MA 523: Homework 7

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Problem 7.1

Solve the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \text{in } 1 < |x| < 2, \\ u = x_1 & \text{on } |x| = 1, \\ u = 1 + x_1 x_2 & \text{on } |x| = 2. \end{cases}$$

(*Hint:* Use Laurent series.)

Solution. First, let us make the change of variables $(x_1, x_2) \mapsto r\mathrm{e}^{\mathrm{i}\theta}$ to the Dirichlet problem in question:

$$\begin{cases} \Delta u = 0 & \text{in } 1 < r < 2, \\ u = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) & \text{on } r = 1, \\ u = 1 + \frac{1}{i}(\mathrm{e}^{\mathrm{i}2\theta} - \mathrm{e}^{-\mathrm{i}2\theta}) & \text{on } r = 2. \end{cases} \tag{7.1}$$

Now, suppose u is a solution, of the form

$$u(r\mathrm{e}^{\mathrm{i}\theta}) = b \ln r + \sum_{n \in \mathbb{Z}} (a_n r^n + \overline{a_{-n}} r^{-n}) \mathrm{e}^{in\theta},$$

to the problem (**??**). It is easy to see that u is in fact harmonic:

$$\begin{aligned} \Delta u &= u_{rr} + \tfrac{1}{r} u_r + \tfrac{1}{r^2} u_{\theta\theta} \\ &= -br^{-2} + br^{-2} + \sum_{n \in \mathbb{Z}} \big[\big(n(n-1) + n - n^2\big) a_n r^n \\ &\qquad\qquad + \big(n(n-1) + n - n^2\big) \overline{a_{-n}} r^{-n} \big] \mathrm{e}^{in\theta} \\ &= 0. \end{aligned}$$

Next we use the boundary data to compute the coefficients a_n, $n \in \mathbb{Z}$. Using the data (**??**), on $r = 1$ we have

$$\tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) = \sum_{n \in \mathbb{Z}} (a_n + \overline{a_{-n}}) \mathrm{e}^{in\theta},$$

and on $r = 2$

$$1 + \tfrac{1}{i}(\mathrm{e}^{\mathrm{i}2\theta} - \mathrm{e}^{-\mathrm{i}2\theta}) = b \ln 2 + \sum_{n \in \mathbb{Z}} (2^n a_n + 2^{-n} a_{-n}) \mathrm{e}^{in\theta}.$$

These equations immediately tell us that $b = 1/\ln 2$. Moreover, the following relations on the coefficients hold

$$\begin{cases} \dfrac{1}{2} = a_1 + \overline{a_{-1}} & \dfrac{1}{2} = a_{-1} + \overline{a_1}, \\ \dfrac{1}{i} = 2^2 a_2 + 2^{-2} \overline{a_{-2}}, & -\dfrac{1}{i} = 2^2 a_{-2} + 2^{-2} \overline{a_2}, \\ 0 = a_n + \overline{a_{-n}} & \text{for } n \neq \pm 1, \\ 0 = 2^n a_n + 2^{-n} \overline{a_{-n}} & \text{for } n \neq \pm 2. \end{cases}$$

A little calculation shows that

$$\begin{cases} a_1 = -\dfrac{1}{6}, \quad a_{-1} = \dfrac{2}{3}, \\ a_2 = -\dfrac{4\mathrm{i}}{15}, \quad a_{-2} = -\dfrac{4\mathrm{i}}{15}, \\ a_n = 0 \qquad\qquad \text{for } n \neq \pm 1, \pm 2. \end{cases}$$

Thus,

$$\begin{aligned} u(r\mathrm{e}^{\mathrm{i}\theta}) &= \tfrac{1}{\ln 2}\ln r + \left(-\tfrac{4\mathrm{i}}{15}r^{-2} + \tfrac{4\mathrm{i}}{15}r^2\right)\mathrm{e}^{-\mathrm{i}2\theta} + \left(\tfrac{2}{3}r^{-1} - \tfrac{1}{6}r\right)\mathrm{e}^{-\mathrm{i}\theta} \\ &\quad + \left(-\tfrac{1}{6}r + \tfrac{2}{3}r^{-1}\right)\mathrm{e}^{\mathrm{i}\theta} + \left(-\tfrac{4\mathrm{i}}{15}r^2 + \tfrac{4\mathrm{i}}{15}r^{-2}\right)\mathrm{e}^{\mathrm{i}2\theta} \\ &= \tfrac{1}{\ln 2}\ln r - \tfrac{8}{15}r^{-4}\left(\frac{r^2\mathrm{e}^{\mathrm{i}2\theta} - r^2\mathrm{e}^{-\mathrm{i}2\theta}}{2\mathrm{i}}\right) + \tfrac{8}{15}\left(\frac{r^2\mathrm{e}^{\mathrm{i}2\theta} - r^2\mathrm{e}^{-\mathrm{i}2\theta}}{2\mathrm{i}}\right) \\ &\quad + \tfrac{4}{3}r^{-2}\left(\frac{r\mathrm{e}^{\mathrm{i}\theta} + r\mathrm{e}^{-\mathrm{i}\theta}}{2}\right) - \tfrac{1}{3}\left(\frac{r\mathrm{e}^{\mathrm{i}\theta} + r\mathrm{e}^{-\mathrm{i}\theta}}{2}\right). \end{aligned}$$

Substituting back, we have

$$u(x_1, x_2) = \tfrac{1}{\ln 2}\ln(x_1^2 + x_2^2) - \frac{16x_1x_2}{15(x_1^2+x_2^2)^2} + \frac{16x_1x_2}{15} + \frac{4x_1}{3(x_1^2+x_2^2)} - \frac{x_1}{3}. \tag{7.2}$$

It is easy to see that (**??**) satisfies the boundary data at $|x| = 1$ and $|x| = 2$. ∎

Problem 7.2

Let Ω be a bounded domain with a C^1 boundary, $g \in C^2(\partial\Omega)$ and $f \in C(\bar{\Omega})$. Consider the so called *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where ν is the outer normal on $\partial\Omega$. Show that the solution of $(*)$ in $C^2(\Omega) \cap C^1(\bar{\Omega})$ is unique up to a constant; i.e., if u_1 and u_2 are both solutions of $(*)$, then $u_2 = u_1 + \text{const.}$ in Ω.
(*Hint:* Look at the proof of the uniqueness for the Dirichlet problem by energy methods [E, 2.2.5a].)

Solution. Suppose u_1 and u_2 are solutions to the Neumann problem $(*)$. Define $v := u_1 - u_2$. Then v is harmonic in Ω and $\partial v/\partial\nu = 0$ on $\partial\Omega$. Consider the energy functional

$$E[v] = \frac{1}{2}\int_\Omega |Dv|^2\,dx.$$

By Green's formula version (ii),

$$\begin{aligned} E[v] &= \frac{1}{2}\int_\Omega |Dv|^2\,dx \\ &= -\frac{1}{2}\int_\Omega v\Delta v\,dx + \int_{\partial U} \frac{\partial v}{\partial \nu} v\,dS(x) \\ &= 0. \end{aligned}$$

This implies that $|Dv|^2 = Dv \cdot Dv = 0$ which, since the standard inner product on $\mathbb{R}^n$ is positive-definite, implies that $Dw \equiv 0$. It follows that $u_1 = u_2 + \text{const}$, i.e., the solution u to $(*)$ is unique up to a constant. ∎

Problem 7.3

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta_x u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\}, \end{cases}$$

where $c \in \mathbb{R}$.
(*Hint:* Rewrite the problem in terms of $v(x,t) := \mathrm{e}^{ct} u(x,t)$.)

Solution. Taking the hint, let us rewrite the problem in terms of $v(x,t) = \mathrm{e}^{ct} u(x,t)$:

$$\begin{cases} v_t - \Delta_x v = \mathrm{e}^{ct} f & \text{in } \mathbb{R}^n \times (0,\infty), \\ v = g & \text{on } \mathbb{R}^n \times \{\, t = 0 \,\}. \end{cases} \tag{7.3}$$

By Duhamel's principle, the problem (**??**) is solved by

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) \mathrm{e}^{cs} f(y,s)\, dy ds,$$

where Φ is the fundamental solution to the heat equation. Thus, the formula

$$u(x,t) = \mathrm{e}^{-ct} v(x,t) = \mathrm{e}^{-ct} \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy + \mathrm{e}^{-ct} \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) \mathrm{e}^{cs} f(y,s)\, dy ds$$

solves the original problem. ∎