1. Given n and m, positive integers, compute the number of elements in the following set:

$$S = \{(x_1, x_2, \dots x_m) : 1 \le x_1 \le x_2 \le \dots \le x_m \le n, \text{ where } x_i \text{ are integers.} \}$$

ym ym+1

$$1 \leq \chi_1 \leq \chi_2 \leq \chi_3 \leq \dots \leq \chi_m \leq n$$

$$y_1 \quad y_2 \quad y_m \quad y_{m+1}$$

Let 
$$y_1 = x_1 - 1 \ge 0$$
  
 $y_2 = x_2 - x_1 \ge 0$   
 $y_3 = x_3 - x_2 \ge 0$ 

$$y_m = x_m - x_{m-1} > 0$$

$$y_{m+1} = m - x_m > 0$$

$$y_{1}+y_{2}+-+y_{m}+y_{m+1}=n-1, y_{i}>0$$

Number of solution = 
$$\left(\binom{m-1}{m+1} - 1\right) = \binom{m+n-1}{m}$$

Arower Key MA519, Fall 2016 (Vip)

2. Consider the matching problem for n people with n hats. All the people throw their hats into a container and then each of them picks one at random. Let S be the number of people getting back their own hats. Compute E(S) and Var(S).

(Hint: introduce random variables  $X_i$ , i = 1, 2, ...n, such that  $X_i = 1$  if the *i*-th person gets back his own hat and  $X_i = 0$  if not. Express S in terms of the  $X_i$ 's.)

back his own hat and 
$$X_{i} = 0$$
 if not. Express  $S$  in terms of the  $X_{i}$ 's.)

$$S = X_{1} + X_{2} + \cdots + X_{N}$$

$$S = E(X_{1} + X_{2} + \cdots + X_{N})$$

$$= E(X_{1} + E(X_{2} + \cdots + E(X_{N}))$$

$$= F(X_{i} = 1) + P(X_{2} = 1) + \cdots + F(X_{N} = 1)$$

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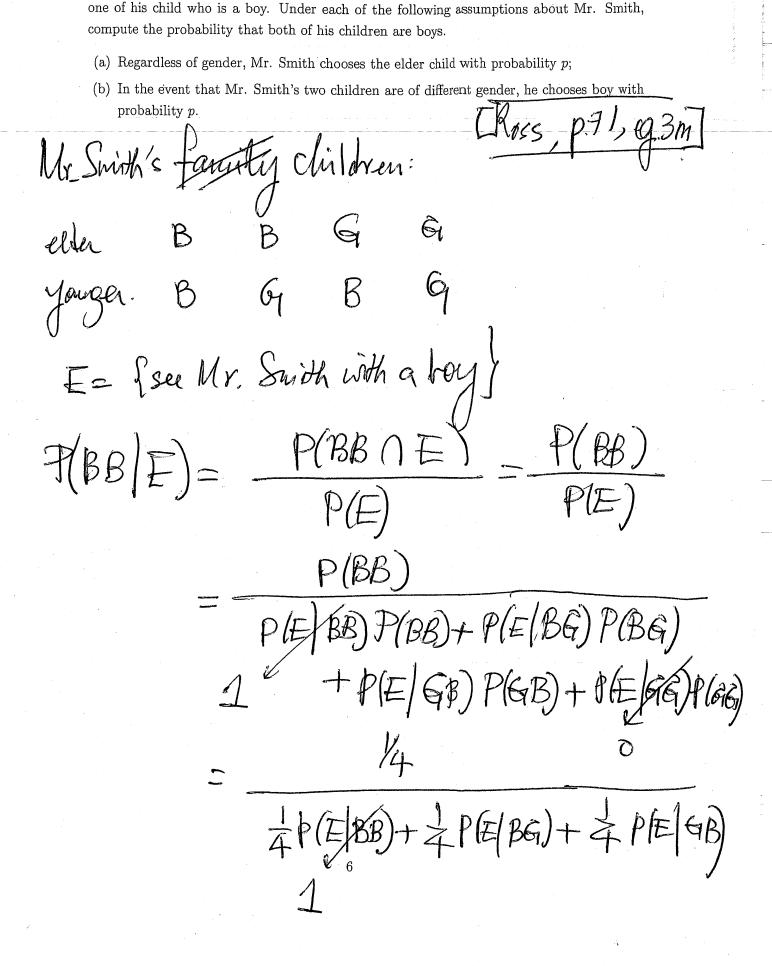
$$= \sum_{i=1}^{n} P(x_{i-1}, x_{j-1})$$

Note

5 is NOT Binomial, even though 5 is the Sum of Bernoulli r.v. Xi. (the Xi's are NOT independent.)

2) We in fact know the exact pot of S  $P(S=k) = \frac{1}{k!} (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{(N-k)!})$ Hence, we have  $ES = \sum_{k=0}^{5} k P(S=k)$ But such a formula is not quite useful.

3) We can also understand the result interms of layen, n>1 or as n->00  $P(S=k) = \frac{1}{R!} (1-\frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{(-1)^{n-k}}{(n-k)!})$ Li ~ Poisson, >=1. Hence ES= = 1 But the above results in facts holds even for finite n.



3. Mr. Smith, new in the neighborhood, is known to have two children. He is seen walking with

$$(a)$$
=
 $1 + p + (1-p)$ 
= 1

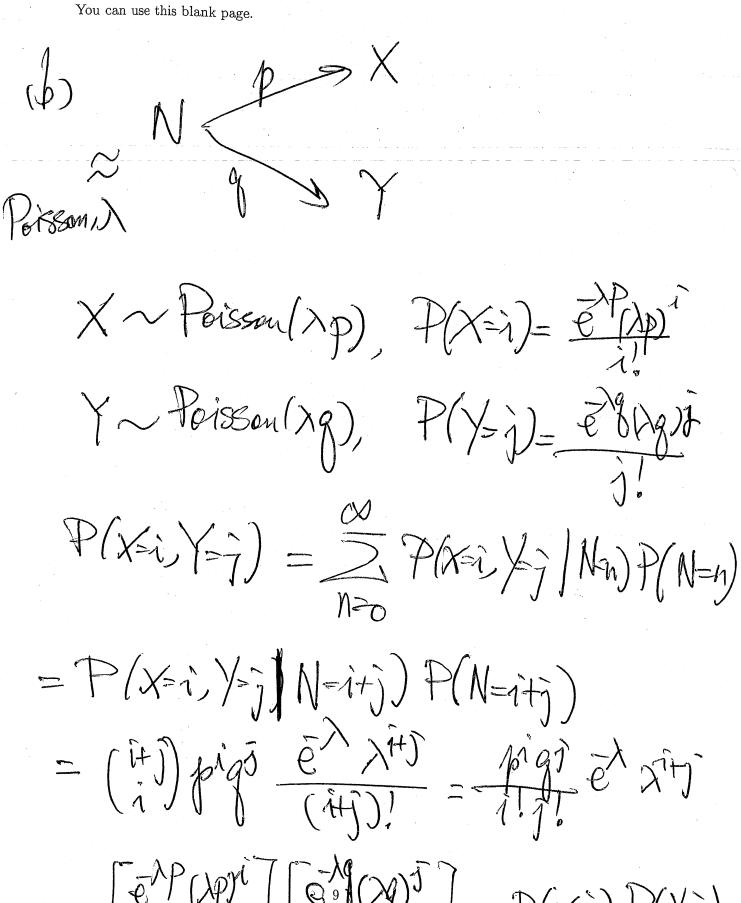
(A goinger) child is equally likely to be boy of girl of theme the answer does not depend on P.T.

$$=\frac{1}{1+p+p}=\frac{1}{1+2p}$$

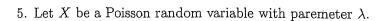
$$[9, P(BB|E)=1 if p=0;$$
 $P(BB|E)=\frac{1}{3} if p=1]$ 

- 4. Consider N interviewees arriving at a company for interviews. Each person is then directed to the first room with probability p or the second room with probability q = 1 p. The decision is done independently for each person. Let X and Y be the number of persons in the first and second room. In the following two situations, find the probability distribution functions of X and Y and determine also if they are independent.
  - (a) N is some fixed deterministic number, i.e. each day the company will only interview certain fixed number of people, say, N = 23.
  - (b) N is Poisson random variable with paramter  $\lambda$ .

In, Analytically, P(X=i, Y=j) = J(X=i) + J(X=i



 $= \left[\frac{e^{\lambda}P(\lambda p)^{i}}{1!}\right] = P(\chi=i)P(\chi=j)$ Hance XdY are independent.



- (a) Show that  $E(X^n) = \lambda E((X+1)^{n-1})$ , for n = 1, 2, ...
- (b) Compute  $E(X^3)$ .
- (c) Compute  $E\left(\frac{1}{X+1}\right)$ .

(a) 
$$= \sum_{i=0}^{\infty} i n \sum_{i=1}^{N} 1^{i}$$

$$=\frac{20}{5}i^{n-1}\frac{e^{2}x^{2}}{(7-0)!}$$

$$= \lambda \frac{2}{1-1} \frac{2^{n-1}}{(n-1)!} \int_{J=n-1}^{\infty}$$

(b) 
$$E[X^3] = \lambda E[X+1)^2$$
  
=  $\lambda E[X^2 + 2X+1]$   
=  $\lambda [E(X^2) + 2EX + 1]$ 

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