MA571 Problem Set 6

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Problem 6.1 (Munkres §25, Ex. 8)

Let $p\colon X \to Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected. [*Hint:* If C is a component of the open set U of Y, show that $p^{-1}(C)$ is a union of components of $p^{-1}(U)$.]

Proof. We will proceed from the hint. Let $U \subset Y$ be open and let C be a component of U. Then we will show that $p^{-1}(C)$ is the union of components of $p^{-1}(U)$. Now, since U is open in Y, $p^{-1}(U)$ is open in X and $p^{-1}(C) \subset p^{-1}(U)$. Let $x \in p^{-1}(C)$ and let C_x be the component of x in $p^{-1}(U)$. Then, by Theorem 25.3, since X is locally connected C_x is open in X. The, we claim that $C_x \subset p^{-1}(C)$. But this claim follows from Theorem 23.5 and Theorem 25.1 since $p(C_x)$ is connected and contains $p(x)$ so $p(C_x) \subset C$ so $C_x \subset p^{-1}(C)$. Taking the union over every component C_x corresponding to a point $x \in p^{-1}(U)$, we have that

$$p^{-1}(U) = \bigcup C_x$$

is a union of components of $p^{-1}(U)$. It follows that $p^{-1}(C)$ is open so, by the definition of the quotient topology, C is open and hence, it follows from Theorem 25.3 that Y is locally path connected. ∎

Problem 6.2 (Munkres §25, Ex. 10(a,b))

Let X be a space. Let us define $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$.

(a) Show this relation is an equivalence relation. The equivalence classes are called *quasicomponents* of X.

(b) Show that each component of X lies in a quasicomponent of X, and that the components and quasicomponents of X are the same if X is locally connected.

(c) Let K denote the set $\{\frac{1}{n} \mid n \in \mathbf{Z}_+\}$ and let $-K$ denote the set $\{-\frac{1}{n} \mid n \in \mathbf{Z}_+\}$. Determine the components, path components, and quasicomponents of the following subspaces of $\mathbf{R}^2$:

$$A = (K \times [0,1]) \cup \{0 \times 0\} \cup \{0 \times 1\}.$$
$$B = A \cup ([0,1] \times \{0\}).$$

Proof. (a) To show that $\sim$ is a equivalence relation, we need to check three things (i) reflexivity ($x \sim x$); (ii) symmetry (if $x \sim y$ then $y \sim x$); and (iii) transitivity (if $x \sim y$ and $y \sim z$ then $x \sim z$). In order:

(i) Seeking a contradiction, suppose that A, B is a separation of x such that $x \in A$ and $x \in B$ then $x \in A \cap B$ but $A \cap B = \emptyset$. Thus, $x \sim x$.
(ii) Suppose that $x \sim y$. Then, if $y \nsim x$ there exists a separation A, B of X such that $y \in A$, $x \in B$, but then $x \nsim y$.
(iii) Suppose $x \sim y$ and $y \sim z$. Seeking a contradiction, if $x \nsim z$ then there exists a separation A, B of X such that $x \in A$ and $z \in B$. Then, since $y \in X$ either $y \in A$ or $y \in B$. In the former case, A, B is a separation with $y \in A$ and $z \in B$ contradicting $y \sim z$ and in the latter case A, B is a separation with $x \in A$ and $y \in B$ contradicting $x \sim y$. Thus, $x \sim z$.

Thus, $\sim$ defines an equivalence relation on X.

(b) Let $x \in X$ and let Q and C denote, respectively, the quasicomponent and component of x. Then, we claim that $C \subset Q$. For if $y \in C$ not in Q, then there exists a separation A, B of X such that $x \in A$ and $y \in B$. But, by Theorem 23.2, either since C is connected either $C \subset A$ or $C \subset A$. In either case, we arrive at a contradiction (it the former $C \subset A$ but $y \notin A$ and in the latter $C \subset B$ but $x \notin A$). Thus, $C \subset Q$.

Keeping the notation the same as in the previous paragraph, suppose X is locally connected. Having shown $C \subset Q$ it suffices holds, it suffices to show that $C \supset Q$. Suppose not, then there exist some $y \in Q$ not in C. Since X is locally connected, then C is open and closed so C and $X \setminus C$ is a separation of X. But then $x \in C$ and $y \in X \setminus C$ which contradicts our assumption that $y \in Q$, i.e, x and y lie in the same quasiconnected component. Thus, $C \supset Q$ and in fact $C = Q$ holds.

(c) ∎

Problem 6.3 (Munkres §26, Ex. 4)

Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

Proof. Let (X, d) be a metric space and $Y \subset X$ a compact subspace. By Theorem 26.3, Y is closed since X is Hausdorff (since for any $x, y \in X$, let $\varepsilon = d(x, y)$, then $B(x, \varepsilon/2) \cap B(y, \varepsilon/2) = \emptyset$). Hence, we need only show Y is bounded.

Recall from Munkres §20 that a set A is bounded if there exist some positive real number M such that $d(a_1, a_2) \leq M$ for every pair $a_1, a_2 \in A$. Fix a $x_0 \in Y$ and consider the collection of open sets

$$\mathcal{A} = \{\, B_d(x_0, M) \mid M \in [0, \infty)\,\}.$$

Then $\mathcal{A}$ is a covering of Y (since for every $x \in Y$, $x \in B_d(x_0, d(x_0, x)+1)$ so it is in the union of all of them). Since Y is compact $\mathcal{A}$, by Theorem 26.1, there is a finite subcollection, say $\{B(x_0, M_n)\}_{n=1}^N$, of $\mathcal{A}$ covering Y. Let $M = \max\{M_1, ..., M_n\}$. Then, for any pair $x, y \in Y$, by the triangle inequality, we have

$$d(x, y) \leq d(x_0, y) + d(x_0, y)$$

but $x \in B(x_0, M_k)$ and $y \in B(x_0, M_\ell)$ for some $1 \leq k, \ell \leq N$ so

$$\begin{aligned} &\leq M_k + M_\ell \\ &\leq 2M. \end{aligned}$$

Thus, Y is bounded. ■

Problem 6.4 (Munkres §26, Ex. 5)

Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exists disjoint open sets U and V containing A and B, respectively.

Proof. Suppose A and B are disjoint compact subpsaces of the Hausdorff space X. By Theorem 26.4, there for every $x \in B$ there exists disjoint open sets $U_x \supset A$ and $V_x \ni x$. Then, the collection of all such V_x, call it $\mathcal{V}$, is a covering of B. By Theorem 26.1, there exists a finite subcollection $\{V_n\}_{n=1}^N$ of $\mathcal{U}$ covering B. Let $\{U_n\}_{n=1}^N$ be the collection of sets U_i corresponding to V_i. Then $U = \bigcap_{i=1}^N U_i$ and $V = \bigcup_{i=1}^N V_i$ are disjoint open subsets containing A and B respectively since, by the distributive property of "$\cup$", we have that

$$U \cap V = U \cap \left(\bigcup V_i \right) = \bigcup U \cap V_i = \bigcup U_i \cap V_i = \emptyset.$$

■

Problem 6.5 (Munkres §26, Ex. 7)

Show that if Y is compact, then the projection $\pi_X \colon X \times Y \to X$ is a closed map.

Proof. We proceed by the tube lemma (Theorem 26.8). Let C be a closed subset of $X \times Y$. Then $N = (X \times Y) \smallsetminus C$ is open in $X \times Y$. Let $x_0 \in X \smallsetminus \pi_X(C)$. Then $x_0 \times Y \subset N$. By the tube lemma, there exists some W neighborhood of x_0 in X such that $W \times Y \subset N$. In particular, $W \subset X \smallsetminus \pi(C)$ for otherwise there is a point $x \in W \cap \pi(C)$ which implies $x \times Y \subset N$ but $x \times Y \cap C \neq \emptyset$ as $(x, y) \in x \times Y \cap C$ for any $y \in \pi_Y(C)$. It follows, by Lemma C, that $X \smallsetminus \pi_X(C)$ is open so $\pi_X(C)$ is closed. Thus, π_X is a closed map. ∎

Problem 6.6 (A)

Let X be a compact space and let $\sim$ be an equivalence relation on X. Suppose that the set

$$S = \{ (x, y) \mid x \sim y \}$$

is a closed subset of $X \times X$. Prove that the quotient map $q\colon X \to X/{\sim}$ is a closed map.

Proof. Put $Y = X/{\sim}$. We claim that:

Lemma 14. *$B \subset Y$ is closed if and only if $q^{-1}(B)$ is closed in X.*

Proof. $B \subset Y$ is closed in Y if and only if $Y \setminus B$ is open in Y if and only if $q^{-1}(Y \setminus B) = Y \setminus q^{-1}(B)$ is open in X, i.e., $q^{-1}(B)$ is closed in X. ♣

Let $C \subset X$ be closed. By Lemma 14, it suffices to show that $q^{-1}(q(C))$ is closed. But note that

$$\begin{aligned} q^{-1}(q(C)) &= \{ x \mid q(x) \in q(C) \} \\ &= \{ x \mid \text{for some } y \in C,\ x \sim y \} \\ &= \pi_1(S \cap (X \times B)). \end{aligned}$$

By Problem 6.5, $\pi_1\colon X \times X \to X$ sending $(x_1, x_2) \mapsto x_1$ is a closed map, therefore it suffices to check that $S \cap (X \times B)$ is closed, in particular, we need to check that $X \times B$ is closed (since S closed is given). But $\pi_2\colon X \times X \to X$ via $(x_1, x_2) \mapsto x_2$ is continuous so $X \times B = \pi_2^{-1}(B)$ is closed. Thus, $S \cap (X \times B)$ is closed. Thus, $q^{-1}(q(C))$ is closed so by Lemma 14 $q(C)$ is closed. ■

Problem 6.7 (B)

Let S^2 be the sphere
$$\{ (x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1 \}.$$
Let S^2_+ be $S^2 \cap \{z \geq 0\}$ (the upper hemisphere), let S^2_- be $S^2 \cap \{z \leq 0\}$ (the lower hemisphere), and let E be $S^2 \cap \{z = 0\}$ (the equator). Recall the definition of Y/S from Homework #4. Prove that S^2/S^2_- is homeomorphic to S^2_+/E. [*Hint:* There are maps in both directions.]

Proof. Let us rewrite ■