MATH 8510, Abstract Algebra I
Fall 2016
Exercises 7
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Exercise 1 (4.4.1)**.** Let G be a group.

(a) Let $\tau \in \mathrm{Aut}(G)$, and let $\sigma_g \in \mathrm{Inn}(G)$ be conjugation by $g \in G$. Prove that $\tau\sigma_g\tau^{-1} = \sigma_{\tau(g)}$.

Proof. Since $\tau \in \mathrm{Aut}(G)$, it is a homomorphism.
Then $\forall\ x, y \in G$, $\tau(xy) = \tau(x)\tau(y)$ and $\tau(x^{-1}) = \tau(x)^{-1}$.
Then $\forall x \in G$,

$$\begin{aligned}\tau\sigma_g\tau^{-1}(x) &= \tau g\tau^{-1}(x)g^{-1}\\ &= \tau(g\tau^{-1}(x)g^{-1})\\ &= \tau(g)\tau\left(\tau^{-1}(x)\right)\tau\left(g^{-1}\right)\\ &= \tau(g)x\tau(g)^{-1}\\ &= \sigma_{\tau(g)}(x)\end{aligned}$$

Thus,

$$\tau\sigma_g\tau^{-1} = \sigma_{\tau(g)}.$$

□

(b) Prove that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

Proof. First we show $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.

(a) By the definition of $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$, we have

$$\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$$

(b) $e_G \in G$ is the identity, $\sigma_{e_G} \in \mathrm{Inn}(G)$.
So

$$\mathrm{Inn}(G) \neq \emptyset.$$

(c) Let $\sigma_f, \sigma_h \in \mathrm{Inn}(G) \subset \mathrm{Aut}(G)$.
We first compute $\sigma_k^{-1}(x)$ for $k, x \in G$.
Let $k, x \in G$, then

$$\sigma_k(x) = kxk^{-1}.$$

Then

$$x = k^{-1}\sigma_k(x)k.$$

So

$$\sigma_k^{-1}(x) = k^{-1}xk.$$

Thus,

$$\sigma_k^{-1} = \sigma_{k^{-1}}.$$

Then $\forall\ f, h, x \in G$,
we have $fh^{-1} \in G$ and

$$\begin{aligned}\sigma_f \sigma_h^{-1}(x) &= \sigma_f(h^{-1}xh) \\ &= f(h^{-1}xh)f^{-1} \\ &= (fh^{-1})x(fh^{-1})^{-1} \\ &= \sigma_{fh^{-1}}(x).\end{aligned}$$

$x \in G$ is arbitrary, so

$$\sigma_f \sigma_h^{-1} = \sigma_{fh^{-1}}.$$

Since $fh^{-1} \in G$,

$$\sigma_f \sigma_h^{-1} = \sigma_{fh^{-1}} \in \mathrm{Inn}(G).$$

Thus,

$$\mathrm{Inn}(G) \leq \mathrm{Aut}(G).$$

Moreover, in (a), we have shown that $\forall\ \tau \in \mathrm{Aut}(G)$ and $\sigma_g \in \mathrm{Inn}(G)$,
we have $\tau\sigma_g\tau^{-1} = \sigma_{\tau(g)}$.
Besides, since $\tau \in \mathrm{Aut}(G)$ and $g \in G$,

$$\tau(g) \in G.$$

So

$$\tau\sigma_g\tau^{-1} = \sigma_{\tau(g)} \in \mathrm{Inn}(G).$$

Therefore,

$$\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G).$$

$\square$

Exercise 2 (4.4.2)**.** Let G be a group.

(a) Prove that if G is abelian of order pq where p and q are distinct primes, then G is cyclic.

Proof. $|G| = pq$ and p, q are primes, then by Cauchy's theorem,
there exists $x, y \in G$ such that $|x| = p$ and $|y| = q$.
Then $x^p = e_G$ and $y^q = e_G$.
Since G is abelian,

$$(xy)^{pq} = (x^p)^q(y^q)^p = (e_G)^p(e_G)^q = e_G.$$

So

$$|xy| \mid pq.$$

Then

$$|xy| = 1 \text{ or } p \text{ or } q \text{ or } pq.$$

(a) If $|xy| = 1$, then $xy = e_G$, so $x^{-1} = y$.
Let $g \in \langle y \rangle$, then there exists $n \in \mathbb{N}$ such that $g = y^n = x^{-n}$.
So

$$\langle y \rangle \subset \langle x \rangle.$$

Similarly, we have

$$\langle x \rangle \subset \langle y \rangle.$$

Thus,

$$\langle x \rangle = \langle y \rangle.$$

Then

$$p = q,$$

which is a contradiction since $p,\ q$ are distinct primes.
Therefore, $|xy| \neq 1$.

(b) If $|xy| = p$, then given G is abelian,

$$(xy)^p = x^p y^p = y^p = e_G.$$

Since $|y| = q$,

$$q \mid p.$$

which is a contradiction since p, q are arbitrary primes.
Therefore, $|xy| \neq p$.

(c) Similarly, we have $|xy| \neq q$.

Hence,

$$|xy| = pq.$$

As a result, G is cyclic. □

(b) Prove that if $|G| = 15$, then $G \cong \mathbb{Z}/15\mathbb{Z}$.

Proof. Since $|G| = 3 \times 5$ and $3 \nmid (5-1)$, by what we have shown in class, we have G is abelian.
Then by part (a), 3 and 5 are primes, so we also have G is cyclic.
Since $|G| = 15 < \infty$,

$$G \cong \mathbb{Z}/15\mathbb{Z}.$$

□

Exercise 3 (4.5.22)**.** Prove that if G is a group with $|G| = 132$, then G is not simple.

Proof. Let $P \in Syl_{11}(G)$, $Q \in Syl_3(G)$ and $R \in Syl_2(G)$.
Since $132 = 2^2 \times 3 \times 5$, by Sylow Theorem 4.5.1,
we have$|P| = 11, |Q| = 3$ and $|R| = 4$.
Suppose $P \ntrianglelefteq G$ and $Q \ntrianglelefteq G$ and $R \ntrianglelefteq G$.
Then $n_{11} \neq 1$ and $n_3 \neq 1$ and $n_2 \neq 1$.

(1) By Sylow Theorem 4.5.4, we have $n_{11} \equiv (1 \mod 11)$ and $n_{11} \mid 12$.
$n_{11} \mid 12$, so $n_{11} = 1$ or 2 or 3 or 4 or 6 or 12.
Besides, $n_{11} \equiv (1 \mod 11)$, so $n_{11} = 1$ or 12.
We already know $n_{11} \neq 1$.
So $n_{11} = 12$.
Then by Lemma 4.5.7, we have

$$|\{x \in G : |x| = 11\}| = 12 \times (11 - 1) = 120.$$

(2) By Sylow Theorem 4.5.4, we have $n_3 \equiv (1 \mod 3)$ and $n_3 \mid 44$.
$n_{11} \mid 44$, so $n_3 = 1$ or 2 or 22 or 4 or 11 or 44.
Besides, $n_3 \equiv (1 \mod 3)$, so $n_3 = 1$ or 4.
We already know $n_3 \neq 1$.
So $n_3 = 4$.
Then by Lemma 4.5.7, we have

$$|\{x \in G : |x| = 3\}| = 4 \times (3 - 1) = 8.$$

(3) Since $n_2 \neq 1$ and $n_2 \in \mathbb{N}$, we have $n_2 \geq 2$.
Then, for distinct $P, P^{'} \in Syl_2(G)$, $P \cap P^{'}$ might be non-trivial.

$$|\{x \in G : |x| = 2\}| \geq |P \setminus \{e\}| + 1 = 3 + 1 = 4.$$

Since $|e| = 1$,

$$|G| \geq 120 + 8 + 4 + 1 = 133,$$

which is a contradiction since $|G| = 132$.
Thus, either P or Q or R is a normal subgroup of G.
We already show

$$1 < |P|, |Q|, |R| < |G|.$$

So either P or Q or R is a non-trivial normal subgroup of G.
So G is not simple.

□

Exercise 4 (4.5.24)**.** Prove that if G is a group with $|G| = 231$, then $Z(G)$ contains a Sylow 11-subgroup of G.

Proof. $|G| = 3 \times 7 \times 11$, and $n_{11} = |Syl_{11}(G)|$.
Then by Sylow Theorem 4.5.4, we have $n_{11} \equiv (1 \mod 11)$ and $n_{11}|21$.
$n_{11}|21$, so $n_{11} = 1$ or 3 or 7 or 21.
Besides, $n_{11} \equiv (1 \mod 11)$, so $n_{11} = 1$.
Let $H \in Syl_{11}(G)$, then by Sylow Theorem 4.5.4, we have $|H| = 11$ and by Corollary 4.5.5, we have $H \trianglelefteq G$.
11 is a prime, so H is cyclic according to what we have shown in class.
Since $C_G(H) \leq G$ and $H \leq G$, and $H \subset C_G(H)$,

$$H \leq C_G(H).$$

So

$$|H| \mid |C_G(H)|.$$

Since $|H| = 11 = |G|/21$,

$$(|G|/21) \mid |C_G(H)|.$$

Then

$$|G| \mid (21 C_G(H)|).$$

Then

$$(|G|/|C_G(H)|) \mid 21.$$

Moreover, since $H \trianglelefteq G$, by Proposition 4.4.2(d), we have $G/C_G(H)$ is isomorphic to a subgroup of $\operatorname{Aut}(H)$.
So

$$(|G|/|C_G(H)|) \mid |\operatorname{Aut}(H)|.$$

Since H is a cyclic group of order 11, $H \cong \mathbb{Z}/11\mathbb{Z}$.
Then

$$\operatorname{Aut}(H) \cong \operatorname{Aut}(\mathbb{Z}/11\mathbb{Z}).$$

Then by Proposition 4.4.5,

$$|\operatorname{Aut}(H)| = \varphi(11) = 10,$$

where φ is $Euler\varphi\text{-}function$.
So

$$(|G|/|C_G(H)|) \mid 10.$$

The common factor(s) of 10 and 21 is just 1.
So

$$|G|/|C_G(H)| = 1.$$

Therefore,

$$C_G(H) = G.$$

Hence,

$$H \subset Z(G).$$

As a result, we have $Z(G)$ contains a Sylow 11-subgroup of G. $\square$