

# MA 523: Homework 3

Carlos Salinas

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## PROBLEM 3.1

Consider the initial value problem

$$u_t = \sin u_x; \quad u(x, 0) = \frac{\pi}{4}x.$$

Verify that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied and obtain the Taylor series of the solution about the origin.

*SOLUTION.* We skip checking that the assumptions of the Cauchy–Kovalevskaya theorem are satisfied (since I cannot decipher Evans’s notation), and show that the Taylor series of  $u$  at  $(0, 0)$ ,

$$\tilde{u}(x, t) = \sum_{(\alpha_1, \alpha_2)} \frac{u_{\alpha_1, \alpha_2}(0)}{\alpha_1! \alpha_2!} x^{\alpha_1} t^{\alpha_2},$$

is a solution to our PDE.

First, we must compute the coefficients

$$\frac{u_{\alpha_1, \alpha_2}(0, 0)}{\alpha_1! \alpha_2!}.$$

To this end, we must find the partial derivatives  $u_{\alpha_1, \alpha_2}$  and potentially, relations among them which will help us to find these coefficients. Naïvely listing the partials with respect to  $t$  and  $x$ , we have

$$\begin{aligned} u(0, 0) &= 0 \\ u_x(0, 0) &= \frac{\pi}{4} \\ u_t(0, 0) &= \sin u_x(0, 0) = \frac{\sqrt{2}}{2} \\ u_{xx}(0, 0) &= 0 \\ u_{tx}(0, 0) &= 0 \\ u_{tt}(0, 0) &= -\cos(u_x(0, 0))u_{xt}(0, 0) = 0 \\ u_{xxx}(0, 0) &= 0 \\ u_{ttt}(0, 0) &= 0, \end{aligned}$$

etc. Thus,

$$\tilde{u} = \frac{\pi}{4}x + \frac{\sqrt{2}}{2}t.$$

Plugging this equation into our PDE, we have

$$\tilde{u}_t - \sin \tilde{u}_x = \frac{\sqrt{2}}{2} - \sin(\pi/4) = 0,$$

as desired. ■

## PROBLEM 3.2

Consider the Cauchy problem for  $u(x, y)$

$$\begin{aligned}u_y &= a(x, y, u)u_x + b(x, y, u) \\ u(x, 0) &= 0\end{aligned}$$

Let  $a$  and  $b$  be analytic functions of their arguments. Assume that  $D^\alpha a(0, 0, 0) \geq 0$  and  $D^\alpha b(0, 0, 0) \geq 0$  for all  $\alpha$ . (Remember by definition, if  $\alpha = 0$  then  $D^\alpha f = f$ .)

- (a) Show that  $D^\beta u(0, 0) \geq 0$  for all  $|\beta| \leq 2$ .
- (b) Prove that  $D^\beta u(0, 0) \geq 0$  for all  $\beta = (\beta_1, \beta_2)$ . (*Hint:* Argue as in the proof of the Cauchy–Kovalevskaya theorem; i.e., use induction in  $\beta_2$ )

SOLUTION. ■

## PROBLEM 3.3

(Kovalevskaya's example) Show that the line  $\{t = 0\}$  is characteristic for the heat equation  $u_t = u_{xx}$ . Show there does not exist an analytic solution of the heat equation in  $\mathbf{R} \times \mathbf{R}$ , with  $u = 1/(1 + x^2)$  on  $\{t = 0\}$ . (*Hint:* Assume there is an analytic solution, compute its coefficients, and show instead that the resulting power series diverges in any neighborhood of  $(0, 0)$ .)

SOLUTION. ■