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MA 26500-215 Quiz 11

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1. In the following question, V is a finite dimensional vector space, W is a subspace of V and $T: V \to V$ is a linear operator (i.e., a linear map for V into itself).
   (a) (2 points) What does it mean for a set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ to be a basis for V?
   (b) (2 points) What is the meaning of $\dim V$?
   (c) (2 points) What is an eigenvalue of T? What is an eigenvector?
   (d) (2 points) When is a linear operator T diagonalizable?
   (e) (2 points) If λ is an eigenvalue of T with respect to W, is λ an eigenvalue of T with respect to V?

2. (4 points) Suppose that A and B are conjugate matrices. Show that if λ is an eigenvalue of A then it is an eigenvalue of B.

Solution: Suppose that λ is an eigenvalue of A and that A is conjugate to B. Then, λ is an eigenvalue of A means that there exists a vector (the associated eigenvector) $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$; while A is conjugate to B means that there exists an invertible matrix P such that $A = PBP^{-1}$. Thus,

$$PBP^{-1}\mathbf{x} = A\mathbf{x} = \lambda\mathbf{x}$$

so

$$\begin{aligned} PBP^{-1}\mathbf{x} &= \lambda\mathbf{x} \\ BP^{-1}\mathbf{x} &= P^{-1}\lambda\mathbf{x} \\ &= \lambda P^{-1}\mathbf{x} \end{aligned}$$

now let $\mathbf{y} = P^{-1}\mathbf{x}$ and we have

$$B\mathbf{y} = \lambda\mathbf{y}.$$

So λ is an eigenvalue of B with associated eigenvector $\mathbf{y} = P^{-1}\mathbf{x}$.

3. (6 points) Suppose V is a finite dimensional complex vector space with an inner product $(\cdot,\cdot)\colon V\times V\to\mathbb{C}$. For an orthogonal 2×2 matrix A, show that

$$(A\mathbf{x}, A\mathbf{y}) = (\mathbf{x}, \mathbf{y})$$

for any $\mathbf{x}, \mathbf{y} \in V$.

Solution: Since A is an orthogonal matrix, we know that $A\bar{A}^{\mathrm{T}} = I$. Now,

$$\begin{aligned}(A\mathbf{x}, A\mathbf{y}) &= (\mathbf{x}, \bar{A}^{\mathrm{T}}A\mathbf{y})\\ &= (\mathbf{x}, I\mathbf{y})\\ &= (\mathbf{x}, \mathbf{y})\end{aligned}$$

as desired.