MA544: Qual Preparation

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Contents

Chapter 1

MA 544 Spring 2016

This is material from the course MA 544 as it was taught in the spring of 2016.

1.1 Homework

These exercises were assigned from Wheeden and Zygmund's *Measure and Integral*, therefore, most of the theorems I reference will be from [5]. Other resources include [1] and [2]. For more elementary results, I cite [3].

Throughout these notes

$\mathbb{R}$	is the set of real numbers
$\mathbb{R}^+$	is the set of positive real numbers, that is, $x \in \mathbb{R}$ with $x \geq 0$
$\mathbb{C}$	is the set of complex numbers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{Z}$	is the set of the integers
$\mathbb{Z}^+$	is the set of positive integers, that is, $x \in \mathbb{Z}$ with $x \geq 0$
$\mathbb{N}$	is the set of the natural numbers $1, 2, \ldots$
$A \setminus B$	is the set difference of A and B, that is, the complement of $A \cap B$ in A
$m^*(E)$	the outer measure of E
$m_*(E)$	the inner measure of E
$m(E)$	the Lebesgue measure of E
$\\|\cdot\\|$	the standard Euclidean norm on $\mathbb{R}^n$
$f \asymp g$	means f is asymptotically equivalent to g, that is, $\lim_{x\to\infty} g(x)/f(x) = 1$

1.1.1 Homework 1

Problem 1 (Wheeden & Zygmund Ch. 2, Ex. 1)**.** Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0,1]$, but that $V[f;0,1] = +\infty$.

Proof. Set $f \coloneqq x\sin(1/x)$. We will show that f is bounded and continuous on $[0,1]$ but that, nevertheless, f is not of bounded variation on $[0,1]$.

To see that f is bounded, we note that both x and $\sin(1/x)$ are bounded by 1 on $[0,1]$. Thus, the absolute value of the product $|x\sin(1/x)|$ is bounded by 1 so f, since it is defined to be the product $x\sin x$, is bounded on $[0,1]$.

To see that f is continuous, we use note that, since $\sin(1/x)$ and x are continuous on $(0,1]$ by properties of continuous functions (you may refer to [3, Ch. 4, p. 87]) f is continuous on $(0,1]$. At a first glance, however, is not obvious that f is continuous at 0 since $\sin(1/x)$ is not continuous at 0.* To show that f is continuous at 0 by Theorem 4.6 from [3, Ch. 4, p. 86] it suffices to show that $\lim_{x\to 0} f(x) = 0$. Let $\{a_n\}$ be a sequence whose limit is 0. Now, given $\varepsilon > 0$, for sufficiently large $N \in \mathbb{Z}^+$, the distance $|a_n - 0| = a_n < \varepsilon$. Thus, for every $n \geq N$ we have

$$|a_n \sin(1/a_n) - 0| \leq a_n < \varepsilon. \tag{1}$$

Thus, $\lim_{a_n \to 0} f(a_n) = 0$. Since a_n was chosen arbitrarily, it follows that f is continuous at 0 and, consequently, on $[0,1]$.

In spite of these nice properties that f possesses f is not b.v. on $[0,1]$. To see this, we note that f is differentiable on $[0,1]$ so by Corollary 2.10 from [5, Ch. 2, p. 23] we have

$$\begin{aligned} V &= \int_0^1 |f'| \,\mathrm{d}x \\ &= \int_0^1 |\sin(1/x) - (1/x)\cos(1/x)| \,\mathrm{d}x \\ &= \int_1^\infty \frac{1}{u^2} |\sin u - u\cos u| \,\mathrm{d}x \\ &\geq \int_M^\infty \frac{1}{2u} \,\mathrm{d}u \\ &= \infty, \end{aligned} \tag{2}$$

where, for sufficiently large $M \in \mathbb{Z}^+$, for $u \geq M$, we have $|\sin u - u\cos u| > u/2$. Thus, f is not b.v. on $[0,1]$. ∎

Problem 2 (Wheeden & Zygmund Ch. 2, Ex. 2)**.** Prove theorem (2.1).

Proof. Recall the statement of theorem (2.1):

(a) If f is of bounded variation on $[a,b]$, then f is bounded on $[a,b]$.

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*Take for example, the sequence of real numbers $\{a_n\}$ and $\{b_n\}$, where we set $a_n \coloneqq (2\pi)^{-1}$ and $b_n \coloneqq (2\pi n + \pi/2)^{-1}$. These sequences approach 0 as $n \to \infty$. However, the sequence $\{\sin(1/a_n)\}$ and $\{\sin(1/b_n)\}$ approach 0 and 1, respectively.

(b) Let f and g be of bounded variation on $[a,b]$. Then cf (for any real constant c), $f+g$, and fg are of bounded variation on $[a,b]$. Moreover, f/g is of bounded variation on $[a,b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for $x \in [a,b]$.

(a) We shall proceed by contradiction. Suppose f is of b.v. on $[a,b]$ with total variation V, but that f is not bounded on $[a,b]$. Then, for every $M \in \mathbb{R}^+$, there exist $x \in [a,b]$ such that $|f(x)| > M$. Hence, there exist $x^* \in [a,b]$ such that $|f(x^*)| > V + |f(a)|$ so by the reverse triangle inequality we have

$$\begin{aligned} |f(a) - f(x^*)| + |f(x^*) - f(b)| &\geq |(V - |f(a)|) - |f(a)|| + |V + (|f(a)| - |f(b)|)| \\ &= V + |V + (|f(a)| - |f(b)|)| \\ &> V. \end{aligned} \tag{3}$$

Since $\Gamma^* := \{a, x^*, b\}$ is a partition of $[a,b]$ and f is b.v. on $[a,b]$, we must have $S_{\Gamma^*} \leq V$. This is a contradiction. Thus, it must be the case that f is bounded on $[a,b]$.

(b) Suppose f and g are b.v. on $[a,b]$ with variation V_f and V_g, respectively. We will show that cf, $f+g$, and fg are b.v. on $[a,b]$. Moreover, we show f/g is b.v. on $[a,b]$ if there exists $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for all $x \in [a,b]$.

<u>(cf is b.v. on $[a,b]$)</u>: Let c be a real number. Given a partition $\Gamma = \{x_0, \dots, x_n\}$ of $[a,b]$, we have

$$\begin{aligned} S_\Gamma &= \sum_{i=0}^{n} |cf(x_i) - cf(x_{i-1})| \\ &= \sum_{i=1}^{n} |c||f(x_i) - f(x_{i-1})| \\ &= |c| \underbrace{\left(\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right)}_{S_{\Gamma,f}}. \end{aligned} \tag{4}$$

But since f is b.v. on $[a,b]$, $S_{\Gamma,f} \leq V_F$ so $S_\Gamma \leq cV_f$. Since Γ was chosen arbitrarily, it follows that cf is b.v. on $[a,b]$. $x \in [a,b]$.

<u>($f+g$ is b.v. on $[a,b]$)</u>: Given a partition $\Gamma = \{x_0, \dots, x_n\}$ of the interval $[a,b]$, we have the sums <u>associated to $f+g$</u>

$$\begin{aligned} S_{\Gamma,f+g} &= \sum_{i=1}^{n} |(f(x_i) + g(x_i)) - (f(x_{i-1}) + g(x_{i-1}))| \\ &= \sum_{i=1}^{n} |(f(x_i) - f(x_{i-1})) + (g(x_i) - g(x_{i-1}))| \\ &\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \\ &\leq V_f + V_g. \end{aligned} \tag{5}$$

Thus, $f+g$ is b.v. on $[a,b]$.

(fg is b.v. on $[a,b]$): First, recall that, since f and g are b.v. on $[a,b]$, f and g are bounded on $[a,b]$ by, say, $M_f > 0$ and $M_g > 0$, respectively. Now, given a partition $\Gamma = \{x_0, \ldots, x_n\}$ of $[a,b]$ consider the sums associated to the product fg

$$\begin{aligned}
S_{\Gamma,fg} &= \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\
&= \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1}) \\
&\qquad +f(x_i)g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^{n} |(f(x_i)g(x_i) - f(x_i)g(x_{i-1})) \\
&\qquad -(f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1}))| \\
&\leq \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_i)g(x_{i-1})| \\
&\quad + \sum_{i=1}^{n} |f(x_{i-1})g(x_{i-1}) - f(x_i)g(x_{i-1})| \\
&= \sum_{i=1}^{n} |f(x_i)||g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} |g(x_{i-1})||f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^{n} M_f|g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} M_g|f(x_i) - f(x_{i-1})| \\
&\leq M_f V_g + M_g V_f.
\end{aligned} \tag{6}$$

Thus, fg is b.v. on $[a,b]$.

(f/g is b.v. on $[a,b]$ if there exists $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for all $x \in [a,b]$.): Suppose there exists $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for all $x \in [a,b]$. Suppose $\Gamma = \{x_0, \ldots, x_n\}$ is a partition of $[a,b]$ and consider the sum associated to the quotient f/g

$$
\begin{aligned}
V_{\Gamma,f/g} &= \sum_{i=1}^{n} |f(x_i)/g(x_i) - f(x_{i-1})/g(x_{i-1})| \\
&= \sum_{i=1}^{n} \left| \frac{f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)}{g(x_i)g(x_{i-1})} \right| \\
&\leq \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)| \\
&= \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1}) \\
&\qquad\qquad - (f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1}))| \\
&\leq \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |g(x_{i-1})||f(x_i) - f(x_{i-1})| + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} |f(x_{i-1})||g(x_i) - g(x_{i-1})| \\
&=
\end{aligned}
\tag{7}
$$

■

Problem 3 (Wheeden & Zygmund Ch. 2, Ex. 3)**.** If $[a', b']$ is a subinterval of $[a, b]$ show that $P[a', b'] \leq P[a, b]$ and $N[a', b'] \leq N[a, b]$.

Proof. ■

Problem 4 (Wheeden & Zygmund Ch. 2, Ex. 11)**.** Show that $\int_a^b f \, \mathrm{d}\varphi$ exists if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|, |\Gamma'| < \delta$.

Proof. ■

Problem 5 (Wheeden & Zygmund Ch. 2, Ex. 13)**.** Prove theorem (2.16).

Proof. Recall the statement of Theorem 2.16:

(i) If $\int_a^b f \, \mathrm{d}\varphi$ exists, then so do $\int_a^b cf \, \mathrm{d}\varphi$ and $\int_a^b f \, \mathrm{d}(c\varphi)$ for any constant c, and

$$\int_a^b cf \, \mathrm{d}\varphi = \int_a^b f \, \mathrm{d}(c\varphi) = c \int_a^b f \, \mathrm{d}\varphi.$$

(ii) If $\int_a^b f_1 \, \mathrm{d}\varphi$ and $\int_a^b f_2 \, \mathrm{d}\varphi$ both exist, so does $\int_a^b (f_1 + f_2) \, \mathrm{d}\varphi$, and

$$\int_a^b (f_1 + f_2) \, \mathrm{d}\varphi = \int_a^b f_1 \, \mathrm{d}\varphi + \int_a^b f_2 \, \mathrm{d}\varphi.$$

(iii) If $\int_a^b f \, \mathrm{d}\varphi_1$ and $\int_a^b f \, \mathrm{d}\varphi_2$ both exist, so does $\int_a^b f \, \mathrm{d}(\varphi_1 + \varphi_2)$, and

$$\int_a^b f \, \mathrm{d}(\varphi_1 + \varphi_2) = \int_a^b f \, \mathrm{d}\varphi_1 + \int_a^b f \, \mathrm{d}\varphi_2.$$

■

1.1.2 Homework 2

Problem 1. Show that the boundary of any interval has outer measure zero.

Proof. ■

Problem 2. Show that a set consisting of a single point has outer measure zero.

Proof. ■

1.1.3 Homework 3

Problem 1 (Wheeden & Zygmund Ch. 3, Ex. 5)**.** Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage each interval removed has length $\delta 3^{-k}$, $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no interval.

Proof. ■

Problem 2 (Wheeden & Zygmund Ch. 3, Ex. 7)**.** Prove (3.15).

Proof. ■

Problem 3 (Wheeden & Zygmund Ch. 3, Ex. 8)**.** Show that the Borel algebra $\mathscr{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.

Proof. ■

Problem 4 (Wheeden & Zygmund Ch. 3, Ex. 9)**.** If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ (and also $\liminf E_k$) has measure zero.

Proof. ■

Problem 5 (Wheeden & Zygmund Ch. 3, Ex. 10)**.** If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

Proof. ■

1.1.4 Homework 4

Problem 1 (Wheeden & Zygmund Ch. 3, Ex. 12)**.** If E_1 and E_2 are measurable sets in $\mathbb{R}^1$, show $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1||E_2|$. (Interpret $0 \cdot \infty$ as 0.) [*Hint:* Use a characterization of measurability.]

Proof. ∎

Problem 2 (Wheeden & Zygmund Ch. 3, Ex. 13)**.** Motivated by (3.7), define the *inner measure* of E by $|E|_i = \sup |F|$, where the supremum is taken over all closed subsets F of E. Show that

(i) $|E|_i \leq |E|_e$, and
(ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$.

[Use (3.22).]

Proof. ∎

Problem 3 (Wheeden & Zygmund Ch. 3, Ex. 15)**.** If E is measurable and A is any subset of E, show that $|E| = |A|_i + |E \setminus A|_e$. (See Exercise 13 for the definition of $|A|_i$.)

Proof. ∎

1.1.5 Homework 5

Problem 1 (Wheeden & Zygmund Ch. 3, Ex. 14)**.** Show that the conclusion of part (ii) of Exercise 13 (Problem) is false if $|E|_e = +\infty$.

Proof. ■

Problem 2 (Wheeden & Zygmund Ch. 3, Ex. 16)**.** Prove (3.34).

Proof. ■

Problem 3 (Wheeden & Zygmund Ch. 3, Ex. 18)**.** Prove that outer measure is *translation invariant*; that is, if $E_{\mathbf{h}} := \{ \mathbf{x} + \mathbf{h} : \mathbf{x} \in E \}$ is the translate of E by $\mathbf{h}$, $\mathbf{h} \in \mathbb{R}^n$, show that $|E_{\mathbf{h}}|_e = |E|_e$. If E is measurable, show that $E_{\mathbf{h}}$ is also measurable. [This fact was used in proving (3.37).]

Proof. ■

Problem 4 (Wheeden & Zygmund Ch. 4, Ex. 1)**.** Prove corollary (4.2) and theorem (4.8)

Proof. ■

Problem 5 (Wheeden & Zygmund Ch. 4, Ex. 2)**.** Let f be a simple function, taking its distinct values on disjoint sets $E_1, ..., E_N$. Show that f is measurable if and only if $E_1, ..., E_N$ are measurable.

Proof. ■

1.1.6 Homework 6

Problem 1 (Wheeden & Zygmund Ch. 4, Ex. 4)**.** Let f be defined and measurable in $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(T\mathbf{x})$ is measurable. [If $E_1 = \{\, \mathbf{x} : f(\mathbf{x}) > a \,\}$ and $E_2 = \{\, \mathbf{x} : f(T\mathbf{x}) > a \,\}$, show $E_2 = T^{-1}E_1$.]

Proof. ∎

Problem 2 (Wheeden & Zygmund Ch. 4, Ex. 7)**.** Let f be usc and less that $+\infty$ on a compact set E. Show that f is bounded above on E. Show also that f assumes its maximum on E, i.e., that there exists $\mathbf{x}_0 \in E$ such that $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x} \in E$.

Proof. ∎

Problem 3 (Wheeden & Zygmund Ch. 4, Ex. 8)**.**

(a) Let f and g be two functions which are usc at $\mathbf{x}_0$. Show that $f + g$ is usc at $\mathbf{x}_0$. Is $f - g$ usc at $\mathbf{x}_0$? When is fg usc at $\mathbf{x}_0$?

(b) If $\{f_k\}$ is a sequence of functions are usc at $\mathbf{x}_0$, show that $\inf f_k(\mathbf{x})$ is usc at $\mathbf{x}_0$.

(c) If $\{f_k\}$ is a sequence of functions which are usc at $\mathbf{x}_0$ and which converge uniformly near $\mathbf{x}_0$, show that $\lim f_k$ is usc at $\mathbf{x}_0$.

Proof. ∎

1.1.7 Homework 7

Problem 1 (Wheeden & Zygmund Ch. 4, Ex. 9)**.** (a) Show that the limit of a decreasing (increasing) sequence of functions usc (lsc) at $\mathbf{x}_0$ is usc (lsc) at $\mathbf{x}_0$. In particular, the limit of a decreasing (increasing) sequence of functions continuous at $\mathbf{x}_0$ is usc (lsc) at $\mathbf{x}_0$.

(b) Let f be usc and less than ∞ on $[a,b]$. Show that there exists continuous f_k on $[a,b]$ such that $f_k \searrow f$.

Proof. ■

Problem 2 (Wheeden & Zygmund Ch. 4, Ex. 11)**.** Let f be defined on $\mathbb{R}^n$ and let $B(\mathbf{x})$ denote the open ball $\{\, \mathbf{y} : |\mathbf{x}-\mathbf{y}| < r \,\}$ with center $\mathbf{x}$ and fixed radius r. Show that the function $g(\mathbf{x}) = \sup\{\, f(\mathbf{y}) : \mathbf{y} \in B(\mathbf{x}) \,\}$ is lsc and the function $h(\mathbf{x}) = \inf\{\, f(\mathbf{y}) : \mathbf{y} \in B(\mathbf{x}) \,\}$ is usc on $\mathbb{R}^n$. Is the same true for the closed ball $\{\, \mathbf{y} : |\mathbf{x}-\mathbf{y}| \le r \,\}$?

Proof. ■

Problem 3 (Wheeden & Zygmund Ch. 4, Ex. 15)**.** Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. If $|f_k(M)| \le M < \infty$ for all k for each $\mathbf{x} \in E$, show that given $\varepsilon > 0$, there is closed $F \subset E$ and finite M such that $|E \setminus F| < \varepsilon$ and $|f_k(\mathbf{x})| \le M$ for all $\mathbf{x} \in F$.

Proof. ■

Problem 4 (Wheeden & Zygmund Ch. 4, Ex. 18)**.** If f is measurable on E, define $\omega_f(a) = |\{\, f > a \,\}|$ for $-\infty < a < \infty$. If $f_k \nearrow f$, show that $\omega_{f_k} \nearrow \omega_f$. If $f_k \to f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f. [For the second part, show that if $f_k \to f$, then $\limsup_{k\to\infty} \omega_{f_k}(a) \le \omega_f(a-\varepsilon)$ and $\liminf_{k\to\infty} \omega_{f_k}(a) \ge \omega_f(a+\varepsilon)$ for every $\varepsilon > 0$.]

Proof. ■

Problem 5 (Wheeden & Zygmund Ch. 5, Ex. 1)**.** If f is a simple measurable function (not necessarily positive) taking values a_j on E_j, $j = 1, ..., N$, show that $\int_E f = \sum_{j=1}^N a_j |E_j|$. [Use (5.24)].

Proof. ■

Problem 6 (Wheeden & Zygmund Ch. 5, Ex. 3)**.** Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \le f$ a.e. on E, show that $\int_E f_k \to \int_E f$.

Proof. ■

1.1.8 Homework 8

Problem 1 (Wheeden & Zygmund Ch. 5, Ex. 2)**.** Show that the conclusion of (5.32) are not true without the assumption that $\varphi \in L(E)$. [In part (ii), for example, take $f_k = \chi_{(k,\infty)}$.]

Proof. ■

Problem 2 (Wheeden & Zygmund Ch. 5, Ex. 4)**.** If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and $\int_0^1 x^k f(x)\,\mathrm{d}x \to 0$.

Proof. ■

Problem 3 (Wheeden & Zygmund Ch. 5, Ex. 6)**.** Let $f(x,y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $\partial f(x,y)/\partial x$ is a bounded function of (x,y). Show that $\partial f(x,y)/\partial x$ is a measurable function of y for each x and

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 f(x,y)\,\mathrm{d}y = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,\mathrm{d}y.$$

Proof. ■

Problem 4 (Wheeden & Zygmund Ch. 5, Ex. 7)**.** Give an example of an f that is not integrable, but whose improper Riemann integral exists and is finite.

Proof. ■

Problem 5 (Wheeden & Zygmund Ch. 5, Ex. 21)**.** If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

Proof. ■

Problem 6 (Wheeden & Zygmund Ch. 6, Ex. 10)**.** Let V_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theorem that

$$V_n = 2V_{n-1}\int_0^1 \left(1-t^2\right)^{(n-1)/2}\,\mathrm{d}t.$$

(We also observe that by setting $w = t^2$, the integral is a multiple of a classical β-function and so can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,\mathrm{d}t$, $s > 0$.)

Proof. ■

Problem 7 (Wheeden & Zygmund Ch. 6, Ex. 11)**.** Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x} = \pi^{n/2}.$$

(For $n = 1$, write $\left(\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x\right)^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2}\,\mathrm{d}x dy$ and use polar. For $n > 1$, use the formula $e^{-|\mathbf{x}|^2} = e^{-{x_1}^2} \cdots e^{-{x_n}^2}$ and Fubini's theorem to reduce the case $n = 1$.)

Proof. ■

1.1.9 Homework 9

Problem 1 (Wheeden & Zygmund Ch. 6, Ex. 1). (a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}$, $\{\, y : (x, y) \in E \,\}$ has $\mathbb{R}$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}$, $\{\, x : (x, y) \in E \,\}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x, y)$ is finite for almost every y. Show that for almost $y \in \mathbb{R}$, $f(x, y)$ is finite for almost every x.

Proof. ∎

Problem 2 (Wheeden & Zygmund Ch. 6, Ex. 3). Let f be measurable and finite a.e. on $[0, 1]$. If $f(x) - f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$, show that $f \in L[0, 1]$.

Proof. ∎

Problem 3 (Wheeden & Zygmund Ch. 6, Ex. 4). Let f be measurable and periodic with period 1: $f(t + 1) = f(t)$. Suppose there is a finite c such that

$$\int_0^1 |f(a + t) - f(b + t)| \,\mathrm{d}t \le c$$

for all a and b. Show that $f \in L[0, 1]$. (Set $a = x$, $b = -x$, integrate with respect to x, and make the change of variables $\xi = x + t$, $\eta = -x + t$.)

Proof. ∎

Problem 4 (Wheeden & Zygmund Ch. 6, Ex. 6). For $f \in L(\mathbb{R})$, define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ixt} \,\mathrm{d}t$$

for $x \in \mathbb{R}$. (For complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i \int F_1$.) Show that if f and g belong to $L(\mathbb{R})$, then

$$(\widehat{f * g})(x) = 2\pi \hat{f}(x) \hat{g}(x).$$

Proof. ∎

Problem 5 (Wheeden & Zygmund Ch. 6, Ex. 7). Let F be a closed subset of $\mathbb{R}$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F, prove that the function

$$\int_{\mathbb{R}} \frac{\delta^\lambda(y) f(y)}{|x - y|^{1+\lambda}} \,\mathrm{d}t$$

is integrable over F and so is finite a.e. in F. (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

Proof. ∎

Problem 6 (Wheeden & Zygmund Ch. 6, Ex. 9). (a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.

(b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subset \mathbb{R}$, and let M_α be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int M_\lambda \leq \lambda^{-1}|G|$, where $G = (a, b) - [c, d]$.

Proof. ∎

1.1.10 Homework 10

Problem 1 (Wheeden & Zygmund Ch. 7, Ex. 1)**.** Let f be measurable in $\mathbb{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(\mathbf{x}) \geq c\|\mathbf{x}\|^{-n}$ for $\|\mathbf{x}\| \geq 1$.

Proof. ■

Problem 2 (Wheeden & Zygmund Ch. 7, Ex. 2)**.** Let $\varphi(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$, be a bounded measurable function such that $\varphi(\mathbf{x}) = 0$ for $\|\mathbf{x}\| \geq 1$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(\mathbf{x}) = \varepsilon^{-n}\varphi(\mathbf{x}/\varepsilon)$. ($\varphi_\varepsilon$ is called an *approximation to the identity.*) If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon \to 0}(f * \varphi_\varepsilon)(\mathbf{x}) = f(\mathbf{x})$$

in the Lebesgue set of f. (Note that $\int \varphi_\varepsilon = 1$, $\varepsilon > 0$, so that

$$(f * \varphi_\varepsilon)(\mathbf{x}) - f(\mathbf{x}) = \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})]\, \varphi_\varepsilon(\mathbf{y})\, \mathrm{d}\mathbf{y}.$$

Use Theorem 7.16.)

Proof. ■

Problem 3 (Wheeden & Zygmund Ch. 7, Ex. 6)**.** Show that if $\alpha > 0$, then x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Proof. ■

Problem 4 (Wheeden & Zygmund Ch. 7, Ex. 8)**.** Prove the following converse of Theorem 7.31: If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Proof. ■

Problem 5 (Wheeden & Zygmund Ch. 7, Ex. 9)**.** If f is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then f is absolutely continuous on $[a, b]$. (For the second part, use Theorems 2.2(ii) and 7.24 to show that $V(x)$ is absolutely continuous and then use the result of Exercise 8).

Proof. ■

Problem 6 (Wheeden & Zygmund Ch. 7, Ex. 12)**.** Use Jensen's inequality to prove that if $a, b \geq 0$, $p, q > 1$, $(1/p) + (1/q) = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

More generally, show that

$$a_1 \cdots a_N = \sum_{j=1}^{N} \frac{{a_j}^{p_j}}{p_j},$$

where $a_j \geq 0$, $p_j > 1$, $\sum_{j=1}^{N}(1/p_j) = 1$. (Write $a_j = e^{x_j/p_j}$ and use the convexity of e^x).

Proof. ∎

Problem 7 (Wheeden & Zygmund Ch. 7, Ex. 13)**.** Prove Theorem 7.36.

Proof. Recall the statement of Theorem 7.36

(i) If φ_1 and φ_2 are convex in (a, b), then $\varphi_1 + \varphi_2$ is convex in (a, b).
(ii) If φ is convex in (a, b) and c is a positive constant, then $c\varphi$ is convex in (a, b).
(iii) If φ_k, $k = 1, 2, \ldots$, are convex in (a, b) and $\varphi_k \to \varphi$ in (a, b), then φ is convex in (a, b).

∎

1.1.11 Homework 11

Problem 1 (Wheeden & Zygmund Ch. 7, Ex. 11)**.** Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t.$$

(Consider the case when f is the characteristic function of an interval, an open set, etc.)

Proof. ■

Problem 2 (Wheeden & Zygmund Ch. 7, Ex. 15)**.** Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\varphi(x) = \int_a^x f(t)\,\mathrm{d}t + \varphi(a)$ in (a, b) and f is monotone increasing, then φ is convex in (a, b). (Use Exercise 14.)

Proof. ■

Problem 3 (Wheeden & Zygmund Ch. 5, Ex. 8)**.** Prove (5.49).

Proof. ■

Problem 4 (Wheeden & Zygmund Ch. 5, Ex. 11)**.** For which p does $1/x \in L^p(0, 1)$? $L^p(1, \infty)$? $L^p(0, \infty)$?

Proof. ■

Problem 5 (Wheeden & Zygmund Ch. 5, Ex. 12)**.** Give an example of a bounded continuous f on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

Proof. ■

Problem 6 (Wheeden & Zygmund Ch. 5, Ex. 17)**.** If $f \geq 0$ and $\omega(\alpha) \leq c(1 + \alpha)^p$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.

Proof. ■

Problem 7 (Wheeden & Zygmund Ch. 8, Thm. 8.3)**.** If $f, g \in L^p(E)$, $p > 0$, then $f + g \in L^p(E)$ and $cf \in L^p(E)$ for any constant c.

Proof. ■

1.1.12 Homework 12

Problem 1 (Wheeden & Zygmund Ch. 8, Ex. 2)**.** Prove the converse of Hölder's inequality for $p = 1$ and ∞. Show also that for $1 \le p \le \infty$, a real-valued measurable f belongs to $L^p(E)$ if $fg \in L^1(E)$ for every $g \in L^{p'}(E)$, $1/p + 1/p' = 1$. The negation is also of interest: if $f \in L^p(E)$ then there exists $g \in L^{p'}(E)$ such that $fg \notin L^1(E)$. (To verify the negation, construct g of the form $\sum a_k g_k$ satisfying $\int_E f g_k \to \infty$.)

Proof. ∎

Problem 2 (Wheeden & Zygmund Ch. 8, Ex. 3)**.** Prove Theorems 8.12 and 8.13. Show that Minkowski's inequality for series fails when $p < 1$.

Proof. ∎

Problem 3 (Wheeden & Zygmund Ch. 8, Ex. 4)**.** Let f and g be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let $1 < p < \infty$. Prove that equality holds in the inequality $|\int fg| \le \|f\|_p \|g\|_{p'}$ if and only if fg has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e.

If $\|f + g\|_p = \|f\|_p + \|g\|_p$ and $g \ne 0$ in Minkowski's inequality, show that f is a multiple of g.

Find analogues of these results for the spaces ℓ^p.

Proof. ∎

Problem 4 (Wheeden & Zygmund Ch. 8, Ex. 5)**.** For $0 < p \le \infty$ and $0 < |E| < \infty$, define

$$N_p[f] := \left(\frac{1}{E} \int_E |f|^p \right)^{1/p},$$

where $N_\infty[f]$ means $\|f\|_\infty$. Prove that if $p_1 < p_2$, then $N_{p_1}[f] \le N_{p_2}[f]$. Prove also that if $1 \le p \le \infty$, then $N_p[f + g] \le N_p[f] + N_p[g]$, $(1/|E|) \int_E |fg| \le N_p[f] N_{p'}[g]$, $1/p + 1/p' = 1$, and $\lim_{p\to\infty} N_p[f] = \|f\|_\infty$. Thus, N_p behaves like $\|\cdot\|_p$ but has the advantage of being monotone in p. Recall Exercise 28 of Chapter 5.

Proof. ∎

Problem 5 (Wheeden & Zygmund Ch. 8, Ex. 6)**.** (a) Let $1 \le p_i, r \le \infty$ and $\sum_{i=1}^k 1/p_i = 1/r$. Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

(b) Let $1 \le p < r < q \le \infty$ and define $\theta \in (0, 1)$ by $1/r = \theta/p + (1-\theta)/q$. Prove the interpolation estimate

$$\|f\|_r \le \|f\|_p{}^\theta \|f\|_q{}^{1-\theta}.$$

In particular, if $A := \max \left\{ \|f\|_p, \|f\|_q \right\}$, then $\|f\|_r \le A$.

Proof. ∎

Problem 6 (Wheeden & Zygmund Ch. 8, Ex. 9)**.** If f is real-valued and measurable on E, $|E| > 0$, define its essential infimum on E by

$$\operatorname{ess\,inf} f := \sup\{\, \alpha : |\{\, x \in E : f(x) < \alpha \,\}| = 0 \,\}\,.$$

If $f \geq 0$, show that $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup} 1/f)^{-1}$.

Proof. ∎

Problem 7 (Wheeden & Zygmund Ch. 8, Ex. 11)**.** If $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty < M$ for all k, prove that $f_k g_k \to fg$ in L^p.

Proof. ∎

1.2 Exam Preparation

1.2.1 Exam 1 Practice

Problem 1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$ and define the set $rE = \{ r\mathbf{x} : \mathbf{x} \in E \}$. Prove that rE is measurable, and that $|rE| = |r|^n|E|$.

Proof. Define a map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by $T\mathbf{x} := r\mathbf{x}$. Note that T is *Lipschitz continuous* since for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the equality

$$|T\mathbf{x} - T\mathbf{y}| = |r\mathbf{x} - r\mathbf{y}| = |r||\mathbf{x} - \mathbf{y}| \tag{1}$$

is satisfied. By Theorem 3.33 from [5, Ch. 3, p.55], the image of E under T is measurable. Moreover, by Theorem 3.35 [5, Ch. 3, p. 56], since T is linear, it follows that $|T(E)| = |\det T||E|$ where $\det T = |r|^n$. Lastly, we note that the image of E under T is precisely the set rE so that $|T(E)| = |rE| = |r|^n|E|$, as was to be shown. ∎

Problem 2. Let $\{E_k\}$, $k \in \mathbf{N}$ be a collection of measurable sets. Define the set

$$\liminf_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n \right).$$

Show that

$$\left| \liminf_{k\to\infty} E_k \right| \leq \liminf_{k\to\infty} |E_k|.$$

Proof. Following the style of [5, Ch. 1, p. 2], particularly, the sets defined after the introduction of equation (1.1), set

$$V_k := \bigcap_{\ell=k}^{\infty} E_\ell. \tag{2}$$

Note that the collection of sets $\{V_k\}$ forms an increasing sequence, that is, if $\mathbf{x} \in V_k$ then, by (2), $\mathbf{x}$ is in the intersection $E_k \cap (\bigcap_{\ell=k+1} E_\ell)$, but, by (2), $\bigcap_{\ell=k+1} E_\ell = V_{k+1}$ thus, $\mathbf{x}$ is in V_{k+1} so $V_{k+1} \supset V_k$. Hence, we have $V_k \nearrow \liminf E_k$.

Now, consider the sequence $\{|V_k|\}$ formed by the Lebesgue measure of the V_k. By Theorem 3.26 from [5, Ch. 3, p. 51], since $V_k \nearrow \liminf E_k$,

$$\lim_{k\to\infty} |V_k| = \lim_{k\to\infty} \left| \bigcap_{\ell=k}^{\infty} E_\ell \right| = \left| \liminf_{k\to\infty} E_k \right|. \tag{3}$$

But note that, by the monotonicity of the Lebesgue measure, we have

$$\left| \bigcap_{\ell=k}^{\infty} E_\ell \right| \leq |E_k|, \tag{4}$$

so, by properties of the lim inf, in particular, by Theorem 19(v) from [2, Ch. 1, p. 23], we have

$$\limsup_{k\to\infty} |V_k| \leq \liminf_{k\to\infty} |E_k|. \tag{5}$$

Hence, by (3) and Proposition 19 (iv), since the sequence $\{|V_k|\}$ converges and is equal to the measure of $\liminf E_k$, by (5), we have

$$\left|\liminf_{k\to\infty} E_k\right| \leq \liminf_{k\to\infty} |E_k| \tag{6}$$

as was to be shown. ■

Problem 3. Consider the function

$$F(x) = \begin{cases} |B(\mathbf{0}, x)| & x > 0 \\ 0 & x = 0 \end{cases}.$$

Here $B(\mathbf{0}, r) = \{\, \mathbf{y} \in \mathbb{R}^n : |\mathbf{y}| < r \,\}$. Prove that F is monotonic increasing and continuous.

Proof. Define the linear map $T\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ by $T(r)\mathbf{x} := r\mathbf{x}$. We claim that $B(\mathbf{0}, r) = T(r, B(\mathbf{0}, 1))$. To reduce notation, set $B_1 := B(\mathbf{0}, 1)$ and $B_r := B(\mathbf{0}, r)$.

Proof of claim. $\subset$: Let $\mathbf{x} \in B_r$. Then $|\mathbf{x}| < r$ so $|\mathbf{x}|/r < 1$. Thus, $|\mathbf{x}|/r \in B_1$ so it is in the image of B_1 under the map $T(r, \cdot)$.

$\supset$: On the other hand, suppose $\mathbf{x} \in T(r, B_1)$. Then $\mathbf{x} = r\mathbf{y}$ for some $\mathbf{y} \in B_1$. Then, since $|\mathbf{y}| < 1$, $|\mathbf{x}| = r|\mathbf{y}| < r$ so $\mathbf{x} \in B_r$. ♣

From the claim, we see that $F(x) = |T(x, B(\mathbf{0}, 1))|$ which, by Problem 1, is nothing more that the polynomial $|B_1|x^n$. It is clear, from this equivalence, that F is monotonically increasing: Take $x, y \in [0, \infty)$ such that $x < y$, then $x^n < y^n$ so

$$F(x) = |B_1|x^n < |B_1|y^n = F(y). \tag{7}$$

Thus, F is monotonically increasing.

In the argument above, since $F(x) = |B_1|x^n$ is a polynomial in $[0, \infty)$ (and polynomials are continuous on $\mathbb{R}$) F is continuous on $[0, \infty)$. ■

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

Proof. (Without much motivation) let us consider the collection of sets $\{E_k\}$ defined by

$$E_k := \left\{ x \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } y, z \in B(x, \delta) \text{ implies } |f(y) - f(z)| < \frac{1}{k} \right\}. \tag{8}$$

We claim that $C = \bigcap_{k=1}^\infty E_k$ and that each E_k is open.

Proof of claim. First, we demonstrate equality. $\subset$: Suppose $x \in C$. Then, by the definition of continuity, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $y \in B(x, \delta)$ implies $|f(x) - f(y)| < \delta$. In particular, for every k, there exists $\delta > 0$ such that for $y \in B(x, \delta)$ the inequality $|f(x) - f(y)| < 1/k$ holds. Thus, x is in $\bigcap_{k=1}^\infty E_k$.

$\supset$: On the other hand, suppose that $x \in \bigcap_{k=1}^\infty E_k$. Then, given $\varepsilon > 0$, by the Archimedean property, there exists a positive integer N such that $1/N < \varepsilon$. Then, since $x \in \bigcap_{k=1}^\infty E_k$, $x \in E_N$ so

$$|f(x) - f(y)| < \frac{1}{N} < \varepsilon. \tag{9}$$

Thus, x is in C and $C = \bigcap_{k=1}^{\infty} E_k$.

All that remains to be shown is that the E_k are open. But this is clear by the way we defined E_k in (8): Let $x \in E_k$, then there exists $\delta > 0$ such that for any $y, z \in B(x, \delta)$, $|f(y) - f(z)| < 1/k$; Let $x' \in B(x, \delta)$ and set $\delta' := \min\{|(x+\delta) - x'|, |(x-\delta) - x|\}$. Then, since $B(x', \delta') \subset B(x, \delta)$, for every $y, z \in B(x', \delta')$, we have $|f(y) - f(z)| < 1/k$. Hence, $x' \in E_k$ for any $x' \in B(x, \delta)$ so $B(x, \delta) \subset E_k$. ♣

Since C can be expressed as the countable intersection of open sets E_k, it follows that C is a G_δ set. ■

Problem 5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{\, f = r \,\}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

Proof. If $\{\, f = r \,\}$ are measurable for all $r \in \mathbb{R}$, it is not necessarily the case that f is measurable. Consider the following construction: Let $E \subset (0,1)$ be an unmeasurable set.[†] Define a map $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{R} \setminus ((0,1) \setminus E), \\ x+1 & \text{if } x \in (0,1) \setminus E. \end{cases} \tag{10}$$

By the definition, it is clear that $\{\, f = r \,\}$ is measurable and $|\{\, f = r \,\}| = 0$ since $\{\, f = r \,\}$ contains at most two elements. However, the set $\{\, 0 < f < 1 \,\} = E$ is not measurable. Thus, f is not measurable. ■

Problem 6. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Prove that the set $\{\, x : \lim_{k\to\infty} f_k(x) \text{ exists} \,\}$ is measurable.

Proof. By Theorem 4.12 from [5, Ch. 4, p. 67], $\liminf_{k\to\infty} f_k$ and $\limsup_{k\to\infty} f_k$ are measurable. By Theorem 4.7 from [5, Ch. 4, p. 66]

$$\left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\} \tag{11}$$

is measurable. Since

$$\left\{ \lim_{k\to\infty} f_k \text{ exists} \right\} = \left\{ \limsup_{k\to\infty} f_k = \liminf_{k\to\infty} f_k \right\} = \mathbb{R} \setminus \left\{ \liminf_{k\to\infty} f_k < \limsup_{k\to\infty} f_k \right\}, \tag{12}$$

by Theorem 3.17 from [5, Ch. 3, p. 48], the set $\{\, \lim_{k\to\infty} f_k \text{ exists} \,\}$ is measurable. ■

Problem 7. A real valued function f on an interval $[a,b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{N}$ of open intervals in (a,b) satisfying $\sum_{k=1}^{N} b_k - a_k < \delta$, one has $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a,b]$ is of bounded variation on $[a,b]$.

[†] It's construction does not concern us. The interested reader such direct their refer to Theorem 3.38 from [5, Ch. 3, p. 57-58] or Theorem 17 from [2, Ch. 2§7, p. 48].

Proof. Suppose f is absolutely continuous on $[a, b]$. Let $\varepsilon := 1$. Then, there exists $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^N$ of open intervals in (a, b) satisfying $\sum_{k=1}^N b_k - a_k < \delta$, one has $\sum_{k=1}^N |f(b_k) - f(a_k)| < 1$. Let $N := \lceil (b-a)/\delta \rceil$, that is, N is the smallest integer greater than $(b-a)/\delta$, and consider the partition $\Gamma = \{x_k\}$ where $x_k := a + k(b-a)/N$, for $k = 0, \dots, N$. Then $x_k - x_{k-1} < (b-a)/N < \delta$ so, by Theorem 2.2(i) from [5, Ch. 2, p. 19], we have $V[f; x_{k-1}, x_k] < 1$ for $k = 0, \dots, N$. In follows by Theorem 2.2(ii) that

$$V[f; a, b] = \sum_{k=1}^{N} V[f; x_{k-1}, x_k] < N. \tag{13}$$

Thus, f is b.v. on $[a, b]$. ∎

Problem 8. Let f be a continuous function from $[a, b]$ into $\mathbb{R}$. Let $\chi_{\{c\}}$ be the characteristic function of a singleton $\{c\}$, that is, $\chi_{\{c\}}(x) = 0$ if $x \neq c$ and $\chi_{\{c\}}(c) = 1$. Show that

$$\int_a^b f d\chi_{\{c\}} = \begin{cases} 0 & \text{if } c \in (a, b), \\ -f(a) & \text{if } c = a, \\ f(b) & \text{if } c = b. \end{cases}$$

Proof. The result follows quite easily from more sophisticated measure theoretic arguments. At this point, however, such language has not been discussed so we shall prove this using nothing but the definition of the Riemann–Stieltjes integral and properties thereof.

Let us consider each case $c \in (a, b)$, $c = a$, and $c = b$ separately.

Recall that the given a partition $\Gamma = \{x_0, \dots, x_m\}$ of $[a, b]$, the Riemann–Stieltjes sum of f with respect to φ is

$$R_\Gamma := \sum_{k=1}^{m} f(\xi_k)[\varphi(x_k) - \varphi(x_{k-1})]. \tag{14}$$

The Riemann–Stieltjes integral is defined as the limit

$$\int_a^b f \, \mathrm{d}\varphi := \lim_{|\Gamma| \to 0} R_\Gamma \tag{15}$$

if it exists.

Suppose $c \in (a, b)$. Then, for any partition Γ of $[a, b]$, either $c \in \Gamma$ or $c \notin \Gamma$. In the latter case, $R_\Gamma = 0$. In the former case c is one of the x_k, say $c = x_\ell$ for $0 < \ell < m$. Then

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^{m} f(\xi_k)[\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= 0 + \cdots + 0 + f(\xi_{\ell-1}) - f(\xi_\ell) + 0 + \cdots + 0 \\ &= f(\xi_{\ell-1}) - f(\xi_\ell). \end{aligned} \tag{16}$$

Since f is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi_\ell - \xi_{\ell-1}| < \delta$ implies $|f(\xi_\ell) - f(\xi_{\ell-1})| < \varepsilon$. It follows that the quantity in (16) approaches 0 as $|\Gamma|$ approaches 0. Therefore, $\int_a^b f \, \mathrm{d}\chi_{\{c\}} = 0$.

Suppose $c = a$. Then, since any partition Γ of $[a, b]$ must contain the point a, we have

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^{m} f(\chi_k)[\chi_{\{c\}}(x_k) - \chi_{\{c\}}(x_{k-1})] \\ &= f(\xi_1)[\chi_{\{c\}}(x_1) - \chi_{\{c\}}(x_0)] + f(\xi_2)[\chi_{\{c\}}(x_2) - \chi_{\{c\}}(x_1)] \\ &\qquad + \cdots + f(\xi_m)[\chi_{\{c\}}(x_m) - \chi_{\{c\}}(x_{m-1})] \\ &= -f(\xi_1) + 0 + \cdots + 0 \\ &= -f(\xi_1) \end{aligned} \tag{17}$$

Taking the limit as $|\Gamma| \to 0$, $\xi_1 \to a$ so, by continuity of f, $f(\xi_1) \to f(a)$. Thus, $\int_a^b f \, \mathrm{d}\chi_{\{c\}} = -f(a)$.

A similar argument to the one above shows that, if $c = b$, the Riemann–Stieltjes integral $\int_a^b f \, \mathrm{d}\chi_{\{c\}} = f(b)$. ■

1.2.2 Exam 1

Problem 1.

Proof. ■

Problem 2.

Proof. ■

Problem 3.

(i) Show that if $B_r := \{ \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < r \}$, then there exists a constant C such that $|B_r| = Cr^n$.

(*Hint*: Think of B_r as $\{ r\mathbf{x} : \mathbf{x} \in B_1 \}$.)

(ii) Let $E \subset \mathbb{R}^n$ be a measurable set and let $\varphi_E \colon \mathbb{R}^n \to \mathbb{R}$ be defined $\varphi_E(\mathbf{x}) := \left| E \cap B_{|\mathbf{x}|} \right|$. Use part (i) to prove that φ_E is continuous.

Proof. (i) To prove this result, we use the map constructed in Problem 1 of the review sheet for Exam 1, the map $T \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Set $T_r \colon \mathbb{R}^n \to \mathbb{R}^n$ to be $T_r := T(r)$. Then, we claim $B_r = T_r(B_1)$ and $|B_r| = |T_r(B_1)|$, which, as we saw in Problem 1 of the review sheet, has measure $|B_1||r|^n$. Setting $C := |B_1|$, we have $|B_r| = C|r|^n$ as desired.

(ii) To prove that φ_E is continuous, we provide an (ε, δ)-argument. Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that $\mathbf{y} \in B(\mathbf{x}, \delta)$ implies

$$|\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| < \varepsilon. \tag{1}$$

First, note that since $\mathbf{x} \mapsto |\mathbf{x}|$ is continuous and polynomials $p \colon \mathbb{R}^n \to \mathbb{R}^n$ are continuous, then the composition $\mathbf{x} \mapsto |\mathbf{x}|^n$ is continuous. Therefore, there exists $\delta > 0$ such that $\mathbf{y} \in B(\mathbf{x}, \delta)$ implies

$$||\mathbf{x}|^n - |\mathbf{y}|^n| < \frac{\varepsilon}{C}, \tag{2}$$

where $C := |B_1|$.

Now, let $x \in \mathbb{R}^n$ and $\mathbf{y} \in B(\mathbf{x}, \delta)$ as above. Then, by (2) we have

$$\begin{aligned} |\varphi_E(\mathbf{x}) - \varphi_E(\mathbf{y})| &= \left| \left| E \cap B_{|\mathbf{x}|} \right| - \left| E \cap B_{|\mathbf{y}|} \right| \right| \\ &\leq \left| |B_{|\mathbf{x}|}| - |B_{|\mathbf{y}|}| \right| \\ &= C| \, |\mathbf{x}|^n - |\mathbf{y}|^n | \\ &\leq C \left[\frac{\varepsilon}{C} \right] \\ &= \varepsilon. \end{aligned} \tag{3}$$

It follows that φ_E is continuous. ■

Problem 4. Assume that $f \colon [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$. Prove that f is measurable.

Proof. By Jordan's theorem (Corollary 2.7 from [5, Ch. 2, p. 21]), the function f is of bounded variation on $[a, b]$ if and only if it can be written as the difference $f_1 - f_2$ of two bounded functions f_1 and f_2 that are monotone increasing on $[a, b]$. Then, f_1 and f_2 are continuous a.e. on $[a, b]$ and hence, are measurable. ∎

1.2.3 Exam 2 Practice Problems

Problem 1. Define for $\mathbf{x} \in \mathbb{R}^n$,

$$f(\mathbf{x}) = \begin{cases} |\mathbf{x}|^{-(n+1)} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Prove that f is integrable outside any ball $B(\mathbf{0}, \varepsilon)$, and that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n \setminus B(\mathbf{0},\varepsilon)} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq \frac{C}{\varepsilon}.$$

Proof. Recall that a real-valued function $f\colon \mathbb{R}^n \to \mathbb{R}$ is Lebesgueintegrable on a subset E of $\mathbb{R}^n$ if

$$\int_E f(\mathbf{x})\,\mathrm{d}\mathbf{x} < \infty. \tag{1}$$

Let f be as given in the statement of the problem and set $B_\varepsilon := B(\mathbf{0}, \varepsilon)$. Consider the change of variables to *hyperspherical coordinates* $(x_1, \dots, x_n) \mapsto (r, \Theta)$ where $\Theta = (\theta_1, \dots, \theta_{n-1})$.[‡] By Theorem 7.26(iii) from [4, Ch. 7, p. 123], we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\varepsilon} f(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^n \setminus B_\varepsilon} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n \setminus B_\varepsilon} \frac{1}{|\mathbf{x}|^{n+1}}\,\mathrm{d}\mathbf{x}. \\ &= \int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}}\,\mathrm{d}r dV, \end{aligned} \tag{2}$$

where S_r^{n-1} is the $(n-1)$-sphere centered at $\mathbf{0}$ with radius r, that is, the subset $\{\, \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1 \,\}$ of $\mathbb{R}^n$ and dV is the *volume element* of S_r^{n-1}. Since $1/|r|^{n+1}$ is nonnegative, by Tonelli's theorem the the iterated integrals in (2) may be exchange, that is,

$$\int_{S_r^{n-1}} \int_\varepsilon^\infty \frac{1}{|r|^{n+1}}\,\mathrm{d}r dV = \int_\varepsilon^\infty \left(\int_{S_r^{n-1}} 1\,\mathrm{d}V \right) \frac{1}{|r|^{n+1}}\,\mathrm{d}r. \tag{3}$$

Now, note that from Problem 1 of the review sheet for Exam 1, we have

$$\int_{S_r^{n-1}} 1\,\mathrm{d}V = |S_r^{n-1}|_{\mathbb{R}^{n-1}} = |S^{n-1}|_{\mathbb{R}^{n-1}} |r|^{n-1}. \tag{4}$$

[‡]The explicit construction of the map $(x_1, \dots, x_n) \mapsto (r, \Theta)$ is of no concern to us for now. What is important is that it exists.

Set $C := |S^{n-1}|_{\mathbb{R}^{n-1}}$. Putting equations (2), (3), and (4) together, we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_\varepsilon} f(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_\varepsilon^\infty C|r|^{n-1}\frac{1}{|r|^{n+1}}\,\mathrm{d}r \\ &= \int_\varepsilon^\infty \frac{C}{|r|^2}\,\mathrm{d}r \\ &= \lim_{x\to\infty}\left[-\frac{C}{x}-\left(-\frac{C}{\varepsilon}\right)\right] \\ &= \frac{C}{\varepsilon},\end{aligned} \tag{5}$$

as was to be shown. ■

Problem 2. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$, and assume that f_k converges pointwise almost everywhere to a function f. If

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty}\int_{\mathbb{R}^n} f_k < \infty,$$

show that

$$\int_E f = \lim_{k\to\infty}\int_E f_k$$

for all measurable subsets E of $\mathbb{R}^n$. Moreover, show that this is not necessarily true if $\int_{\mathbb{R}^n} f = \lim_{k\to\infty} f_k = \infty$.

Proof. Let $E\subset\mathbb{R}^n$ be a measurable subset of $\mathbb{R}^n$. Then, since $f_k\to f$ pointwise a.e. on $\mathbb{R}^n$, then $f_k\to f$ pointwise a.e. on E and $\mathbb{R}^n\setminus E$. To prove that the limit of the sequence of integrals $\left\{\int_E f_k\right\}$ exist and is equal to $\int_E f$, it suffices to prove that

$$\int_E f \le \liminf_{k\to\infty}\int_E f_k \le \limsup_{k\to\infty}\int_E f_k \le \int_E f. \tag{6}$$

The lower bound in (6) follows from an application of Fatou's lemma:

$$\int_E f = \int_E \liminf_{k\to\infty} f \le \liminf_{k\to\infty}\int_E f_k. \tag{7}$$

Also by Fatou's lemma, we have

$$\int_{\mathbb{R}^n\setminus E} f = \int_{\mathbb{R}^n\setminus E}\liminf_{k\to\infty} f \le \liminf_{k\to\infty}\int_{\mathbb{R}^n\setminus E} f_k. \tag{8}$$

Now, since $f \in L^1(\mathbb{R}^n)$, by equation (8) and properties of the lim inf and lim sup[§] we have

$$\begin{aligned}\int_E f = \int_{\mathbb{R}^n} f - \int_{\mathbb{R}^n \setminus E} f &\geq \limsup_{k\to\infty} \int_{\mathbb{R}^n} f - \liminf_{k\to\infty} \int_{\mathbb{R}^n\setminus E} f_k \\ &\geq \limsup_{k\to\infty} \int_{\mathbb{R}^n} f_k - \limsup_{k\to\infty} \int_{\mathbb{R}^n\setminus E} f_k \\ &= \limsup_{k\to\infty} \left[\int_{\mathbb{R}^n} f_k - \int_{\mathbb{R}^n\setminus E} f_k \right] \\ &= \limsup_{k\to\infty} \int_E f_k. \end{aligned} \tag{9}$$

By equations (7) and (9) it follows that $\lim_{k\to\infty} \int_E f_k$ exists and is equal to $\int_E f$.

To see that the result need not be true if $\int_E f = \infty$, consider the following example: Let $f_k\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f_k(x) := \begin{cases} k^2/2 & \text{if } x \in (-1/k, 1/k), \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

and $f = 1$.

It is easy to see that $f_k \to f$ a.e. in $\mathbb{R}$ and that both $\int_{\mathbb{R}} f = \infty$ and $\lim_{k\to\infty} f_k = \infty$. However, if $E := (-1, 1)$ then $\int_E f = 1$, but $\lim_{k\to\infty} \int_E f = \infty$. ■

Problem 3. Assume that E is a measurable set of $\mathbb{R}^n$, with $|E| < \infty$. Prove that a nonnegative function f defined on E is integrable if and only if

$$\sum_{k=0}^{\infty} |\{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}| < \infty.$$

Proof. If f is integrable over a measurable subset E of $\mathbb{R}^n$, then

$$\int_E f(\mathbf{x}) d\mathbf{x} < \infty. \tag{11}$$

Set $E_k = \{\, \mathbf{x} \in E : k+1 > f(\mathbf{x}) \geq k \,\}$ and $F_k = \{\, \mathbf{x} \in E : f(\mathbf{x}) \geq k \,\}$. Note the following properties about the sets we have just defined: first, the E_k's are pairwise disjoint and the F_k's are nested in the following way $F_{k+1} \subset F_k$; second, $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = F_k \setminus F_{k+1}$. By Theorem 3.23, since the E_k's are disjoint, we have

$$|E| = \sum_{k=1}^{\infty} |E_k| < \infty. \tag{12}$$

Now, since $k\chi_{E_k}(\mathbf{x}) \leq f(\mathbf{x}) \leq (k+1)\chi_{E_k}(\mathbf{x})$ on E_k, we have

$$k|E_k| \leq \int_{E_k} f(\mathbf{x}) d\mathbf{x} \leq (k+1)|E_k|. \tag{13}$$

[§]Namely, for any sequence of positive real numbers $\{a_k\}$ the inequality $\liminf a_k \leq \limsup a_k$ holds

Then we have the following upper and lower estimates on the integral of f over E

$$\sum_{k=0}^{\infty} k|E_k| \leq \int_E f(\mathbf{x})d\mathbf{x} \leq \sum_{k=0}^{\infty}(k+1)|E_k|. \tag{14}$$

But note that $|E_k| = |F_k \setminus F_{k+1}| = |F_k| - |F_{k+1}|$ by Corollary 3.25 since the measures of E_k, F_k, and F_{k+1} are all finite. Hence, (14) becomes

$$\sum_{k=0}^{\infty} k\left(|F_k| - |F_{k+1}|\right) \leq \int_E f(\mathbf{x})d\mathbf{x} \leq \sum_{k=0}^{\infty}(k+1)\left(|F_k| - |F_{k+1}|\right). \tag{15}$$

A little manipulation of the series in the leftmost estimate gives us

$$\begin{aligned}
\sum_{k=0}^{\infty} k\left(|F_k| - |F_{k+1}|\right) &= \sum_{k=1}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=2}^{\infty} k|F_k| - \sum_{k=1}^{\infty} k|F_{k+1}| \\
&= |F_1| + \sum_{k=1}^{\infty}(k+1)|F_{k+1}| - \sum_{k=1}^{\infty} k|F_{k+1} \qquad (16) \\
&= |F_1| + \sum_{k=1}^{\infty} |F_{k+1}| \\
&= \sum_{k=1}^{\infty} |F_{k+1}|
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty}(k+1)\left(|F_k| - |F_{k+1}|\right) &= \sum_{k=0}^{\infty}(k+1)|F_k| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=1}^{\infty}(k+1)|F_k| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \\
&= |F_0| + \sum_{k=0}^{\infty}(k+2)|F_{k+1}| - \sum_{k=0}^{\infty}(k+1)|F_{k+1}| \qquad (17) \\
&= |F_0| + \sum_{k=0}^{\infty} |F_{k+1}| \\
&= \sum_{k=0}^{\infty} |F_k|.
\end{aligned}$$

Thus, from (16) and (17)

$$\sum_{k=1}^{\infty} |F_k| \leq \int_E f(\mathbf{x})d\mathbf{x} \leq \sum_{k=0}^{\infty} |F_k| \tag{18}$$

so the integral $\int_E f$ converges if and only if the sum $\sum_{k=0}^{\infty} |F_k|$ converges. ∎

Problem 4. Suppose that E is a measurable subset of $\mathbb{R}^n$, with $|E| < \infty$. If f and g are measurable functions on E, define

$$\rho(f, g) = \int_E \frac{|f-g|}{1+|f-g|}.$$

Prove that $\rho(f_k, f) \to 0$ as $k \to \infty$ if and only if f_k converges to f as $k \to \infty$.

Proof. ∎

Problem 5. Define the *gamma function* $\Gamma \colon \mathbb{R}^+ \to \mathbb{R}$ by

$$\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} \, \mathrm{d}u,$$

and the *beta function* $\beta \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, \mathrm{d}t.$$

(a) Prove that the definition of the gamma function is well-posed, i.e., the function $u \mapsto e^{-u}u^{y-1}$ is in $L(\mathbb{R}^+)$ for all $y \in \mathbb{R}^+$.
(b) Show that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. ∎

Problem 6. Let $f \in L(\mathbb{R}^n)$ and for $\mathbf{h} \in \mathbb{R}^n$ define $f_{\mathbf{h}} \colon \mathbb{R}^n \to \mathbb{R}$ be $f_{\mathbf{h}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{h})$. Prove that

$$\lim_{\mathbf{h} \to \mathbf{0}} \int_{\mathbb{R}^n} |f_{\mathbf{h}} - f| = 0.$$

Proof. ∎

Problem 7. (a) If $f_k, g_k, f, g \in L(\mathbb{R}^n)$, $f_k \to f$ and $g_k \to g$ a.e. in $\mathbb{R}^n$, $|f_k| \leq g_k$ and

$$\int_{\mathbb{R}^n} g_k \longrightarrow \int_{\mathbb{R}^n} g,$$

prove that

$$\int_{\mathbb{R}^n} f_k \longrightarrow \int_{\mathbb{R}^n} f.$$

(b) Using part (a) show that if $f_k, f \in L(\mathbb{R}^n)$ and $f_k \to f$ a.e. in $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \qquad \text{as } k \to \infty$$

if and only if

$$\int_{\mathbb{R}^n} |f_k| \longrightarrow \int_{\mathbb{R}^n} |f| \qquad \text{as } k \to \infty.$$

Proof. (a) $\implies$ (b): Assume part (a) then $\implies$ if

$$\int_{\mathbb{R}^n} |f_k - f| \longrightarrow 0 \tag{19}$$

as $k \to \infty$, we have

(b): ■

1.2.4 Exam 2 (2010)

Problem 1. Suppose $f \in L^1(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Hint: Use the monotone convergence theorem.

Proof. ■

Problem 2. (a) Prove the following generalization of *Chebyshev's inequality*: Let $0 < p < \infty$ and $E \subset \mathbb{R}^n$ be measurable. assume that $|f|^p \in L^1(E)$. Then

$$|\{\, x \in E : f(\mathbf{x}) > \alpha \,\}| \leq \frac{1}{\alpha^p} \int_{\{\, f > \alpha \,\}} f^p,$$

for $\alpha > 0$.

(b) Let p, E, and f be as in part (a). In addition, assume that $\{f_k\}$ is a sequence such that $\int_E |f_k - f|^p \to 0$ as $k \to \infty$. Show that $f_k \to f$ in measure on E.

Recall that $f_k \to f$ in measure on E if and only if for every $\varepsilon > 0$

$$\lim_{k\to\infty} |\{\, \mathbf{x} \in E : |f_k(\mathbf{x}) - f(\mathbf{x})| > \varepsilon \,\}| = 0.$$

Proof. ■

Problem 3. Let $f \in L^1(\mathbb{R})$, and define

$$F(\xi) := \int_{\mathbb{R}} f(x) \cos(2\pi x \xi)\, \mathrm{d}x.$$

Prove that F is continuous and bounded on $\mathbb{R}$.

Proof. ■

Problem 4. Use repeated integration techniques to prove that

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2}\, \mathrm{d}\mathbf{x} = \pi^{n/2}.$$

Hint: Start from the case $n = 1$ by using the polar coordinates in

$$\left[\int_{\mathbb{R}} e^{-x^2}\, \mathrm{d}x\right]^2 = \left[\int_{\mathbb{R}} e^{-x^2}\, \mathrm{d}x\right] \left[\int_{\mathbb{R}} e^{-x^2}\, \mathrm{d}y\right]$$

Proof. ■

Problem 5.

Proof. ■

1.2.5 Exam 2

Problem 1. Assume that $f \in L(\mathbb{R}^n)$. Show that for every $\varepsilon > 0$ there exists a ball B, centered at the origin, such that

$$\int_{\mathbb{R}^n \setminus B} |f| < \varepsilon.$$

Proof. ■

Problem 2. Let $f \in L(E)$, and let $\{E_j\}$ be a countable collection of pairwise disjoint measurable subsets of E, such that $E = \bigcup_{j=1}^{\infty} E_j$. Prove that

$$\int_E f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

Proof. ■

Problem 3. Let $\{f_k\}$ be a family in $L(E)$ satisfying the following property: For any $\varepsilon > 0$ there exits $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |f_k| < \varepsilon$$

for all $k \in \mathbf{N}$. Assume $|E| < \infty$, and $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. $x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(*Hint:* Use Egorov's theorem.)

Proof. ■

Problem 4. Let $I = [0, 1]$, $f \in L(I)$, and define $g(x) = \int_x^1 t^{-1} f(t)\, \mathrm{d}t$ for $x \in I$. Prove that $g \in L(I)$ and

$$\int_I g = \int_I f.$$

Proof. ■

1.2.6 Final Exam Practice Problems

Problem 1. Suppose $f \in L^1(\mathbb{R}^n)$ and that x is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) := \frac{1}{|r|^n} \int_{B(0,r)} |f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{y}.$$

Show that:

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$;
(b) there exists a constant $M > 0$ such that $A(r) \le M$ for all $r > 0$.

Proof. (a) Without loss of generality, we may assume $r < s$. Then, we want to show that as $r \to s$, the quantity

$$|A(s) - A(r)| \longrightarrow 0.$$

Set $F(\mathbf{y}) := |f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x})|$ and consider said quantity

$$\begin{aligned}
|A(s) - A(r)| &= \left| \frac{1}{|s|^n} \int_{B_s} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} - \frac{1}{|r|^n} \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} + \frac{1}{|s|^n} \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} - \frac{1}{|r|^n} \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right| \\
&= \left| \frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} + \left(\frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right| \\
&\le \underbrace{\frac{1}{|s|^n} \int_{B_s \setminus B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y}}_{I_1} + \underbrace{\left(\frac{1}{|s|^n} - \frac{1}{|r|^n} \right) \int_{B_r} F(\mathbf{y}) \, \mathrm{d}\mathbf{y}}_{I_2}.
\end{aligned}$$

Hence, we must show that the quantities $I_1, I_2 \to 0$ as $r \to s$.

To see that $A(r) \to 0$ as $r \to 0$, note that x is a point of the Lebesgue set of f and that

$$0 = \lim_{B_r \searrow \mathbf{x}} \frac{1}{|B_1||r|^n} \int_{B_r} |f(\mathbf{y}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{y} = \frac{1}{|B_1|} \lim_{B_r \searrow \mathbf{x}} \frac{1}{|r|^n} \int_{B_r} |f(\mathbf{t}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{t} = \lim_{r \to 0} A(r).$$

by making the change of variables $\mathbf{t} = \mathbf{x} - \mathbf{y}$.

(b) ■

Problem 2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \le n < \infty$. Assume $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0$$

if and only if

$$\|f_k\|_p \longrightarrow \|f\|_p$$

as $k \to \infty$.

Proof. ■

Problem 3. Let $1 < p < \infty$, $f \in L^p(E)$, $g \in L^{p'}(E)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ and $p = \infty$?

Proof. ■

Problem 4. Let $f \in L(\mathbb{R})$, and let $F(t) := \int_{\mathbb{R}} f(x)\cos(tx)dx$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemann–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Proof. ■

Problem 5. Let f be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular. Show that

$$\int_a^b \varphi \, \mathrm{d}f = \int_a^b \varphi f' \, dx + \int_a^b \varphi \, \mathrm{d}h$$

for all functions φ continuous on $[a, b]$.

Proof. ■

1.2.7 Final Exam 2010

Problem 1. Suppose that $f \in L^1(\mathbb{R}^n)$, and that $\mathbf{x}$ is a point in the Lebesgue set of f. For $r > 0$, let

$$A(r) := \frac{1}{r^n} \int_{B_r} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| \, \mathrm{d}\mathbf{y},$$

where $B_r := B(\mathbf{0}, r)$.

Show that

(a) $A(r)$ is a continuous function of r, and $A(r) \to 0$ as $r \to 0$.
(b) There exists a constant $M > 0$ such that $A(r) \le M$ for all $r > 0$.

Proof. (a)

(b) ■

Problem 2. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \le p < \infty$. assume that $\{f_k\}$ is a sequence in $L^p(E)$ converging pointwise a.e. on E to a function $f \in L^p(E)$. Prove that

$$\|f_k - f\|_p \longrightarrow 0 \iff \|f_k\|_p \longrightarrow \|f\|_p$$

Hint: To prove one of the implications, you can use the following fact without proving it:

$$\left|\frac{a-b}{2}\right| \le \frac{|a|^p + |b|^p}{2}$$

for all $a, b \in \mathbb{R}$.

Proof. ■

Problem 3. Let $0 < p < q < r \le \infty$, $E \subset \mathbb{R}^n$ be a measurable set. Show that each $f \in L^q(E)$ is the sum of a function $g \in L^p(E)$ and a function $h \in L^r(E)$.

Proof. ■

Problem 4. Prove that $f\colon [a, b] \to \mathbb{R}$ is Lipschitz continuous if and only if f is absolutely continuous and there exists a constant $M > 0$ such that $|f'| < M$ a.e. on $[a, b]$.

Proof. ■

Problem 5. Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$.

(a) Prove that $f * g \in C(\mathbb{R}^n)$.
(b) Does this conclusion continue to be valid when $p = 1$ or $p = \infty$?.

Proof. ■

1.2.8 Final Exam

Chapter 2

MA 544 Past Quals

2.1 Danielli: Winter 2012

Problem 1. Let $f(x, y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $\partial f(x, y)/\partial x$ is a bounded function of (x, y). Prove that $\partial f(x, y)/\partial x$ is a measurable function of y for each x and

$$\frac{d}{dx} \int_0^1 f(x, y)\, \mathrm{d}y = \int_0^1 \frac{\partial f(x, y)}{\partial x}\, \mathrm{d}y.$$

Proof. ∎

Problem 2. Let f be a function of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If $f = g + h$, with g absolutely continuous and h singular, show that

$$\int_a^b \varphi\, \mathrm{d}f = \int_a^b \varphi f'\, \mathrm{d}x + \int_a^b \varphi\, \mathrm{d}h.$$

Hint: A function h is said to be singular if $h' = 0$.

Proof. ∎

Problem 3. Let $E \subset \mathbb{R}$ be a measurable set, and let K be a measurable function on $E \times E$. Assume that there exists a positive constant C such that

$$\int_E K(x, y)\, \mathrm{d}x \leq C \tag{1}$$

for a.e. $y \in E$, and

$$\int_E K(x, y)\, \mathrm{d}y \leq C \tag{2}$$

for a.e. $x \in E$.

Let $1 < p < \infty$, $f \in L^p(E)$, and define

$$T_f(x) := \int_E K(x, y) f(y)\, \mathrm{d}y.$$

(a) Prove that $T_f \in L^p(E)$ and
$$\|T_f\|_p \leq C\|f\|_p. \tag{3}$$
(b) Is (3) still valid if $p = 1$ or ∞? If so, are assumptions (1) and (2) needed?

Proof. ■

Problem 4. Let f be a nonnegative measurable function on $[0, 1]$ satisfying
$$|\{ x \in [0, 1] : f(x) > \alpha \}| < \frac{1}{1 + \alpha^2} \tag{4}$$
for $\alpha > 0$.

(a) Determine values of $p \in [1, \infty)$ for which $f \in L^p[0, 1]$.
(b) If p_0 is the minimum value of p for which p may fail to be in L^p, give an example of a function which satisfies (4), but which is not in $L^{p_0}[0, 1]$.

Proof. ■

2.2 Danielli: Summer 2011

Problem 1. Let $f \in L^1(\mathbb{R})$, and let $F(t) := \int_{\mathbb{R}} f(x)\cos(tx)\,\mathrm{d}x$.

(a) Prove that $F(t)$ is continuous for $t \in \mathbb{R}$.
(b) Prove the following *Riemman–Lebesgue lemma*:

$$\lim_{t\to\infty} F(t) = 0.$$

Hint: Start by proving the statement for $f = \chi_{[a,b]}$.

Proof. ■

Problem 2. (a) Suppose that $f_k, f \in L^2(E)$, with E a measurable set, and that

$$\int_E f_k g \longrightarrow \int_E fg \tag{1}$$

as $k \to \infty$ for all $g \in L^2(E)$. If, in addition, $\|f_k\|_2 \to \|f\|_2$ show that f_k converges to f in L^2, i.e., that

$$\int_E |f - f_k|^2 \longrightarrow 0$$

as $k \to \infty$.

(b) Provide an example of a sequence f_k in L^2 and a function f in L^2 satisfying (1), but such that f_k does *not* converge to f in L^2.

Proof. ■

Problem 3. A bounded function f is said to be of bounded variation on $\mathbb{R}$ if it is of bounded variation on any finite subinterval $[a,b]$, and moreover $A := \sup_{a,b} V[a,b;f] < \infty$. Here, $V[a,b;f]$ denotes the total variation of f over the interval $[a,b]$. Show that:

(a) $\displaystyle\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x \le A|h|$ for all $h \in \mathbb{R}$.

Hint: For $h > 0$, write

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} \int_{nh}^{(n+1)h} |f(x+h) - f(x)|\,\mathrm{d}x.$$

(b) $\displaystyle\left|\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x\right| \le A$, where φ is any function of class C^1, of bounded variation, compactly supported, with $\sup_{x\in\mathbb{R}} |\varphi(x)| \le 1$.

Proof. ■

Problem 4. (a) Prove the *generalized Hölder's inequality*: Assume $1 \le p \le \infty$, $j = 1, \dots, n$, with $\sum_{j=1}^{\infty} 1/p_j = 1/r \le 1$. If E is a measurable set and $f_j \in L^{p_j}(E)$ for $j = 1, \dots, n$, then $\prod_{j=1}^{n} f_j \in L^r(E)$ and

$$\|f_1 \cdots f_n\|_r \le \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

(b) Use part (a) to show that that if $1 \leq p, q, r \leq \infty$, with $1/p + 1/q = 1/r + 1$, $f \in L^p(\mathbb{R})$, and $g \in L^p(\mathbb{R})$, then

$$|(f * g)(x)| \leq \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \int |f(y)|^p |g(x-y)|^q \, \mathrm{d}y.$$

(Recall that $(f * g)(x) := \int f(y) g(x-y) \, \mathrm{d}y$.)

(c) Prove *Young's convolution theorem*: Assume that p, q, r, f, and g are as in part (b). Then $f * g \in L^r(\mathbb{R})$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof. ∎

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