MA 519: Homework 10

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Problem 10.1 (Handout 14, # 5)

Approximately find the probability of getting a total exceeding 3600 in 1000 rolls of a fair die.

Solution. Let X_k, $1 \leq k \leq 1000$, denote the roll of a fair die. Then, as we have surely shown before, the mean and variance of the X_k are $\mu = 3.5$ and $\sigma^2 = 2.917$, respectively. By the central limit theorem, we can approximate $P(\sum_{k=1}^{100} X_k \geq 3600)$ by

$$P\left(\sum_{k=1}^{1000} X_k \geq 3600\right) \approx \int_{3600}^{\infty} \frac{\mathrm{e}^{-(x-3500)^2/5833.333}}{\sqrt{2\pi} \cdot 54.006}\, dx$$
$$\approx 0.514.$$

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Problem 10.2 (Handout 14, # 6)

A basketball player has a history of converting 80% of his free throws. Find a normal approximation with a continuity correction of the probability that he will make between 18 and 22 throws out of 25 free throws.

Solution. Let X denote number of free shots (out of 25) the player has made. Since the outcome of the player's free shots is binary (the player can either score or not score the throw) $X \sim \mathrm{Bin}(25, 0.8)$. Therefore, by the de Moivre–Laplace central limit theorem with continuity correction, we have

$$\begin{aligned} P(18 \leq X \leq 22) &\approx \Phi\left(\frac{22.5 - 20}{\sqrt{25 \cdot 0.8 \cdot 0.2}}\right) - \Phi\left(\frac{17.5 - 20}{\sqrt{25 \cdot 0.8 \cdot 0.2}}\right) \\ &= \Phi(1.25) - \Phi(-1.25) \\ &\approx 0.789. \end{aligned}$$

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Problem 10.3 (Handout 14, # 7)

Suppose $X_1, \dots, X_n$ are independent $\mathcal{N}(0,1)$ variables. Find an approximation to the probability that $\sum_{k=1}^n X_k$ is larger than $\sum_{k=1}^n X_k^2$, when $n = 10, 20, 30$.

Solution. We will use the central limit theorem to approximate the probability

$$P\Big(\sum\nolimits_{k=1}^n (X_k^2 - X_k) < 0\Big).$$

But first, we need to find the mean and the variance of the random variables $Y_k := X_k^2 - X_k$. First note than since the Y_k are functions of independent random variables the Y_k are again independent with respect to each other.

Now let us calculate the mean and variance of Y_k. First, the mean of Y_k is

$$\begin{aligned} E(Y_k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x^2 - x) \mathrm{e}^{-x^2/2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \mathrm{e}^{-x^2/2}\, dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \mathrm{e}^{-x^2/2}\, dx \\ &= 1, \end{aligned}$$

and the variance is

$$\begin{aligned} \mathrm{Var}(Y_k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x^2 - x)^2 \mathrm{e}^{-x^2/2}\, dx - 1^2 \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x^4 - 2x^3 + x^2) \mathrm{e}^{-x^2/2}\, dx - 1^2 \\ &= 3 + 0 + 1 - 1 \\ &= 3. \end{aligned}$$

Therefore, by the central limit theorem, we have

$$p_n := P\Big(\sum\nolimits_{k=1}^n (X_k^2 - X_k) < 0\Big) = \frac{1}{\sqrt{2\pi}(3n)^{1/2}} \int_{-\infty}^{0} \mathrm{e}^{-(x-n)^2/6n}.$$

For $n = 10$, we have

$$p_{10} \approx 0.369.$$

For $n = 20$, we have

$$p_{20} \approx 0.369.$$

Lastly, for $n = 30$, we have

$$p_{30} \approx 0.369.$$

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Problem 10.4 (Handout 14, # 8)

(A Product Problem). Suppose $X_1, \dots, X_{30}$ are 30 independent variables, each distributed as $U[0,1]$. Find an approximation to the probability that their *geometric mean* exceeds 0.4; exceeds 0.5.

Solution. Write $Y_k := \ln X_k$ and $\bar{Y} := \frac{1}{30}\sum_{k=1}^{30} Y_k$, $1 \le k \le 30$. Then we can write the geometric mean of $X_1, \dots, X_{30}$ as

$$\sqrt[30]{\prod_{k=1}^{30} X_k} = \exp\left(\frac{1}{30}\sum_{k=1}^{30} Y_k\right).$$

First, let us find the mean and the variance of Y_k, $1 \le k \le 30$.

$$\begin{aligned} E(Y_k) &= \int_0^1 \ln x\, dx \\ &= -1, \end{aligned}$$

and the variance is

$$\begin{aligned} \mathrm{Var}(Y_k) &= \int_0^1 (\ln x)^2\, dx - (-1)^2 \\ &= 2 - 1 \\ &= 1. \end{aligned}$$

Therefore, by the central limit theorem for 0.4 we have

$$\begin{aligned} P(\mathrm{e}^{\bar{Y}} > 0.4) &= 1 - P(\mathrm{e}^{\bar{Y}} \le 0.4) \\ &= 1 - P(\bar{Y} \le -0.916) \\ &= 1 - P\left(\frac{\sqrt{30}\big(\bar{Y} - (-1)\big)}{1} \le \frac{\sqrt{30}(-0.916 - (-1))}{1}\right) \\ &\approx 1 - \Phi(0.458) \\ &\approx 0.323, \end{aligned}$$

and for 0.5 we have

$$\begin{aligned} P(\mathrm{e}^{\bar{Y}} > 0.5) &= 1 - P(\mathrm{e}^{\bar{Y}} \le 0.5) \\ &= 1 - P(S \le -0.693) \\ &= 1 - P\left(\frac{\sqrt{30}\big(\bar{Y} - (-1)\big)}{1} \le \frac{\sqrt{30}(-0.693 - (-1))}{1}\right) \\ &\approx 1 - \Phi(1.681) \\ &\approx 0.046. \end{aligned}$$

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Problem 10.5 (Handout 14, # 9)

(Comparing a Poisson Approximation and a Normal Approximation). Suppose 1.5% of residents of a town never read a newspaper. Compute the exact value, a Poisson approximation, and a normal approximation of the probability that at least one resident in a sample of 50 residents never reads a newspaper.

Solution. Using a Poisson approximation to the binomial distribution $X \sim \mathrm{Bin}(50, 0.015)$, we have

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &\approx 1 - \mathrm{e}^{-0.75} \\ &\approx 0.528. \end{aligned}$$

For the normal approximation, we first compute

$$\begin{aligned} P(X = 0) P(-0.5 \leq X < 0.5) \\ & P\left(\frac{-0.5 - 50 \cdot 0.015}{\sqrt{50 \cdot 0.015(1 - 0.015)}} \leq \frac{X - 50 \cdot 0.015}{\sqrt{50 \cdot 0.015(1 - 0.015)}} < \frac{0.5 - 50 \cdot 0.015}{\sqrt{50 \cdot 0.015(1 - 0.015)}}\right) \\ & \approx \Phi(-0.291) - \Phi(-1.454) \\ & \approx 0.313. \end{aligned}$$

Thus, the normal approximation gives a probability of

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &\approx 0.687 \end{aligned}$$

The exact probability is

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \binom{50}{0} 0.015^0 \cdot (1 - 0.015)^{50} \\ &\approx 0.530. \end{aligned}$$

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Problem 10.6 (Handout 14, # 10)

(Test Your Intuition). Suppose a fair coin is tossed 100 times. Which is more likely: you will get exactly 50 heads, or you will get more than 60 heads?

Solution. Our intuition would say that it is more likely to get exactly 50 heads than it is to get more than 60 heads. Let us approximate these probabilities using the central limit theorem.

The probability that we get exactly 50 heads is approximately

$$\begin{aligned} P(X = 50) &= P(49.5 \leq X < 50.5) \\ &\approx P\left(\frac{49.5 - 50}{\sqrt{100} \cdot 0.5} \leq \frac{X - 50}{\sqrt{100} \cdot 0.5} < \frac{50.5 - 50}{\sqrt{100} \cdot 0.5}\right) \\ &\approx \Phi(0.1) - \Phi(-0.1) \\ &\approx 0.080. \end{aligned}$$

Whereas, the probability that we get more than 60 heads is smaller still

$$\begin{aligned} P(X \geq 60) &\approx 1 - \Phi\left(\frac{60 - 50}{\sqrt{100} \cdot 0.5}\right) \\ &\approx 1 - \Phi(2) \\ &\approx 0.023. \end{aligned}$$

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Problem 10.7 (Handout 14, # 11)

Find the probability that among 10 000 random digits the digit 7 appears not more than 968 times.

Solution. Let X denote the appearance of 7 in 10 000 random digits. Then $X \sim \text{Bin}(10\,000, 0.1)$. By the de Moivre–Laplace central limit theorem, we have

$$\begin{aligned} P(X \geq 968) &\approx 1 - \Phi\left(\frac{968 - 10\,000 \cdot 0.1}{\sqrt{10\,000 \cdot 0.1(1-0.1)}}\right) \\ &\approx 1 - \Phi(-1.067) \\ &\approx 0.857. \end{aligned}$$

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Problem 10.8 (Handout 14, # 12)

Find a number k such that the probability is about 0.5 that the number of heads obtained in 1000 tossings of a coin will be between 490 and k.

Solution. Let X denote the number of heads obtained in 1000 tosses of a fair coin. Then $X \sim \mathrm{Bin}(0.5, 1000)$ so by the de Moivre–Laplace central limit theorem

$$\begin{aligned} 0.5 &\approx \Phi\left(\frac{k + 0.5 - 500}{0.5 \cdot \sqrt{1000}}\right) - \Phi\left(\frac{489.5 - 500}{0.5 \cdot \sqrt{1000}}\right) \\ &\approx \Phi\left(\frac{k - 499.5}{15.811}\right) - \Phi(-0.664) \\ &\approx \Phi\left(\frac{k - 499.5}{15.811}\right) - 0.253. \end{aligned}$$

Therefore, we must find k such that

$$\Phi\left(\frac{k - 499.5}{15.811}\right) \approx 0.753.$$

From the table of values of the CDF for the standard normal distribution,

$$\frac{k - 499.5}{15.811} \approx 0.69.$$

Thus,

$$k \approx 510.$$ ■

Problem 10.9 (Handout 14, # 13)

In 10 000 tossings, a coin fell heads 5400 times. Is it reasonable to assume that the coin is skew?

Solution. Let X denote the number of heads obtained in 10 000 tosses of a coin. By the de Moivre–Laplace central limit theorem, the probability of a fair coin falling heads 5400 times is approximately

$$\begin{aligned} P(X = 5400) &= P(5399.500 \leq X < 5400.500) \\ &\approx P\left(\frac{5399.5 - 5000}{\sqrt{10\,000} \cdot 0.5} \leq \frac{X - 5000}{\sqrt{10\,000} \cdot 0.5} < \frac{5400.5 - 5000}{\sqrt{10\,000} \cdot 0.5}\right) \\ &\approx \Phi(8.01) - \Phi(7.99) \\ &\approx 0. \end{aligned}$$

Since the probability of this event happening is nearly zero, we expect the coin is skewed. ■

Problem 10.10 (Handout 14, # 14)

Interpret in plain words the statement of the problem: *(Normal approximation to the Poisson distribution).* Using Stirling's formula, show that, if $\lambda \to \infty$, then for fixed $\alpha < \beta$

$$\sum_{\lambda+\alpha\sqrt{\lambda}<k<\lambda+\beta\sqrt{\lambda}} p(k;\lambda) \longrightarrow \Phi(\beta) - \Phi(\alpha).$$

Solution. The problem is saying that if $X \sim \text{Poi}(\lambda)$ for very large λ we can approximate the probability

$$P\Big(\lambda + \alpha\sqrt{\lambda} < X < \lambda + \beta\sqrt{\lambda}\Big) \approx \Phi(\beta) - \Phi(\lambda).$$

Let us prove this ■

Problem 10.11 (Handout 14, # 15)

Give a proof that as $x \to \infty$,

$$1 - \Phi(x) \asymp \frac{\varphi(x)}{x}.$$

Remark: This gives the exact rate at which the standard normal right tail area goes to zero. It is even faster than the rate at which the standard normal density goes to zero, because of the extra x in the denominator.

Solution. We show that the limit of the ratios

$$\frac{1-\Phi(x)}{\varphi(x)/x} = \frac{x - x\Phi(x)}{\varphi(x)}$$

tends to 1 as x tends to ∞. By l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{1-\Phi(x)}{\varphi(x)/x} &= \lim_{x\to\infty} \frac{-\varphi(x)}{\varphi'(x)x^{-1} - \varphi(x)x^{-2}} \\
&= \lim_{x\to\infty} \frac{\varphi(x)}{\varphi(x)x^{-2} - \varphi'(x)x^{-1}} \\
&= \lim_{x\to\infty} \frac{\mathrm{e}^{-x^2/2}}{x^{-2}\mathrm{e}^{-x^2/2} + \mathrm{e}^{-x^2/2}} \\
&= \lim_{x\to\infty} \frac{1}{x^{-2}+1} \\
&= \lim_{x\to\infty} \frac{x^2}{x^2+1}
\end{aligned}$$

applying l'Hôpital's rule again gives us

$$\begin{aligned}
&= \lim_{x\to\infty} \frac{2x}{2x} \\
&= 1
\end{aligned}$$

as was to be shown. ∎