

# MA 572: Homework 4

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**PROBLEM 4.1 (HATCHER §2.1, EX. 20)**

Show that  $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$  for all  $n$ , where  $SX$  is the suspension of  $X$ . More generally, thinking of  $SX$  as the union of two cones  $CX$  with their bases identified, compute the reduced homology groups of the union of any finite number of cones  $CX$  with their bases identified.

*Proof.* First note that the reduced suspension of  $X$ ,  $\Sigma X$ , which is homotopy equivalent to  $SX$ , can be realized as the quotient space  $CX/X$ . Given the imbedding  $X \hookrightarrow CX$ , by 2.16 and excision (or 2.22) we have the long exact sequence in

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \tilde{H}_n(X) & \longrightarrow & \tilde{H}_n(CX) & \longrightarrow & \tilde{H}_n(CX, X) & \longrightarrow & \cdots \\ & & & & & & \searrow & & \\ & & & & & & \tilde{H}_{n-1}(X) & \longrightarrow & \tilde{H}_{n-1}(CX) & \longrightarrow & \tilde{H}_{n-1}(CX, X) & \longrightarrow & \cdots \end{array} \quad (1)$$

where  $\tilde{H}_n(CX, X) \cong \tilde{H}_n(CX/X) \cong \tilde{H}_n(SX)$ . Since  $CX$  is contractible, we have  $H_n(CX) = 0$  for all  $n$  and the long exact sequence (1) yields an isomorphism

$$\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X). \quad \blacksquare$$

**PROBLEM 4.2 (HATCHER §2.1, EX. 22)**

Prove by induction on the dimension the following facts about the homology of a finite dimensional CW complex  $X$ , using the observation that  $X^n/X^{n-1}$  is a wedge sum of  $n$ -spheres:

- (a) If  $X$  has dimension  $n$  then  $H_i(X) = 0$  for  $i > n$  and  $H_n(X)$  is free.
- (b)  $H_n(X)$  is free with basis in bijective correspondence with the  $n$ -cells if there are no cells of dimension  $n - 1$  or  $n + 1$ .
- (c) If  $X$  has  $k$   $n$ -cells, then  $H_n(X)$  is generated by at most  $k$  elements.

*Proof.*

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**PROBLEM 4.3 (HATCHER §2.2, EX. 2)**

Given a map  $f: S^{2n} \rightarrow S^{2n}$ , show that there is some point  $x \in S^{2n}$  with either  $f(x) = x$  or  $f(x) = -x$ . Deduce that every map  $\mathbb{RP}^{2n} \rightarrow \mathbb{RP}^{2n}$  has a fixed point. Construct maps  $\mathbb{RP}^{2n-1} \rightarrow \mathbb{RP}^{2n-1}$  without fixed points from linear transformations  $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  without eigenvectors.

*Proof.*

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