4.2 : 2, 8, 10, 12
4.3 : 2, 4, 6, 10

4.2.2| Let V be the set of all 2x2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that the product $abcd = 0$. Let the operation $\oplus$ be standard addition of matrices and the operation $\odot$ be standard scalar multiplication of matrices.
(a) Is V closed under addition? (b) Is V closed under scalar multiplication?
(c) What is the zero vector in the set V?
(d) Does every matrix A in V have a negative that is in V? Explain.
(e) Is V a vector space?

(a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. We need A+B in V. $A+B = \begin{bmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{bmatrix}$ is in V iff $(a+\alpha)(b+\beta)(c+\gamma)(d+\delta) = 0$. As A, B are in V, $abcd = 0$, $\alpha\beta\gamma\delta = 0$. Then notice $a=0,\ b=1,\ c=1,\ d=0$ and $\alpha=1, \beta=0, \gamma=0, \delta=1$ gives $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are in V but $A+B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not. Thus V is not closed under addition.

(b) Let r be a number and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in V so $abcd = 0$. Then $rA = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$ is in V for $(ra)(rb)(rc)(rd) = r^4 abcd = r^4 \cdot 0 = 0$. Thus V is closed under scalar multiplication.

(c) The zero vector is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in V.

(d) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in V. $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ is in V for $(-a)(-b)(-c)(-d) = abcd = 0$. Thus every vector has a negative.

(e) V is not a vector space since it is not closed under vector addition in Defn 4.4a.

In exercises 8 and 10, the given set together with the given operations is not a vector space. List the properties of Defn 4.4 that fail to hold.

4.2.8| The set of all ordered pairs of real numbers with the operations $(x,y) \oplus (x',y') = (x+x', y+y')$ and $r \odot (x,y) = (x, ry)$.

In Defn 4.4, (a) is satisfied with zero vector (0,0). In (b):

(5) $c \odot ((x,y) \oplus (x',y')) = c \odot (x+x', y+y') = (x+x', cy+cy') = (x, cy) \oplus (x', cy') = (c\odot(x,y)) \oplus (c \odot (x',y'))$ is satisfied.

(6) $(c+d) \odot (x,y) = (x, cy+dy)$ is not satisfied for example $(1+2) \odot (1,1) = (1,3)$ while $c \odot (x,y) \oplus d \odot (x,y) = (1,1) \oplus (1,2) = (2,3)$.

(7) and (8) are also satisfied.

4.1.10 The set of all 2×1 matrices $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \le 0$, with the usual operations in $\mathbb{R}^2$.

Going down the list (inverses will fail), let V be this set.

(a) $\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix}$ in V, then $\begin{bmatrix} x+x' \\ y+y' \end{bmatrix}$ is in V for $x+x' \le 0$ if $x, x' \le 0$

(1) $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+x' \\ y+y' \end{bmatrix} = \begin{bmatrix} x'+x \\ y'+y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$ holds.

(2) Similar to (1) so holds

(3) $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has this property.

(4) This fails for if $\begin{bmatrix} x \\ y \end{bmatrix}$ is in V, then $\begin{bmatrix} -x \\ -y \end{bmatrix}$ has $-x \ge 0$.
e.g. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is in V but $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not.

(B) V is not closed under scalar multiplication for $-1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$ is not in V.

(5) fails for negative c: $c=-1$ $-1 \cdot \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

(6) fails for negative $c+d$ similar to (5). (i.e. $c=-1=d$)

(7) fails for $c \cdot d$ negative similar to (5). (i.e. $c=-1$, $d=1$)

(8) holds.

4.1.12 Let V be the set of all positive real numbers; define $\oplus$ by $u \oplus v = uv$ ($\oplus$ is ordinary multiplication) and define $\odot$ by $c \odot v = v^c$. Prove that V is a vector space.

Have to go down the list in Defn 4.4.

(a) Closed under addition for pos. real numbers u, v, then $u \oplus v = uv$ is a pos. real number.

(1) For any pos. real u, v, $u \oplus v = uv = vu = v \oplus u$.

(2) For any pos. real u, v, w, $u \oplus (v \oplus w) = u \oplus vw = uvw = (u \oplus v)w = (u \oplus v) \oplus w$.

(3) The zero vector is 1 for any u pos. real, $1 \oplus u = 1u = u = u1 = u \oplus 1$.

(4) For any pos. real u, $1/u$ is pos. real and $u \oplus \frac{1}{u} = u\frac{1}{u} = 1 = \frac{1}{u}u = \frac{1}{u} \oplus u$
So $-u = 1/u$ as vectors.

(B) Closed under scalar multiplication for any real c and pos. real u, $c \odot u = u^c$ is pos. real.

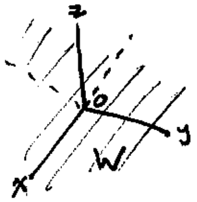
(5) c a real, u, v pos. real, then $c \odot (u \oplus v) = c \odot uv = (uv)^c = u^c v^c = u^c \oplus v^c = (c \odot u) \oplus (c \odot v)$.

(6) c, d reals, u a pos. real, then $(c+d) \odot u = u^{c+d} = u^c u^d = u^c \oplus u^d = (c \odot u) \oplus (d \odot u)$

(7) c, d reals, u a pos. real, then $c \odot (d \odot u) = c \odot u^d = (u^d)^c = u^{cd} = cd \odot u$

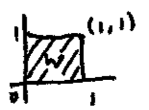
(8) for any pos. real u, then $1 \odot u = u^1 = u$.

4.3.2 | Let W be the set of all points in $\mathbb{R}^3$ that lie in the xy-plane. Is W a subspace of $\mathbb{R}^3$? Explain



Yes. w is the set of vectors $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ So by Thm 4.3 as (a) holds Since $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} x+x' \\ y+y' \\ 0 \end{bmatrix}$ is in W and (b) holds since $c\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ 0 \end{bmatrix}$ is in W we have W is a subspace of $\mathbb{R}^3$.

4.3.4 | Consider the unit square shown in the accompanying figure. Let W be the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $0 \le x \le 1$, $0 \le y \le 1$. Is W a subspace of $\mathbb{R}^2$? Explain.



No. It fails scalar multiplication for $2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is not in W, thus by Thm 4.3 is not a subspace. Further, There is no additive inverse.

Which of the given subsets of $\mathbb{R}^3$ are subspaces?

4.3.6 | The set of all vectors of the form

(a) $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a > 0$ (c) $\begin{bmatrix} a \\ a \\ c \end{bmatrix}$ (d) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $2a - b + c = 1$

(a) This is a subspace by exercise 4.3.2.

(b) Not a subspace by failing Thm 4.3 b as $-\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}$ is not in the subspace.

(c) As $\begin{bmatrix} a \\ a \\ c \end{bmatrix} + \begin{bmatrix} a' \\ a' \\ c' \end{bmatrix} = \begin{bmatrix} a+a' \\ a+a' \\ c+c' \end{bmatrix}$ and $d\begin{bmatrix} a \\ a \\ c \end{bmatrix} = \begin{bmatrix} ad \\ ad \\ cd \end{bmatrix}$ are in W, the subspace, it is a subspace by Thm 4.3.

(d) Not a subspace for if $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in it, then $2\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \\ 2c \end{bmatrix}$ has $2(2a) - (2b) + (2c)$ $= 2(2a - b + c) = 2(1) = 2$, not 1 So fails Thm 4.3 b.

Which of the given subsets of the vector space M_{23}, of a 2x3 matrices are subspaces?

4.3.10 | (a) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = 2c+1$ (b) $\begin{bmatrix} 0 & 1 & a \\ b & c & 0 \end{bmatrix}$ (c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a+c=0$ and $b+d+f=0$.

(a) Fails Thm 4.3 b for $2\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \end{bmatrix}$ has $(2a) = 2(2c+1) = 4c+2 = 2(2c)+2$ which is not $2(2c+1)$.

(b) Fails Thm 4.3 b for $2\begin{bmatrix} 0 & 1 & a \\ b & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2a \\ 2b & 2c & 0 \end{bmatrix}$ is not in the set.

(c) Checking Thm 4.3: (a) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' & c+c' \\ d+d' & e+e' & f+f' \end{bmatrix}$ has $(a+a') + (c+c')$ $= (a+c) + (a'+c') = 0$ and $(b+b') + (d+d') + (f+f') = (b+d+f) + (b'+d'+f') = 0$ is good and (b) $\alpha\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \end{bmatrix}$ has $\alpha a + \alpha c = \alpha(a+c) = 0$ and $\alpha b + \alpha d + \alpha f = \alpha(b+d+f) = 0$ and is thus a subspace.