MATH 8510, Abstract Algebra I
Fall 2016
Exercises 10-2
Due date Thu 03 Nov 4:00PM

Exercise 1 (Binomial Theorem)**.** Let R be a commutative ring with identity. Prove that for all $a, b \in R$ and for all integers $n \geq 1$, we have $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

Proof. We will show it by inducition.
Basic step:
When $n = 1$, since $(R, +)$ is an abelian group and R has multiplicative identity 1_R.

$$\begin{aligned}\binom{1}{0}a^0b + \binom{1}{1}a^1b^0 &= a^0b + a^1b^0 \\ &= 1_Rb + a1_R \\ &= b + a \\ &= a + b.\end{aligned}$$

Inducitve step:
Assume $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.
Then by the distributive law and the commutative and associative law of addition of R,

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n \\ &= (a+b)\sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i} \\ &= a\sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i} + b\sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i} \\ &= \sum_{i=0}^{n}\binom{n}{i}a^{i+1}b^{n-i} + \sum_{i=0}^{n}\binom{n}{i}a^ib^{n+1-i} \\ &= \sum_{i=0}^{n-1}\binom{n}{i}a^{i+1}b^{n-i} + \binom{n}{n}a^{n+1}b^0 + \binom{n}{0}a^0b^{n+1} + \sum_{i=1}^{n}\binom{n}{i}a^ib^{n+1-i} \\ &= \binom{n}{0}a^0b^{n+1} + \sum_{i=0}^{n-1}\binom{n}{i}a^{i+1}b^{n-i} + \sum_{i=1}^{n}\binom{n}{i}a^ib^{n+1-i} + \binom{n}{n}a^{n+1}b^0 \\ &= \binom{n}{0}a^0b^{n+1} + \sum_{i=1}^{n}\binom{n}{i-1}a^ib^{n+1-i} + \sum_{i=1}^{n}\binom{n}{i}a^ib^{n+1-i} + \binom{n}{n}a^{n+1}b^0 \\ &= \binom{n+1}{0}a^0b^{n+1} + \sum_{i=1}^{n}\left(\binom{n}{i-1} + \binom{n}{i}\right)a^ib^{n+1-i} + \binom{n+1}{n+1}a^{n+1}b^0 \\ &= \binom{n+1}{0}a^0b^{n+1} + \sum_{i=1}^{n}\binom{n+1}{i}a^ib^{n+1-i} + \binom{n+1}{n+1}a^{n+1}b^0 \\ &= \sum_{i=0}^{n+1}\binom{n+1}{i}a^ib^{n+1-i}.\end{aligned}$$

So our assumption also holds for the $n+1$ case.
Thus, we have $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

□

Exercise 2 (7.1.14)**.** Let R be a commutative ring with identity. An element $x \in R$ is *nilpotent* if there is an integer $n \geq 1$ such that $x^n = 0$. The *nilradical* of R is the set $N(R) = \{x \in R \mid x \text{ is nilpotent}\}$.

(a) Prove that $N(R)$ is a (two-sided) ideal of R, that is, $N(R)$ is a subring of R such that for all $x \in N(R)$ and all $r \in R$ we have $rx, xr \in N(R)$.

Proof. First we show $N(R)$ is a subring of R.
Since $0 \in R$ and $0^1 = 0$, we have $0 \in N(R)$.
So $N(R) \neq \emptyset$.
Let $x, y \in N(R)$, then $x, y \in R$ and $\exists\ m, n \in \mathbb{N}$ such that $x^m = y^n = 0$.
Besides, $(x+y)^{m+n} \in R$.
Since R is CRW1, its multiplication is commutative.
By the distributive law and the addition associative law of R,

$$\begin{aligned}
(x+y)^{m+n} &= \sum_{i=0}^{m+n} \binom{m+n}{i} x^i y^{m+n-i} \\
&= \sum_{i=0}^{m} \binom{m+n}{i} x^i y^{m+n-i} + \sum_{i=m+1}^{m+n} \binom{m+n}{i} x^i y^{m+n-i} \\
&= y^n \sum_{i=0}^{m} \binom{m+n}{i} x^i y^{m-i} + x^m \sum_{i=m+1}^{m+n} \binom{m+n}{i} x^{i-m} y^{m+n-i} \\
&= 0 \sum_{i=0}^{m} \binom{m+n}{i} x^i y^{m-i} + 0 \sum_{i=m+1}^{m+n} \binom{m+n}{i} x^{i-m} y^{m+n-i} \\
&= 0.
\end{aligned}$$

Then $x+y \in N(R)$ since $m+n \in \mathbb{N}$.
So $N(R)$ is closed under addition.
By the commutative law and associative law of multiplication, we have

$$(xy)^m = (x^m)y^m = 0y^m = 0.$$

Then $xy \in N(R)$.
So $N(R)$ is closed under multiplication.

$$(-x)^m = (-1)^m x^m = (-1)^m 0 = 0,$$

Then $-x \in N(R)$.
So $N(R)$ is closed under taking additive inverses.
Thus, $N(R)$ is a subring of R.
$\forall\ x \in N(R)$ and $\forall\ r \in R$, assume $x^n = 0$ for some $n \in \mathbb{N}$.
Then since the multiplication of R is commutative,

$$(xr)^n = x^n r^n = 0r^n = 0,$$

and

$$(rx)^n = r^n x^n = r^n 0 = 0.$$

So

$$rx, xr \in N(R).$$

Therefore, $N(R)$ is a (two-sided) ideal of R.

□

(b) Prove that for all $x \in N(R)$, the element $1+x$ is a unit of R, that is, $1+x \in R^{\times}$.

Proof. $\forall\ x \in N(R)$, assume $x^n = 0$ for some $n \in \mathbb{N}$.
Then $1 + x \in R$ and $\sum_{i=0}^{n-1}(-1)^i x^i \in R$.
By the distributive law and the associative law of addition of R,

$$\begin{aligned}
(1+x)\sum_{i=0}^{n-1}(-1)^i x^i &= \sum_{i=0}^{n-1}(-1)^i x^i + \sum_{i=0}^{n-1}(-1)^i x^{i+1} \\
&= 1 + \sum_{i=1}^{n-1}(-1)^i x^i - \sum_{i=0}^{n-1}(-1)^{i+1} x^{i+1} \\
&= 1 + \sum_{i=1}^{n-1}(-1)^i x^i - \sum_{i=1}^{n-1}(-1)^i x^i \\
&= 1 + \sum_{i=1}^{n-1}\left((-1)^i x^i - (-1)^i x^i\right) \\
&= 1 + \sum_{i=1}^{n-1} 0 \\
&= 1,
\end{aligned}$$

So $1 + x \in R^{\times}$. □

(c) Prove that for all $x \in N(R)$ and for all $u \in R^{\times}$, we have $u + x \in R^{\times}$.

Proof. For all $x \in N(R)$ and for all $u \in R^{\times}$, there exists some $n \in \mathbb{N}$ such that $x^n = 0$.
By the commutative and associative law of multiplication, we have

$$\left(u^{-1}x\right)^n = u^{-n}x^n = u^{-n}0 = 0.$$

So

$$u^{-1}x \in N(R).$$

By the conclusion from part (b), we have

$$1 + u^{-1}x \in R^{\times}.$$

Since $u \in R^{\times}$, by the distributive law of R,

$$u + x = u + uu^{-1}x = u\left(1 + u^{-1}x\right),$$

and by the associative law of multiplication of R,

$$\begin{aligned}
\left(u\left(1+u^{-1}x\right)\right)\left(\left(1+u^{-1}x\right)^{-1}u^{-1}\right) &= u\left(\left(1+u^{-1}x\right)\left(1+u^{-1}x\right)^{-1}\right)u^{-1} \\
&= u1u^{-1} \\
&= uu^{-1} \\
&= 1,
\end{aligned}$$

we have

$$(u+x)\left(\left(1+u^{-1}\right)^{-1}u^{-1}\right)=1.$$

Thus,

$$u+x\in R^{\times}.$$

$\square$