

MATH 8510, Abstract Algebra I
 Fall 2016
 Exercises 13-1
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Exercise 1. Let R be a commutative ring with identity.

- (a) Let $A, B \subseteq R$ and set $I = (A)R$ and $J = (B)R$. Prove that $I + J$ is generated by $A \cup B$.

Proof. (1) If $A = B = \emptyset$, we have $A \cup B = \emptyset$ and $I = J = \{0\}$.
 Then

$$I + J = \{0\} = (\emptyset)R = (A \cup B)R.$$

So $I + J$ is generated by $A \cup B$.

- (2) If $A = \emptyset$ and $B \neq \emptyset$, we have $A \cup B = B$ and

$$I + J = (\emptyset)R + (B)R = \{0\} + (B)R = (B)R = (A \cup B)R.$$

So $I + J$ is generated by $A \cup B$.

- (3) Assume $A \neq \emptyset$ and $B \neq \emptyset$.
 Since $(A)R = I \leq R$ and $(B)R = J \leq R$,
 we have

$$I + J \leq R.$$

Since $A \subseteq A \cup B$,

$$I = (A)R \subseteq (A \cup B)R.$$

Similarly,

$$J \subseteq (A \cup B)R.$$

By the definition of $I + J$, we have

$$I + J \subseteq (A \cup B)R. \tag{1}$$

Since R is CRW1,

$$(A \cup B)R = \left\{ \sum_i^{\text{finite}} c_i r_i \mid c_i \in A \cup B, r_i \in R \right\}.$$

Let $x \in (A \cup B)R$, then $\exists N \in \mathbb{N}$ and $c_i \in A \cup B$ and $r_i \in R$ for $i = 1, 2, \dots, N$ such that

$$x = \sum_i^N c_i r_i.$$

Without loss of generality, assume $\exists c_i \in A$ for some integer i between 1 and N .

Rearrange $\{c_i r_i, i = 1, \dots, N\}$ such that $c_i \in A$ for $i = 1, \dots, M$ and $c_i \in B$ for $i = M + 1, \dots, N$, where $M \in \mathbb{N}$ and $1 \leq M \leq N$.

Then

$$\begin{aligned} x &= \sum_i^M c_i r_i + \sum_{i=M+1}^N c_i r_i \\ &\in (A)R + (B)R \\ &= I + J. \end{aligned}$$

So

$$(A \cup B)R \subseteq I + J. \quad (2)$$

Thus, by (1) and (2), we have

$$I + J = (A \cup B)R.$$

Therefore, $I + J$ is generated by $A \cup B$.

□

- (b) Prove that if I and J are finitely generated ideals of R , then $I + J$ is also finitely generated.

Proof. Assume the ideal I of R is finitely generated by the set $A = \{a_1, a_2, \dots, a_m\}$, where $a_1, \dots, a_m \in R$, and the ideal J of R is finitely generated by the set $B = \{b_1, b_2, \dots, b_n\}$, where $b_1, \dots, b_n \in R$.

Then

$$I = (a_1, \dots, a_m)R = (A)R$$

and

$$J = (b_1, \dots, b_n)R = (B)R.$$

By part (a), we have $I + J$ is generated by $A \cup B$.

Since $A \cup B$ is a finite set, $I + J$ is finitely generated.

□

Exercise 2. Let R be a non-zero commutative ring with identity, and let $z \in R$. Assume that z is not nilpotent. Use the following steps to prove that there is a prime ideal of R that does not contain z .

- (a) Set $\Sigma := \{I \leq R \mid 1, z, z^2, \dots \notin I\}$, partially ordered by inclusion. Prove that $\Sigma \neq \emptyset$ and that every chain in Σ has an upper bound in Σ . Use Zorn's Lemma to conclude that Σ has a maximal element K .

Proof. Let $I = \{0\}$, then $I \leq R$.

Since z is not nilpotent, $z^n \neq 0, \forall n \in \mathbb{Z}^{\geq 0}$.

So $z^n \notin I, \forall n \in \mathbb{Z}^{\geq 0}$.

Thus, $I \in \Sigma$ and then $\Sigma \neq \emptyset$.

Next we show every chain in Σ has an upper bound in Σ .

Let \mathcal{C} be a chain in Σ .

Set

$$I = \bigcup_{J \in \mathcal{C}} J.$$

Since (Σ, \subseteq) is a poset,

$$I \leq R.$$

Suppose there exists at least one $z^n \in I$ for some $n \in \mathbb{Z}^{\geq 0}$.

Then $z^n \in J$ for some $J \in \mathcal{C} \subseteq \Sigma$.

Since $J \in \Sigma$, $z^n \notin J, \forall n \in \mathbb{Z}^{\geq 0}$.

So there is a contradiction.

Then

$$z^n \notin I, \forall n \in \mathbb{Z}^{\geq 0}.$$

So

$$I \in \Sigma.$$

Also

$$\forall J \in \mathcal{C}, J \subseteq I.$$

Thus, I is an upper bound for \mathcal{C} in Σ .

By Zorn's lemma, Σ has a maximal element K . □

(b) Prove that K is prime as follows.

- (1) Suppose that $r, s \in R - K$ are such that $rs \in K$. Show that $K \subsetneq K + rR \leq R$ and $K \subsetneq K + sR \leq R$.

Proof. Since $0_R \in R$,

$$K = K + r0_R \subseteq K + rR.$$

Assume $K = K + rR$.

Since $1_R \in R$,

$$K + r = K + r1_R \subseteq K + rR = K.$$

By part (a), we already have $K \leq R$, so $r \in K$.

As a result, there is a contradiction since $r \in R - K$ by assumption.

Therefore,

$$K \subsetneq K + rR. \tag{3}$$

Since R is CRW1, $rR = (r)R \leq R$.

Also, $K \leq R$.

So

$$K + rR \leq R. \tag{4}$$

By (3) and (4),

$$K \subsetneq K + rR \leq R.$$

Similarly,

$$K \subsetneq K + sR \leq R.$$

□

- (2) Conclude that there are $m, n \in \mathbb{Z}^{\geq 0}$ such that $z^m \in K + rR$ and $z^n \in K + sR$.

Proof. Assume $z^m \notin K + rR, \forall m \in \mathbb{Z}^{\geq 0}$.

Since $K + rR \leq R$, we have

$$K + rR \in \Sigma.$$

Since K is the maximal element of Σ , $K + rR \subseteq K$.

So there is a contradiction since $K \subsetneq K + rR$.

Thus, $\exists m \in \mathbb{Z}^{\geq 0}$ such that $z^m \in K + rR$.

Similarly, $\exists n \in \mathbb{Z}^{\geq 0}$ such that $z^n \in K + sR$.

□

- (3) Deduce that $z^{m+n} \in K$, derive a contradiction, and conclude that K is prime.

Proof. Since $z^m \in K + rR$ and $z^n \in K + sR$, there exists $k_1, k_2 \in K$ and $p_1, p_2 \in R$ such that $z^m = k_1 + rp_1$ and $z^n = k_2 + sp_2$.

Since R is CRW1,

$$\begin{aligned} z^{m+n} &= (z^m)(z^n) \\ &= (k_1 + rp_1)(k_2 + sp_2) \\ &= k_1k_2 + k_1(sp_2) + k_2(rp_1) + rs(p_1p_2) \end{aligned}$$

Since $r, s, p_1, p_2 \in R$, we have

$$sp_2, rp_1, p_1p_2 \in R.$$

Since $K \leq R$ and $k_1, k_2, rs \in K$, we have

$$k_1k_2, k_1(sp_2), k_2(rp_1), rs(p_1p_2) \in K.$$

Then

$$k_1k_2 + k_1(sp_2) + k_2(rp_1) + rs(p_1p_2) \in K.$$

So for $m, n \in \mathbb{Z}^{\geq 0}$, we have

$$z^{m+n} \in K.$$

Since $K \in \Sigma$, we have $z^n \notin K, \forall n \in \mathbb{Z}^{\geq 0}$, which is contradicted by $z^{m+n} \in K$.

So our assumption does not holds.

Thus, $\forall r, s \in R - K, rs \notin K$.

Henceforth, K is prime.

As a result, there is a prime ideal K of R that does not contain z . □

Exercise 3. Let $i = \sqrt{-1} \in \mathbb{C}$, and consider the following subrings of \mathbb{C} .

$$\mathbb{Z}[i] := \{a + bi \mid a, b \in \mathbb{Z}\}$$

$$\mathbb{Q}[i] := \{a + bi \mid a, b \in \mathbb{Q}\}$$

Prove that $\mathbb{Q}[i]$ is isomorphic to the field of fractions of $\mathbb{Z}[i]$.

Proof. Let $R = \mathbb{Z}[i]$ and $S = \mathbb{Q}[i]$.

By the Theorem 7.5.9, there exists well-defined ring monomorphism

$$\begin{aligned} \rho : R &\rightarrow D^{-1}R \\ r &\rightarrow \frac{rd}{d}, \quad d \in D \end{aligned}$$

We know S is a field in Chapter 1, so S is an integral domain.

By the subring test, we have R is a subring of S .

Also $1_S = 1_R \in R$, we have R is an integral domain.

Let $D = R \setminus 0$, then $D^{-1}S$ is the field of fractions of R .

Define

$$\begin{aligned} \phi : R &\rightarrow S \\ r &\rightarrow r \end{aligned}$$

Since ϕ is an identity map from integral domain R to integral domain S , it is a ring homomorphism.

Also S is CRW1 since it is an integral domain.

Since S is a field, it is a division ring and then $S^\times = S \setminus 0$.

Since $R \subseteq S$,

$$\phi(D) = D = R \setminus 0 \subseteq S \setminus 0 = S^\times.$$

By the Universal Mapping Property, there exists a unique ring homomorphism

$$\Phi : D^{-1}R \rightarrow S$$

such that $\Phi \circ \rho = \phi$ and $\Phi(1) = 1$.

□

Exercise 4. Let R be an integral domain and consider the ring homomorphism $\psi : \mathbb{Z} \rightarrow R$ given by $\psi(n) = n \cdot 1_R$. (You do not need to show that this is a well-defined ring homomorphism.)

- (a) Prove that $\text{Ker}(\psi) = 0$ or $\text{Ker}(\psi) = p\mathbb{Z}$ for some prime number $p \in \mathbb{Z}$.
- (b) Prove that if p is a prime number such that $\text{Ker}(\psi) = p\mathbb{Z}$, then R contains a finite field as a subring.
- (c) Prove that if R is a field and $\text{Ker}(\psi) = 0$, then R has a subring $Q \cong \mathbb{Q}$.