Archimedean Property

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DEFINITION. An ordered field F has the Archimedean Property if, given any positive x and y in F there is an integer $n > 0$ so that $nx > y$.

Theorem 0.1. *The set of real numbers (an ordered field with the Least Upper Bound property) has the Archimedean Property.*

Lemma 0.2. *The set N of positive integers $N = \{0, 1, 2, ...\}$ is not bounded from above.*

Assume N is bounded from above. Since $N \subset R$ and R has the least upper bound property, then N has a least upper bound $a \in R$. Thus $n \leq a$ for all $n \in N$ and is the smallest such real number.

Consequently $a - 1$ is not an upper bound for N (if it were, since $a - 1 < a$, then a would not be the least upper bound). Therefore there is some integer k with $a - 1 < k$. But then $a < k + 1$. This contradicts that a is an upper bound for N.

Proof: Since $x > 0$, the statement that there is an integer n so that $nx > y$ is equivalent to finding an n with $n > y/x$ for some n, But if there is no such n then $n < y/x$ for all integers n. That is, y/x would be an upper bound for the integers. This contradicts the Lemma. □

Think example that will have an ordered field that does not have the Archimedean property.

Summary

If a and b are positive integers, then there exists a positive number n, such that

$$na \geq b$$

Example: Let $a = 7$ and $b = 100$, then we can choose $n = 100$ so that $100(7) > 100$.