Engineering Mathematics-II (MA102)
GBU, April 2014

Assignment 2

Topics: Limit, Continuity, Differentiability.
Definition: Let $f : \mathbb{C} \to \mathbb{C}$ be defined around a point z_0 (not necessarily at z_0) in $\mathbb{C}$. We say that the limit of $f(z)$ at z_0 exists and equal to number $\ell \in \mathbb{C}$, denoted by

$$\lim_{z \to z_0} f(z) = \ell$$

if for each ϵ there is a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon \quad$ whenever $\quad |z - z_0| < \delta$.

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1. Use $\epsilon - \delta$ definition to show that $\lim_{z \to z_0} \alpha = \alpha$. and $\lim_{z \to z_0} z = z_0$. (Here $\alpha \in \mathbb{C}$ is a given constant.)

2. By producing two paths, show that limit of the following functions at a given point z_0 do not exists.

   (a) $f(z) = \left(\frac{z}{\bar{z}}\right)^2$ at $z_0 = 0$

   (b) $f(z) = z^{1/2}$ at $z_0 = -1$. Here $f(z)$ is taken to be the principal branch.

3. Let $\lim_{z \to z_0} f_1(z) = \ell_1$ and $\lim_{z \to z_0} f_2(z) = \ell_2$. Then prove (by using $\epsilon - \delta$ definition) any two of the following facts:

   (a) $\lim_{z \to z_0} (f_1(z) + f_2(z)) = \ell_1 + \ell_2$.

   (b) $\lim_{z \to z_0} (f_1(z) \cdot f_2(z)) = \ell_1 \cdot \ell_2$.

   (c) $\lim_{z \to z_0} \left(\frac{f_1(z)}{f_2(z)}\right) = \frac{\ell_1}{\ell_2} \quad$ provided $\ell_2 \neq 0$.

   (d) For a given $\alpha \in \mathbb{C}$, $\lim_{z \to z_0} (\alpha \cdot f_1(z)) = \alpha \cdot \ell_1$.

   (e) Suppose $f(z) = u(x, y) + \iota v(x, y)$ for $z = x + \iota y \in \mathbb{C}$. Then at $z_0 = x_0 + \iota y_0$ in $\mathbb{C}$

   $$\lim_{z \to z_0} f(z) = a + \iota b \quad \Longleftrightarrow \quad \lim_{(x,y) \to (x_0,y_0)} u(x, y) = a \quad \& \quad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = b.$$

4. Define the continuity of $f(z)$ at a point z_0. Use the above facts to prove the following:

   (a) All polynomials are continuous on entire complex plane $\mathbb{C}$.

   (b) All rationals functions $R(z) = \frac{f(z)}{g(z)}$ are continuous on whole complex plane except where $g(z) = 0$.