MA571 Homework 10

Carlos Salinas

November 4, 2015

Problem 10.1 (Munkres §52, Ex. 2)

Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. By Theorem 52.1, the paths α and β induce a group homomorphism $\hat{\alpha}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $\hat{\beta}\colon \pi_1(X, x_1) \to \pi_1(X, x_2)$, respectively. We want to show therefore that the induced homomorphism $\hat{\gamma} = \widehat{\alpha * \beta}$ is in fact equivalent to the composition $\hat{\beta} \circ \hat{\alpha}$. Let $[f]$ be a loop based at x_0 then

$$\begin{aligned} \hat{\gamma}([f]) &= \widehat{\alpha * \beta}([f]) \\ &= \left[\,\overline{\alpha * \beta}\,\right] * [f] * [\alpha * \beta] \\ &= \left[\bar{\beta} * \bar{\alpha}\right] * [f] * [\alpha] * [\beta] \end{aligned}$$

by the well-definedness of the path product operation, we have

$$= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta]$$

by associativity of the path product,

$$\begin{aligned} &= [\bar{\beta}] * ([\bar{\alpha}] * [f] * [\alpha]) * [\beta] \\ &= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta] \end{aligned}$$

where $\alpha([f])$ is a loop based at x_1 so

$$\begin{aligned} &= \hat{\beta}(\hat{\alpha}([f])) \\ &= \big(\hat{\beta} \circ \hat{\alpha}\big)([f]). \end{aligned}$$

Thus, the following diagram commutes

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\hat{\alpha}"] \arrow[dr, "\hat{\gamma}=\widehat{\alpha*\beta}"'] & \pi_1(X, x_1) \arrow[d, "\hat{\beta}"] \\ & \pi_1(X, x_2). \end{tikzcd}$$

■

Problem 10.2 (Munkres §52, Ex. 3)

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is Abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. $\implies$ Let f and g be loops about x_0. Then, since X is path-connected, we claim that f and g are homotopic to the path product of two paths α and β from x_0 to x_1, say, $\alpha_1 * \bar{\beta}_1$ and $\alpha_2 * \bar{\beta}_2$. More precisely, split f into the path $f_1 = f(t/2)$ from x_0 to x_2 and $f_2 = f((t+1)/2)$ from x_2 to x_0 and let x_2 be the point at $f(1/2)$. Then there exists a path, say α, from x_2 to x_1. Now we construct the homotopy

$$H(x,t) = f_1(x/2) * \alpha(2tx) * \bar{\alpha} * ((2t-1)x) * f_2((x+1)/2)$$

from $f = f_1 * f_2$ to the loop $f = f_1 * \alpha * \bar{\alpha} * f_2$.

$\impliedby$ ■

Problem 10.3 (Munkres §52, Ex. 4)

Let $A \subset X$; suppose $r\colon X \to A$ is continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A.) If $a_0 \in A$, show that

$$r_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

Proof. ∎

PROBLEM 10.4 (MUNKRES §53, EX. 6)

Show that if X is path connected, the homomorphism induced by a continuous map is independent of the base point, up to isomorphisms of the groups involved. More precisely, let $h\colon X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1}) \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps "commutes"

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \hat{\alpha}\Big\downarrow & & \Big\downarrow\hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1). \end{array}$$

Proof. ■

Problem 10.5 (Munkres §55, Ex. 1)

Show that if A is a retract of B^2, then every continuous map $f\colon A \to A$ has a fixed point.

Proof. ∎

Problem 10.6 (Munkres §55, Ex. 2)

Show that if $h\colon S^1 \to S^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof. ∎

Problem 10.7 ((A))

Prove that every m-manifold is locally path-connected.

Proof. ■

Problem 10.8 ((B))

Prove that every m-manifold is regular.

Proof. ■

Problem 10.9 ((C))

Prove that there is no 1-1 continuous function $\iota\colon S^1 \to \mathbf{R}$. You may assume any fact about trigonometric functions. (Note: this shows in particular that there is no $\iota\colon S^1 \to \mathbf{R}$ with $p \circ \iota$ equal to the identity map, where p is the map in the note on the Fundamental Group of the Circle.)

Proof. ■

Problem 10.10 ((D))

Prove Proposition C from the note on the Fundamental Group of the Circle.

Proof. ■