MA571 Homework 9

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Problem 9.1 (Munkres §52, Ex. 2)

Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. By Theorem 52.1, the paths α and β induce a group homomorphism $\hat{\alpha}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $\hat{\beta}\colon \pi_1(X, x_1) \to \pi_1(X, x_2)$, respectively. We want to show therefore that the induced homomorphism $\hat{\gamma} = \widehat{\alpha * \beta}$ is in fact equivalent to the composition $\hat{\beta} \circ \hat{\alpha}$. Let $[f]$ be a loop based at x_0 then

$$\begin{aligned} \hat{\gamma}([f]) &= \widehat{\alpha * \beta}([f]) \\ &= \left[\,\overline{\alpha * \beta}\,\right] * [f] * [\alpha * \beta] \\ &= \left[\bar{\beta} * \bar{\alpha}\right] * [f] * [\alpha] * [\beta] \end{aligned}$$

by the well-definedness of the path product operation, we have

$$= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta]$$

by associativity of the path product,

$$\begin{aligned} &= [\bar{\beta}] * ([\bar{\alpha}] * [f] * [\alpha]) * [\beta] \\ &= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta] \end{aligned}$$

where $\alpha([f])$ is a loop based at x_1 so

$$\begin{aligned} &= \hat{\beta}(\hat{\alpha}([f])) \\ &= \big(\hat{\beta} \circ \hat{\alpha}\big)([f]). \end{aligned}$$

Thus, the following diagram commutes

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\hat{\alpha}"] \arrow[dr, "\hat{\gamma} = \widehat{\alpha * \beta}"'] & \pi_1(X, x_1) \arrow[d, "\hat{\beta}"] \\ & \pi_1(X, x_2). \end{tikzcd}$$

■

Problem 9.2 (Munkres §52, Ex. 3)

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is Abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. ■

Problem 9.3 (Munkres §52, Ex. 4)

Let $A \subset X$; suppose $r\colon X \to A$ is continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A.) If $a_0 \in A$, show that

$$r_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

Proof. ■

Problem 9.4 (Munkres §53, Ex. 6)

Show that if X is path connected, the homomorphism induced by a continuous map is independent of the base point, up to isomorphisms of the groups involved. More precisely, let $h\colon X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1}) \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps "commutes"

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ {\scriptstyle\hat{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle\hat{\beta}} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1). \end{array}$$

Proof. ∎

Problem 9.5 (Munkres §55, Ex. 1)

Show that if A is a retract of B^2, then every continuous map $f\colon A \to A$ has a fixed point.

Proof. ■

PROBLEM 9.6 (MUNKRES §55, EX. 2)

Show that if $h\colon S^1 \to S^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof. ∎

Problem 9.7 ((A))

Prove that every m-manifold is locally path-connected.

Proof. ■

Problem 9.8 ((B))

Prove that every m-manifold is regular.

Proof. ■

Problem 9.9 ((C))

Prove that there is no 1-1 continuous function $\iota \colon S^1 \to \mathbf{R}$. You may assume any fact about trigonometric functions. (Note: this shows in particular that there is no $\iota \colon S^1 \to \mathbf{R}$ with $p \circ \iota$ equal to the identity map, where p is the map in the note on the Fundamental Group of the Circle.)

Proof. ■

Problem 9.10 ((D))

Prove Proposition C from the note on the Fundamental Group of the Circle.

Proof. ■