MA571 Problem Set 2

Carlos Salinas

September 9, 2015

Problem 2.1 (Munkres §17, p. 100, Exercise 3)

Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Proof. Before proceeding with our main result we will prove the following useful set theoretic results which we have taken (and modified) from Munkres §1, p. 14, Exercises 2(n) and 2(o):

Lemma 4. *For sets A, B, C and D we the following equalities hold:*

(a) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(b) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.
(c) $(A \setminus C) \times B = (A \times B) \setminus (C \times B)$.

that is, the Cartesian product distributes over taking complements.

Proof of Lemma 4. (a) The equality follows (rather straightforwardly) from the definition of the Cartesian product and the complement of a set for $x \times y \in (A \times B) \cap (C \times D)$ if and only if $x \times y \in A \times B$ and $x \times y \in C \times D$ if and only if $x \in A$ and $x \in C$ and $y \in B$ and $y \in D$ if and only if $x \in A \cap C$ and $y \in B \cap D$ if and only if $x \times y \in (A \cap C) \times (B \cap D)$.

(b) The point $x \times y \in A \times (B \setminus C)$ if and only if $x \in A$ and $y \in B \setminus C$ if and only if $x \in A$ and $y \in B$ and $y \notin C$ if and only if $x \times y \in A \times B$ and $x \times y \notin A \times C$ if and only if $x \times y \in (A \times B) \setminus (A \times C)$.

(c) The very same argument as part (b) can be used, taking B to be a subset of A and replacing (where appropriate) A by $A \setminus B$ and $B \setminus C$ by C, to prove that

$$(A \setminus B) \times C = (A \times C) \setminus (B \times C). \qquad \clubsuit$$

Now let's turn our attention back to the problem at hand. Since A is closed in X and B is closed in Y, their complements, $X \setminus A$ and $Y \setminus B$, are open in X and Y, respectively (this is by definition cf. Munkres §17, p. 93). Hence, the sets

$$(X \setminus A) \times Y \quad \text{and} \quad X \times (Y \setminus B)$$

are open in $X \times Y$ since they are basis elements of the product topology on $X \times Y$ (cf. definition of the product topology on Munkres §15, p. 86). Hence, their complements are closed. By Lemma 4(b) and 4(c), we may rewrite the complements of $(X \setminus A) \times Y$ and $X \times (Y \setminus B)$ as

$$(X \times Y) \setminus ((X \setminus A) \times Y) = A \times Y \quad \text{and} \quad (X \times Y) \setminus (X \times (Y \setminus B)) = X \times B,$$

respectively. Then, by Theorem 17.(b), the intersection

$$(A \times Y) \cap (X \times B)$$

is closed since $A \times Y$ and $X \times B$ are closed. At last, by Lemma 4(a), we may rewrite the former intersection as

$$(A \times Y) \cap (X \times B) = (A \cap X) \times (Y \cap B) = A \times B.$$

Thus $A \times B$ is closed in $X \times Y$. ■

Problem 2.2 (Munkres §17 p. 101, Exercise 6(b))

Let A, B and A_α denote subsets of a space X. Prove the following:

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. The containment $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ is immediate from the definition of the closure of a set (cf. Munkres §17, p. 95) since $\overline{A} \cup \overline{B}$ is a closed set (by Theorem 17.1(a)) which contains $A \cup B$, hence must contain the closure of $A \cup B$. To see the reverse containment note we will make use of the following lemma (which I was not able to immediately find in Munkres):

Lemma 5. *(a) If $A \subset C$ and $B \subset C$ then $A \cup B \subset C$.*
(b) If $A \subset B$ and C is any set then $A \subset B \cup C$.
(c) If $A \subset C$ and $B \subset D$ then $A \cup B \subset C \cup D$.

Proof of Lemma 5. (a) By the definition of subset and union (cf. Munkres §1, pp. 4-5) if $x \in A \cup B$ then $x \in A$ or $x \in B$. Since $A \subset C$ and $B \subset C$, in either case we have that $x \in C$. Thus $A \cup B \subset C$.

(b) By the definition of subset and union, if $x \in A$ and $A \subset B$, then $x \in B$ so $x \in B \cup C$ (by definition of union).

(c) By part (b) $A \subset C \cup D$ and $B \subset C \cup D$ so by part (a) $A \cup B \subset C \cup D$. ♣

Armed with Lemma 5, note that $A \subset \overline{A \cup B}$ and $B \subset \overline{A \cup B}$ so $\overline{A \cup B}$ contains the closure of A and B so it must contain the union of their respective closures, i.e., $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

Naturally, this result may be extended, by induction, to show that the closure of a finite union of sets is the union of the closure of said sets. ∎

Problem 2.3 (Munkres §17 p. 101, Exercise 6(c))

Let A, B and A_α denote subsets of a space X. Prove the following:

(b) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$; give an example where equality fails.

Proof. The containment $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$ follows immediately from the definition of closure since $\overline{\bigcup A_\alpha}$ is a closed set containing A_α so must contain $\overline{A_\alpha}$ for each α.

The reverse containment is not true in general (in fact, as Theorem 17.1(3) suggests, an arbitrary union of closed sets is not even necessarily closed). As a counter example, consider the family of subsets $A_q = \{q\}$, for $q \in \mathbf{Q}$, of $\mathbf{R}$. Since $\mathbf{R}$ is Hausdorff, by Theorem 17.8, the closure of A_q is itself. Hence, we see that the union

$$\bigcup_{q \in \mathbf{Q}} \overline{A_q} = \mathbf{Q},$$

but, (by Munkres §17, Example 6) $\overline{\mathbf{Q}} = \mathbf{R}$. ∎

Problem 2.4 (Munkres §17 p. 101, Exercise 7)

Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \overline{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so x must belong to the closure of some A_α. Therefore, $x \in \bigcup \overline{A}_\alpha$.

Critique. The claim is false in general as the counterexample in the preceding problem demonstrates. The main problem with this proof lies in the assertion U intersecting some A_α implies "x must belong to the closure of some A_α." But a different neighborhood of x may intersect a different A_α in the union. Recall, by Theorem 17.5(a), if x is in the closure of A_α, then $U \cap A_\alpha \neq \emptyset$ for every neighborhood U of x. That is, the proof is claiming that for every neighborhood U of x there exists some A_α in the union $\bigcup A_\alpha$ such that $U \cap A_\alpha \neq \emptyset$, i.e., $x \in \overline{A_\alpha}$. But for x to be in $\bigcup \overline{A_\alpha}$ we need that for some A_α for every neighborhood U of x, $U \cap A_\alpha \neq \emptyset$. These are not equivalent statements. ♥

Problem 2.5 (Munkres §17, p. 101, 9)

Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

Proof. By Problem 2.1, $\overline{A} \times \overline{B}$ is a closed set which contains $A \times B$ so it must contain the closure of $A \times B$, i.e., $\overline{A \times B} \subset \overline{A} \times \overline{B}$. To see the reverse containment, take a point $x \times y \in \overline{A} \times \overline{B}$. Then, by Theorem 17.5(a), for every neighborhood U of x and every neighborhood V of y, the intersections $U \cap A$ and $V \cap B$ are nonempty. Thus, by Lemma 4(a), the set

$$(V \times U) \cap (A \times B) = (V \cap A) \times (U \cap B)$$

is nonempty. Then, since $U \times V$ is an arbitrary basis element containing $x \times y$, by Theorem 17.5(b) $x \times y \in \overline{A \times B}$. Thus, $\overline{A \times B} = \overline{A} \times \overline{B}$. ∎

Problem 2.6 (Munkres §17, p. 101, 10)

Show that every order topology is Hausdorff.

Proof. Let $(X, <)$ denote a nonempty set equipped with a simple order relation. Then by the definition on Munkres §14, p. 84, a basis for the order topology on X are sets of the following types:

(1) All open intervals (a, b) in X.
(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

Let a and b be two distinct points in X; we may assume, without loss of generality, that $a < b$. Then, we must show that there exists neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$.

If X set with finite cardinality the order topology on X will coincide with the discrete topology so that we may take $\{a\}$ and $\{b\}$ to be neighborhoods of a and b. Then, $\{a\} \cap \{b\} = \emptyset$ so X is Hausdorff.

Now, suppose X is not of finite cardinality. Define the sets

$$C = (a, b), \quad A = \{ x \in X \mid x < a \} \quad \text{and} \quad B = \{ x \in X \mid x > b \}.$$

Then at least one of A, B or C is nonempty and has infinite cardinality.

Suppose A is nonempty, but B and C are empty. Take any element $x \in A$, then (x, b) is a neighborhood of a and b must be a largest element so $(a, b_0] = C \cup \{b\} = \{b\}$ is a neighborhood of b satisfying $(x, b) \cap \{b\} = \emptyset$. Similarly, if B is nonempty, but A and C are empty, $\{a\}$ and (a, x) for some $x \in B$ are neighborhoods of a and b, respectively, with $\{a\} \cap (a, x) = \emptyset$.

If C is nonempty but A and B are empty, a must be a smallest element and b must be a largest element. Then, since X is not finite, there exist at least two distinct elements x and y in C with $x < y$ so $[a, x)$ and $(y, b]$ are neighborhoods of a and b, respectively, with $[a, x) \cap (y, b] = \emptyset$.

Now, suppose at least two of A, B and C are nonempty. If C is empty, but A and B are nonempty. Then the intervals $(x, b) = (x, a]$ and $(a, y) = [b, y)$ are neighborhoods of a and b respectively with $(x, b) \cap (a, y) = \emptyset$. If A is empty, but B and C are nonempty, then a is a smallest element. Then there exists at least two distinct elements x and y with $x < y$ in C so that $[a, x)$ and (y, b) are neighborhoods of a and b, respectively, with $[a, x) \cap (y, b) = \emptyset$. Similarly, if B is empty, but A and C are nonempty, for any $x < y$ in C, (a, x) and $(y, b]$ are neighborhoods of a and b, respectively, with $(a, x) \cap (y, b]$.

Lastly, if A, B and C are nonempty we win! Then, for any $x \in A$, $y \in B$ and $z, w \in C$ with $z < w$ the intervals (x, z) and (w, y) are neighborhoods of a and b, respectively, with $(x, z) \cap (w, y) = \emptyset$.

In every case, X satisfies the Hausdorff property. ∎

***Remarks**.* Perhaps there is a better way to approach this problem. The demonstration is thorough and covers every case, but we still desire a more elegant proof.

Problem 2.7 (Munkres §17, p. 101, 13)

Show that X is Hausdorff if and only if the *diagonal* $\Delta = \{ x \times x \mid x \in X \}$ is closed in $X \times X$.

Proof. $\Longrightarrow$ Suppose X is Hausdorff. The diagonal Δ is closed, by definition, if and only if its complement, $(X \times X) \setminus \Delta$, is open in $X \times X$. Let $x \times y \in (X \times X) \setminus \Delta$. Since X is Hausdorff, there exists neighborhoods U and V of x and y, respectively, such that $U \cap V = \emptyset$. Thus, $U \times V$ is a neighborhood of $x \times y$ contained in $(X \times X) \setminus \Delta$. By the definition (cf. Munkres §13 p. 78), since for every point $x \times y \in (X \times X) \setminus \Delta$ we may find a basis element $U \times V \subset (X \times X) \setminus \Delta$ containing $x \times y$, it follows that $(X \times X) \setminus \Delta$ is open. Thus, Δ is closed.

$\Longleftarrow$ Suppose Δ is closed. Then the complement of Δ is open in $X \times X$. Thus, for every $x \times y$ in the complement of Δ, we may find a basis element $U \times V \subset (X \times X) \setminus \Delta$ containing $x \times y$. Thus, U and V are neighborhoods of x and y, respectively, such that $U \cap V = \emptyset$ (for otherwise $z \times z \in U \times V$ but $U \times V$ is in the complement of Δ). Thus, X is Hausdorff. ∎

Problem 2.8 (Munkres §18, p. 111, 4)

Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f\colon X \to X \times Y$ and $g\colon Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

Proof. Let $Z = \operatorname{im} f$. To show that $f\colon X \to X \times Y$ is an imbedding, we will show that the map $f' : X \to Z$, which is obtained by restricting the codomain of f is a continuous injection with a continuous inverse g. First we shall show injectivity. To see that f is continuous we note that f can be written as the tuple $f'(x) = (f_1, f_2)$ where $f_1 = \mathrm{id}_X$ and f_2 is the constant map $x \mapsto y_0$ for all $x \in X$. The maps f_1 and f_2 are continuous (by Theorem 18.2(a) and (b)) so, by Theorem 18.4, f' is continuous. To prove that f' is bijective it suffices to exhibit an inverse. We claim that the map $F = \pi_X|Z$ is an inverse (continuity of F follows from Theorem 18.2(d) and the fact that projections are continuous as discussed on §15 pp. 87-88). But this claim is clear since

$$\begin{aligned} F \circ f'(x) &= F(f(x)) \\ &= F(x \times y_0) \\ &= x \\ &= \mathrm{id}_X(x) \end{aligned} \qquad\qquad \begin{aligned} f' \circ F(x \times y_0) &= f'(F(x \times y_0)) \\ &= f'(x) \\ &= x \times y_0 \\ &= \mathrm{id}_Z(x \times y_0). \end{aligned}$$

Thus, f is an imbedding.

The proof that g is an imbedding is analogous (it is sufficient to replace f by g, F by G, $x \times y_0$ by $x_0 \times y$, $x \mapsto y_0$ by $y \mapsto x_0$, π_X by π_Y, and id_X by id_Y in the argument above). So as not to be penalized for not providing the proof for g we copy and paste, making the appropriate replacements, here:

Let $Z = \operatorname{im} g$. To show that $g\colon Y \to X \times Y$ is an imbedding, we will show that the map $g' : Y \to Z$, which is obtained by restricting the codomain of g is a continuous injection with a continuous inverse G. First we shall show injectivity. To see that g' is continuous we note that g' can be written as the tuple $g'(x) = (g_1, g_2)$ where $g_1 = \mathrm{id}_Y$ and g_2 is the constant map $y \mapsto x_0$ for all $y \in Y$. The maps g_1 and g_2 are continuous g' is continuous. To prove that g' is bijective it suffices to exhibit an inverse. We claim that the map $G = \pi_Y|Z$ is an inverse (the continuity of G follows from he fact that it is the restriction of a projection). But this claim is clear since

$$\begin{aligned} G \circ g'(x) &= G(g'(x)) \\ &= G(x_0 \times y) \\ &= y \\ &= \mathrm{id}_Y(y) \end{aligned} \qquad\qquad \begin{aligned} g' \circ G(x_0 \times y) &= g'(G(x_0 \times y)) \\ &= g'(y) \\ &= x_0 \times y \\ &= \mathrm{id}_Z(x_0 \times y). \end{aligned}$$

Thus, g is an imbedding. ∎

Problem 2.9 (Munkres §18, p. 111-112, 8(a,b))

Let Y be an ordered set in the order topology. Let $f, g\colon X \to Y$ be continuous.

(a) Show that the set $\{\, x \mid f(x) \leq g(x) \,\}$ is closed in X.
(b) Let $h\colon X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [*Hint:* Use the pasting lemma.]

Proof. (a) Let $A = \{\, x \mid f(x) \leq g(x) \,\}$. To prove that A is closed, we will demonstrate that its complement,

$$X \setminus A = \{\, x \mid f(x) > g(x) \,\},$$

is open. Let $x \in X \setminus A$. Then $f(x) \neq g(x)$. By Problem 2.6, Y is Hausdorff so there exist neighborhoods U and V of $f(x)$ and $g(x)$, respectively, such that $U \cap V = \emptyset$. Without loss of generality, we may assume U and V are basis elements, i.e., $U = (x_3, x_4)$ and $V = (x_1, x_2)$. Then, since f and g are continuous (cf. Munkres §18, p. 102), the intersection $f^{-1}(U) \cap g^{-1}(V)$ in a neighborhood of x contained entirely in $X \setminus A$ (for otherwise there exists a $y \in (f^{-1}(U) \cap g^{-1}(V)) \cap A$ which simultaneously satisfies $x_1 < g(y) < x_2 < x_3 < f(y) < x_4$ and $f(y) \leq g(y)$, but this is absurd).

(b) Define the sets

$$A = \{\, x \mid f(x) \leq g(x) \,\} \quad \text{and} \quad B = \{\, x \mid f(x) \geq g(x) \,\}.$$

By part (a), A and B are closed in X. Lastly, define $f' = f|A$ and $g' = g|B$ (by Theorem 18.2(d) f' and g' are continuous). Since $f' = g'$ on $A \cap B$ (by construction), by the pasting lemma, we have that

$$h(x) = \min\{f(x), g(x)\} = \begin{cases} f'(x) & \text{if } x \in A, \\ g'(x) & \text{if } x \in B \end{cases}$$

is continuous. ∎

Problem 2.10

Given: X is a topological space with open sets $U_1, ..., U_n$ such that $\overline{U}_i = X$ for all i. Prove that the closure of $U_1 \cap \cdots \cap U_n$ is X.

Proof. ****Opening remarks**:** This property of U, that $\overline{U} = X$, is called *density* (and is not defined until Munkres §30, p. 190), but should be recognizable to anyone who has taken a course in real analysis so I don't feel any qualms about using said adjective here. At any rate, we shall proceed by induction on n the number of sets in the intersection.

Consider the base case $n = 2$: Suppose U_1 and U_2 are dense open subsets of X. Let $x \in \overline{U_1} = X$. Then, by Theorem 17.5(a), for any neighborhood U of x, $U \cap U_1 \neq \emptyset$. In particular, note that $U \cap U_1$ is open since it is a finite intersection of open sets (cf. Munkres §13 definition of topology). Let $y \in U \cap U_1$. Then, since $y \in \overline{U_2}$ and $U \cap U_1$ is a neighborhood of y, we have that

$$(U \cap U_1) \cap U_2 = U \cap (U_1 \cap U_2) \neq \emptyset.$$

Hence, x is in the closure of $U_1 \cap U_2$ for any $x \in X$ so $\overline{U_1 \cap U_2} = \emptyset$.

Suppose the property holds for he intersection of $n-1$ such open dense sets. Suppose $U_1, ..., U_n$ are open dense subsets in X. Let $U' = \bigcap_{i=1}^{n-1} U_i$. Then, by the induction hypothesis, U' is an open set with $\overline{U'} = X$. Again, as in the base case, we have $U' \cap U$ is the intersection of open dense subsets of X so

$$\overline{U' \cap U} = X = \overline{U_1 \cap \cdots \cap U_{n-1} \cap U_n}. \qquad \blacksquare$$