MATH 8510, Abstract Algebra I
Fall 2016
Exercises 6-2
Due date Thu 29 Sep 4:00PM

See the text for hints.

Exercise 1 (4.2.10)**.** Prove that every non-abelian group of order 6 has a non-normal subgroup of order 2. Use this to classify all groups of order 6; specifically, provide a list G_1, G_2 of groups of order 6 such that (1) $G_1 \ncong G_2$, and (2) for all groups G if $|G| = 6$, then $G \cong G_1$ or $G \cong G_2$. Justify your answers.

Proof. Let G be a group which is not abelian and $|G| = 6$.
Since $2|6$ and $3|6$ and $2, 3$ are primes,
by Cauthy Theorem, there exists $H, M \leq G$ and $|H| = 2$ and $|H| = 3$.
Assume all the subgroups of G of order 2 are normal subgroups of G.
Consider $H = \{e_G, h\} \leq G$, where $h \in G$ and $h \neq e_G$ and $h^2 = 1$.
Then $|H| = 2$ and $H \trianglelefteq G$ by assumption.
As a result, $N_G(H) = G$.
So for any $g \in G$, $gH = Hg$.
Since $H = \{e_G, h\}$, we have $gh = hg$ for any $g \in G$.
So $h \in Z(G)$.
Since $|G| = 2 \times 3$ and $2, 3$ are primes, we have G is abelian or $Z(G) = \{e_G\}$ by the conclusion from Exercise 1 in homework 5.
Given G is not abelian, we have $Z(G) = \{e_G\}$, which is a contradiction since we already find $h \in Z(G)$ and $h \neq e_G$.
Thus, there exists a non-normal subgroup of G of order 2.
Next we show G is isomorphic to S_3.
Let $H \leq G$ of order 2 be the non-normal subgroup of G.
Then G acts transitively on $G//H$.
Let $\pi_H : G \to S_{G//H}$ be the associated permutation representation.
Then
$$\operatorname{Ker}(\pi_H) = \bigcap_{x \in G} xHx^{-1} \subset H.$$
So $|\operatorname{Ker}(\pi_H)| = 1$ or 2.
If $|\operatorname{Ker}(\pi_H)| = 2$, then $\operatorname{Ker}(\pi_H) = H$, which is a contradiction since $\operatorname{Ker}(\pi_H) \trianglelefteq G$ and H is not a normal subgroup of G by assumption.
So $|\operatorname{Ker}(\pi_H)| = 1$.
Thus, π_H is 1-1.
Since $|G//H| = \frac{|G|}{|H|} = \frac{6}{2} = 3$,
$$S_{G//H} \cong S_3.$$
Then $|S_{G//H}| = |S_3| = 6$.
Since $|G| = 6$, we have π_H is onto since it is 1-1.
Thus, π_H is bijective.
As a result, π_H is an isomorphism.
So
$$G \cong S_{G//H}.$$

Therefore, for each non-abelian group G of order 6, we have

$$G \cong S_3.$$

Next we claim that every abelian group of order 6 is cyclic.
Let G be a ableian group of order 6.
By Cauthy Theorem, there exists $x, y \in G$ and $|x| = 2, |y| = 3$.
Then $x \neq y$.
Besides, $x^{-1} = x$ and $y^{-1} = y^2 \in G$.
Then $x \neq y^2$, otherwise, $x = y$.
So we find $\{e_G, x, y, y^2\} \subset G$.
Similarly, we can verify that two distinct elelments $xy, xy^2 \in G$ but $xy, xy^2 \notin \{e_G, x, y, y^2\}$.
Since $|G| = 6$, we have

$$G = \{e_G, x, y, y^2, xy, xy^2\}.$$

We claim $G = \langle xy \rangle$.
Since G is abelian,

$$\begin{aligned}
(xy)^1 &= xy = e_G. \\
(xy)^2 &= x^2 y^2 = y^2, \\
(xy)^3 &= x^3 y^3 = x, \\
(xy)^4 &= x^4 y^4 = y, \\
(xy)^5 &= x^5 y^5 = xy^2, \\
(xy)^6 &= x^6 y^6 = e_G.
\end{aligned}$$

Thus, G is cyclic.
So every abelian group of order 6 is cyclic.
We know if a group G is cyclic of order 6, then

$$G \cong \mathbb{Z}/6\mathbb{Z}.$$

So for each abelian group G of order 6, we have

$$G \cong \mathbb{Z}/6\mathbb{Z}.$$

Since S_3 is not cyclic,

$$S_3 \ncong \mathbb{Z}/6\mathbb{Z}.$$

As result, we have for any non-abelian group G of order 6,

$$G \cong S_3,$$

and for any abelian group G of order 6,

$$G \cong \mathbb{Z}/6\mathbb{Z},$$

where $S_3 \ncong \mathbb{Z}/6\mathbb{Z}$. □

Exercise 2 (4.3.6)**.** Assume that G is a non-abelian group of order 15. Prove that $Z(G) = \{e\}$. Use the fact that $\langle g \rangle \leq C_G(g)$ to show that there is at most one possible class equation for G; in other words, in the notation of Theorem 4.2.4 of the notes, find r and $|Z(G)|$ and $[G : C_G(g_1)]$, …, $[G : C_G(g_r)]$. Justify your answers.

Proof. Since $|G| = 15 = 3 \times 5$ and $3, 5$ are primes,
G is abelian or $Z(G) = \{e_G\}$ by the conclusion from Exercise 1 in homework 5.
Given G is not abelian, we have $Z(G) = \{e_G\}$.
Let $[r] = \{1, 2, ..., r\}$.
By class equation, we have

$$\sum_{i=1}^{r} [G : C_G(g_i)] = |G| - |Z(G)| = 15 - 1 = 14,$$

where $g_i \in G$ and $g_i \notin Z(G)$ for $i \in [r]$.
Then $C_G(g_i) \neq G$ for $i \in [r]$ and $g_i \neq e_G$ since $Z(G) = \{e_G\}$.
So $[G : C_G(g_i)] \neq 1$.
Then $[G : C_G(g_i)] \in \{3, 5, 15\}$ since $[G : C_G(g_i)] \mid G$.
We know the fact that for $i \in [r]$,

$$\langle g_i \rangle \leq C_G(g_i).$$

Since $g_i \neq e_G$ for $i \in [r]$,

$$C_G(g_i) \neq \{e_G\}.$$

So for $i \in [r]$,

$$[G : C_G(g_i)] < 15.$$

Then for $i \in [r]$,

$$[G : C_G(g_i)] \in \{3, 5\}.$$

So we need to find r and $[G : C_G(g_i)]$, where $\sum_{i=1}^{r} [G : C_G(g_i)] = 14$ and $[G : C_G(g_i)] \in \{3, 5\}$ for $i \in [r]$.
Let m and n be the number of order 3 and order 5 conjugacy classes in G, respectively.
Then $3m + 5n = 14$, where $m, n \in \mathbb{N} \cup \{0\}$.
So

$$3m = 14 - 5n \geq 0.$$

So the possible n can only be 0 or 1 or 2.
To make $3 | (14 - 5n)$, just $n = 1$ is satisfied and then $m = 3$.
So we have $3 + 3 + 3 + 5 = 14$ and then $r = 4$.
Since by class equation, we can just find one $r = 4$ and corresponding $[G : C_G(g_i)]$ for $i = 1, 2, 3, 4$, there is at most one possible class equation for G.

$\square$