Hyperbolic Spaces

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In this talk, I will discuss the irreducible symmetric spaces that are negatively curved. These spaces come in three infinite families associated to the rank one, real, simple Lie groups $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, and $\mathrm{Sp}(n,1)$ along with exceptional rank one, real, simple Lie group $\mathrm{F}_{4,-20}$.

1 Möbius transformations on complex upper half space

The simplest example of a real rank one simple Lie group is the Lie group $\mathrm{SL}(2,\mathbf{R})$ of two-by-two matrices with determinant 1. The group $\mathrm{SL}(2,\mathbf{R})$ is directly related to a symplectic group, a unitary group, and an orthogonal group. The standard symplectic form $\omega\colon \mathbf{R}^2 \to \mathbf{R}$ given by $\omega(v,w) = v_1w_2 - v_2w_1$ and the symplectic group associated to ω is defined to be

$$\mathrm{Sp}(2) = \mathrm{Sp}(\omega) = \left\{A \in \mathrm{Mat}(2,\mathbf{R}) \ : \ \omega(Av,Aw) = \omega(v,w) \text{ for all } v,w \in \mathbf{R}^2\right\}.$$

As $\omega = \det$, it is not hard to see that $\mathrm{SL}(2,\mathbf{R}) = \mathrm{Sp}(\omega)$. We also have the hermitian form H on $\mathbf{C}^2$ given by $H(w,z) = w_1\overline{z_1} - w_2\overline{z_2}$ and an associated symmetry group

$$\mathrm{SU}(1,1) = \mathrm{SU}(H) = \left\{A \in \mathrm{SL}(2,\mathbf{C}) \ : \ H(Aw,Az) = H(w,z) \text{ for all } w,z \in \mathbf{C}^2\right\}.$$

The groups $\mathrm{SL}(2,\mathbf{R})$ and $\mathrm{SU}(H)$ are both subgroups of $\mathrm{SL}(2,\mathbf{C})$ and are conjugate. Finally, we have the symmetric bilinear form $B(v,w) = v_1w_1 + v_2w_2 - v_3w_3$ and the associated symmetry group

$$\mathrm{SO}(2,1) = \mathrm{SO}(B) = \left\{A \in \mathrm{SL}(3,\mathbf{R}) \ : \ B(Av,Aw) = B(v,w) \text{ for all } v,w \in \mathbf{R}^3\right\}.$$

The connected component that contains the identity element in $\mathrm{SO}(B)$ and $\mathrm{SL}(2,\mathbf{R})$ are isomorphic.

Returning to $\mathrm{SL}(2,\mathbf{R})$, acts on the complex upper half plane by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

The subgroup that fixes i can shown to be

$$K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ : \ \theta \in [0,2\pi) \right\}$$

and it is straightforward to see that $K = S^1$ is the circle. Indeed, we have the equation

$$\frac{ai+b}{ci+d} = i$$

and so

$$ai+b = di-b.$$

Therefore, $a=d$ and $b=-c$, and the matrix is of the form

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Since $\det(A) = a^2+b^2$, we see that the matrices are in the above from K. As $\mathrm{SL}(2,\mathbf{R})$ acts transitively on upper half space $\mathbf{H}^2$, for any $z \in \mathbf{H}^2$, we have that $\mathrm{Stab}(z)$ and $\mathrm{Stab}(i)$ are conjugate in $\mathrm{SL}(2,\mathbf{R})$.

Given $A \in \mathrm{SL}(2,\mathbf{R})$, we can solve for $Az = z$, which is given by the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} = z.$$

In particular,

$$cz^2 + (d-a)z - b = 0.$$

Using the quadratic formula, the fact that $ad-bc=1$, and some basic algebra, we obtain

$$z = \frac{(a-d) \pm \sqrt{(a+d)^2-4}}{2c}.$$

We see that there are three different types of outcomes based on $a+d = \mathrm{Tr}(A)$.

Case 1. $|\mathrm{Tr}(A)| > 2$.

In this case, we see that A has two real fixed points given by
$$z = \frac{(a-d) \pm (|a+d|-2)}{2c}.$$
We say that A is **hyperbolic** in this case.

Case 2. $|\mathrm{Tr}(A)| = 2$.

In this case, we see that A has one fixed point and it is given by
$$z = \frac{a-d}{2c}.$$
We say that A is **parabolic** in this case.

Case 3. $|\mathrm{Tr}(A)| < 2$.

In this case, we see that A has two complex conjugate fixed points, and one of them is in the upper half plane. In particular, A is conjugate into $\mathrm{Stab}(i)$ in $\mathrm{SL}(2,\mathbf{R})$. We say that A is **elliptic** in this case.

There is one degenerate case when $A = \pm \mathrm{I}_2$. In this case, A fixes *every* point. Aside from these two elements, every $A \in \mathrm{SL}(2,\mathbf{R})$ is precisely one of the three above types.

Remark 1. If $A \in \mathrm{SL}(2,\mathbf{R})$, then the characteristic polynomial for A is of the form $c_A(t) = t^2 - \mathrm{Tr}(A)t + 1$. In particular, the eigenvalues of A are given by
$$\lambda = \frac{-\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4}}{2}.$$

We have already analyzed the case of elliptic elements. Every elliptic element can be conjugated into $K = S^1$. We now perform this analysis for the hyperbolic and parabolic cases. First, given a parabolic element A, we will assume that A fixes 0. Then we have that $b = 0$ and $ad = 1$. We also have that $|\mathrm{Tr}(A)| = 2$ and so
$$A = \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix}$$
Since $\mathrm{SL}(2,\mathbf{R})$ acts transitively on $\mathbf{R} \cup \{\infty\}$, we see that every parabolic is conjugate into the group
$$N_0 = \left\{ \begin{pmatrix} \pm 1 & 0 \\ x & \pm 1 \end{pmatrix} \ : \ x \in \mathbf{R} \right\}.$$
The stabilizer of ∞ is given by
$$N_\infty = \left\{ \begin{pmatrix} \pm 1 & x \\ 0 & \pm 1 \end{pmatrix} \ : \ x \in \mathbf{R} \right\}.$$

For the hyperbolic case, we will assume that A fixes 0 and ∞ since $\mathrm{SL}(2,\mathbf{R})$ acts transitively on pairs of points in $\mathbf{R}\cup\{\infty\}$. We see that fixing 0 implies that $b=0$ and $ad=1$. Similarly, fixing ∞ implies that $c=0$ and $ad=1$. Since $|\mathrm{Tr}(A)|>2$, we know that neither a,d can be ± 1. Hence

$$A=\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$$

for some $y\neq \pm 1, 0$ and the subgroup of elements in $\mathrm{SL}(2,\mathbf{R})$ that fixes both 0 and ∞ is given by

$$A_{0,\infty}=\left\{ \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \;:\; t\in \mathbf{R}^{\times} \right\}.$$

Note that since $-\mathrm{I}_2$ acts trivially, we get an action of $\mathrm{PSL}(2,\mathbf{R})=\mathrm{SL}(2,\mathbf{R})/\pm\mathrm{I}_2$ on upper half space and this action is faithful. We can then view

$$\begin{aligned}
K &= \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;:\; \theta\in[0,2\pi) \right\} \\
N_0 &= \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \;:\; x\in \mathbf{R} \right\} \\
N_\infty &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;:\; x\in \mathbf{R} \right\} \\
A_{0,\infty} &= \left\{ \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \;:\; y\in \mathbf{R}^{+} \right\}.
\end{aligned}$$

In fact, the QR factorization of a matrix gives a (smooth) bijection of sets

$$\mathrm{SL}(2,\mathbf{R}) \longrightarrow K\times N_\infty \times A_{0,\infty}.$$

As $\mathrm{SL}(2,\mathbf{R})$ acts transitively on the upper half plane, we can identity the upper half plane $\mathbf{H}^2$ with $\mathrm{SL}(2,\mathbf{R})/K \cong N_\infty\times A_{0,\infty}$. Of course, $N_\infty=\mathbf{R}$ and $A_{0,\infty}=\mathbf{R}^+$ and we see that the upper half plane is indeed an upper half plane!

As $\mathrm{SL}(2,\mathbf{R})$ is unimodular, it has a unique bi-invariant Haar measure μ_{Haar} and this measure descends to the quotient $\mathrm{SL}(2,\mathbf{R})/K=\mathbf{H}^2$. This measure gives a method for finding areas and lengths, and endows $\mathbf{H}^2$ with a Riemannian metric of constant -1 sectional curvature. The space $\mathbf{H}^2$ with this Riemannian metric is called the **hyperbolic plane**.

Given a discrete subgroup $\Gamma<\mathrm{SL}(2,\mathbf{R})$, we get an action of Γ on $\mathbf{H}^2$ and the action is properly discontinuous. Specifically, given any compact subset $C\subset\mathbf{H}^2$, the set

$$\{\gamma\in\Gamma \;:\; \gamma C\cap C\neq\emptyset\}$$

is finite. The quotient space $M_\Gamma = \Gamma/\mathbf{H}^2$ is a topological manifold but need not be a smooth manifold. The issue is precisely with elliptic elements in Γ. Indeed, if $\gamma \in \Gamma$ is elliptic with fixed point $z \in \mathbf{H}^2$, then if we take a sufficiently small neighborhood about z, the element γ will act by rotation about z. As Γ is discrete, $\Gamma \cap \mathrm{Stab}(z)$ must be finite and so $\mathrm{Stab}(z)$ is a finite abelian group. The quotient space at z does not have a well defined tangent space and is a cone singularity. If Γ is torsion free, then M_Γ is a 2–dimensional Riemannian manifold. Such manifolds are sometimes called Riemann surfaces though under these general conditions, M_Γ need not be compact or even finite volume. We note that the cone point z, viewed in $\mathbf{H}^2$, is only well defined up to the action of Γ. That is, the orbit of z in $\mathbf{H}^2$ under the action of Γ are all identified with a single point $z_0 \in M_\Gamma$. Likewise, the stabilizers $\mathrm{Stab}(z)$ are only well defined up to conjugation in Γ.

If $\Gamma < \mathrm{SL}(2,\mathbf{R})$ is discrete and $\mu_{\mathrm{Haar}}(M_\Gamma) < \infty$, we say that Γ is a lattice. If M_Γ is compact, we call Γ a **uniform lattice** or **cocompact lattice**. If Γ is also torsion free, the associated manifold is a closed Riemannian 2–manifold. Since M_Γ has constant scalar curvature, by Gauss–Bonnet, the Euler characteristic of M_Γ must be negative. In particular, the topological genus of M_Γ must be at least 2. In fact, every surface of topological genus at least two admits such a Riemannian metric has a moduli space of such metrics $\mathscr{M}_g$ that is a $3g-3$ complex orbifold. If M_Γ is finite volume but not compact, we say that Γ is a **non-uniform lattice** or **non-cocompact lattice**. The non-compactness of M_Γ comes from the presence of parabolic elements in Γ. Given $\varepsilon > 0$, we can decompose M_Γ into two pieces called the thick-thin decomposition. The ε–thick part $M_{\varepsilon,\mathrm{thick}}$ is the subset of points in M_Γ with injectivity radius at least ε and the ε–thin part $M_{\varepsilon,\mathrm{thin}}$ is the subset of points in M_Γ with injectivity radius strictly smaller than ε. Since M_Γ is finite volume, $M_{\varepsilon,\mathrm{thick}}$ is compact for all ε. For ε sufficiently small, $M_{\varepsilon,\mathrm{thin}}$ has finitely many connected components and each component is diffeomorphic to $\mathbf{R}^+ \times S^1$. We refer to these components as the **cusp ends**. Each cusp end has an associate subgroup Δ called the **peripheral subgroup** which corresponds to the stabilizer of some $z \in \mathbf{R} \cup \{\infty\}$. The subgroup Δ is comprised of parabolic elements that fix the point z. Both the point z and the subgroup Δ are only defined up to conjugacy since they require the choice of a lift to the universal cover $\mathbf{H}^2$. This issue is identical to the similar issue we saw for elliptic elements and cone singularities.

If Γ is discrete and $\gamma \in \Gamma$ is a hyperbolic element, then there is a unique bi-infinite geodesic in $\mathbf{H}^2$ that is fixed (as a set) by γ. The selection of the geodesic depends on the choice of a base point and is again only well defined up to the action of Γ. The element γ acts on the geodesic ray by translation and so the quotient of the geodesic by the action of γ is a circle. This circle gives rise to a closed geodesic on the associated orbifold M_Γ. Every closed geodesic on the orbifold M_Γ can be associated to a Γ–conjugacy class of hyperbolic elements $[\gamma]$.

(Elliptic) Elliptic elements in Γ give rise to cone type singularities on the orbifold M_Γ.

(Parabolic) Parabolic elements in Γ give rise to cusp ends on the orbifold M_Γ.

(Hyperbolic) Hyperbolic elements in Γ give rise to closed geodesics on the orbifold M_Γ.

One alternative way to obtain the trichotomy of elliptic, parabolic, and hyperbolic is by translation distance. Given $\gamma \in \mathrm{SL}(2,\mathbf{R})$ and $z \in \mathbf{H}^2$, we define the translation distance $\tau_z(\gamma)$ to be the distance between z and $\gamma(z)$. We define the translation distance for γ to be

$$\tau(\gamma) = \inf_z \tau_z(\gamma).$$

We again have three possibilities:

(a) γ has a fixed point in $\mathbf{H}^2$, then $\tau(\gamma) = 0$ and there exists $z \in \mathbf{H}^2$ such that $\tau_z(\gamma) = 0$.

(b) γ does not have a fixed point but $\tau(\gamma) = 0$. This is the case when γ is parabolic.

(c) γ has $\tau(\gamma) > 0$. In this case, $\tau(\gamma) = \tau_z(\gamma)$ for any z on the geodesic stabilized (setwise) by γ.

2 General picture

One can introduce the negatively curved symmetric spaces in a number of different ways. We first introduce the Lie groups and then take an approach somewhat analogous to the one for $\mathbf{H}^2$ as an upper half space.

2.1 Real hyperbolic space

For each $n \in \mathbf{N}$ with $n \geq 2$, we have the symmetric bilinear form

$$B_{n,1}(v,w) = -v_{n+1}w_{n+1} + \sum_{i=1}^{n} v_i w_i.$$

This bilinear form can be defined via

$$B_{n,1}(v,w) = v^T \mathrm{I}_{n,1} w,$$

where

$$\mathrm{I}_{n,1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

The group of symmetries

$$\mathrm{O}(n,1) = \left\{ A \in \mathrm{GL}(n+1,\mathbf{R}) \ : \ B_{n+1}(Av,Aw) = B_{n+1}(v,w) \text{ for all } v,w \in \mathbf{R}^{n+1} \right\}.$$

This is equivalent to $\mathrm{I}_{n,1}A^T\mathrm{I}_{n,1}A = \mathrm{I}_{n+1}$. In fact, associated to any symmetric, invertible matrix $S \in \mathrm{GL}(n+1,\mathbf{R})$, we can define an involution on $\mathrm{Mat}(n+1,\mathbf{R})$ by $A^\star = SA^TS^{-1}$. Taking $S = \mathrm{I}_{n,1}$, we see that $A \in \mathrm{O}(n,1)$ if and only if $A^\star A = \mathrm{I}_{n+1}$.

The subgroup $\mathrm{SO}(n,1) < \mathrm{O}(n,1)$ is defined to be the elements of $\mathrm{O}(n,1)$ that have determinant 1. One can check that if $A \in \mathrm{O}(n,1)$, then $\det(A) = \pm 1$ and so $\mathrm{SO}(n,1)$ is an index two subgroup of $\mathrm{O}(n,1)$. All maximal compact subgroups K of $\mathrm{SO}(n,1)$ are conjugate in $\mathrm{SO}(n,1)$ and so we take $K = \mathrm{S}(\mathrm{O}(n)\times\mathrm{O}(1))$. The group $\mathrm{SO}(n,1)$ is not connected and so we pass to $\mathrm{SO}_0(n,1)$, the connected component of $\mathrm{SO}(n,1)$.

Formally, hyperbolic n–space $\mathbf{H}^n$ is defined to be $\mathrm{SO}(n,1)/K$ or $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$. The Iwasawa decomposition of $\mathrm{SO}(n,1)$ is a generalization of QR factorization of matrices. Namely, $\gamma = kan$ where k is a unitary matrix, a is diagonalizable, and n is a unipotent matrix. However, one must do some work to prove that $k,a,n \in \mathrm{SO}(n,1)$. The result of this work is a decomposition, but only as a manifold and not as a group, of $\mathrm{SO}(n,1)$ called the Iwasawa decomposition

$$\mathrm{SO}(n,1) = K \times A \times N.$$

The splitting gives again a trichotomy of elements in $\mathrm{SO}(n,1)$ into elliptic, parabolic, and hyperbolic. The elliptic elements fix a point in $\mathbf{H}^n$ and so by definition are in some conjugate of K (or equivalently, can be conjugated into K). Parabolic elements have one fixed point on $\partial\mathbf{H}^n$, the boundary of $\mathbf{H}^n$, which is an $(n-1)$–sphere S^{n-1}. Hyperbolic elements have two fixed points on the boundary and fix (set wise) the unique bi-infinite geodesic that connects the points. However, it should be noted that hyperbolic elements can also rotate about the geodesic, something that could not happen when $n = 2$. Sometimes these hyperbolic elements are called loxodromic. Parabolic elements can also be more complicated. In $\mathbf{H}^2$, parabolic elements translate along horocircles. If the parabolic has a fixed point at ∞, then horocircles centered at ∞ are lines of the form $y = r$ where y is the imaginary part of $z \in \mathbf{H}^2$. In particular, parabolics that fix ∞ translate along these horozontial lines. As the level of the line increase, the translation distance, computed at say ir, tends to zero. $\mathbf{H}^n$, via the Iwasawa decomposition, can be viewed as $\mathbf{R}^{n-1} \times \mathbf{R}^+$ which is the upper half space in $\mathbf{R}^n$. Here, we have

$$A \cong \mathbf{R}^+, \quad N \cong \mathbf{R}^{n-1}.$$

The boundary of $\mathbf{H}^n$ in this model is the hyperplane $x_n = 0$ for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ together with a point at infinity. Parabolic elements that fix ∞ can translate in the horoballs which in this case are the hyperplanes of the form $x_n = r > 0$ by some direction in the boundary $v \in \mathbf{R}^n$ where this is the hyperplane $x_n = 0$. In fact, they can act by a Euclidean transformation, which means that they can translate by some vector and then apply a transformation in the orthogonal group $\mathrm{O}(n-1)$. We can think of this as acting on the horoball by $(v,S)w = Sw + v$. Applying this transformation again, we obtain $(v,S)(Sw+v) = S^2w + Sw + v$. Applying the transformation m times to w yields

$$(v,S)^m(w) = S^m w + \sum_{i=0}^{m-1} S^i v.$$

If such a transformation resides in a discrete subgroup of $\mathrm{SO}(n,1)$, then S must be finite order. If the order is m, then

$$(v,S)^m(w) = w + \sum_{i=0}^{m-1} S^i v = w + v'$$

where

$$v' = \sum_{i=1}^{m-1} S^i v.$$

In particular, this is a pure translation and has the form (v', I_{n-1}). There are other models for hyperbolic n–space like the paraboloid model, the hyperboloid model, and the ball model. Each has its own merits. The formation of an upper half plane model requires the choice of a boundary point since one of the points at ”infinity” is actually at infinity unlike the ball model where all points on the boundary look the same.

If $\Gamma < \mathrm{SO}(n,1)$ is a discrete subgroup, we can again form a quotient orbifold $M_\Gamma = \Gamma/\mathbf{H}^n$. Elliptic elements in Γ give rise to singularities by the type can be more complicated than isolated cone singularities. However, they are codimension at least two. The parabolic isometries give rise to cusps thought the peripheral subgroup fixing the point at infinity needs to be a compact lattice in the associated Euclidean group. These manifolds are called flat manifolds and they have a finite cover that is diffeomorphic to $\mathbf{R}^{n-1}/\mathbf{Z}^{n-1}$, the $(n-1)$–torus. For $n = 2$, note that we get the circle $S^1 = \mathbf{R}/\mathbf{Z}$. When the parabolic elements have the form (v,S) for some non-trivial S, these flat manifolds are not tori and one does indeed have to pass to a finite cover to get to the torus. The hyperbolic elements again give rise to closed geodesics on the quotient orbifold.

2.2 Complex hyperbolic space

Complex hyperbolic space is constructed in a similar way as real hyperbolic space. However, instead of taking a bilinear form of signature $(n,1)$, we instead start with a hermitian form on $\mathbf{C}^{n+1}$ of signature $(n,1)$. Specifically, define the hermitian form

$$H(w,z) = -w_{n+1}\overline{z_{n+1}} + \sum_{j=1}^{n} w_j\overline{z_j}.$$

We again have symmetry groups of this form

$$\mathrm{U}(n,1) = \left\{A \in \mathrm{GL}(n+1,\mathbf{C}) \ : \ H(Aw,Az) = H(w,z) \text{ for all } w,z \in \mathbf{C}^{n+1}\right\}.$$

The determinant one subgroup is denoted by $\mathrm{SU}(n,1)$. Note that in this case, $\mathrm{SU}(n,1)$ is not finite index in $\mathrm{U}(n,1)$ since the determinant can take any complex number of modulus one. That is, $|\det(A)| = 1$ for $A \in \mathrm{U}(n,1)$. The maximal compact subgroup of $K = \mathrm{SU}(n,1)$ is $\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$. In particular, $\mathrm{SU}(n,1)$ is connected. We define complex hyperbolic space $\mathbf{H}^n_{\mathbf{C}}$ to be $\mathrm{SU}(n,1)/K$, metrized with a bi-invariant Haar measure. We again have an Iwasawa decomposition $K \times A \times N$ where K is as above. The group $A = \mathbf{R}^+$ as before. However, N is now the $(2n-1)$–dimensional Heisenberg group. The Heisenberg groups can be defined in many ways. As a set $N_{2n-1} = \mathbf{C}^{n-1} \times \mathbf{R}$ with the short exact sequence

$$1 \longrightarrow \mathbf{R} \longrightarrow N_{2n-1} \longrightarrow \mathbf{C}^{n-1} \longrightarrow 1.$$

This is not a split exact sequence and has the following group law

$$(s,w)*(t,z) = (s+t+2\mathrm{Im}(H_{n-1}(w,z)), w+z)$$

where H_{n-1} is the standard hermitian inner product on $\mathbf{C}^{n-1}$ given by

$$H_{n-1}(w,z) = \sum_{j=1}^{n-1} w_j\overline{z_j}.$$

The imaginary part of this hermitian form is a symplectic form on $\mathbf{C}^{n-1}$, viewed as $\mathbf{R}^{2n-2}$. As such, we will denote it by ω, which in this group extension problem, is a 2–cocycle for the extension.

Via the Iwasawa decomposition, we get an upper half space model for $\mathbf{H}^n_{\mathbf{C}}$ given by $\mathbf{R}^+ \times N_{2n-1} \cong \mathbf{R}^+ \times \mathbf{R} \times \mathbf{C}^{n-1}$. Topologically, we can think of this as just $\mathbf{R}^{2n-1} \times \mathbf{R}^+$ though this suppress important information. The boundary of $\mathbf{H}^n_{\mathbf{C}}$ is the 1-point compactifaction of N_{2n-1}, which is a sphere of dimension $(2n-1)$. We have a similar trichotomy of types for elements in

$\mathrm{SU}(n,1)$. Elliptic elements fix a point in $\mathbf{H}^n_{\mathbf{C}}$ and so are conjugate into K. Parabolic elements fix one boundary points. The horoballs in $\mathbf{H}^n_{\mathbf{C}}$ can now be thought of as topologically just N_{2n-1} and are given by some level. Parabolic elements can act by translations in N_{2n-1}. That is, they translate by right multiplication. However, we can again perform some rotations. We can rotate about the central line given by $\mathbf{R} \subset N_{2n-1}$ but we must preserve the symplectic form ω. In particular, we have a description by (v,S) where $v \in N_{2n-1}$ and $S \in \mathrm{Sp}(\omega)$. As N_{2n-1} is a Lie group, we can take a right invariant metric and with this metric, $N_{2n-1} \rtimes \mathrm{Sp}(\omega)$ plays the role of the Euclidean isometry group $\mathbf{R}^n \rtimes \mathrm{O}(n)$. As in the hyperbolic setting, parabolic elements can do more than simply translate in the Heisenberg group. However, in a lattice, they must have a power that is a pure translation and so are virtually unipotent. The hyperbolic elements again fix two points on the boundary but can again have some twisting behavior coming from rotations about the geodesic fixed by the hyperbolic element.

As before, given a discrete group $\Gamma < \mathrm{SU}(n,1)$, we have an associated quotient orbifold $M_\Gamma = \Gamma/\mathbf{H}^n_{\mathbf{C}}$.