MA 519: Homework 6

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Problem 6.1 (Handout 8, # 2)

Identify the parameters n and p for each of the following binomial distributions:

(a) # boys in a family with 5 children;
(b) # correct answers in a multiple choice test if each question has a 5 alternatives, there are 25 questions, and the student is making guesses at random.

Solution. For part (a), the distribution is binomial with k being the number of children in a given family and p the probability that a child is born, say, male. In this case, we can reasonably assume that $p = 0.5$. Thus, the binomial distribution is given by $\mathrm{Binom}(5, 0.5)$.

For part (b), we use similar reasoning and we have $\mathrm{Binom}(25, 0.2)$ where $k = 25$ is the number of questions and $p = 1/5 = 0.2$ the probability of guessing a question correctly. ■

Problem 6.2 (Handout 8, # 10)

A newsboy purchases papers at 20 cents and sells them for 35 cents. He cannot return unsold papers. If the daily demand for papers is modeled as a Binom$(50, 0.5)$ random variable, what is the optimum number of papers the newsboy should purchase?

Solution. Let $X \sim \text{Binom}(50, 0.5)$ denote the daily demand for papers and n the number of copies bought by the newsboy. Then, the random variable $S = \min\{X, n\}$ denotes the number of copies actually sold by the newsboy. His daily profit is, therefore, measured by the random variable

$$Y = 0.35S - 0.25n.$$

Now let us compute the average sales of the newsboy. By the linearity of expected value, we have

$$\begin{aligned}
E(S) &= \sum_{k=0}^{n} kP(S = k) \\
&= \sum_{k=0}^{n-1} kP(\min\{X, n\} = k) + nP(\min\{X, n\} = n),
\end{aligned}$$

where $P(\min\{X, n\} = k) = P(X = k)$ the probability that there is a demand for k copies, and $P(\min\{X, n\} = n) = P(X \geq n)$ the probability that the demand exceeds the number of copies the newsboy bought, giving us

$$\begin{aligned}
&= \sum_{k=0}^{n-1} kP(X = k) + nP(X \geq n) \\
&= \sum_{k=0}^{n-1} kP(X = k) + n\big(1 - P(X < n)\big) \\
&= n + \sum_{k=0}^{n-1} kP(X = k) - nP(X < n) \\
&= n + \sum_{k=0}^{n-1} kP(X = k) - n\sum_{k=0}^{n-1} P(X = k) \\
&= n + \sum_{k=0}^{n-1} k\binom{n-1}{k} 0.5^k 0.5^{n-1-k} - n\sum_{k=0}^{n-1} \binom{n-1}{k} 0.5^k 0.5^{n-1-k} \\
&= n + \sum_{k=0}^{n-1} k\binom{n-1}{k} 0.5^{n-1} - n\sum_{k=0}^{n-1} \binom{n-1}{k} 0.5^{n-1} \\
&= n + (n-1)0.5 - n0.5^{n-1}2^{n-1} \\
&= \frac{n-1}{2}.
\end{aligned}$$

Thus,

$$E(Y) = 0.35E(S) - 0.25n = 0.175(n-1) - 0.25n =$$

■

Problem 6.3 (Handout 8, # 12)

How many independent bridge dealings are required in order for the probability of a preassigned player having four aces at least once to be 1/2 or better? Solve again for some player instead of a given one.

Solution. Without loss of generality, let us assume it is North that is the preassigned player. Let X be the number of dealings required in order for North to have four aces with probability greater than or equal to 1/2. ■

PROBLEM 6.4 (HANDOUT 8, # 13)

A book of 500 pages contains 500 misprints. Estimate the chances that a given page contains at least three misprints.

SOLUTION. Let X be the number of misprints on the given page. The probability that a given misprint is on that page is 1/500. Now, $P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$. Also,

$$\begin{aligned}
P(X = 0) &= \left(\frac{499}{500}\right)^{500} \\
P(X = 1) &= 500\left(\frac{1}{500}\right)\left(\frac{499}{500}\right)^{499} \\
P(X = 2) &= \frac{500 \cdot 499}{2}\left(\frac{1}{500}\right)^{2}\left(\frac{499}{500}\right)^{498}
\end{aligned}$$

So that

$$\begin{aligned}
P(X \geq 3) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\
&= 1 - \left(\frac{499}{500}\right)^{500} - 500\left(\frac{1}{500}\right)\left(\frac{499}{500}\right)^{499} - \frac{500 \cdot 499}{2}\left(\frac{1}{500}\right)^{2}\left(\frac{499}{500}\right)^{498} \\
&\approx 0.08
\end{aligned}$$

that is, the probability that the given page has at least 3 misprints is about 8 percent. ■

Problem 6.5 (Handout 8, # 14)

Colorblindness appears in 1 per cent of the people in a certain population. How large must a random sample (with replacements) be if the probability of its containing a colorblind person is to be 0.95 or more?

Solution. Let n be the sample size. The probability of the sample containing no colorblind people is 0.99^n. Solving the equation $0.99^n = 0.05$, we see that (for the naturals) taking $n = 299$ is (minimally) sufficient for 0.99^n to be less than 0.05.

The probability that the sample has some colorblind person is equal to $1 - 0.99^n$. This is at least 95 percent if 0.99^n is less than 0.05. That is, having 299 people in the sample is (minimally) sufficient for there to be a 95 percent chance of having some colorblind person. ■

Problem 6.6 (Handout 8, # 15)

Two people toss a true coin n times each. Find the probability that they will score the same number of heads.

Solution. ■

Problem 6.7 (Handout 8, # 16)

Binomial approximation to the hypergeometric distribution. A population of TV elements is divided into red and black elements in the proportion $p : q$ (where $p + q = 1$). A sample of size n is taken without replacement. The probability that it contains exactly k red elements is given by the hypergeometric distribution of II, 6. Show that as $n \to \infty$ this probability approaches $\text{Binom}(n, p)$.

Solution. ■

Problem 6.8 (Handout 9, # 3)

Suppose X, Y, Z are mutually independent random variables, and $E(X) = 0$, $E(Y) = -1$, $E(Z) = 1$, $E(X^2) = 4$, $E(Y^2) = 3$, $E(Z^2) = 10$. Find the variance and the second moment of $2Z - Y/2 + eX$, where e is the number such that $\ln e = 1$.

Solution. Let $W = 2Z - Y/2 + eX$. Then

$$\begin{aligned}
E(W^2) &= E((2Z - Y/2 + eX)^2) \\
&= E(4Z^2 - 2ZY + 4eZX + Y^2/4 - eYX + e^2X^2) \\
&= 4E(Z^2) - 2E(Z)E(Y) + 4eE(Z)E(X) + E(Y^2)/4 - eE(Y)E(X) + e^2E(X^2) \\
&= 4 \cdot 10 + 2 + 3/4 + e^2 \cdot 4 \\
&= \frac{171}{4} + 4e^2 \\
&\approx 72.31
\end{aligned}$$

and

$$\begin{aligned}
Var(W) &= E(W^2) - E(W)^2 \\
&= \frac{171}{4} + 4e^2 - (2E(Z) + E(Y)/2 + eE(X))^2 \\
&= \frac{171}{4} + 4e^2 - (2 - 1/2)^2 \\
&= \frac{171}{4} + 4e^2 - (3/2)^2 \\
&= \frac{81}{2} + 4e^2 \\
&\approx 70.06
\end{aligned}$$

That is, the second moment is “about 72.31” and the variance is “about 70.06”

■

Problem 6.9 (Handout 9, # 14)

(*Variance of Product*). Suppose X, Y are independent random variables. Can it ever be true that $\operatorname{Var}(XY) = \operatorname{Var}(X)\operatorname{Var}(Y)$?
If it can, when?

Solution. ∎