Fall 2016 Notes

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Contents

Chapter 1

Probability

We will devote this chapter to the material that is covered in MA 51900 (discrete probability) as it was covered in DasGupta's class. We will, for the most part, reference Feller's *An introduction to probability theory and its applications, Volume 1* [**?**] (especially for the discrete noncalculus portion of the class) and DasGupta's own book *Fundamentals of Probability: A First Course* [**?**].

1.1 Discrete Probability

The material in this chapter is mostly pulled from Sheldon Ross's *A First Course in Probability Theory* [**?**] with some examples from [**?**] and [**?**]. I find Ross's book to be better structured than the latter two.

Combinatorial Analysis

These are the main results from this section.

Theorem 1.1 (The basic principle of counting)**.** *Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.*

Theorem 1.2 (The generalized principle of counting)**.** *If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes for the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes for the third experiment; etc. ..., then there is a total of $n_1 n_2 \cdots n_r$ possible outcomes of the r experiments.*

Using notation as in [**?**], the number

$$(n)_r = n(n-1)\cdots(n-r+1)$$

represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{(n)_r}{r!} = \frac{n!}{(n-r)!r!}$$

for which we reserve the notation

$$\binom{n}{r}$$

read n choose r. (This is called a binomial coefficient since it appears in the binomial expansion $(a+b)^n$.)

A useful combinatorial identity on binomial coefficients is the following

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

for $1 \le r \le n$.

Theorem 1.3 (The binomial theorem)**.**

$$(a+b)^n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}.$$

Proof. We provide a combinatiorial proof of the theorem. Consider the product

$$(a_1 + b_1) \cdots (a_n + b_n).$$

Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Furthermore, each of the 2^n terms in the sum will contain as a factor either a_i or b_i for each $1 \le i \le n$. Now, how many of the 2^n terms in the sum will have k of the a_i and $n-k$ of the b_i as factors? As each term consisting of k of the a_i and $n-k$ of the b_i correspond to a choice of a group of k from the values $a_1, \ldots, a_n$, there are $\binom{n}{k}$ such terms. Thus, letting $a_i = a$, $b_i = b$, $1 \le i \le n$, we see that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

■

Chapter 2

Introduction to Partial Differential Equations

Here we summarize some important points about PDEs. The material is mostly taken from Evans's *Partial Differential Equations* [**?**] with occasional detours to Strauss's *Partial Differential Equations: An Introduction* [**?**]. We will be following Dr. Petrosyan's Course Log which can be found here `https://www.math.purdue.edu/~arshak/F16/MA523/courselog/`, i.e., summarizing the appropriate chapters from [**?**].

2.1 Introduction

Partial differential equations

Definition 2.1. An expression of the form

$$F\big(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\big) = 0, \quad x \in U, \tag{2.1}$$

is called a *kth-order partial differential equation* (*PDE*), where

$$F\colon \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$$

is given, and

$$u\colon U \longrightarrow \mathbb{R}$$

is the unknown.

Here are some more definitions,

Definition 2.2.

(i) The partial differential equation (2.1) is called *linear* if it has the form

$$\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u = f(x)$$

for given functions $a_\alpha(|\alpha| \le k)$, f. This linear PDE is *homogeneous* if $f = 0$.

(ii) The PDE (2.1) is *semilinear* if it has the form

$$\sum_{|\alpha|=k} a_\alpha D^\alpha u + a_0\big(D^{k-1}u, \dots, Du, u, x\big) = 0.$$

(iii) The PDE (2.1) is *quasilinear* if it has the form

$$\sum_{|\alpha|=k} a_\alpha\big(D^{k-1}u, \dots, Du, u, x\big) D^\alpha u + a_0\big(D^{k-1}u, \dots, Du, u, x\big) = 0.$$

(iv) The PDE (2.1) is *fully nonlinear* if it depends upon the highest order derivatives.

A *system* of partial differential equations is, informally speaking, a collection of several PDEs for several unknown functions.

Definition 2.3. An expression of the form

$$\mathbf{F}\big(D^k\mathbf{u}(x), D^{k-1}\mathbf{u}(x), \dots, D\mathbf{u}(x), \mathbf{u}(x), x\big) = 0, \quad x \in U, \tag{2.2}$$

is called a *kth-order system of PDEs*, where

$$\mathbf{F}\colon \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \longrightarrow \mathbb{R}^m$$

is given and

$$\mathbf{u}\colon U \longrightarrow \mathbb{R}^m, \quad \mathbf{u} = (u^1, \dots, u^m)$$

is the unknown.

Remark 2.4. We haven't talked much about systems of PDEs and I suspect we will not do so very much in this course.

Examples

This is only a fraction of the PDEs listed in Evan's chapter.

Linear equations

1. Laplace's equation
$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0.$$
2. Helmholtz's (or eigenvalue) equation
$$-\Delta u = \lambda u.$$
3. Linear transport equation
$$u_t + \sum_{i=1}^n b^i u_{x_i} = 0.$$

4. Liouville's equation
$$u_t - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$
5. Heat (or diffusion) equation
$$u_t - \Delta u = 0.$$
6. Wave equation
$$u_{tt} - \Delta u = 0.$$
7. Telegraph equation
$$u_{tt} + d u_t - u_{xx} = 0.$$

Nonlinear equations

1. Eikonal equation
$$|Du| = 1.$$
2. Nonlinear Poisson equation
$$-\Delta u = f(u).$$
3. Inviscid Burgers' equation
$$u_t + u u_x = 0.$$

and so on.

2.2 The transport equation

We begin our study with one of the simplest PDEs, the *transport equation* with constant coefficients. This is the PDE
$$u_t + b \cdot Du = 0, \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{2.3}$$
where b is a fixed vector in $\mathbb{R}^n$, $b = (b_1, \dots, b_n)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is a typical point in space, $t \geq 0$ denotes a typical time and $u \colon \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is the unknown, $u = u(x, t)$. We write $Du = D_x u = (u_{x_1}, \dots, u_{x_n})$ for the gradient of u with respect to the spatial variable x.

So, which functions solve (2.3)? Well, let us suppose for a moment that u is a smooth solution to the PDE and let us try to compute it. To do so, we first recognize that (2.3) asserts that a particular directional derivative of u vanishes, namely, $D_b u = 0$. We exploit this by fixing a point $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and defining
$$z(s) := u(x + sb, t + s), \quad s \in \mathbb{R}.$$
Then we calculate
$$\begin{aligned} \dot{z}(s) &= Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) \\ &= 0, \end{aligned}$$
the second equality holding by (2.3). Thus, z is a constant function of s, and consequently for each (x, t), u is constant on the line through (x, t) with direction $(b, 1) \in \mathbb{R}^{n+1}$. Hence, if we know the value of u at any point on ecah such line, we know its value everywhere in $\mathbb{R}^n \times (0, \infty)$.

2.3 Characteristics

Derivation of characteristic ODEs

Consider the nonlinear first-order PDE

$$F(Du, u, x) = 0 \quad \text{in } U, \tag{2.4}$$

subject now to the boundary condition

$$u = g \quad \text{on } \Gamma, \tag{2.5}$$

where $\Gamma \subset \partial U$ and $g \colon \Gamma \to \mathbb{R}$ are given. We hereafter suppose that F, g are smooth functions.

We now develop the method of *characteristics*, which solves (2.4) and (2.5) by converting the PDE into an appropriate system of ODEs. Suppose u solves the (2.4), (2.5) and fix any point $x \in U$. We would like to calculate $u(x)$ by finding some curve lying within U, connecting x with a point $x^0 \in \Gamma$ and along which we can compute u. Since (2.5) says $u = g$ on Γ, we know the value of u at the one end x^0. We hope then to be able to calculate u all along the curve, and so in particular at x.

Finding the characteristic ODEs

How can we choose the curve so all this will work? Let us suppose it is described parametrically by the function $\mathbf{x}(s) = \big(x^1(s), \dots, x^n(s)\big)$, the parameter s lying in some subinterval of $\mathbb{R}$. Assuming u is a C^2 solution of (2.4), we define also

$$z(s) := u\big(\mathbf{x}(s)\big).$$

In addition, set

$$\mathbf{p}(s) := Du(\mathbf{x}(s));$$

that is, $\mathbf{p}(s) = \big(p^1(s), \dots, p^n(s)\big)$, where

$$p^i(s) = u_{x_i}\big(\mathbf{x}(s)\big), \tag{2.6}$$

$1 \le i \le n$. So z gives the values of u along the curve and $\mathbf{p}$ records the values of the gradient Du. We must choose a function $\mathbf{x}$ in such a way that we can compute z and $\mathbf{p}$.

For this, first differentiate (2.6)

$$\dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}\big(\mathbf{x}(s)\big)\dot{x}^j(s)$$

This expression is not too promising, since it involves the second derivatives of u. On the other hand, we can also differentiate the PDE (2.4) with respect to x_i to get

$$\sum_{j=1}^{n} \frac{\partial}{\partial p_j} F(Du, u, x) u_{x_j x_i} + \frac{\partial}{\partial z} F(Du, u, x) u_{x_i} + \frac{\partial}{\partial x_i} F(Du, u, x) = 0.$$

We are able to employ this identity to get rid of the *dangerous* second derivative terms provided we first set

$$\dot{x}^j(s) = \frac{\partial}{\partial p_j} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big).$$

Assuming now that the above equation holds, we can evaluate the partials

$$\sum_{j=1}^{n} \frac{\partial}{\partial p_j} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) \\ + \frac{\partial}{\partial z} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) p^i(s) + \frac{\partial}{\partial x_i} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) = 0.$$

Substitute this expression and the previous one into the derivative for $\dot{p}^i$ and we get

$$\dot{p}^i(s) = \frac{\partial}{\partial x_i} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big)$$

Finally, we differentiate z to get

$$\dot{z}(s) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} u\big(\mathbf{x}(s)\big)\dot{x}^j(s) = \sum_{j=1}^{n} p^j(s) \frac{\partial}{\partial p_j} F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big),$$

the second equality holding by –fuck this guy for numbering every expression–(5) and (8)–whatever they are.

We summarize by rewriting equations (8)–(10) in vector notation:

$$\begin{cases} \text{(a) } \dot{\mathbf{p}}(s) = -D_x F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) - D_z F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big)\mathbf{p}(s), \\ \text{(b) } \dot{z}(s) = D_p F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) \cdot \mathbf{p}(s), \\ \text{(c) } \dot{\mathbf{x}}(s) = D_p F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big). \end{cases} \tag{2.7}$$

This important system of $2n+1$ first-order ODEs comprises the *characteristic equations* of the nonlinear first-order PDE (2.4). The functions $\mathbf{p} = (p^1, \ldots, p^n)$, z, $\mathbf{x} = (x^1, \ldots, x^n)$ are a called the *characteristics*. We will sometimes refer to $\mathbf{x}$ as the *projected characteristics*: it is the projection of the full characteristics $(\mathbf{p}, z, \mathbf{x}) \subset \mathbb{R}^{2n+1}$ onto the physical region $U \subset \mathbb{R}^n$.

Theorem 2.5 (Structure of characteristic ODEs)**.** *Let $u \in C^2(U)$ solve the nonlinear, first-order partial differential equation* (2.4) *in U. Assume $\mathbf{x}$ solves the ODEs* (2.7)(c)*, where $\mathbf{p} = Du$, $z = u$. Then $\mathbf{p}$ solves the ODE* (2.7)(a) *and z solves the ODE* (2.7)(b)*, for those s such that $\mathbf{x} \in U$.*

Examples

F linear

Consider first the situation that (2.4) is linear and homogeneous, and thus has the form

$$F(Du, u, x) = \mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0, \qquad x \in U.$$

Then $F(p, z, x) = \mathbf{b}(x) \cdot p + c(x)z$, and so

$$D_p F = \mathbf{b}(x).$$

In this circumstance (2.7)(c) becomes

$$\mathbf{x}(s) = \mathbf{b}\big(\mathbf{x}(s)\big),$$

an ODE involving only the function $\mathbf{x}$. Furthermore (2.7)(b) becomes

$$\dot{z}(s) = \mathbf{b}\big(\mathbf{x}(s)\big) \cdot \mathbf{p}(s). \tag{2.8}$$

Since $\mathbf{p}(\cdot) = Du\big(\mathbf{x}(\cdot)\big)$, the PDE simplifies the above to

$$\dot{z}(s) = -c\big(\mathbf{x}(s)\big)z(s).$$

This ODE is linear in z, once we know the function $\mathbf{x}$ by solving its ODE.

In summary,

$$\begin{cases} \text{(a) } \dot{\mathbf{x}}(s) = \mathbf{b}\big(\mathbf{x}(s)\big) \\ \text{(b) } \dot{z}(s) = -c\big(\mathbf{x}(s)\big)z(s) \end{cases} \tag{2.9}$$

comprise the characteristic equations for the linear, first-order PDE (2.8).

Example 2.6. We demonstrate the utility of equations (2.9) by explicitly solving the problem

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } U \\ \qquad\qquad\quad u = g & \text{on } \Gamma, \end{cases} \tag{2.10}$$

where U is the quadrant $\{\, x_1 > 0, x_2 > 0 \,\}$ and $\Gamma = \{\, x_1 > 0, x_2 = 0 \,\} \subset \partial U$. The PDE in (2.10) is of the form (2.8), for $\mathbf{b} = (-x_2, x_1)$ and $c = -1$. Thus the equations (2.9) read

$$\begin{cases} (x^1, x^2)(s) = (x^0 \cos s, x^0 \sin s) \\ \qquad\quad z(s) = z^0 \mathrm{e}^s = g(x^0)\mathrm{e}^s, \end{cases} \tag{2.11}$$

where $x^0 \geq 0$, $0 \leq s \leq \pi/2$. Fix a point $(x_1, x_2) \in U$. We select $s > 0$, $x^0 > 0$ so that $(x_1, x_2) = \big(x^1(s), x^2(s)\big) = (x^0 \cos s, x^0 \sin s)$. That is, $x^0 = \big({x_1}^2 + {x_2}^2\big)^{1/2}$, $s = \arctan(x_2/x_1)$. Therefore,

$$\begin{aligned} u(x_1, x_2) &= u\big(x^1(s), x^2(s)\big) \\ &= z(s) \\ &= g(x^0)\mathrm{e}^s \\ &= g\big(({x_1}^2 + {x_2}^2)^{1/2}\big)\mathrm{e}^{\arctan(x_2/x_1)}. \end{aligned}$$

$\boldsymbol{F}$ quasilinear

The PDE (2.4) is quasilinear if it has the form

$$F(Du, u, x) = \mathbf{b}\big(x, u(x)\big) \cdot Du(x) + c\big(x, u(x)\big) = 0. \tag{2.12}$$

In this circumstance $F(p, z, x) = \mathbf{b}(x, z) \cdot p + c(x, z)$; whence

$$D_p F = \mathbf{b}(x, z).$$

Hence equation (2.9)(c) reads

$$\dot{\mathbf{x}} = \mathbf{b}\big(\mathbf{x}(s), z(s)\big),$$

and (2.9)(b) becomes

$$\begin{aligned} \dot{z}(s) &= \mathbf{b}\big(\mathbf{x}(s), z(s)\big) \cdot \mathbf{p}(s) \\ &= -c\big(\mathbf{x}(s), z(s)\big) \end{aligned}$$

by (2.12). Consequently

$$\begin{cases} \text{(a) } \dot{\mathbf{x}}(s) = \mathbf{b}\big(\mathbf{x}(s), z(s)\big), \\ \text{(b) } \dot{z}(s) = -c\big(\mathbf{x}(s), z(s)\big) \end{cases} \tag{2.13}$$

are the characteristic equations for the quasilinear first-order PDE.

Example 2.7. The characteristic ODEs (2.13) are in general difficult to solve, and so we work out in this example the simpler case of a boundary-value problem for a semilinear PDE:

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 \quad \text{in } U \\ \qquad\quad u = g \quad \text{on } \Gamma. \end{cases} \tag{2.14}$$

Now U is the half-space $\{\, x_2 > 0 \,\}$ and $\Gamma = \{\, x_2 = 0\, h\} = \partial U$. Here $\mathbf{b} = (1, 1)$ and $c = -z^2$. Then (2.13) becomes

$$\begin{cases} (\dot{x}^1, \dot{x}^2) = 1 \\ \qquad\quad \dot{z} = z^2. \end{cases}$$

Consequently

$$\begin{cases} (x^1, x^2)(s) = (x^0 + s, s) \\ \qquad\quad z(s) = \dfrac{z^0}{1 - sz^0} \\ \qquad\qquad\quad = \dfrac{g(x^0)}{1 - sg(x^0)}, \end{cases}$$

where $x^0 \in \mathbb{R}$, $s \geq 0$, provided the denominator is not zero.

Fix a point $x_1, x_2 \in U$. We select $s > 0$ and $x^0 \in \mathbb{R}$ so that $(x_1, x_2) = \big(x^1(s), x^2(s)\big) = (x^0 + s, s)$; that is, $x^0 = x_1 - x_2$, $s = x_2$. Then

$$\begin{aligned} u(x_1, x_2) &= u\big(x^1(s), x^2(s)\big) \\ &= z(s) \\ &= \frac{g(x^0)}{1 - sg(x^0)} \\ &= \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}. \end{aligned}$$

This solution of course make sense only if $1 - x_2 g(x_1 - x_2) \neq 0$.

F fully nonlinear

In the general case, the full characteristic equations (2.7) must be integrated, if possible.

Example 2.8. Consider the fully nonlinear problem

$$\begin{cases} u_{x_1}u_{x_2} = u & \text{on } U \\ u = {x_2}^2 & \text{on } \Gamma \end{cases} \tag{2.15}$$

where $U = \{\, x_1 > 0 \,\}$, $\Gamma = \{\, x_1 = 0 \,\} = \partial U$. Here $F(p, z, x) = p_1 p_2 - z$, and hence the characteristic ODEs (2.7) become

$$\begin{cases} (\dot{p}^1, \dot{p}^2) = (p^1, p^2) \\ \dot{z} = 2p^1p^2 \\ (\dot{x}^1, \dot{x}^2) = (p^2, p^1). \end{cases}$$

We integrate these equations to find

$$\begin{cases} (x^1, x^2)(s) = \big(p_2^0(\mathrm{e}^s - 1), p_1^0(\mathrm{e}^s - 1)\big) \\ z(s) = z^0 + p_1^0 p_2^0(\mathrm{e}^{2s} - 1) \\ (p^1, p^2)(s) = (p_1^0\mathrm{e}^s, p_2^0\mathrm{e}^s), \end{cases}$$

where $x^0 \in \mathbb{R}$, $s \in \mathbb{R}$, and $z^0 = \left(x^0\right)^2$.

We must determine $p^0 = (p_1^0, p_2^0)$. Since $u = {x_2}^2$ on Γ, $p_2^0 = u_{x_2}(0, x^0) = 2x^0$. Furthermore the PDE $u_{x_1}u_{x_2} = u$ itself implies $p_1^0 p_2^0 = z^0 = \left(x^0\right)^2$, and so $p_1^0 = x^2/2$. Consequently the formulas above become

$$\begin{cases} (x^1, x^2)(s) = \big(2x^9(\mathrm{e}^s - 1), x^0(\mathrm{e}^s + 1)/2\big) \\ z(s) = \left(x^0\right)^2 \mathrm{e}^{2s} \\ (p^1, p^2)(s) = (x^0\mathrm{e}^s/2, 2x^0\mathrm{e}^s). \end{cases}$$

Fix a point $(x_1, x_2) \in U$. Select s and x^0 so that

$$(x_1, x_2) = \big(x^1(s), x^2(s)\big) = \big(2x^0(\mathrm{e}^s - 1), x^0(\mathrm{e}^s + 1)/2\big).$$

This equality implies $x^0 = (4x_2 - x_1)/4$, $\mathrm{e}^s = (x_1 + 4x_2)/(4x_2 - x_1)$; and so

$$\begin{aligned} u(x_1, x_2) &= u\big(x^1(s), x^2(s)\big) \\ &= \frac{(x_1 + 4x_2)^2}{16}. \end{aligned}$$

2.4 Boundary conditions

Straightening the boundary

We intend in the following section to invoke the characteristic ODE (2.7) to actually solve the boundary-value problem (2.4), (2.5), at least in a small region near an appropriate portion Γ of ∂U. In order to simplify the relevant calculations, it is convenient first fix any point $x^0 \in \partial U$. Then utilizing the notation from the appendix §C.1 of [**?**], we find smooth mappings $\mathbf{\Phi}, \mathbf{\Psi} \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{\Psi} = \mathbf{\Phi}^{-1}$ and $\mathbf{\Phi}$ straightens ∂U near x^0.

Given a function $u \colon U \to \mathbb{R}$, let us write $V := \mathbf{\Phi}(U)$ and set

$$v(y) := u\big(\mathbf{\Psi}(y)\big) \qquad y \in V. \tag{2.16}$$

Then

$$u(x) = v\big(\mathbf{\Phi}(x)\big) \qquad x \in U. \tag{2.17}$$

Now suppose that u is a C^1 solution of our boundary-value problem (2.4), (2.5) in U. What PDE does v then satisfy in V?

According to (2.17), we have

$$u_{x_i}(x) = \sum_{k=1}^{n} v_{y_k}\big(\mathbf{\Phi}(x)\big)\Phi^k_{x_i}(x)$$

i.e.,

$$Du(x) = Dv(y)D\mathbf{\Phi}(x).$$

Thus,

$$\begin{aligned} 0 &= F\big(Du(x), u(x), x\big) \\ &= F\big(Dv(y), D\mathbf{\Phi}\big(\mathbf{\Psi}(y)\big), v(y), \Psi(y)\big). \end{aligned}$$

In addition $v = h$ on Δ, where $\Delta := \mathbf{\Phi}(\Gamma)$ and $h(y) := g\big(\mathbf{\Psi}(y)\big)$.

In summary, our problem (2.4), (2.5) converts into a problem having the same form.

Compatibility conditions on boundary

In view of the foregoing computations, if we are given a point $x^0 \in \Gamma$ we may as well assume that the outset that Γ is flat near x^0, lying in the plane $\{\, x_n = 0 \,\}$.

We intend now to utilize the characteristic ODE to construct a solution to (2.4), at least near x^0, and for this we must discover appropriate initial conditions

$$\mathbf{p}(0) = p^0, \quad z(0) = z^0, \quad \mathbf{x}(0) = x^0. \tag{2.18}$$

Now clearly if the curve $\mathbf{x}$ passes through x^0, we should insist that

$$z^0 = g(x^0). \tag{2.19}$$

What should we require concerning $\mathbf{p}(0) = p^0$? Since (2.5) implies $u(x_1, \ldots, x_{n-1}, 0) = g(x_1, \ldots, x_{n-1})$ near x^0, we may differentiate to find

$$u_{x_i}(x^0) = g_{x_i}(x^0) \quad i = 1, \ldots, n-1.$$

As we also want the PDE (2.4) to hold, we should therefore insist $p^0 = (p^0_1, \ldots, p^0_n)$ satisfies these relations

$$\begin{cases} p^0_i = g_{x_i}(x^0) \qquad i = 1, \ldots, n-1 \\ F(p^0, z^0, x^0) = 0. \end{cases} \tag{2.20}$$

These identities provide n equations for the n quantities $p^0 = (p^0_1, \ldots, p^0_n)$.

We call (2.19) and (2.20) the *compatibility conditions*. A triple $(p^0, z^0, x^0) \in \mathbb{R}^{2n+1}$ verifying (2.19), (2.20) is *admissible*. Note that z^0 is uniquely determined by the boundary condition and our choice of the point x^0, but a vector p^0 satisfying (2.20) may not exist and it may not be unique.

Noncharacteristic boundary data

So now assume as above that $x^0 \in \Gamma$, that Γ near x^0 lies in the plane $\{ x_n = 0 \}$, and that the triple (p^0, z^0, x^0) is admissible. We are planning to construct a solution u of (2.4), (2.5) in U near x^0 by integrating by parts the characteristic ODE (2.7). So far we have ascertained $\mathbf{x}(0) = x^0$, $z(0) = z^0$, $\mathbf{p}(0) = p^0$ are appropriate boundary conditions for the characteristic ODE, with $\mathbf{x}$ intersecting Γ at x^0. But we will need in fact to solve these ODEs for *nearby* initial points as well, and must consequently now ask if we can somehow appropriately perturb (p^0, z^0, x^0), keeping the compatibility conditions.

In other words, given a point $y = (y_1, \ldots, y_{n-1}, 0) \in \Gamma$, with y close to x^0, we intend to solve the characteristic ODE

$$\begin{cases} \text{(a) } \dot{\mathbf{p}}(s) = -D_x F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) - D_z F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) \\ \text{(b) } \dot{z}(s) = D_p F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big) \cdot \mathbf{p}(s) \\ \text{(c) } \dot{\mathbf{x}}(s) = D_p F\big(\mathbf{p}(s), z(s), \mathbf{x}(s)\big), \end{cases} \tag{2.21}$$

with the initial conditions

$$\mathbf{p}(0) = \mathbf{q}(y), \quad z(0) = g(y), \quad \mathbf{x}(0) = y. \tag{2.22}$$

Our task then is to find a function $\mathbf{q} = (q^1, \ldots, q^n)$, so that

$$\mathbf{q}(x^0) = p^0 \tag{2.23}$$

and $\big(\mathbf{q}(y), g(y), y\big)$ is admissible; that is, the compatibility conditions

$$\begin{cases} \qquad q^i(y) = g_{x_i}(y) \qquad 1 \le i \le n-1 \\ F\big(\mathbf{q}(y), g(y), y\big) = 0 \end{cases} \tag{2.24}$$

hold for all $y \in \Gamma$ close to x^0.

Lemma 2.9 (Noncharacteristic boundary conditions)**.** *There exists a unique solution* $\mathbf{q}$ *of* (2.23), (2.24) *for all* $y \in \Gamma$ *sufficiently close to* x^0*, provided*

$$F_{p_n}(p^0, z^0, x^0) \ne 0. \tag{2.25}$$

We say the admissible triple (p^0, z^0, x^0) is *noncharacteristic* if (2.25) holds. We henceforth assume this condition.

Proof. To simplify notation, let us now temporarily write $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We apply the implicit function theorem to the mapping

$$\mathbf{G}\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad \mathbf{G}(p, y) = \big(G^1(p, y), \ldots, G^n(p, y)\big),$$

where

$$\begin{cases} G^i(p, y) = p_i - g_{x_i}(y) \qquad 1 \le i \le n-1, \\ G^n(p, y) = F\big(p, g(y), y\big). \end{cases}$$

Now $\mathbf{G}(p^0, x^0) = 0$, according to (2.19), (2.18). Also

$$D_p\mathbf{G}(p^0, x^0) = \begin{bmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ F_{p_1}(p^0, z^0, x^0) & \cdots & F_{p_{n-1}}(p^0, z^0, x^0) & F_{p_n}(p^0, z^0, x^0) \end{bmatrix}$$

and thus

$$\det D_p\mathbf{G}(p^0, x^0) = F_{p_n}(p^0, z^0, x^0) \neq 0,$$

in view of the noncharacteristic condition (2.25). ■

2.5 The One-Dimensional Wave Equation

We now turn our attention to second-order partial differential equations. In particular, the *wave equation*

$$u_{tt} - \Delta u = 0 \tag{2.26}$$

and the *nonhomogeneous wave equation*

$$u_{tt} - \Delta u = f, \tag{2.27}$$

subject to the appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open. The unknown is $u\colon \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x, t)$, and the Laplacian is taken with respect to the spatial variables $x = (x_1, \ldots, x_n)$. In (2.27) the function $f\colon U \times [0, \infty) \to \mathbb{R}$ is given. It is common to abbreviate (2.27) as

$$\Box u = u_{tt} - \Delta u.$$

D'Alembert's formula

We first focus our attention on the initial-value problem for the one-dimensional wave equation in all of $\mathbb{R}$:

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ \qquad u = g, \quad u_t = h, & \text{on } \mathbb{R} \times \{\, t = 0 \,\}, \end{cases} \tag{2.28}$$

where g, h are given. We desire to derive a formula for u in terms of g and h.

Let us first note the PDE in (2.28) can be factored, to read

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u = u_{tt} - u_{xx} = 0. \tag{2.29}$$

Write

$$v(x, t) :== \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u(x, t). \tag{2.30}$$

Then (2.29) says

$$v_t(x, t) + v_x(x, t) = 0$$

for $x \in \mathbb{R}$, $t > 0$. This is a transport equation with constant coefficient. Applying this formula, we find

$$v(x, t) = a(x - t) \tag{2.31}$$

for $a(x) := v(x,0)$. Combining these equations together, we obtain

$$u_t(x,t) - u_x(x,t) = a(x-t)$$

in $\mathbb{R} \times (0,\infty)$. This is a nonhomogeneous transport equation; and so formula (5) from §2.1.2 implies

$$\begin{aligned} u(x,t) &= \int_0^t a\big(x + (t-s) - s\big)\,ds + b(x+t) \\ &= \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy + b(x,t), \end{aligned} \tag{2.32}$$

where we have $\_(x) := u(x,0)$.

Chapter 3

Algebraic Geometry

A summary to a course on an introduction to sheaf cohomology. We will mostly reference Donu's notes available here `https://www.math.purdue.edu/~dvb/classroom.html`, but also cite Ravi Vakil's *Fundamentals of Algebraic Geometry* [?] available here `https://math216.wordpress.com/`.

3.1 The statement of de Rham's theorem

These are almost verbatim Arapura's notes on the de Rham Complex and cohomology.

Before doing anything fancy, let's start at the beginning. Let $U \subseteq \mathbb{R}^3$ be an open set. In calculus class, we learn about operations

$$\{\,\text{functions}\,\} \xrightarrow{\nabla} \{\,\text{vector fields}\,\} \xrightarrow{\nabla\times} \{\,\text{vector fields}\,\} \xrightarrow{\nabla\cdot} \{\,\text{functions}\,\}$$

such that $(\nabla\times)(\nabla) = 0$ and $(\nabla\cdot)(\nabla\times) = 0$. This is a prototype for a *complex.* An obvious question: does $\nabla \times v = 0$ imply that v is a gradient? Answer: sometimes yes (e.g. if $U = \mathbb{R}^3$) and sometimes no (e.g. if $U = \mathbb{R}^3$ minus a line). To quantify the failure we introduce the first de Rham cohomology

$$H^1_{\mathrm{dR}}(U) = \frac{\{\, v \text{ a vector field on } U : \nabla \times v = 0 \,\}}{\{\, \nabla f \,\}}.$$

Contrary to first appearances, for reasonable U this is finite dimensional and computable. This follows from the de Rham's theorem, which we now explain. First, let's generalize this to an open set $U \subset \mathbb{R}^n$. Once $n > 3$ vector calculus is useless, but there is a good replacement. A differential form of degree p, or p-form, is an expression

$$\alpha = \sum f_{i_1,\ldots,i_p}(x_1,\ldots,x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

such that the x_i are coordinates, the f are C^∞ functions, $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ are symbols where $\wedge$ is an anticommutative product. Let $\mathcal{E}^p(U)$ denote the vector space of p-forms. Define the exterior derivative by

$$d\alpha = \sum \sum_j \frac{\partial f_{i_1,\ldots,i_p}}{\partial x_j}\, dx_j \wedge \cdots \wedge dx_{i_p}.$$

This is a $(p+1)$-form.

Lemma 3.1. $d^2 = 0$.

Proof. We prove it for $p = 0$. In this case, we have

$$
\begin{aligned}
df &= \sum_i \frac{\partial f}{\partial x_i}\, dx_i \\
d(df) &= \sum_{i,j}\sum \frac{\partial^2}{\partial x_j \partial x_i}\, dx_j \wedge dx_i.
\end{aligned}
$$

Using anticommutativity, we can rewrite this as

$$\sum_{j<i}\left(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j}\right) dx_j \wedge dx_i = 0.$$

∎

A cochain complex is a collection of Abelian groups M^i and homomorphisms $d\colon M^i \to M^{i+1}$ such that $d^2 = 0$. We define the p^{th} cohomology of this by

$$H^p_{\text{dR}}(M^\bullet, d) = \frac{\operatorname{Ker} d\colon M^p \to M^{p+1}}{\operatorname{Im} d\colon M^{p-1} \to M^p}.$$

So we have an example of a complex $(\mathcal{E}^\bullet(U), d)$ called the de Rham complex of U. It's cohomology is the de Rham cohomology $H^p_{\text{dR}}(U) = H^p\big(\mathcal{E}^\bullet(U), d\big)$. Here is a basic computation.

Theorem 3.2 (Poincaré's lemma)**.**

$$H^p_{\text{dR}}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \textit{if } p = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We show this for $n \leq 2$. We first treat the case $n = 1$. Clearly $H^p_{\text{dR}}(\mathbb{R})$ consists of constant functions. If $\alpha = f(x)\, dx$, then

$$d\left(\int_0^x f(t)\, dt\right) = \alpha.$$

There are no p-forms for $p > 1$.

Next, we treat $n = 2$ which contains all of the ideas of the general case. Let x, y be coordinates. We define some operators

$$
\begin{tikzcd}
\mathcal{E}^\bullet(\mathbb{R}^2) \arrow[r, bend left, "s*"] & \mathcal{E}^\bullet(\mathbb{R}), \arrow[l, bend left, "\pi^*"]
\end{tikzcd}
$$

where π^* is the pullback along the projection $\mathbb{R}^2 \to \mathbb{R}$. It takes a form in x and treats it as a form in x, y. The pullback along the zero section s^* sets y and dy to zero. Note that $s^* \circ \pi^*$ is the identity. Although $\pi^* \circ s^*$ is not the identity, we will show that it induces the identity on cohomology. This will show that $H^*_{\mathrm{dR}}(\mathbb{R}^2) \cong H^*_{\mathrm{dR}}(\mathbb{R})$, which is all we need. This involves a new concept. We introduce an operator $H\colon \mathcal{E}^p(\mathbb{R}^2) \to \mathcal{E}^{p-1}(\mathbb{R}^2)$ of degree -1 called a *homotopy*. It integrates y as follows:

$$\begin{aligned} H\big(f(x,y)\big) &= 0 \\ H\big(f(x,y)\,dx\big) &= 0 \\ H\big(f(x,y)\,dy\big) &= \int_0^y f(x,t)\,dt \\ H\big(f(x,y)\,dx \wedge dy\big) &= \left[\int_0^y f(x,t)\,dt\right] dx. \end{aligned}$$

A computation using nothing more than the fundamental theorem of calculus shows that

$$1 - \pi^* s^* = \pm(Hd - dH).$$

This implies that the left side induces 0 on $H^*_{\mathrm{dR}}(\mathbb{R}^2)$, or equivalently $\pi^* \circ s^*$ acts like the identity on cohomology. The

$$H^*_{\text{ét}}$$

■

Chapter 4

Algebraic Topology

From my meetings with Mark. We reference Hatcher's *Algebraic Topology* [**?**] freely available here `https://www.math.cornell.edu/~hatcher/#ATI`.

4.1 Cohomology

Let's look at some examples to get an idea of what cohomolgy is all about. Take the simplest case: Let X be a 1-dimensional Δ-complex, i.e., an oriented graph. For a fixed abelian group G, the set of all functions from vertices of X to G forms an abelian group, which we denote by $\Delta^0(X;G)$ in the natural sense, i.e., by point-wise addition. Similarly the set of all functions assigning an element of G to each edge of X forms an abelian group $\Delta^1(X;G)$. We are concerned about homomorphisms $\delta\colon \Delta^0(X;G) \to \Delta^1(X;G)$ sending $\varphi \in \Delta^0$ to the function $\delta\varphi \in \Delta^1(X;G)$ whose value on an oriented edge $[v_0, v_1]$ is the difference $\varphi(v_1) - \varphi(v_0)$. For example, X here might be the graph formed by a system of trails on a mountain, with vertices at the junctions between trails. The function φ could assign to each junction its elevation above sea level, in which case $\delta\varphi$ would measure the net change in elevation along the trail from one junction to the next.

Regarding the map $\delta\colon \Delta^0(X;G) \to \Delta^1(X;G)$ as a chain complex with 0s before and after the two terms, the homology of groups of this chain complex are by definition the simplicial cohomolgy groups of X, namely $H^0(X;G) = \operatorname{Ker}\delta \subset \Delta^0(X;G)$ and $H^1(X;G) = \Delta^1(X;G)/\operatorname{Im}\delta = \operatorname{Coker}\delta$. For simplicity we are using here the same notation as will be used for singular cohomolgy; we later prove that for Δ-complexes, the two theories in fact coincide.

The group $H^0(X;G)$ is easy to describe explicitly. A function $\varphi \in \Delta^0(X;G)$ has $\delta\varphi = 0$ if and only if φ takes the same value at both ends of each edge of X. This is equivalent to saying that φ is constant on each component of X. So $H^0(X;G)$ is the group of all functions from the set of components of X to G. This is a direct product of copies of G, one for each component of X.

The cohomology group $H^1(X:G) = \Delta^1(X;G)/\operatorname{Im}\delta$ will be trivial if and only if $\delta\varphi = \psi$ has a solution $\varphi \in \Delta^0(X;G)$ for each $\varphi \in \Delta^1(X;G)$. Solving this equation means deciding whether specifying the change in φ across each edge of X determines an actual function $\varphi \in \Delta^0(X;G)$. This is rather like the calculus problem of finding a function having a specified derivative, with the difference operator δ playing the role of differentiation. As in calculus, if a solution of $\delta\varphi = \psi$ exists, it will be unique up to adding an element of the kernel of δ, i.e., a f unction constant on each component of X.

The equation $\delta\varphi = \psi$ is always solvable if X is a tree since if we choose arbitrarily a value for φ at a base point vertex v_0, then if the change in φ across each edge of X is specified, this uniquely determines the value of φ at ever other vertex v by induction along the unique path from v_0 to v in a tree. Then, since every vertex lies in one of these maximal trees, the values of ψ on the edges of the maximal trees determine φ uniquely up to a constant on each component of X. But in order for the equation $\delta\varphi = \psi$ to hold, the value of ψ on each edge is not in any of the maximal trees must equal the difference in the already-determined values of φ at the two ends of the edge. This condition need not be satisfied since ψ can have arbitrary values on these edges. Thus we see that the cohomology group $H^1(X;G)$ is a direct product of copies of the group G, one copy for each edge of X not in one of the chosen maximal trees. This can be compared with the homology group $H_1(X;G)$ which consists of a direct *sum* of copies of G, one for each edge of X not in one of the maximal trees. Note that the relation between $H^1(X;G)$ and $H^1(X;G)$ is the same as the relation between $H^0(X;G)$ and $H_0(X;G)$, with $H^0(X;G)$ being a direct product of copies of G and $H_0(X;G)$ a direct sum, with one copy for each component of X in either case.

Now let us move up a dimensino, taking X to be a 2-dimensional Δ-complex. Define $\Delta^0(X;G)$ and $\Delta^1(X;G)$ as before, as functions from vertices and edges of X to be Abelian group G, and define $\Delta^2(X;G)$ to be functions from 2-simplices of X to G, and define $\Delta^2(X;G)$ to be functions from 2-simplices of X to G. A homomorphism $\delta\colon \Delta^1(X;G) \to \Delta^2(X;G)$ is defined by $\delta\psi([v_0,v_1,v_2]) = \psi([v_0,v_1]) + \psi([v_1,v_2]) - \psi([v_0,v_2])$, a signed sum of values of ψ on the three edges in the boundary of $[v_0,v_1,v_2]$, just as $\delta\varphi([v_0,v_1])$ for $\varphi \in \Delta^0(X;G)$ we have $\delta\delta\varphi = \big(\varphi(v_1) - \varphi(v_0)\big) + \big(\varphi(v_2) - \varphi(v_2)\big) - \big(\varphi(v_2) - \varphi(v_0)\big) = 0$. Extending this chain complex by 0s on each end, the resulting homology groups are by definition the cohomology groups $H^i(X;G)$.

Chapter 5

Group Theory and Differential Equations

This is a summary of Kuga's *Galois' Dream: Group Theory and Differential Equations* book.