MA 519: Homework, Midterms and Practice Problems Solutions

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1 Homework Solutions

1.1 Homework 1

Exercise 1.1 (Handout 1, # 5)**.** A closet contains five pairs of shoes. If four shoes are selected at random, what is the probability that there is at least one complete pair among the four?

Solution. ■

Exercise 1.2 (Handout 1, # 7)**.** A gene consists of 10 subunits, each of which is normal or mutant. For a particular cell, there are 3 mutant and 7 normal subunits. Before the cell divides into 2 daughter cells, the gene duplicates. The corresponding gene of cell 1 consists of 10 subunits chosen from the 6 mutant and 14 normal units. Cell 2 gets the rest. What is the probability that one of the cells consists of all normal subunits.

Solution. ■

Exercise 1.3 (Handout 1, # 9)**.** From a sample of size n, r elements are sampled at random. Find the probability that none of the N prespecified elements are included in the sample, if sampling is

(a) with replacement;

(b) without replacement.

Compute it for $r = N = 10$, $n = 100$.

Solution. ■

Exercise 1.4 (Handout 1, # 11)**.** Let E, F, and G be three events. Find expressions for the following events:

(a) only E occurs;
(b) both E and G occur, but not F;
(c) all three occur;
(d) at least one of the events occurs;
(e) at most two of them occur.

Solution. ■

Exercise 1.5 (Handout 1, # 12)**.** Which is more likely:

(a) Obtaining at least one six in six rolls of a fair die, or
(b) Obtaining at least one double six in six rolls of a pair of fair dice.

Solution. ∎

Exercise 1.6 (Handout 1, # 13)**.** There are n people are lined up at random for a photograph. What is the probability that a specified set of r people happen to be next to each other?

Solution. ∎

Exercise 1.7 (Handout 1, # 16)**.** Consider a particular player, say North, in a Bridge game. Let X be the number of aces in his hand. Find the distribution of X.

Solution. ∎

Exercise 1.8 (Handout 1, # 20)**.** If 100 balls are distributed completely at random into 100 cells, find the expected value of the number of empty cells.

Replace 100 by n and derive the general expression. Now approximate it as n tends to ∞.

Solution. ∎

1.2 Homework 2

Exercise 1.9 (Handout 2, # 5). Four men throw their watches into the sea, and the sea brings each man one watch back at random. What is the probability that at least one man gets his own watch back?

Solution. ∎

Exercise 1.10 (Handout 2, # 7). Calculate the probability that in Bridge, the hand of at least one player is void in a particular suit.

Solution. ∎

Exercise 1.11 (Handout 2, # 12). If n balls are placed at random into n cells, find the probability that exactly one cell remains empty.

Solution. ∎

Exercise 1.12 (Handout 2, # 13). *Spread of rumors.* In a town of $n+1$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, etc. At each step the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor told r times without:

(a) returning to the originator,
(b) being repeated to any person.

Do the same problem when at each step the rumor told by one person to a gathering of N randomly chosen people. (The first question is the special case $N = 1$).

Solution. ∎

Exercise 1.13 (Handout 2, # 14). *A family problem.* In a certain family four girls take turns at washing dishes. Out of a total of four breakages, three were caused by the youngest girl, and she was thereafter called clumsy. Was she justified in attributing the frequency of breakages to chance? Discuss the connection with random placement of balls.

Solution. ∎

Exercise 1.14 (Handout 2, # 15). A car is parked among N cars in a row, not at either end. On his return the owner finds exactly r of the N places still occupied. What is the probability that both neighboring places are empty?

Solution. ∎

Exercise 1.15 (Handout 2, # 16)**.** Find the probability that in a random arrangement of 52 bridge card no two aces are adjacent.

Solution. ■

Exercise 1.16 (Handout 2, # 17)**.** Suppose $P(A) = 3/4$, and $P(B) = 1/3$.
Prove that $P(A \cap B) \geq 1/12$. Can it be equal to $1/12$?

Solution. ■

Exercise 1.17 (Handout 2, # 18)**.** Suppose you have infinitely many events $A_1, A_2, \ldots$, and each one is sure to occur, i.e., $P(A_i) = 1$ for any i.
Prove that $P\big(\bigcap_{i=1}^{n} A_i\big) = 1$.

Solution. ■

Exercise 1.18 (Handout 2, # 19)**.** There are n blue, n green, n red, and n white balls in an urn. Four balls are drawn from the urn with replacement. Find the probability that there are balls of at least three different colors among the four drawn.

Solution. ■

2 Midterms, Exams, and Qualifying Exams

2.1 Qualifying Exams, August '99

Exercise 2.1. The number of fish that Anirban catches on any given day has a Poisson distribution with mean 20. Due to the legendary softness of his heart, he sets free, on average, 3 out of the 4 fish he catches. Find the mean and the variance of the number of fish Anirban takes home on a given day.

Solution. ■

Exercise 2.2. A fair die is rolled and at the same time a fair coin is tossed. This is done repeatedly. Find the probability that head occurs (strictly) before six occurs.

Solution. ■

Exercise 2.3. X, Y are independent random variables with a common density $f(x) = e^{-|x|}/2$, $x \in (-\infty, \infty)$. Find the density function of $X + Y$.

Solution. ■

Exercise 2.4. Let X_n denote the distance between two points chosen independently at random from the unit cube in $\mathbb{R}^n$. Evaluate

$$\lim_{n\to\infty} \frac{E(X_n)}{\sqrt{n}}.$$

Solution. ■

Exercise 2.5. Let X be distributed as Uniform$[0, 1]$. What is the probability that the digit 5 does not occur in the decimal expansion of X?

Solution. ■

2.2 Qualifying Exam, January '06

Exercise 2.6. The birthdays of 5 people are known to fall in exactly 3 calendar months. What is the probability that exactly two of the 5 were born in January?

Solution. ■

Exercise 2.7. Coupons are drawn, independently, with replacement, one at a time, from a set of 10 coupons. Find, explicitly, the expected number of draws

(a) until the first draw coupon is drawn again;
(b) until a duplicate occurs.

Solution. ■

Exercise 2.8. Let N be a positive integer. Choose an integer at random from $\{1, \ldots, N\}$. Let E be the event that your chosen random number is divisible by 3, and divisible by at least one of 4 and 6, but not divisible by 5. Find, explicitly, $\lim_{N\to\infty} P(E)$.

Solution. ■

Exercise 2.9. Anirban is driving his Dodge on a highway with 4 lanes each way. He is wired to change lanes every minute on the minute. He changes with equal probability to either adjacent lane if there are two adjacent lanes, and the successive changes are mutually independent. Find, explicitly, the probability that after 4 minutes, Anirban is back to the lane he started from

(a) if he started at an outside lane;
(b) if he started at an inside lane.

Solution. ■

Exercise 2.10. Burgess is going to Moose Pass, Alaska. He is driving his Dodge. He puts his car on cruise control at 70 mph. Gas stations are located every 30 miles, starting from his home. His car runs out of gas at a time distributed as an exponential with mean 4 hours. When that happens, he gets out, takes his bik out of his trunk, and bikes to the next gas station say M, at 10 mph. Let the time elapsed between when Burgess starts his trip and when he arrives at the gas station M be T. Find $E(T)$.

Solution. ■

Exercise 2.11. A fair coin is tossed n times. Suppose X heads are obtained. Given $X = x$, let Y be generated according to the Poisson distribution with mean x. Find the unconditional variance of Y, and then find the limit of the probability $P\big(|Y - n/2| > n^{3/4}\big)$, as $n \to \infty$.

Solution. ■

Exercise 2.12. Anirban plays a game repeatedly. On each play he wins an amount uniformly distributed in $(0, 1)$ dollars, and then he tips the lady in charge of the game the square of the amount he has won. Then he plays again, tips again, and so on. Approximately calculate the probability that if he plays and tips six hundred times, his total winnings minus his total tips will exceed \$105.

Solution. ■

Exercise 2.13. Anirban's dog got mad at him and broke his walking cane, first uniformly into two peices, and then the long piece again uniformly into two pieces. Find the probability that Anirban can make a triangle out of the three pieces of his cane.

Solution. ■

Exercise 2.14. Suppose X, Y, Z are identically independently distributed Exp(1) random variables. Find the joint density of (X, XY, XYZ).

Solution. ■

Exercise 2.15. Let X be the number of Kings and Y the number of Hearts in a Bridge hand. Find the correlation between X and Y.

Solution. ■

2.3 Qualifying Exam, August '14

Exercise 2.16.

(a) 3 balls are distributed one by one and at random in 3 boxes. What is the probability that exactly one box remains empty?
(b) n balls are distributed one by one and at random in n boxes. Find the probability that exactly one box remains empty.
(c) n balls are distributed one by one and at random in n boxes Find the probability that exactly two boxes remain empty.

Solution. ■

Exercise 2.17. n players each roll a fair die. For any pair of players i, j, $i < j$, who roll the same number, the group is awarded one point.

(a) Find the mean of the total points of the group.
(b) Find the variance of the total points of the group.

Solution. ■

Exercise 2.18. Suppose $X_1, X_2, \ldots$, is an infinite sequence of independently identically distributed Uniform$[0, 1]$ random variables. Find the limit

$$\lim_{n\to\infty} P\left[\frac{\left(\prod_{i=1}^{n} X_i\right)^{1/n}}{\left(\sum_{i=1}^{n} X_i\right)/n} > \frac{3}{4}\right].$$

Solution. ■

Exercise 2.19. Suppose X is an exponential random variable with density $e^{-x/\sigma_1}/\sigma_1$ and Y is another exponential random variable with density $e^{-y/\sigma_2}/\sigma_2$, and that X, Y are independent.

(a) Find the CDF of $X/(X + Y)$.
(b) In the case $\sigma_1 = 2$, $\sigma_2 = 1$, find the mean of $X/(X + Y)$.

Solution. ■

Exercise 2.20. Ten independently picked Uniform$[0, 100]$ numbers are each rounded to the nearest integer. Use the central limit theorem to approximate the probability that the sum of the ten rounded numbers equals the rounded value of the sum of the ten original numbers.

Solution. ■

Exercise 2.21. Suppose for some given $m \geq 2$, we choose m independently identically distributed Uniform$[0,1]$ random variables $X_1, \ldots, X_m$. Let $X_{\min}$ denote their minimum and $X_{\max}$ denote their maximum. Now continue sampling $X_{m+1}, \ldots,$ from the Uniform$[0,1]$ density. Let N be the first index k such that X_{m+k} falls outside the interval $[X_{\min}, X_{\max}]$.

(a) Find a formula for $P(N > n)$ for a general n.
(b) Hence, explicitly find $E(N)$.

Solution. ■

Exercise 2.22. A $G_{n,p}$ graph on n vartices is obtained by adding each of the $\binom{n}{2}$ possible edges into the graph mutually independently with probability p. If vertex subsets A, B both have k vertices, and each vertex A shares an edge with each vertex in B, but there are no edges among the vertices within A or within B, then A, B generate a complete bipartate subgraph of order k denoted as $K_{k,k}$.

(a) For a given n and p, find an expression for the expected number of complete bipartate subgraphs $K_{3,3}$ of order $k = 3$ in a $G_5 n, p$ graph.
(b) Let p_n denote the value of p for which the expected value in part (a) equals one. Identify constants α, β such that $\lim_{n\to\infty} n^\alpha p_n = \beta$.

Solution. ■