

MA 544: Homework 6

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PROBLEM 6.1 (WHEEDEN & ZYGMUND §4, EX. 4)

Let f be defined and measurable in \mathbf{R}^n . If T is a nonsingular linear transformation of \mathbf{R}^n , show that $f(T\mathbf{x})$ is measurable. [If $E_1 = \{\mathbf{x} \mid f(\mathbf{x}) > a\}$ and $E_2 = \{\mathbf{x} \mid f(T\mathbf{x}) > a\}$, show $E_2 = T^{-1}E_1$.]

Proof. We shall proceed as in the hint. Suppose f is measurable. Then for every finite a in \mathbf{R} , the set $\{f > a\}$ is measurable. Since $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is nonsingular it is invertible. Let $E_1 := \{\mathbf{x} \mid f(\mathbf{x}) > a\}$ and $E_2 := \{\mathbf{x} \mid f(T\mathbf{x}) > a\}$. Then, we show that $E_2 = T^{-1}E_1$.

Let $\mathbf{x} \in E_2$. Then $f(T\mathbf{x}) > a$. Since T is nonsingular, it is surjective so for some $\mathbf{x}' \in \mathbf{R}^n$, $\mathbf{x} = T^{-1}\mathbf{x}'$ so $f(TT^{-1}\mathbf{x}') = f(\mathbf{x}') > a$. Thus, $\mathbf{x} \in T^{-1}E_1$. Similarly, if $\mathbf{x} \in T^{-1}E_1$ then $\mathbf{x} = T^{-1}\mathbf{x}'$ for some $\mathbf{x}' \in E_1$. Then $f(T\mathbf{x}) = f(TT^{-1}\mathbf{x}') = f(\mathbf{x}') > a$. Thus, $\mathbf{x} \in E_2$. It follows that since E_1 is measurable for all a and T^{-1} is Lipschitz, by 3.33, E_2 is measurable for all a so $f \circ T$ is measurable. ■

PROBLEM 6.2 (WHEEDEN & ZYGMUND §4, EX. 7)

Let f be usc and less than $+\infty$ on a compact set E . Show that f is bounded above on E . Show also that f assumes its maximum on E , i.e., that there exists $\mathbf{x}_0 \in E$ such that $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x} \in E$.

Proof. First we shall demonstrate boundedness. Suppose that f is usc on E . By 4.14 (i), the set $\{\mathbf{x} \mid f(\mathbf{x}) < a\}$ is relatively open. Set $\mathcal{G} := \{G_k\}_{k=1}^{\infty}$ where $G_k := \{\mathbf{x} \mid f(\mathbf{x}) < k\}$. Then \mathcal{G} is a cover of E by relatively open sets so there exists a finite subcover $\{G_{k_\ell}\}_{\ell=1}^N$ of E by relatively open sets. Set $M := \max\{k_1, \dots, k_N\}$. Then $f(\mathbf{x}) < M$ for all $\mathbf{x} \in E$.

Also by 4.14, the set $\{\mathbf{x} \mid f(\mathbf{x}) \geq a\}$ is relatively closed in E . Consider the collection of relatively closed sets $\{H_{\mathbf{x}}\}_{\mathbf{x} \in E}$ where $H_{\mathbf{x}} := \{\mathbf{x}' \mid f(\mathbf{x}') \geq f(\mathbf{x})\}$. Moreover, if we take any finite collection of the $H_{\mathbf{x}}$'s, say $H_{\mathbf{x}_1}, \dots, H_{\mathbf{x}_N}$, then $\bigcap_{k=1}^N H_{\mathbf{x}_k} \neq \emptyset$ since $f(\mathbf{x}) \geq M$ for all $\mathbf{x} \in H_{\mathbf{x}_i}$ where $M := \min\{f(\mathbf{x}_1), \dots, f(\mathbf{x}_N)\}$. Thus, $\{H_{\mathbf{x}}\}$ has the finite intersection property, so, since E is compact, the intersection $H := \bigcap_{\mathbf{x} \in E} H_{\mathbf{x}} \neq \emptyset$. Let $\mathbf{x}_0 \in H$. Then $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x} \in E$, as desired. ■

PROBLEM 6.3 (WHEEDEN & ZYGMUND §4, EX. 8)

- (a) Let f and g be two functions which are usc at \mathbf{x}_0 . Show that $f + g$ is usc at \mathbf{x}_0 . Is $f - g$ usc at \mathbf{x}_0 ? When is fg usc at \mathbf{x}_0 ?
- (b) If $\{f_k\}$ is a sequence of functions are usc at \mathbf{x}_0 , show that $\inf f_k(\mathbf{x})$ is usc at \mathbf{x}_0 .
- (c) If $\{f_k\}$ is a sequence of functions which are usc at \mathbf{x}_0 and which converge uniformly near \mathbf{x}_0 , show that $\lim f_k$ is usc at \mathbf{x}_0 .

Proof. (a) Suppose that f and g are usc at \mathbf{x}_0 . Then, by definition, given $M > f(\mathbf{x}_0), g(\mathbf{x}_0)$ there exists $\delta_f, \delta_g > 0$ such that $f(\mathbf{x}_1), g(\mathbf{x}_2) < M/2$ for all $|\mathbf{x}_1 - \mathbf{x}_0| < \delta_f, |\mathbf{x}_2 - \mathbf{x}_0| < \delta_g$. Set $\delta := \min\{\delta_f, \delta_g\}$. Then for any $\mathbf{x} \in B(\delta, \mathbf{x}_0)$ we have

$$\begin{aligned} |f(\mathbf{x}) + g(\mathbf{x}) - (f(\mathbf{x}_0) + g(\mathbf{x}_0))| &= |(f(\mathbf{x}) - f(\mathbf{x}_0)) + (g(\mathbf{x}) - g(\mathbf{x}_0))| \\ &\leq |f(\mathbf{x}) - f(\mathbf{x}_0)| + |g(\mathbf{x}) - g(\mathbf{x}_0)| \\ &< \frac{M}{2} + \frac{M}{2} = M. \end{aligned}$$

Thus, $f + g$ is usc.

For the second part of this problem, we provide a counter example. Recall the functions u_1 and u_2 from the text

$$u_1(x) := \begin{cases} 0 & \text{if } x < x_0 \\ 1 & \text{if } x \geq x_0 \end{cases} \quad u_2(x) := \begin{cases} 0 & \text{if } x \neq x_0 \\ 1 & \text{if } x = x_0 \end{cases}. \quad (1)$$

Consider the difference $u := u_1 - u_2$ which is piecewise defined to be

$$u(x) := \begin{cases} 0 & \text{if } x \leq x_0 \\ 1 & \text{if } x > x_0 \end{cases}. \quad (2)$$

Now, note that u being usc at x_0 implies that for $1/2 > f(x_0) = 0$ there exists $\delta > 0$ such that $f(x) < 1/2$ for all $x \in B(\delta, x_0)$. But for any $x = x_0 + \delta'$ in $B(\delta, x_0)$, for some $0 < \delta' < \delta$, $u(x) = 1 > 1/2$. Thus, u is not usc at x_0 .

(b) Suppose that $\{f_k\}$ is a sequence of functions that are usc at \mathbf{x}_0 . Then, for every $M > f_k(\mathbf{x}_0)$ there exists $\delta_k > 0$ such that $f_k(\mathbf{x}) < M$ for all $\mathbf{x} \in B(\delta_k, \mathbf{x}_0)$. Set $\delta := \inf_k \{\delta_k\}$. Then $f_k(\mathbf{x}) < M$ for all $\mathbf{x} \in B(\delta, \mathbf{x}_0)$ for all k . Thus, $\inf_k f_k(\mathbf{x}) < M$ for all $\mathbf{x} \in B(\delta, \mathbf{x}_0)$ so $\inf_k f_k(\mathbf{x})$ is usc at \mathbf{x}_0 .

(c) By near, we will assume topological proximity, i.e., on some small neighborhood U of \mathbf{x}_0 . Suppose $f_k \rightarrow f$ uniformly on U . Then for every $\varepsilon > 0$, there exists a positive integer K such that $k > K$ implies $|f_k(\mathbf{x}) - f(\mathbf{x})| < \varepsilon$. Since the f_k 's are usc at \mathbf{x}_0 , for every $M > f_k(\mathbf{x}_0)$ there exists $\delta_k > 0$ such that $f_k(\mathbf{x}) < M$ for all $\mathbf{x} \in B(\delta_k, \mathbf{x}_0)$. ■