MA 54400 - Midterm 1 Practice Problems
Spring 2016
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1. Let $E \subset \mathbb{R}^n$ be a measurable set, $r \in \mathbb{R}$, and define the set $rE = \{rx \mid x \in E\}$. Prove that rE is measurable, and that $|rE| = |r|^n |E|$.

2. Let $\{E_k\},\ k \in \mathbb{N}$, be a collection of measurable sets. Define the set
$$\liminf_{k\to\infty} E_k = \cup_{k=1}^{\infty} \left(\cap_{n=k}^{\infty} E_n\right).$$
Show that
$$|\liminf_{k\to\infty} E_k| \leq \liminf_{k\to\infty} |E_k|.$$
State and prove an analogous result for $\limsup_{k\to\infty} E_k$.
(Recall that for a sequence of real numbers $\{a_k\}$, $\liminf_{k\to\infty} a_k = \lim_{k\to\infty} \inf_{n\geq k} a_n$.)

3. Let $E \subset \mathbb{R}^n$ be a measurable set, with $|E| = \infty$. Show that for any $C > 0$ there exists a measurable set $F \subset E$ such that $C < |F| < \infty$.

4. Consider the function
$$F(x) = \begin{cases} |B(0,x)| & x > 0, \\ 0 & x = 0. \end{cases}$$
Here $B(0,r) = \{y \in \mathbb{R}^n \mid |y| < r\}$. Prove that F is monotone increasing and continuous.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let C be the set of all points at which f is continuous. Show that C is a set of type G_δ.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Is it true that if the sets $\{f = r\}$ are measurable for all $r \in \mathbb{R}$, then f is measurable?

7. Let $\{f_k\}$ be a sequence of measurable function on $\mathbb{R}$. Prove that the set $\{x \mid \exists \lim_{k\to\infty} f_k(x)\}$ is measurable.

8. A real valued function f on an interval $[a,b]$ is said to be *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite disjoint collection $\{(a_k,b_k)\}_{k=1}^n$ of open intervals in (a,b) satisfying $\sum_{k=1}^n (b_k - a_k) < \delta$, one has $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$. Show that an absolutely continuous function on $[a,b]$ is of bounded variation on $[a,b]$.

9. Let f be a continuous function from $[a,b]$ into $\mathbb{R}$. Let χ_c be the characteristic function of the singleton $\{c\}$, i.e. $\chi_c(x) = 0$ if $x \neq c$, and $\chi_c(c) = 1$. Show that
$$\int_a^b f\, d\chi_c = \begin{cases} 0 & \text{if } c \in (a,b) \\ -f(a) & \text{if } c = a \\ f(b) & \text{if } c = b \end{cases}$$