MATH 8510, Abstract Algebra I
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Exercises 10-1
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Exercise 1. Let R be a commutative ring with identity, and let $f = \sum_{i=0}^{n} r_i X^i \in R[X]$.

(a) Prove that f is nilpotent in $R[X]$ if and only if all of its coefficients $r_0, \ldots, r_n$ are nilpotent, that is, $f \in N(R[X])$ if and only if $r_0, \ldots, r_n \in N(R)$.

Proof. Since R is CRW1, $R[x]$ is CRW1.
Then by the conclusion from Exercise 10-2#2(a), $N(R[x]) \leq R[x]$.
"$\Leftarrow$". Assume $r_i \in N(R)$ for $i = 1, 2, \cdots, n$.
Since $R \subset R[x]$, we have $N(R) \subset N(R[x])$.
For $i = 1, 2, \cdots, n$, since $r_i \in N(R) \subseteq N(R[x])$, and $x^i \in R[x]$,
by the definition of the ideal, we have

$$r_i x^i \in N(R[x]).$$

Since $N(R[x])$ is closed under addition,

$$\sum_{i=1}^{n} r_i x^i \in N(R[x]).$$

"$\Rightarrow$". We will show it by induction.
Let $f = a_R \in R \subset R[x]$.
If $f \in N(R[x])$, it is obvious that the coefficient $a \in N(R)$.
Assume for any $f = \sum_{i=0}^{n} r_i x^i \in N(R[x])$, where $r_i \in R$ and $r_n \neq 0_R$, then $r_i \in N(R)$ for $i = 1, 2, \cdots, n$.
Let $g = \sum_{i=0}^{n+1} s_i x^i \in N(R[x])$, where $s_i \in R$ and $s_{n+1} \neq 0_R$.
Then $\exists\, m \in \mathbb{N}$ such that $g^m = 0$.
So

$$\begin{aligned}
0 &= g^m \\
&= \left(\sum_{i=0}^{n+1} s_i x^i\right)^m \\
&= \left(\sum_{i=0}^{n} s_i x^i + s_{n+1} x^{n+1}\right)^m \\
&= s_{n+1}^m x^{mn+m} + \sum_{j=0}^{m} \binom{m}{j} \left(\sum_{i=0}^{n} s_i x^i\right)^j \left(s_{n+1} x^{n+1}\right)^{m-j}.
\end{aligned}$$

The degree of $\sum_{j=0}^{m} \binom{m}{j} \left(\sum_{i=0}^{n} s_i x^i\right)^j \left(s_{n+1} x^{n+1}\right)^{m-j}$ is $mn + m - 1$, which is less than the degree of $s_{n+1} x^{mn+m}$.
So

$$s_{n+1}^m = 0.$$

Thus,

$$s_{n+1} \in N(R).$$

Then
$$0 = g^m = \left(\sum_{i=1}^{n} s_i x^i\right)^m \in N(R[x]).$$
So by the assumption, we have for $i = 1, 2, \cdots, n$,
$$s_i \in N(R).$$
Therefore, when $g = \sum_{i=1}^{n+1} s_i x^i \in N(R[x])$, $s_i \in N(R)$ for $i = 1, 2, \cdots, n+1$.
As a result, our assumption also holds when $g \in R[x]$ and $\deg(g) = n+1$.
Thus, for any $f = \sum_{i=1}^{n} r_i x^i \in N(R[x])$, we have $r_i \in N(R)$ for $i = 1, 2, \cdots, n$.
□

(b) Prove that f is a unit in $R[X]$ if and only if r_0 is a unit in R and $r_1, \ldots, r_n$ are nilpotent in R.

Proof. "$\Rightarrow$". Assume r_0 is a unit in R and $r_1, \cdots, r_n \in N(R)$.
Since $r_1, \cdots, r_n$ are nilpotent, by the conclusion from Exercise 11-1#1(a),
$$\sum_{i=1}^{n} r_i x^i \in N(R[x]).$$
Since $r_0 \in R^\times \subset (R[x])^\times$ by what we have shown in class,
using the conclusion from Exercise 11-1#1(a), we have
$$r_0 + \sum_{i=1} r_i x^i \in (R[x])^\times.$$
Since R is CRW1, there exists multiplicative identity 1_R, then $x^0 = 1_R$ for any $x \in R$.
So
$$r_0 = r_0 x^0.$$
By the associative law of multiplication of $R[x]$, we have
$$f = \sum_{i=0}^{n} r_i x^i \in (R[x])^\times.$$
"$\Leftarrow$". Assume $f \in (R[x])^\times$.
We will show it by induction.
Let $g = \sum_{j=0}^{p} s_j X^j$ be the non-zero multiplicative inverse for f.
We claim for $0 \le q \le p-1$, we have $r_n^{q+1} s_{p-q} = 0$.
Since $fg = \left(\sum_{i=0}^{n} r_i x^i\right)\left(\sum_{j=0}^{p} s_j x^j\right) = 1$,
$$r_n s_p = 0.$$
Then we have shown the basic case for $q = 1$ holds.
Inductive step: Consider the case $p - 1 \ge k \ge 0$ and $k \in \mathbb{N}$.
Assume for $0 \le q \le k$, we have $r_n^{q+1} s_{p-q} = 0$, where $p - 2 \ge k \ge 0$ and $k \in \mathbb{N}$.
Since $fg = \left(\sum_{i=0}^{n} r_i x^i\right)\left(\sum_{j=0}^{p} s_j x^j\right) = 1$, the coefficient of x^{n+k-q} satisfies
$$r_{n-q-1} s_{k+1} + r_{n-q} s_k + \cdots + r_{n-1} s_{k+1-q} + r_n s_{k-q} = 0.$$
Since R is CRW1, its multiplication is commutative.
Multiply by r_n^{q+1} on both sides of the above equation,
$$r_{n-q-1} r_n^{q+1} s_{k+1} + r_{n-q} r_n^{q+1} s_k + \cdots + r_{n-1} r_n^{q+1} s_{k+1-q} + r_n^{q+2} s_{k-q} = 0.$$

Namely,

$$\left(r_{n-q-1}r_n^q\right)\left(r_n s_{k+1}\right)+\left(r_{n-q}r_n^{q-1}\right)\left(r_n^2 s_k\right)+\cdots+\left(r_{n-1}r_n^0\right)\left(r_n^{q+1}s_{k+1-q}\right)+r_n^{q+2}s_{k-q}=0.$$

By assumption, for $0 \le q \le k-1$, we have $r_n^{q+1}s_{k+1-q}=0$.
Then

$$\left(r_{n-q-1}r_n^q\right)0+\left(r_{n-q}r_n^{q-1}\right)0+\cdots+\left(r_{n-1}r_n^0\right)0+r_n^{q+2}s_{k-q}=0.$$

So

$$r_n^{q+2}s_{k-q}=0.$$

Thus, for $0 \le q \le k$, we have $r_n^{q+1}s_{k+1-q}=0$.
Therefore, for $0 \le q \le p-1$, we have $r_n^{q+1}s_{p-q}=0$.
Assume $r_n^k \neq 0$ for $k=1,2,\cdots,p$, then $s_1,s_2,\cdots,s_p=0$.
Then

$$fg=\left(\sum_{i=0}^{n} r_i x^i\right)s_0=\sum_{i=0}^{n} s_0 r_i x^i=1.$$

Since $g=s_0 \neq 0$, we have $r_0=0$, which is a contradiction since $r_0 \in R$ is arbitrary.
So $\exists\ 0 \le q \le p-1$ such that $r_n^{q+1}=0$.
Thus,

$$r_n \in N(R).$$

Then by the conclusion from Exercise 11-1#1(a), $r_n x^n \in N(R[x])$.
$N(R[x])$ is a subring of $R[x]$, so $N(R[x])$ is closed under taking inverses.
So

$$-r_n x^n \in N(R[x]).$$

Since $\sum_{i=0}^{n} r_i x^i \in (R[x])^\times$, by the conclusion from Exercise 10-2#2(c),

$$\sum_{i=0}^{n} r_i x^i+(-r_n x^n)=\sum_{i=0}^{n-1} r_i x^i \in (R[x])^\times.$$

We will show the $r_i \in N(R)$ by induction for $i=1,2\cdots,n-1$.
Assume $\sum_{i=0}^{k} r_i x^i \in (R[x])^\times$, where $1 \le k \le n$.
We have just shown the basic case of $r_n \in N(R)$.
Then when $k<n$, repeat the totally same process we find $r_k \in N(R)$ and $\sum_{i=0}^{k-1} r_i x^i \in (R[x])^\times$.
So our assumption also holds for the $k-1$ case.
Thus, $r_i \in N(R)$ for $i=1,2,\cdots,n$.

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