MA 166: Exam 1 Solutions

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Here are the solutions to Exam 1. I've provided as much detail as I can muster. If you find any mistake or you feel that I am skipping a step, please be sure to let me know. The green booklet and red booklet are similar so I've highlighted the corresponding problem number in green for the green booklet and in red for the red booklet.

Problem 1 (#1, #11)**.** If a and b are the values of k for which the angle between $\langle 1, 2, 2\rangle$ and $\langle 1, 0, k\rangle$ equals $\pi/4$, then $a + b =$?

Solution. All you needed to do for this problem was remember the law of cosines for vectors which tells us that

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta. \tag{1}$$

Applying the equation above to our vectors, we get

$$\begin{aligned}
\langle 1, 2, 2\rangle \cdot \langle 1, 0, k\rangle &= \left(\sqrt{1^2 + 2^2 + 2^2}\right)\left(\sqrt{1^2 + 0^2 + k^2}\right)\cos(\pi/4) \\
1 \cdot 1 + 2 \cdot 0 + 2 \cdot k &= \left(\sqrt{1 + 4 + 4}\right)\left(\sqrt{1 + 0 + k^2}\right)1/\sqrt{2} \\
1 + 2k &= \left(\sqrt{1 + 4 + 4}\right)\left(\sqrt{1 + 0 + k^2}\right) \\
\sqrt{2}(1 + 2k) &= \sqrt{9}\left(\sqrt{1 + k^2}\right) \\
&= 3\sqrt{1 + k^2},
\end{aligned}$$

now, squaring both sides, we get

$$\begin{aligned}
2(1 + 2k)^2 &= 9\left(1 + k^2\right) \\
2\left(1 + 4k + 4k^2\right) &= 9 + 9k^2 \\
2 + 8k + 8k^2 &= 9 + 9k^2
\end{aligned}$$

now move everything on the right to the left and reorder by the highest exponent of k

$$0 = k^2 - 8k + 7. \tag{2}$$

Can you see what $a+b$ is already? No? Well consider the following quadratic polynomial $(x-a)(x-b)$. What are the roots of $(x - a)(x - b)$? Well, they are a and b of course. Now, expand $(x - a)(x - b)$ like so

$$(x - a)(x - b) = x^2 - ax - bx + ab = x^2 - (a + b)x + ab$$

so $a+b$ is the same as the negative of the coefficient in front of x in our quadratic polynomial. In this case, it would be $\boxed{a+b=-(-8)=8}$.

Is that not satisfying? Well, we can go ahead and compute the roots of equation (2) by using the quadratic formula. From doing that, we get the roots

$$\begin{aligned} x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 7}}{2} \\ &= 4 \pm \sqrt{\frac{8^2 - 4 \cdot 7}{4}} \\ &= 4 \pm \sqrt{\frac{8^2 - 4 \cdot 7}{4}} \\ &= 4 \pm \sqrt{\frac{4 \cdot 2 \cdot 8 - 4 \cdot 7}{4}} \\ &= 4 \pm \sqrt{2 \cdot 8 - 7} \\ &= 4 \pm \sqrt{16 - 7} \\ &= 4 \pm \sqrt{9} \\ &= 4 \pm 3 \end{aligned}$$

so $a=7$ and $b=1$. Hence, $\boxed{a+b=8}$ like we said before. ■

Problem 2 (#2, #1)**.** Let $\langle a,b,c\rangle$ be the vector projection of $\vec{u}=\langle 2,-1,9\rangle$ onto $\vec{v}=\langle 1,2,2\rangle$. Compute $a+b+c$.

Solution. Recall the formula for the projection of $\vec{u}$ onto $\vec{v}$:

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}. \tag{3}$$

Plugging in our values of $\vec{u}$ and $\vec{v}$ into this equation, we have

$$\begin{aligned} \operatorname{proj}_{\vec{v}} \vec{u} &= \frac{\langle 2,-1,9\rangle \cdot \langle 1,2,2\rangle}{|\langle 1,2,2\rangle|^2} \langle 1,2,2\rangle \\ &= \frac{2 \cdot 1 - 1 \cdot 2 + 9 \cdot 2}{\left(\sqrt{1^2+2^2+2^2}\right)^2} \langle 1,2,2\rangle \\ &= \frac{2-2+18}{\left(\sqrt{9}\right)^2} \langle 1,2,2\rangle \\ &= \frac{18}{9} \langle 1,2,2\rangle \\ &= 2\langle 1,2,2\rangle \text{ or } \langle 2,4,4\rangle. \end{aligned}$$

So $\boxed{a+b+c=2+4+4=10}$. ■

Problem 3 (#3, #3)**.** Let $\vec{u}=\langle 0,1,2\rangle$, $\vec{v}=\langle 3,1,0\rangle$, and $\vec{w}=\langle a,b,c\rangle$. Suppose $\vec{w}$ is a unit vector with $c>0$ that is perpendicular to both $\vec{u}$ and $\vec{v}$. Compute $a+b+c$.

Solution. Now, you all remember that to find a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$ all we need to do is find their cross product $\vec{u} \times \vec{v}$, right? Let's start by finding this

$$\begin{aligned}
\vec{u} \times \vec{v} &= \langle 0,1,2\rangle \times \langle 3,1,0\rangle \\
&= \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \hat{\imath} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \hat{\jmath} + \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \hat{k} \\
&= (1 \cdot 0 - 2 \cdot 1)\hat{\imath} + (2 \cdot 3 - 0 \cdot 0)\hat{\jmath} + (0 \cdot 1 - 1 \cdot 3)\hat{k} \\
&= -2\hat{\imath} + 6\hat{\jmath} - 3\hat{k} \\
&= \langle -2, 6, -3\rangle.
\end{aligned}$$

We are not quite done yet since we want a unit vector. All we need to do is divide by the magnitude of $\vec{u} \times \vec{v}$ and we are one step closer to the solution

$$\begin{aligned}
\frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} &= \frac{\langle -2,6,-3\rangle}{\sqrt{(-2)^2 + 6^2 + (-3)^2}} \\
&= \frac{\langle -2,6,-3\rangle}{\sqrt{4+36+9}} \\
&= \frac{\langle -2,6,-3\rangle}{\sqrt{49}} \\
&= \frac{\langle -2,6,-3\rangle}{7} \\
&= \left\langle -\frac{2}{7}, \frac{6}{7}, -\frac{3}{7} \right\rangle.
\end{aligned}$$

Notice that the third entry $-3/7$ on the vector above is negative so we need to take the negative of the vector above and we call this $\vec{w}$

$$\vec{w} = \langle a,b,c\rangle = -\left\langle -\frac{2}{7}, \frac{6}{7}, -\frac{3}{7} \right\rangle = \left\langle -\left(-\frac{2}{7}\right), -\frac{6}{7}, -\left(-\frac{3}{7}\right) \right\rangle = \left\langle \frac{2}{7}, -\frac{6}{7}, \frac{3}{7} \right\rangle.$$

Now, all we need to do is take the sum of the entries of the vector $\vec{w}$ above and we are done!

$$\boxed{a+b+c = \frac{2}{7} - \frac{6}{7} + \frac{3}{7} = \frac{2-6+3}{7} = -\frac{1}{7}.} \qquad \blacksquare$$

Problem 4 (#4, #5)**.** Find the area of the region by the curves $y = x^2 - 2$ and $y = |x|$.

Solution. Alright! Remember the definition of the absolute value of a function anyone? Here it is: If $f(x)$ is a function of x, i.e., $f(x) = \cos x$ or $f(x) = e^x + \cos \pi x - x^\pi$ or what have you, then

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}. \tag{4}$$

All this is saying is that if we plug in a value into our function $f(x)$ and it returns a negative value we make it positive and if the value is positive we leave it positive. What does this mean for $y = |x|$? It means that

$$|x| = \begin{cases} x & \text{if } f(x) \geq 0 \\ -x & \text{if } f(x) < 0 \end{cases},$$

i.e., $|x|$ is $-x$ from $-\infty$ to 0 and x from 0 to $+\infty$, if this notation makes sense to you. This means that we must consider two cases when solving for the intersection of $|x|$ with $x^2 - 2$: We must consider the possibility that $x^2 - 2$ intersects $|x|$ for some value $x < 0$ and $x^2 - 2$ intersects $|x|$ for some value $x \geq 0$. So we must solve both equations

$$\begin{aligned} x &= x^2 - 2 \\ -x &= x^2 - 2. \end{aligned}$$

By some simple algebra, we can rearrange the equations above into

$$0 = x^2 - x - 2 \tag{5}$$
$$0 = x^2 + x - 2 \tag{6}$$

and solve for x. By the quadratic equation on equation (5) we have

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-2)}}{2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = -1 \text{ or } 2.$$

Since we are only looking at positive values of x, -1 makes no sense so we throw it out and 2 remains behind.

We do the same thing for equation (6)

$$x = \frac{-1 \pm \sqrt{1^2 - 4(-2)}}{2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2 \text{ or } 1.$$

Since we are only only looking at negative values of, 1 makes no sense so we throw it out and keep -2.

Now all we need to do is observe that for $0 \leq x \leq 2$ the equation $x > x^2 - 2$ and for $-2 \leq x \leq 0$ the equation $-x > x^2 - 2$ so the area enclosed by $y = |x|$ and $y = x^2 - 2$ is given by the integral

$$\begin{aligned} \int_{-2}^{2} \big||x| - \left(x^2 - 2\right)\big| \,\mathrm{d}x &= \int_{-2}^{0} \big||x| - \left(x^2 - 2\right)\big| \,\mathrm{d}x + \int_{0}^{2} \big||x| - \left(x^2 - 2\right)\big| \,\mathrm{d}x \\ &= \underbrace{\int_{-2}^{0} -x - \left(x^2 - 2\right) \mathrm{d}x}_{\text{Int. 1}} + \underbrace{\int_{0}^{2} x - \left(x^2 - 2\right) \mathrm{d}x}_{\text{Int. 2}}. \end{aligned}$$

Let's compute Int. 1 and Int. 2 separately

$$\begin{aligned} \text{Int. 1} &= \int_{-2}^{0} -x - \left(x^2 - 2\right) \mathrm{d}x \\ &= \int_{-2}^{0} -x - x^2 + 2 \,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \int_{-2}^{0} -x^2 - x + 2 \,\mathrm{d}x \\
&= -\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + 2x\Big|_{-2}^{0} \\
&= -\tfrac{1}{3}\cdot 0^3 - \tfrac{1}{2}\cdot 0^2 + 2\cdot 0 \\
&\quad - \left(-\tfrac{1}{3}(-2)^3 - \tfrac{1}{2}(-2)^2 + 2(-2)\right) \\
&= 6 - \tfrac{8}{3} \\
&= \frac{6\cdot 3 - 8}{3} \\
&= \frac{18-8}{3} \\
&= \frac{10}{3}.
\end{aligned}
$$

Now, you can either compute Int. 2 and add that quantity to Int. 1 to get the area of your bounded region, or you can plot the curves and notice that the areas are symmetric and double Int. 2 to get our total area $\boxed{2(10/3) = 20/3}$.

Let's compute Int. 2 just to make sure

$$
\begin{aligned}
\text{Int. } 2 &= \int_{0}^{2} x - \left(x^2 - 2\right) \mathrm{d}x \\
&= \int_{0}^{2} x - x^2 + 2 \,\mathrm{d}x \\
&= \int_{0}^{2} -x^2 + x + 2 \,\mathrm{d}x \\
&= -\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 2x\Big|_{0}^{2} \\
&= -\tfrac{1}{3}2^3 + \tfrac{1}{2}2^2 + 2\cdot 2 \\
&\quad - \left(-\tfrac{1}{3}\cdot 0^3 + \tfrac{1}{2}\cdot 0^2 + 2\cdot 0\right) \\
&= -\frac{8}{3} + 2 + 4 \\
&= -\frac{8}{3} + 6 \\
&= \frac{-8 + 2\cdot 6}{3} \\
&= \frac{-8+18}{3} \\
&= \frac{10}{3}.
\end{aligned}
$$

Then

$$
\int_{-2}^{2} \left| |x| - \left(x^2 - 2\right) \right| \mathrm{d}x = \text{Int. } 1 + \text{Int. } 2 = \frac{10}{3} + \frac{10}{3} = \boxed{\frac{20}{3}.} \qquad \blacksquare
$$

Problem 5 (#5, #4)**.** What is the radius of the sphere $x^2 + y^2 + z^2 + 8x - 2y - 4z = 15$?

Solution. Remember the standard equation for the sphere of radius r with center $C = (a, b, c)$? Here it is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \tag{7}$$

So all we need to do is to manipulate our equation $x^2 + y^2 + z^2 + 8x - 2y - 4z = 15$ until it looks like the equation (7)

$$\begin{aligned} x^2 + y^2 + z^2 + 8x - 2y - 4z &= 15 \\ \left(x^2 + 8x\right) + \left(y^2 - 2y\right) + \left(z^2 - 4z\right) &= 15 \end{aligned}$$

now we complete the square, not forgetting to balance the equation on the right-hand side

$$\begin{aligned} \left(x^2 + 8x + 16\right) + \left(y^2 - 2y + 1\right) + \left(z^2 - 4z + 4\right) &= 15 + 16 + 1 + 4 \\ (x+4)^2 + (y-1)^2 + (z-2)^2 &= 36. \end{aligned}$$

We didn't need to factor the left-hand side, but why not do it anyway? Now, looking at our equation (7) we see that our radius $r^2 = 36$ so $\boxed{r = \sqrt{36} = 6}$. ■

Problem 6 (#6, #6)**.** Consider the region enclosed by the graph of the function $y = x^4$ and the x-axis between $x = 0$ and $x = 1$. Find the volume of the solid obtained by rotating the region about the x-axis using the disks/washers method.

Solution. This one is easy enough. All we need to do is find an equation for the area of the perpendicular cross section A which (as a rule of thumb, you want to express in terms of the axis which is perpendicular to your cross section) will be in terms of x

$$A(x) = \pi y^2 = \pi\left(x^4\right)^2 = \pi x^8.$$

Now, compute

$$\begin{aligned} \int_0^1 \pi A(x)\,\mathrm{d}x &= \int_0^1 \pi x^8\,\mathrm{d}x \\ &= \pi \int_0^1 x^8\,\mathrm{d}x \\ &= \pi\,\tfrac{1}{9}x^9\Big|_0^1 \\ &= \pi\left(\tfrac{1}{9}1^9 - \left(\tfrac{1}{9}0^9\right)\right) \\ &= \boxed{\frac{\pi}{9}.} \end{aligned}$$

■

Problem 7 (#7, #9)**.** Consider the region enclosed by the graph of the function $y = x - x^4$ and the x-axis. Find the volume of the solid obtained by rotating the region about the y-axis using the cylindrical shells method.

Solution. This one is also easy. All we need to do is find the cylindrical area. Since we are revolving about the y-axis, the length of our cylinder will be x and point along the x-axis so we probably want to express our cross sectional area A in terms of x like so

$$A(x) = 2\pi x(x - x^4).$$

Now, we need to find when $x - x^4$ intersects the line $y = 0$. This happens when $x = 0$ since $0 - 0^4 = 0$ and, factoring, $x(1 - x^3)$ when $x = 1$. Using the formula for our cross section, we integrate $A(x)$ from 0 to 1 to find our volume

$$\begin{aligned}
V &= \int_0^1 2\pi x\big(x - x^4\big)\,\mathrm{d}x \\
&= 2\pi \int_0^1 x^2 - x^5\,\mathrm{d}x \\
&= 2\pi\Big(\tfrac{1}{3}x^3 - \tfrac{1}{6}x^6\Big|_0^1\Big) \\
&= 2\pi\big(\tfrac{1}{3}1^3 - \tfrac{1}{6}1^6 - \big(\tfrac{1}{3}0^3 - \tfrac{1}{6}0^6\big)\big) \\
&= 2\pi\left(\frac{2-1}{6}\right) \\
&= \frac{2\pi}{6} \\
&= \boxed{\frac{\pi}{3}.}
\end{aligned}$$

■

Problem 8 (#8, #8)**.** Let $\langle a, b, c\rangle$ be the unit vector of length 6 in the opposite direction to $\langle -2, 1, -2\rangle$. Compute $a + b + c$.

Solution. The wording of this question is very wonky. What was meant, I believe, was "Let $\langle a, b, c\rangle$ be the vector which is 6 times as long as the unit vector pointing in the opposite direction to $\langle -2, 1, -2\rangle$".

First, make turn the vector $\langle -2, 1, -2\rangle$ into a unit vector like so

$$\frac{\langle -2,1,-2\rangle}{|\langle -2,1,-2\rangle|} = \frac{\langle -2,1,-2\rangle}{\sqrt{(-2)^2 + 1^2 + (-2)^2}} = \frac{\langle -2,1,-2\rangle}{\sqrt{4+1+4}} = \frac{\langle -2,1,-2\rangle}{\sqrt{9}} = \left\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle.$$

To get a unit vector pointing in the opposite way, we just multiply by -1.[1] Thus, we have

$$\left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle.$$

We are told that our vector must be 6 times as long as the unit vector so

$$\langle a, b, c\rangle = 6\left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle = \langle 4, -2, 4\rangle,$$

and $\boxed{a + b + c = 4 - 2 + 4 = 6}$. ■

Problem 9 (#9, #10)**.** A force of 8 lb is required to hold a spring stretched 4 in beyond its natural length. How much work is done in stretching the same spring from its natural length to 6 in?

[1]Why? Well, recall the law of cosines, equation (1), which says that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$. If two vectors are pointing in opposite directions, that means that the angle between them is π or 180° so $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$. Now, divide on both sides and we get $(\vec{a}/|\vec{a}|) \cdot (\vec{b}/|\vec{b}|) = -1$. It can be shown that, in fact, $\vec{a}/|\vec{a}| = -\vec{b}/|\vec{b}|$, but you have to solve a system of equations and that requires a bit more math that I am willing to write on this footnote.

Solution. Recall the definition for the force required to move a spring a distance x from its natural length

$$F(x) = kx. \tag{8}$$

This is called Hooke's law and, without a doubt, you will see it in physics and, should you decide to become a mechanical engineer, you will see it again[2] Now, to find the work needed to move the spring from x_1 to x_2 is given by taking the integral

$$W(x_1, x_2) = \int_{x_1}^{x_2} kx \, \mathrm{d}x = \tfrac{1}{2}kx^2\Big|_{x_1}^{x_2} = \tfrac{1}{2}k\big({x_2}^2 - {x_1}^2\big). \tag{9}$$

Since they want the work in terms of lb-ft, it would be best to convert from in to ft now. Let's do that: So initially the spring is stretched to 4 in which is $4/12 = 1/3$ ft and we want to know how much work is required to stretch it from its natural length 0 ft to $6/12 = 1/2$ ft. To proceed, we need to find out what the value of k is

$$k = \frac{8}{1/3} = 3 \cdot 8 = 24 \text{ lb/ft}.$$

Now, plug in our values into equation (9) and we have

$$\begin{aligned} W(0, 1/2) &= \tfrac{1}{2} \cdot 24\big((1/2)^2 - 0^2\big) \\ &= 12 \cdot \frac{1}{4} \\ &= \boxed{3 \text{ lb-ft.}} \end{aligned}$$

■

Problem 10 (#10, #12)**.** Evaluate $\int_1^e x \ln x \, \mathrm{d}x$ using integration by parts.

Solution. Since it's hard to take the integral of $\ln x$ and easy to take the integral of x take $\mathrm{d}v = x$ and $u = \ln x$, then $\mathrm{d}u = x^{-1} \, \mathrm{d}x$ and $v = \frac{1}{2}x^2$ so

$$\begin{aligned} \int_1^e u \, \mathrm{d}v &= \tfrac{1}{2}x^2 \ln x\Big|_1^e - \int \tfrac{1}{2}x^{-1}x^2 \, \mathrm{d}x \\ &= \tfrac{1}{2}x^2 \ln x\Big|_1^e - \int \tfrac{1}{2}x \, \mathrm{d}x \\ &= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2\Big|_1^e \\ &= \tfrac{1}{4}x^2(2\ln x - 1)\Big|_1^e \\ &= \tfrac{1}{4}e^2(2 - 1) - \tfrac{1}{4}1(2\ln 1 - 1) \\ &= \tfrac{1}{4}e^2 + \tfrac{1}{4} \\ &= \boxed{\frac{e^2 + 1}{4}.} \end{aligned}$$

■

Problem 11 (#11, #2)**.** Evaluate $\int_0^\pi \sin^3 x \, \mathrm{d}x$.

[2]The analogue of the spring in electrical engineering is the inductor. By analogue, I mean that, mathematically, the inductor and the spring behave the same way in their appropriate contexts.

Solution. Using the Pythagorean identity

$$\cos^2 x + \sin^2 x = 1, \tag{10}$$

we get $\sin^2 x = 1 - \cos^2 x$ so

$$\begin{aligned}\int_0^\pi \sin^3 x \,\mathrm{d}x &= \int_0^\pi \sin^2 x \sin x \,\mathrm{d}x \\ &= \int_0^\pi \left(1 - \cos^2 x\right) \sin x \,\mathrm{d}x.\end{aligned}$$

Now, what is a good substitution to use at this point? We want to get rid of the $\sin x$ so let's do $u = \cos x$. Then $\mathrm{d}u = -\sin x \,\mathrm{d}x$ and the integral above turns into

$$\int_1^{-1} \left(1 - u^2\right) \sin x \frac{\mathrm{d}u}{-\sin x} = -\int_{-1}^{1} 1 - u^2 \,\mathrm{d}u.$$

Now, remember the identity about the integral that says that $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$, so the above turns into

$$\begin{aligned}-\int_1^{-1} 1 - u^2 \,\mathrm{d}u &= \int_1^{-1} 1 - u^2 \,\mathrm{d}u \\ &= u - \tfrac{1}{3}u^3\Big|_{-1}^{1} \\ &= 1 - \tfrac{1}{3} - \left(-1 - \tfrac{1}{3}(-1)^3\right) \\ &= \tfrac{2}{3} - \left(-1 + \tfrac{1}{3}\right) \\ &= \frac{2}{3} - \left(-\tfrac{2}{3}\right) \\ &= \frac{2}{3} + \frac{2}{3} \\ &= \boxed{\frac{4}{3}.}\end{aligned}$$

■

Why did the limits of integration change from $0 \leq x \leq \pi$ to $1 \geq u \geq -1$, well u is the new variable we are integration with respect to and the relation ship between u and x is that $u = \cos x$ so the limits of u will be from $\cos 0 = 1$ to $\cos \pi = -1$. Makes sense, right?

Problem 12 (#12, #7)**.** Evaluate $\int_0^{\pi/4} \cos^2 x \,\mathrm{d}x$. Hint: $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

Solution. All we need to do is modify the hint to express $\cos^2 x$ in terms of $\cos 2x$ like so

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1 \\ \cos 2x + 1 &= 2\cos^2 x \\ \frac{\cos 2x + 1}{2} &= \cos^2 x\end{aligned}$$

so our integral turns into

$$
\begin{aligned}
\int_0^{\pi/4} \cos^2 x \,\mathrm{d}x &= \int_0^{\pi/4} \frac{\cos 2x + 1}{2} \,\mathrm{d}x \\
&= \frac{1}{2} \int_0^{\pi/4} \cos 2x + 1 \,\mathrm{d}x \\
&= \frac{1}{2} \left(\tfrac{1}{2} \sin 2x + x \Big|_0^{\pi/4} \right) \\
&= \frac{1}{2} \big(\tfrac{1}{2} \sin 2(\pi/4) + \tfrac{pi}{4} - (\tfrac{1}{2} \sin 2 \cdot 0 - 0) \big) \\
&= \frac{1}{2} \big(\tfrac{1}{2} \cdot 1 + \tfrac{\pi}{4} - (0 - 0) \big) \\
&= \tfrac{1}{2} \left(\frac{2}{4} + \frac{\pi}{4} \right) \\
&= \tfrac{1}{2} \left(\frac{2 + \pi}{4} \right) \\
&= \frac{2 + \pi}{8} \\
&= \boxed{\frac{1}{4} + \frac{\pi}{8} \text{ or } \frac{\pi}{8} + \frac{1}{4}.}
\end{aligned}
$$

■