MA553: Qual Preparation

Carlos Salinas

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1 Ulrich

1.1 Ulrich: Winter 2002

Problem 1. Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G of finite index with $N \subset H$.

Solution. ► Let $n = [G : H]$ and $X = \{ H, g_1H, \dots, g_{n-1}H \}$ the set of left-cosets of H in G with representatives $g_0 = e, g_1, \dots, g_{n-1}$. Let G act on X by left multiplication, i.e., $g \mapsto gg_iH$; this is indeed an action since $e(g_iH) = eg_iH = g_iH$ for all $g_iH \in X$ and for $k_1, k_2 \in G$ $k_2(k_1g_iH) = k_2k_1g_iH = (k_2k_1)g_iH$. By Cayley's theorem, this induces a homomorphism $\varphi\colon G \to S_n$. Note that the action is not necessarily faithful. However, by the first isomorphism theorem, the kernel of φ, $N = \operatorname{Ker}\varphi$, is a normal subgroup of G with index $[G : N] \leq |S_n| = n!$ and $N \subset H$ since $g \in N$ if and only if $gg_iH = g_iH$ which, in particular, implies that $gH = H$. Thus, $N \subset H$ and $[G : N] < \infty$. ◄

Problem 2. Show that every group of order 992 (= 32 · 31) is solvable.

Solution. ► Suppose G is a group with order $|G| = 992 = 2^5 \cdot 3$. By Sylow's theorem, there ◄

Problem 3. Let G be a group of order 56 with a normal 2-Sylow subgroup Q, and let P be a 7-Sylow subgroup of G. Show that either $G \simeq P \times Q$ or $Q \simeq \mathbb{Z}/(2) \times \mathbb{Z}/(2) \times \mathbb{Z}/(2)$.

[*Hint*: P acts on $Q \setminus \{e\}$ via conjugation. Show that this action is either trivial or transitive.]

Solution. ► ◄

Problem 4. Let R be a commutative ring and $\operatorname{Rad}(R)$ the intersection of all maximal ideals of R.

(a) Let $a \in R$. Show that $a \in \operatorname{Rad}(R)$ if and only if $1 + ab$ is a unit for every $b \in R$.
(b) Let R be a domain and $R[X]$ the polynomial ring over R. Deduce that $\operatorname{Rad}(R[X]) = 0$.

Solution. ► ◄

Problem 5. Let R be a unique factorization domain and P a prime ideal of $R[X]$ with $P \cap R = 0$.

(a) Let n be the smallest possible degree of a nonzero polynomial in P. Show that P contains a primitive polynomial f of degree n.
(b) Show that P is the principal ideal generated by f.

Solution. ► ◄

Problem 6. Let k be a field of characteristic zero. assume that every polynomial in $k[X]$ of odd degree and every polynomial in $k[X]$ of degree two has a root in k. Show that k is algebraically closed.

Solution. ► ◄

Problem 7. Let $k \subset K$ be a finite Galois extension with Galois group $\mathrm{Gal}(K/k)$, let L be a field with $k \subset L \subset K$, and set $H = \{\, \sigma \in \mathrm{Gal}(K/k) : \sigma(L) = L \,\}$.

(a) Show that H is the normalizer of $\mathrm{Gal}(K/L)$ in $\mathrm{Gal}(K/k)$.
(b) Describe the group $H/\mathrm{Gal}(K/L)$ as an automorphism group.

Solution. ► ◄