Math 527 - Homotopy Theory
Spring 2013
Homework 1 Solutions

Problem 1. Show that the following conditions on a topological space X are equivalent.

1. X is contractible.
2. The identity map $\mathrm{id}_X\colon X \to X$ is null-homotopic.
3. For any space Y, every continuous map $f\colon X \to Y$ is null-homotopic.
4. For any space W, every continuous map $f\colon W \to X$ is null-homotopic.
5. For any space W, the projection map $p_W\colon W \times X \to W$ is a homotopy equivalence.

Solution. Throughout this problem, let c_z denote any constant map with value z (with appropriate source and target).

($1 \Leftrightarrow 2$) Consider the following equivalent statements.

X is contractible.

$\Leftrightarrow$ The unique map $f\colon X \to *$ is a homotopy equivalence, i.e. has a homotopy inverse $g\colon * \to X$.

$\Leftrightarrow$ There is a point $g(*) = x_0 \in X$ such that the constant map $c_{x_0} = g \circ f\colon X \to X$ is homotopic to the identity map id_X.

$\Leftrightarrow$ The map id_X is null-homotopic.

($3 \Rightarrow 2$) Particular case $Y = X$ and $f = \mathrm{id}_X$.

($2 \Rightarrow 3$) Assume $\mathrm{id}_X \simeq c_{x_0}$ for some point $x_0 \in X$. Writing f as the composite $f = f \circ \mathrm{id}_X$, we obtain

$$f = f \circ \mathrm{id}_X \simeq f \circ c_{x_0} = c_{f(x_0)}$$

so that f is null-homotopic.

($4 \Rightarrow 2$) Particular case $W = X$ and $f = \mathrm{id}_X$.

($2 \Rightarrow 4$) Assume $\mathrm{id}_X \simeq c_{x_0}$ for some point $x_0 \in X$. Writing f as the composite $f = \mathrm{id}_X \circ f$, we obtain

$$f = \mathrm{id}_X \circ f \simeq c_{x_0} \circ f = c_{x_0}$$

so that f is null-homotopic.

($5 \Rightarrow 1$) Particular case $W = *$.

($1 \Rightarrow 5$) Assume $\mathrm{id}_X \simeq c_{x_0}$ for some point $x_0 \in X$. Then the map

$$(\mathrm{id}_W, c_{x_0})\colon W \to W \times X$$

is homotopy inverse to the projection $p_W\colon W \times X \to W$. Indeed, one composite is already the identity

$$p_W \circ (\mathrm{id}_W, c_{x_0}) = \mathrm{id}_W$$

while the other composite satisfies

$$(\mathrm{id}_W, c_{x_0}) \circ p_W = \mathrm{id}_W \times c_{x_0} \simeq \mathrm{id}_W \times \mathrm{id}_X = \mathrm{id}_{W \times X}.$$

$\square$

Problem 2. Let $\mathcal{C}$ be a locally small category with finite products, including a terminal object. Let G be a group object in $\mathcal{C}$. Show that for any object X of $\mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, G)$ is naturally a group.

In other words, the structure maps of G induce a group structure on $\mathrm{Hom}_{\mathcal{C}}(X, G)$, and this assignment

$$\mathrm{Hom}_{\mathcal{C}}(-, G) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Gp}$$

is a functor.

Solution. Since the functor $\mathrm{Hom}_{\mathcal{C}}(X, -) \colon \mathcal{C} \to \mathbf{Set}$ preserves limits (in particular finite products), the set $\mathrm{Hom}_{\mathcal{C}}(X, G)$ inherits a group object structure in $\mathbf{Set}$ from G, so that $\mathrm{Hom}_{\mathcal{C}}(X, G)$ is a group. Explicitly, its multiplication map is

$$\mathrm{Hom}_{\mathcal{C}}(X, G) \times \mathrm{Hom}_{\mathcal{C}}(X, G) \cong \mathrm{Hom}_{\mathcal{C}}(X, G \times G) \xrightarrow{\mu_*} \mathrm{Hom}_{\mathcal{C}}(X, G)$$

and likewise for the unit and inverse structure maps.

To prove naturality, it suffices to prove that for any morphism $f \colon X \to Y$ in $\mathcal{C}$, the induced map of sets

$$f^* \colon \mathrm{Hom}_{\mathcal{C}}(Y, G) \to \mathrm{Hom}_{\mathcal{C}}(X, G)$$

is a group homomorphism. This follows from commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, G) \times \mathrm{Hom}_{\mathcal{C}}(X, G) & \mathrm{Hom}_{\mathcal{C}}(X, G \times G) \arrow[l, "\cong"'] \arrow[r, "\mu_*"] & \mathrm{Hom}_{\mathcal{C}}(X, G) \\
\mathrm{Hom}_{\mathcal{C}}(Y, G) \times \mathrm{Hom}_{\mathcal{C}}(Y, G) \arrow[u, "f^* \times f^*"] & \mathrm{Hom}_{\mathcal{C}}(Y, G \times G) \arrow[l, "\cong"'] \arrow[r, "\mu_*"] \arrow[u, "f^*"] & \mathrm{Hom}_{\mathcal{C}}(X, G). \arrow[u, "f^*"]
\end{tikzcd}$$

□

Problem 3. Consider S^1 as the unit circle in $\mathbb{R}^2$ with basepoint $(1,0)$, and consider the "pinch" map

$$p\colon S^1 \to S^1/S^0 \cong S^1 \vee S^1$$

which collapses the equator $S^0 \subset S^1$, i.e. identifies the points $(1,0)$ and $(-1,0)$.

a. Show that the pinch map is (pointed) homotopy coassociative. More precisely, the diagram

$$\begin{tikzcd}
S^1 \arrow[r, "p"] \arrow[d, "p"'] & S^1 \vee S^1 \arrow[d, "p\vee\mathrm{id}"] \\
S^1 \vee S^1 \arrow[r, "\mathrm{id}\vee p"'] & S^1 \vee S^1 \vee S^1
\end{tikzcd}$$

commutes up to pointed homotopy.

Solution. Denote the two summand inclusions by $\iota_i\colon S^1 \to S^1 \vee S^1$ for $i = 1,2$. Using the model $S^1 \cong I/\partial I$, the pinch map can be explicitly written as

$$p(s) = \begin{cases} \iota_1(2s) & \text{if } 0 \le s \le \frac{1}{2} \\ \iota_2(2s-1) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

The composite via the top right is

$$(p \vee \mathrm{id}) \circ p(s) = \begin{cases} \iota_1(4s) & \text{if } 0 \le s \le \frac{1}{4} \\ \iota_2(4s-1) & \text{if } \frac{1}{4} \le s \le \frac{1}{2} \\ \iota_3(2s-1) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

whereas composite via the bottom left is

$$(\mathrm{id} \vee p) \circ p(s) = \begin{cases} \iota_1(2s) & \text{if } 0 \le s \le \frac{1}{2} \\ \iota_2(4s-2) & \text{if } \frac{1}{2} \le s \le \frac{3}{4} \\ \iota_3(4s-3) & \text{if } \frac{3}{4} \le s \le 1. \end{cases}$$

The formula

$$H(s,t) = \begin{cases} \iota_1(\frac{4s}{1+t}) & \text{if } 0 \le s \le \frac{1+t}{4} \\ \iota_2(4s-1-t) & \text{if } \frac{1+t}{4} \le s \le \frac{2+t}{4} \\ \iota_3(\frac{4s-2-t}{2-t}) & \text{if } \frac{2+t}{4} \le s \le 1 \end{cases}$$

defines a pointed homotopy $H\colon S^1 \times I \to S^1 \vee S^1 \vee S^1$ from $(p \vee \mathrm{id}) \circ p$ to $(\mathrm{id} \vee p) \circ p$. $\square$

In fact, a very similar argument shows that S^1 is a homotopy cogroup object in $\mathbf{Top}_*$. (Do not show this.) Comultiplication is the pinch map $S^1 \to S^1 \vee S^1$, the counit is the constant map $S^1 \to *$, and the coinverse $S^1 \to S^1$ reverses the last component (viewed in $\mathbb{R}^2$).

b. Conclude that for any pointed space (X, x_0), the set $\pi_1(X, x_0)$ is naturally a group.

More precisely, the structure maps of S^1 as homotopy cogroup object induce a group structure on $\pi_1(X, x_0)$, and moreover this assignment defines a (covariant) functor $\pi_1 \colon \mathbf{Top}_* \to \mathbf{Gp}$.

Solution. Since S^1 is a homotopy cogroup object in $\mathbf{Top}_*$, it becomes a cogroup object in the homotopy category $\mathrm{Ho}(\mathbf{Top}_*)$, and therefore a group object in the opposite category $\mathrm{Ho}(\mathbf{Top}_*)^{\mathrm{op}}$. We have

$$\begin{aligned}
\pi_1(X, x_0) &= [S^1, (X, x_0)]_* \\
&= \mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top}_*)}\big(S^1, (X, x_0)\big) \\
&= \mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top}_*)^{\mathrm{op}}}\big((X, x_0), S^1\big)
\end{aligned}$$

which is naturally a group, by Problem 2. □

Problem 4. For pointed spaces, show that the smash product distributes over the wedge. More precisely, there is a natural isomorphism

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z).$$

of pointed spaces. Don't forget to argue that the isomorphism is natural.

Solution. Recall that the wedge is the coproduct in $\mathbf{Top}_*$. Applying the functor $X \wedge -$ to the natural summand inclusions $Y \to Y \vee Z$ and $Z \to Y \vee Z$ produces maps

$$X \wedge Y \to X \wedge (Y \vee Z)$$

$$X \wedge Z \to X \wedge (Y \vee Z)$$

which together define a natural map

$$\varphi \colon (X \wedge Y) \vee (X \wedge Z) \to X \wedge (Y \vee Z)$$

of pointed spaces. It remains to show that φ is an homeomorphism, thus an isomorphism in $\mathbf{Top}_*$.

To construct the inverse of φ, consider the map ψ in the commutative diagram

$$\begin{tikzcd}
X \wedge (Y \vee Z) \arrow[r, dashed, "\widetilde{\psi}"] & (X \wedge Y) \vee (X \wedge Z) \\
X \times (Y \vee Z) \arrow[u, two heads] & (X \wedge Y) \amalg (X \wedge Z) \arrow[u] \\
X \times (Y \amalg Z) \arrow[u, two heads] \arrow[uur, "\psi"] & (X \times Y) \amalg (X \amalg Z) \arrow[l, "\cong"'] \arrow[u]
\end{tikzcd}$$

in $\mathbf{Top}$. The bottom is an isomorphism because the functor $X \times - \colon \mathbf{Top} \to \mathbf{Top}$ preserves (arbitrary) coproducts.

Now the map

$$X \times (Y \amalg Z) \to X \times (Y \vee Z)$$

is a quotient map, hence so is the composite

$$X \times (Y \amalg Z) \to X \times (Y \vee Z) \to X \wedge (Y \vee Z).$$

on the left-hand side of the diagram. Since ψ is constant on the equivalence classes defined by this quotient map, it induces a unique *continuous* map

$$\widetilde{\psi} \colon X \wedge (Y \vee Z) \to (X \wedge Y) \vee (X \wedge Z)$$

making the diagram commute.

By construction, $\widetilde{\psi}$ is clearly the set-theoretic inverse of φ. $\square$