MA 572: Homework 5

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Problem 5.1 (Hatcher §2.2, Ex. 3)

Let $f\colon S^n \to S^n$ be a map of degree zero. Show that there exists points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbf{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D where F points radially outward and another point on ∂D^n where F points radially inward.

Proof. Since $\deg f = 0 \neq (-1)^n = \deg a$, then $f \not\simeq a$ and so must have a fixed point $x \in S^n$. Now, consider the map $g \coloneqq a \circ f$. Since $\deg g = \deg a \circ f = (\deg a)(\deg f) = 0$, g must have a fixed point $y \in S^n$. Since $g(y) = a \circ f(y) = y$, then $f(y) = -y$.

Suppose F is a continuous nonzero vector field on S^n, i.e., a map $S^n \to \mathbf{R}^n$ which assigns to each point $x \in S^n$ a tangent vector $\mathbf{v}(x)$ at x. Then, the map $f\colon \partial D^n \to \mathbf{R}^n$ given by $f(\mathbf{v}(x)) = \mathbf{v}(x)/\|\mathbf{v}(x)\|$ is well defined and nowhere zero. ■

PROBLEM 5.2 (HATCHER §2.2, EX. 7)

For an invertible linear transformation $f\colon \mathbf{R}^n \to \mathbf{R}^n$ show that the induced map $H_n(\mathbf{R}^n, \mathbf{R}^n \setminus \{\mathbf{0}\}) \cong \widetilde{H}_{n-1}(\mathbf{R}^n \setminus \{\mathbf{0}\}) \cong \mathbf{Z}$ is id or $-$ id according to whether the determinant of f is positive or negative. [Use Gaußian elimination to show that the matrix of f can be joined by a path of invertible matrices to a diagonal matrix with ± 1's on the diagonal.]

Proof. We show that $\mathrm{O}_n(\mathbf{R})$ is a deformation retraction of $\mathrm{GL}_n(\mathbf{R})$ and prove the result there. This procedure is adapted from a hint in *Элементарная топология* by Виро, Нецветаев и Харламов, стр. 338, номер 39.11. Suppose $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is an invertible linear transformation. Let $\{\mathbf{f}_1, ..., \mathbf{f}_n\}$ be the set of columns vectors of the matrix representation F of f. By Gram–Schmidt orthogonalization construct the vectors

$$\begin{aligned} \mathbf{e}_1 &\coloneqq \lambda_{11}\mathbf{f}_1 \\ \mathbf{e}_2 &\coloneqq \lambda_{21}\mathbf{f}_1 + \lambda_{22}\mathbf{f}_2 \\ &\ \vdots \\ \mathbf{e}_n &\coloneqq \lambda_{n1}\mathbf{f}_1 + \cdots + \lambda_{nn}\mathbf{f}_n \end{aligned} \tag{5.1}$$

where the $\lambda_{kk} > 0$ for $k = 1, ..., n$. Now set

$$\mathbf{g}_k(t) \coloneqq t(\lambda_{n1}\mathbf{f}_1 + \lambda_{n2}\mathbf{f}_2 + \cdots + \lambda_{kk-1}\mathbf{f}_{k-1}) + (t\lambda_{kk} + 1 - t)\mathbf{f}_k. \tag{5.2}$$

Let $g(t, A)$ be the matrix whose columns are the vectors $\mathbf{g}_k(t)$ and define a homotopy $f_t\colon I \times \mathrm{GL}_n(\mathbf{R}) \to \mathrm{GL}_n(\mathbf{R})$ by mapping the pair $(t, A) \mapsto g(t, A)$. Continuity of H follows from the fact that H it is multiplication in $\mathbf{R}^n$ followed by a linear mapping. It's not hard to see that f_t stays in $\mathrm{GL}_n(\mathbf{R})$ for all t and $f_1(A)$ is in $\mathrm{O}_n(\mathbf{R})$.

Last but not least, we show that $\mathrm{O}_n(\mathbf{R})$ consists of two connected components and that membership of f to one of these components is determined by $\det f$. First note that $\det(\mathrm{O}_n(\mathbf{R})) = \{-1, 1\}$ which is disconnected in $\mathbf{R}$. Hence, $\mathrm{O}_n(\mathbf{R})$ is disconnected. Now, if $f \in \mathrm{O}_n(\mathbf{R})$, either $\det f = 1$ or $\det f = -1$. Without loss of generality, we may assume $\det f = 1$ since if r is a reflection. Consider the map $k\colon I \times \mathrm{O}_n(\mathbf{R}) \to \mathrm{O}_n(\mathbf{R})$ given by

$$k(t, A) \coloneqq e^{At} \tag{5.3}$$

■

Problem 5.3 (Hatcher §2.2, Ex. 13)

Let X be the 2-complex obtained from S^1 with its usual cell structure by attaching two 2-cells by maps of degrees 2 and 3, respectively.

(a) Compute the homology groups of all the subcomplexes $A \subset X$ and the corresponding quotient complexes X/A.

(b) Show that $X \simeq S^2$ and that the only subcomplex $A \subset X$ for which the quotient map $X \to X/A$ is a homotopy equivalence is the trivial subcomplex, the 0-cell.

Proof. ■

Problem 5.4

Proof. ∎