# MA 523: Homework 9

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## Problem 9.1

(a) Show that for $n = 3$ the general solution to the wave equation $u_{tt} - \Delta u = 0$ with spherical symmetry about the origin has the form

$$u = \tfrac{1}{r}F(r+t) + \tfrac{1}{r}G(r-t), \quad r = |x|,$$

with suitable $F$ and $G$.

(b) Show that the solution with initial data of the form

$$u(r,0) = 0, \quad u_t(r,0) = h(r)$$

($h$ is an even function of $r$) is given by

$$u = \frac{1}{2r}\int_{r-t}^{r+t} \rho h(\rho)\, d\rho.$$

*Solution.* For part (a): We show that

$$u = \tfrac{1}{r}F(r+t) + \tfrac{1}{r}G(r-t)$$

is in fact a solution to the wave equation in $\mathbb{R}^3 \times (0,\infty)$. By direct calculation, we have

$$\begin{aligned}
u_{tt} &= \tfrac{1}{r}F''(r+t) + \tfrac{1}{r}G''(r-t),\\
u_r &= \tfrac{1}{r}F'(r+t) + \tfrac{1}{r}G'(r-t) - \tfrac{1}{r^2}F(r+t) - \tfrac{1}{r^2}G(r-t),\\
u_{rr} &= \tfrac{1}{r}F''(r+t) + \tfrac{1}{r}G''(r-t)\\
&\qquad - \tfrac{2}{r^2}F'(r+t) - \tfrac{2}{r^2}G'(r-t)\\
&\qquad + \tfrac{2}{r^3}F(r+t) + \tfrac{2}{r^3}G(r-t),\\
\Delta_{S^2}u &= 0
\end{aligned}$$

and lastly,

$$\begin{aligned}
\Delta u &= \tfrac{\partial}{\partial r^2}u + \tfrac{2}{r}\tfrac{\partial}{\partial r}u + \tfrac{1}{r^2}\Delta_{S^2}u\\
&= \tfrac{1}{r}F''(r+t) + \tfrac{1}{r}G''(r-t)\\
&\qquad - \tfrac{2}{r^2}F'(r+t) - \tfrac{2}{r^2}G'(r-t) + \tfrac{2}{r^3}F(r+t) + \tfrac{2}{r^3}G(r-t)\\
&\qquad + \tfrac{2}{r^2}F'(r+t) + \tfrac{2}{r^2}G'(r-t) - \tfrac{2}{r^3}F(r+t) - \tfrac{2}{r^3}G(r-t)\\
&\quad + 0\\
&= \tfrac{1}{r}F''(r+t) + \tfrac{1}{r}G''(r-t).
\end{aligned}$$

Therefore, we indeed have

$$u_{tt} - \Delta u = 0;$$

i.e., $u(r,t) := \tfrac{1}{r}F(r+t) + \tfrac{1}{r}G(r-t)$ is a solution to the wave equation with $F$ and $G$ at least $C^2$ such that they satisfy the initial conditions.

We still need to show that if $u$ is a solution to the wave equation with spherical symmetry it has the form prescribed above. We trust this can be done for now and return to this problem as time permits.

For part (b): Suppose $h$ has even symmetry ■

## PROBLEM 9.2

Show that the solution $w(x_1, t)$ of the initial-value problem for the *Klein–Gordon equation*

$$\begin{cases} w_{tt} = w_{x_1x_1} - \lambda^2 w, \\ \qquad w(x_1, 0) = 0, \quad w_t(x_1, 0) = h(x_1) \end{cases} \tag{9.1}$$

is given by

$$w(x_1, t) = \frac{1}{2}\int_{x_1-t}^{x_1+t} J_0(\lambda s)h(y_1)\, dy_1.$$

Here $s^2 = t^2 - (x_1 - y_1)^2$, while $J_0$ denotes the Bessel function defined by

$$J_0(z) := \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(z\sin\theta)\, d\theta.$$

(*Hint:* Descend to (9.1) from the two-dimensional wave equation satisfied by

$$u(x_1, x_2, t) = \cos(\lambda x_2)w(x_1, t).)$$

*SOLUTION.* ■

## Problem 9.3

Let $u$ solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u = g, \quad u_t = h & \text{on } \mathbb{R}^3 \times \{\, t = 0 \,\} \end{cases}$$

where $g$ and $h$ are smooth and have compact support. Show there exists a constant $C$ such that

$$|u(x,t)| \leq Ct^{-1} \quad (x \in \mathbb{R}^3,\, t > 0).$$

*Solution.* ■