

# MA 523: Homework 6

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## PROBLEM 6.1

For  $n = 2$  find Green's function for the quadrant  $U := \{x_1, x_2 > 0\}$  by repeated reflection.

*SOLUTION.* Taking the hit, set  $x' := (x_1, -x_2)$ ,  $x'' := (-x_1, x_2)$ ,  $x''' := (-x_1, -x_2)$ , and define

$$\varphi^x(y) := \Phi(y - x') + \Phi(y - x'') - \Phi(y - x'''). \quad (6.1)$$

We claim that  $\varphi^x$ , as defined above, solves

$$\begin{cases} \Delta \varphi^x = 0 & \text{in } U, \\ \varphi^x(y) = \Phi(y - x) & \text{on } \partial U. \end{cases}$$

It is clear that  $\Delta \varphi^x = 0$  since it is built up from the fundamental solutions on  $\mathbb{R}^n$  (this follows from the linearity of the Laplace operator). To see that  $\varphi^x(y) = \Phi(y - x)$  on  $\partial U$ , we do a case by case analysis.

Note that on  $\{x_1 = 0\} \subset \partial U$ , we have

$$\varphi^x(y_1, 0) = \Phi(-x_1, y_2 + x_2) + \Phi(-x_1, y_2 - x_2) - \Phi(x_1, y_2 + x_2),$$

where, since the fundamental solution is radial, we have  $\Phi(-x_1, y_2 + x_2) = \Phi(x_1, y_2 + x_2)$ , and hence the above equals

$$\begin{aligned} &= \Phi(-x_1, y_2 - x_2) \\ &= \Phi(y - x) \end{aligned}$$

and on  $\{x_2 = 0\} \subset \partial U$ , we have

$$\varphi^x(0, y_2) = \Phi(y_1 - x_1, x_2) + \Phi(y_1 + x_1, -x_2) - \Phi(y_1 + x_1, x_2)$$

where, again because  $\Phi$  is radial,  $\Phi(y_1 + x_1, -x_2) = \Phi(y_1 + x_1, x_2)$ , thus the above equals

$$\begin{aligned} &= \Phi(y_1 - x_1, x_2) \\ &= \Phi(y - x). \end{aligned}$$

Thus,  $\varphi^x(y) = \Phi(y - x)$  on  $\partial U$ .

Therefore, Green's function on  $U$  is

$$G(x, y) = \Phi(y - x) - \varphi^x(y) = \Phi(y - x) - \Phi(y - x') - \Phi(y - x'') + \Phi(y - x'''). \quad \blacksquare$$

## PROBLEM 6.2

(Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}}u(0)$$

whenever  $u$  is positive and harmonic in  $B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$ .

*SOLUTION.* Recall Poisson's formula for the ball

$$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0, r)} \frac{g(y)}{|x - y|^n} dS(y), \quad (6.2)$$

where  $x \in B(0, r)$  and  $u$  solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0, r), \\ u = g & \text{on } \partial B(0, r). \end{cases}$$

For fixed  $x \in B(0, r)$ , write

$$u(x) = r^{n-2}(r+|x|)(r-|x|) \left[ \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0, r)} \frac{g(y)}{|x - y|^n} dS(y) \right].$$

Now, since  $r + |x| \geq |x - y| \geq r - |x|$  for all  $y \in \partial B(0, r)$ , we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}} \int_{\partial B(0, r)} g(y) dS(y) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}} \int_{\partial B(0, r)} g(y) dS(y). \quad (6.3)$$

Since  $u = g$  on the boundary  $\partial B(0, r)$ , by applying the mean-value property on (6.3) we have

$$\frac{r^{n-2}(r-|x|)}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{r^{n-2}(r+|x|)}{(r-|x|)^{n-1}}u(0),$$

as desired. ■

## PROBLEM 6.3

Let  $P_k(x)$  and  $P_m(x)$  be homogeneous harmonic polynomials in  $\mathbb{R}^n$  of degrees  $k$  and  $m$  respectively; i.e.,

$$\begin{cases} P_k(\lambda x) = \lambda^k P_k(x), & P_m(\lambda x) = \lambda^m P_m(x) & \text{for every } x \in \mathbb{R}^n, \lambda > 0, \\ \Delta P_k = 0, & \Delta P_m = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

(a) Show that

$$\begin{cases} \frac{\partial P_k}{\partial \nu} = k P_k(x), & \frac{\partial P_m}{\partial \nu} = m P_m(x) & \text{on } \partial B(0, 1), \end{cases}$$

where  $B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$  and  $\nu$  is the outward normal on  $\partial B(0, 1)$ .

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0,1)} P_k(x) P_m(x) d\sigma = 0, \quad \text{if } k \neq m.$$

*SOLUTION.* For part (a), let

$$P_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha.$$

Then, since  $\nu = (x_1, \dots, x_n)$ , the derivative along  $\nu$  is given by

$$\begin{aligned} \frac{\partial P_k(x)}{\partial \nu} &= \sum_{i=1}^n (P_k)_{x_i} x_i \\ &= \sum_{i=1}^n \left( \sum_{|\alpha|=k} a_\alpha x^\alpha \right)_{x_i} x_i \\ &= \sum_{i=1}^n \left( \sum_{j=1}^m a_\alpha x_1^{\alpha_1^j} \cdots x_i^{\alpha_i^j} \cdots x_n^{\alpha_n^j} \right)_{x_i} x_i \end{aligned}$$

where  $\sum_{i=1}^n \alpha_i^j = k$  and  $1 \leq j \leq \binom{n+k-1}{n} =: m$  (by the stars and bars theorem)

$$\begin{aligned} &= \sum_{i=1}^n \sum_{j=1}^m \left( \alpha_i^j a_\alpha x_1^{\alpha_1^j} \cdots x_i^{\alpha_i^j-1} \cdots x_n^{\alpha_n^j} \right) x_i \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i^j a_\alpha x_1^{\alpha_1^j} \cdots x_i^{\alpha_i^j} \cdots x_n^{\alpha_n^j} \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i^j a_\alpha x^\alpha \end{aligned}$$

switching the order of summation, we have

$$\begin{aligned}
 &= \sum_{j=1}^m \sum_{i=1}^n \alpha_i^j a_\alpha x^\alpha \\
 &= \sum_{j=1}^m k a_\alpha x^\alpha \\
 &= k \sum_{j=1}^m a_\alpha x^\alpha \\
 &= k P_k(x).
 \end{aligned}$$

This argument, of course, applies to every  $k \in \mathbb{N}$ .

For part (b), by Green's theorem, we have

$$\begin{aligned}
 \int_{B(0,r)} P_k(x) \Delta P_m(x) - (\Delta P_k(x)) P_m(x) dx &= \int_{\partial B(0,r)} P_k(x) \frac{\partial}{\partial \nu} P_m(x) - \frac{\partial}{\partial \nu} P_k(x) P_m(x) dS(x) \\
 &= \int_{\partial B(0,r)} (m - k) P_k(x) P_m(x) dS(x),
 \end{aligned}$$

where the left-hand side is equal to 0 since both  $\Delta P_k$  and  $\Delta P_m$  are 0. Since  $m \neq k$ , it must be the case that

$$\int_{\partial B(0,r)} P_k(x) P_m(x) dS(x) = 0.$$

■