MA571 Problem Set 5

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Problem 5.1 (Munkres §23, Ex. 3)

Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \emptyset$ for all α, then $A \cup (\bigcup A_\alpha)$ is connected.

Proof. We shall aim to prove this result by using Theorem 23.3 from Munkres. Define the collection $\{B_\alpha\}$ by setting $B_\alpha = A \cup A_\alpha$. Note that by Theorem 23.3, B_α is connected for all α, since $A \cap A_\alpha \neq \emptyset$ and both A and A_α are connected. Next observe that the intersection $B_\alpha \cap B_\beta \neq \emptyset$ for all α and β, in particular, the subspace A is contained in the intersection since $A \subset B_\alpha$ and $A \subset B_\beta$ for all α and β. Therefore, $\{B_\alpha\}$ is a collection of connected subspaces of X that have a point in common. Applying Theorem 23.3 one last time, we see that the union

$$\bigcup B_\alpha = \bigcup (A \cup A_\alpha) = A \cup \left(\bigcup A_\alpha\right)$$

is connected. ■

Problem 5.2 (Munkres §23, Ex. 6)

Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X \setminus A$, then C intersects ∂A.

Proof. We shall proceed by contradiction. Suppose that $C \cap \partial A = \emptyset$, then we shall show that the pair $C \cap A$ and $C \cap (X \setminus A)$ forms a separation of C. Recall that by definition (see Munkres §17, p. 102) the boundary $\partial A = \overline{A} \cap \overline{X \setminus A}$. Then we claim that $\overline{A} = \partial A \cup \operatorname{int} A$:

Lemma 13. *Let X be a topological space and $A \subset X$. Then ∂A and $\operatorname{int} A$ are disjoint and $\overline{A} = \partial A \cup \operatorname{int} A$.*

Proof of lemma. The point $x \in \partial A$ if and only if $x \in \overline{A}$ and $x \in \overline{X \setminus A}$. Thus, for every neighborhood U of x, the intersection $U \cap X \setminus A \neq \emptyset$, in particular $U \not\subset A$ so x is not an interior point of A. Hence, we see that $\partial A \cap \operatorname{int} A = \emptyset$. To prove the last statement note that $\partial A \subset \overline{A}$ and $\operatorname{int} A \subset A \subset \overline{A}$ (cf. Munkres §17, p. 95), so that $\partial A \cup \operatorname{int} A \subset \overline{A}$ hence, it suffices to show the reverse inclusion, namely, $\overline{A} \subset \partial A \cup \operatorname{int} A$. Let $x \in \overline{A}$. If $x \in \operatorname{int} A$, then clearly $x \in \partial A \cup \operatorname{int} A$. Suppose $x \notin \operatorname{int} A$. Then, by Theorem 17.5(a), for every neighborhood U of x, the intersection $U \cap A \neq \emptyset$ and $U \not\subset A$. Thus, $U \cap (X \setminus A) \neq \emptyset$ so $x \in \overline{X \setminus A}$. It follows that $x \in \overline{A} \cap \overline{X \setminus A} = \partial A$. ♣

Lemma 14. *Let X be a topological space and $A \subset X$. Then $\partial A = \partial(X \setminus A)$.*

Proof of lemma. Replace A by $X \setminus A$ in the definition of the boundary of A. Then we have:

$$\begin{aligned}
\partial(X \setminus A) &= \overline{X \setminus A} \cap \overline{X \setminus (X \setminus A)} \\
&= \overline{X \setminus A} \cap \overline{A} \\
&= \overline{A} \cap \overline{X \setminus A} \\
&= \partial A.
\end{aligned}$$

♣

Now, by Theorem 17.4, we have that $\overline{C \cap A} = C \cap \overline{A}$ and $\overline{C \cap (X \setminus A)} = C \cap \overline{X \setminus A}$. But by Lemma 13 and Lemma 14, the latter sets are equivalent to $\overline{C \cap A} = C \cap (\partial A \cup \operatorname{int} A)$ and $\overline{C \cap (X \setminus A)} = C \cap (\partial A \cup \operatorname{int}(X \setminus A))$. But since $C \cap \partial A = \emptyset$ by assumption, we have

$$\begin{aligned}
\overline{C \cap A} \cap (C \cap (X \setminus A)) &= (C \cap (\partial A \cup \operatorname{int} A)) \cap (C \cap (X \setminus A)) \\
&= ((C \cap \partial A) \cup (C \cap \operatorname{int} A)) \cap (C \cap (X \setminus A)) \\
&= (C \cap \operatorname{int} A) \cap (C \cap (X \setminus A)) \\
&= \emptyset
\end{aligned}$$

since $C \cap \operatorname{int} A \subset A$ and $C \cap (X \setminus A) \subset X \setminus A$. Similarly, we have that the intersection $\overline{C \cap (X \setminus A)} \cap (C \cap A) = \emptyset$. So by Lemma 23.1, $C \cap A$ and $C \cap (X \setminus A)$ form a separation of C. This contradicts the assumption that C is connected. Therefore, we conclude that $C \cap \partial A \neq \emptyset$. ■

PROBLEM 5.3 (MUNKRES §23, EX. 7)

Is the space $\mathbf{R}_\ell$ connected? Justify your answer.

Proof. No. The space $\mathbf{R}_\ell$ is not connected and we may exhibit an explicit separation. Namely, consider the basis elements $(-\infty, 0)$ and $[0, \infty)$. Then $\mathbf{R} = (-\infty, 0) \cup [0, \infty)$, hence $(-\infty, 0)$ and $[0, \infty)$ form a separation of $\mathbf{R}$ with the lower limit topology.

Alternatively, one may note that $\mathbf{R} \setminus (-\infty, 0) = [0, \infty)$ is open in $\mathbf{R}_\ell$ so $(-\infty, 0)$ is both open and closed. Hence, by Munkres's alternative formulation of connectedness (cf. Munkres §23, p. 148 the italicized paragraph), $\mathbf{R}_\ell$ is disconnected. ∎

Problem 5.4 (Munkres §23, Ex. 9)

Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that

$$(X \times Y) \smallsetminus (A \times B)$$

is connected.

Proof. Consider the family of embeddings $\{i_\alpha\}$ where $i_\alpha \colon X \hookrightarrow X \times Y$ maps $x \mapsto x \times y_\alpha$ for $y_\alpha \notin B$, for all α. By Theorem 23.5, $i_\alpha(X) = X \times y_\alpha$ is connected subspace of $X \times Y$. Moreover $X \times y_\alpha \subset (X \times Y) \smallsetminus (A \times B)$ so $X \times y_0$, in particular, we have that is a connected subspace of $(X \times Y) \smallsetminus (A \times B)$. Similarly, consider the family of embeddigs $\{j_\alpha\}$ where $j_\alpha \colon Y \hookrightarrow X \times Y$ maps $y \mapsto x_\alpha \times y$ for $x_\alpha \notin A$. We similarly have that $j_\alpha(Y) = x_\alpha \times Y$ is a connected subspace of $(X \times Y) \smallsetminus (A \times B)$. Then we claim that

$$(X \times Y) \smallsetminus (A \times B) = \bigcup (X \times y_\alpha) \cup (x_\beta \times Y).$$

It is clear that the union on the right is a subset of $(X \times Y) \smallsetminus (A \times B)$ since each $X \times y_\alpha$ and $x_\beta \times Y$ is a subset of $(X \times Y) \smallsetminus (A \times B)$. To see the reverse containment, take $x \times y$ in the union $\bigcup (X \times y_\alpha) \cup (x_\beta \times Y)$. Then $x \times y$ is in some $(X \times y_\alpha) \cup (x_\beta \times Y)$ so $x \times y \in X \times y_\alpha$ or $x \times y \in x_\beta \times Y$. If $x \times y \in \bigcup X \times y_\alpha$, then $y_\alpha \notin B$ so $x \times y \notin A \times B$, hence $x \times y \in (X \times Y) \smallsetminus (A \times B)$. If $x \times y \in \bigcup x_\beta \times Y$ then $x \notin A$, hence $x \times y \notin A \times B$ so $x \times y \in (X \times Y) \smallsetminus (A \times B)$. Thus, we have that $(X \times Y) \smallsetminus (A \times B) = \bigcup (X \times y_\alpha) \cup (x_\beta \times Y)$. Then, note that by Theorem 23.3, since $X \cap y_\alpha \cap x_\beta \cap Y \neq \emptyset$, in particular, $x_\beta \times y_\alpha$ is in the intersection, $(X \times y_\alpha) \cup (x_\beta \times Y)$ is connected for all α and all β. Thus, the subspace $(X \times Y) \smallsetminus (A \times B)$ is connected. ∎

Problem 5.5 (Munkres §24, Ex. 1(ac))

(a) Show that no two of the spaces $(0,1)$, $(0,1]$ and $[0,1]$ are homeomorphic. [*Hint:* What happens if you remove a point from each of these spaces?]
(c) Show $\mathbf{R}^n$ and $\mathbf{R}$ are not homeomorphic if $n > 1$.

Proof. (a) Suppose $\varphi\colon (0,1] \to (0,1)$ is a homeomorphism. We claim that the restriction of φ to $(0,1) \subset (0,1]$ gives a homeomorphism to $(0,1) \setminus \{\varphi(1)\}$, more generally, the following result holds:

Lemma 15. *Suppose $\varphi\colon X \to Y$ is a homeomorphism and $U \subset X$. Then the restriction $\varphi|_U\colon U \to \varphi(U)$ is a homeomorphism.*

Proof of lemma. The restriction $\varphi_U = \varphi|_U\colon U \to \varphi(U)$ has a canonical inverse, namely, $\varphi_U^{-1} = \varphi^{-1}|_{\varphi(U)}\colon \varphi(U) \to U$ since φ is a bijection. By Theorem 18.2(d,e) both φ_U and φ_U^{-1} are continuous hence, $U \approx \varphi(U)$. ♣

Now remove 1 from $(0,1]$. Then, since $\varphi(1)$ is bijective, there exists $y \in (0,1)$ such that $\varphi(1) = y$ with $0 < y < 1$. Then $(0,1) \setminus \{y\} = (0,y) \cup (y,1)$ is disconnected, but $(0,1] \setminus \{1\} = (0,1)$ is connected. This contradicts Theorem 23.5 that the image of $(0,1]$ under a continuous map is connected. The same argument shows that $(0,1) \not\approx [0,1]$ (in fact, if we allow ourselves results from §26 and §27 we have that $[0,1]$ is compact by 27.3 (Heine–Borel), but $(0,1)$ is not compact, by 26.5 it follows that they are not homeomorphic).

Similarly, if $[0,1] \approx (0,1]$ via φ then $[0,1] \setminus \{0,1\} \approx (0,1] \setminus \{\varphi(0), \varphi(1)\}$.

(b) From Example 4 of §24, the punctured Euclidean space $\mathbf{R} \setminus \{\mathbf{0}\}$ is path-connected, in particular, connected. But $\mathbf{R}$ minus a point is disconnected. More precisely, if $\mathbf{R}^n \approx \mathbf{R}$ via φ, by Lemma 15, $\mathbf{R}^n \setminus \{0\} \approx \mathbf{R} \setminus \{\varphi(0)\}$, but $\mathbf{R} \setminus \{\varphi(0)\}$ is disconnected, contradicting Theorem 23.5. ■

***Remarks**.* I realized too late that Lemma 15 here is the same as Lemma A given to us in lecture.

Problem 5.6 (Munkres §24, Ex. 2)

Let $f\colon S^1 \to \mathbf{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

Proof. Consider the map $g\colon S^1 \to \mathbf{R}$ given by $g(x) = f(x) - f(-x)$. This map is continuous by Lemma 9(i) (proved on Homework 4 which showed that if f, g are continuous real valued maps on a metric space X then (i) $f + g$ and (ii) fg are continuous; moreover S^1 is naturally a metric space as a subspace of $\mathbf{R}^2$ which is how Munkres defines it in Example 5 on §24). Fix $x_0 \in S^1$ and suppose, without loss of generality, that $g(x_0) > 0$ (for if $g(x_0) = 0$ we are done, i.e, $f(x_0) = f(-x_0)$ and if $g(x_0) < 0$ we reverse the direction of $<$ in the following argument). Then

$$g(-x_0) = f(-x_0) - f(-(-x_0)) = -f(x_0) + f(-x_0) = -g(x_0).$$

Then $g(-x_0) = -g(x_0) < g(x_0)$ and by the Intermediate Value Theorem (Theorem 24.3) there exists $y \in S$ such that $g(y) = 0$, i.e, $f(y) = f(-y)$. ∎

Problem 5.7 (Munkres §25, Ex. 2(b))

(b) Consider $\mathbf{R}^\omega$ in the uniform topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same component of $\mathbf{R}^\omega$ if and only if the sequence
$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, ...)$$
is bounded. [*Hint:* It suffices to consider the case where $\mathbf{y} = \mathbf{0}$.]

Proof. $\Longrightarrow$ Suppose that $\mathbf{x}$ and $\mathbf{y}$ lie in the same connected component, call it C. Consider the set
$$U = \{\, \mathbf{z} \mid \mathbf{x} - \mathbf{z} \text{ is bounded} \,\} \quad \text{and} \quad X \smallsetminus U = \{\, \mathbf{z} \mid \mathbf{z} - \mathbf{z} \text{ is unbounded} \,\}.$$
These sets are open in the uniform topology since, in the case of U: for any $\mathbf{z} \in U$ for $\varepsilon = 1/2$, the open ball $B_{\bar\rho}(\mathbf{z}, \varepsilon) \subset U$ since for every $\mathbf{z}' \in B_{\bar\rho}(\mathbf{z}, \varepsilon)$ the difference $\mathbf{x} - \mathbf{z}'$ is bounded, i.e.,
$$\begin{aligned}\bar\rho(\mathbf{x}, \mathbf{z}') &\le \bar\rho(\mathbf{x}, \mathbf{z}) + \bar\rho(\mathbf{z}, \mathbf{z}') \\ &\le M + \varepsilon \\ &= M + \frac{1}{2}\end{aligned}$$
for some positive real number $M < \infty$. In the case of $X \smallsetminus U$: Take $\mathbf{z} \in X \smallsetminus U$ and take the open ball of radius $1/2$, $B_{\bar\rho}(\mathbf{z}, 1/2)$ then if $\mathbf{z}' \in B_{\bar\rho}(\mathbf{z}, 1/2)$ then, by the Reverse Triangle Inequality, the sequence
$$\begin{aligned}|x_n - z_n'| &\ge ||x_n - z_n| - |x_n - z_n'|| \\ &\ge \left| |x_n - z_n| - \frac{1}{2} \right| \\ &> \left| M - \frac{1}{2} \right|\end{aligned}$$
for every positive real number M, hence $\mathbf{x} - \mathbf{z}'$ is unbounded. Then, by Theorem 23.2, $C \subset U$ or $C \subset (X \smallsetminus U)$. But $\mathbf{x} \in U$ so $C \subset U$, i.e., $\mathbf{x} - \mathbf{y}$ is bounded.

$\Longleftarrow$ Suppose $\mathbf{x} - \mathbf{y}$ is bounded. Then for every $\varepsilon > 0$, there exists a positive integer N such that $|x_n - y_n| < \varepsilon$ for all $n \ge N$. We will show that there exists a path $f\colon [0, 1] \to \mathbf{R}^\omega$ from $\mathbf{x}$ to $\mathbf{y}$. Consider the path
$$f(t) = \mathbf{x} - t(\mathbf{x} - \mathbf{y}).$$
Note that $f(0) = \mathbf{x}$ and $f(1) = \mathbf{y}$. Now we show that f is indeed continuous. By Lemma 8 (the equivalence of the ε-δ definition to the metric topology which I proved on Homework 4) it suffices to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $s, s' \in B(t, \delta)$ implies $f(s), f(s') \in B_{\bar\rho}(\mathbf{z}, \varepsilon)$. In particular, chose $\delta = \varepsilon/M$ where $M > |x_n - y_n|$ for all n, then we have
$$\begin{aligned}|f(s) - f(s)'| &= |\mathbf{x} - s(\mathbf{x} - \mathbf{y}) - (\mathbf{x} - s'(\mathbf{x} - \mathbf{y}))| \\ &= |(s - s')(\mathbf{x} - \mathbf{y})| \\ &= |(s - s')||\mathbf{x} - \mathbf{y}| \\ &< \delta|\mathbf{x} - \mathbf{y}| \\ &< \varepsilon.\end{aligned}$$

Thus x and y are path-connected, in particular, by Theorem 25.2, x and y lie in the same connected component. ■

Problem 5.8 (Munkres §25, Ex. 4)

Let X be locally path connected. Show that every connected open set in X is path connected.

Proof. First we prove the following claim:

Claim. *If U is an open subset of X, then it is locally path-connected.*

Proof of claim. Let $x \in U$ and let $V \subset U$ be a neighborhood of x then, by Lemma 16.2, since V is open in X and X is locally path-connected, there exists path-connected neighborhood W of x contained in V, hence contained in U. Thus, U is locally path-connected. ♠

Now, suppose U is a connected open subset of X. Then U has one component by Theorem 25.2. Moreover, by Theorem 25.5, since U is locally path-connected the components of U and path-components are equivalent. Thus, U has exactly one path component, i.e, U is path-connected. ■

Problem 5.9 (Munkres §25, Ex. 6)

A space X is said to be *weakly locally path connected at x* if for every neighborhood U of x, there is a connected subspace of X contained in U that contains a neighborhood of x. Show that if X is weakly locally connected at each of its points, then X is locally connected. [*Hint:* Show that components of open sets are open.]

Proof. By Theorem 25.5, it suffices to show that for every open set U of X, each component of U is open in X. Let $x \in U$. Then, by Theorem 25.2, x lies in some component of U, say C. Since X is weakly locally path-connected, there is a connected subspace, say C_x, contained in U that contains a neighborhood V_x of x. Then by Theorem 25.2, $C_x \subset C$. In particular, for every $x \in C$ we have a neighborhood V_x of x contained in C,i.e., C is the union $C = \bigcup_{x\in C} V_x$ of open subsets. Thus, C is open in X. ■

Problem 5.10 (A)

Let X be a topological space. The quotient space $(X \times [0,1])/(X \times 0)$ is called the *cone* of X and denoted CX.

Prove that if X is homeomorphic to Y then CX is homeomorphic to CY (*Hint:* There are maps in both directions).

Proof. Let $\varphi\colon X \to Y$ be a homeomorphism and let p and q denote the quotient maps the pairs $(X \times [0,1], CX)$ and $(Y \times [0,1], CY)$, respectively. Then we get a canonical homeomorphism $\Phi\colon X \times [0,1] \to Y \times [0,1]$ given by the map $(x,z) \mapsto (\varphi(x), z)$. Note that Φ is continuous, by Theorem 18.4, since φ and $\mathrm{id}_{[0,1]}$ are continuous and its inverse is given by $\Phi^{-1} = (\varphi^{-1}, \mathrm{id}_{[0,1]})$ (which is continuous by 18.4). Now, we claim that the map $\Phi^*\colon CX \to CY$ given by $[(x,z)] \mapsto [\Phi(x,z)] = [(\varphi(x), z)$ defines a homeomorphism $CX \approx CY$.

First we will prove that Φ^* is well-defined. Fix an equivalence class $[(x,z)]$ in CX and choose two representatives (x_1, z_1) and (x_2, z_2) of $[(x,z)]$ in $X \times [0,1]$. Then, by the definition of the quotient space (cf. Homework 4, Problem F), $(x_1, z_1) \sim (x_2, z_2)$ if and only if $(x_1, z_1) = (x_2, z_2)$ or $z_1 = z_2 = 0$, i.e, $\{(x_1, z_1), (x_2, z_2)\} \subset X \times 0$. In the former case $\Phi(x_1, z_1) = \Phi(x_2, z_2) = (\varphi(x_1), z_1)$ and we see that

$$\Phi^*([(x_1, z_1)] = [\Phi(x_1, z_1)] = [(\varphi(x_1), z_1)] = [\Phi(x_2, z_2)] = \Phi^*([(x_2, z_2)])$$

and in the latter $\Phi(x_1, 0) = (\varphi(x_1), 0)$ and $\Phi(x_2, 0) = (\varphi(x_2), 0)$ so $(\varphi(x_1), 0) \sim (\varphi(x_2), 0)$, hence

$$\Phi^*([(x_1, 0)] = [\Phi(x_1, 0)] = [(\varphi(x_1), 0)] = [\Phi(x_2, 0)] = \Phi^*([(x_2, 0)]).$$

Thus Φ is well-defined.

Now we will show that Φ^* is a continuous bijection and with a continuous inverse. To show bijectivity we construct an explicit inverse, namely, define $(\Phi^*)^{-1}\colon CY \to CX$ by $[(y,z)] \mapsto [\Phi^{-1}(y,z)] = [\varphi^{-1}(x), z]$. The map $(\Phi^*)^{-1}$ is clearly well-defined (by a similar argument to showing that Φ is well-defined) and we have that

$$\begin{aligned} \Phi^* \circ (\Phi^*)^{-1}([y,z]) &= \Phi^*([\Phi^{-1}(y,z)] \\ &= [\Phi(\Phi^{-1}(y,z)] \\ &= [(y,z)] \\ &= \mathrm{id}_{CY} \end{aligned} \qquad \begin{aligned} (\Phi^*)^{-1} \circ \Phi^*([x,z]) &= (\Phi^*)^{-1}([\Phi(x,z)]) \\ &= [\Phi^{-1}(\Phi(x,z))] \\ &= [(x,z)] \\ &= \mathrm{id}_{CX}\,. \end{aligned}$$

It is clear that Φ^* is continuous since, by Theorem Q.2, $\Phi^* \circ p = q \circ \Phi$ is continuous. Let $U_\sim$ be open in CX. Then $U = p^{-1}(U_\sim)$ is open in $X \times [0,1]$ then $\Phi(U)$ is open in $Y \times [0,1]$ since Φ is a homeomorphism. The same argument applies to showing that $(\Phi^*)^{-1}$ is continuous in the reverse direction, that is, consider the composition $(\Phi^*)^{-1} \circ q = p \circ \Phi^{-1}$ and apply Theorem Q.2. ■