

MA 523: Homework, Midterms and Practice Problems Solutions

Carlos Salinas

Last compiled: October 5, 2016

Contents

1	Midterms and Qualifying Exams	2
1.1	Qualifying Exam, August '04	2
1.2	Qualifying Exam, August '05	4
1.3	Qualifying Exam, January '14	6

1 Midterms and Qualifying Exams

1.1 Qualifying Exam, August '04

EXERCISE 1.1. Consider the initial value problem

$$\begin{cases} a(x, y)u_x + b(x, y)u_y = -u, \\ u = f \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\},$$

where a and b satisfy

$$a(x, y) + b(x, y)y > 0$$

for any $x, y \in \mathbb{R}^n \setminus \{(0, 0)\}$.

- (a) Show that the initial value problem has a unique solution in a neighborhood of S^1 . Assume that a , b , and f are smooth.
- (b) Show that the solution of the initial value problem actually exists in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

SOLUTION. ■

EXERCISE 1.2. Let $u \in C^2(\mathbb{R} \times [0, \infty))$ be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where f and g have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dx,$$

and the potential energy by

$$P(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) dx.$$

Show that

- (a) $K(t) + P(t)$ is constant in t ,
- (b) $K(t) = P(t)$ for all large enough times t .

SOLUTION. ■

EXERCISE 1.3. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = e^{-t}g(x) & \text{for } x \in \mathbb{R}^3, t > 0, \\ u(x, 0) = u_t(x, 0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution $u = e^{-t} + t - 1$ when $g(x) = 1$.

SOLUTION. ■

EXERCISE 1.4. Let Ω be a bounded open set in \mathbb{R}^n and $g \in C_0^\infty(\Omega)$. Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega, t > 0, \\ u(x, 0) = g(x) & \text{for } x \in \Omega, \\ u(x, t) = 0 & \text{for } x \in \partial\Omega, t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n, t > 0, \\ v(x, 0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put $g = 0$ outside Ω .

(a) Show that

$$-v(x, t) \leq u(x, t) \leq v(x, t)$$

for any $x \in \Omega, t > 0$.

(b) Use (a) to conclude that

$$\lim_{t \rightarrow \infty} u(x, t) = 0,$$

for any $x \in \Omega$.

SOLUTION. ■

EXERCISE 1.5. Let $P_k(x)$ and $P_m(x)$ be homogeneous harmonic polynomials in \mathbb{R}^n of degrees k and m respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \quad P_m(\lambda x) = \lambda^m P_m(x),$$

for any $x \in \mathbb{R}^n, \lambda > 0$,

$$\Delta P_k = 0, \quad \Delta P_m = 0$$

in \mathbb{R}^n .

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = k P_k(x), \quad \frac{\partial P_m(x)}{\partial \nu} = m P_m(x)$$

on ∂B_1 , where $B_1 = \{ |x| < 1 \}$ and ν is the outward normal on ∂B_1 .

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x) dS = 0,$$

if $k \neq m$.

SOLUTION. ■

1.2 Qualifying Exam, August '05

EXERCISE 1.6.

- (a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ u = 1 \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\}.$$

- (b) Is the solution unique in a neighborhood of the point $(1, 0)$? Justify your answer.

SOLUTION. ■

EXERCISE 1.7. Consider the second order PDE in $\{x > 0, y > 0\} \subset \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

- (a) Classify the equation and reduce it to the canonical form.
 (b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy}G(x/y).$$

SOLUTION. ■

EXERCISE 1.8. Let Φ be the fundamental solution of the Laplace equation in \mathbb{R}^3 and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y)f(y) dy$$

is a solution of the Poisson equation $-\Delta u = f$ in \mathbb{R}^n . Show that if f is radial (i.e., $f(y) = f(|y|)$) and supported in $B_R = \{|x| < R\}$, then

$$u(x) = c\Phi(x),$$

for any $x \in \mathbb{R}^n \setminus B_R$, where

$$c = \int_{\mathbb{R}^n} f(y) dy.$$

[Hint: Use spherical (polar) coordinates and the mean value property.]

SOLUTION. ■

EXERCISE 1.9. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where $g, h \in C_0^\infty(\mathbb{R}^2)$ and $\alpha \geq 0$ is a constant.

- (a) Show that for an appropriate choice of constant λ and μ the function

$$v(x_1, x_2, x_3, t) := e^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation $v_{tt} - \Delta v = 0$.

- (b) Use (a) to prove the following domain of dependence result: for any point $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$ the value $u(x_0, t_0)$ is uniquely determined by values of g and h in $\overline{B_{t_0}(x_0)} := \{|x - x_0| \leq t_0\}$. (You may use the corresponding result for the wave equation without proof.)

SOLUTION. ■

EXERCISE 1.10. Let $u(x, t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0, \infty)$ with the initial condition

$$u(x, 0) = u_0(x)$$

for $x \in \mathbb{R}$, where $u_0 \in C^\infty$ is 2π -periodic, i.e., $u_0(x + 2\pi) = u_0(x)$. Show that

$$\lim_{t \rightarrow \infty} u(x, t) = a_0,$$

uniformly in $x \in \mathbb{R}$, where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x) dx.$$

SOLUTION. ■

1.3 Qualifying Exam, January '14

EXERCISE 1.11. Consider the first order equation in \mathbb{R}^2

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

- (a) Find the characteristic curves of the equation.
- (b) Consider the Cauchy problem for this equation prescribed on the line $x_1 = 1$:

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on f so that the problem is solvable in a neighborhood of the point $(1, 0)$.

SOLUTION. ■

EXERCISE 1.12. Let u be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{x_n > 0\}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where g is a continuous function with compact support in \mathbb{R}^{n-1} . Here $n \geq 2$. Prove that

$$u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty,$$

for $x \in \{x_n > 0\}$.

SOLUTION. ■

EXERCISE 1.13. Let u be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \quad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions $u(x, 0) = g(x)$, where g is a bounded continuous function on \mathbb{R} satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^\alpha, \quad x, y \in \mathbb{R}$$

with a constant $\alpha \in (0, 1]$. Show that

$$\begin{aligned} |u(x, t) - u(y, t)| &\leq M|x - y|^\alpha, & x, y \in \mathbb{R}, t > 0, \\ |u(x, t) - u(x, s)| &\leq C_\alpha M|t - s|^{\alpha/2}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of $u(x, t)$ as a convolution with the heat kernel $\Phi(y, t)$, make a change of variables $z = y/\sqrt{t}$ and use that $|\sqrt{t} - \sqrt{s}| \leq \sqrt{|t - s|}$.]

SOLUTION. ■

EXERCISE 1.14. Let u be a positive harmonic function in the unit ball B_1 in \mathbb{R}^n . Show that

$$|D(\ln u)| \leq M \quad \text{in } B_{1/2}$$

for a constant M depending only on the dimension n .

[*Hint:* Use the interior derivative estimate $|Du(x)| \leq (C_n/r) \sup_{B_r(x)} |u|$ for $B_r(x) \subset B_1$ as well as the Harnack inequality for harmonic functions.]

SOLUTION. ■

EXERCISE 1.15. Let u be a C^2 solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = 0, \quad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some $k \geq 0$. Prove that there exists a function $\varphi(r)$ such that

$$u(x, t) = t^{k+2} \varphi(|x|/t).$$

[*Hint:* As one of the steps show that u is $(k+2)$ -homogeneous in (x, t) variables, i.e., $u(\lambda x, \lambda t) = \lambda^{k+2} u(x, t)$ for any $\lambda > 0$.]

SOLUTION. ■