MA 572: Homework 2

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Problem 2.1 (Hatcher §2.1, Ex. 16)

(a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.

(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof. (a) Let $i\colon A \hookrightarrow X$ denote the inclusion map. Then, the map i can be extended to a chain map between chain complexes so, by proposition 2.9, induces a homomorphism $i_*\colon H_*(A) \to H_*(X)$ on homology. Similarly, the map $j_\#\colon C_*(X) \to C_*(X, A)$ induces a map $j_*\colon H_*(X) \to H_*(X, A)$ so, by theorem 2.16, we have a long exact sequence

$$\cdots \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0 \tag{1}$$

on homology. Thus, we see that $H_0(X, A) = 0$ if and only if i_* is injective which, by proposition 2.6, happens if and only if A meets each path-component of X.

(b) Let us extend to the left the long exact sequence of homology groups in (1) as follows

$$\cdots \xrightarrow{\partial_*} H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \xrightarrow{0} 0. \tag{2}$$

Hence, $H_1(X, A) = 0$ if and only if $j_* = 0$ and $\partial_* = 0$ if and only if i_* is surjective and i_* is injective on $H_0(A) \to H_0(X)$, i.e, each path-component of X contains at most one path-component of A. ■

Problem 2.2 (Hatcher §2.1, Ex. 18)

Show that for the subspace $\mathbf{Q} \subset \mathbf{R}$, the relative homology group $H_1(\mathbf{R}, \mathbf{Q})$ is free abelian and find a basis.

Proof. Consider the long exact sequence of homology groups

$$\cdots \xrightarrow{\partial_*} H_1(\mathbf{Q}) \xrightarrow{i_*} H_1(\mathbf{R}) \xrightarrow{j_*} H_1(\mathbf{R}, \mathbf{Q}) \xrightarrow{\partial_*} H_0(\mathbf{Q}) \xrightarrow{i_*} H_0(\mathbf{R}) \xrightarrow{j_*} H_0(\mathbf{R}, \mathbf{Q}) \xrightarrow{0} 0. \tag{3}$$

Since the space $\mathbf{R}$ is contractible, $H_*(\mathbf{R}) = 0$ which implies that the map $i_* = 0$ and $j_* = 0$ on $H_0(\mathbf{Q}) \to H_0(\mathbf{R})$ and $H_1(\mathbf{R}) \to H_1(\mathbf{R}, \mathbf{Q})$, respectively. Hence, $\partial_* \colon H_1(\mathbf{R}, \mathbf{Q}) \to H_0(\mathbf{Q})$ is surjective. Thus, $H_1(\mathbf{R}, \mathbf{Q}) \cong H_0(\mathbf{Q})$. Since, $\mathbf{Q}$ is totally disconnected, i.e., every connected component and hence, path-component of $\mathbf{Q}$ is a singleton set, we have $H_0(\mathbf{Q}) \cong \mathbf{Z}[\mathbf{Q}] \cong H_1(\mathbf{R}, \mathbf{Q})$. ■

Problem 2.3

Homotopy invariance of homology.

Proof. The proof of this follows immediately from corollary 2.10 for if $f\colon X \to Y$ and $g\colon Y \to X$ are maps with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$ then by corollary 2.10 we have $(g \circ f)_* = \mathrm{id}_{H_*(Y)}$ and $(f \circ g)_* = \mathrm{id}_{H_*(X)}$, but $(f \circ g)_* = f_* \circ g_*$ and $(g \circ f)_* = g_* \circ f_*$ so $g_* = f_*^{-1}$ and we see that $H_*(X) \cong H_*(Y)$. ■