MA52300 FALL 2016

Homework Assignment 1 – *Solutions*

1 (Taylor's formula)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, $n \geq 2$. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}) \quad \text{as } x \to 0$$

for each $k = 1, 2, \ldots$, assuming that you know this formula for $n = 1$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$. Prove that

$$\frac{d^m}{dt^m} g(t) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(tx) x^\alpha,$$

by induction in m.

Solution. Fix, $x \in \mathbb{R}^n$ and apply the Taylor's formula for $g(t) = f(tx)$. We have

$$g(t) = \sum_{m=0}^{k} \frac{g^{(m)}(0)}{m!} t^m + \frac{g^{(k+1)}(\theta t)}{(k+1)!} t^{k+1}.$$

Hence, the formula for f will follow, once we show that

$$g^{(m)}(0) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha f(0) x^\alpha.$$

To prove this, observe that formally we have

$$\frac{d}{dt} = \sum_{i=1}^{n} x_i \partial_{x_i},$$

and consequently

$$\frac{d^m}{dt^m} = \left(\sum_{i=1}^{n} x_i \partial_{x_i} \right)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha D^\alpha.$$

The last equality is from the multinomial theorem, which is applicable here as the operators $x_i \partial_{x_i}$, $i = 1, \ldots, n$ commute with each other, i.e., $x_i \partial_{x_i}(x_j \partial_{x_j} u) = x_j \partial_{x_j}(x_i \partial_{x_i} u)$. This can be formally justified by induction in m. $\square$

2. Write down the characteristic equation for the PDE

$$(*) \qquad u_t + b \cdot Du = f \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

Where $b \in \mathbb{R}^n$, $f = f(x,t)$. Using the characteristic equation, solve $(*)$ subject to the initial condition

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}.$$

Make sure your answer agrees with formula (5) in §2.1.2 of [E].

Solution. Recall that we treat t as x_{n+1}. Then the characteristic ODE for $(*)$ is

$$\dot{x}^i = b_i, \quad i = 1, \dots, n \quad \dot{x}^{n+1} = 1, \quad \dot{z} = f(x^1, \dots, x^{n+1})$$

subject to the initial conditions

$$x^i(0) = x_i^0, \quad i = 1, \dots, n, \quad x^{n+1}(0) = 0, \quad z(0) = g(x^0).$$

This easily gives

$$x^i(s) = x_i^0 + b_i s, \quad i = 1, \dots, n, \quad x^{n+1}(s) = s$$

and

$$z(s) = g(x^0) + \int_0^s f(x^0 + b\tau, \tau) d\tau.$$

For given (x,t) now let (x^0, s) be such that

$$x_i = x^i(s) = x_i^0 + b_i s, \quad i = 1, \dots, n, \quad t = x^{n+1}(s) = s.$$

We then have $s = t$ and $x_i^0 = x_i - b_i t$ and therefore

$$u(x,t) = z(s) = g(x - bt) + \int_0^t f(x + b(\tau - t), \tau) d\tau.$$

□

3. Solve using characteristics:

(a) $x_1^2 u_{x_1} + x_2^2 u_{x_2} = u^2, \quad u = 1$ on the line $x_2 = 2x_1$.
(b) $u u_{x_1} + u_{x_2} = 1, \quad u(x_1, x_1) = \frac{1}{2} x_1$.
(c) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \quad u(x_1, x_2, 0) = g(x_1, x_2)$

Solution. **a.** The characteristic ODE for this quasi-linear equation is

$$\dot{x}^1 = (x^1)^2, \quad \dot{x}^2 = (x^2)^2, \quad \dot{z} = z^2$$

with the initial conditions

$$x^1(0) = x^0, \quad x^2(0) = 2x^0, \quad z(0) = z^0 = 1.$$

Using the separation of variables, we find the following formulas for the solution

$$x^1(s) = \frac{x^0}{1 - x^0 s}, \quad x^2(s) = \frac{2x^0}{1 - 2x^0 s}, \quad z(s) = \frac{1}{1 - s}.$$

Let now (x^0, s) be such that $(x_1, x_2) = (x^1(s), x^2(s))$. Then (assuming $x^0 \neq 0$)

$$\frac{1}{x_1} = \frac{1}{x^0} - s, \quad \frac{1}{x_2} = \frac{1}{2x^0} - s$$

and consequently
$$s = \frac{1}{x_1} - \frac{2}{x_2}.$$
This gives
$$u(x_1, x_2) = z(s) = \frac{1}{1 - \frac{1}{x_1} + \frac{2}{x_2}} = \frac{x_1 x_2}{x_1 x_2 - x_2 + 2x_1}$$
at least near the line $x_2 = 2x_1$, away from the origin.

b. The characteristic ODE is
$$\dot{x}^1 = z, \quad \dot{x}^2 = 1, \quad \dot{z} = 1$$
and the initial conditions are
$$x^1(0) = x^0, \quad x^2(0) = x^0, \quad z(0) = z^0 := x^0/2.$$
Then we have
$$\begin{aligned} x^2(s) &= x^0 + s \\ z(s) &= z^0 + s = x^0/2 + s \end{aligned}$$
and consequently
$$x^1(s) = x^0 + (x^0/2)s + s^2/2.$$
Let now (x^0, s) be such that $(x_1, x_2) = (x^1(s), x^2(s))$. We need to express z in terms of x_1 and x_2, by eliminating x^0 and s. In fact, we first express x^0 and s in terms of x_2 and z:
$$x^0 = 2(x_2 - z), \quad s = 2z - x_2.$$
Then, plugging this into the formula for $x^1(s)$, we obtain
$$\begin{aligned} x_1 &= 2(x_2 - z) + (x_2 - z)(2z - x_2) + \frac{1}{2}(2z - x_2)^2 \\ &= -\frac{1}{2}x_2(x_2 - 4) + (x_2 - 2)z. \end{aligned}$$
Thus,
$$u(x_1, x_2) = \frac{2x_1 + x_2^2 - 4x_2}{2(x_2 - 2)}$$
is a solution, provided $x_2 \neq 2$.

c. The characteristic ODE is
$$\dot{x}^1 = x^1, \quad \dot{x}^2 = 2x^2, \quad \dot{x}^3 = 1, \quad \dot{z} = 3z$$
and initial conditions are
$$x^1(0) = x_1^0, \quad x^2(0) = x_2^0, \quad x^3(0) = 0, \quad z(0) = z^0 := g(x_1^0, x_2^0).$$
We have
$$x^1(s) = x_1^0 e^s, \quad x^2(s) = x_2^0 e^{2s}, \quad x^3(s) = s, \quad z(s) = z^0 e^{3s}.$$

Let now (x^0, s) be such that $x^i(s) = x_i$, $i = 1, 2, 3$. We want to express z in terms of x_i. We have

$$s = x_3, \quad x_1^0 = x_1 e^{-x_3}, \quad x_2^0 = x_2 e^{-2x_3}.$$

Thus,

$$z = g(x^0)e^{3s} = g(x_1 e^{-x_3}, x_2 e^{-2x_3})e^{3x_3}$$

and

$$u(x) = h(x_1 e^{-x_3}, x_2 e^{-2x_3})e^{3x_3}$$

is the solution. $\square$

4. For the equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2)$$

find a solution with $u(x_1, 0) = \frac{1}{2}(1 - x_1^2)$.

Solution. The equation can be written as $F(Du, u, x) = 0$, where $F(p, z, x) = x_1 p_1 + x_2 p_2 + \frac{1}{2}(p_1^2 + p_2^2) - z$. Thus, the system of characteristic ODEs is

$$\begin{aligned}
&\dot{x}^1 = F_{p_1} = x^1 + p^1, \quad \dot{x}^2 = F_{p_2} = x^2 + p^2 \\
&\dot{p}^1 = -F_z p^1 - F_{x_1} = 0, \quad \dot{p}^2 = -F_z p^2 - F_{x_2} = 0 \\
&\dot{z} = F_{p_1} p^1 + F_{p_2} p^2 = (x^1 + p^1)p^1 + (x^2 + p^2)p^2
\end{aligned}$$

subject to the initial conditions

$$\begin{aligned}
&x^1(0) = x^0, \quad x^2(0) = 0, \quad z(0) = z^0 := \frac{1}{2}(1 - (x^0)^2) \\
&p^1(0) = p_1^0, \quad p^2(0) = p_2^0,
\end{aligned}$$

where p_1^0 and p_2^0 are found from the compatibility conditions

$$p_1^0 = g_{x_1}(x^0), \quad F(p^0, z^0, (x^0, 0)) = x^0 p_1^0 + \frac{1}{2}((p_1^0)^2 + (p_2^0)^2) - z^0 = 0.$$

The compatibility conditions imply

$$p_1^0 = -x^0, \quad p_2^0 = \pm 1.$$

Thus, we are going to have two solutions, by taking $+$ or $-$ sign for p_2^0. Now, solving the characteristic ODE, we obtain

$$\begin{aligned}
&p^1(s) = -x^0, \quad p^2(s) = \pm 1 \\
&x^1(s) = x^0, \quad x^2(s) = \pm(e^s - 1) \\
&z(s) = \frac{1}{2}(1 - (x^0)^2) + (e^s - 1).
\end{aligned}$$

Now, given (x_1, x_2), let (x^0, s) be such that $x_i = x^i(s)$, $i = 1, 2$. Then, eliminating x^0 and s, we obtain

$$u(x_1, x_2) = z = \frac{1}{2}(1 - x_1^2) \pm x_2. \quad \square$$