

The Big Little Book of Topology Notes

The Students of Math 147
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CHAPTER 1

Definitions and Examples

1. Introduction

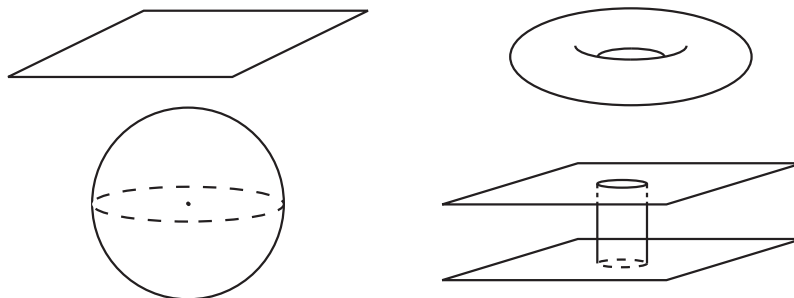
What is geometry? Geometry is the study of rigid shapes that can be distinguished with measurements (length, angle, area, ...).

What is topology? Topology is the study of shapes which are equivalent via deformations.

Topology versus Geometry: Objects that have the same topology do not necessarily have the same geometry. For instance, a square and a triangle have different geometries but the same topology.



Motivation: Our goal is to understand the shape of our universe. Consider the following examples of two-dimensional universes: a plane, a sphere, a torus, and planes connected by tubes.



These are topologically distinct universes. Intuitively, we can see that their “holes” distinguish them. Hence we seek a mathematical way to describe the holes; one familiar concept we will use is continuity, which is related to a lack of holes.

2. Some Review from Analysis

DEFINITION. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$. We say f is continuous at a if $\forall \epsilon > 0 \exists \delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

DEFINITION. Let M be a set and $d : M \times M \rightarrow \mathbb{R}$ be a function such that

$$(1) \ d(a, b) = 0 \text{ iff } a = b \text{ (nondegeneracy)}$$

$$(2) \ \forall a, b, c \in M, \ d(b, c) \leq d(a, b) + d(a, c) \text{ (triangle inequality)}.$$

Then we say d is a **metric** or distance and (M, d) denotes a **metric space**.

Compare this to the usual definition of a metric space. The above definition is equivalent to the usual definition of a metric space, but does not explicitly state the properties of positivity or symmetry:

$$(1) \ d(a, b) \geq 0 \ \forall a, b \in M \text{ (positivity)}$$

$$(2) \ d(a, b) = d(b, a) \ \forall a, b \in M \text{ (symmetry)}.$$

DEFINITION. The **usual metric** on \mathbb{R}^n is

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}.$$

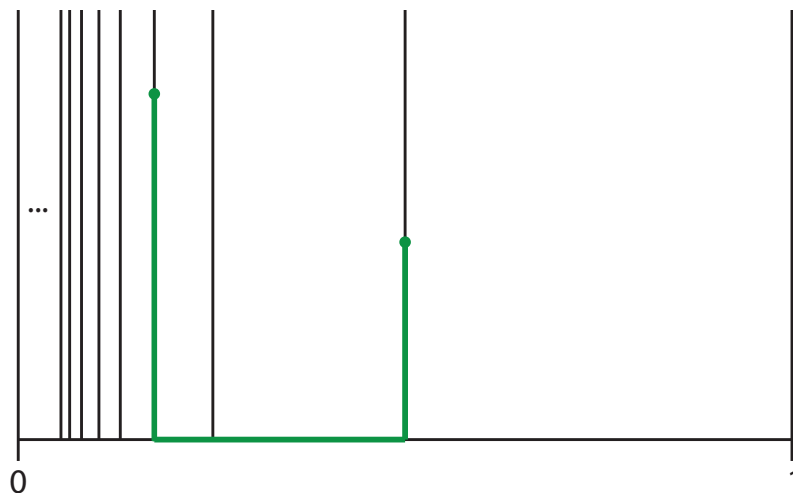
DEFINITION. Let M be any set. Then the **discrete metric** is defined as

$$d(a, b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

The discrete metric can be useful for testing conjectures as it does not rely on \mathbb{R}^n .

EXAMPLE. The Comb Metric for \mathbb{R}^2 .

Let $X_0 = \{0\} \times [0, 1]$, $Y_0 = [0, 1] \times \{0\}$; and $\forall n \in \mathbb{N}$, let $X_n = \{\frac{1}{n}\} \times [0, 1]$. Let $M = (\cup_{n=0}^{\infty} X_n) \cup Y_0$. The distance is the distance measured along the comb in \mathbb{R}^2 .



Using the comb metric,

- Does the sequence $\{(\frac{1}{n}, 0)\}$ converge? Yes, to the origin.
- Does the sequence $\{(\frac{1}{n}, a)\}$ converge when $a \in (0, 1]$? No, since $d((\frac{1}{n}, a), (0, a)) > 2a \forall n$.

EXAMPLE. A Non-Example of a Metric Space.

Consider the line of real numbers with $d(a, b) = a - b$. Clearly, $d(a, b) = 0$ only when $a = b$, satisfying the first property (nondegeneracy). However, the second property (triangle inequality) is not satisfied: if $a = 0$, $b = 1$, and $c = -1$, then $2 = d(b, c) \not\leq d(a, b) + d(a, c) = -1 + 1 = 0$. Notice that this non-example also fails to satisfy the other two properties listed in the usual definition of a metric, positivity and symmetry.

EXAMPLE. Let M be a set and $d : M \times M \rightarrow \mathbb{R}$ a function which satisfies property 2 but not property 1 in the definition of a metric space. Then

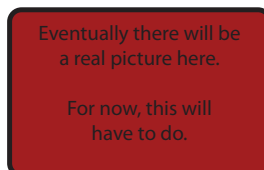
$$d(a, b) = 0 \quad \forall a, b \in M$$

and M has at least 2 points.

3. Continuity, metric spaces, and open balls

We'd like to be able to talk about continuity in metric spaces (not just the reals).

DEFINITION. Let (M_1, d_1) and (M_2, d_2) be metric spaces and $a \in M_1$. Let $f : M_1 \rightarrow M_2$. We say f is **continuous** at a if $\forall \epsilon > 0$ there exists a $\delta > 0$ such that $\forall x \in M_1$ with $d_1(x, a) < \delta$ then $d_2(f(x), f(a)) < \epsilon$.



DEFINITION. Let (M, d) be a metric space and $a \in M$. The **open ball of radius $\epsilon > 0$ about a** is defined to be

$$B_\epsilon(a) = \{x \in M \mid d(x, a) < \epsilon\}$$

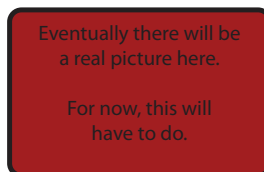
We can redefine ‘continuous’ in terms of open balls:

DEFINITION. A function $f : X \rightarrow Y$ is **continuous** at $a \in X$ iff $\forall \epsilon > 0$ there exists a $\delta > 0$ such that if $x \in B_\delta(a)$ then $f(x) \in B_\epsilon(f(a))$.

In other words, f is continuous at a iff for every $\epsilon > 0$ there is some $\delta > 0$ such that $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$.

It should be noted that open balls don’t always look like open balls...

EXAMPLE. $M =$ upper half of plane with usual distance



An open ball can look like a semicircle if it rests on the x-axis.

EXAMPLE. \mathbb{R} with the discrete metric.

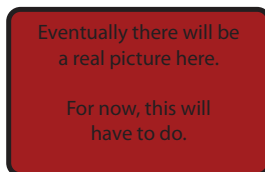
$$B_1(47) = 47$$

$$B_2(47) = \mathbb{R}$$

EXAMPLE. Comb metric

$B_{\frac{1}{2}}((0, 1))$ is just an interval along the y-axis starting at $(0, 1)$

$B_2((0,1))$ is everything on the comb below the line of slope -1 that connects points $(0,1)$ and $(1,0)$. Since it's an open ball, the line is not contained in the ball.



Note: Balls are not closed under \bigcup and \bigcap

4. Open sets

DEFINITION. Let (M, d) be a metric space. A set $U \subseteq M$ is said to be **open** if $\forall a \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(a) \subseteq U$.

EXAMPLE. Every subset of a set M under the discrete metric is open.

SMALL FACT. In a metric space, open balls are open sets.

PROOF. Let $a \in M$ and $\epsilon > 0$. We want to show that $B_\epsilon(a)$ is open. That is, we want to show that $\forall x \in B_\epsilon(a)$ there exists an $r > 0$ such that $B_r(x) \subseteq B_\epsilon(a)$.

Let $x \in B_\epsilon(a)$ and let $r < \epsilon - d(x, a)$ and $r > 0$. We claim that $B_r(x) \subseteq B_\epsilon(a)$.

Let $y \in B_r(x)$. Then $d(y, a) \leq d(y, x) + d(x, a) < r + d(x, a) < \epsilon - d(x, a) + d(x, a) = \epsilon$.

Therefore, $B_r(x) \subseteq B_\epsilon(a)$

DEFINITION. Let

$$\bigcup_{i \in I} U_i = \{x \in U_i \mid i \in I\}$$

THEOREM 1. (the “open sets are nice” theorem)

Let F be the family of open sets in a metric space (M, d) . Then:

- (1) $M, \emptyset \in F$
- (2) If $U, V \in F$ then $U \cap V \in F$
- (3) If $\forall i \in I, U_i \in F$ then $\bigcup_{i \in I} U_i \in F$

PROOF. (1) Let $x \in M$, and $\epsilon = 47$. By definition,

$$B_{47}(x) = \{y \in M \mid d(x, y) < 47\} \subseteq M$$

Also, trivially $\forall x \in \emptyset$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \emptyset$. Therefore, M and \emptyset are open, and so $M, \emptyset \in F$.

- (2) Let $x \in U \cap V$, let $r_1 > 0$ such that $B_{r_1}(x) \subseteq U$, and let $r_2 > 0$ such that $B_{r_2}(x) \subseteq V$. Let $r = \min\{r_1, r_2\}$. Then clearly $B_r(x) \subseteq B_{r_1}(x) \subseteq U$ and $B_r(x) \subseteq B_{r_2}(x) \subseteq V$. Thus $B_r(x) \subseteq U \cap V$ as desired.
- (3) Let $x \in \bigcup_{i \in I} U_i$. We want to show that there exists an $r > 0$ such that $B_r(x) \subseteq \bigcup_{i \in I} U_i$.

Because $x \in \bigcup_{i \in I} U_i$, there exists an $i_0 \in I$ such that $x \in U_{i_0}$. Then there must exist an $r > 0$ such that $B_r(x) \subseteq U_{i_0}$. Then

$$B_r(x) \subseteq U_{i_0} \subseteq \bigcup_{i \in I} U_i$$

and hence $\bigcup_{i \in I} U_i \in F$.

□

DEFINITION. If $f : X \rightarrow Y$ and $U \subseteq Y$ then $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$.

THEOREM 2. Let (M_1, d_1) and (M_2, d_2) be metric spaces and $f : M_1 \rightarrow M_2$. Then f is continuous if and only if for every $U \subseteq M_2$ that is open in (M_2, d_2) , $f^{-1}(U)$ is open in (M_1, d_1) .

PROOF. (\Rightarrow) Suppose f is continuous. Let $U \subseteq M_2$ be open. We want to show that $\forall x \in f^{-1}(U)$ there exists an $r > 0$ such that $B_r(x) \subseteq f^{-1}(U)$.

Let $x \in f^{-1}(U)$. Since $f(x) \in U$ and U is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subseteq U$. Since f is continuous there exists an $r > 0$ such that

$$f(B_r(x)) \subseteq B_\epsilon(f(x)) \subseteq U$$

Thus, $f^{-1}(f(B_r(x))) \subseteq f^{-1}(U)$. Therefore $B_r(x) \subseteq f^{-1}(f(B_r(x))) \subseteq f^{-1}(U)$ as desired.

(\Leftarrow) Suppose that for all open $U \subseteq M_2$, $f^{-1}(U)$ is open in M_1 . Let $p \in M_1$. We want to show that f is continuous at p .

Let $\epsilon > 0$. $B_\epsilon(f(p))$ is open, so $f^{-1}(B_\epsilon(f(p)))$ is open. $p \in f^{-1}(B_\epsilon(f(p)))$, so $\exists \delta > 0$ such that $B_\delta(p) \subseteq f^{-1}(B_\epsilon(f(p)))$.

$f(B_\delta(p)) \subseteq f(f^{-1}(B_\epsilon(f(p)))) = B_\epsilon(f(p))$, so f is continuous at p , and thus everywhere. ■

□

We've proven a small fact:

SMALL FACT. If $f : M_1 \rightarrow M_2$, then f is continuous if for all $p \in M_1$, and for all $\epsilon > 0$ $f^{-1}(B_\epsilon(f(p)))$ is open.

5. Topologies

We haven't actually defined what a topology is yet. Let's fix that.

DEFINITION. Let X be a set and F be some collection of subsets of X such that

- 1) $X, \emptyset \in F$.
- 2) If $U, V \in F$ then $U \cap V \in F$.
- 3) If for all $i \in I$, $U_i \in F$, then $\bigcup U_i \in F$.

Then we say that (X, F) is a **topological space** with **open sets** the elements of F . We also say F is the **topology** on X .

EXAMPLE. Let (M, d) be a metric space and F be the set of open sets in M . Then (M, F) is a topological space.

DEFINITION. Let M be a set and d be the discrete metric, then we say (M, F) is the **discrete topology**.

DEFINITION. Let X be a set with at least 2 points. Let $F = \{X, \emptyset\}$. Then we say (X, F) is the **indiscrete**, or **concrete** topology.

DEFINITION. If F_1 and F_2 are topologies on X and $F_1 \subseteq F_2$ then we say that F_1 is **weaker** than F_2 , or F_2 is **stronger** than F_1 .

Two notes:

- weaker = fewer = coarser and stronger = more = finer.
- The discrete topology is the strongest topology on M and the indiscrete topology is the weakest topology on X .

EXAMPLE. Let $X = \mathbb{R}$ and $U \in F$ iff U is the union of sets of the form $[a, b)$ such that $a, b \in \mathbb{R}$. (This is called the **half-open interval topology**).

Is this weaker, stronger, or neither compared to the usual topology?

If the interval $(a, b) \in F$, then F is stronger. Let $a, b \in \mathbb{R}$, and $a < b$. Then, $(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b)$, so F is stronger.

EXAMPLE. Let $X = \mathbb{R}^2$ have the "dictionary order". This means that $(a, b) < (c, d)$ if either $a < c$ or $a = c$ and $b < d$.

Note that $U \in F$ iff U is a union of "open intervals", i.e. $\{(x, y) \mid (a, b) < (x, y) < (c, d)\}$.

- Is a vertical line open? Yes.
- Is a horizontal line open? No.
- Is this topology finer or coarser than the usual topology on \mathbb{R}^2 ?

Well, any point in a ball in the usual topology can be found in a ball of the dictionary topology, which is contained in the usual ball. Open balls are open in the dictionary topology, so the dictionary topology is finer than the usual topology.

Note: In \mathbb{R} , $\bigcup_{n \in F} (n, n + 1)$ is open.

Question: Are the topologies on a set linearly ordered? No!

EXAMPLE. Consider \mathbb{R} with the usual topology and (\mathbb{R}, F) with $F = \{\mathbb{R}, \phi, 47\}$. We say these two topologies are **incomparable**.

EXAMPLE. Consider (\mathbb{R}, F) Define $U \in F$ if and only if either $U = \phi$ or $U = \mathbb{R}$, or $\mathbb{R} - U$ is finite. We see that if U is open in (\mathbb{R}, F) , then U is open in the usual topology. Therefore the usual topology is finer. This topology is called the **finite complement topology on \mathbb{R}** .

DEFINITION. A set C in a topological space (X, F) is **closed** iff its complement $X \setminus C$ is open.

Note that a set can be both open and closed as well as neither open nor closed. (It is not a door!)

EXAMPLE. Let us consider \mathbb{R} with the half-open interval topology. Then $[0, \infty) = \bigcup_{n \in \mathbb{N}} [0, n)$ is open. On the other hand, the complement of this set is $(-\infty, 0) = \bigcup_{n \in \mathbb{N}} [-n, 0)$, which is also open. Thus this is a **clopen** set.

EXAMPLE. A "vertical line" in \mathbb{R}^2 with the dictionary order is both open and closed. The proof is left as exercise.

LEMMA. Let (X, F) be a topological space and A be the set of all closed sets in X . Then:

- (1) $X, \phi \in A$
- (2) If $C, D \in A$, then $C \cap D \in A$
- (3) If $C_i \in A$ for every $i \in I$, then $\bigcap_{i \in I} C_i \in A$

The proof is basically the same as that for open sets back in section 4.

DEFINITION. Let (X, F) be a topological space and $A \subseteq X$. Let $\{U_j \mid j \in J\}$ be the set of all open sets contained in A . Then we define $\overset{\circ}{A} = \text{Int}(A) = \bigcup_{j \in J} U_j$, and we say $\overset{\circ}{A}$ is the **interior** of A .

SMALL FACT. Let (X, F) be a topological space and $A \subseteq X$. Then

- (1) $\overset{\circ}{A} \subseteq A$
- (2) $\overset{\circ}{A}$ is open
- (3) If $U \subseteq A$ is open, then $U \subseteq \overset{\circ}{A}$
- (4) A is open iff $A = \overset{\circ}{A}$.

PROOF. (1) Since $\overset{\circ}{A} = \bigcup_{j \in J} U_j$ and $U_j \subseteq A$ for every $j \in J$, $\overset{\circ}{A} \subseteq A$.

(2) By definition, $\overset{\circ}{A}$ is a union of open sets, so $\overset{\circ}{A}$ is open.

(3) Since U is open in A , $U \in \{U_j \mid j \in J\}$. Therefore $U \subseteq \bigcup_{j \in J} U_j = \overset{\circ}{A}$.

Note that this means that $\overset{\circ}{A}$ is the “largest” open set in A .

(4) Suppose that $A = \overset{\circ}{A}$. Then $\overset{\circ}{A}$ is open by part (2), hence A is open.

Conversely, suppose that A is open. Then $A \in \{U_j \mid j \in J\}$. Hence $A \subseteq \overset{\circ}{A}$. From (1), $A = \overset{\circ}{A}$.

□

EXAMPLE. In the half-open interval topology on \mathbb{R} , $\text{Int}((0, 1]) = (0, 1)$. To prove this, assume that $1 \in \text{Int}((0, 1])$ and show that it leads to a contradiction.

EXAMPLE. In the finite-complement topology on \mathbb{R} , $\text{Int}((0, 1]) = \emptyset$. This follows from the fact that \mathbb{R} is infinite and all subsets of $(0, 1]$ are finite.

EXAMPLE. In the dictionary-order topology on \mathbb{R}^2 , $\text{Int}([0, 1] \times [0, 1]) = [0, 1] \times (0, 1)$.

Eventually there will be
a real picture here.

For now, this will
have to do.

DEFINITION. Let A be a subset of a topological space (X, F) , and let $\{F_j \mid j \in J\}$ be the set of all closed sets containing A . Then the **closure** of A is defined as $\bar{A} = \text{cl}(A) = \bigcap_{j \in J} F_j$.

SMALL FACT. Small facts about closures:

- (1) $A \subseteq \bar{A}$
- (2) \bar{A} is closed
- (3) If $A \subseteq C$ and C is closed, then $\bar{A} \subseteq \bar{C}$
- (4) $\bar{A} = A$ if and only if A is closed.

The proofs are left as exercises.

EXAMPLE. In \mathbb{R} with the usual topology, $\text{cl}(\{\frac{1}{n} \mid n \in \mathbb{N}\}) = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

EXAMPLE. In \mathbb{R} with the half-open interval topology, $\text{cl}((0, 1]) = [0, 1]$

LEMMA. (*an Important Lemma*)

Let (X, F) be a topological space and $Y \subseteq X$. Then $p \in \bar{Y}$ if and only if for every open set $U \subseteq X$ containing p , $U \cap Y \neq \emptyset$.

PROOF. Let $p \in \bar{Y}$ and $U \subseteq X$ be open with $p \in U$. Suppose $U \cap Y = \emptyset$ and let $C = X \setminus U$. Then $p \notin C$ because $p \in U$. It follows that $\bar{Y} \subseteq C$ because $Y \subseteq C$ and C is closed. We obtain that $p \in C$, which is a contradiction.

Conversely, suppose that for every open set $U \subseteq X$ such that $p \in U$, $U \cap Y \neq \emptyset$. Let $C \subseteq X$ be closed such that $Y \subseteq C$. Suppose $p \notin C$. Let $U = X \setminus C$, which is open with $p \in U$. Now $U \cap Y \neq \emptyset$, so there exists some $x \in U \cap Y$. This means that $x \in X \setminus C$, i.e. $x \notin C$. But since $Y \subseteq C$, we have $x \notin Y$, a contradiction. Therefore we conclude that $p \in \bar{Y}$. \square

COROLLARY. Suppose that U is an open set in a topological space (X, F) and $Y \subseteq X$. If $U \cap \bar{Y} \neq \emptyset$, then $U \cap Y \neq \emptyset$.

PROOF. The proof is immediate from the important lemma.

Now we can use the important lemma to prove that in \mathbb{R} in the usual topology, $\text{cl}(\{1/n \mid n \in \mathbb{N}\}) = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$. Here, we can show that for all open sets U containing 0, $U \cap \{1/n \mid n \in \mathbb{N}\} \neq \emptyset$.

6. Continuity in Topological Spaces

DEFINITION. Let (X_1, F_1) and (X_2, F_2) be topological spaces and $f : X_1 \rightarrow X_2$. We say f is **continuous** if and only if for every $U \in F_2$, $f^{-1}(U) \in F_1$.

SMALL FACT. Let (X_1, F_1) and (X_2, F_2) and (X_3, F_3) be topological spaces, $f : X_1 \rightarrow X_2$ and $g : X_2 \rightarrow X_3$ be continuous functions. Then $g \circ f : X_1 \rightarrow X_3$ is continuous.

PROOF. Let $U \in F_3$. Then $g^{-1}(U) \in F_2$ because g is continuous, and $f^{-1}(g^{-1}(U)) \in F_1$ because f is continuous. Therefore $(g \circ f)^{-1}(U) \in F_1$. Thus $g \circ f$ is continuous. \square

THEOREM 3. Let X, Y be topological spaces and $f : X \rightarrow Y$. Then f is continuous if and only if for every closed set C in Y , $f^{-1}(C)$ is closed in X .

(\Rightarrow) Suppose C is closed in Y . Then $Y \setminus C$ is open, implying that $f^{-1}(Y \setminus C)$ is open in X . Since $f^{-1}(Y \setminus C) = \{x \in X \mid f(x) \in Y \setminus C\} = \{x \in X \mid f(x) \notin C\}$ and $X \setminus f^{-1}(C) = \{x \in X \mid f(x) \notin C\}$, we have the desired results.

\Leftarrow This is exactly like the analogous proof for open sets.

This is nice, because sometimes it's easier to work with closed sets than with open sets.

DEFINITION. Let X and Y be topological spaces, and let $f : X \rightarrow Y$.

- (1) If for every open set $U \subseteq X$, $f(U)$ is open in Y , then f is **open**.
- (2) If for every closed set $U \subseteq X$, $f(U)$ is closed in Y , then f is **closed**.

This is sort of like continuity, except that we care about the image of sets instead of their preimages.

Example time!

EXAMPLE. Let F be the half-open topology on \mathbb{R} , and define a function

$$f : (\mathbb{R}, F) \rightarrow (\mathbb{R}, \text{usual}) \quad \text{by } f(x) = x$$

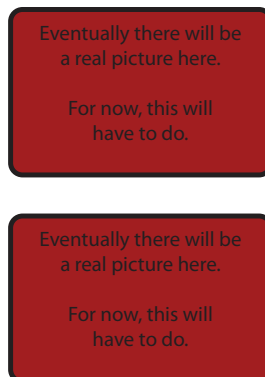
- Is f continuous? Yes! An open set in $(\mathbb{R}, \text{usual})$ is a union of intervals of the form (a, b) . We know that $f^{-1}((a, b)) = (a, b)$, which is open in F .
- Is f open? No. Take any $U = [a, b) \in F$. Then $f(U) = [a, b)$, which is not open in $(\mathbb{R}, \text{usual})$.
- Is f closed? Also no, since $f([a, b)) = [a, b)$ is not closed in $(\mathbb{R}, \text{usual})$.

7. Homeomorphisms

Let's define a notion of equivalency for topological spaces.

DEFINITION. Let (X_1, F_1) and (X_2, F_2) be topological spaces, and let $f : X_1 \rightarrow X_2$ be continuous, bijective, and open. Then f is a **homeomorphism**, and X_1 and X_2 are **homeomorphic** (denoted by $X_1 \cong X_2$).

EXAMPLE. Let $I = [0, 1]$ and $X = I \times I \subset \mathbb{R}^2$, under the usual metric for \mathbb{R}^2 . Let $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} = D^2$ be the unit disk in \mathbb{R}^2 . Are X and Y homeomorphic?



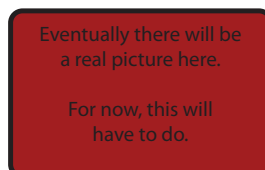
Yes! Let's see how.

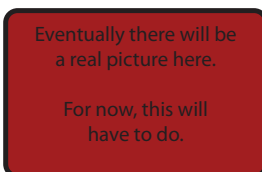
Define the centers of X and Y to be x and y , respectively. Fix some point a on the boundary of X , and some b on the boundary of Y (that is, $a \in \partial X$ and $b \in \partial Y$).

Define a function f as follows:

- $f(x) = y$
- $f(a) = b$
- For $t \in \partial X$, look at the distance along the boundary from a to t . Then $f(t)$ is a point proportionally far along the boundary of Y .
- For $s \in \text{int}(X)$, draw the ray connecting x and s . Let t be the point at which this ray intersects ∂X . Now, in Y , draw the ray connecting y and $f(s)$. Then s is mapped to a point on this ray that is proportionally far from y .

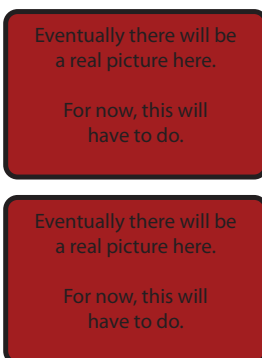
The last two, in pictures:





- Is this well-defined?
 - Yes! There is always exactly one point in Y that can be mapped to be a point in X^1 .
- Is this a bijection?
 - Yes! The inverse is defined identically, so it would make sense for this to be a bijection. Also, consider the images of concentric squares centered on x under f : they are mapped to disjoint concentric circles centered on y .
- Is f continuous and open?
 - Yes! Intuitively, it's easy to see that an open set in X is mapped to an open set in Y , and that the preimage of an open set in Y is open.

Can we extend this to (some) non-convex regions:



Sure. Just divide up the non-convex region into smaller regions. Actually, a subregion will work as long as there is some point in its interior such that any ray from that point intersects the subregion's boundary exactly once.

However, there are limits to this: for example, a donut is *not* homeomorphic to a circle. This is hard to show; we'll see a proof later.

There are also some homeomorphisms that we might find unsettling. For example, a knot in \mathbb{R}^3 and the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ are homeomorphic; the argument works exactly like the one used for ∂X when showing that a square and circle

¹This works because both a square and a circle are *convex* shapes - for example, for all points $p, q \in X$, the line \overline{pq} that connects p and q lies entirely within X . This also implies that x and y didn't actually have to be the exact *centers* of X and Y respectively.

are homeomorphic (above). It turns out that, while the knot and circle are homeomorphic, their complements are not.

EXAMPLE. Define a function $f : [0, 1) \rightarrow S^1$ by

$$f(t) = (\cos(2\pi t), \sin(2\pi t))$$

This takes the interval and wraps it counterclockwise around the circle.

- Is this a bijection?
 - Yes! The interval wraps once around the circle; one end is open, so there is no overlap.
- Is f continuous?
 - Yes! First we define an open ball in S^1 as $B_{\epsilon, S^1}(a) = \{x \in S^1 \mid d(x, a) < \epsilon\}$. Then the preimage of every open ball in S^1 is an open interval, so f is continuous.
- Is f open?
 - No. Let $U = [0, 1/2)$, which is open in $[0, 1)$. Sadly, its image is not open in S^1 .

So this f is not a homeomorphism.

We will show that $\mathbb{R}^2 \not\cong \mathbb{R}^3$ near the end of Chapter 3. This is difficult.

EXAMPLE. Some non-examples of homeomorphisms:

- \mathbb{Q} under the usual topology is not homeomorphic to \mathbb{R} under the usual topology, since there is no bijection between \mathbb{Q} and \mathbb{R} .
- \mathbb{R} with the finite-complement topology (FCT) is not homeomorphic to \mathbb{R} with the usual topology:
 - Suppose that there is some homeomorphism $f : (\mathbb{R}, FCT) \rightarrow (\mathbb{R}, \text{usual})$. Let $U = (0, 1)$.
 - Look at $f^{-1}(U)$ - it must be open, since f is a homeomorphism.
 - It is definitely not equal to \emptyset , since $U \neq \emptyset$.
 - It is also not \mathbb{R} , since f is a bijection.
 - So $\mathbb{R} \setminus f^{-1}(U)$ must be finite. But $\mathbb{R} \setminus U$ is not finite.
 - Because f is a bijection, this is a contradiction.
 - Therefore, $(\mathbb{R}, FCT) \not\cong (\mathbb{R}, \text{usual})$.

8. Subspaces

DEFINITION. Let (X, F_X) be a topological space, and let $Y \subseteq X$. Let $F_Y = \{U \cap Y \mid U \in F_X\}$. Then F_Y is the **subspace topology** or **induced topology** on Y .

EXAMPLE. Consider the topological space \mathbb{R}^2 with the dictionary order.

Q: What are the open sets in the subspace $\mathbb{Z} \times \mathbb{Z}$?

A: All sets are open. We can draw a small open interval about any point, so the set of any one point is open, and any set is the union of such sets. Because of all sets are open, we can consider this topology the discrete topology on $\mathbb{Z} \times \mathbb{Z}$.

Example: consider $\mathbb{Z} \times \mathbb{Z}$ as a subspace of \mathbb{R}^2 , but this time with the usual topology. Again, all sets are open, as we can construct an open ball of radius $\frac{1}{2}$ about any point that doesn't intersect any others in the same way that we can construct an open interval.

Small facts about subspaces:

Let (X, F_X) and (Y, F_Y) be topological subspaces with (S, F_S) a subspace of X and (T, F_T) a subspace of Y . Then

- (1) If $S \in F_X$, then $F_S \subseteq F_X$.
- (2) $C \leq S$ iff \exists a closed set A in X such that $C = A \cap S$.
- (3) Suppose $f : X \rightarrow Y$ is continuous. Then $f|_S : S \rightarrow Y$ is continuous.
- (4) Suppose $f : X \rightarrow Y$ is continuous and $f(X) \subseteq T$. Let $g : X \rightarrow T$ be such that $g(x) = f(x)$ for every $x \in X$. Then g is continuous.

Proofs:

- (1) If $S \in F_X$, then S is open in X . By definition $F_S = \{U \cap S : U \in F_X\}$
- (2) (\Rightarrow) Let $C \subseteq S$ be closed. Because $S \setminus C \in F_S$, there is some $U \in F_X$ such that $U \cap S = S \setminus C$. Since U is open in X , we know that $X \setminus U$ is closed in X . We claim that the closed set we desire is $X \setminus U$. Note that

$$X \setminus (U \cap S) = (X \cap S) \setminus (U \cap S) = S \setminus (U \cap S) = S \setminus (S \setminus C) = C$$

and so we are done

- (\Leftarrow) Suppose that there is a closed set $A \subseteq X$ such that $C = A \cap S$. Then $X \setminus A \in F_X$, so $(X \setminus A) \cap S \in F_S$. But $X \setminus (A \cap S) = (X \cap S) \setminus (A \cap S) = S \setminus C$ is open in S , so C is closed in S .
- (3) Let $U \in F_Y$. Since f is continuous, $f^{-1}(U) \in F_X$. Because $(f|_S)^{-1}(U) = f^{-1}(U) \cap S$, $(f|_S)^{-1}(U)$ is the intersection of an open set with S and so is open. Therefore, $f|_S$ is continuous.

- (4) Let $U \in F_T$. Then there is some $V \in F_Y$ such that $U = V \cap T$. Because $f : X \rightarrow Y$, $f^{-1}(V) \in F_X$. Since $f(X) \subseteq T$,

$$f^{-1}(U) = f^{-1}(V) \cap T = f^{-1}(V \cap T) = f^{-1}(V)$$

Therefore, $f^{-1}(U)$ is open in X . But $f^{-1}(U) = g^{-1}(U)$, so $g^{-1}(U)$ is open and so g is continuous.

9. Bases

You may have noticed that, in metric spaces, the idea of open sets was quite useful. We'd like to extend this idea to more general topological spaces.

DEFINITION. Let (X, F) be a topological space and $\beta \subseteq F$ such that for every $U \in F$, U is a union of elements in β . Then we say that β is a **basis** for F .

Note that this basis is not necessarily minimal (like the basis of a vector space), but it is more useful the more specific it is.

EXAMPLE. \mathbb{R} with the half-open interval topology was *defined* by the basis $\{[a, b) \mid a < b\}$.

EXAMPLE. \mathbb{R}^2 with the dictionary topology was similarly defined by a basis of open intervals.

THEOREM 4. Let X be a set and β be a collection of subsets such that

- (1) $X = \bigcup_{B \in \beta} B$
- (2) $\forall B_1, B_2$, if $x \in B_1 \cap B_2$, then there is some $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
- (3) Let $F =$ collection of elements of β . Then F is a topology on X with basis β .

THEOREM 5. Let X be a set and β a collection of subsets of X such that

- (1) $X = \bigcup_{B \in \beta} B$
- (2) For all $B_1, B_2 \in \beta$ and $x \in B_1 \cap B_2$, $\exists B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Let $F = \{\text{unions of elements of } \beta\}$. Then F is a topology for X with basis β .

PROOF. First we prove that F is a topology for X .

- (1) Since $X = \bigcup_{B \in \beta} B$, by definition $X \in F$. Since \emptyset is the union of zero elements of the β , we also have $\emptyset \in F$.
- (2) Let $U, V \in F$. Hence we know that there exist index sets I and J such that $U = \bigcup_{i \in I} B_i$ and $V = \bigcup_{j \in J} B_j$. Consider $U \cap V = (\bigcup_{i \in I} B_i) \cap (\bigcup_{j \in J} B_j)$. Let $x \in U \cap V$. Then there exists $i_x \in I$ and $j_x \in J$ such that $x \in B_{i_x} \cap B_{j_x}$. From our second assumption we know there exists a $B_x \in \beta$ such that $x \in B_x \subseteq B_{i_x} \cap B_{j_x}$. Let $W = \bigcup_{x \in U \cap V} B_x$. Since W is a union of elements of β it is clearly in F .

WTS: $W = U \cap V$

(\supseteq) For all $x \in U \cap V$, we know that $x \in B_x \subseteq \cup_{x \in U \cap V} B_x = W$. Therefore $U \cap V \subseteq W$.

(\subseteq) For all $x \in U \cap V$, we have a $B_x \subseteq B_{i_x} \cap B_{j_x} \subseteq U \cap V$. Therefore, $W = \cup_{x \in U \cap V} B_x \subseteq U \cap V$.

We have containment in both directions, so $W = U \cap V$.

(3) Suppose $\forall k \in K, U_k \in F$.

WTS: $\cup_{k \in K} U_k \in F$.

For all $k \in K, U_k = \cup_{i \in I_k} B_i$. Hence $\cup_{k \in K} U_k = \cup_{k \in K} (\cup_{i \in I_k} B_i) \in F$, since it is a union of elements of β .

Therefore F is a topology. By definition, β is also a basis of F . □

SMALL FACT. Let (X, F_X) and (Y, F_Y) be topological spaces with bases of β_X and β_Y respectively, and $f : X \rightarrow Y$.

(1) f is continuous iff $\forall B \in \beta_Y, f^{-1}(B) \in F_X$.

(2) f is open iff $\forall B \in \beta_X, f(B) \in F_Y$.

PROOF. (of (1) only. (2) is virtually identical.)

(\Rightarrow) Suppose f is continuous. Then $\forall U \in F_Y, f^{-1}(U) \in F_X$ by the definition of continuity. In particular, if $B \in \beta_Y$, then $B \in F_Y$ and $f^{-1}(B) \in F_X$.

(\Leftarrow) Suppose $B \in \beta_Y$ implies $f^{-1}(B) \in F_X$. Let $U \in F_Y$. Hence $U = \cup_{i \in I} B_i$ for some index set I . Therefore,

$$f^{-1}(U) = f^{-1}(\cup_{i \in I} B_i) = \cup_{i \in I} f^{-1}(B_i) \in F_X,$$

since we know $f^{-1}(B_i) \in F_X$ for all i and unions of elements of F_X are in F_X . □

CHAPTER 2

Making new spaces from old

1. Quotient Spaces

First we will consider quotients of sets.

DEFINITION. Let X be a set and \sim a relation on X . We say \sim is an **equivalence relation** if

- (1) $\forall x \in X, x \sim x$ (reflexivity)
- (2) $\forall x, y \in X$ if $x \sim y$ then $y \sim x$ (symmetry)
- (3) If $x, y, z \in X$ and $x \sim y$ and $y \sim z$, then $x \sim z$ (transitivity)

DEFINITION. Let X be a set with an equivalence relation \sim . Then $\forall a \in X$ define the **equivalence class** of a as

$$[a] = \{x \in X : x \sim a\}.$$

DEFINITION. Let X be a set with an equivalence relation \sim . Then the **quotient** of X by \sim is

$$X/\sim = \{[p] : p \in X\}.$$

NOTE: The equivalence classes partition X (i.e. $\forall x \in X, x \in$ exactly one equivalence class)

EXAMPLE. Similarity (i.e. $A \sim B$ iff there exists a P such that $A = P^{-1}BP$) is an equivalence relationship on the set of $n \times n$ matrices. However, this is not the “kind” of equivalence relationships that we will be studying.

EXAMPLE. Let $X = [0, 1]$ and $x \sim y$ iff $x = y$ or $x, y \in \{0, 1\}$. This “glues” the interval $[0, 1]$ into a circle.

DEFINITION. Let X be a set and \sim an equivalence relation. We define

$$\pi : X \rightarrow X/\sim$$

such that $\pi(x) = [x] \forall x \in X$. We say π is the **projection map**.

TINY FACT. Let X be a set with equivalence relation \sim . Then

- (1) π is onto

(2) π is one-to-one iff “ \sim ” is “ $=$ ”.

PROOF. (1) Let $[x] \in X/\sim$. Then $\pi(x) = [x]$. Therefore π is onto.

(2) (\Rightarrow) Suppose π is one-to-one. Let $x, y \in X$ and $x \sim y$. Then $[x] = [y]$ and $\pi(x) = \pi(y)$, implying $x = y$ since π is one-to-one.

(\Leftarrow) Suppose “ \sim ” is “ $=$ ”. Let $x, y \in X$ such that $\pi(x) = \pi(y)$. Then $\{x\} = [x] = [y] = \{y\}$, and $x = y$.

□

Now we would like to define a topology such that π is continuous.

DEFINITION. Let (X, F_X) be a topological space and \sim be an equivalence relation on X . We define

$$F_\sim = \{U \subseteq X/\sim : \pi^{-1}(U) \in F_X\}$$

and call $(X/\sim, F_\sim)$ the **quotient space** of X with respect to \sim .

SMALL FACT. Let (X, F_X) be a topological space and \sim be an equivalence relation on X . $(X/\sim, F_\sim)$ is a topological space and π is continuous.

PROOF. First let us prove that F_\sim is a topology on X/\sim .

(1) Since $\pi^{-1}(X/\sim) = X \in F_X$, clearly $X/\sim \in F_\sim$. Since $\pi^{-1}(\emptyset) = \emptyset \in F_X$, $\emptyset \in F_\sim$.

(2) Let $U, V \in F_\sim$. Now

$$\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V) \in F_X$$

since $\pi^{-1}(U) \in F_X$ and $\pi^{-1}(V) \in F_X$. Therefore $U \cap V \in F_\sim$.

(3) Consider $\pi^{-1}(\cup_{i \in I} U_i) = \cup_{i \in I} \pi^{-1}(U_i)$.

Since $\pi^{-1}(U_i) \in F_X \forall i$, the arbitrary union of such sets must also be open. Thus by the above equality, $\pi^{-1}(\cup_{i \in I} U_i) \in F_X$, completing the proof.

Note that the continuity of π follows directly from the quotient topology. □

Question: Is π necessarily open? Answer: No

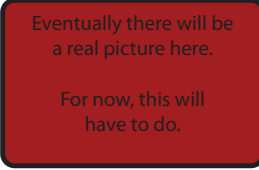
Consider the interval $[0,1]$ mapped to a circle under π . Then the interval $[0,1/2]$, which is open in $[0,1]$ is mapped to a half circle which is not open in the whole circle.

Eventually there will be
a real picture here.

For now, this will
have to do.

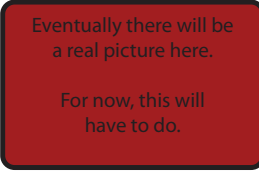
EXAMPLE. Let $X = I \times I$, where I is the unit interval. Define an equivalence relation on X as follows: $(x, y) \sim (x', y')$ if and only if either $(x, y) = (x', y')$ or $x = x'$ and $y, y' \in \{0, 1\}$.

Topologically, X/\sim gives us a cylinder.



EXAMPLE. Let $X = \mathbb{R}^2$ and suppose $(x, y) \sim (x', y')$ if and only if $\exists n, m \in \mathbb{Z}$ such that $x = x' + n, y = y' + m$.

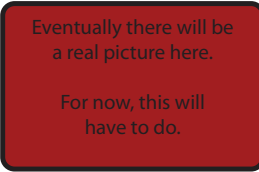
Note that X may be divided into integer side length squares, such that under \sim all of the given squares are equivalent. Thus we need only consider one such square, noting that the opposite sides are equivalent, yielding a torus. We may also generalize X to higher dimensions, yielding the analogous torus for that dimension (i.e. $X = \mathbb{R}^3$ yields a 3-torus and so on).



EXAMPLE. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. Define $x \sim y \Leftrightarrow x = \pm y$.

Note that for $n=1$ we obtain the unit circle such that points connected by a diameter are equivalent. Thus any semi-circle forms a fundamental domain. Such a semicircle has its endpoints as equivalent, and is thus topologically equivalent to the original circle.

We denote the quotient space S^n/\sim by $\mathbb{R}P^n$, or the real projective space. The above argument concludes that $\mathbb{R}P^1 \cong S^1$.

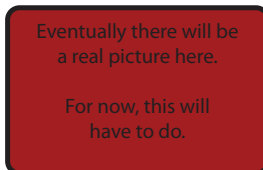


EXAMPLE. $\mathbb{R}P^2 = S^2/\sim$.

Here we simply note that the fundamental domain is a hemisphere whose boundary takes on the same topology of example 3 since it is a circle under \sim .

EXAMPLE. Let $X = \mathbb{R}^2$ and define $(x, y) \sim (x', y') \Leftrightarrow x^2 + y^2 = x'^2 + y'^2$.

Under \sim , any points on a circle centered at the origin are equivalent. Collapsing all such circles, we find that X/\sim is a ray emanating from the origin.



DEFINITION. Let X, Y be sets and $f : X \rightarrow Y$ be a function. We define the relation \sim **induced by** f as follows:

$$\forall p, q \in X, p \sim q \Leftrightarrow f(p) = f(q)$$

It is clear that \sim is an equivalence relation on X because $=$ is an equivalence relation on Y .

DEFINITION. Suppose (X, F_X) is a topological space and Y is a set. Let $f : X \rightarrow Y$ be onto. Then the **quotient topology** on Y with respect to f is given by:

$$F_f = \{U \subseteq Y \mid f^{-1}(U) \in F_X\}$$

If Y has this topology we say that f is a **quotient map**.

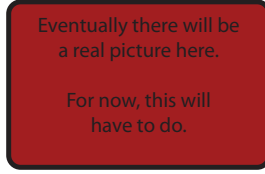
TINY FACT. (1) (Y, F_f) is a topological space.

(2) $f : (X, F_X) \rightarrow (Y, F_f)$ is continuous.

PROOF. Left as an exercise.

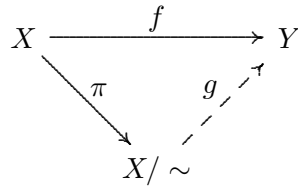
EXAMPLE. Let f map the closed segment $[0, 1]$ to a figure-eight, where $f(0) = f(1) = f(\frac{1}{2})$ at the intersection of the two sides of the figure-eight. What is open in the quotient topology (Y, F_f) ?

- Any open-looking interval on the eight that does not contain the intersection point is open since its preimage is clearly open on the segment; for the same reason, any appropriate union or intersection of these open intervals is also open.
- Furthermore, any open set in the figure-eight that includes the point at the cross must also contain an open interval of nonzero length extending along each of the four legs of the figure-eight. This is necessary because the definition of F_f requires that the preimage of anything open must be open itself; hence the only way for the preimage of a set containing the intersection point to be open is if the preimage is an open set in $[0, 1]$ containing the three preimages of the intersection point (0, $\frac{1}{2}$, and 1).



THEOREM 6. Let (X, F_X) be a topological space, $f : X \rightarrow Y$ onto, and \sim induced by f . Then $(Y, F_f) \cong (X/\sim, F_{\sim})$.

If a diagram commutes, then the path taken does not affect the result. Note that the arrow representing g is dashed because we have not yet shown that the diagram commutes. In the proof, we will define g so that the diagram does commute.



PROOF. Define $g : X/\sim \rightarrow Y$ by $g([\pi]) = f(x)$ where x is a representative of equivalence class $[\pi]$. We want to show that f is well-defined, one-to-one, onto, continuous, and open.

- Well-defined: WTS if $[x] = [y]$, then $g([x]) = g([y])$. That is, any representative of a given equivalence class maps to the same value. Suppose $[x] = [y]$. Then $x \sim y$; since \sim is the relation induced by f , $f(x) = f(y)$. $\therefore g([x]) = g([y])$.
- One-to-one: Suppose $g([x]) = g([y])$. Then $f(x) = f(y) \Rightarrow x \sim y \Rightarrow [x] = [y]$.
- Onto: Suppose $y \in Y$. f is onto, so $\exists x \in X$ such that $f(x) = y$. So $g([x]) = f(x) = y$.
- Continuous: WTS $U \in F_f \Rightarrow g^{-1}(U) \in F_{\sim}$. Suppose $U \in F_f$. Recalling that $F_{\sim} = \{O \subseteq X/\sim \text{ st. } \pi^{-1}(O) \in F_X\}$, we equivalently WTS that $\pi^{-1}(g^{-1}(U)) \in F_X$. Since $U \in F_f \Leftrightarrow f^{-1}(U) \in F_X$, we WTS $f^{-1}(U) = \pi^{-1}(g^{-1}(U))$.
 - (\subseteq) Let $x \in f^{-1}(U)$. Hence $f(x) = g([x]) \in U$. Since $\pi^{-1}(g^{-1}(U)) = \{p \text{ st. } g \circ \pi(p) \in U\}$, $g \circ \pi(x) = g([x]) \in U \Rightarrow x \in \pi^{-1}(g^{-1}(U))$.
 - (\supseteq) Let $x \in \pi^{-1}(g^{-1}(U))$. Hence $g(\pi(x)) \in U \Rightarrow g([x]) \in U \Rightarrow f(x) \in U$. So $x \in f^{-1}(U)$. $\therefore f^{-1}(U) = \pi^{-1}(g^{-1}(U))$. Since $f^{-1}(U) \in F_X$, $\pi^{-1}(g^{-1}(U)) \in F_X$. Thus $g^{-1}(U) \in F_{\sim}$.
- Open: WTS $U \in F_{\sim} \Rightarrow g(U) \in F_f$. Suppose $U \in F_{\sim}$. Recalling that $F_f = \{O \subseteq Y \text{ st. } f^{-1}(O) \in F_X\}$ and that $U \in F_{\sim} \Leftrightarrow \pi^{-1}(U) \in F_X$, we equivalently WTS $f^{-1}(g(U)) = \pi^{-1}(U)$.

(\subseteq) Let $x \in f^{-1}(g(U))$. So $f(x) = g([x]) \in g(U)$. Since g is bijective (from the earlier parts of this proof), $[x] \in U \Rightarrow \pi(x) \in U \Rightarrow x \in \pi^{-1}(U)$.

(\supseteq) Let $x \in \pi^{-1}(U)$. So $\pi(x) \in U$, implying that $g(\pi(x)) \in g(U)$. Then $g(\pi(x)) = g([x]) = f(x) \Rightarrow x \in f^{-1}(g(U))$.

Therefore g is a homeomorphism, and $(Y, F_f) \cong (X/\sim, F_\sim)$. \square

THEOREM 7. *Let (X, F_X) be a topological space and \sim an equivalence relation. Let $Y = X/\sim$ and let $f : X \rightarrow Y$ be π . Then $F_\sim = F_f$ and f is a quotient map.*

PROOF. Recall that:

$$F_\sim = \{U \subseteq X/\sim \text{ st. } \pi^{-1}(U) \in F_X\}$$

$$F_f = \{U \subseteq Y \text{ st. } f^{-1}(U) \in F_X\}.$$

We know that $X/\sim = Y$ and $f = \pi$. $\therefore F_\sim = F_f$. Also, $f = \pi$ is onto since it is a projection, and by definition $F_\sim = F_f$ is a quotient topology. $\therefore f$ is a quotient map. \square

LEMMA. (*Important Lemma about Quotients*)

Let (X, F_X) , (Z, F_Z) be topological spaces and let Y be a set. Suppose $f : X \rightarrow Y$ is onto and let $g : (Y, F_f) \rightarrow (Z, F_Z)$. Then g is continuous if and only if $g \circ f$ is continuous.

PROOF. (\Rightarrow) : Suppose that g is continuous. Since f is a quotient map, it is continuous. Therefore, $g \circ f$ is a composition of continuous functions, and so $g \circ f$ is continuous.

(\Leftarrow) : Suppose, now, that $g \circ f$ is continuous. Let $U \in F_Z$. Then $(g \circ f)^{-1}(U) \in F_X$. Therefore, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in F_X$.

Recall that $F_f = \{V \subseteq Y : f^{-1}(V) \in F_X\}$. So, $g^{-1}(U) \in F_f$, and g is continuous. ■

\square

EXAMPLE. (A Non-Example)

Let $f : \mathbb{R} - \{1\} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & x \in (-\infty, 1) \\ \frac{-1}{x-1} & x \in (1, \infty) \end{cases}$$

f is continuous. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

g is not continuous. But $g \circ f : \mathbb{R} - \{1\} \rightarrow \mathbb{R}$ is defined by

$$(g \circ f)(x) = \begin{cases} -1 & x > 1 \\ 1 & x < 1 \end{cases}$$

$g \circ f$ is continuous.

Question: Does this contradict the Important Lemma?

Answer: No! The solution is, of course, that g actually *is* continuous - the topology of its domain isn't the usual topology. Instead, it has the topology induced by f . That is,

$$F_f = \{U \subseteq \mathbb{R} : f^{-1}(U) \text{ open in } \mathbb{R} \setminus \{1\}\}$$

More precisely, we have $g^{-1}(\{1\}) = [0, \infty)$ and $f^{-1}([0, \infty)) = (-\infty, 1)$, which is open.

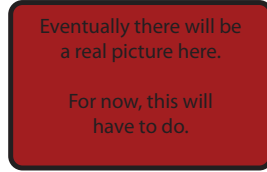
Also, $g^{-1}(\{-1\}) = (-\infty, 0)$ and $f^{-1}((-\infty, 0)) = (1, \infty)$, which again is open.

Therefore, g is continuous (as we only need to look at the range of g , which is $\{-1, 1\}$).

THEOREM 8. *Let (A, F_A) and (B, F_B) be topological spaces and $f : A \rightarrow B$ be a homeomorphism. Let \sim_A and \sim_B be equivalence relations on A and B , respectively, such that $x \sim_A x'$ if and only if $f(x) \sim_B f(x')$.*

Then $A/\sim_A \cong B/\sim_B$.

PROOF. We want to show that there is some function g such that the following diagram commutes:



Define $g : A/\sim_A \rightarrow B/\sim_B$ by $g([x]_A) = [f(x)]_B$. To show it's a homeomorphism:

- **Well-defined:** Suppose $[y]_A = [z]_A$. Then

$$y \sim_A z \quad \Rightarrow \quad f(y) \sim_B f(z) \quad \Rightarrow \quad [f(y)]_B = [f(z)]_B$$

and so g is well-defined.

- **1-to-1:** Suppose that $[x]_A$ and $[y]_A$ are such that $g([x]_A) = g([y]_A)$. Then

$$[f(x)]_B = [f(y)]_B \quad \Rightarrow \quad f(x) \sim_B f(y) \quad \Rightarrow \quad x \sim_A y \quad \Rightarrow \quad [x]_A = [y]_A$$

and so g is 1-to-1.

- **Onto:** Let $[y]_B \in B/\sim_B$, and let $x = f^{-1}(y)$. Then

$$g([x]_A) = [f(x)]_B = [y]_B$$

and g is onto.

- **Continuous:** We want to show that $g \circ \pi_A$ is continuous, so we can use the Important Lemma. By the definition of g , $g \circ \pi_A = \pi_B \circ f$. Since f is a homeomorphism, it is continuous; π_B is a quotient map and is thus continuous. So $\pi_B \circ f$ is continuous. Since $g \circ \pi_A = \pi_B \circ f$, we know that $g \circ \pi_A$ is continuous. As π_A is a quotient map, it is continuous, so from the Important Lemma, g is continuous.
- **Open:** We can show equivalently that g^{-1} is continuous instead, since g is a bijection.

$$\begin{aligned} g \circ \pi_A &= \pi_B \circ f \\ g^{-1} \circ (g \circ \pi_A) \circ f^{-1} &= g^{-1} \circ (\pi_B \circ f) \circ f^{-1} \\ \pi_A \circ f^{-1} &= g^{-1} \circ \pi_B \end{aligned}$$

From here, the argument is eerily similar to the argument for continuity above.

Note that π_A is a quotient map and f^{-1} is a homeomorphism, so both are continuous. As such, $\pi_A \circ f^{-1}$ is continuous, and therefore so is $g^{-1} \circ \pi_B$. Since π_B is a quotient map, it is continuous; from the Important Lemma, g^{-1} is continuous. Therefore, g is open.

Therefore, g is a homeomorphism. Hurrah! □

2. The Product Topology

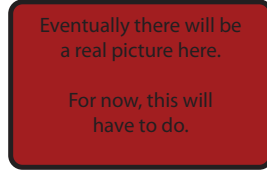
For the past several pages, we've been building new topological spaces from old by using equivalence relations to form quotient spaces. Here, we will change directions and build new topological spaces from old by taking products of spaces. To that end, we wish to define the product of two sets:

DEFINITION. Let X, Y be sets. The product of X and Y is given by:

$$X \times Y \equiv \{(x, y) : x \in X, y \in Y\}$$

This is a topology class, so our intrinsic urge is to find a natural topology for the product of two topological spaces. While our first instinct might be to take products of open sets in our original spaces, this approach will give unsatisfactory results:

EXAMPLE. Consider the sets $A = (0, 1) \times (0, 1)$ and $B = (1/2, 3/2) \times (1/2, 3/2)$ as subsets of $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ with the usual topology. Then $A \cup B$ in \mathbb{R}^2 is NOT a product of an open set in \mathbb{R} with an open set in \mathbb{R} as we would like. To intuitively see that this is not the case, see a picture! On the other hand, $A \cup B$ is open in the usual topology on \mathbb{R}^2 .



Although products of open sets will not work because they are not closed under unions, we can use products of open sets to construct our topology. Define the following:

DEFINITION. Let (X, F_x) and (Y, F_y) be topological spaces. Then the set $\beta_{X \times Y}$ is defined by:

$$\beta_{X \times Y} = \{A \times B : A \in F_x, B \in F_y\}$$

and define:

$$F_{X \times Y} = \{\cup_{i \in I} U_i : I \text{ is some index set, and } U_i \in \beta_{X \times Y}\}$$

In other words, $\beta_{X \times Y}$ is the set of products of an open set in X and an open set in Y , and $F_{X \times Y}$ is the set of unions of elements in $\beta_{X \times Y}$. The motivation for defining our sets this way is that we want $F_{X \times Y}$ to be a topology on $X \times Y$ and for β to be its basis. We will now verify this claim with the following *small fact*:

THEOREM 9. *If (X, F_x) and (Y, F_y) are topological spaces, the space $(X \times Y, F_{X \times Y})$ is a topological space with basis $\beta_{X \times Y}$.*

PROOF. We will apply our basis theorem; i.e. we want to show that

- (1) $\bigcup_{U \in \beta_{X \times Y}} U = X \times Y$
- (2) Given $B_1, B_2 \in \beta_{X \times Y}$, for each $x \in B_1 \cap B_2$, there exists $B_3 \in \beta_{X \times Y}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

For the first statement, we know that $X \in F_x$ and $Y \in F_y$ by definition of a topology so that $X \times Y \in \beta_{X \times Y}$. It follows then that because each $U \in \beta_{X \times Y}$ is subset of $X \times Y$:

$$X \times Y \subseteq \bigcup_{U \in \beta_{X \times Y}} U \subseteq X \times Y$$

so that $X \times Y = \bigcup_{U \in \beta_{X \times Y}} U$, as desired.

For the second statement, let $B_1, B_2 \in \beta_{X \times Y}$ and let $(x, y) \in B_1 \cap B_2$. Then by definition of $\beta_{X \times Y}$, there exist $U_1, U_2 \in F_x$ and $V_1, V_2 \in F_y$ such that $B_1 \cap B_2 = (U_1 \times V_1) \cap (U_2 \times V_2)$.

Our aim is to find $B_3 \in \beta_{X \times Y}$ such that B_3 contains (x, y) and $B_3 \subseteq B_1 \cap B_2$, so define:

$$B_3 = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Thus $U_1, U_2 \in F_x \Rightarrow U_1 \cap U_2 \in F_x$ by the closure of topologies under finite intersections and similarly, $V_1 \cap V_2 \in F_y$, so $B_3 \in \beta_{X \times Y}$. Since $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$, then

$(x, y) \in (U_1 \times V_1)$ and $(x, y) \in (U_2 \times V_2)$. Therefore, $x \in U_1, U_2$ and $y \in V_1, V_2$, so $x \in U_1 \cap U_2$, and $y \in V_1 \cap V_2$. It immediately follows by the definition of the intersection of sets that:

$$(x, y) \in (U_1 \cap U_2) \times (V_1 \cap V_2) = B_3$$

The only thing left to show is that $B_3 \subseteq B_1 \cap B_2$. To that end, let $(a, b) \in B_3$. Then $a \in U_1 \cap U_2$ and $b \in V_1 \cap V_2$ by definition of B_3 . It follows that:

$$a \in U_1, U_2 \text{ and } b \in V_1, V_2 \Rightarrow (a, b) \in U_1 \times V_1 \text{ and } (a, b) \in U_2 \times V_2 \Rightarrow a \in (U_1 \times V_1) \cap (U_2 \times V_2).$$

Therefore, $(a, b) \in B_1 \cap B_2$ so that $B_3 \subseteq B_1 \cap B_2$ because (a, b) is an arbitrary element of B_3 .

Therefore, $\beta_{X \times Y}$ satisfies the hypotheses of our basis theorem, so the set of unions of elements of $\beta_{X \times Y}$, $F_{X \times Y}$, is a topology for $X \times Y$ and $\beta_{X \times Y}$ is a basis for the topology $F_{X \times Y}$. \square

We have successfully devised a topology for product spaces as unions of products of open sets.

2.1. Examples.

EXAMPLE. The set $S^1 \times [0, 1]$ looks like a cylinder! What kinds of sets are open in the cylinder? A particular example is an open disc projected on the face of the cylinder and unions thereof.

Eventually there will be
a real picture here.

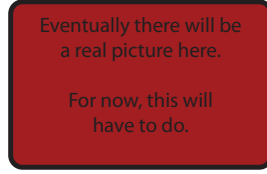
For now, this will
have to do.

EXAMPLE. The set $S^1 \times S^1$ looks like a torus! What kinds of sets are open in the torus? Similar to the previous example, open discs projected onto the torus surface are examples of open sets in $S^1 \times S^1$.

Eventually there will be
a real picture here.

For now, this will
have to do.

EXAMPLE. The set $S^1 \times S^1 \times S^1$ is actually a three torus!



Not every space is a product of more than one space. For example the sphere S^2 is not a product of more than one space. An intuitive (non-rigorous) justification is that the natural axes of a sphere are the great circles; however, every pair of distinct great circles intersect twice which makes it hard to define a coordinate system on the sphere.

2.2. The Product Projection Map. Now that we have a product topology to impose on product spaces, we want a way to relate the product topology to the topologies of the constituent spaces. In order to do so, we will define the projection maps as follows:

DEFINITION. Let (X, F_x) and (Y, F_y) be topological spaces and create $X \times Y$ endowed with the product topology $F_{X \times Y}$. Define $\pi_X : (X \times Y, F_{X \times Y}) \rightarrow (X, F_x)$ and $\pi_Y : (X \times Y, F_{X \times Y}) \rightarrow (Y, F_y)$ by:

$$\pi_X((x, y)) = x \quad \pi_Y((x, y)) = y.$$

The map π_X is the projection onto X and π_Y is the projection onto Y .

A small fact which we will derive now is that the product projection maps are continuous:

THEOREM 10. *Let (X, F_X) and (Y, F_Y) be topological spaces and $(X \times Y, F_{X \times Y})$ be their product with the induced product topology. Then the projection maps π_X and π_Y onto X and Y are continuous.*

PROOF. Suppose $O \subseteq X$ and $O \in F_x$. We see that $\pi^{-1}(O) = O \times Y \in F_{X \times Y}$. Therefore, the preimage of any open set in X under π_X is open in $X \times Y$ with the product topology. A similar argument shows that π_Y is continuous. \square

Recall that the quotient projection map is *not necessarily* an open map. It turns out that the product projection map *is* an open map. Accidentally assuming that the quotient map is open is a very common mistake that one should be aware of! We will now prove that the product projection map is open:

THEOREM 11. *Let (X, F_X) and (Y, F_Y) be topological spaces and $(X \times Y, F_{X \times Y})$ be their product with the induced product topology. Then the projection maps π_X and π_Y onto X and Y are open.*

PROOF. Let $O = U \times V$ such that $U \in F_x$ and $V \in F_y$ and $O \in F_{X \times Y}$. Therefore:

$$\pi_X(O) = U \in F_x$$

So if C is some collection of sets in $\beta_{X \times Y}$, then:

$$\bigcup_{V \in C} V$$

is an arbitrary element of $F_{X \times Y}$ by the definition of a basis and:

$$\pi_X \left(\bigcup_{V \in C} V \right) = \bigcup_{V \in C} \pi_X(V)$$

which is open because the preceding statements indicate that $V \in \beta_{X \times Y} \Rightarrow \pi_X(V) \in F_X$. Consequently, the image of the arbitrary open set in $(X \times Y, F_{X \times Y})$ is a union of open sets in (X, F_x) and is thus open.

The proof is similar for π_Y . □

2.3. Using Bases More Effectively. In metric spaces, open balls are bases for the metric topology. By proving properties about open balls, we were able to say they apply to the entire set. We would like to prove the following *important tiny lemma* so that we can use bases in topological spaces like open balls in metric spaces:

LEMMA. *Let (X, F_x) be a topological space with basis β . If $W \subseteq X$ then $W \in F_x$ if and only if for all $p \in W$, there exists $B_p \in \beta$ such that $p \in B_p \subseteq W$.*

PROOF. (\Rightarrow) Suppose $W \in F_x$. Let $p \in W$. Since $W \in F_x$, for some index set I :

$$W = \bigcup_{i \in I} B_i \quad \forall i \in I, B_i \in \beta$$

because every element of a topology can be written as a union of basis elements. Therefore, by definition of the union, $p \in W$ implies that there exists $i_p \in I$ such that $p \in B_{i_p}$, and $B_{i_p} \subseteq W$. If we let $B_p = B_{i_p}$, we are finished.

(\Leftarrow) Suppose that for all $p \in W$, there exists $B_p \in \beta$ such that $p \in B_p \subseteq W$. We want to show that $W \in F_x$. We see that:

$$V = \bigcup_{p \in W} B_p \subseteq W$$

because each of the constituent $B_p \subseteq W$, and every $p \in W$ is an element of B_p , so the union of B_p over $p \in W$, contains every $p \in W$. Consequently, $V \subseteq W \Rightarrow V = W$. Since β is a basis and V is a union of basis elements, $V \in F_x \Rightarrow W \in F_x$. □

In other words, if we have a topological space with a basis, then every point in an open set U is an element of a basis element contained in U , giving us a structure very similar to the metric topology.

2.4. Finding and Constructing Continuous Maps to Product Spaces. Now that we have product spaces and have addressed their basic topological properties, we would like a way to easily find and construct continuous maps to the product space. To that end we introduce the following *important lemma*:

LEMMA. Let $(X, F_X), (Y, F_Y)$ and (A, F_A) be topological space and let $(X \times Y, F_{X \times Y})$ be the product space of X, Y with the induced product topology. Suppose $f : A \rightarrow X$ and $g : A \rightarrow Y$ and define

$$h : A \rightarrow (X \times Y) \quad \text{by} \quad h(a) = (f(a), g(a)),$$

then h is continuous if and only if f, g are continuous.

PROOF. (\Rightarrow) Suppose h is continuous: then we see that:

$$f = \pi_X \circ h \quad g = \pi_Y \circ h$$

so f, g are continuous because they are compositions of continuous functions.

(\Leftarrow) Suppose that f and g are continuous. We prove that the preimage of every basis element under h is open. From a theorem we proved some time ago, this shows that h is continuous.

Let $U \times V \in \beta_{X \times Y}$. Then $U \in F_X$ and $V \in F_Y$. Since f and g are continuous, $f^{-1}(U) \in F_A$ and $g^{-1}(V) \in F_A$. Then

$$\begin{aligned} h^{-1}(U \times V) &= \{a \in A \mid h(a) \in U \times V\} \\ &= \{a \in A \mid (f(a), g(a)) \in U \times V\} \\ &= \{a \in A \mid f(a) \in U, g(a) \in V\} \\ &= f^{-1}(U) \cap g^{-1}(V) \end{aligned}$$

Then $h^{-1}(U \times V)$ is the intersection of two open sets, hence it is open. Therefore, h is continuous.

□

The following example illustrates how this lemma makes it very easy to define continuous functions to the product space:

EXAMPLE. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = x^2 + 3x$ and $g(x) = \sin(x)$. Then the map $h : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $h(x) = (x^2 + 3x, \sin(x))$ is continuous because f, g are.

2.5. Relating Products to Subspaces.

SMALL FACT. Let (X, F_X) and (Y, F_Y) be topological spaces with subspaces (A, F_A) and (B, F_B) respectively. Let F_S denote the subspace topology on $A \times B \subseteq X \times Y$ and let $F_{A \times B}$ denote the product topology on $A \times B$. Then $F_S = F_{A \times B}$.

This means that we can either look at $A \times B$ as a subspace of $X \times Y$, then induce the subspace topology, or we can induce the subspace topologies on A and B , then take the cross product. Essentially, taking the cross product and creating subspaces “commute.”

PROOF. We start by reviewing which sets are open under each topology:

$$F_S = \{(A \times B) \cap U \mid U \in F_{X \times Y}\}$$

$$F_{X \times Y} = \text{unions of sets of the form } U \times V, \text{ with } U \in F_A, V \in F_B$$

Let $U \cap (A \times B) \in F_S$. Then

$$U \cap (A \times B) = \bigcup_{i \in I} (U_i \times V_i) \cap (A \times B) \quad (\text{where } \forall i \in I, U_i \in F_X, V_i \in F_Y)$$

Then

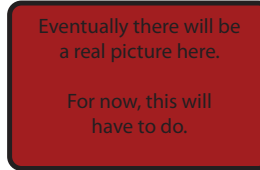
$$\begin{aligned} U \cap (A \times B) &= \bigcup_{i \in I} (U_i \times V_i) \cap (A \times B) \\ &= \{(x, y) \in A \times B \mid (x, y) \in U_{i'} \times V_{i'} \text{ for some } i' \in I\} \\ &= \{(x, y) \in X \times Y \mid x \in U_{i'} \cap A, y \in V_{i'} \cap B \text{ for some } i' \in I\} \\ &= \bigcup_{j \in J} (U_j \cap A) \times (V_j \cap B) \quad \text{for some index set } J \\ &\in F_{X \times Y} \end{aligned}$$

The same argument run backwards shows that $F_{X \times Y} \subseteq F_S$.

Therefore, $F_S = F_{X \times Y}$. □

2.6. Products and Quotient Spaces. Taking products and quotients does not “commute” in the same way that taking products and subspaces does. We demonstrate a (lengthy) counterexample to this idea.

Consider \mathbb{R} with the usual topology, with $x \sim Y$ iff $x = y$ or $x, y \in \mathbb{N}$. This looks something like a “sideways infinite flower.”



If a set open in \mathbb{R}/\sim contains a (“the”) natural number, then its preimage must contain every natural number, so it must contain an open interval about “the” natural number in every direction (all infinity of them).

More formally, let $\pi : \mathbb{R} \rightarrow \mathbb{R}/\sim$ be the projection map. Let $id_{\mathbb{Q}} : \mathbb{Q} \rightarrow \mathbb{Q}$ be the identity map, where each copy of \mathbb{Q} has the usual topology. Note that $id_{\mathbb{Q}}$ is a quotient map (where \sim is '=').

Now, consider $\pi \times id_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \rightarrow (\mathbb{R}/\sim) \times \mathbb{Q}$ by $(\pi \times id_{\mathbb{Q}})(x, y) = (\pi(x), i(y))$. We claim that $\pi \times id_{\mathbb{Q}}$ is not a quotient map. To prove this, we show that the product topology on $(\mathbb{R}/\sim) \times \mathbb{Q}$ is not the same as the quotient topology

$$\{U \in (\mathbb{R}/\sim) \times \mathbb{Q} \mid (\pi \times id_{\mathbb{Q}})(U) \in F_{\mathbb{R} \times \mathbb{Q}}\}$$

To do this, let's find some $U' \in F_{\pi \times i}$ such that $U' \notin F_{(\mathbb{R}/\sim) \times \mathbb{Q}}$. We want to construct U' as a union.

For all $n \in \mathbb{N}$ define U_n to be the interior of the region of $\mathbb{R} \times \mathbb{Q}$ bounded by the vertical lines $n - \frac{1}{4}$ and $n + \frac{1}{4}$ and above and below by the lines through $(n, \frac{\sqrt{2}}{n})$ with slope ± 1 . Each one of these sets has a vertical line through the natural number.

So $\forall n \in \mathbb{N}, \{n\} \times \mathbb{Q} \subseteq U_n$. Observe that $\forall n \in \mathbb{N}, U_n \in F_{\mathbb{R} \times \mathbb{Q}}$ (it is an interior, so it is open).

Let $\bigcup_{n \in \mathbb{N}} U_n \in F_{\mathbb{R} \times \mathbb{Q}}$ (also open in $\mathbb{R} \times \mathbb{Q}$ because it is a union).

Let $U' = (\pi \times i)(U)$. This will glue together the strips along the vertical lines with \mathbb{R} coordinates in \mathbb{N} . This can be pictured as a fan.



Claim: $U' \in F_{\pi \times i}$

PROOF. : $(\pi \times i)^{-1}(U') = U$ and U is a union of U_n 's

Therefore, by definition, $U' \in F_{\pi \times i}$.

Claim: $U' \notin F_{(\mathbb{R}/\sim) \times \mathbb{Q}}$.

PROOF. : Let $([n], 0) \in U'$. Suppose $\exists W \in F_{\mathbb{R}/\sim}$ and $V \in F_{\mathbb{Q}}$ such that $([n], 0) \in W \times V \subseteq U'$.

Then $\pi^{-1}(W) \in F_{\mathbb{R}}$ and of course $i^{-1}(V) = V$.

So $\pi^{-1}(W) \times V$ is open in $\mathbb{R} \times \mathbb{Q}$ such that $\pi^{-1}(W) \times V \subseteq (\pi \times i)^{-1}(U') = U$.

Also, $\pi^{-1}(W)$ is open in \mathbb{R} and contains \mathbb{N} and V is open in \mathbb{Q} and contains $\{0\}$.

So $\exists \delta > 0$ such that $(-\delta, \delta) \cap \mathbb{Q} \subseteq V$. There exists an $n \in \mathbb{N}$ such that $\delta > \frac{\sqrt{2}}{n}$. $\pi^{-1}(W)$ is open in \mathbb{R} , so it's on the x-axis.

So $\exists \epsilon > 0$ such that $(n - \epsilon, n + \epsilon) \subseteq \pi^{-1}(W)$.

So the interval $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq \pi^{-1}(W) \times V$. So $\epsilon < \frac{1}{4}$ from our previous definition of U_n .

Let $x = n + \frac{\epsilon}{2}$. Question: Where does $x = n + \frac{\epsilon}{2}$ meet the boundaries of U_n ? It meets at $y = \frac{\sqrt{2}}{n} \pm \frac{\epsilon}{2}$.

$\exists y \in \mathbb{Q}$ such that $\frac{\sqrt{2}}{n} - \frac{\epsilon}{2} < y < \frac{\sqrt{2}}{n} + \frac{\epsilon}{2}$.

So $(x, y) \notin U_n$ and $\forall m \neq n, (x, y) \notin U_m$ since $(x, y) \in (n - \epsilon, n + \epsilon) \subseteq (n - \frac{1}{4}, n + \frac{1}{4})$.

But $(x, y) \in (n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq \pi^{-1}(W) \times V \subseteq U \Rightarrow (x, y) \notin U$. Therefore we have a contradiction.

$\therefore U' \notin F_{(\mathbb{R}/\sim) \times \mathbb{Q}}$

So we have shown that $F_{\pi \times i} \neq F_{(\mathbb{R}/\sim) \times \mathbb{Q}}$.

2.7. Infinite Products. We would of course like to be able to generalize this to deal with infinite products. Initially, we may simply want to define such products as follows: $X_1 \times X_2 \times \dots = \{(x_1, x_2, \dots) | x_i \in X_i \forall i \in \mathbb{N}\}$. However, such a definition limits us to countable products, so we look more generally.

DEFINITION. $j \in J$, let X_j be a set. Define the product $\Pi_{j \in J} X_j = \{f : J \rightarrow \cup_{j \in J} X_j | f(j) \in X_j\}$. We refer to $f(j)$ as the j^{th} coordinate of the point f .

EXAMPLE. Suppose $J = \{1, 2\}$. Then $\Pi_{j \in \{1, 2\}} X_j = \{f : \{1, 2\} \rightarrow X_1 \cup X_2 | f(j) \in X_j\} = \{(f(1), f(2)) | f(j) \in X_j\} = \{(x_1, x_2) | x_j \in X_j\} = X_1 \times X_2$. Thus we see that this definition agrees with our previous definition for finite products.

EXAMPLE. Consider $\Pi_{j \in \mathbb{R}} \{1, 2\}$. By definition this is equivalent to $\{f : \mathbb{R} \rightarrow \{1, 2\} | f(j) \in \{1, 2\}, j \in \mathbb{R}\}$, which precisely correspond to subsets of \mathbb{R} if we simply think of the preimage of 1 under f as the elements in the set and the preimage of 2 under f as those elements outside the set. Commonly this product is then denoted by $\{1, 2\}^{\mathbb{R}}$.

Now we wish to define a topology on these products which, as in example 1, agrees with our prior definition for a product of two sets if the indexing set $J = \{1, 2\}$. Perhaps the most natural way of doing this is to define the following basis:

$$\beta_{\square} = \{\Pi_{j \in J} U_j | U_j \in F_j\}, \text{ where } X_j \text{ has topology } F_j.$$

$$F_{\square} = \{\text{Unions of elements of } \beta_{\square}\}.$$

This topology is not what we desire, but is a topology, aptly named the box topology on the product.

EXAMPLE. The product topology on $\prod_{j \in J} X_j$ is given by the basis $\beta_\Pi = \{\prod_{j \in J} U_j \mid U_j = X_j \text{ for all but finitely many } j \text{ and } \forall j \in J, U_j \in F_j\}$.

Remarks:

(1) $\beta_\Pi \subseteq \beta_\square$

(2) Both are bases for topologies on the product.

DEFINITION. Define the projection map $\pi_j : \prod_{j \in J} X_j \rightarrow X_j$ by $\pi(f) = f(j)$.

LEMMA. For all $j \in J$, Suppose (X_j, F_j) is a topological space. Then π_j as defined above is continuous for all $j \in J$.

PROOF. Let $j_0 \in J$ and consider $U \in F_{j_0}$. We wish to show that $\pi_{j_0}^{-1}(U) \in F_\Pi$. Note that $\pi_{j_0}^{-1}(U) = \{f \in \prod_{j \in J} X_j \mid \pi_{j_0}(f) \in U\} = \{f \in \prod_{j \in J} X_j \mid f(j_0) \in U\} = \{f \in \prod_{j \in J} X_j \mid f(j_0) \in U, \forall j \neq j_0, f(j) \in X_j\} = \prod_{j \in J} U_j$ such that $U_{j_0} = U$ and $\forall j \neq j_0, U_j = X_j$. It then follows from definitions that $\pi_{j_0}^{-1}(U) \in \beta_\Pi \subseteq F_\Pi$ so the projection map is continuous. \square

We now prove an Important Lemma for Infinite Products.

LEMMA. Let (X_j, F_j) and (Y, F_Y) be topological spaces for all $j \in J$. Moreover, for each such j let $g_j : Y \rightarrow X_j$ be a function. Define $h : Y \rightarrow \prod_{j \in J} X_j$ by: $h(y) = f$ such that $\forall j \in J, f(j) = g_j(y)$. Then h is continuous if and only if g_j is continuous for all $j \in J$.

PROOF. (\Rightarrow) Suppose h is continuous and let $j \in J$. Then $\pi_j \circ h = g_j$. Thus g_j is the composition of continuous functions and must itself be continuous.

(\Leftarrow) Suppose that g_j is continuous for all $j \in J$. Let $U \in \beta_\Pi$. We wish to show that $h^{-1}(U) \in F_Y$.

Note that $h^{-1}(U) = \{y \in Y \mid h(y) \in U\}$. As an open set in the product topology, $U = \prod_{j \in J} U_j$ where $U_j \in F_j$. Thus, $h^{-1}(U) = \{y \in Y \mid h(y) \in \prod_{j \in J} U_j\} = \{f \in \prod_{j \in J} X_j \text{ such that } \forall j \in J, f(j) = g_j(y)\} = \{y \in Y \mid g_j(y) \in U_j\} = \{y \in Y \mid y \in g_j^{-1}(U_j) \forall j \in J\} = \bigcap_{j \in J} g_j^{-1}(U_j)$.

Since g_j is continuous for all j , $g_j^{-1}(U_j)$ is open in Y for all j . Also, by definition of the product topology, $U_j = X_j$ for all but at most finitely many j . Thus, $g_j^{-1}(U_j) = Y$ for all but at most finitely many j . It follows that the intersection $\bigcap_{j \in J} g_j^{-1}(U_j)$ is a finite intersection of open sets since removing all trivial indices will not change the intersection. Thus, $h^{-1}(U) \in F_Y$ so h is continuous, completing the proof. \square

CHAPTER 3

Distinguishing Spaces

DEFINITION. A **topological property** (or ‘top. prop.’) is a property of a topological spaces that is preserved by homeomorphisms.

List of Topological Properties thus Far

- (1) Cardinality of X
- (2) Cardinality of F_x
- (3) Metrizable (we proved this in the homework)
- (4) Discreteness
- (5) Indiscreteness

LEMMA. (**Rachel’s Lemma**)

Let (X, F_X) and (Y, F_Y) be topological spaces and $f : X \rightarrow Y$ be an open bijection. If F_X is the discrete topology, then F_Y is the discrete topology.

PROOF. Let $y \in Y$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Since $\{x\} \in F_X$ and f open, $f(\{x\}) \in F_Y$. Thus all singletons in Y are elements of F_Y and all sets in Y are unions of singletons and hence elements of F_Y , so every set is an open set and F_Y is the discrete topology. \square

LEMMA. (**Daniel’s Lemma**)

Let (X, F_X) and (Y, F_Y) be topological spaces and $f : X \rightarrow Y$ be a continuous bijection. If F_X is the indiscrete topology, then F_Y is the indiscrete topology.

PROOF. Let $U \in F_Y$. WTS $U = Y$ or $U = \emptyset$. Since f is continuous, $f^{-1}(U) \in F_X$. Therefore $f^{-1}(U) = X$ or \emptyset .

Suppose $f^{-1}(U) = X$. Then $U = f(f^{-1}(U)) = f(X) = Y$ since f surjective.

Suppose $f^{-1}(U) = \emptyset$. Then $U = \emptyset$. Therefore F_Y is the indiscrete topology. \square

Corollaries If (X, F_X) and (Y, F_Y) topological spaces and $f : X \rightarrow Y$ a homeomorphism, then

- (1) F_X the discrete topology $\implies F_Y$ the discrete topology

- (2) F_X the indiscrete topology $\implies F_Y$ the indiscrete topology

Unfortunately, even with these five lovely Topological Properties, we can't yet distinguish a circle from a line. Clearly there is more work to do.

EXAMPLE (a non-example). Distance is not a topological property. A big circle *is* homeomorphic to a little circle.

1. Compactness

DEFINITION. Let (X, F_X) be a topological space and $S \subseteq X$.

We say $\{U_j : j \in J\}$ is an **open cover** of S if for all $j \in J$, $U_j \in F_X$ and $S \subseteq \bigcup_{j \in J} U_j$.

We say $\{U_j : j \in K\}$ is a **subcover** if $K \subseteq J$ and $S \subseteq \bigcup_{j \in K} U_j$.

We say S is **compact** if every open cover of S has a finite subcover.

IMPORTANT WARNING!

We learned in Math 131 that in \mathbb{R}^n , a set is compact \iff it is closed and bounded. THIS IS NOT TRUE IN TOPOLOGICAL SPACES. DO NOT TRY TO USE IT.

EXAMPLE. Is the set $(0, 1)$ compact in

- (1) \mathbb{R} with the discrete topology?
- (2) \mathbb{R} with the half-open topology?
- (3) \mathbb{R} with the finite complement topology?

Answers:

- (1) No. Take the open covering $\{B_{1/2}(x) : x \in (0, 1)\}$. If we remove any of these balls, the corresponding x will no longer be covered. However, there are clearly uncountably infinitely many balls.
- (2) No. Take the open covering $\{[\frac{1}{n}, 1) : n \in \mathbb{N}\}$ so that $(0, 1) = \bigcup_{n \in \mathbb{N}} ([\frac{1}{n}, 1)$. There is no finite subcover.
- (3) Yes. Suppose we have a cover of $(0, 1)$. Take any element of the cover, say U_{47} . U_{47} is missing at most finitely many elements of $(0, 1)$ since its complement is finite. For each element $x \in \mathbb{R} \setminus U_{47}$, select one U_{j_x} containing x . The set $\{U_{47}\} \cup \{U_{j_x} : x \in \mathbb{R} \setminus U_{47}\}$ is a finite open subcover.

THEOREM 12. (*analogous to a theorem from 131*)

Let (X, F_X) and (Y, F_Y) be topological spaces, $S \subseteq X$, and $f : X \rightarrow Y$ continuous. If S is compact then $f(S)$ is compact.

- PROOF. (1) Take a covering of $f(S)$.
- (2) Applying f^{-1} to these sets, we pull back to an open covering of S .
- (3) Take a finite subcover.
- (4) Push these forward again. We have a finite cover of $f(S)$.

□

As a corollary we have **Topological Property 6: Compactness**.

THEOREM 13. *Any closed subset of a compact space is compact.*

- PROOF. (1) Take an open cover of S
- (2) Add $X \setminus S$. Now we have an open cover of X .
- (3) Take a finite subcover.
- (4) Remove $X \setminus S$. Now we have a finite subcover of S .

□

THEOREM 14. (*also analogous to a 131 theorem*)

Let (X, F_X) be compact and $f : X \rightarrow \mathbb{R}$ be continuous. Then f has a max value and a min value.

- PROOF. (1) $f(X)$ is compact by Analogous Theorem 1
- (2) Therefore $f(X)$ is closed and bounded
- (3) Therefore $f(X)$ has a lub and a glb (ie a max and a min)

□

SMALL FACT. Let (X, F_X) be a topological space. Suppose every open cover of $S \subseteq X$ made up of basis elements has a finite subcover. Then X is compact.

PROOF. Let $\{U_j : j \in J\}$ be an open cover of S and β be a basis of S . For every $j \in J$, $U_j = \bigcup_{i \in I_j} B_i$ where for every $i \in I_j$ $B_i \in \beta$. Hence $\{B_i : i \in I_j \text{ and } j \in J\}$ is an open cover of S . By hypothesis, we have a finite subcover, $\{B_i : i \in K_j \text{ and } j \in K\}$ where

$K \subseteq J$ and K is finite, and for all $j \in K$, $K_j \subseteq I_j$ and K_j is finite. Now note that given any $j \in K$, for every $i \in K_j$ $B_i \subseteq U_j$. Hence

$$X = \bigcup_{j \in K} \left[\bigcup_{i \in K_j} B_i \right] \subseteq \bigcup_{j \in K} U_j \subseteq X$$

and $\{U_j : j \in K\}$ is a finite subcover. □

THEOREM 15. (*Bolzano-Weierstrass Theorem*).

Let (X, F_X) be a compact topological space. Then for all infinite $S \subseteq X$, $\exists p \in X$ such that $\forall U \in F_X$ with $p \in U$, U contains infinitely many points of S .

PROOF. Suppose that for some such $S \subseteq X$, no such $p \in X$ exists. Therefore, for all $p \in X$ there exists some $U_p \in F_X$ such that $p \in U_p$ and $|S \cap U_p|$ is finite. So, as $\bigcup_{p \in X} U_p = X$, $\{U_p \mid p \in X\}$ is an open cover of X . As X is compact, there exists some finite subcover $\{U_{p_1}, \dots, U_{p_n}\}$. Hence, from the definition of a subcover, $X = \bigcup_{i=1}^n U_{p_i}$. Intersecting both sides with S gives that:

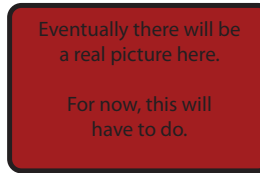
$$S = \left(\bigcup_{i=1}^n U_{p_i} \right) \cap S = \bigcup_{i=1}^n (U_{p_i} \cap S)$$

However, as $|U_p \cap S|$ is finite for all $p \in X$, $|U_{p_i} \cap S|$ is finite for $1 \leq i \leq n$. The union of finite sets is finite, and therefore $|\bigcup_{i=1}^n (U_{p_i} \cap S)|$ is finite. However, S is infinite by assumption. So this is a contradiction, and no such S exists. □

THEOREM 16. (*Finite Tychonoff Theorem*)

Let (X, F_X) and (Y, F_Y) be compact topological spaces. Then $(X \times Y, F_{X \times Y})$ is also a compact topological space.

PROOF. (Comic book style) (various figures...)



So the original sets that covered $\{x\} \times Y$ corresponding to the chosen U_x for the desired finite cover of $X \times Y$. □

Now let's do a real proof.

PROOF. Consider an open cover of $X \times Y$ consisting of basis elements, $\{U_i \times V_i \mid i \in I\}$, where $U_i \in F_X$ and $V_i \in F_Y$ for all $i \in I$.

Let $x \in X$. It is clear that $\{x\} \times Y \cong Y$, so $\{x\} \times Y$ is compact. So the original open cover is a cover of $\{x\} \times Y$, and there exists some finite subcover $\{U_{i(x)} \times V_{i(x)} \mid i(x) \in K_x\}$ where $K_x \subseteq I$ and K_x is finite. If $x \notin U_{i(x)}$, removing the set $U_{i(x)} \times V_{i(x)}$ does not change the fact that the remaining set is a finite subcover of $\{x\} \times Y$. Hence, without loss of generality, we assume that $x \in U_{i(x)}$ for all $i(x) \in K_x$.

Next, define $U_x = \bigcap_{i(x) \in K_x} U_{i(x)}$. As $U_{i(x)} \in F_X$ for all $i(x) \in K_x$, U_x is the intersection of finitely many open sets, and so is open. Additionally, as $x \in U_{i(x)}$ for all $i(x) \in K_x$, $x \in U_x$.

Claim 1: $\{U_x \times V_{i(x)} \mid i(x) \in K_x\}$ is an open cover of $\{x\} \times Y$.

Proof: Let $(x, y) \in \{x\} \times Y$. Then for some $i(x) \in K_x$, $(x, y) \in U_{i(x)} \times V_{i(x)}$. So $y \in V_{i(x)}$. Then, as $x \in U_x$, $(x, y) \in U_x \times V_{i(x)}$. As such an $i(x) \in K_x$ exists for all $(x, y) \in \{x\} \times Y$, $\{U_x \times V_{i(x)} \mid i(x) \in K_x\}$ is an open cover of $\{x\} \times Y$.

Now, as $x \in U_x$ and $U_x \in F_X$ for all $x \in X$, $\{U_x \mid x \in X\}$ is an open cover of X . As X is compact, this must have a finite subcover, $\{U_x \mid x \in L\}$, where $L \subseteq X$ and L is finite.

Claim 2: $\{U_x \times V_{i(x)} \mid x \in L, i(x) \in K_x\}$ is a cover of $X \times Y$.

Proof: Let $(x_0, y_0) \in X \times Y$. As $\{U_x \mid x \in L\}$ covers X , there exists some $x \in L$ such that $x_0 \in U_x$. As $\{U_x \times V_{i(x)} \mid i(x) \in K_x\}$ covers $\{x\} \times Y$, there exists some $i(x) \in K_x$ such that $(x, y_0) \in U_x \times V_{i(x)}$, so $y_0 \in V_{i(x)}$. Hence $(x_0, y_0) \in U_x \times V_{i(x)}$, and (x_0, y_0) is in the cover. As this holds for all such $(x_0, y_0) \in X \times Y$, $\{U_x \times V_{i(x)} \mid x \in L, i(x) \in K_x\}$ is a cover of $X \times Y$.

Claim 3: $\{U_{i(x)} \times V_{i(x)} \mid x \in L, i(x) \in K_x\}$ is a cover of $X \times Y$.

Proof: Let $(x_0, y_0) \in X \times Y$. As $\{U_x \times V_{i(x)} \mid x \in L, i(x) \in K_x\}$ is a cover of $X \times Y$, there exists some $x \in L$, $i(x) \in K_x$ such that $(x_0, y_0) \in U_x \times V_{i(x)}$. From the definition of U_x , $U_x \subseteq U_{i(x)}$. Therefore, as $x_0 \in U_x$, $x_0 \in U_{i(x)}$, and $(x_0, y_0) \in U_{i(x)} \times V_{i(x)}$. As such a $x \in L$, $i(x) \in K_x$ exists for all $(x_0, y_0) \in X \times Y$, this is a cover of $X \times Y$.

Finally, let $J = \bigcup_{x \in L} K_x$. As L is finite, and K_x is finite for all $x \in X$, J is finite. Additionally, as $K_x \subseteq I$ for all $x \in X$, $J \subseteq I$. So $\{U_i \times V_i \mid i \in J\}$ is a finite subcover of $\{U_i \times V_i \mid i \in I\}$. Also, as this has $\{U_i \times V_i \mid i \in J\} = \{U_{i(x)} \times V_{i(x)} \mid x \in L, i(x) \in K_x\}$, this is a cover of $X \times Y$.

So $\{U_i \times V_i \mid i \in J\}$ is a finite subcover of $X \times Y$. As such a subcover exists for any cover of basis elements, $X \times Y$ is compact. \square

2. Hausdorffness

DEFINITION. Let (X, F_X) be a topological space. We say that X is **Hausdorff** if $\forall p, q \in X$ such that $p \neq q$, \exists disjoint sets $U, V \in F_X$ such that $p \in U, q \in V$.

EXAMPLE. Metric spaces **are** Hausdorff.

EXAMPLE. (a non-example) Any space containing at least two points under the indiscrete topology is **not** Hausdorff.

Hausdorff is important because any “reasonable” space/any space that we want to be in will be Hausdorff.

Note: Continuous functions *may not* preserve Hausdorff-ness.

EXAMPLE. Define $f: (\mathbb{R}, \text{discrete}) \rightarrow (\mathbb{R}, \text{indiscrete})$ by the identity map.

This is continuous because the domain has the discrete topology, but the domain is Hausdorff while the range clearly is not.

SMALL FACT. Suppose $f: (X, F_X) \rightarrow (Y, F_Y)$ is a homeomorphism and (X, F_X) is Hausdorff. Then (Y, F_Y) is also Hausdorff.

PROOF. Let $p \neq q \in Y$. Because f is a bijection, $f^{-1}(p)$ and $f^{-1}(q)$ are distinct points in X .

Since X is Hausdorff, $\exists U, V \in F_X$ such that $U \cap V = \emptyset$, $f^{-1}(p) \in U$, $f^{-1}(q) \in V$.

Consider $f(U), f(V)$. Since f is a homeomorphism and, thus, open, $f(U)$ and $f(V)$ are open. Then, because f is a bijection, we have $p \in f(U)$, $q \in f(V)$, and $f(U) \cap f(V) = \emptyset$ □

We are now interested in how Hausdorff-ness and compactness are related.

Flapan Says... “Hausdorff-ness and compactness are like two people who love each other, because when two people love each other, they can do so much more together than they can alone.”

THEOREM 17. Let (X, F_X) be Hausdorff. Let $A \subseteq X$ be compact. Then A is closed.

PROOF. (Compare the following to the proof that all compact subsets of metric spaces are closed. That is, we are going to show that we don’t need a metric, just Hausdorff-ness, for closed subsets to be compact.)

Rather than show A is closed, we will show $X - A$ is open.

Let $p \in X - A$. We want to show $\exists U \in F_X$ such that $p \in U \subseteq X \setminus A$. Let $a \in A$. Because X is Hausdorff, $\exists U_a, V_a \in F_X$ such that $p \in U_a$, $a \in V_a$, $U_a \cap V_a = \emptyset$.

Consider $\{V_a | a \in A\}$. This is an open cover of A because V_a open and $a \in V_a \forall a \in A$. So, because A is compact, \exists finite subcover $\{V_{a_1}, V_{a_2}, \dots, V_{a_n}\}$

Let $U = \bigcap_{i=1}^n U_{a_i}$. $U \in F_X$ since it is a finite intersection of elements of F_X and $p \in U$ since $p \in U_{a_i} \forall i = 1, 2, \dots, n$.

Claim: $U \subseteq X - A$

Proof. $\forall i = 1, 2, \dots, n$, $U_{a_i} \cap V_{a_i} = \emptyset$

$$A \subseteq \bigcup_{i=1}^n V_{a_i}$$

$$U \cap A \subseteq U \cap \bigcup_{i=1}^n V_{a_i}$$

However, $U \cap \bigcup_{i=1}^n V_{a_i} = \emptyset$ because if $\exists x \in U \cap \bigcup_{i=1}^n V_{a_i}$, then $\exists i_o$ such that $x \in V_{a_{i_o}} \cap U_{a_{i_o}}$. But, by definition, $V_{a_{i_o}} \cap U_{a_{i_o}} = \emptyset$, so no such x exists. Thus,

$U \cap A \subseteq U \cap \bigcup_{i=1}^n V_{a_i} = \emptyset$, meaning $U \cap A = \emptyset$. So, because $U \subseteq X$ but $U \cap A = \emptyset$, $U \subseteq X - A$

Thus, we have found $U \in F_X$ such that $p \in U$, $U \subseteq X - A$. Thus, $X - A$ is open.

Thus, **A is closed**. □

COROLLARY. *In any Hausdorff space, (finite sets of) points are closed sets.*

PROOF. Let $p \in (X, F_X)$ and let (X, F_X) be Hausdorff. Let $\{U_i | i \in I\}$ be an open cover of $\{p\}$. Then $\exists i_o \in I$ such that $p \in U_{i_o}$. Thus $\{U_{i_o}\}$ is a finite subcover. Hence, $\{p\}$ is compact.

Thus, by Theorem above, $\{p\}$ is closed. □

LEMMA. *(it's important!)*

Let $f : (X, F_X) \rightarrow (Y, F_Y)$ be continuous. Let X be compact, and Y be Hausdorff. Then f is a homeomorphism if and only if it is a bijection.

PROOF. (\Rightarrow) Since f is a homeomorphism, f is a bijection.

(\Leftarrow) Suppose f is a continuous bijection. We want to show that f is open. Since f is a bijection, this is equivalent to showing that f is closed.

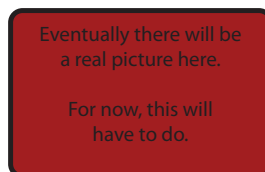
Let $A \subseteq X$ be closed. Since X is compact, by the second analogous theorem above, A is compact. By the first analogous theorem, $f(A)$ is compact. By the above theorem, $f(A)$ is closed. Thus, f is closed, and, thus, open. Thus, f is an open, continuous bijection.

Thus, f is a homeomorphism.

Thus, f is a homeomorphism if and only if it is a bijection.

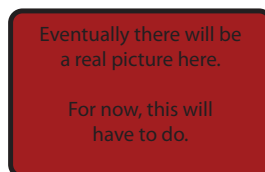
□

EXAMPLE. Let $X = [0, 1] \times [0, 1]$, and let $(a, b) \sim (c, d)$ if either $(a, b) = (c, d)$ or $b = d \neq 0$. (See Homework 4, Problem 3.) Points on the x -axis are not separable.



So this is a Hausdorff space X and an equivalence relation \sim such that X/\sim is not Hausdorff.

EXAMPLE. Let $X = \mathbb{R}$, and set $x \sim y$ if and only if $x = y$ or $x, y > 0$.



Let $p > 0$. The $[p], [0] \in X/\sim$. Let $U \subseteq X/\sim$ be an open set containing $[0]$. We claim that $[p] \in U$. To see why, note that $\pi^{-1}(U)$ is an open subset of \mathbb{R} containing 0. Hence, there exists $q \in \pi^{-1}(U)$ such that $q > 0$. Therefore, $q \sim p \Rightarrow [q] = [p] \in U$, and X/\sim is not Hausdorff.

Note: this proves that the continuous images of a Hausdorff space need not be Hausdorff!

3. Normalness

DEFINITION. Let (X, F_x) be a topological space. We say X is **normal** if for every pair of disjoint closed sets $A, B \subseteq X$, there exist disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

On Homework 2, we proved that metric spaces are normal.

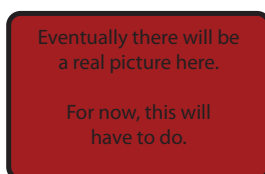
EXAMPLE. Consider \mathbb{R} with a topology F defined by a basis of all sets of the form (a, b) plus all sets of the form $(a, b) \cap \mathbb{Q}$. This is known as the **rational topology**, and it is finer than the usual topology. As an exercise, you can prove that that this collection of sets really form the basis of a topology. (Use the basis lemma.)

We want to show that (\mathbb{R}, F) is Hausdorff. This is easy, because F contains the usual topology on \mathbb{R} , which is Hausdorff.

Next, we will show that (\mathbb{R}, F) is not normal. To see why, let $A = \mathbb{R} \setminus \mathbb{Q}$. Then A is closed, because \mathbb{Q} is contained in F . Let $B = \{47\}$. Suppose there exist $U, V \in F$ such that $A \subseteq U$, and $B \subseteq V$. Then there exists $\epsilon > 0$ such that $(47 - \epsilon, 47 + \epsilon) \cap \mathbb{Q} \subseteq V$. Let $p \in (47 - \epsilon, 47 + \epsilon)$ such that $p \notin \mathbb{Q}$. Then $p \in U$ because $p \notin \mathbb{Q}$. So, there exists $\delta > 0$ such that $(p - \delta, p + \delta) \subseteq U$. Now, there must exist a point in $x \in (p - \delta, p + \delta) \cap ((47 - \epsilon, 47 + \epsilon) \cap \mathbb{Q})$. But then $x \in U \cap V$, and so $U \cap V \neq \emptyset$. This proves that (\mathbb{R}, F) is not a normal space.

In particular, we have shown that a space can be Hausdorff but not normal.

LEMMA. *If (X, F_x) is Hausdorff and compact, then X is normal.*



(This is a comic-book type proof above)

Now for the real proof.

PROOF. Let A and B be disjoint closed subsets of X . Then A and B are compact because X is compact. Let $a \in A$. For every $b \in B$, there exist open sets U_b and V_b such that $a \in U_b$ and $b \in V_b$, and $U_b \cap V_b = \emptyset$. Then $\{V_b \mid b \in B\}$ is an open cover of B . Since B is compact, we can choose a finite subcover $\{V_{b_1}, \dots, V_{b_n}\}$. Let $U_a = \cap_{i=1}^n U_{b_i}$. For every $a \in A$, $U_a \in F_x$ and $a \in U_a$. Let $V_a = \cup_{i=1}^n V_{b_i}$. Then $B \subseteq V_a \in F_x$.

We claim that $U_a \cap V_a = \emptyset$ for all $a \in A$. To see why, note that $(\cap_{i=1}^n U_{b_i}) \cap (\cup_{i=1}^n V_{b_i}) = \emptyset$ because for all i , $U_{b_i} \cap V_{b_i} = \emptyset$.

Now, $\{U_a \mid a \in A\}$ is an open cover of A . Since A is compact, this cover has a finite subcover $\{U_{a_1}, \dots, U_{a_m}\}$. Now, $V = \cap_{i=1}^m V_{a_i}$ is open and $B \subseteq V$. For all $i = 1, \dots, m$, $U_{a_i} \cap (\cap_{i=1}^m V_{a_i}) = \emptyset$ because for all $i = 1, \dots, m$, $U_{a_i} \cap V_{a_i} = \emptyset$.

Let $U = \cup_{i=1}^m U_{a_i} \in F_x$. Then $A \subseteq U$ and $U \cap V = \emptyset$ because $U_{a_i} \cap V_{a_i} = \emptyset$ for all i .

Therefore, X is normal. □

THEOREM 18. *Let (X, F_X) be a compact and Hausdorff topological space. If (Y, F_Y) is a topological space and $f : X \rightarrow Y$ is continuous, onto, and closed, then (Y, F_Y) is compact and Hausdorff.*

PROOF. From previous notes, as f is continuous and X is compact, Y is compact.

So now consider Hausdorff:

Let $p, q \in Y$ such that $p \neq q$. As f is onto, $\exists a, b \in X$ such that $f(a) = p$ and $f(b) = q$. As X is Hausdorff, $\{a\}$ and $\{b\}$ are closed. As f is closed, $f(\{a\}) = \{p\}$ and $f(\{b\}) = \{q\}$ are closed. Therefore, as f is continuous, $f^{-1}(\{p\})$ and $f^{-1}(\{q\})$ are closed, and are clearly disjoint.

As X is both compact and Hausdorff, it is also normal. So $\exists U, V \in F_X$ such that $f^{-1}(\{p\}) \subseteq U$, $f^{-1}(\{q\}) \subseteq V$, and $U \cap V = \emptyset$. As these sets are open, $A = U^c$ and $B = V^c$ are closed. As f is closed, $f(A)$ and $f(B)$ are also closed, and hence $f(A)^c$ and $f(B)^c$ are open.

Suppose that $p \in f(A)$. Then for some $x \in A$, $f(x) = p$. But then $x \in f^{-1}(\{p\}) \subseteq U = A^c$. So $x \in A \cap A^c$, which is a contradiction. So $p \in f(A)^c$, and by a similar argument, $q \in f(B)^c$.

Now suppose that $c \in f(A)^c \cap f(B)^c$. So, as f is onto, there exists some $d \in X$ such that $f(d) = c$. As $c \notin f(A)$, $d \notin A$, so $d \in U$. Similarly, as $c \notin f(B)$, $d \in V$. But then $d \in U \cap V = \emptyset$. This is a contradiction, so no such c exists, and $f(A)^c \cap f(B)^c = \emptyset$.

So $f(A)^c$ and $f(B)^c$ are disjoint open sets with $p \in f(A)^c$ and $q \in f(B)^c$. As such open sets exist for all $p, q \in Y$, Y is Hausdorff.

Therefore Y is compact and Hausdorff, as desired. \square

4. Connectedness

4.1. Connected Sets and Spaces.

DEFINITION. (X, F_X) is a topological space. Then X is **disconnected** if X has a proper, nonempty, clopen subset. Otherwise, X is **connected**.

Equivalently, X is disconnected if there exist nonempty, disjoint $A, B \in F_X$ such that $A \cup B = X$. A and B are called a separation of X .

Recall from Math 131 that $A \subseteq \mathbb{R}$ is connected iff A is an interval.

THEOREM 19. *Let $f : (X, F_X) \rightarrow (Y, F_Y)$ be continuous. Then X is connected only if $f(X)$ is connected.*

PROOF. Suppose that $U, V \in F_Y$ are separating sets of $f(X)$. Then, as f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open. As U and V cover $f(X)$, $f^{-1}(U)$ and $f^{-1}(V)$ cover X . Furthermore, suppose that $x \in f^{-1}(U) \cap f^{-1}(V)$. Then $f(x) \in U \cap V$, and $f(x) \in f(X)$. However, this contradicts the assumption that U and V are separating sets. So $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets that cover X , so they are a separating set. But X is connected, so this is a contradiction. Hence, no such U and V exist, and $f(X)$ is also connected. \square

Are the following connected?

- \mathbb{R} with the finite complement topology - Yes. Let U be some proper, nonempty open subset of \mathbb{R} . Then it is infinite, so its complement does not have finite complement. Therefore, its complement is not open. So there are no proper, nonempty clopen sets, and the topology is connected.
- \mathbb{R} with the half-open interval topology - No. Let $U = [0, \infty)$ and $V = (-\infty, 0)$. These are open as $U = \bigcup_{n \in \mathbb{N}} [0, n)$ and $V = \bigcup_{n \in \mathbb{N}} [-n, 0)$, and the arbitrary union of basis elements is open. Furthermore, they are clearly disjoint, and cover \mathbb{R} .

We can also define (dis)connectedness for subsets of a topological space.

DEFINITION. (X, F_X) is a topological space. $S \subseteq X$ is **disconnected** if and only if there exist $U, V \in F_X$ such that $(U \cap S) \cap (V \cap S) = \emptyset$, $U \cap S, V \cap S \neq \emptyset$ and $S \subseteq U \cup V$. Note that this does not require $U \cap V = \emptyset$.

SMALL FACT. (X, F_X) is connected if and only if $\forall f : X \rightarrow Y$ where Y has the discrete topology and f is continuous, then f constant.

PROOF. (\Rightarrow) Suppose (X, F_X) is connected and that $\exists f : X \rightarrow Y$ where Y has the discrete topology, such that f is continuous and not constant. So there exist $p, q \in Y$ such that $p \neq q$ and $f^{-1}(\{p\}), f^{-1}(\{q\}) \neq \emptyset$. Therefore, $U = f^{-1}(\{p\})$ and $V = f^{-1}(\{p\}^c) \supseteq f^{-1}(\{q\})$ have $U, V \neq \emptyset$. As every set is open in the discrete topology and f is continuous, $U, V \in F_X$. Furthermore, as $\{p\} \cap \{p\}^c = \emptyset$, $U \cap V = \emptyset$, and $U \cup V = f^{-1}(Y) = X$. So U and V separate X . However, X is connected, so this is a contradiction, and no such f exists.

(\Leftarrow) Suppose that $f : X \rightarrow Y$ is constant for all continuous f where Y has the discrete topology and that X is disconnected. Then there exist separating sets $A, B \in F_X$. Define $Y = \{0, 1\}$ with the discrete topology and $f : X \rightarrow Y$ as

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

Then, as $f^{-1}(\{0\}) = A \in F_X$, $f^{-1}(\{1\}) = B \in F_X$, $f^{-1}(\emptyset) = \emptyset \in F_X$ and $f^{-1}(Y) = X \in F_X$, $f^{-1}(U) \in F_X$ for all $U \subseteq Y$ such that U is open. So f is continuous. Furthermore, as $A, B \neq \emptyset$, there exist $a \in A$ and $b \in B$, so $f(a) = 0$ and $f(b) = 1$. Hence f is not constant. Therefore $f : X \rightarrow Y$ is a continuous function from X to a space with the discrete topology that is not constant. This contradicts the initial assumption, so X is connected.

□

We now have an Important Lemma about Flowers.

LEMMA. Suppose $\{y_j \mid j \in J\}$ is a collection of connected subspaces of $X = \bigcup_{j \in J} y_j$, and

$\bigcap_{j \in J} y_j \neq \emptyset$. Then X is connected.

PROOF. Suppose that X is *disconnected*. Then X has a separation, U, V .

Eventually there will be
a real picture here.

For now, this will
have to do.

(draw a flower)

Let $j_0 \in J$. We know that

- $(U \cap y_{j_0}) \cup (V \cap y_{j_0}) = y_{j_0}$,
- $(U \cap y_{j_0}) \cap (V \cap y_{j_0}) = \emptyset$,
- $(U \cap y_{j_0})$ and $(V \cap y_{j_0})$ are open in y_{j_0} ,
- y_{j_0} is connected, so either $U \cap y_{j_0} = \emptyset$ or $V \cap y_{j_0} = \emptyset$.

Without loss of generality, say $U \cap y_{j_0} = \emptyset$. Then $V \cap y_{j_0} = y_{j_0}$. By the same argument, $\forall j \in J, y_j \subseteq U$ or $y_j \subseteq V$.

But $\bigcap_{j \in J} y_j \neq \emptyset$, so there exists $p \in \bigcap_{j \in J} y_j$ such that $p \in y_{j_0} \subseteq V$. Thus $\forall j \in J, y_j \subseteq V$.

Since $X = \bigcup_{j \in J} y_j \subseteq V$, U, V is not a separation of X . Thus, X is connected. \square

THEOREM 20. *Let (X, F_X) and (Y, F_Y) be topological spaces. Then $(X \times Y, F_{X \times Y})$ is connected if and only if both X and Y are connected.*

PROOF. (\Rightarrow) $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are continuous surjections. Continuous functions preserve connectedness, so X and Y are connected.

(\Leftarrow) First, a picture proof.

Eventually there will be
a real picture here.

For now, this will
have to do.

The axes are connected, because X and Y are connected. If we make a copy of the axes, shifted over a little bit, that copy is connected too. The union of these shifted axis crosses will be the entirety of $X \times Y$, and their intersection will be nonempty, so $X \times Y$ is connected.

Now for the actual proof. Let $(x_0, y_0) \in X \times Y$. Since $\{x_0\} \times Y \cong Y$, it is connected (as connectedness is a top. prop.). Similarly, $\{y_0\} \times X$ is connected.

Observe that $(x_0, y_0) \in (\{x_0\} \times Y) \cap (\{y_0\} \times X)$. That is, $(\{x_0\} \times Y) \cap (\{y_0\} \times X) \neq \emptyset$. Then $(x_0, y_0) \in (\{x_0\} \times Y) \cup (\{y_0\} \times X)$ is connected by the Important Flower Lemma. By the same process, we see that $\forall x \in X$, $(X \times \{y_0\}) \cup (\{x\} \times Y)$ is connected.

Note that $(X \times \{y_0\}) \subseteq \bigcap_{x \in X} (X \times \{y_0\}) \cup (\{x\} \times Y)$.

Claim. $\bigcup_{x \in X} (X \times \{y_0\}) \cup (\{x\} \times Y) = X \times Y$ **Proof of claim.**

(\subseteq) Clearly $\bigcup_{x \in X} (X \times \{y_0\}) \cup (\{x\} \times Y) \subseteq X \times Y$ because $X \times Y$ is the entire space.

(\supseteq) Let $(x_1, y_1) \in X \times Y$. Then $(x_1, y_1) \in (\{x_1\} \times Y)$. Thus, $\forall x \in X$ and $y \in Y$, $(x, y) \in (\{x\} \times Y) \subseteq \bigcup_{x \in X} (X \times \{y_0\}) \cup (\{x\} \times Y)$.

Since $(X \times \{y_0\}) \cup (\{x\} \times Y)$ is connected $\forall x \in X$, and $\bigcap_{x \in X} (X \times \{y_0\}) \cup (\{x\} \times Y) \neq \emptyset$, $X \times Y$ is connected by the Important Flower Lemma.

□

4.2. Connected Components.

DEFINITION. Let (X, F_X) be a topological space and let $p \in X$. Let $\{C_j \mid j \in J\}$ be the set of all connected subspaces of X containing p . Then $\bigcup_{j \in J} C_j$ is said to be the **connected component**, C_p , of p .

And now, a series of tiny facts about connected components.

SMALL FACT. Let (X, F_X) be a topological space. Then,

- (1) $\forall p \in X$, C_p is connected.
- (2) If C_p and C_q are connected components, then either $C_p \cap C_q = \emptyset$ or $C_p = C_q$. (That is, connected components partition a space.)

PROOF. (1) $p \in \bigcap_{j \in J} C_j$, so $C_p = \bigcup_{j \in J} C_j$ is connected by the Important Flower Lemma.

- (2) Suppose $C_p \cap C_q \neq \emptyset$ and let $x \in C_p \cap C_q$. Then $C_p \cup C_q$ is connected by the Important Flower Lemma.

$p \in C_p \cup C_q$, so $C_p \cup C_q \in \{C_j \mid j \in J\}$. Also, $C_p \cup C_q \subseteq C_p$, since $C_p = \bigcup_{j \in J} C_j$.

So, $C_q \subseteq C_p$. Similarly, $C_p \subseteq C_q$.

Thus $C_p = C_q$, so we can say X is partitioned by its connected components. \square

EXAMPLE. Let $X = \mathbb{R}^2$ with the dictionary topology. Connected components are vertical lines. (Proof is an exercise.) Define $x \sim y$ ($x, y \in \mathbb{R}^2$) if x and y are in the same connected component. Then $X/\sim = \mathbb{R}$ with the discrete topology.

(Draw your dictionary topology components and X/\sim here!)

Eventually there will be
a real picture here.

For now, this will
have to do.

Connectedness is not always intuitive, though. Here's a large example showing that: **The Flea and the Comb**.

Define the Comb, the Flea, and the space X as follows.

Comb: $\bigcup_{n=0}^{\infty} y_n$ in \mathbb{R}^2 where $y_0 = [0, 1] \times \{0\}$ and $\forall n \in \mathbb{N}$, $y_n = \{1/n\} \times [0, 1]$.

Flea: $\{(0, 1)\}$.

X : Flea \cup Comb with the subspace topology.

EXAMPLE. X is connected.

PROOF. First, we claim that the Comb is connected. Proof:

$\forall n \in \{0\} \cup \mathbb{N}$, y_n is connected because $y_n \cong [0, 1]$. Also, $\forall n \in \mathbb{N}$, $y_0 \cap y_n \neq \emptyset$ (because $(1/n, 0)$ is in both), so $y_0 \cup y_n$ is connected.

Thus $\bigcap_{n \in \mathbb{N}} y_0 \cup y_n \neq \emptyset$, so $\bigcup_{n=0}^{\infty} y_0 \cup y_n = \bigcup_{n=0}^{\infty} y_n$ and so the Comb is connected by the Important Flower Lemma.

\square

Now we want to show that X is connected. Suppose U, V are a separation of X . Since the Comb is connected, we can say, WLOG, that Comb $\subseteq U$. Then it must be that Flea $\subseteq V$ because U, V are a separation. V is open in X , so $\exists \epsilon > 0$ s.t. $B_\epsilon((0, 1); X) \subseteq V$. Let $n \in \mathbb{N}$ s.t. $n > \frac{1}{\epsilon}$. Then $d((\frac{1}{n}, 1), (0, 1)) = \frac{1}{n} < \epsilon \Rightarrow (1/n, 1) \in B_\epsilon((0, 1); X) \subseteq V$. Observe

that since $Y_n = \{1/n\} \times [0, 1]$, $(1/n, 1) \in Y_n$. Thus, $(1/n, 1) \in V \cap Y_n \subseteq V \cap Y \subseteq V \cap U \Rightarrow V \cap U \neq \emptyset$. But this is a contradiction, since we assumed that U, V was a separation of X . Thus, X is connected. \square

5. Path-Connectedness

DEFINITION. Let (X, F_X) be a topological space and let $f : [0, 1] \rightarrow X$ be continuous. Then we say that f is a **path** from $f(0)$ to $f(1)$ (Note: it is handy to think of $t \in [0, 1]$ as time).

DEFINITION. Let (X, F_X) be a topological space. If $\forall p, q \in X$, there exists a path in X from p to q then we say that X is **path-connected**.

EXAMPLE. \mathbb{R}^n is path-connected:

Let $a, b \in \mathbb{R}^n$ be given. Let $f : I \rightarrow \mathbb{R}^n$ be $f(t) = (1 - t)a + bt$. Then $f(0) = a$, and $f(1) = b$. f can be shown to be continuous by performing an involved ϵ - δ proof.

Some remarks:

- (1) Connected is a negative definition and path-connected is a positive definition (for connectedness, we are trying to show that a separation *doesn't* exist, so it is easiest to do connectedness proofs by contradiction. For path-connectedness, we are trying to show that a path exists, so path-connectedness proofs are more easily done constructively).
- (2) In general, it is easier to prove that a space is disconnected than to prove that it is not path-connected.
- (3) In general, it is easier to prove that a space is path-connected than to prove that it is connected.

THEOREM 21. *If (X, F_X) is path-connected, then (X, F_X) is connected.*

PROOF. Suppose there exists a separation U, V of X . Since U, V is a separation of X , $U \neq \emptyset$, $V \neq \emptyset$. Let $p \in U$, $q \in V$ be given. Since X is path-connected, \exists a path f from p to q . Since paths are continuous by definition, f is continuous, so $f^{-1}(U)$, $f^{-1}(V)$ are open in $[0, 1]$.

Claim 1: $f^{-1}(U) \cup f^{-1}(V) = [0, 1]$.

Proof of Claim 1: Let $x \in [0, 1]$ be given. Then $f(x) \in X = U \cup V \Rightarrow f(x) \in U$ or $f(x) \in V \Rightarrow x \in f^{-1}(U)$ or $x \in f^{-1}(V) \Rightarrow x \in f^{-1}(U) \cup f^{-1}(V)$. Thus, $[0, 1] \subseteq f^{-1}(U) \cup f^{-1}(V) \Rightarrow [0, 1] = f^{-1}(U) \cup f^{-1}(V)$ (since $f^{-1}(U), f^{-1}(V) \subseteq [0, 1]$).

Claim 2: $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

proof of Claim 2: Suppose $\exists x \in f^{-1}(U) \cap f^{-1}(V)$. Then $x \in f^{-1}(U) \Rightarrow f(x) \in U$, and

$x \in f^{-1}(V) \Rightarrow f(x) \in V$, so $f(x) \in U \cap V$, which is impossible since we are assuming that U and V are a separation of X . Thus, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

Observe that since f is a path from p to q , $f(0) = p$ and $f(1) = q$, so $0 \in f^{-1}(U)$, $1 \in f^{-1}(V)$ so $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty and proper. Thus, since $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = [0, 1]$, $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of $[0, 1]$. But this is a contradiction, since we know from Math 131 that $[0, 1]$ is connected. Thus, (X, F_X) must be connected. \square

Recall the example of the flea and the comb. We showed that the flea and comb is connected.

THEOREM 22. *The flea and comb is not path-connected.*

PROOF. We want to show that \nexists a path from the flea to $(0, 0)$.

Suppose \exists a path f from the flea to $(0, 0)$. Let $p = \text{flea}$. Then $f^{-1}(\{p\})$ is not empty because $f(0) = p$ by the definition of f . Similarly, by the definition of f , $f(1) = (0, 0) \neq p$, so $f^{-1}(\{p\})$ is proper as well. Observe that since $\{p\}$ is closed in X and f is continuous, $f^{-1}(\{p\})$ is closed.

We'd like to show that $f^{-1}(\{p\})$ is open, since then it would be a nontrivial clopen subset of the flea and the comb. This implies that the set is not connected, a contradiction.

Let $y \in f^{-1}(\{p\})$ be given.

Observe that $B_{\frac{1}{2}}(p; X)$ is open in X . Since f is a path, f is continuous, so $f^{-1}(B_{\frac{1}{2}}(p; X))$ is open in $[0, 1]$. Since $y \in f^{-1}(\{p\})$, $f(y) = p \in B_{\frac{1}{2}}(p; X) \Rightarrow y \in f^{-1}(B_{\frac{1}{2}}(p; X))$. Thus, since $f^{-1}(B_{\frac{1}{2}}(p; X))$ is open in $[0, 1]$, $\exists \epsilon > 0$ s.t. $B_\epsilon(y; [0, 1]) \subseteq f^{-1}(B_{\frac{1}{2}}(p; X))$.

Let $z \in B_\epsilon(y; [0, 1])$. If $f(z) = p$, then $B_\epsilon(y; [0, 1]) \subseteq f^{-1}(p)$ and so $f^{-1}(p)$ is open, completing the proof.

Suppose $f(z) \neq p$. Since $z \in B_\epsilon(y; [0, 1])$, $z \in f^{-1}(B_{\frac{1}{2}}(p; X))$, so $f(z) \in B_{\frac{1}{2}}(p; X)$ (so $d(f(z), p) < \frac{1}{2}$). Since $Y_0 = [0, 1] \times \{0\}$, for each $q \in Y_0$, $d(p, q) \geq 1 \Rightarrow q \notin B_{\frac{1}{2}}(p; X)$. Thus, $f(z) \notin Y_0$. Thus since $f(z) \neq p$, $\exists n \in \mathbb{N}$ s.t. $f(z) \in Y_n$.

There exists $r \in \mathbb{R} - \mathbb{Q}$ such that $0 < r < \frac{1}{n}$. Define sets $A, B \subseteq f(B_\epsilon(y; [0, 1]))$, as

$$A = \{(x, y) \in f(B_\epsilon(y; [0, 1])) \mid x < r\} \quad \text{and} \quad B = \{(x, y) \in f(B_\epsilon(y; [0, 1])) \mid x > r\}$$

We next show that A and B are a separation for $f(B_\epsilon(y; [0, 1]))$. To this end, we'll need to show that A and B are disjoint, proper and non-empty, that their union is the entire set, and that they're both clopen.

We note that $A \cap B = \emptyset$ by definition. We know that $f(z) \in B$, and $p \in A$, so neither set will be empty and both will be proper.

Next, we want to show that $A \cup B = f(B_\epsilon(y; [0, 1]))$. Certainly $A \cup B \subseteq f(B_\epsilon(y; [0, 1]))$. Now let $(x, f(x)) \in f(B_\epsilon(y; [0, 1]))$. We know that $f(x) \neq 0$, and in fact $x = \frac{1}{m}$ for some $m \in \mathbb{N}$. Then $x \neq r$, so $x \in A \cup B$ and we're done.

Next we want to show that A is open in $f(B_\epsilon(y; [0, 1]))$. We know that $\{(x, y) | x < r\}$ is open in \mathbb{R}^2 , the half plane to the left of r . It follows using the subspace topology that $A = \{(x, y) | x < r\} \cap f(B_\epsilon(y; [0, 1]))$ is open in $f(B_\epsilon(y; [0, 1]))$. Similarly, B is open in $f(B_\epsilon(y; [0, 1]))$. Since each is the other's complement in $f(B_\epsilon(y; [0, 1]))$, then both are also closed.

We've hence shown that A and B are a separation of $f(B_\epsilon(y; [0, 1]))$. This is a contradiction, since $B_\epsilon(y; [0, 1])$ is connected and f is continuous, and thus we've disconnected $B_\epsilon(y; [0, 1])$. We conclude that $f(z) = p$, for all such $z \in B_\epsilon(y; [0, 1])$.

Therefore $B_\epsilon(y; [0, 1]) \subseteq f^{-1}(\{p\})$, so $f^{-1}(\{p\})$ is open. So $f^{-1}(\{p\})$ is a clopen, non-empty, proper subset of $[0, 1]$. This is a contradiction, so we conclude that there does not exist a path in X from p to $(0, 0)$ (or anywhere). \square

The point of the whole example is to show that path-connected is stronger than connected. We knew already that path-connected implies connected, and now we see that connected doesn't necessarily imply path-connected.

It turns out that it's useful to have a way to combine paths.

DEFINITION. Let f, g be paths in a topological space (X, F_X) such that $f(1) = g(0)$. Then we define $f * g : I \rightarrow X$ by

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Intuitively what's happening is that we're connecting two paths while speeding things up, creating a new single path parametrized from 0 to 1 from two paths which were parametrized from 0 to 1.

SMALL FACT. $f * g$ is a path from $f(0)$ to $g(1)$.

PROOF. By the Pasting Lemma¹, since $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed subsets under $[0, 1]$, and $f(2(\frac{1}{2})) = f(1) = g(0) = g(2(\frac{1}{2}) - 1)$, $f * g$ is continuous. So $f * g$ is a path. Since $(f * g)(0) = f(0)$ and $(f * g)(1) = g(1)$, then $f * g$ is a path from $f(0)$ to $g(1)$. \square

This definition also allows us to create an analogue to the flower lemma.

THEOREM 23 (Flower Lemma for Path-connected). *Let $X = \cup_{i \in I} Y_i$ such that $\forall i \in I$, Y_i is path-connected, and $\cap_{i \in I} Y_i \neq \emptyset$. Then X is path-connected.*

¹HW3 #1, which states that if we have two continuous functions with closed sets as domains, and they agree over the intersection of these domains, then the combined function is continuous

PROOF. Let $a, b \in X$. If $\exists n \in I$ such that $a, b \in Y_n$, then there exists a path from a to b in $Y_n \subseteq X$. So without loss of generality, suppose that $a \in Y_n, b \in Y_m$, and $m \neq n$. Let $x \in \cap_{i \in I} Y_i$. Then there exists a path f from a to x in Y_n , and a path g from x to b in Y_m . So $f * g$ is a path in X from $a \rightarrow b$, and we're done. \square

COROLLARY. *The product of path-connected spaces is path-connected.*

PROOF. The proof is identical to the connected one; the key is the flower lemma. \square

DEFINITION. Let (X, F_X) be a topological space, and $p \in X$. Let $\{C_j | j \in J\}$ be the set of all path-connected subspaces of X containing p . Then $\cup_{j \in J} C_j$ is said to be the **path-connected component** C_p .

Some tiny facts about path-connected components:

Let (X, F_X) be a topological space. Then

- (1) $\forall p \in X, C_p$ is path-connected
- (2) If C_p, C_q are path-connected components, then either $C_p \cap C_q = \emptyset$, or $C_p = C_q$.
In other words, path-connected components partition the set.

We'll omit this proof, as it is identical to the one presented on connectedness.

6. Homotopies

We now have enough background to start talking about geometric (or algebraic) topology.

DEFINITION. Let f be a path in (X, F_X) , and define $\bar{f} : I \rightarrow X$ by $\bar{f}(t) = f(1 - t)$.

A few remarks:

- (1) \bar{f} is a path, because it is a composition of continuous functions.
- (2) \bar{f} is a path from $f(1)$ to $f(0)$.
- (3) $f * \bar{f} \neq$ the "identity". We haven't created an inverse.

DEFINITION. Let (X, F_X) be a topological space, and $a \in X$. We define $e_a : I \rightarrow X$ by $e_a(t) = a$ for all $t \in I$.

However, note that $f * e_a \neq f$. This isn't an identity.

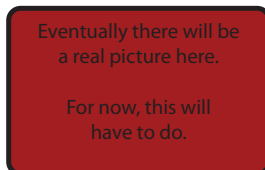
DEFINITION. Let (X, F_X) and (Y, F_Y) be top spaces, and let $f_0 : X \rightarrow Y$ and $f_1 : X \rightarrow Y$ be continuous. Then we say that f_0 is **homotopic** to f_1 if there exists a function $F : X \times I \rightarrow Y$, continuous, such that $F(x, 0) = f_0(x)$, and $F(x, 1) = f_1(x)$. We say that F is a *homotopy* from f_0 to f_1 , and we write $f_0 \simeq f_1$.

This is probably the most important concept in the rest of these notes.

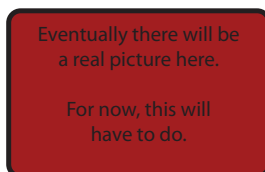
To illustrate homotopies, a couple of pictures.

$f_0 : S^1 \rightarrow \mathbb{R}^2$ (such that the image looks like a circle)

$f_1 : S^1 \rightarrow \mathbb{R}^2$ (such that the image looks like a square)



It is visually apparent that one image can be warped to make the other, so f_0 and f_1 are homotopic. However, homotopic functions need not be homeomorphic. These are homotopic, for example (a circle space and an infinity-shaped space).

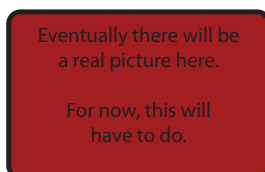


EXAMPLE. Let $X = I$ and $Y = \mathbb{R}^2$. Define $f_0 : I \rightarrow \mathbb{R}^2$ by $f_0(x) = (x, 0)$, $f_1 : I \rightarrow \mathbb{R}^2$ by $f_1(x) = (x, x^2)$, and $F : I \times I \rightarrow \mathbb{R}^2$ by $F(x, t) = t(x, x^2) + (1 - t)(x, 0)$.

Observe that F is continuous. Note also that $F(x, 0) = (x, 0) = f_0(x)$ and $F(x, 1) = (x, x^2) = f_1(x)$.

Thus F is a homotopy.

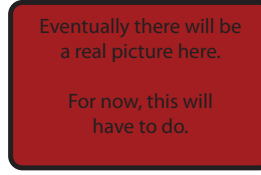
6.1. Drawing Homotopies. How to draw a homotopy:



The image of a vertical segment in the square is the path taken by the x coordinate of that segment during the homotopy.

A few remarks.

- (1) Suppose Y is not path connected and $f_0(x)$ and $f_1(x)$ are in different path components. Then there does not exist a homotopy from f_0 to f_1 .



- (2) If Y is path connected and f_0 and f_1 are paths in Y , then $f_0 \simeq f_1$. Homotope (the verb!) f_0 to its initial point, move it to the initial point of f_1 and then stretch it back out into f_1 .)



EXAMPLE. Let $Y = \mathbb{R}^2 - \{(\frac{1}{2}, \frac{1}{3})\}$. Define $f_0 : I \rightarrow Y$ by $f_0(s) = (s, s^2)$, and $f_1 : I \rightarrow Y$ by $f_1(s) = (s, s)$. Because there's a hole in Y , we won't be able to just bend f_0 over to f_1 .

Define $G : I \times I \rightarrow Y$ by $G(s, t) = f_0((1-t)s)$. Observe that G is continuous, and $G(s, 0) = f_0(s)$ and $G(s, 1) = f_0(0)$.

Now define $H : I \times I \rightarrow Y$ by $H(s, t) = f_1(ts)$. Observe that H is continuous and $H(s, 0) = f_1(0)$ and $H(s, 1) = f_1(s)$.

Finally, define $F : I \times I \rightarrow Y$ by

$$F(s, t) = \begin{cases} G(s, 2t) & : t \in [0, \frac{1}{2}] \\ H(s, 2t - 1) & : t \in [\frac{1}{2}, 1] \end{cases}.$$

F is continuous: Let $A = I \times [0, \frac{1}{2}]$ and $B = I \times [\frac{1}{2}, 1]$. Both are closed in $I \times I$.
 $A \cap B = I \times \{\frac{1}{2}\}$. $G(s, 2(\frac{1}{2})) = G(s, 1) = f_0(0) = (0, 0)$ and $H(s, 2(\frac{1}{2}) - 1) = H(s, 0) = f_1(0) = (0, 0)$.

Since G and H are continuous and agree at $A \cap B$, by the Pasting Lemma, F is continuous.

F is a homotopy :

$$F(s, 0) = G(s, 0) = f_0(s) \quad \text{and} \quad F(s, 1) = H(s, 1) = f_1(s)$$

Thus F is indeed an homotopy, and f_0 is homotopic to f_1 !

What path does $(1, 1)$ take during the aforementioned homotopy?

- Informally put, it moves down along f_0 and climbs back up f_1 to its old position.

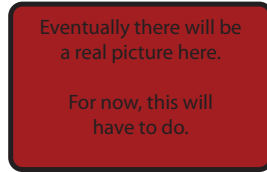
More formally,

$$F(1, t) = \begin{cases} G(1, 2t) & t \in [0, \frac{1}{2}] \\ H(1, 2t - 1) & t \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} f_0(1 - 2t) & t \in [0, \frac{1}{2}] \\ f_1(2t - 1) & t \in [\frac{1}{2}, 1] \end{cases} = \overline{f_0} * f_1.$$

6.2. Path Homotopies.

DEFINITION. Let f_0 and f_1 be paths in (X, F_X) from a to b . We say f_0 is *path-homotopic* if there exists a homotopy F from f_0 to f_1 s.t. $\forall t \in I$, $F(0, t) = a$ and $F(1, t) = b$. We say F is a **path homotopy** and write $f_0 \sim f_1$.

(insert picture of a path homotopy here)



EXAMPLE. Let X be a convex region of \mathbb{R}^n , let $a, b \in X$, and let f_0 and f_1 be paths in X from a to b .

Claim: $f_0 \sim f_1$.

PROOF. Let $F(s, t) = (1 - t)f_0(s) + tf_1(s)$ (Note: we call this the **straight line homotopy**). Observe that $\forall s \in I$, $F(s, 0) = f_0(s)$ and $F(s, 1) = f_1(s)$, and that F is continuous, so F is a homotopy from f_0 to f_1 . Now let $t \in I$ be given. Observe that $F(0, t) = (1 - t)f_0(0) + tf_1(0) = f_0(0) = a$, and that $F(1, t) = f_0(1) = b$. Thus $\forall t \in I$, $F(0, t) = a$ and $F(1, t) = b$, so F is a path homotopy, and thus $f_0 \sim f_1$. \square

EXAMPLE. Let $X \cong D^2$, let $a, b \in X$, and let f_0 and f_1 be paths in X from a to b . Then $f_0 \sim f_1$.

PROOF. Let $g : X \rightarrow D^2$ be a homeomorphism. Let $F : (I \times I) \rightarrow D^2$ be the straight line homotopy in D^2 from $g \circ f_0$ to $g \circ f_1$ (Note: D^2 is a convex region of \mathbb{R}^2 , so by the last example we can use the straight line homotopy here).

First, we need to show that $g^{-1} \circ F$ is continuous. Note that F is continuous since F is a homotopy. Note also that since g is a homeomorphism, g^{-1} is continuous. Thus $g^{-1} \circ F$ is the composition of continuous functions, and hence $g^{-1} \circ F$ is continuous.

Now we need to show that $g^{-1} \circ F$ is a homotopy from f_0 to f_1 .

First, observe that $\forall s \in I$, $(g^{-1} \circ F)(s, 0) = g^{-1}((1 - 0)g(f_0(s)) + (0)g(f_1(s))) = g^{-1}(g(f_0(s))) = f_0(s)$ since g is a bijection, and similarly $(g^{-1} \circ F)(s, 1) = f_1(s)$. Thus, since $g^{-1} \circ F$ is continuous, $g^{-1} \circ F$ is a homotopy from f_0 to f_1 .

Lastly, we need to show that $g^{-1} \circ F$ is a *path* homotopy.

Note that $\forall t \in I$, $(g^{-1} \circ F)(0, t) = g^{-1}((1 - t)g(f_0(0)) + tg(f_1(0))) = g^{-1}((1 - t)g(a) + tg(a)) = g^{-1}(g(a)) = a$ since g is a bijection. Similarly, $\forall t \in I$, $(g^{-1} \circ F)(1, t) = b$. Thus, since $g^{-1} \circ F$ is a homotopy from f_0 to f_1 , $g^{-1} \circ F$ is a path homotopy, and thus $f_0 \sim f_1$. \square

THEOREM 24. *Let (X, F_X) be a topological space and let $a, b \in X$ be given. Then \sim is an equivalence relation of paths in X from a to b .*

PROOF. In order to show \sim is an equivalence relation, we need to show that \sim is reflexive, symmetric, and transitive.

Reflexive: If f is a path in X from a to b , let $F : (I \times I) \rightarrow X$ be given by $F(s, t) = f(s)$. Note that F is a homotopy from f to f since, $\forall s \in I$, $F(s, 0) = f(s)$ and $F(s, 1) = f(s)$ and F is continuous since f is continuous. Observe that $\forall t \in I$, $F(0, t) = f(0) = a$ and $F(1, t) = f(1) = b$, so F is a path homotopy and $f \sim f$. Thus, \sim is reflexive.

Symmetric: Suppose that $f_1 \sim f_2$. Then there exists a path homotopy F from f_1 to f_2 . Define $F' : (I \times I) \rightarrow X$ given by $F'(s, t) = F(s, 1 - t)$, $\forall (s, t) \in (I \times I)$. Note that F' is continuous since F is continuous so F' is a composition of continuous functions. Recall that F is a homotopy from f_1 to f_2 , and thus F' is a homotopy from f_2 to f_1 since, $\forall s \in I$, $F'(s, 0) = F(s, 1) = f_2(s)$ and $F'(s, 1) = F(s, 0) = f_1(s)$. Now observe that $\forall t \in I$, $F'(0, t) = F(0, 1 - t) = a$ and $F'(1, t) = F(1, 1 - t) = b$, since F is a path homotopy. Thus, F' is a path homotopy, and thus $f_2 \sim f_1$, so \sim is symmetric.

Transitive: Suppose that f_1, f_2 , and f_3 are paths in X from a to b s.t. $f_1 \sim f_2$ and $f_2 \sim f_3$. Since $f_1 \sim f_2$, there exists a path homotopy F_1 from f_1 to f_2 , and since $f_2 \sim f_3$, there exists a path homotopy F_2 from f_2 to f_3 . Define $F_3 : (I \times I) \rightarrow X$ by

$$F_3(s, t) = \begin{cases} F_1(s, 2t) & t \in [0, \frac{1}{2}] \\ F_2(s, 2t - 1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Now we must show that F_3 is a path homotopy from f_1 to f_3 . Observe that $A = I \times [0, \frac{1}{2}]$ and $B = I \times [\frac{1}{2}, 1]$ are closed in $I \times I$, and F_1 and F_2 are continuous, so if $F_1(s, t) = F_2(s, t) \forall (s, t) \in A \cap B$, then by the pasting lemma F_3 is continuous. Since $A \cap B = I \times \{\frac{1}{2}\}$, and, $\forall s \in I$, $F_1(s, 2(\frac{1}{2})) = F_1(s, 1) = f_2(s)$ and $F_2(s, 2(\frac{1}{2}) - 1) = F_2(s, 0) = f_2(s)$, F_3 is continuous by the pasting lemma. Now, observe that $\forall s \in I$, $F_3(s, 0) = F_1(s, 2(0)) = F_1(s, 0) = f_1(s)$ and $F_3(s, 1) = F_2(s, 2(1) - 1) = F_2(s, 1) = f_3(s)$, so F_3 is a homotopy from f_1 to f_3 . Now, observe that $\forall t \in I$,

$$\begin{aligned} F_3(0, t) &= \begin{cases} F_1(0, 2t) & \text{if } t \in [0, \frac{1}{2}] \\ F_2(0, 2t - 1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \\ &= \begin{cases} a & t \in [0, \frac{1}{2}] \\ a & t \in [\frac{1}{2}, 1] \end{cases} \end{aligned}$$

(since F_1 and F_2 are path homotopies), and thus $F_3(0, t) = a, \forall t \in I$. Similarly, $\forall t \in I, F_3(1, t) = b$. Thus, F_3 is a path homotopy from f_1 to f_3 , so $f_1 \sim f_3$, and thus \sim is transitive.

Thus, \sim is an equivalence relation of paths in X from a to b . \square

Note that the same proof (using only the parts related to continuity and homotopy) works for \simeq .

DEFINITION. Let (X, F_X) be a topological space and let $a, b \in X$ be given. For each path f from a to b in X , define $[f]$ to be the **path homotopy class of f** .

DEFINITION. Let f be a path in X from a to b and g be a path in X from b to c . Define an INVISIBLE SYMBOL by $[f][g] = [f * g]$.

Some remarks:

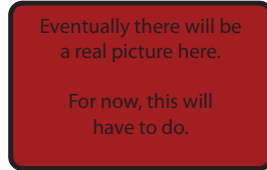
- (1) $[f]$, $[g]$, and $[f * g]$ are not elements of the same quotient ‘world’ unless $a = b = c$.
- (2) We have to prove that invisible multiplication is well-defined, i.e. if $f \sim f'$ and $g \sim g'$, then we want $[f * g] = [f' * g']$ because $[f] = [f']$ and $[g] = [g']$.

LEMMA (Important). *The product of path homotopy classes is well-defined. Formally, let (X, F_x) be a topological space. Let f, f' be paths in X from a to b and let g, g' be paths in X from b to c . Then:*

$$f * g \sim f' * g' \Rightarrow [f][g] = [f'][g']$$

PROOF. We know $f * g$ and $f' * g'$ are paths in X from a to c by a previous result. Let F be the path homotopy from f to f' , and let G be the path homotopy from g to g' .

At this point, it may be wise to draw a homotopy diagram:



Define $H : I \times I \rightarrow X$ by:

$$H(s, t) = \begin{cases} F(2s, t) & s \in [0, 1/2] \\ G((2s - 1), t) & s \in [1/2, 1] \end{cases}$$

We claim H is a homotopy from $f * g$ to $f' * g'$. We will now verify the claim.

We see that:

$$F(2 \cdot (1/2), t) = F(1, t) = b \quad G(2 \cdot (1/2) - 1, t) = G(0, t) = b$$

so by the pasting lemma, it follows that H is continuous.

We now show H gives the desired paths by calculating:

$$H(0, t) = F(0, t) = a \quad H(1, t) = G(1, t) = c$$

so $H(s, [0, 1])$ gives a set of paths from a to c .

Finally, we need to prove that H gives a homotopy between the intended paths:

$$H(s, 0) = \begin{cases} F(2s, 0) & s \in [0, 1/2] \\ G(2s - 1, 0) & s \in [1/2, 1] \end{cases} = \begin{cases} f(2s) & s \in [0, 1/2] \\ g(2s - 1) & s \in [1/2, 1] \end{cases} = f * g$$

Similarly:

$$H(s, 1) = \begin{cases} F(2s, 1) & s \in [0, 1/2] \\ G(2s - 1, 1) & s \in [1/2, 1] \end{cases} = \begin{cases} f'(2s) & s \in [0, 1/2] \\ g'(2s - 1) & s \in [1/2, 1] \end{cases} = f' * g'$$

Consequently, H satisfies the definition of a path homotopy from $f * g$ to $f' * g'$. We then have that $f * g \sim f' * g'$ so that by definition of the product of paths:

$$[f][g] = [f * g] = [f' * g'] = [f'][g']$$

as desired. □

7. Loops

We have shown that the product of two path homotopy classes is well defined. For the purposes of defining a group of path homotopy classes, we would like a set of paths whose products have the same endpoints as the original paths. This simplification motivates the two definitions which follow:

DEFINITION. Let f be a path in X such that $f(0) = f(1) = x_0 \in X$. Then f is a **loop** in X based at x_0 .

Note that if f, g are loops in X based at some point $x_0 \in X$, then their product $f * g$ is also a loop based at x_0 . In particular, we then have that $[f], [g]$ and $[f][g] = [f * g]$ are all path-homotopic classes of loops based at x_0 .

DEFINITION. Let (X, F_x) be a topological space, and let $x_0 \in X$.

Define $\pi_1(X, x_0)$ as the set of path homotopy classes of loops based at x_0 endowed with the product of path homotopy classes. We call $\pi_1(X, x_0)$ the **fundamental group of X based at x_0** .

7.1. The Fundamental Group is a Group. Our ultimate goal is to harness the power of group theory from abstract algebra to study topological spaces. We defined the fundamental group of X based at x_0 with the path homotopy class product operation. Naturally, we would like to prove that $\pi_1(X, x_0)$ endowed with the product operation is actually a group. In other words, if (X, F_x) is a topological space with $x_0 \in X$, we must prove the following:

- (1) $\pi_1(X, x_0)$ is closed under the operation $[\]$. In other words, for $[f], [g] \in \pi_1(X, x_0)$, $[f][g] \in \pi_1(X, x_0)$.
- (2) The operation $[\]$ is associative. In other words, given $[f], [g], [h] \in \pi_1(X, x_0)$:

$$([f][g])[h] = [f]([g][h])$$

- (3) $\pi_1(X, x_0)$ contains an identity element. In other words, there exists $[e] \in \pi_1(X, x_0)$ such that for all $[f] \in \pi_1(X, x_0)$:

$$[e][f] = [f][e] = [f]$$

- (4) Every element of $\pi_1(X, x_0)$ has an inverse. In other words, given $f \in \pi_1(X, x_0)$, there exists $g \in \pi_1(X, x_0)$ such that:

$$[f][g] = [g][f] = [e]$$

If $\pi_1(X, x_0)$ satisfies all of these requirements, $\pi_1(X, x_0)$ is a group.

LEMMA (Closure). *Let (X, F_x) be a topological space, and let $x_0 \in X$. Let $[f], [g] \in \pi_1(X, x_0)$. Then:*

$$[f][g] \in \pi_1(X, x_0)$$

PROOF. We see that:

$$[f][g] = [f * g]$$

We know by a previous result that $f * g$ is a path in X from x_0 to x_0 since x_0 is both the starting point of $[f]$ and the endpoint of $[g]$. Consequently, $f * g$ is a loop in X based at x_0 , so $[f * g] \in \pi_1(X, x_0)$. \square

Next, we will show that path homotopy path products of loops based at a point are associative.

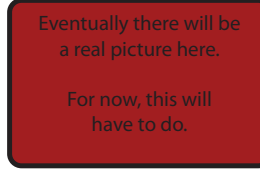
LEMMA (Associativity). *Let (X, F_x) be a topological space, and let $x_0 \in X$. Let $[f], [g], [h] \in \pi_1(X, x_0)$. Then:*

$$([f][g])[h] = [f]([g][h])$$

PROOF. Before actually proving the result, we will explicitly write the formulas for $(f * g) * h$ and $f * (g * h)$:

$$(f * g) * h = \begin{cases} f(4s) & s \in [0, \frac{1}{4}] \\ g(4s - 1) & s \in [\frac{1}{4}, \frac{1}{2}] \\ h(2s - 1) & s \in [\frac{1}{2}, 1] \end{cases} \quad f * (g * h) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(4s - 2) & s \in [\frac{1}{2}, \frac{3}{4}] \\ h(4s - 3) & s \in [\frac{3}{4}, 1] \end{cases}$$

We need to construct a homotopy from $(f * g) * h$ to $f * (g * h)$. Before reading the proof, the reader is encouraged to produce a homotopy diagram in the space provided below:



Based on our formulas for $(f * g) * h$ and $f * (g * h)$, we parameterize the function $F : I \times I \rightarrow X$ by:

$$F(s, t) = \begin{cases} f\left(\frac{4s}{1+t}\right) & s \in [0, \frac{1+t}{4}] \\ g(4s - 1 - t) & s \in [\frac{1+t}{4}, \frac{2+t}{4}] \\ h\left(\frac{4s}{2-t} - \frac{2+t}{2-t}\right) & s \in [\frac{2+t}{4}, 1] \end{cases}$$

We claim and will verify that F is a homotopy from $(f * g) * h$ to $f * (g * h)$. We first prove continuity using the pasting lemma as usual. The proof that each of the piecewise parts of F is defined on a closed set is left as a (relatively trivial but tedious) exercise.

We will, however, show that on the boundaries between these regions, the piecewise parts agree. We see:

$$\begin{aligned} f\left(\frac{4}{1+t} \cdot \left(\frac{1+t}{4}\right)\right) &= f(1) = x_0 \\ g\left(4 \cdot \left(\frac{1+t}{4}\right) - 1 - t\right) &= g(0) = x_0 \\ g\left(4 \cdot \left(\frac{2+t}{4}\right) - 1 - t\right) &= g(1) = x_0 \\ h\left(\frac{4}{2-t} \cdot \left(\frac{2+t}{4}\right) - \frac{2+t}{2-t}\right) &= h(0) = x_0 \end{aligned}$$

so continuity follows by the pasting lemma. Note that we assume f, g, h are continuous so that things like $h\left(\frac{4s}{2-t} - \frac{2+t}{2-t}\right)$ are because compositions of continuous functions are continuous and over $t \in I$, $\frac{4s}{2-t} - \frac{2+t}{2-t}$ is continuous. The justification is similar for the other piecewise parts in the definition of F .

We now verify that F gives a path from x_0 to x_0 for every fixed $t \in I$. We see that:

$$\begin{aligned} F(0, t) &= f(0) = x_0 \\ F(1, t) &= h(1) = x_0 \end{aligned}$$

as desired.

Finally, we need to show that F provides a homotopy from $(f * g) * h$ to $f * (g * h)$. We have that:

$$\begin{aligned} F(s, 0) &= \begin{cases} f(4s) & s \in [0, \frac{1+t}{4}] \\ g(4s-1) & s \in [\frac{1+t}{4}, \frac{2+t}{4}] \\ h(2s-1) & s \in [\frac{2+t}{4}, 1] \end{cases} = (f * g) * h \\ F(s, 1) &= \begin{cases} f(2s) & s \in [0, \frac{1+t}{4}] \\ g(4s-2) & s \in [\frac{1+t}{4}, \frac{2+t}{4}] \\ h(4s-3) & s \in [\frac{2+t}{4}, 1] \end{cases} = f * (g * h) \end{aligned}$$

as desired, so F satisfies the three properties of a homotopy from $(f * g) * h$ to $f * (g * h)$.

We have constructed $F(s, t)$ such that F is a homotopy from $(f * g) * h$ to $f * (g * h)$ which implies that $(f * g) * h \sim f * (g * h) \Rightarrow [(f * g) * h] = [f * (g * h)]$. It immediately follows that

$$([f][g])[h] = [f]([g][h])$$

□

LEMMA (Identity). *Let f be a path in X which begins at x_0 and ends at x_1 . Then $f * e_{x_1} \sim f$ and $e_{x_0} * f \sim f$.*²

PROOF. We prove that $f * e_{x_1} \sim f$. The other case is similar.

In order to define a homotopy, at t , we will “do” f for $s \in [0, t(1) + (1-t)\frac{1}{2}]$ or, by simplification, $s \in [0, \frac{t+1}{2}]$ and $e_{x_1}(t)$ for $s \in [\frac{t+1}{2}, 1]$.

Define $F : I \times I \rightarrow X$ by

$$F(s, t) = \begin{cases} f\left(\frac{2s}{t+1}\right) & s \in [0, \frac{t+1}{2}] \\ e_{x_1} & s \in [\frac{t+1}{2}, 1] \end{cases}$$

This is:

Continuous: By the Pasting lemma, as f, e_{x_1} are continuous on closed domains it suffices to check that they agree for $s = \frac{t+1}{2}$. This follows easily, as $f\left(\frac{2 \cdot \frac{t+1}{2}}{t+1}\right) = f(1) = x_1$, and $e_{x_1}\left(\frac{t+1}{2}\right) = x_1$.

²Recall that \sim denotes “path homotopy.”

A homotopy:

$$F(s, 0) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ e_{x_1}(t) & s \in [\frac{1}{2}, 1] \end{cases} = f * e_{x_1}$$

and

$$F(s, 1) = \begin{cases} f(s) & s \in [0, 1] \\ e_{x_1} & s \in [1, 1] \end{cases} = f.$$

A path: $F(0, t) = f(0) = x_0$, and $F(1, t) = e_{x_1} = x_1$, hence F is a path.

Thus $f * e_{x_1} \sim f$. □

We've now shown that $\pi_1(X, x_0)$ is closed, has an identity, and the operation is associative, so just showing that inverses exist proves that it's a group.

LEMMA (Inverses). *Let f be a path in X from x_0 to x_1 . Then $f * \bar{f} \sim e_{x_0}$ and $\bar{f} * f \sim e_{x_1}$.*

The “wrong” approach: increase the speed over f and \bar{f} and wait at x_1 . The “right” approach: travel successively smaller distances along f .

PROOF. We show only that $f * \bar{f} \sim e_{x_0}$, and the other case follows similarly.

Define $F : I \times I \rightarrow X$ by

$$F(s, t) = \begin{cases} f(2s) & s \in [0, \frac{1-t}{2}] \\ f(1-t) & s \in [\frac{1-t}{2}, \frac{1+t}{2}] \\ \bar{f}(2s-1) & s \in [\frac{1+t}{2}, 1] \end{cases}$$

This is:

Continuous: Using the pasting lemma, it suffices to check that these functions agree on their (shared) endpoints. As $f(2(\frac{1-t}{2})) = f(1-t)$, and $f(2(\frac{1+t}{2})-1) = \bar{f}(t) = f(1-t)$, we conclude that F is continuous.

A homotopy: Let $t = 0$,

$$F(s, 0) = \begin{cases} f(2s) & s \in [0, 1/2] \\ f(1) & s \in [1/2, 1/2] \\ \bar{f}(2s-1) & s \in [1/2, 1] \end{cases} = f * \bar{f}(s)$$

For $t = 1$,

$$F(s, 1) = \begin{cases} f(2s) & s \in [0, 0] \\ f(0) & s \in [0, 1] \\ \bar{f}(2s-1) & s \in [1, 1] \end{cases} = e_{x_0}.$$

A path: For $s = 0$, $F(0, t) = f(0) = x_0$. And for $s = 1$, $F(1, t) = \bar{f}(1) = x_0$.

For each loop f based at x_0 , \bar{f} is a loop based at x_0 . Thus we have shown closure under inverses. \square

It follows that the action $*$ defines a group on the set of loops based at x_0 .

Observe that the set of paths does not have a group structure, as there is no definition of multiplication between arbitrary paths.

EXAMPLE. Let $X = \mathbb{R}^n$ and x_0 be a point in X . Then $\pi_1(X, x_0) = \langle [e_{x_0}] \rangle$.

The following theorem relates the fundamental group at a point within a path component to the fundamental group of that point in the ambient space:

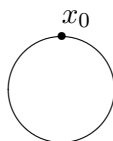
THEOREM 25. *Let A be a path component of a topological space X , and let $x_0 \in A$. Then:*

$$\pi_1(A, x_0) \cong \pi_1(X, x_0)$$

(Note that \cong denotes a group isomorphism and not a homeomorphism)

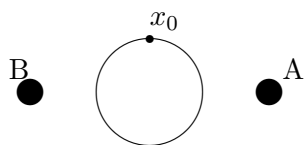
Before proving the theorem, we will cover two quick non-examples of cases where the theorem could break down.

Recall the circle S^1 which we can consider as a subspace of \mathbb{R}^2 . Since \mathbb{R}^2 is a simply connected space, the fundamental group at every point is trivial. On the other hand, picking some point $x_0 \in S^1$, the loop around the circle cannot be deformed in S^1 to the point x_0 , so $\pi_1(S^1, x_0)$ is non-trivial and hence not isomorphic to $\pi_1(\mathbb{R}^2, x_0)$ if we embed S^1 in \mathbb{R}^2 . The following picture illustrates S^1 with the point x_0 :



This may seem like a counterexample to the theorem, but in \mathbb{R}^2 , S^1 is not a distinct path component of \mathbb{R}^2 , so the theorem does not apply. Intuitively, by embedding S^1 in \mathbb{R}^2 , the interior of the circle is part of \mathbb{R}^2 , so we can deform a loop around S^1 based at x_0 to the trivial loop by “pulling” the loop through the middle of the circle, which we could not do when S^1 was considered as a space in its own right.

Consider the following diagram:



where we have the space \mathbb{R}^2 with A, B removed, so loops based at x_0 that run around A, B cannot be deformed into the trivial loop. However, when we consider this space embedded in \mathbb{R}^2 as a subspace, it is not an entire path component in \mathbb{R}^2 , so our theorem does not apply.

We now prove the theorem:

PROOF. Define $\varphi : \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ by:

$$\varphi([f]_A) = [f]_X$$

We claim that φ is a *group isomorphism*, i.e. that φ is a bijection and a group homomorphism. In other words, we need to show that φ is injective, surjective and that if $a, b \in \pi_1(A, x_0)$, then $\varphi(ab) = \varphi(a)\varphi(b)$. For those who have not had algebra, the existence of an isomorphism between two groups means that while the two groups may not have the same elements, they have the same algebraic structure.

Before we actually prove that φ satisfies the properties of an isomorphism, we have to show φ is well defined because φ is defined in terms of equivalence classes. First, let f, g be loops in A based at $x_0 \in A$ such that $f \sim_A g$. Thus there exists a path homotopy in A , $F : I \times I \rightarrow A$, from f to g . Recall the inclusion map $i : A \rightarrow X$ where i is the identity map on X restricted to A and is trivially continuous. Thus we can extend F to the continuous map $i \circ F : I \times I \rightarrow X$, and it is easy to see that $i \circ F$ is a path homotopy in X from f to g , so $f \sim_X g$. Thus if $[f]_A = [g]_A$, $\varphi([f]_A) = \varphi([g]_A)$, and φ is well defined.

We now prove φ is injective. Suppose $[f]_A, [g]_A \in \pi_1(A, x_0)$ such that $\varphi([f]_A) = \varphi([g]_A) \Rightarrow [f]_X = [g]_X$. Then there exists $F : I \times I \rightarrow X$, a path homotopy from f to g in X . Suppose toward a contradiction there exists $(s_0, t_0) \in I \times I$ such that $F(s_0, t_0) \notin A$. Now take a restriction of F , $\alpha : [0, s_0] \rightarrow X$ defined by $\alpha(s) = F(s, t_0)$. Since $[0, s_0] \cong I$, we can easily reparameterize α over the unit interval as $\beta : I \rightarrow X$, giving a path from x_0 to $F(s_0, t_0)$, since $F(0, t_0) = x_0$. Consequently, $\alpha(s_0) = \beta(1) \in A$ because A is a path component, so we have a contradiction. Thus every point in $F(I \times I)$ is contained in A , so we can define $G : I \times I \rightarrow A$ by $G(s, t) = F(s, t)$, so G is a homotopy from f to g in A . Consequently, $f \sim_A g \Rightarrow [f]_A = [g]_A$. Thus φ is injective.

Next we prove φ is surjective. Let $[f]_X \in \pi_1(X, x_0)$, so f is a loop in X based at x_0 . In other words, $f : I \rightarrow X$ is a path containing x_0 . Since A is a path component, $f(I) \subseteq A$ since we can easily use f to find a path between any two points in $f(I)$. Consequently, f is a loop in A based at x_0 , so we may state $\varphi([f]_A) = [f]_X$ and φ is surjective.

Finally, we prove that φ is a homomorphism. Let $[f]_A, [g]_A \in \pi_1(A, x_0)$. We see that:

$$\varphi([f]_A [g]_A) = \varphi([f * g]_A) = [f * g]_X = [f]_X [g]_X = \varphi([f]_A) \varphi([g]_A)$$

and φ satisfies the definition of a homomorphism.

We have proved that φ is a bijective, homomorphism, and is therefore an isomorphism between $\pi_1(A, x_0)$ and $\pi_1(X, x_0)$, so we conclude the two groups are isomorphic, as desired. \square

We now wish to prove a similar result:

THEOREM 26. *Let X be a topological space and let $x, y \in X$. Suppose $f : I \rightarrow X$ is a path from x to y , then $\pi_1(X, x) \cong \pi_1(X, y)$.*

Before providing the proof, we will define a key map which we will use again later in the course:

DEFINITION. Define $u_f : \pi_1(X, x) \rightarrow \pi_1(X, y)$ by $u_f([g]) = [\bar{f} * g * f]$.³

We now prove the theorem by showing that u_f is an isomorphism from $\pi_1(X, x)$ to $\pi_1(X, y)$:

PROOF. Again, we need to show that u_f is a bijective homomorphism. As usual for functions defined in terms of equivalence classes, we need to show that u_f is actually well defined.

Suppose that g, h are loops in X based at x such that $g \sim h$, so there exists $F : I \times I \rightarrow X$, a path homotopy from g to h in X . We want to show that:

$$\bar{f} * g * f \sim \bar{f} * h * f$$

Since we know trivially $f \sim f$, $g \sim h$ and $\bar{f} \sim \bar{f}$, using our inverse and identity lemmas for multiplication of path homotopy classes, it is easy to see that $\bar{f} * g * f \sim \bar{f} * h * f$, so u_f is well defined.

Next, we show that u_f is injective. Suppose g, h are loops in X based at x such that $u_f([g]_X) = u_f([h]_X)$. Therefore:

$$[\bar{f} * g * f] = [\bar{f} * h * f] \Rightarrow \bar{f} * g * f \sim \bar{f} * h * f$$

so by our important lemma for path homotopies and our inverse/identity lemmas, we have that $g \sim h$ and u_f is injective.

Now we wish to show that u_f is surjective. Let $[g] \in \pi_1(X, y)$. Then $[f * g * \bar{f}] \in \pi_1(X, x)$, and $u_f([f * g * \bar{f}]) = u_f([\bar{f} * (f * g * \bar{f}) * f])$, which by associativity, inverses, and identity, is precisely $[g]$. Therefore u_f is onto.

Finally, u_f is a homomorphism: let $g, h \in \pi_1(X, x)$. Then

$$\begin{aligned} u_f([g])u_f([h]) &= [\bar{f} * g * f][\bar{f} * h * f] \\ &= [\bar{f} * g * h * f] \\ &= u_f([g * h]) \end{aligned}$$

³Technically, $\bar{f} * g * f$ should have some indication of ordering of operations, but we dispense with these designations because we previously proved that $*$ is associative.

Therefore, u_f is a group isomorphism. \square

Keep in mind that our overall goal is to show that the fundamental group is a topological invariant. In general, a *topological invariant* is a “thing” that takes a value on topological spaces such that every homeomorphic space takes the same value. These invariants are used to distinguish topological spaces.

7.2. Induced Maps.

DEFINITION. Let $\phi : X \rightarrow Y$ be continuous, and $\phi(x_0) = y_0$. Define $\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ by $\phi_*([f]_X) = [\phi(f)]_Y$.

We say that ϕ_* is induced by ϕ .

SMALL FACT (about induced maps). Let $\phi : X \rightarrow Y$ and $\phi(x_0) = y_0$. Then ϕ_* is well defined.

PROOF. Let f and g be loops in X based at x_0 such that $f \sim g$. Then there exists $F : I \times I \rightarrow X$, a path homotopy from f to g .

Consider $\phi(F) : I \times I \rightarrow Y$. We see that ϕ is a composition of continuous functions, so is itself continuous.

Also, observe that

$$\begin{aligned}\phi(F(s, 0)) &= \phi \circ f(s) \\ \phi(F(s, 1)) &= \phi \circ g(s) \\ \phi(F(0, t)) &= \phi(x_0) = y_0 \\ \phi \circ F(1, t) &= \phi(x_0) = y_0\end{aligned}$$

Thus $\phi(F)$ is a homotopy between $\phi \circ g$ and $\phi \circ f$. So $\phi \circ g \sim \phi \circ f$ — and ϕ_* is well defined. \square

LEMMA. Let $\phi : X \rightarrow Y$ be continuous and $\phi(x_0) = y_0$. Then ϕ_* is a homomorphism.

PROOF. Let $[f], [g] \in \pi_1(X, x_0)$. We want to show that $\phi_*([f]_X [g]_X) = \phi_*([f]_X) \phi_*([g]_X)$.

Observe that

$$\begin{aligned}
 \phi_x([f]_X[g]_X) &= \phi_*([f * g]_X) = [\phi(f * g)]_Y \\
 \phi \circ (f * g)(x) &= \phi_* \begin{cases} f(2s), & s \in [0, 1/2] \\ g(2s - 1) & s \in [1/2, 1] \end{cases} \\
 &= \begin{cases} \phi \circ f(2s) & s \in [0, 1/2] \\ \phi \circ g(2s - 1) & s \in [1/2, 1] \end{cases} \\
 &= (\phi \circ f) * (\phi \circ g)(s).
 \end{aligned}$$

So $\phi \circ f * g = \phi \circ f * \phi \circ g$ and, in particular,

$$\begin{aligned}
 \phi(f * g)_Y &= [(\phi \circ f) * (\phi \circ g)]_Y \\
 &= [\phi \circ f]_Y [\phi \circ g]_Y \\
 &= \phi_*([f]_X) \phi_*([g]_X)
 \end{aligned}$$

This concludes the proof of the lemma. \square

THEOREM 27. *Let $\phi : X \rightarrow Y$ be a homeomorphism and $\phi(x_0) = y_0$. Then ϕ_* is an isomorphism.*

With the previous lemma we showed that ϕ_* is a homomorphism. It remains to show that ϕ_* is a bijection.

PROOF. 1-1: Let $[f]_X, [g]_X \in \pi_1(X, x_0)$ such that $\phi_*([f]_X) = \phi_*([g]_X)$. Then by definition of ϕ_* , $[\phi \circ f]_Y = [\phi \circ g]_Y$.

Observe that $\phi \circ f \sim_Y \phi \circ g$, so there exists a path homotopy F from $\phi \circ f$ to $\phi \circ g$. Also note that, as ϕ is a homeomorphism, $\phi^{-1} : Y \rightarrow X$ is continuous.

Hence $\phi^{-1} \circ F : I \times I \rightarrow X$ is a path homotopy from $\phi^{-1} \circ \phi \circ f$ to $\phi^{-1} \circ \phi \circ g$. Since ϕ is bijective, this is a path homotopy from f to g .

Onto: Recall that $\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$. Let $[f] \in \pi_1(Y, y_0)$. Then $[\phi^{-1}(f)]_X \in \pi_1(X, x_0)$, because ϕ^{-1} is continuous.

$$\phi_*([\phi^{-1}(f)]_X) = [\phi \circ \phi^{-1}(f)]_Y, \text{ and as } \phi \text{ is a bijection, this is } [f]_Y.$$

This illustrates that ϕ_* is surjective, and hence is an isomorphism between X and Y . \square

SMALL FACT (about induced homomorphisms). (1) If $\phi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ are continuous, then $(\psi \circ \phi)_* = \psi_* \circ \phi_*$

(2) If $i : X \rightarrow X$ is the identity, then i_* is the identity isomorphism

(3) Let $\phi : X \rightarrow Y$ be continuous and f a path in X from p to q . Then $\phi_* \circ u_f = u_{\phi(f)} \circ \phi_*$. (recall, $u_f : \pi_1(X, p) \rightarrow \pi_1(X, q)$ by $u_f([g]) = [\bar{f} * g * f]$)

- PROOF. (1) Note that $(\psi \circ \phi)_* : \pi_1(X, x_0) \rightarrow \pi_1(Z, (\psi \circ \phi)(x_0))$, a mapping from equivalence classes of loops in X based at x_0 to the equivalence classes of loops in Z based at $(\psi \circ \phi)(x_0)$. Let $[f]_X \in \pi_1(X, x_0)$. We then have that $(\psi \circ \phi)_*([f]_X) = [(\psi \circ \phi)(f)]_Z$. Similarly, we note that $(\psi_* \circ \phi_*)([f]_X) = \psi_*([\phi \circ f]_Y) = [\psi \circ \phi \circ f]_Z = [(\psi \circ \phi)(f)]_Z$.
- (2) Note that $i_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$. Let $[f]_X \in \pi_1(X, x_0)$. Then $i_*([f]_X) = [i(f)]_X = [f]_X$.
- (3) We want to show that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X, p) & \xrightarrow{u_f} & \pi_1(X, q) \\ \downarrow \phi_* & & \downarrow \phi_* \\ \pi_1(Y, \phi(p)) & \xrightarrow{u_{\phi(f)}} & \pi_1(Y, \phi(q)) \end{array}$$

Let $[g]_X \in \pi_1(X, p)$. Then note that

$$\begin{aligned} (\phi_* \circ u_f)([g]_X) &= \phi_*([\bar{f} * g * f]_X) \\ &= [\phi \circ (\bar{f} * g * f)]_Y \\ &= [(\phi \circ \bar{f}) * (\phi \circ g) * (\phi \circ f)]_Y \end{aligned}$$

It is easy to show that $\phi \circ \bar{f} = \overline{\phi \circ f}$. To see this, note that $(\phi \circ \bar{f})(s) = \phi \circ \bar{f}(s) = \phi(f(1-s)) = (\phi \circ f)(1-s) = \overline{\phi \circ f}(s)$. So continuing the above expression, we have that:

$$\begin{aligned} [(\phi \circ \bar{f}) * (\phi \circ g) * (\phi \circ f)]_Y &= [(\overline{\phi \circ f}) * (\phi \circ g) * (\phi \circ f)]_Y \\ &= u_{\phi \circ f}([\phi \circ g]_Y) \\ &= (u_{\phi(f)} \circ \phi_*)([g]_X) \end{aligned}$$

We then conclude that $\phi_* \circ u_f = u_{\phi(f)} \circ \phi_*$.

□

LEMMA. Suppose X is path-connected, and $x_0 \in X$. Then $\pi_1(X, x_0)$ is trivial if and only if $\forall p, q \in X$ and paths f, g in X from p to q , then $f \sim g$.

PROOF.

(\Rightarrow) Suppose that $\pi_1(X, x_0) = \langle [e_{x_0}] \rangle$. Let $p, q \in X$, and f, g paths in X from p to q . Then $f * \bar{g}$ is a loop based at p . So it is trivial to see that $\pi_1(X, p) \cong \pi_1(X, x_0)$ by an earlier theorem, from which we can see that $f * \bar{g} \sim e_p$. Using our multiplication and inverse lemmas for path multiplication, we conclude that $f \sim g$.

(\Leftarrow) Suppose that $\forall p, q \in X$ and paths f, g from p to q , we have that $f \sim g$. Let $p = q = x_0$, let $f = e_{x_0}$, and let g be a loop in X based at x_0 . Then $g \sim e_{x_0}$. Hence, $\pi_1(X, x_0) = \langle [e_{x_0}] \rangle$. □

7.3. A Technical Lemma. We'd like to try to prove that spaces that are homotopy equivalent have isomorphic fundamental groups. First, we'll need a technical lemma.

LEMMA (A Technical Theorem). *Let $\phi, \psi : X \rightarrow Y$ be continuous, and $\phi \simeq \psi$ by a homotopy F . Let $x_0 \in X$, and a path $f : I \rightarrow Y$ be given by $f(t) = F(x_0, t)$. Then $u_f \circ \phi_* = \psi_*$.*

PROOF. If $[g] \in \pi_1(X, x_0)$, then $u_f \circ \phi_*([g]_X) = u_f([\phi \circ g]_X) = [\bar{f} * (\phi \circ g) * f]_Y$. We want to show that this equals $\psi_*([g]_X)$. We'll prove this using the "fishing rod" f , which represents the image of the line $\{x_0\} \times I$. We "reel in" the fishing rod, and create the path homotopy between $\bar{f} * (\phi \circ g) * f$ and g .

More formally, we want to show that for all $[g] \in \pi_1(X, x_0)$, $u_f(\phi([g])) = \psi([g])$.

Let $[g] \in \pi_1(X, x_0)$. We want to show that $\bar{f} * \phi(g) * f \sim \psi(g)$. More specifically, we will do so by showing that $(\bar{f} * \phi(g)) * f \sim (e_{\psi(x_0)} * \psi(g)) * e_{\psi(x_0)}$.

Define $G : I \times I \rightarrow Y$ by

$$G(s, t) = \begin{cases} e_{\psi(x_0)} & s \in [0, \frac{t}{4}] \\ \bar{f}(4s - t) & s \in [\frac{t}{4}, \frac{1}{4}] \\ F(g(4s - 1), t) & s \in [\frac{1}{4}, \frac{1}{2}] \\ f(2s - 1 + t) & s \in [\frac{1}{2}, \frac{2-t}{2}] \\ e_{\psi(x_0)} & s \in [\frac{2-t}{2}, 1] \end{cases}$$

To prove that $(\bar{f} * \phi(g)) * f \sim (e_{\psi(x_0)} * \psi(g)) * e_{\psi(x_0)}$, we must show that G is path homotopy from $(\bar{f} * \phi(g)) * f$ to $(e_{\psi(x_0)} * \psi(g)) * e_{\psi(x_0)}$.

Continuous. G is defined differently over the five intervals. Over each interval, G is the composition of continuous functions and hence continuous. For G to be continuous everywhere, the value of G at the intersection of any two intervals must agree.

- $s = \frac{t}{4}$: $(\bar{f})(4\frac{t}{4} - t) = \bar{f}(0) = \psi(x_0) = e_{\psi(x_0)}$.
- $s = \frac{1}{4}$: $\bar{f}(4\frac{1}{4} - 1) = \bar{f}(1 - t) = f(t)$.
 $F(g(4\frac{1}{4} - 1), t) = F(g(0), t) = F(x_0, t) = f(t)$.
- $s = \frac{1}{2}$: $F(g(4\frac{1}{2} - 1), t) = F(g(1), t) = F(x_0, t) = f(t)$.
 $f(2\frac{1}{2} - 1 + t) = f(t)$.
- $s = \frac{2-t}{2}$: $f(2(\frac{2-t}{2}) - 1 + t) = f(1) = \psi(x_0) = e_{\psi(x_0)}$.

By the Pasting Lemma, the function G is continuous.

Homotopy. Consider $G(s, 0)$:

$$G(s, 0) = \begin{cases} e_{\psi(x_0)} & s \in [0, 0] \\ \bar{f}(4s) & s \in [0, \frac{1}{4}] \\ F(g(4s - 1), 0) & s \in [\frac{1}{4}, \frac{1}{2}] \\ f(2s - 1) & s \in [\frac{1}{2}, 1] \\ e_{\psi(x_0)} & s \in [1, 1] \end{cases}$$

Thus, $G(s, 0) = \bar{f} * \phi(g) * f$. Consider $G(s, 1)$:

$$G(s, 1) = \begin{cases} e_{\psi(x_0)} & s \in [0, \frac{1}{4}] \\ \bar{f}(4s - 1) & s \in [\frac{1}{4}, \frac{1}{4}] \\ F(g(4s - 1), 1) & s \in [\frac{1}{4}, \frac{1}{2}] \\ f(2s) & s \in [\frac{1}{2}, \frac{1}{2}] \\ e_{\psi(x_0)} & s \in [\frac{1}{2}, 1] \end{cases}$$

Thus, $G(s, 1) = (e_{\psi(x_0)} * \psi(g)) * e_{\psi(x_0)}$.

Path: $G(0, t) = \psi(x_0)$, and $G(1, t) = \psi(x_0)$.

Hence G is a path homotopy from $(\bar{f} * \phi(g)) * f$ to $(e_{\psi(x_0)} * \psi(g)) * e_{\psi(x_0)}$. Therefore, $u_f(\phi([g])) = \psi([g])$ for all $[g] \in \pi_1(X, x_0)$ and $u_f \circ \phi_* = \psi_*$. \square

COROLLARY. *Let $\phi : X \rightarrow Y$ and $\psi : Y \rightarrow X$ be continuous such that $\phi \circ \psi \simeq 1_Y$ and $\psi \circ \phi \simeq 1_X$. Let $\phi(x_0) = y_0$. Then $\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is an isomorphism.*

PROOF. We have already proven that ϕ_* is a homomorphism. For ϕ_* to be an isomorphism, it must also be a bijection.

Let F be a homotopy in Y from 1_Y to $\phi \circ \psi$. Let G be a homotopy in Y from 1_Y to $\phi \circ \psi$. Let G be a homotopy in X from 1_X to $\psi \circ \phi$. Let $f : I \rightarrow Y$ be $f(t) = F(y_0, t)$ and let $g : I \rightarrow X$ be $g(t) = G(x_0, t)$. f is a path in Y from y_0 to $\phi(\psi(y_0))$, and g is a path in X from x_0 to $\psi(\phi(x_0))$.

By the technical theorem proven above, $u_f \circ 1_{Y_*} = (\phi \circ \psi)_*$ and $u_g \circ 1_{X_*} = (\psi \circ \phi)_*$. Therefore, $u_f \circ 1_{Y_*} = \psi_* \circ \phi_*$, implying that $u_f = \phi_* \circ \psi_*$. Similarly, $u_g = \psi_* \circ \phi_*$. We have already established that u_f is an isomorphism and therefore onto — ϕ_* must be onto as well. Because u_g is 1-1 and $u_g = \psi_* \circ \phi_*$, ϕ_* is 1-1. ϕ_* is a bijection and hence an isomorphism. \square

8. Covering Maps

Our ultimate goal is to find a space with an interesting fundamental group.

DEFINITION. Let X and \tilde{X} be topological spaces, and $p : \tilde{X} \rightarrow X$ be a continuous surjection. An open set $U \subseteq X$ is said to be **evenly covered** by p if $p^{-1}(U)$ is the disjoint union of open sets V_α , $\alpha \in A$ for some index set A , such that for all $\alpha \in A$, $p|V_\alpha : V_\alpha \rightarrow U$ is a homeomorphism.

In this case, we say that each V_α is a **sheet** covering U .

EXAMPLE. Let $X = D^2$ with the usual topology, and $\tilde{X} = D^2 \times \mathbb{N}$ with the usual product topology. Define $p : \tilde{X} \rightarrow X$ by $p(x, n) = x$. We can take U to be any open set in X ;

$$p^{-1}(U) = \bigcup_{i=1}^{\infty} p^{-1}U \cap V_i, \text{ where } V_i = D^2 \times i.$$

(Draw a picture here!)



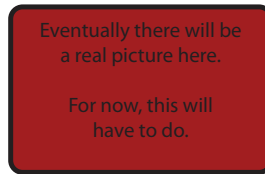
EXAMPLE (a non-example). Let $\tilde{X} = S^1$, and let $X = S^1 \vee S^1$. That is, X is two copies of S^1 , which agree at a point.

Let \sim be the equivalence relation on S^1 given by $x \sim y$ if and only if $x, y \in \{x_1, x_2\}$ or $x = y$. Let p be the quotient map from \tilde{X} to X , corresponding to this relation, and let U be an open set in X containing x_1, x_2 , as shown. Is U evenly covered?

The answer is no. To see why, note that $p^{-1}(U) = V_1 \cup V_2$, where V_1, V_2 are disjoint open sets in \tilde{X} . But $p|V_1 : V_1 \rightarrow U$ is not a homeomorphism because it is not onto.

DEFINITION. Let $p : \tilde{X} \rightarrow X$ be a continuous surjection. Suppose for all $x \in X$, there exists an evenly covered open set U containing x . We say p is a **covering map**, \tilde{X} is the **covering space** and X is the **base space**.

EXAMPLE (another non-example). Let $\tilde{X} = \mathbb{R}^2$, $X = \mathbb{R}$, and $P : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $p(x, y) = x$. Then for each open $U \subseteq \mathbb{R}$, $p^{-1}(U) = \bigcup_{\alpha \in A} V_\alpha$. (We can think of $p^{-1}(U)$ as a horizontal stack of uncountably many copies of U). For each $\alpha \in A$, $p|V_\alpha : V_\alpha \rightarrow U$ is a homeomorphism. However, each V_{α} is not open in \tilde{X} .



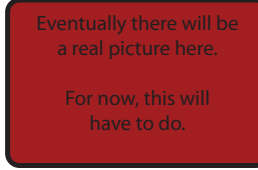
LEMMA (Important Lemma on Covering Maps). *Let $p : \tilde{X} \rightarrow X$ be a covering map. Let $x \in X$. Then the subspace topology on $p^{-1}(\{x\})$ is the discrete topology.*

PROOF. Let $y \in p^{-1}(\{x\})$. We want to show that $\{y\}$ is open in $p^{-1}(\{x\})$. Since p is a covering map, there exists an evenly covered open set U containing x . Then $p^{-1}(U) = \cup_{\alpha \in A} V_\alpha$, where the V_α are pairwise-disjoint open sets such that $p|V_\alpha : V_\alpha \rightarrow U$ is a homeomorphism for every $\alpha \in A$. Hence, there exists $\alpha_0 \in A$ such that $y \in V_{\alpha_0}$. Now, V_{α_0} is open in \tilde{X} . Note that $y \in V_{\alpha_0} \cap p^{-1}(\{x\})$. Let $y' \in V_{\alpha_0} \cap p^{-1}(\{x\})$ be given. We know that $p|V_{\alpha_0}$ is a homeomorphism, so it is injective. Since $p(y') = x$, and $p(y) = x$, we see that $y = y'$. Thus, $\{y\} = V_{\alpha_0} \cap p^{-1}(\{x\})$, both of which are open in $p^{-1}(\{x\})$. So $\{y\}$ is open in $p^{-1}(\{x\})$ with the subspace topology. This completes the proof. \square

The take-home message is that in a covering space, points in the pre-image of a single point are “spread out.”

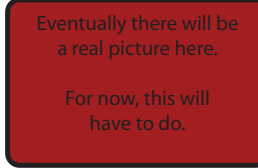
EXAMPLE (Important). Let $p : \mathbb{R} \rightarrow S^1$ be defined by $p(x) = (\cos(2\pi x), \sin(2\pi x))$. (The “slinky” space.)

(Draw a picture, and you will win fortune and fame!)



For each $s \in S^1$, an “open interval” around s is evenly covered, and so \mathbb{R} is a covering space.

EXAMPLE. $\tilde{X} = S^1$, $X = S^1$ by $p : \tilde{X} \rightarrow X$ is $p((\cos(2\pi x), \sin(\pi x))) = (\cos(4\pi x), \sin(4\pi x))$.



Not all quotient maps are covering maps. But we will prove that all covering maps are quotient maps. (Remember our important example?)

THEOREM 28. Let $p : \tilde{X} \rightarrow X$ be a covering map. Then

- (1) p is an open map
- (2) X is a quotient space, and p is a quotient map.

PROOF. Let U be open in \tilde{X} . We want to show that $p(U)$ is open. Let $x \in p(U)$. Then there exists an evenly covered open set V containing x . Now, there exists $y \in U$ such that $p(y) = x$. Then $p^{-1}(V) = \cup_{\alpha \in A} V_\alpha$ such that the V_α are disjoint open sets and $p|V_\alpha$ is a

homeomorphism for all $\alpha \in A$. Let $\alpha_0 \in A$ such that $y \in V_{\alpha_0}$. There exists such an α_0 because $x \in V$.

Because U and V_{α_0} are both open, $U \cap V_{\alpha_0}$ is open in \tilde{X} . Recall $p|_{V_{\alpha_0}} : V_{\alpha_0} \rightarrow V$ is a homeomorphism, so

$$\begin{aligned} p(U \cap V_{\alpha_0}) &= p(V_{\alpha_0} \cap p(U)) \\ &= V \cap p(U) \end{aligned}$$

where $V \cap p(U)$ is open in V and U is open in X . Therefore, $V \cap p(U)$ is open in X . Because our x is in both V and $p(U)$, $x \in V \cap p(U) \subseteq p(U)$. That is, x is an element of an open set contained in $p(U)$. Hence, $p(U)$ is open and p is an open map.

To show X has the quotient topology with regard to p , we want to show that $F_X = \{U \subseteq X \mid p^{-1}(U) \in F_{\tilde{X}}\}$.

(\subseteq) Let $V \in F_X$. Because p is continuous, $p^{-1}(V) \in F_{\tilde{X}}$. Thus, $V \in \{U \subseteq X \mid p^{-1}(U) \in F_{\tilde{X}}\}$.

(\supseteq) Let $U \subseteq X$ such that $p^{-1}(U) \in F_{\tilde{X}}$. Because p is open, $p(p^{-1}(U))$ is open in X . Because p is onto $p(p^{-1}(U)) = U$. Thus, $U \in F_X$.

$F_X = \{U \subseteq X \mid p^{-1}(U) \in F_{\tilde{X}}\}$, so p is a quotient map and X is a quotient space.

□

8.1. Lifts.

DEFINITION. Let $p : \tilde{X} \rightarrow X$ be a covering map and $f : Y \rightarrow X$ be continuous. We define a *lift* of f to be any continuous function $\tilde{f} : Y \rightarrow \tilde{X}$ such that $p \circ \tilde{f} = f$.

EXAMPLE. Let $\tilde{X} = \mathbb{R}$, $X = S^1$ and $p(x) = (\cos(2\pi x), \sin(2\pi x))$. Let $f : I \rightarrow S^1$ by $f(x) = (\cos(\pi x), \sin(\pi x))$. Define $\tilde{f} : I \rightarrow \mathbb{R}$ by $\tilde{f}(x) = \frac{x}{2}$.

Eventually there will be
a real picture here.

For now, this will
have to do.

LEMMA. **Uniqueness of Lifts.** Let $p : \tilde{X} \rightarrow X$ be a covering map and $f : Y \rightarrow X$ be a covering and $f : Y \rightarrow X$ be continuous and Y be connected. Let \tilde{f}_0 and \tilde{f}_1 be lifts of f . Suppose there exists $y_0 \in Y$ such that $\tilde{f}_0(y_0) = \tilde{f}_1(y_0)$ then $\tilde{f}_0 = \tilde{f}_1$.

PROOF. Let $Y' = \{y \in Y \mid \tilde{f}_0(y) = \tilde{f}_1(y)\}$. $Y \neq \emptyset$. We want to show that $Y' = Y$, we will accomplish by showing Y' is clopen in Y .

Open: Let $y \in Y'$. There exists an evenly covered open set V containing $f(y)$. $p^{-1}(V) = \bigcup_{\alpha \in A} V_\alpha$ such that V_α 's are disjoint and open; also, $p|_{V_\alpha} : V_\alpha \rightarrow V$ is a homeomorphism.

Let $q = \tilde{f}_0(y) = \tilde{f}_1(y)$. There exists $\alpha_0 \in A$ such that $q \in V_{\alpha_0}$. $\tilde{f}_0^{-1}(V_{\alpha_0})$ and $\tilde{f}_1^{-1}(V_{\alpha_0})$ are open in Y , and $y \in \tilde{f}_0^{-1}(V_{\alpha_0}) \cap \tilde{f}_1^{-1}(V_{\alpha_0})$. We claim that $\tilde{f}_0^{-1}(V_{\alpha_0}) \cap \tilde{f}_1^{-1}(V_{\alpha_0}) \subseteq Y'$.

Let $z \in \tilde{f}_0^{-1}(V_{\alpha_0}) \cap \tilde{f}_1^{-1}(V_{\alpha_0})$, implying $\tilde{f}_0(z) \in V_{\alpha_0}$ and $\tilde{f}_1(z) \in V_{\alpha_0}$. Because $\tilde{f}_0(z)$ and $\tilde{f}_1(z)$ are lifts of f , $p \circ \tilde{f}_0(z) = f(z)$ and $p \circ \tilde{f}_1(z) = f(z)$. Since p is 1-1 (because p is a homeomorphism) and $\tilde{f}_0(z) \in V_{\alpha_0}$ and $\tilde{f}_1(z) \in V_{\alpha_0}$, $\tilde{f}_0(z) = \tilde{f}_1(z)$. Thus, $z \in Y'$.

Hence, $\tilde{f}_0^{-1}(V_{\alpha_0}) \cap \tilde{f}_1^{-1}(V_{\alpha_0})$ is an open subset of Y' containing y , so Y' is open.

Closed: To show Y' is closed, we will show $Y - Y'$ is open. Let $y \in Y - Y'$. There exists an evenly covered open set V containing $f(y)$. $p^{-1}(V) = \bigcup_{\alpha \in A} V_\alpha$ such that V_α 's are disjoint and open; also, $p|_{V_\alpha} : V_\alpha \rightarrow V$ is a homeomorphism.

Note that $\tilde{f}_0(y) \neq \tilde{f}_1(y)$. There exists $\alpha_1, \alpha_2 \in A$ such that $\tilde{f}_0(y) \in V_{\alpha_1}$ and $\tilde{f}_1(y) \in V_{\alpha_2}$. The set $\tilde{f}_0^{-1}(V_{\alpha_1}) \cap \tilde{f}_1^{-1}(V_{\alpha_2})$ is open and contains y . We claim that $\tilde{f}_0^{-1}(V_{\alpha_1}) \cap \tilde{f}_1^{-1}(V_{\alpha_2}) \subseteq Y - Y'$.

Let $z \in \tilde{f}_0^{-1}(V_{\alpha_1}) \cap \tilde{f}_1^{-1}(V_{\alpha_2})$, implying $\tilde{f}_0(z) \in V_{\alpha_1}$ and $\tilde{f}_1(z) \in V_{\alpha_2}$. We now want to show that $\tilde{f}_0(z) \neq \tilde{f}_1(z)$. Recall that $p|_{V_{\alpha_1}}$ is 1-1 and $p \circ \tilde{f}_0(y) = f(y) = p \circ \tilde{f}_1(y)$: because $\tilde{f}_0(y) \neq \tilde{f}_1(y)$ where $\tilde{f}_0(y) \in V_{\alpha_1}$ and $\tilde{f}_1(y) \in V_{\alpha_2}$, implying that $\alpha_1 \neq \alpha_2$. Otherwise, $\tilde{f}_0(y)$ and $\tilde{f}_1(y)$ would be two points in V_{α_1} that both map to $f(y)$. Therefore, $V_{\alpha_1} \cap V_{\alpha_2} = \emptyset$.

Therefore $\tilde{f}_0(z) \neq \tilde{f}_1(z)$, implying that $z \in Y - Y'$. This implies y is contained in the open set $\tilde{f}_0^{-1}(V_{\alpha_1}) \cap \tilde{f}_1^{-1}(V_{\alpha_2}) \subseteq Y - Y'$, making $Y - Y'$ open.

Therefore Y' is clopen in Y . Because Y' is non-empty and Y is connected, Y' must be all of Y . By the definition of Y' , $\tilde{f}_0 = \tilde{f}_1$. \square

LEMMA (Lebesgue Number Lemma). *Let X be a compact metric space and let Ω be an open cover of X . Then $\exists r > 0$ such that $\forall A \subseteq X$ with $\text{lub}\{d(p, q) | p, q \in A\} < r$, A is contained in a single element of Ω .*

(r is said to be a Lebesgue Number for Ω)

We will now use this to prove the existence of lifts.

THEOREM 29 (Very Important Homotopy Path Lifting Theorem). *Let $p : \tilde{X} \rightarrow X$ be a covering map. Then,*

- (1) Given a path f in X and $a \in \tilde{X}$ such that $p(a) = f(0)$, then $\exists!$ (exists unique) path \tilde{f} in \tilde{X} such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = a$.
- (2) Given a continuous map $F : I \times I \rightarrow X$ and $a \in \tilde{X}$ with $p(a) = f(0,0)$, $\exists!$ continuous map $\tilde{F} : I \times I \rightarrow \tilde{X}$ such that $p \circ \tilde{F} = F$ and $\tilde{F}(0,0) = a$.

PROOF. (1) $\forall x \in f(I)$, $\exists V_x$ an evenly covered open set containing x . $\forall x \in f(I)$, $f^{-1}(V_x)$ is open in I , so $\{f^{-1}(V_x) | x \in f(I)\}$ is an open cover of I . So, \exists Lebesgue number r for this cover. $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < r$. $\forall k \leq n$, $[\frac{k-1}{n}, \frac{k}{n}]$ is contained entirely in some $f^{-1}(V_x)$. So, $\exists \{V_1, V_2, \dots, V_n\} \subseteq \{V_x\}$ such that $\forall k \leq n$, $f([\frac{k-1}{n}, \frac{k}{n}]) \subseteq V_k$. First, V_1 is evenly covered and $f(0) \in V_1$, so $a \in p^{-1}(V_1) = \bigcup_{\alpha \in A_1} V_\alpha$. So, $\exists \alpha_1 \in A_1$ such that $a \in V_{\alpha_1}$. $p|_{V_{\alpha_1}} : V_{\alpha_1} \rightarrow V_1$ is a homeomorphism. So $\forall s \in [0, \frac{1}{n}]$, define $\tilde{f}(s) = (p|_{V_{\alpha_1}})^{-1}f(s)$. Note that $\tilde{f} : [0, \frac{1}{n}] \rightarrow \tilde{X}$ continuous because $(p|_{V_{\alpha_1}})^{-1}$ is a homeomorphism. Now note that, as above, $p^{-1}(V_2) = \bigcup_{\alpha \in A_2} V_\alpha$. $f(\frac{1}{n}) \in V_2$ by definition. $\exists \alpha_2 \in A_2$ such that $\tilde{f}(\frac{1}{n}) \in V_{\alpha_2}$. So, as above, define $\tilde{f} : [\frac{1}{n}, \frac{2}{n}] \rightarrow \tilde{X}$ by $\tilde{f}(s) = (p|_{V_{\alpha_2}})^{-1}f(s)$. $\tilde{f} : [0, \frac{2}{n}] \rightarrow \tilde{X}$ is therefore continuous by Pasting Lemma.

- (2) $\forall x \in F(I \times I)$ \exists evenly covered open set V_x . $\{F^{-1}(V_x)\}$ is an open cover of $I \times I$, so it has a Lebesgue number r . $\exists n > \frac{\sqrt{2}}{r}$. $\forall i \leq n$, let $A_i = [\frac{i-1}{n}, \frac{i}{n}]$, $B_i = [\frac{i-1}{n}, \frac{i}{n}]$. $\forall i, j$, $F(A_i \times B_j) \subseteq V_{ij}$ for some $V_{ij} \in \{V_x\}$. By Part 1, we can lift $F|(I \times \{0\} \cup \{0\} \times I)$ to $\tilde{F} : (I \times \{0\} \cup \{0\} \times I) \rightarrow \tilde{X}$ such that $\tilde{F}(0,0) = a \in \tilde{X}$.

Begin by observing that V_{11} is evenly covered by hypothesis, and $F(I_1 \times J_1) \subseteq V_{11}$. This means that $p^{-1}(V_{11}) = \bigcup_{\alpha \in A_{11}} V_\alpha$ where the V_α are disjoint open sets and A_{11} is some index set. Since $F(0,0) \in V_{11}$, and since $\tilde{F}(0,0) = a$, it follows that there exists some $\alpha_{11} \in A_{11}$ such that $a \in V_{\alpha_{11}}$.

Now we worry that this choice of $V_{\alpha_{11}}$ will agree with how we defined \tilde{F} on the set L . Worry not! For L is connected, and $L \cap (I_1 \times J_1)$ is connected, and since \tilde{F} is continuous, $\tilde{F}(L \cap (I_1 \times J_1))$ is connected. Since the V_α are open and disjoint, we may therefore conclude that $\tilde{F}(L \cap (I_1 \times J_1)) \subseteq V_{\alpha_{11}}$ (otherwise it would be disconnected).

Since V_{11} was evenly covered, we know that $p|_{V_{\alpha_{11}}}$ is a homeomorphism, so we may define $\tilde{F} : I_1 \times J_1 \rightarrow \tilde{X}$ by:

$$\tilde{F}(s, t) = (p|_{V_{\alpha_{11}}})^{-1} \circ F(s, t)$$

This is a composition of continuous functions, so is continuous. Furthermore, $\tilde{F} : L \cup (I_1 \times J_1) \rightarrow \tilde{X}$ is continuous since we showed that $\tilde{F}(L \cap (I_1 \times J_1)) \subseteq V_{\alpha_{11}}$, so we apply the Pasting Lemma.

Now we want to extend \tilde{F} to the rest of $I \times I$, and so we move to $I_2 \times J_1$. Here we have an analogous situation as before: We want to choose the appropriate V_α associated with V_{21} so that our extension of \tilde{F} agrees with what we had previously. But again, $\tilde{F}((I_2 \times J_1) \cap (L \cup (I_1 \times J_1)))$ is connected, so following the argument from above there will be an appropriate choice of V_α to make it “work”. So we inductively define $\tilde{F}: I \times I \rightarrow \tilde{X}$ such that it is continuous as before, and $p \circ \tilde{F} = F$ and $\tilde{F}(0, 0) = a$. That \tilde{F} is unique follows from our Uniqueness of Lifts Lemma above.

We iterate this argument finitely many times for each tile $I_i \times J_j$, and so inductively define a unique lift of F : $\tilde{F}: I \times I \rightarrow \tilde{X}$ such that $\tilde{F}(0, 0) = a$. \square

The natural intuition is that our new function \tilde{F} is a path homotopy when F is a path homotopy. This intuition provides a delightful segue to the next theorem:

THEOREM 30 (Monodromy Theorem). *Let $p: \tilde{X} \rightarrow X$ be a covering map, and let $a \in \tilde{X}$. Let $x_1, x_2 \in X$. Suppose that $p(a) = x_1$, and that f, g are paths in X from x_1 to x_2 . Let \tilde{f}, \tilde{g} be the unique lifts of f, g beginning at a . Then if $f \sim g$, $\tilde{f}(1) = \tilde{g}(1)$ and $\tilde{f} \sim \tilde{g}$.*

Before beginning the proof, we observe with relish the etymology of monodromy. Mono being the prefix for one, and dromy being some sort of Greek for a race track. E.g. hippodrome, airdrome, palindrome (examples courtesy of dictionary.com). Let us race towards the proof!

PROOF. $f \sim g$ means that there exists a path homotopy $F: I \times I \rightarrow X$, and so by the previous theorem there exists a unique lifting of F , whose name is $\tilde{F}: I \times I \rightarrow \tilde{X}$, and \tilde{F} has the property that $\tilde{F}(0, 0) = a$ and $p \circ \tilde{F} = F$. Now \tilde{f}, \tilde{g} are lifts of f, g respectively. Consider $\tilde{F} | (I \times \{0\})$. This is a path in \tilde{X} from a to $\tilde{F}(1, 0)$. Observe that:

$$p \circ \tilde{F} | (I \times \{0\}) = F | (I \times \{0\}) = f$$

since F was a path homotopy, and on the other hand:

$$p \circ \tilde{F} | (I \times \{1\}) = F | (I \times \{1\}) = g$$

The first observation allows us to conclude that $\tilde{F} | (I \times \{0\})$ is a lift of f beginning at a . By the uniqueness of lifts, we conclude that $\tilde{F} | (I \times \{0\}) = \tilde{f}$. We want to say the same for $\tilde{F} | (I \times \{1\})$, but we do not know that $\tilde{F}(0, 1) = a$, so we cannot immediately conclude that this is equal to \tilde{g} since it could possibly be a lift of g originating at some other point.

We claim: $\tilde{F} | (\{0\} \times I) = a$. To see that this is the case, we know:

$$p \circ \tilde{F} | (\{0\} \times I) = F | (\{0\} \times I) = x_1.$$

which implies that

$$\tilde{F} | (\{0\} \times I) \subseteq p^{-1}(F | (\{0\} \times I) = p^{-1}(x_1))$$

Now we know that since p is a covering map, $p^{-1}(x_1)$ has the discrete topology. Also, $\tilde{F}(\{0\} \times I)$ is connected, so must contain only a single point of $p^{-1}(x_1)$. Certainly $a \in \tilde{F}(\{0\} \times I)$, so we may say that $a = \tilde{F}(\{0\} \times I)$ as desired.

The previous consideration tells us that $\tilde{F} \mid (I \times \{1\}) = \tilde{g}$, since the left hand side is a lift of g originating at a , and by the uniqueness of lifts this must be \tilde{g} . To finish off proving that \tilde{F} is a path homotopy, we need to show that the endpoints are constant as well. That is, we want to show that $\tilde{F}(\{1\} \times I) = a'$ for some $a' \in \tilde{X}$. But for this, the same argument as above applies, replacing every instance of x_1 with x_2 . So we conclude that:

$$\tilde{F}(1, 0) = \tilde{f}(1) = \tilde{g}(1) = \tilde{F}(1, 1)$$

which was part of what we were trying to prove. All these considerations together tell us that \tilde{F} is a path homotopy between \tilde{f} and \tilde{g} , so $\tilde{f} \sim \tilde{g}$ and we are done. \square

8.2. S^1 and \mathbb{Z} . Apparently our ultimate goal is to prove that the fundamental group of S^1 is isomorphic to \mathbb{Z} (with addition). But we need just a teensy bit more machinery, and introduce a new function.

DEFINITION. Let $p: \mathbb{R} \rightarrow S^1$ be the covering map $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Let $x_0 = (1, 0)$, let f be a loop in S^1 with base point x_0 . Define the **degree** of f , denoted $\deg(f)$, as $\tilde{f}(1)$ where \tilde{f} is the unique lift of f starting at 0.

First, observe that $\deg(f)$ is well defined. That is, it doesn't matter which lift we select, because there is only one lift! Also, convince yourself that $p^{-1}(\{x_0\}) = \mathbb{Z}$.

This leads us to the theorem we have been clamoring for:

THEOREM 31. *Let $x_0 \in S^1$. Then $\pi_1(S^1, x_0) \cong (\mathbb{Z}, +)$.*

PROOF. We assume WLOG that $x_0 = (1, 0)$ since S^1 is simply connected. Let $\phi: \pi_1(S^1, s_0) \rightarrow \mathbb{Z}$ by $\phi([f]) = \deg(f)$. Recall from last lecture that we defined $\deg(f) = \tilde{f}(1)$ where \tilde{f} was the lift of f based at 0 with respect to the covering map $p: \mathbb{R} \rightarrow S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Our aim is to show that ϕ is an isomorphism.

Well-Defined: Suppose $[f] = [g]$. Then $f \sim g$ are homotopic loops in S^1 based at x_0 . By the Monodromy theorem, we may say that $\tilde{f}(1) = \tilde{g}(1)$ (from here on out we assume that \tilde{f}, \tilde{g} are the lifts of f, g respectively based at 0). This implies that $\deg(f) = \deg(g)$, which means that $\phi([f]) = \phi([g])$ and so ϕ is well-defined.

1-1: Let $[f], [g] \in \pi_1(S^1, x_0)$ be such that $\phi([f]) = \phi([g])$. This means that $\deg(f) = \deg(g)$, and this means that $\tilde{f}(1) = \tilde{g}(1)$. But \mathbb{R} is simply connected, so since \tilde{f}, \tilde{g} share an endpoint we may say that $\tilde{f} \sim \tilde{g}$, and there exists a path homotopy $\tilde{F}: I \times I \rightarrow \mathbb{R}$ such that $\tilde{F}(0, t) = \tilde{f}$ and $\tilde{F}(1, t) = \tilde{g}$. Consider $p \circ \tilde{F}$. This is

continuous because it is a composition of continuous functions, and $p \circ \tilde{F}(0, t) = p \circ \tilde{f} = \bar{f}$; $p \circ \tilde{F}(1, t) = p \circ \tilde{g} = \bar{g}$. Finally, we know that for all $s \in I$ we have $\tilde{F}(s, 0) = \tilde{f}(1)$, so $p \circ \tilde{F}(s, 0) = p \circ \tilde{f}(1) = f(1) = x_0$, and the same goes for $t = 1$, so $p \circ \tilde{F}$ goes to a loop homotopy F which takes \bar{f} to \bar{g} , but since these are loops we may deduce that $f \sim g$. This means that $[f] = [g]$ and we are done with proving one to one.

Onto: Let $n \in \mathbb{Z}$. Since \mathbb{R} is path connected there exists a path \tilde{f} from 0 to n . Then $p \circ \tilde{f}$ is a loop in S^1 based at x_0 (since $p(\cos 2\pi n, \sin 2\pi n) = (1, 0) = x_0$), and $\deg(p \circ \tilde{f}) = n$. Therefore $\phi([p \circ \tilde{f}]) = n$. This proves that ϕ is onto.

Homo: Let $[f], [g] \in \pi_1(S^1, x_0)$. We want to show that $\phi([f][g]) = \phi([f]) + \phi([g])$, and the right hand side is equal to $\deg(f) + \deg(g)$, while the left hand side is equal to $\deg(f * g)$. So we want to show that $\deg(f * g) = \deg(\tilde{f}) + \deg(\tilde{g})$. Let $\tilde{f * g}$ be the lift of $f * g$ beginning at 0. We want to show that $\tilde{f * g}(1) = \tilde{f}(1) + \tilde{g}(1)$. Let $m = \tilde{f}(1), n = \tilde{g}(1)$. Let us define a function $h: I \rightarrow \mathbb{R}$ by:

$$h(s) = \begin{cases} \tilde{f}(2s) & s \in [0, \frac{1}{2}] \\ \tilde{g}(2s - 1) + m & s \in [\frac{1}{2}, 1] \end{cases}$$

Now h is a path in \mathbb{R} from 0 to $m + n$. This is because \tilde{f}, \tilde{g} are continuous, and when $s = \frac{1}{2}$ we have $\tilde{f}(1) = m$, and $\tilde{g}(0) + m = 0 + m = m$, so this is continuous by the pasting lemma. Also, $h(0) = 0$ and $h(1) = \tilde{g}(1) + m = n + m$. Also we know:

$$p \circ h(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ p(\tilde{g}(2s - 1) + m) & s \in [\frac{1}{2}, 1] \end{cases}$$

Now observe that the second half of this is:

$$\begin{aligned} p(\tilde{g}(2s - 1) + m) &= (\cos 2\pi(\tilde{g}(2s - 1) + m), \sin 2\pi(\tilde{g}(2s - 1) + m)) \\ &= (\cos 2\pi\tilde{g}(2s - 1), \sin 2\pi\tilde{g}(2s - 1)) \\ &= p(\tilde{g}(2s - 1)) \\ &= g(2s - 1) \end{aligned}$$

Thus we see that $p \circ h = f * g$, which implies that $h = \tilde{f * g}$ since h begins at 0 and we know that lifts are unique w/r/t their starting points. Now $h(1) = m + n = \tilde{f}(1) + \tilde{g}(1)$, so putting it together:

$$\deg(f * g) = \deg(f) + \deg(g) \Rightarrow \phi([f][g]) = \phi([f]) + \phi([g])$$

So we conclude that ϕ is a homomorphism. This plus one to one and onto means that ϕ is in fact an isomorphism, so we conclude that $\mathbb{Z} \cong \pi_1(S^1, x_0)$. \square

We rejoice at the fact that we have now seen a non trivial fundamental group.

8.3. More *Fundamental Groups*. Check out this theorem:

THEOREM 32. *Let $x_0 \in X$ and $p: \tilde{X} \rightarrow X$ be a covering map. If \tilde{X} is simply connected, then there exists a bijection from $\pi_1(X, x_0)$ to $p^{-1}(\{x_0\})$.*

PROOF. We only needed simple connectedness in the proof that ϕ was 1-1 and onto, so we use an analogous proof to show that $\phi: \pi_1(X, x_0) \rightarrow p^{-1}(\{x_0\})$ by $\phi([f]) = \tilde{f}(1)$, where \tilde{f} is the unique lift originating at some $y_0 \in p^{-1}(\{x_0\})$. \square

THEOREM 33. *Let $p: S^2 \rightarrow \mathbb{R}P^2$ be the quotient map. Then p is a covering map.*

PROOF. Recall that the equivalence relation for this quotient map was defined as $x \sim y$ iff $x = \pm y$. We use one of the most powerful proof techniques known to mathematics here: proof by picture. Consider any open ball in $\mathbb{R}P^2$ containing some point. Then its pre image will be a pair of disjoint balls such that p restricted to the balls will be a homeomorphism. Just think about it... or look at the pictures below.



\square

$\mathbb{R}P^2$ provides us with another example of a space with a nontrivial fundamental group. From one of our theorems, we know that given $x_0 \in \mathbb{R}P^2$ there is a bijection from $\pi_1(\mathbb{R}P^2, x_0)$ to $p^{-1}(\{x_0\})$ (since $\mathbb{R}P^2$ is simply connected), and we know that $p^{-1}(\{x_0\})$ has precisely two elements, so $\pi_1(\mathbb{R}P^2, x_0) \cong \mathbb{Z}_2$ (alternatively written by algebraists as $\mathbb{Z}/2\mathbb{Z}$).

9. $\mathbb{R}^2 \not\cong \mathbb{R}^3$

THEOREM 34. $\mathbb{R}^2 \not\cong \mathbb{R}^3$

PROOF. First, recall some previous results:

- $\mathbb{R}^{n+1} \setminus \{p\}$ is homotopy equivalent to S^n
- $\pi_1(S^2, x_0)$ is trivial
- $\pi_1(S^1, x_0) \cong \mathbb{Z}$

Now suppose that $\mathbb{R}^2 \cong \mathbb{R}^3$. Then there exists some homeomorphism $h: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

Let $p \in \mathbb{R}^2$ and $h(p) = q \in \mathbb{R}^3$. Therefore $f = h|_{\mathbb{R}^2 \setminus \{p\}}: \mathbb{R}^2 \setminus \{p\} \rightarrow \mathbb{R}^3 \setminus \{q\}$ is a homeomorphism. Also, from the previous results:

- $\pi_1(\mathbb{R}^2 \setminus \{p\}, x_0) \cong \pi_1(S^1, y_0) \cong \mathbb{Z}$
- $\pi_1(\mathbb{R}^3 \setminus \{q\}, z_0) \cong \pi_1(S^2, w_0) \cong \{1\}$

But f is a homeomorphism, so $f_* : \pi_1(\mathbb{R}^2 \setminus \{p\}, x_0) \rightarrow \pi_1(\mathbb{R}^3 \setminus \{q\}, z_0)$ is an isomorphism. However, $\mathbb{Z} \not\cong \{1\}$, so this is a contradiction.

Therefore $\mathbb{R}^2 \not\cong \mathbb{R}^3$, as desired. \square

10. One Last Thought

EXAMPLE (Not all Fundamental Groups are Abelian). Let $X = S^1 \vee S^1$ with wedge point x_0 . Then $\pi_1(X, x_0)$ is not abelian.

PROOF. Define $Y \subseteq \mathbb{R}^2$ as $Y = \{(x, y) \mid x = 0 \text{ and/or } y = 0\}$.

Define \tilde{X} as Y with a copy of S^1 wedged at every $(z, 0)$ and $(0, z)$ with $z \in \mathbb{Z} \setminus \{0\}$

Define $p : \tilde{X} \rightarrow X$ such that $(x, 0)$ goes to the first copy of S^1 for all $x \in \mathbb{R}$, $(0, x)$ goes to the second, and $(z, 0)$, $(0, z)$ goes to x_0 for all $z \in \mathbb{Z}$. Copies of S^1 wedged at $(z, 0)$ go with the usual projection to the second copy of S^1 in X , and the copies wedged at $(0, z)$ go to the first copy in X , as show in the image.

Eventually there will be
a real picture here.

For now, this will
have to do.

Now define $\tilde{f}(t) = (t, 0)$, $\tilde{g}(t) = (0, t)$, and $f = p \circ \tilde{f}$, $g = p \circ \tilde{g}$. So f is a single loop on the first circle and g is a single loop on the second circle.

Next, lift $f * g$ and $g * f$ at the origin. The construction of \tilde{X} gives that $\widetilde{f * g}(1) = (1, 0)$ and $\widetilde{g * f}(1) = (0, 1)$. So, by the Monodromy theorem, as this are lifts with the same fixed point, $f * g \not\sim g * f$. Therefore $[f]$ and $[g]$ satisfy $[f], [g] \in \pi_1(X, x_0)$ and $[f][g] \neq [g][f]$, so $\pi_1(X, x_0)$ is not abelian, as desired. \square

Well, that's everything there is to know about topology.