

# MA 523: Homework 3

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## PROBLEM 3.1

Consider the initial value problem

$$u_t = \sin u_x; \quad u(x, 0) = \frac{\pi}{4}x.$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

*SOLUTION.* The initial value problem certainly satisfies the assumptions of the Cauchy–Kovalevskaya theorem, that is, setting  $\mathbf{u} := (u, u_x, u_t, t)$ , the  $\mathbf{B}$  are all identically 0 and  $\mathbf{c}(\mathbf{u}, x) = \sin u_x(x, t)$  is analytic. Next we show that the Taylor series of  $u$  at  $(0, 0)$ ,

$$\tilde{u}(x, t) = \sum_{(\alpha_1, \alpha_2)} \frac{u_{\alpha_1, \alpha_2}(0)}{\alpha_1! \alpha_2!} x^{\alpha_1} t^{\alpha_2},$$

is a solution to our PDE.

First, we must compute the coefficients

$$\frac{u_{\alpha_1, \alpha_2}(0, 0)}{\alpha_1! \alpha_2!}.$$

To this end, we must find the partial derivatives  $u_{\alpha_1, \alpha_2}$  and potentially, relations among them which will help us to find these coefficients. Naïvely listing the partials with respect to  $t$  and  $x$ , we have

$$\begin{aligned} u(0, 0) &= 0 \\ u_x(0, 0) &= \frac{\pi}{4} \\ u_t(0, 0) &= \sin u_x(0, 0) = \frac{\sqrt{2}}{2} \\ u_{xx}(0, 0) &= 0 \\ u_{tx}(0, 0) &= 0 \\ u_{tt}(0, 0) &= -\cos(u_x(0, 0))u_{xt}(0, 0) = 0 \\ u_{xxx}(0, 0) &= 0 \\ u_{ttx}(0, 0) &= 0, \end{aligned}$$

etc. Thus,

$$\tilde{u} = \frac{\pi}{4}x + \frac{\sqrt{2}}{2}t.$$

Plugging this equation into our PDE, we have

$$\tilde{u}_t - \sin \tilde{u}_x = \frac{\sqrt{2}}{2} - \sin(\pi/4) = 0,$$

as desired. ■

## PROBLEM 3.2

Consider the Cauchy problem for  $u(x, y)$

$$\begin{aligned} u_y &= a(x, y, u)u_x + b(x, y, u) \\ u(x, 0) &= 0 \end{aligned}$$

Let  $a$  and  $b$  be analytic functions of their arguments. Assume that  $D^\alpha a(0, 0, 0) \geq 0$  and  $D^\alpha b(0, 0, 0) \geq 0$  for all  $\alpha$ . (Remember by definition, if  $\alpha = 0$  then  $D^\alpha f = f$ .)

- (a) Show that  $D^\beta u(0, 0) \geq 0$  for all  $|\beta| \leq 2$ .
- (b) Prove that  $D^\beta u(0, 0) \geq 0$  for all  $\beta = (\beta_1, \beta_2)$ . (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in  $\beta_2$ )

*SOLUTION.* Write

$$a(x, y, u) = \sum_{\alpha, \beta, \gamma} a_{\alpha, \beta, \gamma} x^\alpha y^\beta u^\gamma, \quad b(x, y, u) = \sum_{\alpha, \beta, \gamma} b_{\alpha, \beta, \gamma} x^\alpha y^\beta u^\gamma$$

where the right-hand side of the expressions above converge to the left-hand side for  $|x| + |y| + |u| < r$  for some sufficiently small  $r$ .

For part (a) we show this explicitly by considering all cases. The case  $\beta = (0, 0)$  is obvious as are the cases  $\beta = (0, 1)$  and  $\beta = (1, 0)$  since  $u_x(0, 0) = 0$  and

$$\begin{aligned} u_y(0, 0) &= a(0, 0, u(0, 0))u_x(0, 0) + b(0, 0, u(0, 0)) \\ &= a(0, 0, 0)u_x(0, 0) + b(0, 0, 0) \\ &= b(0, 0, 0) \\ &\geq 0 \end{aligned}$$

since  $b$  is a series of strictly positive numbers. For  $\beta = (2, 0)$ , we have  $u_{xx}(0, 0) = 0$ . For  $\beta = (1, 1)$ , we have

$$\begin{aligned} u_{xy}(0, 0) &= a(0, 0, u(0, 0))u_{xx}(0, 0) + \frac{\partial}{\partial x}a(0, 0, u(0, 0))u_x(0, 0) + \frac{\partial}{\partial x}b(0, 0, u(0, 0)) \\ &= \frac{\partial}{\partial x}b(0, 0, 0) \\ &\geq 0. \end{aligned}$$

For  $\beta = (0, 2)$ , we have

$$\begin{aligned} u_{yy}(0, 0) &= a(0, 0, u(0, 0))u_{xy}(0, 0) + \frac{\partial}{\partial y}a(0, 0, u(0, 0))u_x(0, 0) + \frac{\partial}{\partial y}b(0, 0, u(0, 0)) \\ &= a(0, 0, 0)\frac{\partial}{\partial y}b(0, 0, 0) + \frac{\partial}{\partial y}b(0, 0, 0) \\ &\geq 0 \end{aligned}$$

since the latter is a sum of positive numbers.

For part (b), in the proof of the Cauchy–Kovalevskaya theorem, for  $\beta_2 = 0$ , we have

$$D^\beta u(0, 0) = 0$$

since  $u$  is constant on the hypersurface  $\{y = 0\}$ . In particular,  $D^\beta u(0, 0) \geq 0$ .

Now, suppose  $D^\beta u(0, 0) \geq 0$  for all  $\beta_2 \leq n - 1$ . Then, for  $\beta = (m, n)$ , we have

$$\begin{aligned} D^\beta u(0, 0) &= D^{(m, n-1)} u_y(0, 0) \\ &= D^{(m, n-1)} (au_x + b)(0, 0) \\ &= \end{aligned}$$

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## PROBLEM 3.3

(Kovalevskaya's example) Show that the line  $\{t = 0\}$  is characteristic for the heat equation  $u_t = u_{xx}$ . Show there does not exist an analytic solution  $u$  of the heat equation in  $\mathbf{R} \times \mathbf{R}$ , with  $u = 1/(1 + x^2)$  on  $\{t = 0\}$ . (*Hint:* Assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of  $(0, 0)$ .)

*SOLUTION.* First we show that the line  $\Gamma := \{t = 0\}$  is characteristic for the heat equation. With  $\nu = (1, 0)$  the normal to the line  $\Gamma$ , the noncharacteristic condition reads

$$\sum_{|\alpha|=2} a_\alpha \nu^\alpha \neq 0.$$

However,

$$\sum_{|\alpha|=2} a_\alpha \nu^\alpha = 1 \cdot 1 + a_{0,2} \cdot 0 = 1 \neq 0.$$

Thus,  $\Gamma$  is characteristic for  $u_t = u_{xx}$ .

Next suppose  $u$  is an analytic solution to the heat equation near  $(0, 0)$ . From the boundary condition, we have

$$u(x, 0) = \sum_{j=1}^{\infty} (-1)^j x^{2j}.$$

Taking the  $n$ th  $x$ -partial derivative at  $(0, 0)$ , we have

$$D_x^n u(0, 0) = \begin{cases} 0 & \text{if } n = 2k - 1 \text{ is odd} \\ (-1)^k (2k)! & \text{if } n = 2k \text{ is even.} \end{cases}$$

Using the PDE to find the  $m$ th  $y$ -partial derivative at  $(0, 0)$ , we have

$$D_y^n u(0, 0) = D_y^{n-1} u(0, 0) = D_y^{n-1} u_{xx}(0, 0) = 0$$

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