

MA 26500-215 Quiz 7

July 29, 2016

1. For the following problems write **T** for true, **F** for false. You do not need to justify your answers.
- (a) (3 points) For all $m \times n$ matrices A and B , $\text{nullity}(A + B) = \text{nullity } A + \text{nullity } B$.
 - (b) (3 points) For all $n \times n$ matrices A and B , $\text{nullity}(AB) = (\text{nullity } A)(\text{nullity } B)$.
 - (c) (3 points) For all $n \times n$ matrices A and B , where A is an elementary matrix, $\text{nullity}(AB) = \text{nullity } B$.
 - (d) (3 points) If \mathbf{x}_p is a solution to the system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{y} + \mathbf{x}_p$ is also a solution to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{y} \in \text{Nullspace } A$.

Solution: The answers for part (a), (b), (c) and (d) are F, F, T and T respectively. For part (c) and (d) you should refer to Kolman and Hill (particularly Ch. 4.7 on *Homogeneous Systems*).

To see that (a) is false consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The nullity of A and B is both 0, but

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which has nullity 2 and $0 + 0$ is by no means equal to 2.

To see that (b) is false, consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then nullity of A is 1 whereas the nullity of B is 0, but

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which has nullity 1. Again, $0 \cdot 1$ is not equal to 1.

2. (8 points) Prove that if \mathbf{u} , \mathbf{v} and \mathbf{w} are in \mathbb{R}^3 and \mathbf{u} is orthogonal to both \mathbf{v} and \mathbf{w} , then \mathbf{u} is orthogonal to every vector in $\text{span}\{\mathbf{v}, \mathbf{w}\}$.

[Hint: What does it mean for a vector \mathbf{x} to be in $\text{span}\{\mathbf{v}, \mathbf{w}\}$ and what does it mean for two vectors to be orthogonal?]

Solution: Starting from the top. We know that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \cdot \mathbf{w} = 0$. Now, what does it mean for \mathbf{x} to be in $\text{span}\{\mathbf{v}, \mathbf{w}\}$? It means that there exists scalars $a_1, a_2 \in \mathbb{R}$ (both can possibly be 0) such that $\mathbf{x} = a_1\mathbf{v} + a_2\mathbf{w}$. Thus

$$\begin{aligned}\mathbf{u} \cdot \mathbf{x} &= \mathbf{u} \cdot (a_1\mathbf{v} + a_2\mathbf{w}) \\ &= \mathbf{u} \cdot (a_1\mathbf{v}) + \mathbf{u} \cdot (a_2\mathbf{w}) \\ &= a_1(\mathbf{u} \cdot \mathbf{v}) + a_2(\mathbf{u} \cdot \mathbf{w}) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

so \mathbf{u} is orthogonal to \mathbf{x} . Since the choice of \mathbf{x} was arbitrary, we conclude that \mathbf{u} is orthogonal to every vector in $\text{span}\{\mathbf{v}, \mathbf{w}\}$.