MA 523: Homework 1

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Problem 1.1 (Taylor's formula)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)$$

as $x \to \mathbf{0}$ for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) \coloneqq f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on m.

Solution. ▸ Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) \coloneqq f(tx)$. We claim that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Proof of claim. We shall proceed by induction on m. The case $m = 1$ follows easily from the chain rule:

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt} f(tx) \\ &= D^{(1,0,\ldots,0)} f(tx) x_1 + \cdots + D^{(0,\ldots,0,1)} f(tx) x_n \\ &= \big(D^{(1,0,\ldots,0)} x_1 + \cdots + D^{(0,\ldots,0,1)} x_n\big) f(tx) \end{aligned}$$

which we can write compactly as

$$= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^\alpha f(tx) x^\alpha.$$

Now, assume the result for $n \leq m - 1$. Then

$$\begin{aligned} \frac{d^m}{dt^m} g(t) &= \frac{d}{dt}\left[\frac{d^{m-1}}{dt^{m-1}} g(t)\right] \\ &= \frac{d}{dt}\left[\sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} D^\alpha f(tx) x^\alpha\right] \end{aligned}$$

since the derivative is a linear operator, we have

$$\begin{aligned} &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \frac{d}{dt}\big[D^\alpha f(tx) x^\alpha\big] \\ &= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \sum_{|\beta|=1} D^{\alpha+\beta} f(tx) x^{\alpha+\beta} \end{aligned}$$

but since f is smooth, the order in which we take derivatives does not matter and, hence the operators commute giving us

$$= \left[\sum_{|\beta|=1}(Dx)^\beta \sum_{|\alpha|=m-1}\frac{(m-1)!}{\alpha!}(Dx)^\alpha\right] f(tx). \tag{1.1}$$

From here it suffices to do some combinatorics on the operators and reduce it to the desired expression. By the multinomial theorem, we have

$$\left(\textstyle\sum_{|\alpha'|=1}(Dx)^{\alpha'}\right)^{m-1} = \sum_{|\alpha|=m-1}\binom{|\alpha|}{\alpha}(Dx)^\alpha = \sum_{|\alpha|=m-1}\frac{(m-1)!}{\alpha!}(Dx)^\alpha.$$

Thus (1.1) becomes

$$\begin{aligned}
\left[\sum_{|\beta|=1}(Dx)^\beta \sum_{|\alpha|=m-1}\frac{(m-1)!}{\alpha!}(Dx)^\alpha\right] f(tx) &= \left[\sum_{|\beta|=1}(Dx)^\beta \left(\textstyle\sum_{|\alpha'|=1}(Dx)^{\alpha'}\right)^{m-1}\right] f(tx) \\
&= \left[\left(\textstyle\sum_{|\beta|=1}(Dx)^\beta\right)^m\right] f(tx) \\
&= \sum_{|\alpha|=m}\frac{m!}{\alpha!}(Dx)^\alpha f(tx) \\
&= \sum_{|\alpha|=m}\frac{m!}{\alpha!}D^\alpha f(tx)x^\alpha,
\end{aligned}$$

as desired. ■

Now, applying Taylor's formula in 1 variable to $g(t)$ and evaluating at $t=1$ we have

$$\begin{aligned}
f(x) &= g(1) \\
&= \sum_{i=0}^k \frac{g^{(i)}(0)}{i!}1^i + \mathrm{O}\big(|x|^{k+1}\big) \\
&= \sum_{i=0}^k \frac{1}{i!}\sum_{|\alpha|=i}\frac{i!}{\alpha!}D^\alpha f(tx)x^\alpha + \mathrm{O}\big(|x|^{k+1}\big) \\
&= \sum_{i=0}^k \sum_{|\alpha|=i}\frac{1}{\alpha!}D^\alpha f(0)x^\alpha + \mathrm{O}\big(|x|^{k+1}\big) \\
&= \sum_{|\alpha|\le k}\frac{1}{\alpha!}D^\alpha f(0)x^\alpha + \mathrm{O}\big(|x|^{k+1}\big)
\end{aligned}$$

as desired. ◂

Problem 1.2

Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{$*$}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ▸ For reference, formula (5) in §2.1.2 of [E] is the solution to the nonhomogeneous problem

$$u(\mathbf{x}, t) = g(\mathbf{x} - t\mathbf{b}) + \int_0^1 f\big(\mathbf{x} + (s - t)\mathbf{b}, s\big)\, ds$$

where $\mathbf{x} \in \mathbb{R}^n$, $t > 0$.

To make the notation more bearable, we will use $\mathbf{b}$ and $\mathbf{x}$ to denote the original vectors in $(*)$. First, we write $(*)$ as the directional derivative along $(\mathbf{b}, 1)$, (note the abuse of notation)

$$\begin{aligned} f &= u_t + \mathbf{b} \cdot Du \\ &= (\mathbf{b}, 1) \cdot Du. \end{aligned}$$

Using the structure of characteristic ODE, we have the PDE

$$F(p, z, x) = (\mathbf{b}, 1) \cdot p$$

with characteristics

$$\dot{x} = b, \qquad \dot{t} = 1, \qquad \dot{z} = f.$$

Now, given a point $(\mathbf{x}, t) \in \mathbb{R}^n \times (0, \infty)$ we can solve the ODEs $\dot{x}$ and $\dot{t}$ easily as the lines $x(s) = \mathbf{x} - \mathbf{b}t + \mathbf{b}s$ and $t = s$. Substituting these solutions into $\dot{z}$, we have

$$\dot{z} = f\big(x(s), t(s)\big) = f\big(\mathbf{x} + \mathbf{b}(s - t), s\big)$$

so

$$\begin{aligned} \int_0^t f\big(x(s), t(s)\big) = f\big(\mathbf{x} + \mathbf{b}(s - t), s\big)\, ds = \int_0^t \dot{z}\, ds \\ &= z(t) - z(0) \\ &= u(\mathbf{x}, t) - u(\mathbf{x} - \mathbf{b}t, 0). \end{aligned}$$

Thus,

$$\begin{aligned} u(\mathbf{x}, t) &= u(\mathbf{x} - \mathbf{b}t, 0) + \int_0^t f\big(x(s), t(s)\big) = f\big(\mathbf{x} + \mathbf{b}(s - t), s\big)\, ds \\ &= g(\mathbf{x} - \mathbf{b}) + \int_0^t f\big(x(s), t(s)\big) = f\big(\mathbf{x} + \mathbf{b}(s - t), s\big)\, ds \end{aligned}$$

as desired. ◂

Problem 1.3

Solve using the characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$, $u = 1$ on the line $x_2 = 2x_1$.

(b) $uu_{x_1} + u_{x_2} = 1$, $u(x_1, x_2) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution. ▸ For part (a), employing the method of characteristics, we write

$$F(p, z, x) = \left(x_1^2, x_2^2\right) \cdot p = z^2.$$

From here, we have

$$\dot{x} = \left(x_1^2, x_2^2\right), \qquad \dot{z} = z^2.$$

Now say $x(0) = (2t', t')$ and solve for $\dot{x}$,

$$\begin{aligned} \int_0^t ds &= \int_0^t \frac{1}{x_1(s)^2}\, dx_1(s) \\ t &= -\frac{1}{x_1(t)} + \frac{1}{x_1(0)} \\ x_1(t) &= \frac{1}{1/x_1(0) - t} \\ &= \frac{2t'}{1 - 2tt'} \end{aligned} \qquad \begin{aligned} \int_0^t ds &= \int_0^t \frac{1}{x_1(s)^2}\, dx_1(s) \\ t &= -\frac{1}{x_2(s)} + \frac{1}{x_2(0)} \\ x_2(t) &= \frac{1}{1/x_2(0) - t} \\ &= \frac{t'}{1 - tt'}. \end{aligned}$$

Thus,

$$x(t) = \left(\frac{2t'}{1 - 2tt'}, \frac{t'}{1 - tt'}\right)$$

and solving for z similarly yields

$$z(t) = \frac{z(0)}{1 - tz(0)} = \frac{1}{1 - t}$$

since $z = u^2 = 1$ on the line $x_2 = 2x_1$. Lastly, solving for t in terms of x_1 and x_2, we have

$$\begin{aligned} t' &= \frac{x_1}{2(tx_1 + 1)} \\ &= \frac{x_2}{tx_2 + 1} \end{aligned}$$

which, with a little algebra, can be turned into

$$t = \frac{1 - 2x_2/x_1}{x_2} = \frac{x_2 - 2x_1}{x_1 x_2}.$$

Now substituting this into the solution $z(t)$, we have

$$\begin{aligned} u(x,y) &= z(t) \\ &= \frac{1}{1-t} \\ &= \frac{1}{1-(x_2-2x_1)/(x_1x_2)} \\ &= \frac{x_1x_2}{x_1x_2 - x_2 + 2x_1}. \end{aligned}$$

For part (b), write

$$F(p,z,x) = (x_1,1)\cdot p = 1.$$

Then, we have

$$\dot{x} = (z,1), \qquad \dot{z} = 1.$$

As before, pick a line $(x_1,0)$ along which the solution is constant. Then we can solve easily for $\dot{z}$,

$$\begin{aligned} z(t) &= z(t') + \int_{t'}^{t} 1\,ds = t - t' \\ &= \frac{x_1}{2} + t - t' \end{aligned}$$

so

$$\dot{x} = \left(\frac{x_1}{2} + t, 1\right)$$

which, in the x_1-coordinate, we can solve using integrating factors giving us

$$x_1(t) = \int_{t'}^{t}$$

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Problem 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}\left(u_{x_1}^2 + u_{x_2}^2\right)$$

find a solution with $u(x_1, 0) = \left(1 - x_1^2\right)/2$.

Solution. ► ◄