MA52300 FALL 2016

Homework Assignment 8 – *Solutions*

1. Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t), \quad \text{where} \quad \Phi(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

is positive for $|x| < 1$, $t > 0$.

Hint: Show that u satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{aligned} u = 0 \quad &\text{on} \quad \{|x| = 1\} \times \{t \geq 0\} \\ u = \delta_0 \quad &\text{on} \quad \{|x| \leq 1\} \times \{t = 0\} \end{aligned}$$

Then, carefully apply maximum (minimum) principle in a domain $\{|x| \leq 1\} \times \{\epsilon \leq t \leq T\}$ for small $\epsilon > 0$ and large $T > 0$ and pass to the limit as $\epsilon \to 0+$ and $T \to \infty$.

Solution. 1) *Convergence*. Fix small $\epsilon > 0$ and large $T > 0$. We claim that the series is uniformly absolutely convergent for $t \in [\epsilon, T]$. Indeed, without loss of generality we may assume that $|x| \leq 2$, since the series is 4-periodic in x. Then

$$\sum_{k=-\infty}^{\infty} |\Phi(x-2k,t)| \leq \frac{1}{\sqrt{4\pi\epsilon}} \left(3 + 2\sum_{k=2}^{\infty} e^{-(k-2)^2/4T}\right) < \infty.$$

Moreover, arguing in a similar fashion, we can show that the series consisting of partial derivatives will also be convergent in $t > 0$. Thus, $u(x,t)$ is a solution of the wave equation $u_t - u_{xx} = 0$ in $t > 0$.

2) *Symmetry*. From the construction of u, it is immediate to verify that

$$u(2-x,t) = -u(x,t) \quad \text{and} \quad u(-2-x,t) = -u(x,t)$$

for all $x \in \mathbb{R}$ and $t > 0$. In particular, this implies that

$$u(1,t) = 0 \quad \text{and} \quad u(-1,t) = 0$$

for all $t > 0$.

3) *Initial condition.* Heuristically, we can argue as follows: since $\Phi(x,t)$ satisfies $\Phi(x,0) = \delta_0(x)$ in a generalized sense, we must also have

$$u(x,0) = \sum_{k=-\infty}^{\infty} (-1)^k \delta_0(x-2k).$$

Thus, restricted to $|x| \le 1$, this gives $u(x,0) = \delta_0(x)$.

A more rigorous argument is as follows. Let $|x| \le 1$ and $t = \epsilon$. Then

$$\begin{aligned} u(x,\epsilon) &= \Phi(x,\epsilon) + \sum_{|k|\ge 1} (-1)^k \Phi(x-2k,\epsilon) \\ &\ge -\frac{2}{\sqrt{4\pi\epsilon}} \sum_{k=1}^{\infty} e^{-(2k-1)^2/4\epsilon} \ge -\frac{1}{\sqrt{4\pi\epsilon}} \sum_{k=1}^{\infty} \frac{4\epsilon}{(2k-1)^2} \ge -C\sqrt{\epsilon}, \end{aligned}$$

where in the last inequality we have used that $e^{-1/s} \le s$ for $s > 0$.

4) *Minimum principle.* Now consider u in the parabolic cylinder

$$U_{\epsilon,T} := \{|x| < 1\} \times \{\epsilon < t \le T\}.$$

Then we have already established that

$$u \ge -C\sqrt{\epsilon} \quad \text{on } \Gamma_{\epsilon,T} := \overline{U}_{\epsilon,T} \setminus U_{\epsilon,T}.$$

Thus, by the minimum principle for solutions of the heat equation

$$u \ge -C\sqrt{\epsilon} \quad \text{in } U_{\epsilon,T}.$$

Letting $\epsilon \to 0+$ and $T \to \infty$, we obtain that

$$u(x,t) \ge 0 \quad \text{for } |x| < 1 \text{ and } t > 0.$$

Finally, if $u = 0$ at some point (x_0, t_0) in that region, by the strict minimum principle we would have $u(x,t) = 0$ for all $|x| \le 1$ and $0 < t \le t_0$. Heuristically, this is not possible, since $u(\cdot,t) \to \delta_0$ as $t \to 0+$. It can be rigorously justified by arguing as in 3) above. $\square$

2 (Tikhonov's example)**.** Let

$$g(t) := \begin{cases} \exp\left(-t^{-2}\right), & t > 0 \\ 0, & t \le 0 \end{cases}.$$

Then $g \in C^\infty(\mathbb{R})$ and we define

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that $u(x,t)$ solves the heat equation in $\mathbb{R} \times (0,\infty)$ with the initial condition $u(x,0) = 0$, $x \in \mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

Solution. 1) Assuming that we can differentiate the series term-wise, we obtain

$$u_t(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k}$$

$$u_{xx}(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2k(2k-1)x^{2k-2} = \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k}.$$

Hence, u is a solution of the heat equation. To verify the initial condition, observe that $g(t) = o(t^k)$ for any integer $k \geq 0$, which implies that $g^{(k)}(0) = 0$ for all k. Hence

$$u(x,0) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{(2k)!} x^{2k} = 0.$$

2) The uniqueness theorem for the solutions of the initial value problem for the heat equation says that if $u \in C^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ satisfies $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,T)$ and $u(x,0) = 0$ for all $x \in \mathbb{R}^n$, then $u = 0$ in $\mathbb{R}^n \times (0,T)$, provided u satisfies a *growth condition* $|u(x,t)| \leq Ce^{a|x|^2}$. Tikhonov's example highlights the necessity of such condition.

Remark. The rigorous proof of convergence can be found in [John, Partial Differential Equations, 4th ed., pp. 212–213]. □

3. Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax)\, e^{-x^2} dx \qquad (a > 0).$$

Hint: Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.

Solution. Consider the initial-value problem

$$(*) \qquad u_t - u_{xx} = 0 \quad \text{for } t > 0; \qquad u(x,0) = \cos ax.$$

The bounded solution of $(*)$ is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \cos(ay) e^{-(x-y)^2/4t}.$$

Thus,

$$\int_{-\infty}^{\infty} \cos(ax)\, e^{-x^2} dx = \sqrt{\pi} u\left(0, \frac{1}{4}\right).$$

To find the solution of $(*)$, we use the separation of variables. Let $u(x,t) = X(x)T(t)$. Then we must have

$$X''(x) + \lambda X(x) = 0, \quad T'(t) + \lambda T(t) = 0, \quad X(x)T(0) = \cos(ax).$$

Normalize by setting $T(0) = 1$. Then $X(x) = \cos(ax)$, $\lambda = a^2$, $T(t) = T(0)e^{-a^2 t} = e^{-a^2 t}$. Hence, $u(x,t) = \cos(ax)e^{-a^2 t}$ and

$$\int_{-\infty}^{\infty} \cos(ax)\, e^{-x^2} dx = \sqrt{\pi} u\left(0, \frac{1}{4}\right) = \sqrt{\pi} e^{-a^2/4}. \qquad \square$$