MA 572: Homework 4

Carlos Salinas

February 13, 2016

Problem 4.1 (Hatcher §2.1, Ex. 20)

Show that $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Proof. Write SX as the disjoint union of two cones $CX \sqcup CX$ mod the relation $x \sim y$ if x, y in the base of CX. By excision (2.20), we have ■

Problem 4.2 (Hatcher §2.1, Ex. 22)

Prove by induction on the dimension the following facts about the homology of a finite dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:

(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.

(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n - 1$ or $n + 1$.

(c) If X is has k n-cells, then $H_n(X)$ is generated by at most k elements.

Proof. ■

Problem 4.3 (Hatcher §2.2, Ex. 2)

Given a map $f\colon S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ has a fixed point. Construct maps $\mathbb{R}\mathrm{P}^{2n-1} \to \mathbb{R}\mathrm{P}^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Proof. ∎