MA166: Recitation 5 Notes

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1. Fun with Euler's Formula

In class today, I talked a little bit about Euler's formula

$$e^{i\theta} = \cos\theta + i\theta, \tag{1}$$

where $i = \sqrt{-1}$. Later in the semester (once we define Taylor series) you will learn why equation (1) is true, but for now we leave it as a black box. We'll use equation (1) to derive some trigonometric identities and compute some integrals.

First, let us derive some results about cosine and sine in terms of the complex exponential $e^{i\theta}$. Associated with every complex number $z = x + iy$ is a (usually complex) number called the *real part* $\Re(z) = x$ and a number called the *imaginary part* $\Im(z) = y$ and a number $\bar{z} = x - iy$ called the *conjugate* of z.

Naturally, if we think of $z = x + iy$ as a vector in the complex plane $\mathbb{C}$, conjugation of z with itself will give you the magnitude of z squared

$$\begin{aligned} z\bar{z} &= (x+iy)(x-iy) \\ &= x^2 - ixy + ixy - i^2y^2 \\ &= x^2 - i^2y^2 \\ &= x^2 - \left(\sqrt{-1}\right)^2 y^2 \\ &= x^2 - (-1)y^2 \\ &= x^2 + y^2. \end{aligned} \tag{2}$$

So $|z| = \sqrt{z\bar{z}}$.

Naturally, the conjugate of $\bar{z} = x - iy$ is again z since

$$\bar{\bar{z}} = \overline{x - iy} = x + iy = z. \tag{3}$$

Moreover, for any complex number z the conjugate of z, $\bar{z}$, has the property that

$$\begin{aligned} z + \bar{z} &= x + iy + x - iy \\ &= 2x \\ &= 2\Re(z) \end{aligned} \qquad\qquad \begin{aligned} z - \bar{z} &= x + iy - (x - iy) \\ &= 2iy \\ &= 2i\Im(z). \end{aligned}$$

Hence, we have a nice formula for expressing $\Re(z)$ and $\Im(z)$ in terms of z alone with no reference to x, y in $z = x + iy$

$$\Re(z) = \frac{z+\bar{z}}{2} \qquad\qquad \Im(z) = \frac{z-\bar{z}}{2i} = \frac{z-\bar{z}}{2i}\frac{-i}{-i} = -\frac{z-\bar{z}}{2}i \tag{4}$$

Date: February 11, 2016.

Now, we may ask, given the complex function $f(z)$ which takes a complex number $z = x + iy$, what is $\Re f(z)$ and $\Im f(z)$? Well, $f(z)$ is just another complex number so equation (4) works and we have

$$\Re f(z) = \frac{f(z) + \bar{f}(z)}{2} \qquad \Im f(z) = -\frac{f(z) - \bar{f}(z)}{2} i$$

Why go through all of this trouble? Well, we want to be able to express sine and cosine in terms of the complex exponential $e^{i\theta}$ so we can do useful things with it like simplify all of our integral calculations.

1.1. Complex sin and cos.

Using equation (4) together with Euler's formula, equation (1), we have

$$\begin{aligned} \Re\left(e^{i\theta}\right) &= \frac{e^{i\theta} + e^{-i\theta}}{2} & \qquad \Im\left(e^{i\theta}\right) &= -\frac{e^{i\theta} - e^{-i\theta}}{2} i \\ \Re(\cos\theta + i\sin\theta) &= \cos\theta & \Im(\cos\theta + i\sin\theta) &= \sin\theta \end{aligned}$$

so we have the amazing identity

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = -\frac{e^{i\theta} + e^{-i\theta}}{2} i. \tag{5}$$

Now, let's verify some trigonometric identities using equation (5).

Examples 1 (Sum of angles formula)**.** Consider the sum of angles formula for either sine or cosine

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi \qquad \sin(\theta + \varphi) = \sin\theta\cos\varphi + \sin\varphi\cos\theta \tag{6}$$

for some angle $0 \le \theta, \varphi \le 2\pi$. Using equation (5) we can express $\sin(\theta + \varphi)$ and $\cos(\theta + \varphi)$ as the the sum of complex exponentials

$$\begin{aligned} \cos(\theta + \varphi) &= \frac{e^{i(\theta+\varphi)} + e^{-i(\theta+\varphi)}}{2} & \qquad \sin(\theta + \varphi) &= -\frac{e^{i(\theta+\varphi)} - e^{-i(\theta+\varphi)}}{2} i \\ &= \frac{e^{i\theta}e^{i\varphi} + e^{-i\theta}e^{-i\varphi}}{2} & &= -\frac{e^{i(\theta+\varphi)} - e^{-i(\theta+\varphi)}}{2} i \\ &= \end{aligned}$$

Examples 2 (Derivative of cos, sin)**.** Here is a fun one I bet you haven't seen yet. We know what the derivative of $e^{i\theta}$ is with respect to θ, right? It's just $ie^{i\theta}$. By Euler's formula, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,\theta}\left(e^{i\theta}\right) &= \frac{\mathrm{d}}{\mathrm{d}\,\theta}(\cos\theta + i\sin\theta) \\ ie^{i\theta} &= \frac{\mathrm{d}}{\mathrm{d}\,\theta}\cos\theta + i\frac{\mathrm{d}}{\mathrm{d}\,\theta}\sin\theta, \end{aligned}$$

but by using equation (1) on the left we have

$$\begin{aligned} i(\cos\theta + i\sin\theta) &= \frac{\mathrm{d}}{\mathrm{d}\,\theta}\cos\theta + i\frac{\mathrm{d}}{\mathrm{d}\,\theta}\sin\theta \\ i\cos\theta + i^2\sin\theta) &= \frac{\mathrm{d}}{\mathrm{d}\,\theta}\cos\theta + i\frac{\mathrm{d}}{\mathrm{d}\,\theta}\sin\theta \\ -\sin\theta + i\cos\theta &= \frac{\mathrm{d}}{\mathrm{d}\,\theta}\cos\theta + i\frac{\mathrm{d}}{\mathrm{d}\,\theta}\sin\theta \\ -\sin\theta + i\cos\theta &= \frac{\mathrm{d}}{\mathrm{d}\,\theta}\cos\theta + i\frac{\mathrm{d}}{\mathrm{d}\,\theta}\sin\theta \end{aligned}$$

Putting real with real and complex with complex, we see that
$$\frac{\mathrm{d}}{\mathrm{d}\,\theta}\cos\theta = -\sin\theta \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}\,\theta}\sin\theta = \cos\theta.$$
Nice, right?