# MA 523: Homework 8

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## Problem 8.1

Show that the function

$$u(x,t) := \sum_{k=-\infty}^{\infty} (-1)^k \Phi(x-2k,t)$$

where

$$\Phi(x,t) = \frac{\mathrm{e}^{-x^2/(4t)}}{\sqrt{4\pi t}}$$

is positive for $|x| < 1$, $t > 0$.
(*Hint:* Show that $u$ satisfies $u_t = u_{xx}$ for $t > 0$,

$$\begin{cases} u = 0 & \text{on } \{\, |x| = 1 \,\} \times \{\, t \geq 0 \,\}, \\ u = \delta_0 & \text{on } \{\, |x| = 1 \,\} \times \{\, t = 0 \,\}. \end{cases}$$

Then, carefully apply the maximum/minimum principle in a domain $\{\, |x| \leq 1 \,\} \times \{\, \varepsilon \leq t \leq T \,\}$ for small $\varepsilon > 0$ and large $T > 0$ pass to the limit as $\varepsilon \to 0^+$ and $T \to \infty$.)

*Solution.* Taking the hint, let us verify that $u_t = u_{xx}$, for $t > 0$. By direct computation, we have

$$\begin{aligned} \Phi_x(x,t) &= \frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{-x^2/(4t)}}{\sqrt{4\pi t}}\right) \\ &= -\frac{x\mathrm{e}^{-x^2/(4t)}}{2\sqrt{4\pi}t^{\frac{3}{2}}}, \end{aligned} \qquad \begin{aligned} \Phi_{xx}(x,t) &= \frac{\partial}{\partial x}\left(-\frac{x\mathrm{e}^{-x^2/(4t)}}{2\sqrt{4\pi}t^{\frac{3}{2}}}\right) \\ &= \frac{x^2\mathrm{e}^{-x^2/(4t)}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{\mathrm{e}^{-x^2/(4t)}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)\mathrm{e}^{-x^2/(4t)}}{4\sqrt{4\pi}t^{\frac{5}{2}}}, \end{aligned}$$

and

$$\begin{aligned} \Phi_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-x^2/(4t)}}{\sqrt{4\pi t}}\right) \\ &= \frac{x^2\mathrm{e}^{-x^2/(4t)}}{4\sqrt{4\pi}t^{\frac{5}{2}}} - \frac{\mathrm{e}^{-x^2/(4t)}}{2\sqrt{4\pi}t^{\frac{3}{2}}} \\ &= \frac{(x^2-2t)\mathrm{e}^{-x^2/(4t)}}{4\sqrt{4\pi}t^{\frac{5}{2}}}. \end{aligned}$$

Since $\Phi_t = \Phi_{xx}$ it follows that $u_t = u_{xx}$ (for $t > 0$).

Next we show that $u = 0$ on $\{\,|x| = 1\,\} \times \{\,t \geq 0\,\}$ and $u = \delta_0$ on $\{\,|x| = 1\,\} \times \{\,t = 0\,\}$. To show $u = 0$ fix a $t \geq 0$ and, after relabeling if necessary, assume that $x = 1$ which gives us

$$\begin{aligned} u(1,t) &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{-k^2/(4t)}}{\sqrt{4\pi t}} \\ &= \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} (-1)^k \big(\mathrm{e}^{-1/(4t)}\big)^{k^2} \\ &= \frac{2}{\sqrt{4\pi t}} \sum_{k=0}^{\infty} (-1)^k \big(\mathrm{e}^{-1/(4t)}\big)^{k^2} \\ &= \end{aligned}$$

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## Problem 8.2 (Tikhonov's example)

Let

$$g(t) := \begin{cases} \mathrm{e}^{-t^2} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then $g \in C^\infty(\mathbb{R})$ and we define

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Assuming that the series is convergent, show that $u(x,t)$ solves the heat equation in $\mathbb{R} \times (0,\infty)$ with the initial condition $u(x,0) = 0$, $x \in \mathbb{R}$. Why doesn't this contradict the uniqueness theorem for the initial value problem?

*Solution.* Let $u$ be as above. Then

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}\right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} \\ &= \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}, \end{aligned}$$

and

$$\begin{aligned} u_x(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}\right) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2k x^{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}, \end{aligned}$$

$$\begin{aligned} u_{xx}(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}\right) \\ &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-1)!} (2k-1) x^{2k-2} + \tfrac{\partial}{\partial x} g^{(0)}(t) \\ &= \sum_{k=2}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2}. \end{aligned}$$

Thus, $u_t - \Delta u = 0$; i.e., $u$ solves the heat equation. ∎

## Problem 8.3

Evaluate the integral

$$\int_{-\infty}^{\infty} \cos(ax)\mathrm{e}^{-x^2}\,dx, \qquad (a > 0).$$

(*Hint:* Use the separation of variables to find the solution of the corresponding initial-value problem for the heat equation.)

*Solution.* Write

$$\cos(ax) = \tfrac{1}{2}\big(\mathrm{e}^{\mathrm{i}ax} + \mathrm{e}^{-\mathrm{i}ax}\big).$$

Then,

$$\cos(ax)\mathrm{e}^{-x^2} = \tfrac{1}{2}\big(\mathrm{e}^{\mathrm{i}ax-x^2} + \mathrm{e}^{-\mathrm{i}ax-x^2}\big)$$

so

$$\begin{aligned}\int_{-\infty}^{\infty} \cos(ax)\mathrm{e}^{-x^2}\,dx &= \frac{1}{2}\int_{-\infty}^{\infty}\big(\mathrm{e}^{\mathrm{i}ax-x^2} + \mathrm{e}^{-\mathrm{i}ax-x^2}\big)dx\\ &=\end{aligned}$$

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