

# MA 523: Homework, Midterms and Practice Problems Solutions

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## Contents

<b>1</b>	<b>Homework Solutions</b>	<b>2</b>
1.1	Homework 1 . . . . .	2
1.2	Homework 2 . . . . .	3
1.3	Homework 3 . . . . .	4
1.4	Homework 4 . . . . .	5
1.5	Homework 5 . . . . .	6
1.6	Homework 6 . . . . .	7
<b>2</b>	<b>Midterms and Qualifying Exams</b>	<b>8</b>
2.1	Qualifying Exam, August '04 . . . . .	8
2.2	Qualifying Exam, August '05 . . . . .	10
2.3	Qualifying Exam, January '14 . . . . .	12

# 1 Homework Solutions

## 1.1 Homework 1

PROBLEM 1.1 (Taylor's formula). Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be smooth,  $n \geq 2$ . Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$$

as  $x \rightarrow \mathbf{0}$  for each  $k = 1, 2, \dots$ , assuming that you know this formula for  $n = 1$ .

*Hint:* Fix  $x \in \mathbb{R}^n$  and consider the function of one variable  $g(t) := f(tx)$ . Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction on  $m$ .

*SOLUTION.* ■

PROBLEM 1.2. Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{*}$$

on  $\mathbb{R}^n \times (0, \infty)$ , where  $b \in \mathbb{R}^n$ . Using the characteristic equation, solve (\*) subject to the initial condition

$$u = g$$

on  $\mathbb{R}^n \times \{t = 0\}$ . Make sure the answer agrees with formula (5) in §2.1.2 of [E].

*SOLUTION.* ■

PROBLEM 1.3. Solve using the characteristics:

- (a)  $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$ ,  $u = 1$  on the line  $x_2 = 2x_1$ .
- (b)  $u u_{x_1} + u_{x_2} = 1$ ,  $u(x_1, x_1) = x_1/2$ .
- (c)  $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$ ,  $u(x_1, x_2, 0) = g(x_1, x_2)$ .

*SOLUTION.* ■

PROBLEM 1.4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with  $u(x_1, 0) = (1 - x_1^2)/2$ .

*SOLUTION.* ■

## 1.2 Homework 2

PROBLEM 1.5. Verify assertion (36) in [E, §3.2.3], that when  $\Gamma$  is not flat near  $x^0$  the noncharacteristic condition is

$$D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here  $\nu(x^0)$  denotes the normal to the hypersurface  $\Gamma$  at  $x^0$ ).

*SOLUTION.* ■

PROBLEM 1.6. Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions  $u(x, 0) = g(x)$  is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some  $t > 0$ , unless  $a(g(x))$  is a nondecreasing function of  $x$ .

*SOLUTION.* ■

PROBLEM 1.7. Show that the function  $u(x, t)$  defined for  $t \geq 0$  by

$$u(x, t) = \begin{cases} -\frac{2}{3} \left( t + \sqrt{3x + t^2} \right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law  $u_t + (u^2/2)_x = 0$  (*inviscid Burgers' equation*).

*SOLUTION.* ■

### 1.3 Homework 3

PROBLEM 1.8. Consider the initial value problem

$$u_t = \sin u_x; \quad u(x, 0) = \frac{\pi}{4}x.$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

*SOLUTION.* ■

PROBLEM 1.9. Consider the Cauchy problem for  $u(x, y)$

$$\begin{aligned} u_y &= a(x, y, u)u_x + b(x, y, u) \\ u(x, 0) &= 0 \end{aligned}$$

let  $a$  and  $b$  be analytic functions of their arguments. Assume that  $d^\alpha a(0, 0, 0) \geq 0$  and  $d^\alpha b(0, 0, 0) \geq 0$  for all  $\alpha$ . (Remember by definition, if  $\alpha = 0$  then  $D^\alpha f = f$ .)

- (a) Show that  $D^\beta u(0, 0) \geq 0$  for all  $|\beta| \leq 2$ .
- (b) Prove that  $D^\beta u(0, 0) \geq 0$  for all  $\beta = (\beta_1, \beta_2)$ . (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in  $\beta_2$ )

*SOLUTION.* ■

PROBLEM 1.10. (Kovalevskaya’s example) show that the line  $\{t = 0\}$  is characteristic for the heat equation  $u_t = u_{xx}$ . Show there does not exist an analytic solution  $u$  of the heat equation in  $\mathbb{R} \times \mathbb{R}$ , with  $u = 1/(1 + x^2)$  on  $\{t = 0\}$ . (*Hint:* assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of  $(0, 0)$ .)

*SOLUTION.* ■

## 1.4 Homework 4

PROBLEM 1.11 (Legendre transform). Let  $u(x_1, x_2)$  be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of  $\mathbb{R}^2$ , where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u(\mathbf{x}(p)),$$

where  $\mathbf{x} = (x^1, x^2)$ ,  $p = (p_1, p_2)$ . Show that  $v$  satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

(*Hint*: See [Evans, 4.4.3b], prove the identities (29)).

SOLUTION. ■

PROBLEM 1.12. Find the solution  $u(x, t)$  of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant  $x > 0, t > 0$  for which

$$\begin{cases} u(x, 0) = f(x), & u_t(x, 0) = g(x), & \text{for } x > 0, \\ u_t(0, t) = \alpha u_x(0, t), & & \text{for } t > 0, \end{cases}$$

where  $\alpha \neq -1$  is a given constant. Show that generally no solution exists when  $\alpha = -1$ . (*Hint*: Use a representation  $u(x, t) = F(x - t) + G(x + t)$  for the solution.)

SOLUTION. ■

PROBLEM 1.13. (a) Let  $u$  be a solution of the wave equation  $u_{tt} - c^2 u_{xx} = 0$  for  $0 < x < \pi$ ,  $t > 0$  such that  $u(0, t) = u(\pi, t) = 0$ . Show that the *energy*

$$E(t) = \frac{1}{2} \int_0^\pi (u_t^2 + c^2 u_x^2) dx, \quad t > 0$$

is independent of  $t$ ; i.e.,  $\frac{d}{dt}E = 0$  for  $t > 0$ . Assume that  $u$  is  $C^2$  up to the boundary.

(b) Express the energy  $E$  of the Fourier series solution

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(nct) + b_n \sin(nct)) \sin(nx)$$

in terms of coefficients  $a_n, b_n$ .

SOLUTION. ■

## 1.5 Homework 5

PROBLEM 1.14. Prove that Laplace's equation  $\Delta u = 0$  is rotation invariant; that is, if  $O$  is an orthogonal  $n \times n$  matrix and we define  $v(x) := u(Ox)$ ,  $x \in \mathbb{R}^n$ , then  $\Delta v = 0$ .

SOLUTION. ■

PROBLEM 1.15. Let  $n = 2$  and  $U$  be the halfplane  $\{x_2 > 0\}$ . Prove that

$$\sup_U u = \sup_{\partial U} u$$

for  $u \in C^2(U) \cap C(\bar{U})$  which are harmonic in  $U$  under the additional assumption that  $u$  is bounded from above in  $\bar{U}$ . (The additional assumption is needed to exclude examples like  $u = x_2$ .)

[Hint: Take for  $\varepsilon > 0$  the harmonic function

$$u(x_1, x_2) - \varepsilon \ln \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle to a region  $\{x_1^2 + (x_2 + 1)^2 < a^2, x_2 > 0\}$  with large  $a$ . Let  $\varepsilon \rightarrow 0$ .]

SOLUTION. ■

PROBLEM 1.16. Let  $U \subset \mathbb{R}^n$  be an open set. We say  $v \in C^2(U)$  is subharmonic if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Let  $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}$  be smooth and convex. Assume  $u^1, \dots, u^m$  are harmonic in  $U$  and

$$v := \varphi(u_1, \dots, u_m).$$

Prove  $v$  is subharmonic.

[Hint: Convexity for a smooth function  $\varphi(z)$  is equivalent to  $\sum_{j,k=1}^m \varphi_{z_j, z_k}(z) \xi_j \xi_k \geq 0$  for any  $\xi \in \mathbb{R}^m$ .]

(b) Prove  $v := |Du|^2$  is subharmonic, whenever  $u$  is harmonic. (Assume that harmonic functions are  $C^\infty$ .)

SOLUTION. ■

## 1.6 Homework 6

PROBLEM 1.17. For  $n = 2$  find Green's function for the quadrant  $\{x_1 > 0, x_2 > 0\}$  by repeated reflection.

SOLUTION. ■

PROBLEM 1.18. (Precise form of Harnack's inequality) Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}}u(0) \leq u(x) \leq \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}}u(0)$$

whenever  $u$  is positive and harmonic in  $B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$ .

SOLUTION. ■

PROBLEM 1.19. Let  $P_k(x)$  and  $P_m(x)$  be homogeneous harmonic polynomials in  $\mathbb{R}^n$  of degrees  $k$  and  $m$  respectively; i.e.,

$$\begin{aligned} P_k(\lambda x) &= \lambda^k P_k(x), & P_m(\lambda x) &= \lambda^m P_m(x) & \text{for every } x \in \mathbb{R}^n, \lambda > 0, \\ \Delta P_k &= 0, & \Delta P_m &= 0 & \text{in } \mathbb{R}^n. \end{aligned}$$

(a) Show that

$$\frac{\partial P_k}{\partial \nu} = k P_k(x), \quad \frac{\partial P_m}{\partial \nu} = m P_m(x) \quad \text{on } \partial B(0, 1)$$

where  $B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$  and  $\nu$  is the outward normal on  $\partial B(0, 1)$ .

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B(0, 1)} P_k(x) P_m(x) d\sigma = 0, \quad \text{if } k \neq m.$$

SOLUTION. ■

## 2 Midterms and Qualifying Exams

### 2.1 Qualifying Exam, August '04

PROBLEM 2.1. Consider the initial value problem

$$\begin{cases} a(x, y)u_x + b(x, y)u_y = -u, \\ u = f \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\},$$

where  $a$  and  $b$  satisfy

$$a(x, y) + b(x, y)y > 0$$

for any  $x, y \in \mathbb{R}^n \setminus \{(0, 0)\}$ .

- (a) Show that the initial value problem has a unique solution in a neighborhood of  $S^1$ . Assume that  $a$ ,  $b$ , and  $f$  are smooth.
- (b) Show that the solution of the initial value problem actually exists in  $\mathbb{R}^2 \setminus \{(0, 0)\}$ .

SOLUTION. ■

PROBLEM 2.2. Let  $u \in C^2(\mathbb{R} \times [0, \infty))$  be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where  $f$  and  $g$  have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dx,$$

and the potential energy by

$$P(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) dx.$$

Show that

- (a)  $K(t) + P(t)$  is constant in  $t$ ,
- (b)  $K(t) = P(t)$  for all large enough times  $t$ .

SOLUTION. ■

PROBLEM 2.3. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = e^{-t}g(x) & \text{for } x \in \mathbb{R}^3, t > 0, \\ u(x, 0) = u_t(x, 0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution  $u = e^{-t} + t - 1$  when  $g(x) = 1$ .



*SOLUTION.* ■

PROBLEM 2.4. Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and  $g \in C_0^\infty(\Omega)$ . Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega, t > 0, \\ u(x, 0) = g(x) & \text{for } x \in \Omega, \\ u(x, t) = 0 & \text{for } x \in \partial\Omega, t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n, t > 0, \\ v(x, 0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put  $g = 0$  outside  $\Omega$ .

(a) Show that

$$-v(x, t) \leq u(x, t) \leq v(x, t)$$

for any  $x \in \Omega, t > 0$ .

(b) Use (a) to conclude that

$$\lim_{t \rightarrow \infty} u(x, t) = 0,$$

for any  $x \in \Omega$ .

*SOLUTION.* ■

PROBLEM 2.5. Let  $P_k(x)$  and  $P_m(x)$  be homogeneous harmonic polynomials in  $\mathbb{R}^n$  of degrees  $k$  and  $m$  respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \quad P_m(\lambda x) = \lambda^m P_m(x),$$

for any  $x \in \mathbb{R}^n, \lambda > 0$ ,

$$\Delta P_k = 0, \quad \Delta P_m = 0$$

in  $\mathbb{R}^n$ .

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = k P_k(x), \quad \frac{\partial P_m(x)}{\partial \nu} = m P_m(x)$$

on  $\partial B_1$ , where  $B_1 = \{ |x| < 1 \}$  and  $\nu$  is the outward normal on  $\partial B_1$ .

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x) dS = 0,$$

if  $k \neq m$ .

*SOLUTION.* ■

## 2.2 Qualifying Exam, August '05

PROBLEM 2.6.

- (a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ u = 1 \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\}.$$

- (b) Is the solution unique in a neighborhood of the point  $(1, 0)$ ? Justify your answer.

SOLUTION. ■

PROBLEM 2.7. Consider the second order PDE in  $\{x > 0, y > 0\} \subset \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

- (a) Classify the equation and reduce it to the canonical form.  
 (b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy}G(x/y).$$

SOLUTION. ■

PROBLEM 2.8. Let  $\Phi$  be the fundamental solution of the Laplace equation in  $\mathbb{R}^3$  and  $f \in C_0^\infty(\mathbb{R}^n)$ . Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y)f(y) dy$$

is a solution of the Poisson equation  $-\Delta u = f$  in  $\mathbb{R}^n$ . Show that if  $f$  is radial (i.e.,  $f(y) = f(|y|)$ ) and supported in  $B_R = \{|x| < R\}$ , then

$$u(x) = c\Phi(x),$$

for any  $x \in \mathbb{R}^n \setminus B_R$ , where

$$c = \int_{\mathbb{R}^n} f(y) dy.$$

[Hint: Use spherical (polar) coordinates and the mean value property.]

SOLUTION. ■

PROBLEM 2.9. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where  $g, h \in C_0^\infty(\mathbb{R}^2)$  and  $\alpha \geq 0$  is a constant.

- (a) Show that for an appropriate choice of constant  $\lambda$  and  $\mu$  the function

$$v(x_1, x_2, x_3, t) := e^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation  $v_{tt} - \Delta v = 0$ .

- (b) Use (a) to prove the following domain of dependence result: for any point  $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$  the value  $u(x_0, t_0)$  is uniquely determined by values of  $g$  and  $h$  in  $\overline{B_{t_0}(x_0)} := \{|x - x_0| \leq t_0\}$ . (You may use the corresponding result for the wave equation without proof.)

*SOLUTION.* ■

PROBLEM 2.10. Let  $u(x, t)$  be a bounded solution of the heat equation  $u_t = u_{xx}$  in  $\mathbb{R} \times (0, \infty)$  with the initial condition

$$u(x, 0) = u_0(x)$$

for  $x \in \mathbb{R}$ , where  $u_0 \in C^\infty$  is  $2\pi$ -periodic, i.e.,  $u_0(x + 2\pi) = u_0(x)$ . Show that

$$\lim_{t \rightarrow \infty} u(x, t) = a_0,$$

uniformly in  $x \in \mathbb{R}$ , where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x) dx.$$

*SOLUTION.* ■

## 2.3 Qualifying Exam, January '14

PROBLEM 2.11. Consider the first order equation in  $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

- (a) Find the characteristic curves of the equation.
- (b) Consider the Cauchy problem for this equation prescribed on the line  $x_1 = 1$ :

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on  $f$  so that the problem is solvable in a neighborhood of the point  $(1, 0)$ .

*SOLUTION.* ■

PROBLEM 2.12. Let  $u$  be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{x_n > 0\}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where  $g$  is a continuous function with compact support in  $\mathbb{R}^{n-1}$ . Here  $n \geq 2$ . Prove that

$$u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty,$$

for  $x \in \{x_n > 0\}$ .

*SOLUTION.* ■

PROBLEM 2.13. Let  $u$  be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \quad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions  $u(x, 0) = g(x)$ , where  $g$  is a bounded continuous function on  $\mathbb{R}$  satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^\alpha, \quad x, y \in \mathbb{R}$$

with a constant  $\alpha \in (0, 1]$ . Show that

$$\begin{aligned} |u(x, t) - u(y, t)| &\leq M|x - y|^\alpha, & x, y \in \mathbb{R}, t > 0, \\ |u(x, t) - u(x, s)| &\leq C_\alpha M|t - s|^{\alpha/2}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of  $u(x, t)$  as a convolution with the heat kernel  $\Phi(y, t)$ , make a change of variables  $z = y/\sqrt{t}$  and use that  $|\sqrt{t} - \sqrt{s}| \leq \sqrt{|t - s|}$ .]

*SOLUTION.* ■

PROBLEM 2.14. Let  $u$  be a positive harmonic function in the unit ball  $B_1$  in  $\mathbb{R}^n$ . Show that

$$|D(\ln u)| \leq M \quad \text{in } B_{1/2}$$

for a constant  $M$  depending only on the dimension  $n$ .

[*Hint:* Use the interior derivative estimate  $|Du(x)| \leq (C_n/r) \sup_{B_r(x)} |u|$  for  $B_r(x) \subset B_1$  as well as the Harnack inequality for harmonic functions.]

SOLUTION. ■

PROBLEM 2.15. Let  $u$  be a  $C^2$  solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = 0, \quad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some  $k \geq 0$ . Prove that there exists a function  $\varphi(r)$  such that

$$u(x, t) = t^{k+2} \varphi(|x|/t).$$

[*Hint:* As one of the steps show that  $u$  is  $(k+2)$ -homogeneous in  $(x, t)$  variables, i.e.,  $u(\lambda x, \lambda t) = \lambda^{k+2} u(x, t)$  for any  $\lambda > 0$ .]

SOLUTION. ■