# $\operatorname{MA}$ 598 (Algebraic Geometry): Homework 1

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## **PROBLEM 1.1 (5-LEMMA)**

Given a commutative diagram with exact rows

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$\downarrow f \qquad \downarrow g \qquad \downarrow h$$

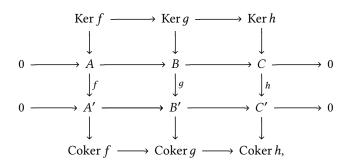
$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0,$$

suppose f and h are isomorphisms. Prove that g is an isomorphism.

Solution. ▶

#### PROBLEM 1.2 (SNAKE LEMMA)

Given the diagram with exact rows



show that the sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} h \longrightarrow 0$$

is exact.

Solution. ►

CARLOS SALINAS PROBLEM 1.3

#### PROBLEM 1.3

Given an exact sequence of modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

fix any  $M \in R$ -Mod.

(a) Show that

$$0 \longrightarrow \operatorname{Hom}(M,A) \longrightarrow \operatorname{Hom}(M,B) \longrightarrow \operatorname{Hom}(M,C)$$

is exact.

(b) Show that

$$0 \longrightarrow \operatorname{Hom}(C, M) \longrightarrow \operatorname{Hom}(B, M) \longrightarrow \operatorname{Hom}(A, M)$$

is exact.

Solution. ►

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#### PROBLEM 1.4

A short exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{p} 0$$

splits if theere is a homomorphism :  $C \to B$  called the *splitting* such that  $p \circ s = \mathrm{id}_C$ . In which case, we can put

$$0 \longrightarrow \operatorname{Hom}(C, M) \longrightarrow \operatorname{Hom}(B, M) \operatorname{Hom}(A, M) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Hom}(M,A) \longrightarrow \operatorname{Hom}(M,B)\operatorname{Hom}(M,C) \longrightarrow 0.$$

Solution. ►

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### PROBLEM 1.5

Find an example which shows that Hom(-, M) is *not* exact.

**Solution**.  $\blacktriangleright$  Consider the following example: let  $A=B=M=\mathbb{Z}$  and  $C=\mathbb{Z}/2\mathbb{Z}$  then the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f=2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact, however