

MA 572: Homework 2

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PROBLEM 2.1 (HATCHER §2.1, EX. 16)

- (a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X .
- (b) Show that $H_1(X, A) = 0$ iff $H_1(A) \rightarrow H_1(X)$ is surjective and each path-component of X contains at most one path-component of A .

Proof. (a) Let $i: A \hookrightarrow X$ denote the inclusion map. Then, the map i can be extended to a chain map between chain complexes so, by proposition 2.9, induces a homomorphism $i_*: H_*(A) \rightarrow H_*(X)$ on homology. Similarly, the map $j_\#: C_*(X) \rightarrow C_*(X, A)$ induces a map $j_*: H_*(X) \rightarrow H_*(X, A)$ so, by theorem 2.16, we have a long exact sequence

$$\cdots \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0 \quad (1)$$

on homology. Thus, we see that $H_0(X, A) = 0$ if and only if i_* is injective which, by proposition 2.6, happens if and only if A meets each path-component of X .

(b) Let us extend to the left the long exact sequence of homology groups in (1) as follows

$$\cdots \xrightarrow{\partial_*} H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0. \quad (2)$$

Hence, $H_1(X, A) = 0$ if and only if $j_* = 0$ and $\partial_* = 0$ if and only if i_* is surjective and i_* is injective on $H_0(A) \rightarrow H_0(X)$, i.e, each path-component of X contains at most one path-component of A . ■

PROBLEM 2.2 (HATCHER §2.1, EX. 18)

Show that for the subspace $\mathbf{Q} \subset \mathbf{R}$, the relative homology group $H_1(\mathbf{R}, \mathbf{Q})$ is free abelian and find a basis.

Proof. Consider the long exact sequence of homology groups

$$\cdots \xrightarrow{\partial_*} H_1(\mathbf{Q}) \xrightarrow{i_*} H_1(\mathbf{R}) \xrightarrow{j_*} H_1(\mathbf{R}, \mathbf{Q}) \xrightarrow{\partial_*} H_0(\mathbf{Q}) \xrightarrow{i_*} H_0(\mathbf{R}) \xrightarrow{j_*} H_0(\mathbf{R}, \mathbf{Q}) \xrightarrow{0} 0. \quad (3)$$

Since the space \mathbf{R} is contractible, $H_1(\mathbf{R}) = 0$ which implies that the map $j_* = 0$ on $H_1(\mathbf{R}) \rightarrow H_1(\mathbf{R}, \mathbf{Q})$, respectively. Hence, $\partial_*: H_1(\mathbf{R}, \mathbf{Q}) \rightarrow H_0(\mathbf{Q})$ is surjective. Thus, $H_1(\mathbf{R}, \mathbf{Q}) \cong H_0(\mathbf{Q})$. Since, \mathbf{Q} is totally disconnected, i.e., every connected component and hence, path-component of \mathbf{Q} is a singleton set, we have $H_0(\mathbf{Q}) \cong \mathbf{Z}[\mathbf{Q}] \cong H_1(\mathbf{R}, \mathbf{Q})$. \blacksquare

PROBLEM 2.3

Homotopy invariance of homology.

Proof. The proof of this follows immediately from corollary 2.10 for if $f: X \rightarrow Y$ and $g: Y \rightarrow X$ are maps with $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$ then by corollary 2.10 we have $(g \circ f)_* = \text{id}_{H_*(X)}$ and $(f \circ g)_* = \text{id}_{H_*(Y)}$, but $(f \circ g)_* = f_* \circ g_*$ and $(g \circ f)_* = g_* \circ f_*$ so $g_* = f_*^{-1}$ and we see that $H_*(X) \cong H_*(Y)$. ■