MA571 Homework 9

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Problem 9.1 (Munkres §46, Ex. 6)

Show that the compact-open topology, $\mathcal{C}(X,Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular. [*Hint:* If $\overline{U} \subset V$, then $\overline{S(C,U)} \subset S(C,V)$.]

Proof. Suppose that Y is regular. We shall proceed by the hint and Lemma 31.1(b). Consider the subbasis element $S(C,U)$. Since Y is regular, there exists a neighborhood $V \supset U$ such that $V \supset \overline{U}$. Let $f \in \overline{S(C,U)}$. Then, we claim that $f \in S(C,V)$. For suppose not, then there exists an element $x_0 \in C$ such that $f(x_0) \notin V$. Then, since $\overline{U} \subset V$, by hypothesis, $f(x_0) \notin \overline{U}$. Consider the subbasic neighborhood $S\big(\{x_0\}, Y - \overline{U}\big)$ of f. Then, $S\big(\{x_0\}, Y - \overline{U}\big) \cap S(C,U)$ is nonempty. Let g be in the aforementioned intersection. Then $g(x_0) \in g(C) \subset U$, but $g(x_0) \in Y - \overline{U}$. This is a contradiction. It follows by Lemma 31.1(b) that $\mathcal{C}(X,Y)$ is regular. ■

Problem 9.2 (Munkres §46, Ex. 9(a,b,c))

Here is a (unexpected) application of Theorem 46.11 to quotient maps. (Compare Exercise 11 of §29.)

Theorem. *If $p\colon A \to B$ is a quotient map and X is locally compact Hausdorff, then $(\mathrm{id}_X, p)\colon X \times A \to X \times B$ is a quotient map.*

Proof. (a) Let Y be the quotient space induced by (id_X, p); let $q\colon X \times A \to Y$ be the quotient map. Show there is a bijective continuous map $f\colon Y \to X \times B$ such that $f \circ q = (\mathrm{id}_X, p)$.

(b) Let $g = f^{-1}$. Let $G\colon B \to \mathcal{C}(X, Y)$ and $Q\colon A \to \mathcal{C}(X, Y)$ be the maps induced by g and q, respectively. Show that $Q = G \circ p$.

(c) Show that Q is continuous; conclude that G is continuous, so that g is continuous.

Proof. ∎

Problem 9.3 (Munkres §52, Ex. 1)

Show that if $h, h' \colon X \to Y$ are homotopic and $k, k' \colon Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof. ■

Problem 9.4 (Munkres §52, Ex. 2)

Given spaces X and Y, let $[X, Y]$ denote the homotopy classes of maps of X into Y

(a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.

(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

Proof. ∎

Problem 9.5 (Munkres §52, Ex. 3(a,b,c,))

A space X is said to be *contractible* if the identity map $\mathrm{id}_X\colon X \to X$ is nullhomotopic.

(a) Show that I and $\mathbf{R}$ are contractible.

(b) Show that a contractible space is path connected.

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

Proof. ■