MA571 Homework 11

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Problem 11.1 (Munkres §53, Ex. 7(abcd))

Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]

(d) Show that $\pi_1(G, x_0)$ is Abelian.

Proof. For part (a) we need to show that the operation (i) $\otimes$ is associtive, (ii) $\Omega(G, x_0)$ is closed under $\otimes$, (iii) $\Omega(G, x_0)$ contains an identity element e and (iv) for every $f \in \Omega(G, x_0)$ there exists an element f^{-1} such that $f \otimes f^{-1} = f^{-1} \otimes f = e$. We shall proceed in order: (i) is easy since the operation $\otimes$ is associative so for any triple $f, g, h \in \Omega(G, x_0)$ we have

$$(f \otimes g) \otimes h = (f(s) \cdot g(s)) \otimes h = (f(s) \cdot g(s)) \cdot h(s)$$

which one clearly sees, by asociativity of $\cdot$, is the same as $f \otimes (g \otimes h)$. (ii) Let $f, g \in \Omega(G, x_0)$ then, since $\cdot$ is continuous, the map $f \otimes g \colon I \to G$ is continuous and

$$(f \otimes g)(0) = f(0) \cdot g(0) = x_0 \cdot x_0 = f(1) \cdot g(1) = (f \otimes g)(1).$$

Thus, $f \otimes g \in \Omega(G, x_0)$. Next, for (iii) consider the constant loop $e_{x_0}(s)$. This map is clearly the identity on $\Omega(G, x_0)$ for if $f \in \Omega(G, x_0)$ then $e_{x_0} \otimes f = e_{x_0}(s) \cdot f(s) = x_0 \cdot f(s) = f(s)$ for all s; similarly for $f \otimes e_{x_0}$. Lastly, (iv) consider the map $f^{-1}(s) \coloneqq (f(s))^{-1} \colon I \to G$. This map is continuous since taking the inverse in G is continuous and composition of continuous maps is continuous by Theorem 18.2(c). Lastly, note that $f^{-1}(0) = x_0^{-1} = x_0$ similarly for $f^{-1}(1)$. Thus, f^{-1} is a loop and

$$f^{-1} \otimes f = (f(s))^{-1} \cdot f(s) = x_0 = f(s) \cdot (f(s))^{-1} = f \otimes f^{-1}$$

so $\Omega(G, x_0)$ is closed under inverses. Thus, $\Omega(G, x_0)$ is a group.

(b) The map $\otimes \colon \Omega(G, x_0) \times \Omega(G, x_0) \to \Omega(G, x_0)$ clearly induces a group operation on $\Pi_1(X, x_0)$ given by $[f] \otimes [g] = [f \otimes g]$. All we need to check is that this operation is in fact well defined on the equivalence class of loops based at x_0, i.e., if $f_1 \simeq_p f_2$ with path homotopy H and $g_1 \simeq_p g_2$ with path homotopy K we want that $f_1 \otimes g_1 \simeq_p f_2 \otimes g_2$. But this is immediate via the homotopy $L(s, t) \coloneqq H(s, t) \cdot K(s, t) \colon G \times I \to G$. This map is continuous since it can be realized as the sequence of compositions

$$(s, t) \longmapsto (H(s, t), K(s, t)) \longmapsto H(s, t) \cdot H(s, t)$$

where the intermediate step in the composition, i.e., the map from $I \times I \to G \times G$, is continuous by Theorem 18.4 since H and K are continuous and lastly $L(s,0) = H(s,0) \cdot K(s,0) = f_1(s) \cdot g_1(s) = f_1 \otimes g_1$ and $L(s,1) = H(s,1) \cdot K(s,1) = f_2(s) \cdot g_2(s) = f_2 \otimes g_2$. Thus, $f_1 \otimes g_1 \simeq_p f_2 \otimes g_2$. It follows that $\otimes$ is a well-defined binary operation on $\pi_1(X, x_0)$.

(c) Following the hint, we shall compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$. Recall that

$$f * e_{x_0} = \begin{cases} f(2s) & \text{for } s \in [0,1/2] \\ e_{x_0}(2s-1) & \text{for } s \in [1/2,1] \end{cases} \qquad \text{and} \qquad e_{x_0} * g = \begin{cases} e_{x_0}(2s) & \text{for } s \in [0,1/2] \\ g(2s-1) & \text{for } s \in [1/2,1] \end{cases}.$$

Then

$$\begin{aligned} (f * e_{x_0}) \otimes (e_{x_0} * g) &= (f * e_{x_0})(s) \cdot (e_{x_0} * g)(s) \\ &= \begin{cases} f(2s) \cdot e_{x_0}(2s) & \text{for } s \in [0,1/2] \\ e_{x_0}(2s-1) \cdot g(2s-1) & \text{for } s \in [1/2,1] \end{cases} \\ &= \begin{cases} f(2s) & \text{for } s \in [0,1/2] \\ g(2s-1) & \text{for } s \in [1/2,1] \end{cases} \\ &= f * g. \end{aligned}$$

Since $f * e_{x_0} \simeq_p f$ and $e_{x_0} * g \simeq_p g$, we have at last that

$$[f \otimes g] = [(f * e_{x_0}) \otimes (e_{x_0} * g)] = [f * g].$$

(d) Lastly, we show that $\pi_1(X, x_0)$ must be Abelian. It suffices to show that given a class of loop $[f]$ at x_0, the conjugacy class of $[f]$ consists of the singleton $\{[f]\}$. First, note that if $[g] \in \pi_1(X, x_0)$ then $[g^{-1}] \otimes [g] = [e_{x_0}] = [\bar{g}] * [g]$ so $[g^{-1}] = [g^{-1}] \otimes ([g] * [\bar{g}]) = [\bar{g}]$. Thus, we have

$$\begin{aligned} [\bar{g}] * [f] * [g] &= \big([g^{-1}] * [f]\big) * [g * e_{x_0}] \\ &= \big([g^{-1}] * [f]\big) \otimes [g * e_{x_0}] \\ &= \begin{cases} g^{-1}(2s) \cdot g(2s) & \text{for } s \in [0,1/2] \\ f(2s-1) \cdot e_{x_0}(2s-1) & \text{for } s \in [1/2,1] \end{cases} \\ &= [e_{x_0} * f] \\ &= [f]. \end{aligned}$$

It follows that $\pi_1(X, x_0)$ is Abelian. ■

Problem 11.2 ((A))

Prove Proposition F from the note on the Fundamental Group of the Circle.

Proof. Recall proposition F:

Proposition F. (i) *W takes the class of the path $f_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ to n (and therefore W is onto).*

(ii) *W is one-to-one.*

(iii) *W is a homomorphism.*

(i) Now, recall that $W\colon \pi_1(S^1, x_0) \to \mathbf{Z}$ defined by $W([f]) \coloneqq w(f)$ where $w(f) = \tilde{f}(1)$ where $\tilde{f}\colon I \to \mathbf{R}$ is the lift of f, i.e. $p \circ \tilde{f} = f$. Now, let f_n be a path as above. Now, by Proposition C, since

$$f_n(s) = (\cos(2\pi ns), \sin(2\pi ns)) = (\cos(2\pi \tilde{f}_n(s)), \sin(2\pi \tilde{f}_n(s)))$$

and $\tilde{f}_n(0) = 0 = n \cdot 0$, by Proposition C, it follows that $f_n(s) = ns$. Thus, $\tilde{f}(1) = n$.

(ii) Let $f\colon I \to S^1$ be a loop at x_0. It suffices to show that if $W([f]) = n$ then $f \simeq_p f_n$. Let $\tilde{f}\colon I \to \mathbf{R}$ be the lift of f and consider the path-homotopy $H\colon I \times I \to \mathbf{R}$ defined by $H(s,t) \coloneqq (1-t)\tilde{f}(s) + t(ns)$. This map is continuous by Theorem 21.5 since it is multiplication from a topological space $I \times I$ into $\mathbf{R}$. Moreover, we have that

$$H(0, s) = \tilde{f}(s) \qquad\qquad H(1, s) = ns$$

and H fixes the endpoints, i.e.,

$$\begin{aligned} H(t, 0) &= (1-t) \cdot 0 + t \cdot 0 \\ &= 0 \end{aligned} \qquad\qquad \begin{aligned} H(t, 1) &= (1-t)n + tn \\ &= n. \end{aligned}$$

At last, define the map $K \coloneqq p \circ H\colon I \times I \to S^1$. We claim that this map is a path-homotopy from the class $f \simeq_p f_n$. First, this map is continuous by Theorem 18.2(c) since it is a composition of continuous maps. Secondly, we have that

$$\begin{aligned} K(0, s) &= p(H(0, s)) \\ &= (\cos(2\pi \tilde{f}(s)), \sin(2\pi \tilde{f}(s))) \\ &= f(s) \end{aligned} \qquad\qquad \begin{aligned} K(1, s) &= p(H(0, s)) \\ &= (\cos(2\pi ns), \sin(2\pi ns)) \\ &= f_n(s). \end{aligned}$$

and K fixes the endpoints, i.e.,

$$\begin{aligned} K(t, 0) &= p(H(t, 0)) \\ &= (\cos 0, \sin 0) \\ &= x_0 \end{aligned} \qquad\qquad \begin{aligned} K(t, 1) &= p(H(t, 1)) \\ &= (\cos 2\pi n, \sin 2\pi n) \\ &= x_0. \end{aligned}$$

Thus, $f \simeq_p f_n$. It follows that if $W([f]) = W([g])$ then $[f] = [g]$, i.e., W is one-to-one.

(iii) Last but not least, we will show that W is in fact a homomorphism thus, proving that W is an isomorphism $\pi_1(X, x_0) \cong \mathbf{Z}$. Let f and g be loops in S^1 at x_0 and let $\tilde{f}$ and $\tilde{g}$ be their respective lifts to paths on $\mathbf{R}$. Let $n \coloneqq W([f])$ and $m \coloneqq W([g])$. Let $G \colon I \to \mathbf{R}$ be the path $G(s) \coloneqq n + \tilde{g}(s)$. Now, since $p(n+s) = p(s)$ for all $s \in \mathbf{R}$, the path G is a lifting of g begining at n. Thus, $\tilde{f} * G$ is defined, and it is the lifting of $f * g$ that begins at 0 and ends at $n + m$ so

$$W([f] * [g]) = n + m = W([f]) + W([g]).$$

Thus, W is a homomorphism. ∎

Problem 11.3 ((B))

Prove Lemma G from the note on the Fundamental Group of the Circle. (Hint: one way to do this is to use the fact, which you don't have to prove, that if $\sim$ is the equivalence relation on $[a, a+1]$ which identifies a and $a+1$ then the restriction of p induces a homeomorphism $[a, a+1]/\sim \to S^1$.)

Proof. Recall the statement of Lemma G:

Lemma G. *For each $a \in \mathbf{R}$, the map*

$$p_a \colon (a, a+1) \longrightarrow S^1 - \{p(a)\}$$

given by $p_a(u) = p(u)$ is a homomorphism.

We shall proceed by the hint. If $\sim$ is the equivalence relation on $[a, a+1]$ which identifies a and $a+1$ then the restriction* of p induces a homeomorphism $[a, a+1]/\sim \to S^1$. Then, we claim that $[a, a+1]/\sim - \{[a]\} \approx (a, a+1)$ via the restriction of $q \colon [a, a+1] \to [a, a+1]/\sim$ to $(a, a+1)$. Note that $q|_{(a,a+1)}$ is bijective: for every $[x] \in [a, a+1]/\sim - \{[a]\}$, $x \in (a, a+1)$ so $q|_{(a,a+1)}(x) = [x]$ and if $[x] = [y]$ and $x \neq a$ or $a+1$ then $x = y$. Lastly, we show that $q|_{(a,a+1)}$ is an open map. By Lemma C, it suffices to prove this for basic open subsets of $(a, a+1)$ so let us consider the open interval $(b, c) \subset (a, a+1)$. Then

$$\left(q|_{(a,a+1)}\right)^{-1}\Big(q|_{(a,a+1)}((b,c))\Big) = \Big\{\, x \in (a, a+1) \;\Big|\; q|_{(a,a+1)}(x) \in q|_{(a,a+1)}((b,c)) \,\Big\} = (b, c)$$

since (b, c) is a saturated set, that is, it contains every $q|_{(a,a+1)}^{-1}([x])$ where $[x] \in q|_{(a,a+1)}(a, b)$ since the equivalence class of x for $x \in (a, a+1)$ consists of the singleton $\{x\}$. It follows that $(a, a+1) \approx [a, a+1]/\sim - \{[a]\}$ which, by Lemma A, is in turn homeomorphic to $S^{-1} - \{\bar{p}([a])\} = S^{-1} - \{p(a)\}$. ■

*What restriction? I'm not sure this actually makes sense.

Problem 11.4 ((C))

Show that for every point $x \in S^n$ the space $S^n - \{x\}$ is homeomorphic to $\mathbf{R}^n$. You may use the fact, shown in Step 1 of the proof of Theorem 59.3, that S^n with the *north pole* removed is homeomorphic to $\mathbf{R}^n$. (Hint: linear algebra.)

Proof. By elementary linear algebra, extend the sets $\{x\}$ and $\{(0, ..., 0, 1)\}$ to orthonormal bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathbf{R}^{n+1}$ (by Gram-Schmidt). Then, the change of basis matrices from the strandard basis on $\mathbf{R}^{n+1}$, say A_1 and A_2 corresponding to $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, are unitary linear transformations. Then the composition $A \coloneqq A_2 A_1^{-1} \colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is a continuous map since it is a bounded linear operator on $\mathbf{R}^{n+1}$ with inverse $A_1 A_2^{-1}$ (which is also continuous since it is a bounded linear operator on $\mathbf{R}^{n+1}$) hence is a homeomorphism. Moreover, since A_1 and A_2 are unitary then $A_2 A_1^{-1} = A$ is unitary, i.e., preserves the Euclidean norm. It follows that the restriction $\varphi \coloneqq A|_{S^n}$ is a map from S^1 to S^1 is a homeomorphism by Lemma A with $\varphi(x) = (0, ..., 0, 1)$. Now, by Lemma A, the restriction $\varphi|_{S^n - \{x\}} \colon S^n - \{x\} \to S^n - \{(0, ..., 0, 1)\}$ is a homeomorphism. Thus, the composition $\Phi \circ \left(\varphi|_{S^1 - \{x\}}\right) \colon S^n - \{x\} \to \mathbf{R}^n$, where $\Phi \colon S^n - \{(0, ..., 0, 1)\} \to \mathbf{R}^n$ is the stereographic projection from the north pole, gives a homeomorphism from $S^n - \{x\}$ to $\mathbf{R}^n$. ∎

Problem 11.5 ((D))

Show that every loop in S^n which is not onto is path-homotopic to a constant path. (Hint: use Problem C).

Proof. Fix $x_0 \in S^n$ and let $f\colon I \to S^n$ be a loop at x_0. If f is not surjective, then there exists a point $x \in S^n$ that does not get hit by f, i.e., $f(t) \neq x$ for all $t \in I$. By Problem C, $S^n - \{x\} \approx \mathbf{R}^n$ with homeomorphism φ. Then $\varphi \circ f\colon I \to \mathbf{R}^n$ is a loop based at $\varphi(x_0)$. Since $\mathbf{R}^n$ is simply-connected, $\phi \circ f \simeq_p e_{\phi(x_0)}$ via the homotopy H. Then $\varphi^{-1} \circ H\colon I \times I \to S^n - \{x\}$ is a path homotopy from f to e_{x_0}. Thus, if $f\colon I \to S^n$ is not surjective, $[f] = [e_{x_0}]$. ∎

Problem 11.6 ((E))

Let X be a topological space and let $A \subset X$ be a deformation retract. In the space X/A, the set A is a point (because it is an equivalence class). Show that this point is a deformation retract of X/A. (Hint: use p. 289 # 9.)

Proof. Let $H\colon X \times I \to X$ be a deformation retraction from X to A, that is, $H(0,x) = \mathrm{id}_X$ and $H(1,x) = r(x)$ where $r\colon X \to A$ is a retraction of X onto A and $\iota\colon A \hookrightarrow X$ is the inclusion of A into X. Let $p\colon X \to X/A$ be a quotient map. Now, we want to construct a deformation retraction $h\colon X/A \times I \to X/A$ from the quotient X/A to $*$, which we shall use to denote the image of A in X/A under p, and what better candidate than the map induced by $p \circ H\colon X \times I \to X/A$ on the quotient $X/A \times I$ into X/A. Consider the map $(p, \mathrm{id}_I)\colon X \times I \to X/A \times I$. This map is a quotient map by Problem 9.2 (Munkres §46, x. 9). Moreover, the map $p \circ H$ preserves the equivalence relation on $X/A \times I$ since for any two representatives (x_1,t) and (x_2,t) of $[(x,t)]$ in $X/A \times I$, we have $H(x_1,t) = H(x_2,t)$ if $x \in X - A$ and $H(x_1,t) = H_2(x_2,t)$ so $p(H(x_1,t)) = p(H(x_2,t))$ and if $x_1, x_2 \in A$ then $H(x_1,t), H(x_2,t) \in A$ so $p(H(x_1,t)) = p(H(x_2,t))$. Thus, by Theorem Q.3 the map $h\colon X/A \times I \to X/A$ induced by H, i.e., the map defined by $h(x,t) \coloneqq [H(x,t)]$, is continuous and the diagram

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "{(p,\mathrm{id}_I)}"'] & X \arrow[d, "p"] \\ X/A \times I \arrow[r, "h"'] & X/A \end{tikzcd}$$

commutes. We claim that h is a deformation retraction from X/A to $*$. To that end, it suffices to show that $h(x,0) = \mathrm{id}_{X/A}$ and, using suggestive notation, $h(x,1) = \bar{r}$ where $\bar{r}\colon X/A \to *$ is a retraction of X/A onto A and $\bar{\iota}\colon * \hookrightarrow X/A$ is the inclusion of $*$ into X/A. The first is easy to verify since $h(x,0) = [H(x,0)] = [x] = \mathrm{id}_{X/A}$. Next, $h(x,1) = [H(x,1)] = [r(x)]$ and we claim that $\bar{r}(x) \coloneqq [r(x)]$ is a retraction of X/A into $*$. The map $\bar{r}$ is continuous since h is continuous (by Lemma 1 from Hw. #9 Munkres §18, Ex. 11) and $\bar{r}\colon X/A \to *$ since $r(x) \in A$ for every $x \in X$. It follows that $*$ is a deformation retract of X/A. ∎