MA166 Recitation Notes and Exercises

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1 Notes: Vectors and the Geometry of Spaces

Material found in Stewart §12.

1.1 Three-Dimensional Coordinate Systems

Here are some of the most important concepts, equations, and theorems from this section. I know. I know. These are very boring concepts that you have probably seen all your life and you know how to do. But we must start somewhere and here is a perfect place.

The distance between two points $P_1(x, y, z)$ and $P_2(x, y, z)$ in $\mathbb{R}^3$ is given by the formula

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1}$$

This is also called the *Euclidean norm* and generalizes to all dimensions. Note that equation (1) is equivalent to

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = |P_2P_1|$$

so that the distance between point does not depend on your point-of-view, i.e, whether you think of the line starting connecting P_1 and P_2 as starting at P_1 and ending at P_2 or vice-a-versa.

We often refer to the point $P_1(x, y, z)$ as the tuple (x_1, y_1, z_2) and $P_2(x, y, z)$ as (x_2, y_2, z_2), $P_3(x, y, z)$ as (x_3, y_3, z_3) and so on.

The equation of a sphere with $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2. \tag{2}$$

In particular, if the center is the origin O, then the equation (2) reduces to

$$x^2 + y^2 + z^2 = r^2.$$

1.2 Vector

A particle moves along a line segment from point A to point B. The corresponding displacement vector $\mathbf{v}$ has initial point A and terminal point B and is written $\mathbf{v} = \overrightarrow{AB}$.

Combining Vectors

If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the sum $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

1.3 The Cross Product

Definition 1. If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, then the *cross product* of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle.$$

Theorem 1. *The vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, i.e., $\mathbf{w} \cdot \mathbf{v} = 0$ and $\mathbf{w} \cdot \mathbf{u}$.*

Theorem 2. *If θ is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *(so $0 \leq \theta \leq \pi$), then*

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta.$$

Theorem 3. *Two nonzero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel if and only if*

$$\mathbf{v} \times \mathbf{u} = \mathbf{0}$$

The length of the product $\mathbf{v} \times \mathbf{u}$ is equal to the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

Theorem 4. *If* $\mathbf{u}$, $\mathbf{v}$, *and c are vectors and c is a scalar, then*

(a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

(b) $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v})$.

(c) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

(d) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.

(e) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

(f) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

The volume of a parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v}$ and is the magnitude of their scalar triple product:

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|. \tag{3}$$

Torque

Consider a force $\mathbf{F}$ acting on a rigid body at a point given by the position vector $\mathbf{r}$. The torque (relative to the origin) is defined to be

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \tag{4}$$

2 Exercises Week 2

Interesting exercises (by that, the professor usually means, tricky or had exercises).

Exercise 2.1 (Stewart §12.3, 31)**.** Find the acute angles at the points of intersection

$$y = x^2, \quad y = x^3.$$

Solution. Remember that the angle between two curves at a point of intersection is the angle made by the tangent lines to those curves at that point.

Let me explain how to do this. In the given exercise we are asked to find the acute angles at the points of intersection of the curves $y = x^2$ and $y = x^3$. We first need to find the points where they intersect, i.e., where

$$x^3 = x^2. \tag{5}$$

A little algebra and we can turn (5) into $0 = x^3 - x^2 = x^2(x-1)$ so the solutions to (5) are $x = 0$ or $x = 1$. Now we need to find the tangent line to the curve $y = x^3$ and $y = x^2$ at $x = 0$ and $x = 1$.

Remember that to find the tangent line of a function $f(x)$ whose derivative exists, at a point x_0 we have to first, compute its derivative, and second, compute $f(x_0)$. After that, the tangent line of $f(x)$ at x_0 will have the form

$$y - f(x_0) = f'(x_0)(x - x_0).$$

To avoid confusing $y = x^3$, $y = x^2$ with their tangent lines, let's call the first one $f_1(x)$ and the second one $f_2(x)$; I mean, $f_1(x) = x^3$ and $f_2(x) = x^2$; this is function notation and you should be familiar with it.

Moving on, we'll start with the intersection at $x = 1$. At $x = 1$, $f_1(1) = 1$ and $f_2(1) = 1$. Moreover, the derivatives of f_1 and f_2 are

$$\begin{aligned} f_1'(x) &= 3x^2 & \qquad f_2'(x) &= 2x \\ f_1'(1) &= 3 \cdot 1^2 & \qquad f_2'(1) &= 2 \cdot 1. \end{aligned}$$

Hence, $f_1'(1) = 3$ and $f_2'(1) = 2$. So the tangent lines to f_1 and f_2 at 1 are

$$y - 1 = 3(x-1) \qquad\qquad y - 1 = 2(x-1)$$

which, in more standard form, is

$$y = 3x - 2 \qquad\qquad y = 2x - 1. \tag{6}$$

As if that wasn't enough we still have to find the y-intercept for these tangent lines and find the vector pointing from the their y-intercept to the point of intersection. To find the y-intercept of the tangent lines in 6 we simply plug in 0 for x and we have $y = -2$ and $y = -1$ so the y-intercepts are $(0, -2)$ and $(0, -1)$ for the tangent line of $f_1(x)$ at 1 and the tangent line of $f_2(x)$ at 1, respectively. Then, we have the vectors $\langle 1, 1\rangle - \langle 0, -2\rangle = \langle 1, 3\rangle$ and $\langle 1, 1\rangle - \langle 0, -1\rangle = \langle 1, 2\rangle$ so we compute the angle between them by the dot-product formula, i.e.,

$$\cos\theta = \frac{\langle\langle 1,3\rangle \cdot \langle 1,2\rangle\rangle}{|\langle 1,3\rangle| \cdot |\langle 1,2\rangle|} = \frac{1+6}{\sqrt{10}\cdot\sqrt{5}} = \frac{7}{5\sqrt{2}}.$$

Now just take $\cos^{-1}$ of both sides and we have $\theta = \tan^{-1}(7/5\sqrt{2}) \approx 8.13°$. ■