MA 519: Homework 6

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Problem 6.1 (Handout 8, # 2)

Identify the parameters n and p for each of the following binomial distributions:

(a) # boys in a family with 5 children;
(b) # correct answers in a multiple choice test if each question has a 5 alternatives, there are 25 questions, and the student is making guesses at random.

Solution. For part (a), the distribution is binomial with k being the number of children in a given family and p the probability that a child is born, say, male. In this case, we can reasonably assume that $p = 0.5$. Thus, the binomial distribution is given by $\text{Binom}(5, 0.5)$.

For part (b), we use similar reasoning and we have $\text{Binom}(25, 0.2)$ where $k = 25$ is the number of questions and $p = 1/5 = 0.2$ the probability of guessing a question correctly. ∎

Problem 6.2 (Handout 8, # 10)

A newsboy purchases papers at 20 cents and sells them for 35 cents. He cannot return unsold papers. If the daily demand for papers is modeled as a Binom$(50, 0.5)$ random variable, what is the optimum number of papers the newsboy should purchase?

Solution. Let $X \sim \text{Binom}(50, 0.5)$ denote the daily demand for papers and n the number of copies bought by the newsboy. Then, we must find the ℓ such that the sum

$$\begin{aligned}
35\sum_{k=\ell}^{50}\binom{50}{k}0.5^{50} - 20\sum_{k=0}^{\ell-1}\binom{50}{k}0.5^{50} &= 35\left(1-\sum_{k=0}^{\ell-1}\binom{50}{k}0.5^{50}\right) - 20\sum_{k=0}^{\ell-1}\binom{50}{k}0.5^{50} \\
&= 35 - 55\sum_{k=0}^{\ell-1}\binom{50}{k}0.5^{50} \\
&= 35 - 55\cdot 0.5^{51-\ell}
\end{aligned}$$

equals 0. This can be computed experimetally ■

Problem 6.3 (Handout 8, # 12)

How many independent bridge dealings are required in order for the probability of a preassigned player having four aces at least once to be 1/2 or better? Solve again for some player instead of a given one.

Solution. ■

Problem 6.4 (Handout 8, # 13)

A book of 500 pages contains 500 misprints. Estimate the chances that a given page contains at least three misprints.

Solution. Let X be the number of misprints on the given page. The probability that a given misprint is on that page is 1/500. Now, $P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$. Also,

$$\begin{aligned} P(X = 0) &= \left(\frac{499}{500}\right)^{500} \\ P(X = 1) &= 500\left(\frac{1}{500}\right)\left(\frac{499}{500}\right)^{499} \\ P(X = 2) &= \frac{500 \cdot 499}{2}\left(\frac{1}{500}\right)^{2}\left(\frac{499}{500}\right)^{498} \end{aligned}$$

So that

$$\begin{aligned} P(X \geq 3) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ &= 1 - \left(\frac{499}{500}\right)^{500} - 500\left(\frac{1}{500}\right)\left(\frac{499}{500}\right)^{499} - \frac{500 \cdot 499}{2}\left(\frac{1}{500}\right)^{2}\left(\frac{499}{500}\right)^{498} \\ &\approx 0.08 \end{aligned}$$

that is, the probability that the given page has at least 3 misprints is about 8 percent. ■

Problem 6.5 (Handout 8, # 14)

Colorblindness appears in 1 per cent of the people in a certain population. How large must a random sample (with replacements) be if the probability of its containing a colorblind person is to be 0.95 or more?

Solution. Let n be the sample size. The probability of the sample containing no colorblind people is 0.99^n. Solving the equation $0.99^n = 0.05$, we see that (for the naturals) taking $n = 299$ is (minimally) sufficient for 0.99^n to be less than 0.05.

The probability that the sample has some colorblind person is equal to $1 - 0.99^n$. This is at least 95 percent if 0.99^n is less than 0.05. That is, having 299 people in the sample is (minimally) sufficient for there to be a 95 percent chance of having some colorblind person. ■

Problem 6.6 (Handout 8, # 15)

Two people toss a true coin n times each. Find the probability that they will score the same number of heads.

Solution. Let X denote the number of heads that, say, person 1 gets. Then

$$P(X = k) = \frac{\binom{n}{k}}{2^n}.$$

Then, assuming independnece, the probability that they score the same number of heads is given by the expression

$$\sum_{k=0}^{n} \left(\frac{\binom{n}{k}}{2^n} \cdot \frac{\binom{n}{k}}{2^n} \right) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{n}{k}^2,$$

which, by the binomial identity of the sum of squares of binomial coefficients, gives us

$$= \frac{1}{2^{2n}} \binom{2n}{n}.$$

∎

Problem 6.7 (Handout 8, # 16)

(*Binomial approximation to the hypergeometric distribution.*) A population of TV elements is divided into red and black elements in the proportion $p : q$ (where $p + q = 1$). A sample of size n is taken without replacement. The probability that it contains exactly k red elements is given by the hypergeometric distribution of II, 6. Show that as $n \to \infty$ this probability approaches Binom(n, p).

Solution. The hypergeometric distribution has the following PMF

$$\begin{aligned} P(X = k) = \binom{r}{k}\binom{n-r}{pn-k}\Big/\binom{n}{pn} &= \frac{r!(n-r)!(pn)!(qn)!}{k!(r-k)!(pn-k)!(qn-(r-k))!n!} \\ &= \binom{pn}{k}\left(\frac{r!(n-r)!(qn)!}{(r-k)!(qn-(r-k))!n!}\right) \\ &= \binom{pn}{k}\left(\frac{r\cdots(r-k+1)qn\cdots(qn-(r-k)+1)}{n\cdots(n-r+1)}\right) \\ &= \end{aligned}$$

Now, take the limit as $n \to \infty$ ■

Problem 6.8 (Handout 9, # 3)

Suppose X, Y, Z are mutually independent random variables, and $E(X) = 0$, $E(Y) = -1$, $E(Z) = 1$, $E(X^2) = 4$, $E(Y^2) = 3$, $E(Z^2) = 10$. Find the variance and the second moment of $2Z - Y/2 + eX$, where e is the number such that $\ln e = 1$.

Solution. Let $W = 2Z - Y/2 + eX$. Then

$$\begin{aligned}
E(W^2) &= E((2Z - Y/2 + eX)^2) \\
&= E(4Z^2 - 2ZY + 4eZX + Y^2/4 - eYX + e^2X^2) \\
&= 4E(Z^2) - 2E(Z)E(Y) + 4eE(Z)E(X) + E(Y^2)/4 - eE(Y)E(X) + e^2E(X^2) \\
&= 4 \cdot 10 + 2 + 3/4 + e^2 \cdot 4 \\
&= \frac{171}{4} + 4e^2 \\
&\approx 72.31
\end{aligned}$$

and

$$\begin{aligned}
Var(W) &= E(W^2) - E(W)^2 \\
&= \frac{171}{4} + 4e^2 - (2E(Z) + E(Y)/2 + eE(X))^2 \\
&= \frac{171}{4} + 4e^2 - (2 - 1/2)^2 \\
&= \frac{171}{4} + 4e^2 - (3/2)^2 \\
&= \frac{81}{2} + 4e^2 \\
&\approx 70.06
\end{aligned}$$

That is, the second moment is “about 72.31” and the variance is “about 70.06”

■

Problem 6.9 (Handout 9, # 14)

(*Variance of Product*). Suppose X, Y are independent random variables. Can it ever be true that $\operatorname{Var}(XY) = \operatorname{Var}(X)\operatorname{Var}(Y)$?
If it can, when?

Solution. Yes. In fact, if X and Y are constant random variables $\operatorname{Var} X = \operatorname{Var} Y = 0$ so clearly

$$\operatorname{Var}(XY) = \operatorname{Var} X \operatorname{Var} Y = 0.$$

The problem is finding sufficient conditions for this to be true.

Since X and Y are independent, we know that

$$E(XY) = E(X)E(Y)$$

and therefore

$$\begin{aligned}\operatorname{Var}(XY) &= E(X^2Y^2) - E(XY)^2\\ &= E(X^2)E(Y^2) - E(X)^2E(Y)^2\end{aligned}$$

and we want this to be equal to

$$\begin{aligned}\operatorname{Var} X \operatorname{Var} Y &= \big(E(X^2) - E(X)^2\big)\big(E(Y^2) - E(Y)^2\big)\\ &= E(X^2)E(Y^2) + E(X)^2E(Y)^2 - E(X)^2E(Y^2) - E(X^2)E(Y)^2\\ &= E(X^2)E(Y^2) - E(X)^2E(Y^2)\\ &\quad - \big(E(X)^2E(Y^2) - E(X)^2E(Y)^2 + E(X^2)E(Y^2) - E(X)^2E(Y^2)\big)\\ &= \operatorname{Var}(XY) - \operatorname{Var}(Y)E(X^2) - \operatorname{Var}(X)E(Y^2).\end{aligned}$$

This happens precisely when the quantity

$$\operatorname{Var}(Y)E(X^2) + \operatorname{Var}(X)E(Y^2) = 0.$$

Since $\operatorname{Var} X, \operatorname{Var} Y, E(X^2), E(Y^2) \geq 0$, this forces the equality

$$\operatorname{Var}(Y)E(X^2) = \operatorname{Var}(X)E(Y^2) = 0.$$

This happens when both $X = Y = 0$ or, more generally, if and only if X and Y are constant. ■