"What is a great man does, is follows by others; people go by the example he sets"

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Definition and some examples, relative topology, weak topology, open basis, open sub basis, continuity and homeomorphism, compact spaces, product spaces, compactness in a metric space.

Separation axioms (T_1, T_2, T_3, Housdorff spaces), connected spaces, components, totally disconnected space, locally connected space.

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Remark: $\bigcap\limits_{\lambda\in\phi} A_\lambda = X$, $\bigcup\limits_{\lambda\in\phi} A_\lambda = \phi$.

Topology: Let X be a nonempty set. A collection T of subsets of X is called a topology on X if

(1) $X \in T$,
(2) $\phi \in T$,
(3) $A \in T$, $B \in T \Rightarrow A \cap B \in T$,

(Intersection of any two members of T is also a member of T),

(1) If $\{A_\lambda | \lambda \in \Lambda\}$ is a collection of members of T, then $\bigcup\limits_{\lambda\in\Lambda} A_\lambda \in T$

(Union of any collection of members of T is a member of T).

Remark: If $T$is a topology on X, then (X, T) is called a topological space. Members of T are called T-*open sets* or simply *open sets*.

Examples

(1) $X = \{1, 2, 3\}$, collection of all subsets of $X = P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Then $T_1 = \{\phi, \{1\}, \{2\}\}$, $T_2 = \{\{1, 2, 3\}, \phi, \{1, 2\}, \{2, 3\}\}$ and $T_3 = \{\{1, 2, 3\}, \phi, \{1\}, \{2\}\}$ are not topologies on X.
(2) *Indiscrete topology* $I = \{\mathrm{X}, \phi\}$
(3) *Discrete topology* $D =$ collection of all subsets of X.
(4) *Cofinite topology:* Let X be a infinite set. Let T consists of ϕ and all those subsets of X whose complements are finite. Then T is a topology called the cofinite topology.

Remarks

(1) Intersection of any family of topologies on X is a topology on X.
(2) Union of two topologies on X is not necessarily a topology on X.

Closed Set: Let (X, T) be a topological space and A be a subset of X. Then A is said to be $T-$ closed or simply closed if $X - A$ is T- open.

Example: $X = \{1, 2, 3\}$, $T = \{\{1, 2, 3\}, \phi, \{1\}, \{2, 3\}$. $\{1\}$ is open as well as closed. $\{1, 2\}$ is neither open nor closed.

Properties of Closed Sets: **(i)** X is closed, **(ii)** ϕ is closed, **(iii)** Union of any two closed sets is closed, **(iv)** Arbitrary intersection of closed sets is closed.

Neighbourhood: Let (X, T) be a topological space. N be a subset of$X$ and, a be an element of X, N is called a $T-$ neighbourhood of 'a' if there exists a T-open set G such that $a \in G \subset N$.

Example: $X = \{1, 2, 3, 4\}$, $T = \{X, \phi, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$, $N = \{1, 2, 3\}$. N is *nbd* of 1, N is not a *nbd* of 4. N is *nbd* of 2.

Remarks

(1) Set of all neighbourhoods of x is denoted by N_x.

(2) N

$\in$

(3)
(4) N_x, $M \supset N$

$\Rightarrow$

(5)
(6) $M \in N_x$.
(7) $N \in N_x$, $M \in N_x N \cup M \in N_x$, $N \cap M \in N_x$.

Theorem: A set G is open if and only if G is a neighbourhood of each of its points.

Limit Point or Accumulation Point: Let (X, T) be a topological space and A be a subset of X. Let p be a point of X. p is called a limit point of A if every open set G containing $p$contains at least one point of A other than p.

Or

If for every open set G containing p, $(G \cap A) - \{p\}$, or $(G - - \{p\}) \cap A$.

Remark

(1) Collection of all limit points of the set A is called the derived set of A and is denoted by A' or $d(A)$ or $D(A)$.
(2) p is said to be an isolated point of A if $p \in A$ but p is not a limit point of A.
(3) A is said to be isolated set if each point of A is an isolated point of A.

Example: $X = \{1, 2, 3, 4\}$, $T = \{X, , \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$, $A = \{2, 3, 4\}$. $4 \in D(A)$, $1 \notin D(A)$, $2 \notin D(A)$. 2 and 3 are isolated points of A. 1 is neither a point of A nor a limit point of A. Thus $D(A) = \{4\}$.

Theorem: **(1)** $A \subset B \Rightarrow D(A) \subset D(B)$, **(2)** $D(A \cap B) \subset D(A) \cap D(B)$, **(3)** $D(A \cup B) \quad = D(A) \cup D(B)$.

Theorem: A subset F of a topological space X is closed iff F contains all its limit points, i.e., $D(F) \subset F$.

Adherent point: A point p is called an adherent point of A if every open set containing p contains at least one point of A. Set of all adherent points of A is called adherence of A and is denoted by $adh(A)$.

Remark:

(1) $adh\ (A) \quad = A \quad \cup D(A)$.
(2) Every point of A is an adherent point of A.
(3) If $p \notin A$ and p is an adherent point of A then p is also a limit point of A.

Example: $X = \{1, 2, 3\}$, $T = \{X, , \{1\}, \{2\}, \{1, 2\}\}$, $A = \{2, 3\}$. 2 is not a limit point of A but 2 is an adherent point of A.

Closure: Let (X, T) be a topological space and A be a subset of X. Closure of A is the intersection of all closed supersets of A and is denoted by $\overline{A}$ or $C(A)$.

Theorem

(1) $\overline{A}$ is the smallest closed superset of A.
(2) A$\subset$B$\Rightarrow \overline{A} \subset \overline{B}$.
(3) A set A is closed iff $A = \quad \overline{A}$.
(4) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(5) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(6) $\overline{A} \quad = \quad A \cup D(A) = adh\ A$.

Interior: Let (X, T) be a topological space and A be a subset of X. Let a be any point of X. a is called an interior point of A if A is *nbd* of a. Collection of all interior points of A is called the interior of A and is denoted by A^o or $i(A)$.

Theorem

(1) $A^o = \cup\{\ G|G$ is open and $G \subset A\}$.
(2) A^o is the largest open subset of A.
(3) A is open iff $A^o = A$.
(4) $A \subset B \Rightarrow A^o \subset B^o$.
(5) $(A \cap B)^o = A^o \cap B^o$.
(6) $A^o \cup B^o \quad \subset (A \cup B)^o$.

Exterior: Interior of complement of A is called exterior of A and is denoted by $e(A)$ or $ext(A)$, i.e., $ext(A) = (X - A)^o$

Theorem: **1.** $(X - A)^o \quad = X - \overline{A}$. **2.** $\overline{X - A}$ = X - A^o. **3.** $\overline{A}$ = X − (X − A)^o.

Boundary of A or Frontier of A: Let (X, T) be a topological space, and A be a subset of X. Boundary of A written as $b(A)$ is the set of those points of X which are neither interior point of A nor interior point of $X - A$. Therefore $b(A) \quad = X - (A^o \cup (X - A)^o) \quad = X - (A^o \cup ext(A)) = (X - A^o) \cap (X - (X - A)^o) = \overline{X - A} \cap \overline{A}$.

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Theorem: Let R be the set of real numbers. Let U consists of null set and those subsets A of R such that for every $x \in A$there exists a real number p ¿0 such that $]\ x - p\ x + p\ [\subset A$. Then U is a topology for R called the *usual topology.*

Remarks

(1) Every open interval is an open set.
(2) Every closed interval is a closed set.
(3) In the usual topology no nonempty finite set is open.
(4) In (R, U), if p is a limit point of A then every open set containing p contains infinite points of A.
(5) If p is a limit point of A then every open interval containing p contains infinite points of A.
(6) In usual topology no finite set can have a limit point.
(7) Every finite set is closed in usual topology.

Definition:

(1) A is dense − in − itself if $A \subset A'$.
(2) A is closed if $A' \subset A$.
(3) A is perfect if $A = A'$

Example

(1) Every open interval is dense − in − itself. But no open interval is a closed set.
(2) Every closed interval is a perfect set.

Exercise

(1) Give an example to illustrate that intersection of members of a topology may not be a member of topology.
(2) Give an example to show that union of closed sets may not be a closed set.

(3) Give an example in which $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

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Subspace: A topological space (Y, T_Y) is called a subspace of (X, T) if $Y \subset X$ and for every $H \in T_Y$ there exists G in T such that $H = Y \cap G$.

Example: $X = \{1, 2, 3, 4\}$, $T = \{X, \phi, \{1, 2\}, \{2, 3\}, \{2\}, \{1, 2, 3\}\}$, $Y = \{2, 3, 4\}$, $T_Y = \{\{2, 3, 4\}, \phi, \{2\}, \{2, 3\}\}$.

Remark: T_Y is called *relative topology on* Y or *topology relativized by* T *on* Y and (Y, T_Y) is called a subspace of (X, T).

Theorem: Let (X, T) be a topological space and Y be a subspace of X. Let $T_Y = \{Y \cap G | G \in T\}$. Then T_Y is a topology for Y. (Y, T_Y) is called subspace of (X, T).

Theorem: Let (Y, V) be a subspace of (X, T). Let F be a subset of Y. Then F is V-closed iff $F = Y \cap H$ for some T-closed set H.

Theorem: If (Z, W) is a subspace of (Y, V) and (Y, V) is a subspace of (X, T). Then (Z, W) is a subspace of (X, T).

Theorem: Let (X^*, T^*) be a subspace of (X, T). Let A be a subset of X^*. Let $y \in X^*$. Then $y$is a T^*-limit point of A iff y is a T-limit point of A.

Lemma: y is a T^*-adherent point of A iff y is a T-adherent point of A.

Theorem: Let (X^*, T^*) be a subspace of (X, T). Let A be a subset of X^*. Then T^*-closure of $A = X^* \cap (T$-closure of $A)$, or, $C^*(A) \quad = X^* \cap C(A)$.

Weak topology: Let (X, T_1) and (X, T_2) be two topological spaces. If $T_1 \subset T_2$, then T_1 is weaker than T_2, or T_2 is stronger than T_1, or T_1 is coarser than T_2, or T_2 is finer than T_1, or T_1 is smaller than T_2, or T_2 is larger than T_1. If $T_1 \not\subset T_2$ and $T_2 \not\subset T_1$, we say that T_1 and T_2 are not comparable.

Example: $X = \{1, 2, 3\}$, $T_1 = \{X, \phi, \{1\}\}$, $T_2 = \{X, \phi, \{2\}\}$, $T_3 = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$. T_1 and T_2 are not comparable, $T_1 \subset T_3$, $T_2 \subset T_3$.

Complete lattice: A complete lattice is a partially ordered set in which every nonempty subset has a least upper bound and greatest lower bound.

Theorem: The set of all topologies on a set X is a 'complete lattice' with respect to 'inclusion' relation.

Base: Let (X, T) be a topological space. A collection B of subsets of X is called a base for T if (i) B $\subset T$, (ii) for every G in T and x in G, there exists B in B such that $x \in B \subset G$.

Example: $X = \{a, b, c, d\}$, $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$, B $= \{\{a\}, \{b\}, \{a, b\}, \{c, d\}\}$. For $G = \{a, c, d\}$, $c \in \{c, d\} \subset G$.

Theorem: Let (X, T) be a topological space. A collection B of members of T is a base iff each member of T is expressible as a union of some subfamilies of B.

Theorem: Let (X, T) be a topological space and B be a base for T. Then T is identical with the collection of unions of all subfamilies of B.

Example: X = {1, 2, 3}, T = {X, ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}}, B = {{1}, {2}, {3}}. Family of unions of subfamilies of B = T, therefore B is a base for T

Example: Is B = {{0, 1}, {1, 2}} a base for some topology on X = {1, 2, 3}?

Solution: If B is a base for a topology T on X, then T must be the collection of unions of all subfamilies of B, i.e., T = {ϕ, {0, 1}, {1, 2}, {0, 1, 2}}. But T is not a topology for X and hence B is not a base for any topology on X.

Theorem: Let X be a nonempty set. A collection B of subsets of X is a base for some topology T on X if and only if (i) $\cup\{B|B \in \text{B}\ \} = X$, (ii) For $U, V \in$ B and $x \in U \cap V$ there exists W in B such that $x \in W \subset U \cap V$.

Or

What is the necessary and sufficient condition for a family to become a base for a topology.

Theorem: Let B be a collection of subsets of X such that (i) $\cup\{B|B \in \text{B}\ \} = X$, (ii) $U, V \in \text{B} \Rightarrow \quad U \cap V \in$ B. Then B is a base for some topology on X.

Sub-base: $X = \{0, 1, 2\}$, $S = \{\{0, 1\}, \{1, 2\}\}$, B = {X, {{0, 1}, {1, 2}, {1}}, $T = \{\phi$, X , {1}, {0, 1}, {1, 2}}.

Definition: Let X be any nonempty set and S be a collection of subsets of X. Let B be the collection of intersections of all finite subfamilies of S. Then ∪{B|B∈ B } $= X$ and intersection of any two members of B is a member of B. Hence B is a base for some topology on X, say T. S is called a sub-base for T and T is called the topology generated by S.

Remark: If S is a sub-base for topology T. Then T is the smallest topology containing S.

Continuity

If $X = \{1, 2, 3, 4, 5, 6, 7\}$, $Y = \{a,\ b,\ c,\ d,\ p,\ q,\ r,\ s,\ t\}$, $f : X \to Y$ such that $f(1) = a$, $f(2) = a$, $f(3) = a$, $f(4) = d$, $f(5) = d$, $f(6) = p$, $f(7) = q$, $A = \{2, 3, 4, 6\}$, $f(A) = \{a, d, p\}$, $f^{-1}(f(A)) = f^{\ -1}\{a, d, p\} = \{1, 2, 3, 4, 5, 6\} \neq A$. $A \subset f^{-1}(f(A))$ and $A = \quad f^{-1}(f(A))$ if f is one-one. Similarly $f(f^{-1}(E)) = E$ if $f$is onto.

Continuous function: Let (X, T) and (Y, V) be two topological spaces and f be a function from $X$to Y. Let $a \in X$. f is said to be *T-V continuous at a* if for every $V-$open set H containing $f(a)$ there exists at least one $T-$open set G containing a such that $f(G) \subset H.f$ is said to be a *T-V continuous* if f is continuous at each point of X.

Theorem: Let (X, T) and (Y, V) be two topological spaces and f be a function from X to Y. Then f is $T - V$continuous iff inverse image of each $V-$open set under f is $T-$open.

Theorem: Let (X, T) and (Y, V) be two topological spaces and f be a function from X to Y. Then f is $T - V$ continuous iff for every $V-$closed set E, f^{-1}(E) is $T-$closed.

Theorem: Let (X, T) and (Y, V) be two topological spaces and f be a function from X to Y. Then f is $T - V$continuous iff $f(\overline{E}) \quad \subset \overline{f(E)}$ for every $E \subset X$.

Open and Closed functions: Let (X, T) and (Y, V) be two topological spaces and f be a function from X to Y.

(1) f is called an *open function* if for every T-open set G, $f(G)$ is $V-$open.
(2) f is called a *closed function* if for every T-closed set H, $f(H)$ is V- closed.

Example: Give an example of a function which is continuous, open but not closed.

Solution: $X = \{a, b, c\}$, $T = \{X,\ \phi,\ \{a\},\ \{b, c\}\}$, $Y = \{1,\ 2,\ 3\}$, $V = \{Y,\ \phi,$ $\{1\}\}$. f:$X \to Y$ such that $f(a) \quad = f(b) \quad = f(c) = 1$. Then f is the required function.

Example: Give an example of a function which is continuous and closed but not open.

Theorem: Let (X,T) and (Y,V) be two topological spaces and f be a function from X to Y. Then f is $T-V$closed mapping iff $\overline{f(E)} \subset f(\overline{E})$ for every $E \subset X$.

Theorem: Let (X,T) and (Y,V) be two topological spaces and f be a function from X to Y. Then f is an open function iff $f(E^o) \subset (f(E))^o$ for every $E \subset X$.

Homeomorphism: Let (X,T) and (Y,V) be two topological spaces and f be a function from X to Y. f is called a homeomorphism if (i) f is one-one and onto, (ii) f is $T-V$continuous, (iii) f^{-1} is $V-T$continuous.

(X,T) and (Y,V) are said to be homeomorphic spaces if there exists a homeomorphism between X and Y.

Bi-continuous: f is called *bi-continuous* if f is continuous and open.

Theorem: Let (X,T) and (Y,V) be two topological spaces and f is a one-one function from Xonto Y. Then the following conditions are equivalent.

(i) f is homeomorphism, **(ii)** f is continuous and open, **(iii)** f is continuous and closed.

Theorem: Homeomorphism is an equivalence relation in the set of all topological spaces.

Theorem: Let (X,T) and (Y,V) be two topological spaces and f be a 1-1 function from X onto Y. Then f is is a homeomorphism iff $\overline{f(E)} = f(\overline{E})$ for every $E \subset X$.

Theorem: Let (X,T) and (Y,V) be two topological spaces and f be a 1-1 function from X onto Y. Then f is a homeomorphism iff $f(A^o) = (f(A))^o$for every $A \subset X$.

Theorem: Let (X,T) and (Y,V) be two topological spaces and f be a function from X to Y. Then f is continuous iff $\overline{f^{-1}(E)} \subset f^{-1}(\overline{E})$ for every $E \subset Y$.

Theorem: Let (X,T) and (Y,V) be two topological spaces and f be a function from X to Y. Then f is continuous iff $f^{-1}(E^o) \subset (f^{-1}(E))^o$ for every $E \subset Y$.

Theorem: Let (X,T) and (Y,V) be two topological spaces and B be a base for V. Let f be a function from X to Y. Then f is continuous iff inverse image of each member of B under f is in T.

Theorem: Let (X,T) and (Y,V) be two topological spaces and S be a base for V. Let f be a function from X to Y. Then f is continuous iff inverse image of each member of S under f is in T, i. e., $f^{-1}(S) \in T$ for every $S \in$S.

Open Cover: Let (X,T) be a topological space and A be a subset of X. A collection $\{G\lambda| \quad \lambda \in \Lambda\}$ of T-open sets is called a *T-open cover* of A if $A \subset \bigcup_{\lambda\in\Lambda} G_\lambda$.

Compact Space: Let (X,T) be a topological space and A be a subset of X. A is said to be *T-compact* set if every T-open cover of A has a finite subscover.

X is called *compact space* if every T-open cover of X has a finite subcover.

Example: $A = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$, $G_1 = \{1,\ 2,\ 3\}$, $G_2 = \{2,\ 3,\ 4\}$, $G_3 = \{3,\ 4,\ 5\}$, $G_4 = \{4,\ 5,\ 6\}$, $G_5 = \{5,\ 6,\ 7\}$. Then $A \subset G_1 \quad \cup G_2 \cup \ldots$

Example: Let $C_1 = \{$] - n, n [$|n \in N\}$, $C_2 = \{$] -3n, 3n[$| \quad n \in N\}$, $C_3 = \{$] $2n-1$, $2n \quad +1$[,]$2n$, $2n+2$[$| \quad n \in N\}$. Then C_1, C_2, C_3 are all U-open covers of R and C_2 is a subcover of C_1.

Theorem: Let $(Y,\ T_Y)$ be a subspace of $(X,\ T)$, and A be a subset of Y. Then A is T_Y-compact if and only if A is T-compact, or, Compactness is an absolute property of the set.

Theorem: Every closed set in a compact space is compact.

Finite Intersection property (FIP): A family of sets is said to have *finite intersection property* if for every finite subfamily has a nonempty intersection.

Example: $A_1 = \{1, 2, 3, \ldots\}$, $A_2 = \{2, 3, 4, \ldots\}$, ... Consider $\{A_1, A_2, A_3, \ldots\}$. Then $A_2 \cap A_7 \cap A_{123} = A_{123}$ and $A_1 \cap A_2 \cap \ldots = \phi$

Theorem: A topological space X is compact iff every family of T-closed sets having *FIP* have a nonempty intersection.

Example: Consider the following class of open intervals $A = \{(0, 1), (0, 1/2), (0, 1/3), \ldots\}$. Now A has the FIP because $(0, a_1) \cap (0, a_2) \cap \ldots \cap (0, a_n) = (0, b)$, where $b = min(a_1, a_2, \ldots, a_m) > 0$. Observe that A itself has an empty intersection.

Topological Property: A property P possessed by a topological space X is called a *topological property* if P is possessed by every homeomorphic image of X.

Theorem: Continuous image of a compact space is compact, or, Compactness is a topological property, Or Compactness is preserved under continuity.

Theorem: (R, U) is not compact.

Theorem: Every closed interval is a compact set.

Heine - Borel Theorem: A subset A of R is compact iff it is closed and bounded.

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Metric Spaces: Let X be a nonempty set. A mapping d: X×X→R is called a metric on X if d satisfies the following conditions:

(1) d(x, y) $\geqslant$0;
(2) d(x,y) = 0 $\Leftrightarrow$x = y;
(3) d(x, y) = d(y, x); Symmetric property
(4) d(x, y) $\leqslant$d(x, z) + d(z, y) for all x, y, z∈X. Triangle inequality

If d is a metric on X, then (X, d) is called the metric space.

Examples

(1) $d\ (x, y) = |x - -y|$ is a metric on R.
(2) $d\ (z_1, z_2) = |z_1 - z_2|$ is a metric on C.
(3) $d: \quad R^n \to R^n : d(x, y) = \sum_{i=1}^{n} [(x_i - y_i)^2]^{1/2}$is a metric on R^n
(4) Let X be a-nonempty set. Then the mapping d: X×X→R defined by d(x, y) = 0, if x = y, and d(x, y) = 1 for x≠y is a metric for X called the *discrete metric* for X.

Examples

1. Show that $|d(x, z) - d(y, z)| \leqslant d(x, y)$.

2. Prove that |d(x, y) − d(x$_1$, y$_1$) $\leqslant$ d(x, x$_1$) + d(y, y$_1$).

Open Sphere, opern ball, open neighbourhood (centered on x and radius r) is S$_r$ (x) = S(x, r) = B(x, r) = N$_r$(x) = { y∈ $X \quad | \quad d(y, x)$ ¡r}.

Remark: x belongs to S$_r$(x), i.e., ball is always nonempty.

Example: For a discrete metric space X = {1, 2, 3, 4, 5, 6, 7, 8}, $S_{3/4}(3)$ = {3}, $S_{13}(3)$ = {1, 2, 3, 4, 5, 6, 7, 8}, $S_{1010}(3)$ = {1, 2, 3, 4, 5, 6, 7, 8}.

Closed Sphere: (centered on x and radius r) is $S_r\ [x]$ = { y∈ $X \quad | \quad d(y, x) \quad \leqslant r$}.

Example: Find the open sphere in R and C.

Open Sets: A subset A of a metric space X is said to be open if for all x∈X there exists a real number r ¿0 such that $S_r(x) \quad \subset A$.

Closed Sets: A subset A of X is closed if its complement is open.

Remarks:

(1) Every open sphere is an open set and every closed sphere is a closed set.

(2) Empty set and the full space X are open.
(3) Arbitrary union of a family of open sets is open.
(4) Finite intersection of a family of open sets is open.

Sequence in a metric space: Let X be a non-empty set. A mapping f: N$\to$X is called a sequence in X.

Subsequence: A sequence $< x_{n_k} >$ is a subsequence of $<$x$_n >$ if (i) $n_k \geqslant k$, (ii) n_1 ¡$n_2 < \dots$

Convergent sequence: A sequence $\{x_n\}$ in X is said to be convergent if there exists x in X such that for every $\in$¿0 there exists a positive integer n_0 such that $n \geqslant n_0 \Rightarrow d(x_n,x) < \in$.

Remarks:

(1) Every convergent sequence in a metric space X has a unique limit.
(2) If ¡x_n¿ is an infinite sequence then its limit is also its limit point.
(3) Let (X, d) be a metric space and A be a subset of X and x be a limit point of A then there is an infinite sequence $<$x$_n >$ in A such that x$_n \to$x.

Cauchy Sequence: A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in X if for every $\in$¿0 there exists a positive integer n_0 such that m, $n \geqslant n_0 \Rightarrow d(x_m, x_n) < \in$.

Theorem: every convergent sequence in a metric space X is a cauchy sequence but the converse is not true.

Complete metric space: A metric space X is called complete if every Cauchy sequence in X is convergent in X.

Remarks

(1) If a sequence $<$x$_n >$ converges to x then every subsequence of $<$x$_n >$ converges to x.
(2) A non convergent sequence may have convergent subsequence.
(3) If $<$x$_n >$ is a Cauchy sequence and $< x_{n_k} >$ is a subsequence of $<$x$_n >$ such that $x_{n_k} \to$a then x$_n \to$a.

Topology induced by a metric: Let (X, d) be a metric space and T be the collection of d-open sets. Then T is a topology on X induced by the metric d.

A topological space (X, T) is called a metrizable topological space if there exists a metric d on X such that the collection of d-open sets is same as T.

Theorem: Let (X, d) be any metric space. Let T$_d$ consists of null set and all those subsets G of X such that for every x in G there is a real number r $>$ 0 such that S$_r$(x)$\subset$X. Then T$_d$ is a topology for X induced by d.

Example: (R, U) is a metrizable topological space.

Remarks: In a metric space X

(1) If x is a limit point of A then every open set containing x contains infinite points of A.
(2) A finite set has no limit points.
(3) Every finite set is closed.

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Bolzano Weierstrass Property (BWP): A metric space X is said to have BWP if every infinite subset of X has at least one limit point in X.

Sequentially Compact Metric Space: A metric space X is said to be a *sequentially compact* if every sequence in X has a convergent subsequence.

Theorem: Every compact space has BWP.

Theorem: Every metric space having BWP is sequentially compact.

Theorem: A metric space X is sequentially compact iff X has BWP.

$\in -$ **net:** A finite set of points, say $\{a_1,\ a_2,\ \ldots,\ a_n\}$ in a metric space X is called an $\in$- *net* if $X = \bigcup\limits_{i=1}^{n} S_{\in}(a_i)$.

Totally Bounded or Precompact: A metric space X is said to be *totally bounded* if there exists an $\in$- *net* in X for every $\in > 0$.

Theorem: Every sequentially compact metric space is totally bounded.

Diameter: Diameter of $A = diam\ A = d(A) = \delta_A = sup\ \{d(x,y) \quad | \quad x,y \in A\}$.

Remark: $\delta_(S_r(a) \leqslant 2r$.

Lebesgue Number: Let $\{G\lambda | \lambda \in \Lambda\}$ be an open cover of a metric space X. A positive real number r is called a *lebesgue number* of $\{G\lambda | \lambda \in \Lambda\}$ if for every subset A of X such that $diam\ A < r$ there is a $\lambda \in \Lambda$ such that $A \quad \subset G_\lambda$.

Remark: If r is a lebesgue number and 0 ¡r_1¡$r$then r_1 is also a lebesgue number.

Big Set: If B is a subset of X which is not contained in any ofG_λ's then B is called a '*Big Set*' and B contains at least two elements and $diam\ B > 0$.

Lebesgue Covering Lemma: In a sequentially compact metric space every open cover has a lebesgue number.

Theorem: Every sequentially compact metric space is compact.

Uniformly Continuous Function: Let $(X_1,\ d_1)$ and $(X_2,\ d_2)$ be two metric spaces and f be a function from X_1 to X_2. f is said to be *uniformly continuous* if for every $\in > 0$ there exists $\delta > 0$ such that $d_1(a,\ b) \quad < \delta \Rightarrow \quad d_2(f(a),\ f(b)) < \in$.

Theorem: Every continuous function defined on a compact metric space is uniformly continuous.

Theorem: A metric space X is compact iff X is complete and totally bounded.

Theorem: Let (X,d) be a complete metric space and Y be a closed subspace of X. Then Y is compact iff Y is totally bounded.

Separable Space: X is *separable* if X has a countable dense subset, i.e., there is a countable subset A of X such that $\overline{A} = X$.

Theorem: Every compact metric space is separable, Or, Every totally bounded metric space is separable.

Second Countable Space: A space X is second countable space (or second axiom space) if X has a countable base.

Remark: A space X is said to satisfy second axiom of countability if X has a countable base.

Theorem: Every second axiom (or second countable) space is separable.

Theorem: Every separable metric space is a second axiom space.

Theorem: A topological space X is compact iff every basic open cover of X has a finite subcover.

Local Bases: Let p be any arbitrary point of a topological space X. A class B_p of open sets containing p is called a local base at p iff for each open set G containing p there exists G_p in B_p such that $p \in G_p \subset G$.

Example: $X = \{1,\ 2,\ 3,\ 4,\ 5\}$, $T_1 = \{\ X,\ \phi,\ \{1\}\ ,\ \{2,\ 3\},\ \{1,\ 2,\ 3\}\}$, $B_1 = \{\{1,\ 2\},\ X\}$, $B_2 = \{\{2,\ 3\},\ X\}$, $B_3 = \{\{1,\ 2,\ 3\}\}$. Find whether B_1, B_2, B_3 form local base at 1, 2, 3 respectively or not?

Answer: B_1 is not a local base at 1, B_2 is a local base at 2, and B_3 is not a local base at 3.

First Countable Space: A topological space X is said to be first countable space (or first axiom space) if there exists a countable base at every point p of X.

Theorem: Every second countable space is first countable space.

Theorem: Let A be any subset of a second countable space X. If G is an open cover of A, then G is reducible to a countable cover.

Or

Let X be a second countable space. If a nonempty open set G in X is represented as the union of a class $\{G_i\}$ of open sets, then G can be represented as a countable union of G_i's.

Theorem: Let G be a base for a second countable space X. Then G is reducible to a countable base for X.

Or

Let X be a second countable space. Then any open base for X has a countable subclass which is also an open base.

Lindelof Space: A space X is said to be a Lindelof space if every open cover of X has a countable subcover.

Lindelof's Theorem: Every second countable space is a Lindelof space.

SEPARATION AXIOMS

T$_o$-space: A space (X,T) is called a T_o-*space* if for any two distinct points x and y there exists an open set G containing one of x and y but not the other.

T$_1$ − Space: A topological space is said to be a T_1 *space* if given any two distinct points x an y of X there exists open sets G and H such that $x \in G$, $y \notin G$, and $y \in H$, $x \notin H$.

T$_2$- space or Housdorff space: X is said to be a T_2 *space* or *Housdorff* if given any two distinct points x an y of X there exists open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \phi$.

Regular Space: A space X is called a *regular* space if given any closed set E and a point $a \notin E$ there exist open sets G and H such that $E \subset G$, $a \in H$ and $G \cap H = \phi$.

T$_3$-space: A regular T_1- *space* is called a T_3 - *space*.

Normal Space: A space X is *normal* if given any two disjoint closed sets E and F in X there exists open sets G and H such that $E \subset G$, $F \subset H$ and $G \cap H = \phi$.

T$_4$-space: A normal T_1-space is called a T_4-*space*.

Completely Regular Space: A topological space X is said to be completely regular if for every closed subset F of X and every point $x \in X - F$ ($x \notin F$) there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(F) = 1$.

Tychonoff Space or $T_{3\frac{1}{2}}$- Space: A completely regular T_1- space is called a *Tychonoff space*.

Remark: Every T_2 − space is T_1 but the converse is not necessarily true.

Theorem: (X,T) is T_1 iff each singleton set in X is closed.

Convergent sequence: A sequence $< x_n >$ in a topological space (X,T) is said to be convergent to a in X if given any open set G containing a there exists a positive integer m such that $n \geqslant m \Rightarrow x_n \in G$.

Example: $X = \{a,b,c\}$, $T = \{X, \phi, \{c\}, \{a,b\}\}$. Then the sequence $<a, b, a, b, a, b, \ldots>$ converges to a and b. $< a, b, a, b, c, c, c, \ldots >$ converges to c.

Theorem: In a hausdorff topological space every convergent sequence has a unique limit.

Theorem: If (X, T) is a hausdorff space and E is a compact subset of X and a is a point of X such that a$\notin$E then there are open sets G and H such that E$\subset$G, a$\in$H, G$\cap$H $= \phi$.

Or

In a Hausdorff space any point and disjoint compact subspace can be separated by open sets, in the sense that they have disjoint neighbourhoods.

Theorem: Every compact set in a hausdorff space is closed.

Theorem: If (X, T) is a compact space and (Y, V) is a hausdorff space then every one-to-one continuous function f from X onto Y is a homeomorphism.

Theorem: Every compact hausdorff space is normal.

Hereditary property: A property P possesses by a topological space is called a hereditary property if P is possessed by every subspace of X.

Theorem:

(1) The property of being a T_o-space is hereditary, i.e., every subspace of a T_o-space is also a T_o- space
(2) The property of being a T_1-space is hereditary.
(3) The property of being a T_2-space is hereditary.
(4) The property of being a regular space is hereditary.
(5) A closed subspace of a normal space is normal.

Theorem: $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_o$.

Examples:

(1) $X = \{1, 2, 3\}$, $T = \{X, \phi, \{1\}, \{1, 2\}, \{2\}\}$. Then (X, T) is T_1.
(2) Every metric space is a T_1, and Housdorff space.
(3) Every finite subset of a T_1-space is a closed set.
(4) If (X, T) is a finite T_1-space, then T is the discrete topology.
(5) Every discrete space is T_2.
(6) Every finite T_2-space is discrete.
(7) Every finite subset of a T_2-space is closed.
(8) If f: $X \to Y$ is continuous then $x_n \to x$ in X implies $f\ (x_n) \to f\ (x)$.
(9) (R, U) is T_1, T_2, T_3, and T_4.
(10) Let X be an infinite set and let T be co-finite topology on X, then (X, T) is T_1 but not T_2.

Connectedness

Separated sets: Let (X, T) be a topological space and A, Bbe subsets of X, then A and B are called separated if $A \cap \overline{B} = \overline{A} \cap B = \phi$.

Or

Two subsets A and B in a topological space X are separated if$(A \cap \overline{B}) \cup (\overline{A} \cap B) = \phi$.

Remark: Separated sets are always disjoint but disjoint sets may not be separated.

Example: If $A =]\ 0,\ 1]$, $B =]\ 1,\ 2]$, $A \cap B = \phi$, then A and B are disjoint sets. $\overline{A}= [0,\ 1]$, $\overline{B} = [1,\ 2]$, $A \quad \cap \overline{B}= \{1\} \neq \phi$. $\therefore A$ and B are not separated.

Separation of two sets: Let E be a subset of X. Two subsets A and B of X are said to form a partition or separation of E if $E = A \cup B$, $A \cap \overline{B} = \overline{A} \cap B = \phi$, $A \neq \phi$, $B \neq \phi$. In this case we write $E = A|B$.

Disconnected space: (X, T) is disconnected if there exists two nonempty sets A and B such that X $= A \cup B$, $A \cap \overline{B} = \overline{A} \cap B = \quad \phi$.

Connected space: A topological space is said to be connected if it is not disconnected.

Theorem: Let (X, T) be a topological space. Then the following conditions are equivalent:

(1) X is disconnected
(2) There exist two nonempty closed sets A and B such that $X = A \cup B$, $A \cap B = \phi$.
(3) There exists a nonempty proper subset of X which is both open and closed in X.
(4) There exist two nonempty open sets A and B such that $X = A \cup B$, $A \cap B = \phi$.

Example: X =$\{1, 2, 3\}$, $T = \{X, \phi, \{1\}, \{2, 3\}\}$. Then X is disconnected, because $X = \{1\} \cup \{2, 3\}$, $\{1\} \cap \{2, 3\} = \phi$.

Theorem: Continuous image of a connected space is connected.

Or

Connectedness is a topological property.

Interval: A subset X of R is called an interval if $x, z \in X$, $x < y < z \Rightarrow y \in X$.

Theorem: A subspace X of R is an interval iff X is connected.

Remark: Real line is connected.

Theorem: Range of every continuous real valued function defined on a connected space X is an interval.

Theorem: A topological space X is disconnected iff there exists a continuous function from X onto discrete space on $\{0, 1\}$.

Disconnected set: Let (X, T) be a topological space and $E \subset X$. E is said to be T-disconnected if there exist two nonempty set A, B in X such that $E = A \cup B$, $A \cap \overline{B} = \overline{A} \cap B = \phi$.

Theorem: Connectedness is an absolute property of the set.

Or

If (Y, T_Y) is a subspace of (X, T), and $E \subset Y$, then E is T_Y- connected iff E is $T-$ connected.

Theorem: **(1)** If A and B forms a separation of a space X and E is a connected subset of X then either $E \subset A$ or $E \subset B$. **(2)** Let Ebe any connected subset of a topological space such that $E \subset G \subset \overline{E}$, then G is connected. In particular $\overline{E}$ is connected (i.e. closure of a connected set is connected).

Theorem: If X is a topological space such that given any pair $x, y$of distinct points of X, there is a connected set E in X containing x and y .Then X is connected.

Theorem: Let (X, T) be a topological space. If $\{A\lambda | \lambda \in \Lambda\}$ is a family of connected sets in X such that $\underset{\lambda \in \Lambda}{\cap} A_\lambda \neq \phi$, then $\underset{\lambda \in \Lambda}{\cup} A_\lambda$is connected.

Components: A subset E of a topological space X is called a component if (i) E is connected (ii) E is not properly contained in any connected set in X.

Or

A maximal connected set in X is called a component.

Remark: If X is connected space then X is the only component of X.

Totally disconnected space: A topological space X is said to be a totally disconnected space if given any two distinct points a, b in X there exist two nonempty open sets A, B in X such that $a \in A$, $b \in B$, $X = A \cup B$, $A \cap B = \phi$.

Remark: Every totally disconnected space is a Hausdorff space.

Example: $X = \{a\}$, $T = \{X, \phi\}$. Then X is not disconnected but X is totally disconnected.

Remark: If a totally disconnected space contains more than one point then the space is disconnected also.

Theorem: If X is totally disconnected space then its components are its singleton sets.

Or

Components of a totally disconnected space are its points.

Theorem: Let X be a compact Hausdorff space. X is totally disconnected iff X has an open base whose each member is closed also.

Locally Connected Space: A topological space X is said to be locally connected if given open set G and x in G there exists an open set H in X such that $x \in H \subset G$.

Theorem: Let (X, T) be a locally connected space and Y be an open subspace of X. Every component C in Y is open in X. In particular, every component of X is open in X.

Theorem: For any arbitrary topological space X

(1) Every point in a topological space X is contained in one and only one component of X.
(2) Every nonempty connected set is always contained in a component.
(3) Every nonempty connected set which is both open and closed is a component.
(4) Every component in a topological space X is closed.

PRODUCT SPACE

Theorem: Let (X, T) and (Y, V) be two topological spaces. Then the collection $\mathrm{B} = \{G \times H | G \in T,\ H \in V\}$ is a base for some topology of $X \times Y$.

Product Space: Let (X, T) and (Y, V) be two topological spaces. Then the topology W whose base is $\mathrm{B} = \{G \times H | G \in T,\ H \in V\}$ is called the product topology of $X \times Y$ and $(X \times Y,\ W)$ is called the product space of X and Y.

Theorem: Let (X, T) and (Y, V) be two topological spaces and B_1 a base for T and B_2 a base for V. Then $\mathrm{B} = \{B_1 \times B_2 | \quad B_1 \in \mathrm{B}_1,\ B_2 \in \mathrm{B}_2\}$ is a base for the product topology W of $X \times Y$.

Projection Mapping: The mapping p_x: $X \times Y \to X : p_x((x, y)) = x$ and p_y: $X \times Y \to Y : p_y((x, y)) = y$ for all (x, y) in X×Y are called the projections of $X \times Y$ onto X and Y respectively.

Theorem: Projections are continuous and open. The product topology T is the coarsest topology for which the projections are continuous.

Example: Let $X = \{a, b, c\}$ and $T = \{X,\ \phi,\ \{a\}\}$. $Y = \{p,\ q,\ r,\ s\}$, $V = \{Y,$ $\{p\}$, $\{q\}$, $\{p, q\}$, $\{r, s\}$, $\{p, r, s\}$, $\{q, r, s\}\}$. Find a base for the product topology of $X \quad \times Y$.

Theorem: The collection $\mathrm{S} = \{p_1^{-1}(U) | U$ is open in $X\} \cup \{p_2^{-1}(V) | V$ is open in $Y\}$ is a subbasis for the product topology on X×Y

Theorem: Let $y_o \in Y$ and let $A = X \times \{y_o\}$. Then the restriction of p_x to A is a homeomorphism of the subspace A of $X \times Y$ onto X. Similarly the restriction p_y to $\mathrm{B} = \{x_o\} \times Y$ is a homeomorphism.

Projections: Let $X = \prod_{i \in I} X_i = \times \{X_i | i \in I\}$. Then the mapping p_i: $X \to X_i$ defined by $p_i(x) \quad = x_i$ for all x in X is called the i^{th} projection.

Theorem: Each projection map p_i is continuous, open but not closed.

Theorem: Let f be a mapping of a space Y into a product space $X = \prod_i X_i$. Then f is continuous $\Leftrightarrow p_i \mathrm{o} f$:$Y \to X_i$ is continuous.

Product Topology and Product Space: Let $(X_i,\ T_i)$ be a collection of topological spaces and $X = \prod_{i\in I} X_i$. Then the topology T for X which has subbase the collection S $= \{p_i^{-1}(G_i) | i \in I, G_i \in T_i\}$ is called the product topology (or Tychonoff Topology) for X and (X,T) is called the product space .

Remarks: (1) The collection S is called the defining subbase for T. The collection B of all finite intersections of elements of S would then form a base for T.

(2) Since $p_i^{-1}(G_i)$ are open sets with respect to the product topology where G_i is any open set in X_i. It follows that the projection p_i is a continuous map for each i in I.

(3) For countable collection of topological spaces $\{\mathrm{X}_i |\ \ i = 1,\ 2, \ldots ,\ \}$ the product space $X = \prod_{n\in N} X_n = \{x = (x_1,\ \ldots,\ x_n,\ \ldots)\}$. Moreover $p_a^{-1}(G_a\} = X_1 \times X_2 \times \ldots \times X_{a-1} \times G_a \times X_{a+1} \times \ldots$.

(4) A base for this topology is a set $B = \cap \{p_i^{-1}(G_i) | i \in I'\}$ where I' is a finite subset of I and $G_i \in T_i$ for some I in I'.

Theorem: Let $(X_i,\ T_i)$ be an arbitrary collection of topological spaces and let$X = \prod_{i\in I} X_i$. Let T be the topology for X. Then the following statements are equivalent: (**1**) T is the product topology for X; (**2**) T is the smallest topology for which the projections are continuous.

EXERCISE - 1

(1) Define discrete topological space, homeomorphism, second countable space, sequentially compact metric space. 05
(2) State and prove Lindelof's theorem. 05
(3) Continuous image of a connected space is connected. 05
(4) One to one continuous mapping of a compact space onto a hausdorff space is a homeomorphism. 02, 05
(5) Compact hausdorff space is normal. 02, 05
(6) Define Topological space, compactness in metric space,

locally connected space, hausdorff space. 04

(1) Continuous image of a compact space is compact. 04
(2) Every second countable space is separable. 04
(3) Components of a totally disconnected space X are the singleton subsets of X. 04
(4) Homeomorphism is an equivalence relation. 04
(5) A metric space is separable if it is second countable. 04
(6) Let T be the collection of subsets of N consisting of empty set and all subsets of the form $G_m = \{m,\ m+1,\ m+2,\ \ldots),\ m \in N$. Show that T is a topology for N. What are open sets containing 5. 04
(7) Every compact subset in a hausdorff space X is closed. 03
(8) Every subspace of a hausdorff space is a hausdorff. 03
(9) If (Z,W) is a subspace of (Y,V) and (Y,V) is a subspace of (X,T).

Then (Z,W) is a subspace of (X,T). 03

(1) Give an example of a T_2-space which is not T_3. 03
(2) Let (X,T) and (Y,V) be two topological spaces and f be a 1-1 function from X onto Y.

Then f is a homeomorphism iff $\overline{f(E)} = f(\overline{E})$ for every $E \subset X$. 03

(1) Define a regular space and prove that the property of a space being regular space is a topological property. 03
(2) Prove that the closure of a set is union of that set and its derived set. 02
(3) Show that a space whose topology has a countable base is separable. 02
(4) The property of a space being a T_2-space is a topological property. 02
(5) X is topological space, Y is a hausdorff space. If $f$and g are continuous functions of X into Y.

Then the set $\{x \in X| \quad f(x) \quad = g(x)\}$ is closed. 02

(1) Let (X, T) and (Y, V) be two topological spaces and f be a function from X to Y.

Then f is $T - V$continuous iff $f(\overline{E}) \quad \subset \overline{f(E)}$ for every $E \subset X$. 02

(1) Let $d_1(x, y) = min\ \{1, d(x, y)\}$. Show that d_1 is a metric and d and d_1 are equivalent. 02
(2) A family B of sets is a base for a topology for the set $X = \quad \cup\{G|G \in B\}$ iff for every G_1, G_2 in B and every x in $G_1 \cap G_2$ there exists G in B such that $x \in G \subseteq G_1 \cap G_2$. 01

True/False

(1) Every topological space is metrizable.
(2) A subset of a topological space is open if it is closure of each of its points.
(3) Every component of a topological space is open.
(4) Every component of a locally connected space is a closed set.

....
....

EXTRA
EXERCISE - 2

(1) **Indiscrete topology:** $I = \{\text{X}, \phi\}$
(2) **Discrete topology:** $D =$ collection of all subsets of X.
(3) **Particular point topology:** $p \in X$(fixed). Then $T = \{A \subset X : p \in A\}\cup\{\phi\}$.
(4) **Excluded point topology:** $x \in X$(fixed). $T_x = \{A \subset X : x \notin A\}\cup\{X\}$.
(5) **Usual topology for R:** Let R be the set of real numbers. Let U consists of null set and those subsets A of R such that for every $x \in A$there exists a real number p ¿0 such that] x – p, x + p [$\subset A$. Then U is a topology for R called the *usual topology.*
(6) **Usual topology on a metric space:** Let (X, d) be any metric space. Let T_d consists of null set and all those subsets G of X such that for all x in G there is a real r¿ 0 such that $S_r(x) \subset \quad G$. then T_d is a topology called the metric topology.
(7) **Co-finite topology or finite complement topology:** Let X be a infinite set. Let T consists of ϕ and all those subsets of X whose complements are finite. Then T is a topology called the cofinite topology.
(8) **Co-countable topology:** Let X be a set; let $T_C = \{A \subset X| \quad X - -A$is either countable or is all of $X\}$.
(9) **Lower limit topology or right half open interval topology or RHO topology on R (R_l):** Let R_l consists of null set and all those subsets A

of R such that for every x in A there exists a right half open interval $[a, b[$, a¡$b$such that $x \in [a, b[\subset A$.

(10) **Upper limit topology:** Same defined as above.

(11) **Left ray topology for R:** For each a in R define $L_a = \{x \in R|x$¡$a\}$ = (-∞, a) = open left ray of real numbers. The point a is called right and point of L_a. Let T consist all possible left rays together with ϕ and R.

(12) $T = \{\phi\} \cup \{R\} \cup \{\ ]a,\infty[\ :a \in R\}$ is a topology for R called the **right ray topology** but $T_1 = \{\phi\} \cup \{R\} \cup \{\ [a,\infty[\ :a \in R\}$ is not a topology for R.

(13) Let T be the collection of subsets of N consisting of empty set and all subsets $G_m = \{m,\ m+ 1,\ m+ 2,\ \dots\}$, $m \in N$. Then T is a topology for N.

(14) **Metrizable Spaces:** A topological space $(X,\ T)$ is said to be metrizable if there is a metric d for X such that $T_d = T$.

(15) **Example:** Give an example of a topological space which is not metrizable.

(16) **Answer:** Let $X = \{a, b\}$, $a \neq b$. Define $T = \{X,\ \phi,\ \{a\}\}$. Then (X,T) is a topological space. Let d be any metric on X and let $d(a, b) = r$. Since $a \neq b$, $r > 0$. Then $S_r(b) = \{b\}$ and hence $\{b\}$ is d open set but $\{b\}$ is not T-open. Hence (X,T) is not metrizable.

EXERCISE – 3

(1) Co-finite topology on a finite set is the same as the discrete topology.

(2) Let $f : X \to Y$ be a function from a nonempty set X into a topological space Y. If T is a topology on Y. Then $\{f^{-1}(G)|G \in T\}$ is a topology on X.

(3) $\overline{A} = \{x \mid$ each neighbourhood of x intersects $A\}$.

(4) A^o, $extA$ and $Fr(A)$ are disjoint and $X = A^o \cup ext(A) \cup Fr(A)$.

(5) A subset A of a topological space X is closed iff it contains its boundary.

(6) A subset of a topological space has empty boundary iff it is both open and closed.

(7) A set A is perfect iff it is closed and has no isolated points.

(8) A closed set is nowhere dense iff its complement is everywhere dense.

(9) The boundary of a closed set is nowhere dense.

(10) For (R, U), B= $\{(a, b)|a, b \in R\}$ is a base for U.

(11) For the discrete space (X, D), B= $\{\{x\}|x \in X\}$ is a base for D.

(12) Indiscrete topology is the weakest topology on a set X and the discrete topology is the strongest topology on X.

(13) For (X, D), $B_x = \{\{x\}\}$ is local base at x in X.

(14) For (R, U) and x in R, $\{(x - -r,\ x + r) \mid r$¿ $0\}$ and $\{(x - - 1/n,\ x+ 1/n) \mid n \in N\}$ are local bases at x.

(15) (R, U), (X, D) are first countable.

(16) (R, U) is second countable. B= $\{(r, s)| \ r, s$ are rational$\}$ is a countable base for U.

(17) (X, D) is not second countable.

(18) Let T and T' be topologies for X which has a common base B. Then $T = T'$.

(19) Every metric space is first countable because $B_x = \{S_{1/n}(x) \mid n \in N\}$ is a countable local base at x.

(20) $f : (R, T) \to (R, V)$ such that $f(x) = 1$ for all x in R is continuous function.

(21) If f is a mapping from a discrete space to any topological space, then f is continuous.
(22) A mapping f from a topological space X to indiscrete space I is always continuous.
(23) If $f : (X, T) \to (Y, V)$ is continuous and $T*$ is finer than T, then f is $T*-V$ continuous.
(24) If $f : (X, T) \to (Y, V)$ is continuous and $V*$ is coarser than V, then f is $T - V*$ continuous.
(25) Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.
(26) If B is a basis for the topology of X then the collection B Y = {B∩Y|B∈B } is a basis for the subspace topology on Y.
(27) Is the collection T∞ = {U | X − U is infinite or empty or all of X} a topology on X?
(28) If X = {a, b, c}, T_1 = {X,ϕ, {a}, {a, b}} and T_2 = { X, ϕ, {a}, {b, c}}. Find the smallest topology containing T_1 and T_2,and the largest topology contained in T_1 and T_2.
(29) Let { T_α} be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collectionsT_α, and a unique largest topology contained in allT_α.
(30) A topological space (X, T) is said to be a **door space** if every subset of X is either open or closed. Give an example of door space.
(31) $\{a\}$ is closed in usual topology for R.
(32) Give an example of a proper nonempty subset of a topological space such that it is both open and closed.
(33) Give an example of a topological space different from the discrete spaces in which open sets are exactly the same as closed sets.
(34) Hint: Let T consists of all those subsets G of R having the property that x in G implies − x in G. Then (R, T) is the required space.
(35) Which of the following subsets of R are U-neighbourhoods of 1? (i)]0, 2], (ii)]0, 2[, (iii) [1, 2], (iv) [0, 2] − {1/2}
(36) Consider the topology $T = \{X, \phi, \{a\}, \{b, c\}\}$ on X and $V = \{Y, \phi, \{r\}, \{p, q\}\}$ on $Y = \{p, q, r\}$. Find which of the mappings defined as follows are

(i) continuous, **(ii)** open, **(iii)** closed, **(iv)** bicontinuous, **(v)** homeomorphism
(**a**) $f(a) = r$, $f(b) = r$, $f(c) = r$, (**b**) $g(a) = p$, $g(b) = q$, $g(c) = p$,
(**c**) $h(a) = r$, $h(b) = p$, $h(c) = q$, (**d**) $I(a) = r$, $I(b) = q$, $I(c) = p$,
(**e**) $j(a) = p$, $j(b) = q$, $j(c) = r$.

EXERCISE – 4 CONNECTED SPACE

(1) Let $\{A_\alpha\}$ be the collection of connected subsets of a space X such that no two members of $\{A_\alpha\}$ are mutually separated, then $\cup_\alpha A_\alpha$ is also connected.
(2) Any two distinct components are mutually disjoint.
(3) Every space is the disjoint union of its components.
(4) Indiscrete space (X, I) is connected.
(5) If (X, D) has more than one point, then (X, D) is connected.
(6) If (X, T) is disconnected and $T \subset T_1$ (i.e., T_1 is finer than T) then (X, T_1) is disconnected.
(7) If (X, T) is connected and T_1 is coarser than T, then (X, T_1) is connected.

(8) In every topological space singleton sets are connected, therefore components are nonempty.
(9) (X, D) is totally disconnected.
(10) (X, D) is locally connected.
(11) Give two examples of locally connected spaces which are not connected.

EXERCISE – 5 SEPERATION AXIOMS

T_o Space

(1) (X, D) is T_o but (X, I) is not T_o.
(2) If T* is finer that T, then T is T_o topology implies T* is To topology.
(3) The property of a space being a T_o space is a topological property.
(4) X is T_o iff for any distinct arbitrary points x, y of X, the closure of {x} and {y} are distinct.

T_1-Space

(1) (R, U) is T_1.
(2) For a topological space X following are equivalent: (1) X is T_1; (2) Every singleton set in X is closed; (3) Every finite set in X is closed; (4) The intersection of all neighbourhoods of an arbitrary point of X is a singleton.
(3) Every metric space is T_1.
(4) Every finite T_1 space is discrete.
(5) A space (X ,T) is T_1 iff T contaons the cofinite topology on X.
(6) Every topology finer than a T_1-topology on a set X is a T_1-topology.
(7) For any set X there exists a unique smallest topology T such that (X, T) is T_1.
(8) Let A be any subset of a T_1 space X. Then x is accumulation point of A iff every open set containing x contains infinitely many distinct points of A.

T_2-Space

(1) Show that (X, D) is T_2.
(2) No indiscrete space consisting of at least two points is Hausdorff.
(3) (R, U), (R, S) are Hausdorff.
(4) Co-finite topology on an infinite set X is not $T_{2.}$
(5) Co-countable topology on an uncountable set is not T_2.
(6) Every metric space is T_2.
(7) T* is finer than a T_2-topology T, then T* is T_2.
(8) Every singleton set in a T_2 space is closed.
(9) Every finite hausdorff space is discrete.
(10) Every T_2-space is T_1 but not converse.
(11) X is topological space, Y is a hausdorff space. If $f$and g are continuous functions of X into Y. Then the set $\{x \in X| \quad f(x) \quad = g(x)\}$ is closed.
(12) The property of being a T_2 space is a topological property.
(13) Let (X, T) be a topological space and (Y, V) be a hausdorff space. Let f: X→Y be a one-one continuous mapping, then (X, T) is also a hausdorff.
(14) Let (X, T) be a T_2 space and f be a continuous map of X into itself. Then the set A = {x∈X| f(x) = x} is T-closed.

Regular Space, T_3-space

(1) X = {a, b, c}, T = {X, ϕ, {a}, {b, c}}. Then (X, T) is regular but not T_3.
(2) (R, U) is T_3.
(3) Every T_3 space is T_2 but not converse.

(4) A topological space X is regular iff every point x in X and every nbd N of x, there exists nbd M of x such that $\overline{M}$ ⊃N. In other words, a topological space is regular iff the collection of all closed nbds of x forms a local base at x.

(5) The property of being a regular space is a topological property.

(6) Every compact hausdorff space is a T_3-space.

(7) Every metric space is regular (T_3).

(8) Let A be any compacet subset of a regular space X. If G is an open set containing A, then there exists a closed set H such that A⊂H⊂G.

Normal Space, T_4 space

(1) X = {a, b, c}, T = {X, ϕ, {a}, {b, c}}. Then X is normal but not T_2.

(2) X is normal iff for any closed set F and open set G containing F, there exists an open set V such that F⊂V and $\overline{V}$ ⊃G.

(3) Normality is a topological property.

(4) (R, U) is T_4.

(5) Every T_4 space is T_3.

(6) X = {a, b, c}, T = {X,ϕ, {a}, {b}, {a, b}}. Then X is normal but not regular.

(7) Every compact regular space is normal.

(8) The property of being a T_4 space is a topological property.

(9) Closed subspace of a normal space is normal

(10) Every metric space is normal.