MA52300 FALL 2016

HOMEWORK ASSIGNMENT 4 – *Solutions*

1 (Legendre transform)**.** Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2,x_2} = 0$$

is some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u(\mathbf{x}(p)),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies a *linear* equation

$$a^{22}(p)v_{p_1p_1} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_2p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29))

Solution. We start with a claim that

$$x^i(p) = v_{p_i}(p), \quad i = 1, 2.$$

Indeed, from the definition of v we will have

$$\begin{aligned} v_{p_i}(p) &= x^i(p) + \mathbf{x}_{p_i}(p) \cdot p - \mathbf{x}_{p_i}(p) \cdot D_x u(\mathbf{x}(p)) \\ &= x^i(p), \quad i = 1, 2, \end{aligned}$$

where we have used that $p = D_x u(\mathbf{x}(p))$. Now, if we denote the mapping $x \mapsto D_x u(x)$ by Φ and its inverse $p \mapsto D_p v(p)$ by Ψ, we will have that

$$D_p \Psi = (D_x \Phi)^{-1} \iff D_p^2 v = (D_x^2 u)^{-1}.$$

Componentwise, using the formula for the inverse of a 2×2 matrix, we will have

$$u_{x_1x_1} = J v_{p_2p_2}, \quad u_{x_1x_2} = -J v_{p_1p_2}, \quad u_{x_2x_2} = J v_{p_1p_1},$$

where $J = \det D_x^2 u \neq 0$ is the Jacobian of the mapping Φ. Plugging these identities into the equation for u and dividing by J, we obtain the equation for v. $\square$

2. Find the solution $u(x,t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$\begin{aligned} &u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for } x > 0 \\ &u_t(0,t) = \alpha u_x(0,t), \quad \text{for } t > 0, \end{aligned}$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$. (*Hint:* use a representation $u(x,t) = F(x-t) + G(x+t)$ for the solution)

Solution. Following the hint we will seek a solution in the form

$$u(x,t) = F(x-t) + G(x+t), \quad \text{for } x, t \geq 0.$$

Observe that we need to know the values of $G(y)$ only for $y \geq 0$, since $x + t \geq 0$ for $x, t \geq 0$. However, for $F(y)$ we need to know values for all $-\infty < y < \infty$, since $x - t$ can take any value for $x, t \geq 0$.

1) Verify the initial conditions:

$$\begin{aligned} u(x,0) &= F(x) + G(x) = g(x) \\ u_t(x,0) &= -F'(x) + G'(x) = h(x). \end{aligned}$$

Thus, arguing as in the derivation of D'Alambert's formula, we obtain

$$F'(x) = \frac{1}{2}(g'(x) - h(x)), \quad G'(x) = \frac{1}{2}(g'(x) + h(x)), \quad \text{for } x \geq 0$$

and integrating

$$\begin{aligned} F(x) &= \frac{g(x)}{2} - \frac{1}{2}\int_0^x h(y)dy + C_1 \\ G(x) &= \frac{g(x)}{2} + \frac{1}{2}\int_0^x h(y)dy + C_2 \end{aligned}$$

for any $x \geq 0$. Besides, checking the initial condition again, we must have $C_1 + C_2 = 0$ and without loss of generality we can assume $C_1 = C_2 = 0$. $\square$

2) The boundary condition gives:

$$-F'(-t) + G'(t) = \alpha(F'(-t) + G'(t)), \quad t > 0$$

and solving for $F'(-t)$, we obtain

$$F'(-t) = \frac{1-\alpha}{1+\alpha} G'(t), \quad t > 0$$

if $\alpha \neq -1$. Note that if $\alpha = -1$ this condition cannot be satisfied unless $G'(t) = 0$ for $t > 0$, i.e., $h(t) = -g'(t)$ which is not generally true. So, proceeding under the hypothesis $\alpha \neq -1$, we have

$$F(x) = -\frac{1-\alpha}{1+\alpha}\left(\frac{g(-x)}{2} + \frac{1}{2}\int_0^{-x} h(y)dy\right) + C, \quad \text{for } x < 0.$$

If we require the continuity of u we will have a specific value $C = C_0 = \frac{g(0)}{1+\alpha}$. However, if we don't require the continuity, any value of the constant C will lead to a *weak solution*.

3) Summarizing, we have the following formula for u

$$u(x,t) = \frac{g(x-t) + g(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy,$$

if $x \geq t$ and

$$u(x,t) = \frac{-C_\alpha g(t-x) + g(x+t)}{2} + \frac{1}{2}\left\{\int_0^{x+t} - C_\alpha \int_0^{t-x}\right\} h(y)dy + C$$

if $x < t$, where $C_\alpha = \frac{1-\alpha}{1+\alpha}$ and C is arbitrary. Taking $C = C_0 = \frac{g(0)}{1+\alpha}$ will guarantee the continuity across the charcteristic line $x = t$.

3. (a) Let u be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0,t) = u(\pi,t) = 0$. Show that the "energy"

$$E(t) = \frac{1}{2}\int_0^\pi (u_t^2 + c^2 u_x^2)dx, \quad t > 0$$

is independent of t; i.e., $\frac{d}{dt}E = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(nct) + b_n \sin(nct)) \sin nx$$

in terms of coefficients a_n, b_n.

Solution. (a) Differentiating under the sign of the integral, we obtain

$$\frac{d}{dt}E(t) = \int_0^\pi (u_t u_{tt} + c^2 u_x u_{xt})dx.$$

Now, integrating by parts the term $u_x u_{xt}$, we arrive at

$$\frac{d}{dt}E(t) = \int_0^\pi (u_t u_{tt} - c^2 u_{xx} u_t)dx + (c^2 u_x u_t)\Big|_{x=0}^{x=\pi} = 0.$$

We have used that u satisfies $u_{tt} - c^2 u_{xx} = 0$ and that $u_t = 0$ on $x = 0, \pi$. The latter identity follows by differentiation from the boundary condition $u = 0$ on $x = 0, \pi$. This completes the proof of (a).

(b) Even though a more direct derivation is possible, we will use the result of part (a) to simplify the computations. So we will compute $E(0)$.

We have

$$u_t(x,0) = c\sum_{n=1}^{\infty} nb_n \sin nx$$
$$u_x(x,0) = \sum_{n=1}^{\infty} na_n \cos nx$$

Now, using that

$$\int_0^\pi \sin nx \sin kx\, dx = \begin{cases} \frac{\pi}{2}, & n = k \\ 0, & n \neq k \end{cases}$$

for $n, k = 1, 2, \ldots$, we obtain

$$\int_0^\pi u_t(x,0)^2 dx = c^2 \sum_{n,k=1}^{\infty} nk\, b_n b_k \int_0^\pi \sin nx \sin kx\, dx$$
$$= \frac{\pi}{2}\, c^2 \sum_{n=1}^{\infty} n^2 b_n^2$$

Using a similar formula for the integrals of cosines, we obtain

$$\int_0^\pi u_x(x,0)^2 dx = \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 a_n^2.$$

Thus,

$$E = E(0) = \frac{\pi}{4}\, c^2 \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2).$$

Finally, all the computations above are justified, since C^2 regularity of u implies that $|a_n|, |b_n| \leq C/n^2$. $\square$