MA166: Recitation 6 Prep

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1 Recitation 6 Prep

Recitation average for Exam 1

Table 1.1 – Section averages for Exam 1.

section	average
294	71.0
151	76.66
112	69.82

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Section 1.1: Homework Solutions

Here are the homework solutions for this week.

Homework 12

Problem 1.1 (WebAssign, HW 12, # 1)**.** Evaluate the integral

$$\int_{3\sqrt{2}}^{6} \frac{1}{t^3\sqrt{t^2-9}}\,dt.$$

Solution. Make the substitution

$$3\sec\theta = t, \tag{1}$$

then $3\sec\theta\tan\theta\,d\theta = dt$ and substituting this and (1) into the integral, making sure to solve for the appropriate values of θ, i.e., the lower bound is $\sec^{-1}(\sqrt{2}) = \pi/4$ and the upper bound is $\sec^{-1}(2) = \pi/3$

$$\begin{aligned}
\int_{3\sqrt{2}}^{6} \frac{1}{t^3\sqrt{t^2-9}}\,dt &= \int_{\pi/4}^{\pi/3} \frac{3\sec\theta\tan\theta}{3^3\sec^3\theta\sqrt{3^2\sec^2\theta-9}}\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{\tan\theta}{3^2\sec^3\theta\sqrt{3^2\sec^2\theta-3^2}}\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{\tan\theta}{3^2\sec^2\theta\sqrt{3^2(\sec^2\theta-1)}}\,d\theta \\
&= \frac{1}{27}\int_{\pi/4}^{\pi/3} \frac{\tan\theta}{\sec^2\theta\sqrt{\tan^2\theta}}\,d\theta \\
&= \frac{1}{27}\int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\,d\theta \\
&= \frac{1}{27}\int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta \\
&= \frac{1}{27}\int_{\pi/4}^{\pi/3} \frac{1+\cos 2\theta}{2}\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{54}\int_{\pi/4}^{\pi/3} 1+\cos 2\theta \; d\theta \\
&= \frac{1}{54}\left(\theta + \frac{\sin 2\theta}{2}\Big|_{\pi/4}^{\pi/3}\right) \\
&= \boxed{\frac{\pi}{648} + \frac{\sqrt{3}-2}{108}.}
\end{aligned}
$$

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Problem 1.2 (WebAssign, HW 12, # 2)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int \sqrt{1-25x^2}\; dx.$$

Solution. First, make the substitution $u = 5x$. Then the integral above turns into

$$\frac{1}{5}\int \sqrt{1-u^2}\; du.$$

Now we make the trig substitution $\cos\theta = u$ so $-\sin\theta\; d\theta = du$ and the integral above turns into

$$
\begin{aligned}
\frac{1}{5}\int \sqrt{1-u^2}\; du &= \frac{1}{5}\int \sin\theta(-\sin\theta)\; d\theta \\
&= -\frac{1}{5}\int \sin^2\theta\; d\theta \\
&= -\frac{1}{5}\int \left(\frac{1-\cos 2\theta}{2}\right) d\theta \\
&= -\frac{1}{10}\int 1-\cos 2\theta\; d\theta \\
&= \frac{1}{10}\int \cos 2\theta - 1\; d\theta \\
&= \frac{1}{10}\left(\frac{1}{2}\sin 2\theta - \theta\right) \\
&= \frac{1}{20}\sin 2\theta - \frac{1}{2}\theta + C \\
&= \frac{1}{20} 2\sin\theta\cos\theta - \frac{1}{2}\theta + C
\end{aligned}
$$

substituting back u then x, we have

$$
\begin{aligned}
&= \frac{1}{10}\sqrt{1-u^2}u - \frac{1}{2}\cos^{-1}(u) + C \\
&= \frac{1}{10}\sqrt{1-25x^2}(5x) - \frac{1}{2}\cos^{-1}(5x) + C \\
&= \boxed{\frac{\sqrt{1-25x^2}x - \cos^{-1}(5x)}{2} + C.}
\end{aligned}
$$

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Problem 1.3 (WebAssign, HW 12, # 3)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int \sqrt{16+6x-x^2}\, dx.$$

Solution. First we need to complete the square

$$\begin{aligned}(x-3)^2-9 &= x^2-6x+9-9\\ &= x^2-6x.\end{aligned}$$

Then, the integral above turns into

$$\int \sqrt{16+x-x^2}\, dx = \int \sqrt{25+(x-3)^2}\, dx$$

and now we can use the substitution $5u = x-3$ to simplify our integral into

$$\frac{1}{5}\int \sqrt{25-(5u)^2}\, du = \frac{1}{5}\int \sqrt{5^2-5^2u^2}\, du = \int \sqrt{1-u^2}\, du.$$

Now we can use the substitution $\cos\theta = u$ and $-\sin\theta\, d\theta = du$ to get

$$\begin{aligned}\int \sqrt{1-u^2}\, du &= \int \sqrt{1-\cos^2\theta}(-\sin\theta)\, d\theta\\ &= -\int \sqrt{\sin^2\theta}\sin\theta\, d\theta\\ &= -\int \sin^2\theta\, d\theta\end{aligned}$$

which, from our previous problem, we know to be

$$= \frac{1}{2}\sin 2\theta - \theta + C.$$

Substututing back first u then x we get

$$\begin{aligned}\frac{1}{2}\sin 2\theta - \theta + C &= \sin\theta\cos\theta - \theta + C\\ &= \sqrt{1-u^2}u - \cos^{-1}(u) + C\\ &= \sqrt{1-\left(\frac{x-3}{5}\right)^2}\left(\frac{x-3}{5}\right) - \cos^{-1}\left(\frac{x-3}{5}\right) + C\\ &= \boxed{\frac{x-3}{25}\sqrt{16+6x-x^2} - \cos^{-1}\left(\frac{x-3}{5}\right) + C.}\end{aligned}$$

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Problem 1.4 (WebAssign, HW 12, # 4)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int \frac{1}{\sqrt{t^2-12t+40}}\, dt.$$

Solution. This problem is similar to the last one. The first thing we need to do is to complete the square

$$\begin{aligned}(t-6)^2 - 36 &= t^2 - 12t + 36 - 36 \\ &= t^2 - 12t.\end{aligned}$$

This turns our integral into

$$\int \frac{1}{\sqrt{t^2 - 12t + 40}}\, dt = \int \frac{1}{\sqrt{(t-6)^2 - 36 + 40}}\, dt = \int \frac{1}{\sqrt{(t-6)^2 + 4}}\, dt.$$

Then, making the substitution $u = (t-6)/2$ we have $2\, du = dt$ and our integral turns into

$$\begin{aligned}\int \frac{1}{\sqrt{(t-6)^2 + 4}}\, dt &= 2\int \frac{1}{\sqrt{2^2u^2 + 2^2}}\, du \\ &= \int \frac{1}{\sqrt{u^2+1}}\, du.\end{aligned}$$

From here we can make the substitution $\tan\theta = u$ so $\sec\theta\tan\theta\, d\theta = du$ and our integral turns into

$$\begin{aligned}\int \frac{1}{\sqrt{u^2+1}}\, du &= \int \frac{\sec\theta\tan\theta}{\sec\theta}\, d\theta \\ &= \int \tan\theta\, d\theta \\ &= \ln|\cos\theta| + C.\end{aligned}$$

Now we substitute u back into the equation by $\cos\theta = 1/\sqrt{1+u^2}$ and $u = (t-6)/2$ giving us

$$\begin{aligned}\ln|\cos\theta| + C &= \ln\left|\frac{1}{\sqrt{1+u^2}}\right| + C \\ &= \ln\left|\frac{1}{\sqrt{1+((t-6)/2)^2}}\right| + C \\ &= \boxed{\ln\left|\frac{2}{\sqrt{t^2-12t+40}}\right| + C.}\end{aligned}$$

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Problem 1.5 (WebAssign, HW 12, # 5)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int \sqrt{x^2 + 6x}\, dx.$$

Solution. Again, nothing special is going on here, all we need to do is first complete the square and then proceed to make a u-substitution or trigonometric substitution where appropriate

$$(x+3)^2 - 9 = x^2 + 6x + 9 - 9$$

$$= x^2 + 6x.$$

Then our integral turns into

$$\int \sqrt{(x+3)^2 - 9}\, dx.$$

Now, making the $u = (x+3)/3$ we have $3\ du = dx$ and

$$\begin{aligned}
\int \sqrt{(x+3)^2 - 9} &= \int \sqrt{(x+3)^2 - 3^2}\ dx \\
&= 3 \int \sqrt{3^2 u^2 - 3^2}\ du \\
&= 3 \int \sqrt{3^2(u^2 - 1)}\ du \\
&= 9 \int \sqrt{u^2 - 1}\ du.
\end{aligned}$$

To continue we need to make a trigonometric substitution. The following is the easiest substitution to make $\sec\theta = u$ then $\sec\theta\tan\theta\ d\theta = du$ and we have

$$\begin{aligned}
9 \int \sqrt{u^2 - 1}\ du &= 9 \int \tan\theta \sec\theta \tan\theta\ d\theta \\
&= 9 \int \tan^2\theta \sec\theta\ d\theta \\
&= 9 \int \left(\sec^2\theta - 1\right) \sec\theta\ d\theta \\
&= 9 \int \sec^3\theta - \sec\theta\ d\theta \\
&= 9 \Bigg(\underbrace{\int \sec^3\ d\theta}_{I_1} - \underbrace{\int \sec\theta\ d\theta}_{I_2} \Bigg).
\end{aligned}$$

Let's compute I_1 and I_2 separately. For I_1 we use integration by parts to get to

$$\begin{aligned}
I_1 &= \int \sec^3\theta\ d\theta \\
&= \int \sec^2\theta \sec\theta\ d\theta
\end{aligned}$$

let $u = \sec\theta$, $dv = \sec^2\theta\ d\theta$ so $du = \sec\theta\tan\theta\ d\theta$ and $v = \tan\theta$ giving us

$$\begin{aligned}
&= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\ d\theta \\
&= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\ d\theta \\
&= \sec\theta\tan\theta - \int \sec^3\theta + \int \sec\theta\ d\theta \\
&= \sec\theta\tan\theta - I_1 + I_2
\end{aligned}$$

$$
\begin{aligned}
I_1 + I_1 &= \sec\theta\tan\theta + I_2 \\
I_1 &= \frac{1}{2}(\sec\theta\tan\theta + I_2) + C_1.
\end{aligned}
$$

This depends on I_2 so let's compute that

$$I_2 = \int \sec\theta \, d\theta$$

rewrite the integral as

$$
\begin{aligned}
&= \int \sec\theta \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \, d\theta \\
&= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}
\end{aligned}
$$

and use the substitution $u = \sec\theta + \tan\theta$ since $du = (\sec\theta\tan\theta + \sec^2\theta)\, d\theta$

$$
\begin{aligned}
&= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{u} \frac{1}{\sec^2\theta + \sec\theta\tan\theta} \, du \\
&= \int \frac{1}{u} \, du \\
&= \ln|u| + C_2 \\
&= \ln|\sec\theta + \tan\theta| + C_2
\end{aligned}
$$

Then

$$I_1 = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C_2) + C_1$$

and we have

$$
\begin{aligned}
9(I_1 - I_2) &= 9\left(\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C_2) + C_1 - \ln|\sec\theta + \tan\theta|\right) \\
&= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + \underbrace{9C_1 - \frac{9}{2}C_2}_{\text{call this } C} \\
&= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + C.
\end{aligned}
$$

Now we substitute back our value of u then x as follows

$$
\begin{aligned}
\frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + C &= \frac{9}{2}u\sqrt{u^2 - 1} - \frac{9}{2}\ln\left|u + \sqrt{u^2 - 1}\right| + C \\
&= \frac{9}{2}\frac{x+3}{3}\sqrt{\left(\frac{x+3}{3}\right)^2 - 1} - \frac{9}{2}\ln\left|\frac{x+3}{3} + \sqrt{\left(\frac{x+3}{3}\right)^2 - 1}\right| + C \\
&= \boxed{\frac{1}{2}(x+3)\sqrt{x^2 + 6x} + \frac{9}{2}\ln\left|\frac{1}{3}\left(x + 3 + \sqrt{x^2 + 6x}\right)\right| + C.} \quad ☺
\end{aligned}
$$

Homework 13

Problem 1.6 (WebAssign, HW 13, # 1)**.** Write out the form of the partial fraction decomposition of the function. Do not determine the numerical values of the coefficients.

(a) $\dfrac{x}{x^2+x-2}$

(b) $\dfrac{x^2}{x^2+x+3}$

Solution. **(a)** Let's start by factoring the denominator

$$x^2+x-2=(x+2)(x-1).$$

This factorization is very easy to see. If you don't believe me, use the quadratic equation and find the roots and you'll see that I'm right. Now, we know that the partial fraction decomposition will look like

$$\boxed{\frac{A}{x+2}+\frac{B}{x-1}.}$$

(b) The polynomial x^2+x+3 has no real roots. Let's confirm this. If we put the coefficients of x^2+x+3 into the quadratic equation we have

$$\begin{aligned}\frac{-1\pm\sqrt{1^2-4\cdot 3}}{2}&=\frac{-1\pm\sqrt{1-12}}{2}\\&=\frac{-1\pm\sqrt{-11}}{2}.\end{aligned}$$

$\sqrt{-11}$ is not a real number so we cannot proceed. In this case, the best we cannot get rid of the x^2+x+3 at the bottom. Now, since the degree of the top x^2 is not less that the degree of x^2+x+3 we need to remove enough terms to get something of lower degree on top

$$\begin{aligned}\frac{x^2}{x^2+x+3}&=\frac{(x^2+x+3)-x-3}{x^2+x+3}\\&=\frac{x^2+x+3}{x^2+x+3}-\frac{x-3}{x^2+x+3}\\&=\boxed{1+\frac{Ax+B}{x^2+x+3}.}\end{aligned}$$

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Problem 1.7 (WebAssign, HW 13, # 2)**.** Write out the form of the partial fraction decomposition of the function. Do not determine the numerical values of the coefficients.

(a) $\dfrac{x^4+1}{x^5+7x^3}$

(b) $\dfrac{3}{(x^2-25)^2}$

Solution. **(a)** Let's start by factoring the denominator into something simpler

$$x^5 + 7x^3 = x^3(x^2 + 7).$$

Then, since the top has degree less than the bottom and we cannot factor $x^2 + 7$, we have the partial fraction decomposition

$$\frac{3}{x^5 + 7x^3} = \boxed{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + 7}.}$$

(c) Let's factor the denominator

$$\left(x^2 - 25\right)^2 = (x - 5)^2(x + 5)^2.$$

Then we have the partial fraction decomposition

$$\frac{3}{\left(x^2 - 25\right)^2} = \boxed{\frac{A}{x + 5} + \frac{B}{(x + 5)^2} + \frac{C}{x - 5} + \frac{D}{(x - 5)^2}.}$$

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Problem 1.8 (WebAssign, HW 13, # 3)**.** Evaluate the integral

$$\int_0^1 \frac{x - 8}{x^2 - 7x + 10}\, dx.$$

Solution. First let's find the partial fraction decomposition of

$$\frac{x - 8}{x^2 - 7x + 10}.$$

To that end, we need to factor the denominator

$$x^2 - 7x + 10 = (x - 5)(x - 2).$$

Then

$$\frac{x - 8}{(x - 5)(x - 2)} = \frac{A}{x - 5} + \frac{B}{x - 2}.$$

Solving for A we have

$$A = \frac{5 - 8}{5 - 2} + \frac{B}{5 - 2}(5 - 5) = \frac{-3}{3} = -1$$

and

$$B = \frac{2 - 8}{2 - 5} + \frac{A}{2 - 5}(2 - 2) = -2 = 2.$$

Hence, the integral turns into

$$\int_0^1 \frac{x - 8}{x^2 - 7x + 10}\, dx = \int_0^1 -\frac{1}{x - 5} + \frac{2}{x - 2}\, dx$$

$$
\begin{aligned}
&= -\ln|x-5| + 2\ln|x-2|\Big|_0^1 \\
&= -\ln|4| + 2\ln|1| + \ln|5| - \ln|2| \\
&= -2\ln|2| - \ln|2| + \ln|5| \\
&= \boxed{\ln|5| - 3\ln|2|.}
\end{aligned}
$$

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Problem 1.9 (WebAssign, HW 13, # 4)**.** Evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{ax}{x^2 - bx}\, dx.$$

Solution. First, let's write down the partial fraction decomposition for

$$\frac{ax}{x^2 - bx}.$$

The denominator factors as $x^2 - bx = x(x-b)$ so

$$\frac{ax}{x(x-b)} = \frac{A}{x} + \frac{B}{x-b}.$$

Then we have

$$ax = A(x-b) + Bx = (A+B)x - Ab.$$

This tells us that $A = 0$ and $B = a$. Well, actually, we didn't have to do partial fractions since $x^2 - bx$ factors neatly. Let's proceed

$$\int \frac{ax}{x^2 - bx}\, dx = \int \frac{a}{x-b}\, dx$$

use the substitution $u = x - 5$ then $du = dx$ and we have

$$
\begin{aligned}
&= a\int \frac{1}{u}\, du \\
&= a\ln|u| + C \\
&= \boxed{a\ln|x-5| + C.}
\end{aligned}
$$

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Problem 1.10 (WebAssign, HW 13, # 5)**.** Evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{7x^2 + 2x - 7}{x^3 - x}\, dx$$

Solution. First note that the denominator factors nicely as

$$x^3 - x = x(x^2 - 1) = x(x+1)(x-1).$$

Then we can find the partial fraction decomposition of the quotient by

$$\begin{aligned}\frac{7x^2 + 2x - 7}{x(x+1)(x-1)} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}\\ 7x^2 + 2x - 7 &= A(x+1)(x-1) + Bx(x-1) + Cx(x+1)\\ &= Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx\\ &= (A+B+C)x^2 + Cx - A.\end{aligned}$$

Solving for A, B and C we have $A = 7$, $C = 2$ and $B = -C - A = -2 - 7 = -10$ so

$$\frac{7x^2 + 2x - 7}{x(x+1)(x-1)} = \frac{7}{x} - \frac{10}{x+1} + \frac{2}{x-1}.$$

Now, integrating this we have

$$\begin{aligned}\int \frac{7x^2 + 2x - 7}{x(x+1)(x-1)}\, dx &= \int \frac{7}{x} - \frac{10}{x+1} + \frac{2}{x-1}\\ &= \boxed{7\ln|x| - \ln|x+1| + 2\ln|x-1| + C.}\end{aligned}$$

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Problem 1.11 (WebAssign, HW 13, # 6)**.** Evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{3x^2 - 20x + 33}{(2x+1)(x-2)^2}\, dx.$$

Solution. First, let's find the partial fraction decomposition

$$\frac{3x^2 - 20x + 33}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Then, we solve for A, B, C and D by

$$\begin{aligned}3x^2 - 20x + 33 &= A(x-2)^2 + B(2x+1)(x-2) + C(2x+1)\\ &= Ax^2 - 4Ax + 4A + 2Bx^2 - 3Bx - 2B + 2Cx + C\\ &= (A+2B)x^2 + (-4A - 3B + 2C)x + (4A - 2B + C)\end{aligned}$$

Then we have

$$\begin{aligned}A + 2B &= 3\\ -4A - 3B + 2C &= -20\\ 4A - 2B + C &= 33\end{aligned}$$

Now to solve this linear system of equations we use Gaussian elimination on the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ -4 & -3 & 2 & -20 \\ 4 & -2 & 1 & 33 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This tells us that $A = 7$, $B = -2$ and $C = 1$ so our partial fraction decomposition is

$$\frac{3x^2 - 20x + 33}{(2x+1)(x-2)^2} = \frac{7}{2x+1} - \frac{2}{x-2} + \frac{1}{(x-2)^2}$$

and our integral becomes

$$\begin{aligned} \int \frac{3x^2 - 20x + 33}{(2x+1)(x-2)^2}\, dx &= \int \frac{7}{2x+1} - \frac{2}{x-2} + \frac{1}{(x-2)^2} \\ &= \boxed{\frac{7}{2}\ln|2x+1| + 2\ln|x-2| - \frac{1}{x-2} + C.} \end{aligned}$$

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Homework 14

Problem 1.12 (WebAssign, HW 14, # 1)**.** Evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{5}{(x-1)(x^2+4)}\, dx$$

Solution. Write

$$\frac{5}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+4}.$$

Then

$$\begin{aligned} 5 &= A(x^2+4) + (Bx + C)(x-1) \\ &= (A+B)x^2 + (C-B)x + (4A - C) \end{aligned}$$

and we have

$$A + B = 0 \qquad C - B = 0 \qquad 4A - C = 5.$$

This tells us that $A = -B = -C = 5 - 4A$ so $A = 1$, $B = -1$ and $C = -1$. Hence, our integral becomes

$$\begin{aligned} \int \frac{5}{(x-1)(x^2-4)}\, dx &= \int \frac{1}{x-1} - \frac{x+1}{x^2+4}\, dx \\ &= \int \frac{1}{x-1} - \frac{x}{x^2+4} - \frac{1}{x^2+4}\, dx \\ &= \underbrace{\int \frac{1}{x-1}\, dx}_{I_1} - \underbrace{\int \frac{x}{x^2+4}\, dx}_{I_2} - \underbrace{\int \frac{1}{x^2+4}\, dx}_{I_3}. \end{aligned}$$

Let's compute these separately. We know what I_1 and I_3 are, they are

$$I_1 = \ln|x-1| + C_1 \qquad\qquad I_3 = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

from our handy integral table. That leaves us only I_2 to figure out. We do this by substitution letting $u = x^2$ so $du = 2x\ dx$ giving us

$$\begin{aligned} I_2 &= \int \frac{x}{x^2+4}\ dx \\ &= \int \frac{x}{u+4}\frac{dx}{2x} \\ &= \frac{1}{2}\int \frac{1}{u+4} \\ &= \frac{1}{2}\ln\left|x^2+4\right| + C_3. \end{aligned}$$

Hence, setting $C \coloneqq C_1 - C_2 - C_3$, our integral is

$$I_1 - I_2 - I_3 = \boxed{\ln|x-1| + \frac{1}{2}\ln\left|x^2+4\right| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.}$$ ☺

Problem 1.13 (WebAssign, HW 14, # 2)**.** Evaluate the integral. (Use C for the constant of integration.)

$$\int \frac{4x^2+3x+4}{(x^2+1)^2}\ dx.$$

Solution. Instead of finding the partial fraction decomposition split the numerator into

$$4(x^2+1) + 3x.$$

Then we can split the integral into

$$\begin{aligned} \int \frac{4x^2+3x+4}{(x^2+1)^2}\ dx &= \int \frac{4(x^2+1)+3x}{(x^2+1)^2}\ dx \\ &= \int \frac{4(x^2+1)+3x}{(x^2+1)^2}\ dx \\ &= 4\int \frac{x^2+1}{(x^2+1)^2}\ dx + 3\int \frac{x}{(x^2+1)^2}\ dx \\ &= 4\underbrace{\int \frac{1}{x^2+1}\ dx}_{I_1} + 3\underbrace{\int \frac{x}{(x^2+1)^2}\ dx}_{I_2}. \end{aligned}$$

Calculating I_1 is easy from the table of integrals $I_1 = \tan(x) + C_1$. I_2 takes a little more work and requires the substitution $u = x^2$ with $du = 2x\ dx$

$$I_2 = \int \frac{x}{(x^2+1)^2}\ dx$$

$$
\begin{aligned}
&= \int \frac{x}{(u+1)^2} \frac{du}{2x} \\
&= \frac{1}{2} \int \frac{1}{(u+1)^2}\, du \\
&= -\frac{1}{2(u+1)} + C_2 \\
&= -\frac{1}{2(x^2+1)} + C_2.
\end{aligned}
$$

Then, letting $C \coloneqq 4C_1 - 3C_2$, the integral is

$$4I_1 + 3I_2 = \boxed{4\tan^{-1} x - \frac{3}{2(x^2+1)}.}$$

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Problem 1.14 (WebAssign, HW 14, # 3)**.** Evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{3}{x(x^2+4)^2}\, dx.$$

Solution. For this problem we need to express the quotient in its partial fraction decomposition. First let's rewrite the integral, putting the coefficient 3 aside

$$\int \frac{3}{x(x^2+4)^2}\, dx = 3\int \frac{1}{x(x^2+4)^2}\, dx.$$

Now the denominator factors as x and $(x^2+4)^2$. But x^2+4 cannot be factored into a polynomial of lower degree so our decomposition will look like

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}.$$

Let's solve for A first. Multiply across by x on the left and right and plug in 0 gives us

$$A = \frac{1}{(0+4)^2} = \frac{1}{16}.$$

Now, multiply by $x(x^2+4)^2$ across and we have

$$
\begin{aligned}
1 &= \frac{1}{16}(x^2+4)^2 + (Bx^2+Cx)(x^2+4) + Dx + E \\
&= \frac{1}{16}x^4 + \frac{1}{2}x^2 + 1 + \left(Bx^4 + Cx^3 + 4Bx^2 + 4Cx\right) + (Dx+E)x \\
&= \left(\frac{1}{16} + B\right)x^4 + Cx^3 + \left(\frac{1}{2} + 4B + D\right)x^2 + (4C+E)x + 1.
\end{aligned}
$$

This tells us that $C = 0$ so $E = 0$, $B = -1/16$ so $D = -1/2 + 1/4 = -1/4$.

Now we can at last compute the integral

$$3\int \frac{1}{x(x^2+4)^2}\, dx = 3\int \left(\frac{1}{16x} - \frac{1}{16(x^2+4)} - \frac{x}{4(x^2+4)^2}\right) dx$$

$$
\begin{aligned}
&= 3\underbrace{\int \frac{1}{16x}\,dx}_{I_1} - 3\underbrace{\int \frac{1}{16(x^2+4)}\,dx}_{I_2} - 3\underbrace{\int \frac{x}{4(x^2+4)^2}\,dx}_{I_3} \\
&= 3(I_1 - I_2 - I_3).
\end{aligned}
$$

Let's compute these separately. It's easy to see that

$$I_1 = \frac{1}{16}\ln|x| + C_1.$$

The integral I_2 is also easy by u-substitution using $u = x^2 + 4$

$$
\begin{aligned}
I_2 &= \frac{1}{16}\int \frac{x}{x^2+4}\,dx \\
&= \frac{1}{32}\int \frac{1}{u}\,du \\
&= \frac{1}{32}\ln|u| + C_2 \\
&= \frac{1}{32}\ln\left|x^2+4\right| + C_2.
\end{aligned}
$$

The integral I_3 can also be computed using the substitution $u = x^2 + 4$ and we have

$$
\begin{aligned}
I_3 &= \int \frac{x}{4(x^2+4)^2}\,dx \\
&= \frac{1}{4}\int \frac{1}{u^2}\,du \\
&= -\frac{1}{4u} + C_3 \\
&= -\frac{1}{4}\frac{1}{x^2+4} + C_3.
\end{aligned}
$$

Then, putting $C = 3(C_1 - C_2 - C_3)$, the integral is

$$3(I_1 - I_2 - I_3) = \boxed{\frac{3}{16}\ln|x| - \frac{3}{32}\ln\left|x^2+4\right| + \frac{3}{8}\frac{1}{x^2+4} + C.}$$

☺

Problem 1.15 (WebAssign, HW 14, # 4)**.** Make a substitution to express the integrand as a rational function and then evaluate the integral. (Remember to use $\ln|u|$ where appropriate. Use C for the constant of integration.)

$$\int \frac{\sqrt{x+49}}{x}\,dx.$$

Solution. Let's make the substitution $u = \sqrt{x+49}$ so we have $u^2 = x + 49$ and $2u\,du = dx$. Then our integral turns into

$$\int \frac{\sqrt{x+49}}{x}\,dx = \int \frac{u}{u^2-49}2u\,du$$

$$
\begin{aligned}
&= \int \frac{2u^2}{u^2-49}\,du \\
&= \int \frac{2u^2-98+98}{u^2-49}\,du \\
&= \int \left(2+\frac{98}{u^2-49}\right)du \\
&= \int 2\,du + \int \frac{98}{u^2-49}\,du \\
&= 2u + C_1 + 98\underbrace{\int \frac{1}{u^2-49}\,du}_{I_2}.
\end{aligned}
$$

Here's a good time to do some partial fraction decomposition. We can break up $u^2-49=(u-7)(u+7)$ and we have

$$\frac{1}{(u-7)(u+7)} = \frac{A}{u-7}+\frac{B}{u+7}.$$

Then

$$1 = A(u+7)+B(u-7) = (A+B)u+7A-7B.$$

This tells us that $A-B=1/7$ and $A+B=0$ so $A=-B$ and $A-(-A)=1/7$ implies $A=1/14$ and $B=-1/14$. Thus, we have

$$
\begin{aligned}
I_2 &= \int \frac{1}{u^2-49}\,du \\
&= \frac{1}{14}\int \frac{1}{u-7}\,du - \frac{1}{14}\int \frac{1}{u+7}\,du \\
&= \frac{1}{14}\ln|u-7| - \frac{1}{14}\ln|u+7| + C_2.
\end{aligned}
$$

So, letting $C=C_1+98C_2$ and substituting back $u=\sqrt{x+49}$ our integral is

$$2\sqrt{x+49}+C_1+98I_2 = \boxed{2\sqrt{x+49}+7\ln\left|\sqrt{x+49}-7\right|-7\ln\left|\sqrt{x+49}+7\right|+C.}$$ ☺

Problem 1.16 (WebAssign, HW 14, # 5)**.** Find the area of the region under the given curve from 1 to 2

$$y=\frac{13}{x^3+3x}.$$

Solution. Using the partial fraction decomposition write $x^3+3x=x(x^2+3)$ and we have

$$\frac{1}{x(x^2+3)} = \frac{A}{x}+\frac{Bx+C}{x^2+3}.$$

Multiplying on both sides by $x(x^2+3)$ we get

$$1 = A(x^2+3)+(Bx+C)x = (A+B)x^2+Cx+3A$$

giving us that $A = 1/3$, $C = 0$ and $B = -1/3$. Hence, we can write the integral as

$$\begin{aligned}
\int_1^2 \frac{13}{x^3+3x}\,dx &= 13\int_1^2 \frac{1}{3x} - 13\frac{x}{3(x^2+3)}\,dx \\
&= \frac{13}{3}\ln|x| - \frac{13}{6}\ln\left|x^2+3\right|\Big|_1^2 \\
&= \frac{13}{3}\ln 2 - \frac{13}{6}\ln|7| - \left(\frac{13}{3}\ln 1 - \frac{13}{6}\ln|4|\right) \\
&= \frac{13}{3}\ln 2 - \frac{13}{6}\ln|7| + \frac{13}{6}\ln|4| \\
&= \frac{13}{6}\ln|4| - \frac{13}{6}\ln|7| + \frac{13}{6}\ln|4| \\
&= \frac{13}{6}\ln|16| - \frac{13}{6}\ln|7| \\
&= \boxed{\frac{13}{6}\ln\left|\frac{16}{17}\right|.}
\end{aligned}$$

☺

Section 1.2: Exam 2 Problems

Problem 1.17 (Spring 2015, # 5)**.** We would like to compute

$$\int \sec^3\theta\,d\theta = \int \frac{\cos\theta}{\cos^4\theta}\,d\theta$$

by using the substitution $u = \sin\theta$.

Solution. Let's make the substitution $u = \sin\theta$. Then $du = \cos\theta\,d\theta$ and since $\cos^2\theta + \sin^2\theta = 1$ we have $\cos^2\theta = 1 - \sin^2\theta$ our integral turns into

$$\begin{aligned}
\int \sec^3\theta\,d\theta &= \int \frac{\cos\theta}{\cos^4\theta}\,d\theta \\
&= \int \frac{\cos\theta}{\left(1-\sin^2\theta\right)^2}\,d\theta \\
&= \int \frac{1}{(1-u^2)^2}\,du \\
&= \int \frac{1}{(1-u)^2(1+u)^2}\,du.
\end{aligned}$$

By the method of partial fractions we have

$$\frac{1}{\left(1-u^2\right)^2} = \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{(1-u)^2} + \frac{D}{(1+u)^2}.$$

Multiply across by $(1+u)^2$ and plug in -1 for u and we get

$$D = \frac{1}{(1-u)^2} = \frac{1}{4}.$$

Do the same for C and plug in 1 for u and we get

$$C = \frac{1}{(1+1)^2} = \frac{1}{4}.$$

Now we can solve for A and B by noting that

$$\begin{aligned}
1 &= A(1-u)(1+u)^2 + B(1-u)^2(1+u) + \frac{1}{4}(1+u)^2 + \frac{1}{4}(1-u)^2 \\
&= A(-u^3 - u^2 + u + 1) + B(u^3 - u^2 - u + 1) + \frac{1}{4}(u^2 + 2u + 1) + \frac{1}{4}(u^2 - 2u + 1) \\
&= (-A+B)u^3 + \left(-A - B + \frac{1}{4} + \frac{1}{4}\right)u^2 + (A-B)u + A + B + \frac{1}{4} + \frac{1}{4} \\
&= (-A+B)u^3 + \left(-A - B + \frac{1}{2}\right)u^2 + (A-B)u + A + B + \frac{1}{2}
\end{aligned}$$

This gives us that $A = B$ and $A + B = 2A = 1/2$ implies $A = B = 1/4$. Hence, the integral transforms into

$$\begin{aligned}
\frac{1}{4}\int\left[\frac{1}{1-u} + \frac{1}{(1-u)^2} + \frac{1}{1+u} + \frac{1}{(1+u)^2}\right] &= \frac{1}{4}\int\left[-\frac{1}{u-1} + (-1)^2\frac{1}{(u-1)^2} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\right] \\
&= \boxed{\frac{1}{4}\int\left[-\frac{1}{u-1} + \frac{1}{(u-1)^2} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\right]}
\end{aligned}$$

which is the desired solution. ☺

Problem 1.18 (Spring 2014, # 5)**.** It is know that

$$\int \frac{2x-3}{x(x^2+1)}\,dx = a\ln x + b\ln(x^2+1) + c\tan x + C.$$

for some constants a, b, c and C. What is b?

Solution. Let's work more abstractly, consider the integral

$$\int \frac{ax+b}{x(x^2+c^2)}.$$

Then, by the method of partial fractions, since the degree of $ax+b$ is less than the degree of $x(x^2+c^2)$, we can rewrite the integral as

$$\int \frac{A}{x} + \frac{Bx+C}{x^2+c^2}\,dx.$$

Let's split the integral and find a general formula for it

$$\int \frac{A}{x} + \frac{Bx+C}{x^2+c^2}\,dx = A\underbrace{\int\frac{1}{x}\,dx}_{I_1} + B\underbrace{\int\frac{x}{x^2+c^2}\,dx}_{I_2} + C\underbrace{\int\frac{1}{x^2+c^2}\,dx}_{I_3}, \tag{2}$$

and compute I_1, I_2 and I_3 separately.

Computing I_1 is easy enough, we've done it so many times $I_1 = \ln x + C_1$.

I_2 is a tad more complicated, but we have seen this before. Making the u-substitution $u = x^2 + c^2$ then $du = 2x\ dx$ and we have

$$\begin{aligned} I_2 &= \int \frac{x}{x^2+c^2}\ dx \\ &= \int \frac{x}{u}\frac{dx}{2x} \\ &= \frac{1}{2}\int \frac{1}{u}\ dx \\ &= \frac{1}{2}\ln\left|x^2+c^2\right| + C_2. \end{aligned}$$

Last but not least we compute I_3 by making the substitution $u = x/c$ so $du = c\ dx$ and

$$\begin{aligned} I_3 &= \int \frac{1}{x^2+c^2}\ dx \\ &= \int \frac{c}{c^2(u^2+1)}\ du \\ &= \frac{1}{c}\int \frac{1}{u^2+1}\ du \end{aligned}$$

and $1/(u^2+1)$ is the well known integral of $\tan^{-1}(us)$

$$\begin{aligned} &= \frac{1}{c}\tan^{-1}(u) + C_3 \\ &= \frac{1}{c}\tan^{-1}\left|\frac{x}{c}\right| + C_3. \end{aligned}$$

Putting everything together into (2), we get

$$I_1 + I_2 + I_3 = A\ln x + \frac{B}{2}\ln\left|x^2+c^2\right| + \frac{C}{c}\tan^{-1}\left|\frac{x}{c}\right| + C_1 + C_2 + C_3. \tag{3}$$

Now we can say with certainty that b in the coefficient in the problem will be $B/2$ where B is the term derived from the partial fraction decomposition of $2x - 3/(x(x^2+1))$. Now all we need to do is find this partial fraction decomposition. We have

$$2x - 3 = A(x^2+1) + (Bx + C)x = (A+B)x^2 + Cx + A$$

so $A = -3$, $B = -A = 3$ and $C = 2$ giving us

$$\boxed{b = \frac{B}{2} = \frac{3}{2}}.$$

☺