Bearings-Only and Doppler-Bearing Tracking Using Instrumental Variables

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Bearings-only tracking (BOT) or Doppler and bearing tracking (DBT) are common passive sonar problems. The measurement equations are nonlinear and in order to apply the Kalman filter, it is either necessary to linearize the equations or embed the nonlinearities into the noise terms. The former sometimes leads to filter divergence while the latter produces biased estimates. A new formulation of BOT and DBT is given which has a constant state vector, and simplifies the tracking problem to one of constant parameter estimation. The solution is by the instrumental variable method. The instrumental variables are obtained from predictions based on past measurements and are therefore independent of the present noisy measurements. The result is a recursive, unbiased estimator. The theoretical developments are verified by simulation, which also shows that the new formulation leads to near optimal estimators whose errors are close to the Cramer-Rao lower bound (CRLB).

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I. INTRODUCTION

Passive tracking is the determination of the trajectory of a target solely from measurements of signals originating from the target. These signals, for example, could be radar or sonar transmissions or machinery noise from a vessel and their detection is usually indicated by an increase in energy above the ambient at a certain bearing. The energy is mostly broadband but in some instances the signal spectrum may contain a few tonals as well. When only bearing measurements are available, the problem is known as bearings-only tracking (BOT) and if there are also frequency measurements on the tonals, Doppler-bearing tracking (DBT). The former, BOT, has been well investigated [1–4], while DBT is a relatively recent development [5, 6].

For successful BOT, the course and speed of the target must be constant, i.e., nonmaneuvering, while the observer must maneuver to ensure a determinant solution. The measurement equation relating target bearing to position is nonlinear and [4] reported divergence when the extended Kalman filter is applied to BOT in the Cartesian coordinates. The pseudolinear estimator (PLE) formulation proposed in [1] lumps the nonlinearities into the noise term, resulting in a linear measurement equation in a standard Kalman filter form. However, the measurement matrix contains elements that are functions of noisy bearings and are correlated with the noise terms. As a result, the PLE exhibits a bias which can be severe under certain target-observer geometries [2, 3]. To overcome this bias, [3] developed maximum likelihood and instrumental variable methods that gave satisfactory results. These are gradient-search based batch processors. They require an expanding memory to store the increasing number of measurements and their proper convergence is sensitive to initial conditions and step sizes.

In DBT, the observer can be stationary and still the tracking will be unique provided the target is not moving radially [5]. Intuitively, the additional frequency measurements provide extra target course and speed information, making DBT more observable than BOT. The measurement equations are also nonlinear but by an appropriate choice of the state vector elements [5], linear equations similar to the PLE were obtained. The same difficulty, of having measurement matrix elements correlated with the noise, is also present in DBT. Thus the DBT-PLE displayed a strong bias, resulting in an underestimated range [5]. The modified instrumental variable estimator was shown in [5] to be effective in removing the bias. But it is a batch processor and its convergence has not been proven.

We give an alternate instrumental variable (IV) solution to BOT and DBT. By choosing, in BOT, the initial target $x - y$ position and speed as the states, which are constants, the estimator need only estimate a constant parameter vector. Estimates for target positions at any other time are obtained by projection of the target equation of motion from the initial position. The same problem of correlation between measurement matrix elements (regressors) and noise persists here. By using the so-called instruments [7], which in our scheme are the predictions of the values of the present regressors based on past measurements, and hence are contemporaneously uncorrelated with the present noise, consistent estimates can be obtained. The same principle is applied to DBT. The estimator is recursive and is demonstrated to be able to attain the Cramér-Rao lower bound (CRLB), given a sufficient tracking time, in the simulation examples.

II. CONSTANT STATE-VECTOR FORMULATION

A. Bearings-Only Tracking

Let $x_T(i)$, $\dot{x}_T$, $y_T(i)$ and, $\dot{y}_T$ be the target position and speed in Cartesian coordinates at time iT, with T (constant) the time interval between measurements. The corresponding observer states are $x_0(i)$, $\dot{x}_0(i)$, $y_0(i)$ and $\dot{y}_0(i)$. Note that $\dot{x}_T$ and $\dot{y}_T$ are constants while $\dot{x}_0(i)$ and $\dot{y}_0(i)$ can be time dependent. Let

$$\beta_i = \overline{\beta}_i + e_i \tag{1}$$

be the measured bearing, relative to the y-axis of the observer, at i, with $\overline{\beta}_i$ the true bearing and e_i a zero mean random variable of variance $E\{e_i e_j\} = \delta_{ij}\sigma^2$. Then

$$\tan\overline{\beta}_i = \frac{x_T(i) - x_0(i)}{y_T(i) - y_0(i)} = \frac{\sin\overline{\beta}_i}{\cos\overline{\beta}_i}. \tag{2}$$

Substituting (1) into (2) and simplifying results in

$$x_0(i)\cos\beta_i - y_0(i)\sin\beta_i = x_T(i)\cos\beta_i - y_T(i)\sin\beta_i + \epsilon_i \tag{3}$$

where

$$\epsilon_i = r_i \tan e_i \tag{4}$$

and

$$r_i = [x_T(i) - x_0(i)]\sin\beta_i + [y_T(i) - y_0(i)]\cos\beta_i \tag{5}$$

is the target range relative to the observer. Equations (2) to (5) are similar to those in [1] that lead to the PLE.

Since

$$x_T(i) = iT\dot{x}_T + x_T(0) \tag{6}$$

and

$$y_T(i) = iT\dot{y}_T + y_T(0) \tag{7}$$

putting (6) and (7) into (3), for $i = 0,1,\ldots,k$, yields the matrix equation

$$\xi_k = A_k\theta + d_k. \tag{8}$$

In (8),

$$\zeta_k^T = [x_0(0)\cos\beta_0 - y_0(0)\sin\beta_0,\ldots,x_0(k)\cos\beta_k - y_0(k)\sin\beta_k] \tag{9}$$

$$A_k = \begin{bmatrix} \cos\beta_0 & 0 & -\sin\beta_0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \cos\beta_k & kT\cos\beta_k & -\sin\beta_k & -kT\sin\beta_k \end{bmatrix} \tag{10}$$

$$\theta^T = [x_T(0) \quad \dot{x}_T \quad y_T(0) \quad \dot{y}_T] \tag{11}$$

is the 4×1 constant state vector, and

$$d_k^T = [\epsilon_0 \cdots \epsilon_k]. \tag{12}$$

Equation (8) is now in the form where any parameter estimation algorithm can be applied to obtain $\hat{\theta}_k$, an estimate of θ at k. Of course, as discussed previously, due to the correlation of the elements of A_k and d_k, the standard least squares technique [7] will give a biased estimate. The IV method described in Section III is aimed at eliminating the bias. After $\hat{\theta}_k$ is obtained, the position estimates for any other time are simply computed from (6) and (7). It is well known that the linear least squares estimator (LLSE) is linear [8], i.e., if $\hat{x}_T(0)$ and $\hat{\dot{x}}_T$ are the LLSE of $x_T(0)$ and $\dot{x}_T$, then $\hat{x}_T(k) = kT\hat{\dot{x}}_T + \hat{x}_T(0)$ is also the LLSE of $x_T(k)$.

B. Doppler-Bearing Tracking

In sonar, it is common that a broadband noise is also accompanied by one or more tonals. These tonals are constant in frequency and because of Doppler shift, a particular frequency measured by the observer is given by

$$f_i^{(j)} = f_s^{(j)}\left(1 + \frac{V}{c}\right) + \epsilon_{f_i}^{(j)}. \tag{13}$$

In (13), $f_i^{(j)}$ denotes the jth ($j = 1,\ldots,p$) frequency measured by the observer at the ith instant, $f_s^{(j)}$ is the jth constant unknown source frequency, c is the speed of propagation of the signal, V is the relative radial velocity between the source and the observer (V is positive if source approaching, negative if receding) and $\epsilon_{f_i}^{(j)}$ is the random frequency measurement error. It has zero mean and variance $E\{\epsilon_{f_i}^{(j)}\epsilon_{f_m}^{(j)}\} = \sigma^2_{f^{(j)}}\delta_{im}$. Referring to Fig. 1 and after some trigonometry, (13) becomes

$$f_i^{(j)} = f_s^{(j)}\left(1 - \frac{\dot{x}_T - \dot{x}_0(i)}{c}\sin\overline{\beta}_i - \frac{\dot{y}_T - \dot{y}_0(i)}{c}\cos\overline{\beta}_i\right) + \epsilon_{f_i}^{(j)}. \tag{14}$$

Let

$$f_i = \sum_{j=1}^{p} f_i^{(j)} \tag{15}$$

$$f_s = \sum_{j=1}^{p} f_s^{(j)} \tag{16}$$

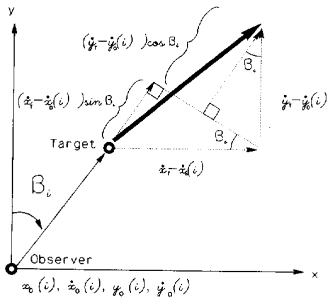

Fig. 1. Doppler-bearing tracking.

and

$$\epsilon_{f_i} = \sum_{j=1}^{p} \epsilon_{f_i}^{(j)}. \qquad (17)$$

Summing (14) from $j = 1$ to p and substituting in (1) results in

$$f_i = f_s - \frac{f_s \dot{x}_T}{c}\sin\beta_i + \frac{f_s \dot{x}_0(i)}{c}\sin\beta_i - \frac{f_s \dot{y}_T(i)}{c}\cos\beta_i + \frac{f_s \dot{y}_0(i)}{c}\cos\beta_i + \eta_i \qquad (18)$$

where

$$\eta_i = \epsilon_{f_i} - [q_i \sin\beta_i - \cos\beta_i \sin e_i]\left[\frac{f_s\dot{x}_T - f_s\dot{x}_0(i)}{c}\right] - [q_i\cos\beta_i + \sin\beta_i \sin e_i]\left[\frac{f_s\dot{y}_T - f_s\dot{y}_0(i)}{c}\right] \qquad (19)$$

and

$$q_i = \cos e_i - 1. \qquad (20)$$

Now define the state vector

$$\mu^T = [f_s \vdots f_s x_T(0) \vdots f_s \dot{x}_T \vdots f_s y_T(0) \vdots f_s \dot{y}_T] \qquad (21)$$

which is a constant. Combining the bearing and frequency measurement equations (3) and (18) gives

$$F_k = C_k \mu + \phi_k \qquad (22)$$

where

$$F_k^T = [f_0 \vdots 0 \cdots f_k \vdots 0] \qquad (23)$$

and

$$C_k = \begin{bmatrix} 1 + \frac{\dot{x}_0(0)\sin\beta_0}{c} + \frac{\dot{y}_0(0)\cos\beta_0}{c} & 0 & -\frac{\sin\beta_0}{c} & 0 & -\frac{\cos\beta_0}{c} \\ -x_0(0)\cos\beta_0 + y_0(0)\sin\beta_0 & \cos\beta_0 & 0 & -\sin\beta_0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 + \frac{\dot{x}_0(k)\sin\beta_k}{c} + \frac{\dot{y}_0(k)\cos\beta_k}{c} & 0 & -\frac{\sin\beta_k}{c} & 0 & -\frac{\cos\beta_k}{c} \\ -x_0(k)\cos\beta_k + y_0(k)\sin\beta_k & \cos\beta_k & kT\cos\beta_k & -\sin\beta_k & -kT\sin\beta_k \end{bmatrix} \qquad (24)$$

$$\phi_k^T = [\eta_0 \vdots \epsilon_0 \cdots \eta_k \vdots \epsilon_k]. \qquad (25)$$

The DBT formulation above is different from that in [5] which uses

$$\left[x_T(k) \vdots \dot{x}_T \vdots y_T(k) \vdots \dot{y}_T \vdots \frac{1}{f_s^{(1)}} \cdots \frac{1}{f_s^{(p)}}\right]$$

as the state vector. Besides not being constant, the dimension of the vector also increases with the number of tonals. The disadvantage of (22) is that individual estimates of the tonal frequencies are not available. However, [5] found that as the number of tonal increases, the accuracy of the frequency estimates also decreases. Similar to (8), (22) is now in the form such that an IV recursive algorithm can be applied to avoid biased estimates.

III. INSTRUMENTAL VARIABLE SOLUTION

Assuming that A_k in (8) is full rank, i.e., the observer has made a maneuver, then the ordinary least squares solution is

$$\hat{\hat{\theta}}_k = (A_k^T A_k)^{-1} A_k^T \xi_k. \qquad (26)$$

Substituting (8) into (26) results in

$$\hat{\hat{\theta}}_k = \theta + (A_k^T A_k)^{-1} A_k^T d_k. \qquad (27)$$

Now $E\{\hat{\hat{\theta}}_k\} \neq \theta$ because $E\{A_k^T d_k\} \neq 0$.

The optimal IV estimate for (8) is [9]

$$\hat{\theta}_k^* = (Z_k^{*T} A_k)^{-1} Z_k^{*T} \xi_k \qquad (28)$$

where

$$Z_k^* = \Omega_k^{-1} \overline{A}_k \qquad (29)$$

and

$$\Omega_k = E\{d_k d_k^T\} \qquad (30)$$

and $\overline{A}_k$ is the matrix A_k with the quantity β_i replaced by $\overline{\beta}_i$.

The covariance matrix of (28) is

$$\sum_k^* = E\{(\hat{\theta}_k^* - \theta)(\hat{\theta}_k^* - \theta)^T\} = (\overline{A}_k^T \Omega_k^{-1} \overline{A}_k)^{-1}. \qquad (31)$$

This estimate is optimal in the sense that $\sum_k - \sum_k^*$ is nonnegative definite, where $\sum_k$ is the covariance matrix of any other unbiased estimator. It is also shown in [7] that

$$\lim_{k \to \infty} E\{\hat{\theta}_k^*\} = \theta \tag{32}$$

so that $\hat{\theta}_k^*$ is consistent.

In practice, $\overline{A}_k$ and Ω_k are not available. Instead we use the suboptimal IV matrix

$$Z_k = \tilde{\Omega}_k^{-1} \tilde{A}_k \tag{33}$$

where $\tilde{A}_k$ is the A_k matrix with the β_j elements replaced by

$$\tilde{\beta}_j = \tan^{-1}\left[\frac{\hat{x}_T(0) + jT\hat{\dot{x}}_T - x_0(k)}{\hat{y}_T(0) + jT\hat{\dot{y}}_T - y_0(k)}\right]. \tag{34}$$

In (34), the estimates $\hat{x}_T(0)$, $\hat{\dot{x}}_T$, $\hat{y}_T(0)$, and $\hat{\dot{y}}_T$ are those obtained at $(j-1)T$, based on the measurements $i = 0,\ldots,j-1$. Hence $\tilde{\beta}_j$ is uncorrelated with β_j. It is easy to show, by using (4) in (30), that

$$\Omega_k = \mathrm{diag}(\sigma^2 r_0^2, \sigma^2 r_1^2 \cdots \sigma^2 r_k^2). \tag{35}$$

For $\tilde{\Omega}_k$, we replace r_j by the estimated range based on the $j-1$ past measurements. Thus it is clear that

$$\hat{\theta}_k = (Z_k^T A_k)^{-1} Z_k^T \xi_k \tag{36}$$

is consistent. In long range tracking scenarios, r_i is relatively constant so that $\Omega_k \simeq \sigma^2 r^2 I$, I is the identity matrix, and (36) reduces to

$$\hat{\theta}_k = (\tilde{A}_k^T A_k)^{-1} \tilde{A}_k^T \xi_k. \tag{37}$$

The question of the existence of $(\tilde{A}_k^T A_k)^{-1}$ has not been answered. Certainly, after an observer maneuver, $\overline{A}_k$ is full rank but there is no guarantee that $\tilde{A}_k^T A_k$ is invertible. Nevertheless, its invertibility is checked at every k by the recursive algorithm [8]

$$\hat{\theta}_k = \hat{\theta}_{k-1} + P_k z_k^T(\xi_k - a_k \hat{\theta}_{k-1}) \tag{38}$$

$$P_k^{-1} = P_{k-1}^{-1} + z_k^T a_k. \tag{39}$$

In (38) z_k is the kth row of the matrix Z_k, ξ_k the kth element of the vector ξ_k and a_k is the kth row of the matrix A_k. It is noted that (39) performs the computation

$$Z_k^T A_k = Z_{k-1}^T A_k + z_k^T a_k \tag{40}$$

so that the inverse $P_k = (Z_k^T A_k)^{-1}$ is taken at each k. The strategy taken in the simulation is that if the inverse does not exist (but this has not occured in any of the simulation runs), then the measured β_i is substituted for $\tilde{\beta}_i$ to ensure that Z_k is full rank.

The DBT-IV estimator is similar to (36) and is given by

$$\hat{\mu}_k = (\tilde{C}_k^T \tilde{\omega}_k^{-1} C_k)^{-1} \tilde{C}_k^T \tilde{\omega}_k^{-1} F_k \tag{41}$$

where $\tilde{C}_k$ and $\tilde{\omega}_k$ are the matrices C_k and ω_k with the elements that contain β_j replaced by $\tilde{\beta}_j$ that are

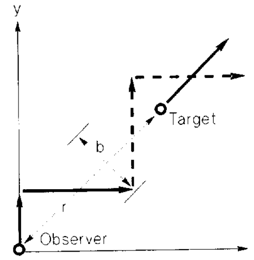

Fig. 2. Geometry for BOT simulations.

estimated from $j-1$ past measurements. The same recursive algorithm, (38) and (39), is used except that the measurement data come at two rows at a time for $\tilde{C}_k$ and two elements (one element is always zero (see (23)) at a time for F_k. The covariance matrix

$$\omega_k = E\{\phi_k \phi_k^T\} = \begin{bmatrix} \sigma_{\eta_0}^2 & \sigma_{\eta_0\epsilon_0} & & & 0 \\ \sigma_{\eta_0} & \sigma^2 r_0^2 & & & \\ & & \ddots & & \\ & & & \sigma_{\eta_k}^2 & \sigma_{\eta_k\epsilon_k} \\ & 0 & & \sigma_{\eta_k\epsilon_k} & \sigma^2 r_k^2 \end{bmatrix} \tag{42}$$

where, from (25), it is seen that

$$\sigma_{\eta_i}^2 = E\{\eta_i \eta_i\} \tag{43}$$

and

$$\sigma_{\eta_i\epsilon_i} = E\{\eta_i \epsilon_i\}. \tag{44}$$

IV. SIMULATION RESULTS

Several simulation experiments were conducted to verify the IV approach to BOT and DBT. For BOT, the tracking scenario is similar to the one given in [3]. In this geometry (Fig. 2), the observer is on a zig-zag path whereby constant course and speed legs are interrupted by maneuvers at $k = 50, 150, 250, 350$ such that the mean line-of-advance is along the target path. The range (r) to baseline (b) ratio is 10. The observer speed is constant at 12.7 m/s with 90° turns as indicated for a total of five maneuvers. Target speed is constant at 9 m/s on a constant bearing of 45°. The measurement period $T = 2$ s and there is a total of 400 measurements at the end of the track. The bearing measurements are corrupted by additive zero mean independent Gaussian random variables of variance σ^2. The recursive algorithm of (38) and (39) was used with the initial conditions

$$\hat{\theta}_0^T = [x_0(0) + 30\sin\beta_0 \vdots 0 \vdots y_0 + 30\cos\beta_0 \vdots 0] \tag{45}$$

and

$$P_0 = \mathrm{diag}\left[\frac{(60\sin\beta_0)^2}{3}\ \frac{40^2}{3}\ \frac{(60\cos\beta_0)^2}{3}\ \frac{40^2}{3}\right]. \tag{46}$$

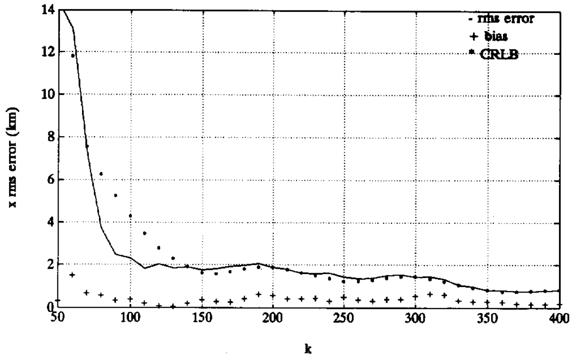

Fig. 3(a). BOT-IV filter performance, x-position, $\sigma = 1°$.

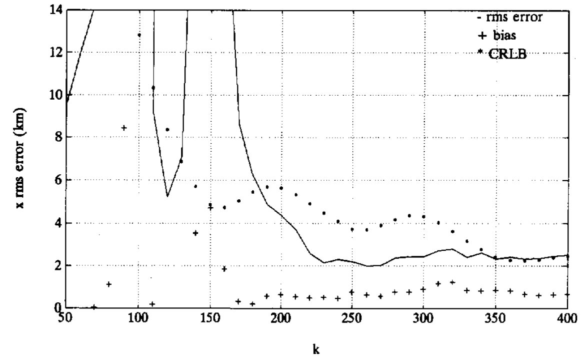

Fig. 4(a). BOT-IV filter performance, x-position, $\sigma = 3°$.

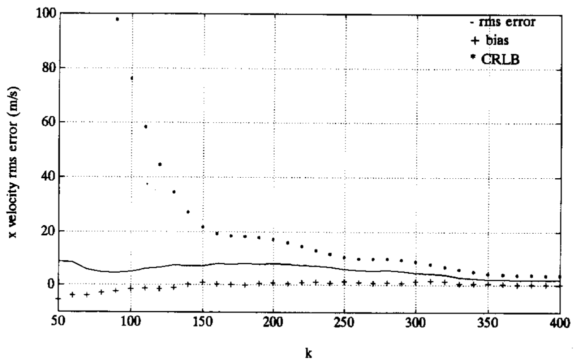

Fig. 3(b). BOT-IV filter performance, x-velocity, $\sigma = 1°$.

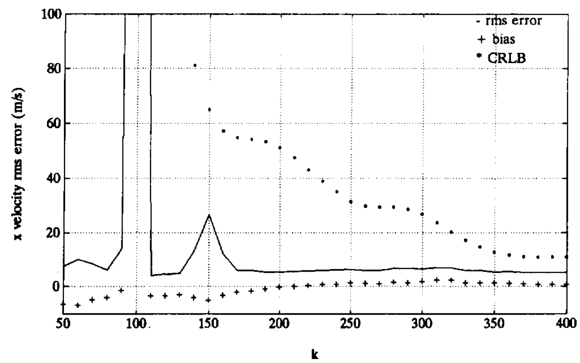

Fig. 4(b). BOT-IV filter performance, x-velocity, $\sigma = 3°$.

The initial value selections are based on a typical sonar tracking situation. The target relative position is assumed to be uniformly distributed between 0 and 60 km, the first convergence zone distance; the target relative speed is assumed to be uniformly distributed between ± 40 m/s. Hence the elements of $\hat{\theta}_0$ and P_0 are assigned the means and variances of their respective variables.

The root mean square errors (RMSE) and bias of the estimates are calculated over 60 independent runs and are plotted in Fig. 3 for $\sigma = 1°$. The CRLB, derived in the Appendix, is also included. Only the x-coordinate results are shown as the y-coordinate results are similar. The plots start at $k = 50$ because the track is not determinant until after the first observer maneuver at $k = 50$. It is seen in Fig. 3 that the RMSE are very close to the CRLB and the bias is approximately zero. For $\sigma = 3°$, the estimator exhibited divergence at the beginning of the track although it manages to bring the RMSE close to the CRLB at the end of the track. The results are given in Fig. 4.

In the previous experiment, the target range was relatively constant and the simpler IV solution (37) was used. The next tracking experiment has a geometry in which the range varies considerably during the track. The target trajectory in Fig. 2 is altered to have the target closing in toward the observer. Other

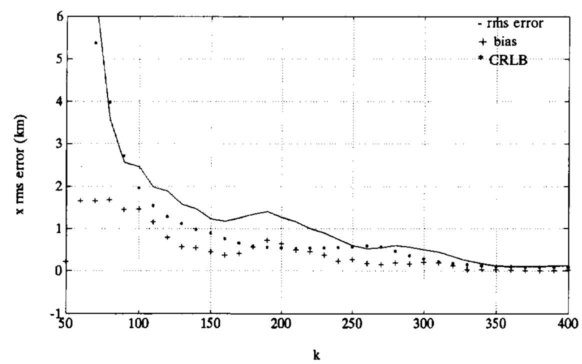

Fig. 5. BOT-IV filter, high range rate, $\sigma = 1°$.

observer-target parameters are identical to the first experiment. The IV of (33) is shown in Fig. 5 to give a near-optimal performance by having RMSE close to the CRLB.

Next, the DBT-IV estimator is tested by the scenario described in [5]. The observer is stationary at the origin and the target commences the run at $x_T(0) = 20$ km and $y_T(0) = -1.8$ km and proceeds due North at 9 m/s radiating a single 300 Hz tone. The frequency measurements are corrupted by random errors of $\sigma_f = 0.3$ Hz. The measurement interval $T = 5$ s and there is a total of 300 measurements at

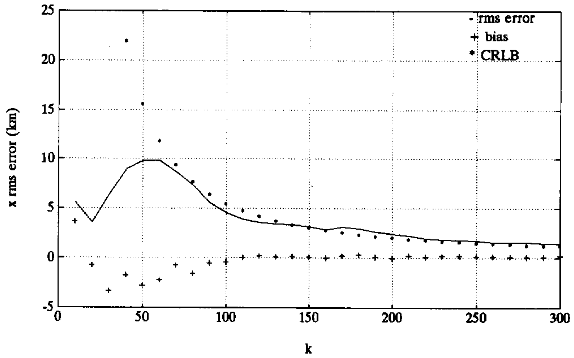

Fig. 6. DBT-IV filter performance, x-position, 1 sinusoid.

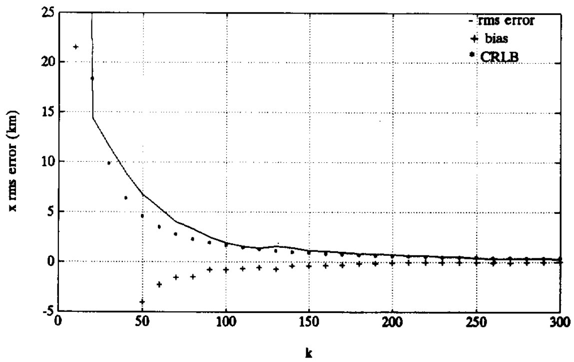

Fig. 7. DBT-IV filter performance, x-position, 3 sinusoids.

the end of the run. The initial conditions for target position and speed are the same as previous. The initial estimate of the frequency is taken as the first measurement. The initial frequency estimate variance is easily computed from (13), based on some maximum relative target velocity, which gives a bound on the frequency. The results in Fig. 6 show that the DBT-IV has negligible bias and attains the CRLB. For the final experiment, the target radiates 3 tones at 300, 900, and 1500 Hz. The frequency measurement of each frequency has an independent error variable of $\sigma_f = 0.3$ Hz. The state vector dimension remains at 5. With 3 sinusoids, there is a considerable improvement in performance. Comparing Figs. 6 and 7, the CRLB is much lower for the 3 sinusoid case. The method of summing the frequency measurements is verified by the proximity of the RMSE to the CRLB in Fig. 7.

V. CONCLUSIONS

We have introduced a constant parameter estimation approach to BOT and DBT. In BOT, the constant parameters are the initial target position and speed. Estimates for other positions are through projection of the target equation of motion. In DBT, the constant parameters are the initial target position and speed multiplied by f_s, the sum of the p source frequencies, and f_s. The parameter vector dimension does not change with the number of source frequencies. The main contribution of the formulation is that recursive IV techniques can be readily applied to give an unbiased estimate. The instrumental variables are computed from target position and speed estimates based on past measurements, which are independent of the present measurements. Thus the IV method avoids the difficulty of correlation between the regressors and the disturbances and produces an unbiased estimate. By formulating the BOT and DBT problems as estimation of constant parameters, the solution is readily obtained via many standard least squares recursive algorithms. Simulation results have demonstrated the effectiveness of the scheme. The estimates are unbiased and their RMSE are close to the CRLB.

APPENDIX. CRAMÉR-RAO LOWER BOUNDS

A. BOT

The CRLB gives the minimum variance that an unbiased estimator can achieve [8] from a series of noisy measurements. The target state vector

$$x^T = [x_T(j) \quad \dot{x}_T \quad y_T \quad (j) \quad \dot{y}_T] \tag{47}$$

is to be estimated from measurements

$$b^T = [\beta_0 \cdots \beta_i \cdots \beta_j] \tag{48}$$

with

$$\beta_i = \overline{\beta}_i + e_i \tag{49}$$

where the e_i are zero mean independent Gaussian random variables of variance σ^2. The Fisher information matrix is

$$\text{FIM} = \frac{1}{\sigma^2}\left(\frac{\partial b}{\partial x}\right)^T \left(\frac{\partial b}{\partial x}\right) \tag{50}$$

where

$$\frac{\partial b}{\partial x} = \begin{bmatrix} \dfrac{\partial \beta_0}{\partial x_T(j)} & \dfrac{\partial \beta_0}{\partial \dot{x}_T} & \dfrac{\partial \beta_0}{\partial y_T(j)} & \dfrac{\partial \beta_0}{\partial \dot{y}_T} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial \beta_j}{\partial x_T(j)} & \dfrac{\partial \beta_j}{\partial \dot{x}_T} & \dfrac{\partial \beta_j}{\partial y_T(j)} & \dfrac{\partial \beta_j}{2\dot{y}_T} \end{bmatrix}. \tag{51}$$

From the relationship

$$\beta_i = \tan^{-1}\left[\frac{x_T(j) - (j-i)T\dot{x}_T - x_0(i)}{y_T(j) - (j-i)T\dot{y}_T - y_0(i)}\right] \tag{52}$$

we obtain the partial derivatives

$$\frac{\partial \beta_i}{\partial x_T(i)} = \frac{y_T(i) - y_0(i)}{r_i^2} \tag{53}$$

$$\frac{\partial \beta_i}{\partial \dot{x}_T} = -(j-1)T\left(\frac{\partial \beta_i}{\partial x_T(j)}\right) \tag{54}$$

$$\frac{\partial \beta_i}{\partial y_T(j)} = \frac{-(x_T(i) - x_0(i))}{r_i^2} \tag{55}$$

$$\frac{\partial \beta_i}{\partial \dot{y}_T} = -(j-i)T\left(\frac{\partial \beta_i}{\partial y_T(j)}\right) \qquad (56)$$

where

$$r_i^2 = [x_T(i) - x_0(i)]^2 + [y_T(i) - y_0(i)]^2. \qquad (57)$$

The CRLB for the first element of (47) then is $(\mathrm{FIM})^{-1}_{1,1}$, which is the first main diagonal element of the inverse of the FIM.

B. DBT

The state vector is

$$y^T = [f_s \mid x_T(j) \mid \dot{x}_T \mid y_T(j) \mid \dot{y}_T] \qquad (58)$$

and the measurement vectors are

$$b^T = [\beta_0 \cdots \beta_i \cdots \beta_j] \qquad (59)$$

$$f^{(\ell)^T} = [f_0^{(\ell)} \cdots f_i^{(\ell)} \cdots f_j^{(\ell)}], \qquad \ell = 1,\ldots,p. \qquad (60)$$

Each frequency measurement contains a noise term which is a zero mean Gaussian random variable of variance $\sigma^2_{f^{(\ell)}}$. The FIM now is

$$\mathrm{FIM}(y) = \frac{1}{\sigma^2}\left(\frac{\partial b}{\partial y}\right)^T\left(\frac{\partial b}{\partial y}\right) + \sum_{\ell=1}^{p}\frac{1}{\sigma^2_{f^{(\ell)}}}\left(\frac{\partial f^{(\ell)}}{\partial y}\right)\left(\frac{\partial f^{(\ell)}}{\partial y}\right) \qquad (61)$$

from which the CRLB for y is found.

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