

MA571 Homework 11

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PROBLEM 11.1 (MUNKRES §53, EX. 7(ABCD))

Let G be a topological group with operation \cdot and identity element x_0 . Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0 . If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

- (a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.
- (b) Show that this operation induces a group operation \otimes on $\pi_1(G, x_0)$.
- (c) Show that the two group operations $*$ and \otimes on $\pi_1(G, x_0)$ are the same. [Hint: Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]
- (d) Show that $\pi_1(G, x_0)$ is Abelian.

Proof. For part (a) we need to show that the operation (i) \otimes is associative, (ii) $\Omega(G, x_0)$ is closed under \otimes , (iii) $\Omega(G, x_0)$ contains an identity element e and (iv) for every $f \in \Omega(G, x_0)$ there exists an element f^{-1} such that $f \otimes f^{-1} = f^{-1} \otimes f = e$. We shall proceed in order: (i) is easy since the operation \otimes is associative so for any triple $f, g, h \in \Omega(G, x_0)$ we have

$$(f \otimes g) \otimes h = (f(s) \cdot g(s)) \otimes h = (f(s) \cdot g(s)) \cdot h(s)$$

which one clearly sees, by associativity of \cdot , is the same as $f \otimes (g \otimes h)$. (ii) Let $f, g \in \Omega(G, x_0)$ then, since \cdot is continuous, the map $f \otimes g: I \rightarrow G$ is continuous and

$$(f \otimes g)(0) = f(0) \cdot g(0) = x_0 \cdot x_0 = f(1) \cdot g(1) = (f \otimes g)(1).$$

Thus, $f \otimes g \in \Omega(G, x_0)$. Next, for (iii) consider the constant loop $e_{x_0}(s)$. This map is clearly the identity on $\Omega(G, x_0)$ for if $f \in \Omega(G, x_0)$ then $e_{x_0} \otimes f = e_{x_0}(s) \cdot f(s) = x_0 \cdot f(s) = f(s)$ for all s ; similarly for $f \otimes e_{x_0}$. Lastly, (iv) consider the map $f^{-1}(s) := (f(s))^{-1}: I \rightarrow G$. This map is continuous since taking the inverse in G is continuous and composition of continuous maps is continuous by Theorem 18.2(c). Lastly, note that $f^{-1}(0) = x_0^{-1} = x_0$ similarly for $f^{-1}(1)$. Thus, f^{-1} is a loop and

$$f^{-1} \otimes f = (f(s))^{-1} \cdot f(s) = x_0 = f(s) \cdot (f(s))^{-1} = f \otimes f^{-1}$$

so $\Omega(G, x_0)$ is closed under inverses. Thus, $\Omega(G, x_0)$ is a group.

(b) The map $\otimes: \Omega(G, x_0) \times \Omega(G, x_0) \rightarrow \Omega(G, x_0)$ clearly induces a group operation on $\Pi_1(X, x_0)$ given by $[f] \otimes [g] = [f \otimes g]$. All we need to check is that this operation is in fact well defined on the equivalence class of loops based at x_0 , i.e., if $f_1 \simeq_p f_2$ with path homotopy H and $g_1 \simeq_p g_2$ with path homotopy K we want that $f_1 \otimes g_1 \simeq_p f_2 \otimes g_2$. But this is immediate via the homotopy $L(s, t) := H(s, t) \cdot K(s, t): G \times I \rightarrow G$. This map is continuous since it can be realized as the sequence of compositions

$$(s, t) \mapsto (H(s, t), K(s, t)) \mapsto H(s, t) \cdot K(s, t)$$

where the intermediate step in the composition, i.e., the map from $I \times I \rightarrow G \times G$, is continuous by Theorem 18.4 since H and K are continuous and lastly $L(s, 0) = H(s, 0) \cdot K(s, 0) = f_1(s) \cdot g_1(s) = f_1 \otimes g_1$ and $L(s, 1) = H(s, 1) \cdot K(s, 1) = f_2(s) \cdot g_2(s) = f_2 \otimes g_2$. Thus, $f_1 \otimes g_1 \simeq_p f_2 \otimes g_2$. It follows that \otimes is a well-defined binary operation on $\pi_1(X, x_0)$.

(c) Following the hint, we shall compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$. Recall that

$$f * e_{x_0} = \begin{cases} f(2s) & \text{for } s \in [0, 1/2] \\ e_{x_0}(2s - 1) & \text{for } s \in [1/2, 1] \end{cases} \quad \text{and} \quad e_{x_0} * g = \begin{cases} e_{x_0}(2s) & \text{for } s \in [0, 1/2] \\ g(2s - 1) & \text{for } s \in [1/2, 1] \end{cases}.$$

Then

$$\begin{aligned} (f * e_{x_0}) \otimes (e_{x_0} * g) &= (f * e_{x_0})(s) \cdot (e_{x_0} * g)(s) \\ &= \begin{cases} f(2s) \cdot e_{x_0}(2s) & \text{for } s \in [0, 1/2] \\ e_{x_0}(2s - 1) \cdot g(2s - 1) & \text{for } s \in [1/2, 1] \end{cases} \\ &= \begin{cases} f(2s) & \text{for } s \in [0, 1/2] \\ g(2s - 1) & \text{for } s \in [1/2, 1] \end{cases} \\ &= f * g. \end{aligned}$$

Since $f * e_{x_0} \simeq_p f$ and $e_{x_0} * g \simeq_p g$, we have at last that

$$[f \otimes g] = [(f * e_{x_0}) \otimes (e_{x_0} * g)] = [f * g].$$

(d) Lastly, we show that $\pi_1(X, x_0)$ must be Abelian. It suffices to show that given a class of loop $[f]$ at x_0 , the conjugacy class of $[f]$ consists of the singleton $\{[f]\}$. First, note that if $[g] \in \pi_1(X, x_0)$ then $[g^{-1}] \otimes [g] = [e_{x_0}] = [\bar{g}] * [g]$ so $[g^{-1}] = [g^{-1}] \otimes ([g] * [\bar{g}]) = [\bar{g}]$. Thus, we have

$$\begin{aligned} [\bar{g}] * [f] * [g] &= ([g^{-1}] * [f]) * [g * e_{x_0}] \\ &= ([g^{-1}] * [f]) \otimes [g * e_{x_0}] \\ &= \begin{cases} g^{-1}(2s) \cdot g(2s) & \text{for } s \in [0, 1/2] \\ f(2s - 1) \cdot e_{x_0}(2s - 1) & \text{for } s \in [1/2, 1] \end{cases} \\ &= [e_{x_0} * f] \\ &= [f]. \end{aligned}$$

It follows that $\pi_1(X, x_0)$ is Abelian. ■

PROBLEM 11.2 ((A))

Prove Proposition F from the note on the Fundamental Group of the Circle.

Proof. Recall proposition F:

Proposition F. (i) W takes the class of the path $f_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ to n (and therefore W is onto).

(ii) W is one-to-one.

(iii) W is a homomorphism.

(i) Now, recall that $W: \pi_1(S^1, x_0) \rightarrow \mathbf{Z}$ defined by $W([f]) := w(f)$ where $w(f) = \tilde{f}(1)$ where $\tilde{f}: I \rightarrow \mathbf{R}$ is the lift of f , i.e. $p \circ \tilde{f} = f$. Now, let f_n be a path as above. Now, by Proposition C, since

$$f_n(s) = (\cos(2\pi ns), \sin(2\pi ns)) = (\cos(2\pi \tilde{f}_n(s)), \sin(2\pi \tilde{f}_n(s)))$$

and $\tilde{f}_n(0) = 0 = n \cdot 0$, by Proposition C, it follows that $\tilde{f}_n(s) = ns$. Thus, $\tilde{f}_n(1) = n$.

(ii) Let $f: I \rightarrow S^1$ be a loop at x_0 . It suffices to show that if $W([f]) = n$ then $f \simeq_p f_n$. Let $\tilde{f}: I \rightarrow \mathbf{R}$ be the lift of f and consider the path-homotopy $H: I \times I \rightarrow \mathbf{R}$ defined by $H(s, t) := (1-t)\tilde{f}(s) + t(ns)$. This map is continuous by Theorem 21.5 since it is multiplication from a topological space $I \times I$ into \mathbf{R} . Moreover, we have that

$$H(0, s) = \tilde{f}(s) \qquad H(1, s) = ns$$

and H fixes the endpoints, i.e.,

$$\begin{aligned} H(t, 0) &= (1-t) \cdot 0 + t \cdot 0 \\ &= 0 \end{aligned} \qquad \begin{aligned} H(t, 1) &= (1-t)n + tn \\ &= n. \end{aligned}$$

At last, define the map $K := p \circ H: I \times I \rightarrow S^1$. We claim that this map is a path-homotopy from the class $f \simeq_p f_n$. First, this map is continuous by Theorem 18.2(c) since it is a composition of continuous maps. Secondly, we have that

$$\begin{aligned} K(0, s) &= p(H(0, s)) \\ &= (\cos(2\pi \tilde{f}(s)), \sin(2\pi \tilde{f}(s))) \\ &= f(s) \end{aligned} \qquad \begin{aligned} K(1, s) &= p(H(1, s)) \\ &= (\cos(2\pi ns), \sin(2\pi ns)) \\ &= f_n(s). \end{aligned}$$

and K fixes the endpoints, i.e.,

$$\begin{aligned} K(t, 0) &= p(H(t, 0)) \\ &= (\cos 0, \sin 0) \\ &= x_0 \end{aligned} \qquad \begin{aligned} K(t, 1) &= p(H(t, 1)) \\ &= (\cos 2\pi n, \sin 2\pi n) \\ &= x_0. \end{aligned}$$

Thus, $f \simeq_p f_n$. It follows that if $W([f]) = W([g])$ then $[f] = [g]$, i.e., W is one-to-one.

(iii) Last but not least, we will show that W is in fact a homomorphism thus, proving that W is an isomorphism $\pi_1(X, x_0) \cong \mathbf{Z}$. Let f and g be loops in S^1 at x_0 and let \tilde{f} and \tilde{g} be their respective lifts to paths on \mathbf{R} . Let $n := W([f])$ and $m := W([g])$. Let $G: I \rightarrow \mathbf{R}$ be the path $G(s) := n + \tilde{g}(s)$. ■

PROBLEM 11.3 ((B))

Prove Lemma G from the note on the Fundamental Group of the Circle. (Hint: one way to do this is to use the fact, which you don't have to prove, that if \sim is the equivalence relation on $[a, a + 1]$ which identifies a and $a + 1$ then the restriction of p induces a homeomorphism $[a, a + 1]/\sim \rightarrow S^1$.)

Proof. Recall the statement of Lemma G:

Lemma G. *For each $a \in \mathbf{R}$, the map*

$$p_a : (a, a + 1) \longrightarrow S^1 - p(a)$$

given by $p_a(u) = p(u)$ is a homomorphism.

We shall proceed by the hint. ■

PROBLEM 11.4 ((C))

Show that for every point $x \in S^n$ the space $S^n - x$ is homeomorphic to \mathbf{R}^n . You may use the fact, shown in Step 1 of the proof of Theorem 59.3, that S^n with the *north pole* removed is homeomorphic to \mathbf{R}^n . (Hint: linear algebra.)

Proof.



PROBLEM 11.5 ((D))

Show that every loop in S^n which is not onto is path-homotopic to a constant path. (Hint: use Problem C).

Proof.



PROBLEM 11.6 ((E))

Let X be a topological space and let $A \subset X$ be a deformation retract. In the space X/A , the set A is a point (because it is an equivalence class). Show that this point is a deformation retract of X/A . (Hint: use p.289 # 9.)

Proof. Let $H: X \times I \rightarrow X$ be a deformation retraction from X to A , that is, $H(0, x) = \text{id}_X$ and $H(1, x) = r(x)$ where $r: X \rightarrow A$ is a retraction of X onto A and $\iota: A \hookrightarrow X$ is the inclusion of A into X . Let $p: X \rightarrow X/A$ be a quotient map. Now, we want to construct a deformation retraction $h: X/A \times I \rightarrow X/A$ from the quotient X/A to $*$, which we shall use to denote the image of A in X/A under p , and what better candidate than the map induced by $p \circ H: X \times I \rightarrow X/A$ on the quotient $X/A \times I$ into X/A . Consider the map $(p, \text{id}_I): X \times I \rightarrow X/A \times I$. This map is a quotient map by Problem 9.2 (Munkres §46, x. 9). Moreover, the map $p \circ H$ preserves the equivalence relation on $X/A \times I$ since for any two representatives (x_1, t) and (x_2, t) of $[(x, t)]$ in $X/A \times I$, we have $H(x_1, t) = H(x_2, t)$ if $x \in X - A$ and $H(x_1, t) = H_2(x_2, t)$ so $p(H(x_1, t)) = p(H(x_2, t))$ and if $x_1, x_2 \in A$ then $H(x_1, t), H(x_2, t) \in A$ so $p(H(x_1, t)) = p(H(x_2, t))$. Thus, by Theorem Q.3 the map $h: X/A \times I \rightarrow X/A$ induced by H , i.e., the map defined by $h(x, t) := [H(x, t)]$, is continuous and the diagram

$$\begin{array}{ccc} X \times I & \xrightarrow{H} & X \\ (p, \text{id}_I) \downarrow & & \downarrow p \\ X/A \times I & \xrightarrow{h} & X/A \end{array}$$

commutes. We claim that h is a deformation retraction from X/A to $*$. To that end, it suffices to show that $h(x, 0) = \text{id}_{X/A}$ and, using suggestive notation, $h(x, 1) = \bar{r}$ where $\bar{r}: X/A \rightarrow *$ is a retraction of X/A onto A and $\bar{\iota}: * \hookrightarrow X/A$ is the inclusion of $*$ into X/A . The first is easy to verify since $h(x, 0) = [H(x, 0)] = [x] = \text{id}_{X/A}$. Next, $h(x, 1) = [H(x, 1)] = [r(x)]$ and we claim that $\bar{r}(x) := [r(x)]$ is a retraction of X/A into $*$. The map \bar{r} is continuous since h is continuous (by Lemma 1 from Hw. #9 Munkres §18, Ex. 11) and $\bar{r}: X/A \rightarrow *$ since $r(x) \in A$ for every $x \in X$. It follows that $*$ is a deformation retract of X/A . ■