MATH 8510, Abstract Algebra I
Fall 2016
Exercises 11-2
Due date Thu 10 Nov 4:00PM

Exercise 1. Let R be a commutative ring with identity. Prove that $R/N(R)$ has no non-zero nilpotent elements, that is, that $N(R/N(R)) = 0$.

Proof. We have shown in Exercise 10-1#2(a) that $N(R) \leq R$.
By the Theorem 7.3.6., we have $R/N(R)$ is a ring and $\forall\ r, s \in R$, we have

$$(r + N(R))\,(s + N(R)) = rs + N(R).$$

Let $r + N(R) \in N(R/N(R))$, then $\exists\ n \in \mathbb{N}$ such that

$$(r + N(R))^n = N(R).$$

Then

$$r^n + N(R) = N(R).$$

So

$$r^n \in N(R).$$

Therefore, $\exists\ m \in \mathbb{N}$ such that

$$(r^n)^m = 1.$$

Since R is CRW1, we have

$$(r^n)^m = r^{mn} = 1.$$

As a result,

$$r \in N(R).$$

Thus, $N(R/N(R)) = 0$. $\square$

Exercise 2 (Division Algorithm)**.** Let R be a commutative ring with identity. Let $f, g \in R[x]$ such that f is monic. Prove that there exist unique polynomials $q, r \in R[x]$ such that $g = qf + r$ and $\deg(r) < \deg(f)$.
Hint: For existence, argue by induction on $\deg(g)$. In the induction step, reduce the degree of g by subtracting an appropriate multiple of f to clear the top degree term.

Proof. First we show the existence.
Basic step:
Let $\deg(g) = 0$.
Since $\deg(f) > \deg(r)$, we have $f = 1_R \in R[x]$ which is monic and $r = 0_R \in R[x]$ with $\deg(r) = -\infty$.
Let $q = g \in R[x]$, then we have $g = qf + r$.
Inductive step:
Assume when $\deg(g) = n$, there exists $q, r \in R[x]$ such that $g = qf + r$.
Let $g = \sum_{i=0}^{n+1} a_i x^i \in R[x]$ with $a_{n+1} \neq 0_R$.
Let $h = \sum_{i=1}^{n+1} a_i x^{i-1}$, then $\deg(h) = n$.

By assumption, there exists $q, r \in R[x]$ such that $h = qf + r$.
Then

$$\begin{aligned} g &= hx + a_0 \\ &= (qf + r)x + a_0 \\ &= qxf + (rx + a_0). \end{aligned}$$

(a) When $\deg(r) + 1 < \deg(f)$, we have $rx + a_0 \in R[x]$ and $\deg(rx + a_0) < \deg(f)$. Then after letting $q_1 = qx \in R[x]$ and $r_1 = rx + a_0 \in R[x]$, we have $g = q_1 f + r_1$.

(b) When $\deg(r) + 1 = \deg(f)$, assume $\deg(r) = k$ and $r = \sum_{j=1}^{k} r_i x^i \in R[x]$ with $r_k \neq 0_R$.
Then $\deg(f) = k + 1$ and

$$\begin{aligned} g &= qxf + rx + a_0. \\ &= qxf + \sum_{j=1}^{k} r_i x^{i+1} + a_0 \\ &= (r_k + qx)f + \left(r_k \left(x^{k+1} - f \right) + \sum_{j=1}^{k-1} r_i x^{i+1} + a_0 \right). \end{aligned}$$

Since $\deg(f) = k + 1$ and f is monic, we have $\deg \left(x^{k+1} - f \right) \leq k$.
So after letting $r_2 = r_k \left(x^{k+1} - f \right) + \sum_{j=1}^{k-1} r_i x^{i+1} + a_0 \in R[x]$,
we have

$$\deg(r_2) \leq k < \deg(f).$$

Let $q_2 = r_k + qx \in R[x]$, then we have

$$g = q_2 f + r_2.$$

Therefore, our assumption also holds when $\deg(g) = n + 1$.
As a result, when $\deg(g) = n$, there exists $q, r \in R[x]$ such that $g = qf + r$.
Next, we show the uniqueness.
Suppose for $g, f \in R[x]$ and f is monic, $\exists\ q_1, q_2, r_1, r_2$ such that $g = q_1 f + r_1 = q_2 f + r_2$ with $\deg(r_1) < \deg(f)$ and $\deg(r_2) < \deg(f)$.
Then

$$(q_1 - q_2)f = r_2 - r_1.$$

So

$$\deg(f) \leq \deg(r_2 - r_1) \leq \max\{\deg(r_1), \deg(r_2)\}. \tag{1}$$

Since $\deg(r_1) < \deg(f)$ and $\deg(r_2) < \deg(f)$, we have

$$\max\{\deg(r_1), \deg(r_2)\} < \deg(f),$$

which is contradicted by (1).
Thus, let $g, f \in R[x]$ and f is monic, there exist unique $q, r \in R[x]$ such that $g = qf + r$. $\square$