1.3: 12, 14, 16, 18, 19, 26, 28, 30
1.4: 3, 8, 10, 12, 22, 32

Consider the following matrices: $A=\begin{bmatrix}1&2&3\\2&1&4\end{bmatrix}$, $B=\begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix}$, $C=\begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix}$, $D=\begin{bmatrix}3&-2\\2&5\end{bmatrix}$, $E=\begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix}$, and $F=\begin{bmatrix}-1&2\\0&4\\3&5\end{bmatrix}$.

1.3.12 | If possible, compute the following:
(a) $DA+B$ (b) EC (c) CE (d) $EB+F$ (e) $FC+D$.

(a) $DA+B = \begin{bmatrix}3&-2\\2&5\end{bmatrix}\begin{bmatrix}1&2&3\\2&1&4\end{bmatrix} + \begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix} = \begin{bmatrix}-1&4&5\\12&9&26\end{bmatrix} + \begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix}$

which you cannot add due to differing matrix sizes.

(b) $EC = \begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix}\begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix} = \begin{bmatrix}0&-1&1\\12&5&17\\21&0&22\end{bmatrix}$

(c) $CE = \begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix}\begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix} = \begin{bmatrix}15&-7&14\\23&-5&29\\13&-1&17\end{bmatrix}$

(d) $EB+F = \begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix}\begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix} + \begin{bmatrix}-1&2\\0&4\\3&5\end{bmatrix} = \begin{bmatrix}9&6\\14&9\\10&4\end{bmatrix} + \begin{bmatrix}-1&2\\0&4\\3&5\end{bmatrix} = \begin{bmatrix}8&8\\14&13\\13&9\end{bmatrix}$

(e) $FC+D$ cannot multiply FC as F is a 3×2 matrix and C is a 3×3 matrix.

1.3.14 If possible, compute the following:

(a) $A(BD)$ (b) $(AB)D$ (c) $A(C+E)$ (d) $AC+AE$

(e) $(2AB)^T$ and $2(AB)^T$ (f) $A(C-3E)$.

(a) $A(BD) = A\left(\begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix}\begin{bmatrix}3&-2\\2&5\end{bmatrix}\right) = A\begin{bmatrix}3&-2\\8&1\\13&4\end{bmatrix} = \begin{bmatrix}1&2&3\\2&1&4\end{bmatrix}\begin{bmatrix}3&-2\\8&1\\13&4\end{bmatrix} = \begin{bmatrix}58&12\\66&13\end{bmatrix}$

(b) $(AB)D = \left(\begin{bmatrix}1&2&3\\2&1&4\end{bmatrix}\begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix}\right)D = \begin{bmatrix}14&8\\16&9\end{bmatrix}\begin{bmatrix}3&-2\\2&5\end{bmatrix} = \begin{bmatrix}58&10\\66&13\end{bmatrix}$

(c) $A(C+E) = A\left(\begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix} + \begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix}\right) = \begin{bmatrix}1&2&3\\2&1&4\end{bmatrix}\begin{bmatrix}5&-5&8\\4&2&9\\5&3&4\end{bmatrix} = \begin{bmatrix}28&8&38\\34&4&41\end{bmatrix}$

(d) $AC+AE = \begin{bmatrix}1&2&3\\2&1&4\end{bmatrix}\begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix} + \begin{bmatrix}1&2&3\\2&1&4\end{bmatrix}\begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix} = \begin{bmatrix}17&4&22\\18&3&23\end{bmatrix} + \begin{bmatrix}11&4&16\\16&1&18\end{bmatrix} = \begin{bmatrix}28&8&38\\34&4&41\end{bmatrix}$

(e) $(2AB)^T = \left(2\begin{bmatrix}1&2&3\\2&1&4\end{bmatrix}\begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix}\right)^T = \left(2\begin{bmatrix}14&8\\16&9\end{bmatrix}\right)^T = \begin{bmatrix}28&16\\32&18\end{bmatrix}^T = \begin{bmatrix}28&32\\16&18\end{bmatrix}$

$2(AB)^T = 2\begin{bmatrix}14&8\\16&9\end{bmatrix}^T = 2\begin{bmatrix}14&16\\8&9\end{bmatrix} = \begin{bmatrix}28&32\\16&18\end{bmatrix}$

(f) $A(C-3E) = A\left(\begin{bmatrix}3&-1&3\\4&1&5\\2&1&3\end{bmatrix} - 3\begin{bmatrix}2&-4&5\\0&1&4\\3&2&1\end{bmatrix}\right) = \begin{bmatrix}1&2&3\\2&1&4\end{bmatrix}\begin{bmatrix}-3&11&-12\\4&-2&-7\\-7&-5&0\end{bmatrix} = \begin{bmatrix}-16&-8&-26\\-30&0&-31\end{bmatrix}$

1.3.16 Let $A = [1\ 2\ -3]$, $B = [-1\ 4\ 2]$, and $C = [-3\ 0\ 1]$. If possible, compute the following:

(a) AB^T (b) CA^T (c) $(BA^T)C$ (d) A^TB (e) CC^T (f) C^TC (g) B^TCAA^T

(a) $AB^T = [1\ 2\ -3]\begin{bmatrix}-1\\4\\2\end{bmatrix} = 1$ (b) $CA^T = [-3\ 0\ 1]\begin{bmatrix}1\\2\\-3\end{bmatrix} = -6$

(c) $(BA^T)C = \left([-1\ 4\ 2]\begin{bmatrix}1\\2\\-3\end{bmatrix}\right)C = 1C = [-3\ 0\ 1]$

(d) $A^TB = \begin{bmatrix}1\\2\\-3\end{bmatrix}[-1\ 4\ 2] = \begin{bmatrix}-1&4&2\\-2&8&4\\3&-12&-6\end{bmatrix}$ (e) $CC^T = [-3\ 0\ 1]\begin{bmatrix}-3\\0\\1\end{bmatrix} = 10$

(f) $C^TC = \begin{bmatrix}-3\\0\\1\end{bmatrix}[-3\ 0\ 1] = \begin{bmatrix}9&0&-3\\0&0&0\\-3&0&1\end{bmatrix}$

(g) $B^TCAA^T = B^TC[1\ 2\ -3]\begin{bmatrix}1\\2\\-3\end{bmatrix} = B^TC14 = 14\begin{bmatrix}-1\\4\\2\end{bmatrix}[-3\ 0\ 1] = 14\begin{bmatrix}3&0&-1\\-12&0&4\\-6&0&2\end{bmatrix} = \begin{bmatrix}42&0&-14\\-168&0&56\\-84&0&28\end{bmatrix}$

1.3.18 If $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, compute DI_2 and $I_2 D$.

$$DI_2 = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \text{ and } I_2 D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}.$$

1.3.19 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$. Show $AB \neq BA$.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 7 \\ 0 & 5 \end{bmatrix},\quad BA = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 9 & 2 \end{bmatrix} \text{ so } AB \neq BA.$$

1.3.26 (a) Find a value of r so that $AB^T = 0$, where $A = [r \ 1 \ -2]$ and $B = [1 \ 3 \ -1]$.
(b) Give an alternative way to write this product.

(a) $AB^T = [r \ 1 \ -2]\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = r + 3 + 2 = r + 5 = 0$ So $r = -5$.

(b) This is the dot product $A \cdot B = AB^T$.

1.3.28 (a) Let A be an $m \times n$ matrix with a row consisting entirely of zeroes. Show that if B is an $n \times p$ matrix, then AB has a row of zeroes.
(b) Let A be an $m \times n$ matrix with a column consisting entirely of zeroes and let B be $p \times m$. Show that BA has a column of zeroes.

(a) Let row i of A be zero so $\text{row}_i(A) = 0$. Then for any column j of B the element c_{ij} of AB is $(\text{row}_i(A))^T \cdot \text{Col}_j(B) = 0^T \cdot \text{Col}_j(B) = 0$ showing that row i of AB is also zero.

(b) Let column j of A be zero so $\text{Col}_j(A) = 0$. Then for any row i of B the element c_{ij} of BA is $(\text{row}_i(B))^T \cdot \text{Col}_j(A) = (\text{row}_i(B))^T \cdot 0 = 0$ showing that column j of BA is also zero.

1.3.30 Consider the following linear system:

$$\begin{aligned} 2x_1 + 3x_2 - 3x_3 + x_4 + x_5 &= 7 \\ 3x_1 \qquad + 2x_3 \qquad\quad + 3x_5 &= -2 \\ 2x_1 + 3x_2 \qquad - 4x_4 \qquad &= 3 \\ x_3 + x_4 + x_5 &= 5 \end{aligned}$$

(a) Find the coefficient matrix. $A = \begin{bmatrix} 2 & 3 & -3 & 1 & 1 \\ 3 & 0 & 2 & 0 & 3 \\ 2 & 3 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

(b) Write the linear system in matrix form. $\begin{bmatrix} 2 & 3 & -3 & 1 & 1 \\ 3 & 0 & 2 & 0 & 3 \\ 2 & 3 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 5 \end{bmatrix}$

(c) Find the augmented matrix. $\left[\begin{array}{ccccc|c} 2 & 3 & -3 & 1 & 1 & 7 \\ 3 & 0 & 2 & 0 & 3 & -2 \\ 2 & 3 & 0 & -4 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 5 \end{array}\right]$.

1.4.3 Verify Theorem 1.2(a) for the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix}.$$

We need to verify $A(BC) = (AB)C$ (which is true since the proof is provided).

$$A(BC) = A\left(\begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 10 & 1 \\ -4 & 11 \end{bmatrix} = \begin{bmatrix} -2 & 34 \\ 24 & -9 \end{bmatrix}$$

$$(AB)C = \left(\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix}\right)C = \begin{bmatrix} 2 & -6 & 14 \\ -3 & 9 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 34 \\ 24 & -9 \end{bmatrix}$$

so $A(BC) = (AB)C$ holds for these matrices.

1.4.8 Let $A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$.

(a) Determine a simple expression for A^2.

$$A^2 = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\cos(\theta)\sin(\theta) \\ -2\cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}$$ as $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

(b) Determine a simple expression for A^3

$$A^3 = A^2 A = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) & \sin(\theta)\cos(2\theta) + \sin(2\theta)\cos(\theta) \\ -\sin(2\theta)\cos(\theta) - \cos(2\theta)\sin(\theta) & -\sin(2\theta)\sin(\theta) + \cos(2\theta)\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(3\theta) & \sin(3\theta) \\ -\sin(3\theta) & \cos(3\theta) \end{bmatrix}$$ by looking up trig. identities.

(c) Conjecture the form of a simple expression for A^k, k a positive integer.

$$A^k = \begin{bmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{bmatrix}$$

(d) Prove or disprove your conjecture in part (c).

Proof by induction on k (see handout). Base cases $k = 2, 3$ already done above.

Suppose $A^k = \begin{bmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{bmatrix}$. Need to show $A^{k+1} = \begin{bmatrix} \cos((k+1)\theta) & \sin((k+1)\theta) \\ -\sin((k+1)\theta) & \cos((k+1)\theta) \end{bmatrix}$.

$$A^{k+1} = A^k A = \begin{bmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{bmatrix}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta) & \cos(k\theta)\sin(\theta) + \sin(k\theta)\cos(\theta) \\ -\sin(k\theta)\cos(\theta) - \cos(k\theta)\sin(\theta) & \cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos((k+1)\theta) & \sin((k+1)\theta) \\ -\sin((k+1)\theta) & \cos((k+1)\theta) \end{bmatrix}$$

as needed by the same trig. identities in (b).

1.4.10) Find two different 2x2 matrices A such that $A^2 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$.

There are many. $A = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $A = \begin{bmatrix}-1&0\\0&-1\end{bmatrix}$, $A = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$, $A = \begin{bmatrix}-1&0\\0&1\end{bmatrix}$.

1.4.12) Find two different 2x2 matrices A such that $A^2 = 0$.

Again, there are many. $A = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ is the trivial one. Note for, $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$.

$A^2 = \begin{bmatrix}a^2+bc & b(a+d)\\ c(a+d) & d^2+bc\end{bmatrix}$ So $a=-d$ and $a^2 = -bc$. Then we can see

$A = \begin{bmatrix}1&-2\\1/2&-1\end{bmatrix}$, $\begin{bmatrix}-1&1/3\\-3&1\end{bmatrix}$, $\begin{bmatrix}2&-2\\2&-2\end{bmatrix}$

1.4.22) Determine a scalar r such that $A\vec{x} = r\vec{x}$ where $A = \begin{bmatrix}2&1\\1&2\end{bmatrix}$ and $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$.

$A\vec{x} = \begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}r\\r\end{bmatrix} = r\vec{x}$ So $r = 3$.

1.4.32) Find three 2x2 matrices $A, B,$ and C such that $AB = AC$ with $B \neq C$ and $A \neq 0$.

Note this is $A(B-C) = 0$ by Thm 1.2 and 1.1.

Note that $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix} = 0$ So set $A = \begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $B - C = \begin{bmatrix}0&0\\0&1\end{bmatrix}$.

Then Take $B = \begin{bmatrix}1&0\\0&2\end{bmatrix}$ and $C = \begin{bmatrix}1&0\\0&1\end{bmatrix}$.