MA52300 FALL 2016

Homework Assignment 3 – *Solutions*

1. Consider the initial value problem

$$u_t = \sin u_x; \quad u(x,0) = \frac{\pi}{4}\, x.$$

Verify that the assumptions of the Cauchy-Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

Solution. The equation is a first order fully nonlinear equation $F(Du) = 0$, where $F(p) = \sin p_1 - p_2$. Now verify that the line $\Gamma = \{t = 0\}$ is noncharacteristic near the origin. The noncharactericity condition reads

$$F_p \cdot \nu \neq 0,$$

where ν is a normal vector to Γ at the origin. In our case we have $\nu = (0,1)$, $F_p = (\cos p_1, -1)$, thus

$$F_p \cdot \nu = (\cos p_1, -1) \cdot (0,1) = -1 \neq 0.$$

Moreover, the curve Γ is analytic, as well as the Cauchy data on Γ. Thus, the conditions of the Cauchy-Kovalevskaya theorem are satisfied and we can find an analytic solution

$$u(x,t) = \sum_{n,m=1}^{\infty} \frac{\partial_x^n \partial_t^m u(0,0)}{n!m!} x^n t^m$$

near the origin. Next we compute the derivatives $\partial_x^n \partial_t^m u(0,0)$.

Differentiating the initial data along Γ (x-axis), we will have

$$u_x(x,0) = \frac{\pi}{4}, \quad \partial_x^n u(x,0) = 0 \quad (n \geq 2).$$

Next, using the equation, we will have

$$u_t(x,0) = \frac{1}{\sqrt{2}}, \quad \partial_x^n \partial_t u(x,0) = 0 \quad (n \geq 1)$$

Differentiating the equation for u with respect to t, we will have

$$u_{tt}(x,0) = \cos(u_x(x,0)) u_{xt}(x,0) = 0$$

and consequently also

$$\partial_x^n \partial_{tt} u(x,0) = 0 \quad (n \geq 1).$$

Then by induction in m we can show that

$$\partial_x^n \partial_t^m u(x,0) = 0 \quad (n \geq 0, m \geq 2).$$

Summarizing, we obtain that the only nonzero derivatives of u at the origin are $u_x = \pi/4$ and $u_t = 1/\sqrt{2}$. Thus,

$$u(x,t) = \frac{\pi}{4}\,x + \frac{1}{\sqrt{2}}\,t. \qquad \square$$

2. Consider the Cauchy problem for $u(x,y)$

$$\begin{aligned} &u_y = a(x,y,u)u_x + b(x,y,u) \\ &u(x,0) = 0 \end{aligned}$$

Let a and b be analytic functions of their arguments. Assume that $D^\alpha a(0,0,0) \geq 0$ and $D^\alpha b(0,0,0) \geq 0$ for all α. (Remember by definition, if $\alpha = 0$ then $D^\alpha f = f$.)

(a) Show that $D^\beta u(0,0) \geq 0$ for all $|\beta| \leq 2$.
(b) Prove that $D^\beta u(0,0) \geq 0$ for all β. (*Hint.* Argue as in the proof of the Cauchy-Kovalevskaya theorem; i.e., use induction in β_2)

Solution. (a) All partial derivatives below are taken at $(0,0,0)$ for a and b and at $(0,0)$ for u. We obtain consecutively

$$\begin{aligned} &u = u_x = u_{xx} = 0 \quad \text{(taking derivative along } x\text{-axis)} \\ &u_y = au_x + b = b \geq 0 \\ &u_{yx} = (a_x + a_z u_x)u_x + au_{xx} + b_x + b_z u_x \geq 0 \\ &u_{yy} = (a_y + a_z u_y)u_x + au_{xy} + b_y + b_z u_y \geq 0 \end{aligned}$$

(b) Denote by $S(m)$ the statement that $D^\beta u(0,0) \geq 0$ for any multiindex $\beta = (\beta_1, \beta_2)$ with $0 \leq \beta_2 \leq m$.

Step 1. $S(0)$ is true.

Indeed, taking the derivative $D^{(\beta_1,0)}u = \partial_x^{\beta_1} u$ of the initial data along the x-axis, we obtain that actually $D^{(\beta_1,0)}u = 0$ identically on the x-axis. Thus, $S(0)$ follows.

Step 2. $S(m-1) \Longrightarrow S(m)$ for any $m \geq 1$.

Assume $S(m-1)$ holds and let $\beta = (\beta_1, m)$. Then from the equation for u we will have

$$\begin{aligned} D^\beta u &= D^{(\beta_1,m-1)}u_y = D^{(\beta_1,m-1)}[a(x,y,u)u_x + b(x,y,u)] \\ &= P_\beta(D^\gamma u, D^\delta a, D^\epsilon b), \end{aligned}$$

where P_β is a polynomial with *nonnegative* coefficients, depending only on certain $D^\gamma u$ with $|\gamma| \leq |\beta|$ and $\gamma_2 \leq m-1$ (important!) and also on $D^\delta a$, $D^\epsilon b$ with $|\delta|, |\epsilon| \leq |\beta|$. By the induction assumption, all partial derivatives involved are nonnegative at the origin and since P_β has nonnegative coefficients, we will obtain that $D^\beta u(0,0) \geq 0$. Thus, $S(m)$ holds.

This completes the proof of (b). $\square$

3 (Kovalevskaya's example)**.** Show that the line $\{t = 0\}$ is characteristic for the heat equation $u_t = u_{xx}$. Show there does not exist an analytic solution u of the heat equation in $\mathbb{R} \times \mathbb{R}$, with $u = \frac{1}{1+x^2}$ on $\{t = 0\}$. (*Hint:* Assume there is an analytic solution, compute its coefficients, and show that the resulting power series diverges in any neighborhood of $(0, 0)$.)

Solution. Given a curve Γ and a normal vector $\nu = (\nu_1, \nu_2)$ at $x_0 \in \Gamma$, the non-charactericity condition with respect to the heat equation reads as $\nu_1^2 \neq 0$. In our case $\Gamma = \{t = 0\}$ and $\nu = (0, 1)$ and therefore $\nu_1^2 = 0$. This precisely means that Γ is characteristic at every its point.

Suppose now we have a real analytic solution 0 near the origin. Differentiating the equation $u_t = u_{xx}$ in t we obtain

$$u_{tt} = u_{xxt} = (u_t)_{xx} = (u_{xx})_{xx} = u_{xxxx}$$

near the origin, and by induction

$$\partial_t^n u = \partial_x^{2n} u, \quad \text{for } n = 0, 1, 2, \dots.$$

Now, since $u(x, 0) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$, we will have that

$$\partial_x^{2n} u(0, 0) = (-1)^n (2n)!, \quad \text{for } n = 0, 1, 2, \dots,$$

implying that

$$\partial_t^n u(0, 0) = (-1)^n (2n)!.$$

Thus, if u were real analytic near the origin, then we would have that

$$u(0, t) = \sum_{n=0}^{\infty} \frac{\partial_t^n u(0, 0)}{n!} t^n = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!} t^n = \sum_{n=0}^{\infty} a_n t^n$$

is convergent for $|t| < \epsilon$ for small $\epsilon > 0$. However, it is easy to see that the radius of convergence R of this power series is 0, by the application of the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2n+2)!}{(n+1)!} \frac{n!}{(2n)!} = \frac{(2n+2)(2n+1)}{n+1} = 2(2n+1) \to \infty = 1/R.$$

Hence, we arrive at a contradiction, which proves that there is no power series solution to this Cauchy problem near the origin. $\square$