MA557 Problem Set 3

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Problem 3.1

Find an example of a finitely generated ring extension $R \subset S$ where S is a Noetherian ring, but R is not.

Proof. Let k be a field and consider its polynomial ring $k[X, Y]$ in two variables. Then we claim that the subring $k[XY, XY^2, ...]$ is non-Noetherian but its extension to $k[X, Y]$ (by adjoining the indeterminants X and Y) is Noetherian by Hilbert's basis theorem. Consider the increasing chain of ideals

$$(XY) \subsetneq (XY, XY^2) \subsetneq (XY, XY^2, XY^3) \subsetneq \cdots .$$

This chain does not stabilize for suppose that it did, then for some positive integer n, we have $(XY, XY^2, ..., XY^n) = (XY, XY^2, ..., XY^n, XY^{n+1})$ so $XY^{n+1} \in (XY, XY^2, ..., XY^n)$. But this implies that $XY^{n+1} = p(XY, XY^2, ...)q(XY, ..., XY^n)$ for some polynomials $q \in k[XY, XY^2, ...]$, $q \in (XY, ..., XY^n)$. Thus, we have that

$$\begin{aligned} \deg_Y(XY^{n+1}) &= n + 1 \\ &= \deg_Y p + \deg_Y q \end{aligned} \qquad \begin{aligned} \deg_X(XY^{n+1}) &= 1 \\ &= \deg_X p + \deg_X q. \end{aligned}$$

Since $\deg_Y q \leq n$, $\deg_Y p \geq 1$. Thus, $\deg_X p = 1$ so $q \in k$, i.e., q is a unit. This is a contradiction since $(XY, ..., XY^n)$ is a proper ideal. ■

Problem 3.2

Consider the homomorphism of rings

$$\begin{array}{ccc} & & S \\ & & \big\downarrow{\scriptstyle\psi} \\ R & \xrightarrow{\varphi} & T. \end{array}$$

The *fiber product* of R and S over T is the subring $R \times_T S = \{ (r, s) \mid \varphi(t) = \psi(s) \}$ of $R \times S$. Assume φ and ψ are surjective. Show that if R and S are Noetherian rings then so is $R \times_T S$.

Proof. We first prove the following result:

Lemma (Matsumura, Ex. 3.1). *Let $\mathfrak{a}_1, ..., \mathfrak{a}_n$ be ideals of a ring A such that $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = 0$. If each $A/\mathfrak{a}_i$ is a Noetherian ring then so is A.*

$$0 \longrightarrow R \times_T S \overset{\iota}{\hookrightarrow} R \times S \overset{\Phi^*}{\twoheadrightarrow} \frac{T \times T}{\Delta_T} \longrightarrow 0.$$

■

Problem 3.3

Let M be an R-module. Show that M is a flat R-module if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.

Proof. $\implies$ ■

Problem 3.4

Let M be an R-module and $\mathfrak{a}$ an R-ideal.

(a) Show that if $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, then $M = \mathfrak{a}M$.
(b) Show that the converse holds in case M is finite.

Proof. (a) Suppose that $M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. ■

Problem 3.5

Prove that every power of a maximal ideal is primary.

Proof. ■

Problem 3.6

(a) Show that the radical of a primary ideal is prime.
(b) Find an example of a power of a prime ideal that is not primary.
(c) Let $\mathfrak{p}$ be a prime ideal of a ring R and $n \in \mathbf{N}$. The R-ideal $\mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$ s called the *nth symbolic power of* $\mathfrak{p}$. Show that $\mathfrak{p}^{(n)}$ is primary.

Proof. ■