Conrad's Notes

Carlos Salinas

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Contents

Chapter 1

Algebraic geometry

1.1 The Nötherian condition

Let R be a commutative ring. In class we saw that the ascending chain condition on ideals of R is equivalent to the condition that all ideals in R are finitely generated. The aim of this handout is to prove the equivalence of these two conditions with yet another "finiteness" condition, on the module theory of R:

Theorem 1.1. *If all ideals of R are finitely generated then every submodule of a finitely generated R-module is also finitely generated.*

In class we noted that the converse of this theorem is true (and elementary), since ideals of R are precisely submodules of the module $M = R \cdot 1$. The subtlety with the implication in the theorem is that if we try to induct on the number of generators of M then we run into the problem that this does *not* control the number of generators needed for submodules of M, in contrast with the case of p.i.d.s. (For example, the ideal (X, Y) in the ring $R = k[X, Y]$ does not have a single generator as an R-module, even though $M = R$ does have a single generator.) For this reason, we will not argue by induction on the number of generators. We will argue by induction in another way.

Proof. Let M be a finitely generated R-module, say with n generators. That is, there is a surjective map $q\colon R^n \twoheadrightarrow M$. Hence, any submodule $M' \subseteq M$ is the image of a submodule $q^{-1}(M') \subseteq R^n$, so to show that M' is finitely generated it suffices to prove that $q^{-1}(M')$ is finitely generated. In other words, we are reduced to proving finite generation for submodules of each R^n. *This* we will prove by induction on n. The case $n = 1$ is precisely our hypothesis that all ideals of R are finitely generated!

Now assume $n > 1$ and that the case $n - 1$ is known.

$$(r_1, \ldots, r_{n-1}) \longmapsto (r_1, \ldots, r_{n-1}, 0)$$

so $R^n/(R^{n-1}) = R$ via projection to the n-th coordinate. For a submodule $N \subseteq R^n$, we get a submodule $N' = N \cap R^{n-1} \subseteq R^{n-1}$ and a quotient

$$N'' = \frac{N}{N'} \subseteq \frac{R^n}{R^{n-1}} \simeq R.$$

Thus, by induction N' is finitely generated, and byt the base case N'' is finitely generated (we have even identified with an ideal of R), so we have reached the following situation: we have a module N over R and a finitely generated submodule N' such that the quotient N/N' is also finitely generated. Then we claim that N is finitely generated.

Explicitly, suppose $\{e'_1, \ldots, e'_m\}$ is a generating set of N', and let $\{e_1, \ldots, e_r\}$ be a subset of N that lifts a generating set of N/N'. Then $\{e'_1, \ldots, e'_m, e_1, \ldots, e_r\}$ is a generating set of N. ■

If you think about it, this method of proof is essentially constructs a finite chain of submodules that the successive quotients in the chain are each identified with either an ideal of R or the quotient of R by an ideal. If you look back at how one proves the existence of bases of finite-dimensional vector spaces over a field or the structure theorem for modules over a p.i.d., you'll see that those earlier arguments are essentially special cases of the kind of analysis we have just carried out (except that in those cases we get more prcise structure theorems for the modules, so we have to do some more work which is not relevant over general Nötherian R).

1.2 Transendence

Transendence bases

Let K be an extension field of a field k. Let S be a subset of K. We recall that S (or the elements of S) are said to be algebraically independent over k, if whenever we have arelation

$$0 = \sum a_{(v)} M_{(v)}(S) = \sum a_{(v)} \prod_{x \in S} X^{v(X)}$$

with coefficients $a_{(v)} \in k$, almost all $a_{(v)} = 0$, then we must necessarily have $a_{(v)} = 0$.

We can introduce an ordering among algebraically independent subsets of K, by ascending inclusion. These subsets are obviously inductively ordered, and thus there exists maximal elements. If S is a subset of K which is algebraically independent over k, and if the cardinality of S is greatest among all such subsets, then we call this cardinality the *transendence degree or dimension* of K over k. Actually, we shall need to distinguish only between finite transendence degree or infinite transendence degree. We observe that the notion of transendence degree bears to the notion of algebraic independence the same relation as the notion of dimension bears to the notion of independence.

We frequently deal with familiar elements of K, say a family $\{x_i : i \in I\}$, and say that such a family is algebraically independent over k if its elements are distinct (in other words, $x_i \neq x_j$ if $i \neq j$) and if the set consisting of the elements in this family is algebraically independent over k.

A subset S of K which is algebraically independent over k and is maximal with respect to the inclusion ordering will be called a *transendence base* of K over k. From the maximality, it is clear that if S is a transendence base of K over k, then K is algebraic over $k(S)$.

Chapter 2

Differential Geometry

Interior, closure and boundary

We wish to develop some basic geometric concepts in metric spaces which make precise intuitive ideas centered on the themes of "interior" and "boundary" of a subset of a metric space. One warning must be given. Although there are a number of results proven in this handout, none of it is particularly deep. If you carefully study the proofs, then you'll see that none of this requires going much beyond the basic definitions. We will certainly encounter some serious ideas and nontrivial proofs in due course, but at this point the central aim is to acquire some linguistic ability when discussing some basic geometric ideas in a metric space. Thus, the main goal is to familiarize ourselves with some very convenient geometric terminology in terms of which we can discuss more sophisticated ideas later on.

Interior and closure

Let X be a metric space and $A \subseteq X$ a subset. We define the *interior* of A to be the set

$$A^\circ = \operatorname{Int} A = \left\{ a \in A : \text{there exists } r > 0 \text{ such that } B(a,) \subseteq A \right\}$$

consisting of points for which A is a "neighborhood". We define the *closure* of A to be the set

$$\bar{A} = \operatorname{Cls} A = \left\{ x \in X : x = \lim_{n\to\infty} a_n \text{ with } a_n \in A \text{ for all } n \in \mathbb{N} \right\}.$$

In words, the interior consists of points in A for which all nearby points of X are also in A whereas the closure allows for "points on the edge of A". Note that obviously

$$A^\circ \subseteq \bar{A}.$$

We will see shortly (after some examples) that A° is the largest open set inside of A — that is, it is open and contains any open lying inside of A (so in fact A is open if and only if $A = A^\circ$) — while $\bar{A}$ is the smallest closed set containing A; i.e., $\bar{A}$ is closed and lies inside any closed set containing A (so in fact A is closed if and only if $\bar{A} = A$).

Beware that we have to prove that the closure is actually closed! Just because we call something the "closure" does not mean the concept is automatically endowed with linguistically similarly sounding properties. The proof won't be particularly deep, as we'll see.

Example 2.1. Let's work out the interior and closure of the "half-open" square

$$A = \left\{ (x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 < y < 1 \right\} = [-1, 1] \times (-1, 1)$$

inside the metric space $X = \mathbb{R}^2$ (the phrase "half-open" is purely intuitive; it has no precise meaning, but the picture should make it clear why we use this terminology). Intuitively, this is a square region whose horizontal edges are "left out". The interior of A should be $(-1, 1) \times (-1, 1)$ and the closure should be $[-1, 1] \times [-1, 1]$, as drawing a picture should convince you. Of course, we want to see that such conclusions really do follow from our precise definitions.

First we check that A° is correctly described. If $-1 < x < 1$ and $-1 < y < 1$ then for

$$r = \min\left\{|-1-x|, |1-x|, |-1-y|, |1-y|\right\} > 0$$

it is easy to check that $B\big((x, y), r\big) \subseteq (-1, 1) \times (-1, 1)$ (since a square box with side-length r contains the disc of radius r with the same center). Thus, $(-1, 1) \times (-1, 1) \subseteq A$ is an open subset of $X = \mathbb{R}^2$. To check it is the full interior of A, we just have to show that the "missing points" of the form $\pm 1, y$ do not lie in the interior. But for any such point $p = (\pm 1, y) \in A$, for any positive small $r > 0$ there is always a point in $B(p, r)$ with the same y-coordinate but with the x-coordinate either slightly larger than 1 or slightly less than -1. Such a point is not in A. Thus, $p \notin A^\circ$.

Now we check that $\bar{A} = [-1, 1] \times [-1, 1]$. Since convergence in $\mathbb{R}^2$ forces convergence in coordinates, to see

$$\bar{A} \subseteq [-1, 1] \times [-1, 1]$$

it suffices to check that $[-1, 1]$ is closed in $\mathbb{R}$ (since certainly $A \subseteq [-1, 1] \times [-1, 1]$). But this is clear (either by using sequences or by explicitly showing its complement in $\mathbb{R}$ to be open). To see that $\bar{A}$ fills up all of $[-1, 1] \times [-1, 1]$, we have to show that each point in $[-1, 1] \times [-1, 1]$ can be obtained as a limit of a sequence in A. We just have to deal with points not in $A = [-1, 1] \times (-1, 1)$ since points in A are limits of constant sequences. That is, we're faced with studying points of the form $(x, \pm 1)$ with $x \in [-1, 1]$. Such a point is a limit of a sequence (x, q_n) with $q_n \in (-1, 1)$ having limit ± 1.

Example 2.2. What happens if we work with the same set A but view it inside of a metric space $X = A$ (with the Euclidean metric)? In this case, $A^\circ = A$ and $\bar{A} = A$! Indeed, quite generally for any metric space X we have $X^\circ = X$ and $\bar{X} = X$. These are easy consequences of thee definitions. Likewise, the empty subset $\emptyset$ in any metric space has interior and closure equal to the subset $\emptyset$.

The moral is that one has to always keep in mind what ambient metric space one is working in when forming interiors and closures! One could imagine that perhaps our notation for interior and closure should somehow incorporate a designation of the ambient metric space. But just as we freely use the same symbols "+" and "0" to denote the addition and additive identity in any vector space, even when working with several spaces at once, it would simply make life too cumbersome (and the notation too cluttered) to always write things like $\mathrm{Int}_X A$ or $\mathrm{Cls}_X A$. One just has to pay careful attention to what is going on so as to keep track of the ambient metric space with respect to which one is forming interiors and closures. The context will usually make it obvious what one is using as the ambient metric space, though if considering several ambient spaces at once it is sometimes helpful to use more precise notation such as $\mathrm{Int}_X A$.

Theorem 2.1. *Let A be a subset of a metric space X. Then A° is open and is the largest open set of X inside of A (i.e., it contains all others).*

Proof. We first show that A° is open. By definition, if $x \in A^\circ$, then some $B(x, r) \subseteq A$. But then since $B(x, r)$ is itsefl an open set we see that any $y \in B(x, r)$ has some $B(y, s) \subseteq B(x, r) \subseteq A$, which forces $y \in A^\circ$. That is, we have shown $B(x, r) \subseteq A^\circ$, whence A° is open.

If $U \subseteq A$ is an open set in X, then for each $u \in U$ there is some $r > 0$ such that $B(u, r) \subseteq U$ whence $B(u, r) \subseteq A$, so $u \in A^\circ$. This is true for all $u \in U$, so $U \subseteq A^\circ$. ■

Corollary 2.2. *A subset A in a metric space X is open if and only if $A = A^\circ$.*

Proof. By the theorem, A° is the unique largest open subset of X contained in A. But obviously A is open if anf only if such a unique maximal open subset of X lying in A is actually equal to A. This establishes the corollary. ■

We next want to show that the closure of a subset A in X is related to closed subsets of X containing A in a manner very similar to the way in which the interior of A is related to open subsets of X which lie inside of A. This goes along with the general idea that openness and closedness are "complementary" points to view (recall that a subset S in a metric space X is open (resp. closed) if and only if its complement $X \smallsetminus S$ is closed (resp. open)). It is actually more convenient for us to first show that closures and interiors have complementary relationship, and to then use this to deduce our desired properties of closure from already-established properties of interior.

Theorem 2.3. *Let A be a subset of a metric space X. Then $X \smallsetminus \bar{A} = (X \smallsetminus A)^\circ$ and $X \smallsetminus A^\circ = \overline{X \smallsetminus A}$.*

Before proving this theorem, we illustrate with an example. Consider $X = \mathbb{R}^2$ with the usual metric, and let $A = [-1, 1] \times (-1, 1)$ be the "half-open" square as considered above. By drawing pictures of $X \smallsetminus A$ and of the complements of $\bar{A}$ and A°, you should convince yourself intuitively that the assertions in this theorem make sense in this case.

Now we prove Theorem 2.3.

Proof. We begin by proving $X \smallsetminus \bar{A} = (X \smallsetminus A)^\circ$. If $x \in X$ is not in $\bar{A}$, there must exist some $B(x, 1/2^n)$ not meeting A, for otherwise we'd have some $x_n \in B(x, 1/2^n) \cap A$ for all $n \in \mathbb{N}$, so clearly $x_n \to x$, contrary to the fact that $x \notin \bar{A}$ is not a limit of a sequence of elements of A. This shows

$$X \smallsetminus \bar{A} \subseteq (X \smallsetminus A)^\circ.$$

Conversely, if x is in the interior of $X \smallsetminus A$ then some $B(x, r)$ lies in $X \smallsetminus A$ and hence is disjoint from A. It follows that no sequnece in A can possibly converge to x because for $\varepsilon = r > 0$ we'd run into problems (i.e., there's nothing in A within a distance of less that ε from x, since $B(x, \varepsilon) \subseteq X \smallsetminus A$).

Applying the *general* equality

$$X \smallsetminus \bar{A} = (X \smallsetminus A)^\circ$$

for arbitrary subsets A to X to the subset $X \smallsetminus A$ in the role of A, we get

$$X \smallsetminus \overline{X \smallsetminus A} = A^\circ.$$

Taking complements of both sides within X yields

$$\overline{X \smallsetminus A} = X \smallsetminus A^\circ,$$

as desired. ■

Corollary 2.4. *Let A be a subset of a metric space X. Then $\bar{A}$ is closed and is contained inside of any closed subset of X which contains A.*

Proof. Since the complement of $\bar{A}$ is equal to $(X \smallsetminus A)^\circ$, which we know to be open, it follows that $\bar{A}$ is closed. If Z is any closed set containing A, we want to prove that Z contains $\bar{A}$ (so $\bar{A}$ is "minimal" among closed sets containing A). But this is clear for several reasons. On the one hand, by definition every point $x \in \bar{A}$ is the limit of a sequnece of elements in $A \subseteq Z$, so by closedness of Z such limit points x are also in Z. This shows $\bar{A} \subseteq Z$. On the other hand, one can argue by noting that passage to complement takes Z to an open set $X \smallsetminus Z$ contained inside of $X \smallsetminus A$, so by maximality this open $X \smallsetminus Z$ must lie inside the interior of $X \smallsetminus A$, which we have seen is the complement $X \smallsetminus \bar{A}$ of $\bar{A}$. Passage back to complements then gives

$$\bar{A} = X \smallsetminus (X \smallsetminus \bar{A}) = X \smallsetminus (X \smallsetminus A)^\circ \subseteq X \smallsetminus (X \smallsetminus Z) \subseteq Z$$

as desired. ∎

Corollary 2.5. *For subsets $A_1, \ldots, A_n$ in a metric space X, the closure of $A_1 \cup \cdots \cup A_n$ is equal to $\bigcup_{i=1}^n \bar{A}_i$; that is, the formation of a finite union commutes with the formation of a closure.*

Proof. A closed set Z contains $\bigcup_{i=1}^n A_i$ if and only if it contains each A_i, and so if and only if it contains $\bar{A}_i$ for every i. Since $\bigcup_{i=1}^n \bar{A}_i$ is a finite union of closed sets, it is closed. We conclude that this closed set is minimal among all closed sets containing $\bigcup_{i=1}^n A_i$, so it is the closure of $\bigcup_{i=1}^n A_i$. ∎

Further aspects of interior and closure

The "interior" and "closure" constructions have been seen to be well-behaved with respect to the formation of complements within a metric space. However, these notions are not well-behaved with respect intersections within a metric space. Also, one cannot capture the closure of a set just from knowing its interior. For example, a set can have empty interior and yet the closure equal to the whole space: think about the subset $\mathbb{Q}$ in $\mathbb{R}$.

Here is one mildly positive result.

Theorem 2.6. *The formation of closures is local in the sense that if U is open in a metric space X and A is an arbitrary subset of X, then the closure of $A \cap U$ in X meets U in $\bar{A} \cap U$ (where $\bar{A}$ denotes the closure of A in X). In particular, if Z is closed in X then $U \cap \overline{Z \cap U} = Z \cap U$.*

Also if U is the interior of a closed set Z in X, then $\bar{U}^\circ = U$.

After proving the theorem, we'll present an interesting example of an open subset of a metric space which is *not* equal to the interior of its closure (and hence, by the second part of the theorem, cannot be expressed as the interior of any closed set at all). It is probably not immediately obvious to you how to find such open sets, since typical open sets one writes down in $\mathbb{R}$ or $\mathbb{R}^2$ tend to be the interior of their closures.

Proof. Since $\bar{A} \cap U$ is a closed set in U that contains $A \cap U$, for the first part of the theorem we need to prove that every point $x \in \bar{A} \cap U$ is a limit of a sequence of points $x_n \in A \cap U$. Since $x \in \bar{A}$ we can write $x = \lim_{n\to\infty} x_n$ with $x_n \in A$. By hypothesis $x \in U$, so by the openness of U we must have some $B(x, r) \subseteq U$, and so since $x_n \to x$ by considering just sufficiently large n we have $x_n \in U$. Thus, for large n the sequence $\{x_n\}$ lies in $A \cap U$ and converges to x.

Now we assume that U is the interior of a closed set Z and we wish to prove U is the interior of $\bar{U}$. Since Z is a closed set containing U, it also contains the closure of U, and by openness of U the open subset U inside of $\bar{U}$ must lie inside the interior of $\bar{U}$. To summarize we have

$$U \subseteq \bar{U}^\circ \subseteq Z^\circ = U,$$

so equality is forced throughout. ∎

Let's give a counterexample to the equality $\bar{U}^{\circ} = U$ if one only requires U to be an open subset of X (rather than even the interior of a closed set). The basic problem is that the closure of U can be quite a lot bigger than U. In fact, we'll find a rather "small" open subset $U \subseteq \mathbb{R}$ with closure equal to $\mathbb{R}$ (whose interior is $\mathbb{R}$, and hence larger than U).

Let $S \subseteq \mathbb{Q}$ denote the set of elements of the form $q = a/10^n$ with $a \in \mathbb{Z}$ and $n \geq 0$ (i.e., finite decimal expansions). We define $n(q) \geq 0$ to be the exponent of 10 in the denominator of q. In words, the base 10 decimal expansion of $q \in S$ is finite and (if $q \notin 10\mathbb{Z}$) begins on the right with a nonzero digit in the $10^{-n(q)}$-slot. Define U to be the union of the intervals $B_{1/10^{n(q)+2}}(q)$ for $q \in S$. This union is certainly open, as it is the union of open intervals. This union U is certainly open, as it is the union of open intervals. Try to draw a picture where U meets $[0, 1]$; it's pretty big though, but after working out a bunch of intervals in U you'll get a sense for what U looks like: it's very "sparse", yet somehow all over the place since certainly $S \subseteq U$. The problem is that "most" elements of S have pretty big denominators, and the tiny interval from U surrounding a choice of $q \in S$ is *really* tiny (depending on how big the denominator of q is).

Since $q \in S$ has a decimal expansion which terminates at the $n(q)$th digit past the decimal point, all points in $B_{1/10^{n(q)+2}}(q)$ have a $10^{-n(q)-1}$th digit equal to either 0 or 9 (think about $.253 \pm .00000998$), but not $2, \ldots, 8$. In particular, if we consider real numbers whose fractional parts consist *entirely* of digits from 2 to 8, such numbers cannot lie in U. Actually, a lot more can't lie in U (as your picture should convince you), but one needs measure theory to give a precise description of how sparse U is. In any case, we have at least shown that U is a proper open subset of $\mathbb{R}$. But every real number is a limit of a sequence in $S \subseteq U$, so the closure of U is equal to $\mathbb{R}$.

Let us conclude with considerations related to the local nature of closedness.

Theorem 2.7. *Let X be a metric space, and $A \subseteq X$ a subset. Let $\{U_i\}$ be an open cover of X. The set A is closed in X if and only if the subset $A \cap U_i$ is closed in U_i for all i.*

Proof. Replacing A with $X \setminus A$, it is equivalent to say that X is open in X if and only if $A \cap U_i$ is open in U_i for all i. However, a subset of U_i is open in U_i if and only if it is open in X (as U_i is open in X), so it is equivalent to say that A is open in X if and only if $A \cap U_i$ if open in X for all i. This implication $\Longrightarrow$ is clear, and the converse follows from the observation that A is the union of the overlaps $A \cap U_i$. ■

Note that it is crucial in the preceding theorem that the U_i cover all of X, and not just A. For example, if $A = [0, 1)$ in $X = \mathbb{R}$ and we take $U = (-\infty, 1)$ and $V = (1, \infty)$ then the open sets U and V barely fail to cover X ($U \cup V = X \setminus \{1\}$), and although $U \cap A = A$ is closed in U and $V \cap A = \emptyset$ is closed in V, clearly A is not closed in X. Sometimes when we are trying to analyze the geometry of A rather than locally near arbitrary points of X. In such cases we clearly cannot hope to prove that A is closed in X, and so the best we can hope to do is verify the conditions in the following definitions:

Definition 2.1. A subset A in a metric space X is *locally closed* if for all $a \in A$ there exists an open set $U_a \subseteq X$ containing a such that $U_a \cap A$ is closed in U_a.

The point of this definition is that the union of the U_a may fail to equal X, though it does contain A. As an example, $A = [0, 1)$ is locally closed in $X = \mathbb{R}$: for every $a \in A$ distinct from 1, we can take $U_a = (0, 1)$ and for $a = 1$ we can take $U_a = (-1, 1)$. More interesting examples are $(-1, 1) \times \{0\}$ in $\mathbb{R}^2$ (using $U_a = (-1, 1) \times \mathbb{R}$ for all a) and any open subset U of a topological space X (taking $U_a = U$ for all U).

The reason for interest in locally closed sets is that they naturally arise when trying to prove closedness of A in X in situations where one is only able to study the situation locally near elements of A. The point worth noting is that locally closed sets look closed if we replace the ambient set with a suitable open around A:

Theorem 2.8. *Let A be a subset of a metric space X. The subset A is locally closed in X if and only if there exists an open set $U \subseteq X$ containing A with A a closed subset of U; in other words, $A = C \cap U$ for a closed subset $C \subseteq X$.*

Proof. If $A = C \cap U$ for closed C in X and open U in X then we verify the definition of local closedness by taking $U_a = U$ for all $a \in A$. Conversely, if A is locally closed in X then let $U = \bigcup_{a \in A} U_a$ where $U_a \subseteq X$ is an open set containing $a \in A$ such that $A \cap U_a$ is closed in U_a. Clearly A is a subset of the metric space U and the U_a continue an open covering U. Thus, the local nature of closedness in metric spaces (applied to U) implies that A is closed in U. ■

Boundary

We now introduce a notion that sits somewhere between the closure and the interior: the boundary.

Definition 2.2. Let A be a subset of a metric space X. We define the *boundary* ∂A of A to be $\bar{A} \setminus A^\circ$.

As with the concepts of interior and closure, the boundary depends on the ambient space (though we suppress this in the notaton).

Example 2.3. If $A = [-1, 1] \times (-1, 1)$ inside $X = \mathbb{R}^2$, then $\partial A = \bar{A} \setminus A^\circ$ consists of points (x, y) on the edge of the unit square: it is equal to

$$\big(\{-1, 1\} \times [-1, 1]\big) \cup \big([-1, 1] \times \{-1, 1\}\big),$$

as you can check.

Example 2.4. Consider the subset $A = \mathbb{Q} \subseteq \mathbb{R}$. We have $A^\circ = \emptyset$ because no nonempty open interval can fail to contain irrationals (i.e., to be contained inside of $A = \mathbb{Q}$), and $\bar{A} = \mathbb{R}$ since every real number is a limit of a sequence of rationals. Thus, in this case $\partial A = \mathbb{R}$.

Just as it is geometrically reasonable that an open subset of a metric space is one which is equal to its interior, a closed subset ought to be exactly one which is equal to its own interior, a closed subset ought to be exactly one which contains its boundary. This is the first part of:

Theorem 2.9. *Let A be a subset of a metric space X. Then A is closed if and only if it contains ∂A, and in general*

$$\partial A = \bar{A} \cap \overline{X \setminus A} = \partial(X \setminus A).$$

If on again considers our friend the half-open square $A = [-1, 1] \times (-1, 1)$ in $\mathbb{R}^2$, it is instructive to recall our earlier determinations of the closures of A and $X \setminus A$ and to see that, sure enough, their intersection is just what the boundary ought to be.

Proof. The boundary ∂A is *defined* as $\bar{A} \setminus A^\circ$. Thus,

$$\bar{A} = A^\circ \cup \partial A \subseteq A \cup \partial A,$$

so when $\partial A \subseteq A$ we get $\bar{A} \subseteq A$ and therefore (the reverse inclusion being obvious) that $A = \bar{A}$, so A is indeed closed. Conversely, if A is closed then since $\partial A \subseteq \bar{A}$ by definition and $\bar{A} = A$ for closed A we get $\partial A \subseteq A$.

One we establish that $\partial A = \bar{A} \cap \overline{X \smallsetminus A}$, then since the right hand side is unaffected by replacing A with $X \smallsetminus A$ everywhere (because $X \smallsetminus (X \smallsetminus A) = A$), it follows that $\partial A = \partial(X \smallsetminus A)$. As for verifying that ∂A is the intersection of the closures of A and $X \smallsetminus A$, we use the definition of ∂A to rewrite this as:

$$\bar{A} \smallsetminus A^\circ = \bar{A} \cap \overline{X \smallsetminus A}.$$

Since $\bar{A} \smallsetminus A^\circ = \bar{A} \cap (X \smallsetminus A^\circ)$, it suffices to check that

$$X \smallsetminus A^\circ = \overline{X \smallsetminus A}.$$

But this was one of the "complementary" relationships proved earlier between interiors and closures. ■

We conclude with a geometrically pleasing corollary.

Corollary 2.10. *Let A be a subset of a metric space X. Then X can be expressed as a disjoint union*

$$A = A^\circ \cup \partial A \cup (X \smallsetminus A)^\circ.$$

In other words, every point of X satisfies exactly one of the following properties: it is the interior to A, interior to $X \smallsetminus A$, or on the common boundary $\partial A = \partial(X \smallsetminus A)$.

The disjointness in this corollary "justifies" the idea that $\partial A = \partial(X \smallsetminus A)$ is sort of a "common interface" between A and $X \smallsetminus A$. For ugly subsets $A \subseteq X$ one can't take this intuition too seriously.

Proof. Since $X \smallsetminus A^\circ = \overline{X \smallsetminus A}$ by an earlier theorem, the assertion of the corollary is exactly the statement

$$\overline{X \smallsetminus A} = \partial A \cup (X \smallsetminus A)^\circ$$

with ∂A disjoint from $(X \smallsetminus A)^\circ$. But by definition of the boundary for $X \smallsetminus A$ we have a *disjoint* union decomposition

$$\overline{X \smallsetminus A} = \partial(X \smallsetminus A) \cup (X \smallsetminus A)^\circ.$$

This, it suffices to show $\partial A = \partial(X \smallsetminus A)$. But this latter equality was shown in the preceding theorem. ■

2.1 Product topology

The aim of this handout is to address two points: metrizability of finite products of metric spaces, and the abstract characterization of the product topology in terms of the universal mapping properties among topological spaces. This latter issue is related to explaining why the definition of the product topology is not merely *ad hoc* but in a sense the "right" definition. In particular, when you study topology more systematically and encounter the problem of topologizing infinite products of topological spaces, if you think in terms of the universal property to be discussed below then you will be inexorably led to the right definition of the product topology for a product of infinitely many topological spaces (it is not what one would naively expect to be, based on experience with the case of finite products).

Metrics of finite products

Let $X_1, \ldots, X_d$ be metrizable topological spaces. The product set

$$X = \prod_{i=1}^{d} X_i$$

admits a natural product topology. It is natural to ask if, upon choosing metrics ρ_j inducing the given topology on each X_j, we can define a metric ρ on X in terms of ρ_j such that ρ induces the product topology on X. The basic idea is to find a metric which describes the idea of "coordinate-wise closedness", but several natural candidates leap out, none of which are evidently better than others

$$\begin{aligned}
\rho^{\max}\big((x_1, \ldots, x_d), (x_1', \ldots, x_d')\big) &= \max_{1 \le j \le d} \rho_j(x_j, x_j') \\
\rho^{\mathrm{Euc}}\big((x_1, \ldots, x_d), (x_1', \ldots, x_d')\big) &= \sqrt{\sum_{i=1}^{d} \rho_j(x_j, x_j')^2} \\
\rho_1\big((x_1, \ldots, x_d), (x_1', \ldots, x_d')\big) &= \sum_{i=1}^{d} \rho_j(x_j, x_j') \\
\rho_p\big((x_1, \ldots, x_d), (x_1', \ldots, x_d')\big) &= \left[\sum_{j=1}^{d} \rho_j(x_j, x_j')^p\right]^{\frac{1}{p}}, \qquad p \ge 1.
\end{aligned}$$

When $X_j = \mathbb{R}$ for all j, with ρ_j the usual absolute value metric, these recover the various concrete norms we've seen on $X = \mathbb{R}^d$. Our first aim will be to show that all of these rather different-looking metrics are at least bounded above and below by a positive multiple of each other (which is the best we can expect, since they sure aren't literally the same), and so in particular they all define the same topology. In fact, we will see that the common topology they define is the product topology.

We first axiomatize the preceding examples. Let $N\colon \mathbb{R}^d \to \mathbb{R}$ be any norm which satisfies the property that on the orthant $[0, \infty)^d$ with nonnegative coordinates it is monotonically increasing function in each individual coordinate when all others are held fixed. Examples such as N include old friends

$$\|-\|_{\max}, \qquad \|-\|_{\mathrm{Euc}}, \qquad \|-\|_1, \qquad \|-\|_p, \quad p \ge 1$$

where we recall that

$$\|(a_1, \ldots, a_n)\|_p = \left[\sum_{j=1}^{d} |a_j|^p\right]^{\frac{1}{p}}.$$