MA571 Problem Set 3

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September 13, 2015

Problem 3.1 (Munkres §18, p. 111, #7(a))

(a) Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is "continuous from the right," that is,

$$\lim_{x \to a+} f(x) = f(a).$$

for each $a \in \mathbf{R}$. Show that f is continuous when considered as a function from $\mathbf{R}_\ell$ to $\mathbf{R}$.

Proof. Recall the definition of "right-hand limit,":

Definition (Rudin §4, p. 94, Def. 4.25)**.** Let f be defined on (a, b). Consider any point x such that $a \leq x < b$. We write $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$.

This definition is not well suited for our purposes since it is defined in terms of limits of sequences, which Rudin validates in Theorem 4.6 (cf. Rudin, §4, p. 86) by proving that the limit-point formulation of continuity coincides with the ε-δ formulation. We shall, therefore, reformulate Rudin's definitions in terms of ε's and δ's as follows:

Definition. $f\colon \mathbf{R} \to \mathbf{R}$ is right-continuous at $x_0 \in \mathbf{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in [x_0, x_0 + \delta)$ implies $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

We will prove that $f\colon \mathbf{R}_\ell \to \mathbf{R}$ is continuous in the sense: "for each open subset V of $\mathbf{R}$, the set $f^{-1}(V)$ is an open subset of $\mathbf{R}_\ell$" (cf. Munkres, §18, p. 102) and we shall do so in the spirit of Example 1 in Munkres §18, p. 103 and employ Theorem 18.1(4). Recall what basic open sets look like in the lower-limit topology (defined in Munkres §13, pp. 81-82), they are intervals of the form $[a, b) \subset \mathbf{R}$. Without loss of generality, consider the basic open set $V = (a, b)$ in $\mathbf{R}$. Let $x_0 \in f^{-1}(V)$. Then, since f is right-continuous, for $\varepsilon \leq \min\{f(x_0) - a, b - f(x_0)\}$, there exists $\delta > 0$ such that $x \in [x_0, x_0 + \delta)$ implies $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, i.e.,

$$f([x_0, x_0 + \delta)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subset V.$$

By Theorem 18.1(4), f is continuous. ∎

Problem 3.2 (Munkres §18, p. 112, #13)

Let $A \subset X$; let $f\colon A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g\colon \overline{A} \to Y$, then g is uniquely determined by f.

Proof. We shall proceed by contradiction. Suppose that g_1 and g_2 are distinct continuous extensions of f to the closure of A, i.e., $g_1(x) \neq g_2(x)$ for some $x \in \overline{A} \setminus A$. Recall from Problem 2.7 (Munkres §13, p. 101, #13) that Y is Hausdorff if and only the diagonal $\Delta = \{\, y \times y \mid y \in Y \,\}$ is closed in $Y \times Y$. Now, consider the product map $G = g_1 \times g_2 \colon X \to Y \times Y$. This map is continuous by Theorem 18.4 so, by Theorem 18.1(2), $G(\overline{A}) \subset \overline{G(A)}$. However, since $g_1 = g_2$ on A we have that $G(A) \subset \Delta$ so, by Lemma B (from Prof. McClure's lectures), we have that

$$G(\overline{A}) \subset \overline{G(A)} \subset \Delta.$$

But by assumption $g_1(x) \times g_2(x) \notin \Delta$. This is a contradiction. Therefore, $g_1 = g_2$ on $\overline{A}$, i.e., the extension of f to a continuous function g on $\overline{A}$ is unique. ∎

Problem 3.3 (Munkres §19, p. 118, #2)

Prove Theorem 19.3.

Proof. Recall the exact statement of Theorem 19.3 from Munkres §19, p. 116:

Theorem. *Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

Our goal is to show that the box topology or the product topology on $\prod A_\alpha$ corresponds to subspace topology on $\prod A_\alpha$ as a subset of $\prod X_\alpha$ with the product topology or the box topology, respectively. That is, we will show that U is open in the box (or product) topology on $\prod A_\alpha$ if and only if it can be expressed in the form $V \cap \prod A_\alpha$ where V is open in $\prod X_\alpha$ with the box (or product) topology (cf. Munkres §16, p. 88 and Munkres §19, pp. 114-116 for the relevant definitions). We will attempt to prove these simultaneously, taking note of where the proof for the box topology may differ from the proof for the product topology.

$\Longrightarrow$ By Lemma 13.3(2), it is enough to consider basic open sets in $\prod A_\alpha$. Recall from Theorem 19.2 that U is a basic open set in $\prod A_\alpha$ if it is of the form $U = \prod U_\alpha$ where U_α is a basic open set in A_α (or, in the case of the product topology, equal to A_α for all but finitely many α). Then, by Lemma 16.1, U_α is of the form $U_\alpha = V_\alpha \cap A_\alpha$ for V_α open in X_α (in the case of the product topology, we simply demand that $V_\alpha = X_\alpha$ for all but finitely many α). Then, by Theorem 19.2, the set $V = \prod V_\alpha$ is a basic open set in $\prod X_\alpha$. We claim that

$$U = V \cap \prod A_\alpha.$$

But this follows from the remarks made by Munkres in §19, p. 144, namely that

$$V \cap \prod A_\alpha = \prod V_\alpha \cap \prod A_\alpha = \prod (V_\alpha \cap A_\alpha) = \prod U_\alpha.$$

This fact is one which Munkres does not prove, but which nevertheless can be effortlessly proven:

Lemma 6.

$$\prod (A_\alpha \cap B_\alpha) = \prod A_\alpha \cap \prod B_\alpha.$$

Proof. The proof is just a matter of chasing definitions so we will be a little informal. The point $\mathbf{x} \in \prod (A_\alpha \cap B_\alpha)$ if and only if $\pi_\alpha(\mathbf{x}) = x_\alpha \in A_\alpha \cap B_\alpha$ for all α if and only if $x_\alpha \in A_\alpha$ and $x_\alpha \in B_\alpha$ for all α if and only if $\mathbf{x} \in \prod A_\alpha$ and $\mathbf{x} \in \prod B_\alpha$ if and only if $\mathbf{x} \in \prod A_\alpha \cap \prod B_\alpha$. ♣

$\Longleftarrow$ Conversely, suppose $V \cap \prod A_\alpha$ for some basic open set $V = \prod V_\alpha$ (where, in the case of the product topology, $V_\alpha = X_\alpha$ for all but finitely many α). Then, by Lemma 6, we have

$$V \cap \prod A_\alpha = \prod (V_\alpha \cap A_\alpha)$$

where $U_\alpha = V_\alpha \cap A_\alpha$ (with $U_\alpha = A_\alpha$ for all but finitely many αin the case of the product topology) is a basic open set in A_α with the subspace topology. Hence, $\prod U_\alpha$ is a basic open set in the box (or product) topology on $\prod A_\alpha$. We have shown that a basic open set in the box (or product) topology on $\prod A_\alpha$ corresponds to a basic open set in the subspace topology on $\prod A_\alpha$. Therefore, by Lemma 13.3(2), the box (or product) topology on $\prod A_\alpha$ is equivalent to the subspace topology on $\prod A_\alpha$. ■

Problem 3.4 (Munkres §19, p. 118, #3)

Prove Theorem 19.4.

Proof. Recall the exact statement of Theorem 19.4 from Munkres §19, p. 116:

Theorem. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

To show that $\prod X_\alpha$ equipped with the box (or the product) topology we will proceed in the following way: let $\mathbf{x}, \mathbf{y} \in \prod X_\alpha$, it is sufficient, although not necessary, to show that there exists basic open sets $U = \prod U_\alpha$ and $V = \prod V_\alpha$ neighborhoods of $\mathbf{x}$ and $\mathbf{y}$, respectively, such that $U \cap V = \emptyset$.

We will first demonstrate this for the product topology on $\prod X_\alpha$. Since X_α is Hausdorff for each α, there exists basic open sets U_α and V_α neighborhoods of x_α and y_α, respectively, such that $U_\alpha \cap V_\alpha = \emptyset$. For a finite collection of α's, say A, let $U'_\alpha = U_\alpha$ and $V'_\alpha = V_\alpha$ for all $\alpha \in A$ and $U'_\alpha = X_\alpha$, $V'_\alpha = X_\alpha$ otherwise. Then $U = \prod U'_\alpha$ and $V = \prod V'_\alpha$, by Theorem 19.2, are open in $\prod X_\alpha$ with the product topology and, by Lemma 6,

$$U \cap V = \prod U'_\alpha \cap \prod V'_\alpha = \prod_{\alpha \in A} (U_\alpha \cap V_\alpha) \times \prod_{\alpha \notin A} X_\alpha = \prod_{\alpha \in A} \emptyset \times \prod_{\alpha \notin A} X_\alpha = \emptyset.$$

Thus $\prod X_\alpha$ with the product topology is Hausdorff.

In the case of $\prod X_\alpha$ with the box topology, we take U and V to be the basis elements $U = \prod U_\alpha$ and $V = \prod V_\alpha$ such that $U_\alpha \cap V_\alpha = \emptyset$ for all α. Then, by Lemma 6, we have

$$U \cap V = \prod U_\alpha \cap \prod V_\alpha = \prod (U_\alpha \cap V_\alpha) = \prod \emptyset = \emptyset.$$

Hence $\prod X_\alpha$ with the box topology is Hausdorff. ■

Problem 3.5 (Munkres §19, p. 118, #6)

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Proof. $\implies$ Suppose that the sequence $\{\mathbf{x}_n\}$ converges to $\mathbf{x} \in \prod X_\alpha$ then, from Munkres §17, p. 98, for every neighborhood U of $\mathbf{x}$, there is a positive integer N such that $\mathbf{x}_n \in U$ for $n \geq N$. We claim that

Lemma 7. *$\pi_\beta \colon \prod X_\alpha \to X_\beta$ is an open map on $\prod X_\alpha$ with the product or the box topology.*

Proof. This is *forced* by the basis definition for the product topology (cf. Theorem 19.2, Munkres §19, p. 114). That is, the U is a basic open set if and only if it is of the form $\prod U_\alpha$ where U_α is open in X_α (and $U_\alpha = X_\alpha$ for all but finitely many α in the case of the product topology). Then clearly $\pi_\beta(\prod U_\alpha) = U_\beta$ is open in X_α. This extends to arbitrary open sets in $\prod X_\alpha$ equipped with the product or box topology. ♣

Now, since π_α is an open map $\pi_\alpha(U)$ is a neighborhood of $\pi_\alpha(\mathbf{x})$ such that $\pi_\alpha(\mathbf{x}_n) \in \pi_\alpha(U)$ for all $n \geq N$ for all α. Thus, the sequence $\{\pi_\alpha(\mathbf{x}_n)\}$ converges to $\pi_\alpha(\mathbf{x})$ in X_α for all α.

$\impliedby$ Conversely, suppose that the sequence $\{\pi_\alpha(\mathbf{x})\}$ converges to $\pi_\alpha(\mathbf{x})$ for all α. Then, for every neighborhood U_α of $\pi_\alpha(\mathbf{x})$ there exists a positive integer N such that $n \geq N$ implies $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$ for all $n \geq N$ for all α. Let U be a basic neighborhood of $\mathbf{x}$. Then, by Theorem 19.2, $U = \prod U_\alpha$ where U_α is an open set in X_α and $U_\alpha = X_\alpha$ for all but finitely many α. Let $B = \{\beta_1, \ldots, \beta_k\}$ be the collection of α's for which $U_\alpha \neq X_\alpha$. Then, since $\pi_{\beta_i}(\mathbf{x}_n) \to \pi_{\beta_i}(\mathbf{x})$ converges and $\pi_{\beta_i}(U)$ is a neighborhood of $\pi_{\beta_i}(\mathbf{x})$, there exists a positive integer N_i such that $\pi_{\beta_i}(\mathbf{x}_n) \in \pi_{\beta_i}(U)$ for all $n \geq N_i$ (note that we are not ignoring the case where $U_\alpha = X_\alpha$ since in that case the sequence $\{\pi_\alpha(\mathbf{x})\} \in X_\alpha$ for all n and everything works out nicely). Take $N = \max\{N_1, \ldots, N_k\}$. Then we claim that $\mathbf{x}_n \in U$ for every $n \geq N$. But this follows from construction for $\mathbf{x}_n \in U$ if and only if $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$ (cf. Munkres §19, p. 113) for all α if $n \geq N_i$, but $n \geq N \geq N_i$ for all i so $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$. Thus, the sequence $\{\mathbf{x}_n\}$ converges to $\mathbf{x}$ in the product topology on $\prod X_\alpha$.

The argument $\mathbf{x}_n \to \mathbf{x} \implies \pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ carries over to the box topology on $\prod X_\alpha$ since π_α is an open map. That is, for every open neighborhood U of $\mathbf{x}$ there exists a positive integer N such that $\mathbf{x}_n \in U$ for all $n \geq N$. Then, $\pi_\alpha(U)$ is a neighborhood of $\pi_\alpha(\mathbf{x})$ with $\mathbf{x}_n \in \pi_\alpha(U)$ for all $n \geq N$ for all α. Thus, the sequence $\{\pi_\alpha(\mathbf{x}_n)\}$ converges to $\pi_\alpha(\mathbf{x})$ in X_α.

The reverse implication, however, does not hold. It is not too difficult to see why the reverse implication may not hold since we relied on the property that for a basic open set U in $\prod X_\alpha$, $\prod_\alpha(U) = X_\alpha$ for all but finitely many α and from this we were able to pick a maximum positive integer N that made the sequence $\mathbf{x}$ converge in $\prod X_\alpha$. Because in the product topology, we do not have the condition that $\pi_\alpha(U) = X_\alpha$ for all but finitely many α, the set $\{N_\alpha\}$ of positive integers does not necessarily posses a maximum element. More concretely, consider $X = \mathbf{R}^\omega$ in the box topology from Example 2 in Munkres §19, p. 117. Consider the sequence $\{x_n\}$ where $x_n = 1/n$ in $\mathbf{R}$. It is clear that $x_n \to 0$, but we claim that the sequence $\mathbf{x}_n = (x_n, x_n, \ldots) \not\to (0, 0, \ldots) = \mathbf{0}$ in $\mathbf{R}^\omega$. That is, we must show that there exist an open neighborhood U of $\mathbf{0}$ such that for every positive integer N, $\mathbf{x}_n \notin U$ for some $n \geq N$. Then consider the neighborhood $U = \prod_{k=1}^\infty U_k$ of $\mathbf{0}$ for open neighborhoods $U_k = (-1/k, 1/k)$ of 0. Thus, if $\mathbf{x}_n \in U$, $\pi_k(\mathbf{x}_n) = 1/n > 1/N \in U_k$ for all

k, but $\pi_{N+1}(\mathbf{x}_N) = 1/N \notin U_{N+1}$. Thus, $\mathbf{x}_N \in U$ not so the sequence $\{\mathbf{x}_n\}$ does not converge to $\mathbf{0}$ and we see that it is not sufficient in the box topology that $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ in X_α for all α to deduce that $\mathbf{x}_n \to \mathbf{x}$ in $\prod X_\alpha$. ■

Problem 3.6 (Munkres §19, p. 118, #7)

Let $\mathbf{R}^\infty$ be the subset of $\mathbf{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, ...)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbf{R}^\infty$ in $\mathbf{R}^\omega$ in the box and product topologies? Justify your answer.

Proof. We first consider the case when $\mathbf{R}^\omega$ is equipped with the product topology. We claim that $\overline{\mathbf{R}^\infty} = \mathbf{R}^\omega$. Let $\mathbf{x} \in \mathbf{R}^\omega$. It is enough to show that for every basic neighborhood $U = \prod U_\alpha$ of $\mathbf{x}$, where $U_\alpha = \mathbf{R}$ for all but finitely many α (by Theorem 19.2), the intersection $U \cap \mathbf{R}^\omega \neq \emptyset$. Without loss of generality, we may assume $U_i \neq \mathbf{R}$ for $i \in \{1, .., n\}$. Then the point $\mathbf{y} = (y_1, ..., y_n, 0, 0, ...) \in \mathbf{R}^\infty$ where $y_i \in U_i$ for $i \in \{1, .., n\}$. In particular, $U \cap \mathbf{R}^\infty \neq \emptyset$ so $\overline{\mathbf{R}^\infty} = \mathbf{R}^\omega$.

Now let us consider the case when $\mathbf{R}^\omega$ is equipped with the box toplogy. It this case, we claim that $\overline{\mathbf{R}^\infty} = \mathbf{R}^\infty$, in particular, we prove that $\mathbf{R}^\omega \setminus \mathbf{R}^\infty$ is open. Let $\mathbf{x} \in \mathbf{R}^\omega \setminus \mathbf{R}^\infty$. Then $\mathbf{x}$ is a point such that $x_n \neq 0$ for infinitely many n. Since $\mathbf{R}$ is Hausdorff, choose neighborhoods U_i of x_i such that $0 \notin U_i$ whenever $x_i \neq 0$ (and whenever $x_i = 0$ choose arbitrary neighborhoods of 0). Then the set $U = \prod U_i$ is a basic open set in $\mathbf{R}^\omega$ such that $\prod U_i \cap \mathbf{R}^\infty = \emptyset$ since every point $\mathbf{y}' \in U$ is zero for only finitely many y_n. Therefore, $U \subset \mathbf{R}^\omega \setminus \mathbf{R}^\infty$ so $\mathbf{R}^\infty$ is closed, i.e., $\overline{\mathbf{R}^\infty} = \mathbf{R}$. ∎

Problem 3.7 (Munkres §20, p. 126, #3(b))

Let X be a metric space with metric d.

(b) Let X' denote a space having the same underlying set as X. Show that if $d\colon X' \times X' \to \mathbf{R}$ is continuous, then the topology of X' is finer than the topology of X.

Proof. Suppose that $d\colon X' \times X' \to \mathbf{R}$ is continuous. Fix $x_0 \in X$ and $\varepsilon > 0$. Recall from by Problem 2.8 (Munkres §18, Ex. #4) the map $\iota(x) = x \times x_0$ is an imbedding, in particular it is continuous so by theorem 18.2(c), the composite map $d_0 = d \circ \iota\colon X \to \mathbf{R}$ is continuous. Then, we claim

$$B_d(x_0, \varepsilon) = \{\, x \in X \mid d(x_0, y) < \varepsilon \,\} = d_0^{-1}((-\infty, \varepsilon)).$$

To see this let $y \in B_d(x_0, \varepsilon)$. Then $d_0(y) = d \circ \iota(y) = d(x_0, y) < \varepsilon$ so $y \in d_0^{-1}((-\infty, \varepsilon))$. To see the reverse containment take $y \in d_0^{-1}((-\infty, \varepsilon))$. Then, similarly, we have that $d(x_0, y) = d_0(y) = d \circ \iota(y) < \varepsilon$ so $y \in B_d(x_0, \varepsilon)$. Hence, by Theorem 13.3(2), the topology on X' is finer than the topology on X induced by d. ■

Problem 3.8 (Munkres §20, p. 127, #4(b))

Consider the product, uniform and box topologies on $\mathbf{R}^\omega$

(b) In which topologies do the following sequences converge?

$$
\begin{aligned}
\mathbf{w}_1 &= (1,1,1,1,...), & \mathbf{x}_1 &= (1,1,1,1,...),\\
\mathbf{w}_2 &= (0,2,2,2,...), & \mathbf{x}_2 &= \left(0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},...\right),\\
\mathbf{w}_3 &= (0,0,3,3,...), & \mathbf{x}_3 &= \left(0,0,\tfrac{1}{3},\tfrac{1}{3},...\right),\\
&\vdots & &\vdots\\
\mathbf{y}_1 &= (1,0,0,0,...) & \mathbf{z}_1 &= (1,1,0,0,...),\\
\mathbf{y}_2 &= \left(\tfrac{1}{2},\tfrac{1}{2},0,0,...\right) & \mathbf{z}_2 &= \left(\tfrac{1}{2},\tfrac{1}{2},0,0,...\right),\\
\mathbf{y}_3 &= \left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,...\right) & \mathbf{z}_3 &= \left(\tfrac{1}{3},\tfrac{1}{3},0,0,...\right),\\
&\vdots & &\vdots
\end{aligned}
$$

Proof. We were asked to work only on $\mathbf{w}$. We claim that $\mathbf{w}_n \to \mathbf{0}$ in the product topology on $\mathbf{R}^\omega$ and that $\mathbf{w}$ does not converge in the uniform topology and the box topology.

To see that $\mathbf{w}_n \to \mathbf{0}$ in $\mathbf{R}^\omega$ with the product topology we appeal to Problem 3.5. That is, $\mathbf{w}_n \to \mathbf{0}$ if and only if $w_n \to 0$ for all n. But this is clear since for the sequence $w_k^{(n)} = 0$ for all $n \geq k$ so $w_k^{(n)} \in U$ for any neighborhood U of 0 for all $n \geq k$. Thus, by Problem 3.5, we have that $\mathbf{w}_n \to \mathbf{0}$.

By Theorem 20.4, it is enough to prove that $\mathbf{w}_n \nrightarrow \mathbf{0}$ in the uniform topology since a counterexample in the uniform topology provides a counterexample in the box topology on $\mathbf{R}^\omega$. By the definition of the uniform topology (cf. Munkres §20, p. 124). Consider the ball of radius 1, $U = B_{\text{unif}}(\mathbf{0}, 1)$, in the uniform topology on $\mathbf{R}^\omega$, that is, the set

$$U = \{\, \mathbf{x} \in \mathbf{R}^\omega \mid \overline{\rho}(\mathbf{0}, \mathbf{x}) < 1 \,\}.$$

Then

$$
\begin{aligned}
\overline{\rho}(\mathbf{0}, \mathbf{w}_n) &= \sup\{\overline{d}(0,0), ..., \overline{d}(0,0), \overline{d}(0,n), \overline{d}(0,n), ...\}\\
&= \sup\{0, ..., 0, 1, 1, ...\}\\
&= 1
\end{aligned}
$$

for all n. Hence the sequence $\{\mathbf{w}_n\}$ is never in U for any n so $\{\mathbf{w}\}$ does not converge to $\mathbf{0}$. ■

Problem 3.9 (A)

Given: X a metric space, A a countable subset of X, and $\overline{A} = X$. To prove: the topology of X has a countable basis.

Proof. Let $d\colon X \times X \to \mathbf{R}$ denote the metric on X. We claim that the collection

$$\mathcal{B} = \bigcup_{a \in A} B_a \quad \text{where} \quad B_a = \{\, B_d\big(a, \tfrac{1}{k}\big) \mid k \in \mathbf{Z} \,\}$$

is a countable basis for the topology on X induced by the metric d. First B_a is countable, by Theorem 7.1(3), since there is an injection, namely the map $B_d\big(a, \frac{1}{k}\big) \xmapsto{f} k$ (since if $f\big(B_d\big(a, \frac{1}{k}\big)\big) = k = \ell = f\big(B_d\big(a, \frac{1}{\ell}\big)\big)$ then $B_d\big(a, \frac{1}{k}\big) = B_d\big(a, \frac{1}{\ell}\big)$), from B_a to $\mathbf{Z}_+$. Then $\mathcal{B}$ is countable by Theorem 7.5 since it is a countable union of countable sets.

Now we will prove that $\mathcal{B}$ is indeed a basis. Let $y \in X$. Then, since A is dense in X, for any open ball $B_d(y, \varepsilon)$, for $\varepsilon > 0$, there is $B_d(y, \varepsilon) \cap A \neq \emptyset$. By the Archimedean property of $\mathbf{R}$ (cf. Munkres §4, Theorem 4.2), we may choose $N > 2/\varepsilon$ so that we have $B_d\big(y, \frac{1}{N}\big) \subset B_d(y, \varepsilon)$. Then, since A is dense in X, $B_d\big(y, \frac{1}{N}\big) \cap A \neq \emptyset$. Let $x \in B_d\big(y, \frac{1}{N}\big) \cap A$. Then $y \in B_d\big(x, \frac{1}{N}\big) \subset B_d(y, \varepsilon)$ so by Lemma 13.2, $\mathcal{B}$ is a basis for the topology of X. Hence, the topology of X has a countable basis. ∎

Problem 3.10 (B)

Given: Y is an ordered set, (a, b) and (c, d) are disjoint open intervals, and there are elements $x \in (a, b)$ and $y \in (c, d)$ with $x < y$. To prove: every element of (a, b) is less than every element of (c, d).

Proof. First, it is evident that $a < b$ and $c < d$. Let $w \in (a, b)$ and $z \in (c, d)$. Then $w < x$ or $w > x$ and $z < y$ or $z > y$ (there are four cases in total). In most of the cases we shall proceed by contradiction.

Case 1: Suppose $w < x$ and $z < y$. If $z < w$, then the interval $(a, b) \cap (c, d) \neq \emptyset$ since $x \in (z, y) \subset (c, d)$ and $x \in (a, b)$ by hypothesis. This is a contradiction.

Case 2: Suppose $w > x$ and $z < y$. If $z < w$, then the interval $(a, b) \cap (c, d) \neq \emptyset$ since $(z, y) \subset (c, d)$ and $w \in (z, y)$. This is a contradiction.

Case 3: Suppose $w < x$ and $z > y$. Then by the transitivity of the order relation (cf. Munkres §1, p. 24), $w < x < y < z$ so $w < z$.

Case 4: Suppose $w > x$ and $z > y$. If $z < w$, then $(a, b) \cap (c, d) \neq \emptyset$ since $y \in (x, w) \subset (a, b)$ and $y \in (c, d)$ by hypothesis. This is a contradiction.

In every case, we have that $w < z$. ∎

Problem 3.11 (C)

(This problem will be used when we discuss quotient spaces). Let S and T be sets and let $f\colon S \to T$ be a function. Let $A \subset S$.

(i) Give an example to show that the equation

$$f^{-1}(f(A)) = A \tag{*}$$

isn't always valid.

(ii) Define an equivalence relation $\sim$ on S by $s \sim s'$ if and only if $f(s) = f(s')$. Using this equivalence relation, describe the subsets A of S for which (*) is true. Prove that your answer is correct.

Proof. (i) Problem 1.1 (Munkres §2, Ex. 1(b)) gives a hint as to what sort of map f might be. In particular, we are looking for a map between sets $f\colon S \to T$ that is not surjective. Consider the map $f\colon x \mapsto 0$ from $\mathbf{R} \subset \mathbf{R}$ to $\mathbf{R}$. Then $f(A) = \{0\}$ for any subset $A \subset \mathbf{R}$, but $f^{-1}(f(A)) = f^{-1}(\{0\}) = \mathbf{R} \neq A$ if $A \subsetneq \mathbf{R}$.

(ii) The subsets of A for which (*) is true are the equivalence classes $A = \{\, s \sim s' \,\}$. By Problem 1.1 (Munkres §2, Ex. 1(b)) it is enough to show that the map $f\colon s \mapsto f(s)$ is surjective onto $\{f(s)\}$. But this is immediate since $s' \in A$ if and only if $f(s') = f(s)$. ∎