

# MA 523: Homework, Midterms and Practice Problems Solutions

Carlos Salinas

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## **Contents**

# 1 Midterms and Qualifying Exams

## 1.1 Qualifying Exam, August '04

PROBLEM 1.1. Consider the initial value problem

$$\begin{cases} a(x, y)u_x + b(x, y)u_y = -u, \\ u = f \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\},$$

where  $a$  and  $b$  satisfy

$$a(x, y) + b(x, y)y > 0$$

for any  $x, y \in \mathbb{R}^n \setminus \{(0, 0)\}$ .

- (a) Show that the initial value problem has a unique solution in a neighborhood of  $S^1$ . Assume that  $a$ ,  $b$ , and  $f$  are smooth.
- (b) Show that the solution of the initial value problem actually exists in  $\mathbb{R}^2 \setminus \{(0, 0)\}$ .

SOLUTION. ■

PROBLEM 1.2. Let  $u \in C^2(\mathbb{R} \times [0, \infty))$  be a solution of the initial value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } \mathbb{R} \times (0, \infty), \\ u = f, \quad u_t = g & \text{in } \mathbb{R} \times 0, \end{cases}$$

where  $f$  and  $g$  have compact support. Define the kinetic energy by

$$K(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dx,$$

and the potential energy by

$$P(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) dx.$$

Show that

- (a)  $K(t) + P(t)$  is constant in  $t$ ,
- (b)  $K(t) = P(t)$  for all large enough times  $t$ .

SOLUTION. ■

PROBLEM 1.3. Use Kirchhoff's formula and Duhamel's principle to obtain an integral representation of the solution of the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = e^{-t}g(x) & \text{for } x \in \mathbb{R}^3, t > 0, \\ u(x, 0) = u_t(x, 0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Verify that the integral representation reduces to the obvious solution  $u = e^{-t} + t - 1$  when  $g(x) = 1$ .

*SOLUTION.* ■

PROBLEM 1.4. Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and  $g \in C_0^\infty(\Omega)$ . Consider the solutions of the initial boundary value problem

$$\begin{cases} \Delta u - u_t = 0 & \text{for } x \in \Omega, t > 0, \\ u(x, 0) = g(x) & \text{for } x \in \Omega, \\ u(x, t) = 0 & \text{for } x \in \partial\Omega, t \geq 0, \end{cases}$$

and the Cauchy problem

$$\begin{cases} \Delta v - v_t = 0 & \text{for } x \in \mathbb{R}^n, t > 0, \\ v(x, 0) = |g(x)| & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where we put  $g = 0$  outside  $\Omega$ .

(a) Show that

$$-v(x, t) \leq u(x, t) \leq v(x, t)$$

for any  $x \in \Omega, t > 0$ .

(b) Use (a) to conclude that

$$\lim_{t \rightarrow \infty} u(x, t) = 0,$$

for any  $x \in \Omega$ .

*SOLUTION.* ■

PROBLEM 1.5. Let  $P_k(x)$  and  $P_m(x)$  be homogeneous harmonic polynomials in  $\mathbb{R}^n$  of degrees  $k$  and  $m$  respectively; i.e.,

$$P_k(\lambda x) = \lambda^k P_k(x), \quad P_m(\lambda x) = \lambda^m P_m(x),$$

for any  $x \in \mathbb{R}^n, \lambda > 0$ ,

$$\Delta P_k = 0, \quad \Delta P_m = 0$$

in  $\mathbb{R}^n$ .

(a) Show that

$$\frac{\partial P_k(x)}{\partial \nu} = k P_k(x), \quad \frac{\partial P_m(x)}{\partial \nu} = m P_m(x)$$

on  $\partial B_1$ , where  $B_1 = \{ |x| < 1 \}$  and  $\nu$  is the outward normal on  $\partial B_1$ .

(b) Use (a) and Green's second identity to prove that

$$\int_{\partial B_1} P_k(x) P_m(x) dS = 0,$$

if  $k \neq m$ .

*SOLUTION.* ■

## 1.2 Qualifying Exam, August '05

PROBLEM 1.6.

- (a) Find a solution of the Cauchy problem

$$\begin{cases} yu_x + xu_y = xy, \\ u = 1 \end{cases} \quad \text{on } S^1 = \{x^2 + y^2 = 1\}.$$

- (b) Is the solution unique in a neighborhood of the point  $(1, 0)$ ? Justify your answer.

SOLUTION. ■

PROBLEM 1.7. Consider the second order PDE in  $\{x > 0, y > 0\} \subset \mathbb{R}^2$

$$x^2 u_{xx} - y^2 u_{yy} = 0.$$

- (a) Classify the equation and reduce it to the canonical form.  
 (b) Show that the general solution of the equation is given by the formula

$$u(x, y) = F(x, y) + \sqrt{xy}G(x/y).$$

SOLUTION. ■

PROBLEM 1.8. Let  $\Phi$  be the fundamental solution of the Laplace equation in  $\mathbb{R}^3$  and  $f \in C_0^\infty(\mathbb{R}^n)$ . Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y)f(y) dy$$

is a solution of the Poisson equation  $-\Delta u = f$  in  $\mathbb{R}^n$ . Show that if  $f$  is radial (i.e.,  $f(y) = f(|y|)$ ) and supported in  $B_R = \{|x| < R\}$ , then

$$u(x) = c\Phi(x),$$

for any  $x \in \mathbb{R}^n \setminus B_R$ , where

$$c = \int_{\mathbb{R}^n} f(y) dy.$$

[Hint: Use spherical (polar) coordinates and the mean value property.]

SOLUTION. ■

PROBLEM 1.9. Consider the so-called 2-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ u(x, 0) = g(x), \quad u_t(x, 0) = h(x) & \text{for } x \in \mathbb{R}^2, \end{cases}$$

where  $g, h \in C_0^\infty(\mathbb{R}^2)$  and  $\alpha \geq 0$  is a constant.

- (a) Show that for an appropriate choice of constant  $\lambda$  and  $\mu$  the function

$$v(x_1, x_2, x_3, t) := e^{\lambda t + \mu x_3} u(x_1, x_2, t)$$

solves the 3-dimensional wave equation  $v_{tt} - \Delta v = 0$ .

- (b) Use (a) to prove the following domain of dependence result: for any point  $(x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$  the value  $u(x_0, t_0)$  is uniquely determined by values of  $g$  and  $h$  in  $\overline{B_{t_0}(x_0)} := \{|x - x_0| \leq t_0\}$ . (You may use the corresponding result for the wave equation without proof.)

*SOLUTION.* ■

PROBLEM 1.10. Let  $u(x, t)$  be a bounded solution of the heat equation  $u_t = u_{xx}$  in  $\mathbb{R} \times (0, \infty)$  with the initial condition

$$u(x, 0) = u_0(x)$$

for  $x \in \mathbb{R}$ , where  $u_0 \in C^\infty$  is  $2\pi$ -periodic, i.e.,  $u_0(x + 2\pi) = u_0(x)$ . Show that

$$\lim_{t \rightarrow \infty} u(x, t) = a_0,$$

uniformly in  $x \in \mathbb{R}$ , where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x) dx.$$

*SOLUTION.* ■

### 1.3 Qualifying Exam, January '14

PROBLEM 1.11. Consider the first order equation in  $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

- (a) Find the characteristic curves of the equation.
- (b) Consider the Cauchy problem for this equation prescribed on the line  $x_1 = 1$ :

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on  $f$  so that the problem is solvable in a neighborhood of the point  $(1, 0)$ .

*SOLUTION.* ■

PROBLEM 1.12. Let  $u$  be a continuous bounded solution of the initial value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \{x_n > 0\}, \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where  $g$  is a continuous function with compact support in  $\mathbb{R}^{n-1}$ . Here  $n \geq 2$ . Prove that

$$u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty,$$

for  $x \in \{x_n > 0\}$ .

*SOLUTION.* ■

PROBLEM 1.13. Let  $u$  be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \quad \text{in } \mathbb{R} \times (0, \infty),$$

with the initial conditions  $u(x, 0) = g(x)$ , where  $g$  is a bounded continuous function on  $\mathbb{R}$  satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^\alpha, \quad x, y \in \mathbb{R}$$

with a constant  $\alpha \in (0, 1]$ . Show that

$$\begin{aligned} |u(x, t) - u(y, t)| &\leq M|x - y|^\alpha, & x, y \in \mathbb{R}, t > 0, \\ |u(x, t) - u(x, s)| &\leq C_\alpha M|t - s|^{\alpha/2}, & x \in \mathbb{R}, t, s > 0. \end{aligned}$$

[*Hint:* For the last inequality, in the representation formula of  $u(x, t)$  as a convolution with the heat kernel  $\Phi(y, t)$ , make a change of variables  $z = y/\sqrt{t}$  and use that  $|\sqrt{t} - \sqrt{s}| \leq \sqrt{|t - s|}$ .]

*SOLUTION.* ■

PROBLEM 1.14. Let  $u$  be a positive harmonic function in the unit ball  $B_1$  in  $\mathbb{R}^n$ . Show that

$$|D(\ln u)| \leq M \quad \text{in } B_{1/2}$$

for a constant  $M$  depending only on the dimension  $n$ .

[*Hint:* Use the interior derivative estimate  $|Du(x)| \leq (C_n/r) \sup_{B_r(x)} |u|$  for  $B_r(x) \subset B_1$  as well as the Harnack inequality for harmonic functions.]

SOLUTION. ■

PROBLEM 1.15. Let  $u$  be a  $C^2$  solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = |x|^k & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = 0, \quad u_t = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

for some  $k \geq 0$ . Prove that there exists a function  $\varphi(r)$  such that

$$u(x, t) = t^{k+2} \varphi(|x|/t).$$

[*Hint:* As one of the steps show that  $u$  is  $(k+2)$ -homogeneous in  $(x, t)$  variables, i.e.,  $u(\lambda x, \lambda t) = \lambda^{k+2} u(x, t)$  for any  $\lambda > 0$ .]

SOLUTION. ■