MA166: Recitation 12

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1 Homework

1.1 This Week's Summary

Here's a summary of the material that was (presumably) covered this week. Sections from Stewart

§10.1: Parametric Equations and Polar Coordinates

Suppose that x and y are defined in terms of another third variable t (called the *parameter*) by the equations

$$x = f(x) \quad y = g(t).$$

(called *parametric equations*). Each value of t determines a point (x, y) which we can plot in the coordinate plane $\mathbb{R}^2$. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call the *parametric curve*. The parameter f does not necessarily represent time, and in fact,

1.2 Homework Problems

Solutions to selected problems:

Homework 31

Problem 1 (WebAssign HW 31, # 1)**.** Select the curve generated by the parametric equations. Indicate with an arrow the direction in which the curve is traced as t increases.

$$x = e^{-t} + t, \quad y = e^t - t, \quad -2 \leq t \leq 2.$$

Problem 2 (WebAssign HW 31, # 2)**.** Consider the following equations.

$$x = 1 - t^2, \quad y = t - 3, \quad -2 \leq t \leq 2.$$

(a) Sketch the curve using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced out as t increases.

(b) Eliminate the parameter to find a Cartesian equation of the curve for $-5 \leq y \leq -1$.

Problem 3 (Solution)**.** To eliminate the parameter, make $t = y + 3$, then

$$\begin{aligned} x &= 1 - t^2 \\ &= 1 - (y + 3)^2 \\ &= -y^2 - 6y - 8 \end{aligned}$$

and this holds for values $-5 \leq y \leq -1$.

Problem 4 (WebAssign HW 31, # 3)**.** Consider the parametric equations below.

$$x = \sqrt{t}, \quad y = 11 - t.$$

(a) Sketch the curve using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced out as t increases.

(b) Eliminate the parameter to find a Cartesian equation of the curve for $x \geq 0$.

Problem 5 (Solution)**.** The same idea works for this problem. Set $t = 11 - y$.

Problem 6 (WebAssign HW 31, # 4)**.** Consider the following.

$$x = \sin \tfrac{1}{2}\theta, \quad y = \cos \tfrac{1}{2}\theta, \quad -\pi \leq \theta \leq \pi.$$

(a) Sketch the curve using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced out as t increases.

(b) Eliminate the parameter to find a Cartesian equation of the curve for $-5 \leq y \leq -1$.

Problem 7 (WebAssign HW 31, # 5)**.** Consider the following.

$$x = \sin t, \quad y = \csc t \quad 0 < t < \pi/2.$$

(a) Eliminate the parameter to find a Cartesian equation of the curve.
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

Problem 8 (WebAssign HW 31, # 6)**.** Describe the motion of a particle with position (x, y) as t varies in the given interval.

$$x = 2 + 2\cos t, \quad y = 1 + 2\sin t, \quad \pi/2 \leq t \leq 3\pi/2.$$

Problem 9 (WebAssign HW 31, # 7)**.** Describe the motion of a particle with position (x, y) as t varies in the given interval.

$$x = 2\sin t, \quad y = 1 + \cos t, \quad 0 \leq t \leq 3\pi/2.$$

Problem 10 (WebAssign HW 31, # 8)**.** Describe the motion of a particle with position (x, y) as t varies in the given interval.

$$x = 4\sin t, \quad y = 9\cos t, \quad -\pi \leq t \leq 9\pi.$$

Problem 11 (WebAssign HW 31, # 9)**.** Match the graphs of the parametric equations $x = f(t)$ and $y = g(t)$ in (a)–(d) with the parametric curves labeled I–IV.

Problem 12 (WebAssign HW 31, # 10)**.** Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.

Homework 32

Problem 13 (WebAssign HW 32, # 1)**.** Find dy/dx.

$$x = t\sin t, \quad y = t^2 + 3t.$$

Solution. Find dy/dt and dx/dt and then take their quotient

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx}.$$

☺

Problem 14 (WebAssign HW 32, # 2)**.** Find dy/dx.

$$x = 7/t, \quad y = \sqrt{t}e^{-t}.$$

Problem 15 (WebAssign HW 32, # 3)**.** Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

$$x = t - t^{-1}, \quad y = 9 + t^2, \quad t = 1.$$

Solution. First we find dy/dx

$$\frac{dy}{dt} = 2t \quad \frac{dx}{dt} = \frac{t^2+1}{t^2}$$

so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{(t^2+1)/t^2} = \frac{2t^3}{t^2+1}.$$

So when $t = 1$, $dy/dx = 2(1)^3/(1^2+1) = 1$ and the value of the parametric equation will be $x = 1 - 1 = 0$ and $y = 9 + 1^2 = 10$. Recall that the equation for the tangent line at the point (x_1, x_2) is

$$y - y_1 = m(x - x_1)$$

where m is the derivative at that point. Hence, our tangent line will look like

$$\begin{aligned} y - 10 &= 1(x - 0) \\ y &= x + 10. \end{aligned}$$

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Problem 16 (WebAssign HW 32, # 4)**.** Find dy/dx and d^2y/dx^2.

$$x = e^t, \quad y = te^{-t}.$$

For which values of t is the curve concave upward?

Solution. First we find the derivatives with respect to t

$$\begin{aligned} \frac{dy}{dt} &= -te^{-t} + e^{-t} \quad \frac{dx}{dt} = e^t \\ &= (1-t)e^{-t} \end{aligned}$$

so

$$\frac{dy}{dx} = \frac{(1-t)e^{-t}}{e^t} = (1-t)e^{-2t}.$$

Now to find the second partial, we need to find

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

and quotient by dx/dt

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = e^{-2t}(-1) + (1-t)(-2e^{-2t})$$

so

$$\frac{d^2y}{dx^2} = \frac{e^{-2t}(-1-2+2t)}{e^t} = e^{-3t}(2t-3).$$

The curve is concave up when the second derivative is greater that 0, so when $t > 3/2$. ☺

Problem 17 (WebAssign HW 32, # 5)**.** Find dy/dx and d^2y/dx^2.

$$x = 2\sin t, \quad y = 3\cos t, \quad 0 < t < 2\pi.$$

For which values of t is the curve concave upward?

Problem 18 (WebAssign HW 32, # 6)**.** Find the exact length of the curve.

$$x = 3 + t^2, \quad y = 3 + 2t^3, \quad 0 \le t \le 2.$$

Problem 19 (WebAssign HW 32, # 7)**.** Find the exact length of the curve

$$x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \le t \le 4.$$

Problem 20 (WebAssign HW 32, # 8)**.** Find the distance traveled by a particle with position (x, y) as t varies in the given time interval.

$$x = 3\sin^2 t, \quad y = 3\cos^2 t, \quad 0 \le t \le 3\pi.$$

What is the length of the curve?

Solution. Some students had trouble computing the length of the curve. For that you simply had to observe that t traverses the whole curve when $0 \le t \le \pi/2$, because the segment of $x + y = 3$ lies in the first quadrant. Thus

$$L = \int_0^{\pi/3} 3\sin 2t = 3\sqrt{2}.$$

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Problem 21 (WebAssign HW 33, # 1)**.** Find two other pairs of polar coordinates of the given polar coordinate, one with $r > 0$ and one with $r < 0$. Then plot the point.

(a) $(5, \pi/4)$

(b) $(4, -2\pi/3)$

(c) $(-4, \pi/6)$

Problem 22 (WebAssign HW 33, # 2). Find the Cartesian coordinates of the given polar coordinates. Then plot the point.

(a) $(5, \pi)$

(b) $(6, -2\pi/3)$

(c) $(-6, 3\pi/4)$

Problem 23 (WebAssign HW 33, # 3). The Cartesian coordinates of a point are given.

(a) $(4, -4)$

  (i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leq \theta < 2\pi$.
  (ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leq \theta < 2\pi$.

(b) $(-1, \sqrt{3})$

  (i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leq \theta < 2\pi$.
  (ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leq \theta < 2\pi$.

Problem 24 (WebAssign HW 33, # 4). The Cartesian coordinates of a point are given.

(a) $(2\sqrt{3}, 2)$

  (i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leq \theta < 2\pi$.
  (ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leq \theta < 2\pi$.

(b) $(1, -3)$

  (i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leq \theta < 2\pi$.
  (ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leq \theta < 2\pi$.

Problem 25 (WebAssign HW 33, # 5). Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions

$$2 < r < 5, \quad 3\pi/2 \leq \theta \leq 5\pi/2.$$

Problem 26 (WebAssign HW 33, # 6). Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions

$$r \geq 5, \quad \pi \leq \theta \leq 2\pi$$

Problem 27 (WebAssign HW 33, # 7). Find a Cartesian equation for the curve and identify it.

$$r^2 \cos 2\theta = 1.$$

Problem 28 (WebAssign HW 33, # 8). Find a Cartesian equation for the curve and identify it.

$$r = 4 \tan\theta \sec\theta.$$

Problem 29 (WebAssign HW 33, # 9)**.** Find a polar equation for the curve represented by the given Cartesian equation.

$$x^2 + y^2 = 10cx.$$

Problem 30 (WebAssign HW 33, # 10)**.** Find a polar equation for the curve represented by the given Cartesian equation.

$$xy = 11.$$

2 Exam Problems

2.1 Exam 3: Spring 2014

Problem 1. Compute

$$\sum_{n=1}^{\infty} \frac{2^{n-1} - (-3)^{n+1}}{4^n}.$$

Solution. To begin with, perform the following manipulations to the sum

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^{n-1} - (-3)^{n+1}}{4^n} &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{4^n} - \sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n} \\
&= \sum_{n=1}^{\infty} \frac{2^{n-1}}{4^n} - \sum_{n=1}^{\infty} \frac{3 \cdot 3^{n+1-1}}{4^n} \\
&= \sum_{n=1}^{\infty} \frac{2^{n-1}}{(2^2)^n} - \sum_{n=1}^{\infty} 3\left(\frac{3}{4}\right)^n \\
&= \sum_{n=1}^{\infty} \frac{2^{n-1}}{2^{2n}} - 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \\
&= \sum_{n=1}^{\infty} 2^{n-1-2n} - 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \\
&= \sum_{n=1}^{\infty} 2^{-n-1} - 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \\
&= \sum_{n=1}^{\infty} 2^{-(n+1)} - 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \\
&= \sum_{n=1}^{\infty} 2^{-1(n+1)} - 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \\
&= \sum_{n=1}^{\infty} (2^{-1})^{n+1} - 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+1} - 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} - 3\sum_{n=0}^{\infty} \frac{3}{4}\left(\frac{3}{4}\right)^{n} \\
&= \underbrace{\frac{1}{4}\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n}}_{S1} - \underbrace{\frac{9}{4}\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{n}}_{S_2}.
\end{aligned}
$$

Now that we have rewritten the series into a sum of geometric series, it is easy to compute it's value

$$
\begin{aligned}
S_1 &= \frac{1/4}{1-1/2} & S_2 &= \frac{9/4}{1-3/4} \\
&= \frac{2}{4} & &= 9 \\
&= \frac{1}{2}
\end{aligned}
$$

Thus,

$$
S_1 - S_2 = \frac{1}{2} - 9 = \frac{1}{2} - \frac{18}{2} = -\frac{17}{2}.
$$

Answer: **D**. ☺

Problem 2. This series

$$
\sum_{n=1}^{\infty} \frac{1}{(n^{2p}+1)^{1/6}}
$$

is convergent if and only if

Problem 3. Test the following series for convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \arctan n$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$.

Proof. ☺

Problem 4. Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{3n+5}$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^3}$.

(c) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$.

Problem 5. Test the following series for convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$.

(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+2}{2n+3}\right)^n$.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{5^n}$.

Problem 6. Which of the following statements are *always true*?

(I) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(II) If $\lim_{n\to\infty} n^3|a_n| = 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

(III) $\sum_{n=1}^{\infty}(e^n + c)/e^{2n}$ converges for any positive value c.

Problem 7. Given the following series

$$\sum_{n=1}^{\infty}\frac{3}{2^n+n-1}.$$

Mark, Nancy and David provide the following ingredient of the arguments for convergence or divergence of the series:

(a) the name of the test to use,

(b) the conclusion for convergence or divergence

Mark: (a) $b_n = 3/2^n$, comparison test ($0 \le a_n \le b_n$); (b) convergent

Nancy: (a) $b_n = 1/n$, limit comparison test ($\lim_{n\to\infty} a_n/b_n = 3$); (b) divergent

David: (a) ratio test ($\lim_{n\to\infty}|a_{n+1}|/|a_n| = 1/2$); (b) convergent.

Choose the name(s) of the person(s) with correct arguments.

Problem 8. Consider the Maclaurin series for e^x

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

By plugging in $x = -1$, one obtains the alternating series

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots.$$

If we compute the sum of the *fewest* terms necessary to guarantee that the error is less than 0.05, *using the estimation theorem for alternating series*, then what is the estimate for e^{-1}?

Problem 9. Suppose the power series

$$\sum_{n=0}^{\infty} c_n (x-3)^n$$

converges when $x = 1$, but diverges when $x = 7$.

From the above information, which of the following statements can we conclude to be true?

(I) The radius of convergence is $R \geq 2$.

(II) The power series converges at $x = 4.5$.

(III) The power series diverges at $x = 6.5$.

Problem 10. What is the coefficient of x^6 in the power series expansion $2/(1+2x^2)$?

Problem 11. Determine the interval of convergence for the following power series

$$\sum_{n=0}^{\infty} \frac{(-5)^n}{\sqrt{n+2}} (x-3)^n.$$

Problem 12. The power series representation (centered at $a = 0$) for $g(x) = x/(4-x^2)$ is given by

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{4^{n+1}}$$

with the interval of convergence $(-2, 2)$.

Find

(a) the power series representation (centered at $a = 0$), and

(b) the interval of convergence

for the function

$$f(x) = \ln|4 - x^2|.$$

2.2 Exam 3 Spring 2013