CS 5785 - Applied Machine Learning - Lec. 4

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1 Conditional Expectation is the Best Regression Model

As we can see, kNN and OLS(ordinary least squares) are some regression model, but which one is the best regression model? We should consider one that minimize $R(f) = E[\ell(Y, f(x))]$,in which X,Y are random variables representing new random examples.
We can start by doing a preliminary warm up:Let's consider given a random variable Y, which $c \in R$ minimize $E[(Y-c)^2]$?

$$\begin{aligned}\frac{\partial}{\partial c}E[(Y-c)^2] &= E[\frac{\partial}{\partial c}(Y-c)^2] \\ &= E[2(c-Y)] \\ &= 2c - 2EY\end{aligned}$$

So that we need to solve $2c - 2EY = 0$,which leads to$c* = EY$. In a nutshell, the mean(average) is the single constant number that's simultaneously closest to all values of a random Y in average squared distance.
Now we can consider minimizing the risk based on our preliminary method, which is:

$$R(f) = E[(Y - f(x))^2] = E[E[((Y - f(x))^2|x]]$$

And what should $f(x)$ be? We can tell according to the warm up conclusion that$f*(x) = E[Y|X = x]$ is the optimal prediction.
OLS regression estimating the conditional mean, and conditional expectations, probabilities and modes are the primary targets of Supervised Learning.

2 Linear Model for Classification

2.1 Log Odds

Focus on binary classifcation $G = \{0, 1\}$.
Recall, Bayes classifier declares $\hat{Y} = 1$ when $Pr(Y = 1|X = x) > Pr(Y = 0|X = x)$.

Look at the odds ratio:

$$OR = \frac{\Pr(Y=1|X=x)}{\Pr(Y=0|X=x)} \in [0, \infty]$$

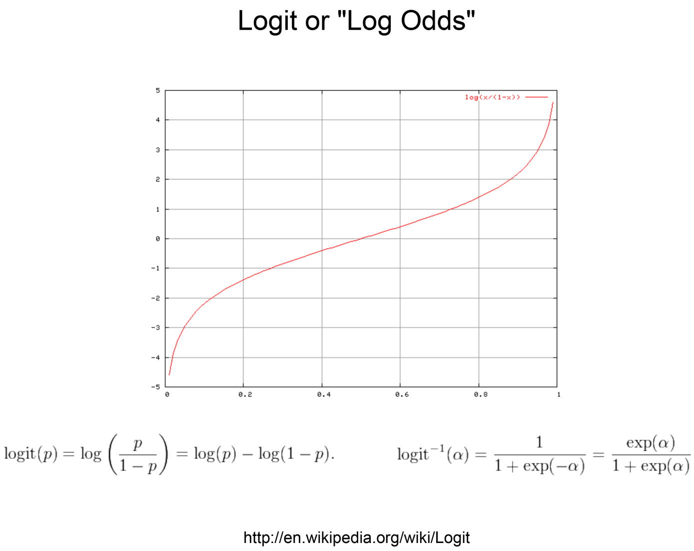
Then, to transform it to a number in $(-\infty, \infty)$:
Log odds:

$$log(\frac{\Pr(Y=1|X=x)}{\Pr(Y=0|X=x)})$$

$$= log(\frac{\Pr(Y=1|X=x)}{1-\Pr(Y=1|X=x)})$$

$$= logit(\Pr(Y=1|X=x))$$

where $logit(P) = log(\frac{P}{1-P}) = log(\frac{1}{\frac{1}{P}-1}) = log(P) - log(1-p)$.
So, the domain of the logit is $P \in [0,1]$, and the co-domain is $[-\infty, \infty]$.
This means that $logit(\Pr(Y=1|X=x))$ is a score for declaring $\hat{Y}=1$ that is symmetric. When it is positive, we can declare $\hat{Y}=1$; when it is negative, we can declare $\hat{Y}=0$.



2.2 Logistic Regression

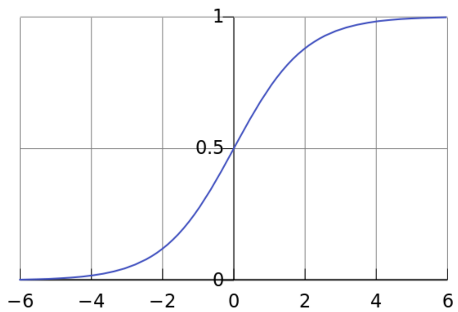
Posit $Logit(P(Y=1|X=x)) = \beta^\top X \Rightarrow P(Y=1|X=x) = \sigma(\beta^\top X)$
where σ is the logistic sigmoid: $\sigma(z) = logit^{-1}(z) = \frac{1}{1+exp(-z)} = \frac{exp(z)}{1+exp(z)}$
σ gives a way to transform a score $\beta^\top X$ into a probability. You can see:

$$\sigma(0) = \frac{1}{2}$$

$$\sigma(\infty) = 1$$

$$\sigma(-\infty) = 0$$
$$\sigma(-z) = 1 - \sigma(z)$$
$$\frac{\partial \sigma}{\partial z} = \frac{-1}{(1+e^{-z})^2} \cdot (-e^{-z}) = \frac{e^-z}{(1+e^{-z})^2}$$
$$= \frac{1}{1+e^{-z}} \cdot \frac{e^{-z}}{1+e^{-z}} = \frac{1}{1+e^{-z}} \cdot (1 - \frac{1}{1+e^{-z}}) = \sigma(z) \cdot (1 - \sigma(z))$$

Standard logistic sigmoid function



$$P(t) = \frac{1}{1+e^{-t}}$$
$$\frac{d}{dt}P(t) = P(t) \cdot (1 - P(t)).$$
$$1 - P(t) = P(-t).$$

http://en.wikipedia.org/wiki/Logistic_regression

2.3 Fitting Logistic Regression: Maximum Likelihood

Logistic regression provides a generative model for the data. That is, it provides a model that specifies how the data was generated given x. Logistic regression specifies $P(Y = 1|X = x) = \sigma(\beta^\top X)$. That is, draw an X, get a $\sigma(\beta^\top X)$; flip a biased coin with that probability of a head and label the example heads or tails accordingly. Alternatively, it can be thought of a such: generate $\mathcal{U} \sim Uniform[0, 1]$ and label 1 if $\mathcal{U} < \sigma(\beta^\top X)$, and otherwise label 0.

Assuming that the data is independent, for every β there is a particular likelihood for observing the data we did:

$$Lik(\beta) = P(X_1, Y_1, ..., X_n, Y_n; \beta)$$
$$= \prod_{i=1}^{N} P(X_i, Y_i; \beta)$$
$$= \prod_{i=1}^{N} P(Y_i|X_i; \beta)P(X_i)$$
$$= \prod_{i=1}^{N} P(X_i)(\begin{cases} \sigma(\beta^\top X_i) & ifY_i = 1 \\ 1 - \sigma(\beta^\top X_i) & ifY_i = 0 \end{cases})$$

Max likelihood's principle is choose parameters that maximize the likelihood of observing the data we observed.
Since log is monotonic increasing, so we can conclude that:

$$argmaxLik(\beta) = argmaxlog(Lik(\beta)) = argmin[-log(Lik(\beta))]$$

And:

$$-log(Lik(\beta)) = \sum_{i=1}^{N}(-logP(X_i) - (\begin{cases} \sigma(\beta^\top X_i) & ifY_i = 1 \\ 1 - \sigma(\beta^\top X_i) & ifY_i = 0 \end{cases}))$$

$$= -\sum_{i=1}^{N} logP(X_i) + \sum_{i=1}^{N}(Y_i(-log\sigma(\beta^\top X_i) + (1 - Y_i)(-log(1 - \sigma(\beta^\top X_i))))$$

In this part, we can define negative log likelihood function as:

$$\mathcal{L}(\beta) = \sum_{i=1}^{N}(Y_i(-log\sigma(\beta^\top X_i) + (1 - Y_i)(-log(1 - \sigma(\beta^\top X_i))))$$

So basically $argmaxLik(\beta)$ is equal to $argmin\mathcal{L}(\beta)$now
Since:

$$log(\sigma(\beta^\top X_i)) = log(\frac{e^{\beta^\top X_i})}{1 + e^{\beta^\top X_i}}$$
$$= \beta^\top X_i - log(1 + e^{\beta^\top X_i})$$

And:

$$log(1 - \sigma(\beta^\top X_i)) = log(\sigma(-\beta^\top X_i))$$
$$= log(\frac{1}{1 + e^{\beta^\top X_i}}$$
$$= -log(1 + e^{\beta^\top X_i})$$

So finally we get:

$$\mathcal{L}(\beta) = \sum_{i=1}^{N}(Y_i log(1 + e^{\beta^\top X_i}) - Y_i\beta^\top X_i + (1 - Y_i)log((1 + e^{\beta^\top X_i}))$$
$$= \sum_{i=1}^{N}(log(1 + e^{\beta^\top X_i}) - Y_i\beta^\top X_i)$$