MA 544: Homework 3

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Problem 3.1 (Wheeden & Zygmund §3, Ex. 5)

Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage each interval removed has length $\delta 3^{-k}$, $0<\delta<1$. Show that the resulting set is perfect, has measure $1-\delta$, and contains no interval.

Proof. Put $C_0 := [0,1]$. We begin constructing our desired set by removing the open set $\left(\frac{\delta}{3}, 1-\frac{\delta}{3}\right)$ from the closed interval $[0,1]$. This separates $[0,1]$ into the union of two disjoint closed (and bounded therefore, compact) intervals $\left[0,\frac{\delta}{3}\right]$ and $\left[1-\frac{\delta}{3},1\right]$ which we shall call C_1. Next, we remove the open interval $\left(\frac{\delta}{9},\frac{\delta}{3}-\frac{\delta}{9}\right)$ from $\left[0,\frac{\delta}{3}\right]$ and the open interval $\left(1-\frac{\delta}{3}+\frac{\delta}{9},1-\frac{\delta}{9}\right)$ and end up with the union C_2 of four disjoint closed intervals. Continue in this fashion ad infinitum. Note that each $C_{k+1}\subset C_k$ and each C_k is a finite union of closed subsets thus, by theorem 1.7 and the Cantor's intersection theorem, the set $C_\delta := \bigcap_{i=1}^\infty C_i$ is closed, compact, and nonempty.

Now we show that the set we have constructed, C_δ, is perfect. Since C_δ is closed, it remains to show that C_δ contains no isolated points. Note that, in the construction of C_δ, we never removed the endpoints the intervals which union to C_k. Thus, the endpoints of the intervals which union to C_k are in C_δ. Before continuing, we need to figure out what the length of each interval at the kth stage in the construction is and in the process, we shall prove that $|C_\delta| = 1-\delta$.

At each stage in the construction (except for $k=0$) we removed 2^{k-1} open intervals of length $\delta 3^{-k}$. Thus, at the kth stage of the construction, the measure of C_k will be

$$|C_k| = 1-\sum_{i=1}^k \frac{2^{i-1}\delta}{3^i} = 1-\frac{\delta}{3}\sum_{i=1}^k\left(\frac{2}{3}\right)^{i-1}.$$

We immediately recognize the right-hand side as a geometric sum so letting $k\to\infty$, by theorem 3.26(ii), we have

$$|C_\delta| = \lim_{k\to\infty} 1-\frac{\delta}{3}\sum_{i=1}^k\left(\frac{2}{3}\right)^{i-1} = 1-\lim_{k\to\infty}\frac{\delta}{3}\sum_{i=1}^k\left(\frac{2}{3}\right)^{i-1} = 1-\frac{\delta}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1-\delta.$$

Now, let $\varepsilon>0$ be given. By the Archimedean principle, we may choose a sufficiently large natural number N so that $|C_N|2^{-N} < 2^{-N} < \varepsilon$. Let x be a point in C_δ, then $x\in C_N$ since $x\in C_k$ for all k. In particular, x is in one of he 2^N disjoint closed intervals that union to C_k, call it I. Let x' be the closest endpoint of I to x (if x is itself an endpoint, choose x' to be the opposite endpoint). Then, by the triangle inequality, we have $|x-x'|\le |C_N|2^{-N}<\varepsilon$. Hence, the open neighborhood $B(x,\varepsilon)\setminus\{x\}\neq\emptyset$ for any ε. Thus, C_δ is perfect.

Last but not least, we show that C_δ contains no interval. Suppose that (a,b) is an interval contained in C_δ. Hence, $(a,b)\subset C_k$ for all k. (I don't know how to finish the proof without using a fact about connected 1-manifolds). Then, since (a,b) is connected, it must be contained in an connected component C of C_δ. However, the connected components of C_k, i.e., the closed intervals, have measure less than 2^{-k} so $b-a\le 2^{-k}$. Letting $k\to\infty$, we have $b-a\le 0$ which leads to a contradiction since the measure of an interval is strictly greater than 0. ∎

Problem 3.2 (Wheeden & Zygmund §3, Ex. 7)

Prove (3.15).

Proof.

Lemma (Wheeden & Zygmund (3.15))**.** *If $\{I_k\}_k^N$ is a finite collection of nonoverlapping intervals, then $\bigcup I_k$ is measurable and $|\bigcup I_k| = \sum |I_k|$.*

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Problem 3.3 (Wheeden & Zygmund §3, Ex. 8)

Show that the Borel algebra $\mathcal{B}$ in $\mathbf{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbf{R}^n$.

Proof. ∎

Problem 3.4 (Wheeden & Zygmund §3, Ex. 9)

If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ (and also $\liminf E_k$) has measure zero.

Proof. Define $E \coloneqq \limsup E_k$ and $E'_\ell \coloneqq \bigcup_{k=\ell}^{\infty} E_k$. Then E'_ℓ is a decreasing (with respect to inclusion) sequence of sets with $\lim_\ell E'_\ell = E$. ∎

Problem 3.5 (Wheeden & Zygmund §3, Ex. 10)

If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

Proof. ■