*“What is a great man does, is follows by others; people go by the example he sets”*

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Definition and some examples, relative topology, weak topology, open basis, open sub basis, continuity and homeomorphism, compact spaces, product spaces, compactness in a metric space.

Separation axioms (T1, T2, T3, Housdorff spaces), connected spaces, components, totally disconnected space, locally connected space.

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**Remark:**  *= X*, *=*.

**Topology:** Let *X* be a nonempty set. A collection *T* of subsets of *X* is called a topology on *X* if

1. *X* *T*,
2. *T*,
3. *A* *T*, *B**T* *A**B**T*,

(Intersection of any two members of *T* is also a member of *T* ),

1. If {*A*|} is a collection of members of *T*, then*A**T*

(Union of any collection of members of *T* is a member of *T* ).

**Remark:** If *T* is a topology on *X*, then (*X*, *T* ) is called a topological space. Members of *T* are called *T*-*open* *sets* or simply *open sets*.

**Examples**

1. *X* = {1, 2, 3}, collection of all subsets of *X = P* (*X*) = {, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}. Then *T*1 = {, {1}, {2}}, *T*2 = {{1, 2, 3}, , {1, 2}, {2, 3}} and *T*3 = {{1, 2, 3}, , {1}, {2}} are not topologies on *X*.
2. *Indiscrete topology* *I =* {X, }
3. *Discrete topology* *D =* collection of all subsets of *X*.
4. *Cofinite topology:* Let *X* be a infinite set. Let *T* consists of  and all those subsets of *X* whose complements are finite. Then *T* is a topology called the cofinite topology.

**Remarks**

1. Intersection of any family of topologies on *X* is a topology on *X*.
2. Union of two topologies on *X* is not necessarily a topology on *X*.

**Closed Set:** Let (*X*, *T*) be a topological space and *A* be a subset of *X*. Then *A* is said to be *T* – closed or simply closed if *X* – *A* is *T*- open.

**Example:** *X =* {1, 2, 3}, *T =* {{1, 2, 3}, , {1}, { 2, 3}. {1} is open as well as closed. {1, 2} is neither open nor closed.

**Properties of Closed Sets:** **(i)** *X* is closed, **(ii)**  is closed, **(iii)** Union of any two closed sets is closed, **(iv)** Arbitrary intersection of closed sets is closed.

**Neighbourhood:** Let (*X*, *T*) be a topological space. *N* be a subset of *X* and, *a* be an element of *X*, *N* is called a *T* – neighbourhood of ‘*a*’ if there exists a *T*-open set *G* such that *a**G**N*.

**Example:** *X =* {1, 2, 3, 4}, *T =* {*X*, , {2}, {1, 2}, {2, 3}, {1, 2, 3}}, *N =* {1, 2, 3}. *N* is *nbd* of 1, *N* is not a *nbd* of 4. *N* is *nbd* of 2.

**Remarks**

1. Set of all neighbourhoods of *x* is denoted by *N*x.
2. *N* *N*x, *M* *N* *M* *N*x.
3. *N* *N*x, *M**N*x*N* *M* *N*x, *N* *M* *N*x.

**Theorem:** A set *G* is open if and only if *G* is a neighbourhood of each of its points.

**Limit Point or Accumulation Point:** Let (*X, T*) be a topological space and *A* be a subset of *X*. Let *p* be a point of *X*. *p* is called a limit point of *A* if every open set *G* containing *p* contains at least one point of *A* other than *p*.

Or

If for every open set *G* containing *p*, (*GA*) – {*p*}, or (*G –* {*p*})*A*.

**Remark**

1. Collection of all limit points of the set *A* is called the derived set of *A* and is denoted by *A*’ or *d*(*A*) or *D*(*A*).
2. *p* is said to be an isolated point of *A* if *p**A* but *p* is not a limit point of *A*.
3. *A* is said to be isolated set if each point of A is an isolated point of *A*.

**Example:** *X =* {1, 2, 3, 4}, *T =* {*X*, , {1}, {1, 2}, {1, 3}, {1, 2, 3}}, *A =* {2, 3, 4}. 4*D* (*A*), 1*D* (*A*), 2 *D* (*A*). 2 and 3 are isolated points of *A*. 1 is neither a point of *A* nor a limit point of *A*. Thus *D* (*A*) = {4}.

**Theorem:** **(1)** *A**B**D* (*A*)*D* (*B*), **(2)** *D* (*A**B*)*D* (*A*)*D* (*B*), **(3)** *D* (*A**B*) *= D* (*A*)*D*(*B*).

**Theorem:** *A* subset *F* of a topological space *X* is closed iff *F* contains all its limit points, i.e., *D* (*F* )*F*.

**Adherent point:** A point *p* is called an adherent point of *A* if every open set containing *p* contains at least one point of *A*. Set of all adherent points of *A* is called adherence of *A* and is denoted by *adh*(*A*).

**Remark:**

1. *adh* (*A*) *= A* *D*(*A*).
2. Every point of *A* is an adherent point of *A*.
3. If *p**A* and *p* is an adherent point of *A* then *p* is also a limit point of *A*.

**Example:** *X =* {1, 2, 3}, *T =* {*X*, , {1}, {2}, {1, 2}}, *A =* {2, 3}. 2 is not a limit point of *A* but 2 is an adherent point of *A*.

**Closure:** Let (*X, T*) be a topological space and *A* be a subset of *X*. Closure of *A* is the intersection of all closed supersets of *A* and is denoted by  or *C* (*A*).

**Theorem**

1.  is the smallest closed superset of *A*.
2. AB.
3. A set *A* is closed iff *A =* .
4. .
5. .
6.  = *A**D*(*A*) *= adh A*.

**Interior:** Let (*X*, *T*) be a topological space and *A* be a subset of *X*. Let *a* be any point of *X*. *a* is called an interior point of *A* if *A* is *nbd* of *a*. Collection of all interior points of *A* is called the interior of *A* and is denoted by *A*o or *i* (*A*).

**Theorem**

1. *A*o = { *G*|*G* is open and *GA*}.
2. *A*o is the largest open subset of *A*.
3. *A* is open iff *A*o = *A*.
4. *A**B* *A*o *Bo*.
5. (*A**B*)o = *A*o *B o*.
6. *Ao* *Bo* (*A**B*)o .

**Exterior:** Interior of complement of *A* is called exterior of *A* and is denoted by *e*(*A*) or *ext*(*A*), i.e., *ext*(*A*) = (*X* – *A*)o

**Theorem: 1.** (*X* – *A*)o *= X* - . **2.**  = X - Ao. **3.**  = X – (X – A)o.

**Boundary of A or Frontier of A:** Let (*X, T*) be a topological space, and *A* be a subset of *X*. Boundary of *A* written as *b*(*A*) is the set of those points of *X* which are neither interior point of *A* nor interior point of *X* – *A*. Therefore *b* (*A*) *= X* – (*A*o(*X* – *A*)o) *= X* – (*A*o*ext*(*A*)) *=* (*X* – *A*o)(*X* – (*X* – *A*)o) = .

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**Theorem:** Let *R* be the set of real numbers. Let *U* consists of null set and those subsets *A* of *R* such that for every *x A* there exists a real number *p* *> 0* such that ] *x – p x + p* [ *A*. Then *U* is a topology for *R* called the *usual topology*.

**Remarks**

1. Every open interval is an open set.
2. Every closed interval is a closed set.
3. In the usual topology no nonempty finite set is open.
4. In (*R, U*), if *p* is a limit point of *A* then every open set containing *p* contains infinite points of *A*.
5. If *p* is a limit point of *A* then every open interval containing *p* contains infinite points of *A*.
6. In usual topology no finite set can have a limit point.
7. Every finite set is closed in usual topology.

**Definition:**

1. *A* is dense – in – itself if *AA’*.
2. *A* is closed if *A’A*.
3. *A* is perfect if *A = A’*

**Example**

1. Every open interval is dense – in – itself. But no open interval is a closed set.
2. Every closed interval is a perfect set.

**Exercise**

1. Give an example to illustrate that intersection of members of a topology may not be a member of topology.
2. Give an example to show that union of closed sets may not be a closed set.
3. Give an example in which.

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**Subspace:** A topological space (*Y*, *T*Y) is called a subspace of (*X, T*) if *YX* and for every *H**T*Y there exists *G* in *T* such that *H = Y**G*.

**Example:** *X =* {1, 2, 3, 4}, *T =* {*X*, , {1, 2}, {2, 3}, {2}, {1, 2, 3}}, *Y =* {2, 3, 4}, *T*Y = {{2, 3, 4}, , {2}, {2, 3}}.

**Remark:** *T*Y is called *relative topology on Y* or *topology relativized by T on Y* and (*Y, T*Y) is called a subspace of (*X, T*).

**Theorem:** Let (*X, T*) be a topological space and *Y* be a subspace of *X*. Let *T*Y = {*Y**G*|*G**T*}. Then *T*Y is a topology for *Y*. (*Y, T*Y) is called subspace of (*X, T*).

**Theorem:** Let (*Y, V*) be a subspace of (*X, T*). Let *F* be a subset of *Y*. Then *F* is *V*-closed iff *F = Y**H* for some *T*-closed set *H*.

**Theorem:** If (*Z, W*) is a subspace of (*Y, V*) and (*Y, V*) is a subspace of (*X, T*). Then (*Z, W*) is a subspace of (*X, T*).

**Theorem:** Let (*X*\*, *T*\*) be a subspace of (*X, T*). Let *A* be a subset of *X*\*. Let *yX*\*. Then *y* is a *T*\*-limit point of *A* iff *y* is a *T*-limit point of *A*.

**Lemma:** *y* is a *T*\*-adherent point of *A* iff *y* is a *T*-adherent point of *A*.

**Theorem:** Let (*X*\*, *T*\*) be a subspace of (*X, T*). Let *A* be a subset of *X*\*. Then *T*\*-closure of *A = X*\*(*T*-closure of *A*), or, *C*\*(*A*) *= X*\**C*(*A*).

**Weak topology:** Let (*X*, *T*1) and (*X*, *T*2) be two topological spaces. If *T*1*T*2, then *T*1 is weaker than *T*2, or *T*2 is stronger than *T*1, or *T*1 is coarser than *T*2, or *T*2 is finer than *T*1, or *T*1 is smaller than *T*2, or *T*2 is larger than *T*1. If *T*1*T*2 and *T*2*T*1, we say that *T*1 and *T*2 are not comparable.

**Example:** *X =* {1, 2, 3}, *T*1 = {*X*, , {1}}, *T*2 = {*X*, , {2}}, *T*3 = {*X*, , {1}, {2}, {1, 2}}. *T*1 and *T*2 are not comparable, *T*1*T*3, *T*2*T*3.

**Complete lattice:** A complete lattice is a partially ordered set in which every nonempty subset has a least upper bound and greatest lower bound.

**Theorem:** The set of all topologies on a set *X* is a ‘complete lattice’ with respect to ‘inclusion’ relation.

**Base:** Let (*X, T*) be a topological space. A collection B of subsets of *X* is called a base for *T* if (i) B *T*, (ii) for every *G* in *T* and *x* in *G*, there exists *B* in B such that *xBG*.

**Example:** *X =* {*a, b, c, d*}, *T =* {*X*, , {*a*}, {*b*}, {*a, b*}, {*c, d*}, {*a, c, d*}, {*b, c, d*}}, B = {{*a*}, {*b*}, {*a, b*}, {*c, d*}}. For *G =* {*a, c, d*}, *c*{*c, d*}*G*.

**Theorem:** Let (*X, T*) be a topological space. A collection B of members of *T* is a base iff each member of *T* is expressible as a union of some subfamilies of B*.*

**Theorem:** Let (*X, T*) be a topological space and B be a base for *T*. Then *T* is identical with the collection of unions of all subfamilies of B.

**Example:** X = {1, 2, 3}, T = {X, , {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}}, B = {{1}, {2}, {3}}. Family of unions of subfamilies of B = T, therefore B is a base for T

**Example:** Is B = {{0, 1}, {1, 2}} a base for some topology on X = {1, 2, 3}?

**Solution:** If B is a base for a topology T on X, then T must be the collection of unions of all subfamilies of B, i.e., T = {, {0, 1}, {1, 2}, {0, 1, 2}}. But T is not a topology for X and hence B is not a base for any topology on X.

**Theorem:** Let *X* be a nonempty set. A collection B of subsets of *X* is a base for some topology *T* on *X* if and only if (i) {*B*|*B* B } = *X*, (ii) For *U, V* B and *xU V* there exists *W* in B such that *xW U V*.

Or

What is the necessary and sufficient condition for a family to become a base for a topology.

**Theorem:** Let B be a collection of subsets of X such that (i) {*B*|*B* B } *= X*, (ii) *U, V* B  *UV* B. Then B is a base for some topology on *X*.

**Sub-base:** *X =* {0, 1, 2}, *S* = {{0, 1}, {1, 2}} , B = {X, {{0, 1}, {1, 2}, {1}}, *T =* {, X , {1}, {0, 1}, {1, 2}}.

**Definition:** Let *X* be any nonempty set and *S* be a collection of subsets of *X*. Let B be the collection of intersections of all finite subfamilies of S. Then {B|B B } = *X* and intersection of any two members of B is a member of B. Hence B is a base for some topology on *X*, say *T*. *S* is called a sub-base for *T* and *T* is called the topology generated by *S*.

**Remark:** If *S* is a sub-base for topology *T*. Then *T* is the smallest topology containing *S*.

**Continuity**

If *X =* {1, 2, 3, 4, 5, 6, 7}, *Y =* {*a, b, c, d, p, q, r, s, t*}, *f : X**Y* such that *f* (1) *= a*, *f* (2) *= a*, *f* (3) *= a*, *f* (4) *= d*, *f* (5) *= d*, *f* (6) *= p*, *f* (7) *= q*, *A =* {2, 3, 4, 6}, *f* (*A*) = {*a, d, p*}, *f*-1(*f* (*A*)) *= f* -1{*a, d, p*} = {1, 2, 3, 4, 5, 6}*A*. *A**f*-1(*f* (*A*)) and *A =* *f*-1(*f* (*A*)) if *f* is one-one. Similarly *f* (*f*-1(*E*)) *= E* if *f* is onto.

**Continuous function:** Let (*X, T*) and (*Y, V*) be two topological spaces and *f* be a function from *X* to *Y*. Let *a**X*. *f* is said to be *T-V continuous at* *a* if for every *V-*open set *H* containing *f* (*a*) there exists at least one *T-*open set *G* containing *a* such that *f* (*G*)*H. f* is said to be a *T-V continuous* if *f* is continuous at each point of *X*.

**Theorem:** Let (*X, T* ) and (*Y, V* ) be two topological spaces and *f* be a function from *X* to *Y*. Then *f* is *T-V* continuous iff inverse image of each *V-*open set under *f* is *T-*open.

**Theorem:** Let (*X, T* ) and (*Y, V* ) be two topological spaces and *f* be a function from *X* to *Y*. Then *f* is *T-V* continuous iff for every *V-* closed set *E*, *f -*1(E) is *T-* closed.

**Theorem:** Let (*X, T* ) and (*Y, V* ) be two topological spaces and *f* be a function from *X* to *Y*. Then *f* is *T-V* continuous iff *f* ()  for every *E X*.

**Open and Closed functions:** Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* be a function from *X* to *Y*.

1. *f* is called an *open function* if for every *T*-open set *G*, *f* (*G*) is *V-* open.
2. *f* is called a *closed function* if for every *T*-closed set *H*, *f* (*H*) is *V*- closed.

**Example:** Give an example of a function which is continuous, open but not closed.

**Solution:** *X =* {*a, b, c*}, *T =* {*X*, , {*a*}, {*b, c*}}, *Y =* {1, 2, 3}, *V =* {*Y*, , {1}}. *f*:*X**Y* such that *f*(*a*) *= f*(*b*) *= f*(*c*) = 1. Then *f* is the required function.

**Example:** Give an example of a function which is continuous and closed but not open.

**Theorem:** Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* be a function from *X* to *Y*. Then *f* is *T-V* closed mapping iff *f*() for every *E**X*.

**Theorem:** Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* be a function from *X* to *Y*. Then *f* is an open function iff *f* (*E*o)(*f* (*E*))o for every *E* *X*.

**Homeomorphism:** Let (*X, T* ) and (*Y, V* ) be two topological spaces and *f* be a function from *X* to *Y*. *f* is called a homeomorphism if (i) *f* is one-one and onto, (ii) *f* is *T-V* continuous, (iii) *f*-1 is *V-T* continuous.

(*X, T*) and (*Y, V*) are said to be homeomorphic spaces if there exists a homeomorphism between *X* and *Y*.

**Bi-continuous:** *f* is called *bi-continuous* if *f* is continuous and open.

**Theorem:** Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* is a one-one function from *X* onto *Y*. Then the following conditions are equivalent.

**(i)** *f* is homeomorphism, **(ii)** *f* is continuous and open, **(iii)** *f* is continuous and closed.

**Theorem:** Homeomorphism is an equivalence relation in the set of all topological spaces.

**Theorem:** Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* be a 1-1 function from *X* onto *Y*. Then *f* is is a homeomorphism iff  = *f*() for every *E**X*.

**Theorem:** Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* be a 1-1 function from *X* onto *Y*. Then *f* is a homeomorphism iff *f* (*A*o) = (*f* (*A*))ofor every *A* *X*.

**Theorem:** Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* be a function from *X* to *Y*. Then *f* is continuous iff *f*-1() for every *E**Y*.

**Theorem:** Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* be a function from *X* to *Y*. Then *f* is continuous iff *f*-1(*E*o)(*f*-1(*E*))o for every *EY*.

**Theorem:** Let (*X, T* ) and (*Y, V*) be two topological spaces and B be a base for *V*. Let *f* be a function from *X* to *Y*. Then *f* is continuous iff inverse image of each member of B under *f* is in *T*.

**Theorem:** Let (*X, T* ) and (*Y, V*) be two topological spaces and S be a base for *V*. Let *f* be a function from *X* to *Y*. Then *f* is continuous iff inverse image of each member of S under *f* is in *T*, i. e., *f*-1(*S*)*T* for every *S*S.

**Open Cover:** Let (*X, T*) be a topological space and *A* be a subset of *X*. A collection {*G*| } of T-open sets is called a *T*-*open cover* of *A* if *A*.

**Compact Space:** Let (*X, T*) be a topological space and *A* be a subset of *X*. A is said to be *T*-*compact* set if every *T*-open cover of *A* has a finite subscover.

X is called *compact space* if every *T*-open cover of *X* has a finite subcover.

**Example:** *A =* {1, 2, 3, 4, 5, 6}, *G*1 = {1, 2, 3}, *G*2 = {2, 3, 4}, *G*3 = {3, 4, 5}, *G*4 = {4, 5, 6}, *G*5 = {5, 6, 7}. Then *A**G*1 *G*2…

**Example:** Let *C*1 = { ] - *n*, *n* [ |*n**N*}, *C*2 = { ] -3*n*, 3*n*[ | *nN*}, *C*3 = { ] 2*n* – 1, 2*n* *+* 1[, ]2*n* , 2*n* + 2[ | *n**N*}. Then *C*1, *C*2, *C*3 are all *U*-open covers of *R* and *C*2 is a subcover of *C*1.

**Theorem:** Let (*Y*, *T*Y) be a subspace of (*X*, *T*), and *A* be a subset of *Y*. Then *A* is *T*Y-compact if and only if *A* is *T*-compact, or, Compactness is an absolute property of the set.

**Theorem:** Every closed set in a compact space is compact.

**Finite Intersection property (FIP):** A family of sets is said to have *finite intersection property* if for every finite subfamily has a nonempty intersection.

**Example:** *A*1 = {1, 2, 3, …}, *A*2 = {2, 3, 4, …}, …. Consider {*A*1, *A*2, *A*3, …}. Then *A*2*A*7*A*123 *=* *A*123 and *A*1*A*2… = 

**Theorem:** A topological space *X* is compact iff every family of *T*-closed sets having *FIP* have a nonempty intersection.

**Example:** Consider the following class of open intervals *A =* {(0, 1), (0, 1/2), (0, 1/3), …}. Now A has the FIP because (0, *a*1)(0, *a*2) …(0, *a*n) = (0, *b*) , where *b = min*(*a*1, *a*2, …, *am*) > 0. Observe that *A* itself has an empty intersection.

**Topological Property:** A property *P* possessed by a topological space *X* is called a *topological property* if *P* is possessed by every homeomorphic image of *X*.

**Theorem:** Continuous image of a compact space is compact, or, Compactness is a topological property, Or Compactness is preserved under continuity.

**Theorem:** (*R, U* ) is not compact.

**Theorem:** Every closed interval is a compact set.

**Heine - Borel Theorem:** A subset *A* of *R* is compact iff it is closed and bounded.

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**Metric Spaces:** Let X be a nonempty set. A mapping d: XXR is called a metric on X if d satisfies the following conditions:

1. d(x, y) 0;
2. d(x,y) = 0 x = y;
3. d(x, y) = d(y, x); Symmetric property
4. d(x, y) d(x, z) + d(z, y) for all x, y, zX. Triangle inequality

If d is a metric on X, then (X, d) is called the metric space.

**Examples**

1. *d* (*x, y*) = |*x – y*| is a metric on *R*.
2. *d* (*z*1, *z*2) = |*z*1 – *z*2| is a metric on *C*.
3. *d:* *R*n*R*n : *d* (*x, y*) = is a metric on *R*n
4. Let X be a-nonempty set. Then the mapping d: XXR defined by d(x, y) = 0, if x = y, and d(x, y) = 1 for xy is a metric for X called the *discrete metric* for X.

**Examples**

**1.** Show that |*d* (*x, z*) – *d* (*y, z*)|*d* (*x, y*).

**2.** Prove that |d(x, y) – d(x1, y1) d(x, x1) + d(y, y1).

**Open Sphere, opern ball, open neighbourhood** (centered on x and radius *r* )is S*r* (*x*) = S(x, r) = B(x, r) = Nr(x) = { y*X* | *d* (*y, x*) *< r* }.

**Remark:** x belongs to Sr(x), i.e., ball is always nonempty.

**Example:** For a discrete metric space X = {1, 2, 3, 4, 5, 6, 7, 8}, S3/4(3) = {3}, S13(3) = {1, 2, 3, 4, 5, 6, 7, 8}, S1010(3) = {1, 2, 3, 4, 5, 6, 7, 8}.

**Closed Sphere:** (centered on x and radius *r* )is *S*r[*x*] = { y*X* | *d* (*y, x*) *r* }.

**Example:** Find the open sphere in R and C.

**Open Sets:** A subset *A* of a metric space *X* is said to be open if for all xX there exists a real number *r* *> 0* such that *S*r(*x*) *A.*

**Closed Sets:** A subset *A* of *X* is closed if its complement is open.

**Remarks:**

1. Every open sphere is an open set and every closed sphere is a closed set.
2. Empty set and the full space X are open.
3. Arbitrary union of a family of open sets is open.
4. Finite intersection of a family of open sets is open.

**Sequence in a metric space:** Let X be a non-empty set. A mapping f: NX is called a sequence in X.

**Subsequence:** A sequence  is a subsequence of <xn> if (i) *n*k *k*, (ii) *n*1 *< n*2 *< …*

**Convergent sequence:** A sequence {*x*n} in *X* is said to be convergent if there exists *x* in *X* such that for every *>0* there exists a positive integer *n*0 such that *nn0* *d*(*x*n,*x*)<.

**Remarks:**

1. Everyconvergent sequence in a metric space X has a unique limit.
2. If *<x*n*>* is an infinite sequence then its limit is also its limit point.
3. Let (X, d) be a metric space and A be a subset of X and x be a limit point of A then there is an infinite sequence <xn> in A such that xnx.

**Cauchy Sequence:** A sequence {*x*n} in *X* is said to be a Cauchy sequence in X if for every *>0* there exists a positive integer *n*0 such that *m*, *nn0* *d*(*x*m, *x*n)<.

**Theorem:** every convergent sequence in a metric space X is a cauchy sequence but the converse is not true.

**Complete metric space:** A metric space X is called complete if every Cauchy sequence in X is convergent in X.

**Remarks**

1. If a sequence <xn> converges to x then every subsequence of <xn> converges to x.
2. A non convergent sequence may have convergent subsequence.
3. If <xn> is a Cauchy sequence and  is a subsequence of <xn> such that a then xna.

**Topology induced by a metric:** Let (X, d) be a metric space and T be the collection of d-open sets. Then T is a topology on X induced by the metric d.

A topological space (X, T) is called a metrizable topological space if there exists a metric d on X such that the collection of d-open sets is same as T.

**Theorem:** Let (X, d) be any metric space. Let Td consists of null set and all those subsets G of X such that for every x in G there is a real number r > 0 such that Sr(x)X. Then Td is a topology for X induced by d.

**Example:** (R, U) is a metrizable topological space.

**Remarks:** In a metric space X

1. If x is a limit point of A then every open set containing x contains infinite points of A.
2. A finite set has no limit points.
3. Every finite set is closed.

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**Bolzano Weierstrass Property (BWP):** A metric space *X* is said to have *BWP* if every infinite subset of *X* has at least one limit point in *X*.

**Sequentially Compact Metric Space:** A metric space *X* is said to be a *sequentially compact* if every sequence in *X* has a convergent subsequence.

**Theorem:** Every compact space has *BWP*.

**Theorem:** Every metric space having *BWP* is sequentially compact.

**Theorem:** A metric space *X* is sequentially compact iff *X* has *BWP*.

**- net:** A finite set of points, say {*a*1, *a*2, …, *an*} in a metric space *X* is called an - *net* if *X =* .

**Totally Bounded or Precompact:** A metric space *X* is said to be *totally bounded* if there exists an - *net* in *X* for every > 0.

**Theorem:** Every sequentially compact metric space is totally bounded.

**Diameter:** Diameter of *A =* *diam* *A = d* (*A*) =  *= sup* {*d* (*x, y*) | *x, y**A*}.

**Remark:** (*S*r(*a*)2*r*.

**Lebesgue Number:** Let {*G*|} be an open cover of a metric space *X*. A positive real number *r* is called a *lebesgue number* of {*G*|} if for every subset *A* of *X* such that *diam A < r* there is a  such that *A* .

**Remark:** If *r* is a lebesgue number and 0 *< r1 < r* then *r*1 is also a lebesgue number.

**Big Set:** If *B* is a subset of *X* which is not contained in any of’s then *B* is called a ‘*Big Set*’ and *B* contains at least two elements and *diam B* > 0.

**Lebesgue Covering Lemma:** In a sequentially compact metric space every open cover has a lebesgue number.

**Theorem:** Every sequentially compact metric space is compact.

**Uniformly Continuous Function:** Let (*X*1, *d*1) and (*X*2, *d*2) be two metric spaces and *f* be a function from *X*1 to *X*2. *f* is said to be *uniformly continuous* if for every > 0 there exists > 0 such that *d*1(*a*, *b*) < *d*2(*f*(*a*), *f*(*b*)) <.

**Theorem:** Every continuous function defined on a compact metric space is uniformly continuous.

**Theorem:** A metric space *X* is compact iff *X* is complete and totally bounded.

**Theorem:** Let (*X, d* ) be a complete metric space and *Y* be a closed subspace of *X*. Then *Y* is compact iff *Y* is totally bounded.

**Separable Space:** *X* is *separable* if *X* has a countable dense subset, i.e., there is a countable subset *A* of *X* such that = *X*.

**Theorem:** Every compact metric space is separable, Or, Every totally bounded metric space is separable.

**Second Countable Space:** *A* space *X* is second countable space (or second axiom space) if *X* has a countable base.

**Remark:** A space X is said to satisfy second axiom of countability if X has a countable base.

**Theorem:** Every second axiom (or second countable) space is separable.

**Theorem:** Every separable metric space is a second axiom space.

**Theorem:** A topological space X is compact iff every basic open cover of *X* has a finite subcover.

**Local Bases:** Let *p* be any arbitrary point of a topological space *X*. A class Bp of open sets containing *p* is called a local base at *p* iff for each open set *G* containing *p* there exists *G*p in Bp such that *p**G*p*G*.

**Example:** *X =* {1, 2, 3, 4, 5}, *T*1 = { *X*, , {1} , {2, 3}, {1, 2, 3}}, *B*1 = {{1, 2}, *X*}, *B*2 = {{2, 3}, *X*}, *B*3 = {{1, 2, 3}}. Find whether *B*1, *B*2, *B*3 form local base at 1, 2, 3 respectively or not?

**Answer:** *B*1 is not a local base at 1, *B*2 is a local base at 2, and *B*3 is not a local base at 3.

**First Countable Space:** A topological space *X* is said to be first countable space (or first axiom space) if there exists a countable base at every point *p* of *X*.

**Theorem:** Every second countable space is first countable space.

**Theorem:** Let *A* be any subset of a second countable space *X*. If *G* is an open cover of *A*, then *G* is reducible to a countable cover.

Or

Let *X* be a second countable space. If a nonempty open set *G* in *X* is represented as the union of a class {*Gi*} of open sets, then *G* can be represented as a countable union of *Gi*’s.

**Theorem:** Let *G* be a base for a second countable space *X*. Then *G* is reducible to a countable base for *X*.

Or

Let X be a second countable space. Then any open base for *X* has a countable subclass which is also an open base.

**Lindelof Space:** A space *X* is said to be a Lindelof space if every open cover of *X* has a countable subcover.

**Lindelof’s Theorem:** Every second countable space is a Lindelof space.

**SEPARATION AXIOMS**

**To-space:** A space (*X, T*) is called a *To-space* if for any two distinct points *x* and *y* there exists an open set *G* containing one of *x* and *y* but not the other.

**T1 – Space:** A topological space is said to be a *T*1 *space* if given any two distinct points *x* an *y* of *X* there exists open sets *G* and *H* such that *x**G*, *y**G*, and *y**H*, *x**H*.

**T2- space or Housdorff space:** *X* is said to be a *T*2 *space* or *Housdorff* if given any two distinct points *x* an y of X there exists open sets *G* and *H* such that *x**G*, *y**H* and *G**H* = .

**Regular Space:** A space *X* is called a *regular* space if given any closed set *E* and a point *a**E* there exist open sets *G* and *H* such that *E**G*, *a**H* and *G**H =* .

**T3-space:** A regular *T*1- *space* is called a *T*3 - *space*.

**Normal Space:** A space *X* is *normal* if given any two disjoint closed sets *E* and *F* in *X* there exists open sets *G* and *H* such that *E**G*, *F**H* and *G**H =* .

**T4-space:** A normal *T*1-space is called a *T*4-*space*.

**Completely Regular Space:** A topological space *X* is said to be completely regular if for every closed subset *F* of *X* and every point *x* *X* – *F* (*x**F*) there exists a continuous function *f* : *X* [0, 1] such that *f* (*x*) = 0 and *f* (*F*) = 1.

**Tychonoff Space or - Space:** A completely regular *T*1- space is called a *Tychonoff space*.

**Remark:** Every *T*2 – space is *T*1 but the converse is not necessarily true.

**Theorem:** (*X, T* ) is *T*1 iff each singleton set in *X* is closed.

**Convergent sequence:** A sequence <*xn*> in a topological space (*X, T*) is said to be convergent to *a* in *X* if given any open set *G* containing a there exists a positive integer *m* such that *n**m**x*n*G*.

**Example:** *X =* {*a, b, c*}, *T =* {*X*, , {*c*}, {*a, b*}}. Then the sequence <*a, b, a, b, a, b, …*> converges to *a* and *b*. <*a, b, a*, *b, c, c, c,* …> converges to *c*.

**Theorem:** In a hausdorff topological space every convergent sequence has a unique limit.

**Theorem:** If (X, T) is a hausdorff space and E is a compact subset of X and a is a point of *X* such that aE then there are open sets G and H such that EG, aH, GH = .

**Or**

In a Hausdorff space any point and disjoint compact subspace can be separated by open sets, in the sense that they have disjoint neighbourhoods.

**Theorem:** Every compact set in a hausdorff space is closed.

**Theorem:** If (*X, T*) is a compact space and (*Y, V*) is a hausdorff space then every one-to-one continuous function *f* from *X* onto *Y* is a homeomorphism.

**Theorem:** Every compact hausdorff space is normal.

**Hereditary property:** A property *P* possesses by a topological space is called a hereditary property if *P* is possessed by every subspace of *X*.

**Theorem:**

1. The property of being a *T*o-space is hereditary, i.e., every subspace of a *T*o-space is also a *T*o- space
2. The property of being a *T*1-space is hereditary.
3. The property of being a *T*2-space is hereditary.
4. The property of being a regular space is hereditary.
5. A closed subspace of a normal space is normal.

**Theorem:** *T*4*T*3*T*2*T*1*T*o.

**Examples:**

1. *X =* {1, 2, 3}, *T =* {*X*, , {1}, {1, 2}, {2}}. Then (*X, T*) is *T*1.
2. Every metric space is a *T*1, and Housdorff space.
3. Every finite subset of a *T*1-space is a closed set.
4. If (*X, T*) is a finite *T*1-space, then *T* is the discrete topology.
5. Every discrete space is *T*2.
6. Every finite *T*2-space is discrete.
7. Every finite subset of a *T*2-space is closed.
8. If *f* : *X**Y* is continuous then *xn**x* in X implies *f* (*xn*)*f* (*x*).
9. (*R, U*) is *T*1, *T*2, *T*3, and *T*4.
10. Let *X* be an infinite set and let *T* be co-finite topology on *X*, then (*X, T*) is *T*1 but not *T*2.

**Connectedness**

**Separated sets:** Let (*X, T*) be a topological space and *A, B* be subsets of *X*, then *A* and *B* are called separated if *A* = *B* =.

**Or**

Two subsets *A* and *B* in a topological space *X* are separated if.

**Remark:** Separated sets are always disjoint but disjoint sets may not be separated.

**Example:** If *A =*] 0, 1], *B* =] 1, 2], *A**B* =, then *A* and *B* are disjoint sets. = [0, 1],  = [1, 2], *A* = {1} . *A* and *B* are not separated.

**Separation of two sets:** Let *E* be a subset of *X*. Two subsets *A* and *B* of *X* are said to form a partition or separation of *E* if *E = A**B*, *A* = *B* =, *A*, *B*. In this case we write *E = A*|*B*.

**Disconnected space:** (*X, T*) is disconnected if there exists two nonempty sets *A* and *B* such that X = *A**B*, *A* = *B =* .

**Connected space:** A topological space is said to be connected if it is not disconnected.

**Theorem:** Let (*X, T*) be a topological space. Then the following conditions are equivalent:

1. *X* is disconnected
2. There exist two nonempty closed sets *A* and *B* such that *X = A**B*, *A**B* =.
3. There exists a nonempty proper subset of *X* which is both open and closed in *X*.
4. There exist two nonempty open sets *A* and *B* such that *X = A**B*, *A**B =*.

**Example:** *X =* {1, 2, 3}, *T =* {*X*, , {1}, {2, 3}}. Then *X* is disconnected, because *X =* {1}{2, 3}, {1}{2, 3} =.

**Theorem:** Continuous image of a connected space is connected.

Or

Connectedness is a topological property.

**Interval:** A subset *X* of *R* is called an interval if *x, z**X*, *x < y < z* *y**X*.

**Theorem:** A subspace *X* of *R* is an interval iff *X* is connected.

**Remark:** Real line is connected.

**Theorem:** Range of every continuous real valued function defined on a connected space *X* is an interval.

**Theorem:** A topological space *X* is disconnected iff there exists a continuous function from *X* onto discrete space on {0, 1}.

**Disconnected set:** Let (*X, T* ) be a topological space and *E**X*. *E* is said to be *T*-disconnected if there exist two nonempty set *A, B* in *X* such that *E = A**B*, *A* = *B =* .

**Theorem:** Connectedness is an absolute property of the set.

**Or**

If (*Y, TY* ) is a subspace of (*X, T* ), and *E**Y*, then *E* is *TY -* connected iff *E* is *T -* connected.

**Theorem: (1)** If *A* and *B* forms a separation of a space *X* and *E* is a connected subset of *X* then either *E**A* or *E**B*. **(2)** Let *E* be any connected subset of a topological space such that *E**G* , then G is connected. In particular  is connected (i.e. closure of a connected set is connected).

**Theorem:** If *X* is a topological space such that given any pair *x, y* of distinct points of *X*, there is a connected set *E* in *X* containing *x* and *y* .Then *X* is connected.

**Theorem:** Let (*X, T*) be a topological space. If {*A*|} is a family of connected sets in *X* such that , then is connected.

**Components:** A subset *E* of a topological space *X* is called a component if (i) *E* is connected (ii) *E* is not properly contained in any connected set in *X*.

Or

A maximal connected set in *X* is called a component.

**Remark:** If *X* is connected space then *X* is the only component of *X*.

**Totally disconnected space:** A topological space *X* is said to be a totally disconnected space if given any two distinct points *a, b* in *X* there exist two nonempty open sets *A, B* in *X* such that *a**A*, *b**B*, *X =* *A**B*, *A**B =* .

**Remark:** Every totally disconnected space is a Hausdorff space.

**Example:** *X =* {*a*}, *T =* {*X*,}. Then *X* is not disconnected but *X* is totally disconnected.

**Remark:** If a totally disconnected space contains more than one point then the space is disconnected also.

**Theorem:** If *X* is totally disconnected space then its components are its singleton sets.

Or

Components of a totally disconnected space are its points.

**Theorem:** Let *X* be a compact Hausdorff space. *X* is totally disconnected iff *X* has an open base whose each member is closed also.

**Locally Connected Space:** A topological space *X* is said to be locally connected if given open set *G* and *x* in *G* there exists an open set *H* in *X* such that *x**H**G*.

**Theorem:** Let (*X, T*) be a locally connected space and *Y* be an open subspace of *X*. Every component *C* in *Y* is open in *X*. In particular, every component of *X* is open in *X*.

**Theorem:** For any arbitrary topological space *X*

1. Every point in a topological space *X* is contained in one and only one component of *X*.
2. Every nonempty connected set is always contained in a component.
3. Every nonempty connected set which is both open and closed is a component.
4. Every component in a topological space *X* is closed.

**PRODUCT SPACE**

**Theorem:** Let (*X, T*) and (*Y, V*) be two topological spaces. Then the collection B = {*G**H*|*G**T*, *H**V*} is a base for some topology of *X**Y*.

**Product Space:** Let (*X, T*) and (*Y, V*) be two topological spaces. Then the topology *W* whose base is B = {*G**H*|*G**T*, *H**V*} is called the product topology of *X**Y* and (*X**Y*, *W*) is called the product space of *X* and *Y*.

**Theorem:** Let (*X, T*) and (*Y, V*) be two topological spaces and B 1 a base for *T* and B 2 a base for V. Then B = {*B*1*B*2| *B*1 B 1, *B*2 B 2} is a base for the product topology *W* of *X**Y*.

**Projection Mapping:** The mapping *p*x : *X**Y* *X* : *p*x((*x, y*)) *= x* and *p*y: *X**Y* *Y* : *p*y((*x, y*)) *= y* for all (*x, y*) in XY are called the projections of *X**Y* onto *X* and *Y* respectively.

**Theorem:** Projections are continuous and open. The product topology *T* is the coarsest topology for which the projections are continuous.

**Example:** Let *X =* {*a, b, c*} and *T =* {*X*, , {*a*}}. *Y =* {*p, q, r, s*}, *V =* {*Y*, {*p*}, {*q*}, {*p, q*}, {*r, s*}, {*p, r, s*}, {*q, r, s*}}. Find a base for the product topology of *X* *Y*.

**Theorem:** The collection S = {*p*1-1(*U*)|*U* is open in *X*}{*p*2-1(*V*)|*V* is open in *Y*} is a subbasis for the product topology on XY

**Theorem:** Let *y*o*Y* and let *A = X* {*y*o}. Then the restriction of *p*x to *A* is a homeomorphism of the subspace *A* of *X**Y* onto *X*. Similarly the restriction *p*y to B = {*x*o}*Y* is a homeomorphism.

**Projections:** Let *X =*  = {*X*i|*i**I*}. Then the mapping *p*i: *X* *X*i defined by *p*i(*x*) *= x*i for all *x* in *X* is called the *ith* projection.

**Theorem:** Each projection map *p*i is continuous, open but not closed.

**Theorem:** Let *f* be a mapping of a space *Y* into a product space *X =* . Then *f* is continuous *p*io*f* :*Y X*i is continuous.

**Product Topology and Product Space:** Let (*X*i, *T*i) be a collection of topological spaces and *X =* . Then the topology *T* for *X* which has subbase the collection S = {*p*i-1(*G*i)|*iI, G*i*T*i} is called the product topology (or Tychonoff Topology) for *X* and (*X, T*) is called the product space .

**Remarks: (1)** The collection S is called the defining subbase for *T*. The collection B of all finite intersections of elements of S would then form a base for *T*.

**(2)** Since *p*i-1(*G*i) are open sets with respect to the product topology where *G*i is any open set in *X*i. It follows that the projection *p*i is a continuous map for each *i* in *I*.

**(3)** For countable collection of topological spaces {Xi| *i* = 1, 2,…, } the product space *X =*  = {*x =* (*x*1, …, *x*n, …)}. Moreover *p*a-1(*G*a} = *X*1*X*2…*X* *GaX*….

**(4)** A base for this topology is a set *B = * {*p*i-1(*G*i)|*i*} where *I’* is a finite subset of *I* and *G*i*T*i for some *I* in *I’.*

**Theorem:** Let (*X*i, *T*i) be an arbitrary collection of topological spaces and let *X =* . Let *T* be the topology for *X*. Then the following statements are equivalent: (**1**) *T* is the product topology for *X*; (**2**) *T* is the smallest topology for which the projections are continuous.

**EXERCISE - 1**

1. Define discrete topological space, homeomorphism, second countable space, sequentially compact metric space. 05
2. State and prove Lindelof’s theorem. 05
3. Continuous image of a connected space is connected. 05
4. One to one continuous mapping of a compact space onto a hausdorff space is a homeomorphism. 02, 05
5. Compact hausdorff space is normal. 02, 05
6. Define Topological space, compactness in metric space,

locally connected space, hausdorff space. 04

1. Continuous image of a compact space is compact. 04
2. Every second countable space is separable. 04
3. Components of a totally disconnected space X are the singleton subsets of X. 04
4. Homeomorphism is an equivalence relation. 04
5. A metric space is separable if it is second countable. 04
6. Let *T* be the collection of subsets of *N* consisting of empty set and all subsets of the form *Gm* = {*m*, *m* + 1, *m* + 2, …), *m**N*. Show that *T* is a topology for *N*. What are open sets containing 5. 04
7. Every compact subset in a hausdorff space X is closed. 03
8. Every subspace of a hausdorff space is a hausdorff. 03
9. If (*Z, W*) is a subspace of (*Y, V*) and (*Y, V*) is a subspace of (*X, T*).

Then (*Z, W*) is a subspace of (*X, T*). 03

1. Give an example of a *T*2-space which is not *T*3. 03
2. Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* be a 1-1 function from *X* onto *Y*.

Then *f* is a homeomorphism iff  = *f*() for every *E* *X*. 03

1. Define a regular space and prove that the property of a space being regular space is a topological property. 03
2. Prove that the closure of a set is union of that set and its derived set. 02
3. Show that a space whose topology has a countable base is separable. 02
4. The property of a space being a *T*2-space is a topological property. 02
5. *X* is topological space, *Y* is a hausdorff space. If *f* and *g* are continuous functions of *X* into *Y*.

Then the set {*x* *X*| *f*(*x*) *= g*(*x*)} is closed. 02

1. Let (*X, T* ) and (*Y, V*) be two topological spaces and *f* be a function from *X* to *Y*.

Then *f* is *T-V* continuous iff *f* ()  for every *E X*. 02

1. Let *d*1(*x, y*) *= min* {1, *d*(*x, y*)}. Show that *d*1 is a metric and *d* and *d*1 are equivalent. 02
2. A family *B* of sets is a base for a topology for the set *X =* {*G*|*G**B*} iff for every *G*1, *G*2 in *B* and every *x* in *G*1*G*2 there exists *G* in *B* such that *xG**G*1*G*2. 01

**True/False**

1. Every topological space is metrizable.
2. A subset of a topological space is open if it is closure of each of its points.
3. Every component of a topological space is open.
4. Every component of a locally connected space is a closed set.

**………………………………… …………………………………… ………………………………..**

**EXTRA**

**EXERCISE - 2**

1. **Indiscrete topology:** *I =* {X, }
2. **Discrete topology:** *D =* collection of all subsets of *X*.
3. **Particular point topology:** *pX* (fixed). Then *T =* {*A**X : pA*}{} .
4. **Excluded point topology:** *xX* (fixed). *T*x = {*AX: x**A*}{*X*}.
5. **Usual topology for R:** Let *R* be the set of real numbers. Let *U* consists of null set and those subsets *A* of *R* such that for every *x A* there exists a real number *p* *> 0* such that ] *x – p, x + p* [*A*. Then *U* is a topology for *R* called the *usual topology*.
6. **Usual topology on a metric space:** Let (*X, d*) be any metric space. Let *Td* consists of null set and all those subsets *G* of *X* such that for all *x* in *G* there is a real *r >* 0 such that *S*r(*x*) *G*. then *T*d is a topology called the metric topology.
7. **Co-finite topology or finite complement topology:**Let *X* be a infinite set. Let *T* consists of  and all those subsets of *X* whose complements are finite. Then *T* is a topology called the cofinite topology.
8. **Co-countable topology:** Let *X* be a set; let *T*C = {*AX*| *X – A* is either countable or is all of *X*}.
9. **Lower limit topology or right half open interval topology or RHO topology on R**(***Rl*):** Let *Rl* consists of null set and all those subsets *A* of *R* such that for every *x* in *A* there exists a right half open interval [*a, b*[ , *a < b* such that *x*[*a, b*[*A*.
10. **Upper limit topology:** Same defined as above.
11. **Left ray topology for R:** For each *a* in *R* define *L*a = {*x**R*|*x < a*} = (-, *a*) = open left ray of real numbers. The point *a* is called right and point of *L*a. Let *T* consist all possible left rays together with  and *R*.
12. *T =* {}{*R*}{ ]*a*,[ :*a**R*} is a topology for *R* called the **right ray topology** but *T*1 = {}{*R*}{ [*a*,[ :*a**R*} is not a topology for *R*.
13. Let *T* be the collection of subsets of *N* consisting of empty set and all subsets *G*m = {*m*, *m +* 1, *m +* 2, …}, *m**N*. Then *T* is a topology for *N*.
14. **Metrizable Spaces:** A topological space (*X*, *T*) is said to be metrizable if there is a metric *d* for *X* such that *T*d = *T*.
15. **Example:** Give an example of a topological space which is not metrizable.
16. **Answer:** Let *X =* {*a, b*}, *ab*. Define *T =* {*X*, , {*a*}}. Then (*X, T*) is a topological space. Let *d* be any metric on *X* and let *d*(*a, b*) *= r*. Since *ab, r >* 0. Then *S*r(*b*) = {*b*} and hence {*b*} is *d* open set but {*b*} is not *T*-open. Hence (*X, T*) is not metrizable.

**EXERCISE – 3**

1. Co-finite topology on a finite set is the same as the discrete topology.
2. Let *f : X* *Y* be a function from a nonempty set *X* into a topological space *Y*. If *T* is a topology on *Y*. Then {*f -*1(*G*)|*G**T*} is a topology on *X*.
3. = {*x* | each neighbourhood of *x* intersects *A*}.
4. *Ao*, *extA* and *Fr*(*A*) are disjoint and *X =* *A*o*ext*(*A*) *Fr*(*A*).
5. A subset *A* of a topological space *X* is closed iff it contains its boundary.
6. A subset of a topological space has empty boundary iff it is both open and closed.
7. A set *A* is perfect iff it is closed and has no isolated points.
8. A closed set is nowhere dense iff its complement is everywhere dense.
9. The boundary of a closed set is nowhere dense.
10. For (*R, U*), B *=* {(*a, b*)|*a, b**R*} is a base for *U*.
11. For the discrete space (*X, D*), B *=* {{*x*}|*xX*} is a base for *D*.
12. Indiscrete topology is the weakest topology on a set *X* and the discrete topology is the strongest topology on *X.*
13. For (*X, D*), *Bx =* {{*x*}} is local base at *x* in *X*.
14. For (*R, U*) and *x* in *R*, {(*x – r*, *x +r*) | *r >* 0} and {(*x –* 1/*n*, *x +* 1/*n*) | *n**N*} are local bases at *x*.
15. (*R, U*), (*X, D*) are first countable.
16. (*R, U*) is second countable. B *=* {(*r, s*)| *r, s* are rational} is a countable base for *U*.
17. (*X, D*) is not second countable.
18. Let *T* and *T’* be topologies for *X* which has a common base *B*. Then *T = T’*.
19. Every metric space is first countable because *B*x *=* {*S*1/n(*x*) | *nN*} is a countable local base at *x*.
20. *f:* (*R, T*)(*R, V*) such that *f* (*x*) = 1 for all *x* in *R* is continuous function.
21. If *f* is a mapping from a discrete space to any topological space, then *f* is continuous.
22. A mapping *f* from a topological space *X* to indiscrete space *I* is always continuous.
23. If *f:* (*X, T*)(*Y, V*) is continuous and *T\** is finer than *T*, then *f* is *T\*-V* continuous.
24. If *f :* (*X, T*)(*Y, V*) is continuous and *V\** is coarser than *V*, then *f* is *T-V\** continuous.
25. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.
26. If B is a basis for the topology of X then the collection B Y = {BY|BB } is a basis for the subspace topology on Y.
27. Is the collection T = {U | X – U is infinite or empty or all of X} a topology on X?
28. If X = {a, b, c}, T1 = {X,, {a}, {a, b}} and T2 = { X, , {a}, {b, c}}. Find the smallest topology containing T1 and T2,and the largest topology contained in T1 and T2.
29. Let { } be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections, and a unique largest topology contained in all.
30. A topological space (X, T) is said to be a **door space** if every subset of X is either open or closed. Give an example of door space.
31. {*a*} is closed in usual topology for *R*.
32. Give an example of a proper nonempty subset of a topological space such that it is both open and closed.
33. Give an example of a topological space different from the discrete spaces in which open sets are exactly the same as closed sets.
34. Hint: Let T consists of all those subsets G of R having the property that x in G implies – x in G. Then (R, T) is the required space.
35. Which of the following subsets of *R* are *U*-neighbourhoods of 1? (i) ]0, 2], (ii) ]0, 2[, (iii) [1, 2], (iv) [0, 2] – {1/2}
36. Consider the topology *T =* {*X*, , {*a*}, {*b, c*}} on *X* and *V =* {*Y*, , {*r*}, {*p, q*}} on *Y =* {*p, q, r*}. Find which of the mappings defined as follows are

**(i)** continuous, **(ii)** open, **(iii)** closed, **(iv)** bicontinuous, **(v)** homeomorphism

(**a**) *f* (*a*) *= r*, *f* (*b*) *= r*, *f* (*c*) *= r*, (**b**) *g* (*a*) *= p*, *g* (*b*) *= q*, *g* (*c*) *= p*,

(**c**) *h* (*a*) *= r*, *h* (*b*) *= p*, *h* (*c*) *= q*, (**d**) *I* (*a*) *= r*, *I* (*b*) *= q*, *I* (*c*) *= p,*

(**e**) *j* (*a*) *= p*, *j* (*b*) *= q*, *j* (*c*) *= r*.

**EXERCISE – 4 CONNECTED SPACE**

1. Let {*A*} be the collection of connected subsets of a space *X* such that no two members of {*A*} are mutually separated, then  is also connected.
2. Any two distinct components are mutually disjoint.
3. Every space is the disjoint union of its components.
4. Indiscrete space (*X, *) is connected.
5. If (*X, D*) has more than one point, then (*X, D*) is connected.
6. If (*X, T*) is disconnected and *T* *T*1 (i.e., *T*1 is finer than *T*) then (*X, T*1) is disconnected.
7. If (*X, T*) is connected and *T*1 is coarser than *T*, then (*X, T*1) is connected.
8. In every topological space singleton sets are connected, therefore components are nonempty.
9. (*X, D*) is totally disconnected.
10. (*X, D*) is locally connected.
11. Give two examples of locally connected spaces which are not connected.

**EXERCISE – 5 SEPERATION AXIOMS**

**To Space**

1. (X, D) is To but (X, I) is not To.
2. If T\* is finer that T, then T is To topology implies T\* is To topology.
3. The property of a space being a To space is a topological property.
4. X is To iff for any distinct arbitrary points x, y of X, the closure of {x} and {y} are distinct.

**T1-Space**

1. (R, U) is T1.
2. For a topological space X following are equivalent: (1) X is T1; (2) Every singleton set in X is closed; (3) Every finite set in X is closed; (4) The intersection of all neighbourhoods of an arbitrary point of X is a singleton.
3. Every metric space is T1.
4. Every finite T1 space is discrete.
5. A space (X ,T) is T1 iff T contaons the cofinite topology on X.
6. Every topology finer than a T1-topology on a set X is a T1-topology.
7. For any set X there exists a unique smallest topology T such that (X, T) is T1.
8. Let A be any subset of a T1 space X. Then x is accumulation point of A iff every open set containing x contains infinitely many distinct points of A.

**T2-Space**

1. Show that (X, D) is T2.
2. No indiscrete space consisting of at least two points is Hausdorff.
3. (R, U), (R, S) are Hausdorff.
4. Co-finite topology on an infinite set X is not T2.
5. Co-countable topology on an uncountable set is not T2.
6. Every metric space is T2.
7. T\* is finer than a T2-topology T, then T\* is T2.
8. Every singleton set in a T2 space is closed.
9. Every finite hausdorff space is discrete.
10. Every T2-space is T1 but not converse.
11. *X* is topological space, *Y* is a hausdorff space. If *f* and *g* are continuous functions of *X* into *Y*. Then the set {*x**X*| *f*(*x*) *= g*(*x*)} is closed.
12. The property of being a T2 space is a topological property.
13. Let (X, T) be a topological space and (Y, V) be a hausdorff space. Let f: XY be a one-one continuous mapping, then (X, T) is also a hausdorff.
14. Let (X, T) be a T2 space and f be a continuous map of X into itself. Then the set A = {xX| f(x) = x} is T-closed.

**Regular Space, T3-space**

1. X = {a, b, c}, T = {X, , {a}, {b, c}}. Then (X, T) is regular but not T3.
2. (R, U) is T3.
3. Every T3 space is T2 but not converse.
4. A topological space X is regular iff every point x in X and every nbd N of x, there exists nbd M of x such that N. In other words, a topological space is regular iff the collection of all closed nbds of x forms a local base at x.
5. The property of being a regular space is a topological property.
6. Every compact hausdorff space is a T3-space.
7. Every metric space is regular (T3).
8. Let A be any compacet subset of a regular space X. If G is an open set containing A, then there exists a closed set H such that AHG.

**Normal Space, T4 space**

1. X = {a, b, c}, T = {X, , {a}, {b, c}}. Then X is normal but not T2.
2. X is normal iff for any closed set F and open set G containing F, there exists an open set V such that FV and G.
3. Normality is a topological property.
4. (R, U) is T4.
5. Every T4 space is T3.
6. X = {a, b, c}, T = {X,, {a}, {b}, {a, b}}. Then X is normal but not regular.
7. Every compact regular space is normal.
8. The property of being a T4 space is a topological property.
9. Closed subspace of a normal space is normal
10. Every metric space is normal.