MA571 Homework 9

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Problem 1. Let X be a Hausdorff space and let A be a compact subset of X. Prove from the definitions that A is closed.

Proof. This is Theorem 26.3 from Munkers §26, p. 165; we shall paraphrase it.

We show that $X - A$ is open. To that end we will show that, given a point $x_0 \in X - A$, there is neighborhood U of x_0 disjoint from A. For each point $a \in A$, by the Hausdorff property of X, choose disjoint neighborhoods U_a and V_a of x_0 and a, respectively. Then the collection $\{ V_a \mid a \in A \}$ forms an open cover of A so, by Lemma 26.1, only finitely many of the V_a's cover A, say $V_{a_1}, ..., V_{a_n}$. Define $U \coloneqq U_{a_1} \cap \cdots \cap U_{a_n}$. We claim that U is a neighborhood of x_0 disjoint from A. First, it is clear that U is a neighborhood of x_0 since each U_a contains x_0 and U is an intersection of finitely many of these. Second, note that if $z \in U \cap A$ then $z \in U_{a_i}$ for all i and $z \in V_{a_j}$ for some $j \in \{1, ..., n\}$, but $U_{a_j} \cap V_{a_j} = \emptyset$. Therefore, $U \cap A = \emptyset$. By Lemma C, it follows that $X - A$ is open. ■

Problem 2. Let X be a Hausdorff space and let A and B be disjoint compact subsets of X. Prove that there are open sets U and V such that U and V are disjoint, $A \subset U$ and $B \subset V$.

Proof. This is Ex. 5 from Munkres §26, p. 171.

Suppose A and B are disjoint compact subspaces of X. Since X is Hausdorff, by Theorem 26.4, for every $x \in B$ there exists disjoint open sets U_x and V_x where $U_x \supset A$ and V_x is a neighborhood of x. Then the collection $\{ V_x \mid x \in B\}$ is an open cover of B so by Lemma 26.1, only finitely many of the V_x's cover B, say $V_{x_1}, ..., V_{x_n}$. Define $U \coloneqq U_{x_1} \cap \cdots \cap U_{x_n}$ and $V \coloneqq V_{x_1} \cup \cdots \cup V_{x_n}$. We claim that U and V are disjoint neighborhood containing A and B, respectively. It is clear that U and V are open since U is a finite intersection of open sets and V is a union of open sets and that they contain A and B, respectively, since each of the U_x's contain A and $V_{x_1}, ..., V_{x_n}$ is an open cover of B. Lastly, U and V are disjoint since intersection distributes over union, i.e., we have

$$U \cap V = \left(\bigcap_{i=1}^{n} U_{x_i} \right) \cap \left(\bigcup_{j=1}^{n} V_{x_j} \right) = \bigcup \left(\bigcap_{i=1}^{n} U_{x_i} \cap V_{x_j} \right) = \emptyset$$

since $U_{x_i} \cap V_{x_i} = \emptyset$ so $\left(\bigcap_{i=1}^{n} U_{x_j} \right) \cap V_{x_i} = \emptyset$. ■

Problem 3. Prove the Tube Lemma: Let X and Y be topological spaces with Y compact, let $x_0 \in X$, and let N be an open set of $X \times Y$ containing $x_0 \times Y$, then there is an open set W of X containing x_0 with $W \times Y \subset N$.

Proof. This is Lemma 26.8 from Munkres §26, p. 168, but is proved in *Step 1* in the process of showing Theorem 26.7; we paraphrase the proof here.

Let $x_0 \in X$, and let N be an open set of $X \times Y$ containing $x_0 \times Y$. Cover $x_0 \times Y$ by basic open sets $U \times V$ lying in N. Note that $x_0 \times Y$ is compact, since it is an imbedding of Y given by the map $y \mapsto (x_0, y)$ from Y into $X \times Y$ therefore, by Lemma 26.1, only finitely many of the $U \times V$'s, say $U_1 \times V_1, ..., U_n \times V_n$, cover $x_0 \times Y$. Define $W \coloneqq U_1 \cap \cdots \cap U_n$. We claim that W is a neighborhood of x_0 such that $W \times Y \subset N$. First, it is clear that W is a neighborhood of x_0 since it is the finite intersection of open sets and each $U_i \times V_i$ intersects $x_0 \times Y$ hence contains a point of the form (x_0, y) so $U_i = \pi_1(U_i \times V_i)$ contains x_0. Lastly, $W \times Y \subset N$ since $W \times Y \subset \bigcup_{i=1}^{n} U_i \times V_i$.

To see this let $(x, y) \in W \times Y$ and consider the point $(x_0, y) \in x_0 \times Y$. Since (x_0, y) is in $U_i \times V_i$ for some i, we have $y \in V_i$. But $x \in U_j$ for every j since $x \in W$. Thus $(x, y) \in U_i \times V_i$ as desired. It follows that, W is a neighborhood of x_0 with $W \times Y \subset N$ as desired. ■

Problem 4. Show that if Y is compact, then the projection map $X \times X \to X$ is a closed map.

Proof. We shall proceed by the tube lemma, i.e, Theorem 26.8. Let C be a closed subset of $X \times Y$ then $N = (X \times Y) - C$ is open. Choose $x_0 \in X - \pi_1(C)$. Then $x_0 \times Y$ is contained in N so by the tube lemma, there exists a neighborhood W of x_0 such that $W \times Y \subset N$. In particular, $W \subset X - \pi_1(C)$ otherwise if $x \in W \cap \pi_1(C)$ then $x \times Y \subset N$ and $(x, y) \in C$ for some $y \in Y$, but $N \cap C = \emptyset$. It follows by Lemma C that $X - \pi_1(C)$ is open so $\pi_1(C)$ is closed. Since C was chosen arbitrarily we see that π_1 is a closed map. ■

Problem 5. Let X be a compact space and suppose we are given a nested sequence of subsets $C_1 \supset C_2 \supset \cdots$ with all C_i closed. Let U be an open set containing $\bigcap C_i$. Prove that there is an i_0 with $C_{i_0} \subset U$.

Proof. Consider the family of open sets $U_i \coloneqq X - C_i$. Since U is open $X - U$ is closed so by Theorem 26.2 is compact. We claim that U_i forms an open cover of $X - U$. To see note that by De Morgan's laws

$$\bigcup_{i \in \mathbf{N}} U_i = \bigcup_{i \in \mathbf{N}} X - C_i = X - \bigcap_{i \in \mathbf{N}} C_i \supset X - U$$

since $\bigcap_{i \in \mathbf{N}} C_i \subset U$. Therefore by Lemma 26.1 only finitely many of the U_i's cover $X - U$, say $U_{i_1}, ..., U_{i_n}$. Thus, we have that $X - U \subset \bigcup_{i=1}^n U_i$ so $U \supset \bigcap_{j=1}^n C_{i_j} = C_{i_n}$ as desired. ■

Problem 6. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i \colon X \to \mathbf{R}$, $i = 1, ..., n$ with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. Prove that X is homeomorphic to a subspace of $\mathbf{R}^n$.

Proof. Consider the map $f \colon X \to \mathbf{R}^n$ defined by $f \coloneqq (f_1, ..., f_n)$. This map is continuous by Theorem 18.4 since each component f_i is continuous. We claim that $X \approx f(X)$. To prove this it suffices to show that f is injective so that its restriction to $f(X)$ will be surjective and lastly invoke Theorem 26.6. Suppose $f(x) = f(y)$ but $x \neq y$. Then $f_i(x) \neq f_i(y)$ for some i, but this implies that $f(x) \neq f(y)$. This is a contradiction therefore, $x = y$. It follows that f is a bijection from a compact space X into $f(X) \subset \mathbf{R}^n$ so by Theorem 26.6, we have $X \approx f(X)$. ■

Problem 7. Let X be a compact metric space and let $\mathcal{U}$ be a covering of X by open sets. Prove that there is an $\varepsilon > 0$ such that, for each set $S \subset X$ with diameter $< \varepsilon$, there is a $U \in \mathcal{U}$ with $S \subset U$. (This fact is known as the "Lebesgue number lemma.")

Proof. ■

Problem 8. Let S^1 denote the circle $\{ x^2 + y^2 = 1 \}$ in $\mathbf{R}^2$. Define an equivalence relation on S^1 by

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } (x, y) = (-x', -y')$$

(you do not have to prove that this is an equivalence relation). Prove that the quotient space $S^1/{\sim}$ is homeomorphic to S^1.

One way to do this is by using complex numbers.

Proof. ■

Problem 9. Let X be a nonempty compact Hausdorff space and let $f\colon X \to X$ be a continuous function. Suppose f is 1-1. Prove that there is a nonempty closed set A with $f(A) = A$. (The hypothesis that f is 1-1 is not actually needed, but it makes the proof a little easier.)

Proof. ■

Problem 10. Let $\sim$ be the equivalence relation on $\mathbf{R}^2$ defined by $(x, y) \sim (x', y')$ if and only if there is a nonzero t with $(x, y) = (tx', ty')$. Prove that the quotient space $\mathbf{R}^2/\!\sim$ is compact but not Hausdorff.

Proof. ■

Problem 11. Let X be a locally compact Hausdorff space. Explain how to construct the one-point compactification of X and prove that the space you construct is really compact (you do not have to prove anything else for this problem.)

Proof. ■

Problem 12. Show that if $\prod_{n=1}^{\infty} X_n$ is locally compact (and each X_n is nonempty), then each X_n is locally compact and X_n is compact for all but finitely many n.

Proof. ■

Problem 13. Let X be a locally compact Hausdorff space, let Y be any space, and let the function space $\mathcal{C}(X, Y)$ have the compact-open topology. Prove that the map

$$e\colon X \times \mathcal{C}(X, Y) \to Y$$

define by the equation $e(x, f) = f(x)$ is continuous.

Proof. ■

Problem 14. Let I be the unit interval, and let Y be a path-connected space. Prove that any two maps from I to Y are homotopic.

Proof. ■

Problem 15. Let X be a topological space and $f\colon [0, 1] \to X$ any continuous function. Define $\bar{f}$ by $\bar{f}(t) = f(1 - t)$. Prove that $f * \bar{f}$ is path-homotopic to the constant path at $f(0)$.

Proof. ■

Problem 16. LEt X be a path-connected topological space and let $x_0, x_1 \in X$. Recall that any path α from x_0 to x_1 gives an isomorphism $\hat{\alpha}$ from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ (you do not have to prove this.)

Suppose that for every pair of paths α and β from x_0 to x_1 the isomorphisms $\hat{\alpha}$ and $\hat{\beta}$ are the same. Prove that $\pi_1(X, x_0)$ is Abelian.

Proof. ■