

MA 519: Homework 6

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PROBLEM 6.1 (HANDOUT 8, # 2)

Identify the parameters n and p for each of the following binomial distributions:

- (a) # boys in a family with 5 children;
- (b) # correct answers in a multiple choice test if each question has a 5 alternatives, there are 25 questions, and the student is making guesses at random.

SOLUTION. For part (a), the distribution is binomial with k being the number of children in a given family and p the probability that a child is born, say, male. In this case, we can reasonably assume that $p = 0.5$. Thus, the binomial distribution is given by $\text{Binom}(5, 0.5)$.

For part (b), we use similar reasoning and we have $\text{Binom}(25, 0.2)$ where $k = 25$ is the number of questions and $p = 1/5 = 0.2$ the probability of guessing a question correctly. ■

PROBLEM 6.2 (HANDOUT 8, # 10)

A newsboy purchases papers at 20 cents and sells them for 35 cents. He cannot return unsold papers. If the daily demand for papers is modeled as a $\text{Binom}(50, 0.5)$ random variable, what is the optimum number of papers the newsboy should purchase?

SOLUTION. Let $X \sim \text{Binom}(50, 0.5)$ denote the daily demand for papers and n the number of copies bought by the newsboy. Then, we must find the ℓ such that the sum

$$\begin{aligned} 35 \sum_{k=\ell}^{50} \binom{50}{k} 0.5^{50} - 20 \sum_{k=0}^{\ell-1} \binom{50}{k} 0.5^{50} &= 35 \left(1 - \sum_{k=0}^{\ell-1} \binom{50}{k} 0.5^{50} \right) - 20 \sum_{k=0}^{\ell-1} \binom{50}{k} 0.5^{50} \\ &= 35 - 55 \sum_{k=0}^{\ell-1} \binom{50}{k} 0.5^{50} \\ &= 35 - 55 \cdot 0.5^{51-\ell} \end{aligned}$$

equals 0. This can be computed experimentally

■

PROBLEM 6.3 (HANDOUT 8, # 12)

How many independent bridge dealings are required in order for the probability of a preassigned player having four aces at least once to be $1/2$ or better? Solve again for some player instead of a given one.

SOLUTION. ■

PROBLEM 6.4 (HANDOUT 8, # 13)

A book of 500 pages contains 500 misprints. Estimate the chances that a given page contains at least three misprints.

SOLUTION. Let X be the number of misprints on the given page. The probability that a given misprint is on that page is $1/500$. Now, $P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$. Also,

$$\begin{aligned}P(X = 0) &= \left(\frac{499}{500}\right)^{500} \\P(X = 1) &= 500 \left(\frac{1}{500}\right) \left(\frac{499}{500}\right)^{499} \\P(X = 2) &= \frac{500 \cdot 499}{2} \left(\frac{1}{500}\right)^2 \left(\frac{499}{500}\right)^{498}\end{aligned}$$

So that

$$\begin{aligned}P(X \geq 3) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\&= 1 - \left(\frac{499}{500}\right)^{500} - 500 \left(\frac{1}{500}\right) \left(\frac{499}{500}\right)^{499} - \frac{500 \cdot 499}{2} \left(\frac{1}{500}\right)^2 \left(\frac{499}{500}\right)^{498} \\&\approx 0.08\end{aligned}$$

that is, the probability that the given page has at least 3 misprints is about 8 percent. ■

PROBLEM 6.5 (HANDOUT 8, # 14)

Colorblindness appears in 1 per cent of the people in a certain population. How large must a random sample (with replacements) be if the probability of its containing a colorblind person is to be 0.95 or more?

SOLUTION. Let n be the sample size. The probability of the sample containing no colorblind people is 0.99^n . Solving the equation $0.99^n = 0.05$, we see that (for the naturals) taking $n = 299$ is (minimally) sufficient for 0.99^n to be less than 0.05.

The probability that the sample has some colorblind person is equal to $1 - 0.99^n$. This is at least 95 percent if 0.99^n is less than 0.05. That is, having 299 people in the sample is (minimally) sufficient for there to be a 95 percent chance of having some colorblind person. ■

PROBLEM 6.6 (HANDOUT 8, # 15)

Two people toss a true coin n times each. Find the probability that they will score the same number of heads.

SOLUTION. Let X denote the number of heads that, say, person 1 gets. Then

$$P(X = k) = \frac{\binom{n}{k}}{2^n}.$$

Then, assuming independence, the probability that they score the same number of heads is given by the expression

$$\sum_{k=0}^n \left(\frac{\binom{n}{k}}{2^n} \cdot \frac{\binom{n}{k}}{2^n} \right) = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{n}{k}^2,$$

which, by the binomial identity of the sum of squares of binomial coefficients, gives us

$$= \frac{1}{2^{2n}} \binom{2n}{n}.$$

■

PROBLEM 6.7 (HANDOUT 8, # 16)

(*Binomial approximation to the hypergeometric distribution.*) A population of TV elements is divided into red and black elements in the proportion $p : q$ (where $p + q = 1$). A sample of size n is taken without replacement. The probability that it contains exactly k red elements is given by the hypergeometric distribution of II, 6. Show that as $n \rightarrow \infty$ this probability approaches $\text{Binom}(n, p)$.

SOLUTION. The hypergeometric distribution has the following PMF

$$\begin{aligned}
 P(X = k) &= \binom{r}{k} \binom{n-r}{pn-k} / \binom{n}{pn} = \frac{r!(n-r)!(pn)!(qn)!}{k!(r-k)!(pn-k)!(qn-(r-k))!n!} \\
 &= \binom{pn}{k} \left(\frac{r!(n-r)!(qn)!}{(r-k)!(qn-(r-k))!n!} \right) \\
 &= \binom{pn}{k} \left(\frac{r!/(r-k)!}{n!/(n-k)!} \cdot \frac{(n-r)!/(n-pn-(r-k))!}{(n-pn+(pn-k))!/(qn)!} \right) \\
 &= \binom{pn}{k} \prod_{j=1}^k \frac{r-k+j}{n-k+j} \prod_{j=1}^{np-k} \frac{n-r-(np-k)+j}{n-np+j}.
 \end{aligned}$$

Now, take the limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \prod_{j=1}^k \frac{r-k+j}{n-k+j} =$$

and

$$\lim_{n \rightarrow \infty} \prod_{j=1}^{np-k} \frac{n-r-(np-k)+j}{n-np+j}$$

■

PROBLEM 6.8 (HANDOUT 9, # 3)

Suppose X, Y, Z are mutually independent random variables, and $E(X) = 0, E(Y) = -1, E(Z) = 1, E(X^2) = 4, E(Y^2) = 3, E(Z^2) = 10$. Find the variance and the second moment of $2Z - Y/2 + eX$, where e is the number such that $\ln e = 1$.

SOLUTION. Let $W = 2Z - Y/2 + eX$. Then

$$\begin{aligned}
 E(W^2) &= E((2Z - Y/2 + eX)^2) \\
 &= E(4Z^2 - 2ZY + 4eZX + Y^2/4 - eYX + e^2X^2) \\
 &= 4E(Z^2) - 2E(Z)E(Y) + 4eE(Z)E(X) + E(Y^2)/4 - eE(Y)E(X) + e^2E(X^2) \\
 &= 4 \cdot 10 + 2 + 3/4 + e^2 \cdot 4 \\
 &= \frac{171}{4} + 4e^2 \\
 &\approx 72.31
 \end{aligned}$$

and

$$\begin{aligned}
 \text{Var}(W) &= E(W^2) - E(W)^2 \\
 &= \frac{171}{4} + 4e^2 - (2E(Z) + E(Y)/2 + eE(X))^2 \\
 &= \frac{171}{4} + 4e^2 - (2 - 1/2)^2 \\
 &= \frac{171}{4} + 4e^2 - (3/2)^2 \\
 &= \frac{81}{2} + 4e^2 \\
 &\approx 70.06
 \end{aligned}$$

That is, the second moment is “about 72.31” and the variance is “about 70.06”

■

PROBLEM 6.9 (HANDOUT 9, # 14)

(*Variance of Product*). Suppose X, Y are independent random variables. Can it ever be true that $\text{Var}(XY) = \text{Var}(X) \text{Var}(Y)$?

If it can, when?

SOLUTION. Yes. In fact, if X and Y are constant random variables $\text{Var } X = \text{Var } Y = 0$ so clearly

$$\text{Var}(XY) = \text{Var } X \text{Var } Y = 0.$$

The problem is finding sufficient conditions for this to be true.

Since X and Y are independent, we know that

$$E(XY) = E(X)E(Y)$$

and therefore

$$\begin{aligned} \text{Var}(XY) &= E(X^2Y^2) - E(XY)^2 \\ &= E(X^2)E(Y^2) - E(X)^2E(Y)^2 \end{aligned}$$

and we want this to be equal to

$$\begin{aligned} \text{Var } X \text{Var } Y &= (E(X^2) - E(X)^2)(E(Y^2) - E(Y)^2) \\ &= E(X^2)E(Y^2) + E(X)^2E(Y)^2 - E(X)^2E(Y^2) - E(X^2)E(Y)^2 \\ &= E(X^2)E(Y^2) - E(X)^2E(Y^2) \\ &\quad - (E(X)^2E(Y^2) - E(X)^2E(Y)^2 + E(X^2)E(Y^2) - E(X)^2E(Y^2)) \\ &= \text{Var}(XY) - \text{Var}(Y)E(X^2) - \text{Var}(X)E(Y^2). \end{aligned}$$

This happens precisely when the quantity

$$\text{Var}(Y)E(X^2) + \text{Var}(X)E(Y^2) = 0.$$

Since $\text{Var } X, \text{Var } Y, E(X^2), E(Y^2) \geq 0$, this forces the equality

$$\text{Var}(Y)E(X^2) = \text{Var}(X)E(Y^2) = 0.$$

This happens when both $X = Y = 0$ or, more generally, if and only if X and Y are constant. ■