

MA 572: Homework 5

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Problem 5.1 (Hatcher §2.2, Ex. 3)

Let $f: S^n \rightarrow S^n$ be a map of degree zero. Show that there exists points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in \mathbb{R}^n such that $F(x) \neq 0$ for all x , then there exists a point on ∂D where F points radially outward and another point on ∂D^n where F points radially inward.

Proof. Since $\deg f = 0 \neq (-1)^n = \deg a$, then $f \not\approx a$ and so must have a fixed point $x \in S^n$. Now, consider the map $g := a \circ f$. Since $\deg g = \deg a \circ f = (\deg a)(\deg f) = 0$, g must have a fixed point $y \in S^n$. Since $g(y) = a \circ f(y) = y$, then $f(y) = -y$.

Suppose F is a continuous nonzero vector field on S^n , i.e., a map $S^n \rightarrow \mathbb{R}^n$ which assigns to each point $x \in S^n$ a tangent vector $\mathbf{v}(x)$ at x . Then, the map $f: \partial D^n \rightarrow \mathbb{R}^n$ given by $f(\mathbf{v}(x)) = \mathbf{v}(x)/\|\mathbf{v}(x)\|$ is well defined and nowhere zero. ■

Problem 5.2 (Hatcher §2.2, Ex. 7)

For an invertible linear transformation $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ show that the induced map $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\}) \cong \tilde{H}_{n-1}(\mathbb{R}^n \setminus \{\mathbf{0}\}) \cong \mathbb{Z}$ is id or $-\text{id}$ according to whether the determinant of f is positive or negative. [Use Gaußian elimination to show that the matrix of f can be joined by a path of invertible matrices to a diagonal matrix with ± 1 's on the diagonal.]

Proof. We show that $O_n(\mathbb{R})$ is a deformation retraction of $GL_n(\mathbb{R})$ and prove the result there. This procedure is adapted from a hint in *Элементарная топология* by Виро, Нецвегаев и Харламов, стр. 338, номер 39.11. Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an invertible linear transformation. Let $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be the set of columns vectors of the matrix representation F of f . By Gram–Schmidt orthogonalization construct the vectors

$$\begin{aligned} \mathbf{e}_1 &:= \lambda_{11}\mathbf{f}_1 \\ \mathbf{e}_2 &:= \lambda_{21}\mathbf{f}_1 + \lambda_{22}\mathbf{f}_2 \\ &\vdots \\ \mathbf{e}_n &:= \lambda_{n1}\mathbf{f}_1 + \dots + \lambda_{nn}\mathbf{f}_n \end{aligned} \tag{5.1}$$

where the $\lambda_{kk} > 0$ for $k = 1, \dots, n$. Now set

$$\mathbf{g}_k(t) := t(\lambda_{n1}\mathbf{f}_1 + \lambda_{n2}\mathbf{f}_2 + \dots + \lambda_{kk-1}\mathbf{f}_{k-1}) + (t\lambda_{kk} + 1 - t)\mathbf{f}_k. \tag{5.2}$$

Let $g(t, A)$ be the matrix whose columns are the vectors $\mathbf{g}_k(t)$ and define a homotopy $f_t: I \times GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$ by mapping the pair $(t, A) \mapsto g(t, A)$. Continuity of H follows from the fact that H is multiplication in \mathbb{R}^n followed by a linear mapping. It's not hard to see that f_t stays in $GL_n(\mathbb{R})$ for all t and $f_1(A)$ is in $O_n(\mathbb{R})$.

Last but not least, we show that $O_n(\mathbb{R})$ consists of two connected components and that membership of f to one of these components is determined by $\det f$. First note that $\det(O_n(\mathbb{R})) = \{-1, 1\}$ which is disconnected in \mathbb{R} . Hence, $O_n(\mathbb{R})$ is disconnected. Now, if $f \in O_n(\mathbb{R})$, either $\det f = 1$ or $\det f = -1$. Without loss of generality, we may assume $\det f = 1$ since if r is a reflection.

Constructing the homotopy is hard. I can't think of a way of doing it and I don't have the time right now, so I'll skip this part. There are other ways to prove this indirectly, but I'm afraid I'm not familiar with Lie groups and I am not willing to state a bunch of results from that subject.

Now that we have established that either $f \simeq \text{id}$ or $-\text{id}$, the map f on \mathbb{R}^n induces a map $f_* = \pm \text{id}_*$ on the homology groups $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{\mathbf{0}\})$. ■

Problem 5.3 (Hatcher §2.2, Ex. 13)

Let X be the 2-complex obtained from S^1 with its usual cell structure by attaching two 2-cells by maps of degrees 2 and 3, respectively.

- Compute the homology groups of all the subcomplexes $A \subset X$ and the corresponding quotient complexes X/A .
- Show that $X \simeq S^2$ and that the only subcomplex $A \subset X$ for which the quotient map $X \rightarrow X/A$ is a homotopy equivalence is the trivial subcomplex, the 0-cell.

Proof. (a) Write X as the union $e^0 \cup e^1$ of a 0-cell and a 1-cell. Let e_1^2, e_2^2 be 2-cells attached to X via maps $f_1, f_2: S^2 \rightarrow X$ of degrees 2 and 3, respectively. We use Lemma 2.34 to compute the cellular homology of the new CW complex X' , it then follows from Theorem 2.35 that the cellular homology is isomorphic to the singular homology of X' . First, we write down the cellular chain complex for $X' = X^2$

$$\cdots \longrightarrow H_3^{\text{CW}}(X^3) \longrightarrow H_2^{\text{CW}}(X^2) \longrightarrow H_1^{\text{CW}}(X^1) \longrightarrow H_0^{\text{CW}}(X^0) \longrightarrow 0. \quad (5.3)$$

Filling in some of the values for $H_n^{\text{CW}}(X^n)$ we have the chain

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0. \quad (5.4)$$

Now recall that by definition a subcomplex of X' , A , is a closed subspace that is the union of cells in X . Since we have the following inclusion $e^0 \subset e^1 \subset e_1^2, e_2^2$ this makes for the following candidates $A_0 := e^0$, $A_1 := e^0 \cup e^1$, $A_{12} := e^0 \cup e^1 \cup e_1^2$, $A_{22} := e^0 \cup e^1 \cup e_2^2$, X' . Let's compute the homology of these spaces.

- Case A_0 : The cellular homology of A_0 is easy enough since it is a 0-cell. It's homology will be that of a point $H_n^{\text{CW}}(A_0) = \mathbb{Z}$ for $n = 0$ and $H_n^{\text{CW}}(A_0) = 0$ otherwise.
- Case A_1 : The subcomplex A_1 is homeomorphic to a circle S^1 so its cellular homology is isomorphic to that of a circle, i.e., $H_n^{\text{CW}}(A_1) = \mathbb{Z}$ if $n = 0, 1$ and $H_n^{\text{CW}}(A_1) = 0$ otherwise.
- Case A_{21} : The cellular homology of A_{21} is more interesting since we have the attaching map of degree 2. This map f_1 induces a map on homology $f_{1*} = 2$ giving us the chain complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0 \quad (5.5)$$

- Case A_{22} :
- Case X :

That concludes this part of the problem. ■

(b)

Problem 5.4

Proof.

