MA 544: Homework 11

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Problem 11.1 (Wheeden & Zygmund §7, Ex. 11)

Prove the following result concerning changes of variable. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt.$$

(Consider the case when f is the characteristic function of an interval, an open set, etc.)

Proof. Recall that, by Theorem 5.21, f is integrable (or in L^1) on $[\alpha, \beta]$ if and only if $|f|$ is integrable on $[\alpha, \beta]$. Therefore, it suffices to prove the result for the case $f \geq 0$. We split the proof of the result into a series of claims and then proceed to show the more general result.

Claim 1. *Let g be as above and G be an open subset of $[\alpha, \beta]$. Then*

$$|g(G)| = \int_G g'(t)dt.$$

Proof of claim 1. Let G be an open subset of (a, b) then, by Theorem 1.10, G can be written as the countable union of disjoint open intervals $\{I_k\}$. By Theorem 5.7, since g' is nonnegative and measurable and $\int_G g'$ is finite (in particular, it is bounded above by $\int_a^b g'$), we have

$$\int_G g'(t)dt = \sum_k \int_{I_k} g'(t)dt. \tag{11.1}$$

But by Theorem 7.27, since g is absolutely continuous on $[\alpha, \beta]$, g is b.v. on $[\alpha, \beta]$ so by Theorem 7.30

$$|g(I_k)| = g(\beta_k) - g(\alpha_k) = V[g; \alpha_k, \beta_k] = \int_{\alpha_k}^{\beta_k} g'(t)dt$$

where α_k is the left-most endpoint of I_k and β_k the right-most. By Equation (11.1), on the right-hand side, we have

$$\int_{I_k} g'(t)dt = |g(I_k)|$$

so, by Theorem 3.23, we have

$$\int_G g'(t)dt = \sum_k |g(I_k)| = |g(\bigcup_k I_k)| = |g(G)| \tag{11.2}$$

as desired. ♣

Claim 2. *Let g be as above and E be a G_δ-subset of $[\alpha, \beta]$. Then*

$$|g(E)| = \int_E g'(t)dt.$$

Proof of claim 2. Suppose E is a G_δ-set, then E is the countable intersection of open subsets $\{G_k\}$ of $[\alpha, \beta]$. We may choose G_k's such that $G_k \searrow E$ (for example, taking our original collection of open subsets $\{G_k\}$ and taking the finite intersection $\bigcap_{j=1}^k G_j$). Hence, we have $\chi_{G_k} \searrow \chi_E$ and consequently $\chi_{G_k} g' \searrow \chi_E g'$. Thus, we have

$$\lim_{k\to\infty} \int_E \chi_{G_k} g'(t)dt = \lim_{k\to\infty} |g(G_k)| = |g(E)| \tag{11.3}$$

by Claim 1 and Theorem 3.10. Thus, by the monotone convergence theorem together with Equation (11.3), we have

$$|g(E)| = \lim_{k\to\infty} \int_E \chi_{G_k} g'(t)dt = \int_E \chi_{G_k} g'(t)dt \tag{11.4}$$

as desired. ♣

Claim 3.

■

Problem 11.2 (Wheeden & Zygmund §7, Ex. 15)

Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\varphi(x) = \int_a^x f(t)dt + \varphi(a)$ in (a,b) and f is monotone increasing, then φ is convex in (a,b). (Use Exercise 14.)

Proof. We will assume the result in Exercise 14. By Exercise 14, since f is monotone increasing, f is b.v. on $[a,b]$ so f is bounded a.e. on (a,b) by a previous exercise. Thus, $f \in L(a,b)$ so ■

Problem 11.3 (Wheeden & Zygmund §5, Ex. 8)

Prove (5.49).

Proof. Recall the content of equation 5.49: For f measurable, we have

$$\omega(\alpha) \leq \frac{1}{\alpha^p} \int_{\{f>\alpha\}} f^p, \quad \alpha > 0. \tag{11.5}$$

■

Problem 11.4 (Wheeden & Zygmund §5, Ex. 11)

For which p does $1/x \in L^p(0,1)$? $L^p(1,\infty)$? $L^p(0,\infty)$?

Proof. ■

PROBLEM 11.5 (WHEEDEN & ZYGMUND §5, EX. 12)

Give an example of a bounded continuous f on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0, \infty)$ for any $p > 0$.

Proof. ■

Problem 11.6 (Wheeden & Zygmund §5, Ex. 17)

If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < \infty$. (Use Exercise 16.)

Proof. ■