Instructor: Tatsunari Watanabe **Name:** ____________________.
TA: Carlos Salinas

MA 26500-215 Quiz 8

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1. Let $\mathscr{P}_2(\mathbb{R})$ be the set of all polynomials of degree less than or equal to 2 with coefficients in $\mathbb{R}$, i.e., if $p(t) = at^2 + bt + c$ is a polynomial in $\mathscr{P}_2(\mathbb{R})$, then $a, b, c \in \mathbb{R}$.

   (a) (4 points) Show that the set $\mathscr{P}_2(\mathbb{R})$ is closed under addition and multiplication by scalars. What is a *zero* for this set?

   Solution: Take $p(t) = a_1t^2 + b_1t + c_1$, $q(t) = a_2t^2 + b_2t + c_2$ in $\mathscr{P}_2(\mathbb{R})$ and $c \in \mathbb{R}$, then

   $$\begin{aligned} p(t) + q(t) &= a_1 + t^2 + b_1t + c_1 + a_2t^2 + b_2t + c_2 \\ &= (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2) \end{aligned} \qquad \begin{aligned} c(pt) &= c(a_1t^2 + b_1t + c_1) \\ &= ca_1 + cb_1t + cc_1. \end{aligned}$$

   More generally, we can show that $\mathscr{P}_2(\mathbb{R})$ satisfies all 8 of the vector space axioms; but they are all trivial calculations that come down to basically these two facts that $\mathscr{P}_2(\mathbb{R})$ is closed under addition and multiplication by scalars.

   (b) (4 points) The set $\mathscr{P}_2(\mathbb{R})$ is in fact a vector space. Find a basis for $\mathscr{P}_2(\mathbb{R})$.

   Solution: The basis I was looking for was $\{ 1, t, t^2 \}$. If your basis had three linearly independent elements, that should be enough.

   (c) (12 points) Define an inner product $\langle -, - \rangle \colon \mathscr{P}_2(\mathbb{R}) \times \mathscr{P}_2(\mathbb{R}) \to \mathbb{R}$ by

   $$\langle p(t), q(t) \rangle \longmapsto \int_0^1 p(t)q(t)\,\mathrm{d}t.$$

   Find a polynomial $q \in \mathscr{P}_2(\mathbb{R})$ such that $\langle p, q \rangle = p(1/2)$ for every $p \in \mathscr{P}_2(\mathbb{R})$. [Hint: You should start by looking at the basis you found in part (b). If you chose a nice basis t^2 should be in your basis. Now for a general $q(t) = at^2 + bt + c \in \mathscr{P}_2(\mathbb{R})$ we have

   $$\langle t^2, p(t) \rangle = \int_0^1 t^2 q(t)\,\mathrm{d}t = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

   Can you come up with enough equations to solve for the unknowns a, b, c?]

Solution: Let $p(t) = at^2 + bt + c$. Using the basis $\{1, t, t^2\}$ we have

$$\begin{aligned}
\int_0^1 at^2 + bt + c \,\mathrm{d}t &= \frac{a}{3} + \frac{b}{2} + c \\
&= 1 \\
\int_0^1 t(at^2 + bt + c) \,\mathrm{d}t &= \int_0^1 at^3 + bt^2 + ct \,\mathrm{d}t \\
&= \frac{a}{4} + \frac{b}{3} + \frac{c}{2} \\
&= \frac{1}{2} \\
\int_0^1 t^2(at^2 + bt + c) \,\mathrm{d}t &= \int_0^1 at^4 + bt^3 + ct^2 \,\mathrm{d}t \\
&= \frac{a}{5} + \frac{b}{4} + \frac{c}{3} \\
&= \frac{1}{4}
\end{aligned}$$

Now, we can write the system above as

$$A = \left[\begin{array}{ccc|c} 1/3 & 1/2 & 1 & 1 \\ 1/4 & 1/3 & 1/2 & 1/2 \\ 1/5 & 1/4 & 1/3 & 1/4 \end{array}\right].$$

Putting A in row reduced echelon form, we have

$$A_{\text{rref}} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & -3/2 \end{array}\right].$$

Thus, the polynomial $q(t) = -15t^2 + 15t - 3/2$.