MATH 8510, Abstract Algebra I
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Exercises 9-1
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Exercise 1 (5.5.1–2)**.** Let H and K be groups, let $\varphi\colon K \to \mathrm{Aut}(H)$ be a homomorphism, and let $\widetilde{H}, \widetilde{K}$ be as in the Theorem 5.5.3. Prove that $C_{\widetilde{K}}(\widetilde{H}) \cong \mathrm{Ker}(\varphi)$ and $C_{\widetilde{H}}(\widetilde{K}) = N_{\widetilde{H}}(\widetilde{K})$.

Proof. Define ϕ as

$$\begin{aligned}\phi : \mathrm{Ker}(\varphi) &\to C_{\widetilde{K}}(\widetilde{H})\\ k &\mapsto (e_H, k)\end{aligned}$$

First we show ϕ is well defined.
Let $k \in \mathrm{Ker}(\varphi)$, then $k \in \mathrm{Ker}(\varphi) \leq K$ by the definition of φ.
$\forall (h, e_K) \in \widetilde{H}$, where $h \in H$, by the Theorem 5.5.3 (d), we have

$$(e_H, k)(h, e_K)(e_H, k)^{-1} = (\varphi(k)h, e_K).$$

Since $k \in \mathrm{Ker}(\varphi)$, $\varphi(k) = e_{\mathrm{Aut}(H)}$, where $e_{\mathrm{Aut}(H)}$ is the identity map from H to H.
Besides, $h \in H$, so

$$\varphi(k)h = e_{\mathrm{Aut}(H)}h = h.$$

Then

$$(e_H, k)(h, e_K)(e_H, k)^{-1} = (h, e_K) \in \widetilde{H}.$$

So $(e_H, k) \in C_{\widetilde{K}}(\widetilde{H})$.
Therefore, ϕ is well defined.
Next we show ϕ is a homomorphism.
$\forall\ k_1, k_2 \in \mathrm{Ker}(\varphi)$,

$$\begin{aligned}\phi(k_1k_2) &= (e_H, k_1k_2)\\ &= (e_H \cdot \varphi(k_1)(e_H), k_1k_2)\\ &= (e_H, k_1)(e_H, k_2)\\ &= \phi(k_1)\phi(k_2),\end{aligned}$$

so ϕ is a homomorphism.
Let $k \in \mathrm{Ker}(\varphi)$.
Then

$$\begin{aligned}k \in \mathrm{Ker}(\phi) &\Leftrightarrow \phi(k) = (e_H, e_K)\\ &\Leftrightarrow (e_H, k) = (e_H, e_K)\\ &\Leftrightarrow k = e_K.\end{aligned}$$

So

$$\mathrm{Ker}(\phi) = \{e_K\} = \{e_{\mathrm{Ker}(\varphi)}\}.$$

Thus, ϕ is 1-1.
Let $(e_H, k) \in C_{\widetilde{K}}(\widetilde{H})$, where $k \in K$.

Then $\forall\ (h, e_K) \in \widetilde{H}$, we have $h \in H$ is arbitrary,
and by the Theorem 5.5.3 (d),

$$\begin{aligned}
(e_H, k) \in C_{\widetilde{K}}(\widetilde{H}) &\Leftrightarrow (e_H, k)(h, e_K)(e_H, k)^{-1} = (h, e_K), \quad \forall\ h \in H \\
&\Leftrightarrow (\varphi(k)h, e_K) = (h, e_K), \quad \forall\ h \in H \\
&\Leftrightarrow \varphi(k)h = h, \quad \forall\ h \in H \\
&\Leftrightarrow \varphi(k) = e_{\mathrm{Aut}(H)} \\
&\Leftrightarrow k \in \mathrm{Ker}(\varphi).
\end{aligned}$$

Namely, for any $(e_H, k) \in C_{\widetilde{K}}(\widetilde{H})$, we have $k \in \mathrm{Ker}(\varphi)$ such that $\phi(k) = (e_H, k)$.
Thus, ϕ is onto.
Next we show

$$C_{\widetilde{H}}(\widetilde{K}) = N_{\widetilde{H}}(\widetilde{K})$$

By the definition of $C_{\widetilde{H}}(\widetilde{K})$ and $N_{\widetilde{H}}(\widetilde{K})$, we have

$$C_{\widetilde{H}}(\widetilde{K}) \subset N_{\widetilde{H}}(\widetilde{K}).$$

Let $(h, e_K) \in N_{\widetilde{H}}(\widetilde{K})$, where $h \in H$.
Let $(e_H, k) \in \widetilde{K}$, where $k \in K$.
Since φ is a homomorphism and $\varphi(e_K) = e_{\mathrm{Aut}(H)}$,

$$\begin{aligned}
(h, e_K)(e_H, k)(h, e_K)^{-1} &= (h \cdot \varphi(e_K)(e_H), e_K k)(h, e_K)^{-1} \\
&= (he_H, k)\left(\varphi(e_K^{-1})(h^{-1}), e_K^{-1}\right) \\
&= (h, k)(\varphi(e_K)h^{-1}, e_K) \\
&= (h, k)\left(h^{-1}, e_K\right) \\
&= \left(h \cdot \varphi(k)(h^{-1}), ke_K\right) \\
&= \left(h \cdot \varphi(k)(h^{-1}), k\right) \\
&\in \widetilde{K}.
\end{aligned}$$

So

$$h \cdot \varphi(k)(h^{-1}) = e_H.$$

Namely, $\forall\ (h, e_K) \in N_{\widetilde{H}}(\widetilde{K})$, $\forall\ (e_H, k) \in \widetilde{K}$, we have

$$(h, e_K)(e_H, k)(h, e_K)^{-1} = (e_H, k)\,.$$

So

$$(h, e_K) \in C_{\widetilde{H}}(\widetilde{K}).$$

Thus,

$$N_{\widetilde{H}}(\widetilde{K}) \subset C_{\widetilde{H}}(\widetilde{K}).$$

As a result,

$$C_{\widetilde{H}}(\widetilde{K}) = N_{\widetilde{H}}(\widetilde{K}).$$

□

Exercise 2 (5.5.11)**.** Classify all groups of order 28. (There are four isomorphism types. Feel free to use Proposition 11 and/or Exercise 6 from this section of the text; You do not need to solve Exercise 6.)
soln: Let G be a group of order 28.

If G is abelian, since $28 = 2^2 \times 7$ and $2 = 2 = 1 + 1$, by the FTFGAG, we have 2 types of isomorphic groups of order 28 and they are

$$G \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$$
$$G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}.$$

The invariant factor decomposition of them are

$$G \cong \mathbb{Z}/28\mathbb{Z}$$
$$G \cong \mathbb{Z}/14\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Next we discuss the case when G is not abelian.
By the Sylow theorem 4.5.4, we have $n_7 \equiv (1 \mod 11)$ and $n_7|4$.
$n_7|4$, so $n_7 = 1$ or 2 or 4.
Besides, $n_7 \equiv (1 \mod 11)$, so $n_7 = 1$.
Let $H \in Syl_7(G)$, then $|H| = 7$.
Then by the Corollary 4.5.5, we have

$$H \trianglelefteq G.$$

Let $K \in Syl_2(G)$, then $|K| = 4$.
Then $KH \leq G$ by the Corollary 3.2.7.
So

$$HK = KH \leq G$$

by the Theorem 3.2.6.
We claim $H \cap K = \{e_G\}$.
Assume $\exists\ x \in G$ and $x \neq e_G$ such that $x \in H \cap K$.
Then $\langle x \rangle \leq H$.
So $|x| \mid |H|$.
Since $|x| \neq 1$ and $|H| = 7$, we have $|x| = 7 > |K|$, which is a contradiction since $x \in K$.
Thus,

$$H \cap K = \{e_G\}.$$

Then by the Theorem 5.5.5, $\exists\ \varphi : K \to \mathrm{Aut}(H)$ such that

$$G = HK \cong H \rtimes_\varphi K.$$

Moreover, let $x \in H$ and $x \neq e_G$, similarly, we have

$$H = \langle x \rangle.$$

We claim K is abelian.
By the Cauchy Theorem, $\exists\ x \in K$ such that $|x| = 2$.
Then $x^{-1} = x$.
So $\{e_G, x\} \subset G$. Since $|K| = 4$, $\exists\ y \in K$ such that $y \neq e_G$ and $y \neq x$.
If $y^{-1} = x$, then $x = x^{-1} = y$, which is a contradiction.
So $y^{-1} \neq x$.
Similarly, we have $y^{-1} \neq e_G$ and $y^{-1} \neq y$.
Then

$$K = \{e_G, x, y, y^{-1}\}.$$

So we have $xy, yx \in K$.
Since $x, y, \neq e_G$ and $y^{-1} \neq x$, we have $xy = y^{-1}$.

Similarly, we have $yx = y^{-1}$.
So
$$y^{-1} = xy = yx.$$
Similarly, we have
$$y = xy^{-1} = y^{-1}x.$$
Thus, K is abelian of order 4.
Since $4 = 2^2$ and $2 = 2 = 1 + 1$, by the FTFGAG,
$$K \cong Z_4.$$
and
$$K \cong Z_2 \times Z_2.$$
Since we consider the types of groups which is isomorphic to non-abelian groups of order 28, we can set $H = \mathbb{Z}/7\mathbb{Z}$ and $K = \mathbb{Z}/4\mathbb{Z}$ or $K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Since
$$\operatorname{Aut}(H) \cong (\mathbb{Z}/7\mathbb{Z})^{\times},$$
we have
$$|\operatorname{Aut}(H)| = |(\mathbb{Z}/7\mathbb{Z})^{\times}| = \varphi(7) = 6.$$
Since $(\mathbb{Z}/7\mathbb{Z})^{\times}$ is abelian and $6 = 2 \times 3$, by the FTFGAG,
$$(\mathbb{Z}/7\mathbb{Z})^{\times} \cong \mathbb{Z}/6\mathbb{Z}.$$
Since $\mathbb{Z}/6\mathbb{Z}$ is cyclic, we have $\operatorname{Aut}(H)$ is also cyclic.
Let $\operatorname{Aut}(H) = \langle \bar{1}_6 \rangle$.

(a) Consider $K = \mathbb{Z}/4\mathbb{Z}$.
Let $K = \mathbb{Z}/4\mathbb{Z} = \langle \bar{1}_4 \rangle$.
Then we consider the homomorphism
$$\varphi : \langle \bar{1}_4 \rangle \to \langle \bar{1}_6 \rangle$$
Since $\mathbb{Z}/4\mathbb{Z} = \langle \bar{1}_4 \rangle$, φ is uniquely determined by $\varphi(\bar{1}_4)$.
Write $\varphi(\bar{1}_4) = \bar{a} \in \langle \bar{1}_6 \rangle$.
Since φ is a homomorphism and $|\bar{1}_4| = 4$,
$$4\varphi(\bar{1}_4) = \varphi(\bar{4}_4) = \varphi(\bar{0}_4) = \bar{0}_6.$$
Then
$$\varphi(\bar{1}_4) = \overline{3k}_6,$$
where $k = \{0\} \cup \mathbb{N}$.
Since $\langle \bar{1}_6 \rangle = \{\bar{0}_6, \bar{1}_6, \bar{2}_6, \bar{3}_6, \bar{4}_6, \bar{5}_6\}$,
$$\varphi(\bar{1}_4) = \bar{0}_6 \text{ or } \bar{3}_6.$$
If $\varphi(\bar{1}_4) = \bar{0}_6$, then φ is a trival homomorphism and then $G \cong H \rtimes_{\varphi} K$ becomes
$$G \cong H \times K = \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},$$
which is a contradiction since G is non-abelian by assumption.
So
$$\varphi(\bar{1}_4) = \bar{3}_6.$$
Thus,
$$G \cong (\mathbb{Z}/7\mathbb{Z}) \rtimes_{\varphi} (\mathbb{Z}/4\mathbb{Z}),$$
with the homomorphism φ determined by $\varphi(\bar{1}_4) = \bar{3}_6$.

(b) Consider $K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Let $K = \langle \bar{1_a} \rangle \times \langle \bar{1}_b \rangle$.
Then φ is uniquely determined by $\varphi(\bar{1}_a)$ and $\varphi(\bar{1}_b)$.
Let $\varphi : \mathbb{Z}/2\mathbb{Z} \to \langle \bar{1}_6 \rangle$ determined by $\varphi(\bar{1}_a)$ be a homomorphism.
Write $\varphi(\bar{1}_a) = \bar{x} \in \langle \bar{1}_6 \rangle$.
Since φ is a homomorphism and $|\bar{1}_a| = 2$,

$$2\varphi(\bar{1}_a) = \varphi(\bar{2}_a) = \varphi(\bar{0}_a) = \bar{0}_6.$$

Then

$$\varphi(\bar{1}_a) = \overline{3k}_6,$$

where $k = \{0\} \cup \mathbb{N}$.
Since $\langle \bar{1}_6 \rangle = \{\bar{0}_6, \bar{1}_6, \bar{2}_6, \bar{3}_6, \bar{4}_6, \bar{5}_6\}$,

$$\varphi(\bar{1}_a) = \bar{0}_6 \text{ or } \bar{3}_6.$$

Let $\varphi : \mathbb{Z}/2\mathbb{Z} \to \langle \bar{1}_6 \rangle$ determined by $\varphi(\bar{1}_b)$ be a homomorphism.
Similarly, we have

$$\varphi(\bar{1}_b) = \bar{0}_6 \text{ or } \bar{3}_6.$$

(i) If $\varphi(\bar{1}_a) = \varphi(\bar{1}_b) = \bar{0}_6$, then ϕ is trivial, similarly, we can find this is contradicted by that G is non-abelian.

(ii) If $\varphi(\bar{1}_a) = \bar{0}_6$ and $\varphi(\bar{1}_b) = \bar{3}_6$, then

$$G \cong \mathbb{Z}/7\mathbb{Z} \rtimes_{\varphi} (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

(iii) If $\varphi(\bar{1}_b) = \bar{0}_6$ and $\varphi(\bar{1}_a) = \bar{3}_6$,
let φ determined by $\varphi(\bar{1}_a) = \bar{0}_6$ and $\varphi(\bar{1}_b) = \bar{3}_6$ be φ_1.
let φ determined by $\varphi(\bar{1}_b) = \bar{0}_6$ and $\varphi(\bar{1}_a) = \bar{3}_6$ be φ_2.
By Symmetry, we have

$$\varphi_1(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = \varphi_2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

Since $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is cyclic, by Exercise 6 from this section of the text, we have

$$\mathbb{Z}/7\mathbb{Z} \rtimes_{\varphi_1} (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/7\mathbb{Z} \rtimes_{\varphi_2} (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$

(iv) If $\varphi(\bar{1}_a) = \bar{3}_6$ and $\varphi(\bar{1}_b) = \bar{3}_6$, it seems no new homomorphism produced.

In summary, we have 4 types of isomorphisms, and they are
(a) $\mathbb{Z}/28\mathbb{Z}$.
(b) $\mathbb{Z}/14\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
(c) $\mathbb{Z}/7\mathbb{Z}) \rtimes_{\varphi} (\mathbb{Z}/4\mathbb{Z})$ with the homomorphism φ determined by $\varphi(\bar{1}_4) = \bar{3}_6$.
(d) $\mathbb{Z}/7\mathbb{Z} \rtimes_{\varphi} (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ with the homomorphism φ determined by $\varphi(\bar{1}_a) = \bar{0}_6$ and $\varphi(\bar{1}_b) = \bar{3}_6$.

Exercise 3 (6.1.1)**.** Let G be a group. Prove that $Z_i(G)$ is a characteristic subgroup of G for all i.

Proof. We will show it by induction.
Basic steps:
Let $\sigma \in \mathrm{Aut}(G)$.
$Z_0(G) = \{e_G\}$.

Since $\sigma \in \mathrm{Aut}(G)$, $\sigma(e_G) = e_G$.
So
$$\sigma(Z_0(G)) = \sigma(Z_0(G)).$$
Thus, $Z_0(G)$ is a characteristic subgroup of G.
$Z_1(G) = Z(G)$.
Since $\sigma \in \mathrm{Aut}(G)$, $\sigma(g^{-1}) = (\sigma(g))^{-1} \in G, \forall\ g \in G$.
Let $z \in Z(G)$
Then $z\sigma(g^{-1}) = \sigma(g^{-1})z, \forall\ g \in G$.
Then
$$\sigma(z\sigma(g^{-1})) = \sigma(\sigma(g^{-1})z), \forall\ g \in G$$
Since $\sigma \in \mathrm{Aut}(G)$, we have
$$\sigma(z)\sigma\left(\sigma(g^{-1})\right) = \sigma\left(\sigma(g^{-1})\right)\sigma(z), \forall\ g \in G.$$
Namely,
$$\sigma(z)g = g\sigma(z), \forall\ g \in G.$$
Then
$$\sigma(z) \in Z(G).$$
So
$$\sigma(Z(G)) \subset Z(G).$$
By the Theorem 4.4.8, $Z(G)$ is a characteristic subgroup of G.
Induction steps:
Assume $Z_i(G)$ is a characteristic subgroup of G.
Let $\sigma \in \mathrm{Aut}(G)$.
Then $\sigma(Z_i(G)) = Z_i(G)$.
Define
$$\begin{aligned} \overline{\sigma} : G/Z_i(G) &\to G/Z_i(G) \\ gZ_i(G) &\to \sigma(g)Z_i(G) \end{aligned}$$
Then $\overline{\sigma}$ is well-defined since $Z_i(G) \trianglelefteq G$ and $\sigma : G \to G$ is a isomomorphism.
Next we show $\overline{\sigma}$ is an automomorphism.
Let $gZ_i(G), hZ_i(G) \in G/Z_i(G)$, where $g, h \in G$.
Since $Z_i(G) \trianglelefteq G$,
$$\begin{aligned} \overline{\sigma}(gZ_i(G)hZ_i(G)) &= \overline{\sigma}(ghZ_i(G)) \\ &= \sigma(gh)Z_i(G) \\ &= \sigma(g)\sigma(h)Z_i(G) \\ &= \sigma(g)Z_i(G)\sigma(h)Z_i(G) \\ &= \overline{\sigma}(gZ_i(G))\overline{\sigma}(hZ_i(G)). \end{aligned}$$
So $\overline{\sigma}$ is a homomorphism.
Since σ is a homomorphism, and $Z_i(G)$ is a characteristic subgroup of G.
$$\begin{aligned} gZ_i(G) \in \mathrm{Ker}\,(\overline{\sigma}) &\Leftrightarrow \overline{\sigma}(gZ_i(G)) = Z_i(G) \\ &\Leftrightarrow \sigma(g)Z_i(G) = Z_i(G) \\ &\Leftrightarrow \sigma(g) \in Z_i(G) \\ &\Leftrightarrow \sigma^{-1}(\sigma(g)) \in Z_i(G) \\ &\Leftrightarrow g \in Z_i(G). \end{aligned}$$

So

$$\mathrm{Ker}(\overline{\sigma}) = Z_i(G).$$

So $\overline{\sigma}$ is 1-1.
Let $kZ_i(G) \in G/Z_i(G)$, where $k \in G$.
We know $\sigma^{-1}(k) \in G$ since $\sigma^{-1} \in \mathrm{Aut}(G)$.
Then

$$\sigma^{-1}(k)Z_i(G) \in G/Z_i(G).$$

Since

$$\overline{\sigma}\left(\sigma^{-1}(k)Z_i(G)\right) = \sigma(\sigma^{-1}(k))Z_i(G) = kZ_i(G),$$

$\overline{\sigma}$ is onto.
Thus,

$$\overline{\sigma} \in \mathrm{Aut}\left(G/Z_i(G)\right).$$

Since in basic steps we have shown the center of a group is a characteristic subgroup of the group,

$$\overline{\sigma}\left(Z\left(G/Z_i(G)\right)\right) = Z\left(G/Z_i(G)\right).$$

By definition, we have

$$Z\left(G/Z_i(G)\right) = Z_{i+1}(G)/Z_i(G).$$

So

$$\overline{\sigma}\left(Z_{i+1}(G)/Z_i(G)\right) = Z_{i+1}(G)/Z_i(G).$$

Let $z \in Z_{i+1}(G)$, then

$$zZ_i(G) \in Z_{i+1}(G)/Z_i(G).$$

Then

$$\overline{\sigma}(zZ_i(G)) \in Z_{i+1}(G)/Z_i(G).$$

Namely,

$$\sigma(z)Z_i(G) \in Z_{i+1}(G)/Z_i(G).$$

So

$$\sigma(z) \in Z_{i+1}(G).$$

Thus,

$$\sigma\left(Z_{i+1}(G)\right) \subset Z_{i+1}(G).$$

By the Theorem 4.4.8, $Z_{i+1}(G)$ is a characteristic subgroup of G.
So the assumption also holds for $Z_{i+1}(G)$.
Thus, we conclude that $Z_i(G)$ is a characteristic subgroup of G for each i. □

Exercise 4 (6.1.6)**.** Let G be a group. Prove that $G/Z(G)$ is nilpotent if and only if G is nilpotent.

Proof. We first find the relationship between $(G/Z(G))^n$ and G^n.
$(G/Z(G))^0 = (G/Z(G)$.

Since $Z(G) \trianglelefteq G$,

$$\begin{aligned}
(G/Z(G))^1 &= [G/Z(G), (G/Z(G))^0] \\
&= [G/Z(G), G/Z(G)] \\
&= \langle [hZ(G), kZ(G)] | hZ(G), kZ(G) \in G/Z(G) \rangle \\
&= \langle (hZ(G))^{-1}(kZ(G))^{-1}(hZ(G))(kZ(G)) | hZ(G), kZ(G) \in G/Z(G) \rangle \\
&= \langle (h^{-1}Z(G))(k^{-1}Z(G))(hZ(G))(kZ(G)) | hZ(G), kZ(G) \in G/Z(G) \rangle \\
&= \langle h^{-1}k^{-1}hkZ(G) | h, k \in G \rangle. \\
&= [G, G]/Z(G) \\
&= G^1/Z(G)
\end{aligned}$$

Then

$$\begin{aligned}
(G/Z(G))^2 &= [G/Z(G), (G/Z(G))^1] \\
&= [G/Z(G), G^1 Z(G)] \\
&= \left\langle [hZ(G), kZ(G)] | hZ(G) \in G/Z(G), kZ(G) \in G^1/Z(G) \right\rangle \\
&= \langle h^{-1}k^{-1}hkZ(G) | h \in G, k \in G^1 \rangle. \\
&= [G, G^1]/Z(G). \\
&= G^2/Z(G).
\end{aligned}$$

We guess $(G/Z(G))^n = G^n/Z(G)$.
We will show it by induction.
We have shown the basic steps.
Induction steps:
Assume $(G/Z(G))^n = G^n/Z(G)$.

$$\begin{aligned}
(G/Z(G))^{n+1} &= [G/Z(G), (G/Z(G))^n] \\
&= [G/Z(G), G^n/Z(G)] \\
&= \langle [hZ(G), kZ(G)] | hZ(G) \in G/Z(G), kZ(G) \in G^n/Z(G) \rangle \\
&= \langle h^{-1}k^{-1}hkZ(G) | h \in G, k \in G^n \rangle. \\
&= [G, G^n]/Z(G). \\
&= G^{n+1}/Z(G).
\end{aligned}$$

So the assumption holds for the $n+1$ case.
Thus, for $n \in \mathbb{N}$,

$$(G/Z(G))^n = G^n/Z(G).$$

"$\Leftarrow$". Assume G is nilpotent.
Then $G^m = e_G$ for some $m \geq 0$.
So

$$(G/Z(G))^m = G^m/Z(G) = Z(G).$$

So $G/Z(G)$ is nilpotent.
"$\Rightarrow$". Assume $G/Z(G)$ is nilpotent.
Then $(G/Z(G))^n = Z_G$ for some $n \geq 0$.
So

$$G^n/Z(G) = (G/Z(G))^n = Z(G).$$

As a result, we have

$$G^n \subset Z(G).$$

Therefore,

$$\begin{aligned}
G^{n+1} &= [G, G^n] \\
&= \langle [h,k] | h \in G, k \in G^n \rangle \\
&= \left\langle h^{-1}k^{-1}hk | h \in G, k \in G^n \right\rangle \\
&= \left\langle h^{-1}k^{-1}kh | h \in G, k \in G^n \right\rangle \\
&= \langle e_G | h \in G, k \in G^n \rangle \\
&= \langle e_G \rangle \\
&= e_G,
\end{aligned}$$

since $G^n \subset Z(G)$.
Thus, G is nilpotent.
Therefore, $G/Z(G)$ is nilpotent if and only if G is nilpotent. $\square$