Representation Theory

Carlos Salinas

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Chapter 1

What is Representation Theory?

Groups arise in nature as "sets of symmetries (of an object), which are closed under composition and under taking inverses". For example, the *symmetric group* S_n is the group of all permutations (symmetries) of $\{1, \ldots, n\}$; the *alternating group* A_n is the set of all symmetries preserving the parity of the number of ordered pairs; the *dihedral group* D_{2n} is the group of symmetries of the regular n-gon in the plane. The *orthogonal group* $O(3)$ is the group of distance-preserving transformations of Euclidean space which fix the origin. There is also the group of *all* distance preserving transformations, which includes the translations along with $O(3)$.*

The official definition is of course more abstract, a group is a set G with a binary operation $*$ which is associative, has a unit element e and for which inverses exist. Associativity allows a convenient abuse of notation, where we write gh for $g * h$; we have $ghk = (gh)k = g(hk)$ and parentheses are unnecessary. I will often write 1 for e, but this is dangerous on rare occasions, such that when studying the group $\mathbb{Z}$ under addition; in that case, $e = 0$.

The abstract definition notwithstanding, the interesting situation involves a group "acting" on a set. Formally, an action of a group G on a set X is an "action map" $a\colon G \times X \to X$ which is *compatible with the group law*, in the sense that

$$a\big(h, a(g, x)\big) = a(hg, x)$$
$$a(e, x) = x.$$

This justifies the abuse of notation $a(g, x) = gx$, for we have $h(gx) = (hg)x$.

From this point of view, geometry asks, "Given a geometric object X, what is its group of symetries?" Representation theory reverses the quostion to "Given a group G, what objects X does it act on?" and attempts to answer this question by classifying such X up to isomorphism.

Before restricting to the linear case, our main concern, let us remember another way to describe an action of G on X. Every $g \in G$ defines a map $a(g)\colon X \to X$ by $x \mapsto gx$. This map is a bijection, with inverse map $a(g^{-1})$: indeed, $\big(a(g^{-1}) \circ a(g)\big)(x) = g^{-1}gx = ex = x$ from the properties of the action. Hence $a(g)$ belongs to the set $\operatorname{Sym} X$ of bijective self-maps of X. This set forms a group under composition, and the properties of an action imply that

Proposition 1.1. *An action of G on X "is the same as" a group homomorphism $\alpha\colon G \to \operatorname{Sym} X$.*

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*This group is isomorphic to the *semi-direct product* $O(3) \ltimes \mathbb{R}^3$.

The formulation of Prop. 1.1 leads to the following observation. For any action a of H on X and group homomorphism $\varphi\colon G \to H$, there is defined a *restricted* or *pulled-back* action φ^*a of G on X, as $\varphi^*a = a \circ \varphi$. In the original definition, the action sends (g, x) to $\varphi(g)(x)$.

Example 1.1 (Tautological action of Sym X on X)**.** This is the obvious action, call it T, sending f, x to $f(x)$, where $f\colon X \to X$ is a bijection and $x \in X$. In this language, the action a of G on X is α^*T with the homomorphism α of the proposition – the pull-back under α of the tautological action.

Example 1.2 (Linearity)**.** The question of classifying all possible X with action of G is hopeless in such generality, but one should recall that, in first approximation, mathematics is linear. So we shall take our X to be a *vector space* over some ground *field*, and ask that the action of G be linear, as well, in other words, that it should preserve the vector space structure. Our interest is mostly confined to the case when the field of scalars is $\mathbb{C}$, although we shall occasionally mention how the picture changes when other fields are studied.

Definition 1.1. A linear representation ρ of G on a complex vector space V is a set-theoretic action on V which preserves the linear structure, i.e.,

$$\rho(g)(\mathbf{v}_1 + \mathbf{v}_2) = \rho(g)\mathbf{v}_1 + \rho(g)\mathbf{v}_2, \qquad\qquad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V,$$
$$\rho(g)(k\mathbf{v}) = k\rho(g)\mathbf{v} \qquad\qquad \text{for all } k \in \mathbb{C}, \mathbf{v} \in V.$$

Unless otherwise mentioned, a *representation* will mean a *finite-dimensional complex representation*.

Example 1.3 (The general linear group)**.** Let V be a complex vector space of dimension $n < \infty$. After choosing a basis, we can identify it with $\mathbb{C}^n$, although we shal lavoid doing so without good reason. Recall that the *endomorphism algebra* End V is the set of all linear maps (or *operators*) $L\colon V \to V$, with the natural addition of linear maps and the composition as multiplication. If V has been identified with $\mathbb{C}^n$, a linear map is uniquely representable by a matrix, and the addition of linear