Representation Theory

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Chapter 1

What is Representation Theory?

Groups arise in nature as "sets of symmetries (of an object), which are closed under composition and under taking inverses". For example, the *symmetric group* S_n is the group of all permutations (symmetries) of $\{1, \ldots, n\}$; the *alternating group* A_n is the set of all symmetries preserving the parity of the number of ordered pairs; the *dihedral group* D_{2n} is the group of symmetries of the regular n-gon in the plane. The *orthogonal group* $\mathrm{O}(3)$ is the group of distance-preserving transformations of Euclidean space which fix the origin. There is also the group of *all* distance preserving transformations, which includes the translations along with $\mathrm{O}(3)$.*

The official definition is of course more abstract, a group is a set G with a binary operation $*$ which is associative, has a unit element e and for which inverses exist. Associativity allows a convenient abuse of notation, where we write gh for $g * h$; we have $ghk = (gh)k = g(hk)$ and parentheses are unnecessary. I will often write 1 for e, but this is dangerous on rare occasions, such that when studying the group $\mathbb{Z}$ under addition; in that case, $e = 0$.

The abstract definition notwithstanding, the interesting situation involves a group "acting" on a set. Formally, an action of a group G on a set X is an "action map" $a \colon G \times X \to X$ which is *compatible with the group law*, in the sense that

$$\begin{aligned} a\big(h, a(g, x)\big) &= a(hg, x) \\ a(e, x) &= x. \end{aligned}$$

This justifies the abuse of notation $a(g, x) = gx$, for we have $h(gx) = (hg)x$.

From this point of view, geometry asks, "Given a geometric object X, what is its group of symetries?" Representation theory reverses the quostion to "Given a group G, what objects X does it act on?" and attempts to answer this question by classifying such X up to isomorphism.

Before restricting to the linear case, our main concern, let us remember another way to describe an action of G on X. Every $g \in G$ defines a map $a(g) \colon X \to X$ by $x \mapsto gx$. This map is a bijection, with inverse map $a(g^{-1})$: indeed, $\big(a(g^{-1}) \circ a(g)\big)(x) = g^{-1}gx = ex = x$ from the properties of the action. Hence $a(g)$ belongs to the set $\operatorname{Sym} X$ of bijective self-maps of X. This set forms a group under composition, and the properties of an action imply that

Proposition 1.1. *An action of G on X "is the same as" a group homomorphism $\alpha \colon G \to \operatorname{Sym} X$.*

*This group is isomorphic to the *semi-direct product* $\mathrm{O}(3) \ltimes \mathbb{R}^3$.

The formulation of Prop. 1.1 leads to the following observation. For any action a of H on X and group homomorphism $\varphi\colon G \to H$, there is defined a *restricted* or *pulled-back* action $\varphi^* a$ of G on X, as $\varphi^* a = a \circ \varphi$. In the original definition, the action sends (g, x) to $\varphi(g)(x)$.

Example 1.1 (Tautological action of $\operatorname{Sym} X$ on X)**.** This is the obvious action, call it T, sending f, x to $f(x)$, where $f\colon X \to X$ is a bijection and $x \in X$. In this language, the action a of G on X is $\alpha^* T$ with the homomorphism α of the proposition – the pull-back under α of the tautological action.

Example 1.2 (Linearity)**.** The question of classifying all possible X with action of G is hopeless in such generality, but one should recall that, in first approximation, mathematics is linear. So we shall take our X to be a*vector space* over some ground *field*, and ask that the action of G be linear, as well, in other words, that it should preserve the vector space structure. Our interest is mostly confined to the case when the field of scalars is $\mathbb{C}$, although we shall occasionally mention how the picture changes when other fields are studied.

Definition 1.1. A linear representation ρ of G on a complex vector space V is a set-theoretic action on V which preserves the linear structure, i.e.,

$$\begin{aligned} \rho(g)(\mathbf{v}_1 + \mathbf{v}_2) &= \rho(g)\mathbf{v}_1 + \rho(g)\mathbf{v}_2, && \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V, \\ \rho(g)(k\mathbf{v}) &= k\rho(g)\mathbf{v} && \text{for all } k \in \mathbb{C}, \mathbf{v} \in V. \end{aligned}$$

Unless otherwise mentioned, a *representation* will mean a *finite-dimensional complex representation*.

Example 1.3 (The general linear group)**.** Let V be a complex vector space of dimension $n < \infty$. After choosing a basis, we can identify it with $\mathbb{C}^n$, although we shal lavoid doing so without good reason. Recall that the *endomorphism algebra* $\operatorname{End}(V)$ is the set of all linear maps (or *operators*) $L\colon V \to V$, with the natural addition of linear maps and the composition as multiplication. If V has been identified with $\mathbb{C}^n$, a linear map is uniquely representable by a matrix, and the addition of linear maps becomes the entrywise addition, while the composition becomes the matrix multiplication.

Inside $\operatorname{End}(V)$, there is contained the group $\mathrm{GL}(V)$ of invertible linear operators; the group operation, of course, is composition.

Proposition 1.2. *V is naturally a representation of* $\mathrm{GL}(V)$.

It is called the *standard* representation of $\mathrm{GL}(V)$. The following corresponds to Prop. 1.1, involvinge the same abuse of language.

Proposition 1.3. *A representation of G on V "is the same as" a group homomorphism from G to* $\mathrm{GL}(V)$.

Proof. Observe that, to give a linear action of G on V, we must assign to each $g \in G$ a linear self-map $\rho(g) \in \operatorname{End}(V)$. Compatibility of the action with the group law requires

$$\rho(h)\big(\rho(g)(\mathbf{v})\big) = \rho(hg)(\mathbf{v}), \qquad \rho(1)(\mathbf{v}) = \mathbf{v},$$

for all $\mathbf{v} \in V$, whence we conclude that $\rho(1) = \mathrm{id}$, $\rho(hg) = \rho(h) \circ \rho(g)$. Taking $h = g^{-1}$ shows that $\rho(g)$ is invertible, hence lands in $\mathrm{GL}(V)$. The first relation then says that we are dealing with a group homomorphism. ∎

Definition 1.2. An *isomorphism* φ between two representation (ρ_1, V_1) and (ρ_2, V_2) of G is a linear isomorphism $\varphi\colon V_1 \to V_2$ which intertwines with the action of G, that is, satisfies

$$\varphi\big(\rho_1(h)(\mathbf{v})\big) = \rho_2(g)\big(\varphi(\mathbf{v})\big).$$

Note that the equality makes sense even if φ is not invertible, in which case it is just called an *intertwining operator* or *G-linear map*. However, if φ is invertible, we can write instead

$$\rho_2 = \varphi \circ \rho_1 \circ \varphi^{-1}, \tag{1}$$

meaning that we have an equality of linear maps after intserting any group element g. Observe that this relation determines ρ_2 if ρ_1 and φ are known. We can finally formulate the basic problem of representation theory: Classify all representation of a given group G, up to isomorphism.

For arbitrary G, this is very hard! We shall concentrate on finite groups, where a very good general theory exists. Later on, we shall study some examples of topological compact groups, such as U(1) and SU(2). The general theory for compact groups is also completely understood, but requires more difficult methods.

I close with a simple observation.