Ch. 5

2. Show that the conclusions of (5.32) are not true without the assumption that $\varphi \in L(E)$.

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<u>Proof</u>: Let $f_K = \chi_{(K,+\infty)} = \begin{cases} 1 & \text{if } x \in (K,+\infty) \\ 0 & \text{if } x \notin (K,+\infty). \end{cases}$

Note that f_K is measurable on $\mathbb{R}$ $\forall K \in \mathbb{N}$ since it is the characteristic function of the measurable (open) set $(K,+\infty)$. Also, $f_{K+1} \le f_K$ $\forall K$, so $f_K \searrow f = 0$ in $\mathbb{R}$.

Now, if φ satisfies $f_K \le \varphi$ a.e. on $\mathbb{R}$ $\forall K \in \mathbb{N}$, then $\int_{\mathbb{R}} f_K \le \int_{\mathbb{R}} \varphi$ $\forall K$ by Theorem 5.10 (f_K, φ are nonnegative). Notice that $\forall K \in \mathbb{N}$, $\int_{\mathbb{R}} f_K = |(K,+\infty)| = \infty$ by Corollary 5.4. So $\int_{\mathbb{R}} \varphi = \infty$, giving $\varphi \notin L(\mathbb{R})$. ✓

Notice that $\int_{\mathbb{R}} f_K = \infty \not\to 0$. Thus $\int_{\mathbb{R}} f_K \not\to \int_{\mathbb{R}} f$ since $\int_{\mathbb{R}} f = \int_{\mathbb{R}} 0 = 0$ by Theorem 5.11. Hence the conclusions of (5.32) are not true without the assumption that $\varphi \in L(E)$. ✓

<u>Note</u>: The conclusion of the first part of (5.32) is not true without $\varphi \in L(E)$ follows from the above proof, considering $-f_K$. ∎

✓

3. Let $\{f_K\}$ be a sequence of nonnegative measurable functions defined on E. If $f_K \to f$ and $f_K \le f$ a.e. on E, show that $\int_E f_K \to \int_E f$.

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<u>Proof</u>: Let $\{f_K\}$ be a sequence of nonnegative measurable functions defined on E. Suppose $f_K \to f$ and $f_K \le f$ a.e. on E. Note that f is measurable by Theorem 4.12. Since f_K is nonnegative $\forall K \in \mathbb{N}$, by Fatou's Lemma (Theorem 5.17), $\int_E f \le \liminf_{K\to\infty} \int_E f_K$.

Now since $f_K \le f$ a.e. on E, by Theorem 5.10, $\int_E f_K \le \int_E f$ $\forall K \in \mathbb{N}$. Taking the limsup, we have that $\limsup_{K\to\infty} \int_E f_K \le \int_E f$.

Then we have that:

$\int_E f \le \liminf_{K\to\infty} \int_E f_K \le \limsup_{K\to\infty} \int_E f_K \le \int_E f$.

Hence $\int_E f_K \to \int_E f$. ■

... $\in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and $\int_0^1 x^k f(x)\,dx \to 0$.

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Proof: Suppose $f \in L(0,1)$. Notice that x^k is continuous and is thus measurable on $(0,1)$ by Theorem 4.3. Also, $|x^k| \leq 1$ in $(0,1)$. Then by Theorem 5.30, $x^k f(x) \in L(0,1)$ $\forall k \in \mathbb{N}$.

Note that by Theorem 5.22, f is finite a.e. in $(0,1)$. Then as $k \to \infty$, $x^k f(x) \to 0$ a.e. on $(0,1)$. Now $|x^k f(x)| \leq |f(x)|$ in $(0,1)$ and by Theorem 5.21, $|f| \in L(0,1)$. Thus by Theorem 5.36, $\int_0^1 x^k f(x) \to \int_0^1 0$. Notice that $\int_0^1 0 = 0$ by Theorem 5.11. Hence $\int_0^1 x^k f(x)\,dx \to 0$.