MATH 8510, Abstract Algebra I
Fall 2016
Exercises 9-2
Due date Thu 27 Oct 4:00PM

Exercise 1 (6.1.1)**.** Let G be a group. Prove that $Z_i(G)$ is a characteristic subgroup of G for all i.

Proof. We will show it by induction.
Basic steps:
Let $\sigma \in \mathrm{Aut}(G)$.
$Z_0(G) = \{e_G\}$.
Since $\sigma \in \mathrm{Aut}(G)$, $\sigma(e_G) = e_G$.
So

$$\sigma(Z_0(G)) = \sigma(Z_0(G)).$$

Thus, $Z_0(G)$ is a characteristic subgroup of G.
$Z_1(G) = Z(G)$.
Since $\sigma \in \mathrm{Aut}(G)$, $\sigma(g^{-1}) = (\sigma(g))^{-1} \in G, \forall\ g \in G$.
Let $z \in Z(G)$
Then $z\sigma(g^{-1}) = \sigma(g^{-1})z, \forall\ g \in G$.
Then

$$\sigma(z\sigma(g^{-1})) = \sigma(\sigma(g^{-1})z), \forall\ g \in G$$

Since $\sigma \in \mathrm{Aut}(G)$, we have

$$\sigma(z)\sigma\left(\sigma(g^{-1})\right) = \sigma\left(\sigma(g^{-1})\right)\sigma(z), \forall\ g \in G.$$

Namely,

$$\sigma(z)g = g\sigma(z), \forall\ g \in G.$$

Then

$$\sigma(z) \in Z(G).$$

So

$$\sigma(Z(G)) \subset Z(G).$$

By Theorem 4.4.8, $Z(G)$ is a characteristic subgroup of G.
Induction steps:
Assume $Z_i(G)$ is a characteristic subgroup of G.
Let $\sigma \in \mathrm{Aut}(G)$.
Then $\sigma(Z_i(G)) = Z_i(G)$.
Define

$$\begin{aligned} \overline{\sigma} : G/Z_i(G) &\to G/Z_i(G) \\ gZ_i(G) &\to \sigma(g)Z_i(G) \end{aligned}$$

Then $\overline{\sigma}$ is well-defined since $Z_i(G) \trianglelefteq G$ and $\sigma : G \to G$ is a isomomorphism.
Next we show $\overline{\sigma}$ is an automomorphism.
Let $gZ_i(G), hZ_i(G) \in G/Z_i(G)$, where $g, h \in G$.

Since $Z_i(G) \trianglelefteq G$,

$$\begin{aligned}\overline{\sigma}(gZ_i(G)hZ_i(G)) &= \overline{\sigma}(ghZ_i(G)) \\ &= \sigma(gh)Z_i(G) \\ &= \sigma(g)\sigma(h)Z_i(G) \\ &= \sigma(g)Z_i(G)\sigma(h)Z_i(G) \\ &= \overline{\sigma}(gZ_i(G))\overline{\sigma}(hZ_i(G)).\end{aligned}$$

So $\overline{\sigma}$ is a homomorphism.
Since σ is a homomorphism, and $Z_i(G)$ is a characteristic subgroup of G.

$$\begin{aligned}gZ_i(G) \in \mathrm{Ker}\,(\overline{\sigma}) &\Leftrightarrow \overline{\sigma}(gZ_i(G)) = Z_i(G) \\ &\Leftrightarrow \sigma(g)Z_i(G) = Z_i(G) \\ &\Leftrightarrow \sigma(g) \in Z_i(G) \\ &\Leftrightarrow \sigma^{-1}(\sigma(g)) \in Z_i(G) \\ &\Leftrightarrow g \in Z_i(G).\end{aligned}$$

So

$$\mathrm{Ker}(\overline{\sigma}) = Z_i(G).$$

So $\overline{\sigma}$ is 1-1.
Let $kZ_i(G) \in G/Z_i(G)$, where $k \in G$.
We know $\sigma^{-1}(k) \in G$ since $\sigma^{-1} \in \mathrm{Aut}(G)$.
Then

$$\sigma^{-1}(k)Z_i(G) \in G/Z_i(G).$$

Since

$$\overline{\sigma}\left(\sigma^{-1}(k)Z_i(G)\right) = \sigma(\sigma^{-1}(k))Z_i(G) = kZ_i(G),$$

$\overline{\sigma}$ is onto.
Thus,

$$\overline{\sigma} \in \mathrm{Aut}\left(G/Z_i(G)\right).$$

Since in basic steps we have shown the center of a group is characteristic subgroup of the group,

$$\overline{\sigma}\left(Z\left(G/Z_i(G)\right)\right) = Z\left(G/Z_i(G)\right).$$

By definition, we have

$$Z\left(G/Z_i(G)\right) = Z_{i+1}(G)/Z_i(G).$$

So

$$\overline{\sigma}\left(Z_{i+1}(G)/Z_i(G)\right) = Z_{i+1}(G)/Z_i(G).$$

Let $z \in Z_{i+1}(G)$, then

$$zZ_i(G) \in Z_{i+1}(G)/Z_i(G).$$

Then

$$\overline{\sigma}(zZ_i(G)) \in Z_{i+1}(G)/Z_i(G).$$

Namely,

$$\sigma(z)Z_i(G) \in Z_{i+1}(G)/Z_i(G).$$

So

$$\sigma(z) \in Z_{i+1}(G).$$

Thus,

$$\sigma\left(Z_{i+1}(G)\right) \subset Z_{i+1}(G).$$

By Theorem 4.4.8, $Z_{i+1}(G)$ is a characteristic subgroup of G.
So the assumption also holds for $Z_{i+1}(G)$.
Thus, we conclude that $Z_i(G)$ is a characteristic subgroup of G for each i. □

Exercise 2 (6.1.6)**.** Let G be a group. Prove that $G/Z(G)$ is nilpotent if and only if G is nilpotent.

Proof. We first find the relationship between $(G/Z(G))^n$ and G^n.
$(G/Z(G))^0 = (G/Z(G)$.
Since $Z(G) \trianglelefteq G$,

$$\begin{aligned}
(G/Z(G))^1 &= [G/Z(G), (G/Z(G))^0] \\
&= [G/Z(G), G/Z(G)] \\
&= \langle [hZ(G), kZ(G)] | hZ(G), kZ(G) \in G/Z(G) \rangle \\
&= \langle (hZ(G))^{-1}(kZ(G))^{-1}(hZ(G))(kZ(G)) | hZ(G), kZ(G) \in G/Z(G) \rangle \\
&= \langle (h^{-1}Z(G))(k^{-1}Z(G))(hZ(G))(kZ(G)) | hZ(G), kZ(G) \in G/Z(G) \rangle \\
&= \langle h^{-1}k^{-1}hkZ(G) | h, k \in G \rangle. \\
&= [G, G]/Z(G) \\
&= G^1/Z(G)
\end{aligned}$$

Then

$$\begin{aligned}
(G/Z(G))^2 &= [G/Z(G), (G/Z(G))^1] \\
&= [G/Z(G), G^1Z(G)] \\
&= \left\langle [hZ(G), kZ(G)] | hZ(G) \in G/Z(G), kZ(G) \in G^1/Z(G) \right\rangle \\
&= \langle h^{-1}k^{-1}hkZ(G) | h \in G, k \in G^1 \rangle. \\
&= [G, G^1]/Z(G). \\
&= G^2/Z(G).
\end{aligned}$$

We guess $(G/Z(G))^n = G^n/Z(G)$.
We will show it by induction.
We have shown the basic steps.
Induction steps:
Assume $(G/Z(G))^n = G^n/Z(G)$.

$$\begin{aligned}
(G/Z(G))^{n+1} &= [G/Z(G), (G/Z(G))^n] \\
&= [G/Z(G), G^n/Z(G)] \\
&= \langle [hZ(G), kZ(G)] | hZ(G) \in G/Z(G), kZ(G) \in G^n/Z(G) \rangle \\
&= \langle h^{-1}k^{-1}hkZ(G) | h \in G, k \in G^n \rangle. \\
&= [G, G^n]/Z(G). \\
&= G^{n+1}/Z(G).
\end{aligned}$$

So the assumption holds for the $n + 1$ case.
Thus, for $n \in \mathbb{N}$,

$$(G/Z(G))^n = G^n/Z(G).$$

"$\Leftarrow$". Assume G is nilpotent.
Then $G^m = e_G$ for some $m \geq 0$.
So
$$(G/Z(G))^m = G^m/Z(G) = Z(G).$$
So $G/Z(G)$ is nilpotent.
"$\Rightarrow$". Assume $G/Z(G)$ is nilpotent.
Then $(G/Z(G))^n = Z_G$ for some $n \geq 0$.
So
$$G^n/Z(G) = (G/Z(G))^n = Z(G).$$
As a result, we have
$$G^n \subset Z(G).$$
Therefore,
$$\begin{aligned}
G^{n+1} &= [G, G^n] \\
&= \langle [h,k] | h \in G, k \in G^n \rangle \\
&= \left\langle h^{-1}k^{-1}hk | h \in G, k \in G^n \right\rangle \\
&= \left\langle h^{-1}k^{-1}kh | h \in G, k \in G^n \right\rangle \\
&= \langle e_G | h \in G, k \in G^n \rangle \\
&= \langle e_G \rangle \\
&= e_G,
\end{aligned}$$
since $G^n \subset Z(G)$.
Thus, G is nilpotent. $\square$