MA 523: Homework 4

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Problem 4.1 (Legendre transform)

Let $u(x_1, x_2)$ be a solution of the quasilinear equation

$$a^{11}(Du)u_{x_1x_1} + 2a^{12}(Du)u_{x_1x_2} + a^{22}(Du)u_{x_2x_2} = 0$$

in some region of $\mathbb{R}^2$, where we can invert the relations

$$p^1 = u_{x_1}(x_1, x_2), \quad p^2 = u_{x_2}(x_1, x_2)$$

to solve for

$$x^1 = x^1(p_1, p_2), \quad x^2 = x^2(p_1, p_2).$$

Define then

$$v(p) := \mathbf{x}(p) \cdot p - u\big(\mathbf{x}(p)\big),$$

where $\mathbf{x} = (x^1, x^2)$, $p = (p_1, p_2)$. Show that v satisfies the *linear* equation

$$a^{22}(p)v_{p_1p_2} - 2a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_1p_2} = 0.$$

(*Hint:* See [Evans, 4.4.3b], prove the identities (29)).

Solution. Assuming the regularity on v prescribed above, we compute $v_{p_1p_1}$, $v_{p_1p_2}$ and $v_{p_2p_2}$.

First, we compute $v_{p_1p_2}$ since in the case of $v_{p_1p_1}$ and $v_{p_2p_2}$, there is some symmetry we can exploit. Taking the first partial with respect to p^1, we have

$$\begin{aligned} v_{p_1} &= \tfrac{\partial}{\partial p_1}\big(x^1(p)p^1 + x^2(p)p^2 - u\big(\mathbf{x}(p)\big)\big) \\ &= x^1(p) + x^1_{p_1}(p)p^1 + x^2_{p_1}(p)p^2 - u_{x_1}\big(\mathbf{x}(p)\big)x^1_{p_1}(p) - u_{x_2}\big(\mathbf{x}(p)\big)x^2_{p_1}(p) \\ &= x^1 + x^1_{p_1}p^1 + x^2_{p_1}p^2 - p^1x^1_{p_1} - p^2x^2_{p_1} \\ &= x^1, \end{aligned} \tag{4.1}$$

since $u_{x_1} = p^1$ and $u_{x_2} = p^2$.

Similarly, for v_{p_2}, we have

$$\begin{aligned} v_{p_2} &= \tfrac{\partial}{\partial p_2}\big(x^1(p)p^1 + x^2(p)p^2 - u\big(\mathbf{x}(p)\big)\big) \\ &= x^1_{p_2}(p)x^1(p) + x^2(p) + x^2_{p_2}(p)p^2 - u_{x_1}\big(\mathbf{x}(p)\big)x^1_{p_2}(p) - u_{x_2}\big(\mathbf{x}(p)\big)x^2_{p_2}(p) \\ &= x^1_{p_2}x^1 + x^2 + x^2_{p_2}p^2 - p^1x^1_{p_2} - p^2x^2_{p_2} \\ &= x^2. \end{aligned} \tag{4.2}$$

Now, taking the partial of (4.1) with respect to p_1 and then p_2, we have

$$v_{p_1p_1} = x^1_{p_1} = x^1_{u_{x_1}}, \qquad v_{p_1p_2} = x^1_{p_2} = x^1_{u_{x_2}},$$

and similarly for (4.2),

$$v_{p_1p_2} = x^2_{p_1} = x^2_{u_{x_1}}, \qquad v_{p_2p_2} = x^2_{p_2} = x^2_{u_{x_2}}.$$

By the inverse function theorem, we have

$$\begin{aligned}\begin{bmatrix} v_{p_1p_1} & v_{p_1p_2} \\ v_{p_1p_2} & v_{p_2p_2} \end{bmatrix} &= \begin{bmatrix} x^1_{u_{x_1}} & x^1_{u_{x_2}} \\ x^2_{u_{x_1}} & x^2_{u_{x_2}} \end{bmatrix} \\ &= \begin{bmatrix} u_{x_1x_1} & u_{x_1x_2} \\ u_{x_1x_2} & u_{x_2x_2} \end{bmatrix}^{-1} \\ &= \frac{1}{J}\begin{bmatrix} u_{x_2x_2} & -u_{x_1x_2} \\ -u_{x_1x_2} & u_{x_1x_1} \end{bmatrix}.\end{aligned}$$

Hence,

$$\begin{cases} u_{x_1x_1} = Jv_{p_2p_2} \\ u_{x_1x_2} = -Jv_{p_1p_2} \\ u_{x_2x_2} = Jv_{p_1p_1}, \end{cases} \tag{4.3}$$

which verifies Equation (29) from [E, 4.4.3b]. Substituting (4.3) into the original equation,

$$\begin{aligned} 0 &= Ja^{11}(p)v_{p_2p_2} - Ja^{12}(p)v_{p_1p_2} + Ja^{22}(p)v_{p_1p_1} \\ &= a^{22}(p)v_{p_1p_1} - a^{12}(p)v_{p_1p_2} + a^{11}(p)v_{p_2p_2}, \end{aligned}$$

as was to be shown. ∎

PROBLEM 4.2

Find the solution $u(x,t)$ of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0$$

in the quadrant $x > 0, t > 0$ for which

$$\begin{cases} u(x,0) = f(x), & u_t(x,0) = g(x), \quad \text{for } x > 0, \\ u_t(0,t) = \alpha u_x(0,t), & \qquad\qquad\qquad \text{for } t > 0, \end{cases}$$

where $\alpha \neq -1$ is a given constant. Show that generally no solution exists when $\alpha = -1$. (*Hint:* Use a representation $u(x,t) = F(x-t) + G(x+t)$ for the solution.)

SOLUTION. Suppose $u(x,t) = F(x-t) + G(x+t)$ is a classical solution to the one-dimensional wave equation. From Evans, we know d'Alembert's formula, i.e.,

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds,$$

is a solution to the PDE with the prescribed initial conditions.

Now, using the initial conditions prescribed above, particularly, the restriction on the derivative of u at $x = 0$, we obtain the following relations on G and F,

$$\begin{aligned} u(x,0) &= F(x) + G(x) \\ &= f(x), \\ u_t(x,0) &= -F'(x) + G'(x) \\ &= g(x), \\ u_t(0,t) &= \alpha u_x(0,t) \\ -F'(-t) + G'(t) &= \alpha\big(F'(-t) + G'(t)\big), \end{aligned}$$

which, more concisely, reads

$$\begin{cases} F(x) = f(x) - G(x) \\ F'(x) = G'(x) - g(x) \\ F'(-t) = -\left(\frac{\alpha-1}{\alpha+1}\right)G'(t). \end{cases} \tag{4.4}$$

Suppose $\alpha = -1$.

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Problem 4.3

(a) Let u be a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $0 < x < \pi$, $t > 0$ such that $u(0,t) = u(\pi,t) = 0$. Show that the *energy*

$$E(t) = \frac{1}{2}\int_0^\pi \left(u_t^2 + c^2 u_x^2\right) dx, \quad t > 0$$

is independent of t; i.e., $\frac{d}{dt}E = 0$ for $t > 0$. Assume that u is C^2 up to the boundary.

(b) Express the energy E of the Fourier series solution

$$u(x,t) = \sum_{n=1}^{\infty}\big(a_n \cos(nct) + b_n \sin(nct)\big)\sin(nx)$$

in terms of coefficients a_n, b_n.

Solution. For part (a), suppose that u is, as above, a solution to the wave equation which is C^2 up to the boundary. We show that its energy is independent of t, i.e., that $\frac{d}{dt}E = 0$. Assuming the energy is bounded, the dominated convergence theorem allows us to permute the order of integration and differentiation like so

$$\begin{aligned}\tfrac{d}{dt}E(t) &= \frac{d}{dt}\left(\frac{1}{2}\int_0^\pi \left(u_t^2 + c^2 u_x^2\right) dx\right)\\ &= \frac{1}{2}\int_0^\pi \tfrac{\partial}{\partial t}\left(u_t^2 + c^2 u_x^2\right) dx\\ &= \frac{1}{2}\int_0^\pi 2u_t u_{tt} + 2c^2 u_x u_{xt}\, dx\end{aligned}$$

which, after using the relation $u_{tt} = c^2 u_{xx}$, becomes

$$\begin{aligned}&= c^2\int_0^\pi u_t u_{xx} + u_x u_{xt}\, dx\\ &= c^2\int_0^\pi \tfrac{\partial}{\partial x}(u_x u_t)\, dx\\ &= c^2\big(u_x(\pi,t)u_t(\pi,t) - u_x(0,t)u_t(0,t)\big)\\ &= 0\end{aligned}$$

since the boundary conditions, i.e., $u = 0$, implies $u_x = u_t = 0$ at the boundary.

For part (b), suppose u is a Fourier series solution to the wave equation, i.e.,

$$u(x,t) = \sum_{n=1}^{\infty}\big(a_n \cos(nct) + b_n \sin(nct)\big)\sin(nx).$$

First we compute u_t and u_x. They are

$$\begin{aligned} u_t(x,t) &= \tfrac{\partial}{\partial t} u(x,t) \\ &= \sum_{n=1}^{\infty} cn\big(b_n \cos(nct) - a_n \sin(nct)\big)\sin(nx) \end{aligned}$$

and

$$\begin{aligned} u_x(x,t) &= \tfrac{\partial}{\partial x} u(x,t) \\ &= \sum_{n=1}^{\infty} n\big(a_n \cos(nct) + b_n \sin(nct)\big)\cos(nx). \end{aligned}$$

Thus,

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^{\pi} \Bigg[& \Bigg(\sum_{n=1}^{\infty} cn\big(b_n \cos(nct) - a_n \sin(nct)\big)\sin(nx)\Bigg)^2 \\ & + c^2\Bigg(\sum_{n=1}^{\infty} n\big(a_n \cos(nct) + b_n \sin(nct)\big)\cos(nx)\Bigg)^2\Bigg] \end{aligned}$$

which, after expanding and using the fact that $\cos(nct)$, $\sin(nct)$, $\cos(nx)$, and $\sin(nx)$ are orthogonal, becomes

$$\begin{aligned} &= \frac{1}{2}\int_0^{\pi} \Bigg[\sum_{n,m=1}^{\infty} c^2 nm\big(b_n b_m \cos(nct)\cos(mct) + a_n a_m \sin(nct)\sin(mct) \\ &\qquad - a_n b_m \cos(mct)\sin(nct) - a_m b_n \cos(nct)\sin(mct)\big)\sin(nx)\sin(mx) \\ &\qquad - c^2 \sum_{n,m=1}^{\infty} n^2\big(a_n a_m \cos(nct)\cos(mct) + b_n b_m \sin(nct)\sin(mct) \\ &\qquad + a_n b_m \cos(nct)\sin(mct) + a_m b_n \cos(mct)\sin(nct)\big)\cos(nx)\cos(mx)\Bigg] \\ &= \frac{1}{2}\int_0^{\pi} \sum_{n=1}^{\infty} \Big(cn^2\big(b_n^2\cos^2(nct) + a_n^2 \sin^2(nct)\big)\sin^2(nx) - cn^2\big(a_n^2\cos^2(nct) + b_n^2\sin^2(nct)\big)\cos^2(nx)\Big) \end{aligned}$$

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