MA 562: Notes

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1 Preliminaries

These set of notes are based off of Boothby's *Differential Geometry* book, chapters 1 through 6.

Definition 1. A *topological space* M is a pair $(X, \mathcal{T})$, where X is a set, $\mathcal{T}$ is a collection of subsets of X such that

(a) $\emptyset, X \in \mathcal{T}$.

(b) The union of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

$$\{U_\alpha\} \subset \mathcal{T} \quad \implies \quad \bigcup_\alpha U_\alpha \in \mathcal{T}.$$

(c) Intersection of a finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

$$\{U_1, ..., U_k\} \subset \mathcal{T} \quad \implies \quad \bigcap_{j=1}^{k} U_j \in \mathcal{T}.$$

$\mathcal{T}$ is called the *topology* of M. Elements of $\mathcal{T}$ are called the *open sets* of M. By abuse of notation, we sometimes refer to X as M.

Definition 2.   (a) A *metric* on X is a function $d \colon X \times X \to \mathbf{R}$ such that

    (1) $d(x, y) \geq 0 \; \forall x, y \in X$ and $d(x, y) = 0 \iff x = y$.

    (2) $d(x, y) = d(y, x)$.

    (3) $d(x, y) + d(y, z) \geq d(x, z)$ (the triangle inequality).

(b) $B_d(x, r) = \{\, y \in X \mid d(x, y) < r \,\}$.

(c) A topological space M is a *metric space* if the set of balls $B_d(x, r)$ form a *basis* of M, i.e., any open set of M can be written as a union of open balls $B_d(x, r)$ for some $x \in X$, $r > 0$.

Definition 3. A topological space X is *Hausdorff* if for any $x_1 \neq x_2$ in X, there exist open sets $U_1 \ni x_1$, $U_2 \ni x_2$ such that $U_1 \cap U_2 = \emptyset$.

Definition 4. A *topological manifold* M of dimension n is a topological space such that

(a) M is Hausdorff.

(b) locally Euclidean, i.e., $\forall x \in M$ there exists a neighborhood U of X which is homeomorphic to $V \subset \mathbf{R}^n$ (there exists a map $f \colon U_x \to V \subset \mathbf{R}^n$ such that f is bijective, continuous and f^{-1} is continuous).

(c) M has a countable basis of open sets.

Theorem 1 (Boothby I.3.6)**.** *A topological manifold is metrizable (also locally connected, locally compact, and normal).*

Definition 5. (a) A *covering* of a topological manifold is a collection of open sets $\{U_\alpha\}$ such that any $x \in M$ is contained in some U_α.

(b) A manifold is *compact* if every open cover contains a finite subcover.

Definition 6. (1) Half space
$$\mathbf{H}^n = \{\, x \in \mathbf{R}^n \mid x_n \geq 0 \,\}.$$

(2) Manifold with boundary. (Similar to definition 4)

    (a) M is Hausdorff.

    (b) M has a countable basis of open sets.

    (c) For any $x \in M$, there exists U open, $x \in U$ such that:

        (i) $\varphi \colon U \to V \subset \mathbf{R}^n$ is a homeomorphism, or

        (ii) $\varphi \colon U \to V \subset \mathbf{H}^n$ is a homeomorphism with x such that $\varphi(x) \in \partial\mathbf{H}^n$ referred to as *boundary points*.

(3)

Example 1 (Unit Quaternions and Rotations in $\mathbf{R}^3$)**.**

$$f(v) = z \wedge z^{-1} \qquad v = (v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$$

where $z = \cos(\alpha/2) + \sin(\alpha/2)\hat{v}$. $\hat{v} = v/\|v\|$. Quaternion multplication:

$$
\begin{aligned}
ij &= k & ji &= -k \\
jk &= i & kj &= -i \\
ki &= j & ik &= -j.
\end{aligned}
$$

Can check that z and $-z$ correspond to the same rotation.

Topologically, unit quaternions $\simeq S^3 = \{\, x \in \mathbf{R}^4 \mid \|x\| = 1 \}$ and rotations $\simeq S^3/\sim$, $z \sim -z$

$$\mathbf{R}P^3 \approx (\mathbf{R}^4 \smallsetminus \{0\})/x \sim \lambda x.$$

for all $\mathbf{R}^{n+1} \smallsetminus \{0\}$ can always find λ such that λx has norm 1. There are precisely 2 such λ which differ by a sign. Therefore, $\mathbf{R}P^n$ can be constructed by identifying antipodal points of S^n in $\mathbf{R}^{n+1}$.

2 Functions of Several Variables

Taken from Boothby Chapter 1.

Definition 7. Let $f\colon U \to \mathbf{R}$ where $U \subset \mathbf{R}^n$ is open, $x = (x^1, ..., x^n)$ (are coordinates).

(a) $f \in C^1(U)$ (continuously differentiable) iff $\partial f/\partial x^j$ exists and is continuous on U.

(b) $f \in C^r(U)$ (r-fold continuously differentiable) iff $\partial f/\partial x^j \in C^{r-1}(U)$ for $j = 1, ..., n$.

(c) $f \in C^\infty(U)$ (smooth) iff $f \in C^r(U)\ \forall r \in \mathbf{N}$.

(d) $f \in C^\omega(U)$ (real analytic) iff f can be expressed as a convergent power series on U.

Theorem 2 (Mean-value theorem). *Suppose U is star-like w.r.t. $a \in U$ (i.e., $\forall x \in U$, the line segment $\overleftrightarrow{ab} \subset U$). Then $\forall x \in U$, $\exists 0 < \theta < 1$ s.t.*

$$g(x) - g(a) = \left[\sum_{i=1}^{n} \left(\frac{\partial g}{\partial x^i} \right) \bigg|_{a + \theta(x-a)} \right] (x^i - a^i).$$

Proof. Consider $f\colon [0,1] \to \overleftrightarrow{ax} \subset U$ given by

$$f(t) = \mathbf{a} + t(\mathbf{x} - \mathbf{a}).$$

Then by $g \circ f\colon [0,1] \to \mathbf{R}$, the usual mean-value theorem implies that $\exists \theta \in (0,1)$ s.t. $\blacksquare$