TOPOLOGY HOMEWORK # 2
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P. 100 # 3, 6(bc), 7, 9, 13

Problem 3. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Solution: Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be defined as on p. 87.

Lemma 1. Let $A \subseteq X$, $B \subseteq Y$, then

$$\pi_1^{-1}(A) = A \times Y \text{ and } \pi_2^{-1}(B) = X \times B.$$

Proof.

$$\begin{aligned}\pi_1^{-1}(A) &= \{x \times y : \pi_1(x \times y) \in A\} \\ &= \{x \times y : x \in A\} \\ &= \{x \times y : x \in A, y \in Y\} \\ &= A \times Y\end{aligned}$$

$$\begin{aligned}\pi_2^{-1}(B) &= \{x \times y : \pi_2(x \times y) \in B\} \\ &= \{x \times y : y \in B\} \\ &= \{x \times y : x \in X, y \in B\} \\ &= X \times B\end{aligned}$$

$\square$

Now, suppose $A \subseteq X$ open and $B \subseteq Y$ open. We show that π_1 and π_2 are continuous. By lemma (1), we may write

$$\pi_1^{-1}(A) = A \times Y \text{ and } \pi_2^{-1}(B) = X \times B.$$

Since A is opened in X and Y is open in Y, we see that $A \times Y$ is open in $X \times Y$ by definition of the product topology on p. 86. Similarly, since X is open in X and B is open in Y, $X \times B$ is open in $X \times Y$. Thus, by definition of continuity on p. 102, we see that π_1 and π_2 are both continuous.

Suppose that A is closed in X and B is closed in Y. Then in particular $X \setminus A$ is open in X and $Y \setminus B$ is open Y. Thus, by the continuity of π_1 and π_2, we have that

$$\pi_1^{-1}(X \setminus A) = (X \setminus A) \times Y \text{ and } \pi_2^{-1}(Y \setminus B) = X \times (Y \setminus B) \tag{1}$$

which are both open.

Lemma 2. Suppose $A \subseteq X$ and $B \subseteq Y$. Then

$$(X \times Y) \setminus (A \times Y) = (X \setminus A) \times Y$$

and

$$(X \times Y) \setminus (X \times B) = X \times (Y \setminus B).$$

Proof. We only show

$$(X \times Y) \setminus (A \times Y) = (X \setminus A) \times Y$$

since

$$(X \times Y) \setminus (X \times B) = X \times (Y \setminus B)$$

is done similarly.

$$\begin{aligned}(X \times Y) \setminus (A \times Y) &= \{x \times y : x \times y \notin A \times Y\} \\ &= \{x \times y : x \notin A \text{ or } y \notin Y\} \\ &= \{x \times y : x \notin A\} \\ &= (X \setminus A) \times Y\end{aligned}$$

$\square$

Therefore, equation (1) becomes

$$\pi_1^{-1}(X \setminus A) = (X \setminus A) \times Y \overset{\text{lemma (2)}}{=} (X \times Y) \setminus (A \times Y)$$

$$\pi_2^{-1}(Y \setminus B) = X \times (Y \setminus B) \overset{\text{lemma (2)}}{=} (X \times Y) \setminus (X \times B).$$

Hence, we have that $(X \times Y) \setminus (A \times Y)$ and $(X \times Y) \setminus (X \times B)$ are both open in $X \times Y$ which means that $A \times Y$ and $X \times B$ are both closed in $X \times Y$.

Finally, we get that

$$(X \times B) \cap (A \times Y) \overset{(*)}{=} (X \cap A) \times (B \cap Y) = A \times B$$

which means that $A \times B$ is closed given that A and B are closed. We need only show $(*)$ to complete the proof.

To see this,

$$\begin{aligned}(X \times B) \cap (A \times Y) &= \{x \times y : x \in X, y \in B\} \cap \{x \times y : x \in A, y \in Y\} \\ &= \{x \times y : x \in A, y \in B\} \\ &= A \times B.\end{aligned}$$

Problem 6. Let A, B, and A_α denote subsets of a space X. Prove the following:

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Solution: $(\subseteq)$ By definition of closure on p. 95 under the section "Closure and Interior of a Set", we see that

$$\overline{A \cup B} = \bigcup_{\substack{C \text{ closed} \\ A \cup B \subseteq C}} C.$$

Further, $A \cup B \subseteq \overline{A} \cup \overline{B}$ since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$. Since $\overline{A}$ and $\overline{B}$ are closed, we see $\overline{A} \cup \overline{B}$ is closed by Theorem 17.1(3). Hence, we must have

$$\overline{A \cup B} = \bigcup_{\substack{C \text{ closed} \\ A \cup B \subseteq C}} C \subseteq \overline{A} \cup \overline{B}.$$

($\supseteq$) $\overline{A \cup B}$ is closed since it is the closure of a set (which is an arbitrary intersection of closed sets and is thus closed by Theorem 17.1(2)). Now, since $A \cup B \subseteq \overline{A \cup B}$, we see that $A \subseteq \overline{A \cup B}$ and $B \subseteq \overline{A \cup B}$. Then, since $\overline{A \cup B}$ is closed, we have $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ because

$$\overline{A} = \bigcap_{\substack{C \text{ closed} \\ A \subseteq C}} C \subseteq \overline{A \cap B}$$

and

$$\overline{B} = \bigcap_{\substack{D \text{ closed} \\ B \subseteq D}} D \subseteq \overline{A \cap B}.$$

Thus,

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$

We conclude then that

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

(c) $\overline{\bigcup_\alpha A_\alpha} \supseteq \bigcup_\alpha \overline{A}_\alpha$; give an example where equality fails.

Solution: We see that $\bigcup_\alpha A_\alpha \subseteq \overline{\bigcup_\alpha A_\alpha}$. Thus, in particular, $A_\alpha \subseteq \overline{\bigcup_\alpha A_\alpha}$ for all α. Then for each α, we may write

$$\overline{A_\alpha} = \bigcap_{\substack{C \text{ closed} \\ A_\alpha \subseteq C}} C \subseteq \overline{\bigcup_\alpha A_\alpha}.$$

Hence, we have

$$\bigcup_\alpha \overline{A}_\alpha \subseteq \overline{\bigcup_\alpha A_\alpha}.$$

For an example where equality fails, consider $\mathbb{R}$ in the order topology (so the usual one from real analysis). Let $A_\alpha = \{\alpha\}$ for $\alpha \in \mathbb{Q}$. Then since singletons are closed in this topology, we see that $\overline{A}_\alpha = \{\alpha\}$ which implies

$$\bigcup_{\alpha \in \mathbb{Q}} \overline{A}_\alpha = \mathbb{Q}.$$

But,

$$\overline{\alpha} A_\alpha = \overline{\mathbb{Q}} = \mathbb{R} \neq \mathbb{Q}.$$

Problem 7. Criticize the following "proof" that $\overline{\bigcup_\alpha A_\alpha} \subseteq \bigcup_\alpha \overline{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup_\alpha A_\alpha}$, then every neighborhood U of x intersects $\bigcup_\alpha A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $a \in \bigcup_\alpha \overline{A}_\alpha$.

Solution: This issue with this proof is the assertion that U intersection some A_α implies that x must be in the closure of some A_α. Different choices for U a neighborhood of x might intersection different A_α. For x to be in the closure of some A_α, we would need U a neighborhood of x to intersection that particular A_α for all U. But by simply saying that U intersects some A_α doesn't imply that a particular A_α intersects every such U.

In other words, the statement is saying $\forall U$ neighborhood of x, $\exists \alpha$ such that $U \cap A_\alpha \neq \varnothing$. But for $x \in \bigcup_\alpha \overline{A_\alpha}$, we would need $\exists \alpha$ such that $\forall U$ neighborhood of x, $u \cap A_\alpha \neq \varnothing$. These are not the same statements.

Problem 9. Let $A \subseteq X$ and $B \subseteq Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

Solution: We assume that $X \times Y$ is the product topology, whose definition is given by the basis $\{U \times V : U \text{ open in } X, V \text{ open in } Y\}$ by p. 86. Theorem 17.5(b) states that $x \times y \in \overline{A \times B}$ if and only if for all $U \times V$ (U open in X, V open in Y) such that $x \times y \in U \times V$, then $(A \times B) \cap (U \times V) \neq \varnothing$.

Lemma 3. If $A, U \subseteq X$ and $B, V \subseteq Y$, then

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V).$$

Proof.

$$\begin{aligned} (A \times B) \cap (U \times V) &= \{x \times y : x \in A, y \in B\} \cap \{x \times y : x \in U, y \in V\} \\ &= \{x \times y : x \in A \cap U, y \in B \cap V\} \\ &= (A \cap U) \times (B \cap V). \end{aligned}$$

□

Hence, we have

$$(A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V) \neq \varnothing.$$

Lemma 4. Let $A \subseteq X$ and $B \subseteq Y$ with $X \times Y$ using the product topology. Then

$$A \times B \neq \varnothing \iff A \neq \varnothing \text{ and } B \neq \varnothing.$$

Proof.

($\Rightarrow$) Suppose $A \times B \neq \varnothing$. Then there exists $x \times y \in A \times B$. Hence

$$x \times y \in A \times B = \{x \times y : x \in A, y \in B\}$$

which implies $x \in A, y \in B$. Thus $A \neq \varnothing$ and $B \neq \varnothing$.

($\Leftarrow$) Suppose $A \neq \varnothing$ and $B \neq \varnothing$. Then there exists $x \in A$ and $y \in B$. Hence $x \times y\{x \times y : x \in A, y \in B\} = A \times B$. This implies $A \times B \neq \varnothing$. □

Therefore, by lemma (4), we see that $A \cap U \neq \varnothing$ and $B \cap V \neq \varnothing$. Since we chose $U \times V$ such that $x \times y \in U \times V$, we have $x \in U$ and $y \in V$. We note that U, V were arbitrary open sets in X, Y respectively containing x, y respectively. Further, they intersect A and B respectively. Theorem 17.5(a) gives that $x \in \overline{A}$ and $y \in \overline{B}$. So, we conclude that $x \times y \in \overline{A} \times \overline{B}$.

We observe that every step is reversible (since everything is an if and only if statement) which means we have shown inclusion both ways.

Problem 10. Show that every order topology is Hausdorff.

Solution: We use the definition of the order topology as given on p.84 where we observe that there must exist at least two distinct elements in the order topology X.

Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since X has a simple order, we may without loss of generality that $x_1 < x_2$.

Let

$$B_1 = \begin{cases} [x_0, x_2) & \text{if } x_0 \text{ is the smallest element of } X \\ (x_0, x_2) & \text{if } X \text{ does not have a smallest element} \end{cases}$$

where we may choose $x_0 < x_1$ if X does not have a smallest element. Similarly, define

$$B_2 = \begin{cases} (x_1, x_3] & \text{if } x_3 \text{ is the largest element of } X \\ (x_1, x_3) & \text{if } X \text{ does not have a largest element} \end{cases}$$

where we may choose $x_3 > x_2$ is X does not have a largest element. We see then that $x_1 \in B_1$ and $x_2 \in B_2$. Then

$$B_1 \cap B_2 = (x_1, x_2).$$

Case 1: Suppose $(x_1, x_2) = \varnothing$. Then B_1, B_2 are two open, disjoint sets by the definition of order topology on p. 84. So we are done.

Case 2: Suppose there exists $\overline{x} \in (x_1, x_2)$. In particular, this means $x_1 \lneq \overline{x} \lneq x_2$. Then take the open sets

$$U_1 = \begin{cases} [x_0, \overline{x}) & \text{if } x_0 \text{ is the smallest element of } X \\ (x_0, \overline{x}) & \text{if } X \text{ does not have a smallest element} \end{cases}$$

and

$$U_2 = \begin{cases} (\overline{x}, x_3] & \text{if } x_3 \text{ is the largest element of } X \\ (\overline{x}, x_3) & \text{if } X \text{ does not have a largest element} \end{cases}.$$

Then, we see that $x_1 \in U_1$ and $x_2 \in U_2$ both open by definition p. 84 with $U_1 \cap U_2 = \varnothing$.

Problem 13. Show that X is Hausdorff if and only if the **diagonal** $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution:

($\Rightarrow$) Suppose X is Hausdorff and suppose $X \times X$ has the product topology as defined on p. 86. Let $N = (X \times X) \setminus \Delta$. We know that $N \neq \varnothing$ since X Hausdorff implies X has at least two distinct points. Let $x \times y \in N$. Since X is Hausdorff, there exists U a neighborhood of x and V a neighborhood of y such that $U \cap V = \varnothing$. By p. 86, $U \times V$ is a basic element of $x \times X$. Now, suppose

$$(U \times V) \cap \Delta \neq \varnothing.$$

Then there exists $z \in X$ such that $z \in U$ and $z \in V$. But, U, V are disjoint. So we must have

$$(U \times V) \cap \Delta = \varnothing.$$

But then

$$x \times y \in U \times V \subseteq N.$$

By the definition of openness on p. 78 (under the definition of a basis), we see that N is open. Therefore, Δ is closed.

($\Leftarrow$) Assume that Δ is closed. Suppose first that $X = \{x\}$. Then the Hausdorff condition holds vacuously and we are done.

Suppose second that X has at least two distinct points. Then, in particular, $N = (X \times X) \setminus \Delta$ is nonempty and open. Since N is open, we can write by Theorem 15.1 that

$$N = \bigcup_{\alpha} (U_\alpha \times V_\alpha)$$

where U_α, V_α are open in X for all α.

Now, if there exists $x \in X$ with $x \in U_\alpha \cap V_\alpha$ for some α, then we would have $x \times x \in U_\alpha \times V_\alpha$. Thus

$$x \times x \in \bigcup_{\alpha} (U_\alpha \cap V_\alpha) = N$$

which implies that $x \neq x$, a contradiction. Hence, for each α, $U_\alpha \cap V_\alpha = \varnothing$.

Take two distinct points in X, say x_1, x_2. Then there exists neighborhoods U_α of x_1 and V_α of x_2 disjoint. Therefore, we conclude that X is Hausdorff.

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Problem 4. Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \text{ and } g(y) = x_0 \times y$$

are imbeddings.

Solution: Let $f(X)$ be considered as a subspace of $X \times Y$. Then by last paragraph p.105, if $f' : X \to f(X)$ defined by $f'(x) = f(x)$ for all $x \in X$ is a homeomorphism, then we say that $f : X \to X \times Y$ is an imbedding of X in $X \times Y$. So we need only show that f' is bijective, continuous, and open.

Bijective: By how we chose f', we see that $f(X) = f'(X)$. By definition of $f'(X)$, we see that f' is onto. So we need only check that it is injective.

Let $x_1, x_2 \in X$ with $f'(x_1) = f'(x_2)$. Then

$$x_1 \times y_0 = f'(x_1) = f'(x_2) = x_2 \times y_0.$$

We see that in $X \times Y$, if $a \times b, u \times v \in X \times V$, then

$$a \times b = u \times v \iff a = u \text{ and } b = v.$$

So we see that $x_1 = x_2$. So f' is injective.

Continuous: By the top of p. 103, it suffices to show that $f'^{-1}(W)$ is open for every basis element W of the subspace $f(X)$. Now, we see that $f'(X) = f(X) = X \times \{y_0\}$. Let $U \times V$ be such that U is open in X and V is open in Y. Then $U \times V$ is a basis element of $X \times Y$ which means in the subspace $f(X)$, $(U \times V) \cap f(X) = (U \times V) \cap (X \times \{y_0\}) = U \times (V \cap \{y_0\})$ is a basic element of $f(X)$ by the first paragraph of the proof of Theorem 16.3 and lemma (3).

So, if $y_0 \notin V$, then

$$f^{-1}((U \times V) \cap (X \times \{y_0\})) = f^{-1}(U \times (V \cap \{y_0\})) = \varnothing.$$

We recall that $\varnothing$ is open in X. If $y_0 \in V$, then

$$f^{-1}((U \times V) \cap (X \times \{y_0\}) = f^{-1}(U \times (V \cap \{y_0\}) = U$$

where U is open in X by assumption. Hence, f' is continuous the top of p. 103.

Open Map: Suppose U is open in X. Then we see that $f'(U) = U \times \{y_0\}$. To see this is open in the subspace topology on $f(X)$, take V a neighborhood of y_0 in Y. Then observe that by lemma (3),

$$(U \times V) \cap (X \times \{y_0\}) = U \times \{y_0\}$$

where $U \times V$ is open in $X \times Y$. Thus, by definition of the subspace topology on p.88, we see that $U \times \{y_0\}$ is open in $f(X)$ which is what we wanted. Therefore, f' is an open map.

We have by the paragraph under the definition of homeomorphism on p.105 that f' is a homeomorphism. Therefore, by the bottom paragraph of p. 105, we conclude that f is an imbedding.

The same proofs work for g by showing that $g' : Y \to g(Y)$ defined by $g'(y) = g(y)$ for $y \in Y$ and changing X to Y.

Problem 8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x : f(x) \leq g(x)\}$ is closed in X.

Solution: Let $A = \{x : f(x) > g(x)\}$. Then $X \setminus A = \{x : f(x) \leq g(x)\}$ and so showing A is open is equivalent to showing that $\{x : f(x) \leq g(x)\}$ is closed.

Since Y is an order topology, we see that Theorem 17.11 gives that Y is Hausdorff.

Lemma 5. Suppose Y has at least two elements. Then Y Hausdorff if and only if for each $y_1, y_2 \in Y$ distinct, there exists basis elements $B_1, B_2 \in \mathcal{B}$ such that $y_1 \in B_1$, $y_2 \in B_2$ with $B_1 \cap B_2 = \varnothing$.

Proof.

($\Rightarrow$) Suppose Y is Hausdorff. Then for $y_1, y_2 \in Y$ distinct, there exist neighborhoods U_1, U_2 of y_1, y_2 respectively such that $U_1 \cap U_2 = \varnothing$. Lemma 13.1 states that

$$U_1 = \bigcup_{\alpha} B_\alpha \text{ and } U_2 = \bigcup_{\beta} C_\beta$$

for $\{B_\alpha\}, \{C_\beta\} \subseteq \mathcal{B}$. Hence, there must exist α_0 and β_0 such that $y_1 \in B_{\alpha_0}, y_2 \in C_{\beta_0}$. Since $U_1 \cap U_2 = \varnothing$, we must have $B_{\alpha_0} \cap C_{\beta_0} = \varnothing$.

($\Leftarrow$) Let $y_1, y_2 \in Y$ distinct. Then there exists basis elements $B_1, B_2 \in \mathcal{B}$ such that $y_1 \in B_1$, $y_2 \in B_2$ with $B_1 \cap B_2 = \varnothing$. Since B_1, B_2 are open sets in Y, we conclude that Y is Hausdorff by definition on p. 98. $\square$

Let $x \in A$. Then we see that $f(x) \neq g(x)$. Since Y is Hausdorff, lemma (**??**) tells us there exists B_1, B_2 basis elements of the order topology as defined on p.84 such that $g(x) \in B_1$ and $f(x) \in B_2$ with $B_1 \cap B_2$. Without loss of generality, let $B_1 = (y_0, y_1)$ and $B_2 = (y_2, y_3)$ (where we note that $g(x) \in (y_0, y_1)$ and $f(x) \in (y_2, y_3)$ implies that $y_1 \leq y_2$, else $B_1 \cap B_2 \neq \varnothing$). Define

$$U = g^{-1}(B_1) \cap f^{-1}(B_2).$$

Since B_1, B_2 are open and f, g are continuous, we have that U is open. Further, $x \in U$ since $g(x) \in B_1$ and $f(x) \in B_2$. We show now that $U \subseteq A$.

Let $z \in U$. Then $g(z) \in B_1$ and $f(z) \in B_2$ by definition. Hence,

$$y_0 < g(z) < y_1 \leq y_2 < f(z) < y_3 \Rightarrow g(z) < f(z)$$

which means $z \in A$.

We have just shown that for each $x \in A$, there exists an open set U with $x \in U \subseteq A$. Since we may always take the topology itself as a basis for the topology, we have that U is a basis element. Therefore, A is open by p.78 under the definition of a basis.

Since A is open, we must have that $\{x : f(x) \leq g(x)\}$ is closed, as desired.

(b) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [**Hint**: Use the pasting lemma.]

Solution: Let $A = \{x : f(x) \leq g(x)\}$ and $B = \{x : g(x) \leq f(x)\}$. We note by part (a) that A, B are both closed in X. Define $f' = f|_A : A \to Y$ and $g' = g|_B : B \to Y$. By Theorem 18.2(d), we see that f', g' are continuous. We note that $f'(x) = g'(x)$ on $A \cap B$. Then

$$h(x) = \min\{f(x), g(x)\} = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}.$$

By the pasting lemma, we conclude that h is continuous.

<u>**Extra Problems**</u>

Problem 1. Given: X is a topological space with open sets $U_1, ..., U_n$ such that $\overline{U_i} = X$ for all $1 \leq i \leq n$. Prove that the closure of $U_1 \cap \cdots \cap U_n$ is X.

Solution: We show this by induction.

Base Case: Let $n = 2$. We show that if $\overline{U_1} = X = \overline{U_2}$ for U_1, U_2 open in X, then $\overline{U_1 \cap U_2} = X$.

Lemma 6. For all nonempty V open in X, $U_1 \cap V \neq \varnothing$ and $U_2 \cap V \neq \varnothing$.

Proof. Let $x \in X$, then $x \in \overline{U_1}$. Suppose V_x is a neighborhood of x. Then by Theorem 17.5(a), $U_1 \cap V_x \neq \varnothing$.

Now suppose $V \neq \varnothing$ is open in X. Then there exists $x \in V$ and V is a neighborhood of x. Hence $U_1 \cap V \neq \varnothing$.

The same proof works for U_2. $\square$

Next, let $x \in X$. Let V be a neighbhood of x in X. Then $U_2 \cap V \neq \varnothing$ with $U_2 \cap V$ open in X since U_2 is open in X. Hence $U_1 \cap (U_2 \cap V) \neq \varnothing$ by lemma (5). So we see that $(U_1 \cap U_2) \cap V \neq \varnothing$. Therefore, by Theorem 17.5(a), we conclude that $x \in \overline{U_1 \cap U_2}$. Since $\overline{U_1 \cap U_2} \subseteq X$, we see that $\overline{U_1 \cap U_2} = X$.

Induction Hypothesis: Suppose that for $n-1$, $\overline{U_1 \cap U_2 \cap \cdots U_{n-1}} = X$ and $\overline{U_n} = X$. Let $S = U_1 \cap U_2 \cap \cdots \cap U_{n-1}$. Then $\overline{S} = X$. By base case we see that $\overline{S \cap U_n} = X$. Hence

$$\overline{U_1 \cap U_2 \cap \cdots U_n} = X.$$