Purdue University
MA 519: Introduction to Differential Equations
Midterm Examination, Fall 2015

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- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, with no electronic device**. One two-sided A11 formula sheet is allowed.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- You can use both sides of the papers to write your answers. But please indicate so if you do.

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Your Name:________________________________

Suppose n balls are distributed at random into r boxes in such a way that each ball chooses a box independently of each other. Let S be the number of *empty boxes*. Compute ES and $Var(S)$.

(Hint: Consider the random variables X_i (for $i = 1, 2, \ldots, r$) which equals 1 if the i-th box is empty and 0 otherwise. Related S and the X_i's.)

$$S = X_1 + X_2 + \cdots + X_r$$

$$ES = E(X_1 + \cdots + X_r) = EX_1 + EX_2 + \cdots + EX_r$$

$$P(X_1 = 1) = \frac{(r-1)^n}{r^n}, \quad P(X_1 = 0) = 1 - \frac{(r-1)^n}{r^n}$$

Hence $EX_1 = 1 \times P(X_1 = 1) + 0 \times P(X_1 = 0)$

$$= \left(\frac{r-1}{r}\right)^n$$

$$\boxed{ES = r \times \left(\frac{r-1}{r}\right)^n}$$

$$Var(S) = E(S^2) - (ES)^2$$

$$E(S^2) = E\left[(X_1 + \cdots + X_r)^2\right]$$

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$$= E\left[\sum_i X_i^2 + \sum_{i \neq j} X_i X_j\right]$$

$$= \sum_{i=1}^{r} E(X_i^2) + \sum_{i \neq j} E[X_i X_j]$$

$$= \sum_{i=1}^{r} E(X_i) + \sum_{i \neq j} E[X_i X_j]$$

takes values 0 or 1.

$$= r\left(\frac{r-1}{r}\right)^n + \sum_{i \neq j} P(X_i = 1, X_j = 1)$$

$$= r\left(\frac{r-1}{r}\right)^n + \sum_{i \neq j} \left(\frac{r-2}{r}\right)^n$$

$$= r\left(\frac{r-1}{r}\right)^n + \binom{r}{2}\left(\frac{r-2}{r}\right)^n$$

Hence $\boxed{Var(S) = r\left(\frac{r-1}{r}\right)^n + \binom{r}{2}\left(\frac{r-2}{r}\right)^n - r^2\left(\frac{r-1}{r}\right)^{2n}}$

2. Your Name:______________________________

Suppose n balls are distributed in n boxes in such a way that each ball chooses a box independently of each other.

a) What is the probability that Box #1 is empty?

b) What is the probability that only Box #1 is empty?

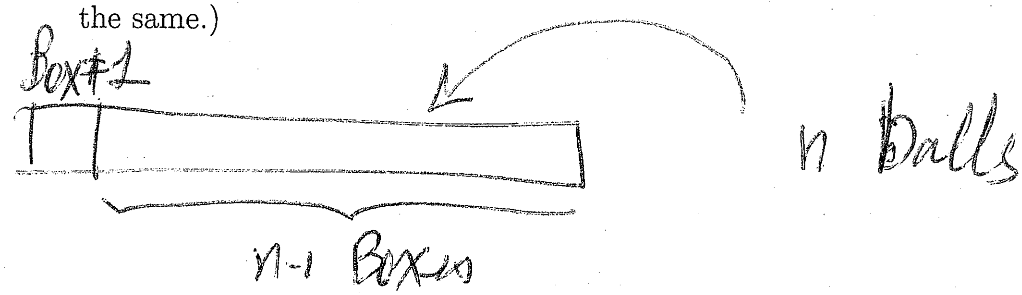
c) What is the probability that only one box is empty?

d) Given that Box #1 is empty, what is the probability that only one box is empty?

e) Given that only one box is empty, what is the probability that Box #1 is empty?

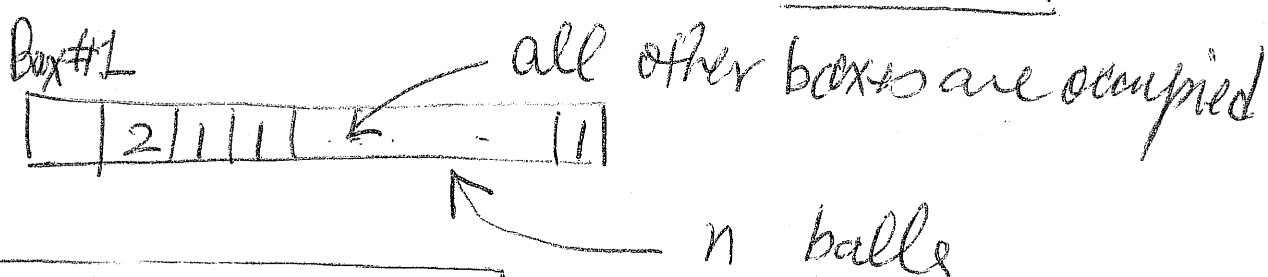
(Hint: make use of the fact that the number of balls and boxes are the same.)

(a)



$$P(\text{Box\#1 is empty}) = \frac{(n-1)^n}{n^n} = \boxed{\left(\frac{n-1}{n}\right)^n}$$

(b)



$$\boxed{P((b)) = \frac{\binom{n}{2}(n-1)!}{n^n}}$$

[$\binom{n}{2}$: choose 2 balls out of n to form a pair.
$(n-1)!$: $(n-1)$ boxes to put in one by one.]

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[Note: what is wrong with the following argument:
Put the $\underbrace{\text{balls}}_{\text{first } (n-1)}$ one by one in Boxes #2, #3, ..., #n,
& ~~then~~ the last ball in any of the $(n-1)$ Boxes.
Then the numerator $= (n-1)! \times (n-1)$]

(c) By symmetry, any box can be empty as as likely as Box #2. Hence

$$P(c) = n \times P(b) = \frac{n \binom{n}{2} (n-1)!}{n^n} = \boxed{\frac{\binom{n}{2}(n-1)!}{n^{n-1}}}$$

(d) Box#1

0	2	1	1

Method 1

$$\boxed{P(d) = \frac{\binom{n}{2}(n-1)!}{(n-1)^n}}$$

Note the new denominator, due to conditioning.

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$\boxed{\text{Method 2}}$ $P(d) = P(\text{only 1 box is empty} \mid \text{Box\#1 is empty})$

$$= \frac{P(\text{only 1 box is empty} \cap \text{Box\#1 is empty})}{P(\text{Box\#1 is empty})}$$

$$= \frac{P(\text{only Box\#1 is empty})}{P(\text{Box\#1 is empty})} = \frac{P(b)}{P(a)}$$

$$= \frac{\binom{n}{2}\frac{(n-1)!}{\cancel{n^n}}}{\left(\frac{n-1}{\cancel{n}}\right)^n} = \boxed{\binom{n}{2}\frac{(n-1)!}{(n-1)^n}}$$

(e) By symmetry again, any Box is just as likely as any other. Hence

$\boxed{\text{Method 1}}$

$$\boxed{P(e) = \frac{1}{n}}$$

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Method 2 for (e)

P(Box#1 is empty | Only 1 box is empty)

$$= \frac{P(\text{Box\#1 is empty} \cap \text{only 1 box is empty})}{P(\text{Only 1 box is empty})}$$

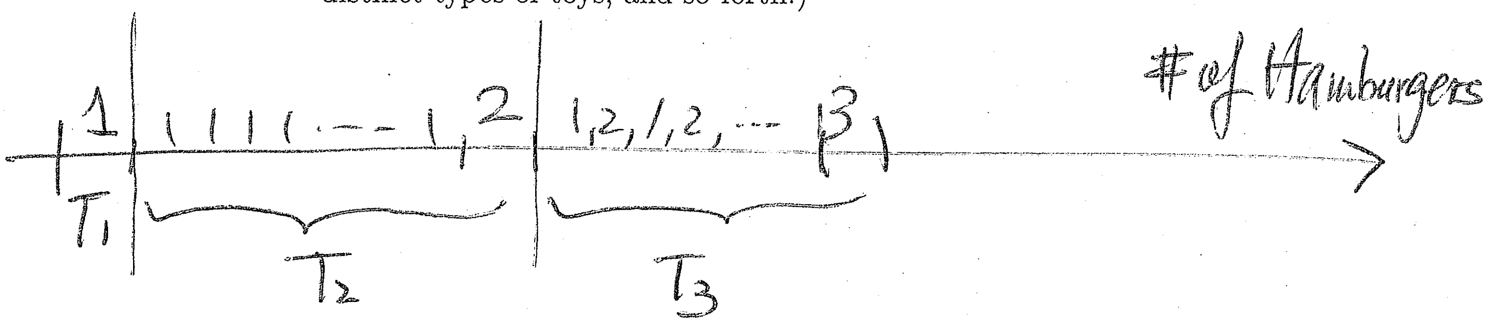
$$= \frac{P(\text{Only Box\#1 is empty})}{P(\text{Only 1 box is empty})} = \frac{P(b)}{P(c)} = \boxed{\frac{1}{n}}$$

(Coupon's Problem)

3. Your Name:______________________________

McDonald's newest promotion is putting a toy inside every one of its hamburger. Suppose there are N distinct types of toys and each of them is equally likely to be put inside any of the hamburger. What is the expected value and variance of the number of hamburgers you need to order (or eat) before you have a complete set of the N toys.

(Hint: consider the number of hamburgers you need to order (or eat) in between getting one and two dinstinct types of toys, two and three distinct types of toys, and so forth.)



$T_1 = 1$

$T_2 = \text{Geometric}, \quad P(\text{Success}) \overset{(p)}{=} \frac{N-1}{N}, \quad P(\text{failure}) \overset{(q)}{=} \frac{1}{N}$

$T_3 = \text{Geometric}, \quad P(\text{Success}) = \frac{N-2}{N}, \quad P(\text{failure}) = \frac{2}{N}$

$\vdots$

$T_N = \text{Geometric}, \quad P(\text{Success}) = \frac{1}{N}, \quad P(\text{failure}) = \frac{N-1}{N}$

$T = \text{total \# of Hamburgers} = T_1 + T_2 + \cdots + T_N$

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$$\boxed{\begin{aligned} E(\text{Geom}) &= \frac{1}{p} \\ \text{Var}(\text{Geom}) &= \frac{q}{p^2} \end{aligned}}$$

$$ET = E(T_1 + T_2 + \cdots + T_N)$$

$$= 1 + \frac{N}{N-1} + \frac{N}{N-2} + \cdots + \frac{N}{1}$$

$$= N\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right]$$

Note: $T_1, T_2, \ldots,$ are independent.

Hence, $\text{Var}(T) = \text{Var}(T_1) + \text{Var}(T_2) + \cdots + \text{Var}(T_N)$

$$= 0 + \frac{(1/N)}{\left(\frac{N-1}{N}\right)^2} + \frac{2/N}{\left(\frac{N-2}{N}\right)^2} + \cdots + \frac{N-1/N}{(1/N)^2}$$

$$= N\left[\frac{1}{(N-1)^2} + \frac{2}{(N-2)^2} + \cdots + \frac{N-1}{1^2}\right]$$

4. Your Name:____________________________

Let X and Y be two independent geometric random variables with parameter p.

a) Find the probability distribution function of $\min(X, Y)$;

b) Find the probability distribution function of $\max(X, Y)$;

c) Find the probability distribution function of $X + Y$;

d) Find $P(X = i \mid X + Y = j)$ for $i = 1, 2, \ldots, j - 1$.

$$\left[P(X=i) = P(Y=i) = pq^{i-1}, \quad i = 1, 2, \cdots \right]$$

(a) $Z = \min(X, Y)$

$$P(Z=i) = P(\min(X,Y)=i)$$

$$= P(X=i, Y>i) + P(X=i=Y) + P(Y=i, X>i)$$

$$= 2P(X=i, Y>i) + P(X=i, \cancel{Y}=i)$$

$$= 2P(X=i)P(Y>i) + P(X=i)^2$$

$$= 2pq^{i-1}\left[\sum_{j=i+1}^{\infty} P(Y=j)\right] + \left[pq^{i-1}\right]^2$$

$$= 2pq^{i-1}\left[\sum_{j=i+1}^{\infty} pq^{j-1}\right] + p^2 q^{2i-2}$$

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$$= 2p^2q^{i-1}\left[q^i + q^{i+1} + \cdots\right] + p^2q^{2i-2}$$

$$= 2p^2q^{2i-1}\left[\underbrace{1+q+q^2+\cdots}_{\frac{1}{1-q}=\frac{1}{p}}\right] + p^2q^{2i-2}$$

$$= \boxed{2pq^{2i-1} + p^2q^{2i-2}} = (2pq+p^2)(q^2)^{i-1}$$

$$= p(2q+p)(q^2)^{i-1}$$

[Note: so happens to be geom. r.v. with parameter $p(2-p)$] $\rightarrow$ $= p(2-p)(q^2)^{i-1}$

(b) $P(\max(X,Y)=i) = P(X=i, Y<i) + P(X=i, Y=i) + P(X<i, Y=i)$

$$= 2P(X=i, Y<i) + P(X=i, Y=i)$$

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$$= 2P(X=i)\,P(Y<i) + P(X=i)^2$$

$$= 2pq^{i-1}\left[\sum_{j=1}^{i-1} pq^{j-1}\right] + \left[pq^{i-1}\right]^2$$

$$= 2p^2q^{i-1}\left[1+q+\cdots+q^{i-2}\right] + p^2q^{2i-2}$$

$$= 2p^2q^{i-1}\left[\frac{1-q^{i-1}}{1-q}\right] + p^2q^{2i-2}$$

$$= 2pq^{i-1} - 2pq^{2i-2} + p^2q^{2i-2}$$

$$\boxed{= 2pq^{i-1} - p(2-p)q^{2i-2}}$$

(c) [Note: $X+Y$ is negative Binomial]

$$P(X+Y=i) = \sum_{j=1}^{i-1} P(X=j, Y=i-j)$$

$(i \geq 2)$

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$$= \sum_{j=1}^{i-1} P(X=j)P(Y=i-j)$$

$$= \sum_{j=1}^{i-1} pq^{j-1} p q^{i-j-1} = \sum_{j=1}^{i-1} p^2 q^{i-2}$$

$$= (i-1)p^2 q^{i-2} \quad \left(= \binom{i-1}{1} p^2 q^{i-2} \right)$$

from neg. Bin dist.

(d) $$P(X=i \mid X+Y=j) = \frac{P(X=i, X+Y=j)}{P(X+Y=j)}$$

$$= \frac{P(X=i, Y=j-i)}{P(X+Y=j)} = \frac{P(X=i)P(Y=j-i)}{P(X+Y=j)}$$

$$= \frac{\cancel{pq^{i-1} p q^{j-i-1}}}{(j-1)\cancel{p^2 q^{j-2}}} = \frac{1}{j-1}$$

(uniformly distributed among $1, 2, \cdots, j-1$).

5. Your Name:________________________

Suppose the events E, F, G are independent, in other words,

$$\begin{aligned} P(E \cap F) &= P(E)P(F), \\ P(F \cap G) &= P(F)P(G), \\ P(G \cap E) &= P(G)P(E), \\ P(E \cap F \cap G) &= P(E)P(F)P(G). \end{aligned}$$

Using the above definition, show that the following events are independent:

a) E and F^c;
b) E and $F \cap G^c$;
c) E and $F^c \cap G^c$;
d) E and $F \cup G$;
e) E and $F \cup G^c$.

$$\boxed{\underline{\text{Note}}:\ P(A \cap B^c) = P(A) - P(A \cap B)}$$

(a) $P(E \cap F^c) = P(E) - P(E \cap F) = P(E) - P(E)P(F)$

$= P(E)[1 - P(F)] = P(E)P(F^c)$

(b) $P(E \cap (F \cap G^c)) = P(E \cap F \cap G^c)$

$= P(E \cap F) - P(E \cap F \cap G) = P(E)P(F) - P(E)P(F)P(G)$

$\cancel{= P(E \cap F) - P(E \cap F)P(G) = P(E \cap F)P(G^c)}$

$= P(E)[P(F) - P(F \cap G)]$

$= P(E)P(F \cap G^c)$

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(c) $P[E \cap (F^c \cap G^c)]$

$= P[E \cap (F \cup G)^c] = P(E) - P(E \cap (F \cup G))$

(by (d)) $= P(E) - P(E)P(F \cup G) = P(E)P((F \cup G)^c)$

$= P(E)P(F^c \cap G^c)$

(d) $P(E \cap (F \cup G)) = P(E \cap F \cup E \cap G)$

$= \cancel{P(E \cap F \cap E \cap G)} + P(E \cap F) + P(E \cap G) - P(E \cap F \cap G)$

$= P(E)[P(F) + P(G) - P(F \cap G)] = P(E)P(F \cup G)$

(e) $P(E \cap (F \cup G^c)) = P(E \cap (F^c \cap G)^c)$

$= P(E) - P(E \cap F^c \cap G) = P(E) - P(E)P(F^c \cap G)$

by (b), & symmetry

$= P(E)P((F^c \cap G)^c) = P(E)P(F \cup G^c)$