MA571 Homework 13

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December 2, 2015

Problem 13.1 (Munkres §68, Ex. 1)

Check the details of Example 1.

Proof. The following is the statement of Example 1 as found in the book:

Examples 1. Consider the group P of bijections of the set $\{0, 1, 2\}$ with itself. For $i = 1, 2$, define an element π_1 of P by setting $\pi_i(i) = i - 1$ and $\pi_i(i-1) = i$ and $\pi_i(j) = j$ otherwise. Then π_i generates a subgroup G_i of P of order 2. The group G_1 and G_2 generate P, as you can check. But P is not their free product. The reduced words (π_1, π_2, π_1) and (π_2, π_1, π_2), for instance, represent the same element of P.

We need to check two claims (i) that G_1 and G_2, as defined above, generate P and (ii) that $P \neq G_1 * G_2$, i.e., show that $(\pi_1, \pi_2, \pi_1) = (\pi_2, \pi_1, \pi_2)$. Let us deal with (i) first. We show that $\langle G_1, G_2 \rangle = P$. Our strategy is the following, by the pigeon-hole principle, it suffices to show that $\langle G_1, G_2 \rangle \subset P$ and that $|\langle G_1, G_2 \rangle| = |P|$. Since $G_1, G_2 < P$, i.e., G_1 and G_2 are subgroups of P, the group generated by G_1 and G_2 will be a subgroup of P hence, $\langle G_1, G_2 \rangle \subset P$. The group P is a well-known group, namely (up to group isomorphism) S_3, and we shall not waste time any time showing that $|P| = |\{0, 1, 2\}| = 3! = 6$, but instead we proceed to showing that $|\langle G_1, G_2 \rangle| = 6$. From the definitions of G_1 and G_2, we have at least 3 in $\langle G_1, G_2 \rangle$, these are the elements 1, π_1 and π_2 (the latter two have order 2, e.g.,

$$\pi_i^2(j) = \pi_i \left(\begin{cases} i-1 & \text{if } j = i \\ i & \text{if } j = i-1 \\ j & \text{otherwise} \end{cases} \right) = \begin{cases} i & \text{if } j = i \\ i-1 & \text{if } j = i-1 \\ j & \text{otherwise} \end{cases}$$

which is the identity on $\{0, 1, 2\}$.) So the elements $1, \pi_1, \pi_2, \pi_1\pi_2, \pi_2\pi_1, \pi_1\pi_2\pi_1 \in \langle G_1, G_2 \rangle$ and all finite strings $\pi_1\pi_2 \cdots \pi_i$, $\pi_2\pi_1 \cdots \pi_i$ for that matter. But as a consequence of Lagrange's theorem, the size of $\langle G_1, G_2 \rangle$ must not exceed the size of P so that we are done when we show that the elements $\pi_1\pi_2$, $\pi_2\pi_1$ and $\pi_1\pi_2\pi_1$ are distinct elements. First, observe that

$$\pi_2\pi_1(j) = \pi_2 \left(\begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ 2 & \text{if } j = 2 \end{cases} \right) = \begin{cases} 2 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} \qquad \pi_1\pi_2(j) = \pi_1 \left(\begin{cases} 0 & \text{if } j = 0 \\ 2 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} \right) = \begin{cases} 1 & \text{if } j = 0 \\ 2 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

and, using the computations above,

$$\pi_1\pi_2\pi_1(j) = \pi_1 \left(\begin{cases} 2 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} \right) = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}.$$

Note that none of these elements are equivalent to any of 1, π_1 or π_2 and are certainly not equal to each other. Moreover, there are six of these elements and there are no more elements in P since $|P| = 6$. Thus, $\langle G_1, G_2 \rangle = P$.

Lastly, we show that $P \neq G_1 * G_2$ since

$$(\pi_1, \pi_2, \pi_1) = \pi_1\pi_2\pi_1(j) = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

and

$$(\pi_2, \pi_1, \pi_2) = \pi_2\pi_1\pi_2(j) = \pi_1\left(\begin{cases} 1 & \text{if } j = 0 \\ 2 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}\right) = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

would imply that $(\pi_1, \pi_2, \pi_1) = (\pi_2, \pi_1, \pi_2)$ in the free product $G_1 * G_2$, but $\pi_1 \neq \pi_2$. ∎

Problem 13.2 (Munkres §68, Ex. 2(a,b,c))

Let $G = G_1 * G_2$, where G_1 and G_2 are nontrivial groups.

(a) Show G is not Abelian.

(b) If $x \in G$, define the *length* of x to be the length of the unique reduced word in the elements of G_1 and G_2 that represents x. Show that if x has even length (at least 2), then x does not have finite order. Show that if x has odd length (at least 3), then x is conjugate to an element of shorter length.

(c) Show that the only elements of G that have finite order are the elements of G_1 and G_2 that have finite order, and their conjugates.

Proof. (i) Suppose G is Abelian. Take an element $x \in G_1$ and $y \in G_2$. Then $(x, y) = (y, x)$. By the definition of a free product (Munkres §68, pp. 413-414) this implies that the word $(x^{-1}, y^{-1}, x, y) = 1$ which implies that $y^{-1}x = 1$, but $y^{-1} \notin G_1$.

(ii) Let $x \in G$ be a word of even length. Then $x = (y_1, y_2, ..., y_{2k})$ for $k \in \mathbf{N}$ where the right hand-side is irreducible, i.e., either $y_i \in G_1$ if $2 \mid i$ and $y_j \in G_2$ if $2 \nmid j$ or vice-versa since two consecutive "letters" in a word must be from distinct groups or else we can reduce the word further. Then $x^2 = (y_1, y_2, ..., y_{2k}, y_1, y_2, ..., y_{2k})$ is again irreducible since $y_{2k} \in G_1$ and $y_1 \in G_2$ or vice-versa. It follows by induction that $x^n \neq 1$ for any finite positive integer n.

Now, suppose that $x \in G$ has odd length. Then $x = (y_1, y_2, ..., y_{2k+1})$ for $k \in \mathbf{N}$ where the right hand-side is irreducible. Without loss of generality, we may assume that $y_1, y_{2k+1} \in G_1$. Then, setting $y'_{2k+1} \coloneqq y_{2k+1}y_1$, we have

$$y_1^{-1}xy_1 = y_1^{-1}(y_1, y_2, ..., y_{2k+1})y_1 = (y_2, y_3, ..., y_{2k+1}y_1) = (y_2, y_3, ..., y'_{2k+1})$$

which has length $2k$. Thus, x is conjugate to a word of shorter length.

(iii) Suppose that $x \in G$ has finite order. By part (i) the length of x cannot be even. Moreover, if x is of finite order, i.e., if $x^n = 1$ for some positive integer n, and y is conjugate to x, i.e., there exist $g \in G$ such that $y = g^{-1}xg$, then

$$y^n = (g^{-1}xg)^n = (g^{-1}xg)(g^{-1}xg) \cdots (g^{-1}xg) = g^{-1}x^n g = 1$$

so y is of finite order. It remains to show that if x has finite order then x is a conjugate of an element y of G_i, where $i = 1, 2$. Let $2k + 1$ be the length of x. By part (ii), x is conjugate to an element y' of shorter length. Since x has finite order y has finite order so by part (i) y' must be of odd length. If y' is of length 1 we are done. If not, then y' is conjugate to a word y'' of shorter length with finite order. Since the length of x is finite, this process must terminate at a word y of length 1 with finite order. ■

Problem 13.3 (Munkres §68, Ex. 3)

Let $G = G_1 * G_2$. Given $c \in G$, let cG_1c^{-1} denote the set of all elements of the form cxc^{-1}, for $x \in G_1$. It is a subgroup of G; show that the intersection with G_2 is the identity alone.

Proof. Suppose $y \in cG_1c^{-1} \cap G_2$. Then $y = cxc^{-1}$ for some $x \in G_1$ and we have, $c = ycx^{-1}$. Let us deal with the trivial case first. If $c = 1$ then, since G is the free group of G_1 and G_2, we have $1 \cdot G_1 \cdot 1^{-1} = G_1$ so $(1 \cdot G_1 \cdot 1^{-1}) \cap G_2 = G_1 \cap G_2 = 1$ by definition of the free product. Now, suppose $c \neq 1$, say that c is represented by the unique reduced word $(y_1, ..., y_k)$, $k \in \mathbf{N}$. We show that for the following cases (i) $y_1, y_k \in G_i$, (ii) $y_1 \in G_1$ and $y_k \in G_2$, or $y_1 \in G_2$ and $y_k \in G_1$, $y = 1$, i.e., the intersection $cG_1c^{-1} \cap G_2 = 1$.

The above three cases can trivially be adapted to just the case where $y_1, y_k \in G_1$ or $y_1, y_k \in G_2$, so we shall prove the aforementioned. Assuming $y_1, y_k \in G_1$, c is represented by $(y_1, ..., y_k)$ and $(y, y_1, ..., y_k, x^{-1})$ where the latter reduces to the word $(y, y_1, ..., y_kx^{-1})$. Now, by the uniqueness of representation by reduced word, $(y_1, ..., y_k) = (y, y_1, ..., y_kx^{-1})$, but the right-hand side has length $k+1$ and cannot be reduced further unless $y_kx^{-1} = 1$. Suppose that y_kx^{-1} then we have, $y_1 = y, y_2 = y_1, ..., y_k = y_{k-1}$ which can only happen if $y = 1$ and $y_i = 1$ for all i. But this contradicts the assumption that $c \neq 1$. Thus, $y = 1$ to begin with.

Now, suppose $y_1, y_k \in G_2$, then c is represented by $(y_1, ..., y_k)$ and $(y, y_1, ..., y_k, x^{-1})$ where the latter reduces to the word $(yy_1, ..., y_k, x^{-1})$. Now, by the uniqueness of representation by reduced word, $(y_1, ..., y_k) = (yy_1, ..., y_k, x^{-1})$, but the right-hand side has length $k+1$ and cannot be reduced further unless the product $yy_1 = 1$. Suppose that $yy_1 = 1$. Then we have, $y_2 = y_1, , ..., y_{k-1} = y_k, y_k = x$. This implies that, since $y_i \notin G_{\alpha_{i+1}}$, $x = 1$ and $y_i = 1$ for all i. But this contradicts the assumption that $c \neq 1$. Thus, y must have been 1 to begin with. ∎

Problem 13.4 (A)

(i) Do the case of p. 367 # 9(e) where h and k take b_0 to b_0. (The proof is similar to the proof of Lemma 55.3, (3) $\implies$ (1), that I gave in class).

(ii) Let G be a path-connected topological group and let $a \in G$. Prove that the map $\varphi \colon G \to G$ defined by $\varphi(g) := ag$ is homotopic to the identity map.

(iii) Use part (ii) to complete the proof of p. 367 # 9(e).

Proof. (i) Set $d := \deg h$. Suppose that $h(b_0) = k(b_0) = b_0$ and that $\deg h = \deg k$. Consider the path $f(s) := (\cos(2\pi s), \sin(2\pi s))$ from the handout on "The fundamental group of S^1." This path is a loop at b_0 ($f(0) = (\cos(2\pi \cdot 0), \sin(2\pi \cdot 0)) = (1, 0) = (\cos(2\pi), \sin(2\pi))f(1))$ of index 1 (i.e., the winding number of f is 1), hence is a generator for $\pi_1(S^1, b_0)$. Thus, we have

$$h_*([f]) = d \cdot [f] = k_*([f])$$

so $h \circ f \simeq_p k \circ f$. Let $H \colon I \times I \to S^1$ denote the homotopy from $h \circ f$ to $k \circ f$, i.e., the continuous map such that $H(s, 0) = h \circ f(s)$ and $H(s, 1) = k \circ f(s)$. Next, by the Problem 9.2 (Munkres §46, Ex. 9), we see that the map $(f, \mathrm{id}_I) \colon I \times I \to S^1 \times I$ is a quotient map so that the following diagram commutes

$$\begin{tikzcd} I \times I \arrow[r, "H"] \arrow[d, "{(f,\mathrm{id}_I)}"'] & S^1 \\ S^1 \times I. \arrow[ur, "\overline{H}"'] & \end{tikzcd}$$

By Theorem Q.2, the map $\overline{H}$ is continuous and H factors through $\overline{H}$, i.e., $H(s, 0) = h \circ f(s) = \overline{H}(f(s), 0)$ and $H(s, 1) = k \circ f(s) = \overline{H}(f(s), 1)$. But since f is onto S^1, setting $x := f(s)$ for $s \in I$, we have $\overline{H}(x, 0) = h(x)$ and $\overline{H}(x, 1) = k(x)$. Thus, $\overline{H}$ is a homotopy from h to k so $h \simeq_p k$.

(ii) Let 1 denote the identity element of G. Since G is path-connected there exists a path $\alpha \colon I \to G$ from a to 1, i.e., $\alpha(0) = a$ and $\alpha(1) = 1$. Define the map $H \colon G \times I \to G$ by $H(g, t) := \alpha(t)g$. Then H is a homotopy from φ to the identity map id_G ($H(g, 0) = \alpha(0)g = ag = \varphi(g)$ and $H(g, 1) = \alpha(1)g = 1 \cdot g = \mathrm{id}_G(g)$; moreover, H is continuous since α is continuous and multiplication in G is continuous). Thus, $\varphi \simeq \mathrm{id}_G$.

(iii) Suppose that $h, k \colon S^1 \to S^1$ have the same degree d. By part (a) of Ex. 9, we know that the degree of a map between circles is independent of the basepoint so we may as well let b_0 be the basepoint for the fundamental group of S^1. Now, the induced maps $h_* \colon \pi_1(S^1, b_0) \to \pi_1(S^1, h(b_0))$ and $k_* \colon \pi_1(S^1, b_0) \to \pi_1(S^1, k(b_0))$ send the generator of $\pi_1(S^1, b_0)$, say γ, to

$$h_*(\gamma) = d \cdot \gamma(h(x_0)) \qquad \text{and} \qquad k_*(\gamma) = d \cdot \gamma(k(x_0)).$$

But since S^1 is a path-connected topological group (viewing S^1 as a subset of $\mathbf{C}$, multiplication is the standard multiplication on the complex numbers, where if $z_1, z_2 \in S^1$ then $z_1 z_2 \in S^1$ since

$\|z_1 z_2\| = 1$) we can define maps $\varphi_h, \varphi_k \colon S^1 \to S^1$ such that $\varphi_h(h(b_0)) = b_0$ and $\varphi_k(k(b_0)) = b_0$ (these are rotation maps/matrices and they can be easily constructed from the argument of $h(b_0)$, $k(b_0)$ etc.) so that the induced maps by the composition $(\varphi_h \circ h)_*, (\varphi_k \circ k)_* \colon \pi_1(S_1, b_0) \to \pi_1(S^1, b_0)$ have the same degree. By part (i), $\varphi_h \circ h \simeq \varphi_k \circ k$ so by part (ii), $h \simeq k$. ■

Problem 13.5 (B)

Let $q\colon S^2 \to P^2$ be the quotient map, where P^2 is the projective plane. Let $x_0 = q(1,0,0)$ and let

$$f(s) = q(\cos(\pi s), \sin(\pi s), 0)$$

for $0 \leq s \leq 1$. Then $f\colon I \to P^2$ is a loop at x_0. Prove that $[f] * [f] = [e_{x_0}]$.

Proof. Recall that the projective plane is constructed from the sphere by identifying antipodal points, i.e., $P^2 \approx S^2/{\sim}$ where $x \sim y$ if and only if $y = x$ or $y = -x$. We will show that $f * f \simeq_p e_{x_0}$ by constructing a homotopy between $f * f$ and e_{x_0}. Define $\tilde{f}\colon S^2 \to I$ by $\tilde{f}(s) \coloneqq (\cos(\pi s), \sin(\pi s), 0)$. Note that $q(\tilde{f}) = f$ and that, by the equivalence relation on S^2, $q(\tilde{f}) = q(-\tilde{f})$. We show that $f * f \simeq_p e_{x_0}$ via homotopy. ■

Problem 13.6 (C)

Let Y be the following subset of $\mathbf{R}^2$: $Y = \{ (s,t) \in I \times I \mid s \in \{0,1\} \text{ or } t \in \{0,1\} \}$ (that is, Y is the boundary of the square $I \times I$). Give Y the equivalence relation $\sim$ that identifies the top and the bottom edges and the left and the right edges: specifically, $\sim$ is the equivalence relation associated to the partition of Y into the following sets:

- for each $s \notin \{0,1\}$, the set $\{(s,0),(s,1)\}$,
- for each $t \notin \{0,1\}$, the set $\{(t,0),(t,1)\}$,
- the set $\{0,1\} \times \{0,1\}$.

Prove that $Y/{\sim}$ is a wedge of two circles.

Proof. Let $q\colon Y \to Y/{\sim}$ denote the quotient map. Recall from the definition at the top of Munkres §71, p. 434, that the wedge of two circles $S^1 \vee S^1$ is the union of two subspaces, say S_1 and S_2A, which are homeomorphic to S^1 such that $S_1 \cap S_2 = \{p\}$. Since translation in $\mathbf{R}^2$ is an isometry, in particular, a homeomorphism, we may as well assume that $p = (0,0)$ and that $S_1 = S^1 - (1,0)$ and $S_2 = S^1 + (1,0)$. The plan of attack is as follows, we define a continuous map $f\colon Y \to S^1 \vee S^1$ and show that the induced map on the quotient, $\bar{f}\colon Y/{\sim} \to S^1 \vee S^1$ is a homeomorphism. Define

$$f(s,t) := \begin{cases} (\cos(2\pi s), \sin(2\pi s)) - (1,0) & \text{if } t = 0 \\ (\cos(2\pi t + \pi), \sin(2\pi t + \pi)) + (1,0) & \text{if } s = 1 \\ (\cos(-2\pi s), \sin(-2\pi s)) - (1,0) & \text{if } t = 1 \\ (\cos(-2\pi t - \pi), \sin(2\pi t - \pi)) + (1,0) & \text{if } s = 0. \end{cases}$$

Note that these maps are continuous on their respective intervals since their projections are trigonometric functions plus constants, and addition is continuous in $\mathbf{R}$ by Theorem 25.1, hence the pieces are continuous by Theorem 18.4. Moreover, note that on the intersection, that is, on the points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ we have

$$\begin{aligned} (\cos(0), \sin(0)) - (1,0) &= (0,0) \\ &= (\cos(-\pi), \sin(-\pi)) + (1,0) \end{aligned} \qquad \begin{aligned} (\cos(2\pi), \sin(2\pi)) - (1,0) &= (0,0) \\ &= (\cos(\pi), \sin(\pi)) + (1,0) \end{aligned}$$

$$\begin{aligned} (\cos(2\pi + \pi), \sin(2\pi + \pi)) + (1,0) &= (0,0) \\ &= (\cos(-2\pi), \sin(-2\pi)) - (1,0) \end{aligned} \qquad \begin{aligned} (\cos(0), \sin(0)) &= \\ &= () \end{aligned}$$

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