MRC 2016: Character Varieties

Tropicalization of Character Varieties

Tropical Geometry Group

Snowbird, 2016

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Using results we got from `Mathematica` and `GFan`, we conjectured that, at least in the case of free groups, the
$\mathrm{Trop}(\mathfrak{X}(F_n, \mathrm{SL}_2\,\mathbb{C})) = \mathrm{Trop}(\mathfrak{X}(F_n, \mathrm{PSL}_2\,\mathbb{C}))$.
Additionally, Charlie Katerba

Generators

Let $F_3 = \langle A, B, C \rangle$. $\mathbb{C}[\mathfrak{X}(F_3, \mathrm{PSL}_2\,\mathbb{C})] = \mathbb{C}[\mathfrak{X}(F_3, \mathrm{SL}_2\,\mathbb{C})]^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ is generated by:

Type χ

$$\begin{aligned} \chi_A &:= (\operatorname{tr} A)^2 & \chi_B &:= (\operatorname{tr} B)^2 & \chi_C &:= (\operatorname{tr} C)^2 \\ \chi_{AB} &:= (\operatorname{tr} AB)^2 & \chi_{AC} &:= (\operatorname{tr} AC)^2 & & \\ \chi_{BC} &:= (\operatorname{tr} BC)^2 & \chi_{ABC} &:= (\operatorname{tr} ABC)^2 & & \end{aligned}$$

Generators cont.

Type τ

$$\tau_{AB} := \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AB \quad \tau_{AC} := \operatorname{tr} A \operatorname{tr} C \operatorname{tr} AC \quad \tau_{BC} := \operatorname{tr} B \operatorname{tr} C \operatorname{tr} BC$$

Generators cont.

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Type Λ

$$\Lambda_A := \operatorname{tr} B \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} AC \qquad \Lambda_B := \operatorname{tr} A \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} BC$$
$$\Lambda_C := \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AC \operatorname{tr} BC$$

Generators cont.

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Lonely Δ

$$\Delta := \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C \operatorname{tr} ABC$$

Generators cont.

Equally lonely Σ

$$\Sigma := \operatorname{tr} AB \operatorname{tr} AC \operatorname{tr} BC$$

Generators cont.

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Type Θ

$$\Theta_A := \operatorname{tr} A \operatorname{tr} BC \operatorname{tr} ABC \qquad \Theta_B := \operatorname{tr} B \operatorname{tr} AC \operatorname{tr} ABC$$
$$\Theta_C := \operatorname{tr} C \operatorname{tr} AB \operatorname{tr} ABC$$

Relations

Explicit example:

$$\Sigma^2 = (\operatorname{tr} AB \operatorname{tr} AC \operatorname{tr} BC)^2 = \chi_{AB}\chi_{AC}\chi_{BC}$$

(Binomial) Relations

$$\tau_{AB}^2 = \chi_A \chi_B \chi_{AB} \qquad \tau_{AC}^2 = \chi_A \chi_C \chi_{AC}$$
$$\tau_{BC}^2 = \chi_B \chi_C \chi_{BC}$$

$$\Lambda_A^2 = \chi_B \chi_C \chi_{AB} \chi_{AC} \qquad \Lambda_B^2 = \chi_A \chi_C \chi_{AB} \chi_{BC}$$
$$\Lambda_C^2 = \chi_A \chi_B \chi_{AC} \chi_{BC}$$

$$\Theta_A^2 = \chi_A \chi_{BC} \chi_{ABC} \qquad \Theta_B^2 = \chi_B \chi_{AC} \chi_{ABC}$$
$$\Theta_C^2 = \chi_C \chi_{AB} \chi_{ABC}$$

$$\Sigma^2 = \chi_{AB} \chi_{AC} \chi_{BC} \qquad \Delta^2 = \chi_A \chi_B \chi_C \chi_{ABC}.$$

. . . and finally the relation coming from $\mathfrak{X}(F_3, \mathrm{SL}_2\,\mathbb{C})$ can be written as

$$\begin{matrix} \chi_A + \chi_B + \chi_C + \chi_{AB} + \chi_{AC} \\ + \chi_{BC} + \chi_{ABC} + \Sigma + \Delta \end{matrix} = \begin{matrix} \tau_{AB} + \tau_{AC} + \tau_{BC} \\ + \Theta_A + \Theta_B + \Theta_C + 4. \end{matrix}$$

$S_{2,g}$ - a recursive approach

Definition

$S_{2,g}$ is the number of words over an alphabet of size g such that no letter appears (considering absolute multiplicities) more than twice.

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$$S_{2,g} = \sum_{d=0}^{g} \sum_{\ell=0}^{2g} \sum_{u=0}^{g} S_{2,g}^{d,\ell,u},$$

and

$$S_{2,0}^{0,0,0} = 1.$$

The recursion

$$
\begin{aligned}
S_{2,g}^{d,\ell,u} &= S_{2,g-1}^{d,\ell,u} \\
&+ \sum_{j=0}^{u-1} \binom{g-1}{j} j! 2^j (\ell - 2j) \left(S_{2,g-1-j}^{d,\ell-1-2j,u-j-1} + S_{2,g-1-j}^{d-1,\ell-1-2j,u-j-1} \right) 2 \\
&+ \sum_{j=0}^{u-1} \sum_{k=0}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \frac{(\ell - 2(j+k) - 1)(\ell - 2(j+k) - 2)}{2} S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&+ \sum_{j=0}^{0} \sum_{k=1}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \frac{(\ell - 2(j+k) - 1)2}{2} S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&+ \sum_{j=1}^{u-1} \sum_{k=0}^{0} \binom{g-1}{j+k} (j+k)! 2^{j+k} \frac{(\ell - 2(j+k) - 1)2}{2} S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&- \sum_{j=1}^{u-1} \sum_{k=1}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \frac{(\ell - 2(j+k) - 1)2}{2} S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&+ \sum_{j=1}^{u-1} \sum_{k=0}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \frac{(\ell - 2(j+k) - 1)2(j-1)}{2} S_{2,g-1-j-k}^{d,\ell-2(j+k)-2,u-1-j-k} 4 \\
&+ \sum_{j=1}^{u-1} \sum_{k=1}^{u-1-j} \binom{g-1}{j+k} (j+k)! 2^{j+k} \frac{(\ell - 2(j+k) - 1)2)}{2} S_{2,g-1-j-k}^{d,-2(j+k)-2,u-1-j-k} 4
\end{aligned}
$$

Some calculations

This has been implemented, with the following results

$$
\begin{aligned}
S_{2,0} &= 1 \\
S_{2,1} &= 5 \\
S_{2,2} &= 105 \\
S_{2,3} &= 6061 \\
S_{2,4} &= 668753 \\
&\vdots
\end{aligned}
$$

Conclusion: it is impractical to consider nave generators when examining representations F_n to $\mathrm{SL}_2\,\mathbb{C}$.

References I

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