

## MA 26500-215 Quiz 4

July 20, 2016

1. (12 points) List all the properties a set  $V$  must satisfy in order to be a vector space.

(*Hint*: there are eight of them.)

**Solution:** The axioms that a *real* vector space (i.e., a vector space with scalar in  $\mathbb{R}$ , because you can have *complex* vector spaces with scalars in the complex or *imaginary* numbers  $\mathbb{C}$ ) must satisfy are

1. associativity of addition:  $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ ;
2. commutativity of addition:  $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{u}$ ;
3. additive identity: there exists some element (called a *zero*)  $\mathbf{0} \in V$  such that for any  $\mathbf{v} \in V$ ,  $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ .
4. additive inverse: for every  $\mathbf{v} \in V$  there exist an inverse element  $\mathbf{w} \in V$  such that  $\mathbf{v} + \mathbf{w} = \mathbf{0}$ .  
**Note** the vector  $\mathbf{w}$  is not always  $-\mathbf{v}$ , it just so happens that in the linear algebra we cover it will almost always be  $-\mathbf{v}$ ;
5. compatibility of scalar multiplication: if  $a, b \in \mathbb{R}$ ,  $a(b\mathbf{v}) = (ab)\mathbf{v}$  for all  $\mathbf{v} \in V$ ;
6. multiplicative identity: there exist element  $1 \in \mathbb{R}$  such that  $1\mathbf{v} = \mathbf{v}$  for all  $\mathbf{v} \in V$ ;
7. distributivity with respect to vector addition: if  $a \in \mathbb{R}$ ,  $\mathbf{v}, \mathbf{w} \in V$  then  $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$ ;
8. distributivity with respect to scalar addition: if  $a, b \in \mathbb{R}$ ,  $\mathbf{v} \in V$  then  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ .

2. Which of the following subsets  $W$  of  $\mathbb{R}^3$  are subspaces?

- (a) (2 points)  $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$ .
- (b) (2 points)  $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}$ .
- (c) (2 points)  $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2 + y^2 + z^2 = 1 \right\}$ .
- (d) (2 points)  $W = \left\{ \begin{bmatrix} x+2y+3z \\ z \\ 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$ .

(By now you should have a feel of what a vector spaces is so you do not need to check all of the conditions; but for those that are not subspaces, give me a reason, e.g., the set is not closed under addition, multiplication by scalars, etc.)

**Solution:** The set in (a) is not subspace and this is easy to see because  $[-1, 0, 1]$  is in the set, but  $-1[-1, 0, 1] = [1, 0, -1]$  is not since  $1 \not\leq 0 \not\leq -1$ .

The set in (b) is a vector space because, first, it is nonempty since it contains  $[0, 0, 0]$  and if  $c \in \mathbb{R}$ ,  $[x_1, y_1, z_1], [x_2, y_2, z_2] \in W$  then

$$c \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

and both satisfy

$$\begin{aligned} cx_1 + cy_1 + cz_1 &= c(x_1 + y_1 + z_1) & x_1 + x_2 + y_1 + y_2 + z_1 + z_2 &= (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) \\ &= 0 & &= 0. \end{aligned}$$

The set in (c) is the sphere. It is clearly not a subspace since  $[1, 0, 0], [0, 1, 0] \in W$ , but  $[1, 0, 0] + [0, 1, 0] = [1, 1, 0]$  has norm  $1^2 + 1^2 + 0^2 = 2 \neq 0$  hence  $[1, 1, 0] \notin W$ .

Lastly, the set in (d) is a subspace because, first, it is nonempty since  $[0, 0, 0] \in W$  and if  $c \in \mathbb{R}$ ,  $[x_1 + 2y_1 + 3z_1, z_1, 0], [x_2 + 2y_2 + 3z_2, z_2, 0] \in W$  then

$$\begin{aligned} c \begin{bmatrix} x_1 + 2y_1 + 3z_1 \\ z_1 \\ 0 \end{bmatrix} &= \begin{bmatrix} c(x_1 + 2y_1 + 3z_1) \\ cz_1 \\ c \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} cx_1 + 2cy_1 + 3cz_1 \\ cz_1 \\ 0 \end{bmatrix} && (\clubsuit) \\ \begin{bmatrix} x_1 + 2y_1 + 3z_1 \\ z_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 + 2y_2 + 3z_2 \\ z_2 \\ 0 \end{bmatrix} &= \begin{bmatrix} x_1 + 2y_1 + 3z_1 + x_2 + 2y_2 + 3z_2 \\ z_1 + z_2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (x_1 + x_2) + 2(y_1 + y_2) + 3(z_1 + z_2) \\ (z_1 + z_2) \\ 0 \end{bmatrix} && (\spadesuit) \end{aligned}$$

Putting  $x = cx_1, y = cy_1, z = cz_1$  for the entries in  $(\clubsuit)$ , we see that the vector in  $(\clubsuit)$  indeed belongs to  $W$  since it has the shape  $[x + 2y + 3z, z, 0]$ . Putting  $x = x_1 + x_2, y = y_1 + y_2, z = z_1 + z_2$  we see that the vector in  $(\spadesuit)$  also belongs to  $W$ . Thus,  $W$  is a subspace.