Exercises on Larson's Problem Solving Through Problems

Carlos Salinas

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1 Heuristics

Typical heuristics that have helped people solve problems in the past

(1) Search for a pattern.
(2) Draw a figure.
(3) Formulate an equivalent problem.
(4) Modify the problem.
(5) Choose effective notation.
(6) Exploit symmetry.
(7) Divide into cases.
(8) Work backward.
(9) Argue by contradiction.
(10) Pursue parity.
(11) Consider extreme cases.
(12) Generalize.

1.1 Search for a Pattern

Virtually all problem solvers begin their analysis by getting a feel for the problem, then convincing themselves of the plausibility of the result. This is best done by examining the most immediate special cases; when this exploration is undertaken in a systematic way, patterns may emerge that will suggest ideas for proceeding with the problem.

Exercise 1.1.1. Prove that the set of n (different) elements has exactly 2^n (different) subsets.

Solution 1. We begin by examining what happens when the set contains $0, 1, 2, 3$ elements; the results are show in the following table:

n	Elements of S	Subsets of S	Number of subsets of S
0	none	$\emptyset$	1
1	x_1	$\emptyset, \{x_1\}$	2
2	x_1, x_2	$\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}$	4
3	x_1, x_2, x_3	$\emptyset, \{x_1\}, \{x_2\}, \{x_3\}$	8
		$\{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}$	

Our purpose in constructing this table is not only to verify the result, but also to look for patterns that might suggest how to proceed in the general case. Thus, we aim to be as systematic as possible. In this case, notice when $n = 3$, we have listed first the subsets of $\{x_1, x_2\}$ and then, in the second line, each of these subsets augmented by the element x_3. This is the key idea that allows us to proceed to higher values of n. For example, when $n = 4$, the subsets of $S = \{x_1, x_2, x_3, x_4\}$ are the eight subsets of $\{x_1, x_2, x_3\}$ (shown in the table) together with the eight formed by adjoining x_4 to each of these. These sixteen subsets constitute the entire collection of possibilities; thus, a set with 4 elements has $2^4 (= 16)$ subsets.

A proof based on this idea is an easy application of mathematical induction. ■

Solution 2. Another way to present the idea of the last solution is to argue as follows. For each n, let A_n denote the number of (different) subsets of a set with n (different) elements. Let S be the set with $n + 1$ elements, and designate one of its elements by x. There is a one-to-one correspondence between those subsets of S which do not contain x and those subsets that do contain x (namely, a subset T of the former type corresponds to $T \cup \{x\}$). The former types are all subsets of $S \setminus \{x\}$, a set with n elements, and therefore, it must be the case that

$$A_{n+1} = 2A_n.$$

This recurrence relation, true for $n = 0, 1, 2, 3, \ldots$, combined with the fact that $A_0 = 1$, implies that $A_n = 2^n$. ($A_n = 2A_{n-1} = 2^2 A_{n-2} = \cdots = 2^n A_0 = 2^n$.) ■

Solution 3. Another systematic enumeration of subsets can be carried out by constructing a "tree." For the case $n = 3$ and $S = \{a, b, c\}$, the tree is as shown below

Each branch of the tree corresponsd to a distinct subset of

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S.$Eachelementof$S$leadstotwopossibilities: eitheritisinthesubsetoritisnot, andthesechoicesarerepresentedbytwobranch$
$elements, thenumberofbranchesis 2 \times 2 \times 2 = 8$. For an n-element set, the number of branches is

$$\underbrace{2 \times \cdots \times 2}_{n} = 2^n;$$

thus, a set with n elements has 2^n subsets. ■

My solution. ■