

# MA571 Homework 13

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### PROBLEM 13.1 (MUNKRES §68, EX. 1)

Check the details of Example 1.

*Proof.* The following is the statement of Example 1 as found in the book:

**Examples 1.** Consider the group  $P$  of bijections of the set  $\{0, 1, 2\}$  with itself. For  $i = 1, 2$ , define an element  $\pi_i$  of  $P$  by setting  $\pi_i(i) = i - 1$  and  $\pi_i(i - 1) = i$  and  $\pi_i(j) = j$  otherwise. Then  $\pi_i$  generates a subgroup  $G_i$  of  $P$  of order 2. The group  $G_1$  and  $G_2$  generate  $P$ , as you can check. But  $P$  is not their free product. The reduced words  $(\pi_1, \pi_2, \pi_1)$  and  $(\pi_2, \pi_1, \pi_2)$ , for instance, represent the same element of  $P$ .

We need to check two claims (i) that  $G_1$  and  $G_2$ , as defined above, generate  $P$  and (ii) that  $P \neq G_1 * G_2$ , i.e., show that  $(\pi_1, \pi_2, \pi_1) = (\pi_2, \pi_1, \pi_2)$ . Let us deal with (i) first. We show that  $\langle G_1, G_2 \rangle = P$ . Our strategy is the following, by the pigeon-hole principle, it suffices to show that  $\langle G_1, G_2 \rangle \subset P$  and that  $|\langle G_1, G_2 \rangle| = |P|$ . Since  $G_1, G_2 < P$ , i.e.,  $G_1$  and  $G_2$  are subgroups of  $P$ , the group generated by  $G_1$  and  $G_2$  will be a subgroup of  $P$  hence,  $\langle G_1, G_2 \rangle \subset P$ . The group  $P$  is a well-known group, namely (up to group isomorphism)  $S_3$ , and we shall not waste time any time showing that  $|P| = |\{0, 1, 2\}| = 3! = 6$ , but instead we proceed to showing that  $|\langle G_1, G_2 \rangle| = 6$ . From the definitions of  $G_1$  and  $G_2$ , we have at least 3 in  $\langle G_1, G_2 \rangle$ , these are the elements 1,  $\pi_1$  and  $\pi_2$  (the latter two have order 2, e.g.,

$$\pi_i^2(j) = \pi_i \left( \begin{cases} i-1 & \text{if } j = i \\ i & \text{if } j = i-1 \\ j & \text{otherwise} \end{cases} \right) = \begin{cases} i & \text{if } j = i \\ i-1 & \text{if } j = i-1 \\ j & \text{otherwise} \end{cases}$$

which is the identity on  $\{0, 1, 2\}$ .) So the elements  $1, \pi_1, \pi_2, \pi_1\pi_2, \pi_2\pi_1, \pi_1\pi_2\pi_1 \in \langle G_1, G_2 \rangle$  and all finite strings  $\pi_1\pi_2 \cdots \pi_i, \pi_2\pi_1 \cdots \pi_i$  for that matter. But as a consequence of Lagrange's theorem, the size of  $\langle G_1, G_2 \rangle$  must not exceed the size of  $P$  so that we are done when we show that the elements  $\pi_1\pi_2, \pi_2\pi_1$  and  $\pi_1\pi_2\pi_1$  are distinct elements. First, observe that

$$\begin{aligned} \pi_2\pi_1(j) &= \pi_2 \left( \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ 2 & \text{if } j = 2 \end{cases} \right) & \pi_1\pi_2(j) &= \pi_1 \left( \begin{cases} 0 & \text{if } j = 0 \\ 2 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} \right) \\ &= \begin{cases} 2 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} & &= \begin{cases} 1 & \text{if } j = 0 \\ 2 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases} \end{aligned}$$

and, using the computations above,

$$\pi_1\pi_2\pi_1(j) = \pi_1 \left( \begin{cases} 2 & \text{if } j = 0 \\ 0 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases} \right) = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}.$$

Note that none of these elements are equivalent to any of 1,  $\pi_1$  or  $\pi_2$  and are certainly not equal to each other. Moreover, there are six of these elements and there are no more elements in  $P$  since  $|P| = 6$ . Thus,  $\langle G_1, G_2 \rangle = P$ .

Lastly, we show that  $P \neq G_1 * G_2$  since

$$(\pi_1, \pi_2, \pi_1) = \pi_1 \pi_2 \pi_1(j) = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

and

$$(\pi_2, \pi_1, \pi_2) = \pi_2 \pi_1 \pi_2(j) = \pi_1 \left( \begin{cases} 1 & \text{if } j = 0 \\ 2 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases} \right) = \begin{cases} 2 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

would imply that  $(\pi_1, \pi_2, \pi_1) = (\pi_2, \pi_1, \pi_2)$  in the free product  $G_1 * G_2$ , but  $\pi_1 \neq \pi_2$ . ■

### PROBLEM 13.2 (MUNKRES §68, EX. 2(A,B,C))

Let  $G = G_1 * G_2$ , where  $G_1$  and  $G_2$  are nontrivial groups.

- (a) Show  $G$  is not Abelian.
- (b) If  $x \in G$ , define the *length* of  $x$  to be the length of the unique reduced word in the elements of  $G_1$  and  $G_2$  that represents  $x$ . Show that if  $x$  has even length (at least 2), then  $x$  does not have finite order. Show that if  $x$  has odd length (at least 3), then  $x$  is conjugate to an element of shorter length.
- (c) Show that the only elements of  $G$  that have finite order are the elements of  $G_1$  and  $G_2$  that have finite order, and their conjugates.

*Proof.* (i) Suppose  $G$  is Abelian. Take an element  $x \in G_1$  and  $y \in G_2$ . Then  $(x, y) = (y, x)$ . By the definition of a free product (Munkres §68, pp. 413-414) this implies that the word  $(x^{-1}, y^{-1}, x, y) = 1$  which implies that  $y^{-1}x = 1$ , but  $y^{-1} \notin G_1$ .

(ii) Let  $x \in G$  be a word of even length. Then  $x = (y_1, y_2, \dots, y_{2k})$  for  $k \in \mathbf{N}$  where the right hand-side is irreducible, i.e., either  $y_i \in G_1$  if  $2 \mid i$  and  $y_j \in G_2$  if  $2 \nmid j$  or vice-versa since two consecutive “letters” in a word must be from distinct groups or else we can reduce the word further. Then  $x^2 = (y_1, y_2, \dots, y_{2k}, y_1, y_2, \dots, y_{2k})$  is again irreducible since  $y_{2k} \in G_1$  and  $y_1 \in G_2$  or vice-versa. It follows by induction that  $x^n \neq 1$  for any finite positive integer  $n$ .

Now, suppose that  $x \in G$  has odd length. Then  $x = (y_1, y_2, \dots, y_{2k+1})$  for  $k \in \mathbf{N}$  where the right hand-side is irreducible. Without loss of generality, we may assume that  $y_1, y_{2k+1} \in G_1$ . Then, setting  $y'_{2k+1} := y_{2k+1}y_1$ , we have

$$y_1^{-1}xy_1 = y_1^{-1}(y_1, y_2, \dots, y_{2k+1})y_1 = (y_2, y_3, \dots, y_{2k+1}y_1) = (y_2, y_3, \dots, y'_{2k+1})$$

which has length  $2k$ . Thus,  $x$  is conjugate to a word of shorter length.

(iii) Suppose that  $x \in G$  has finite order. By part (i) the length of  $x$  cannot be even. Moreover, if  $x$  is of finite order, i.e., if  $x^n = 1$  for some positive integer  $n$ , and  $y$  is conjugate to  $x$ , i.e., there exist  $g \in G$  such that  $y = g^{-1}xg$ , then

$$y^n = (g^{-1}xg)^n = (g^{-1}xg)(g^{-1}xg) \cdots (g^{-1}xg) = g^{-1}x^ng = 1$$

so  $y$  is of finite order. It remains to show that if  $x$  has finite order then  $x$  is a conjugate of an element  $y$  of  $G_i$ , where  $i = 1, 2$ . Let  $2k+1$  be the length of  $x$ . By part (ii),  $x$  is conjugate to an element  $y'$  of shorter length. Since  $x$  has finite order  $y'$  has finite order so by part (i)  $y'$  must be of odd length. If  $y'$  is of length 1 we are done. If not, then  $y'$  is conjugate to a word  $y''$  of shorter length with finite order. Since the length of  $x$  is finite, this process must terminate at a word  $y$  of length 1 with finite order. ■

### PROBLEM 13.3 (MUNKRES §68, EX. 3)

Let  $G = G_1 * G_2$ . Given  $c \in G$ , let  $cG_1c^{-1}$  denote the set of all elements of the form  $cxc^{-1}$ , for  $x \in G_1$ . It is a subgroup of  $G$ ; show that the intersection with  $G_2$  is the identity alone.

*Proof.* Suppose  $y \in cG_1c^{-1} \cap G_2$ . Then  $y = cxc^{-1}$  for some  $x \in G_1$  and we have,  $c = ycx^{-1}$ . Let us deal with the trivial case first. If  $c = 1$  then, since  $G$  is the free group of  $G_1$  and  $G_2$ , we have  $1 \cdot G_1 \cdot 1^{-1} = G_1$  so  $(1 \cdot G_1 \cdot 1^{-1}) \cap G_2 = G_1 \cap G_2 = 1$  by definition of the free product. Now, suppose  $c \neq 1$ , say that  $c$  is represented by the unique reduced word  $(y_1, \dots, y_k)$ ,  $k \in \mathbf{N}$ . We show that for the following cases (i)  $y_1, y_k \in G_i$ , (ii)  $y_1 \in G_1$  and  $y_k \in G_2$ , or  $y_1 \in G_2$  and  $y_k \in G_1$ ,  $y = 1$ , i.e., the intersection  $cG_1c^{-1} \cap G_2 = 1$ .

The above three cases can trivially be adapted to just the case where  $y_1, y_k \in G_1$  or  $y_1, y_k \in G_2$ , so we shall prove the aforementioned. Assuming  $y_1, y_k \in G_1$ ,  $c$  is represented by  $(y_1, \dots, y_k)$  and  $(y, y_1, \dots, y_k, x^{-1})$  where the latter reduces to the word  $(y, y_1, \dots, y_k x^{-1})$ . Now, by the uniqueness of representation by reduced word,  $(y_1, \dots, y_k) = (y, y_1, \dots, y_k x^{-1})$ , but the right-hand side has length  $k + 1$  and cannot be reduced further unless  $y_k x^{-1} = 1$ . Suppose that  $y_k x^{-1} = 1$  then we have,  $y_1 = y, y_2 = y_1, \dots, y_k = y_{k-1}$  which can only happen if  $y = 1$  and  $y_i = 1$  for all  $i$ . But this contradicts the assumption that  $c \neq 1$ . Thus,  $y = 1$  to begin with.

Now, suppose  $y_1, y_k \in G_2$ , then  $c$  is represented by  $(y_1, \dots, y_k)$  and  $(y, y_1, \dots, y_k, x^{-1})$  where the latter reduces to the word  $(yy_1, \dots, y_k, x^{-1})$ . Now, by the uniqueness of representation by reduced word,  $(y_1, \dots, y_k) = (yy_1, \dots, y_k, x^{-1})$ , but the right-hand side has length  $k + 1$  and cannot be reduced further unless the product  $yy_1 = 1$ . Suppose that  $yy_1 = 1$ . Then we have,  $y_2 = y_1, \dots, y_{k-1} = y_k, y_k = x$ . This implies that, since  $y_i \notin G_{\alpha_{i+1}}$ ,  $x = 1$  and  $y_i = 1$  for all  $i$ . But this contradicts the assumption that  $c \neq 1$ . Thus,  $y$  must have been 1 to begin with. ■

### PROBLEM 13.4 (A)

- (i) Do the case of p. 367 # 9(e) where  $h$  and  $k$  take  $b_0$  to  $b_0$ . (The proof is similar to the proof of Lemma 55.3, (3)  $\implies$  (1), that I gave in class).
- (ii) Let  $G$  be a path-connected topological group and let  $a \in G$ . Prove that the map  $\varphi: G \rightarrow G$  defined by  $\varphi(g) := ag$  is homotopic to the identity map.
- (iii) Use part (ii) to complete the proof of p. 367 # 9(e).

*Proof.* (i) Set  $d := \deg h$ . Suppose that  $h(b_0) = k(b_0) = b_0$  and that  $\deg h = \deg k$ . Consider the path  $f(s) := (\cos(2\pi s), \sin(2\pi s))$  from the handout on “The fundamental group of  $S^1$ .” This path is a loop at  $b_0$  ( $f(0) = (\cos(2\pi \cdot 0), \sin(2\pi \cdot 0)) = (1, 0) = (\cos(2\pi), \sin(2\pi)) = f(1)$ ) of index 1 (i.e., the winding number of  $f$  is 1), hence is a generator for  $\pi_1(S^1, b_0)$ . Thus, we have

$$h_*([f]) = d \cdot [f] = k_*([f])$$

so  $h \circ f \simeq_p k \circ f$ . Let  $H: I \times I \rightarrow S^1$  denote the homotopy from  $h \circ f$  to  $k \circ f$ , i.e., the continuous map such that  $H(s, 0) = h \circ f(s)$  and  $H(s, 1) = k \circ f(s)$ . Next, by the Problem 9.2 (Munkres §46, Ex. 9), we see that the map  $(f, \text{id}_I): I \times I \rightarrow S^1 \times I$  is a quotient map so that the following diagram commutes

$$\begin{array}{ccc} I \times I & \xrightarrow{H} & S^1 \\ (f, \text{id}_I) \downarrow & \nearrow \overline{H} & \\ S^1 \times I & & \end{array}$$

By Theorem Q.2, the map  $\overline{H}$  is continuous and  $H$  factors through  $\overline{H}$ , i.e.,  $H(s, 0) = h \circ f(s) = \overline{H}(f(s), 0)$  and  $H(s, 1) = k \circ f(s) = \overline{H}(f(s), 1)$ . But since  $f$  is onto  $S^1$ , setting  $x := f(s)$  for  $s \in I$ , we have  $\overline{H}(x, 0) = h(x)$  and  $\overline{H}(x, 1) = k(x)$ . Thus,  $\overline{H}$  is a homotopy from  $h$  to  $k$  so  $h \simeq_p k$ .

(ii) Let  $1$  denote the identity element of  $G$ . Since  $G$  is path-connected there exists a path  $\alpha: I \rightarrow G$  from  $a$  to  $1$ , i.e.,  $\alpha(0) = a$  and  $\alpha(1) = 1$ . Define the map  $H: G \times I \rightarrow G$  by  $H(g, t) := \alpha(t)g$ . Then  $H$  is a homotopy from  $\varphi$  to the identity map  $\text{id}_G$  ( $H(g, 0) = \alpha(0)g = ag = \varphi(g)$  and  $H(g, 1) = \alpha(1)g = 1 \cdot g = \text{id}_G(g)$ ; moreover,  $H$  is continuous since  $\alpha$  is continuous and multiplication in  $G$  is continuous). Thus,  $\varphi \simeq \text{id}_G$ .

(iii) Suppose that  $h, k: S^1 \rightarrow S^1$  have the same degree  $d$ . By part (a) of Ex. 9, we know that the degree of a map between circles is independent of the basepoint so we may as well let  $b_0$  be the basepoint for the fundamental group of  $S^1$ . Now, the induced maps  $h_*: \pi_1(S^1, b_0) \rightarrow \pi_1(S^1, h(b_0))$  and  $k_*: \pi_1(S^1, b_0) \rightarrow \pi_1(S^1, k(b_0))$  send the generator of  $\pi_1(S^1, b_0)$ , say  $\gamma$ , to

$$h_*(\gamma) = d \cdot \gamma(h(x_0)) \quad \text{and} \quad k_*(\gamma) = d \cdot \gamma(k(x_0)).$$

But since  $S^1$  is a path-connected topological group (viewing  $S^1$  as a subset of  $\mathbf{C}$ , multiplication is the standard multiplication on the complex numbers, where if  $z_1, z_2 \in S^1$  then  $z_1 z_2 \in S^1$  since

$\|z_1 z_2\| = 1$ ) we can define maps  $\varphi_h, \varphi_k: S^1 \rightarrow S^1$  such that  $\varphi_h(h(b_0)) = b_0$  and  $\varphi_k(k(b_0)) = b_0$  (these are rotation maps/matrices and they can be easily constructed from the argument of  $h(b_0)$ ,  $k(b_0)$  etc.) so that the induced maps by the composition  $(\varphi_h \circ h)_*, (\varphi_k \circ k)_*: \pi_1(S_1, b_0) \rightarrow \pi_1(S^1, b_0)$  have the same degree. By part (i),  $\varphi_h \circ h \simeq \varphi_k \circ k$  so by part (ii),  $h \simeq k$ . ■



### PROBLEM 13.5 (B)

Let  $q: S^2 \rightarrow P^2$  be the quotient map, where  $P^2$  is the projective plane. Let  $x_0 = q(1, 0, 0)$  and let

$$f(s) = q(\cos(\pi s), \sin(\pi s), 0)$$

for  $0 \leq s \leq 1$ . Then  $f: I \rightarrow P^2$  is a loop at  $x_0$ . Prove that  $[f] * [f] = [e_{x_0}]$ .

*Proof.* Recall that the projective plane is constructed from the sphere by identifying antipodal points, i.e.,  $P^2 \approx S^2/\sim$  where  $x \sim y$  if and only if  $y = x$  or  $y = -x$ . We will show that  $f * f \simeq_p e_{x_0}$  by constructing a homotopy between  $f * f$  and  $e_{x_0}$ . Define  $\tilde{f}: S^2 \rightarrow I$  by  $\tilde{f}(s) := (\cos(\pi s), \sin(\pi s), 0)$ . Note that  $q(\tilde{f}) = f$  and that, by the equivalence relation on  $S^2$ ,  $q(\tilde{f}) = q(-\tilde{f})$ . Now, we want to show that  $f * f \simeq_p e_{x_0}$  so we shall construct a homotopy  $\tilde{f} * \tilde{f} \simeq_p e_{(1,0,0)}$  ■

### PROBLEM 13.6 (C)

Let  $Y$  be the following subset of  $\mathbf{R}^2$ :  $Y = \{(s, t) \in I \times I \mid s \in \{0, 1\} \text{ or } t \in \{0, 1\}\}$  (that is,  $Y$  is the boundary of the square  $I \times I$ ). Give  $Y$  the equivalence relation  $\sim$  that identifies the top and the bottom edges and the left and the right edges: specifically,  $\sim$  is the equivalence relation associated to the partition of  $Y$  into the following sets:

- for each  $s \notin \{0, 1\}$ , the set  $\{(s, 0), (s, 1)\}$ ,
- for each  $t \notin \{0, 1\}$ , the set  $\{(t, 0), (t, 1)\}$ ,
- the set  $\{0, 1\} \times \{0, 1\}$ .

Prove that  $Y/\sim$  is a wedge of two circles.

*Proof.* Let  $q: Y \rightarrow Y/\sim$  denote the quotient map. Recall from the definition at the top of Munkres §71, p. 434, that the wedge of two circles  $S^1 \vee S^1$  is the union of two subspaces, say  $S_1$  and  $S_2$ , which are homeomorphic to  $S^1$  such that  $S_1 \cap S_2 = \{p\}$ . Since translation in  $\mathbf{R}^2$  is an isometry, in particular, a homeomorphism, we may as well assume that  $p = (0, 0)$  and that  $S_1 = S^1 - (1, 0)$  and  $S_2 = S^1 + (1, 0)$ . The plan of attack is as follows, we define a continuous map  $f: Y \rightarrow S^1 \vee S^1$  and show that the induced map on the quotient,  $\bar{f}: Y/\sim \rightarrow S^1 \vee S^1$  is a homeomorphism. Define

$$f(s, t) := \begin{cases} (\cos(2\pi s), \sin(2\pi s)) - (1, 0) & \text{if } t = 0 \\ (\cos(2\pi t + \pi), \sin(2\pi t + \pi)) + (1, 0) & \text{if } s = 1 \\ (\cos(2\pi s), \sin(2\pi s)) - (1, 0) & \text{if } t = 1 \\ (\cos(2\pi t + \pi), \sin(2\pi t + \pi)) + (1, 0) & \text{if } s = 0. \end{cases}$$

Note that these maps are continuous on their respective intervals since their projections are trigonometric functions plus constants, and addition is continuous in  $\mathbf{R}$  by Theorem 25.1, hence the pieces are continuous by Theorem 18.4. Moreover, note that on the intersection, that is, on the points  $x = (0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$  and  $(0, 1)$  we have  $f(x) = (0, 0)$ . Therefore, by the pasting lemma,  $f$  is continuous.<sup>1</sup> Now, observe that for each  $(s, t) \sim (s', t')$  as in the bullet we have  $f(s, t) = f(s', t')$  (we have already checked the case where  $(s, t)$  is a point in one of the four vertices of  $Y$  so we check only the first two bullets above): either  $s \neq 0, 1$  so  $(s, 0) \sim (s, 1)$  and we have

$$f(s, 0) = (\cos(2\pi s), \sin(2\pi s)) - (1, 0) = f(s, 1)$$

or  $t \neq 0, 1$  so  $(0, t) \sim (1, t)$  and we have

$$f(0, t) = (\cos(2\pi t + \pi), \sin(2\pi t + \pi)) + (1, 0) = f(1, t).$$

Therefore, the induced map on the quotient space  $\bar{f}: Y/\sim \rightarrow S^1 \vee S^1$  is continuous, by Theorem Q.3. It is clear that this map  $\bar{f}$  is surjective since for any  $(x, y)$  in  $S_1$ , setting  $s = \arctan(y/(x - 1))/2\pi$ ,

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<sup>1</sup>What we ought to do is to first define a continuous map counterclockwise from the origin going around the bottom interval  $\{(s, 0)\}_{s \in I} \cup \{(1, t)\}_{t \in I}$  using the pasting lemma and do the same for  $\{(s, 1)\}_{s \in I} \cup \{(0, t)\}_{t \in I}$  then use the pasting lemma again to define a continuous map on  $Y$ . But we trust the grader will sympathize with the writer in that such a display is unnecessary and only adds bulk, but content, to the proof.

we have

$$\begin{aligned} f(s, 0) &= (\cos(\arctan(y/(x-1))), \sin(\arctan(y/(x-1)))) + (1, 0) \\ &= \left( \frac{x-1}{\sqrt{(x-1)^2 + y^2}}, \frac{y}{\sqrt{(x-1)^2 + y^2}} \right) + (1, 0) \end{aligned}$$

but the expression  $\sqrt{(x-1)^2 + y^2} = 1$  since the circle  $S_1$  is centered at  $(1, 0)$

$$\begin{aligned} &= (x-1, y) + (1, 0) \\ &= (x, y). \end{aligned}$$

One can similarly show that  $f$  is onto  $S_2$ . Thus, the map  $\bar{f}$  is onto. To see that this map is one-to-one let  $(x, y), (x', y') \in Y$  and suppose that  $f(x, y) = f(x', y')$ . Then  $f(x, y)$  can be one of three points: (i)  $f(x, y) = (0, 0)$ , (ii)  $f(x, y) \in S_1$  or (iii)  $f(x, y) \in S_2$ . If  $f(x, y) = (0, 0)$  then we know that each of the components of  $f(s)$  is zero at  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$  and  $(0, 1)$  so  $\bar{f}^{-1}((0, 0)) = [0, 0]$  so that indeed, we must have  $(x, y) \sim (x', y')$ . Otherwise,  $f(x, y) \in S_1$  (or  $S_2$  and the demonstration is similar) so  $f^{-1}(x, y) = \{(s, 0), (s, 1)\}$  that in the quotient  $[x, y] = [s, 0] = [x', y']$ . Thus,  $\bar{f}$  is injective. By Theorem 26.6, since  $Y/\sim$  is compact because it is the quotient of a compact space and  $S^1 \vee S^1$  is Hausdorff by the commentary on §71, p. 434,  $\bar{f}: Y/\sim \rightarrow S^1 \vee S^1$  is a homeomorphism. ■