

MA 26500-215 Quiz 9

July 22, 2016

1. Let \mathcal{P}_2 be the set of all polynomials of degree less than or equal to 2. We define an inner product on \mathcal{P}_2 by

$$\langle p(t), q(t) \rangle = \int_{-1}^1 p(t)q(t) dt \quad (\star)$$

for polynomials $p(t), q(t) \in \mathcal{P}_2$.

- (a) (12 points) The set $\{1, t, t^2\}$ is a basis for \mathcal{P}_2 . Use the Gram–Schmidt process to find an orthogonal basis for \mathcal{P}_2 using the inner product (\star) .

Solution: There was a typo in the original. The quiz should read, an orthogonal basis of course, the whole point of the problem was to find an orthonormal basis for \mathcal{P}_2 , so as long as you did that in part (a) you received all of the points for part (b).

Following the general Gram–Schmidt process, define

$$u_1(t) = 1$$

$$\begin{aligned} u_2(t) &= t - \left[\frac{\langle 1, t \rangle}{\langle 1, 1 \rangle} \right] 1 \\ &= t - \left[\frac{\int_{-1}^1 t dt}{\int_{-1}^1 1 dt} \right] 1 \\ &= t - \left[\frac{1^2 - ((-1)^2)}{2} \right] 1 \\ &= t \end{aligned}$$

$$\begin{aligned} u_3(t) &= t^2 - \left[\frac{\langle t, t^2 \rangle}{\langle t, t \rangle} \right] t - \left[\frac{\langle 1, t^2 \rangle}{\langle 1, 1 \rangle} \right] 1 \\ &= t^2 - \left[\frac{\int_{-1}^1 t^3 dt}{\int_{-1}^1 t^2 dt} \right] t - \left[\frac{\int_{-1}^1 t^2 dt}{\int_{-1}^1 1 dt} \right] 1 \\ &= t^2 - \left[\frac{1/4(1)^4 - ((1/4)(-1)^4)}{2/3} \right] t - \left[\frac{1/3(1)^3 - ((1/3)(-1)^3)}{2} \right] 1 \\ &= t^2 - \frac{1}{3} \end{aligned}$$

- (b) (8 points) Find an orthonormal basis for \mathcal{P}_2 . [*Hint*: Use the normal basis you found in part (a).]

Solution: Using $\{u_1(t), u_2(t), u_3(t)\}$ we have

$$\begin{aligned}\frac{u_1(t)}{\|u_1(t)\|} &= \frac{1}{\sqrt{\int_{-1}^1 1 \, dt}} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}.\end{aligned}$$

$$\begin{aligned}\frac{u_2(t)}{\|u_2(t)\|} &= \frac{t}{\sqrt{\int_{-1}^1 t^2 \, dt}} \\ &= \frac{t}{\sqrt{2/3}} \\ &= \sqrt{\frac{3}{2}}t\end{aligned}$$

$$\begin{aligned}\frac{u_3(t)}{\|u_3(t)\|} &= \frac{t^2 - 1/3}{\sqrt{\int_{-1}^1 (t^2 - 1/3)(t^2 - 1/3) \, dt}} \\ &= \sqrt{\frac{45}{8}} \left(t^2 - \frac{1}{3} \right) \\ &= \frac{1}{2} \sqrt{\frac{45}{2}} \left(t^2 - \frac{1}{3} \right).\end{aligned}$$