MA 544: Homework 12

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Problem 12.1 (Wheeden & Zygmund §8, Ex. 2)

Prove the converse of Hölder's inequality for $p = 1$ and ∞. Show also that for $1 \leq p \leq \infty$, a real-valued measurable f belongs to $L^p(E)$ if $fg \in L^1(E)$ for every $g \in L^{p'}(E)$, $1/p + 1/p' = 1$. The negation is also of interest: if $f \in L^p(E)$ then there exists $g \in L^{p'}(E)$ such that $fg \notin L^1(E)$. (To verify the negation, construct g of the form $\sum a_k g_k$ satisfying $\int_E fg_k \to \infty$.)

Proof. In this problem, we finish the proof of Theorem 8.8 for the case $p = 1, \infty$. Therefore, we must show that:

> For f a measurable real-valued function on E and $p = 1, \infty$. Then
> $$\|f\|_p = \sup \int_E fg,$$
> where the supremum is taken over every real-valued g such that $\|g\|_{p'} \leq 1$ and $\int_E fg$ exists.

In both cases, $p = 1$ and $p = \infty$, we may, without loss of generality, assume $\|f\|_p \neq 0$; otherwise, by Hölder's inequality, $\|fg\|_1 \leq \|f\|_p \|g\|_{p'} = 0$ implies $\|fg\|_1 = 0$ so, by Theorem 5.11, $fg = 0$ almost everywhere on E and therefore, $f = 0$ almost everywhere on E.

Let us prove this for $p = 1$. Recall that, by convention, if $p = 1$ its conjugate exponent, p', is ∞ and vice versa. Suppose $\|g\|_\infty \leq 1$ and the integral $\int_E fg$ exists. One direction is trivial, namely, by Hölder's inequality

$$\int_E fg \leq \int_E |fg| \leq \|f\|_1 \|g\|_1 \leq \|f\|_1, \tag{12.1}$$

for all g with $\|g\|_\infty \leq 1$. Hence,

$$\sup \int_E fg \leq \|f\|_1.$$

To get the reverse inequality, consider $g := \operatorname{sgn} f$. The function g is measurable since $g = f/|f|$ for all $f(\mathbf{x}) \neq 0$ and $g = 0$ otherwise. Moreover, g is in $L^\infty(E)$ since $\|g\|_\infty \leq 1$, that is, $|g| \leq 1$ almost everywhere on E. Therefore

$$\|f\|_1 = \int_E |f| = \int_E fg \leq \sup_{\|g'\|_\infty \leq 1} \int_E fg'. \tag{12.2}$$

Thus, $\|f\|_1 = \sup \int fg$ where the supremum is taken over all $g \in L^\infty(E)$ with $\|g\| \leq 1$.

Now, consider the case where $p = \infty$. By Hölder's inequality, it is clear that

$$\sup \int_E fg \leq \|f\|_\infty \tag{12.3}$$

since $\int_E fg \leq \|f\|_\infty \|g\|_1$ for all $g \in L(E)$. To prove the reverse inequality, we consider the cases $\|f\|_\infty < \infty$ and $\|f\|_\infty = \infty$ separately.

Suppose $0 < \|f\|_\infty < \infty$; we may, without loss of generality, assume $\|f\|_\infty = 1$ by normalizing f by its essential supremum. Now, by definition

$$\|f\|_\infty = \inf\{\, \alpha : |\{\, \mathbf{x} \in E : f(\mathbf{x}) > \alpha \,\}| = 0 \,\} = 1. \tag{12.4}$$

Set $E_k \coloneqq \{\, \mathbf{x} \in E : f(\mathbf{x}) > 1 - 1/k \,\} \cap B(\mathbf{0}, k)$. Then $E_k \nearrow \bigcup E_k$ and $|E \setminus \bigcup E_k| = 0$ by Equation (12.4) and the definition of the essential supremum. Therefore, $\int_E fg = \int_{\bigcup E_k} fg$. Moreover, $|E_k| < |B(\mathbf{0}, k)| < \infty$ so we can define the sequence of functions

$$g_k(\mathbf{x}) \coloneqq \begin{cases} \dfrac{1}{|E_k|} & \text{if } x \in E_k \\ 0 & \text{otherwise} \end{cases}. \tag{12.5}$$

Note that $\|g_k\|_1 = 1$ and

$$\int_E f g_k = \int_{E_k} f g_k \geq \int_{E_k} \left(1 - \frac{1}{k}\right) g_k = \left(1 - \frac{1}{k}\right) \int_E g_k = 1 - \frac{1}{k}$$

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Problem 12.2 (Wheeden & Zygmund §8, Ex. 3)

Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when $p < 1$.

Proof. ∎

Problem 12.3 (Wheeden & Zygmund §8, Ex. 4)

Let f and g be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let $1 < p < \infty$. Prove that equality holds in the inequality $\left|\int fg\right| \le \|f\|_p \|g\|_{p'}$ if and only if fg has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e.

If $\|f + g\|_p = \|f\|_p + \|g\|_p$ and $g \neq 0$ in Minkowski's inequality, show that f is a multiple of g.

Find analogues of these results for the spaces ℓ^p.

Proof. ∎

Problem 12.4 (Wheeden & Zygmund §8, Ex. 5)

For $0 < p \leq \infty$ and $0 < |E| < \infty$, define

$$N_p[f] := \left(\frac{1}{E} \int_E |f|^p \right)^{1/p},$$

where $N_\infty[f]$ means $\|f\|_\infty$. Prove that if $p_1 < p_2$, then $N_{p_1}[f] \leq N_{p_2}[f]$. Prove also that if $1 \leq p \leq \infty$, then $N_p[f+g] \leq N_p[f] + N_p[g]$, $(1/|E|) \int_E |fg| \leq N_p[f] N_{p'}[g]$, $1/p + 1/p' = 1$, and $\lim_{p\to\infty} N_p[f] = \|f\|_\infty$. Thus, N_p behaves like $\| \cdot \|_p$ but has the advantage of being monotone in p. Recall Exercise 28 of Chapter 5.

Proof. ∎

Problem 12.5 (Wheeden & Zygmund §8, Ex. 6)

(a) Let $1 \leq p_i$, $r \leq \infty$ and $\sum_{i=1}^{k} 1/p_i = 1/r$. Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

(b) Let $1 \leq p < r < q \leq \infty$ and define $\theta \in (0,1)$ by $1/r = \theta/p + (1-\theta)/q$. Prove the interpolation estimate

$$\|f\|_r \leq \|f\|_p{}^{\theta} \|f\|_q{}^{1-\theta}.$$

In particular, if $A \coloneqq \max\{\|f\|_p, \|f\|_q\}$, then $\|f\|_r \leq A$.

Proof. (a) We will proceed by induction on k the number of measurable f_k whose p_k-norm is finite. When $k = 2$, by applying Hölder's inequality on $|fg|^r$ with $1/(p/r) + 1/(p'/r) = 1$ we have

$$\begin{aligned} \|fg\|_r{}^r &= \left(\int_E |fg|^r \right) \\ &\leq \left(\int_E |f|^{r(p/r)} \right)^{r/p} \left(\int_E |g|^{r(p'/r)} \right)^{r/p'} \\ &= \|f\|_p{}^r \|g\|_{p'}{}^r. \end{aligned}$$

Therefore,

$$\|fg\|_r \leq \|f\|_p \|g\|_{p'}. \tag{12.6}$$

Now, suppose Equation (12.6) holds for $j \leq k-1$ functions measurable functions $f_j \in L^{p_j}(E)$ where $\sum_j 1/p_j = r$. Suppose $\sum_{j=1}^{k} 1/p_j = 1/r$ with $f_j \in L^{p_j}(E)$ and consider

$$\|f_1 \cdots f_{k-1} f_k\|_r{}^r = \int_E |f_1 \cdots f_{k-1} f_k|^r = \int_E |f_1|^r \cdots |f_{k-1}|^r |f_k|^r.$$

(b) ■

Problem 12.6 (Wheeden & Zygmund §8, Ex. 9)

If f is real-valued and measurable on E, $|E| > 0$, define its essential infimum on E by

$$\operatorname{ess\,inf} f := \sup\{\, \alpha : |\{\, x \in E : f(x) < \alpha \,\}| = 0 \,\}.$$

If $f \geq 0$, show that $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup} 1/f)^{-1}$.

Proof. ∎

Problem 12.7 (Wheeden & Zygmund §8, Ex. 11)

If $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty < M$ for all k, prove that $f_k g_k \to fg$ in L^p.

Proof. ■