MA 523: Homework 8

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Problem 8.1

Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t)$$

where

$$\Phi(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}$$

is positive for $|x| < 1$, $t > 0$.
(*Hint:* Show that u satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{cases} u = 0 & \text{on } \{\,|x| = 1\,\} \times \{\,t \geq 0\,\}, \\ u = \delta_0 & \text{on } \{\,|x| \leq 1\,\} \times \{\,t = 0\,\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{\,|x| \leq 1\,\} \times \{\,\varepsilon \leq t \leq T\,\}$ for small $\varepsilon > 0$ and large $T > 0$ pass to the limit as $\varepsilon \to 0+$ and $T \to \infty$.)

Solution. Taking the hint, let us verify that $u_t = u_{xx}$, for $t > 0$. By direct computation, we have

$$\begin{aligned} \Phi_x(x,t) &= \frac{\partial}{\partial x}\left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \\ &= -\frac{xe^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}, \end{aligned}$$

$$\begin{aligned} \Phi_{xx}(x,t) &= \frac{\partial}{\partial x}\left(-\frac{xe^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}}\right) \\ &= \frac{x^2e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}}, \end{aligned}$$

and

$$\begin{aligned} \Phi_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}\right) \\ &= \frac{x^2e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)e^{-\frac{x^2}{4t}}}{4\sqrt{4\pi}t^{\frac{5}{2}}}. \end{aligned}$$

Since $\Phi_t = \Phi_{xx}$ it follows that $u_t = u_{xx}$ (assuming uniform convergence of u).

Next we show that $u = 0$ on $\{\, |x| = 1 \,\} \times \{\, t \geq 0 \,\}$ and $u = \delta_0$ on $\{\, |x| = 1 \,\} \times \{\, t = 0 \,\}$. To show $u = 0$ fix a $t \geq 0$ and, after relabeling if necessary, assume that $x = 1$ which gives us

$$\begin{aligned} u(1,t) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{-\frac{(1-2k)^2}{4t}}}{\sqrt{4\pi t}} \\ &= \frac{1}{\sqrt{4\pi t}} \Big(\cdots - \mathrm{e}^{-\frac{9}{4t}} + \mathrm{e}^{-\frac{1}{4t}} - \mathrm{e}^{-\frac{1}{4t}} + \mathrm{e}^{-\frac{9}{4t}} + \cdots \Big) \\ &= 0. \end{aligned}$$

Similarly for $u(-1,t) = 0$.

For $u(|x| \leq 1, 0)$, we have a

$$\begin{aligned} u(|x| \leq 1, 0) &= \sum_{k=-\infty}^{\infty} (-1)^k \lim_{t \to 0+} \left[\mathrm{e}^{-\frac{(x-2k)^2}{4t}} / \sqrt{4\pi t} \right] \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \delta_0(x - 2k) \\ &= \delta_0(x) \end{aligned}$$

since $|x| \leq 1$ and values δ_0 is zero for values $x - 2k$ outside of the interval $[-1, 1]$.

At last we show that u is positive for $|x| < 1$, $t > 0$. Seeking a contradiction, suppose u is negative on some point (x_0, t_0) in $\{\, |x| < 1 \,\} \times \{\, \varepsilon \leq t \leq T \,\}$. Then by the minimum principle, u achieves its minimum somewhere on the bottom boundary $\{\, |x| = 1 \,\} \times \{\, t = \varepsilon \,\}$. Therefore, there exists a sequence $(x_n, t_n+) \to (x, 0)$, where $|x_n|, |x| < 1$, such that $u(x, 0) < 0$. However, we have shown above that $u(x, 0) = \delta_0(x)$ for $|x| < 1$; i.e., either $u(x, 0) = 0$ or $u(x, 0) = +\infty$. This is a contradiction. Therefore, it must be the case that $u \geq 0$ for $|x| < 1$, $t > 0$. ∎

Problem 8.2 (Tikhonov's example)

Let

$$g(t) := \begin{cases} \mathrm{e}^{-t^{-2}} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then $g \in C^\infty(\mathbb{R})$ and we define

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that $u(x,t)$ solves the heat equation in $\mathbb{R} \times (0, \infty)$ with the initial condition $u(x, 0) = 0$, $x \in \mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

Solution. Let u be as above. Then

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}\right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} \\ &= \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}, \end{aligned}$$

and

$$\begin{aligned} u_x(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}\right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2k x^{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}, \end{aligned}$$

$$\begin{aligned} u_{xx}(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}\right) \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} (2k-1) x^{2k-2} + \tfrac{\partial}{\partial x} g^{(0)}(t) \\ &= \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}. \end{aligned}$$

Thus, $u_t - \Delta u = 0$; i.e., u solves the heat equation. As this example shows, unless some assumptions on u such as subexponential (cf. [E §2.3], Theorem 7) growth is assumed. ∎

Problem 8.3

Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax) e^{-x^2}\, dx, \qquad (a > 0).$$

(*Hint:* Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.)

Solution. By separation of variables,

$$u(x,t) = \cos(ax) e^{-a^2 t}$$

is a solution to the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = \cos(ax) & \text{on } \mathbb{R} \times \{\, t = 0 \,\}. \end{cases}$$

However, the convolution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \cos(ay) e^{-\frac{|x-y|^2}{4t}}\, dy$$

is also a solution to the Cauchy problem. Now note that

$$\begin{aligned} \int_{-\infty}^{\infty} \cos(ay) e^{-y^2}\, dy &= \sqrt{\pi} \cdot u(0, \tfrac{1}{4}) \\ &= \sqrt{\pi} e^{-\frac{a^2}{4}}. \end{aligned}$$

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