

MA 544: Homework 9

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PROBLEM 9.1 (WHEEDEN & ZYGMUND §6, EX. 1)

- (a) Let E be a measurable subset of \mathbb{R}^2 such that for almost every $x \in \mathbb{R}^1$, $\{y : (x, y) \in E\}$ has \mathbb{R}^1 -measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $\{x : (x, y) \in E\}$ has measure zero.
- (b) Let $f(x, y)$ be nonnegative and measurable in \mathbb{R}^2 . Suppose that for almost every $x \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every y . Show that for almost every $y \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every x .

Proof. (a) That E has measure zero is a consequence of Fubini's theorem. Set $E_x := \{y : (x, y) \in E\}$ and $E_y := \{x : (x, y) \in E\}$ then, by Theorem 6.8, we have

$$|E| = \iint_{\mathbb{R}^2} \chi_E \, dx dy = \int_{\mathbb{R}} \left[\int_{E_x} 1 \, dy \right] dx = \int_{\mathbb{R}} \left[\int_{E_y} 1 \, dx \right] dy = 0. \quad (9.1)$$

Hence, E has measure zero. Moreover, we see that $\int_{\mathbb{R}} \left[\int_{E_y} 1 \, dx \right] dy = 0$ which means that for a.e. $y \in \mathbb{R}$, E_y has \mathbb{R}^1 -measure zero.

(b) ■

PROBLEM 9.2 (WHEEDEN & ZYGMUND §6, EX. 3)

Let f be measurable and finite a.e. on $[0, 1]$. If $f(x) - f(y)$ is integrable over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, show that $f \in L[0, 1]$.

Proof. Suppose that f is measurable and finite a.e. on $[0, 1]$ and such that $f(x) - f(y) \in L([0, 1] \times [0, 1])$. Then, by Fubini's theorem we have

$$\iint_{I \times I} f(x) - f(y) \, dx dy = \int_I \left[\int_I f(x) - f(y) \, dy \right] dx \quad (9.2)$$

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PROBLEM 9.3 (WHEEDEN & ZYGMUND §6, EX. 4)

Let f be measurable and periodic with period 1: $f(t+1) = f(t)$. Suppose there is a finite c such that

$$\int_0^1 |f(a+t) - f(b+t)| dt \leq c$$

for all a and b . Show that $f \in L[0, 1]$. (Set $a = x$, $b = -x$, integrate with respect to x , and make the change of variables $\chi = x + t$, $\eta = -x + t$.)

Proof.

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PROBLEM 9.4 (WHEEDEN & ZYGMUND §6, EX. 6)

For $f \in L(\mathbb{R}^1)$, define the *Fourier transform* \hat{f} of f by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-ixt} dt$$

for $x \in \mathbb{R}^1$. (For complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i \int F_1$.) Show that if f and g belong to $L(\mathbb{R}^1)$, then

$$\widehat{(f * g)}(x) = 2\pi \hat{f}(x)\hat{g}(x).$$

Proof.

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PROBLEM 9.5 (WHEEDEN & ZYGMUND §6, EX. 7)

Let F be a closed subset of \mathbb{R}^1 and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F , prove that the function

$$\int_{\mathbb{R}^1} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}} dt$$

is integrable over F and so is finite a.e. in F . (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

Proof.

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PROBLEM 9.6 (WHEEDEN & ZYGMUND §6, EX. 9)

- (a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.
- (b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subset \mathbb{R}^1$, and let M_α be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int M_\lambda \leq \lambda^{-1}|G|$, where $G = (a, b) - [c, d]$.

Proof.

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