MA 519: Homework 14

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PROBLEM 14.1 (HANDOUT 18, # 15)

(X, Y) is distributed uniformly inside of the unit circle. Find the density of $X + Y$ and hence the mean of $X + Y$. Was the value of the mean obvious? Why?

SOLUTION. Suppose the random vector (X, Y) is uniformly distributed in $D := \{\, x^2 + y^2 < 1 \,\}$. It is rather clear that the joint density of (X, Y) is given by the expression

$$p(x, y) = \begin{cases} \frac{1}{\pi} & \text{for } (x, y) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We do not need to find the density of $Z := X + Y$ as the expectation is given by evaluating the integral

$$\begin{aligned} E(Z) &= \int_D (x + y) p(x, y) \, dA(x, y) \\ &= \frac{1}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x + y) \, dy \, dx \\ &= \frac{2}{\pi} \int_{-1}^{1} x \sqrt{1 - x^2} \, dx \\ &= 0. \end{aligned}$$

Yes, the mean was obvious. If we go through the trouble of computing the density of Z we will see that this distribution is symmetric about the origin so we should expect $E(Z)$ to be zero.

Let us now find the PDF of Z. First the CDF of Z is given by the expression

$$F_Z(z) = P(Z \le z)$$

after rotating $\frac{\pi}{4}$ degrees, the problem reduces to geometry, and we have

$$= \begin{cases} 0 & \text{for } -\infty < z \le -\sqrt{2}, \\ \frac{1}{\pi} \left[\cos^{-1}\left(-\frac{z}{\sqrt{2}}\right) + \frac{z}{\sqrt{2}} \sqrt{1 - \frac{z^2}{2}} \right] & \text{for } -\sqrt{2} < z \le 0, \\ 1 - \frac{1}{\pi} \left[\cos^{-1}\left(-\frac{z}{\sqrt{2}}\right) + \frac{z}{\sqrt{2}} \sqrt{1 - \frac{z^2}{2}} \right] & \text{for } 0 < z < \sqrt{2}, \\ 1 & \text{for } \sqrt{2} \ge z < \infty. \end{cases}$$

Now, taking the derivative with respect to z, we have

$$f_Z(z) = \begin{cases} \frac{1}{\sqrt{2}} \sqrt{1 - \frac{z^2}{2}} & \text{for } -\sqrt{2} < z < \sqrt{2}, \\ 0 & \text{otherwise} \end{cases}$$

which is clearly symmetric about the origin. ∎

PROBLEM 14.2 (HANDOUT 18, # 16)

Let X be a random number in $[0, 1]$. What is the probability that the number 5 is completely missing from the decimal expansion of X?

SOLUTION. First let us establish some notation. Let Ω denote the interval $[0, 1]$ and let A denote the set of all real numbers in Ω which do not contain a 5 in their decimal expansion. We show that the probability that X is in A is zero; i.e., that $P(X \in A) = 0$.

To this end, let us consider the following Bernoulli process: Let N_k denote the kth digit in the decimal expansion of X. Then N_k is uniformly distributed on $\{0, \dots, 9\}$ and the probability that N_k is not the number 5 is $\frac{9}{10}$. Let I_k be an indicator random variable given by

$$I_k := \begin{cases} 0 & \text{for } N_k = 5, \\ 1 & \text{otherwise.} \end{cases}$$

Then the sequence of random variables $\{I_k\}$ with $p = \frac{9}{10}$ forms a sequence of independent Bernoulli trials wit expected value

$$E\left(\prod_{k=1}^{n} I_k\right) = \prod_{k=1}^{n} E(I_k) = \prod_{k=1}^{n} p = p^n.$$

The expectation above is in fact the probability every digit of X from the first to the nth is not 5.

Taking the limit of the expression above, the probability that X does not contain a 5 in its decimal expansion is the limit

$$\lim_{n \to \infty} p^n = \lim_{n \to \infty} \left(\frac{9}{10}\right)^n = 0. \qquad \blacksquare$$

PROBLEM 14.3 (HANDOUT 18, # 17)

A foot long stick is broken into three pieces. Find the density functions of the length of the longest part, the smallest part, and the medium part. What are the expected values for each part?

SOLUTION. We should think of breaking the foot long stick into three pieces as choosing a point W randomly from $I := [0, 1]$ and then, given that $W = w$, choosing a point X randomly from $I_w := [0, w]$. Set $Y := W - X$, $Z := 1 - W$. Then if we denote by L_s, L_m, and L_l the smallest, medium, and largest parts of the broken stick, respectively, we have

$$L_s = \min\{X, Y, Z\}, \; L_l = \max\{X, Y, Z\}, \text{ and } L_m = 1 - (L_s + L_l).$$

Now let us find the densities f_s, f_m, and f_l of L_s, L_m, and L_l respectively.   ∎