MA 54400 - Midterm 2 Practice Problems
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1. Define, for $x \in \mathbb{R}^n$,
$$f(x) = \begin{cases} \frac{1}{|x|^{n+1}} & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$
Prove that f is integrable outside any ball $B(0, \varepsilon)$, and that there exists a constant $C > 0$ such that
$$\int_{B(0,\varepsilon)^c} f(x)\ dx \leq \frac{C}{\varepsilon}.$$

2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If
$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k < \infty,$$
show that
$$\int_E f = \lim_{k\to\infty} \int_E f_k$$
for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k = \infty$.

3. Assume that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if, and only if,
$$\sum_{k=0}^{\infty} |\{x \in E \mid f(x) \geq k\}| < \infty.$$

4. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f, g are measurable functions on E, define
$$\rho(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}.$$
Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if, and only if, f_k converges in measure to f as $k \to \infty$.

5. Define the *gamma function* $\Gamma : \mathbb{R}^+ \to \mathbb{R}$ by
$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1}\ du,$$
and the *beta function* $B : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$
$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\ dt.$$
   (a) Prove that the definition of gamma function is well posed, i.e. the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.
   (b) Show that
$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

6. Let $f \in L(\mathbb{R}^n)$, and for $h \in \mathbb{R}^n$ define $f_h : \mathbb{R}^n \to \mathbb{R},\ f_h(x) = f(x-h)$. Prove that

$$\lim_{h\to 0} \int_{\mathbb{R}^n} |f_h - f| = 0.$$

7. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n),\ f_k \to f$ and $g_k \to g$ *a.e.* in $\mathbb{R}^n,\ |f_k| \le g_k$, and

$$\int_{\mathbb{R}^n} g_k \to \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \to \int_{\mathbb{R}^n} f.$$

(b) Using part (a), show that if $f_k, f \in L(\mathbb{R}^n),\ f_k \to f$ *a.e* in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \to 0 \text{ as } k \to \infty$$

if, and only if,

$$\int_{\mathbb{R}^n} |f_k| \to \int_{\mathbb{R}^n} |f| \text{ as } k \to \infty.$$