MA 519: Homework 14

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Problem 14.1 (Handout 18, # 15)

(X, Y) is distributed uniformly inside of the unit circle. Find the density of $X + Y$ and hence the mean of $X + Y$. Was the value of the mean obvious? Why?

Solution. Suppose the random vector (X, Y) is uniformly distributed in $D := \{ x^2 + y^2 < 1 \}$. Let us first find the joint distribution of (X, Y). Fix a number $y \in [0, 1] =: I$, then the PDF of X given $Y = y$ is

$$f_{X \mid Y=y}(x) = \begin{cases} \frac{1}{2}\sqrt{1-y^2} & \text{for } x \in I \smallsetminus A, \\ 0 & \text{otherwise} \end{cases}$$

where $A := \left[-1 + \sqrt{1-y^2}, 1 - \sqrt{1-y^2}\right]$. ∎

Problem 14.2 (Handout 18, # 16)

Let X be a random number in $[0,1]$. What is the probability that the number 5 is completely missing from the decimal expansion of X?

Solution. First let us establish some notation. Let Ω denote the interval $[0,1]$ and let A denote the set of all real numbers in Ω which do not contain a 5 in their decimal expansion. We show that the probability that X is in A is zero; i.e., that $P(X \in A) = 0$.

To this end, let us consider the following Bernoulli process: Let N_k denote the k^{th} digit in the decimal expansion of X. Then N_k is uniformly distributed on $\{0, \ldots, 9\}$ and the probability that N_k is not the number 5 is $\frac{9}{10}$. Let I_k be an indicator random variable given by

$$I_k := \begin{cases} 0 & \text{for } N_k = 5, \\ 1 & \text{otherwise.} \end{cases}$$

Then the sequence of random variables $\{I_k\}$ with $p = \frac{9}{10}$ forms a sequence of independent Bernoulli trials wit expected value

$$E\Big(\prod\nolimits_{k=1}^{n} I_k\Big) = \prod_{k=1}^{n} E(I_k) = \prod_{k=1}^{n} p = p^n.$$

The expectation above is in fact the probability every digit of X from the first to the n^{th} is not 5.

Taking the limit of the expression above, the probability that X does not contain a 5 in its decimal expansion is the limit

$$\lim_{n\to\infty} p^n = \lim_{n\to\infty} \left(\frac{9}{10}\right)^n = 0. \qquad \blacksquare$$

Problem 14.3 (Handout 18, # 17)

A foot long stick is broken into three pieces. Find the density functions of the length of the longest part, the smallest part, and the medium part. What are the expected values for each part?

Solution. ■