MRC 2016 Report: Tropicalization of Character Varieties

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1 Overview

Tropical geometry is a new an exciting area of mathematics that is best described as a piecewise-linear version of algebraic geometry. In tropical geometry the sum of two real numbers $x \oplus y$ is their maximum and the product $x \odot y$ their sum. This, together with a minimum element $-\infty$, gives us the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$. In the tropical setting, polynomials become pice-wise linear functions and algebraic varieties give way to tropical varieties— which are in some sense "skeletons" of the original variety.

During the introductory talks by Manon, we decided to try our hand at *tropicalizing* some of the $\mathrm{SL}(2,\mathbb{C})$-character varieties that were presented Lawton's talk and, further, tropicalize $\mathrm{PSL}(2,\mathbb{C})$-, $\mathrm{SL}(3,\mathbb{C})$-, an $\mathrm{Sp}(4)$-character varieties. We broke out—loosely—into two groups: one concerned with the tropicalization of $\mathrm{PSL}(2,\mathbb{C})$-character varieties and the other with visualization of the Newton polytopes that came about from tropicalization.

Here is a summary of the observations made by the group.

2 Tropicalization of $\mathfrak{X}(F_3, \mathrm{SL}(2,\mathbb{C}))$

From Lawton [2] Corollary 4 and Lemma 5, and some `Mathematica` Gröbner basis magic, we deduced that the coordinate ring of the character variety $\mathfrak{X}(F_3, \mathrm{SL}(2,\mathbb{C}))$ is cut out by the polynomial in 7 indeterminates

$$
\begin{aligned}
f = X_1X_2X_3X_7 - {X_4}^2 + {X_5}^2 + {X_1}^2 \\
+ {X_2}^2 + {X_3}^2 + {X_7}^2 \\
+ X_1X_6X_7 + X_2X_5X_7 + X_3X_4X_7 \\
+ X_1X_2X_4 + X_1X_3X_5 + X_2X_3X_6 - 4.
\end{aligned}
\tag{1}
$$

With the help of `gfan`—a software package for computing Gröbner fans and tropical varieties [1]—and `Mathematica` we were were able to find $\mathrm{Trop}(\mathfrak{X}(F_3, \mathrm{SL}(2,\mathbb{C})))$. Its tropicalization is the codimension 1-cones of the dual fan to the Newton polytope $\mathrm{Newt}(f)$. Figure 1 is a picture made we made using the Ti*k*Z graphics language to draw the edge graph of $\mathrm{Newt}(f)$

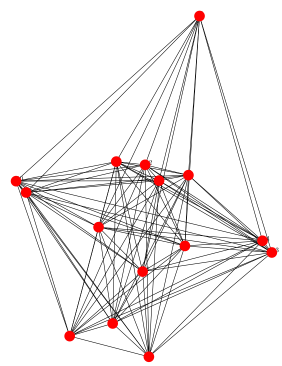

Figure 1: Edge graph of the Newton polytope of $\mathfrak{X}(F_3, \mathrm{SL}(2,\mathbb{C}))$.

More information about its tropicalization was extracted using `gfan`. The relevant file is on the MRC website under the `Tropical Group Files/F_3->SL_2` folder with the tile `hypersurface_F3->SL(2,C)_groebnerfan.txt`.

3 Tropicalization of $\mathfrak{X}(F_2, \mathrm{SL}_3\mathbb{C}))$

The coordinate ring of the character variety $\mathfrak{X}(F_2, \mathrm{SL}_3\mathbb{C}))$ is cut out by the huge polynomial in 9 indeterminates

$$\begin{aligned} f = 9 &+ 3X_9 - 6X_1X_2 \\ &- 6X_3X_4 - 6X_5X_6 - 6X_7X_8 + {X_9}^2 - X_1X_2X_9 \\ &- X_3X_4X_9 - X_5X_6X_9 \ - X_7X_8X_9 + {X_1}^3 \\ &+ {X_3}^3 + {X_5}^3 + {X_7}^3 + {X_2}^3 \\ &+ {X_4}^3 + {X_6}^3 + {X_8}^3 - 3X_2X_8X_6 \\ &- 3X_1X_5X_7 - 3X_3X_5X_8 - 3X_4X_6X_7 + 3X_1X_4X_8 \\ &+ 3X_2X_3X_7 + 3X_1X_3X_6 + 3X_2X_4X_5 - X_1X_2X_3X_4X_9 \\ &+ X_1X_3X_6X_9 + X_2X_4X_5X_9 + X_1X_4X_8X_9 + X_2X_3X_7X_9 \\ &+ X_1X_2X_3X_4 + X_3X_4X_5X_6 + X_1X_2X_7X_8 + X_3X_4X_7X_8 \\ &+ X_1X_2X_5X_6 + X_5X_6X_7X_8 + X_4{X_8}^2X_6 + X_3X_5{X_7}^2 \\ &+ {X_2}^2X_4X_8 + {X_1}^2X_3X_7 + X_1{X_4}^2X_6 + X_2{X_3}^2X_5 \\ &+ {X_1}^2X_8X_6 + {X_2}^2X_5X_7 + X_3X_8{X_6}^2 + X_4{X_5}^2X_7 \\ &+ {X_2}^2X_3X_6 + {X_1}^2X_5X_4 + X_1{X_3}^2X_8 + X_2{X_4}^2X_7 \\ &+ {X_4}^2X_5X_8 + {X_3}^2X_6X_7 + X_1X_5{X_8}^2 + X_2X_6{X_7}^2 \\ &+ X_2{X_5}^2X_8 + X_1{X_6}^2X_7 - 2X_2X_4{X_6}^2 - 2X_1X_3{X_5}^2 \\ &- 2X_2X_3{X_8}^2 - 2X_1X_4{X_7}^2 + {X_2}^2{X_4}^2X_6 + {X_1}^2{X_3}^2X_5 \\ &+ {X_2}^2{X_3}^2X_8 + {X_1}^2{X_4}^2X_7 - X_1X_3{X_4}^2X_8 - X_2{X_3}^2X_4X_7 \\ &- {X_1}^2X_2X_3X_6 - X_1{X_2}^2X_4X_5 - X_1{X_3}^2X_4X_6 - X_2X_3{X_4}^2X_5 \\ &- {X_1}^2X_2X_4X_8 - X_1{X_2}^2X_3X_7 - X_1X_2{X_4}^3 - X_1X_2{X_3}^3 \\ &- {X_2}^3X_3X_4 - {X_1}^3X_3X_4 - X_2X_3X_4X_8X_6 - X_1X_3X_4X_5X_7 \\ &- X_1X_2X_3X_5X_8 - X_1X_2X_4X_6X_7 + {X_1}^2{X_2}^2X_3X_4 + X_1X_2{X_3}^2{X_4}^2 \end{aligned} \tag{2}$$

The edge graph of its Newton polytope is shown in Figure 2.

More information about the tropicalization of was extracted using `gfan`. The file containing this information (is huge) can be found on the MRC website under the `Tropical Group Files/F_2->SL_3` folder with title `F_2SL_3TropicalVariety`. **Warning** the file is huge.

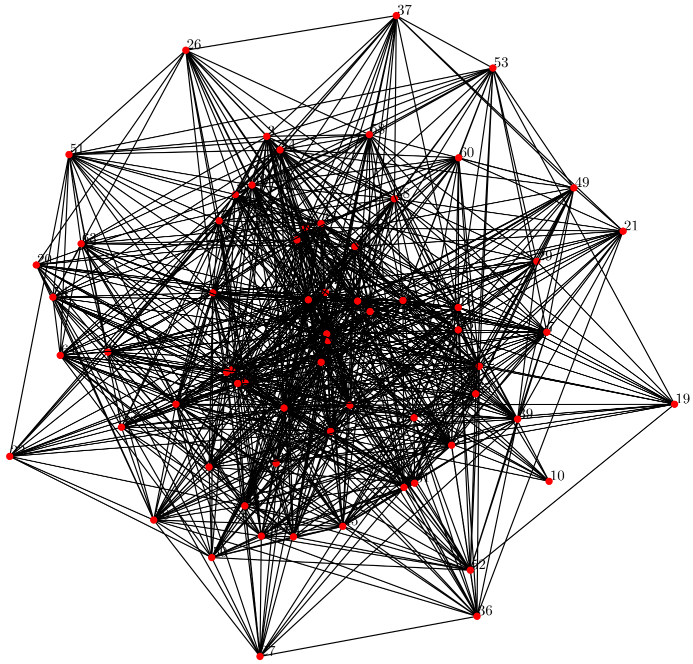

Figure 2: Edge graph of the Newton polytope of $\mathfrak{X}(F_3, \mathrm{SL}(3,\mathbb{C}))$.

4 Tropicalization of $\mathrm{PSL}(2,\mathbb{C})$-character varieties

In addition to tropicalizing plain $\mathrm{SL}(2,\mathbb{C})$- and $\mathrm{SL}(3,\mathbb{C})$-character varieties, we decided to look at $\mathrm{PSL}(2,\mathbb{C})$- and $\mathrm{Sp}(4)$-character varieties.

Let F_3 be the free group on 3 generators A, B, C. Then, we claim that the coordinate ring of $\mathfrak{X}(F_3, \mathrm{PSL}(2,\mathbb{C}))$

$$\mathbb{C}[\mathfrak{X}(F_3, \mathrm{PSL}(2,\mathbb{C})] = \mathbb{C}[\mathfrak{X}(F_3, \mathrm{SL}(2,\mathbb{C}))]^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$$

is cut out the following family of trace-polynomials—which we shall group according to similarity:

- Type χ:

$$\chi_A = (\operatorname{tr} A)^2 \qquad \chi_B = (\operatorname{tr} B)^2 \qquad \chi_C = (\operatorname{tr} C)^2$$
$$\chi_{AB} = (\operatorname{tr} AB)^2 \qquad \chi_{AC} = (\operatorname{tr} AC)^2$$
$$\chi_{BC} = (\operatorname{tr} BC)^2 \qquad \chi_{ABC} = (\operatorname{tr} ABC)^2$$

- Type τ:

$$\tau_{AB} = \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AB \qquad \tau_{AC} = \operatorname{tr} A \operatorname{tr} C \operatorname{tr} AC \qquad \tau_{BC} = \operatorname{tr} B \operatorname{tr} C \operatorname{tr} BC$$

- Type Λ:

$$\Lambda_A = \operatorname{tr} B \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} AC \qquad \Lambda_B = \operatorname{tr} A \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} BC$$
$$\Lambda_C = \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AC \operatorname{tr} BC$$

- Lonely Δ:

$$\Delta = \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C \operatorname{tr} ABC$$

- Equally lonely Σ:

$$\Sigma = \operatorname{tr} AB \operatorname{tr} AC \operatorname{tr} BC$$

- Type Θ:

$$\Theta_A = \operatorname{tr} A \operatorname{tr} BC \operatorname{tr} ABC \qquad \Theta_B = \operatorname{tr} B \operatorname{tr} AC \operatorname{tr} ABC$$
$$\Theta_C = \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} ABC$$

With some further algebraic manipulations, we observe the following relations among the generators

$$\Sigma^2 = (\operatorname{tr} AB \operatorname{tr} AC \operatorname{tr} BC)^2 = \chi_{AB}\chi_{AC}\chi_{BC}$$

and some binomial relations

$$\tau_{AB}^2 = \chi_A\chi_B\chi_{AB} \qquad \tau_{AC}^2 = \chi_A\chi_C\chi_{AC}$$
$$\tau_{BC}^2 = \chi_B\chi_C\chi_{BC}$$

$$\Lambda_A^2 = \chi_B\chi_C\chi_{AB}\chi_{AC} \qquad \Lambda_B^2 = \chi_A\chi_C\chi_{AB}\chi_{BC}$$
$$\Lambda_C^2 = \chi_A\chi_B\chi_{AC}\chi_{BC}$$

$$\Theta_A^2 = \chi_A\chi_{BC}\chi_{ABC} \qquad \Theta_B^2 = \chi_B\chi_{AC}\chi_{ABC}$$
$$\Theta_C^2 = \chi_C\chi_{AB}\chi_{ABC}$$

$$\Sigma^2 = \chi_{AB}\chi_{AC}\chi_{BC} \qquad \Delta^2 = \chi_A\chi_B\chi_C\chi_{ABC}.$$

and finally the relation coming from $\mathfrak{X}(F_3, \mathrm{SL}(2,\mathbb{C}))$

$$\begin{aligned} \chi_A + \chi_B + \chi_C + \chi_{AB} + \chi_{AC} &= \tau_{AB} + \tau_{AC} + \tau_{BC} \\ +\chi_{BC} + \chi_{ABC} + \Sigma + \Delta \quad &+ \Theta_A + \Theta_B + \Theta_C + 4. \end{aligned}$$

On the last day of the MRC we further managed to tropicalize $\mathfrak{X}(F_2, \mathrm{Sp}(4))$. The Gröbner basis and other tropicalization information we obtained from `gfan` can be found on the MRC website under the `Tropical Group Files/F_2->SP_4`.

5 Visualization of character varieties of knot complements

Although `gfan` can make excellent polyhedral representation of the Gröbner fans we obtained from the tropicalization of our character varieties, the documentation is very sparse and so we tried our hand at making these polyhedral representations from scratch using `Mathematica` which, surprisingly enough, does not yet have great support for tropical geometry.

We attempted to create the Newton polytopes of several knot groups whose A-polynomials we obtained from a paper by Eric Chesebro *Formulas for Character Varieties of* 2*-Bridge Knots* which can be found on the University of Montana's Department of Mathematical Sciences, Technical reports page under the Research tab. Here is a link to the document in question `http://hs.umt.edu/math/research/technical-reports/documents/2012/KnotFormulas.pdf`.

With the help of `Mathematica`, we made the following subdivision of the Newton polytopes of these character varieties.

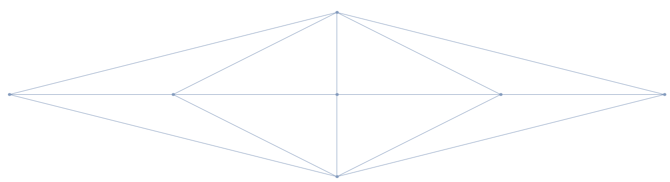

Figure 3: Subdivided Newton polytope for the A-polynomial of the figure-8 knot.

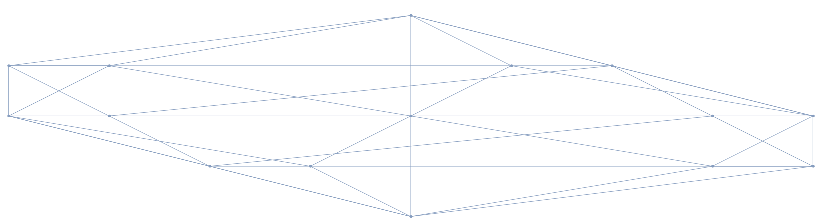

Figure 4: Subdivided Newton polytope for the A-polynomial the 4-twisted knot.

Some more pictures for Newton polytopes of two 2-bridge knots corresponding to $\varphi(1)$ and $\varphi(2)$ of Chesebro's equation are shown in Figures 5 and 6.

The dual of these images is really what we are after and, although we may be on hiatus for the moment, we plan to finish writing the code that does this.

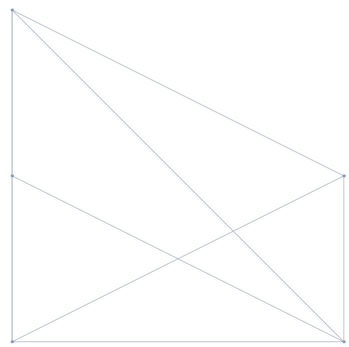

Figure 5: Subdivided Newton polytope for the A-polynomial of the $\varphi(1)$ 2-bridge knot.

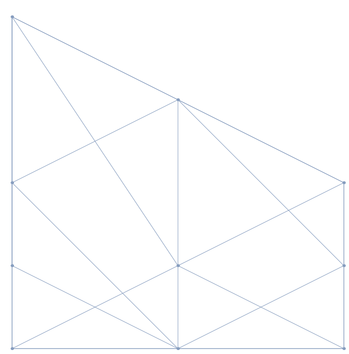

Figure 6: Subdivided Newton polytope for the A-polynomial of the $\varphi(1)$ 2-bridge knot.

References

[1] Jensen, A. N. Gfan, a software system for Gröbner fans and tropical varieties. Available at `http://home.imf.au.dk/jensen/software/gfan/gfan.html`.

[2] Lawton, S. Generators, relations and symmetries in pairs of 3×3 unimodular matrices.