MATH 8510, Abstract Algebra I
Fall 2016
Exercises 1
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Exercise 1. Prove that $\mathbb{Q}(\sqrt{2}) := \left\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\right\}$ is a field under the usual addition and multiplication of $\mathbb{R}$.

Proof. $\forall a + b\sqrt{2}, c + d\sqrt{2}, e + f\sqrt{2} \in \mathbb{Q}(\sqrt{2})$,we have $a, b, c, d, e, f \in \mathbb{Q}$.

(a)

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

since $a + c, b + d \in \mathbb{Q}$.

$$(a + b\sqrt{2})(c + d\sqrt{2}) = a(c + d\sqrt{2}) + b\sqrt{2}(c + d\sqrt{2}),$$

where we use the distributive law of the field $\mathbb{R}$ since $a, b\sqrt{2}, c, d\sqrt{2} \in \mathbb{R}$. Besides, $b\sqrt{2}c = bc\sqrt{2}$ and $(b\sqrt{2})(d\sqrt{2}) = (bd)(\sqrt{2}\sqrt{2}) = bd(2) = 2bd$.
Then

$$\begin{aligned}(a + b\sqrt{2})(c + d\sqrt{2}) &= (ac + ad\sqrt{2}) + (bc\sqrt{2} + 2bd)\\ &= (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Q}(\sqrt{2})\end{aligned}$$

where we use the multiplication associative,commutative and distribution law of the field $\mathbb{R}$.

(b) The commutative law and ssociative law of '+' inherite from $\mathbb{R}$.

$$0_{\mathbb{Q}(\sqrt{2})} = 0_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} = 0_{\mathbb{R}}.$$

$$(0 + 0\sqrt{2}) + (a + b\sqrt{2}) = (0 + a) + (0 + b)\sqrt{2} = a + b\sqrt{2}.$$

$$-(a + b\sqrt{2}) = (-a) + (-b)\sqrt{2} = -a - b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

since $-a, -b \in \mathbb{Q}$.
Check

$$(a + b\sqrt{2}) + (-a - b\sqrt{2}) = (a - a) + (b - b)\sqrt{2} = 0_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} = 0_{\mathbb{Q}(\sqrt{2})} = 0.$$

(c)

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$$

by previous steps .
The commutative law and ssociative law of '$\cdot$' inherite from $\mathbb{R}$.

$$1_{\mathbb{Q}(\sqrt{2})} = 1_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} = 1 + 0\sqrt{2} = 1_{\mathbb{R}}.$$

$$(1 + 0\sqrt{2})(a + b\sqrt{2}) = (a + b\sqrt{2}) + (0a\sqrt{2} + 0(2b)) = a + b\sqrt{2}$$

where we use the multiplication associative, commutative and distributive law of field $\mathbb{R}$. Consider $a + b\sqrt{2} \neq 0$, then $a \neq -b\sqrt{2}$.

(1) If $b = 0$, then $a \neq 0$,

$$(a + b\sqrt{2})^{-1} = \frac{1}{a} = \frac{1}{a} + 0\sqrt{2} \in \mathbb{Q}(\sqrt{2}).$$

(2) If $a = b\sqrt{2} \neq 0$, then $b \neq 0$,

$$\begin{aligned}(a+b\sqrt{2})^{-1} &= (2b\sqrt{2})^{-1} \\ &= \frac{1}{4b}\sqrt{2} = 0 + \frac{1}{4b}\sqrt{2} \in \mathbb{Q}(\sqrt{2}).\end{aligned}$$

(3) If $b \neq 0$ and $a \neq b\sqrt{2}$, then $a^2 \neq 2b^2$ and

$$(a+b\sqrt{2})^{-1} = \frac{1}{a^2-2b^2}(a-b\sqrt{2}) = \frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2}\sqrt{2} \in \mathbb{Q}(\sqrt{2}).$$

since $\frac{a}{a^2-2b^2}, \frac{-b}{a^2-2b^2} \in \mathbb{Q}$. Besides,

$$\begin{aligned}(a+b\sqrt{2})\Big(\frac{1}{a^2-2b^2}(a-b\sqrt{2})\Big) &= (a+b\sqrt{2})\Big(\frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2}\sqrt{2}\Big) \\ &= (a+b\sqrt{2})\Big(\frac{a}{a^2-2b^2}\Big) + (a+b\sqrt{2})\Big(\frac{-b}{a^2-2b^2}\sqrt{2}\Big) \\ &= \frac{a^2}{a^2-2b^2} + \frac{ab\sqrt{2}}{a^2-2b^2} + \frac{-ab\sqrt{2}}{a^2-2b^2} + \frac{-2b^2}{a^2-2b^2} \\ &= \frac{a^2-2b^2}{a^2-2b^2} + \frac{ab-ab}{a^2-2b^2}\sqrt{2} = 1_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} \\ &= 1\end{aligned}$$

where we use the multiplication associative, commutative and distributive law of the field $\mathbb{R}$ since $a, b\sqrt{2}, \frac{a}{a^2-2b^2}, \frac{-b}{a^2-2b^2}\sqrt{2} \in \mathbb{R}$.

(d) The distributive law inherites from $\mathbb{R}$.

(e) Since

$$\begin{aligned}0_{\mathbb{Q}(\sqrt{2})} &= 0_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} \\ 1_{\mathbb{Q}(\sqrt{2})} &= 1_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2},\end{aligned}$$

$$1_{\mathbb{Q}(\sqrt{2})} \neq 0_{\mathbb{Q}(\sqrt{2})}$$

given $1_{\mathbb{Q}} \neq 0_{\mathbb{Q}}$.

□

Exercise 2. Is the set $\mathbb{R}^{2\times 2} := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \mid a, b, c, d \in \mathbb{R}\right\}$ is a field under the usual addition and multiplication of matrices?

If $\mathbb{R}^{2\times 2}$ is a field, prove it. Otherwise, prove the field axioms that do hold, and give specific counterexamples for the axioms that fail.

No, it is not a field.

Proof. $\forall \begin{pmatrix} a & b \\ c & d\end{pmatrix}, \begin{pmatrix} e & f \\ g & h\end{pmatrix}, \begin{pmatrix} i & j \\ k & l\end{pmatrix} \in \mathbb{R}^{2\times 2}$,

(a)

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} + \begin{pmatrix} e & f \\ g & h\end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h\end{pmatrix} \in \mathbb{R}^{2\times 2}$$

since $a+e, b+f, c+d, d+h \in \mathbb{R}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

since $ae+bg, af+bh, ce+dg, cg+dh \in \mathbb{R}$.

(b)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$\begin{aligned} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) + \begin{pmatrix} i & j \\ k & l \end{pmatrix} &= \left(\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}\right) + \begin{pmatrix} i & j \\ k & l \end{pmatrix} \\ &= \begin{pmatrix} a+e+i & b+f+j \\ c+g+k & d+h+l \end{pmatrix} \\ &= \begin{pmatrix} a+(e+i) & b+(f+j) \\ c+(g+k) & d+(h+l) \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left(\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right). \end{aligned}$$

$$0_{\mathbb{R}^{2\times 2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + 0_{\mathbb{R}^{2\times 2}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$-\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Check

$$-\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a+a & -b+b \\ -c+c & -d+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{\mathbb{R}^{2\times 2}}.$$

(c) It does not satisfy mutlipication commutative law. For example,

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix},$$

but

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

$$\begin{aligned} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) \begin{pmatrix} i & j \\ k & l \end{pmatrix} &= \left(\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\right) \begin{pmatrix} i & j \\ k & l \end{pmatrix} \\ &= \begin{pmatrix} (ae+bg)i+(af+bh)k & (ae+bg)j+(af+bh)l \\ (ce+dg)i+(cf+dh)k & (ce+dg)j+(cf+dh)l \end{pmatrix} \\ &= \begin{pmatrix} aei+afk+bgi+bhk & aej+afl+bgj+bhl \\ cei+cfk+dgi+dhk & cej+cfl+dgj+dhl \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{pmatrix} \\ &= \begin{pmatrix} a(ei+fk)+b(gi+hk) & a(ej+fl)+b(gj+hl) \\ c(ei+fk)+d(gi+hk) & c(ej+fl)+d(gj+hl) \end{pmatrix} \\ &= \begin{pmatrix} aei+afk+bgi+bhk & aej+afl+bgj+bhl \\ cei+cfk+dgi+dhk & cej+cfl+dgj+dhl \end{pmatrix} \\ &= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) \begin{pmatrix} i & j \\ k & l \end{pmatrix}. \end{aligned}$$

$$1_{\mathbb{R}^{2\times 2}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}.$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) 1_{\mathbb{R}^{2\times 2}} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+b0 & a0+b1 \\ c1+d0 & c0+d1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

Not every element of $\mathbb{R}^2$ has a multiplicative inverse, for instance, for $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$, asumme we can find an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ such that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then we have $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which is impossible since $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$.

(d)

$$\begin{aligned}
\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right) &= \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right) \\
&= \begin{pmatrix} (a+e)i+(b+f)k & (a+e)j+(b+f)l \\ (c+g)i+(d+h)k & (c+g)j+(d+h)l \end{pmatrix} \\
&= \begin{pmatrix} ai+ei+bk+fk & aj+ej+bl+fl \\ ci+gi+dk+hk & cj+gj+dl+hl \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right) + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right) &= \begin{pmatrix} ai+bk & aj+bl \\ ci+dk & cj+dl \end{pmatrix} + \begin{pmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{pmatrix} \\
&= \begin{pmatrix} ai+ei+bk+fk & aj+ej+bl+fl \\ ci+gi+dk+hk & cj+gj+dl+hl \end{pmatrix} \\
&= \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right).
\end{aligned}$$

(e) Since

$$\begin{aligned}
0_{\mathbb{R}^{2\times 2}} &= \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \\
1_{\mathbb{R}^{2\times 2}} &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),
\end{aligned}$$

$$0_{\mathbb{R}^{2\times 2}} \neq 1_{\mathbb{R}^{2\times 2}}$$

given $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$.

$\square$