

MA 519: Homework 5

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PROBLEM 5.1 (HANDOUT 7, # 6(D, F))

Find the variance of the following random variables

- (d) $X = \#$ of tosses of a fair coin necessary to obtain a head for the first time.
 - (f) $X = \#$ matches observed in random sitting of 4 husbands and their wives in opposite sides of a linear table.
- This is an example of the *matching problem*.

SOLUTION. Recall that the variance of a random variable can be computed as

$$\text{Var}(X) = E(X^2) - E(X)^2.$$

For part (d), let X be as above. First, note that X takes every value on \mathbb{N} . Thus, its PMF is

$$p(n) = P(X = n) = \frac{1}{2^n}$$

and its expectation the value of the series

$$E(X) = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Using a little bit of analysis we can find the value of $E(X)$, e.g., by considering the function $f(x) := \sum_{n=1}^{\infty} nx^{n-1}$, taking its indefinite integral, and noting that it is a geometric series sans the first term. Concretely,

$$\int f(x) dx = \sum_{n=1}^{\infty} x^n = -1 + \sum_{n=0}^{\infty} x^n,$$

which, for $|x| < 1$, converges to the value $x/(1-x)$. Taking the derivative of this, we have $1/(1-x)^2$. Thus,

$$\begin{aligned} E(X) &= \sum_{n=1}^{\infty} \frac{n}{2^n} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \\ &= \frac{1/2}{(1 - (1/2))^2} \\ &= 2. \end{aligned}$$

This is the mean of X .

Next we must compute the mean of X^2 . We have already computed the PMF of X hence,

$$E(X^2) = \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

To find the limit of this series, we can use a similar method to the one in the last paragraph. That is, consider the function $g(x) := \sum_{n=1}^{\infty} n^2 x^{n-1}$. Taking its integral, we have

$$xG(x) = \int g(x) dx = \sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1}$$

and repeat this on G , giving us

$$\int G(x) dx = \sum_{n=1}^{\infty} x^n = -1 + \sum_{n=0}^{\infty} x^n = \frac{x}{1-x}.$$

Tracing back our steps,

$$\int g(x) = \frac{x}{(1-x)^2}$$

so

$$g(x) = \frac{1-x^2}{(1-x)^4}.$$

Thus,

$$\begin{aligned} E(X) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \\ &= \frac{(1/2)(1 - (1/2)^2)}{(1 - (1/2))^4} \\ &= 6. \end{aligned}$$

Putting all of this together, the variance is

$$\boxed{\text{Var}(X) = 6 - (2)^2 = 2.}$$

For part (f), again, we let X be as above. The PMF of X is given by

$$\begin{aligned} P(X=1) &= \frac{1}{12} & P(X=2) &= \frac{1}{24} \\ P(X=3) &= 0 & P(X=4) &= \frac{1}{24}. \end{aligned}$$

Then, the mean is

$$\begin{aligned} \mu &= \frac{1}{12} + 2 \cdot \frac{1}{24} + 3 \cdot 0 + 4 \cdot \frac{1}{24} \\ &= \frac{1}{3} \\ &= 0.333. \end{aligned}$$

The second moment is

$$\begin{aligned} E(X^2) &= \frac{2}{12} + \frac{4}{24} + \frac{16}{24} \\ &= \frac{2 + 4 + 16}{24} \\ &= \frac{22}{24} \\ &= \frac{11}{12} \\ &\approx 0.917. \end{aligned}$$

Thus, the variance is

$$\text{Var}(X) = \frac{11}{12} - \left(\frac{1}{3}\right)^2 \approx 0.806.$$

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PROBLEM 5.2 (HANDOUT 7, # 8)

(Nonexistence of variance).

- (a) Show that for a suitable positive constant c , the function $p(x) = c/x^3$, $x = 1, \dots$, is a valid probability mass function (PMF).
- (b) Show that in this case, the expectation of the underlying random variable exists, but the variance does not!

SOLUTION. For part (a), note that $p(x)$ given above satisfies the requirements to be a probability mass function. First, set $1/c = \sum_{x=1}^{\infty} 1/x^3$, and note that indeed c is well defined (because the relevant series converges, by the p -test.)

This means that $1 = \sum_{x=1}^{\infty} c/x^3 = \sum p(x)$, by definition. Moreover, because $p(x) = c/x^3 > 0$ for all x in our domain, $p(x) \in [0, 1]$. That is, p is a valid probability mass function.

Set X equal to the random variable described by p . Next, note that

$$\begin{aligned} E(X) &= \sum_{n=1}^{\infty} n \frac{c}{n^3} \\ &= \sum_{n=1}^{\infty} \frac{c}{n^2} \end{aligned}$$

which converges (and thus exists), again by the p -test.

However,

$$\begin{aligned} E(X^2) &= \sum_{n=1}^{\infty} n^2 \frac{c}{n^3} \\ &= \sum_{n=1}^{\infty} \frac{c}{n} \end{aligned}$$

which does not converge, again by the p -test. That is, the variance $E(X^2) - E(X)^2$ does not exist. ■

PROBLEM 5.3 (HANDOUT 7, # 9)

In a box, there are 2 black and 4 white balls. These are drawn out one by one at random (without replacement).

- (a) Let X be the draw at which the first black ball comes out. Find the mean and the variance of X .
 (b) Let X be the draw at which the second black ball comes out. Find the mean and the variance of X .

SOLUTION. For part (a), we must first find the PMF of X . This we do explicitly,

$$\begin{aligned} P(X=1) &= \frac{2}{6} = \frac{1}{3}, & P(X=2) &= \frac{2}{5} \cdot \frac{4}{6} = \frac{4}{15}, \\ P(X=3) &= \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} = \frac{1}{5}, & P(X=4) &= \frac{2}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} = \frac{2}{15}, \\ P(X=5) &= 1 \cdot \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} = \frac{1}{15}. \end{aligned}$$

Thus,

$$\begin{aligned} E(X) &= 1 \cdot \frac{1}{3} + 2 \cdot \frac{4}{15} + 3 \cdot \frac{1}{5} + 4 \cdot \frac{2}{15} + 5 \cdot \frac{1}{15} \\ &= \frac{7}{3} \\ &= 2.333. \end{aligned}$$

Similarly, we have

$$\begin{aligned} E(X^2) &= 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{4}{15} + 3^2 \cdot \frac{1}{5} + 4^2 \cdot \frac{2}{15} + 5^2 \cdot \frac{1}{15} \\ &= 7. \end{aligned}$$

Hence,

$$\text{Var}(X) = 7 - \left(\frac{7}{3}\right)^2 \approx 1.556.$$

For part (b) we have a similar setup. We compute the PMF of X explicitly

$$\begin{aligned} P(X=2) &= \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{15}, & P(X=3) &= \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} + \frac{2}{6} \cdot \frac{4}{5} \cdot \frac{1}{4} = \frac{2}{15}, \\ P(X=4) &= 3 \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{3}{15}, & P(X=5) &= 4 \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{4}{15}, \\ P(X=6) &= 5 \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{2}{2} \cdot 1 = \frac{5}{15}. \end{aligned}$$

Thus,

$$\begin{aligned} E(X) &= \frac{2 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4 + 6 \cdot 5}{15} \\ &= \frac{14}{3} \\ &\approx 4.667. \end{aligned}$$

Similarly,

$$\begin{aligned} E(X^2) &= \frac{2^2 + 3^2 \cdot 2 + 4^2 \cdot 3 + 5^2 \cdot 4 + 6^2 \cdot 5}{15} \\ &= \frac{70}{3} \\ &\approx 23.333. \end{aligned}$$

Thus,

$$\text{Var}(X) \approx \frac{70}{3} - \left(\frac{14}{3}\right)^2 \approx 1.556.$$

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PROBLEM 5.4 (HANDOUT 7, # 10)

Suppose X has a *discrete uniform distribution* on the set $\{1, \dots, N\}$.

Find formulas for the mean and the variance of X .

SOLUTION. First, we find the mean:

$$\begin{aligned} E(X) &= \sum_{n=1}^N n \frac{1}{N} \\ &= \frac{1}{N} \frac{N(N+1)}{2} \\ &= \frac{(N+1)}{2} \end{aligned}$$

Next, we find the variance:

$$\begin{aligned} E(X^2) - E(X)^2 &= \sum_{n=1}^N n^2 \frac{1}{N} - \left[\frac{(N+1)}{2} \right]^2 \\ &= \frac{N^2}{3} + \frac{N}{2} + \frac{1}{6} - \left[\frac{(N+1)}{2} \right]^2 \\ &= \frac{N^2 - 1}{12} \end{aligned}$$

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PROBLEM 5.5 (HANDOUT 7, # 11)

(Be Original) Give an example of a random variable with mean 1 and variance 100.

SOLUTION. Let X be the random variable whose PMF is given by

$$\begin{aligned} P(X = -10 - 1) &= 0.5 \\ P(X = 10 - 1) &= 0.5 \\ P(X \neq \pm\sqrt{10} - 1) &= 0 \end{aligned}$$

(Note that those expressions are very easy to simplify ($-10-1=-11$, $10-1=9$), but leaving them in that form makes the arithmetic more obvious.)

Then we see that the mean of X is given by

$$\begin{aligned} E(X) &= 0.5(-10 - 1 + 10 - 1) \\ &= 1 \end{aligned}$$

and the variance of X is given by

$$\begin{aligned} E((X - E(X))^2) &= E((X - 1)^2) \\ &= 0.5(10^2 + (-10)^2) \\ &= 0.5(10^2 + (-10)^2) \\ &= 100 \end{aligned}$$

so that X is such a random variable as described in the problem. ■

PROBLEM 5.6 (HANDOUT 7, # 13)

(*Be Original*). Suppose a random variable X has the property that its second and fourth moment are both 1.

What can you say about the nature of X ?

SOLUTION. By Hölder's inequality,

$$\mu = E(X) \leq E(X^2) < \infty.$$

Similarly, we can show that for any $n \in \mathbb{N}$, X^n has finite mean. ■

PROBLEM 5.7 (HANDOUT 7, # 14)

(Be Original). One of the following inequalities is true in general for all nonnegative random variables. Identify which one!

$$\begin{aligned} E(X)E(X^4) &\geq E(X^2)E(X^3); \\ E(X)E(X^4) &\leq E(X^2)E(X^2). \end{aligned}$$

SOLUTION. We show that the first of these inequalities is not true in general. Consider the probability space $\Omega = \{0, 1\}$ and the random variable $X(0) = -1$, $X(1) = 2$ with PMF $P(X = 0) = P(X = 1) = 1/2$. Then,

$$\begin{aligned} E(X) &= -\frac{1}{2} + 1 = \frac{1}{2}, & E(X^2) &= \frac{1}{2} + 2 = \frac{5}{2}, \\ E(X^3) &= -\frac{1}{2} + 4 = \frac{7}{2}, & E(X^4) &= \frac{1}{2} + 8 = \frac{17}{2}, \end{aligned}$$

so

$$\frac{17}{4} = \frac{1}{2} \cdot \frac{17}{2} \not\geq \frac{5}{2} \cdot \frac{7}{2} = \frac{35}{4}.$$

That leaves the second inequality. How do we see that the second inequality is true? By the Cauchy-Schwartz inequality,

$$E(X^4) \leq E(X^2).$$

Moreover, ■

PROBLEM 5.8 (HANDOUT 7, # 15)

Suppose X is the number of heads obtained in 4 tosses of a fair coin.

Find the expected value of the weird function

$$\log(2 + \sin(\frac{\pi}{4}x)).$$

SOLUTION. First, note that

$$\begin{aligned} P(X=0) &= \frac{1}{16}, & P(X=1) &= \frac{4}{16}, \\ P(X=2) &= \frac{6}{16}, & P(X=3) &= \frac{4}{16}, \\ P(X=4) &= \frac{1}{16}. \end{aligned}$$

Thus, computing the expected value of the function, we get

$$\begin{aligned} E[\log(2 + \sin(\frac{\pi}{4}X))] &= \sum_{x=0}^4 p(X=x) \log(2 + \sin(\frac{\pi}{4}x)) \\ &= \frac{1}{16} \left[\log(2) + 4 \log\left(2 + \frac{\sqrt{2}}{2}\right) + 6 \log(2+1) + 4 \log\left(2 + \frac{\sqrt{2}}{2}\right) + \log(2) \right] \\ &= \frac{1}{16} \left[2 \log(2) + 8 \log\left(2 + \frac{\sqrt{2}}{2}\right) + 6 \log(3) \right] \\ &\approx 0.9966. \end{aligned}$$

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PROBLEM 5.9 (HANDOUT 7, # 16)

In a sequence of Bernoulli trials let X be the length of the run (of either successes or failures) started by the first trial.

- (a) Find the distribution of X , $E(X)$, $\text{Var}(X)$.

SOLUTION. From DasGupta's book, X has the PMF

$$P(X = n) = \binom{N}{n} p^n (1-p)^{N-n}$$

where N is the number of trials.

The mean can be easily computed by taking the series

$$\begin{aligned} E(X) &= \sum_{n=0}^N \left[\binom{N}{n} p^n (1-p)^{N-n} \right] n \\ &= \sum_{n=1}^N \left[n \binom{N}{n} \right] p^n (1-p)^{N-n} \\ &= N \sum_{n=1}^N \binom{N-1}{n-1} p^n (1-p)^{N-n} \\ &= Np \sum_{n=1}^N \binom{N-1}{n-1} p^{n-1} (1-p)^{(N-1)-(n-1)} \end{aligned}$$

now we can use the binomial identity $(x+y)^n = \sum_{n=0}^N \binom{N}{n} x^n y^{N-n}$ to obtain

$$\begin{aligned} &= Np(p + (1-p))^{N-1} \\ &= Np. \end{aligned}$$

To find the variance, we must first compute the second moment of X . But first, note that

$$\begin{aligned} n^2 \binom{N}{n} &= Nn \binom{N-1}{n-1} \\ &= N(n-1) \binom{N-1}{n-1} + N \binom{N-1}{n-1} \\ &= N(N-1) \binom{N-2}{n-2} + N \binom{N-1}{n-1}; \end{aligned}$$

this will be helpful in computing the second moment. Now,

$$\begin{aligned} E(X) &= \sum_{n=0}^N \left[\binom{N}{n} p^n (1-p)^{N-n} \right] n^2 \\ &= \underbrace{N(N-1) \sum_{n=0}^N \binom{N-2}{n-2} p^n (1-p)^{N-n}}_{S_1} + \underbrace{N \sum_{n=0}^N \binom{N-1}{n-1} p^n (1-p)^{N-n}}_{S_2}. \end{aligned}$$

Note that S_2 is exactly the mean of X and hence $S_2 = Np$. To find S_1 we use a similar trick

$$\begin{aligned} S_1 &= N(N-1) \sum_{n=0}^N \binom{N-2}{n-2} p^n (1-p)^{N-n} \\ &= N(N-1)p^2 \sum_{n=2}^N \binom{N-2}{n-2} p^{n-2} (1-p)^{(N-2)-(n-2)} \\ &= N(N-1)p^2. \end{aligned}$$

Thus, the second moment is

$$E(X^2) = N(N-1)p^2 + Np = N^2p^2 + Np(1-p).$$

Lastly, the variance of X is

$$\text{Var}(X) = N^2p^2 + Np(1-p) - (Np)^2 = Np(1-p).$$

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PROBLEM 5.10 (HANDOUT 7, # 17)

A man with n keys wants to open his door and tries the keys independently and at random. Find the mean and variance of the number of trials

- (a) if unsuccessful keys are not eliminated from further selections;
- (b) if they are.

(Assume that only one key fits the door. The exact distributions are given in II, 7, but are not required for the present problem.)

SOLUTION.

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