MA 523: Homework 2

Carlos Salinas

September 9, 2016

Problem 2.1

Verify assertion (36) in [E, §3.2.3], that when Γ is not flat near x^0 the noncharacteristic condition is

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0.$$

(Here $\nu(x^0)$ denotes the normal to the hypersurface Γ at x^0).

Solution. ▸ First, note that the condition

$$D_pF(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0 \tag{2.1}$$

reduces to the standard noncharacteristic boundary condition if Γ is flat near x^0 because in such case we have $\nu(x^0) = (0, \dots, 0, 1)$ so

$$\begin{aligned} 0 &\neq D_pF(p^0, z^0, x^0) \cdot (0, \dots, 0, 1) \\ &= F_{p_n}(p^0, z^0, x^0). \end{aligned}$$

We shall verify the noncharacteristic condition (2.1) by first flattening the boundary near x^0 and then applying the noncharacteristic boundary conditions to the flattened region. Assuming some degree of regularity near x^0, e.g., that the boundary of U be smooth, we may express Γ near x^0 as the graph of a smooth function $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$, i.e., $x = (x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}))$ on Γ and $x_n \geq f(y)$ after reorienting the coordinate axes. Then we flatten out Γ via the map $\Phi(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\begin{cases} y_1 = x_1 = \Phi^1(x), \\ \quad\vdots \\ y_{n-1} = x_{n-1} = \Phi^{n-1}(x), \\ y_n = x_n - f(x_1, \dots, x_{n-1}) = \Phi^n(x) \end{cases}$$

and write $y = \Phi(x)$. Let $\Psi = \Phi^{-1}$ and rewrite our PDE F in terms of y as follows,

$$0 = F\big(Du(\Psi(y)), u(\Psi(y)), \Psi(y)\big). \tag{2.2}$$

Since $\Delta = \Phi(\Gamma)$ is flat near $y^0 = \Phi(x^0) = (y_1^0, \dots, y_{n-1}^0, 0)$, we may apply the standard noncharacteristic condition on (2.2) and get

$$0 \neq F_{u_{y_n}}\big(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)\big).$$

Before we move on to finding an expression for this derivative, let us consider the gradient $Du(\Psi(y))$. By the chain rule, we have

$$\begin{aligned} u_{y_i}(\Psi(y)) &= \sum_{j=1}^n u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_i} \\ &= u_{x_i}(\Psi(y)) + u_{x_n}(\Psi(y)) f_{y_i}(y_1, \dots, y_{n-1}), \\ u_{y_n}(\Psi(y)) &= \sum_{j=1}^n u_{x_j}(\Psi(y)) \frac{\partial x_j}{\partial y_n} \\ &= u_{x_n}(\Psi(y)), \end{aligned}$$

Then, substituting u_{y_n} for u_{x_n}, we have

$$u_{y_i}(\Psi(y)) = u_{x_i}(\Psi(y)) + u_{y_n}(\Psi(y)) f_{y_i}(y_1, \ldots, y_{n-1}),$$

Now, by the chain rule on (2.2), we have

$$\begin{aligned}
0 &\neq F_{u_{y_n}}\big(Du(\Psi(y^0)), u(\Psi(y^0)), \Psi(y^0)\big)\\
&= F_{u_{y_n}}\big(u_{x_1} + u_{y_n} f_{y_1}, \ldots, u_{x_{n-1}} + u_{y_n} f_{y_{n-1}}, u_{y_n}, z^0, x^0\big)\\
&= F_{u_{x_1}} f_{y_1} + \cdots + F_{u_{x_{n-1}}} f_{y_{n-1}} + F_{u_{x_n}}\\
&= D_p F(p^0, z^0, x^0) \cdot (Df(x^0), 1)\\
&= D_p F(p^0, z^0, x^0) \cdot \nu(x^0),
\end{aligned}$$

as we set out to show. ◂

Problem 2.2

Show that the solution of the quasilinear PDE

$$u_t + a(u)u_x = 0$$

with initial conditions $u(x, 0) = g(x)$ is given implicitly by

$$u = g(x - a(u)t).$$

Show that the solution develops a shock (becomes singular) for some $t > 0$, unless $a(g(x))$ is a nondecreasing function of x.

Solution. ▸ The characteristic ODEs of this PDE are

$$\dot{t} = 1, \qquad \dot{x} = a(z), \qquad \dot{z} = 0. \tag{2.3}$$

with initial conditions $t_0 = 0$, $x_0 = x(0)$ and $z(x_0, 0) = g(x_0)$ with $(x_0, 0) \in \mathbb{R} \times (0, \infty)$. Hence, we have

$$t(s) = s, \qquad x(s) = a(g(x_0))s + x_0, \qquad z(s) = g(x_0).$$

Thus, solving for x_0 and s in terms of t, x and z, we have

$$\begin{aligned} x &= a(g(x_0))s + x_0 \\ &= a(z)t + x_0, \end{aligned}$$

so, moving x_0 to the left-hand side

$$x_0 = x - a(z)t$$

hence,

$$z = g(x - a(z)t),$$

i.e.,

$$u = g(x - a(u)t),$$

as desired.

For the latter half of the problem, write

$$u\big(x + a(g(x))t, t\big) = g(x).$$

Suppose that $a(g(x))$ is not a nondecreasing function of x. Then, there exists $0 < x_1 < x_2$ such that $a(g(x_1)) > a(g(x_2))$. Define

$$y = -\frac{x_1 - x_2}{a(g(x_1)) - a(g(x_2))} > 0. \tag{2.4}$$

Then, we have

$$t_0 = x_1 + a(g(x_1))y = x_2 + a(g(x_2))y.$$

Thus,

$$\begin{aligned} u(x, t_0) &= g(x_1) \\ &= u\big(x_1 + a(g(x_1))t_0, t_0\big) \\ &= g(x_2) \\ &= u\big(x_2 + a(g(x_2))t_0, t_0\big). \end{aligned}$$

However, $g(x_1) \neq g(x_2)$ since $a(g(x_1)) > a(g(x_2))$. ◂

Problem 2.3

Show that the function $u(x,t)$ defined for $t \geq 0$ by

$$u(x,t) = \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x+t^2}\right) & \text{for } 4x + t^2 > 0 \\ 0 & \text{for } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution of the conservation law $u_t + (u^2/2)_x = 0$ (*inviscid Burgers' equation*).

Solution. ▸ The shock occurs along the curve C given by $s(t) = -t^2/4$. First, we verify that the equation given by u above is in fact a solution to the inviscid Burgers' equation to the right and to the left of C: to the left of C, $4x + t^2 < 0$, the equation is trivially satisfied whereas to the right, $4x + t^2 > 0$, we have,

$$-\frac{2}{3}\left(1 + \frac{t}{\sqrt{3x+t^2}}\right) + \frac{2}{9}\left(3 + \frac{3t}{\sqrt{3x+t^2}}\right) = 0.$$

So u is indeed a solution to the inviscid Burgers' equation.

Now we examine the behavior of u along the curve C. First, we have

$$\begin{aligned} \sigma &= \dot{s}(t), & [\![u]\!] &= u_\ell - u_r, & [\![F]\!] &= F(u_\ell) - F(u_r) \\ &= -\frac{t}{2} & &= 0 + \frac{2}{3}\left(t + \sqrt{-\tfrac{3}{4}t^2 + t^2}\right) & &= 0 - \frac{[\![u_r]\!]^2}{2} \\ & & &= 0 + \frac{2}{3}\left(\frac{3}{2}t\right) & &= 0 - \frac{t^2}{2} \\ & & &= t & &= -\frac{t^2}{2}. \end{aligned}$$

Thus,

$$[\![F]\!] = -\frac{t^2}{2} = \left(-\frac{t}{2}\right)t = \sigma[\![u]\!]$$

so the PDE satisfies the Rankine–Hugoniot condition and hence, is an integral solution.

Lastly, we verify that u satisfies the entropy condition, i.e., the one-sided jump estimate, Eq. (36) from [E, §3.4.3]. We break this test into two cases, (a) u is to the left of C and (b) u is to the right of C.

First we deal with case (b). When $x > -t^2/4$, $t > 0$, $z > 0$, using linear interpolation, we have

$$\begin{aligned} u(x+z,t) - u(x,t) &\leq \sup_{x > -t^2/4}\left\{u_x(x,t)\right\}(x+z-x) \\ &= \sup_{x>-t^2/4}\left\{\frac{1}{\sqrt{3x+t^2}}\right\}z \end{aligned}$$

and since the function u_x is decreasing on $x > -t^2/4$, it achieves its maximum as $x- \to -t^2/4$

$$\begin{aligned} &= \frac{1}{\sqrt{-(3/4)t^2 + t^2}}z \\ &= \frac{2}{t}z \end{aligned}$$

for all $t > 0$, $z > 0$. Thus, u satisfies the entropy condition on $x > -t^2/4$.

For case (a), we have run into the problem of the characteristic curve of $u(x,t)$ crossing the curve C for sufficiently large values of t. But in that case, we are in the region $x > -t^2/4$ and the argument in provided for case (b), shows that u satisfies the entropy condition. On the other hand, assuming t is less than some threshold $t'(x)$ we stay in the region $x < -t^2/4$ and the entropy condition

$$u(x+z,t) - u(x,t) = u(x+z,t) \leq 0$$

is satisfied. ◀