

MA 572: Homework 3

Carlos Salinas

February 13, 2016

By exactness at $\tilde{H}_1(S^2, A)$, we have $H_1(S^2, A) \cong \tilde{H}_1(S^2, A) \cong \bigoplus_{|A|-1} \mathbb{Z}$ and, last but not least, exactness at $\tilde{H}_0(S^2, A) \cong 0$ gives us $H_0(S^2, A) \cong \mathbb{Z}$. In summary, we have

$$H_n(S^2, A) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \bigoplus_{|A|-1} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

From 2.27 we know that $H_n^\Delta(S^1 \times S^1) \cong H_n(S^1 \times S^1)$ so from 2.3, we know that the homology of the torus $S^1 \times S^1$ is

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2, 0. \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Skipping directly, to our calculation, we have the long exact sequence

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_2(S^1 \times S^1, A) \\ & & & & & & \searrow \\ & & & & & & \longrightarrow 0 \\ & & & & & & \searrow \\ & & & & & & \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ & & & & & & \searrow \\ & & & & & & \longrightarrow H_1(S^1 \times S^1, A) \\ & & & & & & \searrow \\ & & & & & & \longrightarrow \bigoplus_{|A|} \mathbb{Z} \\ & & & & & & \searrow \\ & & & & & & \longrightarrow \mathbb{Z} \\ & & & & & & \searrow \\ & & & & & & \longrightarrow H_0(S^2, A) \\ & & & & & & \searrow \\ & & & & & & \longrightarrow 0. \end{array} \quad (6)$$

It is clear from exactness that $H_2(S^1 \times S^1, A) \cong \mathbb{Z}$ and $H_0(S^1 \times S^1, A) \cong \mathbb{Z}$. What is not clear is what $H_1(S^1 \times S^1, A)$ is. Exactness at $\mathbb{Z} \oplus \mathbb{Z}$ tells us that $\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow H_1(S^1 \times S^1, A)$ and, looking at the reduced homology, exactness at $\bigoplus_{|A|-1} \mathbb{Z}$ tells us that $H_1(S^1 \times S^1, A) \twoheadrightarrow \bigoplus_{|A|-1} \mathbb{Z}$. Thus, we have $\bigoplus_{|A|-1} \mathbb{Z} \cong H_1(S^1 \times S^1, A) / \mathbb{Z} \oplus \mathbb{Z}$ from which we can deduce that $H_1(S^1 \times S^1, A) \cong \bigoplus_{|A|+1} \mathbb{Z}$.¹ In summary, the relative homology of $S^1 \times S^1$ with respect to A is

$$H_n(S^1 \times S^1, A) = \begin{cases} \mathbb{Z} & \text{if } n = 2, 0 \\ \bigoplus_{|A|+1} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

(b) For this problem, we'll freely assume ■

¹I know this is not strictly correct, but the approach I took to solve the problem required me to construct an inverse map $H_1(S^1 \times S^1, A) \leftarrow \bigoplus_{|A|-1} \mathbb{Z}$, but this is difficult.

PROBLEM 3.2 (HATCHER §2.2, EX. 1)

Prove the Brouwer fixed point theorem for maps $f: D^n \rightarrow D^n$ by applying degree theory to the map $S^n \rightarrow S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f . [This was Brouwer's original proof.]

Proof. Seeking a contradiction, suppose $f: D^n \rightarrow D^n$ has no fixed point. Let N and S denote, respectively, the northern and southern hemisphere meeting at the equator of S^n . Now, since the disk $D^n \approx S$, we may as well identify D^n with S and consider the map $f: D^n \rightarrow D^n$ as the map $S \rightarrow S$ by composing with the homeomorphism. Define a map $g: S^n \rightarrow S^n$ by

$$g := \begin{cases} r & \text{on } N \\ \text{id} & \text{on } S \end{cases}. \quad (8)$$

Note that the map g is continuous by the pasting lemma since $g|_S = \text{id}$ and $g|_N = r$ are continuous and r fixes points at the equator $N \cap S$. Now, consider the map $F: S^n \rightarrow S^n$ given by the composition $\iota \circ f \circ g$ where $\iota: S \hookrightarrow S^n$ is the inclusion $S \subset S^n$. This map has no fixed points since f has no fixed point hence, by property (g) of the degree, $\deg F = (-1)^{n+1}$. But F is not onto, therefore $\deg F = 0$. This is a contradiction. ■

PROBLEM 3.3 (HATCHER §2.2, EX. 6)

Show that every map $S^n \rightarrow S^n$ can be homotoped to have a fixed point if $n > 0$.

Proof. The result follows from 4.25 since a map $f: S^n \rightarrow S^n$ without any fixed points is homotopic to the antipodal map. Since the antipodal map has degree -1 or 1 depending on n , it follows that the antipodal map is homotopic to either the identity map or a reflection map, both of which have fixed points. ■

PROBLEM 3.4

Let \mathcal{U} be an open cover of X . Prove that the inclusion of $C_*^{\mathcal{U}}(C)$ into $C_*(X)$ is a chain homotopy equivalence.

Proof. This is proposition 2.21 in the book. I will summarize the proof in four steps here.

- (1) (Barycentric subdivision) Given a simplex $[v_0, \dots, v_n]$ its barycenter is the point $b = \sum_i t_i v_i$ whose barycentric coordinates t_i are all equal, i.e., $t_i = 1/(n+1)$ for all i . The barycentric subdivision of $[v_0, \dots, v_n]$ is the decomposition of $[v_0, \dots, v_n]$ into the n -simplices $[b, w_0, \dots, w_{n-1}]$, where

(2)

(3)

(4)

■