MA 519: Homework 4

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Problem 4.1 (Handout 5, # 2)

In an urn, there are 12 balls. 4 of these are white. Three players: A, B, and C, take turns drawing a ball from the urn, in the alphabetical order. The first player to draw a white ball is the winner. Find the respective winning probabilities: assume that at each trial, the ball drawn in the trial before is put back into the urn (i.e., selection *with replacement*).

Solution. ∎

Problem 4.2 (Handout 5, # 8)

Consider n families with 4 children each. How large must n be to have a 90% probability that at least 3 of the n families are all girl families?

Solution. ∎

Problem 4.3 (Handout 5, # 10)

(*Yahtzee*). In Yahtzee, five fair dice are rolled. Find the probability of getting a Full House, which is three rolls of one number and two rolls of another, in Yahtzee.

Solution. ∎

Problem 4.4 (Handout 5, # 12)

The probability that a coin will show all heads or all tails when tossed four times is 0.25. What is the probability that it will show two heads and two tails?

Solution. ■

Problem 4.5 (Handout 5, # 13)

Let the events $A_1, A_2, \ldots, A_n$ be independent and $P(A_k) = p_k$. Find the probability p that none of the events occurs.

Solution. ■

Problem 4.6 (Handout 6, # 5)

Suppose a fair die is rolled twice and suppose X is the absolute value of the difference of the two rolls. Find the PMF and the CDF of X and plot the CDF. Find a median of X; is the median unique?

Solution. ■

Problem 4.7 (Handout 6, # 7)

Find a discrete random variable X such that $\mathrm{Ex}(X) = \mathrm{Ex}(X^3) = 0$; $\mathrm{Ex}(X^2) = \mathrm{Ex}(X^4) = 1$.

Solution. Set $\Omega = \{0, 1\}$ and define a random variable $X \colon \Omega \to \mathbb{R}$ by $X(0) = -1$, $X(1) = 1$ as well as a probability $P(0) = P(1) = 1/2$. Then

$$\mathrm{Ex}[X] = -1(1/2) + 1(1/2) = 0 = (-1)^3(1/2) + 1^3(1/2) = \mathrm{Ex}[X^3],$$

whereas

$$\mathrm{Ex}[X^2] = (-1)^2(1/2) + 1^2(1/2) = 1 = (-1)^4(1/2) + 1^4(1/2) = \mathrm{Ex}[X^4],$$

as desired. ∎

Problem 4.8 (Handout 6, # 9)

(*Runs*). Suppose a fair die is rolled n times. By using the indicator variable method, find the expected number of times that a six is followed by at least two other sixes. Now compute the value when $n = 100$.

Solution. Let Ω denote the sample space and A denote the event that in a sequence of n rolls, at least three consecutive sixes come up. Define a random variable $X\colon \Omega \to \mathbb{R}$ by $X(x) = \chi_A(x)$, that is, $X(x) = 1$ if $x \in A$ and $X(x) = 1$ otherwise. ■

Problem 4.9 (Handout 6, # 10)

(*Birthdays*). For a group of n people find the expected number of days of the year which are birthdays of exactly k people. (Assume 365 days and that all arrangements are equally probable.)

Solution. Let Ω denote the sample space and let A_k denote the event that exactly k people have the same birthday. We can write A_k as the intersection of events $A_{n,k}$ ■

Problem 4.10 (Handout 6, # 11)

(*Continuation*). Find the expected number of multiple birthdays. How large should n be to make this expectation exceed 1?

Solution. ■

Problem 4.11 (Handout 6, # 12)

(*The blood-testing problem*). A large number, N, of people are subject to a blood test. This can be administered in two ways, (i) Each person can be tested separately. In this case N tests are required, (ii) The blood samples of k people can be pooled and analyzed together. If the test is negative, this one test suffices for the k people. If the test is positive, each of the k persons must be tested separately, and in all $k + 1$ tests are required for the k people. Assume the probability p that the test is positive is the same for all people and that people are stochastically independent.

(b) What is the expected value of the number, X, of tests necessary under plan (ii)?
(c) Find an equation for the value of k which will minimize the expected number of tests under the second plan. (Do not try numerical solutions.)

Solution. ■

Problem 4.12 (Handout 6, # 13)

(*Sample structure*). A population consists of r (classes whose sizes are in the proportion $p_1 : p_2 : \cdots : p_r$. A random sample of size n is taken with replacement. Find the expected number of classes not represented in the sample.

Solution. ■