MATH 8510, Abstract Algebra I
Fall 2016
Exercises 10-1
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Exercise 1 (6.1.14)**.** Let G be a group. Prove that G^i is a characteristic subgroup of G for all i.

Proof. Let $\sigma \in \mathrm{Aut}(G)$.
We will show it by induction.
Basic steps:
$G^0 = G$, $\sigma(G) = G$ as $\sigma \in \mathrm{Aut}(G)$.
So G^0 is a characteristic subgroup of G.
Inductive steps:
Assume G^n is a characteristic subgroup of G for $n \geq 2$ and $n \in \mathbb{N}$.
Then $\sigma(G^n) = G^n$.
Since

$$G^{n+1} = \langle h^{-1}k^{-1}hk | h \in G, k \in G^n \rangle.$$

let $g \in G^{n+1}$, then by the definition of the span $\langle \cdot \rangle$, $\exists\, n \in \mathbb{N}$ and $m_1, m_2, \cdots, m_n \in \mathbb{N}$ and $h_1, h_2, \cdots, h_n \in G$ and $k_1, k_2, \cdots, k_n \in G^n$ such that

$$g = \left(h_1^{-1}k_1^{-1}h_1k_1\right)^{m_1}\left(h_2^{-1}k_2^{-1}h_2k_2\right)^{m_2}\cdots\left(h_n^{-1}k_n^{-1}h_nk_n\right)^{m_n}.$$

Since $\sigma \in \mathrm{Aut}(G)$,

$$\begin{aligned}
\sigma(g) &= \sigma\left(\left(h_1^{-1}k_1^{-1}h_1k_1\right)^{m_1}\left(h_2^{-1}k_2^{-1}h_2k_2\right)^{m_2}\cdots\left(h_n^{-1}k_n^{-1}h_nk_n\right)^{m_n}\right)\\
&= \sigma\left(\left(h_1^{-1}k_1^{-1}h_1k_1\right)^{m_1}\right)\sigma\left(\left(h_2^{-1}k_2^{-1}h_2k_2\right)^{m_2}\right)\cdots\sigma\left(\left(h_n^{-1}k_n^{-1}h_nk_n\right)^{m_n}\right)\\
&= \left(\sigma\left(h_1^{-1}k_1^{-1}h_1k_1\right)\right)^{m_1}\left(\sigma\left(h_2^{-1}k_2^{-1}h_2k_2\right)\right)^{m_2}\cdots\left(\sigma\left(h_n^{-1}k_n^{-1}h_nk_n\right)\right)^{m_n}.
\end{aligned}$$

For $i = 1, 2, \cdots, n$, $h_i \in G$ and $k_i \in G^n$, since $\sigma(G^n) = G^n$ by assumption and $\sigma(G) = G$, we have $\sigma(h_i) \in G$ and $\sigma(k_i) \in G^n$.
For $i = 1, 2, \cdots, n$, let $g_i = \sigma\left(h_i^{-1}k_i^{-1}h_ik_i\right)$, since $\sigma \in \mathrm{Aut}(G)$,

$$\begin{aligned}
g_i &= \sigma({h_i}^{-1})\sigma({k_i}^{-1})\sigma(h_i)\sigma(k_i)\\
&= (\sigma(h_i))^{-1}(\sigma(k_i))^{-1}\sigma(h_i)\sigma(k_i)\\
&\in G^{n+1}.
\end{aligned}$$

So

$$\sigma(g) = g_1^{m_1}g_2^{m_2}\cdots g_n^{m_n} \in G^{n+1}.$$

As a result, $\sigma(G^{n+1}) \subset G^{n+1}$.
So G^{n+1} is a characteristic subgroup of G.
It implies our assumption also holds for the $n+1$ case.
Thus, G^i is characteristic subgroup of G for all i. □

Exercise 2 (6.1.17)**.** Let G be a group. Prove that $G^{(i)}$ is a characteristic subgroup of G for all i.

Proof. Let $\sigma \in \mathrm{Aut}(G)$.
We will show it by induction.
Basic steps:
$G^{(0)} = G$, $\sigma(G) = G$ as $\sigma \in \mathrm{Aut}(G)$.
So $G^{(0)}$ is a characteristic subgroup of G.
Inductive steps:
Assume $G^{(n)}$ is a characteristic subgroup of G for $n \geq 2$ and $n \in \mathbb{N}$.
Then $\sigma(G^{(n)}) = G^{(n)}$.
Since

$$G^{(n+1)} = \langle h^{-1}k^{-1}hk | h \in G^{(n)}, k \in G^{(n)} \rangle.$$

let $g \in G^{(n+1)}$, then by the definition of the span $\langle \cdot \rangle$, $\exists\, n \in \mathbb{N}$ and $m_1, m_2, \cdots, m_n \in \mathbb{N}$ and $h_1, h_2, \cdots, h_n \in G^{(n)}$ and $k_1, k_2, \cdots, k_n \in G^{(n)}$ such that

$$g = \left(h_1^{-1}k_1^{-1}h_1k_1\right)^{m_1} \left(h_2^{-1}k_2^{-1}h_2k_2\right)^{m_2} \cdots \left(h_n^{-1}k_n^{-1}h_nk_n\right)^{m_n}.$$

Since $\sigma \in \mathrm{Aut}(G)$,

$$\begin{aligned}
\sigma(g) &= \sigma\left(\left(h_1^{-1}k_1^{-1}h_1k_1\right)^{m_1} \left(h_2^{-1}k_2^{-1}h_2k_2\right)^{m_2} \cdots \left(h_n^{-1}k_n^{-1}h_nk_n\right)^{m_n}\right) \\
&= \sigma\left(\left(h_1^{-1}k_1^{-1}h_1k_1\right)^{m_1}\right) \sigma\left(\left(h_2^{-1}k_2^{-1}h_2k_2\right)^{m_2}\right) \cdots \sigma\left(\left(h_n^{-1}k_n^{-1}h_nk_n\right)^{m_n}\right) \\
&= \left(\sigma\left(h_1^{-1}k_1^{-1}h_1k_1\right)\right)^{m_1} \left(\sigma\left(h_2^{-1}k_2^{-1}h_2k_2\right)\right)^{m_2} \cdots \left(\sigma\left(h_n^{-1}k_n^{-1}h_nk_n\right)\right)^{m_n}.
\end{aligned}$$

For $i = 1, 2, \cdots, n$, $h_i \in G^{(n)}$ and $k_i \in G^{(n)}$, since $\sigma(G^n) = G^{(n)}$ by assumption, we have $\sigma(h_i) \in G^{(n)}$ and $\sigma(k_i) \in G^{(n)}$.
For $i = 1, 2, \cdots, n$, let $g_i = \sigma\left(h_i^{-1}k_i^{-1}h_ik_i\right)$, since $\sigma \in \mathrm{Aut}(G)$,

$$\begin{aligned}
g_i &= \sigma({h_i}^{-1})\sigma({k_i}^{-1})\sigma(h_i)\sigma(k_i) \\
&= \left(\sigma(h_i)\right)^{-1} \left(\sigma(k_i)\right)^{-1} \sigma(h_i)\sigma(k_i) \\
&\in G^{(n+1)}.
\end{aligned}$$

So

$$\sigma(g) = g_1^{m_1} g_2^{m_2} \cdots g_n^{m_n} \in G^{n+1}.$$

As a result, $\sigma(G^{(n+1)}) \subset G^{(n+1)}$.
So $G^{(n+1)}$ is a characteristic subgroup of G.
It implies our assumption also holds for the $n+1$ case.
Thus, $G^{(i)}$ is characteristic subgroup of G for all i. $\square$