MA 523: Homework 1

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August 31, 2016

Problem 1.1 (Taylor's formula)

Let $f: \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \ge 2$. Prove that

$$f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} D^{\alpha} f(0) x^{\alpha} + O(|x|^{k+1})$$

as $x \to 0$ for each k = 1, 2, ..., assuming that you know this formula for n = 1.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable g(t) := f(tx). Prove that

$$\frac{d^m}{dt^m}g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^{\alpha} f(tx) x^{\alpha},$$

by induction on m.

Solution. \blacktriangleright Taking the hint, fix $x \in \mathbb{R}^n$ and consider the function of one variable g(t) := f(tx). We claim that

$$\frac{d^m}{dt^m}g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^{\alpha} f(tx) x^{\alpha}.$$

Proof of claim. We shall proceed by induction on m. The case m = 1 follows easily from the chain rule:

$$\frac{d}{dt}g(t) = \frac{d}{dt}f(tx)$$

$$= D^{(1,0,\dots,0)}f(tx)x_1 + \dots + D^{(0,\dots,0,1)}f(tx)x_n$$

$$= (D^{(1,0,\dots,0)}x_1 + \dots + D^{(0,\dots,0,1)}x_n)f(tx)$$

which we can write compactly as

$$= \sum_{|\alpha|=1} \frac{1!}{\alpha!} D^{\alpha} f(tx) x^{\alpha}.$$

Now, assume the result for $n \le m - 1$. Then

$$\frac{d^m}{dt^m}g(t) = \frac{d}{dt} \left[\frac{d^{m-1}}{dt^{m-1}}g(t) \right]$$
$$= \frac{d}{dt} \left[\sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} D^{\alpha} f(tx) x^{\alpha} \right]$$

since the derivative is a linear operator, we have

$$= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \frac{d}{dt} \left[D^{\alpha} f(tx) x^{\alpha} \right]$$
$$= \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \sum_{|\beta|=1} D^{\alpha+\beta} f(tx) x^{\alpha+\beta}$$

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but since f is smooth, the order in which we take derivatives does not matter and, hence the operators commute giving us

$$= \left[\sum_{|\beta|=1} (Dx)^{\beta} \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^{\alpha} \right] f(tx). \tag{1.1}$$

From here it suffices to do some combinatorics on the operators and reduce it to the desired expression. By the multinomial theorem, we have

$$\left(\sum_{|\alpha'|=1}(Dx)^{\alpha'}\right)^{m-1} = \sum_{|\alpha|=m-1} \binom{|\alpha|}{\alpha} (Dx)^{\alpha} = \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^{\alpha}.$$

Thus (1.1) becomes

$$\left[\sum_{|\beta|=1} (Dx)^{\beta} \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} (Dx)^{\alpha}\right] f(tx) = \left[\sum_{|\beta|=1} (Dx)^{\beta} \left(\sum_{|\alpha'|=1} (Dx)^{\alpha'}\right)^{m-1}\right] f(tx)$$

$$= \left[\left(\sum_{|\beta|=1} (Dx)^{\beta}\right)^{m}\right] f(tx)$$

$$= \sum_{|\alpha|=m} \frac{m!}{\alpha!} (Dx)^{\alpha} f(tx)$$

$$= \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^{\alpha} f(tx) x^{\alpha},$$

as desired.

Now, applying Taylor's formula in 1 variable to g(t) and evaluating at t = 1 we have

$$f(x) = g(1)$$

$$= \sum_{i=0}^{k} \frac{g^{(i)}(0)}{i!} 1^{i} + O(|x|^{k+1})$$

$$= \sum_{i=0}^{k} \frac{1}{i!} \sum_{|\alpha|=i} \frac{i!}{\alpha!} D^{\alpha} f(tx) x^{\alpha} + O(|x|^{k+1})$$

$$= \sum_{i=0}^{k} \sum_{|\alpha|=i} \frac{1}{\alpha!} D^{\alpha} f(0) x^{\alpha} + O(|x|^{k+1})$$

$$= \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^{\alpha} f(0) x^{\alpha} + O(|x|^{k+1})$$

as desired.

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Problem 1.2

Write down the characteristic equation for the PDE

$$u_t + b \cdot Du = f \tag{*}$$

on $\mathbb{R}^n \times (0, \infty)$, where $b \in \mathbb{R}^n$. Using the characteristic equation, solve (*) subject to the initial condition

$$u = g$$

on $\mathbb{R}^n \times \{t = 0\}$. Make sure the answer agrees with formula (5) in §2.1.2 of [E].

Solution. ► For reference, formula (5) in §2.1.2 of [E] is the solution to the nonhomogeneous problem

$$u(\mathbf{x},t) = g(\mathbf{x} - t\mathbf{b}) + \int_0^1 f(\mathbf{x} + (s-t)\mathbf{b}, s) ds$$

where $\mathbf{x} \in \mathbb{R}^n$, t > 0.

To make the notation more bearable, we will use \mathbf{b} and \mathbf{x} to denote the original vectors in (*). First, we write (*) as the directional derivative along (\mathbf{b} , 1), (note the abuse of notation)

$$f = u_t + \mathbf{b} \cdot Du$$
$$= (\mathbf{b}, 1) \cdot Du.$$

Using the structure of characteristic ODE, we have the PDE

$$F(p, z, x) = (\mathbf{b}, 1) \cdot p$$

with characteristics

$$\dot{x} = b$$
, $\dot{t} = 1$, $\dot{z} = f$.

Now, given a point $(\mathbf{x}, t) \in \mathbb{R}^n \times (0, \infty)$ we can solve the ODEs \dot{x} and \dot{t} easily as the lines $x(s) = \mathbf{x} - \mathbf{b}t + \mathbf{b}s$ and t = s. Substituting these solutions into \dot{z} , we have

$$\dot{z} = f(x(s), t(s)) = f(\mathbf{x} + \mathbf{b}(s - t), s)$$

so

$$\int_0^t f(x(s), t(s)) = f(\mathbf{x} + \mathbf{b}(s - t), s) ds = \int_0^t \dot{z} ds$$
$$= z(t) - z(0)$$
$$= u(\mathbf{x}, t) - u(\mathbf{x} - \mathbf{b}t, 0).$$

Thus,

$$u(\mathbf{x},t) = u(\mathbf{x} - \mathbf{b}t, 0) + \int_0^t f(x(s), t(s)) = f(\mathbf{x} + \mathbf{b}(s-t), s) ds$$
$$= g(\mathbf{x} - \mathbf{b}) + \int_0^t f(x(s), t(s)) = f(\mathbf{x} + \mathbf{b}(s-t), s) ds$$

as desired.

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Problem 1.3

Solve using the characteristics:

(a)
$$x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2$$
, $u = 1$ on the line $x_2 = 2x_1$.

(b)
$$uu_{x_1} + u_{x_2} = 1$$
, $u(x_1, x_1) = x_1/2$.

(c)
$$x_1u_{x_1} + 2x_2u_{x_2} + u_{x_3} = 3u, u(x_1, x_2, 0) = g(x_1, x_2).$$

Solution. ► For part (a), employing the method of characteristics, we write

$$F(p, z, x) = (x_1^2, x_2^2) \cdot p = z^2.$$

From here, we have

$$\dot{x} = (x_1^2, x_2^2), \qquad \dot{z} = z^2.$$

Now say x(0) = (2t', t') and solve for \dot{x} ,

$$\int_{0}^{t} ds = \int_{0}^{t} \frac{1}{x_{1}(s)^{2}} dx_{1}(s)$$

$$\int_{0}^{t} ds = \int_{0}^{t} \frac{1}{x_{1}(s)^{2}} dx_{1}(s)$$

$$t = -\frac{1}{x_{1}(t)} + \frac{1}{x_{1}(0)}$$

$$t = -\frac{1}{x_{2}(s)} + \frac{1}{x_{2}(0)}$$

$$x_{1}(t) = \frac{1}{1/x_{1}(0) - t}$$

$$x_{2}(t) = \frac{1}{1/x_{2}(0) - t}$$

$$= \frac{2t'}{1 - 2tt'}$$

$$= \frac{t'}{1 - tt'}.$$

Thus,

$$x(t) = \left(\frac{2t'}{1 - 2tt'}, \frac{t'}{1 - tt'}\right)$$

and solving for z similarly yields

$$z(t) = \frac{z(0)}{1 - tz(0)} = \frac{1}{1 - t}$$

since $z = u^2 = 1$ on the line $x_2 = 2x_1$. Lastly, solving for t in terms of x_1 and x_2 , we have

$$t' = \frac{x_1}{2(tx_1 + 1)}$$
$$= \frac{x_2}{tx_2 + 1}$$

which, with a little algebra, can be turned into

$$t = \frac{1 - 2x_2/x_1}{x_2} = \frac{x_2 - 2x_1}{x_1 x_2}.$$

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Now substituting this into the solution z(t), we have

$$u(x, y) = z(t)$$

$$= \frac{1}{1 - t}$$

$$= \frac{1}{1 - (x_2 - 2x_1)/(x_1 x_2)}$$

$$= \frac{x_1 x_2}{x_1 x_2 - x_2 + 2x_1}.$$

For part (b), write

$$F(p, z, x) = (z, 1) \cdot p = 1.$$

Then, we have

$$\dot{x} = (z, 1), \qquad \dot{z} = 1.$$

From here, fix $x^0 \in \mathbb{R}$ and reparameterize x(t) such $x(0) = (x^0, x^0)$ then solutions have the form

$$x_1(t) = \frac{t^2}{2} + \frac{x^0}{2}t + x^0,$$

$$x_2(t) = t + x^0,$$

$$x_2(t) = t + x^0,$$

Now, solve for t in terms of x_1, x_2 (using the square root formula), we have

$$t = -\left(\frac{x^0}{2} - 1\right) \pm \sqrt{\left(\frac{x^0}{2} - 1\right)^2 - 2x_2}$$

For part (c), we have

$$F(p, z, x) = (x_1, 2x_2, 1) \cdot p = 3z.$$

Then

$$\dot{x} = (x_1, 2x_2, 1), \qquad \dot{z} = 3z.$$

From here, reparameterize $x_3(t)$ such $x_3(0) = 0$ then solutions have the form

$$x_1(t) = x_1' e^t,$$
 $x_2(t) = x_2' e^{2t},$ $x_3(t) = t,$ $z(t) = g(x_1', x_2') e^{3t}.$

Then the solution has the form

$$u(x_1, x_2, x_3) = z(t)$$

= $g(x'_1, x'_2)e^{3x_3}$

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Problem 1.4

For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2} (u_{x_1}^2 + u_{x_2}^2)$$

find a solution with $u(x_1, 0) = (1 - x_1^2)/2$.

Solution. ► By inspection, the PDE looks like should be separable, so suppose we have a solution of the form

$$u(x_1, x_2) = u_1(x_1) + u_2(x_2).$$

Then

$$u_1(x_1) + u_2(x_2) = x_1 \dot{u}_1(x_1) + x_2 \dot{u}_2(x_2) + \frac{1}{2} \left(\dot{u}_1(x_1)^2 + \dot{u}_2(x_2)^2 \right).$$

Now we solve u_1 and u_2 separately. For u_1 , we have

$$u_1 = x_1 \dot{u}_1(x_1) + \frac{1}{2} \dot{u}(x_1)^2$$

which has a solution of the form

$$ax_1^2 + bx_1 + c.$$

Plugging this in, we have

$$ax_1^2 + bx_1 + c = x_1(2ax_1 + b) + a$$
$$= 2ax_1^2 + bx_1 + a.$$

so a = c = 0 and the solution is of the form $u_1(x_1) = bx_1$. Similarly, for $u_2(x_2)$, we have $u_2(x_2) = b'x_2$.

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