MA557 Problem Set 3

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Problem 3.1

Find an example of a finitely generated ring extension $R \subset S$ where S is a Noetherian ring, but R is not.

Proof. ■

Problem 3.2

Consider the homomorphism of rings

$$\begin{array}{ccc} & & S \\ & & \downarrow{\scriptstyle\psi} \\ R & \xrightarrow{\varphi} & T. \end{array}$$

The *fiber product* of R and S over T is the subring $R\times_T S = \{\,(r,s) \mid \varphi(t) = \psi(s)\,\}$ of $R\times S$. Assume φ and ψ are surjective. Show that if R and S are Noetherian rings then so is $R\times_T S$.

Proof. Suppose that R and S are Noetherian rings with surjective ring maps $\varphi\colon R\to T$ and $\psi\colon S\to T$. Then, by (3.5), the product $R\times S$ is Noetherian. Define the ring map $\Phi\colon R\times S\to T\times T$ by $\Phi = (\varphi,\psi)$. Then the diagonal, $\Delta_T = \{\,(t,t)\mid t\in T\,\}$, of $T\times T$ is exactly the image of the fiber product of R and S under the ring map Φ. And this is not terribly difficult to see: It is clear, by the definition of the fiber product, that $\Phi(R\times_T S)\subset\Delta_T$. To show the reverse containment, take an element $(t,t)\in\Delta_T$. Then, since φ and ψ are surjective, there are corresponding elements r and s of the rings R and S, respectively, such that $\varphi(r)=t$ and $\psi(s)=t$. Hence, (t,t) are in the image $R\times_T S$ under Φ.

Now, consider the quotient $R\times S$-module $T\times T/\Delta_T$. Define the homomorphism $\Phi^*\colon R\times S\to T\times T/\Delta_T$ given by the composition $\Phi^*=\pi\circ\Phi$ where π is the canonical projection map of $T\times T$ onto the quotient module $T\times T/\Delta_T$. This map is an $R\times S$-linear map, i.e., take $(r,s)\in R\times S$ as a ring and $(r_1,s_1),(r_2,s_2)\in R\times S$ as an $R\times S$-module, then

$$\begin{aligned}
\Phi^*((r_1,s_1)+(r,s)(r_2,s_2)) &= \Phi^*((r_1,s_1)+(rr_2,ss_2))\\
&= \Phi^*(r_1+rr_2,s_1+ss_2)\\
&= \pi(\varphi(r_1+rr_2),\psi(s_1+ss_2))\\
&= \pi(\varphi(r_1)+\varphi(r)\varphi(r_2)),\psi(s_1)+\psi(s)\psi(s_2)))\\
&= (\pi\circ\varphi(r_1)+(\pi\circ\varphi(r))(\pi\circ\varphi(r_2))),\pi\circ\psi(s_1)+(\pi\circ\psi(s))(\pi\circ\psi(s_2))))\\
&= \pi\circ\varphi((r_1,s_1))+\pi\circ\varphi((r,s))\pi\circ\varphi((r_2,s_2)),\\
&= \Phi^*((r_1,s_1))+\Phi^*((r,s))\Phi^*((r_2,s_2)),
\end{aligned}$$

with $\ker\Phi^* = R\times_T S$. Therefore, we have the following exact sequence

$$0 \longrightarrow R\times_T S \xhookrightarrow{\iota} R\times S \xrightarrow{\Phi^*} \frac{T\times T}{\Delta_T} \longrightarrow 0.$$

By (3.4), $R\times_T S$ are Noetherian. ∎

Problem 3.3

Let M be an R-module. Show that M is a flat R-module if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.

Proof. $\Longrightarrow$: Suppose that M is a flat R-module.

$\Longleftarrow$: ■

Problem 3.4

Let M be an R-module and $\mathfrak{a}$ an R-ideal.

(a) Show that if $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, then $M = IM$.
(b) Show that the converse holds in case M is finite.

Proof. ■

Problem 3.5

Prove that every power of a maximal ideal is primary.

Proof. ■

Problem 3.6

(a) Show that the radical of a primary ideal is prime.
(b) Find an example of a power of a prime ideal that is not primary.
(c) Let $\mathfrak{p}$ be a prime ideal of a ring R and $n \in \mathbf{N}$. The R-ideal $\mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$ s called the *nth symbolic power of* $\mathfrak{p}$. Show that $\mathfrak{p}^{(n)}$ is primary.

Proof. ■