An example of two random variables X & Y which are uncorrelated but not independent.

① Choose any two random variables X, Y that take values in $\{-1, 0, 1\}$ and $\underline{\underline{EX=0}}$, $\underline{\underline{EY=0}}$.

eg. $P(X=-1)=P(X=1)=\alpha,\ P(X=0)=1-2\alpha$

$P(Y=-1)=P(Y=1)=\beta,\ P(Y=0)=1-2\beta$.

You can choose $\alpha=\beta$ so that X, Y are identically distributed

② Consider

$$E(XY) = \sum_{i,j\in\{-1,0,1\}} ij\, P(X=i, Y=j)$$

$$= \sum_{i,j\in\{-1,1\}} ij\, P(X=i, Y=j)$$

$$= p_{11} + p_{-1,-1} - (p_{1,-1} + p_{-1,1})$$

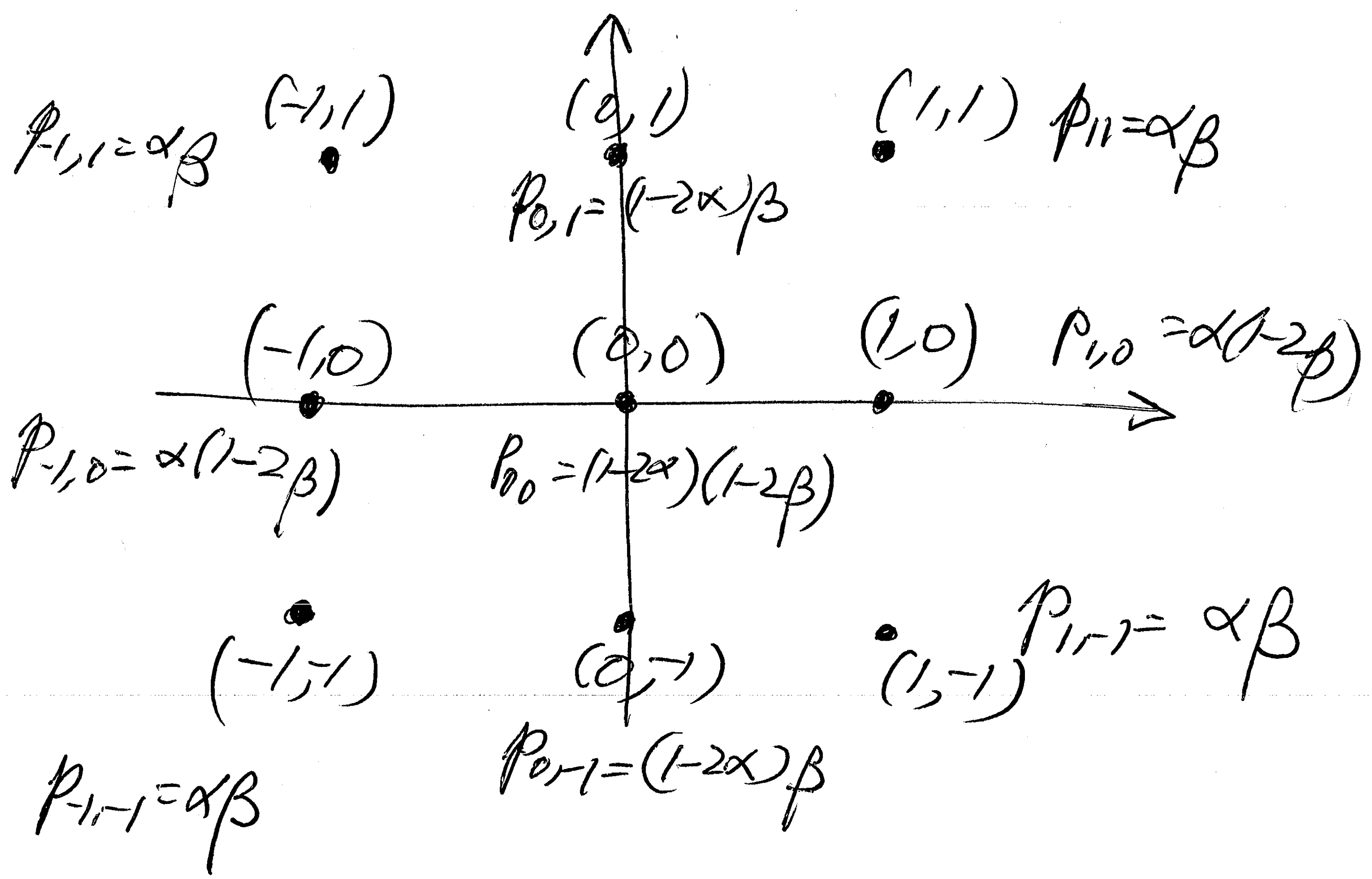
$(p_{ij} = P(X=i, Y=j))$

Hence $EXY=0 \Longleftrightarrow \boxed{p_{1,1}+p_{-1,-1} = p_{-1,1}+p_{1,-1}}$

③ Consider X, Y to be independent:

$\Updownarrow$

$$P(X=i, Y=j) = P(X=i)P(Y=j)$$



④ Modify (Pertub) the above values of $p_{ij} \longrightarrow \tilde{p}_{ij}$. while preserving

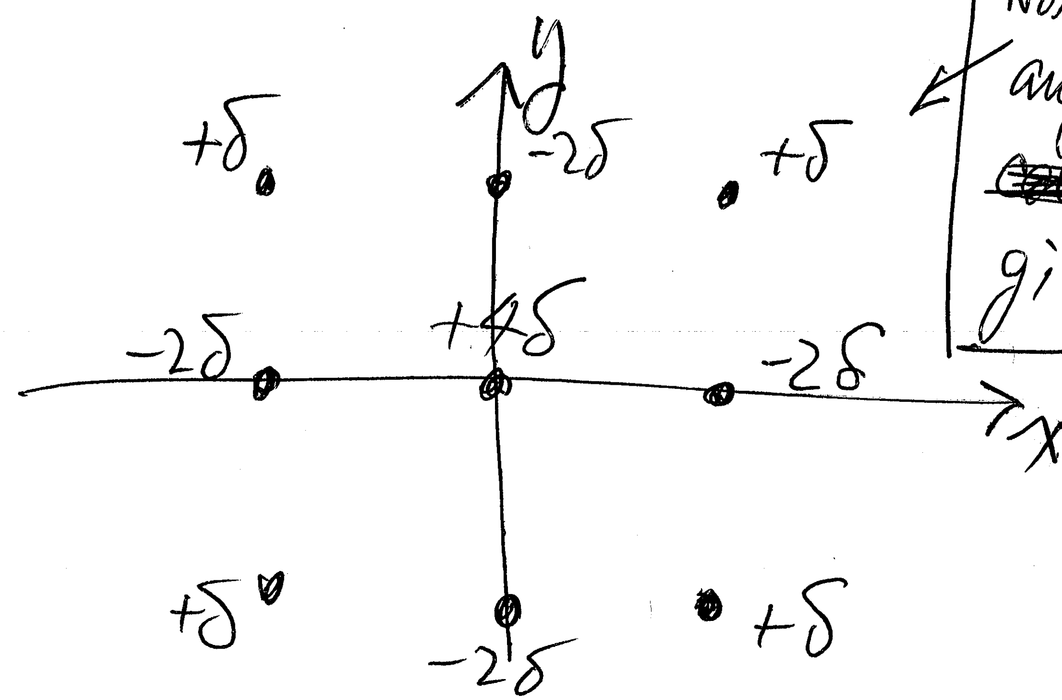
$EXY=0$ ie $\tilde{p}_{11}+\tilde{p}_{-1,-1}=\tilde{p}_{-1,1}+\tilde{p}_{1,-1}$

$EX=0$ ie $p^X_{-1}=p^X_1$ (p^X = marginal of X)

$EY=0$ ie $p^Y_{-1}=p^Y_1$ p^Y = marginal of Y

Note sum of any row & any column gives zero.

eg



$\tilde{p}_{-1,1}=p_{-1,1}+\delta$; $\tilde{p}_{0,1}=p_{0,1}-2\delta$; $p_{1,1}=p_{1,1}+\delta$

$\tilde{p}_{-1,0}=p_{-1,0}-2\delta$; $\tilde{p}_{0,0}=p_{0,0}+4\delta$; $\tilde{p}_{1,0}=p_{1,0}-2\delta$

$\tilde{p}_{-1,-1}=p_{-1,-1}+\delta$; $\tilde{p}_{0,-1}=p_{0,-1}-2\delta$; $\tilde{p}_{1,-1}=p_{1,-1}+\delta$

Then the X,Y with the new $\tilde{p}_{ij}$ are not independent.