MA52300 FALL 2016

Homework Assignment 6 – *Solutions*

1. For $n = 2$ find Green's function for the quadrant $\{x_1 > 0, x_2 > 0\}$ by repeated reflection.

Solution. Let $x^0 = (x_1^0, x_2^0)$ be from the first quadrant $\Omega := \{x_1 > 0, x_2 > 0\}$. Consider the reflections of x^0 with respect to the coordinate axes and the origin,

$$x^1 = (x_1^0, -x_2^0), \quad x^2 = (-x_1^0, x_2^0), \quad x^3 = (-x_1^0, -x_2^0).$$

and set

$$\begin{aligned} g(x) :&= \Phi(x - x^0) - \Phi(x - x^1) - \Phi(x - x^2) + \Phi(x - x^3) \\ &= \frac{1}{2\pi} \log \frac{|x - x^1||x - x^2|}{|x - x^0||x - x^3|}. \end{aligned}$$

By construction, it is straightforward to verify that

$$g(x_1, -x_2) = -g(x_1, x_2), \quad g(-x_1, x_2) = -g(x_1, x_2),$$

consequently $g = 0$ on $\{x_1 = 0\} \cup \{x_2 = 0\} \supset \partial\Omega$. Also, since $x^1, x^2, x^3 \notin \Omega$, we have $\Delta g = -\delta_{x^0}$ in Ω. Thus,

$$G(x^0, x) = g(x). \qquad \square$$

2 (Precise form of Harnack's inequality)**.** Use Poisson's formula for the ball to prove

$$\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \le u(x) \le \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B_r = \{x \in \mathbb{R}^n : |x| < r\}$.

Solution. Fix $\rho < r$ and write Poisson's formula for the ball B_ρ:

$$u(x) = \frac{\rho^2 - |x|^2}{n\alpha_n \rho} \int_{\partial B_\rho} \frac{u(y) dS(y)}{|x - y|^n}.$$

Using now that $u(y) \geq 0$ and $\rho - |x| \leq |y-x| \leq \rho + |x|$ for any $y \in \partial B_\rho$, we obtain

$$\begin{aligned} u(x) &\leq \frac{\rho^2 - |x|^2}{n\alpha_n \rho(\rho - |x|)^n} \int_{\partial B_\rho} u(y) dS(y) \\ &= \frac{\rho^2 - |x|^2}{n\alpha_n \rho(\rho - |x|)^n} n\alpha_n \rho^{n-1} u(0) \\ &= \frac{\rho^{n-2}(\rho + |x|)}{(\rho - |x|)^{n-1}} u(0), \end{aligned}$$

where in the second line we have used the mean value property

$$\int_{\partial B_\rho} u(y) dS(y) = n\alpha_n \rho^{n-1} u(0).$$

Letting $\rho \to r$, we obtain the required bound from above. Using similar arguments, we also obtain the bound from below.

Remark. The inequality is indeed precise: the equality is attained on functions

$$\kappa^y(x) := K(x,y) = \frac{r^2 - |x|^2}{n\alpha_n r |y-x|^n}$$

for any fixed $|y| = r$.

$\square$

3. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in $\mathbb{R}^n$ of degrees k and m respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \quad P_m(\lambda x) = \lambda^m P_k(x), \quad \text{for every } x \in \mathbb{R}^n, \lambda > 0$$
$$\Delta P_k = 0, \quad \Delta P_m = 0 \quad \text{in } \mathbb{R}^n.$$

(a) Show that

$$\frac{\partial P_k}{\partial \nu} = k P_k(x), \quad \frac{\partial P_m}{\partial \nu} = m P_m(x) \quad \text{on } \partial B_1,$$

where $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ and ν is the outward normal on ∂B_1.

(b) Use (a) and Green's formula to prove that

$$\int_{\partial B_1} P_k(x) P_m(x) dS = 0, \quad \text{if } k \neq m$$

Solution. (a) Differentiating the identity $P_k(\lambda x) = \lambda^k P_k(x)$ with respect to λ and evaluating at $\lambda = 1$ we obtain

$$x \cdot DP_k(x) = k P_k(x).$$

(This is known as *Euler's equation* for homogeneous functions.) Next, on the unit sphere ∂B_1, we have that the outward normal ν is given by $\nu(x) = x$, thus

$$\frac{\partial P_k}{\partial \nu} = \nu \cdot DP_k = x \cdot DP_k = kP_k \quad \text{on } \partial B_1$$

Similarly, we establish the identity for P_m.

(b) Now applying Green's formula for functions P_k and P_m we obtain

$$\begin{aligned} 0 &= \int_{B_1} (P_k \Delta P_m - P_m \Delta P_k) dx \\ &= \int_{\partial B_1} \left(P_k \frac{\partial P_m}{\partial \nu} - P_m \frac{\partial P_k}{\partial \nu} \right) dS \\ &= \int_{\partial B_1} (m-k) P_k P_m dS, \end{aligned}$$

where in the last step we have used the identities from part (a). Dividing by $m-k$, we obtain the statement of part (b). □