Math 527 - Homotopy Theory
Spring 2013
Homework 13 Solutions

Definition. A space weakly equivalent to a product of Eilenberg-MacLane spaces is called a **generalized Eilenberg-MacLane space**, or GEM for short.

Problem 1. Show that any topological abelian *group* is a GEM. (It need not be path-connected.)

Solution. Let X be a topological abelian group, with multiplication map $\mu\colon X \times X \to X$. We write the product $\mu(x, y) = xy$ for short.

Basic facts about topological groups.

1. The path component $X_0 \subseteq X$ of the unit element $e \in X$ is a subgroup (in fact a normal subgroup).

Proof: Let $x, y \in X_0$ and let α, β be paths in X from e to x and y respectively. Then pointwise multiplication of β by x yields the path $x\beta$ from $x\beta(0) = xe = x$ to $x\beta(1) = xy$, so that x and xy are in the same path component. This proves $xy \in X_0$.

Likewise, pointwise multiplication of α by x^{-1} yields the path $x^{-1}\alpha$ from $x^{-1}\alpha(0) = x^{-1}e = x^{-1}$ to $x^{-1}\alpha(1) = x^{-1}x = e$, so that x^{-1} and e are in the same path component. This proves $x^{-1} \in X_0$, and thus $X_0 \subseteq X$ is a subgroup.

For any $z \in X$, pointwise conjugation $z\alpha z^{-1}$ yields a path from $z\alpha(0)z^{-1} = zez^{-1} = e$ to $z\alpha(1)z^{-1} = zxz^{-1}$, proving $zxz^{-1} \in X_0$, and thus $X_0 \subseteq X$ is a normal subgroup.

2. The set $\pi_0 X$ of path components is the set of cosets X/X_0.

Proof: Two points $x, y \in X$ lie in the same path component if and only if $x^{-1}x$ and $x^{-1}y$ lie in the same path component, i.e. $x^{-1}y \in X_0$.

3. All path components of X are homeomorphic.

Proof: Let $x \in X$ and denote by $C_x \subseteq X$ its path component (so that $C_e = X_0$ in this awkward notation). By fact (2), left multiplication by x provides a continuous map

$$\mu(x, -)\colon C_e \to C_x$$

and left multiplication by x^{-1} provides its inverse

$$\mu(x^{-1}, -)\colon C_x \to C_e$$

which is also continuous, hence a homeomorphism.

WLOG X is the coproduct of its path components. Consider the natural map

$$\epsilon\colon X' := \coprod_{C \in \pi_0 X} C \to X$$

from the coproduct of the path components of X to X. Thus ϵ is the identity function, but the domain X' has a (possibly) larger topology. Then X' is the coproducts of its own path components, which are the summands $C \subseteq X'$, and moreover $\epsilon\colon X' \xrightarrow{\sim} X$ is a weak equivalence.

Since ϵ is bijective, we can use it to put a group structure on X'. In other words, X' has the same underlying group as X. It remains to check that X' is still a topological group, i.e. that its structure maps are continuous.

Consider the diagram

$$\begin{tikzcd}
X' \times X' \arrow[r, "\mu'"] \arrow[d, "\epsilon\times\epsilon"'] & X' \arrow[d, "\epsilon"] \\
X \times X \arrow[r, "\mu"'] & X.
\end{tikzcd}$$

Since coproducts in **Top** distribute over products, we have the homeomorphism

$$\begin{aligned}
X' \times X' &= \left(\coprod_{C \in \pi_0 X} C \right) \times \left(\coprod_{D \in \pi_0 X} D \right) \\
&\cong \coprod_{C, D \in \pi_0 X} C \times D.
\end{aligned}$$

To check that $\mu'\colon X' \times X' \to X'$ is continuous, it suffices to check that its restriction to each summand

$$\mu'|_{C\times D} \colon C \times D \to X'$$

is continuous. But $\epsilon\mu' = \mu(\epsilon \times \epsilon)$ is continuous and $C \times D$ is path-connected, so that $\epsilon\mu'(C \times D)$ lies inside one path component $E \subseteq X$ of X. Since the inclusion $E \subseteq X$ is an embedding and $E \hookrightarrow X'$ is continuous (in fact also an embedding), it follows that μ' is continuous, as illustrated in the diagram:

$$\begin{tikzcd}
C \times D \arrow[r, hook] \arrow[drr, dashed] & X' \times X' \arrow[rr, "\mu'"] \arrow[dd, "\epsilon\times\epsilon"'] & & X' \arrow[dd, "\epsilon"] \\
& & E \arrow[ur, hook] \arrow[dr, hook] & \\
& X \times X \arrow[rr, "\mu"'] & & X.
\end{tikzcd}$$

Likewise, the inverse structure map $i\colon X' \to X'$ is continuous, and thus X' is a topological group.

Concluding from there. Assume X is a topological abelian group which is the coproduct of its path components. Then we have the homeomorphism:

$$\begin{aligned} X &\cong \coprod_{C \in \pi_0 X} C \\ &\cong \coprod_{C \in \pi_0 X} X_0 \text{ by fact (3)} \\ &\cong (\pi_0 X) \times X_0 \end{aligned}$$

where $\pi_0 X$ is given the discrete topology. But the discrete space $\pi_0 X$ is in particular an Eilenberg-MacLane space $K(\pi_0 X, 0)$, so we can write

$$X \cong K(\pi_0 X, 0) \times X_0.$$

Since X_0 is a *path-connected* topological abelian monoid, it is a GEM:

$$\begin{aligned} X_0 &\simeq \prod_{k \geq 1} K(\pi_k X_0, k) \\ &\cong \prod_{k \geq 1} K(\pi_k X, k). \end{aligned}$$

Since a product of weak equivalences is again a weak equivalence, we obtain the desired weak equivalence:

$$\begin{aligned} X &\simeq K(\pi_0 X, 0) \times \prod_{k \geq 1} K(\pi_k X, k) \\ &= \prod_{k \geq 0} K(\pi_k X, k). \quad \square \end{aligned}$$

Remark. Not every topological abelian group is the coproduct of its path components. For example, the additive group of p-adic integers $\mathbb{Z}_p$ with the p-adic topology is a totally disconnected space (so that its path components are all singletons), yet it is not discrete.

Problem 2. Let X be a pointed CW complex, let $n \geq 0$, and let A be an abelian group. Show that there is a weak equivalence

$$\operatorname{Map}_*(X, K(A,n)) \simeq \prod_{k=0}^{n} K\left(\widetilde{H}^{n-k}(X;A), k\right).$$

Solution. Since X is a CW complex, the functor $\operatorname{Map}_*(X,-)\colon \mathbf{Top}_* \to \mathbf{Top}_*$ preserves weak equivalences. Hence we may choose our favorite model of $K(A,n)$, in particular we may assume that $K(A,n)$ is a topological abelian group (which is an abelian group object in $\mathbf{Top}_*$).

Since $\operatorname{Map}_*(X,-)$ preserves products, $\operatorname{Map}_*(X, K(A,n))$ is naturally a topological abelian group, via pointwise addition.

By Problem 1, $\operatorname{Map}_*(X, K(A,n))$ is a GEM:

$$\operatorname{Map}_*(X, K(A,n)) \simeq \prod_{k\geq 0} K\left(\pi_k \operatorname{Map}_*(X, K(A,n)), k\right). \tag{1}$$

Let us compute its homotopy groups. For any $k \geq 0$, we have:

$$\begin{aligned}
\pi_k \operatorname{Map}_*(X, K(A,n)) &\cong \pi_0 \operatorname{Map}_*\left(S^k, \operatorname{Map}_*(X, K(A,n))\right) \\
&\cong \pi_0 \operatorname{Map}_*\left(S^k \wedge X, K(A,n)\right) \\
&\cong \pi_0 \operatorname{Map}_*\left(\Sigma^k X, K(A,n)\right) \\
&\cong \pi_0 \operatorname{Map}_*\left(X, \Omega^k K(A,n)\right) \\
&\cong \pi_0 \operatorname{Map}_*(X, K(A,n-k)) \quad \text{with the convention } K(A, \text{negative}) = * \\
&\cong [X, K(A,n-k)]_* \\
&\cong \widetilde{H}^{n-k}(X;A)
\end{aligned}$$

including the fact that this group vanishes for $k > n$. Putting this result back into (1), we obtain the desired weak equivalence:

$$\operatorname{Map}_*(X, K(A,n)) \simeq \prod_{k=0}^{n} K\left(\widetilde{H}^{n-k}(X;A), k\right). \quad \square$$

Exercise (for fun, not to be turned in). Consider the circle with a disjoint basepoint $S^1_+ = S^1 \amalg \{e\}$ where e serves as basepoint. Show that the infinite symmetric product $\mathrm{Sym}(S^1_+)$ is *not* weakly equivalent to a product of Eilenberg-MacLane spaces.

This shows that the assumption of path-connectedness in the theorem cannot be dropped in general.

Solution A: Explicit calculation. Let us describe more generally the symmetric products of a coproduct $X \amalg Y$, where Y contains the basepoint $y_0 \in Y$.

Since coproducts distribute over products, we have the homeomorphism

$$(X \amalg Y)^n \cong \coprod_{\epsilon \in \{0,1\}^n} \prod_{i=1}^n W_{\epsilon_i}$$

with $W_0 := X$ and $W_1 := Y$, and the 2^n summands correspond to all possible ways of choosing either X or YZ for each of the n entries of an element in $(X \amalg Y)^n$.

Summands of the same "type", that differ only by a reordering of the factors (e.g. $X \times Y \times X$ and $Y \times X \times X$), are identified homeomorphically by the Σ_n-action. The action then permutes the entries within the summand of each type. Hence there is a homeomorphism

$$\mathrm{Sym}_n(X \amalg Y) \cong \coprod_{i=0}^n \mathrm{Sym}_{n-i}(X) \times \mathrm{Sym}_i(Y).$$

Via this homeomorphism, the embedding

$$\begin{aligned} \iota_n^{X \amalg Y} \colon \mathrm{Sym}_n(X \amalg Y) &\hookrightarrow \mathrm{Sym}_{n+1}(X \amalg Y) \\ (w_1, \ldots, w_n) &\mapsto (w_1, \ldots, w_n, y_0) \end{aligned}$$

corresponds to the composite

$$\begin{tikzcd} \coprod_{i=0}^n \mathrm{Sym}_{n-i}(X) \times \mathrm{Sym}_i(Y) \arrow[r, "\coprod \mathrm{id} \times \iota_i^Y"] \arrow[d, hook] & \coprod_{i=0}^n \mathrm{Sym}_{n-i}(X) \times \mathrm{Sym}_{i+1}(Y) \arrow[d, hook] \\ \coprod_{i=0}^{n+1} \mathrm{Sym}_{n+1-i}(X) \times \mathrm{Sym}_i(Y) \arrow[r, equal] & \mathrm{Sym}_{n+1}(X) \amalg \coprod_{i=0}^n \mathrm{Sym}_{n-i}(X) \times \mathrm{Sym}_{i+1}(Y). \end{tikzcd}$$

Letting n go to infinity yields the homeomorphism:

$$\begin{aligned} \mathrm{Sym}(X \amalg Y) &= \operatorname*{colim}_n \mathrm{Sym}_n(X \amalg Y) \\ &\cong \coprod_{i \geq 0} (\mathrm{Sym}_i(X) \times \mathrm{Sym}(Y)) \\ &\cong \left(\coprod_{i \geq 0} \mathrm{Sym}_i(X) \right) \times \mathrm{Sym}(Y). \end{aligned}$$

In particular, taking the one-point space $Y = *$, we have $\mathrm{Sym}(*) = *$ and therefore

$$\begin{aligned} \mathrm{Sym}(X_+) &= \mathrm{Sym}(X \amalg \{*\}) \\ &\cong \left(\coprod_{i \geq 0} \mathrm{Sym}_i(X) \right) \times \mathrm{Sym}(*) \\ &\cong \coprod_{i \geq 0} \mathrm{Sym}_i(X). \end{aligned}$$

In particular, taking the circle $X = S^1$, we obtain a homotopy equivalence

$$\begin{aligned} \mathrm{Sym}(S^1_+) &\cong \coprod_{i \geq 0} \mathrm{Sym}_i(S^1) \\ &= \{*\} \amalg \coprod_{i \geq 1} \mathrm{Sym}_i(S^1) \\ &\simeq \{*\} \amalg \coprod_{i \geq 1} S^1 \end{aligned}$$

where we used the homotopy equivalence $S^1 \simeq \mathrm{Sym}_i(S^1)$ for all $i \geq 1$.

The space $\mathrm{Sym}(S^1_+)$ has some path components $\{*\} \not\simeq S^1$ that are not weakly equivalent. Therefore $\mathrm{Sym}(S^1_+)$ cannot be weakly equivalent to a space of the form

$$K(\pi_0, 0) \times W$$

for some path-connected W. In particular, $\mathrm{Sym}(S^1_+)$ is not a GEM. $\square$

Solution B: Categorical consideration. Consider the adjoint pairs

$$\mathbf{Top} \underset{U}{\overset{(-)_+}{\rightleftarrows}} \mathbf{Top}_* \underset{U}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathbf{TopAbMon}$$

where both forgetful functors U are the right adjoints. It follows that the composite of left adjoints $\mathrm{Sym} \circ (-)_+$ is left adjoint to the forgetful functor $U\colon \mathbf{TopAbMon} \to \mathbf{Top}$. As sets, the free abelian monoid on X is

$$F(X) := \coprod_{i \geq 0} X^i / \Sigma_i = \coprod_{i \geq 0} \mathrm{Sym}_i(X)$$

where the unit element (the "empty word") is in the zeroth summand $\mathrm{Sym}_0(X) = \{*\}$. One readily checks that endowing $F(X)$ with the coproduct topology makes it into a topological abelian monoid, which moreover satisfies the universal property of a free topological abelian monoid on the unpointed space X. By uniqueness of adjoints, there is a natural isomorphism

$$\mathrm{Sym}(X_+) \cong F(X) = \coprod_{i \geq 0} \mathrm{Sym}_i(X).$$

Conclude as above. $\square$

Problem 3. Let $E = \{E_n\}_{n\in\mathbb{N}}$ be an Ω-spectrum. Show that the assignments

$$h^n(X) := [X, E_n]_*$$

define a reduced cohomology theory $\{h^n\}_{n\in\mathbb{Z}}$. Don't forget to address the abelian group structure of $h^n(X)$.

Here we use the convention $E_{-m} := \Omega^m E_0$ for $m > 0$.

Solution. For each $n \in \mathbb{Z}$, the composite

$$E_n \xrightarrow[\sim]{\omega_n} \Omega E_{n+1} \xrightarrow[\sim]{\Omega\omega_{n+1}} \Omega^2 E_{n+2}$$

provides E_n with the structure of a (weak) homotopy abelian group object. This provides a lift of the functor

$$[-, E_n]_* \colon \mathbf{CW}_*^{\mathrm{op}} \to \mathbf{Set}_*$$

to a functor

$$[-, E_n]_* \colon \mathbf{CW}_*^{\mathrm{op}} \to \mathbf{Ab}.$$

It remains to check that the functors h^* satisfy the axioms of a reduced cohomology theory.

Homotopy invariance. By construction, $[-, E_n]_*$ is homotopy invariant, as it factors through the homotopy category $\mathrm{Ho}(\mathbf{CW}_*)$.

Exactness. For any cofiber sequence $A \xrightarrow{i} X \xrightarrow{p} C$ of well-pointed spaces, we know that the sequence of pointed sets

$$[C, Z]_* \xrightarrow{p^*} [X, Z]_* \xrightarrow{i^*} [A, Z]_*$$

is exact, for any pointed space Z.

Moreover, for any pointed CW complex X, applying $[X, -]_*$ to the weak equivalence $\omega_n \colon E_n \xrightarrow{\sim} \Omega E_{n+1}$ yields a bijection (which is natural in X):

$$\begin{array}{ccc} [X, E_n]_* & \xrightarrow[\cong]{\omega_{n*}} & [X, \Omega E_{n+1}]_* \\ & & \downarrow \cong \\ & & [\Sigma X, E_{n+1}]_*. \end{array}$$

Let us check that the bijection $[X, E_n]_* \xrightarrow{\cong} [\Sigma X, E_{n+1}]_*$ is in fact a group isomorphism.

By definition of the infinite loop space structure on E_n, the structure map ω_n is an infinite loop map, which proves that ω_{n*} is a map of groups. The bijection $[X, \Omega E_{n+1}]_* \xrightarrow{\cong} [\Sigma X, E_{n+1}]_*$ is a group isomorphism if one notes that the group structure can be obtained on both sides from the (let's say) 1-fold loop space structure of $E_{n+1} \xrightarrow{\sim} \Omega E_{n+2}$.

Wedge axiom. Recall that the wedge is the coproduct in $\mathbf{CW}_*$ as well as in the homotopy category $\mathrm{Ho}(\mathbf{CW}_*)$, which yields the natural isomorphism:

$$[\bigvee_\alpha X_\alpha, E_n]_* \xrightarrow{\cong} \prod_\alpha [X_\alpha, E_n]_*. \quad \square$$

Problem 4. Let $h^* = \{h^n\}_{n\in\mathbb{Z}}$ be a reduced cohomology theory. Show that there is an Ω-spectrum E representing h^* in the sense of Problem 3. Explicitly: there are natural isomorphisms of abelian groups

$$h^n(X) \cong [X, E_n]_*$$

for all $n \in \mathbb{Z}$ which are moreover compatible with the suspension isomorphisms, i.e. making the diagram:

$$\begin{tikzcd} h^n(X) \arrow[r, "\cong"] \arrow[d, "\cong"'] & {[X, E_n]_*} \arrow[d, "\cong"] \\ h^{n+1}(\Sigma X) \arrow[r, "\cong"'] & {[\Sigma X, E_{n+1}]_*} \end{tikzcd}$$

commute.

Solution. Step 1. Applying Brown representability.

By definition, h^n satisfies homotopy invariance and the wedge axiom. Recall that the Mayer-Vietoris sequence in (co)homology is a formal consequence of the Eilenberg-Steenrod axioms c.f. Hatcher § 2.3.Axioms for Homology and § 3.1.Cohomology of Spaces. Thus for a CW triad $(X; A, B)$ as illustrated here:

$$\begin{tikzcd} & A \cap B \arrow[dl, "j_A"'] \arrow[dr, "j_B"] & \\ A \arrow[dr, "i_A"'] & & B \arrow[dl, "i_B"] \\ & X & \end{tikzcd}$$

there is a Mayer-Vietoris (long) exact sequence

$$\ldots \longrightarrow h^n(X) \xrightarrow{i_A^* + i_B^*} h^n(A) \oplus h^n(B) \xrightarrow{j_A^* - j_B^*} h^n(A \cap B) \xrightarrow{\delta} h^{n+1}(X) \longrightarrow \ldots$$

The inclusion $\ker(j_A^* - j_B^*) \subseteq \operatorname{im}(i_A^* + i_B^*)$ shows that h^n satisfies the Mayer-Vietoris axiom.

Thus, h^n (and its restriction to *connected* pointed CW complexes) satisfies the hypotheses of Brown representability. Hence, there is a connected pointed CW complex D_n with a universal class $u_n \in h^n(D_n)$ such that the "pullback" natural map

$$[X, D_n]_* \to h^n(X)$$

is a bijection for all *connected* pointed CW complex X.

Since $h^n(X)$ is naturally a group (by assumption), this bijection produces a homotopy abelian group object structure on D_n.

Step 2. Getting rid of the connectedness assumption.

Let X be a pointed CW complex. Then X is the coproduct (in **CW**) of its path components. Denoting the basepoint component by $X_0 \subseteq X$, write X as

$$X = X_0 \amalg \coprod_\alpha X_\alpha$$

where X_α runs over all non basepoint components of X. This can be rewritten (naturally) as a big wedge:

$$X = X_0 \vee \bigvee_\alpha (X_\alpha)_+$$

where as usual $(-)_+$ denotes the disjoint basepoint construction. By the wedge axiom, h^n is determined by its behavior on connected CW complexes and those of the form X_+ for a connected CW complex X.

Consider the natural cofiber sequence in $\mathbf{CW}_*$

$$\begin{tikzcd} S^0 \arrow[r] & X_+ \arrow[r] \arrow[l, bend left, dashed] & X \end{tikzcd}$$

where the inclusion $S^0 \hookrightarrow X_+$ admits a *natural* retraction $X_+ \twoheadrightarrow S^0$ induced by the unique map $X \to *$ in **Top**. In particular, the induced map $h^n(X_+) \twoheadrightarrow h^n(S^0)$ is always an epimorphism. We obtain the exact sequence

$$\begin{tikzcd} h^{n-1}(S^0) \arrow[r, "0"] & h^n(X) \arrow[r] & h^n(X_+) \arrow[r] & h^n(S^0) \arrow[r, "0"] \arrow[l, bend left, dashed] & h^{n+1}(X) \end{tikzcd}$$

which can be rewritten as the *naturally* split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & h^n(X) \arrow[r] & h^n(X_+) \arrow[r] & h^n(S^0) \arrow[r] \arrow[l, bend left, dashed] & 0 \end{tikzcd}$$

or equivalently, the natural isomorphism $h^n(X_+) \cong h^n(X) \oplus h^n(S^0)$.

Therefore h^n is determined by its behavior on connected CW complexes and S^0.

Define the pointed CW complexes

$$E_n := D_n \times h^n(S^0)$$

where $h^n(S^0)$ is viewed as a discrete pointed space. Then for X a connected pointed CW complex, we have isomorphisms of pointed sets

$$\begin{aligned} [X, E_n]_* &= [X, D_n \times h^n(S^0)]_* \\ &\cong [X, D_n]_* \times [X, h^n(S^0)]_* \\ &\cong [X, D_n]_* \text{ since } X \text{ is connected and } h^n(S^0) \text{ is discrete} \\ &\cong h^n(X) \end{aligned}$$

which is in fact an isomorphism of abelian groups, by definition of the H-group structure on D_n.

Likewise for S^0, we have isomorphisms of pointed sets

$$\begin{aligned}
[S^0, E_n]_* &= [S^0, D_n \times h^n(S^0)]_* \\
&= \pi_0(D_n \times h^n(S^0)) \\
&\cong \pi_0(D_n) \times \pi_0(h^n(S^0)) \\
&\cong \pi_0(h^n(S^0)) \;\text{ since } D_n \text{ is connected} \\
&\cong h^n(S^0) \;\text{ since } h^n(S^0) \text{ is discrete.}
\end{aligned}$$

It is in fact an isomorphism of abelian groups, by using pointwise addition in $h^n(S^0)$.

Therefore, for all pointed CW complex X, we have a natural isomorphism of abelian groups $[X, E_n]_* \cong h^n(X)$.

Step 3. Building an Ω-spectrum.

For all pointed CW complex X, consider the diagram of abelian groups

$$\begin{tikzcd}
h^n(X) \arrow[r, "\cong"] \arrow[d, "\cong"'] & {[X, E_n]_*} \arrow[d, dashed, "\cong"] \\
h^{n+1}(\Sigma X) \arrow[r, "\cong"'] & {[\Sigma X, E_{n+1}]_*}
\end{tikzcd}$$

and the unique isomorphism $[X, E_n]_* \xrightarrow{\cong} [\Sigma X, E_{n+1}]_*$ making the diagram commute. As in Problem 3, we have the natural isomorphism of abelian groups

$$[X, E_n]_* \xrightarrow{\cong} [\Sigma X, E_{n+1}]_* \xrightarrow{\cong} [X, \Omega E_{n+1}]_*$$

where the group structure $[X, \Omega E_{n+1}]_*$ comes from the H-group structure of E_{n+1}. By Yoneda – after functorially replacing ΩE_{n+1} by a homotopy equivalent CW complex if needed – we obtain a homotopy equivalence

$$\omega_n \colon E_n \xrightarrow{\simeq} \Omega E_{n+1}$$

which is moreover an H-group map.

The resulting Ω-spectrum E represents h^* in the sense of Problem 3. Indeed, the two group structures on $[X, E_n]_* \cong [X, \Omega E_{n+1}]_*$, one from the H-group structure of E_n and one from loop concatenation in ΩE_{n+1}, satisfy the Eckmann-Hilton interchange law, and are thus equal. Therefore, the abelian group structure of $h^n(X)$ can be recovered using only the structure maps ω_n.

To conclude, note that the "new" suspension isomorphism is induced by ω_n and is thus compatible with the original suspension isomorphism $h^n(X) \cong h^{n+1}(\Sigma X)$, by construction of ω_n. $\square$

Problem 5. Show that every complex vector bundle over the circle S^1 is trivial. Conclude that its reduced K-theory is trivial: $\widetilde{K}(S^1) = 0$.

Solution. Complex vector bundles of dimension $n \geq 1$ over S^1 are classified by

$$\begin{aligned}
[S^1, BU(n)] &\cong [S^1, BU(n)]_* \\
&= \pi_1 BU(n) \\
&\cong \pi_0 U(n) \\
&\cong \pi_0 U(1) \\
&= 0
\end{aligned}$$

where we used Homework 12 Problem 1.

In particular, all complex vector bundles over S^1 are stably equivalent. But since S^1 is compact, Hausdorff, and connected, the natural map

$$\mathrm{Vect}^{\mathbb{C}}(S^1)/\text{stable equivalence} \to \widetilde{K}(S^1)$$

is an isomorphism, from which we conclude $\widetilde{K}(S^1) = 0$. $\square$