# MA 562: Notes

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## 1 Preliminaries

These set of notes are based off of Boothby's *Differential Geometry* book, chapters 1 through 6.

**Definition 1.** A *topological space* $M$ is a pair $(X, \mathcal{T})$, where $X$ is a set, $\mathcal{T}$ is a collection of subsets of $X$ such that

(a) $\emptyset, X \in \mathcal{T}$.

(b) The union of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

$$\{U_\alpha\} \subset \mathcal{T} \quad \implies \quad \bigcup_\alpha U_\alpha \in \mathcal{T}.$$

(c) Intersection of a finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

$$\{U_1, ..., U_k\} \subset \mathcal{T} \quad \implies \quad \bigcap_{j=1}^{k} U_j \in \mathcal{T}.$$

$\mathcal{T}$ is called the *topology* of $M$. Elements of $\mathcal{T}$ are called the *open sets* of $M$. By abuse of notation, we sometimes refer to $X$ as $M$.

**Definition 2.** (a) A *metric* on $X$ is a function $d\colon X \times X \to \mathbf{R}$ such that

(1) $d(x, y) \geq 0\ \forall x, y \in X$ and $d(x, y) = 0 \iff x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) + d(y, z) \geq d(x, z)$ (the triangle inequality).

(b) $B_d(x, r) = \{\, y \in X \mid d(x, y) < r \,\}$.

(c) A topological space $M$ is a *metric space* if the set of balls $B_d(x, r)$ form a *basis* of $M$, i.e., any open set of $M$ can be written as a union of open balls $B_d(x, r)$ for some $x \in X$, $r > 0$.

**Definition 3.** A topological space $X$ is *Hausdorff* if for any $x_1 \neq x_2$ in $X$, there exist open sets $U_1 \ni x_1$, $U_2 \ni x_2$ such that $U_1 \cap U_2 = \emptyset$.

**Definition 4.** A *topological manifold* $M$ of dimension $n$ is a topological space such that

(a) $M$ is Hausdorff.

(b) locally Euclidean, i.e., $\forall x \in M$ there exists a neighborhood $U$ of $X$ which is homeomorphic to $V \subset \mathbf{R}^n$ (there exists a map $f\colon U_x \to V \subset \mathbf{R}^n$ such that $f$ is bijective, continuous and $f^{-1}$ is continuous).

(c) $M$ has a countable basis of open sets.

**Theorem 1** (Boothby I.3.6)**.** *A* topological manifold *is metrizable (also locally connected, locally compact, and normal).*

**Definition 5.** (a) A *covering* of a topological manifold is a collection of open sets $\{U_\alpha\}$ such that any $x \in M$ is contained in some $U_\alpha$.

(b) A manifold is *compact* if every open cover contains a finite subcover.

**Definition 6.** (1) Half space
$$\mathbf{H}^n = \{\, x \in \mathbf{R}^n \mid x_n \geq 0 \,\}.$$

(2) Manifold with boundary. (Similar to definition 4)

   (a) $M$ is Hausdorff.

   (b) $M$ has a countable basis of open sets.

   (c) For any $x \in M$, there exists $U$ open, $x \in U$ such that:

      (i) $\varphi \colon U \to V \subset \mathbf{R}^n$ is a homeomorphism, or

      (ii) $\varphi \colon U \to V \subset \mathbf{H}^n$ is a homeomorphism with $x$ such that $\varphi(x) \in \partial\mathbf{H}^n$ referred to as *boundary points.*

(3)

**Example 1** (Unit Quaternions and Rotations in $\mathbf{R}^3$)**.**

$$f(v) = z \wedge z^{-1} \qquad v = (v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$$

where $z = \cos(\alpha/2) + \sin(\alpha/2)\hat{v}$. $\hat{v} = v/\|v\|$. Quaternion multplication:

$$\begin{aligned} ij &= k & \qquad ji &= -k \\ jk &= i & \qquad kj &= -i \\ ki &= j & \qquad ik &= -j. \end{aligned}$$

Can check that $z$ and $-z$ correspond to the same rotation.

Topologically, unit quaternions $\simeq S^3 = \left\{\, x \in \mathbf{R}^4 \;\middle|\; \|x\| = 1 \right\}$ and rotations $\simeq S^3/{\sim}$, $z \sim -z$

$$\mathbf{R}P^3 \approx \big(\mathbf{R}^4 \smallsetminus \{0\}\big)/x \sim \lambda x.$$

for all $\mathbf{R}^{n+1} \smallsetminus \{0\}$ can always find $\lambda$ such that $\lambda x$ has norm 1. There are precisely 2 such $\lambda$ which differ by a sign. Therefore, $\mathbf{R}P^n$ can be constructed by identifying antipodal points of $S^n$ in $\mathbf{R}^{n+1}$.

## 2 Functions of Several Variables

Taken from Boothby Chapter 1.

**Definition 7.** Let $f\colon U \to \mathbf{R}$ where $U \subset \mathbf{R}^n$ is open, $x = (x^1, ..., x^n)$ (are coordinates).

(a) $f \in C^1(U)$ (continuously differentiable) iff $\partial f/\partial x^j$ exists and is continuous on $U$.

(b) $f \in C^r(U)$ ($r$-fold continuously differentiable) iff $\partial f/\partial x^j \in C^{r-1}(U)$ for $j = 1, ..., n$.

(c) $f \in C^\infty(U)$ (smooth) iff $f \in C^r(U)$ $\forall r \in \mathbf{N}$.

(d) $f \in C^\omega(U)$ (real analytic) iff $f$ can be expressed as a convergent power series on $U$.

**Theorem 2** (Mean-value theorem)**.** *Suppose $U$ is star-like w.r.t. $a \in U$ (i.e., $\forall x \in U$, the line segment $\overleftrightarrow{ab} \subset U$). Then $\forall x \in U$, $\exists 0 < \theta < 1$ s.t.*

$$g(x) - g(a) = \left[\sum_{i=1}^{n} \left(\frac{\partial g}{\partial x^i}\right)\Bigg|_{a+\theta(x-a)}\right](x^i - a^i).$$

*Proof.* Consider $f\colon [0,1] \to \overleftrightarrow{ax} \subset U$ given by

$$f(t) = \mathbf{a} + t(\mathbf{x} - \mathbf{a})$$

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