

# MA 523: Homework 9

Carlos Salinas

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## PROBLEM 9.1

- (a) Show that for  $n = 3$  the general solution to the wave equation  $u_{tt} - \Delta u = 0$  with spherical symmetry about the origin has the form

$$u = \frac{1}{r}F(r+t) + \frac{1}{r}G(r-t), \quad r = |x|,$$

with suitable  $F$  and  $G$ .

- (b) Show that the solution with initial data of the form

$$u(r, 0) = 0, \quad u_t(r, 0) = h(r)$$

( $h$  is an even function of  $r$ ) is given by

$$u = \frac{1}{2r} \int_{r-t}^{r+t} \rho h(\rho) d\rho.$$

*SOLUTION.* For part (a): From Kirchhoff's formula, we have

$$u(x, t) = \oint_{\partial B(x, t)} [th(y) + g(y) + D^\alpha g(y) \cdot (y - x)] dS(y)$$

solves the initial-value problem for the wave equation.

For part (b):

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## PROBLEM 9.2

Show that the solution  $w(x_1, t)$  of the initial-value problem for the *Klein-Gordon equation*

$$\begin{cases} w_{tt} = w_{x_1 x_1} - \lambda^2 w, \\ w(x_1, 0) = 0, \end{cases} \quad w_t(x_1, 0) = h(x_1) \quad (9.1)$$

is given by

$$w(x_1, t) = \frac{1}{2} \int_{x_1-t}^{x_1+t} J_0(\lambda s) h(y_1) dy_1.$$

Here  $s^2 = t^2 - (x_1 - y_1)^2$ , while  $J_0$  denotes the Bessel function defined by

$$J_0(z) := \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(z \sin \theta) d\theta.$$

(*Hint:* Descend to (9.1) from the two-dimensional wave equation satisfied by

$$u(x_1, x_2, t) = \cos(\lambda x_2) w(x_1, t).)$$

SOLUTION. ■

## PROBLEM 9.3

Let  $u$  solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u = g, \quad u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

where  $g$  and  $h$  are smooth and have compact support. Show there exists a constant  $C$  such that

$$|u(x, t)| \leq Ct^{-1} \quad (x \in \mathbb{R}^3, t > 0).$$

*SOLUTION.*

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