MA 523: Homework 3

Carlos Salinas

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Problem 3.1

Consider the initial value problem

$$u_t = \sin u_x; \qquad u(x,0) = \frac{\pi}{4}x.$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

Solution. We skip checking that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied (since I cannot decipher Evans's notation), and show that he Taylor series of u at $(0,0)$,

$$\tilde{u}(x,t) = \sum_{(\alpha_1,\alpha_2)} \frac{u_{\alpha_1,\alpha_2}(0)}{\alpha_1!\alpha_2!} x^{\alpha_1} t^{\alpha_2},$$

is a solution to our PDE.

First, we must compute the coefficients

$$\frac{u_{\alpha_1,\alpha_2}(0,0)}{\alpha_1!\alpha_2!}.$$

To this end, we must find the partial derivatives u_{α_1,α_2} and potentially, relations among them which will help us to find these coefficients. Naïvely listing the partials with respect to t and x, we have

$$\begin{aligned}
u(0,0) &= 0 \\
u_x(0,0) &= \frac{\pi}{4} \\
u_t(0,0) &= \sin u_x(0,0) = \frac{\sqrt{2}}{2} \\
u_{xx}(0,0) &= 0 \\
u_{tx}(0,0) &= 0 \\
u_{tt}(0,0) &= -\cos\big(u_x(0,0)\big) u_{xt}(0,0) = 0 \\
u_{xxx}(0,0) &= 0 \\
u_{ttx}(0,0) &= 0,
\end{aligned}$$

etc. Thus,

$$\tilde{u} = \frac{\pi}{4}x + \frac{\sqrt{2}}{2}t.$$

Plugging this equation into our PDE, we have

$$\tilde{u}_t - \sin \tilde{u}_x = \frac{\sqrt{2}}{2} - \sin(\pi/4) = 0,$$

as desired. ■

Problem 3.2

Consider the Cauchy problem for $u(x, y)$

$$\begin{aligned} u_y &= a(x, y, u)u_x + b(x, y, u) \\ u(x, 0) &= 0 \end{aligned}$$

Let a and b be analytic functions of their arguments. Assume that $D^\alpha a(0, 0, 0) \geq 0$ and $D^\alpha b(0, 0, 0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

(a) Show that $D^\beta u(0, 0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $D^\beta u(0, 0) \geq 0$ for all $\beta = (\beta_1, \beta_2)$. (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in β_2)

Solution. For part (a):

For part (b): ∎

Problem 3.3

(Kovalevskaya's example) Show that the line $\{t = 0\}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution of the heat equation in $\mathbf{R} \times \mathbf{R}$, with $u = 1/(1 + x^2)$ on $\{t = 0\}$. (*Hint:* Assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of $(0, 0)$.)

Solution. ■