3** (Wheeden & Zygmund Ch. 4, Ex. 15). Let  $\{f_k\}$  be a sequence of measurable functions defined on a measurable set  $E$  with  $m(E) < \infty$ . If  $|f_k(x)| \leq M_x < \infty$  for all  $k$  for each  $x \in E$ , show that given  $\varepsilon > 0$ , there is closed  $F \subseteq E$  and finite  $M$  such that  $m(E \setminus F) < \varepsilon$  and  $|f_k(x)| \leq M$  for all  $x \in F$ .

**Solution.** ▶ Set  $f = \sup_{n \in \mathbb{N}} |f_n|$ ; then,  $f$  is measurable since it is the supremum of measurable functions  $|f_n|$ . By Lusin's theorem  $f$  satisfies the  $\mathcal{C}$ -property, i.e., there exists a closed subset  $F'$  of  $E$  with  $m(E \setminus F') < \varepsilon/2$  and a continuous function  $\bar{f} : E \rightarrow \mathbb{R}$  such that  $f|_{F'} = \bar{f}|_{F'}$ . Now, let  $B$  be the closed ball centered at  $\mathbf{0}$  such that  $|E \setminus B| < \varepsilon/2$  (remember, this is all taking place in  $\mathbb{R}^n$ , so we can do this). Thus,  $F' \cap B$  is compact since it is a closed subset of  $B$  the latter being a compact set. Let  $F = F' \cap B$  then,

$$\begin{aligned} |E \setminus F| &= |E \setminus (F' \cap B)| \\ &= |(E \setminus F') \cup (E \setminus B)| \\ &\leq |E \setminus F'| + |E \setminus B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$



so  $F$  has the desired measure. Lastly, by the mean value theorem,  $f$  achieves its maximum, call it  $M$ , on  $F$  since  $F$  is compact. It follows that  $f_n|_F \leq M$  for all  $n \in \mathbb{N}$ . ◀

**Problem 4** (Wheeden & Zygmund Ch. 4, Ex. 18). If  $f$  is measurable on  $E$ , define  $\omega_f(a) = m\{f > a\}$  for  $-\infty < a < \infty$ . If  $f_k \uparrow f$ , show that  $\omega_{f_k} \uparrow \omega_f$ . If  $f_k \rightarrow f$ , show that  $\omega_{f_k} \rightarrow \omega_f$  at each point of continuity of  $\omega_f$ . [For the second part, show that if  $f_k \rightarrow f$ , then  $\limsup_{k \rightarrow \infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$  and  $\liminf_{k \rightarrow \infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$  for every  $\varepsilon > 0$ .]

**Solution.** ▶ For the first part of this problem we will show that the sequence of distribution functions  $\{\omega_{f_n}\}$ ,  $n \in \mathbb{N}$ , is increasing and that its limit is  $\omega_f$ . It is easy to verify that this sequence is in fact increasing: if  $x \in \{f_{n-1} \geq M\}$  then  $x \in \{f_n \geq M\}$  since  $f_n \geq f_{n-1}$  for all  $x \in E$ . Thus,  $\omega_{f_n} \geq \omega_{f_{n-1}}$ . Now we need to show that the limit of this sequence is in fact  $\omega_f$ : fix an  $x \in E$  and let  $\varepsilon > 0$  be given. Then there exists an index  $N'$  such that  $n \geq N'$  implies  $|f(x) - f_n(x)| < \varepsilon$ . Now, we want to use this  $\varepsilon$  and index  $N'$  (with some possible alterations), for some fixed  $M$ , we want to show that the difference

$$|\omega_f(M) - \omega_{f_n}(M)| < \varepsilon.$$

First, by properties of the Lebesgue measure

$$m\{f > M\} - m\{f_n > M\} \leq m(\{f > M\} \setminus \{f_n > M\}).$$

In turn, it is easy to see that the latter set is in fact

$$\begin{aligned} E_{M,n} &= \{x \in E : f(x) > M \text{ and } f_n(x) \leq M\} \\ &= \{x \in E : f(x) > M \text{ and } f(x) - f_n(x) > 0\}. \end{aligned}$$

Then,  $E_{M,n} \subseteq \{x \in E : f(x) - f_n(x) > M\} = E_{0,n}$  and the the measure of the latter set converges to 0 since  $f_n \rightarrow f$  and this implies that  $f_n$  converges to  $f$  in measure (a weaker form of pointwise convergence). Let  $N''$  be the index such that  $n \geq N''$  implies  $m(E_{0,n}) < \varepsilon$ . Then for  $n \geq N$  with  $N = \max\{N', N''\}$ , the difference

$$|\omega_f(M) - \omega_{f_n}(M)| < \varepsilon.$$

Thus, we have shown that  $\omega_{f_n} \uparrow \omega_f$ . ◀

**Problem 5** (Wheeden & Zygmund Ch. 5, Ex. 1). If  $f$  is a simple measurable function (not necessarily positive) taking values  $a_j$  on  $E_j$ ,  $j = 1, \dots, N$ , show that  $\int_E f = \sum_{j=1}^N a_j m(E_j)$ . [Use (5.24)].

**Solution.** ▶ It is enough to consider simple positive measurable functions  $f$  since we can split  $f$  into the difference of two positive simple measurable functions, namely,  $f = f^+ - f^-$ . Now, since  $f$  is a simple function,  $f = \sum_{n=1}^N a_n \chi_{E_n}$  for measurable subsets  $E_n \subseteq E$ . Now, by Theorem 5.24, we

have

$$\begin{aligned}\int_E f \, dx &= \int_E \left[ \sum_{n=1}^N a_n \chi_{E_n} \right] dx \\ &= \sum_{n=1}^N \int_{E_n} a_n \, dx \\ &= \sum_{n=1}^N a_n m(E_n),\end{aligned}$$

as we set out to show. ◀

**Problem 6** (Wheeden & Zygmund Ch. 5, Ex. 3). Let  $\{f_k\}$  be a sequence of nonnegative measurable functions defined on  $E$ . If  $f_k \rightarrow f$  and  $f_k \leq f$  a.e. on  $E$ , show that  $\int_E f_k \rightarrow \int_E f$ .

**Solution.** ▶ The result follows from a simple application of Fatou's lemma. Consider the sequence of integrals  $\{\int_E f_n\}$ ,  $n \in \mathbb{N}$ . By Fatou's lemma

$$\begin{aligned}\int_E \liminf_{n \rightarrow \infty} f_n \, dx &= \int_E f \, dx \\ &\leq \liminf_{n \rightarrow \infty} \int_E f_n \, dx.\end{aligned}$$

By Theorem 5.10, since  $f_n \leq f$ , we have

$$\limsup_{n \rightarrow \infty} \int_E f_n \, dx \leq \int_E f \, dx.$$

Thus, we have

$$\limsup_{n \rightarrow \infty} \int_E f_n \, dx \leq \liminf_{n \rightarrow \infty} \int_E f_n \, dx,$$

which implies that

$$\limsup_{n \rightarrow \infty} \int_E f_n \, dx = \liminf_{n \rightarrow \infty} \int_E f_n \, dx$$

so

$$\lim_{n \rightarrow \infty} \int_E f_n \, dx = \int_E f \, dx$$

as we set out to show. ◀

### 1.1.8 Homework 8

**Problem 1** (Wheeden & Zygmund Ch. 5, Ex. 2). Show that the conclusion of (5.32) are not true without the assumption that  $\varphi \in L(E)$ . [In part (ii), for example, take  $f_k = \chi_{(k,\infty)}$ .]

**Solution.** ► ◀

**Problem 2** (Wheeden & Zygmund Ch. 5, Ex. 4). If  $f \in L(0, 1)$ , show that  $x^k f(x) \in L(0, 1)$  for  $k = 1, 2, \dots$ , and  $\int_0^1 x^k f(x) dx \rightarrow 0$ .

**Solution.** ► ◀

**Problem 3** (Wheeden & Zygmund Ch. 5, Ex. 6). Let  $f(x, y)$ ,  $0 \leq x, y \leq 1$ , satisfy the following conditions: for each  $x$ ,  $f(x, y)$  is an integrable function of  $y$ , and  $\partial f(x, y)/\partial x$  is a bounded function of  $(x, y)$ . Show that  $\partial f(x, y)/\partial x$  is a measurable function of  $y$  for each  $x$  and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial}{\partial x} f(x, y) dy.$$

**Solution.** ► ◀

**Problem 4** (Wheeden & Zygmund Ch. 5, Ex. 7). Give an example of an  $f$  that is not integrable, but whose improper Riemann integral exists and is finite.

**Solution.** ► ◀

**Problem 5** (Wheeden & Zygmund Ch. 5, Ex. 21). If  $\int_A f = 0$  for every measurable subset  $A$  of a measurable set  $E$ , show that  $f = 0$  a.e. in  $E$ .

**Solution.** ► ◀

**Problem 6** (Wheeden & Zygmund Ch. 6, Ex. 10). Let  $V_n$  be the volume of the unit ball in  $\mathbb{R}^n$ . Show by using Fubini's theorem that

$$V_n = 2V_{n-1} \int_0^1 (1-t^2)^{(n-1)/2} dt.$$

(We also observe that by setting  $w = t^2$ , the integral is a multiple of a classical  $\beta$ -function and so can be expressed in terms of the  $\Gamma$ -function:  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ ,  $s > 0$ .)

**Solution.** ► ◀

**Problem 7** (Wheeden & Zygmund Ch. 6, Ex. 11). Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}.$$

(For  $n = 1$ , write  $\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy$  and use polar. For  $n > 1$ , use the formula  $e^{-|x|^2} = e^{-x_1^2} \cdots e^{-x_n^2}$  and Fubini's theorem to reduce the case  $n = 1$ .)

**Solution.** ►

◀

### 1.1.9 Homework 9

**Problem 1** (Wheeden & Zygmund Ch. 6, Ex. 1).

- (a) Let  $E$  be a measurable subset of  $\mathbb{R}^2$  such that for almost every  $x \in \mathbb{R}$ ,  $\{y : (x, y) \in E\}$  has  $\mathbb{R}$ -measure zero. Show that  $E$  has measure zero and that for almost every  $y \in \mathbb{R}$ ,  $\{x : (x, y) \in E\}$  has measure zero.
- (b) Let  $f(x, y)$  be nonnegative and measurable in  $\mathbb{R}^2$ . Suppose that for almost every  $x \in \mathbb{R}$ ,  $f(x, y)$  is finite for almost every  $y$ . Show that for almost  $y \in \mathbb{R}$ ,  $f(x, y)$  is finite for almost every  $x$ .

**Solution.** ►

**Problem 2** (Wheeden & Zygmund Ch. 6, Ex. 3). Let  $f$  be measurable and finite a.e. on  $[0, 1]$ . If  $f(x) - f(y)$  is integrable over the square  $0 \leq x \leq 1, 0 \leq y \leq 1$ , show that  $f \in L[0, 1]$ .

**Solution.** ►

**Problem 3** (Wheeden & Zygmund Ch. 6, Ex. 4). Let  $f$  be measurable and periodic with period 1:  $f(t+1) = f(t)$ . Suppose there is a finite  $c$  such that

$$\int_0^1 |f(a+t) - f(b+t)| dt \leq c$$

for all  $a$  and  $b$ . Show that  $f \in L[0, 1]$ . (Set  $a = x, b = -x$ , integrate with respect to  $x$ , and make the change of variables  $\xi = x + t, \eta = -x + t$ .)

**Solution.** ►

**Problem 4** (Wheeden & Zygmund Ch. 6, Ex. 6). For  $f \in L(\mathbb{R})$ , define the *Fourier transform*  $\hat{f}$  of  $f$  by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt$$

for  $x \in \mathbb{R}$ . (For complex-valued function  $F = F_0 + iF_1$  whose real and imaginary parts  $F_0$  and  $F_1$  are integrable, we define  $\int F = \int F_0 + i \int F_1$ .) Show that if  $f$  and  $g$  belong to  $L(\mathbb{R})$ , then

$$\widehat{(f * g)}(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

**Solution.** ►

**Problem 5** (Wheeden & Zygmund Ch. 6, Ex. 7). Let  $F$  be a closed subset of  $\mathbb{R}$  and let  $\delta(x) = \delta(x, F)$  be the corresponding distance function. If  $\lambda > 0$  and  $f$  is nonnegative and integrable over the complement of  $F$ , prove that the function

$$\int_{\mathbb{R}} \frac{\delta^\lambda(y) f(y)}{|x - y|^{1+\lambda}} dy$$

is integrable over  $F$  and so is finite a.e. in  $F$ . (In case  $f = \chi_{(a,b)}$ , this reduces to Theorem 6.17.)

**Solution.** ►

◀

**Problem 6** (Wheeden & Zygmund Ch. 6, Ex. 9).

- (a) Show that  $M_\lambda(x; F) = +\infty$  if  $x \notin F$ ,  $\lambda > 0$ .
- (b) Let  $F = [c, d]$  be a closed subinterval of a bounded open interval  $(a, b) \subseteq \mathbb{R}$ , and let  $M_\alpha$  be the corresponding Marcinkiewicz integral,  $\lambda > 0$ . Show that  $M_\lambda$  is finite for every  $x \in (c, d)$  and that  $M_\lambda(c) = M_\lambda(d) = \infty$ . Show also that  $\int M_\lambda \leq \lambda^{-1}|G|$ , where  $G = (a, b) - [c, d]$ .

**Solution.** ►

◀

### 1.1.10 Homework 10

**Problem 1** (Wheeden & Zygmund Ch. 7, Ex. 1). Let  $f$  be measurable in  $\mathbb{R}^n$  and different from zero in some set of positive measure. Show that there is a positive constant  $c$  such that  $f^*(x) \geq c\|x\|^{-n}$  for  $\|x\| \geq 1$ .

**Solution.** ►

**Problem 2** (Wheeden & Zygmund Ch. 7, Ex. 2). Let  $\varphi(x), x \in \mathbb{R}^n$ , be a bounded measurable function such that  $\varphi(x) = 0$  for  $\|x\| \geq 1$  and  $\int \varphi = 1$ . For  $\varepsilon > 0$ , let  $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ . ( $\varphi_\varepsilon$  is called an *approximation to the identity*.) If  $f \in L(\mathbb{R}^n)$ , show that

$$\lim_{\varepsilon \rightarrow 0} (f * \varphi_\varepsilon)(x) = f(x)$$

in the Lebesgue set of  $f$ . (Note that  $\int \varphi_\varepsilon = 1, \varepsilon > 0$ , so that

$$(f * \varphi_\varepsilon)(x) - f(x) = \int [f(x-y) - f(x)]\varphi_\varepsilon(y) \, dy.$$

Use Theorem 7.16.)

**Solution.** ►

**Problem 3** (Wheeden & Zygmund Ch. 7, Ex. 6). Show that if  $\alpha > 0$ , then  $x^\alpha$  is absolutely continuous on every bounded subinterval of  $[0, \infty)$ .

**Solution.** ►

**Problem 4** (Wheeden & Zygmund Ch. 7, Ex. 8). Prove the following converse of Theorem 7.31: If  $f$  is of bounded variation on  $[a, b]$ , and if the function  $V(x) = V[a, x]$  is absolutely continuous on  $[a, b]$ , then  $f$  is absolutely continuous on  $[a, b]$ .

**Solution.** ►

**Problem 5** (Wheeden & Zygmund Ch. 7, Ex. 9). If  $f$  is of bounded variation on  $[a, b]$ , show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then  $f$  is absolutely continuous on  $[a, b]$ . (For the second part, use Theorems 2.2(ii) and 7.24 to show that  $V(x)$  is absolutely continuous and then use the result of Exercise 8).

**Solution.** ►

**Problem 6** (Wheeden & Zygmund Ch. 7, Ex. 12). Use Jensen's inequality to prove that if  $a, b \geq 0$ ,  $p, q > 1$ ,  $(1/p) + (1/q) = 1$ , then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

More generally, show that

$$a_1 \cdots a_N = \sum_{j=1}^N \frac{a_j^{p_j}}{p_j},$$

where  $a_j \geq 0$ ,  $p_j > 1$ ,  $\sum_{j=1}^N (1/p_j) = 1$ . (Write  $a_j = e^{x_j/p_j}$  and use the convexity of  $e^x$ .)

**Solution.** ►

◀

**Problem 7** (Wheeden & Zygmund Ch. 7, Ex. 13). Prove Theorem 7.36.

**Solution.** ► Recall the statement of Theorem 7.36

- (i) If  $\varphi_1$  and  $\varphi_2$  are convex in  $(a, b)$ , then  $\varphi_1 + \varphi_2$  is convex in  $(a, b)$ .
- (ii) If  $\varphi$  is convex in  $(a, b)$  and  $c$  is a positive constant, then  $c\varphi$  is convex in  $(a, b)$ .
- (iii) If  $\varphi_k$ ,  $k = 1, 2, \dots$ , are convex in  $(a, b)$  and  $\varphi_k \rightarrow \varphi$  in  $(a, b)$ , then  $\varphi$  is convex in  $(a, b)$ .

◀



### 1.1.11 Homework 11

**Problem 1** (Wheeden & Zygmund Ch. 7, Ex. 11). Prove the following result concerning changes of variable. Let  $g(t)$  be monotone increasing and absolutely continuous on  $[\alpha, \beta]$  and let  $f$  be integrable on  $[a, b]$ ,  $a = g(\alpha)$ ,  $b = g(\beta)$ . Then  $f(g(t))g'(t)$  is measurable and integrable on  $[\alpha, \beta]$ , and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t) dt.$$

(Consider the case when  $f$  is the characteristic function of an interval, an open set, etc.)

**Solution.** ►

◀

**Problem 2** (Wheeden & Zygmund Ch. 7, Ex. 15). Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If  $\varphi(x) = \int_a^x f(t) dt + \varphi(a)$  in  $(a, b)$  and  $f$  is monotone increasing, then  $\varphi$  is convex in  $(a, b)$ . (Use Exercise 14.)

**Solution.** ►

◀

**Problem 3** (Wheeden & Zygmund Ch. 5, Ex. 8). Prove (5.49).

**Solution.** ►

◀

**Problem 4** (Wheeden & Zygmund Ch. 5, Ex. 11). For which  $p$  does  $1/x \in L^p(0, 1)$ ?  $L^p(1, \infty)$ ?  $L^p(0, \infty)$ ?

**Solution.** ►

◀

**Problem 5** (Wheeden & Zygmund Ch. 5, Ex. 12). Give an example of a bounded continuous  $f$  on  $(0, \infty)$  such that  $\lim_{x \rightarrow \infty} f(x) = 0$  but  $f \notin L^p(0, \infty)$  for any  $p > 0$ .

**Solution.** ►

◀

**Problem 6** (Wheeden & Zygmund Ch. 5, Ex. 17). If  $f \geq 0$  and  $\omega(\alpha) \leq c(1 + \alpha)^p$  for all  $\alpha > 0$ , show that  $f \in L^r$ ,  $0 < r < p$ .

**Solution.** ►

◀

**Problem 7** (Wheeden & Zygmund Ch. 8, Thm. 8.3). If  $f, g \in L^p(E)$ ,  $p > 0$ , then  $f + g \in L^p(E)$  and  $cf \in L^p(E)$  for any constant  $c$ .

**Solution.** ►

◀

### 1.1.12 Homework 12

**Problem 1** (Wheeden & Zygmund Ch. 8, Ex. 2). Prove the converse of Hölder's inequality for  $p = 1$  and  $\infty$ . Show also that for  $1 \leq p \leq \infty$ , a real-valued measurable  $f$  belongs to  $L^p(E)$  if  $fg \in L^1(E)$  for every  $g \in L^{p'}(E)$ ,  $1/p + 1/p' = 1$ . The negation is also of interest: if  $f \in L^p(E)$  then there exists  $g \in L^{p'}(E)$  such that  $fg \notin L^1(E)$ . (To verify the negation, construct  $g$  of the form  $\sum a_k g_k$  satisfying  $\int_E fg_k \rightarrow \infty$ .)

**Solution.** ► ◀

**Problem 2** (Wheeden & Zygmund Ch. 8, Ex. 3). Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when  $p < 1$ .

**Solution.** ► ◀

**Problem 3** (Wheeden & Zygmund Ch. 8, Ex. 4). Let  $f$  and  $g$  be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let  $1 < p < \infty$ . Prove that equality holds in the inequality  $|\int fg| \leq \|f\|_p \|g\|_{p'}$  if and only if  $fg$  has constant sign a.e. and  $|f|^p$  is a multiple of  $|g|^{p'}$  a.e.

If  $\|f + g\|_p = \|f\|_p + \|g\|_p$  and  $g \neq 0$  in Minkowski's inequality, show that  $f$  is a multiple of  $g$ .

Find analogues of these results for the spaces  $\ell^p$ .

**Solution.** ► ◀

**Problem 4** (Wheeden & Zygmund Ch. 8, Ex. 5). For  $0 < p \leq \infty$  and  $0 < |E| < \infty$ , define

$$N_p[f] = \left( \frac{1}{|E|} \int_E |f|^p \right)^{1/p},$$

where  $N_\infty[f]$  means  $\|f\|_\infty$ . Prove that if  $p_1 < p_2$ , then  $N_{p_1}[f] \leq N_{p_2}[f]$ . Prove also that if  $1 \leq p \leq \infty$ , then  $N_p[f + g] \leq N_p[f] + N_p[g]$ ,  $(1/|E|) \int_E |fg| \leq N_p[f] N_{p'}[g]$ ,  $1/p + 1/p' = 1$ , and  $\lim_{p \rightarrow \infty} N_p[f] = \|f\|_\infty$ . Thus,  $N_p$  behaves like  $\|\cdot\|_p$  but has the advantage of being monotone in  $p$ . Recall Exercise 28 of Chapter 5.

**Solution.** ► ◀

**Problem 5** (Wheeden & Zygmund Ch. 8, Ex. 6).

- (a) Let  $1 \leq p_i$ ,  $r \leq \infty$  and  $\sum_{i=1}^k 1/p_i = 1/r$ . Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

- (b) Let  $1 \leq p < r < q \leq \infty$  and define  $\theta \in (0, 1)$  by  $1/r = \theta/p + (1 - \theta)/q$ . Prove the interpolation estimate

$$\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}.$$

In particular, if  $A = \max\{\|f\|_p, \|f\|_q\}$ , then  $\|f\|_r \leq A$ .

**Solution.** ►

◀

**Problem 6** (Wheeden & Zygmund Ch. 8, Ex. 9). If  $f$  is real-valued and measurable on  $E$ ,  $|E| > 0$ , define its essential infimum on  $E$  by

$$\operatorname{ess\,inf} f = \sup\{\alpha : |\{x \in E : f(x) < \alpha\}| = 0\}.$$

If  $f \geq 0$ , show that  $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup} 1/f)^{-1}$ .

**Solution.** ►

◀

**Problem 7** (Wheeden & Zygmund Ch. 8, Ex. 11). If  $f_k \rightarrow f$  in  $L^p$ ,  $1 \leq p < \infty$ ,  $g_k \rightarrow g$  pointwise, and  $\|g_k\|_\infty < M$  for all  $k$ , prove that  $f_k g_k \rightarrow f g$  in  $L^p$ .

**Solution.** ►

◀

## 2 Danielli

### 2.1 Danielli: Practice Exams Spring 2016

#### 2.1.1 Exam 1 Practice

**Problem 1.** Let  $E \subseteq \mathbb{R}^n$  be a measurable set,  $r \in \mathbb{R}$  and define the set  $rE = \{rx : x \in E\}$ . Prove that  $rE$  is measurable, and that  $|rE| = |r|^n |E|$ .

**Solution.** ► Define a map a linear map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $T(x) = rx$ . Since a the image of a measurable set  $E$  under linear map is measurable and  $m(T(E)) = |\det T|m(E) = |r|^n m(E)$ , it suffices to show that  $T(E) = rE$ .

Let  $y \in T(E)$  then  $y = rx$  for some  $x \in E$ . Thus,  $y \in rE$ . Let  $y \in rE$ . Then,  $y = rx = T(x)$  for some  $x \in E$ . Thus,  $y \in T(E)$ . It follows that  $m(rE) = |r|^n m(E)$ . ◀

**Problem 2.** Let  $\{E_n\}$ ,  $n \in \mathbb{N}$  be a collection of measurable sets. Define the set

$$\liminf_{n \rightarrow \infty} E_n = \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} E_k \right).$$

Show that

$$m\left(\liminf_{n \rightarrow \infty} E_n\right) \leq \liminf_{n \rightarrow \infty} m(E_n).$$

**Solution.** ► Here's a quick and dirty way of proving this: let  $\chi_{E_n}$  be the characteristic function of  $E_n$ . Then, by Fatou's lemma,

$$\int \liminf_{n \rightarrow \infty} \chi_{E_n}(x) dx \leq \liminf_{n \rightarrow \infty} \int \chi_{E_n}(x) dx. \quad (1)$$

By definition of the characteristic function, it is easy to see that the right hand-side of the Equation (1) is

$$\liminf_{k \rightarrow \infty} m(E_k).$$

But what about the left-hand side of (1)? We claim that

$$\liminf_{n \rightarrow \infty} \chi_{E_n} = \chi_E$$

where  $E = \liminf_{n \rightarrow \infty} E_n$ .

*Proof of claim.* Suppose  $x \in E$ . We must show that  $\liminf_{n \rightarrow \infty} \chi_{E_n}(x) = 1$ . By definition

$$\liminf_{n \rightarrow \infty} \chi_{E_n} = \lim_{n \rightarrow \infty} \left[ \inf_{k \geq n} \chi_{E_k} \right].$$

Now  $x \in E$  if and only if  $x \in \bigcap_{k=N}^{\infty} E_k$  for some  $N \in \mathbb{N}$ . Then for  $k \geq N$

$$\inf_{k \geq n} \chi_{E_k}(x) = 1$$

so  $\liminf_{n \rightarrow \infty} \chi_{E_n}(x) = 1$ .

On the other hand, if  $x \notin E$  then  $x \notin \bigcap_{k=n}^{\infty} E_k$  for all  $n \in \mathbb{N}$ . Thus, for all  $n \in \mathbb{N}$ ,

$$\inf_{k \geq n} \chi_{E_k}(x) = 0$$

so  $\liminf_{n \rightarrow \infty} \chi_{E_n} = 0$ . ■

Having established this equivalence, we have

$$m\left(\liminf_{n \rightarrow \infty} E_n\right) = \int \liminf_{n \rightarrow \infty} \chi_{E_n}(x) \, dx \leq \liminf_{n \rightarrow \infty} \int \chi_{E_n}(x) \, dx = \liminf_{n \rightarrow \infty} m(E_n).$$

◀

**Problem 3.** Consider the function

$$F(x) = \begin{cases} m(B(\mathbf{0}, x)) & x > 0, \\ 0 & x = 0. \end{cases}$$

Here  $B(\mathbf{0}, r) = \{y \in \mathbb{R}^n : |y| < r\}$ . Prove that  $F$  is monotonic increasing and continuous.

**Solution.** ▶ Let  $T: \mathbb{R}^n \times [0, x] \rightarrow \mathbb{R}^n$  be the linear map given by  $T(x, r) = rx$ . By Problem 1, we know that  $T(B(\mathbf{0}, 1), r) = B(\mathbf{0}, r)$  and consequently,  $m(B(\mathbf{0}, 1)) = |r|^n m(B(\mathbf{0}, 1))$ . Interpreting  $B(\mathbf{0}, 0) = \emptyset$ , we have  $F(x) = |r|^n m(B(\mathbf{0}, 1))$  and it is easy to see that  $F$  is both monotonically increasing and continuous since it is a polynomial in  $r$ . ◀

**Problem 4.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $C$  be the set of all points at which  $f$  is continuous. Show that  $C$  is a set of type  $G_\delta$ .

**Solution.** ▶ Let  $C$  be the subset of  $\mathbb{R}$  where  $f$  is continuous, i.e., the set

$$C = \{x \in \mathbb{R} : \text{given } \varepsilon > 0 \text{ there exist } \delta > 0 \text{ such that } |f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta\}.$$

In light of the latter equality, for each  $n \in \mathbb{N}$  define the following family of subsets of  $C$ ,

$$G_n = \left\{x \in \mathbb{R} : \text{there exists } \delta_n > 0 \text{ such that } |f(x) - f(y)| < \frac{1}{n} \text{ whenever } |x - y| < \delta_n\right\}.$$

We claim that (i) the  $G_n$  are open and (ii)  $C = \bigcap_{n \in \mathbb{N}} G_n$ .

The proof of (i) is easy: let  $x \in G_n$  then there exists  $\delta_n > 0$  such that

$$|f(x) - f(y)| < \frac{1}{n}.$$

Then  $B(x, \delta_n) \subseteq G_n$  since  $x' \in B(x, \delta_n)$  implies that  $|x - x'| < \delta$  so

$$|f(x) - f(x')| < \frac{1}{n}.$$

The proof of (ii) is also straight forward: let  $x \in C$  then given  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$|f(x) - f(y)| < \varepsilon$$

whenever  $|x - y| < \delta$ . In particular, if  $\varepsilon = 1/n$  then there exists  $\delta_n$  such that  $|x - y| < \delta_n$  implies

$$|f(x) - f(y)| < \frac{1}{n}$$

for ever  $n \in \mathbb{N}$ . Thus,  $x \in \bigcap_{n \in \mathbb{N}} G_n$ . On the other hand, if  $x \in \bigcap_{n \in \mathbb{N}} G_n$ , then  $x \in G_n$  for all  $n \in \mathbb{N}$ . Thus, given  $\varepsilon > 0$ , by the Archimedean property of the real numbers, there exists a positive integer  $N$  such that  $1/N < \varepsilon$  and hence for  $\delta = \delta_N > 0$  we have

$$|f(x) - f(y)| < \frac{1}{N}$$

whenever  $|x - y| < \delta_N$ . Thus,  $x \in C$ .

It follows that  $C = \bigcap_{n \in \mathbb{N}} G_n$  and hence is a  $G_\delta$  set. ◀

**Problem 5.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function. Is it true that if the sets  $\{f = r\}$  are measurable for all  $r \in \mathbb{R}$ , then  $f$  is measurable?

**Solution.** ▶ The statement is false and, of course, the counterexample involves existence of non-measurable sets. Let  $V \subseteq [0, 1]$  be a Vitali set and consider the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by the rule

$$f(x) = \begin{cases} x & \text{if } x \in V, \\ -x & \text{if } x \in \mathbb{R} \setminus V. \end{cases}$$

Then,  $\{f = r\}$  is measurable for all  $r \in \mathbb{R}$  since the set either consists of a single point or is the empty set. However,  $\{f \geq 0\} = V$  is not measurable. ◀

**Problem 6.** Let  $\{f_k\}$  be a sequence of measurable functions on  $\mathbb{R}$ . Prove that the set

$$\left\{ x : \lim_{k \rightarrow \infty} f_k(x) \text{ exists} \right\}$$

is measurable.

**Solution.** ▶ Suppose  $\{f_n\}$ ,  $n \in \mathbb{N}$ , is a sequence of measurable functions and let

$$E = \left\{ x : \lim_{n \rightarrow \infty} f_n(x) \text{ exists} \right\}.$$

Then, by general properties of the limit supremum and the limit infimum, we know that  $\lim_{n \rightarrow \infty} f_n(x)$  exists if and only if

$$\limsup_{n \rightarrow \infty} f_n(x) = \liminf_{n \rightarrow \infty} f_n(x).$$

Both of these functions are measurable so the set

$$E = \left\{ x : \limsup_{n \rightarrow \infty} f_n(x) = \liminf_{n \rightarrow \infty} f_n(x) \right\}.$$

is measurable. ◀

**Problem 7.** A real valued function  $f$  on an interval  $[a, b]$  is said to be *absolutely continuous* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^N$  of open intervals in  $(a, b)$  satisfying  $\sum_{k=1}^N b_k - a_k < \delta$ , one has  $\sum_{k=1}^N |f(b_k) - f(a_k)| < \varepsilon$ . Show that an absolutely continuous function on  $[a, b]$  is of bounded variation on  $[a, b]$ .

**Solution.** ▶ Let  $\varepsilon = 1$  then, since  $f: [a, b] \rightarrow \mathbb{R}$  is absolutely continuous, there exists  $\delta > 0$  such that  $|f(x) - f(y)| < \varepsilon$  whenever  $y - x < \delta$  (assuming  $x < y$ ). Partition the closed interval  $[a, b]$  into subintervals  $\{[a_n, b_n] : 1 \leq n \leq N\}$  of length less than or equal to  $\delta$ . Then

$$\text{var}(f; [a_n, b_n]) \leq 1.$$

Thus,

$$\text{var}(f; [a, b]) \leq N$$

for every partition  $\Gamma$  of  $[a, b]$ . ◀

**Problem 8.** Let  $f$  be a continuous function from  $[a, b]$  into  $\mathbb{R}$ . Let  $\chi_{\{c\}}$  be the characteristic function of a singleton  $\{c\}$ , that is,  $\chi_{\{c\}}(x) = 0$  if  $x \neq c$  and  $\chi_{\{c\}}(c) = 1$ . Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b), \\ -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

**Solution.** ▶ There are three cases to consider (1)  $c \in (a, b)$ , (2)  $c = a$  and (3)  $c = b$ . Cases (2) and (3) can be handled easily: if  $c = a$  then the Riemann–Stieltjes integral of  $f$  with respect to  $\chi_{\{c\}}$  is the supremum over all sums

$$\sum_{n=1}^N f(\xi_n) [\chi_{\{c\}}(x_n) - \chi_{\{c\}}(x_{n-1})]$$

where  $x_0 = a$  and  $x_N = b$  for all partitions  $\Gamma = \{x_0, \dots, x_N\}$  of  $[a, b]$ . Thus, the sum

$$\sum_{n=1}^N f(\xi_n) [\chi_{\{c\}}(x_n) - \chi_{\{c\}}(x_{n-1})] = \begin{cases} -f(\xi_0) & \text{if } c = a, \\ f(\xi_N) & \text{if } c = b. \end{cases}$$

Letting  $\Delta(\Gamma) \rightarrow 0$ ,  $\xi_0 \rightarrow a$  and  $\xi_N \rightarrow b$  giving us

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

It remains to show that

$$\int_a^b f d\chi_{\{c\}} = 0$$

if  $c \in (a, b)$ . To that end, note that if  $\Gamma_c$  is a partition containing the point  $c$ , say,  $x_m = c$  for some  $1 \leq m \leq N$ , the partial sums yield

$$\sum_{n=1}^N f(\xi_n) [\chi_{\{c\}}(x_n) - \chi_{\{c\}}(x_{n-1})] = f(\xi_{m+1}) - f(\xi_m).$$

Letting  $\Delta(\Gamma_c) \rightarrow 0$ , since  $f$  is continuous,  $f(\xi_{m+1}) \rightarrow f(\xi_m)$ . Thus,

$$\int_a^b f \, d\chi_{\{c\}} = 0.$$

◀



### 2.1.2 Exam 1

I lost this exam. These are the questions I could recall explicitly. For the first problem, we were asked to show that the Dirichlet function  $\mathbf{1}_{\mathbb{Q}}(x)$  is not Riemann integrable and prove something about  $\mathbb{Q}$ . For the second question, we were asked to show that the measure of countable union of disjoint measurable sets  $\{E_n : n \in \mathbb{N}\}$ , is equal to the sum of their individual measures (or something to that effect).

**Problem 1.**

**Problem 2.**

**Problem 3.**

(i) Show that if  $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ , then there exists a constant  $C$  such that  $|B_r| = Cr^n$ .

(Hint: Think of  $B_r$  as  $\{rx : x \in B_1\}$ .)

(ii) Let  $E \subseteq \mathbb{R}^n$  be a measurable set and let  $\varphi_E : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined  $\varphi_E(x) = |E \cap B_{|x|}|$ . Use part (i) to prove that  $\varphi_E$  is continuous.

**Solution.** ►

◀

**Problem 4.** Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation on  $[a, b]$ . Prove that  $f$  is measurable.

**Solution.** ►

◀

### 2.1.3 Exam 2 Practice Problems

**Problem 1.** Define for  $x \in \mathbb{R}^n$ ,

$$f(x) = \begin{cases} |x|^{-(n+1)} & \text{if } x \neq \mathbf{0}, \\ 0 & \text{if } x = \mathbf{0}. \end{cases}$$

Prove that  $f$  is integrable outside any ball  $B(\mathbf{0}, \varepsilon)$ , and that there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n \setminus B(\mathbf{0}, \varepsilon)} f(x) \, dx \leq \frac{C}{\varepsilon}.$$

**Solution.** ► Suppose that the following is true. ◀

**Problem 2.** Let  $\{f_k\}$  be a sequence of nonnegative measurable functions on  $\mathbb{R}^n$ , and assume that  $f_k$  converges pointwise almost everywhere to a function  $f$ . If

$$\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k \rightarrow \infty} \int_E f_k$$

for all measurable subsets  $E$  of  $\mathbb{R}^n$ . Moreover, show that this is not necessarily true if  $\int_{\mathbb{R}^n} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k = \infty$ .

**Solution.** ► ◀

**Problem 3.** Assume that  $E$  is a measurable set of  $\mathbb{R}^n$ , with  $|E| < \infty$ . Prove that a nonnegative function  $f$  defined on  $E$  is integrable if and only if

$$\sum_{k=0}^{\infty} |\{x \in E : f(x) \geq k\}| < \infty.$$

**Solution.** ► ◀

**Problem 4.** Suppose that  $E$  is a measurable subset of  $\mathbb{R}^n$ , with  $|E| < \infty$ . If  $f$  and  $g$  are measurable functions on  $E$ , define

$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$

Prove that  $\rho(f_k, f) \rightarrow 0$  as  $k \rightarrow \infty$  if and only if  $f_k$  converges to  $f$  as  $k \rightarrow \infty$ .

**Solution.** ► ◀

**Problem 5.** Define the *gamma function*  $\Gamma: \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} du,$$

and the *beta function*  $\beta: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

- (a) Prove that the definition of the gamma function is well-posed, i.e., the function  $u \mapsto e^{-u} u^{y-1}$  is in  $L(\mathbb{R}^+)$  for all  $y \in \mathbb{R}^+$ .
- (b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

**Solution.** ►

◀

**Problem 6.** Let  $f \in L(\mathbb{R}^n)$  and for  $\mathbf{h} \in \mathbb{R}^n$  define  $f_{\mathbf{h}}: \mathbb{R}^n \rightarrow \mathbb{R}$  by  $f_{\mathbf{h}}(x) = f(x - \mathbf{h})$ . Prove that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

**Solution.** ►

◀

**Problem 7.** (a) If  $f_k, g_k, f, g \in L(\mathbb{R}^n)$ ,  $f_k \rightarrow f$  and  $g_k \rightarrow g$  a.e. in  $\mathbb{R}^n$ ,  $|f_k| \leq g_k$  and

$$\int_{\mathbb{R}^n} g_k \longrightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f.$$

- (b) Using part (a) show that if  $f_k, f \in L(\mathbb{R}^n)$  and  $f_k \rightarrow f$  a.e. in  $\mathbb{R}^n$ , then

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \quad \text{as } k \rightarrow \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \longrightarrow \int_{\mathbb{R}^n} |f| \quad \text{as } k \rightarrow \infty.$$

**Solution.** ►

◀

### 2.1.4 Exam 2 (2010)

**Problem 1.** Suppose  $f \in L^1(\mathbb{R}^n)$ . Show that for every  $\varepsilon > 0$  there exists a ball  $B$ , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

*Hint:* Use the monotone convergence theorem.

**Solution.** ►

### Problem 2.

- (a) Prove the following generalization of *Chebyshev's inequality*: Let  $0 < p < \infty$  and  $E \subseteq \mathbb{R}^n$  be measurable. assume that  $|f|^p \in L^1(E)$ . Then

$$|\{x \in E : f(x) > \alpha\}| \leq \frac{1}{\alpha^p} \int_{\{f > \alpha\}} f^p,$$

for  $\alpha > 0$ .

- (b) Let  $p$ ,  $E$ , and  $f$  be as in part (a). In addition, assume that  $\{f_k\}$  is a sequence such that  $\int_E |f_k - f|^p \rightarrow 0$  as  $k \rightarrow \infty$ . Show that  $f_k \rightarrow f$  in measure on  $E$ .

Recall that  $f_k \rightarrow f$  in measure on  $E$  if and only if for every  $\varepsilon > 0$

$$\lim_{k \rightarrow \infty} |\{x \in E : |f_k(x) - f(x)| > \varepsilon\}| = 0.$$

**Solution.** ►

**Problem 3.** Let  $f \in L^1(\mathbb{R})$ , and define

$$F(\xi) = \int_{\mathbb{R}} f(x) \cos(2\pi x \xi) dx.$$

Prove that  $F$  is continuous and bounded on  $\mathbb{R}$ .

**Solution.** ►

**Problem 4.** Use repeated integration techniques to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}.$$

*Hint:* Start from the case  $n = 1$  by using the polar coordinates in

$$\left[ \int_{\mathbb{R}} e^{-x^2} dx \right]^2 = \left[ \int_{\mathbb{R}} e^{-x^2} dx \right] \left[ \int_{\mathbb{R}} e^{-y^2} dy \right]$$

Solution. ►



Problem 5.

Solution. ►



### 2.1.5 Exam 2

**Problem 1.** Assume that  $f \in L(\mathbb{R}^n)$ . Show that for every  $\varepsilon > 0$  there exists a ball  $B$ , centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

**Solution.** ►

◀

**Problem 2.** Let  $f \in L(E)$ , and let  $\{E_j\}$  be a countable collection of pairwise disjoint measurable subsets of  $E$ , such that  $E = \bigcup_{j=1}^{\infty} E_j$ . Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

**Solution.** ►

◀

**Problem 3.** Let  $\{f_k\}$  be a family in  $L(E)$  satisfying the following property: For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|A| < \delta$  implies

$$\int_A |f_k| < \varepsilon$$

for all  $k \in \mathbb{N}$ . Assume  $|E| < \infty$ , and  $f_k(x) \rightarrow f(x)$  as  $k \rightarrow \infty$  for a.e.  $x \in E$ . Show that

$$\lim_{k \rightarrow \infty} \int_E f_k = \int_E f.$$

(Hint: Use Egorov's theorem.)

**Solution.** ►

◀

**Problem 4.** Let  $I = [0, 1]$ ,  $f \in L(I)$ , and define  $g(x) = \int_x^1 t^{-1} f(t) dt$  for  $x \in I$ . Prove that  $g \in L(I)$  and

$$\int_I g = \int_I f.$$

**Solution.** ►

◀

### 2.1.6 Final Exam Practice Problems

**Problem 1.** Suppose  $f \in L^1(\mathbb{R}^n)$  and that  $x$  is a point in the Lebesgue set of  $f$ . For  $r > 0$ , let

$$A(r) = \frac{1}{|r|^n} \int_{B(0,r)} |f(x-y) - f(x)| \, dy.$$

Show that:

- (a)  $A(r)$  is a continuous function of  $r$ , and  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ ;
- (b) there exists a constant  $M > 0$  such that  $A(r) \leq M$  for all  $r > 0$ .

**Solution.** ► (a) Without loss of generality, we may assume  $r < s$ . Then, we want to show that as  $r \rightarrow s$ , the quantity

$$|A(s) - A(r)| \rightarrow 0.$$

Set  $F(y) = |f(x-y) - f(x)|$  and consider said quantity

$$\begin{aligned} |A(s) - A(r)| &= \left| \frac{1}{|s|^n} \int_{B_s} F(y) \, dy - \frac{1}{|r|^n} \int_{B_r} F(y) \, dy \right| \\ &= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y) \, dy + \frac{1}{|s|^n} \int_{B_r} F(y) \, dy - \frac{1}{|r|^n} \int_{B_r} F(y) \, dy \right| \\ &= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y) \, dy + \left( \frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(y) \, dy \right| \\ &\leq \underbrace{\frac{1}{|s|^n} \int_{B_s \setminus B_r} F(y) \, dy}_{I_1} + \underbrace{\left( \frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(y) \, dy}_{I_2}. \end{aligned}$$

Hence, we must show that the quantities  $I_1, I_2 \rightarrow 0$  as  $r \rightarrow s$ .

To see that  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ , note that  $x$  is a point of the Lebesgue set of  $f$  and that

$$0 = \lim_{B_r \searrow x} \frac{1}{|B_1||r|^n} \int_{B_r} |f(y) - f(x)| \, dy = \frac{1}{|B_1|} \lim_{B_r \searrow x} \frac{1}{|r|^n} \int_{B_r} |f(t) - f(x)| \, dt = \lim_{r \rightarrow 0} A(r).$$

by making the change of variables  $t = x - y$ .

(b) ◀

**Problem 2.** Let  $E \subseteq \mathbb{R}^n$  be a measurable set,  $1 \leq n < \infty$ . Assume  $\{f_k\}$  is a sequence in  $L^p(E)$  converging pointwise a.e. on  $E$  to a function  $f \in L^p(E)$ . Prove that

$$\|f_k - f\|_p \rightarrow 0$$

if and only if

$$\|f_k\|_p \rightarrow \|f\|_p$$

as  $k \rightarrow \infty$ .

**Solution.** ►

◀

**Problem 3.** Let  $1 < p < \infty$ ,  $f \in L^p(E)$ ,  $g \in L^{p'}(E)$ .

- (a) Prove that  $f * g \in C(\mathbb{R}^n)$ .
- (b) Does this conclusion continue to be valid when  $p = 1$  and  $p = \infty$ ?

**Solution.** ►

◀

**Problem 4.** Let  $f \in L(\mathbb{R})$ , and let  $F(t) = \int_{\mathbb{R}} f(x) \cos(tx) dx$ .

- (a) Prove that  $F(t)$  is continuous for  $t \in \mathbb{R}$ .
- (b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t \rightarrow \infty} F(t) = 0.$$

**Solution.** ►

◀

**Problem 5.** Let  $f$  be of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular. Show that

$$\int_a^b \varphi \, df = \int_a^b \varphi f' \, dx + \int_a^b \varphi \, dh$$

for all functions  $\varphi$  continuous on  $[a, b]$ .

**Solution.** ►

◀



### 2.1.7 Final Exam 2010

**Problem 1.** Suppose that  $f \in L^1(\mathbb{R}^n)$ , and that  $x$  is a point in the Lebesgue set of  $f$ . For  $r > 0$ , let

$$A(r) = \frac{1}{r^n} \int_{B_r} |f(x-y) - f(x)| \, dy,$$

where  $B_r = B(\mathbf{0}, r)$ .

Show that

- (a)  $A(r)$  is a continuous function of  $r$ , and  $A(r) \rightarrow 0$  as  $r \rightarrow 0$ .
- (b) There exists a constant  $M > 0$  such that  $A(r) \leq M$  for all  $r > 0$ .

**Solution.** ► (a)

(b) ◀

**Problem 2.** Let  $E \subseteq \mathbb{R}^n$  be a measurable set,  $1 \leq p < \infty$ . assume that  $\{f_k\}$  is a sequence in  $L^p(E)$  converging pointwise a.e. on  $E$  to a function  $f \in L^p(E)$ . Prove that

$$\|f_k - f\|_p \longrightarrow 0 \iff \|f_k\|_p \longrightarrow \|f\|_p$$

*Hint:* To prove one of the implications, you can use the following fact without proving it:

$$\left| \frac{a-b}{2} \right| \leq \frac{|a|^p + |b|^p}{2}$$

for all  $a, b \in \mathbb{R}$ .

**Solution.** ► ◀

**Problem 3.** Let  $0 < p < q < r \leq \infty$ ,  $E \subseteq \mathbb{R}^n$  be a measurable set. Show that each  $f \in L^q(E)$  is the sum of a function  $g \in L^p(E)$  and a function  $h \in L^r(E)$ .

**Solution.** ► ◀

**Problem 4.** Prove that  $f: [a, b] \rightarrow \mathbb{R}$  is Lipschitz continuous if and only if  $f$  is absolutely continuous and there exists a constant  $M > 0$  such that  $|f'| < M$  a.e. on  $[a, b]$ .

**Solution.** ► ◀

**Problem 5.** Let  $1 < p < \infty$ ,  $f \in L^p(\mathbb{R}^n)$ ,  $g \in L^{p'}(\mathbb{R}^n)$ .

- (a) Prove that  $f * g \in C(\mathbb{R}^n)$ .
- (b) Does this conclusion continue to be valid when  $p = 1$  or  $p = \infty$ ?

**Solution.** ► ◀

### 2.1.8 Final Exam

## 2.2 Danielli: Summer 2011

**Problem 1.** Let  $f \in L^1(\mathbb{R})$ , and let  $\hat{f}(x) = \int_{\mathbb{R}} f(t) \cos(xt) dt$ .

- (a) Prove that  $\hat{f}(x)$  is continuous for  $x \in \mathbb{R}$ .
- (b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{x \rightarrow \infty} \hat{f}(x) = 0.$$

*Hint:* Start by proving the statement for  $f = \mathbf{1}_{[a,b]}$ .

**Solution.** ► For part (a): let  $\varepsilon > 0$  be given. Then, since  $\cos(xt)$  is continuous there exists  $\delta' > 0$  such that  $|x - y| < \delta$  implies

$$|\cos(xt) - \cos(yt)| < \frac{\varepsilon}{\|f\|_1}.$$

Now, let  $\delta = \delta'$ . Then we have

$$\begin{aligned} |\hat{f}(x) - \hat{f}(y)| &= \left| \int_{\mathbb{R}} f(t) \cos(xt) dt - \int_{\mathbb{R}} f(t) \cos(yt) dt \right| \\ &\leq \int_{\mathbb{R}} |f(t)| |\cos(xt) - \cos(yt)| dt \\ &< \frac{\varepsilon}{\|f\|_1} \int_{\mathbb{R}} |f(t)| dt \\ &= \frac{\varepsilon}{\|f\|_1} \|f\|_1 \\ &= \varepsilon. \end{aligned}$$

Since this can be done for any  $x \in \mathbb{R}$ ,  $\hat{f}$  is continuous on  $\mathbb{R}$ .

For part (b): since simple functions are dense in  $L^1(\mathbb{R})$ ,  $f$  there exists a sequence of simple functions  $\{s_n\}$ ,  $n \in \mathbb{N}$ , such that  $\int_{\mathbb{R}} s_n \rightarrow \|f\|_1$ . Therefore, it suffices to prove the result for characteristic functions. Let  $f = \mathbf{1}_{[a,b]}$  and consider the limit

$$\lim_{x \rightarrow \infty} \hat{f}(x) = \lim_{x \rightarrow \infty} \int_{\mathbb{R}} f(t) \cos(xt) dt.$$

Since  $f = \mathbf{1}_{[a,b]}$ , we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \int_{\mathbb{R}} f(t) \cos(xt) dt &= \lim_{x \rightarrow \infty} \int_a^b \cos(xt) dt \\ &= \lim_{x \rightarrow \infty} \left[ \frac{1}{x} (\sin(xa) - \sin(xb)) \right] \\ &= \lim_{x \rightarrow \infty} \left[ \frac{\sin(xa)}{x} - \frac{\sin(xb)}{x} \right] \\ &= \left[ \lim_{x \rightarrow \infty} \frac{\sin(xa)}{x} \right] - \left[ \lim_{x \rightarrow \infty} \frac{\sin(xb)}{x} \right] \\ &= 1 - 1 \\ &= 0, \end{aligned}$$

as we set out to show. ◀

**Problem 2.**

(a) Suppose that  $f_k, f \in L^2(E)$ , with  $E$  a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E f g \quad (\star)$$

as  $k \rightarrow \infty$  for all  $g \in L^2(E)$ . If, in addition,  $\|f_k\|_2 \rightarrow \|f\|_2$  show that  $f_k$  converges to  $f$  in  $L^2$ , i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as  $k \rightarrow \infty$ .

(b) Provide an example of a sequence  $f_k$  in  $L^2$  and a function  $f$  in  $L^2$  satisfying  $(\star)$ , but such that  $f_k$  does *not* converge to  $f$  in  $L^2$ .

**Solution.** ► For part (a): expand the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_E |f - f_n|^2 dx &= \lim_{n \rightarrow \infty} \left[ \int_E (|f|^2 - 2|ff_n| + |f_n|^2) dx \right] \\ &= \lim_{n \rightarrow \infty} \left[ \|f_n\|_2^2 + \|f\|_2^2 - 2 \int_E f_n f dx \right] \\ &= \lim_{n \rightarrow \infty} \|f_n\|_2^2 + \lim_{n \rightarrow \infty} \|f\|_2^2 - 2 \lim_{n \rightarrow \infty} \int_E f_n f dx. \end{aligned} \quad (1)$$

Since

$$\int_E f_n g dx \longrightarrow \int_E f g dx$$

for every  $g \in L^2(E)$ ,

$$\int_E f_n f dx \longrightarrow \int_E f^2 dx = \|f\|_2^2.$$

Moreover,  $\|f_n\|_2 \rightarrow \|f\|_2$  so the limit in (1) converges to

$$\lim_{n \rightarrow \infty} \|f_n\|_2^2 + \lim_{n \rightarrow \infty} \|f\|_2^2 - 2 \lim_{n \rightarrow \infty} \int_E f_n f dx = \|f\|_2^2 + \|f\|_2^2 - 2\|f\|_2^2 = 0$$

as  $n \rightarrow \infty$ .

For part (b), consider the sequence  $\{f_n\}$ ,  $n \in \mathbb{N}$ , where  $f_n(x) = \log(n) \exp(-nx)$ . Then, we claim that  $f_n \xrightarrow{L^2[0,1]} 0$ , but that  $f_n \not\rightarrow 0$  pointwise. To see the former, first note that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[ \int_0^1 f_n(x) dx \right] &= \lim_{n \rightarrow \infty} \left[ \int_0^1 \log(n) \exp(-nx) dx \right] \\ &= \lim_{n \rightarrow \infty} \left[ \log(n) \exp(-nx) \Big|_0^1 \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{1}{n} \log(n) - \frac{1}{n} \log(n) \exp(-n) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \left( \frac{1 - \exp(-n)}{n} \right) \log(n) \right] \\ &= 0. \end{aligned}$$

However,  $f_n$  does not converge to 0 a.e. since, for  $x = 0$  there exist no  $N \in \mathbb{N}$  such that

$$|\log(n)| < 1.$$

for all  $n \geq N$ . ◀

**Problem 3.** A bounded function  $f$  is said to be of bounded variation on  $\mathbb{R}$  if it is of bounded variation on any finite subinterval  $[a, b]$ , and moreover  $A := \sup_{a,b} V[a, b; f] < \infty$ . Here,  $V[a, b; f]$  denotes the total variation of  $f$  over the interval  $[a, b]$ . Show that:

(a)  $\int_{\mathbb{R}} |f(x+h) - f(x)| dx \leq A|h|$  for all  $h \in \mathbb{R}$ .

*Hint:* For  $h > 0$ , write

$$\int_{\mathbb{R}} |f(x+h) - f(x)| dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)| dx.$$

(b)  $\left| \int_{\mathbb{R}} f(x) \varphi'(x) dx \right| \leq A$ , where  $\varphi$  is any function of class  $C^1$ , of bounded variation, compactly supported, with  $\sup_{x \in \mathbb{R}} |\varphi(x)| \leq 1$ .

**Solution.** ▶ For part (a), it suffices to consider only positive  $h$  as, making the change of variables  $u = x + h$  yields

$$\int_{\mathbb{R}} |f(u) - f(u-h)| du = \int_{\mathbb{R}} |f(u+(-h)) - f(u)| du$$

where  $-h$  is positive (and letting  $h = 0$ , we have a trivial inequality). Now, taking the hint, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)| dx = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)| dx.$$

Now, since  $|f((n+1)h) - f(nh)|$  is a sum in the total variation of  $f$  on the interval  $[nh, (n+1)h]$ ,  $|f(x+h) - f(x)|$  is bounded by  $V[nh, (n+2)h; f]$ . Thus, we have

$$\begin{aligned} \int_{\mathbb{R}} |f(x+h) - f(x)| dx &= \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)| dx \\ &\leq \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} V[nh, (n+2)h; f] dx \\ &= \sum_{n=-\infty}^{\infty} V[nh, (n+2)h; f] \int_{nh}^{(n+1)h} dx \\ &= \sum_{n=-\infty}^{\infty} V[nh, (n+2)h; f] |h| \\ &= 2A|h|. \end{aligned}$$

I suspect there is an error here as the most obvious bound we can get is  $2A|h|$  and not the stricter  $A|h|$ .

For part (b),  $f$  is absolutely continuous since it is of bounded variation and  $\varphi$  is absolutely continuous since it is Lipschitz ( $\varphi$  is differentiable on a compact set, thus, by the mean value theorem  $|\varphi(x) - \varphi(y)| \leq \varphi'(\xi)|x - y|$  for some  $\xi \in \text{Supp } \varphi$ ). Assuming  $\text{Supp } \varphi$  has nonempty interior,  $\text{Supp } \varphi$  contains a closed interval  $I = [a, b]$  (in fact,  $\text{Supp } \varphi$  is of the form  $[a, b] \setminus \bigcup_{n \in A} I_n$ ,  $A \subseteq \mathbb{N}$ , where  $I_n = (a_n, b_n)$  with  $a_n, b_n \in \text{Supp } \varphi$ ) and thus, by integration by parts, we have

$$\begin{aligned} \int_a^b f \varphi' dx &= f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b f' \varphi dx \\ &\leq f(b) - f(a) - \int_a^b f' dx \\ &= 2(f(b) - f(a)) \\ &\leq 2V[a, b; f] \end{aligned}$$

Thus, summing over every

$$\sum_{n=0}^{\infty} \int_{a_n}^{b_n} f \varphi' dx \leq 2|A|.$$

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#### Problem 4.

- (a) Prove the *generalized Hölder's inequality*: Assume  $1 \leq p_j \leq \infty$ ,  $j = 1, \dots, n$ , with  $\sum_{j=1}^n 1/p_j = 1/r \leq 1$ . If  $E$  is a measurable set and  $f_j \in L^{p_j}(E)$  for  $j = 1, \dots, n$ , then  $\prod_{j=1}^n f_j \in L^r(E)$  and

$$\|f_1 \cdots f_n\|_r \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

- (b) Use part (a) to show that that if  $1 \leq p, q, r \leq \infty$ , with  $1/p + 1/q = 1/r + 1$ ,  $f \in L^p(\mathbb{R})$ , and  $g \in L^q(\mathbb{R})$ , then

$$|(f * g)(x)|^r \leq \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q dy.$$

(Recall that  $(f * g)(x) = \int f(y)g(x-y) dy$ .)

- (c) Prove *Young's convolution theorem*: Assume that  $p, q, r, f$ , and  $g$  are as in part (b). Then  $f * g \in L^r(\mathbb{R})$  and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

**Solution.** ▶ For (a) we shall proceed by induction on  $n$  the number of measurable functions  $f_j \in L^{p_j}(E)$ ,  $1 \leq j \leq n$ . The case  $n = 2$  holds by using Hölder's inequality on the exponents  $r/p + r/q = 1$ ,

$$\begin{aligned} \left[ \int_E |f_1 f_2|^r dx \right]^{1/r} &= \|f_1^r f_2^r\|_1^{1/r} \\ &\leq \|f_1^r\|_{p/r} \|f_2^r\|_{q/r} \\ &= \|f_1\|_p \|f_2\|_q. \end{aligned}$$

Now, suppose this holds for  $n - 1$  measurable functions  $f_j \in L^{p_j}(E)$ ,  $1 \leq j \leq n - 1$ . Then for  $f_j \in L^{p_j}(E)$  with  $\sum_{j=1}^n 1/p_j = 1/r$ , we have  $r' = \sum_{j=1}^{n-1} 1/p_j = 1/r - 1/p_n$  so by the inductive step

$$\|f_1 \cdots f_{n-1}\|_{r'} \leq \|f_1\|_{p_1} \cdots \|f_{n-1}\|_{p_{n-1}}$$

hence,  $f_1 \cdots f_{n-1} \in L^{r'}(E)$ . Thus,

$$\begin{aligned} \|f_1 \cdots f_{n-1} f_n\|_r &\leq \|f_1 \cdots f_{n-1}\|_{r'} \|f_n\|_{p_n} \\ &\leq \|f_1\|_{p_1} \cdots \|f_{n-1}\|_{p_{n-1}} \|f_n\|_{p_n}, \end{aligned}$$

as we set out to show.

For part (b), applying the generalized Hölder's inequality we proved in part (a),

$$\begin{aligned} |f * g| &= \left| \int_{\mathbb{R}} f(y)g(x-y) \, dy \right| \\ &\leq \int_{\mathbb{R}} |f(y)g(x-y)| \, dy \\ &= \int_{\mathbb{R}} |f(y)|^{1+p/r-p/r} |g(x-y)|^{1+q/r-q/r} \, dy \\ &= \int_{\mathbb{R}} |f(y)|^{p/r} |g(x)|^{q/r} |f(y)|^{1-p/r} |g(x-y)|^{1-q/r} \, dy \\ &= \int_{\mathbb{R}} |f(y)|^{p/r} |g(x)|^{q/r} |f(y)|^{(r-p)/r} |g(x-y)|^{(r-q)/r} \, dy \\ &= \int_{\mathbb{R}} (|f(y)|^p |g(x)|^q)^{1/r} |f(y)|^{(r-p)/r} |g(x-y)|^{(r-q)/r} \, dy \\ &\leq \|(|f(y)|^p |g(x)|^q)^{1/r}\|_r \| |f(y)|^{(r-p)/r} \|_{pr/(r-p)} \| |g(x-y)|^{(r-q)/r} \|_{qr/(r-q)} \\ &= \|f\|_p^{(r-p)/r} \|g\|_q^{(r-q)/r} \left[ \int_{\mathbb{R}} |f(y)|^p |g(x-y)|^q \, dy \right]^{1/r}. \end{aligned}$$

Raising both sides to the power  $r$ , we have

$$|(f * g)(x)|^r \leq \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q \, dy,$$

as desired.

For part (c), using the estimate we worked out in part (b) together with Tonelli's theorem, we

have

$$\begin{aligned}
\|f * g\|_r^r &= \int_{\mathbb{R}} |f * g(x)|^r dx \\
&\leq \int_{\mathbb{R}} \left[ \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}} |f(y)|^p |g(x-y)|^q dy \right] dx \\
&= \|f\|_p^{r-p} \|g\|_q^{r-q} \iint_{\mathbb{R} \times \mathbb{R}} |f(y)|^p |g(x-y)|^q dy dx \\
&= \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}} |f(y)|^p \left[ \int_{\mathbb{R}} |g(x-y)|^q dx \right] dy \\
&\leq \|f\|_p^{r-p} \|g\|_q^{r-q} \|f\|_p^p \|g\|_q^q \\
&= \|f\|_p^r \|g\|_q^r.
\end{aligned}$$

Taking the  $r$ th root on each side, we achieve the desired estimate

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

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## 2.3 Danielli: Winter 2012

**Problem 1.** Let  $f(x, y)$ ,  $0 \leq x, y \leq 1$ , satisfy the following conditions: for each  $x$ ,  $f(x, y)$  is an integrable function of  $y$ , and  $\partial f(x, y)/\partial x$  is a bounded function of  $(x, y)$ . Prove that  $\partial f(x, y)/\partial x$  is a measurable function of  $y$  for each  $x$  and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial f(x, y)}{\partial x} dy.$$

**Solution.** ► The end points can be dealt with separately. Fix a point  $x_0 \in (0, 1)$  and consider the sequence of measurable functions  $\{f'_n\}$  where

$$f'_n(y) = \frac{f(x_0 + h_n, y) - f(x_0, y)}{h_n}$$

where  $\{h_n\}$  is a sequence of numbers converging to 0. Since  $f$  is differentiable as a function of  $x$ , the sequence  $\{f'_n(x_0, y)\}$  converges to  $\partial f/\partial x(x_0, y)$ . Now, since  $|\partial f/\partial x(x, y)| \leq M$  for some  $M \in \mathbb{R}^+$  for all  $(x, y) \in [0, 1] \times [0, 1]$ , by the bounded convergence theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 f'_n(y) dy &= \int_0^1 \lim_{n \rightarrow \infty} f'_n(y) dy \\ &= \int_0^1 \frac{\partial f(x_0, y)}{\partial x} dy. \end{aligned} \tag{1}$$

It remains to show that the left side of (1) is the derivative of the integral of  $f(x_0, y)$  as a function of  $y$ . But this is exactly

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 f'_n(y) dy &= \lim_{n \rightarrow \infty} \int_0^1 \frac{f(x_0 + h_n, y) - f(x_0, y)}{h_n} dy \\ &= \lim_{n \rightarrow \infty} \frac{\int_0^1 f(x_0 + h_n, y) dy - \int_0^1 f(x_0, y) dy}{h_n} \\ &= \frac{d}{dx} \int_0^1 f(x, y) dy. \end{aligned}$$

It follows that for any  $x \in [0, 1]$ ,

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial f(x, y)}{\partial x} dy$$

so  $\partial f/\partial x(x, y)$  is a measurable function of  $y$ . ◀

**Problem 2.** Let  $f$  be a function of bounded variation on  $[a, b]$ ,  $-\infty < a < b < \infty$ . If  $f = g + h$ , with  $g$  absolutely continuous and  $h$  singular, show that

$$\int_a^b \varphi df = \int_a^b \varphi f' dx + \int_a^b \varphi dh.$$

*Hint:* A function  $h$  is said to be singular if  $h' = 0$ .

**Solution.** ► Let

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**Problem 3.** Let  $E \subseteq \mathbb{R}$  be a measurable set, and let  $K$  be a measurable function on  $E \times E$ . Assume that there exists a positive constant  $C$  such that

$$\int_E K(x, y) \, dx \leq C \quad (\star)$$

for a.e.  $y \in E$ , and

$$\int_E K(x, y) \, dy \leq C \quad (\clubsuit)$$

for a.e.  $x \in E$ .

Let  $1 < p < \infty$ ,  $f \in L^p(E)$ , and define

$$T_f(x) = \int_E K(x, y) f(y) \, dy.$$

(a) Prove that  $T_f \in L^p(E)$  and

$$\|T_f\|_p \leq C \|f\|_p. \quad (\spadesuit)$$

(b) Is  $(\spadesuit)$  still valid if  $p = 1$  or  $\infty$ ? If so, are assumptions  $(\star)$  and  $(\clubsuit)$  needed?

**Solution.** ►

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**Problem 4.** Let  $f$  be a nonnegative measurable function on  $[0, 1]$  satisfying

$$m\{x \in [0, 1] : f(x) > \alpha\} < \frac{1}{1 + \alpha^2} \quad (\diamond)$$

for  $\alpha > 0$ .

(a) Determine values of  $p \in [1, \infty)$  for which  $f \in L^p[0, 1]$ .

(b) If  $p_0$  is the minimum value of  $p$  for which  $p$  may fail to be in  $L^p$ , give an example of a function which satisfies  $(\diamond)$ , but which is not in  $L^{p_0}[0, 1]$ .

**Solution.** ►

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