MA 166: Quiz 7 Solutions

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You have **15 minutes** to complete this quiz. You may work in groups, but you are not allowed to use any other resources.

Problem 1. For **one** of the following integrals, determine whether it is convergent or divergent. If it is convergent, evaluate it

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

(b) $\displaystyle\int_1^5 \frac{dx}{(5-x)^2}$

(c) $\displaystyle\int_0^\infty xe^{-x^2}\,dx.$

Problem 2. Find the length of the curve

$$y = \ln\bigl(x^2 - 1\bigr), \qquad 2 \le x \le 5.$$

Solutions

Here are the solutions to the quiz.

Solution. For the following problems, let t be a dummy variable.

(a) Take the limit as $t \to 0$ of the indefinite integral

$$\begin{aligned}
I_1 &= \lim_{t\to 0} \int_t^1 \frac{dx}{\sqrt{x}} \\
&= \lim_{t\to 0} \int_t^1 x^{-1/2}\, dx \\
&= \lim_{t\to 0} \left[\frac{1}{2} x^{1/2}\right]_t^1 \\
&= \lim_{t\to 0} \frac{1}{2} - \frac{1}{2} t^{1/2} \\
&= \frac{1}{2}.
\end{aligned}$$

So the integral converges and its value is 1/2.

(b) Take the limit as $t \to 5$ of the indefinite integral

$$\begin{aligned}
I_2 &= \lim_{t\to 5} \int_t^1 \frac{dx}{(5-x)^2} \\
&= \lim_{t\to 5} \int_1^t (5-x)^{-2}\, dx
\end{aligned}$$

make the substitution $u = 5 - x$, $du = -dx$

$$
\begin{aligned}
&= \lim_{t\to 5} - \int_{5-t}^{4} u^{-2}\, du \\
&= \lim_{t\to 5} \int_{5-t}^{4} u^{-2}\, du \\
&= \lim_{t\to 5} - \left[-u^{-1}\right]_{5-t}^{4} \\
&= \lim_{t\to 5} \left[u^{-1}\right]_{5-t}^{4} \\
&= \lim_{t\to 5} \frac{1}{4} - \frac{1}{5-t} \\
&= \frac{1}{4} - \lim_{t\to 5} \frac{1}{5-t}
\end{aligned}
$$

So the integral does not exist because for any number you can think of N, we can pick a value t such that $(5-t)^{-1}$ is bigger than N, in fact let's see just when $(5-t)^{-1} = N$

$$
\begin{aligned}
\frac{1}{5-t} &= N \\
5 - t &= \frac{1}{N} \\
t &= 5 - \frac{1}{N}.
\end{aligned}
$$

Now let M be a number bigger than N and set $t = 5 - 1/M$ then

$$
\frac{1}{5-t} = \frac{1}{5-(5-1/M)} = \frac{1}{1/M} = M > N.
$$

This is what it means for an integral to not converge.

(c) Take the limit as $t \to 0$ of the indefinite integral

$$
I_3 = \lim_{t\to\infty} \int_0^t xe^{-x^2}\, dx
$$

use the substitution $u = x^2$, $du = 2x\ dx$

$$\begin{aligned}
&= \lim_{t^2\to\infty} \frac{1}{2} \int_0^{t^2} e^{-u}\ du \\
&= \frac{1}{2} \lim_{t^2\to\infty} \int_0^{t^2} e^{-u}\ du \\
&= \frac{1}{2} \lim_{t^2\to\infty} \left[-e^{-u}\right]_0^{t^2} \\
&= \frac{1}{2} \lim_{t^2\to\infty} \left[e^{-u}\right]_{t^2}^{0} \\
&= \frac{1}{2} \lim_{t^2\to\infty} \left(e^{0} - e^{t^2}\right) \\
&= \frac{1}{2} \lim_{t^2\to\infty} \left(1 - e^{-t^2}\right) \\
&= \frac{1}{2} - \frac{1}{2}\left(\lim_{t^2\to\infty} e^{-t^2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

So the integral converges and its value is 1/2. ☺

Solution. Remember the formula for the arc-length of a curve

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \tag{1}$$

So we need to find the derivative of $y = \ln(x^2 - 1)$

$$\frac{d}{dx}\big(\ln\big(x^2 - 1\big)\big) = \frac{2x}{x^2 - 1}.$$

Next we plug this into our equation and we have

$$
\begin{aligned}
L &= \int_2^5 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_2^5 \sqrt{1 + \left(\frac{2x}{x^2-1}\right)^2}\, dx \\
&= \int_2^5 \sqrt{1 + \frac{4x^2}{\left(x^2-1\right)^2}}\, dx \\
&= \int_2^5 \sqrt{\frac{\left(x^2-1\right)^2}{\left(x^2-1\right)^2} + \frac{4x^2}{\left(x^2-1\right)^2}}\, dx \\
&= \int_2^5 \sqrt{\frac{\left(x^2-1\right)^2 + 4x^2}{\left(x^2-1\right)^2}}\, dx \\
&= \int_2^5 \sqrt{\frac{x^4 - 2x^2 + 1 + 4x^2}{\left(x^2-1\right)^2}}\, dx \\
&= \int_2^5 \sqrt{\frac{x^4 + 2x^2 + 1}{\left(x^2-1\right)^2}}\, dx \\
&= \int_2^5 \sqrt{\frac{\left(x^2+1\right)^2}{\left(x^2-1\right)^2}}\, dx \\
&= \int_2^5 \frac{x^2+1}{x^2-1}\, dx \\
&= \int_2^5 \frac{x^2+1-1+1}{x^2-1}\, dx \\
&= \int_2^5 \frac{x^2-1+1+1}{x^2-1}\, dx \\
&= \int_2^5 \frac{(x^2-1)+2}{x^2-1}\, dx \\
&= \int_2^5 \frac{x^2-1}{x^2-1} + \frac{2}{x^2-1}\, dx \\
&= \underbrace{\int_2^5 1\, dx}_{I_1} + \underbrace{\int_2^5 \frac{2}{x^2-1}\, dx}_{I_2}.
\end{aligned}
$$

$I_1 = 3$, I_2 requires a bit more work. First note that $x^2 - 1 = (x-1)(x+1)$ so by the partial fractions decomposition we have

$$\begin{aligned} \frac{2}{(x-1)(x+1)} &= \frac{A}{x-1} + \frac{B}{x+1} \\ 2 &= A(x+1) + B(x-1) \\ 0x + 2 &= (A+B)x(A-B) \end{aligned}$$

so $A + B = 0$ and $A - B = 2$ hence $A - (-B) = 2A = 2$ gives us that $A = 1$ and $B = -1$. Now we can find I_2

$$\begin{aligned} I_2 &= \int_2^5 \frac{2}{x^2-1} \\ &= \int_2^5 \frac{1}{x-1} - \frac{1}{x+1}\, dx \\ &= [\ln|x-1| - \ln|x+1|]_2^5 \\ &= \left[\ln\left|\frac{x-1}{x+1}\right|\right]_2^5 \\ &= \ln\left|\frac{4}{6}\right| - \ln\left|\frac{1}{3}\right| \\ &= \ln 2. \end{aligned}$$

Hence

$$L = I_1 + I_2 = \boxed{3 + \ln 2.}$$

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