MATH 8510, Abstract Algebra I
Fall 2016
Exercises 8-2
Due date Thu 20 Oct 4:00PM

Exercise 1 (5.2)**.** Write a list of the non-isomorphic abelian groups of order 270 in terms of their elementary divisor decompositions. For each group in this list, write its invariant factor decomposition.
Solution:
Let G be an abelian group of order 270.

$$270 = 2 \times 3^3 \times 5.$$

Since $3 = 3$, $3 = 1 + 2$ and $3 = 1 + 1 + 1$, we have 3 non-isomorphic abelian groups of order 270 and they are

$$(\mathbb{Z}/27\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z});$$
$$(\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z});$$
$$(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}).$$

Next we compute their invariant factor decomposition.
(1)

$$(\mathbb{Z}/27\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/270\mathbb{Z}$$

(2)

$$\begin{aligned}(\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}) &\cong (\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \\ &\cong (\mathbb{Z}/90\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})\end{aligned}$$

(3)

$$\begin{aligned}&(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}) \\ &\cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \\ &\cong (\mathbb{Z}/30\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})\end{aligned}$$

Exercise 2 (5.4.11)**.** Let H and K be characteristic subgroups of a group G such that $H \cap K = \{e\}$ and $G = HK$. Prove that $\mathrm{Aut}(G) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K)$.

Proof. Since H and K be characteristic subgroups of a group G,

$$H \trianglelefteq G \text{ and } K \trianglelefteq G.$$

Besides,

$$H \cap K = \{e\}.$$

Then by Theorem 5.4, we have

$$H \times K \cong HK = G.$$

Let $\sigma \in \mathrm{Aut}(H)$ and $\tau \in \mathrm{Aut}(K)$.
Define $\sigma \times \tau$ as

$$\begin{aligned}\sigma \times \tau : H \times K &\to H \times K \\ (h, k) &\mapsto (\sigma(h), \tau(k))\end{aligned}$$

Then we show $\sigma \times \tau \in \mathrm{Aut}(H \times K)$.
Let $(h_1, k_1), (h_2, k_2) \in H \times K$.
Since $\sigma \in \mathrm{Aut}(H)$, and $\tau \in \mathrm{Aut}(K)$, σ, τ are homomorphisms.

$$\begin{aligned} \sigma \times \tau \left((h_1, k_1)(h_2, k_2)\right) &= \sigma \times \tau(h_1 h_2, k_1 k_2) \\ &= (\sigma(h_1 h_2), \tau(k_1 k_2)) \\ &= (\sigma(h_1)\sigma(h_2), \tau(k_1)\tau(k_2)) \\ &= (\sigma(h_1), \tau(k_1))\,(\sigma(h_2), \tau(k_2)) \\ &= (\sigma \times \tau(h_1, k_1))\,(\sigma \times \tau(h_2, k_2))\,, \end{aligned}$$

Therefore, $\sigma \times \tau$ is a homomorphism.
Let $(h, k) \in H \times K$ where $h \in H, k \in K$.
Since $\sigma \in \mathrm{Aut}(H)$, and $\tau \in \mathrm{Aut}(K)$, σ and τ are isomomorphsims.
So $\sigma(h) = e_G$ and $\tau(k) = e_G$ if and only if $h = e_G$ and $k = e_G$.

$$\begin{aligned} (h, k) \in \mathrm{Ker}(\sigma \times \tau) &\Leftrightarrow \sigma \times \tau(h, k) = (e_G, e_G) \\ &\Leftrightarrow (\sigma(h), \tau(k)) = (e_G, e_G) \\ &\Leftrightarrow \sigma(h) = e_G \text{ and } \tau(k) = e_G \\ &\Leftrightarrow h = e_G \text{ and } k = e_G \\ &\Leftrightarrow (h, k) = (e_G, e_G), \end{aligned}$$

so

$$\mathrm{Ker}(\sigma \times \tau) = (e_G, e_G).$$

So $\sigma \times \tau$ is 1-1.
Let $(h^{'}, k^{'}) \in (H, K)$, where $h \in H, k \in K$.
We have shown before that $\sigma^{-1} \in \mathrm{Aut}(H)$ and $\tau^{-1} \in \mathrm{Aut}(K)$ when $\sigma \in \mathrm{Aut}(H)$ and $\tau \in \mathrm{Aut}(K)$.
Then $\sigma^{-1}(h^{'}) \in H$ and $\tau^{-1}(k^{'}) \in K$.
So

$$\left(\sigma^{-1}(h^{'}), \tau^{-1}(k^{'})\right) \in H \times K.$$

Since by the definition of $\sigma \times \tau$,

$$\sigma \times \tau \left(\sigma^{-1}(h^{'}), \tau^{-1}(k^{'})\right) = (h^{'}, k^{'}),$$

$\sigma \times \tau$ is onto.
Therefore,

$$\sigma \times \tau \in \mathrm{Aut}(G).$$

Next we define ϕ as

$$\begin{aligned} \phi : \mathrm{Aut}(H) \times \mathrm{Aut}(K) &\to \mathrm{Aut}(H \times K) \\ (\sigma, \tau) &\mapsto \sigma \times \tau \end{aligned}$$

Since we have show $\sigma \times \tau \in \mathrm{Aut}(H \times K)$, ϕ is well-defined.
Let $(\sigma_1, \tau_1), (\sigma_2, \tau_2) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$, where $\sigma_1, \sigma_2 \in \mathrm{Aut}(H)$ and $\tau_1, \tau_2 \in \mathrm{Aut}(K)$.

Let $(h,k) \in H \times K$ where $h \in H, k \in K$.

$$\begin{aligned}
((\sigma_1\sigma_2) \times (\tau_1\tau_2))\,(h,k) &= ((\sigma_1\sigma_2)(h), (\tau_1\tau_2)(k)) \\
&= (\sigma_1(\sigma_2(h)), \tau_1(\tau_2(k))) \\
&= (\sigma_1 \times \tau_1)(\sigma_2(h), \tau_2(k)) \\
&= (\sigma_1 \times \tau_1)\,((\sigma_2 \times \tau_2)(h,k)) \\
&= ((\sigma_1 \times \tau_1)(\sigma_2 \times \tau_2))\,(h,k),
\end{aligned}$$

so

$$(\sigma_1\sigma_2) \times (\tau_1\tau_2) = (\sigma_1 \times \tau_1)(\sigma_2 \times \tau_2)$$

Then

$$\begin{aligned}
\phi\,((\sigma_1,\tau_1)(\sigma_2,\tau_2)) &= \phi(\sigma_1\sigma_2, \tau_1\tau_2) \\
&= (\sigma_1\sigma_2) \times (\tau_1\tau_2) \\
&= (\sigma_1 \times \tau_1)(\sigma_2 \times \tau_2) \\
&= (\phi(\sigma_1,\tau_1))\,(\phi(\sigma_2,\tau_2))\,.
\end{aligned}$$

So ϕ is a homomorphism.
Let $(\sigma,\tau) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$, where $\sigma \in \mathrm{Aut}(H)$ and $\tau \in \mathrm{Aut}(K)$.
Let id_H and id_K be the identity maps of H and K, respectively.
Let $id_{H\times K}$ be the identity map of $\mathrm{Aut}(H \times K)$.

$$\begin{aligned}
(\sigma,\tau) \in \mathrm{Ker}(\phi) &\Leftrightarrow \phi(\sigma,\tau) = id_{H\times K} \\
&\Leftrightarrow \sigma \times \tau = id_{H\times K} \\
&\Leftrightarrow \sigma \times \tau = id_H \times id_K \\
&\Leftrightarrow (\sigma,\tau) = (id_H, id_K)
\end{aligned}$$

So ϕ is 1-1.
Let $\pi \in \mathrm{Aut}(H \times K)$.
Define two maps $\pi_H : H \to H$ and $\pi_K : K \to K$ by $(\pi_H(h), 1) = \pi(h,1)$ and $(1, \pi_K(k)) = \pi(1,k)$(Not true in general for cartesian products).
Repeat the similar processes as previous ones,
we have π_H and π_K are well-defined and $\pi_H \in \mathrm{Aut}(H)$ and $\pi_K \in \mathrm{Aut}(K)$.
Let $(h,k) \in H \times K$ where $h \in H, k \in K$.

$$\begin{aligned}
\pi(h,k) &= \pi((h,1)(1,k)) \\
&= \pi(h,1)\pi(1,k) \\
&= (\pi_H(h),1)\,(1,\pi_K(k)) \\
&= (\pi_H(h), \pi_K(k)) \\
&= \pi_H \times \pi_K(h,k),
\end{aligned}$$

so $\pi = \pi_H \times \pi_K$.
Thus, ϕ is onto.
As a result,

$$\mathrm{Aut}(H) \times \mathrm{Aut}(K) \cong \mathrm{Aut}(H \times K)$$

Since we have show

$$H \times K \cong G,$$

$$\mathrm{Aut}(G) \cong \mathrm{Aut}(H \times K).$$

Hence,

$$\mathrm{Aut}(G) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K)$$

□

Use this to prove that if G is a finite abelian group, then $\mathrm{Aut}(G)$ is isomorphic to the direct product of the automorphism groups of its Sylow subgroups.

Proof. Let $\{P_i\}_{i=1}^{n}$ be the collection of all the Sylow subgroups of G_n(Use G is better, in the following just use one grp G, not the whole group).
Then $G_n = P_1P_2 \dots P_n$.
Since G_n is abelian and $P_i \leq G_n$ for $1 \leq i \leq n$,

$$P_i \trianglelefteq G_n.$$

So the Sylow $|P_i|$-subgroup is unique for $1 \leq i \leq n$.
Thus, $P_i \cap P_j = \{e_{G_n}\}$ for $1 \leq i, j \leq n$ and $i \neq j$.
Besides, by Corollary 4.5.6, P_i is a characteristic subgroup of G_n for $1 \leq i \leq n$.
We will show it by induction.
Basic steps:
When $n = 1$, it is a trivial case since the only Sylow subgroup is G_n and $\mathrm{Aut}(G_n) \cong \mathrm{Aut}(G_n)$.
We have just showed the case for $n = 2$.
Inductive steps:
Assume

$$\mathrm{Aut}(G_n) = \mathrm{Aut}(P_1P_2 \dots P_n) \cong \prod_{i=1}^{n} \mathrm{Aut}(P_i).$$

Let $\{Q_i\}_{i=1}^{n+1}$ be the collection of all the Sylow subgroups of J_{n+1}.
Then $J_{n+1} = Q_1Q_2 \dots Q_{n+1}$.
Similarly, we have for $i = 1, 2, \dots, n+1$,

$$Q_i \trianglelefteq J_{n+1}.$$

For $1 \leq i, j \leq n+1$ and $i \neq j$,

$$Q_i \cap Q_j = \{e_{J_{n+1}}\}.$$

For $1 \leq i \leq n+1$, Q_i is a characteristic subgroup of J_{n+1}.
Let $J_n = Q_1Q_2 \dots Q_n$.
Then it is obvious that $\{Q_i\}_{i=1}^{n}$ is the collection of all the Sylow subgroups of J_n.
Let $\sigma \in \mathrm{Aut}(J_{n+1})$.
Then for $i = 1, 2, \dots, n$, $\sigma(Q_i) = Q_i$ since Q_i is a characteristic subgroup of J_{n+1}.
Since σ is a homomorphism,

$$\begin{aligned}\sigma(J_n) &= \sigma(Q_1Q_2 \dots Q_n)\\ &= \sigma(Q_1)\sigma(Q_2)\dots\sigma(Q_n)\\ &= Q_1Q_2 \dots Q_n\\ &= J_n.\end{aligned}$$

So J_n is a characteristic subgroup of J_{n+1}.
Since for $1 \leq i, j \leq n+1$ and $i \neq j$,

$$Q_i \cap Q_j = \{e_{J_{n+1}}\},$$

and $J_n = Q_1 Q_2 \dots Q_n$, we have

$$J_n \cap Q_{n+1} = \{e_{J_{n+1}}\}.$$

We already have Q_{n+1} is a characteristic subgroup of J_{n+1}.
Besides,

$$J_{n+1} = J_n Q_{n+1}.$$

By the conclusion we just made,

$$\mathrm{Aut}(J_{n+1}) \cong \mathrm{Aut}(J_n) \times \mathrm{Aut}(Q_{n+1}).$$

By the inductive assumption, we have

$$\mathrm{Aut}(J_n) = \mathrm{Aut}(Q_1 Q_2 \dots Q_n) \cong \prod_{i=1}^{n} \mathrm{Aut}(Q_i).$$

Thus,

$$\mathrm{Aut}(J_{n+1}) \cong \left(\prod_{i=1}^{n} \mathrm{Aut}(Q_i) \right) \times \mathrm{Aut}(Q_{n+1}).$$

Namely,

$$\mathrm{Aut}(J_{n+1}) \cong \prod_{i=1}^{n+1} \mathrm{Aut}(Q_i).$$

Thus, the assumption also holds for J_{n+1}.
As a result, if $\{P_i\}_{i=1}^{n}$ is the collection of all the Sylow subgroups of a finite abelian group G, then

$$\mathrm{Aut}(G) \cong \prod_{i=1}^{n} \mathrm{Aut}(P_i).$$

$\square$