Assumptions&Derivation

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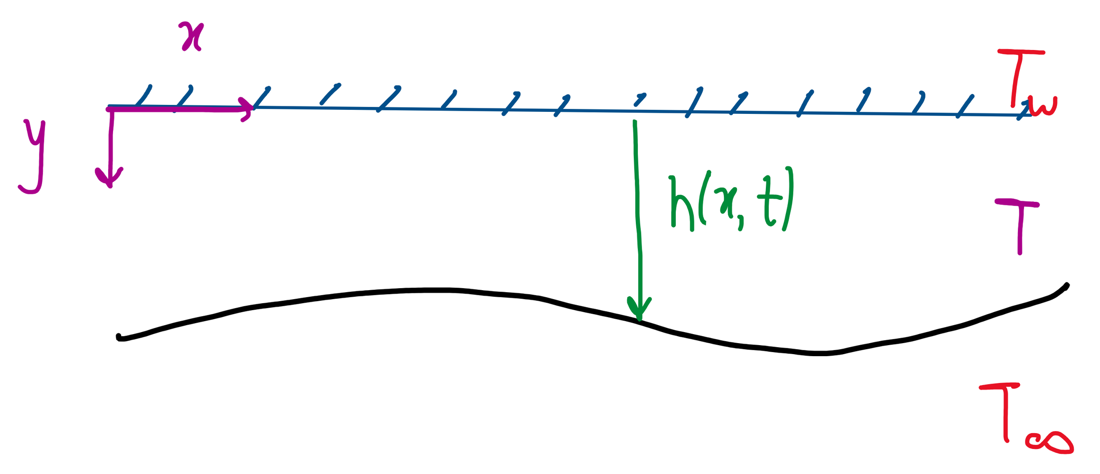

Assumptions :

i) Incompressible Newtonian Fluid

ii) No slip condition at the wall

iii) No applied pressure gradient $[G=0]$

iv) Long Wave approximation $\Rightarrow$ a) Low Reynold's number regime $[Re \sim O(1)]$
b) Length scale of variation along x is much larger than mean width along y

v) Negligible convection and steady state

vi) Temperature varies only along y in the bulk

vii) Wall temperature and ambient temperature are independent of x

Derivation :

Equation of Energy :

$$\rho \hat{C}_p \left[\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \mu \Phi_v$$

Neglecting convection, assuming steady state and that the temperature varies only along y in the bulk,

$$\frac{\partial^2 T}{\partial y^2} = 0 \qquad T = T_w \quad @\, y=0$$

$$\alpha \frac{dT}{dy} = -U(T - T_\infty) \quad @\, y=h$$

[Newton's Law of Cooling]

$$\tilde{T} = \frac{T - T_\infty}{T_w - T_\infty} \quad \text{[Scaling]}$$

$$\Rightarrow \frac{\partial^2 \tilde{T}}{\partial y^2} = 0 \qquad \tilde{T} = 1 \quad @\, y=0$$

$$\frac{\partial \tilde{T}}{\partial y} = -B_i \tilde{T} \quad @\, y=h$$

$$\frac{\partial \tilde{T}}{\partial y} = C_1 \qquad \therefore\ C_1 = -B_i \tilde{T}_h$$

$$\therefore\ \frac{\partial \tilde{T}}{\partial y} = -B_i \tilde{T}(h)$$

$$\Rightarrow \tilde{T} = -B_i \tilde{T}(h)\, y + C_2 \qquad \because\ T = 1 \quad @\, y=0$$

$$C_2 = 1$$

$$\Rightarrow \tilde{T} = 1 - B_i\left[\tilde{T}(h)\, y\right]$$

$$\Rightarrow \tilde{T}(h) = 1 - B_i\,(\tilde{T}(h))\, h$$

$$\Rightarrow \boxed{\tilde{T}(h) = \frac{1}{1 + B_i \cdot h}}$$

$$\mu \left.\frac{du}{dy}\right|_h = \left.\frac{d\gamma}{dx}\right|_h \qquad \gamma = \gamma_\infty\left(1 - \beta(T - T_\infty)\right)$$

[Tangential Stress Balance]

$$\Rightarrow \mu \left.\frac{du}{dy}\right|_h = -\beta\gamma_\infty \frac{dT(h)}{dx}$$

$$\Rightarrow \mu \frac{u_c}{H} \left.\frac{d\tilde{u}}{d\tilde{y}}\right|_h = -\beta\gamma_\infty \frac{(T_w - T_\infty)}{L} \frac{d\tilde{T}(h)}{d\tilde{x}}$$

$$\rightarrow \boxed{\left.\frac{d\tilde{u}}{dy}\right|_h = -M_a \frac{d\tilde{T}(h)}{d\tilde{x}}} \quad \text{B.C. 1.}$$

$$\rightarrow \boxed{\tilde{u} = 0 \quad @\, \tilde{y} = 0} \quad \text{B.C. 2}$$

for $\frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial x}$ [N.S. Equation]

$$\Rightarrow \frac{\partial u}{\partial y} = y \frac{\partial P}{\partial x} + C_1$$

$$\Rightarrow \left.\frac{\partial u}{\partial y}\right|_{y=h} = -M_a \frac{\partial T(h)}{\partial x} = \left.\left(y \frac{\partial P}{\partial x} + C_1\right)\right|_h$$

$$\Rightarrow C_1 = -M_a \frac{\partial T(h)}{\partial x} - y \frac{\partial P}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial y} = y \frac{\partial P}{\partial x} - h \frac{\partial P}{\partial x} - M_a \frac{\partial T(h)}{\partial x}$$

$$\Rightarrow u = \frac{y^2}{2} \frac{\partial P}{\partial x} - hy \frac{\partial P}{\partial x} - M_a\, y \frac{\partial T(h)}{\partial x} + \cancel{C_2}^{\,0}$$

$$\Rightarrow Q = \int_0^h u\, dy = \frac{\partial P}{\partial x}\left[\frac{h^3}{6} - \frac{h^3}{2}\right] - M_a \frac{h^2}{2} \frac{\partial T(h)}{\partial x}$$

$$\Rightarrow Q = -\frac{h^3}{3} \frac{\partial P}{\partial x} - M_a \frac{h^2}{2} \frac{\partial T(h)}{\partial x}$$

$$\Rightarrow Q = -\frac{h^3}{3}\left[-\tilde{B}_o \frac{\partial h}{\partial x} + \frac{\partial P_h}{\partial x}\right] - M_a \frac{h^2}{2} \frac{\partial T(h)}{\partial x}$$

$$\Rightarrow Q = -\frac{h^3}{3}\left[-\tilde{B}_o \frac{\partial h}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial^2 h}{\partial x^2}\right)\right] - M_a \frac{h^2}{2} \frac{\partial T(h)}{\partial x}$$

$$\Rightarrow \boxed{Q = \frac{h^3}{3}\left[\tilde{B}_o \frac{\partial h}{\partial x} + \frac{\partial^3 h}{\partial x^3}\right] - M_a \frac{h^2}{2} \frac{\partial T(h)}{\partial x}}$$

Linear Stability Analysis :

$h = 1 + \delta h'$

where $h' = A e^{\sigma t} e^{ikx}$

$$\frac{\partial h}{\partial t} = -\frac{\partial Q}{\partial x}$$

$$\sigma h' = -\frac{\partial}{\partial x}\left[\frac{1}{3}\left(\tilde{B}_o\, ik\, h' + (-ik)^3 h'\right) + \frac{1}{2} \frac{M_a \cdot B_i\ ikh'}{(1 + B_i + B_i \delta h')^2}\right]$$

$$\boxed{\sigma = \frac{1}{3}\left[\tilde{B}_o k^2 - k^4\right] + \frac{M_a \cdot B_i\ k^2}{2\ (1 + B_i)^2}}$$

Note that $M_a = \frac{\beta\gamma_\infty}{L}(T_w - T_\infty)$

where $\beta\gamma_\infty > 0$

1. If the wall is hotter, $T_w - T_\infty > 0 \Rightarrow \boxed{M_a > 0}$
With increasing $|T_w - T_\infty|$, surface tension γ decreases
Thus, increasing (+ve) Ma destabilises.

2. If the wall is colder, $T_w - T_\infty < 0 \Rightarrow \boxed{M_a < 0}$
With increasing $|T_w - T_\infty|$, surface tension γ increases
Thus, decreasing (-ve) Ma stabilises.